(a) Give Equation (1) $H = X(X^TX)^{-1}X^T$, prove that $\text{trace}(H) = q$. assuming matrices are invertible where indicated. (Hint: First prove the following identity. If A, B are $n \times m$ matrices, then $\text{trace}(AB^T) = \text{trace}(B^TA)$).

Suppose A and B are the same dimension. a_{ij} is the ij^{th} element of A b_{ij} is the ij^{th} element of B. AB^T is $n \times n$ matrix.

Diagonal: $[AB^T] = \sum_{j=1}^{m} a_{ij} b_{ij} \quad i = 1 \dots n$

$$\text{Trace}(AB^T) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} b_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} a_{ij} = \text{trace}(B^TA).$$

Now we need to proof $\text{trace}(H) = q$:

We have $H = \underbrace{X}_{\substack{A \\ n\times q}}\underbrace{(X^TX)^{-1}X^T}_{\substack{B^T \\ q\times n}}$

$$\text{trace}(H) = \text{trace}(X(X^TX)^{-1}X^T)$$
$$= \text{trace}(\underbrace{X^TX}_{\substack{B \\ q\times q}}\underbrace{(X^TX)^{-1}}_{\substack{A \\ q\times q}})$$
$$= \text{trace}(I_q) = q$$

(b). A square matrix A is idempotent if and only if $A = AA$. Show that H is idempotent (i) Use matrix algebra following Equation (1). (ii) Use the fact that Hy is the point in S_q closest to y.

$H y = y$ $\qquad X(X^TX)^{-1}X^T \cdot X(X^TX)^{-1}X^T = \underbrace{X(X^TX)^{-1}X^TX}_{I}(X^TX)^{-1}X^T$

$H(Hy) = Hy$ $\qquad = XI(X^TX)^{-1}X^T = X(X^TX)^{-1}X^T = H.$

$Hy = Hy$ $\qquad$ Since $H = HH$ when matrix H is idempotent $Hy = Hy$ when Hy is the point in S_q closest to y. So H is idempotent.

(c) In each case, describe precisely H', and give its trace.

(i) $\hat{y} = (\bar{y}, \dots \bar{y}) \in \mathbb{R}^n$, where $\bar{y}$ is the sample mean of the elements of y.

$\hat{y} = H'y = (\bar{y}, \dots \bar{y})$

$$\begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \bar{Y} \\ \bar{Y} \\ \bar{Y} \\ \vdots \end{bmatrix}$$

$\text{trace}(H') = \sum_{i=1}^{n} \frac{1}{n} = 1$

(ii) $\hat{y}_1 = \frac{y_1 + y_2}{2}$. $\hat{y}_i = \frac{y_{i-1} + y_i + y_{i+1}}{3}$, $i = 2, 3, \dots n-1$.

$\hat{y}_n = \frac{y_{n-1} + y_n}{2}$.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \frac{y_1 + y_2}{2} \\ \frac{y_{i-1}+y_i+y_{i+1}}{3} \\ \vdots \\ \frac{y_{n-1} + y_n}{2} \end{bmatrix}$$

$\text{trace}(H') = \frac{n+1}{3} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} \times n$

(iii) $\hat{y} = y$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ & & \vdots \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_n \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_n \end{bmatrix}$$

$\text{Trace}(H') = n$