IDEAL 2-BODY MOTION IN THE GRAVITATIONAL FIELD OF CLASSICAL MECHANICS

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Abstract

By establishing an appropriate plane rectangular coordinate system to describe the motion of a particle in a centered gravitational field, and solving the dynamic differential equation from the properties of classical mechanics gravitation and general curve motion, the planar polar coordinate expression of the trajectory is obtained.

And select a specific mathematical method to prove the conservation of angular momentum in a special two-body gravitational system, the expression of gravitational potential energy, and Kepler's law of planetary motion.

1 Derivation of Celestial Orbits

Let a particle of mass m be subject to the gravitational force of $M(m \ll M)$, it is easy to see that the motion path of m determines a plane.

In a known plane, the position of a point can be uniquely determined by two linearly independent degrees of freedom parameters, such as cartesian coordinates $\{\hat{x}, \hat{y}\}$, polar coordinates $\{\rho, \theta\}$, etc.

Now establish the plane rectangular coordinate system Oxy. Let $M(0,0)$[1], $m(r\cos\theta, r\sin\theta)$. Where r, θ satisfy the functional relation $r = r(t)$; $\theta = \theta(t)$. The diameter vector (or position vector) of M to m is denoted as $\boldsymbol{r} = (r\cos\theta, r\sin\theta)$ ($|\boldsymbol{r}|$ is r, similarly hereinafter).

The diameter vector $\boldsymbol{r}$ is satisfied

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{1.0.1}$$

Taking the derivative[2] of time t gives the linear velocity $\boldsymbol{v}$:

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases} \tag{1.0.2}$$

Take the derivative again, and we get the acceleration $\boldsymbol{a}$:

$$\begin{cases} \ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta \\ \ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta \end{cases} \tag{1.0.3}$$

The vector representation of gravity in classical mechanics is:

$$\boldsymbol{F}_{\mathrm{G}} = -G\frac{Mm}{r^2} \cdot \frac{\boldsymbol{r}}{|r|} \tag{1.0.4}$$

1.1 Law of Conservation of Angular Momentum

Define the angular momentum of a particle m: $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}$, corresponding scalar expression $L = m(x\dot{y} - \dot{x}y)$. The function derivative rule $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$ gives

$$L = m \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{x}\right) \cdot x^2 = mr^2 \cdot \dot{\theta} \tag{1.1.1}$$

[1] Unless otherwise specified, study objects are represented and distinguished by their mass.

[2] The derivative $\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ of the coordinate function with respect to time t are denoted as $\dot{x}, \ddot{x}$ respectively.

The rate of change of momentum $\boldsymbol{L}$ with respect to time is the moment $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$. In a centered gravitational field, $\boldsymbol{r} \,//\, \boldsymbol{F}_{\mathrm{G}}$, so $\boldsymbol{M} = \boldsymbol{0}$. Which is $\dfrac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{r} \times \boldsymbol{v}) = \boldsymbol{0} \Rightarrow \mathrm{d}\boldsymbol{L} = \boldsymbol{0}$. $\boldsymbol{L}$ is a conserved quantity. This is the **Law of Conservation of Angular Momentum** in a centered force field.

1.2 The Solution of Orbital Equation

In the direction of x, y, we have

$$\begin{cases} \ddot{x} = -\dfrac{MG}{r^2}\cos\theta \\ \\ \ddot{y} = -\dfrac{MG}{r^2}\sin\,\theta \end{cases} \tag{1.2.1}$$

Thus $\dfrac{\ddot{x}}{\cos\theta} = \dfrac{\ddot{y}}{\sin\theta} = -\dfrac{MG}{r^2}$. Which is $\ddot{r} - r\dot{\theta}^2 - (2\dot{r}\dot{\theta} + r\ddot{\theta})\tan\theta = \ddot{r} - r\dot{\theta}^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cot\theta$

$$(2\dot{r}\dot{\theta} + r\ddot{\theta})(\tan\theta + \cot\theta) = 0 \tag{1.2.2}$$

Where $\tan\theta + \cot\theta = \sec\theta\csc\theta \not\equiv 0$, so $2\dot{r}\dot{\theta} + r\ddot{\theta} \equiv 0$. Whereupon

$$-\frac{MG}{r^2} = \ddot{r} - r\dot{\theta}^2 = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 \tag{1.2.3}$$

Let us call $r' = \dfrac{\mathrm{d}r}{\mathrm{d}\theta}$. From the formula $L = mr^2 \cdot \dot{\theta}$, we get $\dfrac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}r}{\mathrm{d}t}\right) = \dfrac{L}{mr^2} \cdot \dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{L}{m} \cdot \dfrac{1}{r^2} \cdot \dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right) = \dfrac{L^2}{m^2 r^2}\left[\dfrac{r''}{r^2} - \dfrac{2(r')^2}{r^3}\right]$. $r \cdot \left(\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \dfrac{L^2}{m^2 r^3}$, thus

$$\frac{r''}{r^2} - \frac{2(r')^2}{r^3} - \frac{1}{r} = -\frac{MG}{L^2} \cdot m^2 = -P \quad (P = \text{const.}) \tag{1.2.4}$$

$$r'' - \frac{2(r')^2}{r} + P \cdot r^2 - r = 0$$

Take $u = r^{-1}$ to eliminate the quadratic differential term, then $r' = \dfrac{\mathrm{d}r}{\mathrm{d}u} \cdot u'$. The above formula can be reduced to $-\dfrac{u''}{u^2} + \dfrac{2(u')^2}{u^3} - 2u \cdot \dfrac{(u')^2}{u^4} + \dfrac{P}{u^2} - \dfrac{1}{u} = 0$.

$$u'' + u - P = 0 \quad \text{Characteristic equation: } r_0^2 + 1 = 0. \tag{1.2.5}$$

It is easy to get a set of special solutions (the initial phase is 0): $u(\theta) = q\cos\theta + P$ (q is an undetermined coefficient). $r(\theta) = \dfrac{1}{u(\theta)} = \dfrac{1}{P + q\cos\theta}$.

$$r(\theta) = \frac{P^{-1}}{1 + \dfrac{q}{P}\cos\theta} = \frac{L^2}{MG \cdot m^2} \cdot \frac{1}{1 + e\cos\theta} \triangleq \frac{p}{1 + e\cos\theta}{}^{3} \tag{1.2.6}$$

This is the plane polar coordinate equation of a conic curve. Let $\theta = 0$ when $r = R$; $L = mv_0 R$. Then we get $q = \dfrac{v_0^2 R - MG}{v_0^2 R^2}$. $e = \dfrac{v_0^2 R}{MG} - 1$. The following conditions exist:

1) $v_0 = \sqrt{\dfrac{MG}{R}}$, $e = 0$.

   The orbit is a perfectly circular (**First Cosmic Velocity**);

[3] p and e are called semi-diameter (product of eccentricity and focal distance) and eccentricity, respectively.

2) $v_0 \in (\sqrt{\dfrac{MG}{R}}, \sqrt{\dfrac{2MG}{R}})$, $e \in (0,1)$.

The orbit is an ellipse with the long axis on the x axis (**Kepler's First Law**);

3) $v_0 = \sqrt{\dfrac{2MG}{R}}$, $e = 1$.

The orbit is a parabola with the axis of symmetry x axis (**Second Cosmic Velocity**);

4) $v_0 > \sqrt{\dfrac{2MG}{R}}$, $e \in (1, +\infty)$.

The orbit is one (left) branch of the hyperbola with the real axis on the x axis.

2 Expression of Gravitational Potential Energy

Let m be transferred from $\boldsymbol{r} = \boldsymbol{r}_0$ to $\boldsymbol{r} = \boldsymbol{r}_1$ under the gravitational action of M, and the work done by the gravitational force (conservative force) is

$$W = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} G\frac{Mm}{r^3} \cdot \boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} = G\frac{Mm}{r}\Big|_{r_0}^{r_1} = G\frac{Mm}{r_1} - G\frac{Mm}{r_0} \tag{2.0.1}$$

Define the position function **Gravitational Potential Energy** E_p such that $W = -\Delta E_\mathrm{p}$. Choose infinity as the zero potential energy point, i.e $\lim\limits_{r\to\infty} E_\mathrm{p}(r) = 0$, From the above formula

$$E_\mathrm{p}(r) = -G\frac{Mm}{r} \tag{2.0.2}$$

In particular, if m can escape the gravitational influence of M, that is, v_0 reaches escape velocity, then the mechanical energy of the system is conserved as follows

$$\frac{1}{2}mv_0^2 - G\frac{Mm}{R} = 0 \Rightarrow v_0 = \sqrt{\frac{2MG}{R}} \tag{2.0.3}$$

This is consistent with the second cosmic velocity obtained in 1.2.

3 Mathematical Proof of Kepler's Laws of Planetary Motion

For the proof of Kepler's First Law, see section 1.2.

3.1 Kepler's Second Law

By the Law of Conservation of Angular Momentum, the component of m velocity perpendicular to $\boldsymbol{r}$ is satisfied $v_\varphi = \dfrac{L}{mr} = \dfrac{v_0 R}{r}$. Select a time infinitesimal element $t \to r + \mathrm{d}t$. the displacement length in $\mathrm{d}t$ is $\mathrm{d}l = v_\varphi \, \mathrm{d}t$, the area swept by the radial vector $\boldsymbol{r}$ is $\mathrm{d}S = \dfrac{1}{2} r \cdot \mathrm{d}l$. So sweep velocity $u_S = \dfrac{\mathrm{d}S}{\mathrm{d}t} = \dfrac{1}{2}v_0 R$. This is a fixed value for fixed initial conditions. That is, **in a certain time Δt, the area ΔS swept by the radial vector is equal (Kepler's Second Law)**. The same result can be proved by

$$u_S = \frac{\Delta S}{\Delta t} = \frac{\displaystyle\int_{\theta_0}^{\theta_1} \frac{1}{2} r^2 \, \mathrm{d}\theta}{\displaystyle\int_{\theta_0}^{\theta_1} \frac{r^2}{v_0 R} \, \mathrm{d}\theta} = \frac{1}{2} v_0 R \tag{3.1.1}$$

3.2 Kepler's Third Law

When $v_0 \in (\sqrt{\frac{MG}{R}}, \sqrt{\frac{2MG}{R}})$, the motion orbit is an ellipse, and the corresponding plane rectangular coordinate trajectory equation is

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0, a^2 = b^2 + c^2). \tag{3.2.1}$$

eccentricity $e = \frac{c}{a} = \frac{v_0^2 R}{MG} - 1$, $a = R + c$, $b^2 = a^2 - c^2 = R^2 + 2cR$. From this, or from the combination of conservation of angular momentum and conservation of mechanical energy, $v_0 = \sqrt{\frac{MG}{a}} \cdot \frac{a+c}{b}$ is obtained, we can find:

$$\begin{cases} a = R \cdot \dfrac{MG}{2MG - v_0^2 R} \\ b = R \cdot \sqrt{\dfrac{v_0^2 R}{2MG - v_0^2 R}} \\ c = R \cdot \dfrac{v_0^2 R - MG}{2MG - v_0^2 R} \end{cases} \tag{3.2.2}$$

The period of motion is $T = \frac{S}{u_S} = \frac{2\pi ab}{v_0 R}$, $T^2 = \frac{4\pi^2 (MG)^2 R^3}{(2MG - v_0^2 R)^3}$, $a^3 = R^3 \cdot \frac{(MG)^3}{(2MG - v_0^2 R)^3}$, thus

$$\frac{a^3}{T^2} = \frac{MG}{4\pi^2} \tag{3.2.3}$$

For different particles moving around the same object (ignoring gravity and other influences on each other), they have the same $\frac{a^3}{T^2}$. This is the **Kepler's Third Law**.

References

[1] AMZG. Model of Gravitation in A Plane[R/OL]. (2020-8-3)[2023-10-14]. https://zhuanlan.zhihu.com/p/166216476.

[2] SengWong. Two-body Problem[R/OL]. (2020-8-24)[2023-10-14]. https://zhuanlan.zhihu.com/p/193107193.