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Pythagorean fuzzy linear programming technique for multidimensional analysis of preference using a squared-distance-based approach for multiple criteria decision analysis

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ABSTRACT

Pythagorean fuzzy (PF) sets involving Pythagorean membership grades can befittingly manipulate inexact and equivocal information in real-life problems involving multiple criteria decision analysis (MCDA). The linear programming technique for multidimensional analysis of preference (LINMAP) is a prototypical compromising model, and it is widely used to carry on decision-making problems in many down-to-earth applications. In LINMAP, the employment of squares of Euclidean distances is a significant technique that is an effective approach to fit measurements. Taking the advantages of a newly developed Euclidean distance model on the grounds of PF sets, this paper initiates a beneficial concept of squared PF Euclidean distances and studies its valuable and desirable properties. This paper aims to establish a squared Euclidean distance (SED)-based outranking approach and develop a novel PF LINMAP methodology for handling an MCDA problem under PF uncertainty. In the SED-based outranking approach, a novel SED-based dominance index is proposed to reflect an overall balance of a PF evaluative rating between the connection and remotest connection with positive- and negative-ideal ratings, respectively. The properties of the proposed index are also analyzed to exhibit its efficaciousness in determining the dominance relations for intracriterion comparisons. Moreover, this paper derives the comprehensive dominance index to determine the overall dominance relation and defines measurements of rank consistency for goodness of fit and rank inconsistency for poorness of fit. The PF LINMAP model is formulated to seek to ascertain the optimal weight vector that maximizes the total comprehensive dominance index and minimizes the poorness of fit under consideration of the lowest acceptable level and specialized degenerate weighting issues. The practical application concerning bridge-superstructure construction methods is conducted to test the feasibility and practicability of the PF LINMAP model. Over and above that, a generalization of the proposed methodology, along with applications to green supplier selection and railway project investment, is investigated to deal with group decision-making issues. Several comparative studies are implemented to further validate its usefulness and advantages. The application and comparison results display the effectuality and flexibility of the developed PF LINMAP methodology. In the end, the directions for future research of this work are represented in the conclusion.

1. Introduction

Multiple criteria decision analysis (MCDA) scrutinizes making preference judgments among candidate alternatives that are evaluated by multiplex, usually conflicting, criteria (Liang et al., 2019; Lin et al., 2019). MCDA problems are mainly concerned with diverse categories of decision-making issues, such as evaluation, prioritization, and selection, to give assistance in reaching the best compromise solution in many real applications (Akram et al., 2020a; Chen, 2018a; Chen, 2019b). MCDA models and methods play an important role in comparing, choosing, or ranking various competing alternatives based on specific criteria (Haghighi et al., 2019). Nonetheless, in numerous practical MCDA problems, an important matter in question that complicates the decision-making process is uncertainty, especially in intricate and

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unpredictable circumstances (Chen, 2019a; Liang et al., 2019; Lin et al., 2019). Because of a complex nature, a considerable amount of research in the MCDA field has focused on the development of appropriate techniques for dealing with decision-making uncertainty. One of the useful tools employed in addressing uncertain MCDA problems is Pythagorean fuzzy (PF) sets (Yager & Abbasov, 2013; Yager, 2013, 2014, 2016).

The concept of PF sets generalizes the notions of Atanassov's intuitionistic fuzzy sets (Atanassov, 1986) to portray complicated uncertain information more effectively (Akram et al., 2020a, 2020b; Mandal & Ranadive, 2019). The theory of intuitionistic fuzzy sets, in which the sum of degrees of membership and non-membership is less than or equal to one, is the most widespread nonstandard fuzzy theory (Chen, 2019a). In contrast, PF sets composed of Pythagorean membership grades meet the situation that the sum of square degrees of membership and non-membership is less than or equal to one while the sum is not required to be less than one (Yager & Abbasov, 2013; Yager, 2014; Zhang & Xu, 2014a). Owing to the relaxed prerequisite condition, PF sets can furnish more adaptability and flexibility than intuitionistic fuzzy sets for treating imprecise and ambiguous information and modeling the equivocation and vagueness in practice (Chen, 2019a; Mandal & Ranadive, 2019; Wang et al., 2019). Because it is beneficial to exploit PF sets to describe highly complicated uncertain information in realistic MCDA problems, numerous scholars have paid attention to extend the existing decision-making models and techniques to make them adapt a more complicated environment involving Pythagorean fuzziness (Ding et al., 2019; Liang et al., 2019; Wang et al., 2019). For example, Akram et al. (2020b) extended the elimination and choice translating reality method (ELECTRE) to PF contexts and presented a group decision-making approach via the PF ELECTRE procedure. Furthermore, Chen (2020) originated new Chebyshev distance measures in PF settings and carried forward an extended PF ELECTRE approach. Biswas and Sarkar (2019) developed a new PF technique for order preference by similarity to ideal solutions (TOPSIS) to address group decision-aiding issues involving unknown weight information through entropy measures. Ho et al. (2020) advanced a Pearson-like correlation-based TOPSIS method under PF uncertainties with an interval-valued format. Akram et al. (2020a) exploited complex PF TOPSIS and complex PF ELECTRE to manipulate group decision-making problems. Büyüközkan and Göçer (2019) integrated the complex proportional assessment and analytic hierarchy process (AHP) in PF contexts and explored the partner selection issue for digital supply chain. Shete et al. (2020) employed PF AHP to appraise the enablers of sustainable supply chain innovation. Chen (2019a) advanced a PF combinative distance-based precedence procedure and constructed an inventive preference ranking organization method for enrichment evaluations (PROMETHEE). Liang et al. (2020) proposed an expanded linear assignment method through the utility of partitioned fuzzy measures for MCDA with PF information. Mandal and Ranadive (2019) introduced the concept of PF preference relations for managing group decision analysis issues. Wang et al. (2019) established a novel group decision-making technique by use of PF linguistic variable Hamy mean operators and the Vise Kriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method. Yu et al. (2019) utilized an extensive TOPSIS on grounds of interval-valued PF sets to carry out multiple criteria group decision analysis for sustainable supplier selection. Zeb et al. (2019) initiated the concepts of credible extended PF sets and possible extended PF sets and provided an approach to MCDA with risk preference.

There are many studies in the literature for the development and enrichment of the MCDA models and methods in uncertain PF contexts. However, there is limited research on an extension of the linear programming technique for multidimensional analysis of preference (LINMAP) to uncertain environments comprising of Pythagorean fuzziness. The LINMAP, launched by Srinivasan and Shocker (1973), is a renowned MCDA methodology to estimate the importance weights of criteria and decide the best compromise choice. The prominent attribute of LINMAP is the preference structure and information in regards to alternatives provided by the decision maker (Gou et al., 2019; Haghighi et al., 2019; Xue et al., 2018). Numerous MCDA models and techniques require the decision maker to reveal the intensity of relative preferences among multiple criteria and then determine the grades of relative importance via a weight assessment technique. In contrast, LINMAP postulates the decision maker to display subjective preference between two alternatives (Liu et al., 2018; Song et al., 2018). By way of explanation, the input information of the LINMAP methods comprises a set of imposed consequences for paired preference of alternatives, and the weights of criteria and the ideal solutions are a priori unknowns. Supported by the given preference relations between alternatives, an easy-to-use linear programming method that aspires to attain the minimum inconsistency is formulated to determine the optimal criterion weights and the best compromise solution (Gou et al., 2019). The traditional LINMAP methodology has been advanced to a variety of uncertain surroundings, such as in the settings of hesitant fuzzy sets (Wan et al., 2017; Zhang & Xu, 2014b), hesitant fuzzy linguistic term sets (Xu et al., 2016), double hierarchy hesitant fuzzy linguistic term sets (Liu et al., 2018), probabilistic linguistic term sets (Liao et al., 2017), intuitionistic fuzzy sets (Wan & Li, 2014), interval-valued intuitionistic fuzzy sets (Wan & Li, 2015; Zhang, Ju, & Liu, 2016), PF sets (Wan et al., 2018; Xue et al., 2018), interval-valued PF sets (Xue et al., 2018), interval type-2 fuzzy sets (Haghighi et al., 2019; Qin et al., 2017), interval grey numbers (Song et al., 2018), and heterogeneous information (Zhang, Ju, Liu, & Giannakis, 2017).

The LINMAP methodology plays an essential role in information fusion of preference relations on alternatives and identification of the optimal weights and the best compromise solution for decision aiding and support. Nevertheless, the LINMAP methodology has not yet fully explored in the existing MCDA literature in uncertain circumstances on grounds of PF sets. Wan et al. (2018) and Xue et al. (2018), prior to the present time, have developed innovative LINMAP-based methods in the PF context. Based on fuzzy truth degrees, information entropy, and a cross-entropy optimization model, the characteristic of the LINMAP-based method presented by Wan et al. (2018) is that it exploits the inconsistency indices in accordance with the ideal solutions. To be specific, it incorporates the negative-ideal solution into the determination of the criterion weights. Additionally, the prominent feature of the PF LINMAP model originated by Xue et al. (2018) is consideration of the PF entropy that can conjecture the PF uncertain information. Their proposed linear programming model can acquire the solution result with the maximum consistency and reliability towards decision data. By reason of information insufficiency (e.g., lack of complete information, lack of clarity of information, and lack of consistent information), PF sets have been increasingly used in real-world MCDA problems, especially in a complicated and highly uncertain environment. However, except for Wan et al. (2018) and Xue et al. (2018), very few of the studies have focused on the LINMAP models and techniques with complex PF uncertain information. This question constitutes the primary motivation of this paper.

Notably, the PF LINMAP approaches presented by Wan et al. (2018) and Xue et al. (2018) consist of many complicated and interconnecting concepts, which would be intricate and not easily understood by decision makers. As an illustration, the noticeable core concept of their developed techniques is information entropy. Their methods are helpful and comprehensive to address group decision-making issues in PF uncertain circumstances. However, their mathematical programming models are rather troublesome to implement for decision makers because their approaches require a lot of sophisticated concepts, such as cross-entropy and amount of knowledge. Additionally, the influence on the solution results and the technical character in the main LINMAP structure are not always well understood for decision makers, especially for less knowledgeable or less skilled practitioners. On account of these concerns, this paper would like to propose an easy-to-use approach for enriching the LINMAP methodology within the PF context and addressing MCDA problems in complex circumstances of Pythagorean fuzziness. In contrast to the approaches proposed by Wan et al. (2018) and Xue et al. (2018), this paper attempts to exploit some simple concepts such as squared PF Euclidean distances to manipulate Pythagorean membership grades and quantify the amount of rank consistency and rank inconsistency. This paper puts forward a novel squared-distance-based approach in PF settings and develops a new PF LINMAP method that is very clear and comprehensible for the decision maker.

There are several ways to measure separation between PF information (e.g., Pythagorean membership grades or PF sets). The square of the standard Euclidean distance, which is recognized as the squared Euclidean distance (SED), is of central importance in comparing the closeness between various data points. Although the Euclidean distance can measure the ordinary distance between two positions in space, it is troublesome as there pertain to costly square and square root operations. In contrast, SED is not a metric for the reason that it does not satisfy the triangle inequality. Nonetheless, the Pythagorean theorem is simpler through the agency of squared distance because there is no square root, which implies that the square of Euclidean distance is differentiable. SED is a more general notion of distances; it is useful for comparing distances and for measuring similarity. In this regard, this paper utilizes a newly developed Euclidean distance model on grounds of PF sets to present a novel measurement of squared PF Euclidean distances and explores some important and desirable properties. An alternative approach based on squared PF Euclidean distances can be exploited to preclude the computation of square root and save the computational expenses in decision-making processes. Regarding the usefulness of Pythagorean membership grades in describing the ambiguous and uncertain decision-making details as well as the efficacy of the LINMAP-based methods, the basic structure of LINMAP is advanced and enriched to the PF environment, and a new PF LINMAP model is proposed by identifying the SED-based dominance index and providing a SED-based outranking approach. To reduce the computational complexity in manipulating PF data, this paper shows how SED can be effectively incorporated into the central procedure of the PF LINMAP methodology by means of several new concepts of (comprehensive) dominance indices, measures of rank consistency, measures of rank inconsistency, and fit measurements (consisting of goodness of fit and poorness of fit) in PF uncertain surroundings.

The purpose of this research is to launch an efficacious PF LINMAP methodology using the conception of SED-based dominance indices and a SED-based outranking technique for handling an MCDA question containing PF uncertain information. The feature of the classical LINMAP is that the decision maker would reveal a set of paired preference discernments over alternatives as input, while the output comprises a set of the optimal criterion weights. Accordingly, this paper attempts to address an MCDA problem which is formulated by PF evaluative ratings and the decision maker's paired preference judgments towards alternatives within the uncertain environment of Pythagorean fuzziness. The Euclidean distance model is an essential part of the LINMAP methodology because the square of weighted Euclidean distances is employed to appraise the poorness and goodness of fit in the LINMAP model. This research utilizes a newly developed Euclidean distance measure on grounds of Pythagorean membership grades to present the concept of squared PF Euclidean distances and investigate several essential and valuable properties. Based on squared PF Euclidean distances from ideal ratings, this paper defines the SED-based dominance index and constructs a SED-based outranking approach for conducting intracriterion comparisons among alternatives. The outranking relationships based on the SED-based outranking approach builds a solid foundation of the theoretical framework of the PF LINMAP methodology. This paper specifies the comprehensive dominance index to measure comprehensive dominance relations among alternatives in terms of multiple criteria. By use of such comprehensive indices, this paper provides measurements of rank consistency and rank inconsistency with respect to each ordered pair of alternatives for the sake of determining fit measurements. This paper identifies the degrees of goodness of fit and poorness of fit by synthesizing consistency measures and inconsistency measures, respectively, over all ordered pairs. Next, this paper would like to originate a PF LINMAP model, which seeks to ascertain the optimal weights for criteria that maximizes the total comprehensive dominance index and minimizes the degree of poorness of fit under consideration of the lowest acceptable level and specialized degenerate weighting issues. Solving the optimization problem, the optimal criterion weights and degrees of violation for each ordered pair would be derived to generate the priority orders of competing alternatives and identify the best compromise choice. Based on the aforementioned concepts, a new PF LINMAP methodology using a SED-based outranking technique can be formulated to solve complex uncertain MCDA problems comprising Pythagorean fuzziness. Subsequently, a down-to-earth application relating to an MCDA issue of choosing a bridge-superstructure construction method is investigated to corroborate the workableness and effectuality of the PF LINMAP model. Furthermore, this paper implements certain comparative studies consisting of the first sensitivity analysis concerning the non-negative boundary conditions of criterion weights, the second sensitivity analysis concerning the relative significance of the total comprehensive dominance in regards to the poorness of fit under various lowest acceptable levels. Over and above that, the proposed methodology would be generalized to tackle group decision-making problems for enriching more efficaciousness in practice. This paper executes several valuable comparative analyses concerning the application results yielded by other pertinent MCDA methods. The comparison results can verify the flexibility and advantages of the PF LINMAP methodology.

The main contributions of the developed PF LINMAP methodology are outlined as follows: (i) The proposed SED-based dominance index can reflect an overall counterpoise between the association towards the positive-ideal PF evaluative rating and the remotest association towards the negative-ideal PF evaluative rating; (ii) The developed SED-based outranking approach can effectively manipulate complex PF information and determine (comprehensive) dominance relations among PF evaluative ratings and among alternatives; (iii) The new fit measurements in the PF context can synthesize the measures of rank consistency for goodness of fit and rank inconsistency for poorness of fit over all ordered pairs, which are reliable, comprehensive, and in accordance with the paired preference relations; (iv) The proposed PF LINMAP methodology can manage MCDA and group decision-making problems despite the existence of non-transitive or inconsistent relations among pairwise preferences of alternatives; and (v) The novel PF LINMAP optimization model can coordinate the maximal objective of the total comprehensive dominance index and the minimal objective of the degree of poorness of fit, which guarantees that the obtained PF LINMAP rankings can retain the original preference judgments indicated by the decision maker as much as possible.

The remainder of this paper is organized as below. Section 2 delivers certain fundamental concepts that are in connection with PF sets. Based on squared PF Euclidean distances, Section 3 proposes a SED-based dominance index to develop a SED-based outranking approach on grounds of PF sets. Section 4 establishes the PF LINMAP approach based on new concepts of the comprehensive dominance index, the measures of rank consistency/inconsistency, the degree of goodness of fit, and the degree of poorness of fit. An effective linear programming model is formulated to acquire the optimal criterion weights and determine the best compromise alternative. Section 5 depicts a realistic MCDA problem concerning bridge-superstructure construction to display the calculation procedure of the proposed methods and techniques. A comparison analysis with other relevant approaches and some comparative discussions in different parameter settings are executed to perform the applicability and adaptability of the developed PF LINMAP methodology. Furthermore, a generalization to the group decision-making field is further investigated; moreover, two practical applications of green supplier selection and railway project investment are implemented for making more comprehensive comparisons. Section 6 conveys the conclusions and some directions for future research.

2. Preliminary

Yager (2013, 2014, 2016) and Yager and Abbasov (2013) introduced a class of nonstandard membership grades that are named as Pythagorean membership grades, which can be delineated by five parameters: the degrees of membership, non-membership, and indeterminacy, the strength of commitment about membership, and the direction of commitment (Chen, 2019a, 2019b; Li & Zeng, 2018; Yager & Abbasov, 2013; Yager, 2014; Zeng et al., 2018). Zhang and Xu (2014a) depicted a helpful mathematical representation towards PF sets. On the grounds of the representation, Chen (2019a, 2019b) provided more comprehensive definitions in connection with Pythagorean membership grades in PF settings. Due to the duality of the strength of commitment and the degree of indeterminacy, Chen (2019a, 2019b) considered the four parameters (i.e., degrees of membership and non-membership, strength, and direction) to contribute an operationally mathematical representation with relevance to a Pythagorean membership grade. This section delivers several fundamental concepts including PF representations and scalar-valued functions. These notations and conceptions are necessary for the succeeding analysis.

Definition 1. (*(Chen, 2019a, 2019b; Yager & Abbasov, 2013; Yager, 2014)*) A PF set P is delineated by a set of ordered parameters involving the degree of membership $\mu_P(x)$, the degree of non-membership $\nu_P(x)$, the strength of commitment $r_P(x)$, and the direction of commitment $d_P(x)$, in a finite universe of discourse X, where $\mu_P(x), \nu_P(x), r_P(x), d_P(x) \in [0, 1]$; it is expressed as below:

$$P = \{ \langle x, (\mu_P(x), \nu_P(x); r_P(x), d_P(x)) \rangle \, | x \in X \}, \tag{1}$$

subject to the succeeding constraint:

$$0 \leqslant (\mu_P(x))^2 + (\nu_P(x))^2 \leqslant 1 \tag{2}$$

The degree of indeterminacy $\tau_P(x) \in [0, 1]$ of the element $x \in X$ to the set P is stated as below:

$$\tau_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2} \tag{3}$$

Definition 2. (*(Chen, 2019a, 2019b; Yager & Abbasov, 2013; Yager, 2014)*) Let X be a finite universe of discourse. A Pythagorean membership grade, which is denoted by p, of element $x \in X$ in a PF set P is represented as below:

$$p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x)); \tag{4}$$

moreover, the corresponding parameters are real-valued functions depending on x, as below:

$$\mu_P(x) = r_P(x) \cdot cos(\theta_P(x)), \tag{5}$$

$$\nu_P(x) = r_P(x) \cdot sin(\theta_P(x)), \tag{6}$$

$$r_P(x) = \sqrt{(\mu_P(x))^2 + (\nu_P(x))^2}, \tag{7}$$

$$d_P(x) = \frac{\pi - 2 \cdot \theta_P(x)}{\pi}, \tag{8}$$

where $\theta_P(x)$ is remarked in radians in the range $[0, \pi/2]$.

The Pythagorean membership grade p is featured by four parameters, i.e., membership $\mu_P(x)$, non-membership $\nu_P(x)$, strength $r_P(x)$, and direction $d_P(x)$, according to the above definition. Moreover, the radians $\theta_P(x)$ in p can be determined by providing $d_P(x)$, by $\mu_P(x)$ and $r_P(x)$, or by $\nu_P(x)$ and $r_P(x)$, in the following manner:

$$\theta_P(x) = \frac{\pi}{2}(1 - d_P(x)) = arccos\left(\frac{\mu_P(x)}{r_P(x)}\right) = arcsin\left(\frac{\nu_P(x)}{r_P(x)}\right) \tag{9}$$

Fig. 1 depicts a geometrical explication about the Pythagorean membership space. By way of illustration, the Pythagorean membership grade $p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x))$ is regarded as a point (in the upper right quadrant) on a circle of radius $r_P(x)$. The direction $d_P(x)$ signifies on a commitment scale in [0, 1] to reveal the extent of the strength $r_P(x)$ at pointing towards membership. The strength $r_P(x)$ is completely towards membership and non-membership in the cases of $d_P(x) = 1$ and $d_P(x) = 0$, respectively, while an intermediate value of $d_P(x)$ represents partial support towards membership and non-membership. Furthermore, $\mu_P(x)$ and $\nu_P(x)$ actually related through Pythagorean complements with reference to $r_P(x)$ due to $(\mu_P(x))^2 + (\nu_P(x))^2 = (r_P(x))^2 \cdot (cos^2(\theta_P(x)) + sin^2(\theta_P(x))) = (r_P(x))^2$. Over and above that, the complement p^c of p is stated as follows:

$$p^c = (\mu_{P^c}(x), \nu_{P^c}(x); r_{P^c}(x), d_{P^c}(x)) = (\nu_P(x), \mu_P(x); r_P(x), 1 - d_P(x)) \tag{10}$$

To determine a scalar value affiliated with the Pythagorean membership grade p, Yager (2013, 2014) exploited the Takagi–Sugeno approach to initiate the scalar-valued function $V(p)$ on account of fuzzy rule bases. Zeng et al. (2018) investigated four existing comparison methods for ranking Pythagorean membership grades, consisting of the approaches by way of score functions, using score and accuracy functions, using closeness indices, and using scalar-valued functions. They indicated that the ranking result based on the scalar-valued function is more credible than that obtained by the other comparison methods. Because the effectiveness of the scalar-valued function has been demonstrated by Zeng et al. (2018), one can conveniently employ $V(p)$ to compare the magnitude of Pythagorean membership grades. Based on the following definition, the scalar-valued function $V(p)$ is derived by attaining $r_P(x)$ and $d_P(x)$ or by $r_P(x)$ and $\theta_P(x)$.

Definition 3. (*(Yager, 2013, 2014)*) Let X be a finite universe of discourse. Let $p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x))$ be a Pythagorean membership grade in X. The scalar-valued function $V(p)$ is stated as below:

$$V(p) = \frac{1}{2} + r_P(x)\left(d_P(x) - \frac{1}{2}\right) = \frac{1}{2} + r_P(x)\left(\frac{1}{2} - \frac{2 \cdot \theta_P(x)}{\pi}\right) \tag{11}$$

3. A SED-based outranking approach based on PF sets

This section intends to develop a SED-based outranking approach on

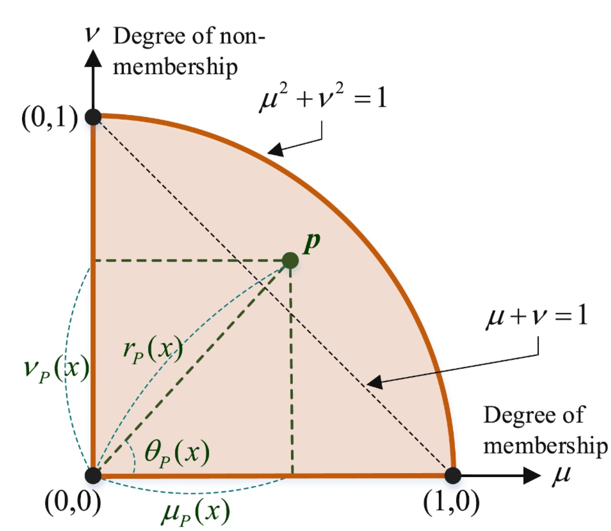


Fig. 1. Interpretation of a Pythagorean membership grade.

the grounds of PF sets. To be specific, this section contrives an MCDA problem involving PF uncertain information and reveals stated preference information between pairs of alternatives. Through the utility of the Euclidean distance measures as regards to Pythagorean membership grades, this section introduces squared PF Euclidean distances and inquires into valuable and desirable properties. This section mentions the concept of SED-based dominance indices and constructs a SED-based outranking approach in PF circumstances. The advanced SED-based outranking approach can supply a substantial theoretical foundation, as revealed in Fig. 2. This figure sketches pertinent concepts of squared PF Euclidean distances and SED-based dominance indices. Some useful properties in the developed theorems facilitate the construction of the core structure in the PF LINMAP methodology in the subsequent study.

3.1. Problem description

This subsection institutes an MCDA problem through the utility of PF sets in which the evaluative ratings of alternatives in relevance to criteria are described as Pythagorean membership grades. As opposed to most MCDA problems that involves preference information about criteria, the developed PF LINMAP methodology requires different types of preference information. To put it another way, the importance weights towards multiple criteria are a priori unknowns and must be determined in the investigated problem. In particular, the proposed techniques require decision makers to provide preference information across the alternatives. Accordingly, this subsection represents the decision maker's paired preference discernments via the stated assessments between alternatives. Simply stated, in the MCDA problem under study, the input of the proposed methods comprises the characteristics of all candidate alternatives by use of PF sets and a set of paired preference discernments in connection with competing alternatives.

Let $Z = \{z_1, z_2, \cdots, z_m\}$ and $C = \{c_1, c_2, \cdots, c_n\}$ indicate a discrete set of m candidate alternatives and a finite set of n evaluative criteria, respectively. To be specific, the set C is separated into two disjoint sets, namely, C_{I} and C_{II}, where $C_{\text{I}} \cap C_{\text{II}} = \varnothing$ and $C_{\text{I}} \cup C_{\text{II}} = C$. Here, C_{I} and C_{II} indicate a collection of benefit criteria and a collection of cost criteria,

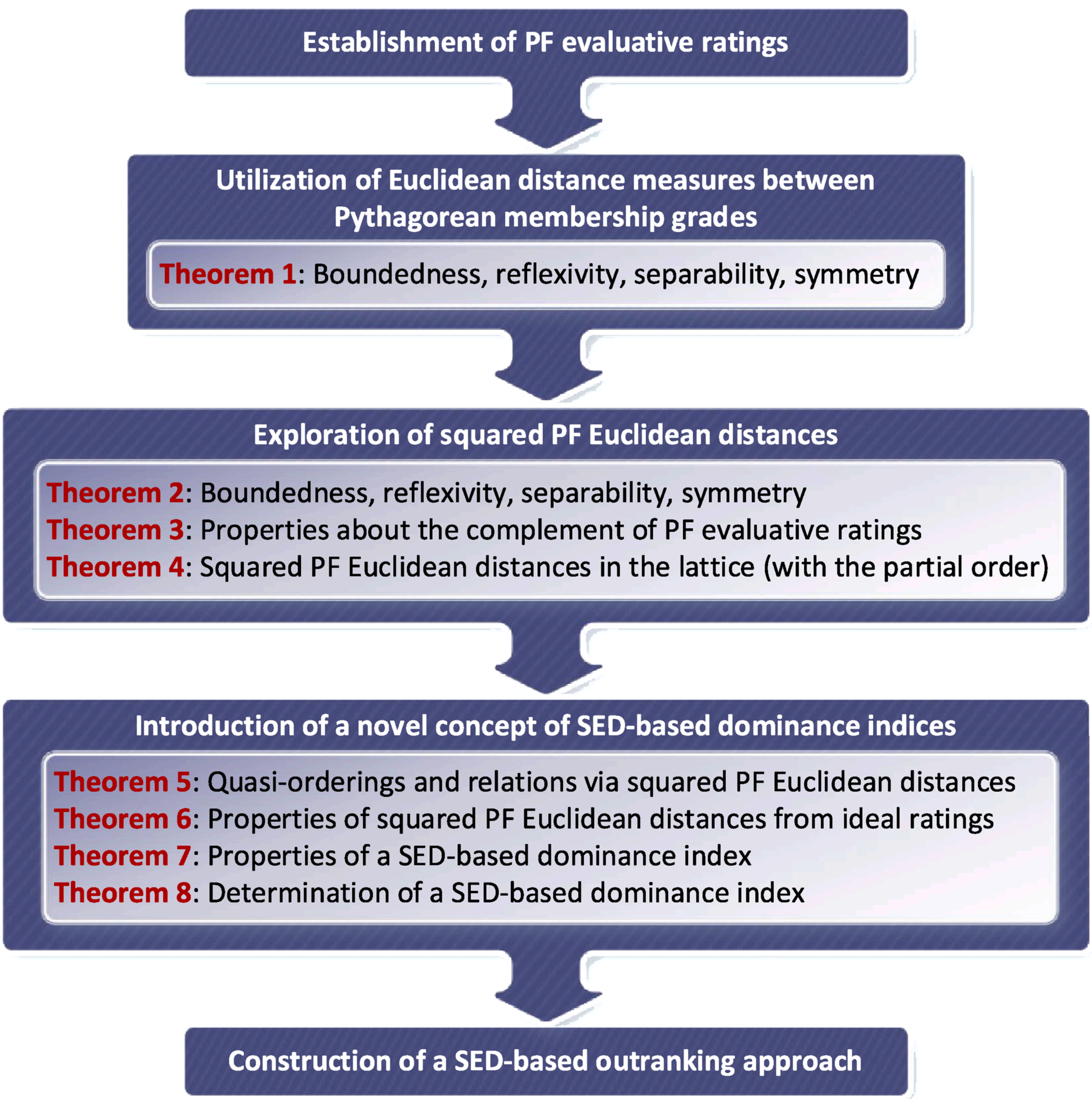


Fig. 2. Theoretical development of the SED-based outranking approach.

respectively. For each $c_j \in C_{\mathrm{I}}$, a higher Pythagorean membership grade of c_j signifies a stronger preference, whereas for each $c_j \in C_{\mathrm{II}}$, a higher Pythagorean membership grade reveals a weaker preference. Let w_j be the importance weight of each criterion $c_j \in C$, which meets the normalization conditions, i.e., $0 \leqslant w_j \leqslant 1$ for all $j \in \{1, 2, \cdots, n\}$ and $\sum_{j=1}^{n} w_j = 1$. Note that the importance weights are a priori unknowns and need to be resolved.

Consider an MCDA problem with convoluted uncertainty involving Pythagorean fuzziness. The PF evaluative rating of an alternative $z_i \in Z$ in the matter of a criterion $c_j \in C$ is denoted as $\mathbb{C}_j(z_i)$ for $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, n\}$. It is portrayed as a Pythagorean membership grade as follows:

$$\mathbb{C}_j(z_i) = \left(\mu_{c_j}(z_i), \nu_{c_j}(z_i); r_{c_j}(z_i), d_{c_j}(z_i)\right) \tag{12}$$

The four parameters of $\mathbb{C}_j(z_i)$ (i.e., the degree of membership $\mu_{c_j}(z_i)$, the degree of non-membership $\nu_{c_j}(z_i)$, the strength of commitment $r_{c_j}(z_i)$, and the direction of commitment $d_{c_j}(z_i)$) are real-valued functions that are evaluated towards z_i in terms of c_j, where $\mu_{c_j}(z_i)$, $\nu_{c_j}(z_i)$, $r_{c_j}(z_i)$, $d_{c_j}(z_i) \in [0, 1]$. The degree of indeterminacy in connection with each $\mathbb{C}_j(z_i)$ is $\tau_{c_j}(z_i) = \sqrt{1 - (\mu_{c_j}(z_i))^2 - (\nu_{c_j}(z_i))^2}$. In accordance with Definition 2, the following relationships are satisfied: $\mu_{c_j}(z_i) = r_{c_j}(z_i) \cdot \cos(\theta_{c_j}(z_i))$, $\nu_{c_j}(z_i) = r_{c_j}(z_i) \cdot \sin(\theta_{c_j}(z_i))$, $r_{c_j}(z_i) = \sqrt{(\mu_{c_j}(z_i))^2 + (\nu_{c_j}(z_i))^2}$, and $d_{c_j}(z_i) = (\pi - 2 \cdot \theta_{c_j}(z_i))/\pi$ for $\theta_{c_j}(z_i) \in [0, \pi/2]$. For notational convenience, let $p_{ij} = \mathbb{C}_j(z_i)$ in the following discussion, where $p_{ij} = (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})$. That is, the notations $\mu_{c_j}(z_i)$, $\nu_{c_j}(z_i)$, $r_{c_j}(z_i)$, and $d_{c_j}(z_i)$ are denoted with μ_{ij}, ν_{ij}, r_{ij}, and d_{ij}, respectively, for simplicity.

By collecting the PF evaluative rating p_{ij} over all $c_j \in C$, the characteristic P_i of the alternative z_i is described as follows:

$$P_i = \left\{ \langle c_j, p_{ij} \rangle \middle| c_j \in C \right\} = \left\{ \langle c_j, (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij}) \rangle \middle| c_j \in C \right\} \tag{13}$$

Let Ω represent a set of ordered pairs (ϕ, φ). The decision maker provides the paired preference relations towards the alternatives in the set Z with Ω, as follows:

$$\Omega = \left\{ (\phi, \varphi) \middle| z_\phi \succeq z_\varphi, \ \phi, \varphi \in \{1, 2, \cdots, m\} \right\}, \tag{14}$$

in which the preference relation $z_\phi \succeq z_\varphi$ indicates that z_ϕ is noninferior to z_φ. In other words, either z_ϕ is preferred to z_φ or z_ϕ and z_φ are identically preferred.

The preference relations $z_\phi \succeq z_\varphi$ in the set Ω are designated by the decision maker's subjective discernments. Theoretically, there are no more than $m(m-1)/2$ paired comparison judgments in Ω. However, in realistic decision-making problems, incomplete judgments often occur because of various factors such as the incomplete information and limited expertise and knowledge. Thus, it is not indispensable to postulate the decision maker to indicate all preference relations for each pairs of alternatives. On the other side, the decision maker sometimes depicts non-transitive paired comparison judgments because of the existence of non-transitive relations among pairwise preferences in practice. Note that the transitivity does not always hold when multiple criteria are evaluated, which implies that the preference relations provided by the decision maker might be inconsistent. For this reason, in addition to the incomplete judgment estimation, the proposed methodology can cope with the inconsistency issue among preference relations in the set Ω. Given the characteristics composed of PF evaluative ratings and the paired preference discernments over the alternatives, this paper advances a SED-based outranking approach using squared PF Euclidean distances and presents a new PF LINMAP method for tackling MCDA problems with uncertainty involving Pythagorean fuzziness.

3.2. Squared PF Euclidean distances

This subsection introduces the current PF distance measures between Pythagorean membership grades on the grounds of the Euclidean distance model. Next, this subsection employs a newly developed Euclidean distance measure on grounds of PF sets to propose the concept of squared PF Euclidean distances and explore several helpful and appealing properties.

On account of the fact that the conventional LINMAP procedure belongs to ideal-solution based models; moreover, it utilizes the square of the weighted Euclidean distance between an alternative and the ideal solution to appraise the measurements on poorness of fit and goodness of fit. Accordingly, the Euclidean distance model is capable of playing a key role in LINMAP-based approaches. To manipulate PF information, this paper briefly reviews the current Euclidean distance models for facilitating the LINMAP analysis. In the PF context, some Euclidean distance measures have been proposed so far for appropriately measuring the separation between PF information. For example, Ren et al. (2016) considered the individual differences between square degrees of membership, between square degrees of non-membership, and between square degrees of indeterminacy to introduce the Euclidean distance for PF sets. By extending their definition, Chen (2018c) presented a generalized PF distance measure for appraising separation magnitudes between Pythagorean membership grades through the agency of the Minkowski distance model with a distance parameter. Analogously, Wan et al. (2018) introduced (weighted) Minkowski distance measures for PF sets. When the distance parameter is equal to two, Chen (2018c) generalized PF distance measure and Wan et al. (2018) Minkowski distance measure reduce to Ren et al. (2016) Euclidean distance measure. However, a key critical issue for the appropriateness of the aforementioned Euclidean distances is the ignorance of the distinct attributes of PF sets. Specifically, the strength and direction of commitment possessed by a Pythagorean membership grade have not been incorporated into the specifications of the PF Euclidean distance model.

In this regard, Li and Zeng (2018) and Zeng et al. (2018) developed more comprehensive generalized PF distance measures with a distance parameter in the PF contexts. Analogously, when the distance parameter is equal to two, both generalized PF distance measures reduce to the Euclidean distance that is different from the measure proposed by Ren et al. (2016). To be precise, Li and Zeng (2018) premeditated the strength and direction of commitment to propose a new generalized distance measure (involving the Euclidean distance) for PF sets. They employed the individual differences between degrees of membership, degrees of non-membership, strengths of commitment, and between directions of commitment to present a normalized measure for Pythagorean membership grades. However, Li and Zeng did not consider the influences of the square degrees of membership and non-membership, as well as the square of strengths of commitment, in their distance measurements. Moreover, their normalization approach would underrate the extent of the normalized distances. On the other side, Zeng et al. (2018) incorporated five elementary parameters featured by Pythagorean membership grades into the identification of PF distance measures that involves the Euclidean distance. The respective differences between the membership degrees, non-membership degrees, indeterminacy degrees, strengths of commitment, and between directions of commitment are simultaneously employed in identifying a normalized generalized distance measure for PF sets. Nonetheless, Zeng et al. (2018) did not consider the squared differences in the five fundamental parameters. Another important issue to address is the double weighting question of the strength of commitment and the degree of indeterminacy. Because of the dual concepts (i.e., the indeterminacy degree is equal to one minus the strength of commitment), the distance measure proposed by Zeng et al. would bring about unreasonable computation results of PF distances. Analogous to Li and Zeng (2018), the normalization technique adopted by Zeng et al. (2018) would underestimate the extent of the normalized distances.

To surmount the above-mentioned difficulty and limitations, Chen (2019a, 2019b) employed the fundamental characteristics of PF sets, namely, membership, non-membership, strength, and direction, to propose an innovative generalized distance measure on grounds of PF sets. In conformity with the application and comparative results, Chen demonstrated the effectiveness and practicability of the developed generalized distance measure. Compared to the other existing PF distance measures, this newly developed measure has the strong points of stipulating an opportune normalization approach, handling the double weighting question, and exploiting the square terms concerning Pythagorean membership degrees; moreover, it provides a flexible and effective measure for quantifying the separation between PF information. Therefore, this paper originates the generalized distance measure proposed by Chen (2019a, 2019b) to put forward the squared PF Euclidean distance for PF sets.

On grounds of the Euclidean distance model in the generalized distance measure by Chen (2019a, 2019b) (i.e., in case when the distance parameter is equal to two), the following definition presents the normalized PF Euclidean distance measure D and the squared PF Euclidean distance measure S between PF evaluative ratings. In particular, the distance measure D is identified through the utility of a four-term representation. That is, the degrees of membership and non-membership, the strength of commitment, and the direction of commitment are considered in quantifying the measure D, which is a positive function (also called metric) from pairs of Pythagorean membership grades.

Definition 4. Let two Pythagorean membership grades $p_{\phi j}$ $(=(\mu_{\phi j},\nu_{\phi j};r_{\phi j},d_{\phi j}))$ and $p_{\varphi j}$ $(=(\mu_{\varphi j},\nu_{\varphi j};r_{\varphi j},d_{\varphi j}))$ be the PF evaluative ratings of alternatives z_ϕ and z_φ, respectively, in connection with a criterion c_j, where $z_\phi, z_\varphi \in Z$ and $c_j \in C$. The normalized PF Euclidean distance $D(p_{\phi j}, p_{\varphi j})$ between $p_{\phi j}$ and $p_{\varphi j}$ is expounded as follows:

$$D(p_{\phi j},p_{\varphi j})=\sqrt{\frac{1}{3}\left[\left((\mu_{\phi j})^2-(\mu_{\varphi j})^2\right)^2+\left((\nu_{\phi j})^2-(\nu_{\varphi j})^2\right)^2+\left((r_{\phi j})^2-(r_{\varphi j})^2\right)^2+\left(d_{\phi j}-d_{\varphi j}\right)^2\right]}. \tag{15}$$

Moreover, the squared PF Euclidean distance $S(p_{\phi j},p_{\varphi j})$ $(=(D(p_{\phi j},p_{\varphi j}))^2)$ is stated as below:

$$S(p_{\phi j},p_{\varphi j})=\frac{1}{3}\left[\left((\mu_{\phi j})^2-(\mu_{\varphi j})^2\right)^2+\left((\nu_{\phi j})^2-(\nu_{\varphi j})^2\right)^2+\left((r_{\phi j})^2-(r_{\varphi j})^2\right)^2+\left(d_{\phi j}-d_{\varphi j}\right)^2\right] \tag{16}$$

Consider a thorough mathematical representation for expressing the PF distance measures. The normalized PF Euclidean distance $D(\mathbb{C}_j(z_\phi),\mathbb{C}_j(z_\varphi))$ (i.e., $D(p_{\phi j},p_{\varphi j})$) and the squared PF Euclidean distance $S(\mathbb{C}_j(z_\phi),\mathbb{C}_j(z_\varphi))$ (i.e., $S(p_{\phi j},p_{\varphi j})$) are stated as below:

$$D(\mathbb{C}_j(z_\phi),\mathbb{C}_j(z_\varphi))=\left\{\left(1/3\right)\left[\left(\left(\mu_{c_j}(z_\phi)\right)^2-\left(\mu_{c_j}(z_\varphi)\right)^2\right)^2+\left(\left(\nu_{c_j}(z_\phi)\right)^2-\left(\nu_{c_j}(z_\varphi)\right)^2\right)^2+\left(\left(r_{c_j}(z_\phi)\right)^2-\left(r_{c_j}(z_\varphi)\right)^2\right)^2+\left(d_{c_j}(z_\phi)-d_{c_j}(z_\varphi)\right)^2\right]\right\}^{0.5}$$

and $S(\mathbb{C}_j(z_\phi),\mathbb{C}_j(z_\varphi))$

$$=(1/3)\left[\left(\left(\mu_{c_j}(z_\phi)\right)^2-\left(\mu_{c_j}(z_\varphi)\right)^2\right)^2+\left(\left(\nu_{c_j}(z_\phi)\right)^2-\left(\nu_{c_j}(z_\varphi)\right)^2\right)^2+\left(\left(r_{c_j}(z_\phi)\right)^2-\left(r_{c_j}(z_\varphi)\right)^2\right)^2+\left(d_{c_j}(z_\phi)-d_{c_j}(z_\varphi)\right)^2\right]$$

.

Example 1. Consider two PF evaluative ratings $p_{\phi j}$=(0.5142, 0.6984; 0.8673, 0.4040) and $p_{\varphi j}$=(0.8277, 0.3815; 0.9114, 0.7250). From Definition 4, the normalized PF Euclidean distance $D(p_{\phi j},p_{\varphi j})$={(1/3) $\cdot$[(0.5142^2 −0.8277^2)^2+(0.6984^2 −0.3815^2)^2+(0.8673^2 −0.9114^2)^2 +(0.4040^2 −0.7250^2)^2)]}$^{0.5}$ = 0.3667. The squared PF Euclidean distance $S(p_{\phi j},\ p_{\varphi j})$=(1/3)$\cdot$[(0.5142^2 −0.8277^2)^2+(0.6984^2 −0.3815^2)^2 +(0.8673^2 −0.9114^2)^2+(0.4040^2 −0.7250^2)^2)] = 0.1344.

Theorem 1. *The normalized PF Euclidean distance measure D between two PF evaluative ratings $p_{\phi j}$ and $p_{\varphi j}$ fulfills the succeeding properties:*

(T1.1) Boundedness: $0\leqslant D(p_{\phi j},p_{\varphi j})\leqslant 1$;
(T1.2) Reflexivity: $D(p_{\phi j},p_{\phi j})=0$;
(T1.3) Separability: $D(p_{\phi j},p_{\varphi j})=0$ if and only if $p_{\phi j}=p_{\varphi j}$;
(T1.4) Symmetry: $D(p_{\phi j},p_{\varphi j})=D(p_{\varphi j},p_{\phi j})$.

Proof. Refer to the proof process in Chen (2019b).

Remark. The squared PF Euclidean distance measure S between two PF evaluative ratings $p_{\phi j}$ and $p_{\varphi j}$ fulfills the succeeding properties:

(R.1) Boundedness: $0\leqslant S(p_{\phi j},p_{\varphi j})\leqslant 1$;
(R.2) Reflexivity: $S(p_{\phi j},p_{\phi j})=0$;
(R.3) Separability: $S(p_{\phi j},p_{\varphi j})=0$ if and only if $p_{\phi j}=p_{\varphi j}$;
(R.4) Symmetry: $S(p_{\phi j},p_{\varphi j})=S(p_{\varphi j},p_{\phi j})$.

Properties (R.1)–(R.4) are fulfilled based on (T1.1)–(T1.4), respectively. The squared PF Euclidean distance measure S is not a metric, but is a semimetric because $S(p_{\phi j},p_{\varphi j})$ satisfies the prerequisite conditions of reflexivity, separability, and symmetry.

In addition to the properties (T1.1)–(T1.4), the main advantage of the normalized PF Euclidean distance measure D is that it can measure the ordinary distance between Pythagorean membership degrees in a PF space. However, the use of the measure D is troublesome as there pertain to costly square and square root operations. For this reason, the concept of squared PF Euclidean distances towards Pythagorean membership degrees will be exploited to keep away from the computation of square root for simplicity. Especially, in the PF context, the Pythagorean theorem is simpler in connection with the squared PF Euclidean distance measure S because there is no square root. One useful application of the measure S is to compare the dissimilarity between various PF evaluative ratings. Therefore, the measurement of squared PF Euclidean distances will be adopted to reduce computational complexity.

Theorem 2. Let $p^c_{\phi j}$ $(=(\mu^c_{\phi j},\nu^c_{\phi j};r^c_{\phi j},d^c_{\phi j}))$ and $p^c_{\varphi j}$ $(=(\mu^c_{\varphi j},\nu^c_{\varphi j};r^c_{\varphi j},d^c_{\varphi j}))$ be the complements of the PF evaluative ratings $p_{\phi j}$ and $p_{\varphi j}$, respectively. The succeeding properties are correct:

(T2.1) $|d_{\phi j}-d_{\varphi j}|=\left|d^c_{\phi j}-d^c_{\varphi j}\right|$ and $\left|d_{\phi j}-d^c_{\varphi j}\right|=\left|d^c_{\phi j}-d_{\varphi j}\right|$;
(T2.2) $|\theta_{\phi j}-\theta_{\varphi j}|=\left|\theta^c_{\phi j}-\theta^c_{\varphi j}\right|$ and $\left|\theta_{\phi j}-\theta^c_{\varphi j}\right|=\left|\theta^c_{\phi j}-\theta_{\varphi j}\right|$;
(T2.3) $S(p_{\phi j},p_{\varphi j})=S(p^c_{\phi j},p^c_{\varphi j})$ and $S(p_{\phi j},p^c_{\varphi j})=S(p^c_{\phi j},p_{\varphi j})$.

Proof. For (T2.1), based on (10), one has $d^c_{\phi j} = 1 - d_{\phi j}$ and $d^c_{\varphi j} = 1 - d_{\varphi j}$ for two complements $p^c_{\phi j}$ and $p^c_{\varphi j}$, respectively. It is easily derived that:

$$\left|d_{\phi j} - d_{\varphi j}\right| = \left|\left(1 - d_{\phi j}\right) - \left(1 - d_{\varphi j}\right)\right| = \left|d^c_{\phi j} - d^c_{\varphi j}\right|$$

$$\left|d_{\phi j} - d^c_{\varphi j}\right| = \left|d_{\phi j} - \left(1 - d_{\varphi j}\right)\right| = \left|\left(1 - d_{\phi j}\right) - d_{\varphi j}\right| = \left|d^c_{\phi j} - d_{\varphi j}\right|$$

Thus, (T2.1) is valid. For (T2.2), the radians $\theta_{\phi j}$ and $\theta_{\varphi j}$ that are associated with $p_{\phi j}$ and $p_{\varphi j}$, respectively, can be gained using (9), i.e., $\theta_{\phi j} = \pi(1 - d_{\phi j})/2$ and $\theta_{\varphi j} = \pi(1 - d_{\varphi j})/2$. According to $d^c_{\phi j} = 1 - d_{\phi j}$ and $d^c_{\varphi j} = 1 - d_{\varphi j}$, it can be inferred that $\theta^c_{\phi j} = \pi(1 - d^c_{\phi j})/2 = \pi \cdot d_{\phi j}/2$ and $\theta^c_{\varphi j} = \pi(1 - d^c_{\varphi j})/2 = \pi \cdot d_{\varphi j}/2$. The following results are derived:

$$\left|\theta_{\phi j} - \theta_{\varphi j}\right| = \left|\frac{\pi}{2}\left(1 - d_{\phi j}\right) - \frac{\pi}{2}\left(1 - d_{\varphi j}\right)\right| = \left|\frac{\pi}{2}\left(d_{\phi j}\right) - \frac{\pi}{2}\left(d_{\varphi j}\right)\right| = \left|\theta^c_{\phi j} - \theta^c_{\varphi j}\right|$$

$$\left|\theta_{\phi j} - \theta^c_{\varphi j}\right| = \left|\frac{\pi}{2}\left(1 - d_{\phi j}\right) - \frac{\pi}{2} d_{\varphi j}\right| = \left|\frac{\pi}{2} d_{\phi j} - \frac{\pi}{2}\left(1 - d_{\varphi j}\right)\right| = \left|\theta^c_{\phi j} - \theta_{\varphi j}\right|$$

Thus, (T2.2) is correct. For (T2.3), it is recognized that $p^c_{\phi j} = (\nu_{\phi j}, \mu_{\phi j}; r_{\phi j}, 1 - d_{\phi j})$ and $p^c_{\varphi j} = (\nu_{\varphi j}, \mu_{\varphi j}; r_{\varphi j}, 1 - d_{\varphi j})$ from (10). Based on (16), it is proven that:

$$\begin{aligned} S(p_{\phi j}, p_{\varphi j}) &= \frac{1}{3}\Big[\left((\nu_{\phi j})^2 - (\nu_{\varphi j})^2\right)^2 + \left((\mu_{\phi j})^2 - (\mu_{\varphi j})^2\right)^2 + \left((r_{\phi j})^2 - (r_{\varphi j})^2\right)^2 \\ &\quad + \left((1 - d_{\phi j}) - (1 - d_{\varphi j})\right)^2\Big] \\ &= S(p^c_{\phi j}, p^c_{\varphi j}), \end{aligned}$$

$$S\left(p_{\phi j}, p^c_{\varphi j}\right) = \frac{1}{3}\left[\left((\mu_{\phi j})^2 - (\nu_{\varphi j})^2\right)^2 + \left((\nu_{\phi j})^2 - (\mu_{\varphi j})^2\right)^2 + \left((r_{\phi j})^2 - (r_{\varphi j})^2\right)^2 + \left(d_{\phi j} - (1 - d_{\varphi j})\right)^2\right]$$
$$= \tfrac{1}{3}\left[\left((\nu_{\phi j})^2 - (\mu_{\varphi j})^2\right)^2 + \left((\mu_{\phi j})^2 - (\nu_{\varphi j})^2\right)^2 + \left((r_{\phi j})^2 - (r_{\varphi j})^2\right)^2 + \left((1 - d_{\phi j}) - d_{\varphi j}\right)^2\right] = S(p^c_{\phi j}, p_{\varphi j}).$$

Thus, (T2.3) is fulfilled, which completes the proof.

Chen (2018c) expounded a complete lattice as a partially ordered set in such a manner that a non-empty PF subset has a supremum and an infimum. Based on the Pythagorean identity, i.e., $sin^2(\theta_{ij}) + cos^2(\theta_{ij}) = 1$, it can be obtained that $(\mu_{ij})^2 + (\nu_{ij})^2 = (r_{ij})^2 \cdot (sin^2(\theta_{ij}) + cos^2(\theta_{ij})) = (r_{ij})^2 \leqslant 1$. Analogous to Chen (2018c), this paper construes the lattice $(L_{PF}, \leqslant_{L_{PF}})$ with the partial order $\leqslant_{L_{PF}}$ for non-empty Pythagorean membership degrees in a PF set as follows:

$$L_{PF} = \left\{\left(\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij}\right) \middle| \mu_{ij}, \nu_{ij}, r_{ij}, d_{ij} \in [0, 1], (\mu_{ij})^2 + (\nu_{ij})^2 \leqslant 1\right\} \tag{17}$$

For any PF evaluative ratings $p_{\phi j} = (\mu_{\phi j}, \nu_{\phi j}; r_{\phi j}, d_{\phi j}) \in L_{PF}$ and $p_{\varphi j} = (\mu_{\varphi j}, \nu_{\varphi j}; r_{\varphi j}, d_{\varphi j}) \in L_{PF}$, the partial order $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ if and only if $\mu_{\phi j} \leqslant \mu_{\varphi j}$ and $\nu_{\phi j} \geqslant \nu_{\varphi j}$. Nonetheless, such orderings regarding PF evaluative ratings do not constantly take place in PF surroundings. As an illustration, consider $p_{\phi j}$=(0.51, 0.47; 0.69, 0.53) and $p_{\varphi j}$=(0.62, 0.73; 0.96, 0.45). Because $\mu_{\phi j}$ (=0.51) < $\mu_{\varphi j}$ (=0.62) and $\nu_{\phi j}$ (=0.47) < $\nu_{\varphi j}$ (=0.73), the partial order does not exist between $p_{\phi j}$ and $p_{\varphi j}$, namely, neither $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ nor $p_{\varphi j} \leqslant_{L_{PF}} p_{\phi j}$ holds. The main issue lies in this example is the differentiation between PF evaluative ratings. It frequently happens that the dominance relationships cannot be ascertained based on inequalities between the membership functions. To differentiate the dominance relationships for PF information, this paper would like to advance a new concept of SED-based dominance indices to implement effectual comparisons on the subject of PF evaluative ratings.

On the other side, the directions of commitment $d_{\phi j}$ and $d_{\varphi j}$ depict on a scale of 0 to 1 how entirely the strengths $r_{\phi j}$ and $r_{\varphi j}$, respectively, are pointing toward membership. It follows that $d_{\phi j} \leqslant d_{\varphi j}$ because $\mu_{\phi j} \leqslant \mu_{\varphi j}$. Based on $\theta_{\phi j} = (\pi/2) \cdot (1 - d_{\phi j})$ and $\theta_{\varphi j} = (\pi/2) \cdot (1 - d_{\varphi j})$, one obtains $\theta_{\phi j} \geqslant \theta_{\varphi j}$ because π is a constant. It is noted that the bottom and top elements of $(L_{PF}, \leqslant_{L_{PF}})$ are (0,1;1,0) and (1,0;1,1), respectively. This indicates that Pythagorean membership degrees (0,1;1,0) and (1,0;1,1) are the lowest and highest PF evaluative ratings, respectively, in $(L_{PF}, \leqslant_{L_{PF}})$.

Theorem 3. Let $(L_{PF}, \leqslant_{L_{PF}})$ be a partially ordered set in the universe of discourse. Let three Pythagorean membership degrees $p_{\phi j}$, $p_{\gamma j}$, and $p_{\varphi j}$ be the PF evaluative ratings, where $p_{\phi j}$, $p_{\gamma j}$, *and* $p_{\varphi j}$ are affiliated with L_{PF} that involves the partial order $\leqslant_{L_{PF}}$. If $p_{\phi j} \leqslant_{L_{PF}} p_{\gamma j} \leqslant_{L_{PF}} p_{\varphi j}$, then $S(p_{\phi j}, p_{\gamma j}) \leqslant S(p_{\phi j}, p_{\varphi j})$ and $S(p_{\gamma j}, p_{\varphi j}) \leqslant S(p_{\phi j}, p_{\varphi j})$.

Proof. Because $p_{\phi j}$, $p_{\gamma j}$, *and* $p_{\varphi j}$ are affiliated with L_{PF} that contains the partial order $\leqslant_{L_{PF}}$, the condition $p_{\phi j} \leqslant_{L_{PF}} p_{\gamma j} \leqslant_{L_{PF}} p_{\varphi j}$ holds if and only if $0 \leqslant \mu_{\phi j} \leqslant \mu_{\gamma j} \leqslant \mu_{\varphi j} \leqslant 1$ and $1 \geqslant \nu_{\phi j} \geqslant \nu_{\gamma j} \geqslant \nu_{\varphi j} \geqslant 0$. Moreover, it is known that $0 \leqslant d_{\phi j} \leqslant d_{\gamma j} \leqslant d_{\varphi j} \leqslant 1$ from $\mu_{\phi j} \leqslant \mu_{\gamma j} \leqslant \mu_{\varphi j}$. The following results are obtained: $\left((\mu_{\phi j})^2 - (\mu_{\gamma j})^2\right)^2 \leqslant \left((\mu_{\phi j})^2 - (\mu_{\varphi j})^2\right)^2$, $\left((\nu_{\phi j})^2 - (\nu_{\gamma j})^2\right)^2 \leqslant \left((\nu_{\phi j})^2 - (\nu_{\varphi j})^2\right)^2$, and $\left((d_{\phi j})^2 - (d_{\gamma j})^2\right)^2 \leqslant \left((d_{\phi j})^2 - (d_{\varphi j})^2\right)^2$. Furthermore, the following is derived:

$$\begin{aligned} &\left((r_{\phi j})^2 - (r_{\gamma j})^2\right)^2 = \left((\mu_{\phi j})^2 + (\nu_{\phi j})^2 - (\mu_{\gamma j})^2 - (\nu_{\gamma j})^2\right)^2 \\ &= \left((\mu_{\phi j})^2 - (\mu_{\gamma j})^2\right)^2 + \left((\nu_{\phi j})^2 - (\nu_{\gamma j})^2\right)^2 + 2\left((\mu_{\phi j})^2 - (\mu_{\gamma j})^2\right)\left((\nu_{\phi j})^2 - (\nu_{\gamma j})^2\right) \\ &\leqslant \left((\mu_{\phi j})^2 - (\mu_{\varphi j})^2\right)^2 + \left((\nu_{\phi j})^2 - (\nu_{\varphi j})^2\right)^2 + 2\left((\mu_{\phi j})^2 - (\mu_{\varphi j})^2\right)\left((\nu_{\phi j})^2 - (\nu_{\varphi j})^2\right) \\ &= \left((\mu_{\phi j})^2 + (\nu_{\phi j})^2 - (\mu_{\varphi j})^2 - (\nu_{\varphi j})^2\right)^2 = \left((r_{\phi j})^2 - (r_{\varphi j})^2\right)^2. \end{aligned}$$

In accordance with these results, it is concluded that $S(p_{\phi j}, p_{\gamma j}) \leqslant S(p_{\phi j}, p_{\varphi j})$ holds. The inequality $S(p_{\gamma j}, p_{\varphi j}) \leqslant S(p_{\phi j}, p_{\varphi j})$ can be proved just the same. This completes the proof.

Example 2. Let three Pythagorean membership grades $p_{\phi j}$ = (0.2124, 0.8436; 0.8699, 0.1570), $p_{\gamma j}$ = (0.3978, 0.7534; 0.8520, 0.3093), and $p_{\phi j}$ = (0.5407, 0.6176; 0.8208, 0.4578) be the PF evaluative ratings. By utilizing Definition 4, it is calculated that $S(p_{\phi j}, p_{\gamma j}) = 0.0192$, $S(p_{\gamma j}, p_{\varphi j}) = 0.0258$, and $S(p_{\phi j}, p_{\varphi j}) = 0.0892$. One can see that the two inequalities $S(p_{\phi j}, p_{\gamma j}) \leqslant S(p_{\phi j}, p_{\varphi j})$ and $S(p_{\gamma j}, p_{\varphi j}) \leqslant S(p_{\phi j}, p_{\varphi j})$ are satisfied.

The squared PF Euclidean distance $S(p_{\phi j}, p_{\varphi j})$ between $p_{\phi j}$ and $p_{\varphi j}$ is similar to metrics, but it does not possess the property of triangle inequality. Yet, in spite of this, it is a more general conception of distance and can be used to describe a natural quasi-ordering between Pythagorean membership grades, namely $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ if and only if $\mu_{\phi j} \leqslant \mu_{\varphi j}$ and $\nu_{\phi j} \geqslant \nu_{\varphi j}$. Moreover, the Pythagorean theorem is simpler through the agency of the squared distance measure. As proven in Theorem 3, it is advisable to employ $S(p_{\phi j}, p_{\varphi j})$ to capture the separation between PF evaluative ratings. There is one more thing to be noted here. Because the most canonical Bregman divergence is the squared Euclidean distance, the proposed $S(p_{\phi j}, p_{\varphi j})$ can facilitate forming an important class of divergences in the PF context for future research.

3.3. SED-based dominance indices

On the strength of the proposed squared PF Euclidean distance, this subsection attempts to construct a SED-based dominance index and present a SED-based outranking approach in the PF decision context.

Because (0,1;1,0) and (1,0;1,1) are the bottom and top elements, respectively, of $(L_{PF}, \leqslant_{L_{PF}})$, they are considered as the lowest and highest Pythagorean membership grades, respectively. Accordingly, the positive- and negative-ideal PF evaluative ratings are designated as (1,0;1,1) and (0,1;1,0), respectively, regarding benefit criteria. On the contrary, the positive- and negative-ideal PF evaluative ratings are designated as (0,1;1,0) and (1,0;1,1), respectively, for cost criteria. Motivated by the idea of closeness coefficient in the TOPSIS, this paper contrives an innovative concept of SED-based dominance indices and inquires into several useful properties. The advanced dominance indices furnish a basis for formulating the succeeding PF LINMAP method and techniques.

Definition 5. Let p_{+j} and p_{-j} denote the positive- and negative-ideal PF evaluative ratings, respectively, in relation to each criterion $c_j \in C$; they are expounded as below:

$$p_{+j} = (\mu_{+j}, \nu_{+j}; r_{+j}, d_{+j}) = \begin{cases} (1,0;1;1) & \text{for } c_j \in C_{\mathrm{I}}, \\ (0,1;1,0) & \text{for } c_j \in C_{\mathrm{II}}; \end{cases} \tag{18}$$

$$p_{-j} = (\mu_{-j}, \nu_{-j}; r_{-j}, d_{-j}) = \begin{cases} (0,1;1,0) & \text{for } c_j \in C_{\mathrm{I}}, \\ (1,0;1;1) & \text{for } c_j \in C_{\mathrm{II}}. \end{cases} \tag{19}$$

Theorem 4. *Consider the characteristic P_i for all $z_i \in Z$, in which $P_i = \{\langle c_j, p_{ij}\rangle | c_j \in C\} = \{\langle c_j, (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij})\rangle | c_j \in C_{\mathrm{I}} \cup C_{\mathrm{II}}\}$. The PF evaluative rating p_{ij}, the positive-ideal PF evaluative rating p_{+j}, and the negative-ideal PF evaluative rating p_{-j} fulfill the following quasi-orderings, and the squared PF Euclidean distances between them have the following relations*:

(T4.1) $p_{-j} \leqslant_{L_{PF}} p_{ij} \leqslant_{L_{PF}} p_{+j}$ for $c_j \in C_{\mathrm{I}}$;
(T4.2) $p_{+j} \leqslant_{L_{PF}} p_{ij} \leqslant_{L_{PF}} p_{-j}$ for $c_j \in C_{\mathrm{II}}$;
(T4.3) $S(p_{ij}, p_{-j}) \leqslant S(p_{-j}, p_{+j})$ and $S(p_{ij}, p_{+j}) \leqslant S(p_{-j}, p_{+j})$ for $c_j \in C$.

Proof. For (T4.1), it is cognized that p_{ij}, p_{+j}, and p_{-j} are affiliated with L_{PF} that involves the partial order $\leqslant_{L_{PF}}$. For each benefit criterion $c_j \in C_{\mathrm{I}}$, inequalities $\mu_{-j} \leqslant \mu_{ij} \leqslant \mu_{+j}$ and $\nu_{-j} \geqslant \nu_{ij} \geqslant \nu_{+j}$ hold because $\mu_{-j} = \nu_{+j} = 0$ and $\mu_{+j} = \nu_{-j} = 1$. As a result, $p_{-j} \leqslant_{L_{PF}} p_{ij} \leqslant_{L_{PF}} p_{+j}$ is valid. For (T4.2), inequalities $\mu_{-j} \geqslant \mu_{ij} \geqslant \mu_{+j}$ and $\nu_{-j} \leqslant \nu_{ij} \leqslant \nu_{+j}$ are satisfied for each cost criterion $c_j \in C_{\mathrm{II}}$ because $\mu_{-j} = \nu_{+j} = 1$ and $\mu_{+j} = \nu_{-j} = 0$, which indicates that $p_{+j} \leqslant_{L_{PF}} p_{ij} \leqslant_{L_{PF}} p_{-j}$. For (T4.3), it is received that $S(p_{-j}, p_{ij}) = S(p_{ij}, p_{-j})$ and $p_{-j} \leqslant_{L_{PF}} p_{ij} \leqslant_{L_{PF}} p_{+j}$ for $c_j \in C_{\mathrm{I}}$ from (R.4) and (T4.1), respectively. Thus, $S(p_{ij}, p_{-j}) \leqslant S(p_{-j}, p_{+j})$ and $S(p_{ij}, p_{+j}) \leqslant S(p_{-j}, p_{+j})$ hold for $c_j \in C_{\mathrm{I}}$ according to Theorem 3. Analogously, the above two relations are valid for each $c_j \in C_{\mathrm{II}}$ because $S(p_{+j}, p_{ij}) = S(p_{ij}, p_{+j})$ and $p_{+j} \leqslant_{L_{PF}} p_{ij} \leqslant_{L_{PF}} p_{-j}$. Therefore, (T4.3) is correct for $c_j \in C$, which completes the proof.

On the subject of each p_{ij}, the square degree of the indeterminacy is derived as $(\tau_{ij})^2 = 1 - (\mu_{ij})^2 - (\nu_{ij})^2 = 1 - (r_{ij})^2$. That is, the two parameters τ_{ij} and r_{ij} are dual concepts. From Definitions 4 and 5, the squared PF Euclidean distances $S(p_{ij}, p_{-j})$ $(1/3)\left[\left((\mu_{ij})^2 - (\mu_{-j})^2\right)^2 + \left((\nu_{ij})^2 - (\nu_{-j})^2\right)^2 + ((r_{ij})^2 - (r_{-j})^2)^2 + (d_{ij} - d_{-j})^2\right]$ and $S(p_{ij}, p_{+j}) = (1/3)\left[\left((\mu_{ij})^2 - (\mu_{+j})^2\right)^2 + \left((\nu_{ij})^2 - (\nu_{+j})^2\right)^2 + \left((r_{ij})^2 - (r_{+j})^2\right)^2 + (d_{ij} - d_{+j})^2\right]$ are rewritten as follows:

$$S(p_{ij}, p_{-j}) = \begin{cases} \frac{1}{3}\left[(\mu_{ij})^4 + (1 - (\nu_{ij})^2)^2 + (\tau_{ij})^4 + (d_{ij})^2\right] \text{ for } c_j \in C_{\mathrm{I}}, \\ \frac{1}{3}\left[(1 - (\mu_{ij})^2)^2 + (\nu_{ij})^4 + (\tau_{ij})^4 + (1 - d_{ij})^2\right] \text{ for } c_j \in C_{\mathrm{II}}; \end{cases} \tag{20}$$

$$S(p_{ij}, p_{+j}) = \begin{cases} \frac{1}{3}\left[(1 - (\mu_{ij})^2)^2 + (\nu_{ij})^4 + (\tau_{ij})^4 + (1 - d_{ij})^2\right] & \text{for } c_j \in C_{\mathrm{I}}, \\ \frac{1}{3}\left[(\mu_{ij})^4 + (1 - (\nu_{ij})^2)^2 + (\tau_{ij})^4 + (d_{ij})^2\right] & \text{for } c_j \in C_{\mathrm{II}}. \end{cases} \tag{21}$$

Theorem 5. *For two PF evaluative ratings $p_{\phi j}$ and $p_{\varphi j}$ in connection with $c_j \in C$ $(= C_{\mathrm{I}} \cup C_{\mathrm{II}})$, their squared PF Euclidean distances from the positive-ideal PF evaluative rating p_{+j} and from the negative-ideal PF evaluative rating p_{-j} meet the succeeding properties*:

(T5.1) $S\left(p_{\phi j}, p_{+j}\right) \geqslant S\left(p_{\varphi j}, p_{+j}\right)$ if $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$ for $c_j \in C_{\mathrm{I}}$;
(T5.2) $S\left(p_{\phi j}, p_{-j}\right) \geqslant S\left(p_{\varphi j}, p_{-j}\right)$ if $p_{\varphi j} \leqslant_{L_{PF}} p_{\phi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$ for $c_j \in C_{\mathrm{I}}$;
(T5.3) $S\left(p_{\phi j}, p_{+j}\right) \geqslant S\left(p_{\varphi j}, p_{+j}\right)$ if $p_{\varphi j} \leqslant_{L_{PF}} p_{\phi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$ for $c_j \in C_{\mathrm{II}}$;
(T5.4) $S\left(p_{\phi j}, p_{-j}\right) \geqslant S\left(p_{\varphi j}, p_{-j}\right)$ if $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$ for $c_j \in C_{\mathrm{II}}$.

Proof. For (T5.1), one has $0 \leqslant \mu_{\phi j} \leqslant \mu_{\varphi j} \leqslant 1$ and $1 \geqslant \nu_{\phi j} \geqslant \nu_{\varphi j} \geqslant 0$ from the condition $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$. It is received that $0 \leqslant d_{\phi j} \leqslant d_{\varphi j} \leqslant 1$ from $\mu_{\phi j} \leqslant \mu_{\varphi j}$. Regarding each benefit criterion $c_j \in C_{\mathrm{I}}$, it can be calculated that $S\left(p_{\phi j}, p_{+j}\right) = (1/3)\left[(1 - (\mu_{\phi j})^2)^2 + (\nu_{\phi j})^4 + (\tau_{\phi j})^4 + (1 - d_{\phi j})^2\right]$ and $S(p_{\varphi j}, p_{+j}) = (1/3)\left[(1 - (\mu_{\varphi j})^2)^2 + (\nu_{\varphi j})^4 + (\tau_{\varphi j})^4 + (1 - d_{\varphi j})^2\right]$ using (21). If $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$, it is acquired that $S(p_{\phi j}, p_{+j}) \geqslant S(p_{\varphi j}, p_{+j})$ because $(1 - (\mu_{\phi j})^2)^2 \geqslant (1 - (\mu_{\varphi j})^2)^2$, $(\nu_{\phi j})^4 \geqslant (\nu_{\varphi j})^4$, $(\tau_{\phi j})^4 \geqslant (\tau_{\varphi j})^4$, and $(1 - d_{\phi j})^2 \geqslant (1 - d_{\varphi j})^2$. Thus, (T5.1) is satisfied. Property (T5.2) can be demonstrated by analogy to (T5.1). For (T5.3), the condition $p_{\varphi j} \leqslant_{L_{PF}} p_{\phi j}$ indicates that $1 \geqslant \mu_{\phi j} \geqslant \mu_{\varphi j} \geqslant 0$ and $0 \leqslant \nu_{\phi j} \leqslant \nu_{\varphi j} \leqslant 1$, which follows that $1 \geqslant d_{\phi j} \geqslant d_{\varphi j} \geqslant 0$. Regarding each cost criterion $c_j \in C_{\mathrm{II}}$, one can yield $S(p_{\phi j}, p_{+j}) = (1/3)\left[(\mu_{\phi j})^4 + \left(1 - (\nu_{\phi j})^2\right)^2 + (\tau_{\phi j})^4 + (d_{\phi j})^2\right]$ and $S(p_{\varphi j}, p_{+j}) = (1/3)\left[(\mu_{\varphi j})^4 + \left(1 - (\nu_{\varphi j})^2\right)^2 + (\tau_{\varphi j})^4 + (d_{\varphi j})^2\right]$ using (21). If $p_{\varphi j} \leqslant_{L_{PF}} p_{\phi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$, it is easily corroborated that $S(p_{\phi j}, p_{+j}) \geqslant S(p_{\varphi j}, p_{+j})$ because $(\mu_{\phi j})^4 \geqslant (\mu_{\varphi j})^4$, $(1 - (\nu_{\phi j})^2)^2 \geqslant (1 - (\nu_{\varphi j})^2)^2$, $(\tau_{\phi j})^4 \geqslant (\tau_{\varphi j})^4$, and $(d_{\phi j})^2 \geqslant (d_{\varphi j})^2$, i.e., (T5.3) is fulfilled. Property (T5.4) is demonstrated by analogy to (T5.3), which completes the proof.

As proven in Theorem 5, the proposed squared PF Euclidean distances in the matter of p_{+j} and p_{-j} can render justified rankings in line with the quasi-orderings defined in $(L_{PF}, \leqslant_{L_{PF}})$. In particular, even if the condition concerning degrees of indeterminacy is not satisfied, the ranking results yielded by the squared PF Euclidean distances are more reasonable than other PF distance measures. Let us examine the following example.

Example 3. Let two Pythagorean membership grades $p_{\phi j}$ and $p_{\varphi j}$ be the PF evaluative ratings of the alternatives z_ϕ and z_φ, respectively, in terms of a benefit criterion c_j, where $p_{\phi j}$=(0.1600, 0.3500; 0.3848, 0.2730) and $p_{\varphi j}$=(0.1610, 0.2000; 0.2568, 0.4315) in Case 1, $p_{\phi j}$=(0.3536, 0.3536; 0.5000, 0.5000) and $p_{\varphi j}$=(0.3537, 0.3500; 0.4976, 0.5033) in Case 2, and $p_{\phi j}$=(0.5000, 0.6000; 0.7810, 0.4423) and $p_{\varphi j}$=(0.5100, 0.4000; 0.6482, 0.5766) in Case 3. It displays that the partial order $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ holds in Cases 1–3. However, the condition $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$ is not satisfied because $\tau_{\phi j} = 0.9230$ and $\tau_{\varphi j} = 0.9665$ in Case 1, $\tau_{\phi j} = 0.8660$ and $\tau_{\varphi j} = 0.8674$ in Case 2, and $\tau_{\phi j} = 0.6245$ and $\tau_{\varphi j} = 0.7615$ in Case 3. Although one of the preconditions in (T5.1) is not satisfied, the concept of scalar-valued functions in Definition 3 can facilitate the comparison of the PF evaluative ratings. Based on (11), one can clearly see that:

$$V(p_{\phi j}) = \frac{1}{2} + r_{\phi j}\left(d_{\phi j} - \frac{1}{2}\right) = \frac{1}{2} + 0.3848\left(0.2730 - \frac{1}{2}\right) = 0.4126$$

or equivalently,

$$V(p_{\phi j}) = \frac{1}{2} + r_{\phi j}\left(\frac{1}{2} - \frac{2\cdot\theta_{\phi j}}{\pi}\right) = \frac{1}{2} + 0.3848\left(\frac{1}{2} - \frac{2\cdot 1.1420}{3.1416}\right) = 0.4126$$

and $V(p_{\varphi j}) = 0.4824$ in Case 1, $V(p_{\phi j}) = 0.5000$ and $V(p_{\varphi j}) = 0.5017$ in Case 2, and $V(p_{\phi j}) = 0.4549$ and $V(p_{\varphi j}) = 0.5496$ in Case 3. Recall that c_j is a benefit criterion, i.e., $c_j \in C_{\mathrm{I}}$. Accordingly, it is anticipated that $p_{\varphi j}$ should have an advantage over $p_{\phi j}$ because $V(p_{\varphi j}) \geqslant V(p_{\phi j})$ in Cases 1–3. According to (21), the squared PF Euclidean distances from p_{+j} are computed as below:

$$S(p_{\phi j}, p_{+j}) = \frac{1}{3}\left[\left(1 - 0.1600^2\right)^2 + 0.3500^4 + 0.9230^4 + (1 - 0.2730)^2\right] = 0.7396$$

and $S(p_{\varphi j},p_{+j}) = 0.7154$ in Case 1, $S(p_{\phi j},p_{+j}) = 0.5313$ and $S(p_{\varphi j},p_{+j}) = 0.5311$ in Case 2, and $S(p_{\phi j},p_{+j}) = 0.3851$ and $S(p_{\varphi j},p_{+j}) = 0.3629$ in Case 3. It is obvious that $p_{\varphi j}$ has a smaller squared PF Euclidean distance from p_{+j} than $p_{\phi j}$ because $S(p_{\phi j},p_{+j}) \geqslant S(p_{\varphi j},p_{+j})$ hold in Cases 1–3. Therefore, $p_{\varphi j}$ is more desirable than $p_{\phi j}$ according to the comparison results via squared PF Euclidean distances from p_{+j}.

Example 4. (*(Continued from Example 3)*) As mentioned previously, Ren et al. (2016) introduced the Euclidean distance measure in PF circumstances. By the agency of the Minkowski distance model, Chen (2018c) and Wan et al. (2018) conveyed the generalized PF distance measure on account of PF sets. Applying the definition proposed by Ren et al. (2016), the Euclidean distances between $p_{\phi j}$ and p_{+j} and between $p_{\varphi j}$ and p_{+j} are respectively:

$$\sqrt{\frac{1}{2}\cdot\left[\left((\mu_{\phi j})^2 - (\mu_{+j})^2\right)^2 + \left((\nu_{\phi j})^2 - (\nu_{+j})^2\right)^2 + \left((\tau_{\phi j})^2 - (\tau_{+j})^2\right)^2\right]}$$
$$= \sqrt{\frac{1}{2}\cdot\left[\left(0.1600^2 - 1^2\right)^2 + \left(0.3500^2 - 0^2\right)^2 + \left(0.9230^2 - 0^2\right)^2\right]} = 0.9193$$

and 0.9547 in Case 1, 0.8197 and 0.8205 in Case 2, and 0.6497 and 0.6743 in Case 3. However, the obtained results are unreasonable because the Euclidean distance between $p_{\phi j}$ and p_{+j} is smaller than that between $p_{\varphi j}$ and p_{+j} in Cases 1–3. This paper makes use of the definitions instituted by Chen (2018c) and Wan et al. (2018) to ascertain the PF distances. Let the distance parameter be equal to five. The generalized PF distances between $p_{\phi j}$ and p_{+j} and between $p_{\varphi j}$ and p_{+j} are respectively:

$$\sqrt[5]{\frac{1}{2}\cdot\left[\left|(\mu_{\phi j})^2 - (\mu_{+j})^2\right|^5 + \left|(\nu_{\phi j})^2 - (\nu_{+j})^2\right|^5 + \left|(\tau_{\phi j})^2 - (\tau_{+j})^2\right|^5\right]}$$
$$= \sqrt[5]{\frac{1}{2}\cdot\left[\left|0.1600^2 - 1^2\right|^5 + \left|0.3500^2 - 0^2\right|^5 + \left|0.9230^2 - 0^2\right|^5\right]} = 0.9212$$

and 0.9549 in Case 1, 0.8219 and 0.8227 in Case 2, and 0.6610 and 0.6784 in Case 3. Nonetheless, such results are unacceptable because the generalized PF distance between $p_{\phi j}$ and p_{+j} is smaller than that between $p_{\varphi j}$ and p_{+j} in Cases 1–3. The comparison results through the agency of the Euclidean distance measure and the generalized PF distance measure violate the partial order $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ and the scalar-valued functions $V(p_{\varphi j}) \geqslant V(p_{\phi j})$. On account of the above discussions, the proposed concept of squared PF Euclidean distances can indeed produce believable and desirable results, i.e., $S(p_{\phi j},p_{+j}) \geqslant S(p_{\varphi j},p_{+j})$ in the three cases.

The positive- and negative-ideal PF evaluative ratings are conceived to be the points of reference since they serve as approachable targets (via positive ideals) and avoidable targets (via negative ideals) and furnish anchors for the ascertainment of dominance intensities. In general, PF evaluative ratings that have closer squared PF Euclidean distances from p_{+j} are more favorable and advantageous. In contrast, PF evaluative ratings that have farther squared PF Euclidean distances from p_{-j} are more recommendable. Consider $p_{\phi j}$ and $p_{\varphi j}$ of $z_\phi, z_\varphi \in Z$, respectively, in regard to $c_j \in C$ as an example. When $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$, it is reasonable to infer that z_ϕ performs worse than z_φ in relation to a benefit criterion $c_j \in C_{\mathrm{I}}$ because of the lower degree of membership (i.e., $\mu_{\phi j} \leqslant \mu_{\varphi j}$), higher degree of non-membership (i.e., $\nu_{\phi j} \geqslant \nu_{\varphi j}$), and higher degree of indeterminacy (i.e., $\tau_{\phi j} \geqslant \tau_{\varphi j}$). When $p_{\varphi j} \leqslant_{L_{PF}} p_{\phi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$, it implies that z_ϕ performs worse than z_φ in relation to a cost criterion $c_j \in C_{\mathrm{II}}$ because of the higher degree of membership (i.e., $\mu_{\phi j} \geqslant \mu_{\varphi j}$), lower degree of non-membership (i.e., $\nu_{\phi j} \leqslant \nu_{\varphi j}$), and higher degree of indeterminacy (i.e., $\tau_{\phi j} \geqslant \tau_{\varphi j}$). As indicated in Theorem 5, (T5.1) and (T5.3) verify the accuracy of the relation $S(p_{\phi j},p_{+j}) \geqslant S(p_{\varphi j},p_{+j})$ for $c_j \in C_{\mathrm{I}}$ and $c_j \in C_{\mathrm{II}}$, respectively. Specifically, for $c_j \in C$, higher squared PF Euclidean distances from p_{+j} indicate a weaker preference, and conversely, lower squared PF Euclidean distances from p_{+j} indicate a stronger preference. It is concluded that z_ϕ performs superior than z_φ as regards to $c_j \in C_{\mathrm{I}}$ and $c_j \in C_{\mathrm{II}}$ if $p_{\varphi j} \leqslant_{L_{PF}} p_{\phi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$ and if $p_{\phi j} \leqslant_{L_{PF}} p_{\varphi j}$ and $(\tau_{\phi j})^2 \geqslant (\tau_{\varphi j})^2$, respectively. Moreover, (T5.2) and (T5.4) examine the validity of the relation $S(p_{\phi j},p_{-j}) \geqslant S(p_{\varphi j},p_{-j})$ in respect to $c_j \in C_{\mathrm{I}}$ and $c_j \in C_{\mathrm{II}}$, respectively. Accordingly, for $c_j \in C$, higher squared PF Euclidean distances from p_{-j} advocate a stronger preference; conversely, lower squared PF Euclidean distances from p_{-j} indicate a weaker preference.

To determine the dominance relations among PF evaluations and conduct intracriterion contrasts and judgments, this study specifies a novel SED-based dominance index for appraising the span to which p_{ij} is far from p_{-j} and close to p_{+j} simultaneously. The advanced SED-based dominance index is capable of incarnating an overall counterpoise of p_{ij} between the remotest connection with p_{-j} and the connection with p_{+j}.

Definition 6. Consider a PF evaluative rating p_{ij} $(= (\mu_{ij}, \nu_{ij}; r_{ij}, d_{ij}))$ in the characteristic P_i. The SED-based dominance index ψ of p_{ij} is expounded as below:

$$\psi(p_{ij}) = \frac{S(p_{ij}, p_{-j})}{S(p_{ij}, p_{+j}) + S(p_{ij}, p_{-j})} \tag{22}$$

Theorem 6. The SED-based dominance index $\psi(p_{ij})$ of the PF evaluative rating p_{ij} fulfills the succeeding properties:

(T6.1) $0\leqslant\psi(p_{ij})\leqslant1$;
(T6.2) $\psi(p_{ij})=0$ if and only if $p_{ij}=p_{-j}$;
(T6.3) $\psi(p_{ij})=1$ if and only if $p_{ij}=p_{+j}$.

Proof. For (T6.1), the squared PF Euclidean distances $S(p_{ij},p_{-j})$ and $S(p_{ij},p_{+j})$ range from 0 to 1 by virtue of the property of boundedness in (R.1). Thus, it can prove directly that $0\leqslant\psi(p_{ij})\leqslant1$. For the necessity in (T6.2), the condition of $\psi(p_{ij})=0$ indicates that $S(p_{ij},p_{-j})=0$. In line with the separability property in (R.3), it is acquired that $p_{ij}=p_{-j}$. Considering the sufficiency in (T6.2), if $p_{ij}=p_{-j}$, then $S(p_{ij},p_{-j})=0$ in conformity with the property of reflexivity in (R.2). It reveals that $\psi(p_{ij})=0$. For the necessity in (T6.3), one has $S(p_{ij},p_{+j})+S(p_{ij},p_{-j})=S(p_{ij},p_{-j})$ because $\psi(p_{ij})=1$. It can be acquired that $S(p_{ij},p_{+j})=0$, which signifies that $p_{ij}=p_{+j}$. Afterwards, for the sufficiency in (T6.3), if $p_{ij}=p_{+j}$, then $S(p_{ij},p_{+j})=0$, which indicates that $\psi(p_{ij})=1$. This completes the proof.

Theorem 7. The SED-based dominance index $\psi(p_{ij})$ in connection with the ideal PF evaluative ratings p_{+j} and p_{-j} can be identified as follows:

$$\psi(p_{ij})=\begin{cases}\dfrac{1-2(\nu_{ij})^2+\Theta_{ij}}{1+2\left((\tau_{ij})^2-d_{ij}+\Theta_{ij}\right)} & \text{for } c_j\in C_{\mathrm{I}},\\[2ex] \dfrac{2\left(1-(\mu_{ij})^2-d_{ij}\right)+\Theta_{ij}}{1+2\left((\tau_{ij})^2-d_{ij}+\Theta_{ij}\right)} & \text{for } c_j\in C_{\mathrm{II}},\end{cases}\tag{23}$$

where $\Theta_{ij}=(\mu_{ij})^4+(\nu_{ij})^4+(\tau_{ij})^4+(d_{ij})^2$ for brevity.

Proof. Concerning a PF evaluative rating p_{ij}, in line with (20), the numerator in $\psi(p_{ij})$ in the matter of each benefit criterion $c_j\in C_{\mathrm{I}}$ becomes:

$$S(p_{ij},p_{-j})=\frac{1}{3}\left[(\mu_{ij})^4+\left(1-(\nu_{ij})^2\right)^2+(\tau_{ij})^4+(d_{ij})^2\right]$$
$$=\tfrac{1}{3}\left((\mu_{ij})^4+1-2(\nu_{ij})^2+(\nu_{ij})^4+(\tau_{ij})^4+(d_{ij})^2\right)=\tfrac{1}{3}\left(1-2(\nu_{ij})^2+\Theta_{ij}\right)$$

Based on (20) and (21), the denominator in $\psi(p_{ij})$ for $c_j\in C_{\mathrm{I}}$ is derived as below:

$$S(p_{ij},p_{+j})+S(p_{ij},p_{-j})$$
$$=\tfrac{1}{3}\left[\left(1-(\mu_{ij})^2\right)^2+(\nu_{ij})^4+(\tau_{ij})^4+\left(1-d_{ij}\right)^2\right]+\tfrac{1}{3}\left[(\mu_{ij})^4+\left(1-(\nu_{ij})^2\right)^2+(\tau_{ij})^4+(d_{ij})^2\right]$$
$$=\frac{1}{3}\left(2-2(\mu_{ij})^2-2(\nu_{ij})^2+2(\mu_{ij})^4+2(\nu_{ij})^4+2(\tau_{ij})^4+1-2d_{ij}+2(d_{ij})^2\right)$$
$$=\tfrac{1}{3}\left(2(\tau_{ij})^2+2(\mu_{ij})^4+2(\nu_{ij})^4+2(\tau_{ij})^4+1-2d_{ij}+2(d_{ij})^2\right)=\tfrac{1}{3}\left[1+2\left((\tau_{ij})^2-d_{ij}+\Theta_{ij}\right)\right]$$

In the case of $c_j\in C_{\mathrm{I}}$, the following holds:

$$\psi(p_{ij})=\frac{1-2(\nu_{ij})^2+\Theta_{ij}}{1+2\left((\tau_{ij})^2-d_{ij}+\Theta_{ij}\right)}$$

Regarding each cost criterion $c_j\in C_{\mathrm{II}}$, the numerator and denominator in $\psi(p_{ij})$ are calculated in the following manner:

$$S(p_{ij},p_{-j})=\frac{1}{3}\left[\left(1-(\mu_{ij})^2\right)^2+(\nu_{ij})^4+(\tau_{ij})^4+\left(1-d_{ij}\right)^2\right]$$
$$=\frac{1}{3}\left(1-2(\mu_{ij})^2+(\mu_{ij})^4+(\nu_{ij})^4+(\tau_{ij})^4+1-2d_{ij}+(d_{ij})^2\right)$$
$$=\frac{1}{3}\left(2-2(\mu_{ij})^2-2d_{ij}+\Theta_{ij}\right)=\frac{1}{3}\left[2\left(1-(\mu_{ij})^2-d_{ij}\right)+\Theta_{ij}\right],$$

$$S(p_{ij},p_{+j})+S(p_{ij},p_{-j})=\frac{1}{3}\left[(\mu_{ij})^4+\left(1-(\nu_{ij})^2\right)^2+(\tau_{ij})^4+(d_{ij})^2\right]$$
$$+\frac{1}{3}\left[\left(1-(\mu_{ij})^2\right)^2+(\nu_{ij})^4+(\tau_{ij})^4+\left(1-d_{ij}\right)^2\right]$$
$$=\frac{1}{3}\left(2-2(\mu_{ij})^2-2(\nu_{ij})^2+2(\mu_{ij})^4+2(\nu_{ij})^4+2(\tau_{ij})^4+2(d_{ij})^2+1-2d_{ij}\right)$$
$$=\frac{1}{3}\left(2(\tau_{ij})^2+2(\mu_{ij})^4+2(\nu_{ij})^4+2(\tau_{ij})^4+2(d_{ij})^2+1-2d_{ij}\right)$$
$$=\frac{1}{3}\left[1+2\left((\tau_{ij})^2-d_{ij}+\Theta_{ij}\right)\right]$$

It should be noted that $S(p_{ij},p_{+j})+S(p_{ij},p_{-j})=(1/3)[1+2((\tau_{ij})^2-d_{ij}+\Theta_{ij})]$ regardless of benefit and cost criteria. In the case of $c_j\in C_{\mathrm{II}}$, the following is derived:

$$\psi(p_{ij})=\frac{2\left(1-(\mu_{ij})^2-d_{ij}\right)+\Theta_{ij}}{1+2\left((\tau_{ij})^2-d_{ij}+\Theta_{ij}\right)}$$

This completes the proof.

Definition 7. On grounds of the SED-based dominance index ψ, the dominance relations between PF evaluative ratings $p_{\phi j}$ and $p_{\varphi j}$ (for ϕ, $\varphi\in\{1,2,\cdots,m\}$ and $j\in\{1,2,\cdots,n\}$) are defined via the following SED-based outranking approach:

(D7.1) $p_{\phi j}\prec_{SED}p_{\varphi j}$ if $\psi(p_{\phi j})<\psi(p_{\varphi j})$;
(D7.2) $p_{\phi j}\sim_{SED}p_{\varphi j}$ if $\psi(p_{\phi j})=\psi(p_{\varphi j})$;
(D7.3) $p_{\phi j}\succ_{SED}p_{\varphi j}$ if $\psi(p_{\phi j})>\psi(p_{\varphi j})$.

Example 5. (*(Continued Example 3)*) According to (23) in Theorem 7, the SED-based dominance indices are computed as follows:

$\psi(p_{\phi j})=\frac{1-2(\nu_{\phi j})^2+\Theta_{\phi j}}{1+2\left((\tau_{\phi j})^2-d_{\phi j}+\Theta_{\phi j}\right)}=\frac{1-2\cdot0.3500^2+0.8160}{1+2(0.9230^2-0.2730+0.8160)}=0.4145$,

where $\Theta_{\phi j}=0.1600^4+0.3500^4+0.9230^4+0.2730^2=0.8160$, and $\psi(p_{\varphi j})=0.4800$ in Case 1, $\psi(p_{\phi j})=0.5000$ and $\psi(p_{\varphi j})=0.5019$ in Case 2, and $\psi(p_{\phi j})=0.4151$ and $\psi(p_{\varphi j})=0.5698$ in Case 3. From (D7.1), it is clear that $p_{\phi j}\prec_{SED}p_{\varphi j}$ because $\psi(p_{\phi j})<\psi(p_{\varphi j})$ in Cases 1–3. Based on Definition 6, the *DI* index can be determined alternately using the squared PF Euclidean distances on the subject of the ideal PF ratings p_{+j} and p_{-j}. The squared PF Euclidean distances from p_{+j} have been provided in Example 3. The squared PF Euclidean distances from p_{-j} are calculated as below: $S(p_{\phi j},p_{-j})=0.5236$ and $S(p_{\varphi j},p_{-j})=0.6603$ in Case 1, $S(p_{\phi j},p_{-j})=0.5313$ and $S(p_{\varphi j},p_{-j})=0.5350$ in Case 2, and $S(p_{\phi j},p_{-j})=0.2733$ and $S(p_{\varphi j},p_{-j})=0.4807$ in Case 3. Observe that $p_{\varphi j}$ has a larger squared PF Euclidean distance from p_{-j} than $p_{\phi j}$ because $S(p_{\phi j},p_{-j})\leqslant S(p_{\varphi j},p_{-j})$ hold in Cases 1–3. This reveals that $p_{\varphi j}$ is more desirable than $p_{\phi j}$ according to the comparison results via squared PF Euclidean distances from p_{-j}. According to Definition 6, the SED-based dominance index $\psi(p_{\phi j})$ in Case 1 is determined as below:

$\psi(p_{\phi j})=\frac{S(p_{\phi j},p_{-j})}{S(p_{\phi j},p_{+j})+S(p_{\phi j},p_{-j})}=\frac{0.5236}{0.7396+0.5236}=0.4145$,

which is the same as that using (23) in Theorem 7. In accordance with the derived result $p_{\phi j}\prec_{SED}p_{\varphi j}$ in Cases 1–3, there is concordance

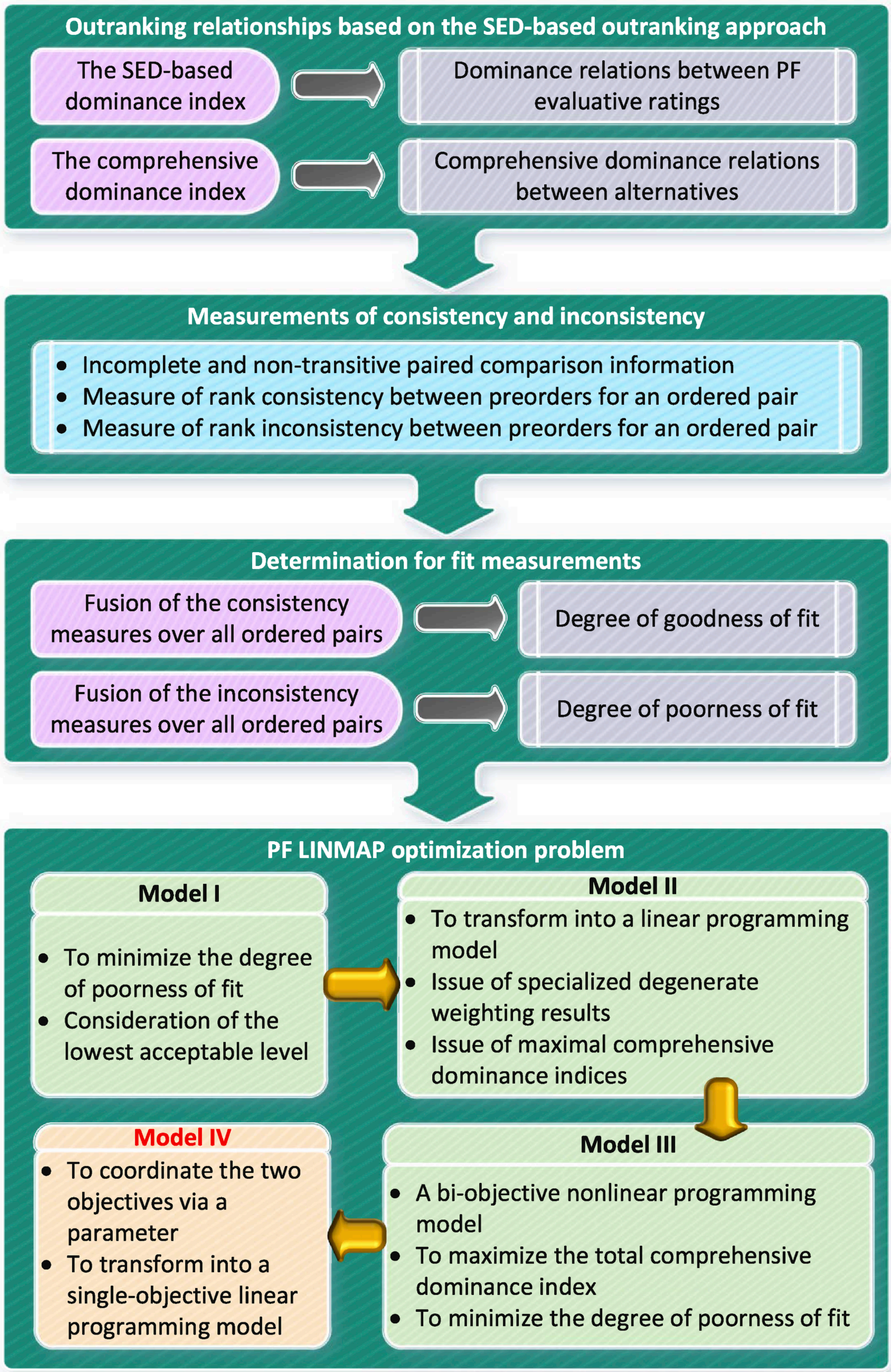


Fig. 3. Theoretical framework of the proposed PF LINMAP methodology.

among the comparison results yielded by the squared PF Euclidean distances and by the scalar-valued functions in Example 3. The use of the SED-based dominance index ψ can identify and differentiate the dominance intensities among PF evaluative ratings. More precisely, the differences between $\psi(p_{\phi j})$ and $\psi(p_{\varphi j})$ are more significant than those between $S(p_{\phi j}, p_{+j})$ and $S(p_{\varphi j}, p_{+j})$ and between $V(p_{\phi j})$ and $V(p_{\varphi j})$ in three cases. Thus, the ψ index can serve a comprehensive comparison tool to determine intracriterion dominance extent among alternatives.

Based on the SED-based dominance index $\psi(p_{ij})$, (D7.1)–(D7.3) provide an easy-to-use and convincing law, namely, a SED-based outranking approach, for differentiating the PF evaluative ratings p_{ij} on the subject of a specific criterion c_j. By way of the SED-based outranking approach, the intracriterion comparison of PF evaluative ratings and the determination of dominance relationships among the alternatives can be implemented in a manner that is uncomplicated and simple to understand. Over and above that, this paper makes use of differences between SED-based dominance indices to ascertain the rank consistency and inconsistency measurements for each ordered pair of alternatives for use in determining goodness and poorness of fit in the PF LINMAP methodology.

4. PF LINMAP methods

This section intends to constitute a beneficial PF LINMAP methodology that uses the proposed SED-based dominance index and the SED-based outranking approach. Some useful measurements of consistency and inconsistency are proposed to appraise the goodness of fit and poorness of fit towards each ordered pair of alternatives with relevance to paired preference relations indicated by the decision maker. A straightforward and efficacious optimization model is formulated to resolve the optimal criterion weights and identify the best compromise alternative in PF decision circumstances.

4.1. Proposed methodology

The decision maker establishes the paired preference relations within a finite set of alternatives in the matter of multiple criteria. To match the subjective preferences of one alternative over another, a SED-based outranking approach using SED-based dominance indices can be employed in the PF LINMAP procedure. This dominance relation is characterized by the squared PF Euclidean distances from the largest and smallest Pythagorean membership grades. Additionally, such a SED-based outranking relation is constructed and an application is presented to ascertain the goodness of fit and poorness of fit from the distributive PF evaluative ratings over the various criteria. To make the conceptual process clear, Fig. 3 depicts the theoretical framework of the advanced PF LINMAP methodology via a SED-based outranking approach.

The outranking relationship defined by the proposed SED-based outranking approach is a dominance relation to identify preference between two PF evaluative ratings. It is conceived that the PF evaluative rating $p_{\phi j}$ outranks $p_{\varphi j}$ if and only if there is adequate evidence to believe that $p_{\phi j}$ is better than $p_{\varphi j}$ or at least $p_{\phi j}$ is as good as $p_{\varphi j}$. The developed concept of SED-based dominance indices can provide tangible evidence based on a solid theoretical basis. More specifically, applying Definition 7, if $\psi(p_{\phi j}) \geqslant \psi(p_{\varphi j})$, it would be concluded that $p_{\phi j}$ outranks $p_{\varphi j}$, i.e., $p_{\phi j} \succsim SED p_{\varphi j}$.

To further capture the dominance relation of one alternative over another with respect to multiple criteria, a concept of comprehensive dominance indices can be used. The comprehensive dominance index is expounded as the weighted sum of the SED-based dominance indices over all criteria. Let $w = (w_1, w_2, \cdots, w_n)$ represent the weight vector relevant to the evaluative criteria, in which $0 \leqslant w_j \leqslant 1$ for each criterion c_j and $\sum_{j=1}^{n} w_j = 1$. It is noted that the exact value of w_j is unknown a priori and should be ascertained in the PF LINMAP approach. For each alternative z_i, the comprehensive dominance index is determined by multiplying the SED-based dominance index $\psi(p_{ij})$ by the unknown weight w_j and then aggregating these products across the n criteria.

Definition 8. Given the weight vector w and the characteristic P_i of each alternative $z_i \in Z$, the comprehensive dominance index $\Psi(P_i)$ for each z_i is described as follows:

$$\Psi(P_i) = \sum_{j=1}^{n} w_j \cdot \psi(p_{ij}) \tag{24}$$

where $w_j \in [0, 1]$ for all $c_j \in C$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 8. For each characteristic P_i in the context of the alternative set Z, the comprehensive dominance index $\Psi(P_i)$ meets the succeeding properties:

(T8.1) $0 \leqslant \Psi(P_i) \leqslant 1$;
(T8.2) $\Psi(P_i) = 0$ if $p_{ij} = p_{-j}$ for all $c_j \in C$;
(T8.3) $\Psi(P_i) = 1$ if $p_{ij} = p_{+j}$ for all $c_j \in C$.

Proof. The properties (T8.1)–(T8.3) are obviously corroborated from (T6.1) to (T6.3), respectively, which completes the proof.

The core techniques of the advanced PF LINMAP methodology involve the comparison among the pairwise preference information and the comprehensive dominance relations. The preference relations between alternatives in the paired comparison information are indicated by the decision maker based on individual opinion or experience in the classical LINMAP approach. Nonetheless, incomplete and non-transitive judgments often occur because of various determinants such as limited knowledge and expertise in making the decision (Kou et al., 2016). To preserve most of the original information provided by the decision maker, this paper defines two measurements based on the perspectives of rank consistency and inconsistency. Consider two alternatives z_ϕ and z_φ in Z. For each ordered pair $(\phi, \varphi) \in \Omega$, there are concordance and discordance if z_ϕ and z_φ in the PF LINMAP result are ranked in the identical order and counter-ranked, respectively, within the two preorders. If z_ϕ and z_φ have the identical rank, ex aequo exists. As mentioned previously, the PF evaluative rating $p_{\phi j}$ outranks $p_{\varphi j}$ (i.e., $p_{\phi j} \succsim SED p_{\varphi j}$) if $\psi(p_{\phi j}) \geqslant \psi(p_{\varphi j})$. Accordingly, it is anticipated that the comprehensive dominance indices $\Psi(P_\phi)$ and $\Psi(P_\varphi)$ of z_ϕ and z_φ, respectively, which are connected with the optimal weight vector, should satisfy the inequality of $\Psi(P_\phi) \geqslant \Psi(P_\varphi)$. Moreover, it can be inferred that a higher value of $\Psi(P_i)$ indicates a stronger preference for the alternative z_i. The proposed PF LINMAP methodology utilizes this rationale to ascertain the consistency measure and the inconsistency measure for a couple of alternatives. More precisely, no error is redounded to the preference relation between z_ϕ and z_φ towards the ordered pair (ϕ, φ) if $\Psi(P_\phi) \geqslant \Psi(P_\varphi)$; in contrast, if $\Psi(P_\phi) < \Psi(P_\varphi)$, then errors exist that will be attributable to the preference relation revealed by the decision maker. In this regard, the following definition provides the measurements of rank consistency and rank inconsistency for each ordered pair (ϕ, φ) based on the comprehensive dominance indices $\Psi(P_\phi)$ and $\Psi(P_\varphi)$.

Definition 9. Given the characteristics P_ϕ and P_φ of the alternatives z_ϕ and z_φ, respectively, and the set of paired preference relations Ω, the measure of rank consistency between the preorders of z_ϕ and z_φ for an ordered pair $(\phi, \varphi) \in \Omega$ is expounded as follows:

$$(\Psi(P_\phi) - \Psi(P_\varphi))^+ = \begin{cases} \Psi(P_\phi) - \Psi(P_\varphi) & \text{if } \Psi(P_\phi) \geqslant \Psi(P_\varphi), \\ 0 & \text{if } \Psi(P_\phi) < \Psi(P_\varphi). \end{cases} \tag{25}$$

The measure of rank inconsistency between the preorders for (ϕ, φ) is expounded as below:

$$(\Psi(P_\phi) - \Psi(P_\varphi))^- = \begin{cases} \Psi(P_\varphi) - \Psi(P_\phi) & \text{if } \Psi(P_\phi) < \Psi(P_\varphi), \\ 0 & \text{if } \Psi(P_\phi) \geqslant \Psi(P_\varphi). \end{cases} \tag{26}$$

Theorem 9. *For each ordered pair* $(\phi,\varphi)\in\Omega$, *the consistency measure* $(\Psi(P_\phi)-\Psi(P_\varphi))^+$ *and the inconsistency measure* $(\Psi(P_\phi)-\Psi(P_\varphi))^-$ *satisfy the following properties*:

(T9.1) $(\Psi(P_\phi)-\Psi(P_\varphi))^+ = max\{0,\Psi(P_\phi)-\Psi(P_\varphi)\}$;
(T9.2) $(\Psi(P_\phi)-\Psi(P_\varphi))^- = max\{0,\Psi(P_\varphi)-\Psi(P_\phi)\}$;
(T9.3) $(\Psi(P_\phi)-\Psi(P_\varphi))^+\geqslant 0$ and $(\Psi(P_\phi)-\Psi(P_\varphi))^-\geqslant 0$;
(T9.4) $(\Psi(P_\phi)-\Psi(P_\varphi))^+ - (\Psi(P_\phi)-\Psi(P_\varphi))^- = \Psi(P_\phi)-\Psi(P_\varphi)$.

Proof. Properties in (T9.1) and (T9.2) can be easily proved from the definitions in (25) and (26), respectively. (T9.3) is trivially true based on (T9.1) and (T9.2). For (T9.4), according to Definition 9, one obtains:

$$(\Psi(P_\phi)-\Psi(P_\varphi))^+ - (\Psi(P_\phi)-\Psi(P_\varphi))^- = \begin{cases} (\Psi(P_\phi)-\Psi(P_\varphi))-0 & \text{if } \Psi(P_\phi)\geqslant\Psi(P_\varphi) \\ 0-(\Psi(P_\varphi)-\Psi(P_\phi)) & \text{if } \Psi(P_\phi)<\Psi(P_\varphi) \end{cases} = \Psi(P_\phi)-\Psi(P_\varphi).$$

Consequently, (T9.4) is correct, which completes the proof.

The consistency measure $(\Psi(P_\phi)-\Psi(P_\varphi))^+$ ascertains the level of consistency between the ranking orders of z_ϕ and z_φ in which the ordered pair (ϕ,φ) is acquired in the light of the subjective preference relationships in Ω, and the other ranking order is identified on the basis of the obtained $\Psi(P_\phi)$ and $\Psi(P_\varphi)$. Next, this paper sums the consistency measure $(\Psi(P_\phi)-\Psi(P_\varphi))^+$ over all $(\phi,\varphi)\in\Omega$; accordingly, the degree of goodness of fit G is quantified in the following manner:

$$G=\sum_{(\phi,\varphi)\in\Omega}(\Psi(P_\phi)-\Psi(P_\varphi))^+ \tag{27}$$

where $G\geqslant 0$ because $(\Psi(P_\phi)-\Psi(P_\varphi))^+\geqslant 0$ from (T9.3).

The inconsistency measure $(\Psi(P_\phi)-\Psi(P_\varphi))^-$ appraises the level of inconsistency towards the ranking orders of z_ϕ and z_φ; it can describe the error for each ordered pair $(\phi,\varphi)\in\Omega$. Specifically, when $\Psi(P_\phi)<\Psi(P_\varphi)$ towards pair (ϕ,φ), the difference between $\Psi(P_\phi)$ and $\Psi(P_\varphi)$ denotes the error. By way of explanation, the difference $\Psi(P_\varphi)-\Psi(P_\phi)$ signifies the degree to which the condition $\Psi(P_\phi)\geqslant\Psi(P_\varphi)$ is violated. This paper sums the inconsistency measure $(\Psi(P_\phi)-\Psi(P_\varphi))^-$ over all $(\phi,\varphi)\in\Omega$; thereupon, the degree of poorness of fit B is measured as below:

$$B=\sum_{(\phi,\varphi)\in\Omega}(\Psi(P_\phi)-\Psi(P_\varphi))^- \tag{28}$$

where $B\geqslant 0$ because $(\Psi(P_\phi)-\Psi(P_\varphi))^-\geqslant 0$ from (T9.3).

In general, a larger B value indicates a greater degree of violation of the conditions associated with the preference relations revealed in the set Ω. To acquire the optimal weight vector for which B is minimal, this paper establishes an optimization model to minimize the B value in the proposed PF LINMAP methodology. Particularly, it is rational and understandable to postulate that the decision maker desires to receive the PF LINMAP solution with high goodness of fit and low poorness of fit. It directly follows that the condition of $G\geqslant B$ should be amalgamated to the proposed optimization model. In what follows, let a non-negative number $\hbar$ denote the lowest acceptable level relating to the difference between G and B. Thereby, the imposed condition in which G must be larger than or equal to B is expressed as follows:

$$G-B\geq\hbar \tag{29}$$

This paper attempts to formulate an optimization problem in order that paired preference judgments compared to the given ordered pairs in Ω are violated as minimally as possible. Specifically, to minimize the poorness of fit B under the consideration that the goodness of fit G is not smaller than B by $\hbar$, the optimization problem Model I is established as below:

$$\textbf{Model I}.\ min\{B\}$$
$$\text{s.t.}\begin{cases} G-B\geqslant\hbar, \\ \sum_{j=1}^{n} w_j=1,\ w_j\geqslant 0 \text{ for all } j. \end{cases} \tag{30}$$

The poorness of fit B is composed of the inconsistency measures $(\Psi(P_\phi)-\Psi(P_\varphi))^-$ for all $(\phi,\varphi)\in\Omega$. It directly follows that B is identified by the comprehensive dominance indices $\Psi(P_\phi)$ and $\Psi(P_\varphi)$, which involve the unknown importance weights of the criteria. Because $(\Psi(P_\phi)-\Psi(P_\varphi))^-= max\{0,\Psi(P_\varphi)-\Psi(P_\phi)\}$ from (T9.2), the objective B in Model I can be represented as below:

$$B=\sum_{(\phi,\varphi)\in\Omega} max\{0,\Psi(P_\varphi)-\Psi(P_\phi)\} = \sum_{(\phi,\varphi)\in\Omega} max\left\{0,\sum_{j=1}^{n}\left(\psi(p_{\varphi j})-\psi(p_{\phi j})\right)\cdot w_j\right\} \tag{31}$$

Let $\Gamma_{\phi\varphi}$ denote the degree of violation concerning the ordered pair (ϕ,φ); it is defined as the maximum of 0 and $\Psi(P_\varphi)-\Psi(P_\phi)$ for each $(\phi,\varphi)\in\Omega$. Specifically, $\Gamma_{\phi\varphi}$ in relation to $\Psi(P_\phi)$ and $\Psi(P_\varphi)$ represents the extent contrary to the decision maker's preference relation $z_\phi\succeq z_\varphi$, as below:

$$\Gamma_{\phi\varphi}=max\{0,\Psi(P_\varphi)-\Psi(P_\phi)\}=max\left\{0,\sum_{j=1}^{n}\left(\psi(p_{\varphi j})-\psi(p_{\phi j})\right)\cdot w_j\right\} \tag{32}$$

Based on (31), it directly follows that $B=\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}$. Obviously, $\Gamma_{\phi\varphi}\geqslant 0$ and $\Gamma_{\phi\varphi}\geqslant\sum_{j=1}^{n}\left(\psi(p_{\varphi j})-\psi(p_{\phi j})\right)\cdot w_j$ in connection to $(\phi,\varphi)\in\Omega$. It is inferred that:

$$\sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j+\Gamma_{\phi\varphi}\geqslant 0 \tag{33}$$

According to Definition 8 and the property (T9.4), one obtains:

$$G-B=\sum_{(\phi,\varphi)\in\Omega}(\Psi(P_\phi)-\Psi(P_\varphi))=\sum_{(\phi,\varphi)\in\Omega}\sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j. \tag{34}$$

In this manner, Model I is transfigured into the succeeding linear programming format:

$$\textbf{ModelII}.\ min\left\{B=\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}\right\}$$
$$\text{s.t.}\begin{cases} \sum_{(\phi,\varphi)\in\Omega}\sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j\geqslant\hbar, \\ \sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j+\Gamma_{\phi\varphi}\geqslant 0 \text{ for } (\phi,\varphi)\in\Omega, \\ \Gamma_{\phi\varphi}\geqslant 0 \text{ for } (\phi,\varphi)\in\Omega,\ \sum_{j=1}^{n} w_j=1,\ w_j\geqslant 0 \text{ for all } j. \end{cases} \tag{35}$$

To acquire an acceptable and convincing result that is in concordance with the paired preference relations (ϕ,φ) offered by decision maker in the set Ω as possible, this paper formulates Model II to determine the optimal $\overline{\Gamma}_{\phi\varphi}$ values for each $(\phi,\varphi)\in\Omega$ and the optimal weight vector $\overline{w}=(\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_n)$. Accordingly, one receives the optimal comprehensive dominance indices $\overline{\Psi}(P_i)$ for all $z_i\in Z$ and next acquire the priority orders of the alternatives in accordance with the obtained $\overline{\Psi}(P_i)$ values. Moreover, the paired preference relations in Ω, as expected, is violated as minimally as possible with the optimal solution. Nevertheless, such an optimization problem in Model II for which $\sum_{(\phi,\varphi)\in\Omega}\overline{\Gamma}_{\phi\varphi}$ is minimal might lead to a trivial solution involving $\overline{w}_j=1$ for a particular criterion c_j and $\overline{w}_{j'}=0$ for the remaining $n-1$ criteria $(c_{j'}\in C)$, which indicates a specialized degenerate weighting result. To overcome such difficulty, this paper modifies the range of the criterion weights by adding some non-negative boundary conditions. More precisely, one can apply the constraints that $w_j \geqslant \varepsilon_j$ for each $c_j \in C$, in which ε_j is a sufficiently small non-negative number. For convenience, without loss of generality, assume that all ε_j values are the same, i.e., $\varepsilon_j = \varepsilon_0$ for each c_j.

The comprehensive dominance index $\Psi(P_i)$ is obtained by adding the weighted SED-based dominance indices (i.e., $w_j \cdot \psi(p_{ij})$) over the n criteria. The alternative with the highest comprehensive dominance index will be the decision maker's preferable option. To be specific, a larger value of $\Psi(P_i)$ gives rise to a greater preference for the alternative z_i. Thus, it is anticipated that the decision maker desires to ascertain a solution of criterion weights in such a way that the total sum of $\Psi(P_i)$ for all m alternatives is maximal. In addition to the minimal B value (i.e., $\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}$), the proposed optimization model should involve another objective to maximize the total sum of the Ψ values, namely $\sum_{i=1}^{m}\Psi(P_i) = \sum_{i=1}^{m}\sum_{j=1}^{n}\psi(p_{ij})\cdot w_j$. The PF LINMAP methodology maximizes the total comprehensive dominance index and minimizes the degree of poorness of fit. More specifically, this paper constitutes the succeeding bi-objective nonlinear programming model:

$$\textbf{Model III}.\ max\left\{\sum_{i=1}^{m}\Psi(P_i) = \sum_{i=1}^{m}\sum_{j=1}^{n}\psi(p_{ij})\cdot w_j\right\},\ min\left\{B = \sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}\right\}$$

$$\text{s.t.}\begin{cases}\displaystyle\sum_{(\phi,\varphi)\in\Omega}\sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j \geqslant \hbar,\\ \displaystyle\sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j + \Gamma_{\phi\varphi} \geqslant 0 \text{ for } (\phi,\varphi)\in\Omega,\\ \displaystyle\Gamma_{\phi\varphi}\geqslant 0 \text{ for } (\phi,\varphi)\in\Omega,\ \sum_{j=1}^{n}w_j = 1,\ w_j\geqslant\varepsilon_j \text{ for all } j.\end{cases} \tag{36}$$

It is noted that the minimal B objective is equivalent to the maximal $-B$ objective, namely, $min\left\{B = \sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}\right\} = max\left\{-B = -\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}\right\}$. The maximal objective $\sum_{i=1}^{m}\sum_{j=1}^{n}\psi(p_{ij})\cdot w_j$ and the constraint $\sum_{(\phi,\varphi)\in\Omega}\sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j \geq \hbar$ can be written as $\sum_{j=1}^{n}\sum_{i=1}^{m}\psi(p_{ij})\cdot w_j$ and $\sum_{j=1}^{n}\sum_{(\phi,\varphi)\in\Omega}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j \geq \hbar$, respectively. By initiating a parameter $\eta\in[0,1]$, the bi-objective optimization problem in Model III is transfigured into a simple single-objective linear programming format. The adoption of the parameter η can efficaciously coordinate the two objectives in Model III. Specifically, the values η and $1-\eta$ represent the weights pertaining to the "total comprehensive dominance index" objective and the "degree of poorness of fit" objective, respectively. Accordingly, the advanced PF LINMAP methodology is unfolded by the succeeding linear programming model:

$$\textbf{Model IV}.\ max\left\{\eta\sum_{j=1}^{n}\sum_{i=1}^{m}\psi(p_{ij})\cdot w_j - (1-\eta)\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}\right\}$$

$$\text{s.t.}\begin{cases}\displaystyle\sum_{j=1}^{n}\sum_{(\phi,\varphi)\in\Omega}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j \geqslant \hbar,\\ \displaystyle\sum_{j=1}^{n}\left(\psi(p_{\phi j})-\psi(p_{\varphi j})\right)\cdot w_j + \Gamma_{\phi\varphi} \geqslant 0 \text{ for } (\phi,\varphi)\in\Omega,\\ \displaystyle\Gamma_{\phi\varphi}\geqslant 0 \text{ for } (\phi,\varphi)\in\Omega,\ \sum_{j=1}^{n}w_j = 1,\ w_j\geqslant\varepsilon_j \text{ for all } j.\end{cases} \tag{37}$$

The optimal $\overline{\Gamma}_{\phi\varphi}$ values for each ordered pair $(\phi,\varphi)\in\Omega$ and the optimal weight vector $\overline{w} = (\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_n)$ are obtained by solving Model IV using the Simplex method. The corresponding comprehensive dominance index $\overline{\Psi}(P_i)$ for each alternative $z_i\in Z$ is determined as follows:

$$\overline{\Psi}(P_i) = \sum_{j=1}^{n}\overline{w}_j\cdot\psi(p_{ij}) \tag{38}$$

In the end, the m candidate alternatives are ranked in accordance with a declining order of the $\overline{\Psi}(P_i)$ values. In mathematical notation, the best compromise solution z^* in $Z^*\subseteq Z$ is ranked the best by $\overline{\Psi}(P_i)$ such that:

$$Z^* = \left\{z_i\middle|\max_{i=1}^{m}\overline{\Psi}(P_i),\ z_i\in Z\right\} = \left\{z_i\middle|\max_{i=1}^{m}\sum_{j=1}^{n}\overline{w}_j\cdot\psi(p_{ij}),\ z_i\in Z\right\} \tag{39}$$

4.2. Proposed algorithm

Under the aegis of SED-based dominance indices, the proposed PF LINMAP method by way of a SED-based outranking technique for tackling an MCDA problem under sophisticated uncertainty of Pythagorean fuzziness can be concretized in the succeeding algorithmic procedure. Additionally, Fig. 4 recapitulates the implementation process of the developed PF LINMAP methodology, comprising of formulation of an MCDA problem, query from the decision maker, determination of SED-based dominance indices, specification of parameter values, construction of a linear programming problem using Model IV, and complete ranking of alternatives.

Algorithm for the PF LINMAP methodology

Step 1: Form the set of candidate alternatives Z $(=\{z_1,z_2,\cdots,z_m\})$ and the set of evaluative criteria C $(=\{c_1,c_2,\cdots,c_n\})$ to construct an MCDA problem. The set C is separated into C_{I} and C_{II}.
Step 2: Use Pythagorean membership grades to constitute the PF evaluative rating p_{ij} in (12) of an alternative $z_i\in Z$ in connection with criterion $c_j\in C$, and then construct the characteristic P_i of z_i using (13).
Step 3: Receive the preference relations between alternatives in Z by conducting an investigation to the decision maker. Sculpture the set of ordered pairs Ω using (14), and set the η value to represent the importance of total comprehensive dominance relative to poorness of fit.
Step 4: Derive the SED-based dominance index $\psi(p_{ij})$ of each p_{ij} on the subject of the ideal PF evaluative ratings p_{+j} and p_{-j} using (23).
Step 5: Set the parameter values. Assign the $\hbar$ value that denotes the lowest acceptable level relating to the difference between goodness and poorness of fit. Specify the ε_j value that represents the non-negative boundary condition of the weight w_j. Alternately, designate the ε_0 value and let $\varepsilon_j=\varepsilon_0$ for each c_j for brevity.
Step 6: Establish a linear programming model using Model IV in (37) by virtue of the degree of violation $\Gamma_{\phi\varphi}$ for each ordered pair $(\phi,\varphi)\in\Omega$ and the weight vector w $(=(w_1,w_2,\cdots,w_n))$.
Step 7: Apply the Simplex method to resolve the model to ascertain the optimal $\overline{\Gamma}_{\phi\varphi}$ values and the optimal vector $\overline{w}=(\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_n)$ of criterion importance.
Step 8: Employ (38) to compute the comprehensive dominance index $\overline{\Psi}(P_i)$ for each $z_i\in Z$, and then rank the m alternatives in Z in the light of the descending order of $\overline{\Psi}(P_i)$. Identify the best compromise solution z^* using (39).

5. Real-world application with comparison studies

To corroborate the feasibility and efficaciousness of the advanced PF LINMAP methodology, this section attempts to investigate a down-to-earth application concerning the assessment of bridge-superstructure construction methods. Moreover, certain comparative analyses with other pertinent techniques, followed by a discussion under various parameter settings, are executed to evaluate the flexibility and rationality of the developed methodology. In the end, the proposed

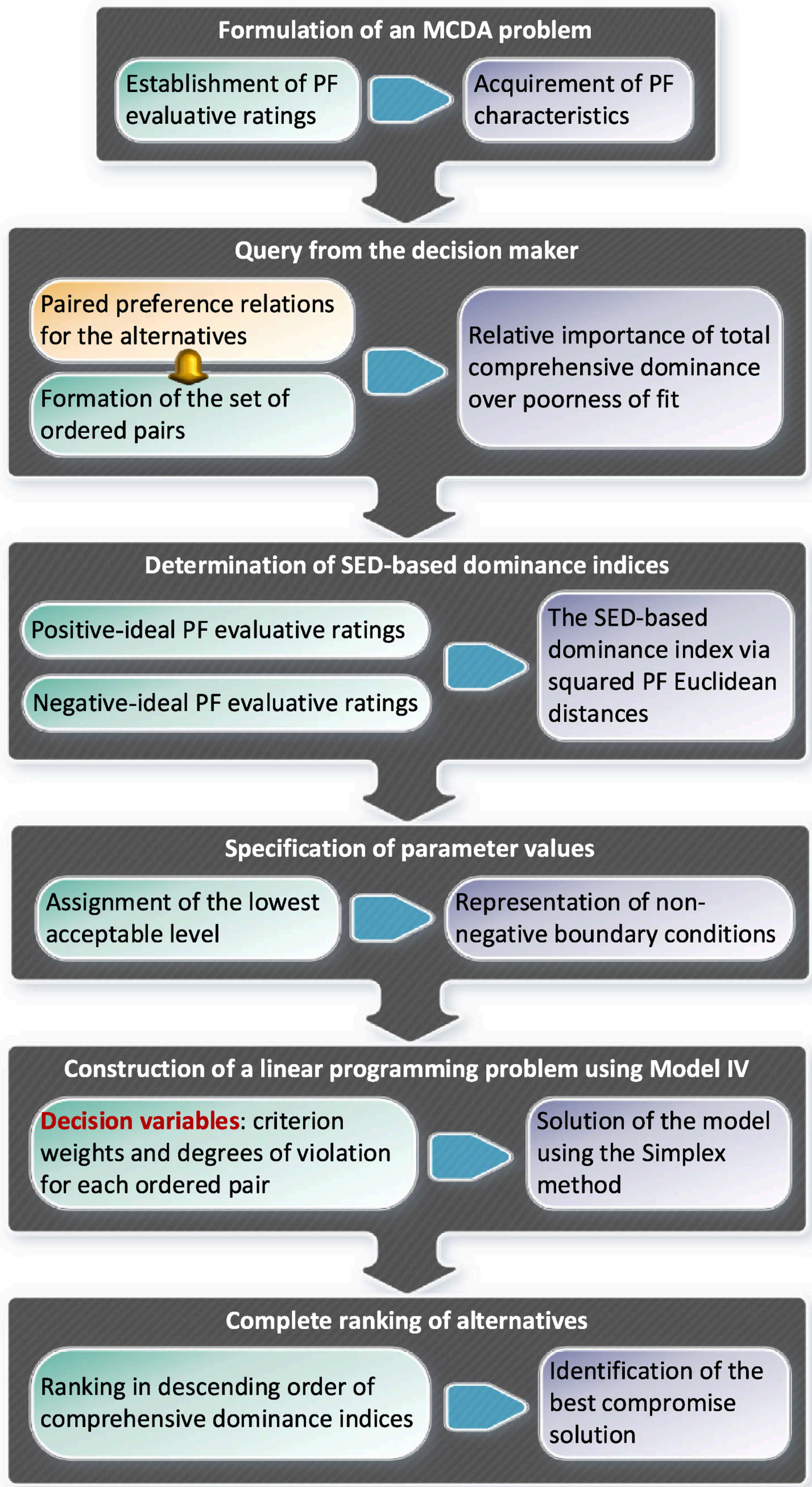


Fig. 4. Procedure of the PF LINMAP methodology.

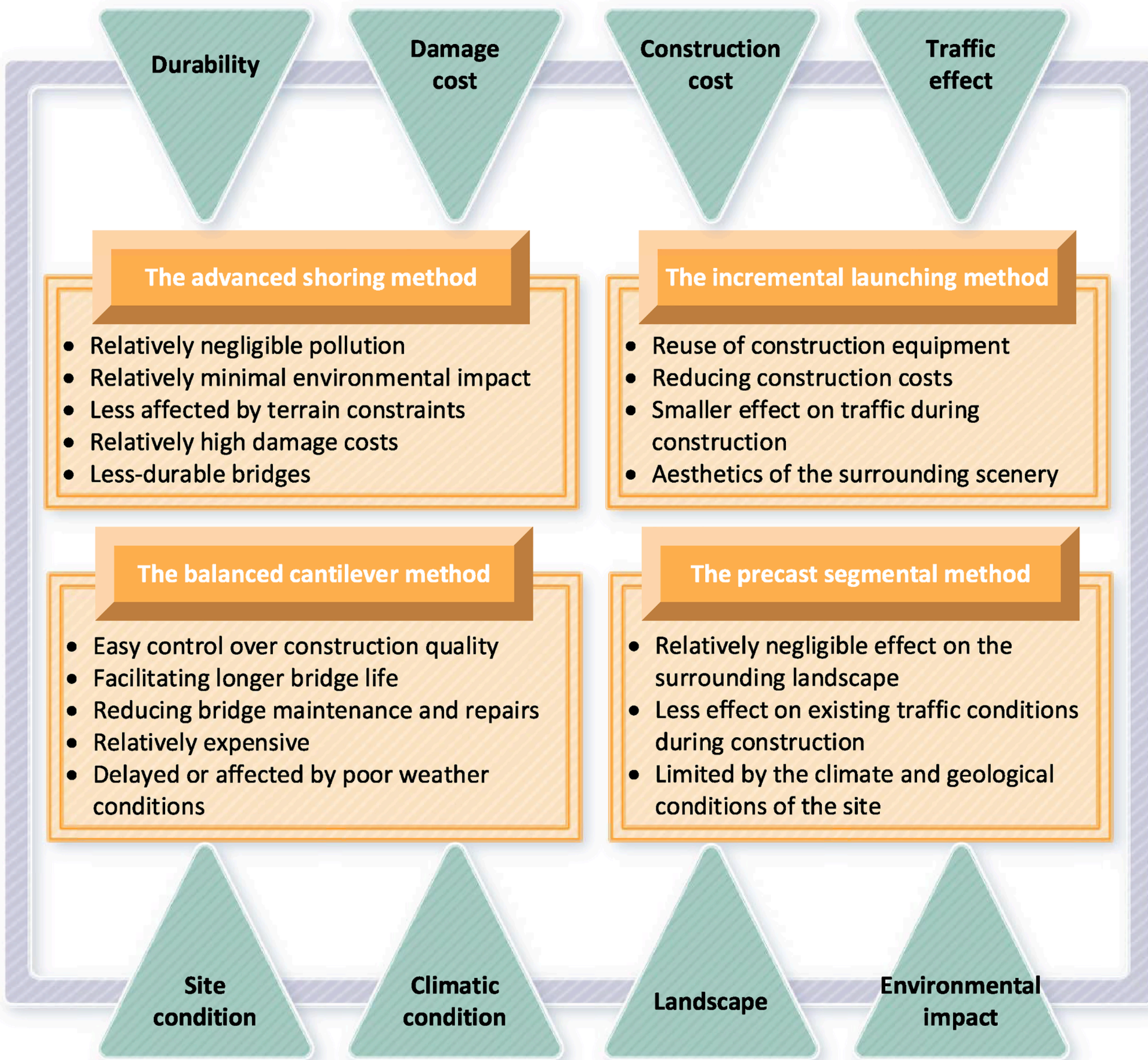


Fig. 5. The selection problem of bridge-superstructure construction methods.

methodology is further extended to manage group decision-making problems in PF settings, along with two applications of green supplier selection and railway project investment.

5.1. *Model illustration and realistic applications*

This subsection intends to explore the selection issue concerning bridge-superstructure construction methods to illustrate the algorithmic procedure and demonstrate the effectuality of the current methods. This realistic problem was originally formulated by Chen (2012, 2014) in interval-valued intuitionistic fuzzy contexts and was further extended by Chen (2018a, 2018b) in interval-valued PF contexts. Recently, Chen (2019a, 2019b) modified the bridge-superstructure construction case to adapt to the PF environments and converted the interval-valued PF data into Pythagorean membership grades. This subsection utilized the decision information based on PF sets for examining the achievability and usefulness of the advanced PF LINMAP methodology.

The practical case under study, which was adopted from Chen (2019a, 2019b), comprises four bridge-construction methods and eight evaluative criteria, as summarized in Fig. 5. As shown in this figure, the candidate alternatives are comprised of the advanced shoring method (z_1), the incremental launching method (z_2), the balanced cantilever method (z_3), and the precast segmental method (z_4). The criteria for evaluating the alternatives consist of durability (c_1), damage cost (c_2), construction cost (c_3), traffic effect (c_4), site condition (c_5), climatic condition (c_6), landscape (c_7), and environmental impact (c_8). Durability and site condition are benefit criteria, whereas the remaining six criteria are cost criteria.

The proposed PF LINMAP methodology using a SED-based outranking approach with SED-based dominance indices was utilized to

Table 1
Results of the SED-based dominance indices.

c_j	p_{+j}	p_{-j}	$\psi(p_{1j})$	$\psi(p_{2j})$	$\psi(p_{3j})$	$\psi(p_{4j})$
c_1	(1,0;1,1)	(0,1;1,0)	0.3613	0.9871	0.9873	0.6435
c_2	(0,1;1,0)	(1,0;1,1)	0.1083	0.9951	0.9913	0.8683
c_3	(0,1;1,0)	(1,0;1,1)	0.9897	0.7319	0.8703	0.7454
c_4	(0,1;1,0)	(1,0;1,1)	0.6800	0.9792	0.5295	0.6565
c_5	(1,0;1,1)	(0,1;1,0)	0.9625	0.6574	0.9625	0.2494
c_6	(0,1;1,0)	(1,0;1,1)	0.9900	0.9881	0.5674	0.0633
c_7	(0,1;1,0)	(1,0;1,1)	0.6722	0.0582	0.9848	0.9979
c_8	(0,1;1,0)	(1,0;1,1)	0.9992	0.4318	0.7319	0.9903

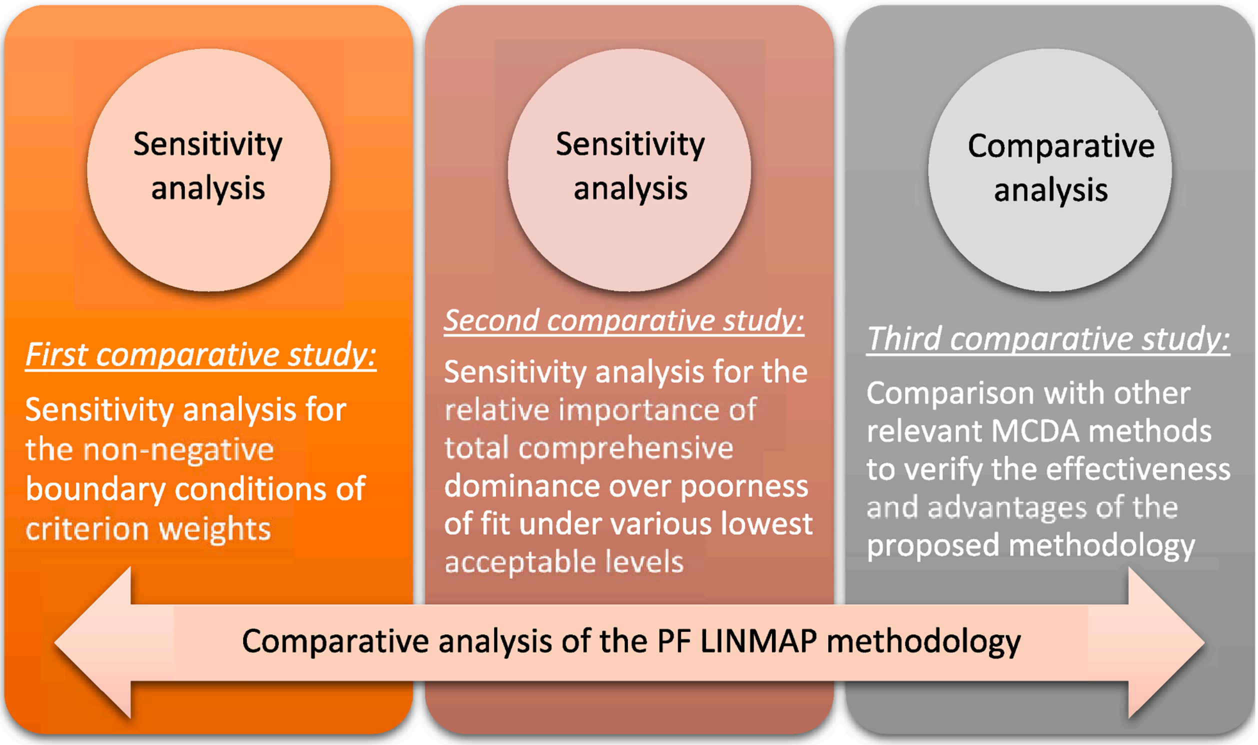


Fig. 6. Framework of the comparative analysis.

solving the selection issue concerning bridge-superstructure construction methods in PF decision situations. In Step 1, the MCDA problem was represented by the agency of the set of candidate alternatives, i.e., $Z = \{z_1, z_2, z_3, z_4\}$, and the set of evaluative criteria, i.e., $C = \{c_1, c_2, \cdots, c_8\}$. Here, the set C is separated into $C_{\text{I}} = \{c_1, c_5\}$ and $C_{\text{II}} = \{c_2, c_3, c_4, c_6, c_7, c_8\}$. In Step 2, based on the data of PF evaluative ratings in Chen (2019a, 2019b), the characteristic P_i of each alternative $z_i \in Z$ can be manifested as a PF format as below:

$$P_1 = \{\langle c_j, p_{1j}\rangle | c_j \in C\}$$
$$= \{\langle c_j, (\mu_{1j}, \nu_{1j}; r_{1j}, d_{1j})\rangle | c_j \in \{c_1, c_2, \cdots, c_8\}\}$$

$=\{\langle c_1,$ (0.5407, 0.6781; 0.8673, 0.4285)$\rangle$, $\langle c_2,$ (0.8075, 0.2638; 0.8495, 0.7990)$\rangle$,
$\langle c_3,$ (0.1654, 0.9508; 0.9651, 0.1096)$\rangle$, $\langle c_4,$ (0.4802, 0.6774; 0.8303, 0.3926)$\rangle$,
$\langle c_5,$ (0.9285, 0.3402; 0.9889, 0.7764)$\rangle$, $\langle c_6,$ (0.1443, 0.9485; 0.9594, 0.0961)$\rangle$,
$\langle c_7,$ (0.2848, 0.5778; 0.6442, 0.2915)$\rangle$, $\langle c_8,$ (0.0727, 0.9961; 0.9987, 0.0464)$\rangle\}$,

$$P_2 = \{\langle c_j, p_{2j}\rangle | c_j \in C\}$$
$$= \{\langle c_j, (\mu_{2j}, \nu_{2j}; r_{2j}, d_{2j})\rangle | c_j \in \{c_1, c_2, \cdots, c_8\}\}$$

$=\{\langle c_1,$ (0.9550, 0.2198; 0.9800, 0.8560)$\rangle$, $\langle c_2,$ (0.1050, 0.9644; 0.9701, 0.0690)$\rangle$,
$\langle c_3,$ (0.2871, 0.6480; 0.7088, 0.2655)$\rangle$, $\langle c_4,$ (0.2504, 0.9371; 0.9700, 0.1662)$\rangle$,
$\langle c_5,$ (0.6526, 0.4723; 0.8056, 0.6012)$\rangle$, $\langle c_6,$ (0.2359, 0.9692; 0.9975, 0.1520)$\rangle$,
$\langle c_7,$ (0.8861, 0.3390; 0.9487, 0.7674)$\rangle$, $\langle c_8,$ (0.4863, 0.3709; 0.6116, 0.5852)$\rangle\}$,

$$P_3 = \{\langle c_j, p_{3j}\rangle | c_j \in C\}$$
$$= \{\langle c_j, (\mu_{3j}, \nu_{3j}; r_{3j}, d_{3j})\rangle | c_j \in \{c_1, c_2, \cdots, c_8\}\}$$

$=\{\langle c_1,$ (0.9665, 0.2400; 0.9959, 0.8450)$\rangle$, $\langle c_2,$ (0.0666, 0.9452; 0.9475, 0.0448)$\rangle$,
$\langle c_3,$ (0.1795, 0.7727; 0.7933, 0.1453)$\rangle$, $\langle c_4,$ (0.6608, 0.6856; 0.9522, 0.4883)$\rangle$,
$\langle c_5,$ (0.9285, 0.3402; 0.9889, 0.7764)$\rangle$, $\langle c_6,$ (0.4213, 0.5224; 0.6711, 0.4321)$\rangle$,
$\langle c_7,$ (0.2496, 0.9579; 0.9899, 0.1623)$\rangle$, $\langle c_8,$ (0.5887, 0.7902; 0.9854, 0.4076)$\rangle\}$,

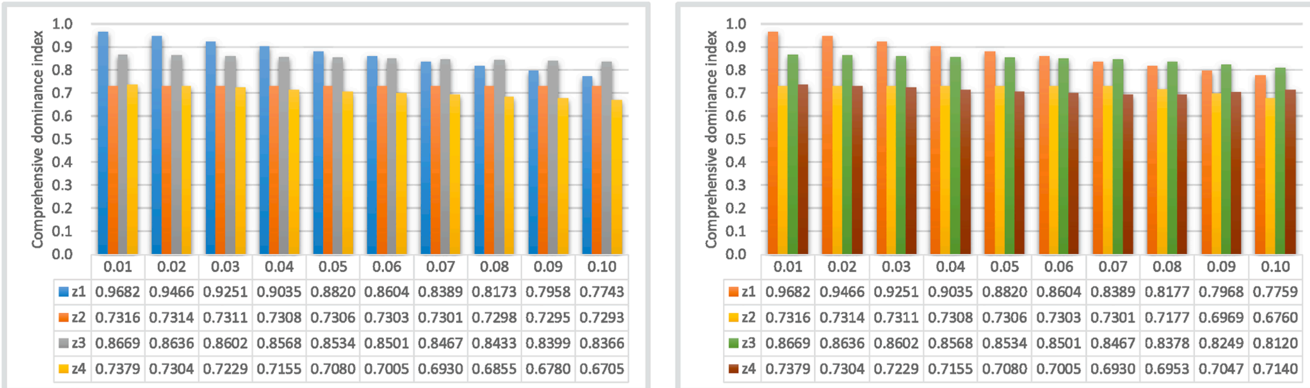
$$P_4 = \{\langle c_j, p_{4j}\rangle | c_j \in C\}$$
$$= \{\langle c_j, (\mu_{4j}, \nu_{4j}; r_{4j}, d_{4j})\rangle | c_j \in \{c_1, c_2, \cdots, c_8\}\}$$

$=\{\langle c_1,$ (0.7324, 0.6090; 0.9525, 0.5584)$\rangle$, $\langle c_2,$ (0.4794, 0.8505; 0.9763, 0.3268)$\rangle$,
$\langle c_3,$ (0.5830, 0.7978; 0.9881, 0.4018)$\rangle$, $\langle c_4,$ (0.4900, 0.6619; 0.8235, 0.4057)$\rangle$,
$\langle c_5,$ (0.3881, 0.7003; 0.8007, 0.3222)$\rangle$, $\langle c_6,$ (0.8540, 0.2077; 0.8789, 0.8481)$\rangle$,
$\langle c_7,$ (0.1127, 0.9902; 0.9966, 0.0721)$\rangle$, $\langle c_8,$ (0.2174, 0.9720; 0.9960, 0.1401)$\rangle\}$.

Regarding the relevant literature of the bridge-superstructure construction case, the inclusion-based LINMAP approach proposed by Chen (2014) belongs to the LINMAP methodology. In Step 3, in accordance with Chen's settings, this paper incorporates the paired preference relations for the alternatives to construct the set of ordered pairs Ω. It is

Table 2
Selected computational results of the components in Model IV.

c_j	$\psi(p_{3j})$-$\psi(p_{4j})$	$\psi(p_{4j})$-$\psi(p_{1j})$	$\psi(p_{1j})$-$\psi(p_{2j})$	$\psi(p_{1j})$-$\psi(p_{3j})$	$\sum(\psi(p_{\phi j})$-$\psi(p_{\varphi j}))$	$\sum\psi(p_{ij})$
c_1	0.3438	0.2823	−0.6258	−0.6260	−0.6258	2.9792
c_2	0.1229	0.7600	−0.8868	−0.8830	−0.8868	2.9631
c_3	0.1249	−0.2443	0.2578	0.1193	0.2578	3.3373
c_4	−0.1270	−0.0235	−0.2991	0.1505	−0.2991	2.8453
c_5	0.7132	−0.7132	0.3051	0.0000	0.3051	2.8318
c_6	0.5041	−0.9267	0.0019	0.4227	0.0019	2.6089
c_7	−0.0131	0.3257	0.6141	−0.3126	0.6141	2.7131
c_8	−0.2584	−0.0090	0.5674	0.2674	0.5674	3.1531



(a) Results of $\bar{\Psi}(P_i)$ for $\hbar$=0.0 (b) Results of $\bar{\Psi}(P_i)$ for $\hbar$=0.1

Fig. 7. Comparison of comprehensive dominance indices under various ε_0 values.

denoted by: $\Omega = \{(3,4),(4,1),(1,2),(1,3)\}$. That is, the decision maker provided the preference relations $z_3 \succeq z_4$, $z_4 \succeq z_1$, $z_1 \succeq z_2$, and $z_1 \succeq z_3$. It is noted that some non-transitive relations exist in the set Ω. Specifically, $z_3 \succeq z_4$ and $z_4 \succeq z_1$ but $z_1 \succeq z_3$. Moreover, the set Ω does not comprise all preference judgments between alternatives in Z. In conformity with the decision maker's specification, the relative importance of total comprehensive dominance in regard to poorness of fit is designated as follows: $\eta = 0.8$. In Step 4, this paper derived the SED-based dominance index $\psi(p_{ij})$ of each p_{ij} in connection with p_{+j} and p_{-j}. The obtained $\psi(p_{ij})$ values are displayed in Table 1.

In Step 5, this paper designated some relevant parameters as below. $\Omega = \{(3,4),(4,1),(1,2),(1,3)\}$. To apply Model IV, the consequences of $\psi(p_{\phi j}) - \psi(p_{\varphi j})$ for $(\phi,\varphi) \in \Omega$, $\sum_{(\phi,\varphi)\in\Omega}\left(\psi(p_{\phi j}) - \psi(p_{\varphi j})\right)$, and $\sum_{i=1}^{m}\psi(p_{ij})$ for each criterion $c_j \in C$ were computed based on the SED-based dominance indices in Table 1. The yielded results are demonstrated in Table 2.

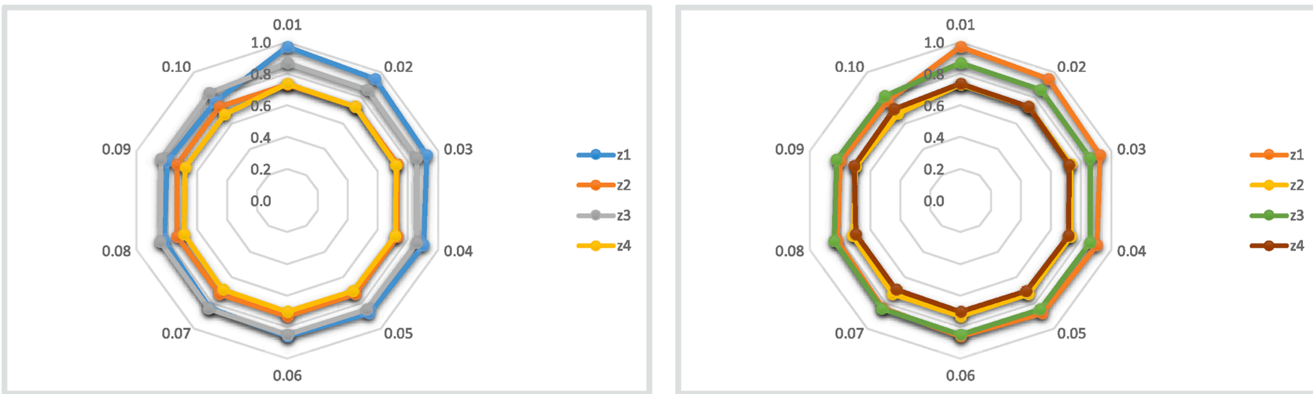
As mentioned before, the relative importance of total comprehensive dominance in regards to poorness of fit was designated as η=0.8. That is, the weight of $\sum_{i=1}^{4}\sum_{j=1}^{8}\psi(p_{ij})\cdot w_j$ is 0.8, while the weight of $-\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}$ is 0.2. Based on Model IV, the PF LINMAP model was established as below:

$$
\begin{aligned}
& max\{0.8\cdot(2.9792w_1 + 2.9631w_2 + 3.3373w_3 + 2.8453w_4 + 2.8318w_5 + 2.6089w_6 \\
& \qquad + 2.7131w_7 + 3.1531w_8) - 0.2\cdot(\Gamma_{34} + \Gamma_{41} + \Gamma_{12} + \Gamma_{13})\} \\
& \text{s.t.} \begin{cases}
-0.6258w_1 - 0.8868w_2 + 0.2578w_3 - 0.2991w_4 + 0.3051w_5 + 0.0019w_6 + 0.6141w_7 \\
\qquad +0.5674w_8 \geqslant 0.25, \\
0.3438w_1 + 0.1229w_2 + 0.1249w_3 - 0.1270w_4 + 0.7132w_5 + 0.5041w_6 - 0.0131w_7 \\
\qquad -0.2584w_8 + \Gamma_{34} \geqslant 0, \\
0.2823w_1 + 0.7600w_2 - 0.2443w_3 - 0.0235w_4 - 0.7132w_5 - 0.9267w_6 + 0.3257w_7 \\
\qquad -0.0090w_8 + \Gamma_{41} \geqslant 0, \\
-0.6258w_1 - 0.8868w_2 + 0.2578w_3 - 0.2991w_4 + 0.3051w_5 + 0.0019w_6 + 0.6141w_7 \\
\qquad +0.5674w_8 + \Gamma_{12} \geqslant 0, \\
-0.6260w_1 - 0.8830w_2 + 0.1193w_3 + 0.1505w_4 + 0.0000w_5 + 0.4227w_6 - 0.3126w_7 \\
\qquad +0.2674w_8 + \Gamma_{13} \geqslant 0, \\
\qquad \Gamma_{34}, \Gamma_{41}, \Gamma_{12}, \Gamma_{13} \geqslant 0, \\
\sum_{j=1}^{8} w_j = 1,\ w_1, w_7 \geqslant 0.1, w_5, w_8 \geqslant 0.075, w_2, w_6 \geqslant 0.05, w_3, w_4 \geqslant 0.025.
\end{cases}
\end{aligned}
\tag{40}
$$

First, concerning the lowest acceptable level ħ, the degree of goodness of fit G is not smaller than the degree of poorness of fit B by ħ=0.25, i.e., $G - B \geqslant 0.25$. Next, consider the non-negative boundary condition ε_j for each $j \in \{1,2,\cdots,8\}$. The boundary conditions were assigned as follows: $\varepsilon_1 = \varepsilon_7 = 0.1$ for c_1 and c_7, $\varepsilon_5 = \varepsilon_8 = 0.075$ for c_5 and c_8, $\varepsilon_2 = \varepsilon_6 = 0.05$ for c_2 and c_6, and $\varepsilon_3 = \varepsilon_4 = 0.025$ for c_3 and c_4. Accordingly, the following constraints were imposed into the PF LINMAP model: $w_1 \geqslant 0.1$, $w_2 \geqslant 0.05$, $w_3 \geqslant 0.025$, $w_4 \geqslant 0.025$, $w_5 \geqslant 0.075$, $w_6 \geqslant 0.05$, $w_7 \geqslant 0.1$, and $w_8 \geqslant 0.075$.

In Step 6, the decision variables consist of the unknown weight w_j for $j \in \{1,2,\cdots,8\}$ and the degree of violation $\Gamma_{\phi\varphi}$ for each ordered pair $(\phi, \varphi) \in \Omega$, i.e., Γ_{34}, Γ_{41}, Γ_{12}, and Γ_{13} according to $\Omega = \{(3,4),(4,1),(1,2),(1,3)\}$.

In Step 7, the linear programming model in (40) was resolved to receive the optimal objective value 2.4368, the optimal weight vector $\overline{w} = (\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_8) = (0.1,\ 0.05,\ 0.1952,\ 0.025,\ 0.075,\ 0.05,\ 0.1,\ 0.4048)$, and the optimal degrees of violation $\overline{\Gamma}_{34} = \overline{\Gamma}_{12} = \overline{\Gamma}_{13} = 0$, and $\overline{\Gamma}_{41} = 0.0529$. In Step 8, the comprehensive dominance indices of the four characteristics were obtained as below: $\overline{\Psi}(P_1) = 0.8451$, $\overline{\Psi}(P_2) = 0.5951$, $\overline{\Psi}(P_3) = 0.8267$, and $\overline{\Psi}(P_4) = 0.7922$. Thus, the priority ranking of the four alternatives is $z_1 \succ z_3 \succ z_4 \succ z_2$. Correspondingly, the best compromise solution z^* is the advanced shoring method (z_1)



(a) Contrast results for $\hbar$=0.0 (b) Contrast results for $\hbar$=0.1

Fig. 8. Distribution chart of $\overline{\Psi}(P_i)$ among alternatives under various ε_0 values.

because $Z^* = \{z_1\}$.

Furthermore, it can be observed that different solution results were determined under various parameter settings. Consider the lowest acceptable level ħ as an example. When ħ=0.00–0.21, the ranking consequence of the four alternatives was $z_3 \succ z_1 \succ z_4 \succ z_2$, which yields $Z^* = \{z_3\}$. The ranking outcomes $z_1 \succ z_3 \succ z_4 \succ z_2$ and $z_1 \succ z_4 \succ z_3 \succ z_2$ were drawn in the cases of ħ=0.22–0.27 and ħ=0.28–0.30, respectively, which renders $Z^* = \{z_1\}$. The ranking consequence of the alternatives was $z_3 \succ z_4 \succ z_1 \succ z_2$ for ħ=0.31 and 0.32; moreover, $Z^* = \{z_3\}$. It is worthwhile to notice that no feasible solutions would be found if a high ħ value was assigned. On this account, the designation of a suitable ħ value is an indispensable issue of utilizing the PF LINMAP methodology in practice.

5.2. Comparative analysis with discussions

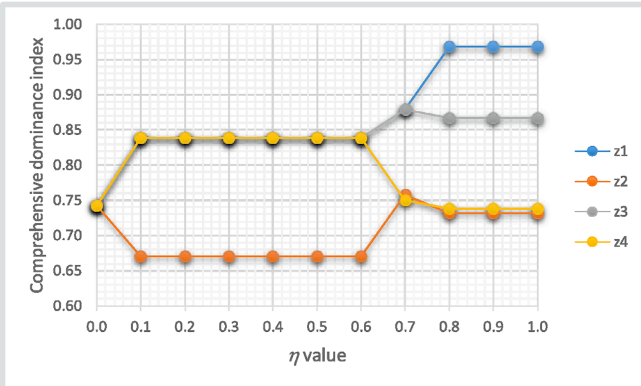
This subsection strives to systematically scrutinize the effect of the PF LINMAP model in distinct parameter settings concerning the comprehensive dominance indices among alternatives. Furthermore, this subsection carries out a comparative study with pertinent MCDA methods to examine the effectuality and strong points of the advanced methodology in practice. The main framework of the comparative analysis is shown in Fig. 6.

The first comparative study aims to conduct a sensitivity analysis for the non-negative boundary conditions of criterion weights. Specifically, this analysis observes the effects of distinct ε_0 values towards the optimal comprehensive dominance indices and their ranking outcomes rendered by the PF LINMAP methodology. Regarding the same the bridge-superstructure construction case, one can assume $\varepsilon_j = \varepsilon_0$ for each $c_j \in C$ in the imposed condition $w_j \geqslant \varepsilon_j$ for convenience. Ten instances of the ε_0 values were concerned, consisting of $\varepsilon_0 = 0.01, 0.02, \cdots$, and 0.10. Two ħ values were designated when assigning the lowest acceptable level towards $G - B$: ħ=0.0 and 0.1. Under the setting of $\eta = 0.8$ inquired from the decision maker, the comparison results of the comprehensive dominance index $\overline{\Psi}(P_i)$ for each alternative $z_i \in Z$ are depicted in Fig. 7.

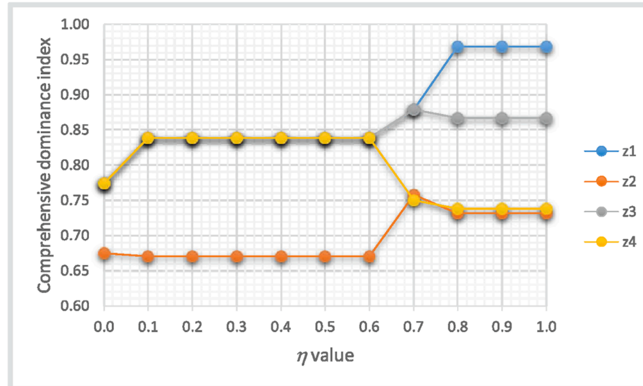
On the strength of the optimal comprehensive dominance index $\overline{\Psi}(P_i)$ for each $z_i \in Z$, several priority rankings among alternatives were acquired when the ε_0 value gradually increases from 0.01 to 0.10. For the comparison results revealed in Fig. 7(a) (i.e., the case of ħ=0.0), three ranking results were determined: $z_1 \succ z_3 \succ z_4 \succ z_2$, $z_1 \succ z_3 \succ z_2 \succ z_4$, and $z_3 \succ z_1 \succ z_2 \succ z_4$ for $\varepsilon_0 = 0.01$, $\varepsilon_0 = 0.02, 0.03, \cdots, 0.06$, and $\varepsilon_0 = 0.07, 0.08, \cdots, 0.10$, respectively. The corresponding best compromise solution z^* would be z_1 and z_3 (i.e., $Z^* = \{z_1\}$ and $Z^* = \{z_3\}$) for $\varepsilon_0 = 0.01, 0.02, \cdots, 0.06$ and $\varepsilon_0 = 0.07, 0.08, \cdots, 0.10$, respectively. In the case of ħ=0.1, as shown in Fig. 7(b), the proposed PF LINMAP model rendered four ranking results of $z_1 \succ z_3 \succ z_4 \succ z_2$, $z_1 \succ z_3 \succ z_2 \succ z_4$, $z_3 \succ z_1 \succ z_2 \succ z_4$, and $z_3 \succ z_1 \succ z_4 \succ z_2$ for $\varepsilon_0 = 0.01$, $\varepsilon_0 = 0.02, 0.03, \cdots, 0.06$, $\varepsilon_0 = 0.07, 0.08$, and $\varepsilon_0 = 0.09, 0.10$, respectively. The best compromise solution z^* would also be z_1 and z_3 for $\varepsilon_0 = 0.01, 0.02, \cdots, 0.06$ and $\varepsilon_0 = 0.07, 0.08, \cdots, 0.10$, respectively. The contrasts of comprehensive dominance indices among the four alternatives under various ε_0 values in the cases of ħ=0.0 and 0.1 are sketched in the two distribution charts of Fig. 8. When the ε_0 value increases from 0.01 to 0.10 under the settings of ħ=0.0 and $\eta = 0.8$ as well as ħ=0.1 and $\eta = 0.8$, the two alternatives z_1 and z_3 perform better than z_2 and z_4 because the values of $\overline{\Psi}(P_1)$ and $\overline{\Psi}(P_3)$ are steadily larger than $\overline{\Psi}(P_2)$ and $\overline{\Psi}(P_4)$.

The second comparative study attempts to implement a sensitivity analysis for the relative importance of total comprehensive dominance in regards to poorness of fit under various lowest acceptable levels. Eleven instances of the η values were concerned: $\eta = 0.0, 0.1, \cdots$, and 1.0. Moreover, this study designated six ħ values (consisting of $0.0, 0.1, \cdots$, and 0.5) and two ε_0 values (consisting of 0.01 and 0.02) to investigate the resulting comprehensive dominance indices among the four alternatives under different settings more comprehensively. Note that the assigned ħ values are appropriate because such parameter settings would not lead to infeasibility of the PF LINMAP model. More specifically, the following instances were considered, consisting of the combinations of $\eta = 0.0, 0.1, \cdots, 1.0$ and $\hbar = 0.0, 0.1, \cdots, 0.5$ under the settings of$\varepsilon_0 = 0.01$ and 0.02.

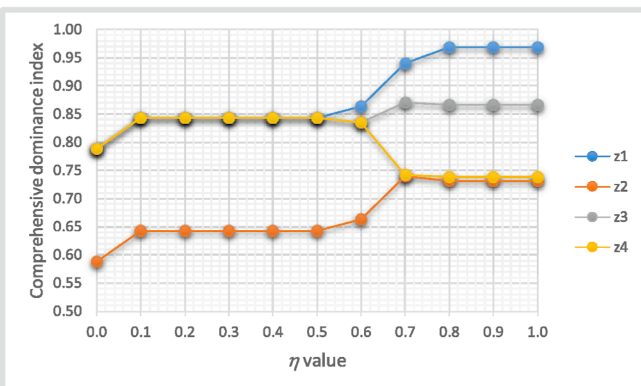
Fig. 9 depicts the comparison results of the obtained $\overline{\Psi}(P_i)$ values under various η values for$\varepsilon_0 = 0.01$. In particular, the optimal comprehensive dominance indices in the cases of $\hbar = 0.0,\ 0.1, \cdots$, and 0.5 are contrasted in the figures (a)–(f), respectively. As a whole, the differentiations among the $\overline{\Psi}(P_i)$ values become much apparent when $\eta > 0.6$, $\eta > 0.5$, and $\eta > 0.4$ in the cases of $\hbar = 0.0,\ 0.1$, $\hbar = 0.2,\ 0.3,\ 0.4$, and $\hbar = 0.5$, respectively. As sketched in Fig. 9(a)–(d), the most frequent ranking outcome was $z_1 \succ z_3 \succ z_4 \succ z_2$ in the situations of $\hbar = 0.0,\ 0.1,\ 0.2$, and 0.3. Based on Fig. 9(e), the two rankings $z_1 \succ z_3 \succ z_4 \succ z_2$ and $z_1 \succ z_4 \succ z_3 \succ z_2$ were rendered frequently in the case of $\hbar = 0.4$. In contrast, from Fig. 9(f), the two rankings $z_4 \succ z_3 \succ z_1 \succ z_2$ and $z_1 \succ z_4 \succ z_3 \succ z_2$ were determined frequently in the case of $\hbar = 0.5$. The consistent set $Z^* = \{z_1\}$ was acquired in most η values, which follows that the best compromise solution z^*, namely the advanced shoring method (z_1), is considerably steady in various settings of the η values in Fig. 9(a)–(e). Nonetheless, distinct results of $Z^* = \{z_4\}$ and $Z^* = \{z_1\}$ were obtained according to the pattern in Fig. 9(f). That is, the precast segmental method (z_4) and the advanced shoring method (z_1) were the best compromise solutions in the cases of $\eta = 0.0, 0.1, \cdots, 0.5$ and $\eta = 0.6, 0.7, \cdots, 1.0$, respectively. However, the differences between the top



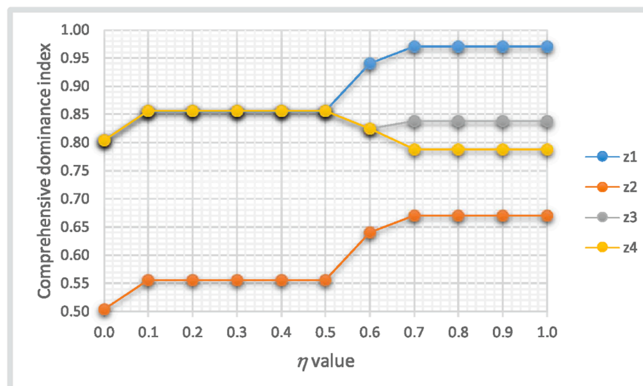
(a) Contrast results for $\hbar$=0.0



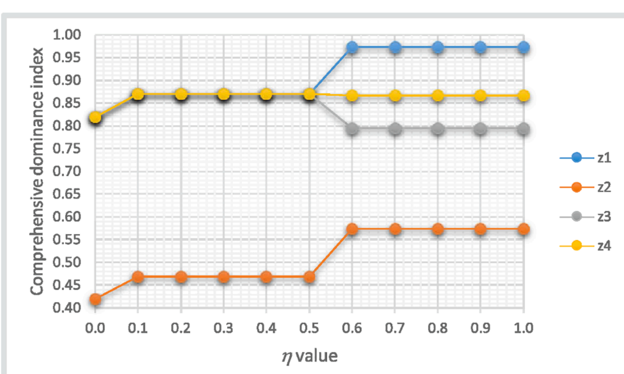
(b) Contrast results for $\hbar$=0.1



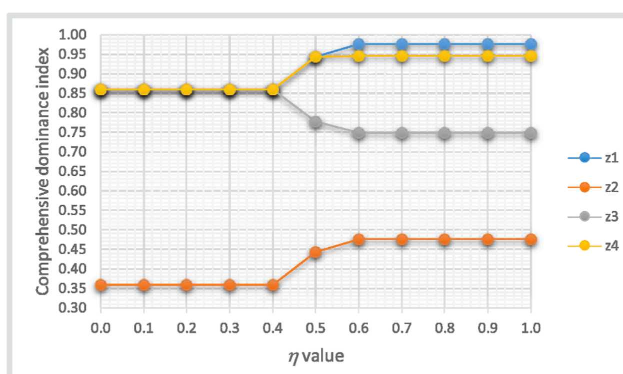
(c) Contrast results for $\hbar$=0.2



(d) Contrast results for $\hbar$=0.3



(e) Contrast results for $\hbar$=0.4

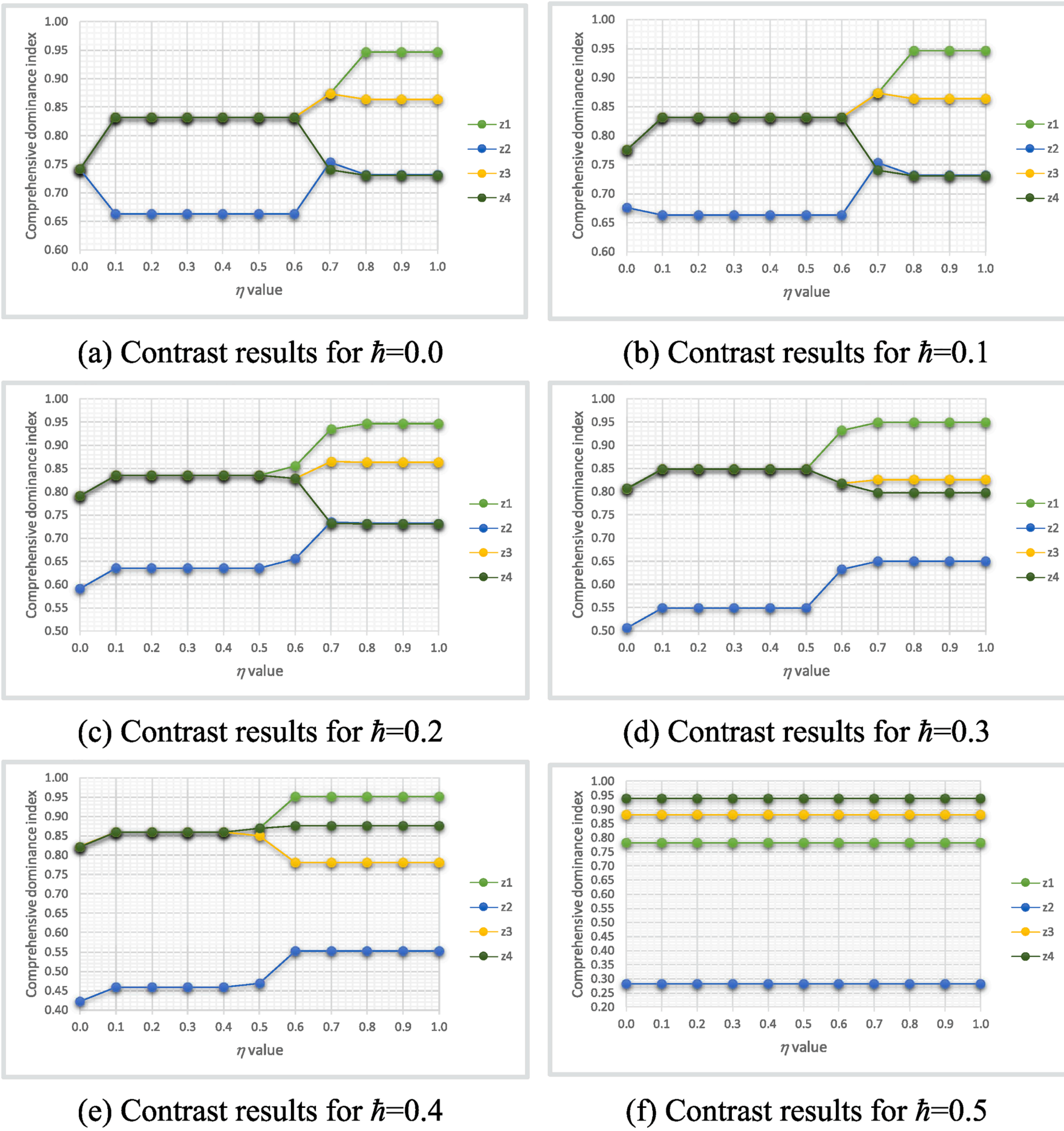


(f) Contrast results for $\hbar$=0.5

Fig. 9. Comparison of the $\overline{\Psi}(P_i)$ values under various η values for $\varepsilon_0 = 0.01$.

two $\overline{\Psi}(P_i)$ values are unnoticeable when $\eta \leqslant 0.6$, $\eta \leqslant 0.5$, and $\eta \leqslant 0.4$ in the cases of $\hbar = 0.0,\ 0.1$, $\hbar = 0.2,\ 0.3,\ 0.4$, and $\hbar = 0.5$, respectively, which implies that the final conclusion about the best compromise solution z^* should be made seriously in these cases.

In the case of $\varepsilon_0 = 0.02$, the solution results of the optimal comprehensive dominance indices under various η values are demonstrated in Fig. 10, in which the $\overline{\Psi}(P_i)$ values for $\hbar = 0.0,\ 0.1,\ \cdots$, and 0.5 are compared in the figures (a)–(f), respectively. Except for $\hbar = 0.5$, the similar observations in the case of $\varepsilon_0 = 0.01$ can roughly apply to the case of $\varepsilon_0 = 0.02$. As shown in Fig. 10(a)–(e), the distinctions among the $\overline{\Psi}(P_i)$ values become noticeable when $\eta > 0.6$ and $\eta > 0.5$ in the cases of $\hbar = 0.0,\ 0.1$ and $\hbar = 0.2,\ 0.3,\ 0.4$, respectively. Nevertheless, from Fig. 10 (f), the obtained $\overline{\Psi}(P_i)$ values for $\hbar = 0.5$ presents unchangeable patterns under different settings of the η value. Concerning the patterns in Fig. 10 (a)–(c) (i.e., $\hbar = 0.0,\ 0.1$, and 0.2), the same ranking result $z_1 \succ z_3 \succ z_2 \succ z_4$ was obtained in the cases of $\eta = 0.7, 0.8, 0.9, 1.0$. Thus, the set $Z^* = \{z_1\}$ was acquired in these cases. The settings of $\eta = 0.1, 0.2, \cdots, 0.6$ rendered the consistent ranking $z_3 \succ z_1 \succ z_4 \succ z_2$ (with $Z^* = \{z_3\}$) for $\hbar = 0.0$ and 0.1. However, for $\hbar = 0.2$, two distinct ranking outcomes $z_1 \tilde{} z_3 \succ z_4 \succ z_2$ (with $Z^* = \{z_1, z_3\}$) and $z_1 \succ z_3 \succ z_4 \succ z_2$ (with $Z^* = \{z_1\}$) were generated for $\eta = 0.1, 0.2, \cdots, 0.5$ and $\eta = 0.6$, respectively. Consider the case of $\hbar = 0.3$ in Fig. 10(d). Except for $\eta = 0.0$, the settings of $\eta = 0.1, 0.2, \cdots, 1.0$ produced the identical ranking result $z_1 \succ z_3 \succ z_4 \succ z_2$, where $Z^* = \{z_1\}$. Based on Fig. 10(e), the two rankings $z_4 \succ z_3 \succ z_1 \succ z_2$ (with $Z^* = \{z_4\}$) and $z_1 \succ z_4 \succ z_3 \succ z_2$ (with $Z^* = \{z_1\}$) were finalized frequently in the case of $\hbar = 0.4$. For $\hbar = 0.5$ in



(a) Contrast results for $\hbar$=0.0 (b) Contrast results for $\hbar$=0.1

(c) Contrast results for $\hbar$=0.2 (d) Contrast results for $\hbar$=0.3

(e) Contrast results for $\hbar$=0.4 (f) Contrast results for $\hbar$=0.5

Fig. 10. Comparison of the $\overline{\Psi}(P_i)$ values under various η values for $\varepsilon_0 = 0.02$.

Fig. 10(f), the settings of $\eta = 0.0, 0.1, \cdots, 1.0$ yielded the same ranking result $z_4 \succ z_3 \succ z_1 \succ z_2$, where $Z^* = \{z_4\}$. Analogous to the situation for$\varepsilon_0 = 0.01$, the differences between the top two $\overline{\Psi}(P_i)$ values in Fig. 10 (a)–(e) are inapparent when $\eta \leqslant 0.6$ and $\eta \leqslant 0.5$ in the cases of $\hbar = 0.0,\ 0.1$ and $\hbar = 0.2,\ 0.3,\ 0.4$, respectively.

The third comparative study concentrates on the exploration of the application outcomes rendered by other pertinent MCDA approaches to demonstrate the flexibility and strong points of the developed PF LINMAP methodology using a SED-based outranking approach. As mentioned before, the MCDA problem concerning bridge-superstructure construction methods has been investigated by Chen (2012, 2014) within interval-valued intuitionistic fuzzy environments, Chen (2018a, 2018b) within interval-valued PF environments, and Chen (2019a, 2019b) within PF environments. Table 3 summarizes the comparison results concerning the bridge-superstructure construction problem.

The priority ranking $z_1 \succ z_3 \succ z_4 \succ z_2$ rendered by the developed PF LINMAP methodology coincides with the ranking outcomes produced by the interval-valued PF outranking method (Chen, 2018b) and the PROMETHEE-based method (Chen, 2019a). Overall, all comparative approaches rendered similar solution results. More precisely, the alternatives z_1 and z_3 belong to the relatively superior choices, whereas z_2 and z_4 belong to the relatively inferior choices in the bridge-superstructure construction case. Accordingly, the priority ranking generated by the current method achieves a consensus in view of the results that were rendered by use of the other comparative methods. It is worthy to note that the core techniques of the inclusion-based LINMAP

Table 3
Contrast of application outcomes in the bridge construction case.

Comparative method	Type of fuzziness	Importance	Ranking outcome
The nonlinear assignment-based model (Chen, 2012)	Interval-valued intuitionistic fuzzy set	Incomplete preference information	$z_1 \succ z_3 \succ z_2 \succ z_4$
The inclusion-based LINMAP method (Chen, 2014)	Interval-valued intuitionistic fuzzy set	A priori unknown	$z_3 \succ z_1 \succ z_4 \succ z_2$
The compromise approach with correlation-based closeness indices (Chen, 2018a)	Interval-valued PF set	Nonfuzzy and normalized weights	$z_3 \succ z_1 \succ z_2 \succ z_4$
The interval-valued PF outranking method (Chen, 2018b)	Interval-valued PF set	Interval-valued PF weights	$z_1 \succ z_3 \succ z_4 \succ z_2$
The PROMETHEE-based method using a PF combinative distance-based precedence approach (Chen, 2019a)	Pythagorean membership grades based on PF sets	Nonfuzzy and normalized weights	$z_1 \succ z_3 \succ z_4 \succ z_2$
The PF compromise approach based on generalized distance measures (Chen, 2019b)	Pythagorean membership grades based on PF sets	Nonfuzzy and normalized weights	$z_3 \succ z_1 \succ z_2 \succ z_4$
The proposed PF LINMAP methodology using a SED-based outranking approach	Pythagorean membership grades based on PF sets	A priori unknown with non-negative conditions	$z_1 \succ z_3 \succ z_4 \succ z_2$

approach (Chen, 2014) and the proposed method are the LINMAP model. Thus, this comparative analysis further examines the effectiveness and validity of the solution results generated by these two LINMAP-based approaches. As mentioned before, the decision maker indicated the preference relations $z_3 \succeq z_4$, $z_4 \succeq z_1$, $z_1 \succeq z_2$, and $z_1 \succeq z_3$; namely, the set of ordered pairs $\Omega = \{(3,4),(4,1),(1,2),(1,3)\}$. The rationale of the LINMAP model is to preclude any violation of the decision maker's preference assessments as minimally as possible. That is, the LINMAP model should produce a consistent evaluation of all alternatives to minimize the degree of violation of the preference relationships in the set Ω. However, it is presumed that certain degrees of violation may be larger than zero because non-transitive relations exist in the set Ω (i.e., the ordered pairs (3,4), (4,1), and (1,3)) in the problem of bridge-superstructure construction. The developed PF LINMAP model generated the ranking $z_1 \succ z_3 \succ z_4 \succ z_2$. Only one ordered pair (4,1) is violated because the obtained partial order $z_1 \succ z_4$ is contrary to the preference relation $z_4 \succeq z_1$ revealed by the decision maker. In conformity with the solution results, the optimal degree of violation for (4,1) was $\overline{\Gamma}_{41} = 0.0529$; moreover, the other optimal degrees of violation were equal to zero (i.e., $\overline{\Gamma}_{34} = \overline{\Gamma}_{12} = \overline{\Gamma}_{13} = 0$). In contrast, the inclusion-based LINMAP approach generated the ranking result $z_3 \succ z_1 \succ z_4 \succ z_2$, in which two ordered pairs (4,1) and (1,3) are violated because the partial orders $z_1 \succ z_4$ and $z_3 \succ z_1$ are contrary to the preference relations $z_4 \succeq z_1$ and $z_1 \succeq z_3$ based on the set Ω. Therefore, the proposed PF LINMAP methodology performs better than the inclusion-based LINMAP approach because it can render more reasonable and desirable results.

To compare the application results more comprehensively, this paper incorporates the aforementioned sensitivity analyses into the comparative discussions. Table 4 summarizes the comparison results of the sensitivity analyses, consisting of the obtained priority rankings and the sets of best compromise solutions. The top part of this table displays the outcomes of the sensitivity analysis for the non-negative boundary conditions (i.e., the ε_0 values) in the cases of $\hbar$=0.0 and 0.1 under the condition $\eta = 0.8$. The bottom part of this table gives details of the outcomes of the sensitivity analysis for various weights of the total comprehensive dominance objective (i.e., the η values) in the cases of $\varepsilon_0 = 0.01$ and 0.02 under the conditions $\hbar = 0.0,\ 0.1,\cdots$, and 0.5.

In accordance with the comparisons in Table 4, the alternatives z_1 and z_3 are the best compromise solutions (i.e., $Z^* = \{z_1\}$ or $\{z_3\}$) in most situations when employing the proposed PF LINMAP methodology, which is in conformity with the obtained results generated by the other comparative approaches. In particular, the nonlinear assignment-based method, the interval-valued PF outranking approach, and the PROMETHEE-based method yielded the best compromise solution z_1, while the inclusion-based LINMAP method, the compromise approach with correlation-based closeness indices, and the PF compromise approach based on generalized distance measures produced the best compromise solution z_3. It is worthwhile to mention that the proposed methodology rendered another best compromise solution z_4 in some specific situations, including the following cases: (i) $\varepsilon_0 = 0.02$, $\hbar$=0.4, and $\eta = 0.1$–0.5; (ii) $\varepsilon_0 = 0.01$, $\hbar$=0.5, and $\eta = 0.0$–0.5; and (iii) $\varepsilon_0 = 0.02$, $\hbar$=0.5, and $\eta = 0.0$–1.0. It implies that the alternative z_4 is peradventure more likely to be the best compromise solution in case of high $\hbar$ values. Except for the cases (i)–(iii), the alternatives z_1 and z_3 are generally the better choices in the case of bridge-superstructure construction. However, unlike the other comparative approaches, the developed PF LINMAP methodology is capable of generating different priority rankings and the best compromise solution by way of distinct settings of relevant parameters. That is, the employment of the parameters η, $\hbar$, and ε_j (or ε_0) empowers the PF LINMAP model to adapt to the realistic particularities.

The discussions and findings in the sensitivity and comparative analyses have demonstrated the applicability and adaptability of the PF LINMAP methodology using a SED-based outranking approach. In the classical LINMAP procedure, the decision maker would indicate a set of preference comparisons about the alternatives as input data. Note that the current methodology has been proven effective in reducing the extent of violation of the decision maker's pairwise preferences for acquiring a consistent evaluation of candidate alternatives. Under the less consistent preference information, the proposed methodology has demonstrated the ability to manage such non-transitive information; moreover, it could still produce a more reasonable ranking result than the inclusion-based LINMAP method. The PF LINMAP method does not inquire the decision maker to indicate all paired preference comparisons; thus, it is more flexible and useful than the comparative approaches. Furthermore, the obtained ranking results are more acceptable and valid than those rendered by the inclusion-based LINMAP approach because of less extent of violation of the preference relationships in the set Ω. The parameters η, $\hbar$, and ε_j (or ε_0) involved in the PF LINMAP methodology have proven effective in adapting to the particularities in practical MCDA situations. Therefore, the sensitivity analyses and comparative discussions have corroborated the applicability and merits of the current methodology.

5.3. Discussion on a generalization to group decision making

There is little investigation of the advancement of LINMAP-based methodology for manipulating PF information over time. As stated earlier, the PF LINMAP approaches proposed by Wan et al. (2018) and Xue et al. (2018) are the most significant LINMAP-based methods in the last few years. The two researches opened out the classical LINMAP to

Table 4
Comprehensive comparison of the sensitivity analyses.

Sensitivity analysis of the ε_0 values in the case of $\eta = 0.8$						
Results for $\hbar$=0.0				Results for $\hbar$=0.1		
ε_0		Ranking	Z^*	ε_0	Ranking	Z^*
0.01		$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$	0.01	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$
0.02–0.06		$z_1 \succ z_3 \succ z_2 \succ z_4$	$\{z_1\}$	0.02–0.06	$z_1 \succ z_3 \succ z_2 \succ z_4$	$\{z_1\}$
0.07–0.10		$z_3 \succ z_1 \succ z_2 \succ z_4$	$\{z_3\}$	0.07, 0.08	$z_3 \succ z_1 \succ z_2 \succ z_4$	$\{z_3\}$
				0.09, 0.10	$z_3 \succ z_1 \succ z_4 \succ z_2$	$\{z_3\}$
Sensitivity analysis of the η values						
	Results for $\varepsilon_0 = 0.01$			Results for $\varepsilon_0 = 0.02$		
$\hbar$	η	Ranking	Z^*	η	Ranking	Z^*
0.0	0.0	$z_3 \succ z_4 \succ z_2 \succ z_1$	$\{z_3\}$	0.0	$z_3 \succ z_4 \succ z_2 \succ z_1$	$\{z_3\}$
	0.1–0.6	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$	0.1–0.6	$z_3 \succ z_1 \succ z_4 \succ z_2$	$\{z_3\}$
	0.7	$z_1 \succ z_3 \succ z_2 \succ z_4$	$\{z_1\}$	0.7–1.0	$z_1 \succ z_3 \succ z_2 \succ z_4$	$\{z_1\}$
	0.8–1.0	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$			
0.1	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$
	0.1–0.6	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$	0.1–0.6	$z_3 \succ z_1 \succ z_4 \succ z_2$	$\{z_3\}$
	0.7	$z_1 \succ z_3 \succ z_2 \succ z_4$	$\{z_1\}$	0.7–1.0	$z_1 \succ z_3 \succ z_2 \succ z_4$	$\{z_1\}$
	0.8–1.0	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$			
0.2	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$
	0.1–1.0	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$	0.1–0.5	$z_1 \sim z_3 \succ z_4 \succ z_2$	$\{z_1, z_3\}$
				0.6	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$
				0.7–1.0	$z_1 \succ z_3 \succ z_2 \succ z_4$	$\{z_1\}$
0.3	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$
	0.1–1.0	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$	0.1–1.0	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$
0.4	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$	0.0	$z_3 \succ z_4 \succ z_1 \succ z_2$	$\{z_3\}$
	0.1–0.5	$z_1 \succ z_3 \succ z_4 \succ z_2$	$\{z_1\}$	0.1–0.4	$z_4 \succ z_3 \succ z_1 \succ z_2$	$\{z_4\}$
	0.6–1.0	$z_1 \succ z_4 \succ z_3 \succ z_2$	$\{z_1\}$	0.5	$z_4 \succ z_1 \succ z_3 \succ z_2$	$\{z_4\}$
				0.6–1.0	$z_1 \succ z_4 \succ z_3 \succ z_2$	$\{z_1\}$
0.5	0.0–0.4	$z_4 \succ z_3 \succ z_1 \succ z_2$	$\{z_4\}$	0.0–1.0	$z_4 \succ z_3 \succ z_1 \succ z_2$	$\{z_4\}$
	0.5	$z_4 \succ z_1 \succ z_3 \succ z_2$	$\{z_4\}$			
	0.6–1.0	$z_1 \succ z_4 \succ z_3 \succ z_2$	$\{z_1\}$			

adapt to complicated uncertain environments encompassing Pythagorean fuzziness and then advanced group decision-making methods in PF settings. To make a further comparison, this subsection provides a practical extension of utilizing the proposed methodology in the group decision-making field and conducts a comparative analysis for deepening more comprehensive discussions.

Wan et al. (2018) employed the concept of information entropy to appraise the subjective weights of criteria for individual decision makers. They applied a cross-entropy optimization model to integrate multiple weight vectors into a collective one. Moreover, each decision maker's weight was determined from the collective subjective criterion weight vector in an objective manner. Based on the essential structure of the classical LINMAP, Wan et al. (2018) defined PF group consistency and inconsistency indices in connection with the positive- and negative-ideal solutions, respectively, to unfold a bi-objective PF mathematical programming model. Next, a linear programming method was productively launched to resolve the bi-objective model for handling group decision-making issues in PF circumstances. Xue et al. (2018) defined the PF entropy grounded on the similarity and hesitancy concepts to measure the amount of knowledge associated with a PF set. They incorporated the PF entropy into the central procedure of LINMAP and developed an innovative PF LINMAP approach for multiple attribute group decision analysis. Their proposed model considered the maximum consistency and the amount of knowledge under the decision makers' given preference relations between alternatives. More specifically, the objective function in Xue et al.'s developed linear programming model is to achieve the highest consistency and receive an amount of knowledge in connection with given preference relations.

Notably, Wan et al. (2018) and Xue et al. (2018) exploited their developed approaches to manage group decision-making issues in PF circumstances. Nonetheless, the proposed methodology is originally suitable to handle single-person decision making. To solve these issues, this paper discusses a generalization of the advanced methodology to manage group decision-making issues for enhancing the efficiency and effectiveness of application merits. What is more, to validate the efficaciousness of the developed methodology as opposed to the existing works in the area involving multiple criteria group decision making, this paper compares the extended methods and techniques with the PF LINMAP approaches by Wan et al. (2018) and Xue et al. (2018) on the same application problems, i.e., green supplier selection and railway project investment, for facilitating a fair and impartial comparative analysis and validating the advantages of the developed techniques.

Let $E = \{e_1, e_2, \cdots, e_K\}$ stipulate a set of K decision makers. Let $\mathbb{C}_j^k(z_i)$ indicate the PF evaluative rating of an alternative $z_i \in Z$ in the matter of criterion $c_j \in C$ $(= C_{\mathrm{I}} \cup C_{\mathrm{II}})$ furnished by the decision maker $e_k \in E$, where $\mathbb{C}_j^k(z_i) = (\mu_{c_j}^k(z_i), \nu_{c_j}^k(z_i); r_{c_j}^k(z_i), d_{c_j}^k(z_i))$. Let $p_{ij}^k = \mathbb{C}_j^k(z_i)$ for notational convenience, in which $p_{ij}^k = (\mu_{ij}^k, \nu_{ij}^k; r_{ij}^k, d_{ij}^k)$. Let Ω^k declare a set of ordered pairs (ϕ, φ) furnished by the decision maker e_k, where $\Omega^k = \{(\phi, \varphi) | z_\phi \succeq z_\varphi,\ \phi, \varphi \in \{1, 2, \cdots, m\}\}$. In keeping with the ideal PF evaluative ratings p_{+j} and p_{-j} in Definition 5, the SED-based dominance index $\psi(p_{ij}^k)$ of p_{ij}^k is acquired through the utility of Definition 6, as follows:

Table 5
Description of the group decision-making problem concerning green supplier selection.

		p_{ij}^1 furnished by e_1	p_{ij}^2 furnished by e_2	p_{ij}^3 furnished by e_3
z_i	c_j	$(\mu_{ij}^1, \nu_{ij}^1; r_{ij}^1, d_{ij}^1)$	$(\mu_{ij}^2, \nu_{ij}^2; r_{ij}^2, d_{ij}^2)$	$(\mu_{ij}^3, \nu_{ij}^3; r_{ij}^3, d_{ij}^3)$
z_1	c_1	(0.20, 0.50; 0.5385, 0.2422)	(0.40, 0.40; 0.5657, 0.5000)	(0.60, 0.40; 0.7211, 0.6257)
	c_2	(0.40, 0.60; 0.7211, 0.3743)	(0.30, 0.20; 0.3606, 0.6257)	(0.50, 0.70; 0.8602, 0.3949)
	c_3	(0.30, 0.60; 0.6708, 0.2952)	(0.40, 0.60; 0.7211, 0.3743)	(0.40, 0.70; 0.8062, 0.3305)
	c_4	(0.60, 0.70; 0.9220, 0.4511)	(0.40, 0.60; 0.7211, 0.3743)	(0.60, 0.20; 0.6325, 0.7952)
	c_5	(0.20, 0.40; 0.4472, 0.2952)	(0.50, 0.40; 0.6403, 0.5704)	(0.60, 0.50; 0.7810, 0.5577)
z_2	c_1	(0.40, 0.40; 0.5657, 0.5000)	(0.50, 0.40; 0.6403, 0.5704)	(0.70, 0.30; 0.7616, 0.7422)
	c_2	(0.50, 0.20; 0.5385, 0.7578)	(0.40, 0.30; 0.5000, 0.5903)	(0.60, 0.50; 0.7810, 0.5577)
	c_3	(0.40, 0.50; 0.6403, 0.4296)	(0.50, 0.40; 0.6403, 0.5704)	(0.60, 0.50; 0.7810, 0.5577)
	c_4	(0.50, 0.80; 0.9434, 0.3556)	(0.30, 0.60; 0.6708, 0.2952)	(0.50, 0.50; 0.7071, 0.5000)
	c_5	(0.30, 0.50; 0.5831, 0.3440)	(0.40, 0.50; 0.6403, 0.4296)	(0.50, 0.60; 0.7810, 0.4423)
z_3	c_1	(0.80, 0.30; 0.8544, 0.7716)	(0.30, 0.20; 0.3606, 0.6257)	(0.60, 0.30; 0.6708, 0.7048)
	c_2	(0.60, 0.10; 0.6083, 0.8949)	(0.50, 0.20; 0.5385, 0.7578)	(0.80, 0.30; 0.8544, 0.7716)
	c_3	(0.60, 0.20; 0.6325, 0.7952)	(0.80, 0.20; 0.8246, 0.8440)	(0.70, 0.10; 0.7071, 0.9097)
	c_4	(0.20, 0.90; 0.9220, 0.1392)	(0.40, 0.40; 0.5657, 0.5000)	(0.10, 0.80; 0.8062, 0.0792)
	c_5	(0.20, 0.30; 0.3606, 0.3743)	(0.20, 0.80; 0.8246, 0.1560)	(0.30, 0.70; 0.7616, 0.2578)
z_4	c_1	(0.60, 0.30; 0.6708, 0.7048)	(0.70, 0.30; 0.7616, 0.7422)	(0.80, 0.40; 0.8944, 0.7048)
	c_2	(0.60, 0.20; 0.6325, 0.7952)	(0.60, 0.30; 0.6708, 0.7048)	(0.70, 0.20; 0.7280, 0.8228)
	c_3	(0.50, 0.20; 0.5385, 0.7578)	(0.70, 0.30; 0.7616, 0.7422)	(0.70, 0.40; 0.8062, 0.6695)
	c_4	(0.40, 0.80; 0.8944, 0.2952)	(0.10, 0.80; 0.8062, 0.0792)	(0.20, 0.50; 0.5385, 0.2422)
	c_5	(0.30, 0.60; 0.6708, 0.2952)	(0.20, 0.70; 0.7280, 0.1772)	(0.40, 0.70; 0.8062, 0.3305)
	Ω^k	Ω^1={(3,2), (4,2)}	Ω^2={(4,1), (4,2)}	Ω^3={(2,1), (3,4)}

Remark: (i) $Z = \{z_1$ (Green supplier #1), z_2 (Green supplier #2), z_3 (Green supplier #3), z_4 (Green supplier #4)$\}$.
(ii) $C = \{c_1$ (resource recovery and utilization), c_2 (green identity), c_3 (environmental impact degree), c_4 (energy consumption), c_5 (use of environmental protection funds)$\}$, where $C_{\mathrm{I}}=\{c_1, c_2, c_3\}$ and $C_{\mathrm{II}}=\{c_4, c_5\}$.
(iii) $E=\{e_1$ (Expert from the materials and equipment department), e_2 (Expert from the quality supervision department), e_3 (Expert from the engineering department)$\}$.

Table 6
Description of the group decision-making problem concerning railway project investment.

		p_{ij}^1 furnished by e_1	p_{ij}^2 furnished by e_2	p_{ij}^3 furnished by e_3
z_i	c_j	$(\mu_{ij}^1, \nu_{ij}^1; r_{ij}^1, d_{ij}^1)$	$(\mu_{ij}^2, \nu_{ij}^2; r_{ij}^2, d_{ij}^2)$	$(\mu_{ij}^3, \nu_{ij}^3; r_{ij}^3, d_{ij}^3)$
z_1	c_1	(0.70, 0.60; 0.9220, 0.5489)	(0.50, 0.40; 0.6403, 0.5704)	(0.40, 0.30; 0.5000, 0.5903)
	c_2	(0.80, 0.60; 1.0000, 0.5903)	(0.30, 0.90; 0.9487, 0.2048)	(0.30, 0.40; 0.5000, 0.4097)
	c_3	(0.50, 0.50; 0.7071, 0.5000)	(0.40, 0.30; 0.5000, 0.5903)	(0.50, 0.40; 0.6403, 0.5704)
	c_4	(0.40, 0.70; 0.8062, 0.3305)	(0.90, 0.40; 0.9849, 0.7338)	(0.60, 0.60; 0.8485, 0.5000)
	c_5	(0.90, 0.40; 0.9849, 0.7338)	(0.30, 0.70; 0.7616, 0.2578)	(0.70, 0.70; 0.9899, 0.5000)
	c_6	(0.40, 0.90; 0.9849, 0.2662)	(0.40, 0.50; 0.6403, 0.4296)	(0.30, 0.70; 0.7616, 0.2578)
z_2	c_1	(0.90, 0.40; 0.9849, 0.7338)	(0.90, 0.30; 0.9487, 0.7952)	(0.80, 0.30; 0.8544, 0.7716)
	c_2	(0.80, 0.60; 1.0000, 0.5903)	(0.80, 0.50; 0.9434, 0.6444)	(0.70, 0.10; 0.7071, 0.9097)
	c_3	(0.70, 0.70; 0.9899, 0.5000)	(0.70, 0.60; 0.9220, 0.5489)	(0.80, 0.20; 0.8246, 0.8440)
	c_4	(0.90, 0.30; 0.9487, 0.7952)	(0.90, 0.20; 0.9220, 0.8608)	(0.90, 0.10; 0.9055, 0.9296)
	c_5	(0.80, 0.20; 0.8246, 0.8440)	(0.90, 0.20; 0.9220, 0.8608)	(0.80, 0.30; 0.8544, 0.7716)
	c_6	(0.90, 0.40; 0.9849, 0.7338)	(0.80, 0.30; 0.8544, 0.7716)	(0.80, 0.10; 0.8062, 0.9208)
z_3	c_1	(0.70, 0.50; 0.8602, 0.6051)	(0.70, 0.60; 0.9220, 0.5489)	(0.20, 0.50; 0.5385, 0.2422)
	c_2	(0.40, 0.30; 0.5000, 0.5903)	(0.50, 0.30; 0.5831, 0.6560)	(0.30, 0.40; 0.5000, 0.4097)
	c_3	(0.90, 0.10; 0.9055, 0.9296)	(0.80, 0.10; 0.8062, 0.9208)	(0.80, 0.60; 1.0000, 0.5903)
	c_4	(0.80, 0.20; 0.8246, 0.8440)	(0.90, 0.10; 0.9055, 0.9296)	(0.90, 0.10; 0.9055, 0.9296)
	c_5	(0.70, 0.40; 0.8062, 0.6695)	(0.70, 0.30; 0.7616, 0.7422)	(0.30, 0.10; 0.3162, 0.7952)
	c_6	(0.60, 0.60; 0.8485, 0.5000)	(0.50, 0.50; 0.7071, 0.5000)	(0.60, 0.40; 0.7211, 0.6257)
z_4	c_1	(0.90, 0.30; 0.9487, 0.7952)	(0.30, 0.90; 0.9487, 0.2048)	(0.60, 0.80; 1.0000, 0.4097)
	c_2	(0.70, 0.20; 0.7280, 0.8228)	(0.20, 0.30; 0.3606, 0.3743)	(0.40, 0.10; 0.4123, 0.8440)
	c_3	(0.40, 0.30; 0.5000, 0.5903)	(0.30, 0.70; 0.7616, 0.2578)	(0.20, 0.40; 0.4472, 0.2952)
	c_4	(0.90, 0.40; 0.9849, 0.7338)	(0.50, 0.80; 0.9434, 0.3556)	(0.70, 0.10; 0.7071, 0.9097)
	c_5	(0.50, 0.40; 0.6403, 0.5704)	(0.70, 0.60; 0.9220, 0.5489)	(0.60, 0.20; 0.6325, 0.7952)
	c_6	(0.60, 0.70; 0.9220, 0.4511)	(0.40, 0.70; 0.8062, 0.3305)	(0.80, 0.10; 0.8062, 0.9208)
	Ω^k	Ω^1={(3,2), (4,1), (3,1)}	Ω^2={(2,1),(4,3),(2,4),(3,1)}	Ω^3={(3,1),(3,4),(2,3),(2,4)}

Remark: (i) $Z=\{z_1$ (Germany), z_2 (Russia), z_3 (Singapore), z_4 (Malaysia)$\}$.
(ii) $C=\{c_1$ (financial internal rate of return), c_2 (net present value), c_3 (investment recovery period), c_4 (debt ratio and current ratio), c_5 (repayment period of loan), c_6 (public benefit and diplomatic influence)$\}$, where $C_{\mathrm{I}}=\{c_1, c_2,\ldots, c_6\}$ and $C_{\mathrm{II}} = \varnothing$.
(iii) $E=\{e_1$ (Decision maker #1), e_2 (Decision maker #2), e_3 (Decision maker #3)$\}$.

$$\psi(p_{ij}^k) = \frac{S(p_{ij}^k, p_{-j})}{S(p_{ij}^k, p_{+j}) + S(p_{ij}^k, p_{-j})} \tag{41}$$

Let $\Gamma_{\phi\varphi}^k$ denote the degree of violation about the ordered pair (ϕ, φ) associated with e_k, where $\Gamma_{\phi\varphi}^k = max\{0, \sum_{j=1}^n (\psi(p_{\varphi j}^k) - \psi(p_{\phi j}^k))\cdot w_j\}$. By generalizing the proposed PF LINMAP methodology to the group decision-making field, this paper formulates an efficacious linear programming model that serves a group MCDA scheme as follows:

$$\textbf{Model V}.\ max\left\{\eta \sum_{k=1}^K \sum_{j=1}^n \sum_{i=1}^m \psi\left(p_{ij}^k\right)\cdot w_j - (1-\eta)\sum_{k=1}^K \sum_{(\phi,\varphi)\in\Omega^k} \Gamma_{\phi\varphi}^k\right\}$$

$$\text{s.t.}\ \begin{cases} \sum_{j=1}^n \sum_{(\phi,\varphi)\in\Omega^k} \left(\psi\left(p_{\phi j}^k\right) - \psi\left(p_{\varphi j}^k\right)\right)\cdot w_j \geqslant \hbar \text{ for all } k, \\ \sum_{j=1}^n \left(\psi\left(p_{\phi j}^k\right) - \psi\left(p_{\varphi j}^k\right)\right)\cdot w_j + \Gamma_{\phi\varphi}^k \geqslant 0 \text{ for } (\phi,\varphi)\in\Omega^k \text{ for all } k, \\ \Gamma_{\phi\varphi}^k \geqslant 0 \text{ for } (\phi,\varphi)\in\Omega^k \text{ and } e_k\in E,\ \sum_{j=1}^n w_j = 1,\ w_j \geqslant \varepsilon_j \text{ for all } j. \end{cases} \tag{42}$$

The optimal $\overline{\Gamma}_{\phi\varphi}^k$ values towards each ordered pair $(\phi, \varphi) \in \Omega^k$ and the optimal weight vector $\overline{w} = (\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_n)$ are received by tackling the extended PF LINMAP model based on the group MCDA scheme in Model V. The corresponding total comprehensive dominance index $\overline{\Psi}(P_i)$ for each alternative z_i is generated by $\overline{\Psi}(P_i) = \sum_{k=1}^K \sum_{j=1}^n \overline{w}_j\cdot\psi(p_{ij}^k)$. Afterwards, the m alternatives are ranked in the light of the decreasing order of the $\overline{\Psi}(P_i)$ values. Additionally, the best compromise solution z^* in $Z^* \subseteq Z$ is ranked the best by $\overline{\Psi}(P_i)$ such that $Z^* = \{z_i | max_{i=1}^m \overline{\Psi}(P_i),\ z_i \in Z\} = \{z_i | max_{i=1}^m \sum_{k=1}^K \sum_{j=1}^n \overline{w}_j\cdot\psi(p_{ij}^k),\ z_i \in Z\}$.

As mentioned earlier, Wan et al. (2018) and Xue et al. (2018) employed their PF LINMAP models to carry on the group decision-making issues concerning green supplier selection and railway project investment, respectively. To confirm the workableness and practicability of the generalized PF LINMAP method in Model V in empirical group MCDA circumstances, this paper also investigates the two realistic problems and makes an equitable and even-handed comparative analysis. In line with Wan et al.'s and Xue et al.'s problem representations, the descriptions of the realistic cases of green supplier selection and railway project investment are displayed in Tables 5 and 6, respectively. There is one point worthy of particular attention in these two tables. Wan et al. (2018) and Xue et al. (2018) revealed the assessment data about the degrees of membership μ_{ij}^k (i.e., $\mu_{c_j}^k(z_i)$) and non-membership ν_{ij}^k (i.e., $\nu_{c_j}^k(z_i)$) for the evaluation of alternatives in the criteria offered by each decision maker. Nonetheless, the proposed extended Model V, i.e., the generalized PF LINMAP method using a SED-based outranking approach, requires the strength of commitment r_{ij}^k (i.e., $r_{c_j}^k(z_i)$) and the direction of commitment d_{ij}^k (i.e., $d_{c_j}^k(z_i)$); to be specific, the four-term representation is a requisite for assessment data by dint of the proposed methodology. Even though the four parameters are necessary for implementing the developed SED-based outranking approach, there is no extra information demanding in comparison with Xue et al.'s and Wan et al.'s approaches. The main reason is that the values of r_{ij}^k and d_{ij}^k can be derived by way of Definition 2. More specifically, $r_{ij}^k = \sqrt{(\mu_{ij}^k)^2 + (\nu_{ij}^k)^2}$ and $d_{ij}^k = (\pi - 2\cdot\theta_{ij}^k)/\pi$, where $\theta_{ij}^k = \arccos(\mu_{ij}^k/r_{ij}^k) = \arcsin(\nu_{ij}^k/r_{ij}^k)$. Therefore, no additional information is needed to ask from the decision makers. This indicates that the four-term representation does not limit practical applications of the proposed methodology. Moreover, the computed results of r_{ij}^k and d_{ij}^k are contained within p_{ij}^k $(= (\mu_{ij}^k, \nu_{ij}^k; r_{ij}^k, d_{ij}^k))$, as depicted in Tables 5 and 6.

Consider the group decision-making issue of green supplier selection. As reported by the remark section in Table 5, it is seen that $Z = \{z_1, z_2, z_3, z_4\}$, $C = \{c_1, c_2, \cdots, c_5\}$ (with $C_{\mathrm{I}} = \{c_1, c_2\}$ and $C_{\mathrm{II}} = \{c_3, c_4, c_5\}$), and $E = \{e_1, e_2, e_3\}$. Moreover, the sets of ordered pairs for three experts (i.e., decision makers e_1–e_3) are mentioned as follows: Ω^1={(3,2), (4,2)} for the expert e_1 from the materials and equipment department, Ω^2={(4,1), (4,2)} for the expert e_2 from the quality supervision department, and Ω^3={(2,1), (3,4)} for the expert e_3 from the engineering department. On account of the PF evaluative rating p_{ij}^k in Table 5, this paper employed

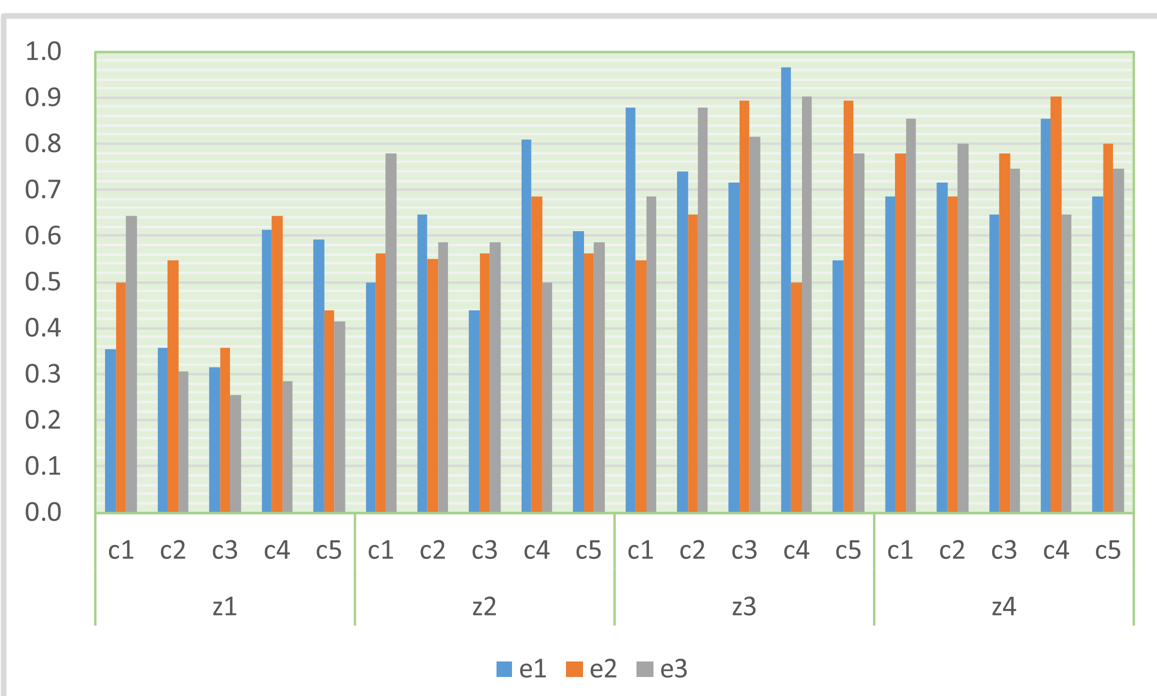


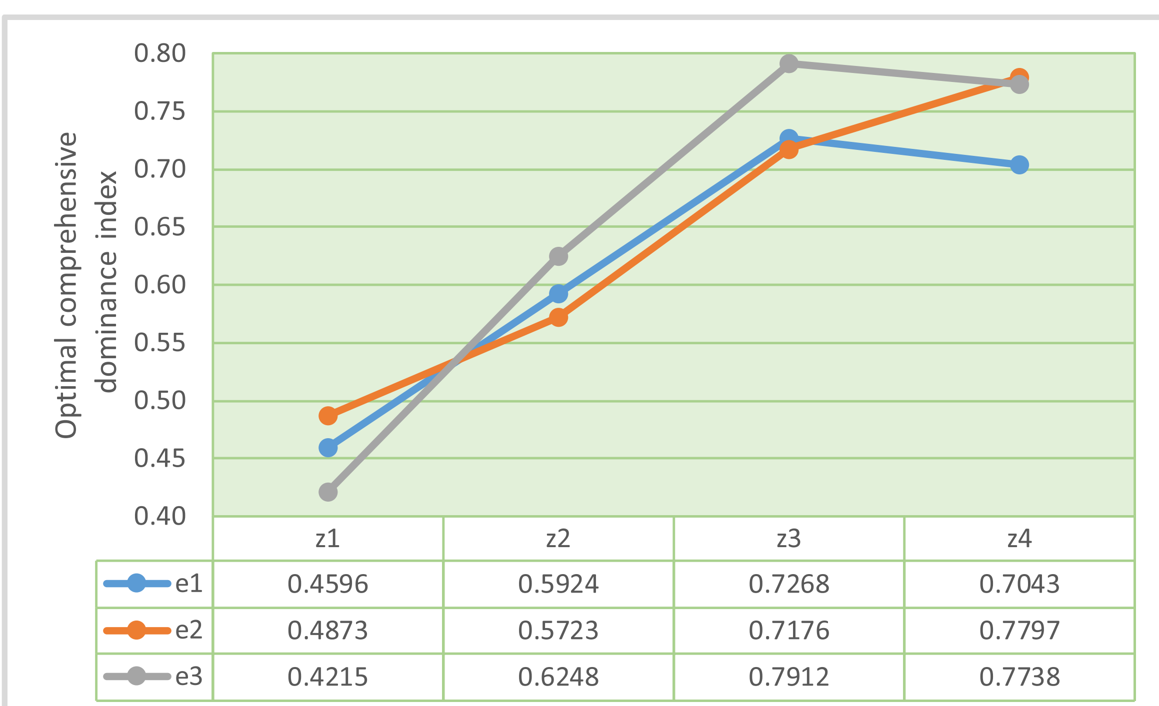
Fig. 11. SED-based dominance indices in the green supplier selection case.

(41) to calculate the SED-based dominance index $\psi(p_{ij}^k)$, and the obtained results are contrasted in Fig. 11. The alternatives z_3 and z_4 performs better, roughly speaking, than z_1 and z_2 in conformity with the comparison patterns of the $\psi(p_{ij}^k)$ values.

Relevant parameter values were set as follows: the lowest acceptable level ħ=0.2, the relative importance $\eta = 0.5$, and the non-negative boundary condition ε_j was replaced with the incomplete weight information that was designated by the three experts. As indicated in Wan et al. (2018), the following incomplete information was imposed into the generalized PF LINMAP model: $w_2 \geqslant 2w_3$, $0.1 \leqslant w_5 - w_1 \leqslant 0.3$, $w_2 - w_3 \geqslant w_5 - w_1$, $w_3 \geqslant w_4$, and $0.1 \leqslant w_3 \leqslant 0.15$. To apply Model V, the decision variables consist of the unknown weight w_j for $j \in \{1, 2, \cdots, 5\}$ and the degree of violation $\Gamma_{\phi\varphi}^k$ towards each ordered pair $(\phi, \varphi) \in \Omega^k$, i.e., Γ_{32}^1, Γ_{42}^1, Γ_{41}^2, Γ_{42}^2, Γ_{21}^3, and Γ_{34}^3. By employing (42), the following extended PF LINMAP model was established:

$$
\begin{aligned}
& max\{0.5\cdot(7.7661w_1 + 7.4548w_2 + 7.1048w_3 + 8.3056w_4 + 7.6503w_5) \\
& \quad - 0.5\cdot\left(\Gamma_{32}^1 + \Gamma_{42}^1 + \Gamma_{41}^2 + \Gamma_{42}^2 + \Gamma_{21}^3 + \Gamma_{34}^3\right)\}
\end{aligned}
$$

$$
\text{s.t.}\begin{cases}
0.5640w_1 + 0.1604w_2 + 0.4872w_3 + 0.2034w_4 + 0.0120w_5 \geqslant 0.2, \\
0.4943w_1 + 0.2742w_2 + 0.6382w_3 + 0.4735w_4 + 0.6006w_5 \geqslant 0.2, \\
-0.0344w_1 + 0.3571w_2 + 0.4008w_3 + 0.4700w_4 + 0.2027w_5 \geqslant 0.2, \\
0.3788w_1 + 0.0917w_2 + 0.2780w_3 + 0.1583w_4 - 0.0635w_5 + \Gamma_{32}^1 \geqslant 0, \\
0.1852w_1 + 0.0687w_2 + 0.2093w_3 + 0.0452w_4 + 0.0755w_5 + \Gamma_{42}^1 \geqslant 0, \\
0.2784w_1 + 0.1390w_2 + 0.4224w_3 + 0.2574w_4 + 0.3629w_5 + \Gamma_{41}^2 \geqslant 0, \\
0.2158w_1 + 0.1352w_2 + 0.2158w_3 + 0.2161w_4 + 0.2377w_5 + \Gamma_{42}^2 \geqslant 0, \\
0.1345w_1 + 0.2785w_2 + 0.3305w_3 + 0.2153w_4 + 0.1698w_5 + \Gamma_{21}^3 \geqslant 0, \\
-0.1689w_1 + 0.0786w_2 + 0.0703w_3 + 0.2546w_4 + 0.0328w_5 + \Gamma_{34}^3 \geqslant 0, \\
\Gamma_{32}^1, \Gamma_{42}^1, \Gamma_{41}^2, \Gamma_{42}^2, \Gamma_{21}^3, \Gamma_{34}^3 \geqslant 0, \\
\sum_{j=1}^{5} w_j = 1, w_2 \geqslant 2w_3, 0.1 \leqslant w_5 - w_1 \leqslant 0.3, w_2 - w_3 \geqslant w_5 - w_1, w_3 \geqslant w_4, 0.1 \leqslant w_3 \leqslant 0.15.
\end{cases}
$$



	z1	z2	z3	z4
e1	0.4596	0.5924	0.7268	0.7043
e2	0.4873	0.5723	0.7176	0.7797
e3	0.4215	0.6248	0.7912	0.7738

Fig. 12. Comparison of the $\overline{\Psi}(P_i^k)$ values in the green supplier selection case.

By resolving the above linear programming model, one received the optimal objective value 3.8256, the optimal weight vector $\overline{w} = (\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_5) = (0.25, 0.2, 0.1, 0.1, 0.35)$, and the optimal degrees of violation $\overline{\Gamma}^1_{32} = \overline{\Gamma}^1_{42} = \overline{\Gamma}^2_{41} = \overline{\Gamma}^2_{42} = \overline{\Gamma}^3_{21} = \overline{\Gamma}^3_{34} = 0$. Under the setting of the vector $(\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_5)$, this paper further plots the optimal comprehensive dominance index $\overline{\Psi}(P^k_i)$ of the alternative $z_i \in Z$ with relevance to each decision maker $e_k \in E$, where $\overline{\Psi}(P^k_i) = \sum_{j=1}^{5} \overline{w}_j \cdot \psi(p^k_{ij})$. The comparison results of the $\overline{\Psi}(P^k_i)$ values are sketched in Fig. 12. On the whole, the optimal comprehensive dominance indices in connection with the three experts demonstrate a consistent and stable pattern, i.e., $min\{\overline{\Psi}(P^k_3), \overline{\Psi}(P^k_4)\} > \overline{\Psi}(P^k_2) > \overline{\Psi}(P^k_1)$ for $k \in \{1, 2, 3\}$. More specifically, it is observed that $\overline{\Psi}(P^1_3) > \overline{\Psi}(P^1_4) > \overline{\Psi}(P^1_2) > \overline{\Psi}(P^1_1)$, $\overline{\Psi}(P^2_4) > \overline{\Psi}(P^2_3) > \overline{\Psi}(P^2_2) > \overline{\Psi}(P^2_1)$, and $\overline{\Psi}(P^3_3) > \overline{\Psi}(P^3_4) > \overline{\Psi}(P^3_2) > \overline{\Psi}(P^3_1)$ for the experts e_1–e_3, respectively. The total comprehensive dominance indices were produced as follows: $\overline{\Psi}(P_1) = \sum_{k=1}^{3}\sum_{j=1}^{5} \overline{w}_j \cdot \psi(p^k_{1j}) = 1.3684$, $\overline{\Psi}(P_2) = 1.7895$, $\overline{\Psi}(P_3) = 2.2356$, and $\overline{\Psi}(P_4) = 2.2577$. In consequence, the priority ranking of the four green suppliers is $z_4 \succ z_3 \succ z_2 \succ z_1$; moreover, the best compromise solution z^* is Green supplier #4 because $Z^* = \{z_4\}$. The yielded outcomes are concordant with those generated via Wan et al. (2018) method.

Next, as attested by the description in Table 6, this paper exploited the current methodology to explore the group decision analysis problem about railway project investment. Referring to the remark section in this table, it is known that $Z = \{z_1, z_2, z_3, z_4\}$, $C = \{c_1, c_2, \cdots, c_6\}$ (with $C_{\mathrm{I}} = \{c_1, c_2, \cdots, c_6\}$ and $C_{\mathrm{II}} = \varnothing$), and $E = \{e_1, e_2, e_3\}$. The sets of ordered pairs for three decision makers are stated as follows: Ω^1={(3,2), (4,1), (3,1)} for e_1, Ω^2={(2,1), (4,3), (2,4), (3,1)} for e_2, and Ω^3={(3,1), (3,4), (2,3), (2,4)} for e_3. On account of the PF evaluative rating p^k_{ij} in Table 6, the determination outcomes of the SED-based dominance index $\psi(p^k_{ij})$ are contrasted in Fig. 13. For the most part, the alternatives z_2 and z_3 performs better than z_1 and z_4 in the light of the comparison patterns of the $\psi(p^k_{ij})$ values.

This paper designated relevant parameter values as follows: the lowest acceptable level ħ=0.2, the relative importance $\eta = 0.5$, and the non-negative boundary condition $\varepsilon_j = \varepsilon_0 = 0.01$ for each $c_j \in C$. Note that $\varepsilon_0 = 0.01$ is in congruence with Xue et al. (2018) setting of the weight constraints. By virtue of Model V, the decision variables consist of the unknown weight w_j for $j \in \{1, 2, \cdots, 6\}$ and the degree of violation $\Gamma^k_{\phi\varphi}$ towards each $(\phi, \varphi) \in \Omega^k$, i.e., Γ^1_{32}, Γ^1_{41}, Γ^1_{31}, Γ^2_{21}, Γ^2_{43}, Γ^2_{24}, Γ^2_{31}, Γ^3_{31}, Γ^3_{34}, Γ^3_{23}, and Γ^3_{24}. This paper constructed the following extended PF LINMAP model:

$$
\begin{aligned}
& max\{0.5\cdot(7.4224w_1 + 7.0754w_2 + 7.4023w_3 + 9.3621w_4 + 8.4005w_5 + 6.6261w_6) \\
& \quad - 0.5\cdot\left(\Gamma^1_{32} + \Gamma^1_{41} + \Gamma^1_{31} + \Gamma^2_{21} + \Gamma^2_{43} + \Gamma^2_{24} + \Gamma^2_{31} + \Gamma^3_{31} + \Gamma^3_{34} + \Gamma^3_{23} + \Gamma^3_{24}\right)\}
\end{aligned}
$$

$$
\text{s.t.}\begin{cases}
0.1845w_1 - 0.2965w_2 + 0.9935w_3 + 1.2542w_4 - 0.7077w_5 + 0.3218w_6 \geqslant 0.2, \\
0.7882w_1 + 1.5312w_2 + 0.1252w_3 + 0.0655w_4 + 1.4911w_5 + 0.8829w_6 \geqslant 0.2, \\
1.0255w_1 + 0.3736w_2 + 1.1397w_3 + 0.7835w_4 + 0.4183w_5 + 0.4224w_6 \geqslant 0.2, \\
-0.2408w_1 - 0.1823w_2 + 0.4718w_3 - 0.0640w_4 - 0.1471w_5 - 0.4344w_6 + \Gamma^1_{32} \geqslant 0, \\
0.3442w_1 + 0.0681w_2 + 0.0500w_3 + 0.6800w_4 - 0.3718w_5 + 0.3218w_6 + \Gamma^1_{41} \geqslant 0, \\
0.0811w_1 - 0.1823w_2 + 0.4718w_3 + 0.6383w_4 - 0.1888w_5 + 0.4344w_6 + \Gamma^1_{31} \geqslant 0, \\
0.3941w_1 + 0.7656w_2 + 0.0626w_3 + 0.0327w_4 + 0.7456w_5 + 0.4415w_6 + \Gamma^2_{21} \geqslant 0, \\
-0.5693w_1 - 0.1558w_2 - 0.6797w_3 - 0.7807w_4 - 0.1659w_5 - 0.2456w_6 + \Gamma^2_{43} \geqslant 0, \\
0.9135w_1 + 0.3550w_2 + 0.3910w_3 + 0.7760w_4 + 0.3546w_5 + 0.6244w_6 + \Gamma^2_{24} \geqslant 0, \\
0.0499w_1 + 0.5664w_2 + 0.3513w_3 + 0.0374w_4 + 0.5569w_5 + 0.0626w_6 + \Gamma^2_{31} \geqslant 0, \\
-0.1966w_1 + 0.0000w_2 + 0.1696w_3 + 0.4718w_4 + 0.0913w_5 + 0.4224w_6 + \Gamma^3_{31} \geqslant 0, \\
0.0856w_1 - 0.1791w_2 + 0.3246w_3 + 0.1558w_4 - 0.1241w_5 - 0.2574w_6 + \Gamma^3_{34} \geqslant 0, \\
0.5255w_1 + 0.3659w_2 + 0.1605w_3 + 0.0000w_4 + 0.2876w_5 + 0.2574w_6 + \Gamma^3_{23} \geqslant 0, \\
0.6111w_1 + 0.1868w_2 + 0.4850w_3 + 0.1558w_4 + 0.1635w_5 + 0.0000w_6 + \Gamma^3_{24} \geqslant 0, \\
\Gamma^1_{32}, \Gamma^1_{41}, \Gamma^1_{31}, \Gamma^2_{21}, \Gamma^2_{43}, \Gamma^2_{24}, \Gamma^2_{31}, \Gamma^3_{31}, \Gamma^3_{34}, \Gamma^3_{23}, \Gamma^3_{24} \geqslant 0, \\
\sum_{j=1}^{6} w_j = 1, \ w_j \geqslant 0.01 \text{ for } j = 1, 2, \cdots, 6.
\end{cases}
$$

The above linear programming model was resolved to receive the optimal objective value 4.2051, the optimal weight vector $\overline{w} = (\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_6) = (0.01, 0.01, 0.01, 0.8872, 0.0728, 0.01)$, the optimal degrees of violation $\overline{\Gamma}^1_{32} = 0.0714$, $\overline{\Gamma}^2_{43} = 0.7212$, and $\overline{\Gamma}^1_{41} = \overline{\Gamma}^1_{31} = \overline{\Gamma}^2_{21} = \overline{\Gamma}^2_{24} = \overline{\Gamma}^2_{31} = \overline{\Gamma}^3_{31} = \overline{\Gamma}^3_{34} = \overline{\Gamma}^3_{23} = \overline{\Gamma}^3_{24} = 0$. Next, this paper pictures the optimal comprehensive dominance index $\overline{\Psi}(P^k_i)$ in conjunction with each $z_i \in Z$

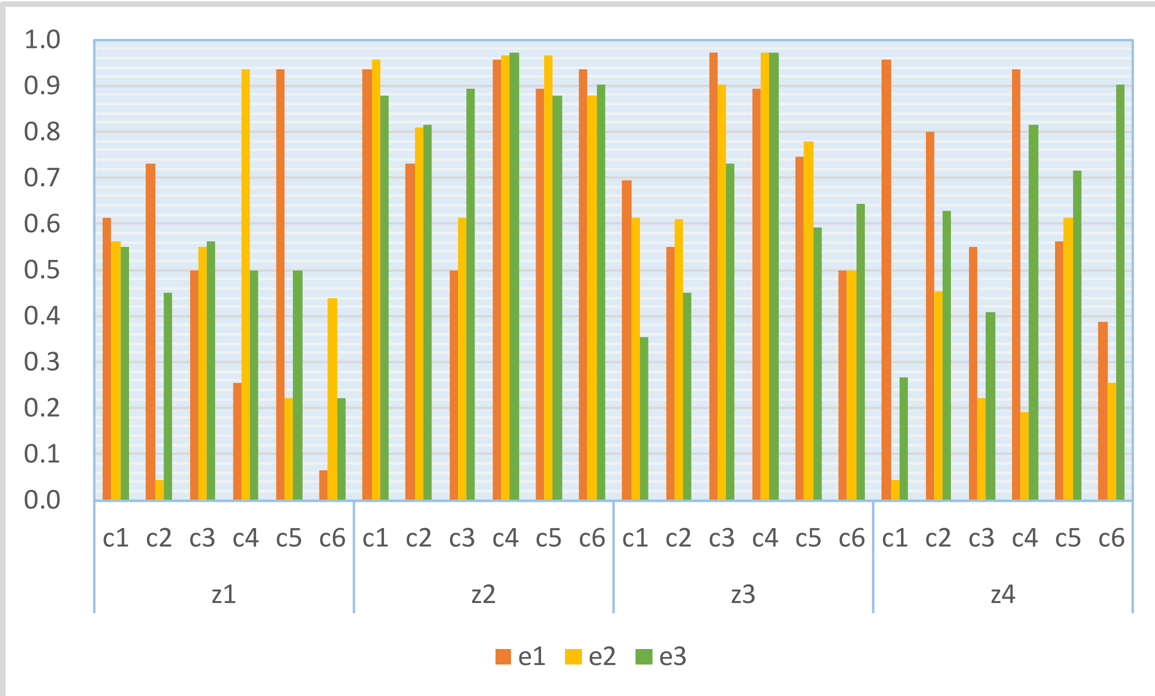


Fig. 13. SED-based dominance indices in the railway project investment case.

and $e_k \in E$ under the setting of the vector $(\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_6)$, where $\overline{\Psi}(P_i^k) = \sum_{j=1}^{6} \overline{w}_j \cdot \psi(p_{ij}^k)$. Fig. 14 delineates the contrast outcomes of the $\overline{\Psi}(P_i^k)$ values. In the railway project investment case, the optimal comprehensive dominance indices with relevance to the three decision makers reveal somewhat different patterns. More specifically, it is known that $\overline{\Psi}(P_2^1) > \overline{\Psi}(P_4^1) > \overline{\Psi}(P_3^1) > \overline{\Psi}(P_1^1)$ for e_1 and $\overline{\Psi}(P_2^k) > \overline{\Psi}(P_3^k) > min\{\overline{\Psi}(P_1^k), \overline{\Psi}(P_4^k)\}$ for $k \in \{2, 3\}$ (i.e., $\overline{\Psi}(P_2^2) > \overline{\Psi}(P_3^2) > \overline{\Psi}(P_1^2) > \overline{\Psi}(P_4^2)$ for e_2 and $\overline{\Psi}(P_2^3) > \overline{\Psi}(P_3^3) > \overline{\Psi}(P_4^3) > \overline{\Psi}(P_1^3)$ for e_3). The total comprehensive dominance indices were determined as follows: $\overline{\Psi}(P_1) = \sum_{k=1}^{3} \sum_{j=1}^{6} \overline{w}_j \cdot \psi(p_{1j}^k) = 1.6717$, $\overline{\Psi}(P_2) = 2.8668$, $\overline{\Psi}(P_3) = 2.7455$, and $\overline{\Psi}(P_4) = 1.9188$. Consequently, the priority ranking of the alternatives is $z_2 \succ z_3 \succ z_4 \succ z_1$, and the best compromise solution z^* is Russia (z_2) because $Z^* = \{z_2\}$. The yielded outcomes are consistent with those produced by Xue et al. (2018) approach.

The application results concerning two practical cases of green supplier selection and railway project investment further exhibit several

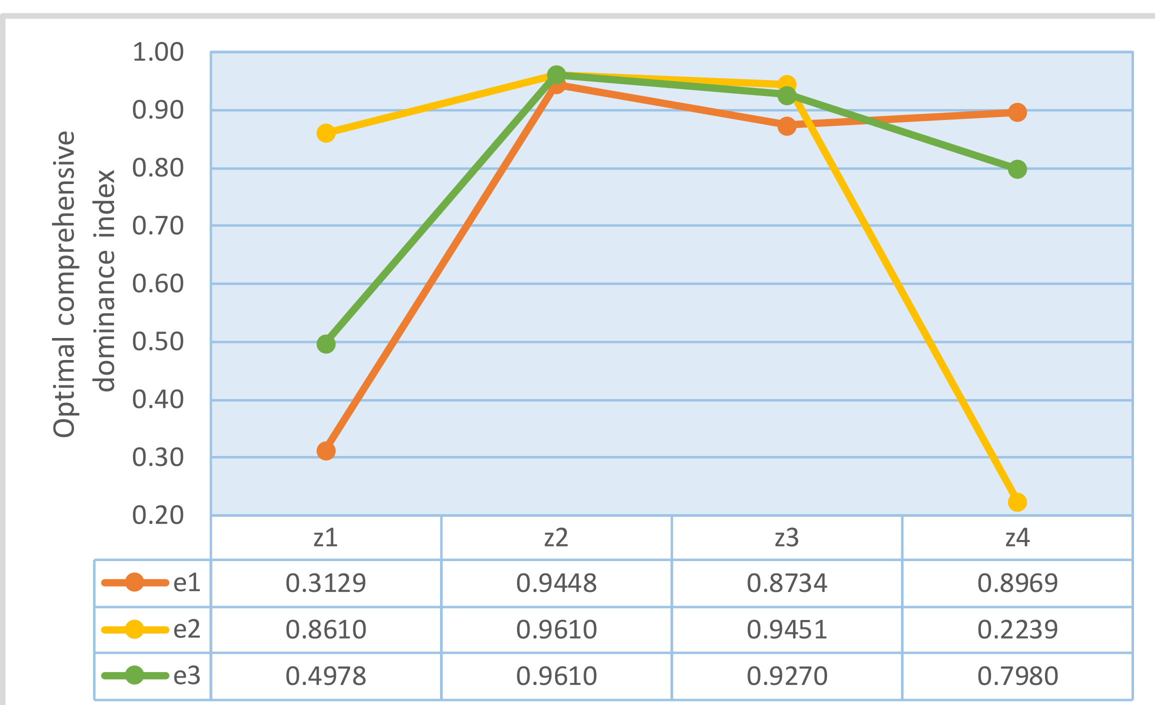


Fig. 14. Comparison of the $\overline{\Psi}(P_i^k)$ values in the railway project investment case.

strong points of the current methodology in comparison to the existing PF LINMAP techniques (e.g., Wan et al. (2018) and Xue et al. (2018) methods). First, the proposed methodology furnishes efficacious and flexible techniques for handling individual decision-making issues via Model IV and group decision-making issues via Model V. Second, the developed models and execution procedures are much easier to understand than Wan et al.'s and Xue et al.'s approaches. On the strength of a uncomplicated and efficacious SED-based outranking technique, the PF LINMAP optimization models (i.e., Models IV and V) supply an effortless and unsophisticated tool to tackle (group) decision-making problems under complex PF uncertainty. Third, the applicability and usefulness of the current methodology have been thoroughly corroborated by the agency of three investigated issues regarding bridge-superstructure construction methods, green supplier selection, and railway project investment. In contrast to the existing methods and techniques, the employment of the proposed methodology (and its generalization) can yield believable and befitting outcomes and thus bring about more application merits in the decision-making field.

6. Conclusions

This paper has addressed MCDA problems with PF evaluative ratings and stated preference information between pairs of alternatives. By the agency of a newly developed Euclidean distance measure in the context of PF sets, this paper has delivered an efficacious concept of squared PF Euclidean distances and explored several beneficial and desirable properties, such as boundedness, reflexivity, separability, symmetry, and concordance of quasi-ordering. Thereafter, this paper has constructed the SED-based dominance index and justified its effectiveness in determining the dominance relations among PF evaluative ratings. To conduct an intracriterion comparison between alternatives, the SED-based outranking approach has been developed in PF decision contexts. Subsequently, this paper has delivered some useful measurements of rank consistency and inconsistency to evaluate the goodness of fit and the poorness of fit, respectively, from the distributive PF evaluative ratings over various criteria for each paired preference relation. This paper has formulated a series of PF LINMAP optimization problems under incomplete and non-transitive paired preference relations of alternatives. Specifically, Model I aims to minimize the degree of poorness of fit under consideration of the lowest acceptable level. Model II addresses the issues of specialized degenerate weighting results and maximal comprehensive dominance indices and provides a linear programming model. Model III presents a bi-objective nonlinear programming model with a focus on maximizing the total comprehensive dominance index and minimizing the degree of poorness of fit. Model IV coordinates the two objectives by means of a parameter to transform Model III into a single-objective linear programming format. By resolving the simple and effective linear programming problem in Model IV, the optimal criterion weights and optimal degrees of violation were derived to receive the ultimate comprehensive dominance indices, generate the priority orders of competing alternatives, and conclude the best compromise choice. Furthermore, the selection issue of bridge-superstructure construction methods has been explored to corroborate the practicability and efficacy of the PF LINMAP methodology. Over and above that, the proposed methodology can be easily generalized to the group decision-making field. Model V exhibits the extended PF LINMAP model that serves a group MCDA scheme for expanding potentials in practice. Finally, some comparative studies with other relevant approaches, along with sensitivity analyses of various parameter settings, have validated the advantages and flexibility of the advanced methodology.

This paper furnishes four recommendations for future research. First, by the agency of the concept of squared PF Euclidean distances, this paper suggests that future research can focus on construction of a novel class of Bregman divergences under uncertainty with Pythagorean fuzzy information. Second, this paper recommends that both the SED-based dominance index and the SED-based outranking approach can be incorporated into other MCDA models that are most often built using a concordance–discordance principle, such as the ELECTRE and PROMETHEE methodologies. Third, a recommendation that research on the challenges of extending the PF LINMAP for tackling group decision analysis issues involving inconsistent preference information, and the consensus-reaching processes used to overcome those challenges can be made. The fourth recommendation is that an extensive range of applications of the proposed methodology can be investigated. Because the SED-based outranking approach with dominance indices can be conveniently combined with other evaluation techniques, which enables various applications in practical fields and industries.

CRediT authorship contribution statement

Ting-Yu Chen: Conceptualization, Methodology, Validation, Formal analysis, Data curation, Writing - original draft, Writing - review & editing, Visualization, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability statement

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Compliance with ethical standards

This article does not contain any studies with human participants or animals that were performed by the author.

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