① SVM Loss.

$L = \sum_{i=1}^{n} L_i$. $\quad L_i = \sum_{j \neq y_i} \max(0, (Wx_i)_j - (Wx_i)_{y_i} + \Delta) \triangleq \sum_{j \neq y_i} \max(0, gap_i(j, y_i))$.

$\forall i \in \{1, \cdots, n\}$, $\quad \frac{\partial L_i}{\partial W_{kl}} = \sum_{j \neq y_i} a_i^{(kl)}(j, y_i)$.

其中 $a_i^{(kl)}(j, y_i) = \begin{cases} 1\{j=k\} x_{il} - 1\{y_i = k\} x_{il}, & gap_i(j, y_i) > 0 \\ 0, & gap_i(j, y_i) < 0 \end{cases}$

$\Downarrow$

$\forall i \in \{1, \cdots, n\}$, $\begin{cases} k = y_i \text{ 时}, \ \frac{\partial L_i}{\partial W_{kl}} = \sum_{j \neq y_i} -x_{il} 1\{gap_i(j, y_i) > 0\} = -x_{il} \sum_{j \neq y_i} 1\{gap_i(j, y_i) > 0\} \\ k \neq y_i \text{ 时}, \ \frac{\partial L_i}{\partial W_{kl}} = \sum_{j \neq y_i} x_{il} 1\{gap_i(j, y_i) > 0\} 1\{j = k\} \\ \qquad = x_{il} 1\{gap_i(j, y_i) > 0\}. \end{cases}$ (*)

记 $1\{gap_i(j, y_i) > 0\} = m(i, j)$ 则 (*) 可表为 $\begin{cases} k = y_i, \ \frac{\partial L_i}{\partial W_{kl}} = -x_{il} \sum_{j \neq y_i} m(i, j) \\ k \neq y_i, \ \frac{\partial L_i}{\partial W_{kl}} = x_{il} m(i, k). \end{cases}$

$\swarrow$ 向量化, 抛掉 l

$\begin{cases} k = y_i, \ \frac{\partial L_i}{\partial W_k} = -x_i \sum_{j \neq y_i} m(i, j) \\ k \neq y_i, \ \frac{\partial L_i}{\partial W_k} = x_i m(i, k) \end{cases}$

$\Downarrow$ 向量化, 构造矩阵 $C(i, k) = \begin{cases} -\sum_{j \neq y_i} m(i, j), & k = y_i \\ m(i, k), & k \neq y_i \end{cases}$

$\frac{\partial L_i}{\partial W_k} = x_i C(i, k) = C(i, k) x_i$ (C(i,k): 数; x_i: 向量)

$$\left(\frac{\partial L_i}{\partial W_k} = C(i,k) X_i \right)$$

⇓ 向量化，抛掉 i

$$\frac{\partial L}{\partial W_k} = (C(1,k), \cdots, C(n,k))\, X$$

⇓ 抛掉 k.

$$\frac{\partial L}{\partial W} = C^T X.$$

② Softmax ~~loss~~ (Cross-entropy loss)

$$L = \frac{1}{N} \sum_i L_i + \lambda \sum_k \sum_l W_{kl}^2 \quad , \quad L_i = -\log \left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}} \right)$$

$$= -s_{y_i} + \log \left(\sum_j e^{s_j} \right). \quad (*)$$

同理，先分析 $\frac{\partial L_i}{\partial W_{kl}}$. 据 (*) 式得 (*) 上的等式不要求，改用 (*) 式)

$$\frac{\partial L_i}{\partial W_{kl}} = -x_{il} \mathbb{1}\{k = y_i\} + \frac{1}{\sum_j e^{s_j}} \cdot \sum_j e^{s_j} \mathbb{1}\{k = j\} x_{il}.$$

($s = Wx_i$, 一般 $\mathbb{1}\{k = y_i\}$ 、$\mathbb{1}\{k = j\}$ 的出现是因为取了 Wx_i 的 y_i 分量或 j 分量)

⇓ 向量化，抛掉 l.

$$\frac{\partial L_i}{\partial W_k} = -x_i \mathbb{1}\{k = y_i\} + \frac{1}{\sum_j e^{s_j}} e^{s_k} \cdot x_i.$$

$$\left(\frac{\partial L_i}{\partial w_k} = -x_i \mathbb{1}\{k=y_i\} + \frac{1}{\sum_j e^{s_j}} e^{s_k} \cdot x_i \right)$$

⇓

$$\frac{\partial L_i}{\partial w_k} = \begin{cases} k=y_i \text{ 时}, & = \left[-1 + \frac{e^{s_k}}{\sum e^{s_j}}\right] x_i \\ k\neq y_i \text{ 时}, & = \left[\frac{e^{s_k}}{\sum e^{s_j}}\right] \cdot x_i \end{cases}$$

⇓ 向量化，构造 $A(i,k) = \begin{cases} \frac{e^{s_k}}{\sum e^{s_j}} - 1, & k=y_i \\ \frac{e^{s_k}}{\sum e^{s_j}}, & k\neq y_i \end{cases}$

$$\frac{\partial L_i}{\partial w_k} = A(i,k)\, x_i$$

（A的计算中涉及exp()，为防止大数溢出，可用上下同乘常数的技巧

$$\frac{Ce^{s_k}}{C\sum e^{s_j}} = \frac{e^{s_k - c'}}{\sum e^{s_j - c'}}$$，令 $c' = \max_j \{s_j\}$，则指数部分 $s_j - c'$ 被中心化了）

⇓ 向量化，抛掉 i

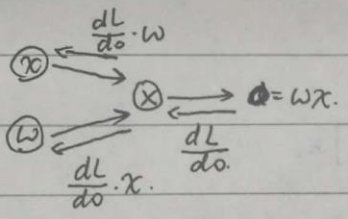
$$\frac{\partial L}{\partial w_k} = (A(1,k), \cdots, A(n,k))\, X.$$

⇓ 抛掉 k.

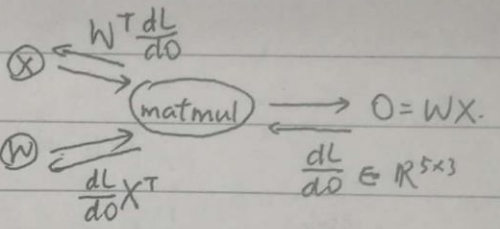
$$\frac{\partial L}{\partial W} = A^T X.$$

③ Two Layers NN

重要1：



向量化



($X \in \mathbb{R}^{10\times3}$, $W \in \mathbb{R}^{5\times10}$, $O \in \mathbb{R}^{5\times3}$).

(根据维度看是否转置).

(维度相同时套用此规律).

$$\frac{\partial L}{\partial X} = \frac{\partial f}{\partial O}\cdot\frac{\partial O}{\partial X}, \quad \frac{\partial L}{\partial X_{kl}} = \sum_{ij}\frac{\partial L}{\partial O_{ij}}\cdot\frac{\partial O_{ij}}{\partial X_{kl}} \quad (\text{因 } O = WX, \text{ 故 } j\neq l \text{ 时 } \frac{\partial O_{ij}}{\partial X_{kl}} = 0)$$

$$= \sum_{i}\frac{\partial L}{\partial O_{il}}\cdot\frac{\partial O_{il}}{\partial X_{kl}}$$

$$= \sum_{i}\frac{\partial L}{\partial O_{il}}\cdot W_{ik}$$

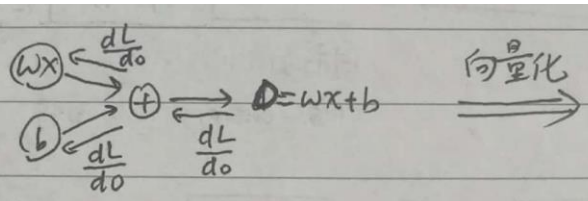
← 此处 l 是自由的！该式意义为 $\frac{dL}{dO}$ 的第 l 列与 W 的第 k 列内积，作为 $\frac{dL}{dX}$ 的第 k 行第 l 列元素.

$$\Downarrow$$

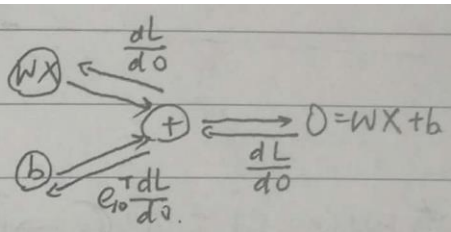
$$\frac{\partial L}{\partial X} = W^T\frac{\partial L}{\partial O}.$$

关于 $\frac{\partial L}{\partial W}$ 可类似推导.)

重要2：



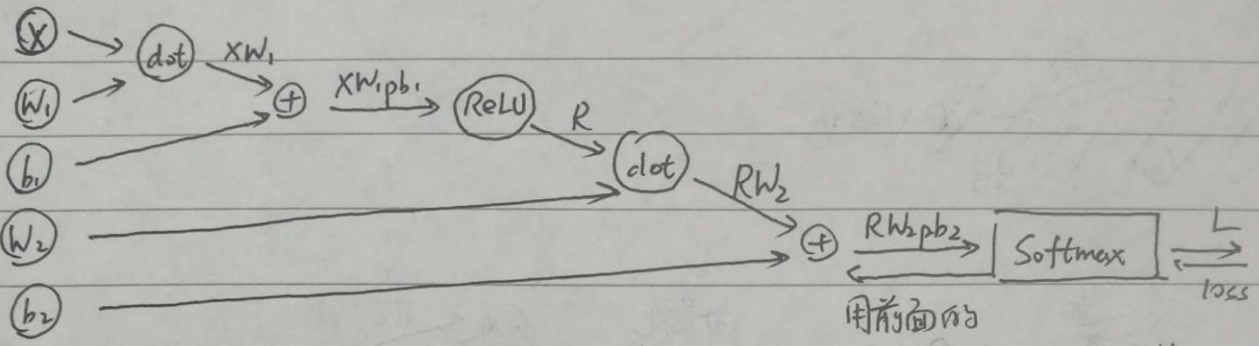
向量化



($WX \in \mathbb{R}^{10\times 3}$, $b \in \mathbb{R}^{1\times 3}$, $O \in \mathbb{R}^{10\times 3}$).

(对于 broadcast 变量，因最终的损失 L 是在行坐标上求和，故梯度也应在行坐标上累加).

由上述两个重要性质，结合下述 computation graph，则可对 Two Layer NN 进行 BP.



用前面的 cross-entropy loss 可得

(若有正则化，则应有 $W_1 \to$ Softmax 与 $W_2 \to$ Softmax，勿漏！)