MAT185 Linear Algebra
Assignment 1

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: ______________________________

Student number: ______________________________

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Student number: ______________________________

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) ______________________________

2) ______________________________

1. A *vector spyce* is a set V together with two operations called *vector addition* and *scalar multiplication* such that the following nine *axioms* hold:

(i) For all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} \in V$

(ii) For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$

(iii) There exists a vector $\mathbf{0} \in V$ with the property that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$

(iv) For each vector $\mathbf{x} \in V$, there exists a vector $-\mathbf{x} \in \mathbf{V}$ with the property that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

(v) For all vectors $\mathbf{x} \in V$, and scalars $c \in \mathbb{R}$, $c\mathbf{x} \in V$

(vi) For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$

(vii) For all vectors $\mathbf{x}, \mathbf{y} \in V$, and scalars $c \in \mathbb{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$

(viii) For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$

(ix) For all vectors $\mathbf{x} \in V$, $c\mathbf{x} = \mathbf{0}$ implies that either $c = 0$ or $\mathbf{x} = \mathbf{0}$

Prove that a *vector spyce* is a vector space.

$$
\begin{aligned}
v \quad & y = 1x \\
Sub \quad & 1x = 1x \\
\mathbb{R} \quad & (1 \cdot 1)x = 1x \\
vi \quad & 1(1x) = 1x \\
Sub \quad & 1y = 1x \\
Cancellation \quad & 1y - 1x = 1x - 1x \\
vii \quad & 1(y - x) = 1(x - x) \\
iv \quad & 1(y - x) = 1(0) \\
ix \quad & 1(y - x) = 1(0) \\
ix \quad & 1 \neq 0 \implies y - x = 0 \\
Cancellation \quad & y - x + x = 0 + x \\
-u + u = 0 \quad & y + 0 = 0 + x \\
iii \quad & y = 0 + x \\
0 + u = u \quad & y = x \\
Sub \quad & 1x = x
\end{aligned}
$$

2. Let $V = \{a_0 + a_1x + a_2x^2 \mid a_0 > 0\}$. Define vector addition in V by

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0b_0) + (a_1 + b_1 + 4)x + (a_2 + b_2)x^2$$

and scalar multiplication in V by

$$c(a_0 + a_1x + a_2x^2) = a_0^c + (ca_1 + 4(c-1))x + ca_2x^2$$

Verify that axiom MIII. (Medici, pp104) holds in V.
MIII a:

$$\begin{aligned}
(c+d)(a_0 + a_1x + a_2x^2) =& a_0^{c+d} + ((c+d)a_1 + 4(c+d-1))x + (c+d)a_2x^2 \\
=& a_0^c a_0^d + (ca_1 + da_1 + 4c + 4d - 4)x + (ca_2 + da_2)x^2 \\
=& a_0^c a_0^d + (ca_1 + 4c - 4 + da_1 + 4d - 4 + 4)x + (ca_2 + da_2)x^2 \\
=& a_0^c a_0^d + (ca_1 + 4(c-1) + da_1 + 4(d-1) + 4)x + (ca_2 + da_2)x^2 \\
=& (a_0^c + (ca_1 + 4(c-1))x + ca_2x^2) + (a_0^d + (da_1 + 4(d-1))x + da_2x^2) \\
=& c(a_0 + a_1x + a_2x^2) + d(a_0 + a_1x + a_2x^2)
\end{aligned}$$

MIII b:

$$\begin{aligned}
c((a_0 + a_1x + a2x^2) + (b_0 + b_1x + b_2x^2)) =& c((a_0b_0) + (a_1 + b_1 + 4)x + (a_2 + b_2)x^2) \\
=& (a_0b_0)^c + (c(a_1 + b_1 + 4) + 4(c-1))x + c(a_2 + b_2)x^2 \\
=& (a_0^c b_0^c) + (ca_1 + cb_1 + 4c + 4c - 4)x + (ca_2 + cb_2)x^2 \\
=& (a_0^c b_0^c) + (ca_1 + 4c - 4 + cb_1 + 4c - 4 + 4)x + (ca_2 + cb_2)x^2 \\
=& (a_0^c b_0^c) + (ca_1 + 4(c-1) + cb_1 + 4(c-1) + 4)x + (ca_2 + cb_2)x^2 \\
=& (a_0^c + (ca_1 + 4(c-1))x + ca_2x^2) + (b_0^c + (cb_1 + 4(c-1))x + cb_2x^2) \\
=& c(a_0 + a_1x + a_2x^2) + c(b_0 + b_1x + b_2x^2)
\end{aligned}$$

3. We asked ChatGPT if $V = \{f \mid f' \text{ is constant}\}$ is a vector space. It's response is given below.

KY **You**

Is V={ f|f' is constant} a vector space?

ChatGPT

To determine if the set $V = \{f \mid f' \text{ is constant}\}$ is a vector space, we need to check the three vector space properties:

1. **Zero vector:** The constant function $f(x) = c$ has a derivative of $f'(x) = 0$, which is a constant. Therefore, the zero function is in V.
2. **Closure under addition:** Let f and g be functions in V with constant derivatives. Consider the function $h(x) = f(x) + g(x)$. The derivative of h is the sum of the derivatives of f and g, which are constants. So, h also has a constant derivative, and V is closed under addition.
3. **Closure under scalar multiplication:** Let f be a function in V with a constant derivative. If we multiply f by a scalar c, the derivative of cf is cf', which is still a constant. Thus, V is closed under scalar multiplication.

Since V satisfies all three vector space properties, it is indeed a vector space.

Identify *two* errors in ChatGPT's response and write *two* paragraphs, one for each error, each of which clearly identifies an error, and how you would correct them.

4. Let $c \in \mathbb{R}$, and consider the subset $W_c = \{ (x, y, z) \in \mathbb{R}^3 \mid x + y + z = c\,|z| \}$ of $\mathbb{R}^3$. Determine *all* values of c such that W is a subspace of $\mathbb{R}^3$. Your answer should clearly demonstrate that you've found *all* values of c such that W_c is a subspace. This includes demonstrating why W_c is not a subspace for certain values of c, if you think no such c exists.

Since W_c is a subset of $\mathbb{R}^3$ and $\mathbb{R}^3$ is a vector space, W_c is a subspace of $\mathbb{R}^3$ if W_c satisfies the conditions of the subspace test (W_c is non-empty and contains the zero vector, W_c is closed under vector addition and scalar multiplication).
The zero vector:

$$\begin{aligned}
&\forall c \in \mathbb{R}, 0 + 0 + 0 = c|0| = 0 \\
&\forall x \in W_c, x + 0 = (x_1, x_2, x_3) + (0, 0, 0) = (x_1 + x_2 + x_3) = x \\
&\therefore (0, 0, 0) \in W_c
\end{aligned}$$

Therefore W_c is non-empty and contains the zero vector for all c.

Inverse: Since the inverse of a vector $-v = (-1)v$, the inverse of a vector should also exist in a vector space due to closure of scalar multiplication.

$$\begin{aligned}
&\forall x = (x_1, x_2, x_3) \in W_c, \exists -x = (-x_1, -x_2, -x_3) \in W_c \mid x + (-x) = 0 \\
&(x_1 + (-x_1), x_2 + (-x_2), x_3 + (-x_3)) = 0 \\
&\because x \in W_c, x_1 + x_2 + x_3 = c|x_3| \\
&\because -x \in W_c, -x_1 + -x_2 + -x_3 = c|-x_3| = c|x_3| \\
&\therefore x_1 + x_2 + x_3 = -x_1 + -x_2 + -x_3 \\
&x_1 + x_2 + x_3 = -(x_1 + x_2 + x_3) \\
&\therefore x_1 + x_2 + x_3 = 0 \\
&\therefore c|x_3| = 0 \\
&\because \forall x_3 \in \mathbb{R} \\
&\therefore c = 0
\end{aligned}$$

Therefore only $c = 0$ satisfies the inverse condition. If $c \neq 0$, then $x_1 + x_2 + x_3 \neq -x_1 + -x_2 + -x_3$.

Since only $c = 0$ satisfies the inverse condition, only W_0 will be checked for closure under vector addition and scalar multiplication.

$$\begin{aligned}
&\forall x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in W_0, \forall c \in \mathbb{R} \\
&cx + y = (cx_1 + y_1, cx_2 + y_2, cx_3 + y_3) \\
&(cx_1 + y_1) + (cx_2 + y_2) + (cx_3 + y_3) = cx_1 + cx_2 + cx_3 + y_1 + y_2 + y_3 \\
&= c(x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) \\
&\because x \in W_0, x_1 + x_2 + x_3 = 0 \\
&\because y \in W_0, y_1 + y_2 + y_3 = 0 \\
&\therefore (cx_1 + y_1) + (cx_2 + y_2) + (cx_3 + y_3) = c(0) + (0) = 0 \\
&\therefore cx + y \in W_0
\end{aligned}$$

Therefore W_0 is closed under vector addition and scalar multiplication.

Therefore W_c is a subspace of $\mathbb{R}^3$ if and only if $c = 0$.