MAT185 Linear Algebra
Assignment 2

Instructions:

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Preamble: An application of linear algebra to calculus.

Recall the technique of partial fractions decomposition to evaluate the integral of rational functions. For example, suppose we would like to evaluate the integral

$$\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx$$

We look for scalars a, b, and c such that

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

After some algebra, we find that $a = 2$, $b = 4$, and $c = 5$, and therefore,

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{2x+4}{x^2+3} + \frac{5}{x-2}$$

Then,

$$\begin{aligned}\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx &= \int \frac{2x+4}{x^2+3}\,dx + \int \frac{5}{x-2}\,dx \\ &= \ln(x^2+3) + \frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + 5\ln(x-2) + C\end{aligned}$$

where C is a constant.

In Question 1, we will use the theory of basis and dimension in linear algebra to explain why the partial fractions decomposition

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

exists, thereby allowing us to solve the integral.

1. Let

$$V = \left\{ \frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(a) Prove that dim $V = 3$. Then, explain why a partial fractions decomposition of the form

$$\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

is consistent with the dimension of V.

Use the page 3 to answer this question.

1(a)

$$V = \{\frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d,e,f \in \mathbb{R}\} \tag{1}$$
$$= \{d\frac{x^2}{(x^2+3)(x-2)} + e\frac{x}{(x^2+3)(x-2)} + f\frac{1}{(x^2+3)(x-2)} \mid d,e,f \in \mathbb{R}\} \tag{2}$$

Since $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = d\frac{x^2}{(x^2+3)(x-2)} + e\frac{x}{(x^2+3)(x-2)} + f\frac{1}{(x^2+3)(x-2)}$, therefore elements of V can be written as linear combinations of $\frac{x^2}{(x^2+3)(x-2)}$, $\frac{x}{(x^2+3)(x-2)}$, and $\frac{1}{(x^2+3)(x-2)}$

$$\therefore V = \text{span}\{\frac{x^2}{(x^2+3)(x-2)}, \frac{x}{(x^2+3)(x-2)}, \frac{1}{(x^2+3)(x-2)}\} \tag{3}$$

$$\text{Let } \frac{x^2}{(x^2+3)(x-2)} = u, \frac{x}{(x^2+3)(x-2)} = v, \frac{1}{(x^2+3)(x-2)} = w$$
$$\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \mid \lambda_1 u + \lambda_2 v + \lambda_3 w = 0$$
$$v = xw, u = x^2 w$$
$$\because u, v, w \in \mathcal{F}(\mathbb{R})$$
$$\Rightarrow \lambda_1 x^2 w + \lambda_2 x w + \lambda_3 w = 0, \forall x \in \mathbb{R}$$
$$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$$
$$\therefore u, v, w \text{ are linearly independent}$$

Since u, v, w are linearly independent and $V = \text{span}\{u, v, w\}$, they form a basis for V. Therefore $\dim V = 3$ because 3 vectors forms the basis of V.

$$\frac{ax+b}{x^2+3} + \frac{c}{x-2} = \frac{(ax+b)(x-2)+c(x^2+3)}{(x^2+3)(x-2)} \tag{4}$$
$$= \frac{ax^2-2ax+bx-2b+cx^2+3c}{(x^2+3)(x-2)} \tag{5}$$
$$= \frac{(a+c)x^2+(-2a+b)x+(-2b+3c)}{(x^2+3)(x-2)} \tag{6}$$
$$= \frac{dx^2+ex+f}{(x^2+3)(x-2)} \tag{7}$$
$$\therefore d = a+c, e = -2a+b, f = -2b+3c \tag{8}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{9}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{4d+2e+f}{7} \\ \frac{6d+3e-2f}{7} \\ \frac{3d-2e-f}{7} \end{bmatrix} \tag{10}$$
$$\tag{11}$$

Any set of d, e, and f may be represented in terms of a, b, and c. As a, b, and c are real numbers; V can once again be written as spanu, v, w. Therefore, the dimension of the form $\frac{ax+b}{x^2+3} + \frac{c}{x-2}$ is also 3.

1. Let

$$V = \left\{ \frac{dx^2 + ex + f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(b) Using that $\dim V = 3$ from part (a), explain why we do not expect a partial fractions decomposition of the form

$$\frac{dx^2 + ex + f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$$

to exist.

$$\begin{aligned}
\frac{a}{x^2+3} + \frac{b}{x-2} &= \frac{a(x-2) + b(x^2+3)}{(x^2+3)(x-2)} && (12)\\
&= \frac{ax - 2a + bx^2 + 3b}{(x^2+3)(x-2)} && (13)\\
&= \frac{bx^2 + ax + (3b - 2a)}{(x^2+3)(x-2)} && (14)\\
&= \frac{dx^2 + ex + f}{(x^2+3)(x-2)} && (15)\\
\therefore d = b, e = a, f &= 3b - 2a && (16)\\
\therefore f &= 3d - 2e && (17)\\
& && (18)
\end{aligned}$$

Since f has to equal to $3d - 2e$, there exists partial fraction decomposition in the form $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ if and only if $f = 3d - 2e$. Therefore the partial fraction decomposition in the form $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ does not exist if $f \neq 3d - 2e$; it does not exist for all d, e and f.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, we will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(a) If $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ all belong to W_1 explain why $W_1 \cap W_2$ contains a plane.

w_1, w_2, and w_3 forms a basis of W_1 and u_1, u_2, and u_3 forms a basis of W_2; by this, dim W_2 = dim W_1 = 3. Since $u_1, u_2, u_3 \in W_2$ and are linearly independent to eachother, then u_1, u_2, and u_3 can form a basis of W_1 aswell; then, by definintion, $W_1 = W_2 = W_1 \cap W_2$. Therefore, as $W_1 \cap W_2$ is a 3d subspace of $\mathbb{R}^4$, then it contains a plane.

(b) Now suppose that not all of $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ belong to W_1. Say $\mathbf{u}_1 \notin W_1$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}_1$ is a basis for $\mathbb{R}^4$.

Since w_1, w_2, and w_3 forms a basis for W_1; dimW_1 = dim span$\{w_1, w_2, w_3\} = 3$, and w_1, w_2, w_3 are linearly independent. $u_1 \notin W_1$; thus, u_1 cannot be expressed as a linear combination of w_1, w_2, w_3. Therefore w_1, w_2, w_3, u_1 are linearly independent.

Since $w_1, w_2, w_3, u_1 \in \mathbb{R}^4$, are linearly independent, and dim$\mathbb{R}^4 = 4$, therefore w_1, w_2, w_3, u_1 forms a basis for $\mathbb{R}^4$.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, you will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(c) Using the assumption and conclusion from part (b), find two vectors in $W_1 \cap W_2$ and then prove that these two vectors span a plane.

$$\because \operatorname{span}\{w_1, w_2, w_3, u_1\} = \mathbb{R}^4 \tag{19}$$
$$W_2 = \operatorname{span}\{u_1, u_2, u_3\} \subseteq \mathbb{R}^4 \tag{20}$$
$$\therefore u_2, u_3 \in \operatorname{span}\{w_1, w_2, w_3, u_1\} \tag{21}$$

$\because w_1, w_2, w_3, u_1$ are linearly independent
$\therefore \exists$ a unique $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R} \mid$

$$u_2 = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1, \tag{22}$$
$$u_3 = b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1 \tag{23}$$

$$\therefore W_2 = \operatorname{span}\{u_1, a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1, b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1\} \tag{24}$$
$$= \{x u_1 + y(a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1) + z(b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1) \mid x, y, z \in \mathbb{R}\} \tag{25}$$
$$= \{x u_1 + y a_1 w_1 + y a_2 w_2 + y a_3 w_3 + y a_4 u_1 + z b_1 w_1 + z b_2 w_2 + z b_3 w_3 + z b_4 u_1 \mid x, y, z \in \mathbb{R}\} \tag{26}$$
$$= \{(x + y a_4 + z b_4) u_1 + (y a_1 + z b_1) w_1 + (y a_2 + z b_2) w_2 + (y a_3 + z b_3) w_3 \mid x, y, z \in \mathbb{R}\} \tag{27}$$
$$\therefore W_1 \cap W_2 \tag{28}$$
$$= \{(x + y a_4 + z b_4) u_1 + (y a_1 + z b_1) w_1 + (y a_2 + z b_2) w_2 + (y a_3 + z b_3) w_3 \mid x, y, z \in \mathbb{R}\} \cap \{c w_1 + d w_2 + e w_3 + 0 u_1 \mid c, d, e \in \mathbb{R}\} \tag{29}$$

For $W_1 \cap W_2$, the coefficients for u_1, w_1, w_2, w_3 must be equal.

$$\therefore W_1 \cap W_2 \tag{30}$$
$$= \{k u_1 + l w_1 + m w_2 + n w_3 \mid k = x + y a_4 + z b_4 = 0, l = y a_1 + z b_1 = c, m = y a_2 + y b_2 = d, n = y a_3 + y b_3 = e\} \tag{31}$$

Since x can be freely chosen, y, z are not bounded by the equation $k = x + y a_4 + z b_4 = 0$.

$$\therefore W_1 \cap W_2 \tag{32}$$
$$= \{0 u_1 + (y a_1 + z b_1) w_1 + (y a_2 + z b_2) w_2 + (y a_3 + z b_3) w_3 \mid y, z \in \mathbb{R}\} \tag{33}$$
$$= \{y a_1 w_1 + z b_1 w_1 + y a_2 w_2 + z b_2 w_2 + y a_3 w_3 + z b_3 w_3 \mid y, z \in \mathbb{R}\} \tag{34}$$
$$= \{y(a_1 w_1 + a_2 w_2 + a_3 w_3) + z(b_1 w_1 + b_2 w_2 + b_3 w_3) \mid y, z \in \mathbb{R}\} \tag{35}$$

If $a_1 w_1 + a_2 w_2 + a_3 w_3$ and $b_1 w_1 + b_2 w_2 + b_3 w_3$ are linearly dependent, then there exist c such that $c(a_1 w_1 + a_2 w_2 + a_3 w_3) = b_1 w_1 + b_2 w_2 + b_3 w_3$.

$$\because u_2 = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1 \tag{36}$$
$$\therefore a_1 w_1 + a_2 w_2 + a_3 w_3 = u_2 - a_4 u_1 \tag{37}$$
$$\because u_3 = b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1 \tag{38}$$
$$\therefore b_1 w_1 + b_2 w_2 + b_3 w_3 = u_3 - b_4 u_1 \tag{39}$$
$$c(a_1 w_1 + a_2 w_2 + a_3 w_3) = b_1 w_1 + b_2 w_2 + b_3 w_3 \tag{40}$$
$$\Rightarrow c(u_2 - a_4 u_1) = u_3 - b_4 u_1 \tag{41}$$
$$u_3 = c u_2 - (a_4 c - b_4) u_1 \tag{42}$$

This contradicts the assumption that u_1, u_2, u_3 are linearly independent, therefore $a_1 w_1 + a_2 w_2 + a_3 w_3$ and $b_1 w_1 + b_2 w_2 + b_3 w_3$ are linearly independent and forms a basis for $W_1 \cap W_2$. Therefore $\dim(W_1 \cap W_2) = 2$ and $W_1 \cap W_2 = \operatorname{span}\{a_1 w_1 + a_2 w_2 + a_3 w_3, b_1 w_1 + b_2 w_2 + b_3 w_3\}$ which spans a plane in $\mathbb{R}^4$.