MAT185 Linear Algebra
Assignment 2

Instructions:

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

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I confirm that:

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- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
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Preamble: An application of linear algebra to calculus.

Recall the technique of partial fractions decomposition to evaluate the integral of rational functions. For example, suppose we would like to evaluate the integral

$$\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx$$

We look for scalars a, b, and c such that

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

After some algebra, we find that $a = 2$, $b = 4$, and $c = 5$, and therefore,

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{2x+4}{x^2+3} + \frac{5}{x-2}$$

Then,

$$\begin{aligned}\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx &= \int \frac{2x+4}{x^2+3}\,dx + \int \frac{5}{x-2}\,dx \\ &= \ln(x^2+3) + \frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + 5\ln(x-2) + C\end{aligned}$$

where C is a constant.

In Question 1, we will use the theory of basis and dimension in linear algebra to explain why the partial fractions decomposition

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

exists, thereby allowing us to solve the integral.

1. Let

$$V = \left\{ \frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(a) Prove that dim $V = 3$. Then, explain why a partial fractions decomposition of the form

$$\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

is consistent with the dimension of V.

Use the page 3 to answer this question.

1(a)

$$V = \{\frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d,e,f \in \mathbb{R}\} \tag{1}$$
$$= \{d\frac{x^2}{(x^2+3)(x-2)} + e\frac{x}{(x^2+3)(x-2)} + f\frac{1}{(x^2+3)(x-2)} \mid d,e,f \in \mathbb{R}\} \tag{2}$$

Since $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = d\frac{x^2}{(x^2+3)(x-2)} + e\frac{x}{(x^2+3)(x-2)} + f\frac{1}{(x^2+3)(x-2)}$, therefore elements of V can be written as linear combinations of $\frac{x^2}{(x^2+3)(x-2)}$, $\frac{x}{(x^2+3)(x-2)}$, and $\frac{1}{(x^2+3)(x-2)}$

$$\therefore V = \text{span}\{\frac{x^2}{(x^2+3)(x-2)}, \frac{x}{(x^2+3)(x-2)}, \frac{1}{(x^2+3)(x-2)}\} \tag{3}$$

$$\text{Let } \frac{x^2}{(x^2+3)(x-2)} = u, \frac{x}{(x^2+3)(x-2)} = v, \frac{1}{(x^2+3)(x-2)} = w$$
$$\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \mid \lambda_1 u + \lambda_2 v + \lambda_3 w = 0$$
$$v = xw, u = x^2 w$$
$$\because u, v, w \in \mathcal{F}(\mathbb{R})$$
$$\Rightarrow \lambda_1 x^2 w + \lambda_2 x w + \lambda_3 w = 0, \forall x \in \mathbb{R}$$
$$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$$
$$\therefore u, v, w \text{ are linearly independent}$$

Since u, v, w are linearly independent and $V = \text{span}\{u, v, w\}$, they form a basis for V. Therefore dim $V = 3$ because 3 vectors forms the basis of V.

$$\frac{ax+b}{x^2+3} + \frac{c}{x-2} = \frac{(ax+b)(x-2)+c(x^2+3)}{(x^2+3)(x-2)} \tag{4}$$
$$= \frac{ax^2-2ax+bx-2b+cx^2+3c}{(x^2+3)(x-2)} \tag{5}$$
$$= \frac{(a+c)x^2+(-2a+b)x+(-2b+3c)}{(x^2+3)(x-2)} \tag{6}$$
$$= \frac{dx^2+ex+f}{(x^2+3)(x-2)} \tag{7}$$
$$\therefore d = a+c, e = -2a+b, f = -2b+3c \tag{8}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{9}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{4d+2e+f}{7} \\ \frac{6d+3e-2f}{7} \\ \frac{3d-2e-f}{7} \end{bmatrix} \tag{10}$$
$$\tag{11}$$

Any set of d, e, and f may be represented in terms of a, b, and c. As a, b, and c are real numbers; V can once again be written as spanu, v, w. Therefore, the dimension of the form $\frac{ax+b}{x^2+3} + \frac{c}{x-2}$ is also 3.

1. Let

$$V = \left\{ \frac{dx^2 + ex + f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(b) Using that $\dim V = 3$ from part (a), explain why we do not expect a partial fractions decomposition of the form

$$\frac{dx^2 + ex + f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$$

to exist.

$$\begin{aligned}
\frac{a}{x^2+3} + \frac{b}{x-2} &= \frac{a(x-2) + b(x^2+3)}{(x^2+3)(x-2)} && (12)\\
&= \frac{ax - 2a + bx^2 + 3b}{(x^2+3)(x-2)} && (13)\\
&= \frac{bx^2 + ax + (3b-2a)}{(x^2+3)(x-2)} && (14)\\
&= \frac{dx^2 + ex + f}{(x^2+3)(x-2)} && (15)\\
\therefore d = b, e = a, f &= 3b - 2a && (16)\\
\therefore f &= 3d - 2e && (17)\\
& && (18)
\end{aligned}$$

Since f has to equal to $3d - 2e$, there exists partial fraction decomposition in the form $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ if and only if $f = 3d - 2e$. Therefore the partial fraction decomposition in the form $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ does not exist if $f \neq 3d - 2e$. Therefore $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ does not exist for all f.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, we will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(a) If $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ all belong to W_1 explain why $W_1 \cap W_2$ contains a plane.

w_1, w_2, and w_3 forms a basis of W_1 and u_1, u_2, and u_3 forms a basis of W_2; by this, dim W_2 = dim W_1 = 3. Since $u_1, u_2, u_3 \in W_2$ and are linearly independent to eachother, then u_1, u_2, and u_3 can form a basis of W_1 aswell; then, by definintion, $W_1 = W_2 = W_1 \cap W_2$. Therefore, as $W_1 \cap W_2$ is a 3d subspace of $\mathbb{R}^4$, then it contains a plane.

(b) Now suppose that not all of $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ belong to W_1. Say $\mathbf{u}_1 \notin W_1$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}_1$ is a basis for $\mathbb{R}^4$.

Since w_1, w_2, and w_3 forms a basis for W_1; dimW_1 = dim span$\{w_1, w_2, w_3\} = 3$, and w_1, w_2, w_3 are linearly independent. $u_1 \notin W_1$; thus, u_1 cannot be expressed as a linear combination of w_1, w_2, w_3. Therefore w_1, w_2, w_3, u_1 are linearly independent.

Since $w_1, w_2, w_3, u_1 \in \mathbb{R}^4$, are linearly independent, and dim$\mathbb{R}^4 = 4$, therefore w_1, w_2, w_3, u_1 forms a basis for $\mathbb{R}^4$.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, you will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(c) Using the assumption and conclusion from part (b), find two vectors in $W_1 \cap W_2$ and then prove that these two vectors span a plane.

$$\because \text{span}\{w_1, w_2, w_3, u_1\} = \mathbb{R}^4 \tag{19}$$
$$W_2 = \text{span}\{u_1, u_2, u_3\} \subseteq \mathbb{R}^4 \tag{20}$$
$$\therefore u_2, u_3 \in \text{span}\{w_1, w_2, w_3, u_1\} \tag{21}$$

$\because w_1, w_2, w_3, u_1$ are linearly independent
$\therefore \exists$ a unique $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R} \mid$

$$u_2 = a_1w_1 + a_2w_2 + a_3w_3 + a_4u_1, \tag{22}$$
$$u_3 = b_1w_1 + b_2w_2 + b_3w_3 + b_4u_1 \tag{23}$$

$$\therefore W_2 = \text{span}\{u_1, a_1w_1 + a_2w_2 + a_3w_3 + a_4u_1, b_1w_1 + b_2w_2 + b_3w_3 + b_4u_1\} \tag{24}$$
$$= \{xu_1 + y(a_1w_1 + a_2w_2 + a_3w_3 + a_4u_1) + z(b_1w_1 + b_2w_2 + b_3w_3 + b_4u_1) \mid x, y, z \in \mathbb{R}\} \tag{25}$$
$$= \{xu_1 + ya_1w_1 + ya_2w_2 + ya_3w_3 + ya_4u_1 + zb_1w_1 + zb_2w_2 + zb_3w_3 + zb_4u_1 \mid x, y, z \in \mathbb{R}\} \tag{26}$$
$$= \{(x + ya_4 + zb_4)u_1 + (ya_1 + zb_1)w_1 + (ya_2 + zb_2)w_2 + (ya_3 + zb_3)w_3 \mid x, y, z \in \mathbb{R}\} \tag{27}$$
$$\therefore W_1 \cap W_2 \tag{28}$$
$$= \{(x + ya_4 + zb_4)u_1 + (ya_1 + zb_1)w_1 + (ya_2 + zb_2)w_2 + (ya_3 + zb_3)w_3 \mid x, y, z \in \mathbb{R}\} \cap \{cw_1 + dw_2 + ew_3 + 0u_1 \mid c, d, e \in \mathbb{R}\} \tag{29}$$

For $W_1 \cap W_2$, the coefficients for u_1, w_1, w_2, w_3 must be equal.

$$\therefore W_1 \cap W_2 \tag{30}$$
$$= \{ku_1 + lw_1 + mw_2 + nw_3 \mid k = x + ya_4 + zb_4 = 0, l = ya_1 + zb_1 = c, m = ya_2 + yb_2 = d, n = ya_3 + yb_3 = e\} \tag{31}$$

Since x can be freely chosen, y, z are not bounded by the equation $k = x + ya_4 + zb_4 = 0$.

$$\therefore W_1 \cap W_2 \tag{32}$$
$$= \{0u_1 + (ya_1 + zb_1)w_1 + (ya_2 + zb_2)w_2 + (ya_3 + zb_3)w_3 \mid y, z \in \mathbb{R}\} \tag{33}$$
$$= \{ya_1w_1 + zb_1w_1 + ya_2w_2 + zb_2w_2 + ya_3w_3 + zb_3w_3 \mid y, z \in \mathbb{R}\} \tag{34}$$
$$= \{y(a_1w_1 + a_2w_2 + a_3w_3) + z(b_1w_1 + b_2w_2 + b_3w_3) \mid y, z \in \mathbb{R}\} \tag{35}$$

If $a_1w_1 + a_2w_2 + a_3w_3$ and $b_1w_1 + b_2w_2 + b_3w_3$ are linearly dependent, then there exist c such that $c(a_1w_1 + a_2w_2 + a_3w_3) = b_1w_1 + b_2w_2 + b_3w_3$.

$$\because u_2 = a_1w_1 + a_2w_2 + a_3w_3 + a_4u_1 \tag{36}$$
$$\therefore a_1w_1 + a_2w_2 + a_3w_3 = u_2 - a_4u_1 \tag{37}$$
$$\because u_3 = b_1w_1 + b_2w_2 + b_3w_3 + b_4u_1 \tag{38}$$
$$\therefore b_1w_1 + b_2w_2 + b_3w_3 = u_3 - b_4u_1 \tag{39}$$
$$c(a_1w_1 + a_2w_2 + a_3w_3) = b_1w_1 + b_2w_2 + b_3w_3 \tag{40}$$
$$\Rightarrow c(u_2 - a_4u_1) = u_3 - b_4u_1 \tag{41}$$
$$u_3 = cu_2 - (a_4c - b_4)u_1 \tag{42}$$

This contradicts the assumption that u_1, u_2, u_3 are linearly independent, therefore $a_1w_1 + a_2w_2 + a_3w_3$ and $b_1w_1 + b_2w_2 + b_3w_3$ are linearly independent and forms a basis for $W_1 \cap W_2$. Therefore $\dim(W_1 \cap W_2) = 2$ and $W_1 \cap W_2 = \text{span}\{a_1w_1 + a_2w_2 + a_3w_3, b_1w_1 + b_2w_2 + b_3w_3\}$ which spans a plane in $\mathbb{R}^4$.