

MAT185 Linear Algebra Assignment 2

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Preamble: An application of linear algebra to calculus.

Recall the technique of partial fractions decomposition to evaluate the integral of rational functions. For example, suppose we would like to evaluate the integral

$$\int \frac{7x^2 + 7}{(x^2 + 3)(x - 2)} dx$$

We look for scalars a, b , and c such that

$$\frac{7x^2 + 7}{(x^2 + 3)(x - 2)} = \frac{ax + b}{x^2 + 3} + \frac{c}{x - 2}$$

After some algebra, we find that $a = 2$, $b = 4$, and $c = 5$, and therefore,

$$\frac{7x^2 + 7}{(x^2 + 3)(x - 2)} = \frac{2x + 4}{x^2 + 3} + \frac{5}{x - 2}$$

Then,

$$\begin{aligned} \int \frac{7x^2 + 7}{(x^2 + 3)(x - 2)} dx &= \int \frac{2x + 4}{x^2 + 3} dx + \int \frac{5}{x - 2} dx \\ &= \ln(x^2 + 3) + \frac{4}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + 5 \ln(x - 2) + C \end{aligned}$$

where C is a constant.

In Question 1, we will use the theory of basis and dimension in linear algebra to explain why the partial fractions decomposition

$$\frac{7x^2 + 7}{(x^2 + 3)(x - 2)} = \frac{ax + b}{x^2 + 3} + \frac{c}{x - 2}$$

exists, thereby allowing us to solve the integral.

1. Let

$$V = \left\{ \frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(a) Prove that $\dim V = 3$. Then, explain why a partial fractions decomposition of the form

$$\frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} = \frac{ax + b}{x^2 + 3} + \frac{c}{x - 2}$$

is consistent with the dimension of V .

Use the page 3 to answer this question.

1(a)

$$V = \left\{ \frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} \mid d, e, f \in \mathbb{R} \right\} \quad (1)$$

$$= \left\{ d \frac{x^2}{(x^2 + 3)(x - 2)} + e \frac{x}{(x^2 + 3)(x - 2)} + f \frac{1}{(x^2 + 3)(x - 2)} \mid d, e, f \in \mathbb{R} \right\} \quad (2)$$

Since $\frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} = d \frac{x^2}{(x^2 + 3)(x - 2)} + e \frac{x}{(x^2 + 3)(x - 2)} + f \frac{1}{(x^2 + 3)(x - 2)}$, therefore elements of V can be written as linear combinations of $\frac{x^2}{(x^2 + 3)(x - 2)}$, $\frac{x}{(x^2 + 3)(x - 2)}$, and $\frac{1}{(x^2 + 3)(x - 2)}$

$$\therefore V = \text{span} \left\{ \frac{x^2}{(x^2 + 3)(x - 2)}, \frac{x}{(x^2 + 3)(x - 2)}, \frac{1}{(x^2 + 3)(x - 2)} \right\} \quad (3)$$

$$\text{Let } \frac{x^2}{(x^2 + 3)(x - 2)} = u, \frac{x}{(x^2 + 3)(x - 2)} = v, \frac{1}{(x^2 + 3)(x - 2)} = w$$

$$\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \mid \lambda_1 u + \lambda_2 v + \lambda_3 w = 0$$

$$v = xw, u = x^2 w$$

$$\therefore u, v, w \in \mathcal{F}(\mathbb{R})$$

$$\Rightarrow \lambda_1 x^2 w + \lambda_2 xw + \lambda_3 w = 0, \forall x \in \mathbb{R}$$

$$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\therefore u, v, w \text{ are linearly independent}$$

Since u, v, w are linearly independent and $V = \text{span}\{u, v, w\}$, they form a basis for V . Therefore $\dim V = 3$ because 3 vectors forms the basis of V .

$$\frac{ax + b}{x^2 + 3} + \frac{c}{x - 2} = \frac{(ax + b)(x - 2) + c(x^2 + 3)}{(x^2 + 3)(x - 2)} \quad (4)$$

$$= \frac{ax^2 - 2ax + bx - 2b + cx^2 + 3c}{(x^2 + 3)(x - 2)} \quad (5)$$

$$= \frac{(a + c)x^2 + (-2a + b)x + (-2b + 3c)}{(x^2 + 3)(x - 2)} \quad (6)$$

$$= \frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} \quad (7)$$

$$\therefore d = a + c, e = -2a + b, f = -2b + 3c \quad (8)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \quad (9)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{4d+2e+f}{7} \\ \frac{6d+3e-2f}{7} \\ \frac{3d-2e-f}{7} \end{bmatrix} \quad (10)$$

$$(11)$$

Any set of d, e , and f may be represented in terms of a, b , and c , and any set of a, b , and c may be represented in terms of d, e , and f . By distribution, all functions in the set $\left\{ \frac{ax+b}{x^2+3} + \frac{c}{x-2} \mid a, b, c \in \mathbb{R} \right\} \subseteq V = \left\{ \frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$ and vice versa. Therefore, all functions in the form of $\left\{ \frac{ax+b}{x^2+3} + \frac{c}{x-2} \mid a, b, c \in \mathbb{R} \right\} = V = \left\{ \frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$, and thus share a dimension of 3.

1. Let

$$V = \left\{ \frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(b) Using that $\dim V = 3$ from part (a), explain why we do not expect a partial fractions decomposition of the form

$$\frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} = \frac{a}{x^2 + 3} + \frac{b}{x - 2}$$

to exist.

$$\frac{a}{x^2 + 3} + \frac{b}{x - 2} = \frac{a(x - 2) + b(x^2 + 3)}{(x^2 + 3)(x - 2)} \quad (12)$$

$$= \frac{ax - 2a + bx^2 + 3b}{(x^2 + 3)(x - 2)} \quad (13)$$

$$= \frac{bx^2 + ax + (3b - 2a)}{(x^2 + 3)(x - 2)} \quad (14)$$

$$= \frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} \quad (15)$$

$$\therefore d = b, e = a, f = 3b - 2a \quad (16)$$

$$\therefore f = 3d - 2e \quad (17)$$

$$(18)$$

Since f has to equal to $3d - 2e$, there exists partial fraction decomposition in the form $\frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} = \frac{a}{x^2 + 3} + \frac{b}{x - 2}$ if and only if $f = 3d - 2e$. Therefore the partial fraction decomposition in the form $\frac{dx^2 + ex + f}{(x^2 + 3)(x - 2)} = \frac{a}{x^2 + 3} + \frac{b}{x - 2}$ does not exist if $f \neq 3d - 2e$; it does not exist for all d, e and f .

2. Suppose that W_1 and W_2 are both three dimensional subspaces of \mathbb{R}^4 . In this question, we will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1 , and let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2 .

(a) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ all belong to W_1 explain why $W_1 \cap W_2$ contains a plane.

A plane is defined as a 2 dimensional space that extends indefinitely, therefore they have a dimension of 2 and are contained in vectorspaces of dimension 2 or higher.

w_1, w_2 , and w_3 forms a basis of W_1 and u_1, u_2 , and u_3 forms a basis of W_2 ; by this, $\dim W_2 = \dim W_1 = 3$. Since $u_1, u_2, u_3 \in W_2$ and are linearly independent to eachother, then u_1, u_2 , and u_3 can form a basis of W_1 aswell; then, by definintion, $W_1 = W_2 = W_1 \cap W_2$. Therefore, as $W_1 \cap W_2$ is a 3d subspace of \mathbb{R}^4 , then it contains a plane.

(b) Now suppose that not all of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ belong to W_1 . Say $\mathbf{u}_1 \notin W_1$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}_1$ is a basis for \mathbb{R}^4 .

Since w_1, w_2 , and w_3 forms a basis for W_1 ; $\dim W_1 = \dim \text{span}\{w_1, w_2, w_3\} = 3$, and w_1, w_2, w_3 are linearly independent. $u_1 \notin W_1$; thus, u_1 cannot be expressed as a linear combination of w_1, w_2, w_3 . Therefore w_1, w_2, w_3, u_1 are linearly independent.

Since $w_1, w_2, w_3, u_1 \in \mathbb{R}^4$, are linearly independent, and $\dim \mathbb{R}^4 = 4$, therefore w_1, w_2, w_3, u_1 forms a basis for \mathbb{R}^4 .

2. Suppose that W_1 and W_2 are both three dimensional subspaces of \mathbb{R}^4 . In this question, you will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1 , and let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2 .

(c) Using the assumption and conclusion from part (b), find two vectors in $W_1 \cap W_2$ and then prove that these two vectors span a plane.

$$\therefore \text{span}\{w_1, w_2, w_3, u_1\} = \mathbb{R}^4 \quad (19)$$

$$W_2 = \text{span}\{u_1, u_2, u_3\} \subseteq \mathbb{R}^4 \quad (20)$$

$$\therefore u_2, u_3 \in \text{span}\{w_1, w_2, w_3, u_1\} \quad (21)$$

$\therefore w_1, w_2, w_3, u_1$ are linearly independent

$\therefore \exists$ a unique $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R} \mid$

$$u_2 = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1, \quad (22)$$

$$u_3 = b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1 \quad (23)$$

$$\therefore W_2 = \text{span}\{u_1, a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1, b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1\} \quad (24)$$

$$= \{xu_1 + y(a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1) + z(b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1) \mid x, y, z \in \mathbb{R}\} \quad (25)$$

$$= \{xu_1 + ya_1 w_1 + ya_2 w_2 + ya_3 w_3 + ya_4 u_1 + zb_1 w_1 + zb_2 w_2 + zb_3 w_3 + zb_4 u_1 \mid x, y, z \in \mathbb{R}\} \quad (26)$$

$$= \{(x + ya_4 + zb_4)u_1 + (ya_1 + zb_1)w_1 + (ya_2 + zb_2)w_2 + (ya_3 + zb_3)w_3 \mid x, y, z \in \mathbb{R}\} \quad (27)$$

$$\therefore W_1 \cap W_2 \quad (28)$$

$$= \{(x + ya_4 + zb_4)u_1 + (ya_1 + zb_1)w_1 + (ya_2 + zb_2)w_2 + (ya_3 + zb_3)w_3 \mid x, y, z \in \mathbb{R}\} \cap \{cw_1 + dw_2 + ew_3 + 0u_1 \mid c, d, e \in \mathbb{R}\} \quad (29)$$

For $W_1 \cap W_2$, the coefficients for u_1, w_1, w_2, w_3 must be equal.

$$\therefore W_1 \cap W_2 \quad (30)$$

$$= \{ku_1 + lw_1 + mw_2 + nw_3 \mid k = x + ya_4 + zb_4 = 0, l = ya_1 + zb_1 = c, m = ya_2 + zb_2 = d, n = ya_3 + zb_3 = e\} \quad (31)$$

Since x can be freely chosen, y, z are not bounded by the equation $k = x + ya_4 + zb_4 = 0$.

$$\therefore W_1 \cap W_2 \quad (32)$$

$$= \{0u_1 + (ya_1 + zb_1)w_1 + (ya_2 + zb_2)w_2 + (ya_3 + zb_3)w_3 \mid y, z \in \mathbb{R}\} \quad (33)$$

$$= \{ya_1 w_1 + zb_1 w_1 + ya_2 w_2 + zb_2 w_2 + ya_3 w_3 + zb_3 w_3 \mid y, z \in \mathbb{R}\} \quad (34)$$

$$= \{y(a_1 w_1 + a_2 w_2 + a_3 w_3) + z(b_1 w_1 + b_2 w_2 + b_3 w_3) \mid y, z \in \mathbb{R}\} \quad (35)$$

If $a_1 w_1 + a_2 w_2 + a_3 w_3$ and $b_1 w_1 + b_2 w_2 + b_3 w_3$ are linearly dependent, then there exist c such that $c(a_1 w_1 + a_2 w_2 + a_3 w_3) = b_1 w_1 + b_2 w_2 + b_3 w_3$.

$$\therefore u_2 = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1 \quad (36)$$

$$\therefore a_1 w_1 + a_2 w_2 + a_3 w_3 = u_2 - a_4 u_1 \quad (37)$$

$$\therefore u_3 = b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1 \quad (38)$$

$$\therefore b_1 w_1 + b_2 w_2 + b_3 w_3 = u_3 - b_4 u_1 \quad (39)$$

$$c(a_1 w_1 + a_2 w_2 + a_3 w_3) = b_1 w_1 + b_2 w_2 + b_3 w_3 \quad (40)$$

$$\Rightarrow c(u_2 - a_4 u_1) = u_3 - b_4 u_1 \quad (41)$$

$$u_3 = cu_2 - (a_4 c - b_4)u_1 \quad (42)$$

This contradicts the assumption that u_1, u_2, u_3 are linearly independent, therefore $a_1 w_1 + a_2 w_2 + a_3 w_3$ and $b_1 w_1 + b_2 w_2 + b_3 w_3$ are linearly independent and forms a basis for $W_1 \cap W_2$. Therefore $\dim(W_1 \cap W_2) = 2$ and $W_1 \cap W_2 = \text{span}\{a_1 w_1 + a_2 w_2 + a_3 w_3, b_1 w_1 + b_2 w_2 + b_3 w_3\}$ which spans a plane in \mathbb{R}^4 .