MAT185 Linear Algebra
Assignment 2

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: ______________________________

Student number: ______________________________

Full Name: ______________________________

Student number: ______________________________

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) ______________________________

2) ______________________________

Preamble: An application of linear algebra to calculus.

Recall the technique of partial fractions decomposition to evaluate the integral of rational functions. For example, suppose we would like to evaluate the integral

$$\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx$$

We look for scalars a, b, and c such that

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

After some algebra, we find that $a = 2$, $b = 4$, and $c = 5$, and therefore,

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{2x+4}{x^2+3} + \frac{5}{x-2}$$

Then,

$$\begin{aligned}\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx &= \int \frac{2x+4}{x^2+3}\,dx + \int \frac{5}{x-2}\,dx \\ &= \ln(x^2+3) + \frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + 5\ln(x-2) + C\end{aligned}$$

where C is a constant.

In Question 1, we will use the theory of basis and dimension in linear algebra to explain why the partial fractions decomposition

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

exists, thereby allowing us to solve the integral.

1. Let

$$V = \left\{ \frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(a) Prove that dim $V = 3$. Then, explain why a partial fractions decomposition of the form

$$\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

is consistent with the dimension of V.

Use the page 3 to answer this question.

1(a)

$$V = \{\frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d,e,f \in \mathbb{R}\} \tag{1}$$

$$= \{d\frac{x^2}{(x^2+3)(x-2)} + e\frac{x}{(x^2+3)(x-2)} + f\frac{1}{(x^2+3)(x-2)} \mid d,e,f \in \mathbb{R}\} \tag{2}$$

Since $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = d\frac{x^2}{(x^2+3)(x-2)} + e\frac{x}{(x^2+3)(x-2)} + f\frac{1}{(x^2+3)(x-2)}$, therefore elements of V can be written as linear combinations of $\frac{x^2}{(x^2+3)(x-2)}$, $\frac{x}{(x^2+3)(x-2)}$, and $\frac{1}{(x^2+3)(x-2)}$

$$\therefore V = \operatorname{span}\{\frac{x^2}{(x^2+3)(x-2)}, \frac{x}{(x^2+3)(x-2)}, \frac{1}{(x^2+3)(x-2)}\} \tag{3}$$

$$\text{Let } \frac{x^2}{(x^2+3)(x-2)} = u, \frac{x}{(x^2+3)(x-2)} = v, \frac{1}{(x^2+3)(x-2)} = w$$

$$\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \mid \lambda_1 u + \lambda_2 v + \lambda_3 w = 0$$

$$v = xw, u = x^2 w$$

$$\because u, v, w \in \mathcal{F}(\mathbb{R})$$

$$\Rightarrow \lambda_1 x^2 w + \lambda_2 x w + \lambda_3 w = 0, \forall x \in \mathbb{R}$$

$$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\therefore u, v, w \text{ are linearly independent}$$

Since u, v, w are linearly independent and $V = \operatorname{span}\{u, v, w\}$, they form a basis for V. Therefore $\dim V = 3$ because 3 vectors forms the basis of V.

$$\frac{ax+b}{x^2+3} + \frac{c}{x-2} = \frac{(ax+b)(x-2)+c(x^2+3)}{(x^2+3)(x-2)} \tag{4}$$

$$= \frac{ax^2-2ax+bx-2b+cx^2+3c}{(x^2+3)(x-2)} \tag{5}$$

$$= \frac{(a+c)x^2+(-2a+b)x+(-2b+3c)}{(x^2+3)(x-2)} \tag{6}$$

$$= \frac{dx^2+ex+f}{(x^2+3)(x-2)} \tag{7}$$

$$\therefore d = a+c, e = -2a+b, f = -2b+3c \tag{8}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \tag{9}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{4d+2e+f}{7} \\ \frac{6d+3e-2f}{7} \\ \frac{3d-2e-f}{7} \end{bmatrix} \tag{10}$$

$$\tag{11}$$

Since the matrix describing the relationship between a, b, c and d, e, f is invertible. Therefore the partial fraction is full rank, which is 3 because there are 3 variables. Therefore the partial fraction decomposition has a dimension of 3, which is consistent with the dimension of V.

1. Let

$$V = \left\{ \frac{dx^2 + ex + f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(b) Using that $\dim V = 3$ from part (a), explain why we do not expect a partial fractions decomposition of the form

$$\frac{dx^2 + ex + f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$$

to exist.

$$\begin{aligned}
\frac{a}{x^2+3} + \frac{b}{x-2} &= \frac{a(x-2) + b(x^2+3)}{(x^2+3)(x-2)} && (12)\\
&= \frac{ax - 2a + bx^2 + 3b}{(x^2+3)(x-2)} && (13)\\
&= \frac{bx^2 + ax + (3b - 2a)}{(x^2+3)(x-2)} && (14)\\
&= \frac{dx^2 + ex + f}{(x^2+3)(x-2)} && (15)\\
\therefore d &= b, e = a, f = 3b - 2a && (16)\\
\therefore f &= 3d - 2e && (17)\\
& && (18)
\end{aligned}$$

Since f have to equal to $3d - 2e$, there exists partial fraction decomposition in the form $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ if and only if $f = 3d - 2e$. Therefore the partial fraction decomposition in the form $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ does not exist if $f \neq 3d - 2e$. Therefore $\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$ does not exist for all f.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, we will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(a) If $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ all belong to W_1 explain why $W_1 \cap W_2$ contains a plane.

Since u_1, u_2, u_3 forms a basis for W_2, $\dim W_2 = 3$, and u_1, u_2, u_3 all belong to W_1, therefore $W_2 \subseteq W_1$. Since $\dim W_1 = \dim W_2 = 3$, therefore $W_1 = W_2$, $W_1 \cap W_2 = W_1$, which is a 3-dimensional subspace of $\mathbb{R}^4$. Therefore $W_1 \cap W_2$ contains a plane.

(b) Now suppose that not all of $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ belong to W_1. Say $\mathbf{u}_1 \notin W_1$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}_1$ is a basis for $\mathbb{R}^4$.

Since w_1, w_2, w_3 forms a basis for W_1, $\dim W_1 = \dim \text{span}\{w_1, w_2, w_3\} = 3$, and w_1, w_2, w_3 are linearly independent. Since $u_1 \notin W_1$, u_1 cannot be expressed as a linear combination of w_1, w_2, w_3. Therefore w_1, w_2, w_3, u_1 are linearly independent. Since $w_1, w_2, w_3, u_1 \in \mathbb{R}^4$, are linearly independent, and $\dim\mathbb{R}^4 = 4$, therefore w_1, w_2, w_3, u_1 forms a basis for $\mathbb{R}^4$.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, you will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(c) Using the assumption and conclusion from part (b), find two vectors in $W_1 \cap W_2$ and then prove that these two vectors span a plane.

$$\because \text{span}\{w_1, w_2, w_3, u_1\} = \mathbb{R}^4 \tag{19}$$
$$W_2 = \text{span}\{u_1, u_2, u_3\} \subseteq \mathbb{R}^4 \tag{20}$$
$$\therefore u_2, u_3 \in \text{span}\{w_1, w_2, w_3, u_1\} \tag{21}$$

$\because w_1, w_2, w_3, u_1$ are linearly independent
$\therefore \exists$ a unique $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R} \mid$

$$u_2 = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1, \tag{22}$$
$$u_3 = b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1 \tag{23}$$

$$\therefore W_2 = \text{span}\{u_1, a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1, b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1\} \tag{24}$$
$$= \{x u_1 + y(a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1) + z(b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1) \mid x, y, z \in \mathbb{R}\} \tag{25}$$
$$= \{x u_1 + y a_1 w_1 + y a_2 w_2 + y a_3 w_3 + y a_4 u_1 + z b_1 w_1 + z b_2 w_2 + z b_3 w_3 + z b_4 u_1 \mid x, y, z \in \mathbb{R}\} \tag{26}$$
$$= \{(x + y a_4 + z b_4) u_1 + (y a_1 + z b_1) w_1 + (y a_2 + z b_2) w_2 + (y a_3 + z b_3) w_3 \mid x, y, z \in \mathbb{R}\} \tag{27}$$
$$\therefore W_1 \cap W_2 \tag{28}$$
$$= \{(x + y a_4 + z b_4) u_1 + (y a_1 + z b_1) w_1 + (y a_2 + z b_2) w_2 + (y a_3 + z b_3) w_3 \mid x, y, z \in \mathbb{R}\} \cap \{c w_1 + d w_2 + e w_3 + 0 u_1 \mid c, d, e \in \mathbb{R}\} \tag{29}$$

For $W_1 \cap W_2$, the coefficients for u_1, w_1, w_2, w_3 must be equal.

$$\therefore W_1 \cap W_2 \tag{30}$$
$$= \{k u_1 + l w_1 + m w_2 + n w_3 \mid k = x + y a_4 + z b_4 = 0, l = y a_1 + z b_1 = c, m = y a_2 + y b_2 = d, n = y a_3 + y b_3 = e\} \tag{31}$$

Since x can be freely chosen, y, z are not bounded by the equation $k = x + y a_4 + z b_4 = 0$.

$$\therefore W_1 \cap W_2 \tag{32}$$
$$= \{0 u_1 + (y a_1 + z b_1) w_1 + (y a_2 + z b_2) w_2 + (y a_3 + z b_3) w_3 \mid y, z \in \mathbb{R}\} \tag{33}$$
$$= \{y a_1 w_1 + z b_1 w_1 + y a_2 w_2 + z b_2 w_2 + y a_3 w_3 + z b_3 w_3 \mid y, z \in \mathbb{R}\} \tag{34}$$
$$= \{y(a_1 w_1 + a_2 w_2 + a_3 w_3) + z(b_1 w_1 + b_2 w_2 + b_3 w_3) \mid y, z \in \mathbb{R}\} \tag{35}$$

If $a_1 w_1 + a_2 w_2 + a_3 w_3$ and $b_1 w_1 + b_2 w_2 + b_3 w_3$ are linearly dependent, then there exist c such that $c(a_1 w_1 + a_2 w_2 + a_3 w_3) = b_1 w_1 + b_2 w_2 + b_3 w_3$. Since $u_2 = a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 u_1$, $u_3 = b_1 w_1 + b_2 w_2 + b_3 w_3 + b_4 u_1$, therefore $a_1 w_1 + a_2 w_2 + a_3 w_3 = u_2 - a_4 u_1$, $b_1 w_1 + b_2 w_2 + b_3 w_3 = u_3 - b_4 u_1$. $c(a_1 w_1 + a_2 w_2 + a_3 w_3) = b_1 w_1 + b_2 w_2 + b_3 w_3 = c(u_2 - a_4 u_1) = u_3 - b_4 u_1$