MAT185 Linear Algebra
Assignment 3

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: ______________________

Student number: ______________________

Full Name: ______________________

Student number: ______________________

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

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2) ______________________

Preamble: Rank factorization.

Suppose that $A \in {}^m\mathbb{R}^k$ has rank $r \geq 1$. Then A has r linearly independent columns that form a basis for $\operatorname{col} A$. Let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_r$ be any basis for $\operatorname{col} A$, and let $B \in {}^m\mathbb{R}^r$ be the matrix whose columns are $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_r$. That is,

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b_2} & \cdots & \mathbf{b}_r \end{bmatrix}.$$

Then, every column of A can be written as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_r$. In other words, if $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are the columns of A, then, for every $j = 1, 2, \dots k$,

$$\mathbf{a}_j = c_{1j}\mathbf{b}_1 + c_{2j}\mathbf{b}_2 + \cdots + c_{rj}\mathbf{b}_r$$

for some scalars $c_{1j}, c_{2j}, \dots, c_{rj} \in \mathbb{R}$.

Then,

$$\begin{aligned} A &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_k \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{b}_1 & \mathbf{b_2} & \cdots & \mathbf{b}_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rk} \end{bmatrix} \\ &= BC \end{aligned}$$

where $B \in {}^m\mathbb{R}^r$, $C \in {}^r\mathbb{R}^k$, and $r = \operatorname{rank} A$. This is called a *rank factorization* of A.

In this assignment, we will see how such a rank factorization can help us investigate the rank of a sum of matrices.

1. Let $A \in {}^m\mathbb{R}^k$, and $B \in {}^k\mathbb{R}^n$.

(a) Suppose that $AB = 0$. Prove that

$$\operatorname{rank} A + \operatorname{rank} B - k \leq \operatorname{rank} AB.$$

Hint: Consider the cases where $A = 0$ and $A \neq 0$.

1. $AB = 0 \Rightarrow \forall \mathbf{b} \in \text{col}(B), A\mathbf{b} = \mathbf{0}$, thus $\text{col}(B) \subseteq \text{null}(A)$.
2. $\text{rank}(B) = \dim \text{col}(B)$.
3. By "Theorem II" $\rightarrow \dim \text{null } A = k - \text{rank } A$, where k is the number of columns of A.
4. Since $\text{col}(B) \subseteq \text{null}(A)$, it follows that $\text{rank}(B) \leq \dim \text{null}(A)$.
5. Applying "Theorem II", we deduce $\text{rank}(B) \leq k - \text{rank}(A) \Rightarrow \text{rank}(A) + \text{rank}(B) - k \leq 0$.
6. Since the rank of any matrix, including the zero matrix AB, is non-negative, we have $\text{rank}(AB) = 0$.
7. Therefore, $\text{rank}(A) + \text{rank}(B) - k \leq \text{rank}(AB)$ by substitution from (6)

1. Let $A \in {}^m\mathbb{R}^k$, and $B \in {}^k\mathbb{R}^n$.

(b) Suppose that $AB \neq 0$. Prove that

$$\operatorname{rank} A + \operatorname{rank} B - k \leq \operatorname{rank} AB.$$

Hint: Suppose that $\operatorname{rank} AB = r \geq 1$ and use a rank factorization $AB = CD$. Let X and Y be the augmented matrices

$$X = \begin{bmatrix} A & C \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} B \\ -D \end{bmatrix}$$

(be sure to note the sizes of C, D, X, Y) then compute XY and use part (a).

1. Assume $AB \neq 0$ with $\operatorname{rank}(AB) = r \geq 1$, implying rank factorization $AB = CD$.

2. Construct $X = \begin{bmatrix} A & C \end{bmatrix}$, $Y = \begin{bmatrix} B \\ -D \end{bmatrix}$, ensuring $XY = 0$ via $CD = AB$.

3. Validate $XY = 0$ through $AB - CD = 0$, leveraging rank factorization.

4. Assess $\operatorname{rank}(X)$, $\operatorname{rank}(Y)$ considering column space overlaps; $\operatorname{rank}(X) \leq \operatorname{rank}(A) + \min\{r, k - \operatorname{rank}(A)\}$, similarly for Y.

5. By invoking part (a), note that $\operatorname{rank}(X) + \operatorname{rank}(Y) - (k + r) \leq 0$ holds due to $XY = 0$. This inequality reflects the principle that the sum of the ranks of two matrices minus the dimension they share cannot exceed the rank of their product, even when that product is the zero matrix.

6. Merge rank bounds, getting $\operatorname{rank}(A) + \operatorname{rank}(B) + 2r - (k + r) \leq 0$, simplified to $\operatorname{rank}(A) + \operatorname{rank}(B) + r - k \leq 0$.

7. Therefore $\operatorname{rank}(A) + \operatorname{rank}(B) - k \leq -\operatorname{rank}(AB) \leq \operatorname{rank}(AB) \Rightarrow \operatorname{rank}(A) + \operatorname{rank}(B) - k \leq \operatorname{rank}(AB)$.

2. Let $A \in {}^m\mathbb{R}^k$, $B \in {}^k\mathbb{R}^n$. Prove that the rank inequality $\operatorname{rank} A + \operatorname{rank} B - k \leq \operatorname{rank} AB$. from Question 1. is equivalent to the inequality

$$\operatorname{nullity} AB \leq \operatorname{nullity} A + \operatorname{nullity} B$$

1. Start by applying the Rank-Nullity Theorem for matrix A:
   $\operatorname{rank}(A) + \operatorname{nullity}(A) = k$.
2. Apply the Rank-Nullity Theorem for matrix B:
   $\operatorname{rank}(B) + \operatorname{nullity}(B) = n$.
3. Apply the Rank-Nullity Theorem for matrix AB:
   $\operatorname{rank}(AB) + \operatorname{nullity}(AB) = n$.
4. Use the inequality provided from Question 1, which can be expressed as:
   $\operatorname{rank}(A) + \operatorname{rank}(B) - k \leq \operatorname{rank}(AB)$.
5. From the inequality in Step 4, replace $\operatorname{rank}(A)$ using Step 1, resulting in:
   $k - \operatorname{nullity}(A) + \operatorname{rank}(B) - k \leq \operatorname{rank}(AB)$.
6. Simplify the inequality obtained in Step 5:
   $\operatorname{rank}(B) - \operatorname{nullity}(A) \leq \operatorname{rank}(AB)$.
7. Replacing $\operatorname{rank}(AB)$ using Step 3 in the inequality from Step 6, we get:
   $\operatorname{rank}(B) - \operatorname{nullity}(A) \leq n - \operatorname{nullity}(AB)$.
8. Rearrange the inequality in Step 7:
   $\operatorname{nullity}(AB) \leq n - \operatorname{rank}(B) + \operatorname{nullity}(A)$.
9. Finally, replace $n - \operatorname{rank}(B)$ using Step 2 in the inequality from Step 8:
   $\operatorname{nullity}(AB) \leq \operatorname{nullity}(B) + \operatorname{nullity}(A)$, which is the goal inequality.

3. Let $A, B \in {}^m\mathbb{R}^n$. Prove that

$$|\operatorname{rank} A - \operatorname{rank} B| \leq \operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$$

Hint: Prove each inequality separately. Assume that $\operatorname{rank} A = r \geq 1$, and $\operatorname{rank} B = s \geq 1$, and use a rank factorization $A = CD$, and $B = EF$. Let X and Y be the augmented matrices

$$X = \begin{bmatrix} C & E \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} D \\ F \end{bmatrix}$$

(be sure to note the sizes of C, D, E, F, X, Y) then compute XY and use previous results.