MAT185 Linear Algebra
Assignment 3

Instructions:

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2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

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Full Name: ______________________________

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

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Preamble: Rank factorization.

Suppose that $A \in {}^m\mathbb{R}^k$ has rank $r \geq 1$. Then A has r linearly independent columns that form a basis for $\operatorname{col} A$. Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$ be any basis for $\operatorname{col} A$, and let $B \in {}^m\mathbb{R}^r$ be the matrix whose columns are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$. That is,

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b_2} & \cdots & \mathbf{b}_r \end{bmatrix}.$$

Then, every column of A can be written as a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r$. In other words, if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are the columns of A, then, for every $j = 1, 2, \ldots k$,

$$\mathbf{a}_j = c_{1j}\mathbf{b}_1 + c_{2j}\mathbf{b}_2 + \cdots + c_{rj}\mathbf{b}_r$$

for some scalars $c_{1j}, c_{2j}, \ldots, c_{rj} \in \mathbb{R}$.

Then,

$$\begin{aligned} A &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_k \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{b}_1 & \mathbf{b_2} & \cdots & \mathbf{b}_r \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rk} \end{bmatrix} \\ &= BC \end{aligned}$$

where $B \in {}^m\mathbb{R}^r$, $C \in {}^r\mathbb{R}^k$, and $r = \operatorname{rank} A$. This is called a *rank factorization* of A.

In this assignment, we will see how such a rank factorization can help us investigate the rank of a sum of matrices.

1. Let $A \in {}^m\mathbb{R}^k$, and $B \in {}^k\mathbb{R}^n$.

(a) Suppose that $AB = 0$. Prove that

$$\operatorname{rank} A + \operatorname{rank} B - k \leq \operatorname{rank} AB.$$

Hint: Consider the cases where $A = 0$ and $A \neq 0$.

1. $AB = 0$ implies that for all $\mathbf{b} \in \text{col}(B)$, we have $A\mathbf{b} = \mathbf{0}$. Hence, $\text{col}(B) \subseteq \text{null}(A)$.
2. By "Theorem II", we have $\dim(\text{null}(A)) = k - \text{rank}(A)$, where k is the number of columns of A.
3. Since $\text{col}(B) \subseteq \text{null}(A)$, it follows that $\text{rank}(B) \leq \dim(\text{null}(A))$.
4. From "Theorem II", we deduce $\text{rank}(B) \leq k - \text{rank}(A)$ which implies $\text{rank}(A) + \text{rank}(B) - k \leq 0$.
5. The rank of any matrix, including the zero matrix AB, is non-negative, hence $\text{rank}(AB) = 0$.
6. Therefore, $\text{rank}(A) + \text{rank}(B) - k \leq \text{rank}(AB)$, by substitution from the previous point.

1. Let $A \in {}^m\mathbb{R}^k$, and $B \in {}^k\mathbb{R}^n$.

(b) Suppose that $AB \neq 0$. Prove that

$$\operatorname{rank} A + \operatorname{rank} B - k \leq \operatorname{rank} AB.$$

Hint: Suppose that $\operatorname{rank} AB = r \geq 1$ and use a rank factorization $AB = CD$. Let X and Y be the augmented matrices

$$X = \begin{bmatrix} A & C \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} B \\ -D \end{bmatrix}$$

(be sure to note the sizes of C, D, X, Y) then compute XY and use part (a).

1. Assume $AB \neq 0$ with $\operatorname{rank}(AB) = r \geq 1$, create rank factorization $AB = CD$.
2. Construct $X = \begin{bmatrix} A & C \end{bmatrix}$, $Y = \begin{bmatrix} B \\ -D \end{bmatrix}$, ensuring $XY = 0$ via $CD = AB$.
3. Validate $XY = 0$ through $AB - CD = 0$, leveraging rank factorization.
4. Since $\operatorname{col}(A) \subseteq \operatorname{col}(X)$, then $\operatorname{rank}(A) \leq \operatorname{rank}(X)$.
5. Since $\operatorname{row}(B) \subseteq \operatorname{row}(Y)$, then $\operatorname{rank}(B) \leq \operatorname{rank}(Y)$.
6. By invoking part (a), note that $\operatorname{rank}(X) + \operatorname{rank}(Y) - (k + r) \leq 0$ holds due to $XY = 0$. This inequality reflects the principle that the sum of the ranks of two matrices minus the dimension they share cannot exceed the rank of their product, even when that product is the zero matrix.
7. By substitution from step 4 to step 5: $\operatorname{rank}(A) + \operatorname{rank}(B) - (k + r) \leq 0$
8. Therefore $\operatorname{rank}(A) + \operatorname{rank}(B) - k \leq r = \operatorname{rank}(AB)$.

2. Let $A \in {}^m\mathbb{R}^k$, $B \in {}^k\mathbb{R}^n$. Prove that the rank inequality $\operatorname{rank} A + \operatorname{rank} B - k \leq \operatorname{rank} AB$. from Question 1. is equivalent to the inequality

$$\operatorname{nullity} AB \leq \operatorname{nullity} A + \operatorname{nullity} B$$

1. Start by applying the Rank-Nullity Theorem for matrix A:
   $\operatorname{rank}(A) + \operatorname{nullity}(A) = k$.
2. Apply the Rank-Nullity Theorem for matrix B:
   $\operatorname{rank}(B) + \operatorname{nullity}(B) = n$.
3. Apply the Rank-Nullity Theorem for matrix AB:
   $\operatorname{rank}(AB) + \operatorname{nullity}(AB) = n$.
4. Use the inequality provided from Question 1, which can be expressed as:
   $\operatorname{rank}(A) + \operatorname{rank}(B) - k \leq \operatorname{rank}(AB)$.
5. From the inequality in Step 4, replace $\operatorname{rank}(A)$ using Step 1, resulting in:
   $k - \operatorname{nullity}(A) + \operatorname{rank}(B) - k \leq \operatorname{rank}(AB)$.
6. Simplify the inequality obtained in Step 5:
   $\operatorname{rank}(B) - \operatorname{nullity}(A) \leq \operatorname{rank}(AB)$.
7. Replacing $\operatorname{rank}(AB)$ using Step 3 in the inequality from Step 6, we get:
   $\operatorname{rank}(B) - \operatorname{nullity}(A) \leq n - \operatorname{nullity}(AB)$.
8. Rearrange the inequality in Step 7:
   $\operatorname{nullity}(AB) \leq n - \operatorname{rank}(B) + \operatorname{nullity}(A)$.
9. Finally, replace $n - \operatorname{rank}(B)$ using Step 2 in the inequality from Step 8:
   $\operatorname{nullity}(AB) \leq \operatorname{nullity}(B) + \operatorname{nullity}(A)$, which is the goal inequality.

3. Let $A, B \in {}^m\mathbb{R}^n$. Prove that

$$|\operatorname{rank} A - \operatorname{rank} B| \leq \operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$$

Hint: Prove each inequality separately. Assume that $\operatorname{rank} A = r \geq 1$, and $\operatorname{rank} B = s \geq 1$, and use a rank factorization $A = CD$, and $B = EF$. Let X and Y be the augmented matrices

$$X = \begin{bmatrix} C & E \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} D \\ F \end{bmatrix}$$

(be sure to note the sizes of C, D, E, F, X, Y) then compute XY and use previous results.

1. To prove $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$:
   - Let $A, B \in \mathbb{R}^{m\times n}$ and assume that $\operatorname{rank}(A) = r$ and $\operatorname{rank}(B) = s$.
   - We can write rank factorizations $A = CD$ and $B = EF$ where $C, E \in \mathbb{R}^{m\times k}$ for some k and $D, F \in \mathbb{R}^{k\times n}$.
   - Consider the augmented matrix $X = [C\ E] \in \mathbb{R}^{m\times 2k}$ and $Y = \begin{bmatrix} D \\ F \end{bmatrix} \in \mathbb{R}^{2k\times n}$.
   - The product XY represents $A+B$ since $CD + EF = A + B$.
   - $\operatorname{rank}(XY)$ cannot exceed the total number of columns in X or the total number of rows in Y, which is $2k$.
   - Since k can be at most $\operatorname{rank}(A)$ or $\operatorname{rank}(B)$, it follows that $\operatorname{rank}(A+B) = \operatorname{rank}(XY) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.
2. To prove $|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A+B)$:
   - Using the same augmented matrices X and Y as in the previous proof, we have $XY = A + B$.
   - There exist the inequality $\operatorname{rank}(X) + \operatorname{rank}(Y) - r - s \leq \operatorname{rank}(A+B)$ from Question 1.
   - Using the factorizations of A and B,
     - Since $\operatorname{col}(C) \subseteq \operatorname{col}(X)$, then $\operatorname{rank}(C) \leq \operatorname{rank}(X)$.
     - Since $\operatorname{col}(E) \subseteq \operatorname{col}(X)$, then $\operatorname{rank}(E) \leq \operatorname{rank}(X)$.
     - Since $\operatorname{row}(D) \subseteq \operatorname{row}(Y)$, then $\operatorname{rank}(D) \leq \operatorname{rank}(Y)$.
     - Since $\operatorname{row}(F) \subseteq \operatorname{row}(Y)$, then $\operatorname{rank}(F) \leq \operatorname{rank}(Y)$.
   - By substituting the inequalities above into the inequality from Question 1, we get $\operatorname{rank}(C) + \operatorname{rank}(D) - r - s \leq \operatorname{rank}(A+B)$, $\operatorname{rank}(E) + \operatorname{rank}(F) - r - s \leq \operatorname{rank}(A+B)$.
   - $\operatorname{rank}(C) + \operatorname{rank}(D) \leq \operatorname{rank}(A+B) + r + s$, $\operatorname{rank}(E) + \operatorname{rank}(F) \leq \operatorname{rank}(A+B) + r + s$.
   - $\operatorname{rank}(C) + \operatorname{rank}(D) \leq \operatorname{rank}(A+B) + \operatorname{rank}(A) + \operatorname{rank}(B)$, $\operatorname{rank}(E) + \operatorname{rank}(F) \leq \operatorname{rank}(A+B) + \operatorname{rank}(A) + \operatorname{rank}(B)$.
   - Since CD is a factorization of A, we have $r = \operatorname{rank}(A) = \operatorname{rank}(C) = \operatorname{rank}(D)$ because C consists of a set of column vetor basis from A, which are independent, therefore $\operatorname{rank}(C) = r = \operatorname{rank}(A)$. D would need to be the same rank as A to ensure that the product of CD is the same rank as A, as C has full colume rank.
   - Similarly, EF is a factorization of B, therefore we have $s = \operatorname{rank}(B) = \operatorname{rank}(E) = \operatorname{rank}(F)$.
   - By substituting the ranks of C, D, E, F into the inequality, we get $\operatorname{rank}(A) + \operatorname{rank}(A) \leq \operatorname{rank}(A+B) + \operatorname{rank}(A) + \operatorname{rank}(B)$, $\operatorname{rank}(B) + \operatorname{rank}(B) \leq \operatorname{rank}(A+B) + \operatorname{rank}(A) + \operatorname{rank}(B)$.
   - $\operatorname{rank}(A) \leq \operatorname{rank}(A+B) + \operatorname{rank}(B)$, $\operatorname{rank}(B) \leq \operatorname{rank}(A+B) + \operatorname{rank}(A)$
   - $\operatorname{rank}(A) - \operatorname{rank}(B) \leq \operatorname{rank}(A+B)$, $\operatorname{rank}(B) - \operatorname{rank}(A) \leq \operatorname{rank}(A+B)$
   - Therefore we can combine the two results above with an absolute value as when
     - $\operatorname{rank}(A) > \operatorname{rank}(B)$, then $\operatorname{rank}(A+B) \geq \operatorname{rank}(A) - \operatorname{rank}(B) > 0 > \operatorname{rank}(B) - \operatorname{rank}(A)$
     - $\operatorname{rank}(A) < \operatorname{rank}(B)$, then $\operatorname{rank}(A+B) \geq \operatorname{rank}(B) - \operatorname{rank}(A) > 0 > \operatorname{rank}(A) - \operatorname{rank}(B)$
   - Therefore $|\operatorname{rank}(A) - \operatorname{rank}(B)| \leq \operatorname{rank}(A+B)$.