First principles in fluids

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Abstract: This note, intended for being used as a quick reference, provides a collection of a wide range of equations in fluid mechanics, from basic equations that can be found in introductory textbooks, to those only left as an exercise or conclusion in graduate textbooks, monographs, or research papers, the detailed derivations of which were typically not provided. We try to use symbols and notations as consistently as possible, and it is noted that is unavoidable that the note is biased according to our preference.

Contents

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1 Conservation laws

1.1 Continuity

Differential form; integral form;

1.1.1 Compressibility

Note on incompressibility condition - Drho/Dt, and the volume change rate relation ,and the thermal effects (EOS).

1.2 Momentum equation

Differential form; integral form;

1.3 Bernoulli theorem

Assumptions:

- Inviscid.
- Barotropic.
- Potential force.
- Steady.

1.3.1 Lamb-Gromyko

Consider the inviscid Euler equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{F} \tag{1.1}$$

with a conservative force

$$\boldsymbol{F} = -\nabla \Phi. \tag{1.2}$$

Using Eq. (A.36) we have

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} \tag{1.3}$$

We defined

$$\boldsymbol{L} = \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{\omega} \tag{1.4}$$

which is called the Lamb vector.

The Euler equation becomes

$$\nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{p}{\rho}\right) - \nabla \Phi \tag{1.5}$$

and hence

$$\nabla \left(\frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi\right) = \boldsymbol{L} \tag{1.6}$$

which is called the Lamb-Gromyko equation. Define

$$H = \frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi \tag{1.7}$$

we have

$$\nabla H = \boldsymbol{L}. \tag{1.8}$$

If the flow is irrotational, i.e., $\boldsymbol{L} = 0$, we recover a special version (isentropic) of Bernoulli's theorem.

1.3.2 Constitution relations

1.3.3 Jump conditions

1.4 Pressure Poisson

Take the divergence of the follow equation for incompressible flows ($\nabla \cdot \boldsymbol{u} = 0$):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{1.9}$$

we have

$$-\frac{1}{\rho}\nabla^2 p = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u}\nabla + \boldsymbol{u} \cdot \nabla(\nabla \cdot \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u}\nabla \tag{1.10}$$

i.e.

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i} = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \tag{1.11}$$

In CFD, the continuity equation ($\nabla \cdot \boldsymbol{u} = 0$) is responsible for solving pressure for the above reason.

1.5 Energy equation

1.5.1 A thermodynamics perspective

1.5.2 Thermodynamics relations

2 Vortex dynamics

2.1 Vorticity transport equation

The vorticity field is the curl of the velocity field:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{2.1}$$

By Eq. (A.27) we know

$$\nabla \cdot \boldsymbol{\omega} = 0 \tag{2.2}$$

i.e., the continuity of vorticity.

The incompressible Navier-Stokes equation in vector form:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F} \tag{2.3}$$

Using Eq. (1.3) we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \frac{\boldsymbol{u}^2}{2} + \boldsymbol{\omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F} \tag{2.4}$$

Using the (A.37) and take the curl of Eq. (2.4)

$$\mathrm{LHS} = \nabla \times (\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \frac{\boldsymbol{u}^2}{2} + \boldsymbol{\omega} \times \boldsymbol{u}) \tag{2.5}$$

$$= \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) \tag{2.6}$$

$$= \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}) - \boldsymbol{u}(\nabla \cdot \boldsymbol{\omega}) \tag{2.7}$$

$$= \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} \tag{2.8}$$

$$\text{RHS} = \nabla \times (-\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F}) \tag{2.9}$$

$$= \nu \nabla^2 \boldsymbol{\omega} + \nabla \times \boldsymbol{F} + \frac{1}{\rho^2}\nabla \rho \times \nabla p \tag{2.10}$$

Equaling both sides we obtain

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t}}_{\substack{\text{rate of} \\ \text{change}}} + \underbrace{\boldsymbol{u} \cdot \nabla \boldsymbol{\omega}}_{\text{advection}} = \underbrace{\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}}_{\substack{\text{vortex} \\ \text{stretching}}} + \underbrace{\nu \nabla^2 \boldsymbol{\omega}}_{\substack{\text{viscous} \\ \text{diffusion}}} + \underbrace{\nabla \times \boldsymbol{F}}_{\substack{\text{external torque in} \\ \text{a non-conservative field}}} + \underbrace{\frac{1}{\rho^2}\nabla \rho \times \nabla p}_{\substack{\text{baroclinic} \\ \text{torque}}} \tag{2.11}$$

Again, the stretching term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ comes from the non-linear advection/inertial term in N-S. It is important in the energy cascade in turbulence.

2.2 Enstrophy equation

$$\mathcal{E} \triangleq \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\omega} = \frac{1}{2}\omega_i \omega_i \tag{2.12}$$

Re-write (2.11) into tensor notation we have

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2} + \epsilon_{ijk}\frac{\partial f_k}{\partial x_j} + \frac{1}{\rho^2}\epsilon_{ijk}\frac{\partial \rho}{\partial x_j}\frac{\partial p}{\partial x_k} \tag{2.13}$$

$\omega_i \times$ (2.13) we have

$$\frac{\partial}{\partial t}(\frac{1}{2}\omega_i\omega_i) + u_j \frac{\partial}{\partial x_j}(\frac{1}{2}\omega_i\omega_i) = \omega_i\omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2}{\partial x_j^2}(\frac{1}{2}\omega_i\omega_i) - \nu \frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} \tag{2.14}$$

$$+ \epsilon_{ijk}\omega_i \frac{\partial F_k}{\partial x_j} + \frac{1}{\rho^2}\epsilon_{ijk}\omega_i \frac{\partial \rho}{\partial x_j}\frac{\partial p}{\partial x_k} \tag{2.15}$$

Note that ϵ_{ijk} is the Levi-Civita symbol, not to be confused with the turbulent kinetic energy rate ε or Reynols stresses dissipation rate ε_{ij}.

Re-write back into vector form:

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{u} \cdot \nabla \mathcal{E} = \boldsymbol{\omega}\boldsymbol{\omega} : \nabla \boldsymbol{u} + \nu \nabla^2 \mathcal{E} \underbrace{-\nu \nabla \boldsymbol{\omega} : \nabla \boldsymbol{\omega}}_{\substack{\text{viscous} \\ \text{dissipation}}} + \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{F}) + \boldsymbol{\omega} \cdot \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{2.16}$$

Note that we are assuming an incompressible flow, hence $\nabla \cdot \boldsymbol{u}$ related terms are not appearing in Eq. (2.16). A new mechanism compared to (2.11) is the viscous dissipation of enstrophy. This term is always negative.

2.3 Velocity gradient tensor, its invariants and dynamics

Meneveau (2011), also Q-R invariant space etc, restricted Euler.

2.4 Vortex models

2.4.1 Point vortex system

2.4.2 Lamb-Oseen similarity solution

3 Turbulent flows

3.1 Mean and fluctuation flows

3.1.1 Reynolds average

We denote time average as $\overline{(\cdot)}$, space or ensemble average as $\langle\cdot\rangle$, and sometimes use these notations interchangeably given that they are equivalent under the ergodicity assumption. The properties proved for one definition are expected to hold for another. Although Reynolds decomposition and RANS modelings are not an accurate way of computing turbulence, they consist the foundation of our understanding of turbulence.

Below we give briefly some properties of Reynolds averaging:

(i) (Definition) The time average of a physical variable A is

$$\overline{A} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T A\,\mathrm{d}t \tag{3.1}$$

In practice, the limit is often neglected and the average window is assumed to be long enough.

(ii) (Definition) The fluctuation of a physical variable A is

$$A' \triangleq A - \overline{A} \tag{3.2}$$

(iii) (Proposition) The average of fluctuation is zero.

$$\overline{A'} = \overline{A - \overline{A}} = \overline{A} - \overline{A} = 0 \tag{3.3}$$

3.1.2 Continuity and momemtum

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3.4}$$

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i}{\partial x_j}) \tag{3.5}$$

Taking the average of Eq. (3.4) we have

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} = 0 \tag{3.6}$$

where

$$\frac{\partial \overline{u}_i}{\partial x_i} = \frac{\partial}{\partial x_i}(\frac{1}{T}\int_0^T u_i\,\mathrm{d}t) = \frac{1}{T}\int_0^T (\frac{\partial u_i}{\partial x_i})\,\mathrm{d}t = \frac{1}{T}\int_0^T 0\,\mathrm{d}t = 0 \tag{3.7}$$

Hence we have the continuity for fluctuating velocity

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{3.8}$$

Taking the average of Eq. (3.5) we have

$$\mathrm{LHS} = (\frac{1}{T}\int_0^T \,\mathrm{d}t) * [\frac{\partial}{\partial t}(\overline{u}_i + u_i') + (\overline{u}_j + u_j')\frac{\partial}{\partial x_j}(\bar{u}_i + u_i')] \tag{3.9}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{\frac{\partial u_i'}{\partial t}} + \overline{\overline{u}_j \frac{\partial}{\partial x_j} \overline{u}_i} + \overline{\overline{u}_j \frac{\partial}{\partial x_j} u_i'} + \overline{u_j' \frac{\partial}{\partial x_j} \overline{u}_i} + \overline{u_j' \frac{\partial}{\partial x_j} u_i'} \tag{3.10}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j} \overline{u}_i + \overline{u_j' \frac{\partial}{\partial x_j} u_i'} \tag{3.11}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j} \overline{u}_i + \overline{\frac{\partial}{\partial x_j} (u_j' u_i')} - \overline{u_i' \frac{\partial u_j'}{\partial x_j}} = \frac{\partial}{\partial x_j} \overline{u_j' u_i'} \tag{3.12}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j} \overline{u_j' u_i'} \tag{3.13}$$

$$\text{RHS} = (\frac{1}{T} \int_0^T \mathrm{d}t)[-\frac{1}{\rho} \frac{\partial}{\partial x_i} (\overline{p} + p') + \frac{\partial}{\partial x_j} [\nu \frac{\partial}{\partial x_j} (\overline{u}_i + u_i')] \tag{3.14}$$

$$= -\frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j} (\nu \frac{\partial \overline{u}_i}{\partial x_j}) \tag{3.15}$$

Equaling both sides yields:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{3.16}$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j} (\nu \frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i' u_j'}) \tag{3.17}$$

where the cross-correlation term having dimension of shear stress

$$\tau_{\text{Rey}} = -\overline{u_i' u_j'} \tag{3.18}$$

is called the Reynolds stress term. It is a rank 2 tensor. It comes from the Reynolds averaging of the non-linear advection term on the LHS of Navier-Stokes, and it distinguishes turbulent flows from laminar ones. It represents the momentum transport due to turbulent motions, in anology to the molecular diffusion.

3.1.3 Transport equation of the fluctuating velocity

Denote the material derivative based on the mean flow advection as

$$\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}_k \frac{\partial}{\partial x_k} \tag{3.19}$$

and subtract the Reynolds equation from N-S equation

$$\frac{\bar{D} u_i'}{Dt} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j} (\nu \frac{\partial u_i'}{\partial x_j} + \overline{u_i' u_j'} - u_i' u_j') - u_j' \frac{\partial \overline{u}_i}{\partial x_j} \tag{3.20}$$

The last term shows the mean-flow stretching of the fluctuation, which is a generation mechanism be shown later related to the shear production of turbulent kinetic energy.

3.1.4 Mean-flow and turbulent kinetic energy

The total kinetic energy of the flow can be devided into the mean kinetic energy (MKE) and the turbulent kinetic energy (TKE)

$$K_{\text{tot}} = \frac{1}{2} \overline{u_i u_i} \tag{3.21}$$

$$= \overline{\frac{1}{2}(\overline{u}_i + u_i')(\overline{u}_i + u_i')} \tag{3.22}$$

$$= \overline{\frac{1}{2}\overline{u}_i\overline{u}_i + \overline{u}_i u_i' + \frac{1}{2}u_i'u_i'} \tag{3.23}$$

$$= \frac{1}{2}\overline{u}_i\overline{u}_i + \frac{1}{2}\overline{u_i'u_i'} \tag{3.24}$$

$$= K + k \tag{3.25}$$

We will show how these two parts are related dynamically.

3.1.5 MKE equation

Multiply the Reynolds equation (3.17) by $\overline{u}_i$ we have

$$\text{LHS} = \overline{u}_i \frac{\bar{D}\overline{u}_i}{Dt} = \frac{\bar{D}K}{Dt} \tag{3.26}$$

$$\overline{u}_i(-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}) = -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_i}{\partial x_i} + \frac{1}{\rho}\frac{\partial \overline{u}_i}{\partial x_i} \tag{3.27}$$

$$= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_i}{\partial x_i} \tag{3.28}$$

$$= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_j}{\partial x_j} \tag{3.29}$$

$$\overline{u}_i \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}) = \frac{\partial}{\partial x_j}[\overline{u}_i(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'})] - \frac{\partial \overline{u}_i}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}) \tag{3.30}$$

$$= -\nu\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_i}{\partial x_j}\overline{u_i'u_j'} + \frac{\partial}{\partial x_j}(\nu\frac{\partial K}{\partial x_j}) - \frac{\partial \overline{u}_i\,\overline{u_i'u_j'}}{\partial x_j} \tag{3.31}$$

Equaling both sides we have

$$\frac{\bar{D}K}{Dt} = \frac{\partial}{\partial x_j}(\underbrace{-\frac{1}{\rho}\overline{p}\,\overline{u}_j}_{\substack{\text{pressure}\\\text{distortion}}} \underbrace{+\nu\frac{\partial K}{\partial x_j}}_{\substack{\text{molecular}\\\text{diffusion}}} \underbrace{-\overline{u}_i\,\overline{u_i'u_j'}}_{\substack{\text{turbulent}\\\text{diffusion}}}) \underbrace{-\frac{1}{2}P_{kk}}_{\substack{\text{production}\\\text{of TKE}}} \underbrace{-\nu(\frac{\partial \overline{u}_i}{\partial x_j})^2}_{\text{dissipation}} \tag{3.32}$$

where the term

$$P_{kk} = -2\overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j} \tag{3.33}$$

is the production term of the turbulent kinetic energy, and, on the other hand, is the sink in MKE.

3.1.6 TKE equation

Similarly, multiply (3.20) by u_i' and then take the average

$$\text{LHS} = \overline{u_i'(\frac{\partial u_i'}{\partial t} + \overline{u_k}\frac{\partial u_i'}{\partial x_k})} \tag{3.34}$$

$$= \frac{\partial \frac{1}{2}\overline{u_i'u_i'}}{\partial t} + \overline{u_k}\frac{\partial \frac{1}{2}\overline{u_i'u_i'}}{\partial x_k} \tag{3.35}$$

$$= \frac{\bar{D}k}{Dt} \tag{3.36}$$

$$\overline{u_i'(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{1}{\rho}\frac{\partial \overline{p'u_i'}}{\partial x_i'} \tag{3.37}$$

$$= -\frac{1}{\rho}\frac{\partial \overline{p'u'_k}}{\partial x'_k} \tag{3.38}$$

$$\overline{u'_i(\frac{\partial}{\partial x_k}\nu\frac{\partial u'_i}{\partial x_k})} = \overline{\frac{\partial}{\partial x_k}(\nu u'_i\frac{\partial u'_i}{\partial x_k}) - \nu\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_k}} \tag{3.39}$$

$$= \overline{\frac{\partial}{\partial x_k}(\nu\frac{\partial \frac{1}{2}u'_iu'_i}{\partial x_k}) - \nu\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_k}} \tag{3.40}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial k}{\partial x_k}) - \nu\overline{(\frac{\partial u'_i}{\partial x_k})^2} \tag{3.41}$$

$$\overline{u'_i(\frac{\partial}{\partial x_k}\overline{u'_iu'_k})} = 0 \tag{3.42}$$

$$\overline{u'_i(\frac{\partial}{\partial x_k}u'_iu'_k)} = \frac{1}{2}\frac{\partial\overline{u'_iu'_iu'_k}}{\partial x_k} \tag{3.43}$$

$$= \frac{1}{2}\frac{\partial\overline{u'_iu'_iu'_k}}{\partial x_k} \tag{3.44}$$

$$\overline{u'_i(-u'_k\frac{\partial \overline{u}_i}{\partial x_k})} = -\overline{u'_iu'_k}\frac{\partial \overline{u}_i}{\partial x_k} \tag{3.45}$$

Equaling both sides we have

$$\frac{\bar{D}k}{Dt} = \frac{\partial}{\partial x_k}(\underbrace{\nu\frac{\partial k}{\partial x_k}}_{\text{molecular diffusion}} + \underbrace{\frac{1}{2}\overline{u'_iu'_iu'_k}}_{\text{turbulent diffusion}} \underbrace{-\frac{1}{\rho}\overline{p'u'_k}}_{\text{pressure distortion}}) + \underbrace{\frac{1}{2}P_{kk}}_{\text{production of TKE}} \underbrace{-\nu\overline{(\frac{\partial u'_i}{\partial x_k})(\frac{\partial u'_i}{\partial x_k})}}_{\text{dissipation}} \tag{3.46}$$

Comments:

(1) The turbulent kinetic energy generation term

$$P_{kk} = -2\overline{u'_iu'_j}\frac{\partial \overline{u}_i}{\partial x_j} \tag{3.47}$$

can be expressed in tensor notation as

$$P = 2\boldsymbol{\tau}_{\text{Rey}} : \nabla\overline{\boldsymbol{u}} = 2\boldsymbol{\tau}_{\text{Rey}} : \boldsymbol{S} \tag{3.48}$$

where the inner product represents the projection of the velocity fluctuation correlation on the mean shear/strain rate.

(2) The dissipation term

$$\varepsilon = \nu\overline{(\frac{\partial u'_i}{\partial x_k})(\frac{\partial u'_i}{\partial x_k})} \tag{3.49}$$

is always positive, representing the dissipation mechanism of turbulence kinetic energy.

3.1.7 Reynolds stress transport equation

The velocity fluctuation transport equation is

$$\frac{\bar{D}u'_i}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u'_i}{\partial x_j} + \overline{u'_iu'_j} - u'_iu'_j) - u'_j\frac{\partial \overline{u}_i}{\partial x_j} \tag{3.50}$$

Or if we exchange the two subscripts we obtain:

$$\frac{\bar{D}u'_j}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \frac{\partial}{\partial x_i}(\nu\frac{\partial u'_j}{\partial x_i} + \overline{u'_i u'_j} - u'_i u'_j) - u'_i\frac{\partial \overline{u}_j}{\partial x_i} \tag{3.51}$$

$u'_j \times (19) + u'_i \times (20)$ and take the time average:

$$\text{LHS} = \frac{\bar{D}\overline{u'_i u'_j}}{Dt} \tag{3.52}$$

$$\text{RHS}_1 = -\frac{1}{\rho}[-2\overline{p's_{ij}} + \frac{\partial}{\partial x_i}(\overline{p'u'_j}) + \frac{\partial}{\partial x_j}(\overline{p'u'_i})] \tag{3.53}$$

$$\text{RHS}_2 = \overline{u'_j\frac{\partial}{\partial x_k}(\nu\frac{\partial u'_i}{\partial x_k}) + u'_i\frac{\partial}{\partial x_k}(\nu\frac{\partial u'_j}{\partial x_k})} \tag{3.54}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u'_i u'_j}}{\partial x_k}) - 2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} \tag{3.55}$$

$$\text{RHS}_3 = \overline{u'_j\frac{\partial}{\partial x_k}\overline{u'_i u'_k} + u'_i\frac{\partial}{\partial x_k}\overline{u'_j u'_k}} \tag{3.56}$$

$$= 0 \tag{3.57}$$

$$\text{RHS}_4 = -\overline{u'_j\frac{\partial}{\partial x_k}(u'_i u'_k) + u'_i\frac{\partial}{\partial x_k}(u'_j u'_k)} \tag{3.58}$$

$$= -\overline{u'_j u'_k\frac{\partial}{\partial x_k}(u'_i) + u'_i u'_k\frac{\partial}{\partial x_k}(u'_j) + u'_i u'_j\frac{\partial}{\partial x_k}(u'_k)} \tag{3.59}$$

$$(Continuity, \frac{\partial u'_k}{\partial x_k} = 0, is\, used\, twice\, here.) \tag{3.60}$$

$$= -\frac{\partial}{\partial x_k}\overline{u'_i u'_j u'_k} \tag{3.61}$$

$$\text{RHS}_5 = \overline{-u'_k u'_j\frac{\partial \overline{u}_i}{\partial x_k} - u'_k u'_i\frac{\partial \overline{u}_j}{\partial x_k}} \tag{3.62}$$

$$= -\overline{u'_k u'_j}\frac{\partial \overline{u}_i}{\partial x_k} - \overline{u'_k u'_i}\frac{\partial \overline{u}_j}{\partial x_k} \tag{3.63}$$

$$\tag{3.64}$$

By equalizing both sides we obtain

$$\frac{\bar{D}\overline{u'_i u'_j}}{Dt} = \frac{2}{\rho}\overline{p's_{ij}} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p'u'_j})\delta_{ik} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p'u'_i})\delta_{jk} + \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u'_i u'_j}}{\partial x_k}) \tag{3.65}$$

$$- 2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} - \frac{\partial}{\partial x_k}\overline{u'_i u'_j u'_k} - \overline{u'_k u'_j}\frac{\partial \overline{u}_i}{\partial x_k} - \overline{u'_k u'_i}\frac{\partial \overline{u}_j}{\partial x_k} \tag{3.66}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u'_i u'_j}}{\partial x_k} - \overline{u'_i u'_j u'_k} - \frac{1}{\rho}\overline{p'u'_i}\delta_{jk} - \frac{1}{\rho}\overline{p'u'_j}\delta_{ik}) \tag{3.67}$$

$$- (\overline{u'_k u'_j}\frac{\partial \overline{u}_i}{\partial x_k} + \overline{u'_k u'_i}\frac{\partial \overline{u}_j}{\partial x_k}) + \frac{2}{\rho}\overline{p's_{ij}} - 2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} \tag{3.68}$$

Equaling both sides we have

$$\frac{\bar{D}\overline{u'_i u'_j}}{Dt} = d_{ij} + P_{ij} + \Phi_{ij} - \varepsilon_{ij} \tag{3.69}$$

where

$$d_{ij} = \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'u_j'u_k'} - \frac{1}{\rho}\overline{p'u_i'}\delta_{jk} - \frac{1}{\rho}\overline{p'u_j'}\delta_{ik}) \tag{3.70}$$

$$P_{ij} = -\overline{u_k'u_j'}\frac{\partial\overline{u}_i}{\partial x_k} - \overline{u_k'u_i'}\frac{\partial\overline{u}_j}{\partial x_k} \tag{3.71}$$

$$\Phi_{ij} = \frac{2}{\rho}\overline{p's_{ij}} \tag{3.72}$$

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{3.73}$$

$$s_{ij} = \frac{1}{2}(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}) \tag{3.74}$$

Comments:

(1) The left hand side term $\frac{\bar{D}\overline{u_i'u_j'}}{Dt}$ is the rate of change of the Reynolds stress along the particle line.

(2) The term d_{ij} is the diffusion term in the equation, appearing in the form of gradient. It includes viscous term, Reynolds stress term and pressure-velocity fluctuation coupling term.The diffusion is resulted by the spatial non-uniformity of these property.

(3) The term P_{ij} is the generation term of Reynolds stress, showed in the form of the product of Reynolds stress and the mean flow strain rate.

(4) The term Φ_{ij} is the redistribution term. We note that the contraction of Reynolds stress transport equation is the transport equation for turbulence kinetic energy. And the contraction of Φ_{ij} is $\Phi_{ii} = \frac{2}{\rho}\overline{p's_{ii}} = 0$ as continuity holds. So the term contributes nothing to the growth of turbulent kinetic energy. It just takes the kinetic energy from one component of fluid motion to another component.

(5) The term ε_{ij}, whose contraction is positive forever, representing the dissipation mechanism of kinetic energy.

3.1.8 Dissipation rate transport equation

The dissipation term in Reynolds stresses transport equation is defined as

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_j'}{\partial x_p}} \tag{3.75}$$

Multiply equation (3.20) by $2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}$ and take the time derivative we have:

$$\text{LHS} = 2\nu\frac{\bar{D}}{Dt}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_p}} = \frac{\bar{D}\varepsilon}{Dt} + 2\nu\frac{\partial\overline{u_k}}{\partial x_p}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} \tag{3.76}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{2\nu}{\rho}\frac{\partial}{\partial x_k}(\overline{\frac{\partial u_k'}{\partial x_p}\frac{\partial p'}{\partial x_p}}) \tag{3.77}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}(\nu\frac{\partial u_i'}{\partial x_k}))} = \frac{\partial}{\partial x_k}(\nu\frac{\partial\varepsilon}{\partial x_k}) - 2\overline{(\nu\frac{\partial^2 u_i'}{\partial x_p\partial x_k})^2} \tag{3.78}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}\overline{u_i'u_k'})} = 0 \tag{3.79}$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}-u_i'u_k')} = -2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} + \frac{\partial}{\partial x_k}\overline{u_k'\nu(\frac{\partial u_i'}{\partial x_p})^2} \tag{3.80}$$

$$= -2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} + \frac{\partial}{\partial x_k}\overline{u_k'\varepsilon'} \tag{3.81}$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-u_k'\frac{\partial \overline{u}_i}{\partial x_k})} = -2\nu\frac{\partial \overline{u}_i}{\partial x_k}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}} - 2\nu\frac{\partial^2\overline{u}_i}{\partial x_k\partial x_p}\overline{u_k'\frac{\partial u_i'}{\partial x_p}} \tag{3.82}$$

By equalizing both side we yield the transport equation for turbulence dissipation rate

$$\frac{\bar{D}\varepsilon}{Dt} = \frac{\partial}{\partial x_k}(-\frac{2\nu}{\rho}\overline{\frac{\partial u_k}{\partial x_p}\frac{\partial p}{\partial x_p}} + \nu\frac{\partial \varepsilon}{\partial x_k} - \overline{u_k'\varepsilon'}) - 2\nu\frac{\partial \overline{u}_i}{\partial x_k}(\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}} + \overline{\frac{\partial u_p'}{\partial x_k}\frac{\partial u_p'}{\partial x_i}}) \tag{3.83}$$

$$-2\nu\overline{u_k'\frac{\partial u_i'}{\partial x_p}}\frac{\partial^2\overline{u}_i}{\partial x_p\partial x_k} - 2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} - 2\overline{(\nu\frac{\partial u_i'}{\partial^2 x_p\partial x_k})^2} \tag{3.84}$$

The final equation of the equation agrees with that given in the turbulence book by Shi (1994). Second moment equation closure problem Chou (1945) could be discussed briefly here.

3.1.9 Scalar flux, its mean and kinetic energy transport equations

Similar to Eq. (3.20) we have the transport equation for the mean and fluctuation of a passive scalar c:

$$\frac{\bar{D}\bar{c}}{\bar{D}t} = \frac{\partial}{\partial x_j}(\nu\frac{\partial \bar{c}}{\partial x_j} - \overline{c'u_j'}) \tag{3.85}$$

and

$$\frac{\bar{D}c'}{Dt} = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial c'}{\partial x_j} + \overline{c'u_j'} - c'u_j') - u_j'\frac{\partial \bar{c}}{\partial x_j} \tag{3.86}$$

where Γ is the molecular diffusion coefficient of c.

Take $c'\times$ (3.20) $+u_i'\times$ (3.86) and apply the average

$$\text{LHS} = \frac{\bar{D}\overline{c'u_i'}}{\bar{D}t} \tag{3.87}$$

$$\text{RHS}_1 = -\frac{1}{\rho}\overline{c'\frac{\partial p'}{\partial x_i}} = -\frac{1}{\rho}(\frac{\partial}{\partial x_j}\overline{p'c'}\delta_{ij} - \overline{p'\frac{\partial c'}{\partial x_i}}) \tag{3.88}$$

$$\text{RHS}_2 = \frac{\partial}{\partial x_j}(\Gamma\overline{u_i'\frac{\partial c'}{\partial x_j}} + \nu\overline{c'\frac{\partial u_i'}{\partial x_j}}) - (\nu+\Gamma)\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{3.89}$$

$$\text{RHS}_3 = -\frac{\partial}{\partial x_j}(\overline{c'u_i'u_j'}) \tag{3.90}$$

$$\text{RHS}_4 = -\overline{c'u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \bar{c}}{\partial x_j} \tag{3.91}$$

then we obtain the transport equation for scalar flux

$$\frac{\bar{D}\overline{c'u_i'}}{\bar{D}t} = d_{jc} + P_{jc} + \Phi_{jc} - \varepsilon_{jc} \tag{3.92}$$

where

$$d_{ic} = \frac{\partial}{\partial x_j}(\Gamma\overline{u_i'\frac{\partial c'}{\partial x_j}} + \nu\overline{c'\frac{\partial u_i'}{\partial x_j}} - \frac{1}{\rho}\overline{p'c'}\delta_{ij} - \overline{c'u_i'u_j'}) \tag{3.93}$$

$$P_{ic} = -\overline{c'u'_j}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}\frac{\partial \overline{c}}{\partial x_j} \tag{3.94}$$

$$\Phi_{ic} = \frac{1}{\rho}\overline{p'\frac{\partial c'}{\partial x_i}} \tag{3.95}$$

$$\varepsilon_{ic} = (\nu + \Gamma)\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{3.96}$$

Comments:

(1) Gradient diffusion: velocity-fluctuation scalar-diffusion correlation, momentum-diffusion scalar-fluctation ccorelation, pressure diffusion, turbulence diffusion.

(2) Production: scalar flux interacting with mean shear, turbulent flux (Reynolds stresses) interacting with mean scalar gradient.

(3) Re-distribution.

(4) Dissipation.

Define scalar mean and fluctuation energy as

$$K_c = \frac{1}{2}\bar{c}^2 \tag{3.97}$$

$$k_c = \frac{1}{2}\overline{c'c'} \tag{3.98}$$

$c'\times$ (3.86) and apply the average

$$\text{LHS} = \frac{\bar{D}k_c}{\bar{D}t} \tag{3.99}$$

$$\text{RHS}_1 = \frac{\partial}{\partial x_j}\Gamma\frac{\partial k_c}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{3.100}$$

$$\text{RHS}_2 = -\frac{1}{2}\frac{\partial}{\partial x_j}\overline{c'c'u_j} \tag{3.101}$$

$$\text{RHS}_3 = -\overline{c'u'_j}\frac{\partial \overline{c}}{\partial x_j} \tag{3.102}$$

then we obtain the transport equation for scalar fluctuation energy

$$\frac{\bar{D}k_c}{\bar{D}t} = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial}{\partial x_j}k_c - \frac{1}{2}\overline{c'c'u'_j}) - \overline{c'u'_j}\frac{\partial \overline{c}}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{3.103}$$

For active scalar (for example, density which appears in the momentum equation as buoyancy force), see section 4.6.

3.1.10 Poisson equation for mean and fluctuation pressure

The Reynolds average equation is

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}) \tag{3.104}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} \tag{3.105}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2\overline{p} - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} \tag{3.106}$$

Poisson equation for mean pressure:

$$-\frac{1}{\rho}\nabla^2\overline{p} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} \tag{3.107}$$

The velocity fluctuation transport equation is

$$\frac{\bar{D}u_i'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i'}{\partial x_j} + \overline{u_i'u_j'} - u_i'u_j') - u_j'\frac{\partial \overline{u}_i}{\partial x_j} \tag{3.108}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u_i'}{\partial x_j} \tag{3.109}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2 p' - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} - \frac{\partial u_j'}{\partial x_i}\frac{\partial u_i'}{\partial x_j} - \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u_j'}{\partial x_i} \tag{3.110}$$

Poisson equation for fluctuation pressure:

$$-\frac{1}{\rho}\nabla^2 p' = \frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u_i'}{\partial x_j} + \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_j'}{\partial x_i}\frac{\partial u_i'}{\partial x_j} - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} \tag{3.111}$$

$$= \frac{\partial u_j'}{\partial x_i}\frac{\partial u_i'}{\partial x_j} - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} + 2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u_i'}{\partial x_j} \tag{3.112}$$

3.1.11 Turbulent vorticity and enstrophy

Similarly, vorticity can be decomposed into the mean and the pertubation. We give the equation of pertuabtion vorticity without derivation:

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j'\overline{S}_{ij} + \overline{\omega_j}S_{ij}' + \omega_j'S_{ij}' - \overline{\omega_j S_{ij}'} - u_j'\frac{\partial \overline{\omega}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_j'\omega_i'} - u_j'\omega_i') + \nu\frac{\partial^2 \omega_i'}{\partial x_j^2} \tag{3.113}$$

where $\overline{S}_{ij}$ and S_{ij}' are the mean and the fluctuation shear, respectively.

We define the fluctuating enstrophy as

$$\mathscr{E} = \frac{1}{2}\overline{\omega_i'\omega_i'} \tag{3.114}$$

$\omega_i' \times$ (3.113) and take the time average

$$\text{LHS} = \frac{\bar{D}\mathscr{E}}{\bar{D}t} \tag{3.115}$$

$$\text{RHS}_1 = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega}_j\overline{\omega_i'S_{ij}'} + \overline{\omega_i'\omega_j'S_{ij}'} \tag{3.116}$$

$$\text{RHS}_2 = -\overline{\omega_i'u_j'}\frac{\partial \overline{\omega_i}}{\partial x_j} \tag{3.117}$$

$$\text{RHS}_3 = -\frac{1}{2}\frac{\partial}{\partial x_j}(\overline{u_j'\omega_i'\omega_i'}) \tag{3.118}$$

$$\text{RHS}_4 = \nu\overline{\frac{\partial^2 \mathscr{E}}{\partial x_j^2}} - \overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} \tag{3.119}$$

Equaling both sides we obtain

$$\frac{\bar{D}\mathscr{E}}{\bar{D}t} = P_{\mathscr{E}} + D_{\mathscr{E}} - \varepsilon_{\mathscr{E}} \tag{3.120}$$

$$P_{\mathscr{E}} = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega_j}\overline{\omega_i' S_{ij}'} + \overline{\omega_i'\omega_j' S_{ij}'} - \overline{\omega_i' u_j'}\frac{\partial\overline{\omega_i}}{\partial x_j} \tag{3.121}$$

$$D_{\mathscr{E}} = \frac{\partial}{\partial x_j}\left(\nu\frac{\partial\mathscr{E}}{\partial x_j} - \frac{1}{2}\overline{u_j'\omega_i'\omega_i'}\right) \tag{3.122}$$

$$\varepsilon_{\mathscr{E}} = \nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}} \tag{3.123}$$

Comment: The energy balance process of fluctuation enstrophy obeys four principle processes in nature (Kolmogorov):

change rate = production + diffusion + dissipation

3.2 Farve average in compressible flows

3.3 LES equations

3.4 Homogeneous turbulence theory

K-H etc.

3.5 Free shear flows

3.5.1 Momentum integral

Similarity solutions (turbulent). Pope (2001).

3.5.2 Similarity solutions

The characteristic velocity and length scales are U_s and δ_s, respectively.

Flow type	U_s	δ_s	$U_s \propto x^m$	$\delta_s \propto x^n$	$f(\eta)$
Round jet	$\bar{u}(x, y = 0)$	$r_{1/2}$	-1	1	$1/(1 + a\eta^2)^2$
Plane jet	$\bar{u}(x, r = 0)$	$y_{1/2}$	$-1/2$	1	$\mathrm{sech}^2(\ln(1 + \sqrt{2})\,\eta)$
Round wake	$U_\infty - \bar{u}(x, y = 0)$	$r_{1/2}$	$-2/3$	$1/3$	$\exp(-\ln 2\,\eta^2)$
Plane wake	$U_\infty - \bar{u}(x, r = 0)$	$y_{1/2}$	$-1/2$	$1/2$	$\exp(-\ln 2\,\eta^2)$
Plane mixing layer	$U_2 - U_1$	$y_{0.9} - y_{0.1}$	0	1	$1/2\,\mathrm{erf}(\eta/\sigma\sqrt{2})$

Table 1: Self-similar solution table.

The example of plane jet is the easiest to understand and derive so we are the most detailed in that case and more loosely on the others. The same principles and machinery apply to all cases.

3.5.3 Round jet

Characteristic scales:
The centerline velocity is

$$U_s(x) = \bar{u}(x, r = 0) \tag{3.124}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = \bar{u}(x, r_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{3.125}$$

Momentum integral constraint:
The boundary layer equation in cylindrical coordinates reads

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{r}\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{3.126}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial (r\bar{v})}{\partial r} = 0 \tag{3.127}$$

by $r\bar{u}$ and add it to (3.126) multiplied by r we obtain

$$\frac{\partial (r\bar{u}\bar{u})}{\partial x} + \frac{\partial (r\bar{u}\bar{v})}{\partial r} = -\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{3.128}$$

Integrate (3.128) in r we obtain

$$\int_0^\infty \frac{\partial (r\bar{u}\bar{u})}{\partial r}\,\mathrm{d}r + r\bar{u}\bar{v}|_0^\infty = -r\overline{u'v'}|_0^\infty \tag{3.129}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^\infty r\bar{u}^2\,\mathrm{d}r\right) = 0 \tag{3.130}$$

which implies the momentum flux

$$\dot{M}(x) = \int_0^\infty \rho\bar{u}^2 2\pi r\,\mathrm{d}r = J_0 \tag{3.131}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.
Self-similar assumptions:

$$\bar{u} = U_s(x)f(\eta),\ \overline{u'v'} = U_s^2(x)g(\eta) \tag{3.132}$$

where $\eta = r/\delta_s(x)$ with $\delta_s = r_{1/2}$. Substitute (3.132) into (3.131) we have

$$\dot{M}(x) = (2\pi\rho)(U_s^2\delta_s^2)\left(\int_0^\infty \eta f^2(\eta)\,\mathrm{d}\eta\right) \tag{3.133}$$

to be a constant and implying

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2\delta_s^2) = 0 \tag{3.134}$$

and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{3.135}$$

Using the continuity equation we have

$$\bar{v} = -\frac{1}{r}\int_0^r \frac{\partial(r\bar{u})}{\partial x}\,\mathrm{d}y = U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{\eta}\int_0^\eta f\eta\,\mathrm{d}\eta\right) \tag{3.136}$$

We note that $\bar{v}$ switch sign from positive to negative when r is greater than a certain value (entrainment).

Next we establish the constant spread rate of the round jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left[f^2\eta + ff'\eta + \left(\frac{f}{\eta}+f'\right)\int_0^\eta f\eta\,\mathrm{d}\eta\right] = g + g'\eta \tag{3.137}$$

and then $\mathrm{d}\delta_s/\mathrm{d}x$ has to be a constant. Combining with momentum integral restriction we have

$$\delta_s \propto x,\, U_s \propto x^{-1}. \tag{3.138}$$

3.5.4 Plane jet

Characteristic scales:
The centerline velocity is

$$U_s(x) = \bar{u}(x, y=0) \tag{3.139}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = \bar{u}(x, y_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{3.140}$$

Momentum integral constraint:
The boundary layer equation for the mean velocity simplifies to

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} = -\frac{\partial\overline{u'v'}}{\partial y}. \tag{3.141}$$

Multiply the continuity equation

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} = 0 \tag{3.142}$$

by $\bar{u}$ and add it to (3.141) we obtain

$$\frac{\partial\bar{u}\bar{u}}{\partial x} + \frac{\partial\bar{u}\bar{v}}{\partial y} = -\frac{\partial\overline{u'v'}}{\partial y}. \tag{3.143}$$

Integrate (3.143) in y we obtain

$$\int_{-\infty}^{\infty}\frac{\partial\bar{u}\bar{u}}{\partial x}\,\mathrm{d}y + \bar{u}\bar{v}|_{-\infty}^{\infty} = -\overline{u'v'}|_{-\infty}^{\infty} \tag{3.144}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty}\bar{u}^2\,\mathrm{d}y\right) = 0 \tag{3.145}$$

which implies the momentum flux

$$\dot{M}(x) = \int_{-\infty}^{\infty}\rho\bar{u}^2\,\mathrm{d}y = J_0 \tag{3.146}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.

Self-similar assumptions:

$$\bar{u} = U_s(x) f(\eta),\ \overline{u'v'} = U_s^2(x) g(\eta) \tag{3.147}$$

where $\eta = y/\delta_s(x)$ and we have

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{\delta_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{3.148}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta_s} \tag{3.149}$$

Substitute (3.147) into (3.146) we have

$$\dot{M}(x) = (U_s^2 \delta_s)\left(\int_{-\infty}^{\infty} f^2(\eta)\,\mathrm{d}\eta\right) \tag{3.150}$$

is a constant. So it must be

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2\delta_s) = 0 \tag{3.151}$$

which gives the momentum flux constraint in terms of characteristic variables, and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{3.152}$$

Using the continuity equation we have

$$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x}\,\mathrm{d}y = U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{2}\int_0^\eta f\,\mathrm{d}\eta\right) \tag{3.153}$$

Next we establish the constant spread rate of the plane jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(f^2 + f'\int_0^\eta f\,\mathrm{d}\eta) = g' \tag{3.154}$$

and then

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x} = \frac{2g'}{f^2 + f'\int_0^\eta f\,\mathrm{d}\eta} = C \tag{3.155}$$

with the LHS only depend on x and RHS only depend on η. Then both sides have to be constant. Combining (3.155) and (3.151) we have

$$\delta_s \propto x,\ U_s \propto x^{-1/2}. \tag{3.156}$$

3.5.5 Round wake

Characteristic scales:
The centerline velocity deficit is

$$U_0(x) = U_\infty - \bar{u}(x, r=0) = U_d(x, 0) \tag{3.157}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = U_\infty - \bar{u}(x, r_{1/2}(x)) = \frac{1}{2}U_0(x). \tag{3.158}$$

Momentum integral constraint:
Here we start from the simplified (see plane wake) momentum equation

$$U_\infty \frac{\partial \bar{u}}{\partial x} = -\frac{1}{r}\frac{\partial (r\overline{u'v'})}{\partial r} \tag{3.159}$$

and the momentum deficit flux conservation

$$\dot{M}(x) = \int_0^\infty \rho U_\infty (U_\infty - \bar{u}) 2\pi r \, \mathrm{d}r. \tag{3.160}$$

Note that we have already replaced the $\bar{u}$ with U_∞ assuming (or by order of magnitude analysis) the convection velocity is U_∞.
Self-similar assumptions:

$$U_\infty - \bar{u} = U_s(x) f(\eta), \; \overline{u'v'} = U_s^2(x) g(\eta) \tag{3.161}$$

We have

$$\dot{M}(x) = (U_s \delta_s^2)(2\pi\rho U_\infty) \int_0^\eta f \, \mathrm{d}\eta \tag{3.162}$$

is a constant and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s \delta_s^2) = 0. \tag{3.163}$$

Consider the momentum equation, the other constraint reads

$$-\frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(2f + f'\eta)\eta = (g'\eta + g) \tag{3.164}$$

We define the spread rate as

$$S = \frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}, \tag{3.165}$$

it has to be a constant. Then

$$-S(2f\eta + f'\eta^2) = (g\eta)' \tag{3.166}$$

and including boundary conditions after integration we get

$$g = -S\eta f \tag{3.167}$$

same as in plane wakes. Combining (3.163) and (3.165) we have

$$\delta_s \propto x^{1/3}, \; U_s \propto x^{-2/3}. \tag{3.168}$$

3.5.6 Plane wake

Characteristic scales:
The centerline velocity deficit is

$$U_s(x) = U_\infty - \bar{u}(x, y = 0) = U_d(x, 0) \tag{3.169}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = U_\infty - \bar{u}(x, y_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{3.170}$$

Momentum integral constraint:
The boundary layer equation:

$$\bar{u}\frac{\partial(\bar{u}-U_\infty)}{\partial x}+\bar{v}\frac{\partial(\bar{u}-U_\infty)}{\partial y}=\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}=-\frac{\partial\overline{u'v'}}{\partial y}. \tag{3.171}$$

Multiply the continuity equation

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}=0 \tag{3.172}$$

by $\bar{u}-U_\infty$ and add it to (3.171) we obtain

$$\frac{\partial\bar{u}(\bar{u}-U_\infty)}{\partial x}+\frac{\partial\bar{v}(\bar{u}-U_\infty)}{\partial y}=-\frac{\partial\overline{u'v'}}{\partial y}. \tag{3.173}$$

Integrate (3.143) in y we obtain

$$\int_{-\infty}^{\infty}\frac{\partial\bar{u}(\bar{u}-U_\infty)}{\partial x}\,\mathrm{d}y+\bar{v}(\bar{u}-U_\infty)|_{-\infty}^{\infty}=-\overline{u'v'}|_{-\infty}^{\infty} \tag{3.174}$$

and since $\overline{u'v'}$ and $\bar{u}-U_\infty$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty}\bar{u}(\bar{u}-U_\infty)\,\mathrm{d}y\right)=0 \tag{3.175}$$

which implies the momentum deficit flux

$$\dot{M}(x)=\int_{-\infty}^{\infty}\rho\bar{u}(U_\infty-\bar{u})\,\mathrm{d}y \tag{3.176}$$

is conserved (we note that we haven't assumed far wake yet).
Self-similar assumptions:

$$U_\infty-\bar{u}=U_s(x)f(\eta),\ \overline{u'v'}=U_s^2(x)g(\eta) \tag{3.177}$$

Substitute (3.177) into (3.176), and assume the far wake is reached ($U_s/U_\infty \ll 1$) we have

$$\dot{M}(x)=\int_{-\infty}^{\infty}\rho(U_\infty-U_sf)U_sf\delta_s\,\mathrm{d}\eta \tag{3.178}$$

$$=U_\infty^2\int_{-\infty}^{\infty}\rho(1-\frac{U_sf}{U_\infty})\frac{U_s}{U_\infty}f\delta_s\,\mathrm{d}\eta \tag{3.179}$$

$$=\rho U_\infty U_s\delta_s\int_{-\infty}^{\infty}f\,\mathrm{d}\eta \tag{3.180}$$

is a constant. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s\delta_s)=0. \tag{3.181}$$

Using the continuity equation we have

$$\bar{v}=-\int_0^y\frac{\partial\bar{u}}{\partial x}\,\mathrm{d}y=-U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}f\eta. \tag{3.182}$$

Note the negative speed corresponding to wake entrainment (of high momentum into low momentum region).

Now we consider another constraint. Since in the far wake, the velocity deficit $U_s/U_\infty \ll 1$, we have the simplification of the momentum equation as

$$\frac{\partial \bar{u}(\bar{u}-U_\infty)}{\partial x} + \frac{\partial \bar{v}(\bar{u}-U_\infty)}{\partial y} = U_\infty \frac{\partial \bar{u}}{\partial x} = -\frac{\partial \overline{u'v'}}{\partial y} \tag{3.183}$$

where

$$\bar{u}(\bar{u}-U_\infty) = (U_\infty - U_s f)(-U_s f) = U_\infty^2(1-\frac{U_s f}{U_\infty})(-\frac{U_s}{U_\infty} f) = -U_s U_\infty f = U_\infty(\bar{u}-U_\infty). \tag{3.184}$$

And the scale for $\partial \bar{u}(\bar{u}-U_\infty)/\partial x$ is

$$\frac{U_\infty U_s}{L_x} \tag{3.185}$$

while the scale for $\partial \bar{v}(\bar{u}-U_\infty)/\partial y$ (from (3.182)) is

$$\frac{U_s}{\delta_s}\left(U_s \frac{\delta_s}{L_x}\right). \tag{3.186}$$

Define the spread rate as

$$S = \frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{3.187}$$

Take $\bar{v}$ into the simplified momentum equation we have

$$(f + f'\eta)\frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} = -g' \tag{3.188}$$

with S depends only on x and the rest on η hence S has to be a constant. Then (3.188) can be rewritten as

$$g' + S(f + f'\eta) = 0 \tag{3.189}$$

which is to say

$$(g + S\eta f)' = 0. \tag{3.190}$$

Integrate from $\eta = 0$ to η and note that $g(0) = 0$, we have

$$g = -S\eta f. \tag{3.191}$$

Combining two conditions (3.181) and (3.187) we have

$$\delta_s \propto x^{1/2},\ U_s \propto x^{-1/2}. \tag{3.192}$$

3.5.7 Plane mixing layer

Characteristic scales:
The two velocities are $U_2 > U_1$ with U_2 on the top. The mean convection velocity is

$$U_c = \frac{1}{2}(U_1 + U_2) \tag{3.193}$$

and the characteristic velocity scale is

$$U_s = U_2 - U_1. \tag{3.194}$$

The characteristic length is the mixing layer width,

$$\delta_s(x) = y_{0.9} - y_{0.1} \tag{3.195}$$

with cross-stream location $y_\alpha(x)$ such that

$$\bar{u}(x, y_\alpha(x)) = U_1 + \alpha U_s. \tag{3.196}$$

a reference position is

$$\hat{y} = \frac{1}{2}(y_{0.1} + y_{0.9}) \tag{3.197}$$

such that the self-similar variable is defined as

$$\eta = \frac{y - \hat{y}}{\delta_s(x)} \tag{3.198}$$

3.6 Wall flows

3.6.1 von Kármán momentum integral

3.6.2 Blasius similarity solution

The references are Schlichting & Gersten (2016); Kundu *et al.* (2015) with the definition of $\delta(x)$ different by a factor of $\sqrt{2}$. Here we will follow the definition in Schlichting & Gersten (2016).

The boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.199}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial x^2} \tag{3.200}$$

The idea of self-similar solutions is that the velocity profile $u(y)$ will be the same under some proper transformation/normalization of u and y. The scale for u is apparently U_∞, while the scale for y is δ. From the viscous scaling $v \sim \nu/\delta$ and the scaling of the continuity equation $v/\delta \sim U_\infty/x$ we have

$$\delta^2 \sim \frac{\nu x}{U_\infty} \tag{3.201}$$

and for the sake of simplification of the final result (ODE) we define

$$\delta(x) = \sqrt{\frac{2x\nu}{U_\infty}} \tag{3.202}$$

such that the similarity transformation is

$$\eta = \frac{y}{\delta(x)} \tag{3.203}$$

such that

$$\frac{u}{U_\infty} = f(\eta) \tag{3.204}$$

where $f(\eta)$ is the similarity function and η is the similarity coordinate.

We note that the streamfunction ψ depends on ν, U_∞, x, y and dimensionally

$$\psi(x, y) = U_\infty \delta(s) f(\eta) = \sqrt{2\nu U_\infty x} f(\eta) \tag{3.205}$$

and hence

$$u = U_\infty f' \tag{3.206}$$

$$v = \sqrt{\frac{U_\infty \nu}{2x}}(\eta f' - f) \tag{3.207}$$

The derivatives are

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2x} f'' \eta \tag{3.208}$$

$$\frac{\partial u}{\partial y} = U_\infty f'' \sqrt{\frac{U_\infty}{2\nu x}} \tag{3.209}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{2\nu x} f''' \tag{3.210}$$

and then

$$u\frac{\partial u}{\partial x} = -\frac{U_\infty^2}{2x} f' f'' \eta \tag{3.211}$$

$$v\frac{\partial u}{\partial y} = \frac{U_\infty^2}{2x} f''(\eta f' - f) \tag{3.212}$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{2x} f''' \tag{3.213}$$

and finally we have the ODE

$$f f'' + f''' = 0 \tag{3.214}$$

with the boundary conditions being

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1, \tag{3.215}$$

corresponding to

$$v(y = 0) = 0,\ u(y = 0) = 0,\ u(y = \infty) = U_\infty. \tag{3.216}$$

It is common to use a Runge-Kutta shooting method to solve (3.215).

3.6.3 Turbulent channel flow

channel basic equation; FIK;

4 Geophysical fluid dynamics

4.1 Basics

4.1.1 Centrifugal and Coriolis forces

4.1.2 Inertial oscillations: buoyancy and Coriolis frequencies

4.2 Boussinesq approximation

4.3 Hydrostatic and geostrophic balances

In balanced flow, there is a background horizontal pressure gradient that balances the Coriolis forces due to horizontal motions and a vertical pressure gradient that balances the background unperturbed density:

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x} + f_c V \tag{4.1}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y} - f_c U \tag{4.2}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial z} - \frac{\rho^* g}{\rho_0} \tag{4.3}$$

with

$$p_g = p - p_0,\ \rho^* = \rho - \rho_0 - \rho_b(z) \tag{4.4}$$

and the background balance

$$0 = -\frac{\partial p_0}{\partial z} - (\rho_0 + \rho_b)g \tag{4.5}$$

already subtracted. Note that the Boussinesq and hydrostatic approximations are already applied.

The above equations in vector form:

$$\boldsymbol{f}_c \times \boldsymbol{U} = -\frac{1}{\rho_0}\nabla p_g + \frac{\rho^*}{\rho_0}\boldsymbol{g}. \tag{4.6}$$

We have

$$\boldsymbol{U} = (U, V, 0) = -\frac{1}{\rho_0 f_c}\left(\frac{\partial p_g}{\partial y}, -\frac{\partial p_g}{\partial x}, 0\right). \tag{4.7}$$

And we have

$$\nabla_h \cdot \boldsymbol{U} = 0. \tag{4.8}$$

4.3.1 Thermal wind relations

In hydrostatic Boussinesq flow. Taking the vertical gradient of (4.6) and using the hydrostatic balance, we have

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial x} + f_c\frac{\partial V}{\partial z} \tag{4.9}$$

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial y} - f_c\frac{\partial U}{\partial z} \tag{4.10}$$

and hence

$$\left(\frac{\partial U}{\partial z}, \frac{\partial V}{\partial z}\right) = \frac{g}{\rho_0 f_c}\left(\frac{\partial \rho^*}{\partial y}, -\frac{\partial \rho^*}{\partial x}\right), \tag{4.11}$$

or in vector form,

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f_c} \times \nabla \rho^* \tag{4.12}$$

In a more general case. Without introducing the hydrostatic balance and Boussinesq approximation, we write

$$\boldsymbol{f}_c \times \boldsymbol{U} = \frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{4.13}$$

Taking its curl:

$$\text{LHS} = \nabla \times (\boldsymbol{f}_c \times \boldsymbol{U}) = -\boldsymbol{f}_c \cdot \nabla \boldsymbol{U} = -f_{\mathrm{c}}\frac{\partial \boldsymbol{U}}{\partial z} \tag{4.14}$$

$$\text{RHS} = -\nabla \times (\frac{1}{\rho}\nabla p) + \nabla \times \boldsymbol{g} = -\frac{1}{\rho^2}(\nabla p \times \nabla \rho) \tag{4.15}$$

Re-introduce hydrostatic ($\partial_z p = -\rho g$) and Boussinesq, we have

$$\nabla p \approx \frac{\partial p}{\partial z}\hat{\boldsymbol{e}}_z = -\rho g \hat{\boldsymbol{e}}_z \tag{4.16}$$

and

$$-\frac{1}{\rho^2}(\nabla p \times \nabla \rho) = \frac{\rho g}{\rho_0^2}(\hat{\boldsymbol{e}}_z \times \nabla \rho^*) \tag{4.17}$$

$$= -\frac{\boldsymbol{g}}{\rho_0} \times \nabla \rho^* \tag{4.18}$$

hence we recover

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f_{\mathrm{c}}} \times \nabla \rho^*. \tag{4.19}$$

Note:

4.4 Governing equations of unbalanced motions

It is reasonable to assume directions of both system rotation and gravity are in $\hat{\boldsymbol{z}}$.

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{4.20}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} - f_{\mathrm{c}}\epsilon_{ij3}(u_j - U_j) = -\frac{1}{\rho_0}\frac{\partial p^*}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\rho^* g}{\rho_0}\delta_{i3}, \tag{4.21}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i}, \tag{4.22}$$

$$\tau_{ij} = \nu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),\ J_{\rho,i} = \kappa\frac{\partial \rho}{\partial x_i}. \tag{4.23}$$

In vector form,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{4.24}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) + f_{\mathrm{c}}\hat{\boldsymbol{e}}_z \times (\boldsymbol{u} - \boldsymbol{U}) = -\frac{1}{\rho_0}\nabla p^* + \nabla \cdot \boldsymbol{\tau} - \frac{\rho^* g}{\rho_0}\hat{\boldsymbol{e}}_z \tag{4.25}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = \nabla \cdot \boldsymbol{J}_\rho \tag{4.26}$$

where the stress and the scalar flux are

$$\boldsymbol{\tau} = \nu(\nabla \boldsymbol{u} + \boldsymbol{u}\nabla),\ \boldsymbol{J}_\rho = \kappa \nabla \rho. \tag{4.27}$$

The total density ρ is decomposed into the reference density ρ_0, the background density $\rho_b(z)$, and the density perturbation ρ^* due to fluid motion,

$$\rho(x, y, z, t) = \rho_0 + \rho_b(z) + \rho^*(x, y, z, t). \tag{4.28}$$

The total pressure is written as

$$p(x, y, z, t) = p_0 + p_g(x, y) + p_a(z) + p^*(x, y, z, t), \tag{4.29}$$

where the reference pressure p_0 is a constant, the hydrostatic (ambient) pressure p_a has a vertical gradient that balances the ambient density ($\rho_a = \rho_0 + \rho_b(z)$), and the geostrophic pressure p_g has a transverse gradient that balances the Coriolis force due to the geostrophic wind $\boldsymbol{U}$. Only the dynamic pressure p^* appears in the momentum equation (4.21).

Instead of using ρ^*, it is also common to express the buoyancy term as

$$b = -\frac{\rho^* g}{\rho_0}, \tag{4.30}$$

and the 'total' buoyancy

$$\tilde{b} = b + \bar{b} = -\frac{(\rho^* + \bar{\rho}(z))g}{\rho_0}, \tag{4.31}$$

where the background linear stratification is $N^2 = \partial \bar{b}/\partial z$ and we have $\tilde{b} = b + N^2 z$ with the reference value $\bar{\rho}(z = 0) = 0$.

Eqn, (4.22) can also be expressed as

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* u_i}{\partial x_i} + w\frac{\partial \bar{\rho}}{\partial z} = \kappa \frac{\partial^2 \rho^*}{\partial x_i^2}, \tag{4.32}$$

and hence we have the buoyancy equation

$$\frac{\partial b}{\partial t} + \frac{\partial b u_i}{\partial x_i} + wN^2 = \kappa \frac{\partial^2 b}{\partial x_i^2}, \tag{4.33}$$

and the equation for the total buoyancy is

$$\frac{\partial \tilde{b}}{\partial t} + \frac{\partial \tilde{b} u_i}{\partial x_i} = \kappa \frac{\partial^2 \tilde{b}}{\partial x_i^2}. \tag{4.34}$$

4.4.1 Incompressibility

Evan though there is a density transport due to the diffusion (due to the special role that ρ plays; this is not the mass conservation equation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i} \neq 0, \tag{4.35}$$

we could still establish incompressible condition

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{4.36}$$

with some additional assumtions.

First we review the integration form of the general conservation equation for an arbitraty scalar (per unit mass)

$$\frac{\partial}{\partial t}\left(\iiint_V \rho\psi \,\mathrm{d}V\right) = -\iint_{\Omega=\partial V} (\rho \boldsymbol{u}\psi) \cdot \,\mathrm{d}\boldsymbol{A} - \iint_{\Omega=\partial V} \rho\kappa(-\nabla\psi) \cdot \,\mathrm{d}\boldsymbol{A} \tag{4.37}$$

$$= -\iiint_V \nabla\cdot(\rho\boldsymbol{u}\psi)\,\mathrm{d}V - \iiint_V \nabla\cdot(\rho\kappa(-\nabla\psi))\,\mathrm{d}V \tag{4.38}$$

and we have

$$\frac{\partial\rho\psi}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u}) = \nabla\cdot(\rho\kappa\nabla\psi). \tag{4.39}$$

It is in a general form of a conservational principle

$$\frac{\partial Q_v}{\partial t} + \nabla\cdot\boldsymbol{F} = 0, \tag{4.40}$$

where $\boldsymbol{F}$ is the flux and $\nabla\cdot\boldsymbol{F}$ is the transport term.

Taking $\psi = 1$ we recover the mass conservation equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = \frac{D\rho}{Dt} + \rho(\nabla\cdot\boldsymbol{u}) = 0. \tag{4.41}$$

Usually, density change along the material lines, is small enough such that $(1/\rho)D\rho/Dt \ll U/L$ and hence $\nabla\cdot\boldsymbol{u} \ll U/L$. That being said, being non-dimensionalised, the velocity field is solenoidal. We note in density-variable flows that

$$\nabla\cdot\boldsymbol{u} = 0 \tag{4.42}$$

is an approximation. See Batchelor (1967), section 3.2, for details as why this is valid.

4.4.2 Scalar transport equation

Taking $\psi = s$ (salinity or temperature) and assume diffusivity κ is constant, we have the scalar transport equation

$$s\frac{\partial\rho}{\partial t} + \rho\frac{\partial s}{\partial t} + u_j\rho\frac{\partial s}{\partial x_j} + u_j s\frac{\partial\rho}{\partial x_j} = \kappa\frac{\partial s}{\partial x_j}\frac{\partial\rho}{\partial x_j} + \rho\kappa\frac{\partial^2 s}{\partial x_j^2}, \tag{4.43}$$

taking into account

$$\frac{\partial\rho}{\partial t} + u_j\frac{\partial\rho}{\partial x_j} = 0 \tag{4.44}$$

we have

$$\rho\frac{\partial s}{\partial t} + u_j\rho\frac{\partial s}{\partial x_j} = \kappa\frac{\partial s}{\partial x_j}\frac{\partial\rho^*}{\partial x_j} + \rho\kappa\frac{\partial^2 s}{\partial x_j^2}. \tag{4.45}$$

We note that under Boussinesq assumption, $\rho^*/\rho = \rho^*/(\rho_0 + \rho*) \ll 1$, we have

$$\frac{\partial s}{\partial t} + u_j\frac{\partial s}{\partial x_j} = \kappa\frac{\partial^2 s}{\partial x_j^2}. \tag{4.46}$$

With some linear equation of state, we can relate s or T to ρ and get a scalar transport equation for ρ as

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u_j\frac{\partial\rho}{\partial x_j} = \kappa\frac{\partial^2\rho}{\partial x_j^2}, \tag{4.47}$$

with the incompressibility being implied by $\nabla\cdot\boldsymbol{u} = 0$ from Eqn. (4.41). We note that Eqns. (4.41) and (4.47) correspond to two different physical principles.

E.g.,

$$p = \rho RT \tag{4.48}$$

$$\ln p = ln(\rho) + \ln T + \ln R \tag{4.49}$$

$$\frac{\delta p}{p} = \frac{\delta\rho}{\rho} + \frac{\delta T}{T}. \tag{4.50}$$

Assuming isobaric process we have

$$\frac{\partial \rho}{\partial z} \propto -\frac{\partial T}{\partial z} \tag{4.51}$$

and $b = -(g/\rho_0)\partial\rho^*/\partial z = \partial T^*/\partial z$. It is also possible to solve or interpret as the temperature equation (4.46) is being solved and density will be obtain using an equation-of-state such as

$$\frac{\rho - \rho_0}{\rho_0} = -\beta(T - T_0),\ \beta = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p \tag{4.52}$$

where β is call the themal expansion coefficient. Then

$$\frac{\partial \rho}{\partial z} = -\beta\frac{\partial T}{\partial z} \tag{4.53}$$

and we can define

$$b = \frac{gT^*}{T_0} \tag{4.54}$$

$$N^2 = \frac{g}{T_0}\frac{\partial \bar{T}}{\partial z} \tag{4.55}$$

4.5 GFD vorticity equations

4.5.1 Absolute vorticity equation

The ‘absolute’ vorticity, defined as $\boldsymbol{\omega}_a = \boldsymbol{\omega} + \boldsymbol{f}_{\mathrm{c}}$, is the ‘relative’ vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ plus the ‘planetary’ vorticity $\boldsymbol{f}_{\mathrm{c}} = f_{\mathrm{c}}\hat{\boldsymbol{e}}_z$ ($f_{\mathrm{c}} = 2\Omega\sin\phi$).

Similar to Eq. (2.11), we can derive the governing equation for $\boldsymbol{\omega}_a$ starting from

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}_{\mathrm{c}} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F} \tag{4.56}$$

without the hydrostatic part separated and/or Boussinesq assumed.

According to identity (A.37) we have, in an f-plane,

$$\nabla \times (\boldsymbol{f}_{\mathrm{c}} \times \boldsymbol{u}) = -\boldsymbol{f}_{\mathrm{c}} \times \nabla\boldsymbol{u}. \tag{4.57}$$

Similar to Eq. (2.4), by taking the curl of (4.56) and taking $\boldsymbol{F} = b\hat{\boldsymbol{e}}_z$ we have

$$\frac{D\boldsymbol{\omega}_a}{Dt} = \frac{\partial \boldsymbol{\omega}_a}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega}_a = \boldsymbol{\omega}_a\cdot\nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}_a + \nabla\times\boldsymbol{F} + \frac{1}{\rho^2}\nabla\rho\times\nabla p \tag{4.58}$$

Although the derivation and properties can be studied under various assumptions, we provide here a more common version.

4.5.2 Potential vorticity equation

Pedlosky (2013) is a good reference for this section.

Assume a scalar λ with a governing operator $D\lambda/Dt = \mathrm{RHS}$. Consider

$$\frac{D}{Dt}\left(\frac{\partial \lambda}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial t} + u_j\frac{\partial \lambda}{\partial x_j}\right) - \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j}, \tag{4.59}$$

i.e.,

$$\frac{D}{Dt}(\nabla\lambda) = \nabla\left(\frac{D\lambda}{Dt}\right) - \nabla\boldsymbol{u}\cdot\nabla\lambda. \tag{4.60}$$

$\nabla\lambda\cdot$ (4.58) + $\boldsymbol{\omega}_a\cdot$ (4.60), with a magic that two opposite-sign $\boldsymbol{\omega}_a\cdot\nabla\boldsymbol{u}\cdot\nabla\lambda$ terms cancel, we have

$$\frac{D}{Dt}\left(\boldsymbol{\omega}_a\cdot\nabla\lambda\right)=\boldsymbol{\omega}_a\cdot\nabla\left(\frac{D\lambda}{Dt}\right)+\nu\nabla^2\boldsymbol{\omega}_a\cdot\nabla\lambda+(\nabla\times\boldsymbol{F})\cdot(\nabla\lambda)+\frac{1}{\rho^2}\nabla\rho\times\nabla p\cdot(\nabla\lambda) \tag{4.61}$$

Take $\lambda=\tilde{b}$ which is the total buoyancy, with its governing equation being (4.34), assuming conservative external force $\boldsymbol{F}$ and barotropic flow, we have the potential vorticity (PV) equation:

$$\frac{D}{Dt}\left(\boldsymbol{\omega}_a\cdot\nabla\tilde{b}\right)=\nu\nabla^2\boldsymbol{\omega}_a\cdot\nabla\tilde{b}+\kappa[\nabla^2(\nabla\tilde{b})]\cdot\boldsymbol{\omega}_a, \tag{4.62}$$

where

$$\Pi=\boldsymbol{\omega}_a\cdot\nabla\tilde{b} \tag{4.63}$$

is called the potential vorticity, which is the component of the absolute vorticity perpendicular to the isosurface (or parallel to the gradient) of $\tilde{b}$. In the absence of dissipation,

$$\frac{D\Pi}{Dt}=0, \tag{4.64}$$

i.e., PV is conserved along the streamlines. Eq. (4.62) is like a double-diffusion problem with one 'passive' scalar diffuse together with vorticity.

Assumptions made for the PV conservation theorem (from Eq. (4.61)):

- Conservative external force: $\nabla\times\boldsymbol{F}=0$.
- Baroclinicity absent: $\nabla\rho\times\nabla p=0$.
- Diffusion-less: $\nu=\kappa=0$.

4.6 Turbulence equations for an active scalar

4.6.1 Mean flow equations

$$\frac{\partial\bar{u}_i}{\partial x_i}=0 \tag{4.65}$$

$$\frac{\partial\bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}-f_c\epsilon_{ij3}(\bar{u}_j-U_j)=-\frac{1}{\rho_0}\frac{\partial\overline{p^*}}{\partial x_i}+\frac{\partial}{\partial x_j}(\nu\frac{\partial\bar{u}_i}{\partial x_j}-\overline{u_i'u_j'})-\frac{\overline{\rho^*}g}{\rho_0}\delta_{i3} \tag{4.66}$$

$$\frac{\partial\bar{\rho}}{\partial t}+\bar{u}_j\frac{\partial\bar{\rho}}{\partial x_j}=\frac{\partial}{\partial x_i}\left(\kappa\frac{\partial\bar{\rho}}{\partial x_j}-\overline{\rho'u_j'}\right), \tag{4.67}$$

We note that

$$\rho'=\rho-\bar{\rho}=\rho^*-\overline{\rho^*}=\rho^{*\prime}. \tag{4.68}$$

4.6.2 Fluctuation equations

$$\frac{\partial u_i'}{\partial x_i}=0 \tag{4.69}$$

$$\frac{\partial u_i'}{\partial t}+\bar{u}_j\frac{\partial u_i'}{\partial x_j}-f_c\epsilon_{ij3}u_j'=-\frac{1}{\rho}\frac{\partial p^{*\prime}}{\partial x_i}+\frac{\partial}{\partial x_j}(\nu\frac{\partial\bar{u}_i}{\partial x_j}+\overline{u_i'u_j'}-u_i'u_j')-u_j'\frac{\partial\bar{u}_i}{\partial x_j}-\frac{\rho^{*\prime}g}{\rho_0}\delta_{i3} \tag{4.70}$$

$$\frac{\partial\rho^{*\prime}}{\partial t}+\bar{u}_j\frac{\partial\rho^{*\prime}}{\partial x_j}=\frac{\partial}{\partial x_i}\left(\kappa\frac{\partial\rho^{*\prime}}{\partial x_j}+\overline{\rho^{*\prime}u_j'}-\rho^{*\prime}u_j'\right)-\rho^{*\prime}\frac{\partial\overline{p^*}}{\partial x_i} \tag{4.71}$$

We will see later the Coriolis term won't appear in the transport equations of MKE, TKE, and Reynolds stresses. Coriolis just bends the direction of the velocity.

4.6.3 MKE, MPE, TKE, TPE, and buoyancy flux equations

Define the mean and turbulent kinetic and potential energy as

$$K = \frac{1}{2}\bar{u}_i\bar{u}_i \tag{4.72}$$
$$K_\rho = \frac{1}{2}\bar{b}^2 \tag{4.73}$$

and

$$k = \frac{1}{2}\overline{u_i'u_i'} \tag{4.74}$$
$$k_\rho = \frac{1}{2}\overline{b'b'} \tag{4.75}$$

where the instantaneous, mean, and fluctuation buoyancy forces are

$$b = -\frac{\rho^* g}{\rho_0},\ \bar{b} = -\frac{\overline{\rho^*} g}{\rho_0},\ b' = -\frac{{\rho^*}' g}{\rho_0}, \tag{4.76}$$

such that k and k_ρ have the same dimension as the kinetic energy.

The **MKE equations** is (repeating (3.32)) :

$$\frac{\partial K}{\partial t} + \bar{u}_j\frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho}\bar{p}\,\bar{u}_j + \nu\frac{\partial K}{\partial x_j} - \bar{u}_i\,\overline{u_i'u_j'}\right) + \overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \nu\frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j} \tag{4.77}$$

The **MPE equations** is:

$$\frac{\partial K_\rho}{\partial t} + \bar{u}_j\frac{\partial K_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}(\kappa\frac{\partial K_\rho}{\partial x_j} - \bar{b}\,\overline{b'u_j'}) + \overline{b'u_j'}\frac{\partial \bar{b}}{\partial x_j} - \kappa\frac{\partial \bar{b}}{\partial x_j}\frac{\partial \bar{b}}{\partial x_j} \tag{4.78}$$

We note that the buoyancy flux $\overline{b'u_j'}\partial\bar{b}/\partial x_j$ is a sink in the MPE equation and is a source in the TPE equation.

The **TKE equations** is:

$$\frac{\partial k}{\partial t} + \bar{u}_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_k}(\underbrace{\nu\frac{\partial k}{\partial x_k}}_{\text{molecular diffusion}} \underbrace{+\frac{1}{2}\overline{u_i'u_i'u_k'}}_{\text{turbulent diffusion}} \underbrace{-\frac{1}{\rho_0}\overline{p'u_k'}}_{\text{pressure distortion}}) \underbrace{-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}}_{\text{production } P} - \underbrace{\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}}_{\text{dissipation } \varepsilon} + \underbrace{\overline{b'w'}}_{\text{buoyancy flux } B} \tag{4.79}$$
$$= \nabla\cdot\boldsymbol{T} + P - \varepsilon + B \tag{4.80}$$

where the turbulent buoyancy flux

$$B = -\frac{g}{\rho_0}\overline{{\rho^*}'w'} = \overline{b'w'} \tag{4.81}$$

consumes TKE and lead to the production of TPE.

The **TPE equation** is:

$$\frac{\partial k_\rho}{\partial t} + \bar{u}_j\frac{\partial k_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}(\kappa\frac{\partial k_\rho}{\partial x_j} - \frac{1}{2}\overline{b'b'u_j'}) - \overline{b'u_j'}\frac{\partial \bar{b}}{\partial x_j} - \kappa\overline{\frac{\partial b'}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{4.82}$$

We can see that the turbulent buoyancy flux B (negative, think $-\overline{u_i'u_j'}$) works with the density distortion $\partial\bar{b}/\partial z$ to remove energy from TKE and MPE to produce TPE.

The **buoyancy flux equation** is:

$$\frac{\partial \overline{b'u_i'}}{\partial t} + \bar{u}_j\frac{\partial \overline{b'u_i'}}{\partial x_j} = d_{b,i} + P_{b,i} + \Phi_{b,i} - \varepsilon_{b,i} \tag{4.83}$$

where

$$d_{b,i} = \frac{\partial}{\partial x_j}(\kappa \overline{u_i' \frac{\partial b'}{\partial x_j}} + \nu \overline{b' \frac{\partial u_i'}{\partial x_j}} - \frac{1}{\rho_0}\overline{p'b'}\delta_{ij} - \overline{b'u_i'u_j'}) \tag{4.84}$$

$$P_{b,i} = -\overline{b'u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \bar{b}}{\partial x_j} \tag{4.85}$$

$$\Phi_{b,i} = \frac{1}{\rho_0}\overline{p'\frac{\partial b'}{\partial x_i}} \tag{4.86}$$

$$\varepsilon_{b,i} = (\nu + \kappa)\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{4.87}$$

4.7 Inertial and buoyancy oscillations

4.7.1 Derivation of Coriolis force

4.7.2 Boussinesq approximation

4.8 Surface and bottom Ekman layer solutions

4.9 Miscellaneous

coriolis frequency; shallow water / wave equations; igw equations;

5 Hydrodynamic stability

5.1 Linearised Navier-Stokes

Consider the incompressible N-S equations

$$\nabla \cdot \boldsymbol{u} = 0 \tag{5.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{5.2}$$

and the decomposition of velocity and pressure into the base and perturbation states:

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}' \tag{5.3}$$

$$p = P + p' \tag{5.4}$$

We note that the base state also satisfies N-S:

$$\nabla \cdot \boldsymbol{U} = 0 \tag{5.5}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \boldsymbol{U} \tag{5.6}$$

hence by plugging in the decomposition to (5.1)-(5.2) we have the perturbation equation:

$$\nabla \cdot \boldsymbol{u}' = 0 \tag{5.7}$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{U} + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' = -\frac{1}{\rho}\nabla p' + \nu \nabla^2 \boldsymbol{u}' \tag{5.8}$$

And we note that the boundary conditions that the perturbation $\boldsymbol{u}', p'$ satisfy is homogeneous, such that $\boldsymbol{U}$ and p_b satisfy the same BC's as $\boldsymbol{u}$ and p in the original equation.

In linear stability, with the assumption that

$$O(\boldsymbol{u}') = \epsilon O(\boldsymbol{U}), \tag{5.9}$$

we neglect the nonlinear term $\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'$ and the primes, and have the linearised perturbation equation

$$\nabla \cdot \boldsymbol{u} = 0 \tag{5.10}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} - \nu \nabla^2 \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho} \nabla p \tag{5.11}$$

or if we define the linear operator as

$$\mathcal{L}_{\boldsymbol{U}} = \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{5.12}$$

there is

$$\mathcal{L}_{\boldsymbol{U}} \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho} \nabla p. \tag{5.13}$$

The linearised equations (5.10)-(5.11), if written in matrix form (Arratia, 2011), is

$$\mathcal{L}_{\mathrm{NS}} \boldsymbol{q} = \begin{bmatrix} \mathcal{L}_{\boldsymbol{U}} + \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial V}{\partial x} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial W}{\partial z} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = 0, \tag{5.14}$$

where $\boldsymbol{q} = [u, v, w, p]^T$. This is in the KKT form that will be described below, where we will see that the same mathematical properties of the operators will be shared in both stability analysis and CFD.

5.1.1 The role of pressure

A separate short note on the pressure being the Lagrangian multiplier in incompressible system. Consider the Stokes flow (actually that can be the linearised equations as described above)

$$-\nabla^2 \boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{5.15}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{5.16}$$

and in the matrix form

$$\begin{bmatrix} -\nabla^2 & \nabla \\ \nabla \cdot & O \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} \tag{5.17}$$

and its discrete version

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{5.18}$$

which is a saddle point problem or a KKT (Karush-Kuhn-Tucker) system (Benzi *et al.*, 2005). The Stokes equations can be interpreted as a constained optimisation problem (section 3.15.5 of Gresho & Sani (1998))

$$\min \ J(\boldsymbol{u}) = \frac{1}{2} \int \|\nabla \boldsymbol{u}\|_2^2 dV - \int \boldsymbol{f} \cdot \boldsymbol{u} dV \tag{5.19}$$

$$\text{subject to } \nabla \cdot \boldsymbol{u} = 0 \tag{5.20}$$

where the variable p, introduced to satisfy an additional constraint, plays the role of a Lagrangian multiplier. We note that the adjoint of the gradient operator is the (negative) divergence operator

$$(\nabla)^\dagger = -\nabla \cdot \tag{5.21}$$

5.1.2 Parallel shear flow

In the case of a parallel shear flow that $U(y)$ is the only nonzero mean flow component, we have

$$\mathcal{L}_U \boldsymbol{u} = \begin{bmatrix} -\partial_x p - U'v \\ -\partial_y p \\ -\partial_z p \end{bmatrix}, \ \mathcal{L}_U \boldsymbol{u} = \frac{\partial}{\partial t} + U\partial_x - \nu \nabla^2 \tag{5.22}$$

5.2 Normal-mode stability theory

5.2.1 K-H instability

and M-H condition.

5.2.2 Rayleigh's criterion

5.2.3 Orr-Sommerfield equations

Viscous instability mechanism.

5.2.4 T-S waves

5.2.5 Centrifugal instability

5.2.6 GFD instabilities

5.3 Non-normal instability

5.3.1 Adjoint matrices, operators, and equations

For a complex matrix $A \in \mathbb{C}^{N\times N}$, define an inner product

$$(u, v)_A = (Au, v) \tag{5.23}$$

where $u, v \in \mathbb{C}^N$ and $(u, v) = v^H u$, $(\cdot)^H$ is the Hermitian transpose. Define the adjoint matrix of A as $A^\dagger$ such that

$$(Au, v) = (u, A^\dagger v). \tag{5.24}$$

We note that if A is Hermitian ($A^H = A$), (5.24) is valid. Such matrix A is also called self-adjoint ($A = A^\dagger$). For operators defined on domains like $\mathbb{C}^N$, Hermitian and self-adjointness imply each other and we don't distinguish these two in what follows.

Consider the following standard Sturm-Liouville eigenvalue problem:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi = -\lambda\sigma(x)\phi, \tag{5.25}$$

where $p(x), w(x)$ are positive, and $\lambda, \phi(x)$ are the eigenvalue and corresponding eigenfunction of the problem. The boundary conditions are

$$\alpha_1\phi(a) + \alpha_2\frac{d\phi}{dx}(a) = 0 \tag{5.26}$$

$$\beta_1\phi(b) + \beta_2\frac{d\phi}{dx}(b) = 0 \tag{5.27}$$

$$\tag{5.28}$$

with $\alpha_1^2 + \alpha_2^2 \neq 0, \beta_1^2 + \beta_2^2 \neq 0$.

The LHS operator is defined as

$$\mathcal{L}(y) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y \tag{5.29}$$

and the S-L eigenvalue problem is

$$\mathcal{L}(\phi) + \lambda\sigma(x)\phi = 0. \tag{5.30}$$

The Lagrange identity is

$$u\mathcal{L}(v) - v\mathcal{L}(u) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right] \tag{5.31}$$

and the Green's formula is

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b \tag{5.32}$$

When u, v satisfy the same set of boundary conditions (either homogeneous or periodic), we have the self-adjointness, i.e.,

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]dx = 0 \tag{5.33}$$

or

$$\int_a^b u\mathcal{L}(v)dx = \int_a^b v\mathcal{L}(u)dx. \tag{5.34}$$

$$= \int_a^b v\mathcal{L}^\dagger(u)dx. \tag{5.35}$$

and we note the definition of adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$ is that

$$(u, \mathcal{L}(v)) = (v, \mathcal{L}^\dagger(u)), \tag{5.36}$$

with the inner product defined based on spatial integral and the adjoint is dependent on the inner product.

Examples.

1. The Laplacian operator.

$$\mathcal{L} = \nabla^2. \tag{5.37}$$

The multidimensional variation of (5.32), with $\mathcal{L} = \nabla^2$, is

$$\iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV = \iiint \nabla\cdot[u\nabla v - v\nabla u]dV = \iint (u\nabla v - v\nabla u)\cdot d\boldsymbol{A} \tag{5.38}$$

and if u, v satisfy the same homogeneous BC,

$$\iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV = \iint (u\nabla v - v\nabla u)\cdot d\boldsymbol{A} = 0, \tag{5.39}$$

and ∇^2 is self-adjoint.

2. The wave equation.

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\nabla^2. \tag{5.40}$$

The Green's formula

$$\int_{t_i}^{t_f}\iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV dt \tag{5.41}$$

$$= \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right) dV \Big|_{t_i}^{t_f} - c^2 \int_{t_i}^{t_f} \left(\iint (u\nabla v - v\nabla u) \cdot d\boldsymbol{A} \right) dt \tag{5.42}$$

And we note that the $\mathcal{L} = \partial_{tt}$ operator alone is self-adjoint if the boundary terms vanish, by

$$\int_{t_i}^{t_f} [u\mathcal{L}(v) - v\mathcal{L}(u)]dt = \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right) \Big|_{t_i}^{t_f}. \tag{5.43}$$

3. Heat equation. Given above, we only consider the temporal derivative $\mathcal{L} = \partial_t$ here.

$$\int_{t_i}^{t_f} u\frac{dv}{dt} + v\frac{du}{dt}dt = uv|_{t_i}^{t_f} \tag{5.44}$$

and the adjoint operator is $\mathcal{L}^\dagger = -\partial_t$. And we note that for the boundary term to vanish, we usually only have one BC for u in a first order problem such as $u(a) = 0$ and need to introduce an adjoint BC as $v(b) = 0$.

5.3.2 Non-self-adjointness and non-normality

A normal matrix $L \in \mathbb{C}^{N\times N}$ is defined as

$$L^H L = LL^H \tag{5.45}$$

and it is unitarily diagonalizable ($L = U\Lambda U^H$). The eigenvectors of L span an orthogonal basis of $\mathbb{C}^N$. More specifically, the eigenvectors corresponding to different eigenvalues are orthogonal, and even for degenerate eigenvalues an orthogonal basis can be found. A normal matrix is Hermitian if and only if all its eigenvalues are real.

The normality of a linear operator $\mathcal{L}$ is defined as

$$\mathcal{L}^\dagger \mathcal{L} = \mathcal{L}\mathcal{L}^\dagger. \tag{5.46}$$

The eigenmodes of $\mathcal{L}$ are normal to each other. A self-adjoint operator is hence normal and a non-normal operator must be non-self-adjoint.

For two eigenmodes $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$, where $\boldsymbol{\Phi}_i = e^{\lambda_i t}\boldsymbol{\phi}_i$, and $\lambda_i, \boldsymbol{\phi}_i$ are the eigenpair. There difference/-cancellation $\boldsymbol{f} = \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2$ decays if both decays and $(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2) = 0$. That said, if the real part of each eigenvalue is negative, the energy of the perturbation will decay. However, for non-self-adjoint operators, there could be a transient growth of the cancellation $\boldsymbol{f}$ (Schmid, 2007), where the decay of individual eigenmodes does not imply the transient decay of the total energy. The idea of optimal perturbations it to find such a transient mode that grows most within a certain period of time.

5.3.3 Adjoint of the linearised N-S equations (Op's)

Op's: optimal linear perturbation solved as an optimal control problem in an optimization formulation constrained by the PDEs with linear operators.

Similar to (5.10)-(5.11), the linearised perturbation equation with buoyancy is

$$\nabla \cdot \boldsymbol{u} = 0 \tag{5.47}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{u} + b\hat{\boldsymbol{e}}_z \tag{5.48}$$

$$\frac{\partial b}{\partial t} + \boldsymbol{U} \cdot \nabla b + \boldsymbol{u} \cdot \nabla B = \kappa\nabla^2 b \tag{5.49}$$

where $b = B + b'$ is buoyancy and the primes are dropped from the equations above.

Assume the adjoints of $(\boldsymbol{u}, p, b)$ are $(\boldsymbol{v}, q, \varphi)$, multiplying each term in (5.47) by (v_1, v_2, v_3), (5.48) by $\boldsymbol{v}$, and (5.49) by φ, we can derive the adjoint equations of (5.47)-(5.49) as

$$\nabla \cdot \boldsymbol{v} = 0 \tag{5.50}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{U} + \varphi \nabla B - \frac{1}{\rho_0} \nabla q - \nu \nabla^2 \boldsymbol{v} \tag{5.51}$$

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{U} \cdot \nabla \varphi = -v_3 - \kappa \nabla^2 \varphi \tag{5.52}$$

We note the cross contribution terms $\varphi \nabla B$ and $-v_3$. The same set of equations can also be derived from a Lagrangian multiplier approach (a more morden method, but now classic), with the total perturbation energy being the Lagrangian and the set of governing equations along with BC's being the constraints enforced as multipliers. Such Lagrangian is in the form of energy gain as (Arratia, 2011; Luchini & Bottaro, 2014; Kaminski *et al.*, 2014)

$$\mathscr{L}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T) \rangle + \langle b(T), b(T) \rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0 \rangle + \langle b_0, b_0 \rangle / N^2} \tag{5.53}$$

$$- \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j^2} - b\delta_{i3}, v_i\right] \tag{5.54}$$

$$- \left[\frac{\partial b}{\partial t} + u_j \frac{\partial B}{\partial x_j} + U_j \frac{\partial b}{\partial x_j} - \kappa \frac{\partial^2 b}{\partial x_j^2}, \varphi\right] - \left[\frac{\partial u_i}{\partial x_i}, q\right] \tag{5.55}$$

$$- \langle u_i(0) - u_{0i}, v_{0i} \rangle - \langle b(0) - b_0, \varphi_0 \rangle \tag{5.56}$$

constrained by the equations through the multipliers $(\boldsymbol{v}, q, \varphi)$ and similarly the BC's. The inner products

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_V \boldsymbol{u} \cdot \boldsymbol{v} dV \tag{5.57}$$

$$[\boldsymbol{u}, \boldsymbol{v}] = \int_0^T \langle \boldsymbol{u}, \boldsymbol{v} \rangle dt \tag{5.58}$$

are defined as respective spatial and spatio-temporal integrations.

In the phraseology of optimal control (with PDE constraints, Ref. section 5 of Manzoni *et al.* (2021)):

- Target functional (finite time transient gain):

$$\mathscr{G}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T) \rangle + \langle b(T), b(T) \rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0 \rangle + \langle b_0, b_0 \rangle / N^2}. \tag{5.59}$$

- State variables: $(\boldsymbol{u}, b)$.
- State equations (PDE constraints): (5.47)-(5.49). We note that (adjoint) pressure only appears as a Lagrangian multiplier that enforces the continuity.
- Control variables: $\boldsymbol{u}(0), b(0)$ and hence $\boldsymbol{u}(\boldsymbol{x}, t), b(\boldsymbol{x}, t)$.
- Admissible constraints on controls: none for now.

Taking the variation of (5.56) w.r.t.:

- The multipliers $(\boldsymbol{v}, q, \varphi)$: we recover the 'direct' equations (5.47)-(5.49).
- The terms $\boldsymbol{v}_0, \varphi_0$: we obtain the definition of IC's $\boldsymbol{u}_0, b_0$.
- The 'direct' variables $(\boldsymbol{u}, p, b)$: we obtain the adjoint equations (5.50)-(5.52). This step will be shown in detail.

Other than deriving from a Lagrangian perspective, the adjoint can also be derived using (multiple) integrations by parts. Starting from (5.14), i.e.,

$$\mathcal{L}_{\mathrm{NS}}\boldsymbol{q} = 0, \tag{5.60}$$

with the direct and adjoint variables being $\boldsymbol{q} = (\boldsymbol{u}, p)$ and $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$, we look for the adjoint $\mathcal{L}^{\dagger}_{\mathrm{NS}}$ such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] - [\mathcal{L}^{\dagger}_{\mathrm{NS}}\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) \tag{5.61}$$

and the boundary conditions that make the RHS boundary terms vanish.

Where the inner product $[\cdot, \cdot]$ is that same as in (5.58) such that

$$[\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \int_T \int_V (\boldsymbol{v} \cdot \boldsymbol{u} + qp) dt dV. \tag{5.62}$$

The weak form of (5.14) is

$$[\mathbb{1}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V (\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + \partial_i p + u_j \partial_j U_i) dt dV = 0. \tag{5.63}$$

We note that (5.63) should also be valid on any arbitrary test function $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$ for (5.14) to hold, such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V v_i(\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + u_j \partial_j U_i) + q \partial_i p \, dt dV = 0. \tag{5.64}$$

By integration by parts we have

$$\begin{aligned}
[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] &= \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p \partial_i q \, dt dV + \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) && (5.65)\\
&= \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p \partial_i q \, dt dV && (5.66)\\
&= [\mathcal{L}^{\dagger}_{\mathrm{NS}}\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] && (5.67)
\end{aligned}$$

that defines the adjoint operator of $\mathcal{L}_{\boldsymbol{U}} = \partial_t + \boldsymbol{U} \cdot \nabla - \nu \nabla^2$:

$$\mathcal{L}^{\dagger}_{\boldsymbol{U}} = -\partial_t - \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{5.68}$$

and the adjoint equation

$$\partial_t v_i + U_j \partial_j v_i + \nu \partial_j^2 v_i = v_j \partial_j U_i - \partial_i q \tag{5.69}$$

and we note that by advancing forward in time, the viscous term is injecting energy into the system. Using the transfrom $\tau = -t$ we have

$$\partial_\tau v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i = -v_j \partial_j U_i + \partial_i q. \tag{5.70}$$

5.4 The Lorenz system

A Vectors, tensors, and their calculus

Aris (1989) is a good reference.

A.1 Levi-Civita symbol

A.1.1 Determinant representation

The matrix determinants can be expressed in terms of the Levi-Civita symbol. Assume A is a matrix

$$\det(A) = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{A.1}$$

where

$$\boldsymbol{a}_1 = (a_{11}, a_{12}, a_{13})^\top,\ \boldsymbol{a}_2 = (a_{21}, a_{22}, a_{23})^\top,\ \boldsymbol{a}_3 = (a_{31}, a_{32}, a_{33})^\top \tag{A.2}$$

Therefore the Levi-Civita symbol can be expressed as

$$\epsilon_{ijk} = \det(\hat{\boldsymbol{e}}_i, \hat{\boldsymbol{e}}_j, \hat{\boldsymbol{e}}_k) = \hat{\boldsymbol{e}}_i \cdot (\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k) \tag{A.3}$$

Similarly, the outer product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \epsilon_{ijk} a_j b_k \hat{\boldsymbol{e}}_i \tag{A.4}$$

Example: $\boldsymbol{\omega}$.

A.1.2 Epsilon identity

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} && \text{(A.5)} \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{jl}(\delta_{in}\delta_{km} - \delta_{im}\delta_{kn}) + \delta_{kl}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) && \text{(A.6)} \\ &= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} && \text{(A.7)} \end{aligned}$$

A.1.3 Contracted epsilon identity

Let $i = l$ and notice $\delta_{ii} = 3$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{A.8}$$

Futhur let $k = m$

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{A.9}$$

Futhermore

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{A.10}$$

A.1.4 Pseudo-vector and associated antisymmetric rotation tensor

The velocity gradient tensor $\nabla \boldsymbol{u}$ is

$$\nabla \boldsymbol{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial w} & \frac{\partial v}{\partial w} & \frac{\partial w}{\partial w} \end{bmatrix} \tag{A.11}$$

and in entity notation

$$(\nabla \boldsymbol{u})_{ij} = \frac{\partial u_j}{\partial x_i}. \tag{A.12}$$

We note the transpose as compared to the Jacobian

$$J_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{A.13}$$

Vorticity

$$\begin{aligned} \boldsymbol{\omega} &= \nabla \times \boldsymbol{u} && \text{(A.14)} \\ &= \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} && \text{(A.15)} \\ &= \begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} && \text{(A.16)} \end{aligned}$$

is a pseudo-vector ($\omega_i = \epsilon_{ijk}\partial_j u_k$) whose sign depends on the coordinate system (the order of i, j, k; left-hand or right-hand; cyclic or anticyclic), and is related to the antisymmetric part of velocity gradient tensor $\nabla \boldsymbol{u}$ (the rotation rate tensor $\boldsymbol{\Omega}$):

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right). \tag{A.17}$$

or

$$\begin{aligned} \boldsymbol{\Omega} &= \frac{1}{2}(\nabla \boldsymbol{u} - \boldsymbol{u}\nabla) && \text{(A.18)} \\ &= \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \\ -\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ -\frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & -\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix} && \text{(A.19)} \end{aligned}$$

Each antisymmetric tensor $\boldsymbol{\Omega}$ can be represented by a pseudo-vector $\boldsymbol{\omega}^*$ (since it just has three independent elements), such that

$$\Omega_{ij} = \epsilon_{ijk}\omega_k^* \tag{A.20}$$

$$\omega_k^* = \frac{1}{2}\epsilon_{ijk}\Omega_{ij} \tag{A.21}$$

and the inner product of the tensor $\boldsymbol{\Omega}$ with and arbitrary vector $\boldsymbol{a}$ can be written as

$$\boldsymbol{\Omega} \cdot \boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{\omega}^*. \tag{A.22}$$

It is easy to verify (A.20) by definition and (A.21) using (A.9).

Element-wise, the rotation tensor can be represented as

$$\Omega_{ij} = \begin{bmatrix} 0 & \omega_z^* & -\omega_y^* \\ -\omega_z^* & 0 & \omega_x^* \\ \omega_y^* & -\omega_x^* & 0 \end{bmatrix} \tag{A.23}$$

with

$$\boldsymbol{\omega}^* = \begin{bmatrix} \omega_x^* \\ \omega_y^* \\ \omega_z^* \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{bmatrix} = \frac{1}{2}\boldsymbol{\omega}. \tag{A.24}$$

Hence we show that $\boldsymbol{\omega} = 2\boldsymbol{\omega}^*$, i.e., vorticity is twice of the angular velocity of the local solid-body rotation motion.

In the context of solid-body rotation (with no translation, $\boldsymbol{u}_T = 0$), the definition of (A.23) becomes

$$\Omega_{ij} = \begin{bmatrix} 0 & -\omega_z^* & \omega_y^* \\ \omega_z^* & 0 & -\omega_x^* \\ -\omega_y^* & \omega_x^* & 0 \end{bmatrix} \tag{A.25}$$

such that

$$\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{\Omega} \cdot \boldsymbol{x} = \boldsymbol{\omega}^* \times \boldsymbol{x} \tag{A.26}$$

where $\boldsymbol{\omega}^*$ is the angular velocity.

A.2 Vector identities

Assume λ is a scalar and $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$,$\boldsymbol{d}$ are vectors in $\mathbb{R}^3$. The identities below might be useful in fluids, some of which have geometric implecations.

$$\nabla \cdot (\nabla \times \boldsymbol{b}) = 0 \tag{A.27}$$
$$\nabla \times (\nabla \boldsymbol{b}) = 0 \tag{A.28}$$
$$\nabla \cdot (\lambda \boldsymbol{b}) = \nabla\lambda \cdot \boldsymbol{b} + \lambda(\nabla \cdot \boldsymbol{b}) \tag{A.29}$$
$$\nabla \times (\lambda \boldsymbol{b}) = \lambda(\nabla \times \boldsymbol{b}) - \boldsymbol{b} \times \nabla\lambda \tag{A.30}$$
$$\nabla(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla\boldsymbol{a} + \boldsymbol{a} \cdot \nabla\boldsymbol{b} + \boldsymbol{b} \times (\nabla \times \boldsymbol{a}) + \boldsymbol{a} \times (\nabla \times \boldsymbol{b}) \tag{A.31}$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \tag{A.32}$$
$$\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla \times \boldsymbol{a}) \cdot \boldsymbol{b} - (\nabla \times \boldsymbol{b}) \cdot \boldsymbol{a} \tag{A.33}$$
$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}) \tag{A.34}$$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) \tag{A.35}$$
$$\boldsymbol{b} \times (\nabla \times \boldsymbol{b}) = \nabla(\frac{1}{2}\boldsymbol{b} \cdot \boldsymbol{b}) - \boldsymbol{b} \cdot \nabla\boldsymbol{b} \tag{A.36}$$
$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) \tag{A.37}$$
$$\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2\boldsymbol{a} \tag{A.38}$$

Their proofs are left as exercises.

Comments:

(1) Eq. (A.27): A curl field is solenoidal (divergence-free).

(2) Eq. (A.28): A gradient field is irrotational (curl-free).

(3) Eq. (A.37): $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$, so its curl is in the space spaned by $\boldsymbol{a}$ and $\boldsymbol{b}$.

(4) Eq. (A.35): $\boldsymbol{a} \times (\cdot)$ is perpendicular to $\boldsymbol{a}$ and $(\cdot) \times (\boldsymbol{b} \times \boldsymbol{c})$ is in the space spaned by $\boldsymbol{b}$ and $\boldsymbol{c}$. This two facts in combinition gives the bases of $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

(5) Eq. (A.32): This is the volume spaned by $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, and the identity is basically the invariance of a determinant with respect to row/column permutation.

(6) Eq. (A.34): By letting $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$ and noticing the inner product with itself is non-negative, we re-discover the Cauchy-Schwartz inequality.

(7) From (A.31) it can be immediately seen that

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla(\boldsymbol{u}^2/2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}). \tag{A.39}$$

A.3 Tensor eigenvalues and invariants

Consider a tensor $\boldsymbol{A}$ in Cartesian coordinate

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{A.40}$$

Its eigenvalues are roots of the characteristic polynomial

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{A.41}$$

with the three coefficients being the three principle invariants of $\boldsymbol{A}$

$$\begin{aligned}
I_1 &= a_{11} + a_{22} + a_{33} && \text{(A.42)} \\
&= \mathrm{tr}(\boldsymbol{A}) && \text{(A.43)} \\
&= a_{ii} && \text{(A.44)} \\
I_2 &= a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31} && \text{(A.45)} \\
&= \frac{\mathrm{tr}(\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)}{2} && \text{(A.46)} \\
&= \frac{1}{2}((a_{ii})^2 - a_{ij}a_{ji}) && \text{(A.47)} \\
I_3 &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) && \text{(A.48)} \\
&= \det(\boldsymbol{A}) && \text{(A.49)}
\end{aligned}$$

in both element-wise and coordinate-independent expression.

Now we consider the factorization of the characteristic polynomial as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\lambda - \lambda_1\lambda_2\lambda_3 = 0, \tag{A.50}$$

and obtain the Vieta's theorem for cubic equations as

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3 \tag{A.51}$$

$$
\begin{aligned}
I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 && \text{(A.52)}\\
I_3 &= \lambda_1\lambda_2\lambda_3 && \text{(A.53)}
\end{aligned}
$$

which are the three principle invariants of tensor $\boldsymbol{A}$.

Additionally, there are more invariants (although not independent) of $\boldsymbol{A}$, such as the main invariants

$$
\begin{aligned}
J_1 &= \lambda_1 + \lambda_2 + \lambda_3 = I_1 = \mathrm{tr}(\boldsymbol{A}) && \text{(A.54)}\\
J_2 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1^2 - 2I_2 = \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{A}) && \text{(A.55)}\\
J_1 &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = I_1^3 - 3I_1I_2 + 3I_3 = \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{A}\cdot\boldsymbol{A}) && \text{(A.56)}
\end{aligned}
$$

which are the coefficients of the characteristic polynomial of the deviatoric part of $\boldsymbol{A}$:

$$
\boldsymbol{A} - \frac{\mathrm{tr}(\boldsymbol{A})}{3}\boldsymbol{I}, \tag{A.57}
$$

which is traceless and has eigenvalues

$$
\lambda_i - \frac{1}{3}. \tag{A.58}
$$

A.3.1 Discriminant of a cubic equation

Consider

$$
ax^3 + bx^2 + cx + d = 0, \tag{A.59}
$$

its determinant is

$$
\begin{aligned}
\Delta &= (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2 && \text{(A.60)}\\
&= 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 && \text{(A.61)}
\end{aligned}
$$

with x_1, x_2, x_3 being the three roots.

1. $\Delta > 0$: Three distinct real roots.
2. $\Delta = 0$: All roots are real with at least two identical.
3. $\Delta < 0$: One real and a pair of complex conjugate roots (proof: assume complex roots are $x \pm iy$).

Proof. The Vieta's theorem for (A.59) and the invariant relations can be used to simplify (A.59) to obtain (A.61).

Note: Eq. (A.61) can also be obtained as follows (with some reasons/meanings in algebraic geometry). Consider a cubic equation in canonical form

$$
f(x, w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = 0. \tag{A.62}
$$

The Hessain matrix is

$$
H(f) = \begin{bmatrix} 6Ax + 6Bw & 6Bx + 6Cw \\ 6Bx + 6Cw & 6Cx + 6Dw \end{bmatrix} \tag{A.63}
$$

and the Hessain

$$
\begin{aligned}
\det(H) &= 36[(AC - B^2)x^2 + (AD - BC)xw + (BD - C^2)w^2] && \text{(A.64)}\\
&= 18[x, w]\begin{bmatrix} 2(AC - B^2) & (AD - BC) \\ (AD - BC) & 2(BD - C^2) \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}, && \text{(A.65)}
\end{aligned}
$$

in quadratic form. Define the Hessain

$$\boldsymbol{H} = \begin{bmatrix} 2(AC-B^2) & (AD-BC) \\ (AD-BC) & 2(BD-C^2) \end{bmatrix} \tag{A.66}$$

The discriminant of the cubic is just the determinant of the Hessain $\boldsymbol{H}$:

$$\Delta = \det(\boldsymbol{H}) = -A^2D^2 + 6ABCD - 4AC^3 - 4B^3D + 3B^2C^2, \tag{A.67}$$

and $\Delta > 0$ for three real roots, $\Delta = 0$ for double or triple real root, and $\Delta < 0$ for single real root.

A.3.2 Examples

Utilizing and the discriminant $\triangle$ or the second invariant Q of $\nabla \boldsymbol{u}$ to identify vortices in fluid flows (Hunt *et al.*, 1988; Chong *et al.*, 1990; Jeong & Hussain, 1995) and the invariants of the Reynolds stress tensor $-\overline{u_i'u_j'}$ to classify turbulent states (Lumley & Newman, 1977; Choi & Lumley, 2001) are useful.

We note that both the rate-of-strain tensor $\boldsymbol{S}$ and the Reynolds stress tensor $-\overline{u_i'u_j'}$ are real symmetric, hence they have three real eigenvalues and three orthogonal eigenvectors (principle axes).

Vortex identification in incompressible flows:

In the case of incompressible flow ($u_{i,i} = 0$) with the invariants being $(P, Q, R) = (I_1, I_2, I_3)$. We have P, the coefficient of the quadratic term being zero and the characteristic polynomial for $\nabla \boldsymbol{u}$ being in the so-called 'depressed' form (an elliptic curve is called in Weierstrass form if it satisfies the Weierstrass equation $y^2 = x^3 + ax + b$)

$$\lambda^3 - P\lambda^2 + Q\lambda - R = \lambda^3 + Q\lambda - R = 0. \tag{A.68}$$

The discriminant for depressed cubic equation

$$x^3 + px + q = 0 \tag{A.69}$$

reduces to

$$\Delta = -4p^3 - 27q^2. \tag{A.70}$$

So we have the discriminant for the gradient of a solenoidal field (with renormalized coefficients; note the flipped sign)

$$\triangle = \left(\frac{1}{3}Q\right)^3 + \left(\frac{1}{2}R\right)^2 \tag{A.71}$$

and if $\triangle > 0$ there will be complex eigenvalues (in complex conjugate pair according to the algebra basic theorem) and so-defined vortical motions.

Lumley triangle and invariant maps:

Consider the anisotropic (deviatoric) tensor of Reynolds stress

$$a_{ij} = \frac{\overline{u_i'u_j'}}{2k} - \frac{1}{3}\delta_{ij} \tag{A.72}$$

and its three principle invariants

$$I = \sigma_1 + \sigma_2 + \sigma_3 \tag{A.73}$$

$$II = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \tag{A.74}$$

$$III = \sigma_1\sigma_2\sigma_3 \tag{A.75}$$

along with its three eigenvalues

$$\sigma_1,\ \sigma_2,\ \sigma_3. \tag{A.76}$$

Since a_{ij} is a deviator, it is traceless and

$$I = a_{ii} = 0. \tag{A.77}$$

Consider turbulence. and has zero determinant

$$\det\left(\frac{\overline{u_i'u_j'}}{2k}\right) = (\sigma_1 + \frac{1}{3})(\sigma_2 + \frac{1}{3})(\sigma_3 + \frac{1}{3}) \tag{A.78}$$

$$= \sigma_1\sigma_2\sigma_3 + \frac{1}{3}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{27}, \tag{A.79}$$

and we define

$$F = 27III + 9II + 1 \tag{A.80}$$

since $I = 0$.

1. Two-dimensional turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ can be diagonalized to

   $$diag(a, k-a, 0)$$

   and has zero determinant (there's a direction that has no turbulence). $F = 0$.

2. Three-dimensional isotropic turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ is

   $$diag(k/3, k/3, k/3)$$

   and we have $F = 1$.

3. Axisymmetric turbulence. Similarly, the characteristic polynomial of a_{ij} is in Weierstrass form and the condition for repeated eigenvalues (same energy in two principle directions) is

   $$\triangle = \left(\frac{1}{3}II\right)^3 + \left(\frac{1}{2}III\right)^2 = 0 \tag{A.81}$$

   and hence

   $$III = \pm 2\left(-\frac{II}{3}\right)^3, \tag{A.82}$$

   corresponding to the negative/left (pancake) and positive/right (cigar) limit curves of the Lumley triangle.

B Matrix and linear transformation

B.1 Unitary matrix

Unitary transformations preserve inner products (and hence length and angle).

B.1.1 Rotation and reflection

B.2 Conformal mapping

B.3 Coordinate transformation

C Coordinate systems

C.1 Cylindrical coordinate

Consider the cylindrical transformation

$$(x, y) \to (r, \theta) \tag{C.1}$$

where

$$x = r\cos\theta \tag{C.2}$$
$$y = r\sin\theta \tag{C.3}$$

or

$$r = \sqrt{x^2 + y^2} \tag{C.4}$$
$$\theta = \text{actan}\left(\frac{y}{x}\right) \tag{C.5}$$

we have the corresponding relation between unit vectors

$$\begin{bmatrix} \hat{\boldsymbol{e}}_x \\ \hat{\boldsymbol{e}}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\theta \end{bmatrix} \tag{C.6}$$

and

$$\begin{bmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_x \\ \hat{\boldsymbol{e}}_y \end{bmatrix}, \tag{C.7}$$

which can be proven graphically. We note that the grid transformation matrix is unitary and has $\det() = 1$ (rotation matrix).

The Jacobian of the forward transformation $(r, \theta) = F(x, y)$ is

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix} \tag{C.8}$$

We note that the directions of the unit vectors $\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta$ depend on space, i.e.,

$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial r} = \frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial r} = 0 \tag{C.9}$$
$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial \theta} = -\sin\theta \hat{\boldsymbol{e}}_x + \cos\theta \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_\theta \tag{C.10}$$
$$\frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial \theta} = -\cos\theta \hat{\boldsymbol{e}}_x - \sin\theta \hat{\boldsymbol{e}}_y = -\hat{\boldsymbol{e}}_r \tag{C.11}$$

which can also be seen graphically. These relations are crucial to later derivations.

Consider the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{C.12}$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{C.13}$$

C.1.1 Operators in cylindrical coordinate

For a scalar function, say $f(x, y) = f(r, \theta)$, the gradient operator can be expressed as

$$\nabla = \hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z} \tag{C.14}$$
$$= \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\right)(\cos\theta \hat{\boldsymbol{e}}_r - \sin\theta \hat{\boldsymbol{e}}_\theta) + \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\right)(\sin\theta \hat{\boldsymbol{e}}_r + \cos\theta \hat{\boldsymbol{e}}_\theta) + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z} \tag{C.15}$$
$$= \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z} \tag{C.16}$$

The factor $r\partial\theta$ can be interpreted as infinitesimal length element in θ direction.

The Laplace operator

$$\nabla^2 = \nabla \cdot \nabla = \left(\hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}\right) \cdot \left(\hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}\right) \tag{C.17}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \hat{\boldsymbol{e}}_\theta \cdot \frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)\right] \tag{C.18}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{C.19}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{C.20}$$

Now consider a vector

$$\boldsymbol{u} = \hat{\boldsymbol{e}}_r u + \hat{\boldsymbol{e}}_\theta v + \hat{\boldsymbol{e}}_z w \tag{C.21}$$

and its derivatives.

Its divergence

$$\nabla \cdot \boldsymbol{u} = \left(\hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}\right) \cdot (\hat{\boldsymbol{e}}_r u + \hat{\boldsymbol{e}}_\theta v + \hat{\boldsymbol{e}}_z w) \tag{C.22}$$

$$= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.23}$$

$$= \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.24}$$

The convection term

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z})(\hat{\boldsymbol{e}}_r u + \hat{\boldsymbol{e}}_\theta v + \hat{\boldsymbol{e}}_z w) \tag{C.25}$$

$$= (u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r})\hat{\boldsymbol{e}}_r \tag{C.26}$$

$$+ (u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r})\hat{\boldsymbol{e}}_\theta \tag{C.27}$$

$$+ (u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z})\hat{\boldsymbol{e}}_z \tag{C.28}$$

Now we deal with $\nabla^2 \boldsymbol{u}$.

$$\nabla^2 \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{\boldsymbol{e}}_r u + \hat{\boldsymbol{e}}_\theta v + \hat{\boldsymbol{e}}_z w) \tag{C.29}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{u}}{\partial r}\right) + \frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\hat{\boldsymbol{e}}_r u + \hat{\boldsymbol{e}}_\theta v) \tag{C.30}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right)\hat{\boldsymbol{e}}_r \tag{C.31}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right)\hat{\boldsymbol{e}}_\theta \tag{C.32}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)\hat{\boldsymbol{e}}_z \tag{C.33}$$

with

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\hat{\boldsymbol{e}}_r u) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \hat{\boldsymbol{e}}_r u}{\partial \theta} = \frac{1}{r^2}(2\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta - u\hat{\boldsymbol{e}}_r + \frac{\partial^2 u}{\partial \theta^2}\hat{\boldsymbol{e}}_r) \tag{C.34}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\hat{\boldsymbol{e}}_\theta v) = \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{\partial\hat{\boldsymbol{e}}_\theta v}{\partial\theta} = \frac{1}{r^2}(-2\frac{\partial v}{\partial\theta}\boldsymbol{e}_r - v\hat{\boldsymbol{e}}_\theta + \frac{\partial^2 v}{\partial\theta^2}\hat{\boldsymbol{e}}_\theta) \tag{C.35}$$

Moreover, the curl can be established as

$$\begin{aligned}
\nabla\times\boldsymbol{u} &= \left(\hat{\boldsymbol{e}}_r\frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{e}}_z\frac{\partial}{\partial z}\right)\times(\hat{\boldsymbol{e}}_r u + \hat{\boldsymbol{e}}_\theta v + \hat{\boldsymbol{e}}_z w) && \text{(C.36)}\\
&= \begin{vmatrix} \hat{\boldsymbol{e}}_r & \hat{\boldsymbol{e}}_\theta & \hat{\boldsymbol{e}}_z \\ \partial_r & \frac{1}{r}\partial_\theta & \partial_z \\ u & v & w \end{vmatrix} + \frac{1}{r}\hat{\boldsymbol{e}}_\theta\times\frac{\partial(v\hat{\boldsymbol{e}}_\theta)}{\partial\theta} && \text{(C.37)}\\
&= (\frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{\partial v}{\partial z})\hat{\boldsymbol{e}}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\hat{\boldsymbol{e}}_\theta + (\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial\theta})\hat{\boldsymbol{e}}_z && \text{(C.38)}\\
&= (\frac{1}{r}\frac{\partial w}{\partial\theta} - \frac{\partial v}{\partial z})\hat{\boldsymbol{e}}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\hat{\boldsymbol{e}}_\theta + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial\theta})\hat{\boldsymbol{e}}_z && \text{(C.39)}\\
&= \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\theta & \hat{\boldsymbol{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix} && \text{(C.40)}
\end{aligned}$$

We note that to the vertical vorticity ω_z, the shear vorticity $\partial_r v$ and the curvature vorticity v/r have equal contributions. The relation (C.40) is generalizable and will be shown in section C.3.

Examples.

1. Rigid body rotation with angular velocity Ω and $v = \Omega r$. Vorticity $\omega_z = 2\Omega$ but there is no vortical motion.

2. Potential point vortex with $v = \Gamma/2\pi r$. Vorticity $\omega_z = 0$.

C.1.2 Navier-Stokes in cylindrical coordinate

The Navier-Stokes equation in cylindrical coordinate reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{C.41}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{C.42}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{C.43}$$

Q.E.D.

C.2 Spherical coordinate

Consider the transformation

$$(x, y, z) \to (r, \phi, \theta) \tag{C.44}$$

where

$$\begin{aligned}
x &= r\sin\phi\cos\theta && \text{(C.45)}\\
y &= r\sin\phi\sin\theta && \text{(C.46)}\\
z &= r\cos\phi && \text{(C.47)}
\end{aligned}$$

or

$$r = \sqrt{x^2 + y^2 + z^2} \tag{C.48}$$

$$\phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \tag{C.49}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{C.50}$$

or thought of as from cylindrical with

$$z = r\cos\phi \tag{C.51}$$

$$r' = r\sin\phi \tag{C.52}$$

$$x = r'\sin\theta \tag{C.53}$$

$$y = r'\cos\theta \tag{C.54}$$

Here θ is the azimuthal angle with x-axis on the equatorial plane and ϕ is the polar angle with z-axis (North), for the convenience of going from cylindrical to polar and backwards.

We have the corresponding relation between unit vectors

$$\begin{bmatrix} \hat{\boldsymbol{e}}_x \\ \hat{\boldsymbol{e}}_y \\ \hat{\boldsymbol{e}}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\phi \\ \hat{\boldsymbol{e}}_\theta \end{bmatrix} \tag{C.55}$$

and

$$\begin{bmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\phi \\ \hat{\boldsymbol{e}}_\theta \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_x \\ \hat{\boldsymbol{e}}_y \\ \hat{\boldsymbol{e}}_z \end{bmatrix}, \tag{C.56}$$

which can be proven graphically. We note that the grid transformation matrix is unitary and has $\det() = 1$ (rotation matrix).

C.2.1 From cylindrical to spherical

We have the transformation

$$\begin{bmatrix} \hat{\boldsymbol{e}}_x \\ \hat{\boldsymbol{e}}_y \\ \hat{\boldsymbol{e}}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\phi \\ \hat{\boldsymbol{e}}_\theta \end{bmatrix} \tag{C.57}$$

that can be factorized as

$$\begin{bmatrix} \hat{\boldsymbol{e}}_x \\ \hat{\boldsymbol{e}}_y \\ \hat{\boldsymbol{e}}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\phi \\ \hat{\boldsymbol{e}}_\theta \end{bmatrix} \tag{C.58}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\phi \\ \hat{\boldsymbol{e}}_\theta \end{bmatrix} \tag{C.59}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_{r'} \\ \hat{\boldsymbol{e}}_{\theta'} \\ \hat{\boldsymbol{e}}_{z'} \end{bmatrix} \tag{C.60}$$

with

$$\begin{bmatrix} \hat{\boldsymbol{e}}_{r'} \\ \hat{\boldsymbol{e}}_{\theta'} \\ \hat{\boldsymbol{e}}_{z'} \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{e}}_{r} \\ \hat{\boldsymbol{e}}_{\phi} \\ \hat{\boldsymbol{e}}_{\theta} \end{bmatrix}. \tag{C.61}$$

C.3 General curvilinear coordinates

Consider the coordinate transformations

$$q_i = q_i(x_1, x_2, x_3),\ x_i = x_i(q_1, q_2, q_3) \tag{C.62}$$

where (x_1, x_2, x_3) is the standard Cartesian coordinates and q_i are mutually independent.

C.3.1 Length, area, and volume

Consider the change of the vector

$$\boldsymbol{x} = x_1\hat{\boldsymbol{e}}_{x_1} + x_2\hat{\boldsymbol{e}}_{x_2} + x_3\hat{\boldsymbol{e}}_{x_3} \tag{C.63}$$
$$= q_1\boldsymbol{h}_1 + q_2\boldsymbol{h}_2 + q_3\boldsymbol{h}_3 \tag{C.64}$$

where $\boldsymbol{x} = \boldsymbol{x}(x_i(q_j))$ as

$$\mathrm{d}\boldsymbol{x} = \hat{\boldsymbol{e}}_{x_1}\mathrm{d}x_1 + \hat{\boldsymbol{e}}_{x_2}\mathrm{d}x_2 + \hat{\boldsymbol{e}}_{x_3}\mathrm{d}x_3 \tag{C.65}$$
$$= \frac{\partial\boldsymbol{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial\boldsymbol{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial\boldsymbol{x}}{\partial q_3}\mathrm{d}q_3 \tag{C.66}$$

and

$$\boldsymbol{h}_i = \frac{\partial\boldsymbol{x}}{\partial q_i}. \tag{C.67}$$

We note that $\boldsymbol{h}_i$ is the change of $\boldsymbol{x}$ with only changing q_i, so it does define direction of coordinate lines of q_i. We denote with $\hat{(\cdot)}$ unit vectors and note that $\boldsymbol{h}_i$ are not necessary unit vectors.

Now consider the length of $\mathrm{d}\boldsymbol{x}$:

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} \tag{C.68}$$
$$= \frac{\partial\boldsymbol{x}}{\partial q_j}\mathrm{d}q_j \cdot \frac{\partial\boldsymbol{x}}{\partial q_k}\mathrm{d}q_k \tag{C.69}$$
$$= \frac{\partial x_i}{\partial q_j}\mathrm{d}q_j \frac{\partial x_i}{\partial q_k}\mathrm{d}q_k \tag{C.70}$$
$$= g_{jk}\mathrm{d}q_j\mathrm{d}q_k \tag{C.71}$$

with

$$g_{ij} = \frac{\partial x_l}{\partial q_i}\frac{\partial x_l}{\partial q_j} \tag{C.72}$$

being the metric tensor. When q_i are orthogonal coordinates,

$$\frac{\partial\boldsymbol{x}}{\partial q_i}\cdot\frac{\partial\boldsymbol{x}}{\partial q_j} = \delta_{ij} \tag{C.73}$$

and g_{ij} only has diagonal elements and

$$\mathrm{d}s^2 = g_{11}(\mathrm{d}q_1)^2 + g_{22}(\mathrm{d}q_3)^2 + g_{33}(\mathrm{d}q_3)^2 \tag{C.74}$$

$$= h_1^2(\mathrm{d}q_1)^2 + h_2^2(\mathrm{d}q_3)^2 + h_3^2(\mathrm{d}q_3)^2 \tag{C.75}$$

Define the Lamé parameters as

$$h_1 = \sqrt{g_{11}} = \sqrt{\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2} = |\boldsymbol{h}_1| \tag{C.76}$$

$$h_2 = \sqrt{g_{22}} = \sqrt{\left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2} = |\boldsymbol{h}_2| \tag{C.77}$$

$$h_3 = \sqrt{g_{33}} = \sqrt{\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2} = |\boldsymbol{h}_3| \tag{C.78}$$

and unit vectors in q_i directions as

$$\hat{\boldsymbol{h}}_i = \frac{\boldsymbol{h}_i}{|\boldsymbol{h}_i|} = \frac{\boldsymbol{h}_i}{h_i}. \tag{C.79}$$

We note that the Lamé parameters can depend on the coordinates as

$$h_i = h_i(q_1, q_2, q_3). \tag{C.80}$$

The increment can be rewritten as

$$\mathrm{d}\boldsymbol{x} = h_1\mathrm{d}q_1\hat{\boldsymbol{h}}_1 + h_2\mathrm{d}q_2\hat{\boldsymbol{h}}_2 + h_3\mathrm{d}q_3\hat{\boldsymbol{h}}_3 \tag{C.81}$$

$$= \mathrm{d}s_1\hat{\boldsymbol{h}}_1 + \mathrm{d}s_2\hat{\boldsymbol{h}}_2 + \mathrm{d}s_3\hat{\boldsymbol{h}}_3 \tag{C.82}$$

with

$$\mathrm{d}s_i \tag{C.83}$$

being the projection of $\mathrm{d}\boldsymbol{x}$ on each coordinate.

Now consider the surface and volume of infinitesimal elements. The (directed) areas of surface elements are

$$\mathrm{d}\sigma_i = \hat{\boldsymbol{h}}_i \cdot (h_j\mathrm{d}q_j\hat{\boldsymbol{h}}_j \times h_k\mathrm{d}q_k\hat{\boldsymbol{h}}_k) = h_jh_k\mathrm{d}q_j\mathrm{d}q_k \tag{C.84}$$

or

$$\mathrm{d}\sigma_1 = h_1h_2\mathrm{d}q_1\mathrm{d}q_2 \tag{C.85}$$

$$\mathrm{d}\sigma_2 = h_1h_3\mathrm{d}q_1\mathrm{d}q_3 \tag{C.86}$$

$$\mathrm{d}\sigma_3 = h_1h_2\mathrm{d}q_1\mathrm{d}q_2 \tag{C.87}$$

The volume element (e.g. in volume integrals) spanned by the vector $\mathrm{d}\boldsymbol{x}$ is

$$\mathrm{d}V = (h_1\mathrm{d}q_1\hat{\boldsymbol{h}}_1) \cdot (h_2\mathrm{d}q_2\hat{\boldsymbol{h}}_2 \times h_3\mathrm{d}q_3\hat{\boldsymbol{h}}_3) \tag{C.88}$$

$$= h_1\mathrm{d}q_1h_2\mathrm{d}q_2h_3\mathrm{d}q_3(\hat{\boldsymbol{h}}_1) \cdot (\hat{\boldsymbol{h}}_2 \times \hat{\boldsymbol{h}}_3) \tag{C.89}$$

$$= h_1h_2h_3\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3 \tag{C.90}$$

when $\hat{\boldsymbol{h}}_i$ mutually orthogonal.

Example.

For cylindrical coordinate, by definition,

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1 \tag{C.91}$$

$$h_2 = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r \tag{C.92}$$

$$h_3 = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1 \tag{C.93}$$

C.3.2 Jacobian

Now we consider the Jacobian of the backward transformation

$$(q_1, q_2, q_3) \to (x_1, x_2, x_3) \tag{C.94}$$

which reads

$$\boldsymbol{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{bmatrix} \tag{C.95}$$

and the Jacobian determinant (with $\exists\, \boldsymbol{J}^{-1}$)

$$\begin{aligned}
J &= \det(\boldsymbol{J}) = \det(\boldsymbol{J}^{\mathrm{T}}) && \text{(C.96)} \\
&= \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} && \text{(C.97)} \\
&= \left(\frac{\partial x_1}{\partial q_1}\hat{\boldsymbol{x}}_1 + \frac{\partial x_2}{\partial q_1}\hat{\boldsymbol{x}}_2 + \frac{\partial x_3}{\partial q_1}\hat{\boldsymbol{x}}_3\right) \cdot \begin{vmatrix} \hat{\boldsymbol{x}}_1 & \hat{\boldsymbol{x}}_2 & \hat{\boldsymbol{x}}_3 \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} && \text{(C.98)} \\
&= \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial q_2} \times \frac{\partial \boldsymbol{x}}{\partial q_3}\right) && \text{(C.99)} \\
&= \boldsymbol{h}_1 \cdot (\boldsymbol{h}_2 \times \boldsymbol{h}_3) && \text{(C.100)} \\
&= h_1 h_2 h_3 && \text{(C.101)} \\
&\neq 0 && \text{(C.102)}
\end{aligned}$$

Hence we have

$$\mathrm{d}V = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3 = J \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3. \tag{C.103}$$

C.3.3 Three major calculus theorems

1. Gradient theorem:

$$\int_{\boldsymbol{l}:\boldsymbol{x}_1 \to \boldsymbol{x}_2} (\nabla f) \cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1) \tag{C.104}$$

   The integral is independent of path since ∇f is potential (conservative, curl-free).

2. Divergence theorem:

$$\iint_{\Omega} (\nabla \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A} = \oint_{l=\partial\Omega} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} \tag{C.105}$$

   Implication: vorticity is circulation per unit area.

3. Curl theorem:

$$\iiint_{V} (\nabla \cdot \boldsymbol{u})\, \mathrm{d}V = \iint_{\Omega=\partial V} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A} \tag{C.106}$$

C.3.4 Differential operators in curvilinear coordinate systems

Next, let's consider differential operators in curvilinear coordinates. Consider a scalar $f = f(q_1, q_2, q_3)$ and its gradient ∇f. Starting from

$$\mathrm{d}f = \frac{\partial f}{\partial q_i}\mathrm{d}q_i, \tag{C.107}$$

due to the displacement $\boldsymbol{x}$. On the other hand,

$$\begin{aligned}\mathrm{d}f &= \nabla f \cdot \mathrm{d}\boldsymbol{x} && \text{(C.108)}\\ &= (\nabla f)_{q_i}\mathrm{d}s_i && \text{(C.109)}\\ &= (\nabla f)_{q_i} h_i \mathrm{d}q_i && \text{(C.110)}\end{aligned}$$

Compare (C.110) and (C.107) we have

$$(\nabla f)_{q_i} = \frac{1}{h_i}\frac{\partial f}{\partial q_i} \tag{C.111}$$

where

$$\begin{aligned}\nabla f &= (\nabla f)_{q_i}\hat{\boldsymbol{h}}_i && \text{(C.112)}\\ &= \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{\boldsymbol{h}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{\boldsymbol{h}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{\boldsymbol{h}}_3 && \text{(C.113)}\end{aligned}$$

Consider the divergence of a vector $\boldsymbol{u}$ in a coordinate-free form:

$$\begin{aligned}\nabla \cdot \boldsymbol{u} &\triangleq \lim_{V\to 0}\frac{\oint_{\Omega=\partial V}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{\sigma}}{V} && \text{(C.114)}\\ &= \frac{1}{V}\left(\frac{\partial(u_1 h_2 h_3 \mathrm{d}q_2 \mathrm{d}q_3)}{\partial q_1}\mathrm{d}q_1 + \frac{\partial(u_2 h_1 h_3 \mathrm{d}q_1 \mathrm{d}q_3)}{\partial q_2}\mathrm{d}q_2 + \frac{\partial(u_3 h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2)}{\partial q_3}\mathrm{d}q_3\right) && \text{(C.115)}\\ &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(u_1 h_2 h_3)}{\partial q_1} + \frac{\partial(u_2 h_1 h_3)}{\partial q_2} + \frac{\partial(u_3 h_1 h_2)}{\partial q_3}\right) && \text{(C.116)}\end{aligned}$$

Consider the curl of a vector $\boldsymbol{u}$ in a coordinate-free form, its compoment along $\hat{\boldsymbol{n}}$ (normal of the surface $\boldsymbol{S} = S\hat{\boldsymbol{n}}$) is

$$(\nabla\times\boldsymbol{u})\cdot\hat{\boldsymbol{n}} \triangleq \lim_{S\to 0}\frac{\int_{l=\partial S}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}}{S} \tag{C.117}$$

and (consider the area spanned by $\mathrm{d}s_2 = h_2\mathrm{d}q_2$ and $\mathrm{d}s_3 = h_3\mathrm{d}q_3$)

$$\begin{aligned}(\nabla\times\boldsymbol{u})\cdot\hat{\boldsymbol{h}}_1 &= \frac{\oint_l \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}}{\mathrm{d}\sigma_1} && \text{(C.118)}\\ &= \frac{1}{h_2 h_3 \mathrm{d}q_2 \mathrm{d}q_3}[u_2 h_2 \mathrm{d}q_2 && \text{(C.119)}\\ &\quad - (u_2 h_2 + \frac{\partial u_2 h_2}{\partial q_3}\mathrm{d}q_3)\mathrm{d}q_2 && \text{(C.120)}\\ &\quad - u_3 h_3 \mathrm{d}q_3 && \text{(C.121)}\\ &\quad + (u_3 h_3 + \frac{\partial u_3 h_3}{\partial q_2}\mathrm{d}q_2)\mathrm{d}q_3] && \text{(C.122)}\\ &= \frac{1}{h_2 h_3}\left(\frac{\partial u_3 h_3}{\partial q_2} - \frac{\partial u_2 h_2}{\partial q_3}\right) && \text{(C.123)}\end{aligned}$$

$$(\nabla\times\boldsymbol{u})\cdot\hat{\boldsymbol{h}}_2=\frac{1}{h_1h_3}\left(\frac{\partial u_1h_1}{\partial q_3}-\frac{\partial u_3h_3}{\partial q_1}\right) \tag{C.124}$$

$$(\nabla\times\boldsymbol{u})\cdot\hat{\boldsymbol{h}}_3=\frac{1}{h_1h_2}\left(\frac{\partial u_2h_2}{\partial q_1}-\frac{\partial u_1h_1}{\partial q_2}\right) \tag{C.125}$$

We note that the Lamé coefficient also changes as the coordinate changes.

In determinant form,

$$\nabla\times\boldsymbol{u}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\boldsymbol{h}}_1 & h_2\hat{\boldsymbol{h}}_2 & h_3\hat{\boldsymbol{h}}_3\\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3}\\ h_1u_1 & h_2u_2 & h_3u_3\end{vmatrix} \tag{C.126}$$

The Laplacian can be obtained by taking the divergence of ∇f as combining (C.113) and (C.116)

$$\nabla^2 f=\nabla\cdot(\nabla f)=\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right) \tag{C.127}$$

C.3.5 Derivatives of unit vectors

In general curvilinear coordinates, the directions of unit vectors could change with coordinate as well. We are basically concerned about

$$\frac{\partial\hat{\boldsymbol{h}}_i}{\partial q_j} \tag{C.128}$$

and we will establish that

$$\frac{\partial\hat{\boldsymbol{h}}_i}{\partial q_j}//\hat{\boldsymbol{h}}_j,\ i\neq j. \tag{C.129}$$

First we have

$$\hat{\boldsymbol{h}}_i\cdot\frac{\partial\hat{\boldsymbol{h}}_i}{\partial q_j}=\frac{\partial\hat{\boldsymbol{h}}_i^2/2}{\partial q_j}=0 \tag{C.130}$$

and hence

$$\frac{\partial\hat{\boldsymbol{h}}_i}{\partial q_j}\perp\hat{\boldsymbol{h}}_i,\ i\neq j. \tag{C.131}$$

According to the orthogonality we have

$$\boldsymbol{h}_1\cdot\boldsymbol{h}_2=\frac{\partial\boldsymbol{x}}{\partial q_1}\cdot\frac{\partial\boldsymbol{x}}{\partial q_2}=0 \tag{C.132}$$

and

$$\begin{aligned}0=\frac{\partial}{\partial q_3}\left(\frac{\partial\boldsymbol{x}}{\partial q_1}\cdot\frac{\partial\boldsymbol{x}}{\partial q_2}\right)&=\frac{\partial\boldsymbol{x}}{\partial q_1}\cdot\frac{\partial^2\boldsymbol{x}}{\partial q_2\partial q_3}+\frac{\partial^2\boldsymbol{x}}{\partial q_1\partial q_3}\cdot\frac{\partial\boldsymbol{x}}{\partial q_2} && \text{(C.133)}\\ &=\frac{\partial\boldsymbol{x}}{\partial q_2}\cdot\frac{\partial^2\boldsymbol{x}}{\partial q_3\partial q_1}+\frac{\partial^2\boldsymbol{x}}{\partial q_2\partial q_1}\cdot\frac{\partial\boldsymbol{x}}{\partial q_3} && \text{(C.134)}\\ &=\frac{\partial\boldsymbol{x}}{\partial q_3}\cdot\frac{\partial^2\boldsymbol{x}}{\partial q_1\partial q_2}+\frac{\partial^2\boldsymbol{x}}{\partial q_3\partial q_2}\cdot\frac{\partial\boldsymbol{x}}{\partial q_1} && \text{(C.135)}\end{aligned}$$

then

$$\frac{\partial\boldsymbol{x}}{\partial q_1}\cdot\frac{\partial^2\boldsymbol{x}}{\partial q_2\partial q_3}+\frac{\partial\boldsymbol{x}}{\partial q_2}\cdot\frac{\partial^2\boldsymbol{x}}{\partial q_3\partial q_1}+\frac{\partial\boldsymbol{x}}{\partial q_3}\cdot\frac{\partial^2\boldsymbol{x}}{\partial q_1\partial q_2}=0 \tag{C.136}$$

and then

$$\frac{\partial\boldsymbol{x}}{\partial q_1}\cdot\frac{\partial^2\boldsymbol{x}}{\partial q_2\partial q_3}=0 \tag{C.137}$$

$$\frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_3} = 0 \tag{C.138}$$

$$\frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} = 0 \tag{C.139}$$

From

$$0 = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial}{\partial q_2}\left(\frac{\partial \boldsymbol{x}}{\partial q_3}\right) \tag{C.140}$$

$$= h_1 \hat{\boldsymbol{h}}_1 \cdot \frac{\partial h_3 \hat{\boldsymbol{h}}_3}{\partial q_2} \tag{C.141}$$

$$= h_1 \hat{\boldsymbol{h}}_1 \cdot \left(h_3 \frac{\partial \hat{\boldsymbol{h}}_3}{\partial q_2} + \hat{\boldsymbol{h}}_3 \frac{\partial h_3}{\partial q_2}\right) \tag{C.142}$$

$$= h_1 h_3 \hat{\boldsymbol{h}}_1 \cdot \frac{\partial \hat{\boldsymbol{h}}_3}{\partial q_2} \tag{C.143}$$

we have (similarly)

$$\frac{\partial \hat{\boldsymbol{h}}_i}{\partial q_j} \perp \hat{\boldsymbol{h}}_k,\ i \neq j \neq k \neq i. \tag{C.144}$$

Combining (C.131) and (C.144) we have

$$\frac{\partial \hat{\boldsymbol{h}}_i}{\partial q_j} // \hat{\boldsymbol{h}}_j,\ i \neq j, \tag{C.145}$$

and using

$$\frac{\partial^2 \boldsymbol{x}}{\partial q_i \partial q_j} = \frac{\partial^2 \boldsymbol{x}}{\partial q_j \partial q_i} \tag{C.146}$$

we have

$$\frac{\partial}{\partial q_j}\left(\frac{\partial \boldsymbol{x}}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\frac{\partial \boldsymbol{x}}{\partial q_j}\right) \tag{C.147}$$

$$\hat{\boldsymbol{h}}_i \frac{\partial h_i}{\partial q_j} + h_i \frac{\partial \hat{\boldsymbol{h}}_i}{\partial q_j} = \hat{\boldsymbol{h}}_j \frac{\partial h_j}{\partial q_i} + h_j \frac{\partial \hat{\boldsymbol{h}}_j}{\partial q_i} \tag{C.148}$$

with repeated indices not implying summation. Since $i \neq j$, $\hat{\boldsymbol{h}}_i$ and $\hat{\boldsymbol{h}}_j$ are linearly independent, we have

$$\frac{\partial \hat{\boldsymbol{h}}_i}{\partial q_j} = \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \hat{\boldsymbol{h}}_j. \tag{C.149}$$

Now we turn back and consider $\partial \hat{\boldsymbol{h}}_i / \partial q_j$.

$$\frac{\partial \hat{\boldsymbol{h}}_i}{\partial q_i} = \frac{\partial (\hat{\boldsymbol{h}}_j \times \boldsymbol{h}_k)}{\partial q_i} \tag{C.150}$$

$$= \frac{\partial \hat{\boldsymbol{h}}_j}{\partial q_i} \times \boldsymbol{h}_k + \boldsymbol{h}_j \times \frac{\partial \hat{\boldsymbol{h}}_k}{\partial q_i} \tag{C.151}$$

$$= \frac{1}{h_j} \frac{\partial h_i}{\partial q_j} \hat{\boldsymbol{h}}_i \times \hat{\boldsymbol{h}}_k + \hat{\boldsymbol{h}}_j \times \hat{\boldsymbol{h}}_i \frac{1}{h_k} \frac{\partial h_i}{\partial q_k} \tag{C.152}$$

$$= -\left(\frac{1}{h_j} \frac{\partial h_i}{\partial q_j} \hat{\boldsymbol{h}}_j + \frac{1}{h_k} \frac{\partial h_i}{\partial q_k} \hat{\boldsymbol{h}}_k\right) \tag{C.153}$$

without repeated indices being summed over.

Using the relations (C.149) and (C.153), gradient, curl, divergence, Laplacian, as well as operators like $\nabla \boldsymbol{u}$ and $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ can be expressed.

Example: $\nabla \cdot \boldsymbol{u}$.
We have before

$$\nabla = \frac{\hat{\boldsymbol{h}}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{\boldsymbol{h}}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{\boldsymbol{h}}_3}{h_3}\frac{\partial}{\partial q_3} \tag{C.154}$$

and now consider $\nabla \boldsymbol{u}$ with

$$\boldsymbol{u} = u_1\hat{\boldsymbol{h}}_1 + u_2\hat{\boldsymbol{h}}_2 + u_3\hat{\boldsymbol{h}}_3 \tag{C.155}$$

and we have

$$\nabla \cdot \boldsymbol{u} = \left(\frac{\hat{\boldsymbol{h}}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{\boldsymbol{h}}_2}{h_2}\frac{\partial}{\partial q_2}\hat{\boldsymbol{h}}_2 + \frac{\hat{\boldsymbol{h}}_3}{h_3}\frac{\partial}{\partial q_3}\hat{\boldsymbol{h}}_3\right) \cdot (u_1\hat{\boldsymbol{h}}_1 + u_2\hat{\boldsymbol{h}}_2 + u_3\hat{\boldsymbol{h}}_3) \tag{C.156}$$

$$= \frac{\hat{\boldsymbol{h}}_1}{h_1}\frac{\partial}{\partial q_1}(u_1\hat{\boldsymbol{h}}_1 + u_2\hat{\boldsymbol{h}}_2 + u_3\hat{\boldsymbol{h}}_3) + \dots \tag{C.157}$$

$$= \frac{1}{h_1}\left(\frac{\partial u_1}{\partial q_1} + \frac{u_2}{h_2}\frac{\partial h_1}{\partial q_2} + \frac{u_3}{h_3}\frac{\partial h_1}{\partial q_3}\right) \tag{C.158}$$

$$+ \frac{1}{h_2}\left(\frac{\partial u_2}{\partial q_2} + \frac{u_3}{h_3}\frac{\partial h_2}{\partial q_3} + \frac{u_1}{h_1}\frac{\partial h_2}{\partial q_1}\right) \tag{C.159}$$

$$+ \frac{1}{h_3}\left(\frac{\partial u_3}{\partial q_3} + \frac{u_1}{h_1}\frac{\partial h_3}{\partial q_1} + \frac{u_2}{h_2}\frac{\partial h_3}{\partial q_2}\right) \tag{C.160}$$

$$= \frac{1}{h_1h_2h_3}\left(\frac{\partial u_1h_2h_3}{\partial q_1} + \frac{\partial u_2h_1h_3}{\partial q_2} + \frac{\partial u_3h_1h_2}{\partial q_3}\right) \tag{C.161}$$

References: Appendices in Batchelor (1967); Griffiths (2013), and text book of Wu (1982).

C.3.6 Examples

1. Cartesian. $(q_1, q_2, q_3) = (x_1, x_2, x_3)$
   Elements:

$$h_1 = h_2 = h_3 = 1 \tag{C.162}$$

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{C.163}$$

$$\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \tag{C.164}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial x_1}\hat{\boldsymbol{e}}_1 + \frac{\partial f}{\partial x_2}\hat{\boldsymbol{e}}_2 + \frac{\partial f}{\partial x_3}\hat{\boldsymbol{e}}_3 \tag{C.165}$$

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{C.166}$$

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{C.167}$$

$$= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\hat{\boldsymbol{e}}_x + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\hat{\boldsymbol{e}}_y + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\hat{\boldsymbol{e}}_z \tag{C.168}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.169}$$

2. Cylindrical. $(q_1, q_2, q_3) = (r, \theta, z)$
   Elements:

$$h_1 = h_3 = 1, h_2 = r \tag{C.170}$$
$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2 \tag{C.171}$$
$$\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \tag{C.172}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{e}}_\theta + \frac{\partial f}{\partial z}\hat{\boldsymbol{e}}_z \tag{C.173}$$
$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\left(\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial(rw)}{\partial z}\right) \tag{C.174}$$
$$= \frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.175}$$
$$\nabla \times \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\theta & \hat{\boldsymbol{e}}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix} \tag{C.176}$$
$$= (\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z})\hat{\boldsymbol{e}}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\hat{\boldsymbol{e}}_\theta + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta})\hat{\boldsymbol{e}}_z \tag{C.177}$$
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.178}$$

3. Spherical. $(q_1, q_2, q_3) = (r, \phi, \theta)$, ϕ is the polar angle and θ is the azimuthal.
   Elements:

$$h_1 = 1, h_2 = r, h_3 = r\sin\phi \tag{C.179}$$
$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + r^2\sin^2\phi\mathrm{d}\theta^2 \tag{C.180}$$
$$\mathrm{d}V = r^2\sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta \tag{C.181}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{e}}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{e}}_\theta \tag{C.182}$$
$$\nabla \cdot \boldsymbol{u} = \frac{1}{r^2\sin\phi}\left(\frac{\partial(r^2\sin\phi u)}{\partial r} + \frac{\partial(r\sin\phi v)}{\partial \phi} + \frac{\partial(rw)}{\partial \theta}\right) \tag{C.183}$$
$$= \frac{1}{r^2}\frac{\partial(r^2 u)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(\sin\phi v)}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial w}{\partial \theta} \tag{C.184}$$
$$\nabla \times \boldsymbol{u} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{e}}_r & r\hat{\boldsymbol{e}}_\phi & r\sin\phi\hat{\boldsymbol{e}}_\theta \\ \partial_r & \partial_\phi & \partial_\theta \\ u & rv & r\sin\phi w \end{vmatrix} \tag{C.185}$$
$$= \frac{1}{r\sin\phi}(\frac{\partial \sin\phi w}{\partial \phi} - \frac{\partial v}{\partial \theta})\hat{\boldsymbol{e}}_r + (\frac{1}{r\sin\phi}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial rw}{\partial r})\hat{\boldsymbol{e}}_\phi + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \phi})\hat{\boldsymbol{e}}_\theta \tag{C.186}$$
$$\nabla^2 f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin\phi}\frac{\partial^2 f}{\partial \theta^2} \tag{C.187}$$

D Hyperbolic functions

D.1 Defining ODEs

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