A note on hydrostatics and geostrophy

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Abstract: In this note the validity and relevant scaling relations for the two axiomatic assumptions in geophysical flows – geostrophic and hydrostatic, are discussed. The first section will define symbols and notations and separate discussions on the two assumptions will follow.

CONTENTS

1. Background and notation definition

1.1. *The ambient pressure and the dymanic pressure*

The density is decomposed as

$$\rho(x,y,z,t) = \rho_0 + \bar{\rho}(z) + \rho'(x,y,z,t) \tag{1.1}$$

z-momentum for a steady state:

$$\frac{\partial p_0}{\partial z} = -\rho g \tag{1.2}$$

where p_0 is the ambient pressure as in

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t) \tag{1.3}$$

For a constantly and stably stratified fluid:

$$\frac{\partial \bar{\rho}(z)}{\partial z} = -\frac{\rho_0}{g} N^2 \tag{1.4}$$

and hence

$$\bar{\rho}(z) = \rho_0 - \int_{z_0}^{z} N^2 \frac{\rho_0}{g}\,\mathrm{d}z = \rho_0 (1 - N^2 \frac{z}{g}) \tag{1.5}$$

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Integration of Eq. (1.3) gives

$$p_0 = p_a - \rho_0 g(z - z_0) + \frac{\rho_0}{2g} N^2 (z - z_0)^2 \tag{1.6}$$

where the three parts contributing to the pressure are: the surface/reference pressure $p_a = p(z = z_0)$, the pressure due to depth increase, and the one due to density variation. Note that $z < z_0$.

1.2. *Bousinessq approximation*

Note that we denote here

$$\rho(z) = \rho_0 + \bar{\rho}(z) \tag{1.7}$$

instead of Eq. (1.1).

z-momentum in Euler equation:

$$\frac{Dw}{Dt} = -\frac{1}{\rho(z) + \rho'} \frac{\partial}{\partial z}(p_0 + p') - g \tag{1.8}$$

$$(\rho(z) + \overset{\text{small}}{\cancel{\rho'}})\frac{Dw}{Dt} = -\frac{\partial}{\partial z}(\overset{\text{ambient}}{\cancel{p_0}} + p') - (\overset{\text{ambient}}{\cancel{\rho(z)}} + \rho')g \tag{1.9}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b \tag{1.10}$$

where the buoyancy is defined as

$$b = -\frac{\rho' g}{\rho_0} \tag{1.11}$$

1.3. *Buoyancy equation*

From Eq. (1.11),

$$\frac{Db}{Dt} = -\frac{g}{\rho_0}\frac{D\rho'}{Dt} \tag{1.12}$$

Continuity:

$$\frac{D\rho}{Dt} = \frac{D\rho'}{Dt} + w\frac{\partial \bar{\rho}}{\partial z} = 0 \tag{1.13}$$

$$\frac{D\rho'}{Dt} = wN^2 \frac{\rho_0}{g} \tag{1.14}$$

and hence

$$\frac{Db}{Dt} + wN^2 = 0 \tag{1.15}$$

1.4. *Hydrostatic*

Hydrostatic assumption is almost axiomatic in ocean models. It is valid in some situations. The sections below follows basically Marshall *et al.* (1997) but steps and notations might defer.

(i) fully hydrostatic:

$$\frac{\partial p_0}{\partial z} = -(\rho_0 + \bar{\rho}(z))g \tag{1.16}$$

$$\frac{\partial p'}{\partial z} = \rho_0 b = -\rho' g \tag{1.17}$$

Summing together Eqs. (1.16) and (1.17) we have

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}(p_0 + p') = -(\rho_0 + \bar{\rho} + \rho')g = -\rho g \tag{1.18}$$

which is the standard hydrostatic assumption.

(ii) non-hydrostatic:

$$\frac{\partial p_0}{\partial z} = -(\rho_0 + \bar{\rho}(z))g \tag{1.19}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b,\ b = -\frac{\rho' g}{\rho_0} \tag{1.20}$$

$$\tag{1.21}$$

Summing together Eqs. (1.19) and (1.20) we have

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{1.22}$$

which is again the full z-momentum equation.

2. Buoyancy scaling for hydrostatic flows

The buoyancy equation (1.15) gives

$$w \sim U\frac{b}{L}\frac{1}{N^2} \tag{2.1}$$

In Eq. (1.10) hydrostatic requires the order-of-magnitude relation

$$-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} \sim b \gg \frac{Dw}{Dt} \tag{2.2}$$

and hence

$$\frac{U^2}{L^2 N^2} \ll 1 \tag{2.3}$$

Here L and h are horizontal and vertical length scales, respectively.

The Froude and Richardson numbers are

$$Fr = \frac{U}{Nh} \tag{2.4}$$

$$Ri = \frac{N^2 h^2}{U^2} \tag{2.5}$$

and the condition for the hydrostatic limit is

$$n \triangleq \frac{U^2}{L^2 N^2} = \frac{\gamma^2}{Ri} \ll 1 \tag{2.6}$$

or equivalently

$$\gamma Fr \ll 1 \tag{2.7}$$

where

$$\gamma = \frac{h}{L} \tag{2.8}$$

is the aspect ratio.

Comments on conditions for hydrostatic flows:

(i) strong stratification / weak flow / low Fr

(ii) small aspect ratio, could be in terms of 'pancake' vortices or 'pancake' turbulence.

3. Quasi-hydrostatic

In Eq. (1.10) assume there is one 'hydrostatic' part of the hydrostatic pressure p_{HY} that balances the buoyancy as in Eq. (1.17) and the remainder p_{NH} – the non-hydrostatic pressure.

$$p' = p_{\mathrm{HY}} + p_{\mathrm{NH}} \tag{3.1}$$

Note that ambient pressure is also 'hydrostatic' but excluded in this part.

The ratio

$$n = \frac{p_{\mathrm{NH}}}{p_{\mathrm{HY}}} \tag{3.2}$$

is call the hydrostatic index, which measures quantitatively to which extent the hydrostatic assumption is valid and will be shown later to appear as a small parameter (in the case of quasi-hydrostatic flows) in the pertubation expansion of the governing equations.

From

$$-\frac{1}{\rho_0}\frac{\partial p_{\mathrm{HY}}}{\partial z} + b = 0 \tag{3.3}$$

we could have

$$p_{\mathrm{HY}} = \int_{z_0}^{z} \rho_0 b \, \mathrm{d}z = \int_{z_0}^{z} -\rho' g \, \mathrm{d}z \tag{3.4}$$

and then

$$p_{\mathrm{NH}} = p' - p_{\mathrm{HY}}. \tag{3.5}$$

The more efficient way might be using

$$n = \frac{\gamma^2}{Ri} \tag{3.6}$$

than using (3.2) which will be shown equivalent in the momentum equation scaling later.

Governing equations are

$$\frac{D\boldsymbol{u}_{\mathrm{h}}}{Dt} = -\frac{1}{\rho}\nabla_{\mathrm{h}}(p_0 + p_{\mathrm{HY}} + p_{\mathrm{NH}}) - f\hat{\boldsymbol{k}} \times \boldsymbol{u}_{\mathrm{h}} \tag{3.7}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p_{\mathrm{NH}}}{\partial z} \tag{3.8}$$

$$\frac{Db}{Dt} + N^2 w = 0 \tag{3.9}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{3.10}$$

where $\boldsymbol{u} = (u, v, w)$ and $\boldsymbol{u}_{\mathrm{h}} = (u, v)$.

4. Quasi-geostrophic

Following the section above, we separate the scalings for the hydrostatic and non-hydrostatic parts in pressure and exam the geostrophic condition.

Scales:

$$u, v : U \tag{4.1}$$
$$w : W \tag{4.2}$$
$$x, y : L \tag{4.3}$$
$$z : h \tag{4.4}$$
$$p_0, p_{\mathrm{HY}} : P_{\mathrm{HY}} \tag{4.5}$$
$$p_{\mathrm{Nh}} : P_{\mathrm{NH}} \tag{4.6}$$
$$f : F \tag{4.7}$$
$$b : g\rho'/\rho_0 \tag{4.8}$$
$$N : -\frac{g}{\rho_0}\frac{\Delta\rho}{h} \tag{4.9}$$

(1) Geostrophy scaling:

$$\frac{1}{\rho_0}\frac{P_{\mathrm{HY}}}{L} \sim FU \to \frac{P_{\mathrm{HY}}}{\rho_0 U^2} \sim \frac{FL}{U} = \frac{1}{Ro} \tag{4.10}$$

(2) Hydrostatic pressure scaling:

$$b = -\frac{\rho' g}{\rho_0} = \frac{1}{\rho_0}\frac{\partial P_{\mathrm{HY}}}{\partial z} \to P_{\mathrm{HY}} \sim \rho' g h \tag{4.11}$$

(3) Continuity scaling: (assumingly, and commented on later)

$$\frac{U}{L} \sim \frac{1}{Ro}\frac{W}{h} \tag{4.12}$$

(4) Buoyancy scaling:

$$\frac{Db}{Dt} = -N^2 w \to \frac{U}{L}\frac{\rho' g}{\rho_0} \sim w\frac{g}{\rho_0}\frac{\Delta\rho}{h} \to \frac{\rho'}{\Delta\rho} \sim \frac{WL}{Uh} \sim Ro \tag{4.13}$$

Combining Eqs. (4.10) – (4.13) we have

$$Ri Ro^2 \sim 1 \to Bu = \frac{Ro^2}{Fr^2} \sim 1 \tag{4.14}$$

which is the celebrated condition for (quasi-) **geostrophy**. Note that by using (4.11) we don't apply hydrostatic assumption but just use the hydrostatic pressure scaling.

Comment on continuity equation scaling:

(i) In shallow water (with rotation) systems, the continuity equation is converted into the equation for the surface water displacement height η, and the scaling in Eq. (4.12) works when the wave height scales as

$$\eta^* = \frac{g}{fUL}\eta, \tag{4.15}$$

according to Vallis (2017).

(ii) In general, we may naively expect

$$\frac{\partial u}{\partial x} \sim \frac{\partial w}{\partial z} \to \frac{U}{L} \sim \frac{W}{h} \tag{4.16}$$

but it doesn't work. Firstly, the Rossby number order zero expansion gives the geostrophic balance (it results from the momentum equation instead of the continuity equation!)

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0. \tag{4.17}$$

From

$$\frac{\partial w}{\partial z} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}), \tag{4.18}$$

we know $\frac{\partial w}{\partial z}$ scales with the horizontal velocity divergence, which is small according to Eq. (4.17), instead of scaling with $\frac{\partial u}{\partial x}$ or $\frac{\partial v}{\partial y}$. So

$$\frac{W}{h} \sim Ro\frac{U}{L} \tag{4.19}$$

could be a safe guess. But it is not guaranteed to work (Vallis 2017).

Comment on density fluctuation in geostrophic flow:

From Eq. (4.13) we learn that

$$\frac{\rho'}{\Delta\rho} \sim Ro \tag{4.20}$$

which means in order to have non-hydrostatic pressure, the density fluctuation should be smaller than what the upwelling of the background density can compensate, in case of geostrophy.

5. Non-dimensional governing equations

Assuming quasi-hydrostatic and quasi-geostrophic, i.e., the hydrostatic index n and Ro are both small, we can rewrite the governing equation into the following non-dimensional form with small parameters:

$$\frac{D\boldsymbol{u}_\mathrm{h}}{Dt} = -\frac{1}{Ro}\nabla_\mathrm{h}(p_0 + p_\mathrm{HY} + np_\mathrm{NH}) - \frac{1}{Ro}f\hat{\boldsymbol{k}} \times \boldsymbol{u}_\mathrm{h} \tag{5.1}$$

$$\frac{Dw}{Dt} = -\frac{\partial p_\mathrm{NH}}{\partial z} \tag{5.2}$$

$$\frac{Db}{Dt} + N^2 w = 0 \tag{5.3}$$

$$\nabla_\mathrm{h} \cdot \boldsymbol{u}_\mathrm{h} + Ro\frac{\partial w}{\partial z} = 0 \tag{5.4}$$

6. Hydrostatic index revisited

Scaling of hydrostatic pressure:

$$\frac{1}{\rho_0}\frac{\partial p_\mathrm{HY}}{\partial z} = b \triangleq -\frac{\rho' g}{\rho_0} \to P_\mathrm{HY} \sim \rho' gh \tag{6.1}$$

Scaling of non-hydrostatic pressure:

$$\frac{Dw}{Dt} \sim \frac{1}{\rho_0}\frac{\partial p_\mathrm{NH}}{\partial z} \to U\frac{W}{L} \sim \frac{1}{\rho_0}\frac{P_\mathrm{NH}}{h} \to P_\mathrm{NH} \sim \frac{UWh}{L}\rho_0 \tag{6.2}$$

6.1. *Geostrophic flow*

Hydrostatic index:

$$n \triangleq \frac{P_{\mathrm{NH}}}{P_{\mathrm{HY}}} = \frac{U^2}{L^2}\frac{h}{\rho'}\frac{\rho_0}{g} = \frac{U^2}{L^2N^2} = \gamma^2 Fr^2 \tag{6.3}$$

Note that the *Ro*-scaling (4.20) for fluctuating density ρ' is used and *Ro* finally cancelled out in both numerator and denominator. Note also the use of velocity scaling (4.19) which is also under geostrophic assunmption.

6.2. *Non-geostrophic flow*

The velocity scaling (4.19) falls back to

$$\frac{W}{h} \sim \frac{U}{L} \tag{6.4}$$

given *Ro* is no longer small. And the buoyancy scaling (4.13) follows as

$$\frac{\rho'}{\Delta\rho} \sim \frac{WL}{Uh} \sim \mathcal{O}(1) \tag{6.5}$$

and we resume

$$n \triangleq \frac{P_{\mathrm{NH}}}{P_{\mathrm{HY}}} = \frac{U^2}{L^2N^2} = \gamma^2 Fr^2 \tag{6.6}$$

Comment:

From Eqs. (6.3) and (6.6) we learn that the hydrostatic assumption is independent on geostrophy, as the opposite is also true.

And the Rossby and the Froude numbers can be regarded as any other relevant time scales devided by f or N, not necessarily the global/large time scale $U/L, W/h$ but could also be some other scales defined locally.

References

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Vallis, Geoffrey K 2017 *Atmospheric and oceanic fluid dynamics*. Cambridge University Press.