First principles in fluids

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Abstract: This note, intended for being used as a quick reference, provides a collection of a wide range of equations in fluid mechanics, from basic equations that can be found in introductory textbooks, to those only left as an exercise or a conclusion in graduate textbooks, monographs, or research papers, the detailed derivations of which are typically not provided. We try to use symbols and notations as consistently as possible, and it is unavoidable that this note is biased according to the author's own preference and taste. These principles are underlying how this World works. A note of caution: it is mostly on mathematical descriptions but not how fluids flow. This note is still under construction.

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1 Conservation laws

1.1 Generic conservation laws

Consider a control volume V enclosed by a surface A and a scalar ψ (per unit mass) carried by the fluid. The rate of change of the total scalar in V, in the absence of any source or sink, can be established as

$$\frac{d}{dt}\left(\int_V \psi(\boldsymbol{x},t)dm\right) = \frac{d}{dt}\left(\int_V \psi(\boldsymbol{x},t)\rho(\boldsymbol{x},t)dV\right) \tag{1.1}$$

$$= -\int_{A=\partial V} \rho\psi\boldsymbol{u}\cdot\boldsymbol{n}dA \tag{1.2}$$

$$= -\int_V \nabla\cdot(\rho\psi\boldsymbol{u})dV \tag{1.3}$$

Hence,

$$\int_V \left[\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u})\right] dV = 0. \tag{1.4}$$

Since the control volume is arbitrary and can be shrunk to infinitesimal, we have the differential form

$$\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u}) = 0. \tag{1.5}$$

Here, the scalar can be taken to be mass ($\psi = 1$), momentum ($\psi := \boldsymbol{u}$), or any other scalar ($\psi = X_i$, where X_i is the species concentration).

Example: Moving control volume.

In what follows, we will discuss several conservation laws. Ref. Chorin *et al.* (1990).

1.2 Mass conservation

Taking $\psi = 1$,

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right] dV = 0. \tag{1.6}$$

In differential form:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0. \tag{1.7}$$

More over, using the definition of material derivative, we have

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = \frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho + \rho(\nabla\cdot\boldsymbol{u}) \tag{1.8}$$

$$= \frac{D\rho}{Dt} + \rho(\nabla\cdot\boldsymbol{u}) \tag{1.9}$$

$$= 0. \tag{1.10}$$

1.2.1 Incompressibility, continuity

It can be seen from (1.9) that solenoidality of the velocity field ($\nabla\cdot\boldsymbol{u} = 0$) is a sufficient and necessary condition for imcompressibility ($D\rho/Dt = 0$, density doesn't change along the material). The physical meaning is as the following.

Consider a cubic finite volume $dV = dxdydz$, the volume change is

$$\delta(dV) = [(u + \partial_x u dx) - u]dydzdt + [(v + \partial_y v dy) - v]dxdzdt + [(w + \partial_z w dz) - w]dxdydt \tag{1.11}$$

$$= (\partial_x u + \partial_y v + \partial_x w)dxdydzdt. \tag{1.12}$$

We have the divergence of a fluid parcel

$$\nabla \cdot \boldsymbol{u} = \partial_x u + \partial_y v + \partial_x w = \frac{1}{dV}\frac{\delta(dV)}{\delta t}, \tag{1.13}$$

which is the volume change rate per unit volumn. The specific volume (not to be confused with viscosity) is defined as $\nu = 1/\rho$. We can also have

$$\nabla \cdot \boldsymbol{u} = \frac{1}{\nu}\frac{D\nu}{Dt}, \tag{1.14}$$

which is stating that the divergence of the velocity filed is the normalized rate of change of specific volume.

One might have already been satisfied with such statements. Actually, there is more to it. Note on incompressibility condition - Drho/Dt, and the volume change rate relation ,and the thermal effects (EOS).

Ref. Batchelor (1967).

1.3 Material derivative and the Reynolds transport theorem

1.3.1 Material derivative

Consider the total time derivative of a scalar ψ:

$$\frac{d}{dt}\psi(\boldsymbol{x}(t), t) = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt} = \frac{\partial \psi}{\partial t} + \boldsymbol{u} \cdot \nabla \psi. \tag{1.15}$$

We define

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \tag{1.16}$$

as the material derivative, with $\partial/\partial t$ being the local rate-of-change and $\boldsymbol{u} \cdot \nabla$ being the convective derivative.

This is actually a bridge between the Eulerian and Lagrangian description of fluids.

1.3.2 Reynolds transport theorem

1.3.3 A Lagrangian perspective

An Eulerian field is

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\varphi(\boldsymbol{x}_0, t), t) \tag{1.17}$$

where $\varphi : \boldsymbol{x}_0 \to \boldsymbol{x}$ is a smooth and invertible flow mapping that maps the initial location of the fluid parcel $\boldsymbol{x}_0$ at $t = 0$ to $\boldsymbol{x}$ at t such that

$$\boldsymbol{x} = \varphi(\boldsymbol{x}_0, t). \tag{1.18}$$

1.4 Momentum conservation

Let $\psi = \boldsymbol{u}$ to be the quantity being transported. The change of momentum in V is equal to the momentum flux in the direction $\boldsymbol{n}$ and volumetric contribution from the external body forcing per unit mass $\boldsymbol{f}$ ($\boldsymbol{f} = \lim_{\Delta V \to 0} \Delta \boldsymbol{F}/\Delta m$):

$$\frac{d}{dt}\int_V \rho \boldsymbol{u} dV = -\int_A (\rho \boldsymbol{u})\boldsymbol{u} \cdot \boldsymbol{n} dA + \int_A \boldsymbol{\sigma} \cdot \boldsymbol{n} dA + \int_V \rho \boldsymbol{f} dV \tag{1.19}$$

$$= -\int_V \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) dV + \int_V (\nabla \cdot \boldsymbol{\sigma}) dV + \int_V \rho \boldsymbol{f} dV, \tag{1.20}$$

where $\boldsymbol{\sigma} = \lim_{\Delta A \to 0} \Delta \boldsymbol{F}/\Delta \boldsymbol{A}$ is the stress tensor.

We have the integration from of the Euler equation:

$$\int_V [\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u})]\, dV = \int_v [\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}] dV \tag{1.21}$$

which is valid for a finite volume V. Hence, (the differential form)

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}. \tag{1.22}$$

For example, if the body force is gravity, $\boldsymbol{f} = \boldsymbol{g}$. Or if the body force is Coriolis, $\boldsymbol{f} = \boldsymbol{u} \times \boldsymbol{f}_c$.
In general, for continuum mechanics, the force balance can generally be written as

$$\rho \ddot{\boldsymbol{u}} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} \tag{1.23}$$

where $-\rho\ddot{\boldsymbol{u}}$ is D'Alembert's force (with $\boldsymbol{u}$ being the displacement instead), $\boldsymbol{\sigma}$ is the stress tensor, and $\boldsymbol{f}$ is the body force per unit volume.

Now we consider the sources of stresses.

- Pressure. It is direction is $-\boldsymbol{n}$ so $\boldsymbol{\sigma}_p = -p\boldsymbol{I}$.
- Viscous stress. For imcompressible Newtonian fluid (see section 1.7), $\boldsymbol{\sigma}_v = \boldsymbol{\tau} = 2\mu \boldsymbol{S}$.
- Hence,
$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu \boldsymbol{S}. \tag{1.24}$$

Hence, the momemtum conservation is

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{f}. \tag{1.25}$$

We note that $\partial_j(2\mu S_{ij}) = \partial_j(\mu \partial_j u_i)$.

1.5 Energy conservation

1.6 Bernoulli equation

It is a combination of momemtum and energy conservations. (?)

Assumptions:

- Inviscid.
- Barotropic.
- Potential force.
- Steady.

Consider the inviscid Euler equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{F} \tag{1.26}$$

with a conservative force

$$\boldsymbol{F} = -\nabla \Phi. \tag{1.27}$$

Using Eq. (A.20) we have

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} \tag{1.28}$$

We defined

$$\boldsymbol{L} = \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{\omega} \tag{1.29}$$

which is called the Lamb vector.

The Euler equation becomes

$$\nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{p}{\rho} \right) - \nabla \Phi \tag{1.30}$$

and hence

$$\nabla \left(\frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi \right) = \boldsymbol{L} \tag{1.31}$$

which is called the Lamb-Gromyko equation. Define

$$H = \frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi \tag{1.32}$$

we have

$$\nabla H = \boldsymbol{L}. \tag{1.33}$$

If the flow is irrotational, i.e., $\boldsymbol{L} = 0$, we recover a special version (isentropic) of Bernuolli's theorem.

1.7 Constitutive relations

A note on angular momentum conservation.

A difference between solid and fluid mechanics is that, in solid mechanics, stress is proportional to strain, while in fluid mechanics, stress is proportional to strain rate.

Some comments about stress–strain relation in the solids.

1.8 Pressure Poisson

Take the divergence of the follow equation for incompressible flows ($\nabla \cdot \boldsymbol{u} = 0$):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} \tag{1.34}$$

we have

$$-\frac{1}{\rho} \nabla^2 p = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u} \nabla + \boldsymbol{u} \cdot \nabla (\nabla \cdot \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u} \nabla \tag{1.35}$$

i.e.

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_i} = -\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \tag{1.36}$$

In CFD, the continuity equation ($\nabla \cdot \boldsymbol{u} = 0$) is responsible for solving pressure for the above reason.

2 Vortex dynamics

Vorticity is defined as the curl of the velocity field:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{2.1}$$

2.1 Kelvin's theorem

Ref. Kundu and Cohen.

The circulation along a closed line is defined as

$$\Gamma = \oint_l \boldsymbol{u} \cdot d\boldsymbol{x} = \iint_A (\nabla \times \boldsymbol{u}) \cdot d\boldsymbol{A}, \tag{2.2}$$

according to Stokes' theorem.

Consider the momentum equation

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}. \tag{2.3}$$

The material (Lagrangian) derivative of Γ is

$$\begin{aligned}
\frac{D\Gamma}{Dt} &= \frac{D}{Dt}\left(\oint_l \boldsymbol{u} \cdot d\boldsymbol{x}\right) && (2.4)\\
&= \oint_l \frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{x} + \oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} && (2.5)\\
&= -\oint_l \left(\frac{1}{\rho}\nabla p\right) \cdot d\boldsymbol{x} + \oint_l (\nabla \cdot \boldsymbol{\sigma}) \cdot d\boldsymbol{x} + \oint_l \boldsymbol{f} \cdot d\boldsymbol{x} + \oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} && (2.6)
\end{aligned}$$

Assumptions:

1. Inviscid fluid: $\nabla \cdot \boldsymbol{\sigma} = 0$.
2. Conservative body force: $\boldsymbol{f} = -\nabla\Phi, \nabla \times \boldsymbol{f} = 0$ (it is the gradient of a potential field). Conservative means the potential difference does not depend on the path:

   $$\int_A^B \boldsymbol{f} \cdot d\boldsymbol{x} = -\int_A^B \nabla\Phi \cdot d\boldsymbol{x} = -\int_A^B d\Phi = \Phi_A - \Phi_B. \tag{2.7}$$

   For a close path, $A = B$ hence the integral is zero.
3. Barotropic flow: $\nabla\rho \times \nabla p = 0$.

Moreover, by

$$\boldsymbol{u} + d\boldsymbol{u} = \frac{D}{Dt}(\boldsymbol{x} + d\boldsymbol{x}) = \frac{D\boldsymbol{x}}{Dt} + \frac{D(d\boldsymbol{x})}{Dt}, \tag{2.8}$$

we have

$$\frac{D(d\boldsymbol{x})}{Dt} = d\boldsymbol{u} = d\boldsymbol{x} \cdot \nabla\boldsymbol{u} \tag{2.9}$$

so the last term in (2.6) is

$$\oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} = \oint_l \boldsymbol{u} \cdot d\boldsymbol{u} = \oint_l d(\boldsymbol{u}^2) = 0. \tag{2.10}$$

Then we are able to prove all RHS terms in (2.6) are zero hence

$$\frac{D\Gamma}{Dt} = 0 \tag{2.11}$$

along any arbitrary closed curves.

2.1.1 Helmholtz's theorems

2.2 Vorticity transport equation

By Eq. (A.11) we know

$$\nabla \cdot \boldsymbol{\omega} = 0 \tag{2.12}$$

i.e., the continuity of vorticity.

The incompressible Navier–Stokes equation in vector form:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f} \tag{2.13}$$

Using Eq. (1.28) we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\frac{\boldsymbol{u}^2}{2} + \boldsymbol{\omega}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F} \tag{2.14}$$

Using the (A.21) and take the curl of Eq. (2.14)

$$\begin{aligned}
\text{LHS} &= \nabla\times(\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\frac{\boldsymbol{u}^2}{2} + \boldsymbol{\omega}\times\boldsymbol{u}) && (2.15)\\
&= \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{u}) && (2.16)\\
&= \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \boldsymbol{\omega}(\nabla\cdot\boldsymbol{u}) - \boldsymbol{u}(\nabla\cdot\boldsymbol{\omega}) && (2.17)\\
&= \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} && (2.18)\\
\text{RHS} &= \nabla\times(-\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}) && (2.19)\\
&= \nu\nabla^2\boldsymbol{\omega} + \nabla\times\boldsymbol{f} + \frac{1}{\rho^2}\nabla\rho\times\nabla p && (2.20)
\end{aligned}$$

Equaling both sides we obtain

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t}}_{\substack{\text{rate of}\\\text{change}}} + \underbrace{\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}}_{\text{advection}} = \underbrace{\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}}_{\substack{\text{vortex}\\\text{stretching}}} + \underbrace{\nu\nabla^2\boldsymbol{\omega}}_{\substack{\text{viscous}\\\text{diffusion}}} + \underbrace{\nabla\times\boldsymbol{f}}_{\substack{\text{external torque in}\\\text{a non-conservative field}}} + \underbrace{\frac{1}{\rho^2}\nabla\rho\times\nabla p}_{\substack{\text{baroclinic}\\\text{torque}}} \tag{2.21}$$

Again, the stretching term $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}$ comes from the non-linear advection/inertial term in N-S. It is important in the energy cascade in turbulence.

2.3 Enstrophy equation

Define enstrophy as:

$$\mathcal{E} \triangleq \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \frac{1}{2}\omega_i\omega_i \tag{2.22}$$

Re-write (2.21) into tensor notation we have

$$\frac{\partial \omega_i}{\partial t} + u_j\frac{\partial \omega_i}{\partial x_j} = \omega_j\frac{\partial u_i}{\partial x_j} + \nu\frac{\partial^2\omega_i}{\partial x_j^2} + \epsilon_{ijk}\frac{\partial f_k}{\partial x_j} + \frac{1}{\rho^2}\epsilon_{ijk}\frac{\partial \rho}{\partial x_j}\frac{\partial p}{\partial x_k} \tag{2.23}$$

$\omega_i \times$ (2.23) we have

$$\frac{\partial}{\partial t}(\frac{1}{2}\omega_i\omega_i) + u_j\frac{\partial}{\partial x_j}(\frac{1}{2}\omega_i\omega_i) = \omega_i\omega_j\frac{\partial u_i}{\partial x_j} + \nu\frac{\partial^2}{\partial x_j^2}(\frac{1}{2}\omega_i\omega_i) - \nu\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} \tag{2.24}$$

$$+\epsilon_{ijk}\omega_i\frac{\partial F_k}{\partial x_j}+\frac{1}{\rho^2}\epsilon_{ijk}\omega_i\frac{\partial\rho}{\partial x_j}\frac{\partial p}{\partial x_k} \tag{2.25}$$

Note that ϵ_{ijk} is the Levi-Civita symbol, not to be confused with the turbulent kinetic energy rate ε or Reynols stresses dissipation rate ε_{ij}.

Re-write back into vector form:

$$\frac{\partial\mathcal{E}}{\partial t}+\boldsymbol{u}\cdot\nabla\mathcal{E}=\boldsymbol{\omega}\boldsymbol{\omega}:\nabla\boldsymbol{u}+\nu\nabla^2\mathcal{E}\underbrace{-\nu\nabla\boldsymbol{\omega}:\nabla\boldsymbol{\omega}}_{\text{viscous dissipation}}+\boldsymbol{\omega}\cdot(\nabla\times\boldsymbol{F})+\boldsymbol{\omega}\cdot\frac{\nabla\rho\times\nabla p}{\rho^2} \tag{2.26}$$

Note that we are assuming an incompressible flow, hence $\nabla\cdot\boldsymbol{u}$ related terms are not appearing in Eq. (2.26). A new mechanism compared to (2.21) is the viscous dissipation of enstrophy. This term is always negative.

2.4 Vortex models

2.4.1 Point vortex system

2.4.2 Lamb-Oseen similarity solution

3 Velocity gradient tensor, its decomposition and dynamics

Following the previous section, we continue to consider the local flow structures represented by the velocity gradients.

The velocity gradient tensor $\boldsymbol{u}\nabla$ is

$$\boldsymbol{u}\nabla=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial w}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial w}\end{bmatrix} \tag{3.1}$$

and in entity notation

$$(\boldsymbol{u}\nabla)_{ij}=\frac{\partial u_i}{\partial x_j}, \tag{3.2}$$

and we note that is a Jacobian such that

$$\delta\boldsymbol{u}=(\boldsymbol{u}\nabla)\cdot\delta\boldsymbol{x}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial w}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial w}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\\ \delta z\end{bmatrix}. \tag{3.3}$$

For an arbitrary tensor $\boldsymbol{A}$, it is always possible to decompose it into symmetric and antisymmetric (skew-symmetric) parts:

$$\boldsymbol{A}=\boldsymbol{S}+\boldsymbol{\Omega} \tag{3.4}$$

where

$$\boldsymbol{S}=\frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T) \tag{3.5}$$

$$\boldsymbol{\Omega}=\frac{1}{2}(\boldsymbol{A}-\boldsymbol{A}^T) \tag{3.6}$$

such that

$$\boldsymbol{S}^T=\boldsymbol{S},\ \boldsymbol{\Omega}^T=-\boldsymbol{\Omega}. \tag{3.7}$$

For the velocity gradient tensor, $\boldsymbol{S}=1/2(\boldsymbol{u}\nabla+\nabla\boldsymbol{u})$ is called the rate-of-strain tensor and $\boldsymbol{\Omega}=1/2(\boldsymbol{u}\nabla-\nabla\boldsymbol{u})$ is called the rotation tensor.

3.1 Pseudo-vector and associated antisymmetric rotation tensor

Vorticity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{3.8}$$

$$= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{3.9}$$

$$= \begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} \tag{3.10}$$

is a pseudo-vector ($\omega_i = \epsilon_{ijk}\partial_j u_k$) whose sign depends on the coordinate system (the order of i, j, k; left-hand or right-hand; cyclic or anticyclic), and is related to the antisymmetric part of velocity gradient tensor $\boldsymbol{u}\nabla$ (the rotation rate tensor $\boldsymbol{\Omega}$):

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{3.11}$$

or

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & -\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & -\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ -\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) & \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix} \tag{3.12}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3.13}$$

$$= \begin{bmatrix} 0 & -\omega_z^* & \omega_y^* \\ \omega_z^* & 0 & -\omega_x^* \\ -\omega_y^* & \omega_x^* & 0 \end{bmatrix} \tag{3.14}$$

where $\boldsymbol{\omega}^*$ is the angular velocity and $\boldsymbol{\omega} = 2\boldsymbol{\omega}^*$ (vorticity is twice of the angular velocity of the local solid-body rotation motion).

Each antisymmetric tensor $\boldsymbol{\Omega}$ can be represented by a pseudo-vector $\boldsymbol{\omega}^*$ (since it just has three independent elements), such that

$$\Omega_{ij} = -\epsilon_{ijk}\omega_k^* \tag{3.15}$$

$$\omega_k^* = -\frac{1}{2}\epsilon_{ijk}\Omega_{ij} \tag{3.16}$$

and according to (3.15)-(3.16) the inner product of the tensor $\boldsymbol{\Omega}$ with and arbitrary vector $\boldsymbol{a}$ can be written as

$$\boldsymbol{\Omega} \cdot \boldsymbol{a} = \boldsymbol{\omega}^* \times \boldsymbol{a}. \tag{3.17}$$

It is easy to verify (3.15) by definition and (3.16) using (A.8). As a corollary, we have

$$\boldsymbol{\Omega} \cdot \boldsymbol{\omega}^{\boldsymbol{*}} = \boldsymbol{\omega}^* \times \boldsymbol{\omega}^* = 0. \tag{3.18}$$

We can also find that

$$\|\boldsymbol{\Omega}\|^2 = \Omega_{ij}\Omega_{ij} = 2\delta_{kl}\omega_k^*\omega_l^* = \frac{1}{2}\boldsymbol{\omega}^2, \tag{3.19}$$

i.e., the norm of the rotation tensor is just the enstrophy.

Note the vorticity is $\boldsymbol{\omega} = 2\boldsymbol{\omega}^*$. We will further look into the eigenvalue decomposition and principle directions of $\boldsymbol{S}$ to further understand how it describes the geometry of the flow.

3.2 Strain rate tensor

$$\boldsymbol{S} = \begin{bmatrix} \epsilon_1 & \frac{1}{2}\gamma_3 & \frac{1}{2}\gamma_2 \\ \frac{1}{2}\gamma_3 & \epsilon_2 & \frac{1}{2}\gamma_1 \\ \frac{1}{2}\gamma_2 & \frac{1}{2}\gamma_1 & \epsilon_3 \end{bmatrix} \tag{3.20}$$

where the normal strain rate is

$$\epsilon_i = \frac{\partial u_{(i)}}{\partial x_{(i)}}, \tag{3.21}$$

indices inside parentheses don't imply summation, and the shear strain rate is

$$\gamma_i = \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right), \, i \neq j \neq k. \tag{3.22}$$

Additionally, it is easy to verify with index notation that

$$\boldsymbol{S} : \boldsymbol{\Omega} = 0, \tag{3.23}$$

where (:) denotes tensor inner-product, which induces the Frobenius norm. But the kinematic alignment between the principal directions of $\boldsymbol{S}$ and the vorticity vector is still an important question in turbulence (Ashurst *et al.*, 1987).

Examples.

1. Consider a pure stretching motion, $\epsilon_1 = \partial_x u \neq 0$ only.

   $$\frac{d(\delta x)}{dt} = u = \epsilon_1 \delta x \tag{3.24}$$

   hence

   $$\epsilon_1 = \frac{1}{\delta x}\frac{d(\delta x)}{dt} \tag{3.25}$$

   is the expansion rate (per unit time and per unit length).

2. Consider a pure rotation, $\gamma_1 \neq 0$ only. According to

   $$\gamma_1 = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{3.26}$$

   and

   $$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \tag{3.27}$$

we have

$$v = \frac{1}{2}\gamma_1 \delta z, \, w = \frac{1}{2}\gamma_1 \delta y \tag{3.28}$$

Define α_{23} as the angle between two line elements that are initially aligned with x_2 and x_3, we have (assumingi infinitesimal angle of rotaion during δt)

$$\frac{d\alpha_{23}}{dt} = \frac{w\delta t/\delta y - v\delta t/\delta z}{\delta t} = (\frac{1}{2}\gamma_1 - (-\frac{1}{2}\gamma_1)) = \gamma_1. \tag{3.29}$$

Hence, γ_i can be understood as the rate of change of two perpendicular elements in the plane normal to $\boldsymbol{e}_i$.

Motivation: during the deformation, can we find a set of axes the mutual angle between each pair is not change? That leads to the eigenvalues/principal strains and principal directions of the strain rate tensor.

3.3 Velocity field decomposition examples

1. Consider a plane constant-rate pure solid body rotation with the position vector being

$$\boldsymbol{r} = \cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y, \tag{3.30}$$

and we have

$$\begin{aligned}
\boldsymbol{u} &= \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &(3.31)\\
&= \dot{\theta}(-\sin\theta \boldsymbol{e}_x + \cos\theta \boldsymbol{e}_y) &(3.32)\\
&= \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \boldsymbol{e}_x \\ \sin\theta \boldsymbol{e}_y \end{bmatrix} &(3.33)\\
&= \boldsymbol{\Omega} \cdot \boldsymbol{r}. &(3.34)
\end{aligned}$$

With $\boldsymbol{\omega}^* = \dot{\theta}\boldsymbol{e}_z$ being the angular velocity, we have

$$\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{\Omega} \cdot \boldsymbol{r} = \boldsymbol{\omega}^* \times \boldsymbol{r} \tag{3.35}$$

where $\boldsymbol{\omega}^*$ is the angular velocity.

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix}, \tag{3.36}$$

In general, the decomposition of velocity gradient tensor can be applied to yield:

$$\begin{aligned}
\delta\boldsymbol{u} &= \boldsymbol{u}\nabla \cdot \delta\boldsymbol{x} &(3.37)\\
&= \boldsymbol{S} \cdot \delta\boldsymbol{x} + \boldsymbol{\Omega} \cdot \delta\boldsymbol{x} &(3.38)\\
&= \boldsymbol{S} \cdot \delta\boldsymbol{x} + \boldsymbol{\omega}^* \times \delta\boldsymbol{x} &(3.39)\\
&= \nabla\varphi + \boldsymbol{\omega}^* \times \delta\boldsymbol{x} &(3.40)
\end{aligned}$$

where the potential is

$$\begin{aligned}
\varphi &= \frac{1}{2}(\epsilon_1\delta x^2 + \epsilon_2\delta y^2 + \epsilon_3\delta z^2 + \gamma_1\delta y\delta z + \gamma_2\delta z\delta x + \gamma_3\delta x\delta y) &(3.41)\\
&= \frac{1}{2}\delta\boldsymbol{x} \cdot \boldsymbol{S} \cdot \delta\boldsymbol{x} &(3.42)
\end{aligned}$$

such that

$$\nabla\varphi = \boldsymbol{S} \cdot \delta\boldsymbol{x}. \tag{3.43}$$

According to the Helmholtz decomposition theorem, a vector field can be decomposed into the sum of an irrotational field (gradient field/curl-free) and a solenoidal field (curl field/divergence free). Here we provide a construction for the velocity field.

2. Consider plain shear flow with $U(y) = ay$ and

$$\boldsymbol{u}\nabla = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a/2 \\ a/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a/2 \\ -a/2 & 0 \end{bmatrix} = \boldsymbol{S} + \boldsymbol{\Omega}. \tag{3.44}$$

The flow is liearly decomposed into a pure deformation/strain motion with $\partial_y u = \partial_x v = a/2$ and a rotation motion with $\partial_y u = -\partial_x v = a/2$.

3.4 Dynamics of the velocity gradient tensor

3.4.1 Dynamics of A and its powers

Let

$$\boldsymbol{A} = \boldsymbol{u}\nabla = \left[\frac{\partial u_i}{\partial x_j}\right]. \tag{3.45}$$

Taking the gradient of Navier–Stokes we have

$$\frac{\partial \boldsymbol{A}}{\partial t} + u_k \frac{\partial \boldsymbol{A}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} + \nu \frac{\partial^2 \boldsymbol{A}}{\partial x_k^2} - \boldsymbol{A}\cdot\boldsymbol{A} \tag{3.46}$$

Contracting the indices in (3.46) we have

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_k^2} = \mathrm{tr}(\boldsymbol{A}^2) = -u_{i,j}u_{j,i} \tag{3.47}$$

and i.e.,

$$\nabla^2 p = 2\rho Q, \tag{3.48}$$

where

$$Q = \frac{1}{2}[(\mathrm{tr}\,\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)] \tag{3.49}$$

is the second invariant of $\boldsymbol{A}$ (see Appendix A.3). Hence, people argue that the sign of Q implies the existence of local pressure minimum and so forth the existence of vortices Hunt *et al.* (1988); Jeong & Hussain (1995).

The next question is the dynamics of its eigenvalues, invariants, and powers. We immediately deal with the latter since $\boldsymbol{A}^2$ appears in the equation of $\boldsymbol{A}$.

Re-write (3.46) as

$$\mathcal{L}(A_{ik}) = 0,\ \mathcal{L}(A_{kj}) = 0 \tag{3.50}$$

and multiply with A_{kj} and A_{ik} we have

$$\frac{\partial \boldsymbol{A}^2}{\partial t} + u_k \frac{\partial \boldsymbol{A}^2}{\partial x_k} = -\frac{1}{\rho}\left(\boldsymbol{A}\cdot\nabla(\nabla p) + \nabla(\nabla p)\cdot\boldsymbol{A}\right) + \nu\frac{\partial^2 \boldsymbol{A}^2}{\partial x_k^2} - 2\boldsymbol{A}^3. \tag{3.51}$$

Here $\nabla(\nabla p)$ is the pressure Hessian. We can see that, similar to the Reynolds average closure problem, the transport equations for $\boldsymbol{A}$ are never closed due to the appearance of even-higher powers.

Contracting the indices in (3.51) and assuming incompressibility ($\mathrm{tr}\,\boldsymbol{A} = 0$) we have

$$\frac{\partial Q}{\partial t} + u_k \frac{\partial Q}{\partial x_k} = \frac{1}{\rho}(\boldsymbol{A} : \nabla(\nabla p)) + \nu\frac{\partial^2 Q}{\partial x_k^2} + \mathrm{tr}(\boldsymbol{A}^3), \tag{3.52}$$

while we note that $-2Q = A_{ik}A_{ki}$.

3.4.2 Dynamics of S, $\boldsymbol{\Omega}$, and the eigenvalues of $\mathbf{S^2} + \boldsymbol{\Omega}^2$

(3.46)$^T \cdot \boldsymbol{A} +$ (3.46) $\cdot \boldsymbol{A}^T$ we have

$$\frac{\partial S_{ij}}{\partial t} + u_k \frac{\partial S_{ij}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} + \nu \frac{\partial^2 S_{ij}}{\partial x_k^2} - (S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj}) \tag{3.53}$$

Consider a balance between the last term in the RHS and pressure Hessain,

$$\nabla(\nabla p) = -\rho(\boldsymbol{\Omega}^2 + \boldsymbol{S}^2), \tag{3.54}$$

which can be, similarly to (3.48) (which can also be obtained by contracting (3.53)), be used to search for pressure minimum (Jeong & Hussain, 1995).

Assume $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\boldsymbol{\Omega}^2 + \boldsymbol{S}^2$. Since

$$Q = -\frac{1}{2}\operatorname{tr}(\boldsymbol{A}^2) = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\Omega}^2 + \boldsymbol{S}^2) = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3), \tag{3.55}$$

a positive Q corresponds to $\lambda_1 + \lambda_2 + \lambda_3 < 0$. Then assume $-\lambda_3 > -\lambda_2 > -\lambda_1$, we have $\lambda_3 < 0, \lambda_1 > 0$. The requirement for p to have a (plane) local minimum (where $\nabla p = 0$ and a sub-matrix of $\nabla(\nabla p)$ is positive definite) is that the second eigenvalue $\lambda < 0$. Hence, a negative λ_2 is a sufficient condition for a pressure minimum that defines a vortex core (Jeong & Hussain, 1995).

On the other hand, the equation for $\boldsymbol{\Omega}$,

$$\frac{\partial \Omega_{ij}}{\partial t} + u_k \frac{\partial \Omega_{ij}}{\partial x_k} = \nu \frac{\partial^2 \Omega_{ij}}{\partial x_k^2} - (S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj}), \tag{3.56}$$

does not involve pressure directly.

3.4.3 Dynamics of the invariant space

In Euler equations (neglecting viscosity), (3.46) is written as

$$\frac{\partial \boldsymbol{A}}{\partial t} + u_k \frac{\partial \boldsymbol{A}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - \boldsymbol{A}\cdot\boldsymbol{A} \tag{3.57}$$

Given

$$-\nabla^2 p = \operatorname{tr}(\boldsymbol{A}) = A_{ik}A_{ki}, \tag{3.58}$$

we have

$$\begin{aligned}
\text{RHS} &= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - A_{ik}A_{kj} && (3.59)\\
&= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - A_{ik}A_{kj} + \frac{1}{3}\left(\frac{\partial^2 p}{\partial x_k x_k} + A_{mk}A_{km}\right)\delta_{ij} && (3.60)\\
&= -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x_i x_j} - \frac{1}{3}\frac{\partial^2 p}{\partial x_k x_k}\delta_{ij}\right) - \left(A_{ik}A_{kj} - \frac{1}{3}A_{mk}A_{km}\delta_{ij}\right) && (3.61)
\end{aligned}$$

Assume the heterogeneous part of the pressure is negligible (at least homogeneous in the small scales), we have

$$\frac{d\boldsymbol{A}}{dt} + \left(A_{ik}A_{kj} - \frac{1}{3}A_{mk}A_{km}\delta_{ij}\right) = 0. \tag{3.62}$$

Multiply (3.62) with A_{ji}, given

$$A_{ji}\frac{dA_{ij}}{dt} = \frac{1}{2}\frac{d}{dt}(A_{ij}A_{ji}) = -\frac{dQ}{dt}, \tag{3.63}$$

and $Q = -1/2\,\mathrm{tr}(\boldsymbol{A}), R = 1/3\,\mathrm{tr}(\boldsymbol{A}^3)$, we have

$$\frac{dQ}{dt} = 3R. \tag{3.64}$$

Multiply (3.62) with $A_{jk}A_{ki}$, given

$$A_{jk}A_{ki}\frac{dA_{ij}}{dt} = \frac{1}{3}\frac{d}{dt}(A_{ij}A_{jk}A_{ki}) = \frac{dR}{dt} \tag{3.65}$$

we have

$$\frac{dR}{dt} + \mathrm{tr}(\boldsymbol{A}^4) - \frac{1}{3}[\mathrm{tr}(\boldsymbol{A})^2]^2 = 0 \tag{3.66}$$

From Appendix A.3 we know (in incompressible flows)

$$P = \mathrm{tr}(\boldsymbol{A}) = 0 \tag{3.67}$$

$$Q = \frac{1}{2}[\mathrm{tr}(\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)] = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{A}^2) \tag{3.68}$$

$$R = \det(\boldsymbol{A}) = \frac{1}{6}\left(\mathrm{tr}(\boldsymbol{A})^3 - 3\,\mathrm{tr}(\boldsymbol{A})\,\mathrm{tr}(\boldsymbol{A}^2) + 2\,\mathrm{tr}(\boldsymbol{A}^3)\right) = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{A}^3) \tag{3.69}$$

where the last relation from the Newton's identity.

According to Cayley–Hamilton theory, matrix $\boldsymbol{A}$ satisfies its characteristic polynomial as

$$\boldsymbol{A}^3 - P\boldsymbol{A}^2 + Q\boldsymbol{A} - R\boldsymbol{I} = 0 \tag{3.70}$$

so

$$\boldsymbol{A}^4 = \boldsymbol{A}\boldsymbol{A}^3 = -Q\boldsymbol{A}^2 + R\boldsymbol{A} \tag{3.71}$$

and

$$\mathrm{tr}(\boldsymbol{A}^4) = -Q\,\mathrm{tr}(\boldsymbol{A}^2) = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{A}^2)^2. \tag{3.72}$$

Finally, we have

$$\frac{dR}{dt} = -\frac{2}{3}Q^2. \tag{3.73}$$

Equations (3.64)-(3.73) form a autonomous system, which can be further integrated:

$$dt = \frac{dQ}{3R} = \frac{dR}{-2Q/3} \tag{3.74}$$

$$Q^3 + \frac{27}{4}R^2 = \mathrm{const.} \tag{3.75}$$

We note that $\triangle = Q^3 + 27/4R^2$ is just the discriminant of $\boldsymbol{A}$. The approximate inviscid dynamics is just for the discriminant to conserve along the path. But from (3.62) on the characters of the N–S is less seen.

Ref. Meneveau (2011).

3.5 Lagrangian representations

Cauchy-Green tensor etc.

4 Turbulent flows

4.1 Mean and fluctuation flows

4.1.1 Reynolds average

We denote time average as $\overline{(\cdot)}$, space or ensemble average as $\langle\cdot\rangle$, and sometimes use these notations interchangeably given that they are equivalent under the ergodicity assumption. The properties proved for one definition are expected to hold for another. Although Reynolds decomposition and RANS modelings are not an accurate way of computing turbulence, they consist the foundation of our understanding of turbulence.

Below we give briefly some properties of Reynolds averaging:

(i) (Definition) The time average of a physical variable A is

$$\overline{A} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T A\,\mathrm{d}t \tag{4.1}$$

In practice, the limit is often neglected and the average window is assumed to be long enough.

(ii) (Definition) The fluctuation of a physical variable A is

$$A' \triangleq A - \overline{A} \tag{4.2}$$

(iii) (Proposition) The average of fluctuation is zero.

$$\overline{A'} = \overline{A - \overline{A}} = \overline{A} - \overline{A} = 0 \tag{4.3}$$

4.1.2 Continuity and momemtum

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{4.4}$$

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i}{\partial x_j}) \tag{4.5}$$

Taking the average of Eq. (4.4) we have

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} = 0 \tag{4.6}$$

where

$$\frac{\partial \overline{u}_i}{\partial x_i} = \frac{\partial}{\partial x_i}(\frac{1}{T}\int_0^T u_i\,\mathrm{d}t) = \frac{1}{T}\int_0^T (\frac{\partial u_i}{\partial x_i})\,\mathrm{d}t = \frac{1}{T}\int_0^T 0\,\mathrm{d}t = 0 \tag{4.7}$$

Hence we have the continuity for fluctuating velocity

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{4.8}$$

Taking the average of Eq. (4.5) we have

$$\mathrm{LHS} = (\frac{1}{T}\int_0^T \mathrm{d}t) * [\frac{\partial}{\partial t}(\overline{u}_i + u_i') + (\overline{u}_j + u_j')\frac{\partial}{\partial x_j}(\overline{u}_i + u_i')] \tag{4.9}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{\frac{\partial u_i'}{\partial t}} + \overline{\overline{u}_j\frac{\partial}{\partial x_j}\overline{u}_i} + \overline{\overline{u}_j\frac{\partial}{\partial x_j}u_i'} + \overline{u_j'\frac{\partial}{\partial x_j}\overline{u}_i} + \overline{u_j'\frac{\partial}{\partial x_j}u_i'} \tag{4.10}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\overline{u}_i + \overline{u'_j \frac{\partial}{\partial x_j} u'_i} \tag{4.11}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\overline{u}_i + \overline{\frac{\partial}{\partial x_j}(u'_j u'_i)} - \overline{u'_i \frac{\partial u'_j}{\partial x_j}} = \frac{\partial}{\partial x_j}\overline{u'_j u'_i} \tag{4.12}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u'_j u'_i} \tag{4.13}$$

$$\mathrm{RHS} = (\frac{1}{T}\int_0^T \mathrm{d}t)[-\frac{1}{\rho}\frac{\partial}{\partial x_i}(\overline{p} + p') + \frac{\partial}{\partial x_j}[\nu \frac{\partial}{\partial x_j}(\overline{u}_i + u'_i)] \tag{4.14}$$

$$= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \overline{u}_i}{\partial x_j}) \tag{4.15}$$

Equaling both sides yields:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{4.16}$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}) \tag{4.17}$$

where the cross-correlation term having dimension of shear stress

$$\tau_{\mathrm{Rey}} = -\overline{u'_i u'_j} \tag{4.18}$$

is called the Reynolds stress term. It is a rank 2 tensor. It comes from the Reynolds averaging of the non-linear advection term on the LHS of Navier–Stokes, and it distinguishes turbulent flows from laminar ones. It represents the momentum transport due to turbulent motions, in anology to the molecular diffusion.

4.1.3 Transport equation of the fluctuating velocity

Denote the material derivative based on the mean flow advection as

$$\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}_k \frac{\partial}{\partial x_k} \tag{4.19}$$

and subtract the Reynolds equation from N-S equation

$$\frac{\bar{D}u'_i}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u'_i}{\partial x_j} + \overline{u'_i u'_j} - u'_i u'_j) - u'_j \frac{\partial \overline{u}_i}{\partial x_j} \tag{4.20}$$

The last term shows the mean-flow stretching of the fluctuation, which is a generation mechanism be shown later related to the shear production of turbulent kinetic energy.

4.1.4 Mean-flow and turbulent kinetic energy

The total kinetic energy of the flow can be devided into the mean kinetic energy (MKE) and the turbulent kinetic energy (TKE)

$$K_{\mathrm{tot}} = \frac{1}{2}\overline{u_i u_i} \tag{4.21}$$

$$= \overline{\frac{1}{2}(\overline{u}_i + u'_i)(\overline{u}_i + u'_i)} \tag{4.22}$$

$$= \overline{\frac{1}{2}\overline{u}_i\overline{u}_i + \overline{u}_i u'_i + \frac{1}{2}u'_i u'_i} \tag{4.23}$$

$$= \frac{1}{2}\overline{u}_i\overline{u}_i + \frac{1}{2}\overline{u_i'u_i'} \tag{4.24}$$

$$= K + k \tag{4.25}$$

We will show how these two parts are related dynamically.

4.1.5 MKE equation

Multiply the Reynolds equation (4.17) by $\overline{u}_i$ we have

$$\text{LHS} = \overline{u}_i\frac{\bar{D}\overline{u}_i}{Dt} = \frac{\bar{D}K}{Dt} \tag{4.26}$$

$$\overline{u}_i(-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x_i}) = -\frac{1}{\rho}\frac{\partial\overline{p}\,\overline{u}_i}{\partial x_i} + \frac{1}{\rho}\frac{\partial\overline{u}_i}{\partial x_i} \tag{4.27}$$

$$= -\frac{1}{\rho}\frac{\partial\overline{p}\,\overline{u}_i}{\partial x_i} \tag{4.28}$$

$$= -\frac{1}{\rho}\frac{\partial\overline{p}\,\overline{u}_j}{\partial x_j} \tag{4.29}$$

$$\overline{u}_i\frac{\partial}{\partial x_j}(\nu\frac{\partial\overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}) = \frac{\partial}{\partial x_j}[\overline{u}_i(\nu\frac{\partial\overline{u}_i}{\partial x_j} - \overline{u_i'u_j'})] - \frac{\partial\overline{u}_i}{\partial x_j}(\nu\frac{\partial\overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}) \tag{4.30}$$

$$= -\nu\frac{\partial\overline{u}_i}{\partial x_j}\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_i}{\partial x_j}\overline{u_i'u_j'} + \frac{\partial}{\partial x_j}(\nu\frac{\partial K}{\partial x_j}) - \frac{\partial\overline{u}_i\,\overline{u_i'u_j'}}{\partial x_j} \tag{4.31}$$

Equaling both sides we have

$$\frac{\bar{D}K}{Dt} = \frac{\partial}{\partial x_j}(\underbrace{-\frac{1}{\rho}\overline{p}\,\overline{u}_j}_{\substack{\text{pressure}\\\text{distortion}}}\ \underbrace{+\nu\frac{\partial K}{\partial x_j}}_{\substack{\text{molecular}\\\text{diffusion}}}\ \underbrace{-\overline{u}_i\,\overline{u_i'u_j'}}_{\substack{\text{turbulent}\\\text{diffusion}}})\ \underbrace{-\frac{1}{2}P_{kk}}_{\substack{\text{production}\\\text{of TKE}}}\ \underbrace{-\nu\left(\frac{\partial\overline{u}_i}{\partial x_j}\right)^2}_{\text{dissipation}} \tag{4.32}$$

where the term

$$P_{kk} = -2\overline{u_i'u_j'}\frac{\partial\overline{u}_i}{\partial x_j} \tag{4.33}$$

is the production term of the turbulent kinetic energy, and, on the other hand, is the sink in MKE.

4.1.6 TKE equation

Similarly, multiply (4.20) by u_i' and then take the average

$$\text{LHS} = \overline{u_i'(\frac{\partial u_i'}{\partial t} + \overline{u}_k\frac{\partial u_i'}{\partial x_k})} \tag{4.34}$$

$$= \frac{\partial\frac{1}{2}\overline{u_i'u_i'}}{\partial t} + \overline{u}_k\frac{\partial\frac{1}{2}\overline{u_i'u_i'}}{\partial x_k} \tag{4.35}$$

$$= \frac{\bar{D}k}{Dt} \tag{4.36}$$

$$\overline{u_i'(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{1}{\rho}\frac{\partial\overline{p'u_i'}}{\partial x_i'} \tag{4.37}$$

$$= -\frac{1}{\rho}\frac{\partial\overline{p'u_k'}}{\partial x_k'} \tag{4.38}$$

$$\overline{u_i'(\frac{\partial}{\partial x_k}\nu\frac{\partial u_i'}{\partial x_k})} = \overline{\frac{\partial}{\partial x_k}(\nu u_i'\frac{\partial u_i'}{\partial x_k}) - \nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} \tag{4.39}$$

$$= \overline{\frac{\partial}{\partial x_k}(\nu \frac{\partial \frac{1}{2} u_i' u_i'}{\partial x_k}) - \nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}} \tag{4.40}$$

$$= \frac{\partial}{\partial x_k}(\nu \frac{\partial k}{\partial x_k}) - \nu \overline{(\frac{\partial u_i'}{\partial x_k})^2} \tag{4.41}$$

$$\overline{u_i'(\frac{\partial}{\partial x_k}\overline{u_i' u_k'})} = 0 \tag{4.42}$$

$$\overline{-u_i'(\frac{\partial}{\partial x_k} u_i' u_k')} = -\frac{1}{2}\overline{\frac{\partial u_i' u_i' u_k'}{\partial x_k}} \tag{4.43}$$

$$= -\frac{1}{2}\frac{\partial \overline{u_i' u_i' u_k'}}{\partial x_k} \tag{4.44}$$

$$\overline{u_i'(-u_k' \frac{\partial \overline{u}_i}{\partial x_k})} = -\overline{u_i' u_k'}\frac{\partial \overline{u}_i}{\partial x_k} \tag{4.45}$$

Equaling both sides we have

$$\frac{\bar{D}k}{Dt} = \frac{\partial}{\partial x_k}(\underbrace{\nu \frac{\partial k}{\partial x_k}}_{\substack{\text{molecular} \\ \text{diffusion}}} \underbrace{- \frac{1}{2}\overline{u_i' u_i' u_k'}}_{\substack{\text{turbulent} \\ \text{diffusion}}} \underbrace{- \frac{1}{\rho}\overline{p' u_k'}}_{\substack{\text{pressure} \\ \text{distortion}}}) + \underbrace{\frac{1}{2} P_{kk}}_{\substack{\text{production} \\ \text{of TKE}}} \underbrace{- \nu \overline{\left(\frac{\partial u_i'}{\partial x_k}\right)\left(\frac{\partial u_i'}{\partial x_k}\right)}}_{\text{dissipation}} \tag{4.46}$$

Comments:

(1) The turbulent kinetic energy generation term

$$P_{kk} = -2\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} \tag{4.47}$$

can be expressed in tensor notation as

$$P = 2\boldsymbol{\tau}_{\text{Rey}} : \nabla \overline{\boldsymbol{u}} = 2\boldsymbol{\tau}_{\text{Rey}} : \boldsymbol{S} \tag{4.48}$$

where the inner product represents the projection of the velocity fluctuation correlation on the mean shear/strain rate.

(2) The dissipation term

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_k}\right)\left(\frac{\partial u_i'}{\partial x_k}\right)} = \nu \overline{S_{ij}' S_{ij}'} + \nu \overline{\Omega_{ij}' \Omega_{ij}'} \tag{4.49}$$

is always positive, representing the dissipation mechanism of turbulence kinetic energy. We can also see that (perturbation) enstrophy is directly linked to dissipation rate of TKE/total KE. We note that the relations

$$\boldsymbol{S} : \boldsymbol{\Omega} = 0 \tag{4.50}$$

and

$$\nabla \boldsymbol{u} : \nabla \boldsymbol{u} = \boldsymbol{S} : \boldsymbol{S} + \boldsymbol{\Omega} : \boldsymbol{\Omega} \tag{4.51}$$

also carries for the perturbation quantities. Also note $\|\boldsymbol{\Omega}\|^2 = \|\boldsymbol{\omega}\|^2/2$ – the perturbation enstrophy is largely related to turbulent dissipation.

4.1.7 Reynolds stress transport equation

The velocity fluctuation transport equation is

$$\frac{\bar{D}u_i'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u_i'}{\partial x_j} + \overline{u_i' u_j'} - u_i' u_j') - u_j' \frac{\partial \overline{u}_i}{\partial x_j} \tag{4.52}$$

Or if we exchange the two subscripts we obtain:

$$\frac{\bar{D}u'_j}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \frac{\partial}{\partial x_i}(\nu\frac{\partial u'_j}{\partial x_i} + \overline{u'_i u'_j} - u'_i u'_j) - u'_i\frac{\partial \overline{u}_j}{\partial x_i} \tag{4.53}$$

$u'_j \times (19) + u'_i \times (20)$ and take the time average:

$$\text{LHS} = \frac{\bar{D}\overline{u'_i u'_j}}{Dt} \tag{4.54}$$

$$\text{RHS}_1 = -\frac{1}{\rho}[-2\overline{p' s_{ij}} + \frac{\partial}{\partial x_i}(\overline{p' u'_j}) + \frac{\partial}{\partial x_j}(\overline{p' u'_i})] \tag{4.55}$$

$$\text{RHS}_2 = \overline{u'_j\frac{\partial}{\partial x_k}(\nu\frac{\partial u'_i}{\partial x_k}) + u'_i\frac{\partial}{\partial x_k}(\nu\frac{\partial u'_j}{\partial x_k})} \tag{4.56}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u'_i u'_j}}{\partial x_k}) - 2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} \tag{4.57}$$

$$\text{RHS}_3 = \overline{u'_j\frac{\partial}{\partial x_k}\overline{u'_i u'_k} + u'_i\frac{\partial}{\partial x_k}\overline{u'_j u'_k}} \tag{4.58}$$

$$= 0 \tag{4.59}$$

$$\text{RHS}_4 = -\overline{u'_j\frac{\partial}{\partial x_k}(u'_i u'_k) + u'_i\frac{\partial}{\partial x_k}(u'_j u'_k)} \tag{4.60}$$

$$= -\overline{u'_j u'_k\frac{\partial}{\partial x_k}(u'_i) + u'_i u'_k\frac{\partial}{\partial x_k}(u'_j) + u'_i u'_j\frac{\partial}{\partial x_k}(u'_k)} \tag{4.61}$$

$$(Continuity, \frac{\partial u'_k}{\partial x_k} = 0, is\, used\, twice\, here.) \tag{4.62}$$

$$= -\frac{\partial}{\partial x_k}\overline{u'_i u'_j u'_k} \tag{4.63}$$

$$\text{RHS}_5 = \overline{-u'_k u'_j\frac{\partial\overline{u}_i}{\partial x_k} - u'_k u'_i\frac{\partial\overline{u}_j}{\partial x_k}} \tag{4.64}$$

$$= -\overline{u'_k u'_j}\frac{\partial\overline{u}_i}{\partial x_k} - \overline{u'_k u'_i}\frac{\partial\overline{u}_j}{\partial x_k} \tag{4.65}$$

$$\tag{4.66}$$

By equalizing both sides we obtain

$$\frac{\bar{D}\overline{u'_i u'_j}}{Dt} = \frac{2}{\rho}\overline{p' s_{ij}} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p' u'_j})\delta_{ik} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p' u'_i})\delta_{jk} + \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u'_i u'_j}}{\partial x_k}) \tag{4.67}$$

$$- 2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} - \frac{\partial}{\partial x_k}\overline{u'_i u'_j u'_k} - \overline{u'_k u'_j}\frac{\partial\overline{u}_i}{\partial x_k} - \overline{u'_k u'_i}\frac{\partial\overline{u}_j}{\partial x_k} \tag{4.68}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u'_i u'_j}}{\partial x_k} - \overline{u'_i u'_j u'_k} - \frac{1}{\rho}\overline{p' u'_i}\delta_{jk} - \frac{1}{\rho}\overline{p' u'_j}\delta_{ik}) \tag{4.69}$$

$$- (\overline{u'_k u'_j}\frac{\partial\overline{u}_i}{\partial x_k} + \overline{u'_k u'_i}\frac{\partial\overline{u}_j}{\partial x_k}) + \frac{2}{\rho}\overline{p' s_{ij}} - 2\nu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}} \tag{4.70}$$

Equaling both sides we have

$$\frac{\bar{D}\overline{u'_i u'_j}}{Dt} = d_{ij} + P_{ij} + \Phi_{ij} - \varepsilon_{ij} \tag{4.71}$$

where

$$d_{ij} = \frac{\partial}{\partial x_k}(\nu \frac{\partial \overline{u_i' u_j'}}{\partial x_k} - \overline{u_i' u_j' u_k'} - \frac{1}{\rho}\overline{p' u_i'}\delta_{jk} - \frac{1}{\rho}\overline{p' u_j'}\delta_{ik}) \tag{4.72}$$

$$P_{ij} = -\overline{u_k' u_j'}\frac{\partial \overline{u}_i}{\partial x_k} - \overline{u_k' u_i'}\frac{\partial \overline{u}_j}{\partial x_k} \tag{4.73}$$

$$\Phi_{ij} = \frac{2}{\rho}\overline{p' s_{ij}} \tag{4.74}$$

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{4.75}$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{4.76}$$

Comments:

(1) The left hand side term $\frac{\bar{D}\overline{u_i' u_j'}}{Dt}$ is the rate of change of the Reynolds stress along the particle line.

(2) The term d_{ij} is the diffusion term in the equation, appearing in the form of gradient. It includes viscous term, Reynolds stress term and pressure-velocity fluctuation coupling term.The diffusion is resulted by the spatial non-uniformity of these property.

(3) The term P_{ij} is the generation term of Reynolds stress, showed in the form of the product of Reynolds stress and the mean flow strain rate.

(4) The term Φ_{ij} is the redistribution term. We note that the contraction of Reynolds stress transport equation is the transport equation for turbulence kinetic energy. And the contraction of Φ_{ij} is $\Phi_{ii} = \frac{2}{\rho}\overline{p' s_{ii}} = 0$ as continuity holds. So the term contributes nothing to the growth of turbulent kinetic energy. It just takes the kinetic energy from one component of fluid motion to another component.

(5) The term ε_{ij}, whose contraction is positive forever, representing the dissipation mechanism of kinetic energy.

4.1.8 Dissipation rate transport equation

The dissipation term in Reynolds stresses transport equation is defined as

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_j'}{\partial x_p}} \tag{4.77}$$

Multiply equation (4.20) by $2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}$ and take the time derivative we have:

$$\text{LHS} = 2\nu \frac{\bar{D}}{Dt}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_p}} = \frac{\bar{D}\varepsilon}{Dt} + 2\nu \frac{\partial \overline{u_k}}{\partial x_p}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} \tag{4.78}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{2\nu}{\rho}\frac{\partial}{\partial x_k}(\overline{\frac{\partial u_k'}{\partial x_p}\frac{\partial p'}{\partial x_p}}) \tag{4.79}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}(\nu \frac{\partial u_i'}{\partial x_k}))} = \frac{\partial}{\partial x_k}(\nu \frac{\partial \varepsilon}{\partial x_k}) - 2\overline{(\nu \frac{\partial^2 u_i'}{\partial x_p \partial x_k})^2} \tag{4.80}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}\overline{u_i' u_k'})} = 0 \tag{4.81}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}-u_i'u_k')}=-2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}}+\frac{\partial}{\partial x_k}\overline{u_k'\nu(\frac{\partial u_i'}{\partial x_p})^2} \tag{4.82}$$

$$=-2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}}+\frac{\partial}{\partial x_k}\overline{u_k'\varepsilon'} \tag{4.83}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-u_k'\frac{\partial \overline{u}_i}{\partial x_k})}=-2\nu\frac{\partial \overline{u}_i}{\partial x_k}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}}-2\nu\frac{\partial^2\overline{u}_i}{\partial x_k\partial x_p}\overline{u_k'\frac{\partial u_i'}{\partial x_p}} \tag{4.84}$$

By equalizing both side we yield the transport equation for turbulence dissipation rate

$$\frac{\bar{D}\varepsilon}{Dt}=\frac{\partial}{\partial x_k}(-\frac{2\nu}{\rho}\overline{\frac{\partial u_k}{\partial x_p}\frac{\partial p}{\partial x_p}}+\nu\frac{\partial\varepsilon}{\partial x_k}-\overline{u_k'\varepsilon'})-2\nu\frac{\partial\overline{u}_i}{\partial x_k}(\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}}+\overline{\frac{\partial u_p'}{\partial x_k}\frac{\partial u_p'}{\partial x_i}}) \tag{4.85}$$

$$-2\nu\overline{u_k'\frac{\partial u_i'}{\partial x_p}}\frac{\partial^2\overline{u}_i}{\partial x_p\partial x_k}-2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}}-2\overline{(\nu\frac{\partial u_i'}{\partial^2 x_p\partial x_k})^2} \tag{4.86}$$

The final equation of the equation agrees with that given in the turbulence book by Shi (1994). Second moment equation closure problem Chou (1945) could be discussed briefly here.

4.1.9 Scalar flux, its mean and kinetic energy transport equations

Similar to Eq. (4.20) we have the transport equation for the mean and fluctuation of a passive scalar c:

$$\frac{\bar{D}\overline{c}}{\bar{D}t}=\frac{\partial}{\partial x_j}(\nu\frac{\partial\overline{c}}{\partial x_j}-\overline{c'u_j'}) \tag{4.87}$$

and

$$\frac{\bar{D}c'}{Dt}=\frac{\partial}{\partial x_j}(\Gamma\frac{\partial c'}{\partial x_j}+\overline{c'u_j'}-c'u_j')-u_j'\frac{\partial\overline{c}}{\partial x_j} \tag{4.88}$$

where Γ is the molecular diffusion coefficient of c.

Take $c'\times$ (4.20) $+u_i'\times$ (4.88) and apply the average

$$\text{LHS}=\frac{\bar{D}\overline{c'u_i'}}{\bar{D}t} \tag{4.89}$$

$$\text{RHS}_1=-\frac{1}{\rho}\overline{c'\frac{\partial p'}{\partial x_i}}=-\frac{1}{\rho}(\frac{\partial}{\partial x_j}\overline{p'c'}\delta_{ij}-\overline{p'\frac{\partial c'}{\partial x_i}}) \tag{4.90}$$

$$\text{RHS}_2=\frac{\partial}{\partial x_j}(\Gamma\overline{u_i'\frac{\partial c'}{\partial x_j}}+\nu\overline{c'\frac{\partial u_i'}{\partial x_j}})-(\nu+\Gamma)\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{4.91}$$

$$\text{RHS}_3=-\frac{\partial}{\partial x_j}(\overline{c'u_i'u_j'}) \tag{4.92}$$

$$\text{RHS}_4=-\overline{c'u_j'}\frac{\partial\overline{u}_i}{\partial x_j}-\overline{u_i'u_j'}\frac{\partial\overline{c}}{\partial x_j} \tag{4.93}$$

then we obtain the transport equation for scalar flux

$$\frac{\bar{D}\overline{c'u_i'}}{\bar{D}t}=d_{jc}+P_{jc}+\Phi_{jc}-\varepsilon_{jc} \tag{4.94}$$

where

$$d_{ic}=\frac{\partial}{\partial x_j}(\Gamma\overline{u_i'\frac{\partial c'}{\partial x_j}}+\nu\overline{c'\frac{\partial u_i'}{\partial x_j}}-\frac{1}{\rho}\overline{p'c'}\delta_{ij}-\overline{c'u_i'u_j'}) \tag{4.95}$$

$$P_{ic} = -\overline{c'u'_j}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}\frac{\partial \overline{c}}{\partial x_j} \tag{4.96}$$

$$\Phi_{ic} = \frac{1}{\rho}\overline{p'\frac{\partial c'}{\partial x_i}} \tag{4.97}$$

$$\varepsilon_{ic} = (\nu + \Gamma)\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{4.98}$$

Comments:

(1) Gradient diffusion: velocity-fluctuation scalar-diffusion correlation, momentum-diffusion scalar-fluctation ccorelation, pressure diffusion, turbulence diffusion.

(2) Production: scalar flux interacting with mean shear, turbulent flux (Reynolds stresses) interacting with mean scalar gradient.

(3) Re-distribution.

(4) Dissipation.

Define scalar mean and fluctuation energy as

$$K_c = \frac{1}{2}\bar{c}^2 \tag{4.99}$$

$$k_c = \frac{1}{2}\overline{c'c'} \tag{4.100}$$

$c' \times$ (4.88) and apply the average

$$\text{LHS} = \frac{\bar{D}k_c}{\bar{D}t} \tag{4.101}$$

$$\text{RHS}_1 = \frac{\partial}{\partial x_j}\Gamma\frac{\partial k_c}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{4.102}$$

$$\text{RHS}_2 = -\frac{1}{2}\frac{\partial}{\partial x_j}\overline{c'c'u_j} \tag{4.103}$$

$$\text{RHS}_3 = -\overline{c'u'_j}\frac{\partial \overline{c}}{\partial x_j} \tag{4.104}$$

then we obtain the transport equation for scalar fluctuation energy

$$\frac{\bar{D}k_c}{\bar{D}t} = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial}{\partial x_j}k_c - \frac{1}{2}\overline{c'c'u'_j}) - \overline{c'u'_j}\frac{\partial \overline{c}}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{4.105}$$

For active scalar (for example, density which appears in the momentum equation as buoyancy force), see section 5.6.

4.1.10 Poisson equation for mean and fluctuation pressure

The Reynolds average equation is

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}) \tag{4.106}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} \tag{4.107}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2\overline{p} - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} \tag{4.108}$$

Poisson equation for mean pressure:

$$-\frac{1}{\rho}\nabla^2\overline{p} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} \tag{4.109}$$

The velocity fluctuation transport equation is

$$\frac{\bar{D}u_i'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i'}{\partial x_j} + \overline{u_i'u_j'} - u_i'u_j') - u_j'\frac{\partial \overline{u}_i}{\partial x_j} \tag{4.110}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u_i'}{\partial x_j} \tag{4.111}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2 p' - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} - \frac{\partial u_j'}{\partial x_i}\frac{\partial u_i'}{\partial x_j} - \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u_j'}{\partial x_i} \tag{4.112}$$

Poisson equation for fluctuation pressure:

$$-\frac{1}{\rho}\nabla^2 p' = \frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u_i'}{\partial x_j} + \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u_j'}{\partial x_i} + \frac{\partial u_j'}{\partial x_i}\frac{\partial u_i'}{\partial x_j} - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} \tag{4.113}$$

$$= \frac{\partial u_j'}{\partial x_i}\frac{\partial u_i'}{\partial x_j} - \frac{\partial^2\,\overline{u_i'u_j'}}{\partial x_i \partial x_j} + 2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u_i'}{\partial x_j} \tag{4.114}$$

4.1.11 Turbulent vorticity and enstrophy

Similarly, vorticity can be decomposed into the mean and the pertubation. We give the equation of pertuabtion vorticity without derivation:

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j'\overline{S}_{ij} + \overline{\omega_j}S_{ij}' + \omega_j'S_{ij}' - \overline{\omega_j'S_{ij}'} - u_j'\frac{\partial \overline{\omega}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_j'\omega_i'} - u_j'\omega_i') + \nu\frac{\partial^2 \omega_i'}{\partial x_j^2} \tag{4.115}$$

where $\overline{S}_{ij}$ and S_{ij}' are the mean and the fluctuation shear, respectively.

We define the fluctuating enstrophy as

$$\mathscr{E} = \frac{1}{2}\overline{\omega_i'\omega_i'} \tag{4.116}$$

$\omega_i' \times$ (4.115) and take the time average

$$\text{LHS} = \frac{\bar{D}\mathscr{E}}{\bar{D}t} \tag{4.117}$$

$$\text{RHS}_1 = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega}_j\overline{\omega_i'S_{ij}'} + \overline{\omega_i'\omega_j'S_{ij}'} \tag{4.118}$$

$$\text{RHS}_2 = -\overline{\omega_i'u_j'}\frac{\partial \overline{\omega_i}}{\partial x_j} \tag{4.119}$$

$$\text{RHS}_3 = -\frac{1}{2}\frac{\partial}{\partial x_j}(\overline{u_j'\omega_i'\omega_i'}) \tag{4.120}$$

$$\text{RHS}_4 = \nu\overline{\frac{\partial^2 \mathscr{E}}{\partial x_j^2}} - \overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} \tag{4.121}$$

Equaling both sides we obtain

$$\frac{\bar{D}\mathscr{E}}{\bar{D}t} = P_{\mathscr{E}} + D_{\mathscr{E}} - \varepsilon_{\mathscr{E}} \tag{4.122}$$

$$P_{\mathscr{E}} = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega_j}\overline{\omega_i'S_{ij}'} + \overline{\omega_i'\omega_j'S_{ij}'} - \overline{\omega_i'u_j'}\frac{\partial\overline{\omega_i}}{\partial x_j} \tag{4.123}$$

$$D_{\mathscr{E}} = \frac{\partial}{\partial x_j}(\nu\frac{\partial\mathscr{E}}{\partial x_j} - \frac{1}{2}\overline{u_j'\omega_i'\omega_i'}) \tag{4.124}$$

$$\varepsilon_{\mathscr{E}} = \nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}} \tag{4.125}$$

Comment: The energy balance process of fluctuation enstrophy obeys four principle processes in nature (Kolmogorov):

change rate = production + diffusion + dissipation

Moreover, in instability problems, it is convenience to consider the linearized inviscid perturbation vorticity equation, which reads

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j'\bar{S}_{ij} + \bar{\omega}_j S_{ij}' - u_j'\frac{\partial\bar{\omega}_i}{\partial x_j}, \tag{4.126}$$

since perturbation enstrophy is related to turbulent dissipation (see 4.51).

4.2 Farve average in compressible flows

4.3 LES equations

4.4 Theory of homogeneous isotropic turbulence

K-H etc.

Taylor microscale (velocity gradient based on that and urms, dissipation) and decorrelation; scale separation v.s. Re;

4.5 Scales of turbulent motions

4.5.1 Kolmogorov scale

The Kolmogorov scales are:

$$\epsilon = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \, u_\eta = (\epsilon\nu)^{1/4}, \tau_\eta = (\nu/\epsilon)^{1/2}. \tag{4.127}$$

Note the scales depend only on viscosity and dissipation rate (hence the expressions above can be derived from a dimensional analysis).

A resulting Reynolds number

$$Re_\eta = \frac{u_\eta\eta}{\nu} = 1 \tag{4.128}$$

indicates that viscous effect is active at the Kolmogorov scale.

4.5.2 Taylor microscale

Also the idea of using urms and lambdaf to estimate dissipation.

Taylor Reynolds number

4.5.3 Other useful scales

Corrsin scale (below which the flow doesn't feel the shear and turbulence in different shear flows are similar).

Ozmidov scale.

4.6 Free shear flows

4.6.1 Momentum integral

Similarity solutions (turbulent). Pope (2001).

4.6.2 Similarity solutions

The characteristic velocity and length scales are U_s and δ_s, respectively.

Flow type	U_s	δ_s	$U_s \propto x^m$	$\delta_s \propto x^n$	$f(\eta)$
Round jet	$\bar{u}(x, y=0)$	$r_{1/2}$	-1	1	$1/(1+a\eta^2)^2$
Plane jet	$\bar{u}(x, r=0)$	$y_{1/2}$	$-1/2$	1	$\operatorname{sech}^2(\ln(1+\sqrt{2})\,\eta)$
Round wake	$U_\infty - \bar{u}(x, y=0)$	$r_{1/2}$	$-2/3$	$1/3$	$\exp(-\ln 2\,\eta^2)$
Plane wake	$U_\infty - \bar{u}(x, r=0)$	$y_{1/2}$	$-1/2$	$1/2$	$\exp(-\ln 2\,\eta^2)$
Plane mixing layer	$U_2 - U_1$	$y_{0.9} - y_{0.1}$	0	1	$1/2\,\operatorname{erf}(\eta/\sigma\sqrt{2})$

Table 1: Self-similar solution table.

The example of plane jet is the easiest to understand and derive so we are the most detailed in that case and more loosely on the others. The same principles and machinery apply to all cases.

4.6.3 Round jet

Characteristic scales:
The centerline velocity is

$$U_s(x) = \bar{u}(x, r=0) \tag{4.129}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = \bar{u}(x, r_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{4.130}$$

Momentum integral constraint:
The boundary layer equation in cylindrical coordinates reads

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{r}\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{4.131}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial (r\bar{v})}{\partial r} = 0 \tag{4.132}$$

by $r\bar{u}$ and add it to (4.131) multiplied by r we obtain

$$\frac{\partial (r\bar{u}\bar{u})}{\partial x} + \frac{\partial (r\bar{u}\bar{v})}{\partial r} = -\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{4.133}$$

Integrate (4.133) in r we obtain

$$\int_0^\infty \frac{\partial(r\bar{u}\bar{u})}{\partial r}\,\mathrm{d}r + r\bar{u}\bar{v}|_0^\infty = -r\overline{u'v'}|_0^\infty \tag{4.134}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^\infty r\bar{u}^2\,\mathrm{d}r\right) = 0 \tag{4.135}$$

which implies the momentum flux

$$\dot{M}(x) = \int_0^\infty \rho\bar{u}^2 2\pi r\,\mathrm{d}r = J_0 \tag{4.136}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.
Self-similar assumptions:

$$\bar{u} = U_s(x)f(\eta),\ \overline{u'v'} = U_s^2(x)g(\eta) \tag{4.137}$$

where $\eta = r/\delta_s(x)$ with $\delta_s = r_{1/2}$. Substitute (4.137) into (4.136) we have

$$\dot{M}(x) = (2\pi\rho)(U_s^2\delta_s^2)\left(\int_0^\infty \eta f^2(\eta)\,\mathrm{d}\eta\right) \tag{4.138}$$

to be a constant and implying

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2\delta_s^2) = 0 \tag{4.139}$$

and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{4.140}$$

Using the continuity equation we have

$$\bar{v} = -\frac{1}{r}\int_0^r \frac{\partial(r\bar{u})}{\partial x}\,\mathrm{d}y = U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{\eta}\int_0^\eta f\eta\,\mathrm{d}\eta\right) \tag{4.141}$$

We note that $\bar{v}$ switch sign from positive to negative when r is greater than a certain value (entrainment).

Next we establish the constant spread rate of the round jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left[f^2\eta + ff'\eta + \left(\frac{f}{\eta} + f'\right)\int_0^\eta f\eta\,\mathrm{d}\eta\right] = g + g'\eta \tag{4.142}$$

and then $\mathrm{d}\delta_s/\mathrm{d}x$ has to be a constant. Combining with momentum integral restriction we have

$$\delta_s \propto x,\ U_s \propto x^{-1}. \tag{4.143}$$

4.6.4 Plane jet

Characteristic scales:
The centerline velocity is

$$U_s(x) = \bar{u}(x, y = 0) \tag{4.144}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = \bar{u}(x, y_{1/2}(x)) = \frac{1}{2} U_s(x). \tag{4.145}$$

Momentum integral constraint:
The boundary layer equation for the mean velocity simplifies to

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{4.146}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{4.147}$$

by $\bar{u}$ and add it to (4.146) we obtain

$$\frac{\partial \bar{u}\bar{u}}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{4.148}$$

Integrate (4.148) in y we obtain

$$\int_{-\infty}^{\infty} \frac{\partial \bar{u}\bar{u}}{\partial x}\, \mathrm{d}y + \bar{u}\bar{v}|_{-\infty}^{\infty} = -\overline{u'v'}|_{-\infty}^{\infty} \tag{4.149}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty} \bar{u}^2\, \mathrm{d}y\right) = 0 \tag{4.150}$$

which implies the momentum flux

$$\dot{M}(x) = \int_{-\infty}^{\infty} \rho \bar{u}^2\, \mathrm{d}y = J_0 \tag{4.151}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.
Self-similar assumptions:

$$\bar{u} = U_s(x) f(\eta),\ \overline{u'v'} = U_s^2(x) g(\eta) \tag{4.152}$$

where $\eta = y/\delta_s(x)$ and we have

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{\delta_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{4.153}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta_s} \tag{4.154}$$

Substitute (4.152) into (4.151) we have

$$\dot{M}(x) = (U_s^2 \delta_s)\left(\int_{-\infty}^{\infty} f^2(\eta)\, \mathrm{d}\eta\right) \tag{4.155}$$

is a constant. So it must be

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2 \delta_s) = 0 \tag{4.156}$$

which gives the momentum flux constraint in terms of characteristic variables, and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{4.157}$$

Using the continuity equation we have

$$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x}\,\mathrm{d}y = U_s \frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{2}\int_0^\eta f\,\mathrm{d}\eta\right) \tag{4.158}$$

Next we establish the constant spread rate of the plane jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(f^2 + f'\int_0^\eta f\,\mathrm{d}\eta) = g' \tag{4.159}$$

and then

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x} = \frac{2g'}{f^2 + f'\int_0^\eta f\,\mathrm{d}\eta} = C \tag{4.160}$$

with the LHS only depend on x and RHS only depend on η. Then both sides have to be constant. Combining (4.160) and (4.156) we have

$$\delta_s \propto x,\, U_s \propto x^{-1/2}. \tag{4.161}$$

4.6.5 Round wake

Characteristic scales:
The centerline velocity deficit is

$$U_0(x) = U_\infty - \bar{u}(x, r=0) = U_d(x, 0) \tag{4.162}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = U_\infty - \bar{u}(x, r_{1/2}(x)) = \frac{1}{2}U_0(x). \tag{4.163}$$

Momentum integral constraint:
Here we start from the simplified (see plane wake) momentum equation

$$U_\infty \frac{\partial \bar{u}}{\partial x} = -\frac{1}{r}\frac{\partial (r\overline{u'v'})}{\partial r} \tag{4.164}$$

and the momentum deficit flux conservation

$$\dot{M}(x) = \int_0^\infty \rho U_\infty (U_\infty - \bar{u}) 2\pi r\,\mathrm{d}r. \tag{4.165}$$

Note that we have already replaced the $\bar{u}$ with U_∞ assuming (or by order of magnitude analysis) the convection velocity is U_∞.
Self-similar assumptions:

$$U_\infty - \bar{u} = U_s(x) f(\eta),\ \overline{u'v'} = U_s^2(x) g(\eta) \tag{4.166}$$

We have

$$\dot{M}(x) = (U_s \delta_s^2)(2\pi\rho U_\infty)\int_0^\eta f\,\mathrm{d}\eta \tag{4.167}$$

is a constant and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s\delta_s^2) = 0. \tag{4.168}$$

Consider the momentum equation, the other constraint reads

$$-\frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(2f + f'\eta)\eta = (g'\eta + g) \tag{4.169}$$

We define the spread rate as

$$S = \frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}, \tag{4.170}$$

it has to be a constant. Then

$$-S(2f\eta + f'\eta^2) = (g\eta)' \tag{4.171}$$

and including boundary conditions after integration we get

$$g = -S\eta f \tag{4.172}$$

same as in plane wakes. Combining (4.168) and (4.170) we have

$$\delta_s \propto x^{1/3},\, U_s \propto x^{-2/3}. \tag{4.173}$$

4.6.6 Plane wake

Characteristic scales:
The centerline velocity deficit is

$$U_s(x) = U_\infty - \bar{u}(x, y=0) = U_d(x, 0) \tag{4.174}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = U_\infty - \bar{u}(x, y_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{4.175}$$

Momentum integral constraint:
The boundary layer equation:

$$\bar{u}\frac{\partial(\bar{u}-U_\infty)}{\partial x} + \bar{v}\frac{\partial(\bar{u}-U_\infty)}{\partial y} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{4.176}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{4.177}$$

by $\bar{u} - U_\infty$ and add it to (4.176) we obtain

$$\frac{\partial \bar{u}(\bar{u}-U_\infty)}{\partial x} + \frac{\partial \bar{v}(\bar{u}-U_\infty)}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{4.178}$$

Integrate (4.148) in y we obtain

$$\int_{-\infty}^{\infty} \frac{\partial \bar{u}(\bar{u}-U_\infty)}{\partial x}\,\mathrm{d}y + \bar{v}(\bar{u}-U_\infty)|_{-\infty}^{\infty} = -\overline{u'v'}|_{-\infty}^{\infty} \tag{4.179}$$

and since $\overline{u'v'}$ and $\bar{u} - U_\infty$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty} \bar{u}(\bar{u}-U_\infty)\,\mathrm{d}y\right) = 0 \tag{4.180}$$

which implies the momentum deficit flux

$$\dot{M}(x) = \int_{-\infty}^{\infty} \rho\bar{u}(U_\infty - \bar{u})\,\mathrm{d}y \tag{4.181}$$

is conserved (we note that we haven't assumed far wake yet).

Self-similar assumptions:

$$U_\infty - \bar{u} = U_s(x) f(\eta),\ \overline{u'v'} = U_s^2(x) g(\eta) \tag{4.182}$$

Substitute (4.182) into (4.181), and assume the far wake is reached ($U_s/U_\infty \ll 1$) we have

$$\dot{M}(x) = \int_{-\infty}^{\infty} \rho(U_\infty - U_s f) U_s f \delta_s \,\mathrm{d}\eta \tag{4.183}$$

$$= U_\infty^2 \int_{-\infty}^{\infty} \rho(1 - \frac{U_s f}{U_\infty}) \frac{U_s}{U_\infty} f \delta_s \,\mathrm{d}\eta \tag{4.184}$$

$$= \rho U_\infty U_s \delta_s \int_{-\infty}^{\infty} f \,\mathrm{d}\eta \tag{4.185}$$

is a constant. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s \delta_s) = 0. \tag{4.186}$$

Using the continuity equation we have

$$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x} \,\mathrm{d}y = -U_s \frac{\mathrm{d}\delta_s}{\mathrm{d}x} f\eta. \tag{4.187}$$

Note the negative speed corresponding to wake entrainment (of high momentum into low momentum region).

Now we consider another constraint. Since in the far wake, the velocity deficit $U_s/U_\infty \ll 1$, we have the simplification of the momentum equation as

$$\frac{\partial \bar{u}(\bar{u} - U_\infty)}{\partial x} + \frac{\partial \bar{v}(\bar{u} - U_\infty)}{\partial y} = U_\infty \frac{\partial \bar{u}}{\partial x} = -\frac{\partial \overline{u'v'}}{\partial y} \tag{4.188}$$

where

$$\bar{u}(\bar{u} - U_\infty) = (U_\infty - U_s f)(-U_s f) = U_\infty^2 (1 - \frac{U_s f}{U_\infty})(-\frac{U_s}{U_\infty} f) = -U_s U_\infty f = U_\infty(\bar{u} - U_\infty). \tag{4.189}$$

And the scale for $\partial \bar{u}(\bar{u} - U_\infty)/\partial x$ is

$$\frac{U_\infty U_s}{L_x} \tag{4.190}$$

while the scale for $\partial \bar{v}(\bar{u} - U_\infty)/\partial y$ (from (4.187)) is

$$\frac{U_s}{\delta_s}\left(U_s \frac{\delta_s}{L_x}\right). \tag{4.191}$$

Define the spread rate as

$$S = \frac{U_\infty}{U_s} \frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{4.192}$$

Take $\bar{v}$ into the simplified momentum equation we have

$$(f + f'\eta) \frac{U_\infty}{U_s} \frac{\mathrm{d}\delta_s}{\mathrm{d}x} = -g' \tag{4.193}$$

with S depends only on x and the rest on η hence S has to be a constant. Then (4.193) can be rewritten as

$$g' + S(f + f'\eta) = 0 \tag{4.194}$$

which is to say

$$(g + S\eta f)' = 0. \tag{4.195}$$

Integrate from $\eta = 0$ to η and note that $g(0) = 0$, we have

$$g = -S\eta f. \tag{4.196}$$

Combining two conditions (4.186) and (4.192) we have

$$\delta_s \propto x^{1/2},\ U_s \propto x^{-1/2}. \tag{4.197}$$

4.6.7 Plane mixing layer

Characteristic scales:
The two velocities are $U_2 > U_1$ with U_2 on the top. The mean convection velocity is

$$U_c = \frac{1}{2}(U_1 + U_2) \tag{4.198}$$

and the characteristic velocity scale is

$$U_s = U_2 - U_1. \tag{4.199}$$

The characteristic length is the mixing layer width,

$$\delta_s(x) = y_{0.9} - y_{0.1} \tag{4.200}$$

with cross-stream location $y_\alpha(x)$ such that

$$\bar{u}(x, y_\alpha(x)) = U_1 + \alpha U_s. \tag{4.201}$$

a reference position is

$$\hat{y} = \frac{1}{2}(y_{0.1} + y_{0.9}) \tag{4.202}$$

such that the self-similar variable is defined as

$$\eta = \frac{y - \hat{y}}{\delta_s(x)} \tag{4.203}$$

4.7 Wall flows

4.7.1 von Kármán momentum integral

4.7.2 Blasius similarity solution

The references are Schlichting & Gersten (2016); Kundu *et al.* (2015) with the definition of $\delta(x)$ different by a factor of $\sqrt{2}$. Here we will follow the definition in Schlichting & Gersten (2016).

The boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.204}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial x^2} \tag{4.205}$$

The idea of self-similar solutions is that the velocity profile $u(y)$ will be the same under some proper transformation/normalization of u and y. The scale for u is apparently U_∞, while the scale for y is δ. From the viscous scaling $v \sim \nu/\delta$ and the scaling of the continuity equation $v/\delta \sim U_\infty/x$ we have

$$\delta^2 \sim \frac{\nu x}{U_\infty} \tag{4.206}$$

and for the sake of simplification of the final result (ODE) we define

$$\delta(x) = \sqrt{\frac{2x\nu}{U_\infty}} \tag{4.207}$$

such that the similarity transformation is

$$\eta = \frac{y}{\delta(x)} \tag{4.208}$$

such that

$$\frac{u}{U_\infty} = f(\eta) \tag{4.209}$$

where $f(\eta)$ is the similarity function and η is the similarity coordinate.

We note that the streamfunction ψ depends on ν, U_∞, x, y and dimensionally

$$\psi(x, y) = U_\infty \delta(s) f(\eta) = \sqrt{2\nu U_\infty x} f(\eta) \tag{4.210}$$

and hence

$$u = U_\infty f' \tag{4.211}$$

$$v = \sqrt{\frac{U_\infty \nu}{2x}} (\eta f' - f) \tag{4.212}$$

The derivatives are

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2x} f'' \eta \tag{4.213}$$

$$\frac{\partial u}{\partial y} = U_\infty f'' \sqrt{\frac{U_\infty}{2\nu x}} \tag{4.214}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{2\nu x} f''' \tag{4.215}$$

and then

$$u\frac{\partial u}{\partial x} = -\frac{U_\infty^2}{2x} f' f'' \eta \tag{4.216}$$

$$v\frac{\partial u}{\partial y} = \frac{U_\infty^2}{2x} f''(\eta f' - f) \tag{4.217}$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{2x} f''' \tag{4.218}$$

and finally we have the ODE

$$ff'' + f''' = 0 \tag{4.219}$$

with the boundary conditions being

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1, \tag{4.220}$$

corresponding to

$$v(y = 0) = 0,\ u(y = 0) = 0,\ u(y = \infty) = U_\infty. \tag{4.221}$$

It is common to use a Runge-Kutta shooting method to solve (4.220).

4.7.3 Similarity solutions to laminar free shear flows of Blasius kind

4.7.4 Turbulent channel flow

channel basic equation; FIK;

5 Geophysical fluid dynamics

5.1 Basics

5.1.1 Non-inertial frames, centrifugal and Coriolis forces

5.1.2 Absolute velocity

In an inertial frame,

$$\boldsymbol{u}_a = \boldsymbol{u}_{(r)} + \boldsymbol{\Omega} \times \boldsymbol{r}, \tag{5.1}$$

where $\boldsymbol{u}_{(r)}$ is the relative (to the non-inertial frame) velocity. Especially, in a cylindrical frame with a plane-normal rotation,

$$u_{\theta,a} = u_\theta + \Omega r. \tag{5.2}$$

5.1.3 Inertial oscillations: buoyancy and Coriolis frequencies

5.2 Boussinesq approximation

5.3 Balanced flows

5.3.1 Hydrostatic and geostrophic balances

In balanced flow, there is a background horizontal pressure gradient that balances the Coriolis forces due to horizontal motions and a vertical pressure gradient that balances the background unperturbed density:

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x} + f_\mathrm{c} V \tag{5.3}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y} - f_\mathrm{c} U \tag{5.4}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial z} - \frac{\rho^* g}{\rho_0} \tag{5.5}$$

with

$$p_g = p - p_0,\ \rho^* = \rho - \rho_0 - \rho_b(z) \tag{5.6}$$

and the background balance

$$0 = -\frac{\partial p_0}{\partial z} - (\rho_0 + \rho_b) g \tag{5.7}$$

already subtracted. Note that the Boussinesq and hydrostatic approximations are already applied.

The above equations in vector form:

$$\boldsymbol{f}_\mathrm{c} \times \boldsymbol{U} = -\frac{1}{\rho_0}\nabla p_g + \frac{\rho^*}{\rho_0}\boldsymbol{g}. \tag{5.8}$$

We have

$$\boldsymbol{U} = (U, V, 0) = -\frac{1}{\rho_0 f_\mathrm{c}}(\frac{\partial p_g}{\partial y}, -\frac{\partial p_g}{\partial x}, 0). \tag{5.9}$$

And we have

$$\nabla_\mathrm{h} \cdot \boldsymbol{U} = 0. \tag{5.10}$$

In the world geostrophic, geo means Coriolis and strophic means cyclone/anticyclone or low-/high-pressure systems.

5.3.2 Thermal wind relations

In hydrostatic Boussinesq flow. Taking the vertical gradient of (5.8) and using the hydrostatic balance, we have

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial x} + f_{\mathrm{c}}\frac{\partial V}{\partial z} \tag{5.11}$$

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial y} - f_{\mathrm{c}}\frac{\partial U}{\partial z} \tag{5.12}$$

and hence

$$(\frac{\partial U}{\partial z}, \frac{\partial V}{\partial z}) = \frac{g}{\rho_0 f_{\mathrm{c}}}(\frac{\partial \rho^*}{\partial y}, -\frac{\partial \rho^*}{\partial x}), \tag{5.13}$$

or in vector form,

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f_{\mathrm{c}}} \times \nabla \rho^* \tag{5.14}$$

In a more general case. Without introducing the hydrostatic balance and Boussinesq approximation, we write

$$\boldsymbol{f}_c \times \boldsymbol{U} = \frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{5.15}$$

Taking its curl:

$$\mathrm{LHS} = \nabla \times (\boldsymbol{f}_c \times \boldsymbol{U}) = -\boldsymbol{f}_c \cdot \nabla \boldsymbol{U} = -f_{\mathrm{c}}\frac{\partial \boldsymbol{U}}{\partial z} \tag{5.16}$$

$$\mathrm{RHS} = -\nabla \times (\frac{1}{\rho}\nabla p) + \nabla \times \boldsymbol{g} = -\frac{1}{\rho^2}(\nabla p \times \nabla \rho) \tag{5.17}$$

Re-introduce hydrostatic ($\partial_z p = -\rho g$) and Boussinesq, we have

$$\nabla p \approx \frac{\partial p}{\partial z}\boldsymbol{e}_z = -\rho g \boldsymbol{e}_z \tag{5.18}$$

and

$$-\frac{1}{\rho^2}(\nabla p \times \nabla \rho) = \frac{\rho g}{\rho_0^2}(\boldsymbol{e}_z \times \nabla \rho^*) \tag{5.19}$$

$$= -\frac{\boldsymbol{g}}{\rho_0} \times \nabla \rho^* \tag{5.20}$$

hence we recover

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f_{\mathrm{c}}} \times \nabla \rho^*. \tag{5.21}$$

Note:

5.3.3 Cyclostrophic wind relations

In analogy to the thermal wind (Coriolis balances the horizontal pressure gradient), we have similarly the cyclostrophic wind (centrifugal balances the radial pressure gradient):

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_0}\frac{\partial p^*}{\partial r} \tag{5.22}$$

$$\frac{\partial p^*}{\partial z} = -\frac{\rho^* g}{\rho_0} \tag{5.23}$$

$$\frac{\partial \rho^*}{\partial r} = -\frac{\rho_0}{g}\frac{\partial}{\partial z}\left(\frac{u_\theta^2}{r}\right) \tag{5.24}$$

where a vertical wind shear is associated with a horizontal density gradient. Cyclo means 'cyclone' or low-pressure system and strophic means 'turning'.

5.3.4 Example: Taylor-Proudman theory

Consider steady flow with negligible convective term (in geostrophic balance, $Ro \ll 1$):

$$0 = -\frac{1}{\rho}\nabla p + \boldsymbol{u} \times \boldsymbol{f}_{\mathrm{c}}. \tag{5.25}$$

Taking the curl of the above equation we have

$$0 = \nabla \times (\boldsymbol{u} \times \boldsymbol{f}_{\mathrm{c}}). \tag{5.26}$$

Using triple product rules (A.21), in an f-plane, we have

$$\boldsymbol{f}_{\mathrm{c}} \times \nabla \boldsymbol{u} = 0. \tag{5.27}$$

Assuming the rotation axis is normal to the plane, $\boldsymbol{f}_{\mathrm{c}} = f_{\mathrm{c}}\boldsymbol{k}$, we have

$$\frac{\partial \boldsymbol{u}}{\partial z} = \boldsymbol{0}. \tag{5.28}$$

That implies very strong rotation negates vertical gradients.

Stratified Taylor column: consider the thermal wind balance

$$0 = -\frac{1}{\rho_0}\nabla p + \boldsymbol{u} \times \boldsymbol{f}_{\mathrm{c}} - \frac{\rho^* g}{\rho_0}\boldsymbol{k}. \tag{5.29}$$

Taking the curl we have

$$\boldsymbol{f}_{\mathrm{c}} \cdot \nabla \boldsymbol{u} = (\boldsymbol{i}\partial_y - \boldsymbol{j}\partial_x)\frac{\rho^* g}{\rho_0}, \tag{5.30}$$

i.e.,

$$[f_3\frac{\partial u}{\partial z}, f_3\frac{\partial v}{\partial z}, f_3\frac{\partial w}{\partial z}] = [\partial_y\frac{\rho^* g}{\rho_0}, -\partial_x\frac{\rho^* g}{\rho_0}, 0], \tag{5.31}$$

i.e.,

$$f_3\frac{\partial u}{\partial z} = \partial_y\frac{\rho^* g}{\rho_0} \tag{5.32}$$

$$f_3\frac{\partial v}{\partial z} = -\partial_x\frac{\rho^* g}{\rho_0} \tag{5.33}$$

$$\frac{\partial w}{\partial z} = 0 \tag{5.34}$$

which implies Q2D flow with

$$\nabla_H \cdot \boldsymbol{u} = \partial_x u + \partial_y v = -\partial_z w = 0. \tag{5.35}$$

The relations (5.32)-(5.33) are essentially thermal wind relations where the vertical wind shear is balanced by (the pressure gradient created by) horizontal density gradients.

5.4 Governing equations of unbalanced motions

It is reasonable to assume directions of both system rotation and gravity are in $\boldsymbol{z}$.

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{5.36}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} - f_{\mathrm{c}}\epsilon_{ij3}(u_j - U_j) = -\frac{1}{\rho_0}\frac{\partial p^*}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\rho^* g}{\rho_0}\delta_{i3}, \tag{5.37}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i}, \tag{5.38}$$

$$\tau_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ J_{\rho,i} = \kappa\frac{\partial \rho}{\partial x_i}. \tag{5.39}$$

In vector form,

$$\nabla \cdot \boldsymbol{u} = 0 \tag{5.40}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) + f_c \boldsymbol{e}_z \times (\boldsymbol{u} - \boldsymbol{U}) = -\frac{1}{\rho_0}\nabla p^* + \nabla \cdot \boldsymbol{\tau} - \frac{\rho^* g}{\rho_0}\boldsymbol{e}_z \tag{5.41}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = \nabla \cdot \boldsymbol{J}_\rho \tag{5.42}$$

where the stress and the scalar flux are

$$\boldsymbol{\tau} = \nu(\nabla \boldsymbol{u} + \boldsymbol{u}\nabla),\ \boldsymbol{J}_\rho = \kappa \nabla \rho. \tag{5.43}$$

The total density ρ is decomposed into the reference density ρ_0, the background density $\rho_b(z)$, and the density perturbation ρ^* due to fluid motion,

$$\rho(x,y,z,t) = \rho_0 + \rho_b(z) + \rho^*(x,y,z,t). \tag{5.44}$$

The total pressure is written as

$$p(x,y,z,t) = p_0 + p_g(x,y) + p_a(z) + p^*(x,y,z,t), \tag{5.45}$$

where the reference pressure p_0 is a constant, the hydrostatic (ambient) pressure p_a has a vertical gradient that balances the ambient density ($\rho_a = \rho_0 + \rho_b(z)$), and the geostrophic pressure p_g has a transverse gradient that balances the Coriolis force due to the geostrophic wind $\boldsymbol{U}$. Only the dynamic pressure p^* appears in the momentum equation (5.37).

Instead of using ρ^*, it is also common to express the buoyancy term as

$$b = -\frac{\rho^* g}{\rho_0}, \tag{5.46}$$

and the 'total' buoyancy

$$\tilde{b} = b + \bar{b} = -\frac{(\rho^* + \bar{\rho}(z))g}{\rho_0}, \tag{5.47}$$

where the background linear stratification is $N^2 = \partial \bar{b}/\partial z$ and we have $\tilde{b} = b + N^2 z$ with the reference value $\bar{\rho}(z=0) = 0$.

Eqn, (5.38) can also be expressed as

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* u_i}{\partial x_i} + w\frac{\partial \bar{\rho}}{\partial z} = \kappa\frac{\partial^2 \rho^*}{\partial x_i^2}, \tag{5.48}$$

and hence we have the buoyancy equation

$$\frac{\partial b}{\partial t} + \frac{\partial b u_i}{\partial x_i} + wN^2 = \kappa\frac{\partial^2 b}{\partial x_i^2}, \tag{5.49}$$

and the equation for the total buoyancy is

$$\frac{\partial \tilde{b}}{\partial t} + \frac{\partial \tilde{b} u_i}{\partial x_i} = \kappa\frac{\partial^2 \tilde{b}}{\partial x_i^2}. \tag{5.50}$$

5.4.1 Incompressibility

Evan though there is a density transport due to the diffusion (due to the special role that ρ plays; this is not the mass conservation equation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i} \neq 0, \tag{5.51}$$

we could still establish incompressible condition

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{5.52}$$

with some additional assumtions.

First we review the integration form of the general conservation equation for an arbitraty scalar (per unit mass)

$$\frac{\partial}{\partial t}\left(\iiint_V \rho\psi \,\mathrm{d}V\right) = -\iint_{\Omega=\partial V} (\rho \boldsymbol{u}\psi)\cdot\,\mathrm{d}\boldsymbol{A} - \iint_{\Omega=\partial V} \rho\kappa(-\nabla\psi)\cdot\,\mathrm{d}\boldsymbol{A} \tag{5.53}$$

$$= -\iiint_V \nabla\cdot(\rho\boldsymbol{u}\psi)\,\mathrm{d}V - \iiint_V \nabla\cdot(\rho\kappa(-\nabla\psi))\,\mathrm{d}V \tag{5.54}$$

and we have

$$\frac{\partial \rho\psi}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u}) = \nabla\cdot(\rho\kappa\nabla\psi). \tag{5.55}$$

It is in a general form of a conservational principle

$$\frac{\partial Q_v}{\partial t} + \nabla\cdot\boldsymbol{F} = 0, \tag{5.56}$$

where $\boldsymbol{F}$ is the flux and $\nabla\cdot\boldsymbol{F}$ is the transport term.

Taking $\psi = 1$ we recover the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = \frac{D\rho}{Dt} + \rho(\nabla\cdot\boldsymbol{u}) = 0. \tag{5.57}$$

Usually, density change along the material lines, is small enough such that $(1/\rho)D\rho/Dt \ll U/L$ and hence $\nabla\cdot\boldsymbol{u} \ll U/L$. That being said, being non-dimensionalised, the velocity field is solenoidal. We note in density-variable flows that

$$\nabla\cdot\boldsymbol{u} = 0 \tag{5.58}$$

is an approximation. See Batchelor (1967), section 3.2, for details as why this is valid.

5.4.2 Scalar transport equation

Taking $\psi = s$ (salinity or temperature) and assume diffusivity κ is constant, we have the scalar transport equation

$$s\frac{\partial \rho}{\partial t} + \rho\frac{\partial s}{\partial t} + u_j\rho\frac{\partial s}{\partial x_j} + u_j s\frac{\partial \rho}{\partial x_j} = \kappa\frac{\partial s}{\partial x_j}\frac{\partial \rho}{\partial x_j} + \rho\kappa\frac{\partial^2 s}{\partial x_j^2}, \tag{5.59}$$

taking into account

$$\frac{\partial \rho}{\partial t} + u_j\frac{\partial \rho}{\partial x_j} = 0 \tag{5.60}$$

we have

$$\rho\frac{\partial s}{\partial t} + u_j\rho\frac{\partial s}{\partial x_j} = \kappa\frac{\partial s}{\partial x_j}\frac{\partial \rho^*}{\partial x_j} + \rho\kappa\frac{\partial^2 s}{\partial x_j^2}. \tag{5.61}$$

We note that under Boussinesq assumption, $\rho^*/\rho = \rho^*/(\rho_0 + \rho*) \ll 1$, we have

$$\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_j} = \kappa \frac{\partial^2 s}{\partial x_j^2}. \tag{5.62}$$

With some linear equation of state, we can relate s or T to ρ and get a scalar transport equation for ρ as

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = \kappa \frac{\partial^2 \rho}{\partial x_j^2}, \tag{5.63}$$

with the incompressibility being implied by $\nabla \cdot \boldsymbol{u} = 0$ from Eqn. (5.57). We note that Eqns. (5.57) and (5.63) correspond to two different physical principles.

E.g.,

$$p = \rho R T \tag{5.64}$$

$$\ln p = ln(\rho) + \ln T + \ln R \tag{5.65}$$

$$\frac{\delta p}{p} = \frac{\delta \rho}{\rho} + \frac{\delta T}{T}. \tag{5.66}$$

Assuming isobaric process we have

$$\frac{\partial \rho}{\partial z} \propto -\frac{\partial T}{\partial z} \tag{5.67}$$

and $b = -(g/\rho_0)\partial\rho^*/\partial z = \partial T^*/\partial z$. It is also possible to solve or interpret as the temperature equation (5.62) is being solved and density will be obtain using an equation-of-state such as

$$\frac{\rho - \rho_0}{\rho_0} = -\beta(T - T_0),\ \beta = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p \tag{5.68}$$

where β is call the themal expansion coefficient. Then

$$\frac{\partial \rho}{\partial z} = -\beta \frac{\partial T}{\partial z} \tag{5.69}$$

and we can define

$$b = \frac{g T^*}{T_0} \tag{5.70}$$

$$N^2 = \frac{g}{T_0}\frac{\partial \bar{T}}{\partial z} \tag{5.71}$$

5.5 GFD vorticity equations

5.5.1 Absolute vorticity equation

The 'absolute' vorticity, defined as $\boldsymbol{\omega}_a = \boldsymbol{\omega} + \boldsymbol{f}_c$, is the 'relative' vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ plus the 'planetary' vorticity $\boldsymbol{f}_c = 2\boldsymbol{\Omega}_c$ ($\Omega_c = \Omega \sin\phi$).

Similar to Eq. (2.21), we can derive the governing equation for $\boldsymbol{\omega}_a$ starting from

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boldsymbol{f}_c \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F} \tag{5.72}$$

without the hydrostatic part separated and/or Boussinesq assumed.

According to identity (A.21) we have, *in an f-plane*,

$$\nabla \times (\boldsymbol{f}_c \times \boldsymbol{u}) = -\boldsymbol{f}_c \cdot \nabla \boldsymbol{u}. \tag{5.73}$$

Similar to Eq. (2.14), by taking the curl of (5.72) and taking $\boldsymbol{F} = b\boldsymbol{e}_z$ we have the vorticity equation in a rotating frame

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}_a + \nabla \times \boldsymbol{F} + \frac{1}{\rho^2} \nabla \rho \times \nabla p, \tag{5.74}$$

i.e., the absolute vorticity equation:

$$\frac{D\boldsymbol{\omega}_a}{Dt} = \frac{\partial \boldsymbol{\omega}_a}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega}_a = \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} \tag{5.75}$$

$$= \boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times \boldsymbol{F} + \frac{1}{\rho^2} \nabla \rho \times \nabla p. \tag{5.76}$$

In a Boussinesq fluid,

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times (b\boldsymbol{e}_z) + \frac{1}{\rho_0^2} \nabla \rho^* \times \nabla p^*, \tag{5.77}$$

where $\nabla \times (b\boldsymbol{e}_z) = \epsilon_{ij3} \partial_j b$.

Additionally, the linearized inviscid evolution equation for the perturbation vorticity, similar to (4.126), reads

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j' \bar{S}_{ij} + (\bar{\omega}_j + f_c \delta_{j3}) S_{ij}' - u_j' \frac{\partial \bar{\omega}_i}{\partial x_j} + \frac{1}{2} \epsilon_{ij3} f_c \omega_j' + \epsilon_{ij3} \frac{\partial b}{\partial x_j}, \tag{5.78}$$

which is convenient for instability considerations.

Example (Taylor-Proudman theorem, 5.3.4; another proof): Assume inviscid, barotropic fluid acted on by conservative force, and that the rotation rate $\boldsymbol{\Omega}_c = \boldsymbol{f}_c/2$ is much greater than other frequencies. Eqn. (5.76) becomes

$$0 = \boldsymbol{f}_c \cdot \nabla \boldsymbol{u} \tag{5.79}$$

and that completes the proof.

5.5.2 Potential vorticity equation; Ertal's theorem

Ref. Pedlosky (2013).

Assume a conserved scalar λ with a governing operator $D\lambda/Dt = S$ where S is a source term for λ. Consider

$$\frac{D}{Dt}\left(\frac{\partial \lambda}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial t} + u_j \frac{\partial \lambda}{\partial x_j}\right) - \frac{\partial u_j}{\partial x_i} \frac{\partial \lambda}{\partial x_j}, \tag{5.80}$$

i.e.,

$$\frac{D}{Dt}(\nabla \lambda) = \nabla \left(\frac{D\lambda}{Dt}\right) - \nabla \boldsymbol{u} \cdot \nabla \lambda. \tag{5.81}$$

$\nabla\lambda \cdot$ (5.76) + $\boldsymbol{\omega}_a \cdot$ (5.81), with a magic that two opposite-sign $\boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u} \cdot \nabla \lambda$ terms cancel, we have

$$\frac{D}{Dt}(\boldsymbol{\omega}_a \cdot \nabla \lambda) = \boldsymbol{\omega}_a \cdot \nabla S + \nu \nabla^2 \boldsymbol{\omega}_a \cdot \nabla \lambda + (\nabla \times \boldsymbol{F}) \cdot (\nabla \lambda) + \frac{1}{\rho^2} (\nabla \rho \times \nabla p) \cdot (\nabla \lambda) \tag{5.82}$$

Take $\lambda = \tilde{b}$ which is the total buoyancy, with its governing equation being (5.50), assuming conservative external force $\boldsymbol{F}$ and barotropic flow*, we have the potential vorticity (PV) equation:

$$\frac{D}{Dt}\left(\boldsymbol{\omega}_a \cdot \nabla \tilde{b}\right) = \nu \nabla^2 \boldsymbol{\omega}_a \cdot \nabla \tilde{b} + \kappa [\nabla^2 (\nabla \tilde{b})] \cdot \boldsymbol{\omega}_a, \tag{5.83}$$

where

$$\Pi = \boldsymbol{\omega}_a \cdot \nabla \tilde{b} \tag{5.84}$$

is called the potential vorticity, which is the component of the absolute vorticity perpendicular to the isosurface (or parallel to the gradient) of $\tilde{b}$. In the absence of dissipation,

$$\frac{D\Pi}{Dt} = 0, \tag{5.85}$$

i.e., PV is conserved along the streamlines. Eq. (5.83) is like a double-diffusion problem with one 'passive' scalar diffuse together with vorticity.

Ertel's theorem: under the follwing assumptions, PV conservation along fluid motion is satisfied (from Eq. (5.82)):

- λ is a conserved quantity that following fluid motion $S = 0$.
- Conservative external force: $\nabla \times \boldsymbol{F} = 0$.
- Either
  1. Baroclinicity absent ($\nabla\rho \times \nabla p = 0$)
  2. λ is only a thermodynamic function of p, ρ, i.e., $\lambda = \lambda(p, \rho)$ so that the last term vanishes when $\cdot(\nabla\lambda)$. For example, $\lambda = s$ (entropy).
- Diffusion-less/inviscid: $\nu = \kappa = 0$.

5.5.3 Relation of PV to Kelvin's theorem in a rotating frame

Similarly, Kelvin's circulation theorem (see (2.6)) in a rotating frame is

$$\frac{D\Gamma_a}{Dt} = \iint_A \frac{\nabla\rho \times \nabla p}{\rho^2} \cdot d\boldsymbol{A}, \tag{5.86}$$

where the absolute circulation is

$$\Gamma_a = \int_A \boldsymbol{\omega}_a \cdot d\boldsymbol{A} = \Gamma + \int_A \boldsymbol{f}_\mathrm{c} \cdot d\boldsymbol{A}. \tag{5.87}$$

When the surface A is specifically chosen to be on $\lambda = \lambda(\rho, p)) =$ constant (and A is enclosed by a contour l that stays on $\lambda =$ constant for all time), what follows is that $\nabla\lambda$ must be in the parameter plane spanned by $\nabla\rho$ and ∇p, hence $\nabla\lambda \cdot (\nabla\rho \times \nabla p) = 0$. Here, we note that the normal vector for A is $\boldsymbol{n} = \nabla\lambda/|\nabla\lambda|$ so $(\nabla\rho \times \nabla p) \cdot \boldsymbol{n} = 0$ on the entire plane and

$$\frac{D\Gamma_a}{Dt} = 0. \tag{5.88}$$

The choice of λ is just to choose a surface/contour (of $\lambda =$ constant) on which $\nabla\rho, \nabla p$ lie in the surface and the baroclinicity term makes zero contribution to the circulation in a baroclinic flow. We can regard PV conservation as a special statement of Kelvin's theorem.

Example: $\lambda(\rho, p) = \rho^2 + p^2$. Think of $\lambda =$ constant as a cylinderical surface.

Example: For a finite flat area A, $\Gamma_a \approxeq \omega_a A$ and its conservation implies angular momentum conservation, and the change of spacing of λ surfaces (associated with change of area) can change relative vorticity — stretching and compressing.

5.6 Turbulence equations for an active scalar

5.6.1 Mean flow equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{5.89}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} - f_c \epsilon_{ij3}(\bar{u}_j - U_j) = -\frac{1}{\rho_0}\frac{\partial \overline{p^*}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \overline{u_i' u_j'}) - \frac{\overline{\rho^*} g}{\rho_0}\delta_{i3} \tag{5.90}$$

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{u}_j \frac{\partial \bar{\rho}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial \bar{\rho}}{\partial x_j} - \overline{\rho' u_j'}\right), \tag{5.91}$$

We note that

$$\rho' = \rho - \bar{\rho} = \rho^* - \overline{\rho^*} = \rho^{*\prime}. \tag{5.92}$$

5.6.2 Fluctuation equations

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{5.93}$$

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} - f_c \epsilon_{ij3} u_j' = -\frac{1}{\rho_0}\frac{\partial p^{*\prime}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \bar{u}_i}{\partial x_j} + \overline{u_i' u_j'} - u_i' u_j') - u_j' \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\rho^{*\prime} g}{\rho_0}\delta_{i3} \tag{5.94}$$

$$\frac{\partial \rho^{*\prime}}{\partial t} + \bar{u}_j \frac{\partial \rho^{*\prime}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial \rho^{*\prime}}{\partial x_j} + \overline{\rho^{*\prime} u_j'} - \rho^{*\prime} u_j'\right) - \rho^{*\prime}\frac{\partial \overline{p^*}}{\partial x_i} \tag{5.95}$$

We will see later the Coriolis term won't appear in the transport equations of MKE, TKE, and Reynolds stresses. Coriolis just bends the direction of the velocity. In other words, Coriolis does not have direct influence on turbulence budgets, but indirectly through the change of the mean flow. On the other hand, it does make a difference in the vorticity/enstrophy equation (see section 5.6.4).

5.6.3 MKE, MPE, TKE, TPE, and buoyancy flux equations

Define the mean and turbulent kinetic and potential energy as

$$K = \frac{1}{2}\bar{u}_i \bar{u}_i \tag{5.96}$$

$$K_\rho = \frac{1}{2}\bar{b}^2 \tag{5.97}$$

and

$$k = \frac{1}{2}\overline{u_i' u_i'} \tag{5.98}$$

$$k_\rho = \frac{1}{2}\overline{b' b'} \tag{5.99}$$

where the instantaneous, mean, and fluctuation buoyancy forces are

$$b = -\frac{\rho^* g}{\rho_0},\ \bar{b} = -\frac{\overline{\rho^*} g}{\rho_0},\ b' = -\frac{\rho^{*\prime} g}{\rho_0}, \tag{5.100}$$

such that k and k_ρ have the same dimension as the kinetic energy.

The **MKE equations** is (repeating (4.32)) :

$$\frac{\partial K}{\partial t} + \bar{u}_j \frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho}\bar{p}\,\bar{u}_j + \nu \frac{\partial K}{\partial x_j} - \bar{u}_i\,\overline{u_i' u_j'}\right) + \overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \nu \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j} \tag{5.101}$$

The **MPE equations** is:

$$\frac{\partial K_\rho}{\partial t} + \bar{u}_j \frac{\partial K_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}(\kappa \frac{\partial K_\rho}{\partial x_j} - \bar{b}\,\overline{b' u_j'}) + \overline{b' u_j'}\frac{\partial \bar{b}}{\partial x_j} - \kappa \frac{\partial \bar{b}}{\partial x_j}\frac{\partial \bar{b}}{\partial x_j} \tag{5.102}$$

We note that the buoyancy flux $\overline{b'u'_j}\partial\bar{b}/\partial x_j$ is a sink in the MPE equation and is a source in the TPE equation.

The **TKE equations** is:

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_k}(\underbrace{\nu \frac{\partial k}{\partial x_k}}_{\substack{\text{molecular}\\ \text{diffusion}}} + \underbrace{\frac{1}{2}\overline{u'_i u'_i u'_k}}_{\substack{\text{turbulent}\\ \text{diffusion}}} \underbrace{- \frac{1}{\rho_0}\overline{p'u'_k}}_{\substack{\text{pressure}\\ \text{distortion}}}) \underbrace{-\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j}}_{\text{production } P} - \underbrace{\nu \overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_k}}}_{\text{dissipation } \varepsilon} + \underbrace{\overline{b'w'}}_{\substack{\text{buoyancy}\\ \text{flux } B}} \tag{5.103}$$

$$= \nabla \cdot \boldsymbol{T} + P - \varepsilon + B \tag{5.104}$$

where the turbulent buoyancy flux

$$B = -\frac{g}{\rho_0}\overline{\rho^{*\prime} w'} = \overline{b'w'} \tag{5.105}$$

consumes TKE and lead to the production of TPE.

The **TPE equation** is:

$$\frac{\partial k_\rho}{\partial t} + \bar{u}_j \frac{\partial k_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}(\kappa \frac{\partial k_\rho}{\partial x_j} - \frac{1}{2}\overline{b'b'u'_j}) - \overline{b'u'_j}\frac{\partial \bar{b}}{\partial x_j} - \kappa \overline{\frac{\partial b'}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{5.106}$$

We can see that the turbulent buoyancy flux B (negative, think $-\overline{u'_i u'_j}$) works with the density distortion $\partial\bar{b}/\partial z$ to remove energy from TKE and MPE to produce TPE.

The **buoyancy flux equation** is:

$$\frac{\partial \overline{b'u'_i}}{\partial t} + \bar{u}_j \frac{\partial \overline{b'u'_i}}{\partial x_j} = d_{b,i} + P_{b,i} + \Phi_{b,i} - \varepsilon_{b,i} \tag{5.107}$$

where

$$d_{b,i} = \frac{\partial}{\partial x_j}(\kappa \overline{u'_i \frac{\partial b'}{\partial x_j}} + \nu \overline{b' \frac{\partial u'_i}{\partial x_j}} - \frac{1}{\rho_0}\overline{p'b'}\delta_{ij} - \overline{b'u'_i u'_j}) \tag{5.108}$$

$$P_{b,i} = -\overline{b'u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \overline{u'_i u'_j}\frac{\partial \bar{b}}{\partial x_j} \tag{5.109}$$

$$\Phi_{b,i} = \frac{1}{\rho_0}\overline{p'\frac{\partial b'}{\partial x_i}} \tag{5.110}$$

$$\varepsilon_{b,i} = (\nu + \kappa)\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{5.111}$$

5.6.4 Perturbation vorticity and enstrophy equations

5.7 Inertial and buoyancy oscillations

5.7.1 Derivation of Coriolis force

5.7.2 Boussinesq approximation

5.8 Surface and bottom Ekman layer solutions

5.9 Miscellaneous

coriolis frequency; shallow water / wave equations; igw equations;

6 Hydrodynamic stability

Due to the fewness of the existence of exact solutions to PDEs, no matter for the N-S or its linearzed from (around some base state), numerical solution of many of the stability problems can not be avoided. Hence we would also not avoid discussing related numerical methods here.

6.1 Linearised Navier–Stokes

Consider the incompressible N-S equations

$$\nabla \cdot \boldsymbol{u} = 0 \tag{6.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{6.2}$$

and the decomposition of velocity and pressure into the base and perturbation states:

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}' \tag{6.3}$$

$$p = P + p' \tag{6.4}$$

We note that the base state also satisfies N-S:

$$\nabla \cdot \boldsymbol{U} = 0 \tag{6.5}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \boldsymbol{U} \tag{6.6}$$

hence by plugging in the decomposition to (6.1)-(6.2) we have the perturbation equation:

$$\nabla \cdot \boldsymbol{u}' = 0 \tag{6.7}$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{U} + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' = -\frac{1}{\rho}\nabla p' + \nu \nabla^2 \boldsymbol{u}' \tag{6.8}$$

And we note that the boundary conditions that the perturbation $\boldsymbol{u}', p'$ satisfy is homogeneous, such that $\boldsymbol{U}$ and p_b satisfy the same BC's as $\boldsymbol{u}$ and p in the original equation.

In linear stability, with the assumption that

$$O(\boldsymbol{u}') = \epsilon O(\boldsymbol{U}), \tag{6.9}$$

we neglect the nonlinear term $\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'$ and the primes, and have the linearised perturbation equation

$$\nabla \cdot \boldsymbol{u} = 0 \tag{6.10}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} - \nu \nabla^2 \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho}\nabla p \tag{6.11}$$

or if we define the linear operator as

$$\mathcal{L}_{\boldsymbol{U}} = \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{6.12}$$

there is

$$\mathcal{L}_{\boldsymbol{U}} \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho}\nabla p. \tag{6.13}$$

The linearised equations (6.10)-(6.11), if written in matrix form (Arratia, 2011), is

$$\mathcal{L}_{\mathrm{NS}} \boldsymbol{q} = \begin{bmatrix} \mathcal{L}_{\boldsymbol{U}} + \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial V}{\partial x} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial W}{\partial z} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = 0, \tag{6.14}$$

where $\boldsymbol{q} = [u, v, w, p]^T$. This is in the KKT form that will be described below, where we will see that the same mathematical properties of the operators will be shared in both stability analysis and CFD.

On the other hand, it is sometimes also convenient to define the RHS operator as

$$\mathcal{A}_{\boldsymbol{U}}\boldsymbol{u} = -\boldsymbol{U}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{U} - \frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} \tag{6.15}$$

such that

$$\frac{\partial\boldsymbol{u}}{\partial t} = \mathcal{A}_{\boldsymbol{U}}\boldsymbol{u}. \tag{6.16}$$

In practice, the pressure gradient can be neglected when calculating the operator $\mathcal{A}_{\boldsymbol{U}}$ and the resulting pressure is projected onto a divergence-free space (along with satisfying the boundary conditions in each numerical iteration/time-step), under the framework of projection methods.

6.1.1 The role of pressure

A separate short note on the pressure being the Lagrangian multiplier in incompressible system. Consider the Stokes flow (actually that can be the linearised equations as described above)

$$-\nabla^2\boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{6.17}$$

$$\nabla\cdot\boldsymbol{u} = 0 \tag{6.18}$$

and in the matrix form

$$\begin{bmatrix} -\nabla^2 & \nabla \\ \nabla\cdot & O \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} \tag{6.19}$$

and its discrete version

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{6.20}$$

which is a saddle point problem or a KKT (Karush-Kuhn-Tucker) system (Benzi *et al.*, 2005). The Stokes equations can be interpreted as a constained optimisation problem (section 3.15.5 of Gresho & Sani (1998))

$$\min\ J(\boldsymbol{u}) = \frac{1}{2}\int \|\nabla\boldsymbol{u}\|_2^2 dV - \int \boldsymbol{f}\cdot\boldsymbol{u}dV \tag{6.21}$$

$$\text{subject to } \nabla\cdot\boldsymbol{u} = 0 \tag{6.22}$$

where the variable p, introduced to satisfy an additional constraint, plays the role of a Lagrangian multiplier. We note that the adjoint of the gradient operator is the (negative) divergence operator

$$(\nabla)^\dagger = -\nabla\cdot \tag{6.23}$$

We not establish this fact. Consider a scalar f and a vector $\boldsymbol{F}$. consider

$$\int_V \nabla\cdot(f\boldsymbol{F})\,dV = \int_V f(\nabla\cdot\boldsymbol{F})\,dV + \int_V \nabla f\cdot\boldsymbol{F}\,dV = \iint_{\Omega=\partial V} f\boldsymbol{F}\cdot d\boldsymbol{A} \tag{6.24}$$

and if we the boundary integral vanishes we have

$$(\nabla f, \boldsymbol{F}) = \int_V (\nabla f\cdot\boldsymbol{F})\,dV = -\int_V f(\nabla\cdot\boldsymbol{F})\,dV = (f, -\nabla\cdot\boldsymbol{F}) = (f, (\nabla)^\dagger(\boldsymbol{F})) \tag{6.25}$$

and hence

$$(\nabla)^\dagger = -\nabla\cdot \tag{6.26}$$

6.1.2 Parallel shear flow

In the case of a parallel shear flow that $U(y)$ is the only nonzero mean flow component, we have

$$\mathcal{L}_U \boldsymbol{u} = \begin{bmatrix} -\partial_x p - U'v \\ -\partial_y p \\ -\partial_z p \end{bmatrix}, \; \mathcal{L}_U \boldsymbol{u} = \frac{\partial}{\partial t} + U\partial_x - \nu \nabla^2 \tag{6.27}$$

6.2 Normal-mode stability theory

6.2.1 K-H instability

and M-H condition.

6.2.2 Rayleigh's criterion

6.2.3 Orr-Sommerfield equations

Viscous instability mechanism.

6.2.4 T-S waves

6.2.5 Centrifugal instability

6.2.6 GFD instabilities

6.3 Non-normal instability

6.3.1 Adjoint matrices, operators, and equations

For a complex matrix $A \in \mathbb{C}^{N\times N}$, define an inner product

$$(u, v)_A = (Au, v) \tag{6.28}$$

where $u, v \in \mathbb{C}^N$ and $(u, v) = v^H u$, $(\cdot)^H$ is the Hermitian transpose. Define the adjoint matrix of A as $A^\dagger$ such that

$$(Au, v) = (u, A^\dagger v). \tag{6.29}$$

We note that if A is Hermitian ($A^H = A$), (6.29) is valid. Such matrix A is also called self-adjoint ($A = A^\dagger$). For operators defined on domains like $\mathbb{C}^N$, Hermitian and self-adjointness imply each other and we don't distinguish these two in what follows.

Consider the following standard Sturm-Liouville eigenvalue problem:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi = -\lambda\sigma(x)\phi, \tag{6.30}$$

where $p(x), w(x)$ are positive, and $\lambda, \phi(x)$ are the eigenvalue and corresponding eigenfunction of the problem. The boundary conditions are

$$\alpha_1\phi(a) + \alpha_2\frac{d\phi}{dx}(a) = 0 \tag{6.31}$$

$$\beta_1\phi(b) + \beta_2\frac{d\phi}{dx}(b) = 0 \tag{6.32}$$

$$\tag{6.33}$$

with $\alpha_1^2 + \alpha_2^2 \neq 0, \beta_1^2 + \beta_2^2 \neq 0$.

The LHS operator is defined as

$$\mathcal{L}(y) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y \tag{6.34}$$

and the S-L eigenvalue problem is

$$\mathcal{L}(\phi) + \lambda\sigma(x)\phi = 0. \tag{6.35}$$

The Lagrange identity is

$$u\mathcal{L}(v) - v\mathcal{L}(u) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right] \tag{6.36}$$

and the Green's formula is

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b \tag{6.37}$$

When u, v satisfy the same set of boundary conditions (either homogeneous or periodic), we have the self-adjointness, i.e.,

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]dx = 0 \tag{6.38}$$

or

$$\int_a^b u\mathcal{L}(v)dx = \int_a^b v\mathcal{L}(u)dx. \tag{6.39}$$

$$= \int_a^b v\mathcal{L}^\dagger(u)dx. \tag{6.40}$$

and we note the definition of adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$ is that

$$(u, \mathcal{L}(v)) = (v, \mathcal{L}^\dagger(u)), \tag{6.41}$$

with the inner product defined based on spatial integral and the adjoint is dependent on the inner product.

Examples.

1. The Laplacian operator.

$$\mathcal{L} = \nabla^2. \tag{6.42}$$

The multidimensional variation of (6.37), with $\mathcal{L} = \nabla^2$, is

$$\iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV = \iiint \nabla\cdot[u\nabla v - v\nabla u]dV = \iint (u\nabla v - v\nabla u)\cdot d\boldsymbol{A} \tag{6.43}$$

and if u, v satisfy the same homogeneous BC,

$$\iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV = \iint (u\nabla v - v\nabla u)\cdot d\boldsymbol{A} = 0, \tag{6.44}$$

and ∇^2 is self-adjoint.

2. The wave equation.

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\nabla^2. \tag{6.45}$$

The Green's formula

$$\int_{t_i}^{t_f} \iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV dt \tag{6.46}$$

$$= \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right) dV \bigg|_{t_i}^{t_f} - c^2 \int_{t_i}^{t_f} \left(\iint (u\nabla v - v\nabla u) \cdot d\boldsymbol{A} \right) dt \tag{6.47}$$

And we note that the $\mathcal{L} = \partial_{tt}$ operator alone is self-adjoint if the boundary terms vanish, by

$$\int_{t_i}^{t_f} [u\mathcal{L}(v) - v\mathcal{L}(u)]dt = \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right) \bigg|_{t_i}^{t_f}. \tag{6.48}$$

3. Heat equation. Given above, we only consider the temporal derivative $\mathcal{L} = \partial_t$ here.

$$\int_{t_i}^{t_f} u\frac{dv}{dt} + v\frac{du}{dt}dt = uv|_{t_i}^{t_f} \tag{6.49}$$

and the adjoint operator is $\mathcal{L}^\dagger = -\partial_t$. And we note that for the boundary term to vanish, we usually only have one BC for u in a first order problem such as $u(a) = 0$ and need to introduce an adjoint BC as $v(b) = 0$.

6.3.2 Non-self-adjointness and non-normality

A normal matrix $L \in \mathbb{C}^{N\times N}$ is defined as

$$L^H L = LL^H \tag{6.50}$$

and it is unitarily diagonalizable ($L = U\Lambda U^H$). The eigenvectors of L span an orthogonal basis of $\mathbb{C}^N$. More specifically, the eigenvectors corresponding to different eigenvalues are orthogonal, and even for degenerate eigenvalues an orthogonal basis can be found. A normal matrix is Hermitian if and only if all its eigenvalues are real.

The normality of a linear operator $\mathcal{L}$ is defined as

$$\mathcal{L}^\dagger \mathcal{L} = \mathcal{L}\mathcal{L}^\dagger. \tag{6.51}$$

The eigenmodes of $\mathcal{L}$ are normal to each other. A self-adjoint operator is hence normal and a non-normal operator must be non-self-adjoint.

For two eigenmodes $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$, where $\boldsymbol{\Phi}_i = e^{\lambda_i t}\boldsymbol{\phi}_i$, and $\lambda_i, \boldsymbol{\phi}_i$ are the eigenpair. There difference/-cancellation $\boldsymbol{f} = \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2$ decays if both decays and $(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2) = 0$. That said, if the real part of each eigenvalue is negative, the energy of the perturbation will decay. However, for non-self-adjoint operators, there could be a transient growth of the cancellation $\boldsymbol{f}$ (Schmid, 2007), where the decay of individual eigenmodes does not imply the transient decay of the total energy. The idea of optimal perturbations it to find such a transient mode that grows most within a certain period of time.

6.3.3 Adjoint of the linearised N-S equations (Op's)

Op's: optimal linear perturbation solved as an optimal control problem in an optimization formulation constrained by the PDEs with linear operators.

Similar to (6.10)-(6.11), the linearised perturbation equation with buoyancy is

$$\nabla \cdot \boldsymbol{u} = 0 \tag{6.52}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{u} + b\boldsymbol{e}_z \tag{6.53}$$

$$\frac{\partial b}{\partial t} + \boldsymbol{U} \cdot \nabla b + \boldsymbol{u} \cdot \nabla B = \kappa\nabla^2 b \tag{6.54}$$

where $b = B + b'$ is buoyancy and the primes are dropped from the equations above.

Assume the adjoints of $(\boldsymbol{u}, p, b)$ are $(\boldsymbol{v}, q, \varphi)$, multiplying each term in (6.52) by (v_1, v_2, v_3), (6.53) by $\boldsymbol{v}$, and (6.54) by φ, we can derive the adjoint equations of (6.52)-(6.54) as

$$\nabla \cdot \boldsymbol{v} = 0 \tag{6.55}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{U} + \varphi \nabla B - \frac{1}{\rho_0} \nabla q - \nu \nabla^2 \boldsymbol{v} \tag{6.56}$$

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{U} \cdot \nabla \varphi = -v_3 - \kappa \nabla^2 \varphi \tag{6.57}$$

We note the cross contribution terms $\varphi \nabla B$ and $-v_3$. The same set of equations can also be derived from a Lagrangian multiplier approach (a more morden method, but now classic), with the total perturbation energy being the Lagrangian and the set of governing equations along with BC's being the constraints enforced as multipliers. Such Lagrangian is in the form of energy gain as (Arratia, 2011; Luchini & Bottaro, 2014; Kaminski *et al.*, 2014)

$$\mathscr{L}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T) \rangle + \langle b(T), b(T) \rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0 \rangle + \langle b_0, b_0 \rangle / N^2} \tag{6.58}$$

$$- [\frac{\partial u_i}{\partial t} + u_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j^2} - b \delta_{i3}, v_i] \tag{6.59}$$

$$- [\frac{\partial b}{\partial t} + u_j \frac{\partial B}{\partial x_j} + U_j \frac{\partial b}{\partial x_j} - \kappa \frac{\partial^2 b}{\partial x_j^2}, \varphi] - [\frac{\partial u_i}{\partial x_i}, q] \tag{6.60}$$

$$- \langle u_i(0) - u_{0i}, v_{0i} \rangle - \langle b(0) - b_0, \varphi_0 \rangle \tag{6.61}$$

constrained by the equations through the multipliers $(\boldsymbol{v}, q, \varphi)$ and similarly the BC's. The inner products

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_V \boldsymbol{u} \cdot \boldsymbol{v} dV \tag{6.62}$$

$$[\boldsymbol{u}, \boldsymbol{v}] = \int_0^T \langle \boldsymbol{u}, \boldsymbol{v} \rangle dt \tag{6.63}$$

are defined as respective spatial and spatio-temporal integrations.

In the phraseology of optimal control (with PDE constraints, Ref. section 5 of Manzoni *et al.* (2021)):

- Target functional (finite time transient gain):

$$\mathscr{G}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T) \rangle + \langle b(T), b(T) \rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0 \rangle + \langle b_0, b_0 \rangle / N^2}. \tag{6.64}$$

- State variables: $(\boldsymbol{u}, b)$.
- State equations (PDE constraints): (6.52)-(6.54). We note that (adjoint) pressure only appears as a Lagrangian multiplier that enforces the continuity.
- Control variables: $\boldsymbol{u}(0), b(0)$ and hence $\boldsymbol{u}(\boldsymbol{x}, t), b(\boldsymbol{x}, t)$.
- Admissible constraints on controls: none for now.

Taking the variation of (6.61) w.r.t.:

- The multipliers $(\boldsymbol{v}, q, \varphi)$: we recover the 'direct' equations (6.52)-(6.54).
- The terms $\boldsymbol{v}_0, \varphi_0$: we obtain the definition of IC's $\boldsymbol{u}_0, b_0$.
- The 'direct' variables $(\boldsymbol{u}, p, b)$: we obtain the adjoint equations (6.55)-(6.57). This step will be shown in detail.

Other than deriving from a Lagrangian perspective, the adjoint can also be derived using (multiple) integrations by parts. Starting from (6.14), i.e.,

$$\mathcal{L}_{\mathrm{NS}}\boldsymbol{q} = 0, \tag{6.65}$$

with the direct and adjoint variables being $\boldsymbol{q} = (\boldsymbol{u}, p)$ and $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$, we look for the adjoint $\mathcal{L}_{\mathrm{NS}}^{\dagger}$ such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] - [\mathcal{L}_{\mathrm{NS}}^{\dagger}\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) \tag{6.66}$$

and the boundary conditions that make the RHS boundary terms vanish.

Where the inner product $[\cdot,\cdot]$ is that same as in (6.63) such that

$$[\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \int_T \int_V (\boldsymbol{v} \cdot \boldsymbol{u} + qp) dt dV. \tag{6.67}$$

The weak form of (6.14) is

$$[\mathbb{1}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V (\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + \partial_i p + u_j \partial_j U_i) dt dV = 0. \tag{6.68}$$

We note that (6.68) should also be valid on any arbitrary test function $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$ for (6.14) to hold, such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V v_i(\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + u_j \partial_j U_i) + q \partial_i p \, dt dV = 0. \tag{6.69}$$

By integration by parts we have

$$\begin{aligned}
[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] &= \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p \partial_i q \, dt dV + \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) && (6.70) \\
&= \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p \partial_i q \, dt dV && (6.71) \\
&= [\mathcal{L}_{\mathrm{NS}}^{\dagger}\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] && (6.72)
\end{aligned}$$

that defines the adjoint operator of $\mathcal{L}_{\boldsymbol{U}} = \partial_t + \boldsymbol{U} \cdot \nabla - \nu \nabla^2$:

$$\mathcal{L}_{\boldsymbol{U}}^{\dagger} = -\partial_t - \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{6.73}$$

and the adjoint equation

$$\partial_t v_i + U_j \partial_j v_i + \nu \partial_j^2 v_i = v_j \partial_j U_i - \partial_i q \tag{6.74}$$

and we note that by advancing forward in time, the viscous term is injecting energy into the system. Using the transfrom $\tau = -t$ we have

$$\partial_\tau v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i = -v_j \partial_j U_i + \partial_i q. \tag{6.75}$$

In another form, the adjoint equation (without density) can be expressed similar to (6.16) as

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\mathcal{A}_{\boldsymbol{U}}^{\dagger}\boldsymbol{v} \tag{6.76}$$

where $\mathcal{A}_{\boldsymbol{U}}^{\dagger}$ is the adjoint operator of $\mathcal{A}_{\boldsymbol{U}}$ in (6.15)

$$\mathcal{A}_{\boldsymbol{U}}^{\dagger}\boldsymbol{v} = \boldsymbol{U} \cdot \nabla \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{U} + \frac{1}{\rho_0}\nabla q + \nu \nabla^2 \boldsymbol{v}. \tag{6.77}$$

6.4 The Lorenz system

7 Compressible flows, gas dynamics

7.1 Conservation laws

Apart from deriving the Euler equation by neglecting the viscous term in the Navier–Stokes, we present another way here that is more to the taste of gas dynamics.

7.2 Thermodynamics

7.3 Kinetic theory, microscopic view of fluid and flow properties

pressure (molecules bouncing back from a flat wall), viscosity, etc.

kinetic theory. Cf. Frank Shu.

AFD notes; the temperature lapse rate example;

Incompressible and isentropic flows ($p/\rho^\gamma = C$ for perfect gas), are barotropic flows.

7.4 Small perturbation linearized equations, sound waves

Simple waves, wave equations, enthalpy, shock wakes, etc. Small perturbation linearized equations, aerodynamics.

7.5 Shock waves, discontinuities, and jump conditions

Conservation form, weak form, weak continuity, test functions.

A Vectors, tensors, and their calculus

Aris (1989) is a good reference.

A.1 Levi-Civita symbol

A.1.1 Determinant representation

The matrix determinants can be expressed in terms of the Levi-Civita symbol. Assume A is a matrix

$$\det(A) = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{A.1}$$

where

$$\boldsymbol{a}_1 = (a_{11}, a_{12}, a_{13})^\top,\ \boldsymbol{a}_2 = (a_{21}, a_{22}, a_{23})^\top,\ \boldsymbol{a}_3 = (a_{31}, a_{32}, a_{33})^\top \tag{A.2}$$

Therefore the Levi-Civita symbol can be expressed as

$$\epsilon_{ijk} = \det(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k) = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) \tag{A.3}$$

Similarly, the outer product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \epsilon_{ijk} a_j b_k \boldsymbol{e}_i \tag{A.4}$$

Example: $\boldsymbol{\omega}$.

A.1.2 Epsilon identity

$$
\begin{aligned}
\epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} && \text{(A.5)} \\
&= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{jl}(\delta_{in}\delta_{km} - \delta_{im}\delta_{kn}) + \delta_{kl}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) && \text{(A.6)} \\
&= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} && \text{(A.7)}
\end{aligned}
$$

A.1.3 Contracted epsilon identity

Let $i = l$ and notice $\delta_{ii} = 3$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{A.8}$$

Futhur let $k = m$

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{A.9}$$

Futhermore

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{A.10}$$

A.2 Vector identities

Assume λ is a scalar and $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$,$\boldsymbol{d}$ are vectors in $\mathbb{R}^3$. The identities below might be useful in fluids, some of which have geometric implecations.

$$
\begin{aligned}
&\nabla \cdot (\nabla \times \boldsymbol{b}) = 0 && \text{(A.11)} \\
&\nabla \times (\nabla \boldsymbol{b}) = 0 && \text{(A.12)} \\
&\nabla \cdot (\lambda \boldsymbol{b}) = \nabla\lambda \cdot \boldsymbol{b} + \lambda(\nabla \cdot \boldsymbol{b}) && \text{(A.13)} \\
&\nabla \times (\lambda \boldsymbol{b}) = \lambda(\nabla \times \boldsymbol{b}) - \boldsymbol{b} \times \nabla\lambda && \text{(A.14)} \\
&\nabla(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla\boldsymbol{a} + \boldsymbol{a} \cdot \nabla\boldsymbol{b} + \boldsymbol{b} \times (\nabla \times \boldsymbol{a}) + \boldsymbol{a} \times (\nabla \times \boldsymbol{b}) && \text{(A.15)} \\
&\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) && \text{(A.16)} \\
&\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla \times \boldsymbol{a}) \cdot \boldsymbol{b} - (\nabla \times \boldsymbol{b}) \cdot \boldsymbol{a} && \text{(A.17)} \\
&(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}) && \text{(A.18)} \\
&\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) && \text{(A.19)} \\
&\boldsymbol{b} \times (\nabla \times \boldsymbol{b}) = \nabla(\frac{1}{2}\boldsymbol{b} \cdot \boldsymbol{b}) - \boldsymbol{b} \cdot \nabla\boldsymbol{b} && \text{(A.20)} \\
&\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) && \text{(A.21)} \\
&\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2\boldsymbol{a} && \text{(A.22)}
\end{aligned}
$$

Their proofs are left as exercises.
Comments:

(1) Eq. (A.11): A curl field is solenoidal (divergence-free).

(2) Eq. (A.12): A gradient field is irrotational (curl-free).

(3) Eq. (A.21): $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$, so its curl is in the space spaned by $\boldsymbol{a}$ and $\boldsymbol{b}$.

(4) Eq. (A.19): $\boldsymbol{a} \times (\cdot)$ is perpendicular to $\boldsymbol{a}$ and $(\cdot) \times (\boldsymbol{b} \times \boldsymbol{c})$ is in the space spaned by $\boldsymbol{b}$ and $\boldsymbol{c}$. This two facts in combinition gives the bases of $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

(5) Eq. (A.16): This is the volume spaned by $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, and the identity is basically the invariance of a determinant with respect to row/column permutation.

(6) Eq. (A.18): By letting $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$ and noticing the inner product with itself is non-negative, we re-discover the Cauchy-Schwartz inequality.

(7) From (A.15) it can be immediately seen that

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla(\boldsymbol{u}^2/2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}). \tag{A.23}$$

A.3 Tensor eigenvalues, invariants, and its application in fluids

Consider a tensor $\boldsymbol{A}$ in Cartesian coordinate

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{A.24}$$

Its eigenvalues are roots of the characteristic polynomial

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} = \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \tag{A.25}$$

with the three coefficients being the three principle invariants of $\boldsymbol{A}$

$$\begin{aligned}
I_1 &= a_{11} + a_{22} + a_{33} && \text{(A.26)} \\
&= \mathrm{tr}(\boldsymbol{A}) && \text{(A.27)} \\
&= a_{ii} && \text{(A.28)} \\
I_2 &= a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31} && \text{(A.29)} \\
&= \frac{\mathrm{tr}(\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)}{2} && \text{(A.30)} \\
&= \frac{1}{2}((a_{ii})^2 - a_{ij}a_{ji}) && \text{(A.31)} \\
I_3 &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) && \text{(A.32)} \\
&= \det(\boldsymbol{A}) && \text{(A.33)}
\end{aligned}$$

in both element-wise and coordinate-independent expression.

Now we consider the factorization of the characteristic polynomial as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\lambda - \lambda_1\lambda_2\lambda_3 = 0, \tag{A.34}$$

and obtain the Vieta's theorem for cubic equations as

$$\begin{aligned}
I_1 &= \lambda_1 + \lambda_2 + \lambda_3 && \text{(A.35)} \\
I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 && \text{(A.36)} \\
I_3 &= \lambda_1\lambda_2\lambda_3 && \text{(A.37)}
\end{aligned}$$

which are the three principle invariants of tensor $\boldsymbol{A}$.

Additionally, there are more invariants (although not independent) of $\boldsymbol{A}$, such as the main invariants

$$J_1 = \lambda_1 + \lambda_2 + \lambda_3 = I_1 = \mathrm{tr}(\boldsymbol{A}) \tag{A.38}$$

$$J_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1^2 - 2I_2 = \mathrm{tr}(\boldsymbol{A} \cdot \boldsymbol{A}) \tag{A.39}$$

$$J_1 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = I_1^3 - 3I_1 I_2 + 3I_3 = \mathrm{tr}(\boldsymbol{A} \cdot \boldsymbol{A} \cdot \boldsymbol{A}) \tag{A.40}$$

which are the coefficients of the characteristic polynomial of the deviatoric part of $\boldsymbol{A}$:

$$\boldsymbol{A} - \frac{\mathrm{tr}(\boldsymbol{A})}{3}\boldsymbol{I}, \tag{A.41}$$

which is traceless and has eigenvalues

$$\lambda_i - \frac{1}{3}. \tag{A.42}$$

A.3.1 Discriminant of a cubic equation

Consider

$$ax^3 + bx^2 + cx + d = 0, \tag{A.43}$$

its determinant is

$$\Delta = (x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2 \tag{A.44}$$

$$= 18abcd - 4b^3 d + b^2 c^2 - 4ac^3 - 27a^2 d^2 \tag{A.45}$$

with x_1, x_2, x_3 being the three roots.

1. $\Delta > 0$: Three distinct real roots.
2. $\Delta = 0$: All roots are real with at least two identical.
3. $\Delta < 0$: One real and a pair of complex conjugate roots (proof: assume complex roots are $x \pm iy$).

Proof. The Vieta's theorem for (A.43) and the invariant relations can be used to simplify (A.43) to obtain (A.45).

Note: Eq. (A.45) can also be obtained as follows (with some reasons/meanings in algebraic geometry). Consider a cubic equation in canonical form

$$f(x, w) = Ax^3 + 3Bx^2 w + 3Cxw^2 + Dw^3 = 0. \tag{A.46}$$

The Hessain matrix is

$$H(f) = \begin{bmatrix} 6Ax + 6Bw & 6Bx + 6Cw \\ 6Bx + 6Cw & 6Cx + 6Dw \end{bmatrix} \tag{A.47}$$

and the Hessain

$$\det(H) = 36[(AC - B^2)x^2 + (AD - BC)xw + (BD - C^2)w^2] \tag{A.48}$$

$$= 18[x, w] \begin{bmatrix} 2(AC - B^2) & (AD - BC) \\ (AD - BC) & 2(BD - C^2) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}, \tag{A.49}$$

in quadratic form. Define the Hessain

$$\boldsymbol{H} = \begin{bmatrix} 2(AC - B^2) & (AD - BC) \\ (AD - BC) & 2(BD - C^2) \end{bmatrix} \tag{A.50}$$

The discriminant of the cubic is just the determinant of the Hessain $\boldsymbol{H}$:

$$\Delta = \det(\boldsymbol{H}) = -A^2 D^2 + 6ABCD - 4AC^3 - 4B^3 D + 3B^2 C^2, \tag{A.51}$$

and $\Delta > 0$ for three real roots, $\Delta = 0$ for double or triple real root, and $\Delta < 0$ for single real root.

A.3.2 Application to the velocity gradient tensor

The cubic curve theory, especially root finding, has relation to the characteristic polynomial of the velocity gradient tensor and so forth the local flow geometry (basically the number of real/complex eigenvalues).

The characteristic polynomial for $\boldsymbol{u}\nabla$ is

$$\lambda^3 - P\lambda^2 + Q\lambda - R = 0, \tag{A.52}$$

where $(P, Q, R) = (I_1, I_2, I_3)$ are the three invariants. In incompressible flow, $P = u_{i,i} = 0$, and the equation above degenerates to

$$\lambda^3 + Q\lambda - R = 0. \tag{A.53}$$

It is in the so-called 'depressed' form (comparatively, an elliptic curve is called in the Weierstrass form if it satisfies the Weierstrass equation $y^2 = x^3 + ax + b$)

Positive second invariant $Q = -1/2u_{i,j}u_{j,i} = 1/2(\|\boldsymbol{\Omega}\|^2 - \|\boldsymbol{S}\|^2)$ is used for identifying vortical motions, or 'eddies' Hunt *et al.* (1988); Jeong & Hussain (1995). Consider the Poisson equation,

$$\frac{1}{\rho}\nabla^2 p = -u_{i,j}u_{j,i}, \tag{A.54}$$

we have positive Q corresponding to a local pressure minimum.

The discriminant for depressed cubic equation

$$x^3 + px + q = 0 \tag{A.55}$$

reduces to

$$\Delta = -4p^3 - 27q^2. \tag{A.56}$$

So we have the discriminant for the gradient of a solenoidal field (with renormalized coefficients; note the flipped sign)

$$\triangle = \left(\frac{1}{3}Q\right)^3 + \left(\frac{1}{2}R\right)^2 \tag{A.57}$$

and if $\triangle > 0$ there will be complex eigenvalues (in complex conjugate pair according to the algebra basic theorem) and so-defined vortical motions. Hence, we can see that the invariants of the velocity gradient tensor is largely related to the local geometry (Chong *et al.*, 1990) of the flow.

A.3.3 Application to the strain rate tensor

We note that both the rate-of-strain tensor $\boldsymbol{S}$ and the Reynolds stress tensor $-\overline{u_i'u_j'}$ are real symmetric, hence they have three real eigenvalues and three orthogonal eigenvectors (principle axes), or, in another word, they are unitarily similar to a diagonal matrix. The rate of strain tensor is diagonal and all strains are normal strains in the principle coordinate.

This part is largely related to eigenvalue decomposition (see section B).

Example. Consider a strain rate tensor

$$\boldsymbol{S} = \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix}. \tag{A.58}$$

Its eigenvalues are $\lambda_1 = -\gamma/2$, $\lambda_2 = \gamma/2$ and associated eigenvectors are $\boldsymbol{x}_1 = 1/\sqrt{2}[1, -1]^T$ and $\boldsymbol{x}_2 = 1/\sqrt{2}[1, 1]^T$. The two principle directions are 45 deg (stretching, the direction that receives the most amplification) and -45 deg (compressing). The similarity transform is

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}} = \begin{bmatrix} -\gamma/2 & 0 \\ 0 & \gamma/2 \end{bmatrix} \tag{A.59}$$

where

$$\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2] \tag{A.60}$$

is a unitary matrix such that $\boldsymbol{AX} = \boldsymbol{X\Lambda}$.

Take $\gamma = \partial_y U$, this is an importan example of plane shear flow. Think of the deformation of a rectangle to a diamond as it moves with the shear.

A.3.4 Application to the Reynolds stress tensor, the invariant map, and the Lumley triangle,

Consider the anisotropic (deviatoric) tensor of Reynolds stress

$$a_{ij} = \frac{\overline{u_i'u_j'}}{2k} - \frac{1}{3}\delta_{ij} \tag{A.61}$$

and its three principle invariants

$$I = \sigma_1 + \sigma_2 + \sigma_3 \tag{A.62}$$

$$II = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \tag{A.63}$$

$$III = \sigma_1\sigma_2\sigma_3 \tag{A.64}$$

along with its three eigenvalues

$$\sigma_1,\ \sigma_2,\ \sigma_3. \tag{A.65}$$

Since a_{ij} is a deviator, it is traceless and

$$I = a_{ii} = 0. \tag{A.66}$$

Consider turbulence. and has zero determinant

$$\det\left(\frac{\overline{u_i'u_j'}}{2k}\right) = (\sigma_1 + \frac{1}{3})(\sigma_2 + \frac{1}{3})(\sigma_3 + \frac{1}{3}) \tag{A.67}$$

$$= \sigma_1\sigma_2\sigma_3 + \frac{1}{3}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{27}, \tag{A.68}$$

and we define

$$F = 27III + 9II + 1 \tag{A.69}$$

since $I = 0$.

1. Two-dimensional turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ can be diagonalized to

   $$diag(a, k-a, 0)$$

   and has zero determinant (there's a direction that has no turbulence). $F = 0$.

2. Three-dimensional isotropic turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ is

   $$diag(k/3, k/3, k/3)$$

   and we have $F = 1$.

3. Axisymmetric turbulence. Similarly, the characteristic polynomial of a_{ij} is in Weierstrass form and the condition for repeated eigenvalues (same energy in two principle directions) is

   $$\triangle = \left(\frac{1}{3}II\right)^3 + \left(\frac{1}{2}III\right)^2 = 0 \tag{A.70}$$

   and hence

   $$III = \pm 2\left(-\frac{II}{3}\right)^3, \tag{A.71}$$

   corresponding to the negative/left (pancake) and positive/right (cigar) limit curves of the Lumley triangle (Lumley & Newman, 1977; Choi & Lumley, 2001).

B Matrix Analysis

B.1 Unitary matrix

Unitary transformations preserve inner products (and hence length and angle). Rotation and reflection.

B.2 Singular value decomposition and eigenvalue decomposition

B.3 Conformal mapping

B.4 Coordinate transformation

C Coordinate systems

C.1 Cylindrical coordinate

Consider the cylindrical transformation

$$(x, y) \to (r, \theta) \tag{C.1}$$

where

$$x = r\cos\theta \tag{C.2}$$
$$y = r\sin\theta \tag{C.3}$$

or

$$r = \sqrt{x^2 + y^2} \tag{C.4}$$
$$\theta = \mathrm{actan}\left(\frac{y}{x}\right) \tag{C.5}$$

we have the corresponding relation between unit vectors

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.6}$$

and

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix}, \tag{C.7}$$

which can be proven graphically. We note that the grid transformation matrix is unitary and has det() = 1 (rotation matrix).

The Jacobian of the forward transformation $(r, \theta) = F(x, y)$ is

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix} \tag{C.8}$$

We note that the directions of the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ depend on space, i.e.,

$$\frac{\partial \boldsymbol{e}_r}{\partial r} = \frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0 \tag{C.9}$$
$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = -\sin\theta\boldsymbol{e}_x + \cos\theta\boldsymbol{e}_y = \boldsymbol{e}_\theta \tag{C.10}$$
$$\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\cos\theta\boldsymbol{e}_x - \sin\theta\boldsymbol{e}_y = -\boldsymbol{e}_r \tag{C.11}$$

which can also be seen graphically. These relations are crucial to later derivations.

Consider the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{C.12}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{C.13}$$

C.1.1 Operators in cylindrical coordinate

For a scalar function, say $f(x,y) = f(r,\theta)$, the gradient operator can be expressed as

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{C.14}$$

$$= \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\right)(\cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta) + \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\right)(\sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta) + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{C.15}$$

$$= \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{C.16}$$

The factor $r\partial\theta$ can be interpreted as infinitesimal length element in θ direction.

The Laplace operator

$$\nabla^2 = \nabla \cdot \nabla = \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \cdot \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \tag{C.17}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \boldsymbol{e}_\theta \cdot \frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)\right] \tag{C.18}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{C.19}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{C.20}$$

Now consider a vector

$$\boldsymbol{u} = \boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w \tag{C.21}$$

and its derivatives.

Its divergence

$$\nabla \cdot \boldsymbol{u} = \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \cdot (\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.22}$$

$$= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.23}$$

$$= \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.24}$$

The convection term

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z})(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.25}$$

$$= (u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r})\boldsymbol{e}_r \tag{C.26}$$

$$+ (u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r})\boldsymbol{e}_\theta \tag{C.27}$$

$$+ (u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z})\boldsymbol{e}_z \tag{C.28}$$

Now we deal with $\nabla^2 \boldsymbol{u}$.

$$\nabla^2 \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.29}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{u}}{\partial r}\right) + \frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v) \tag{C.30}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right)\boldsymbol{e}_r \tag{C.31}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right)\boldsymbol{e}_\theta \tag{C.32}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)\boldsymbol{e}_z \tag{C.33}$$

with

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_r u) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \boldsymbol{e}_r u}{\partial \theta} = \frac{1}{r^2}(2\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta - u\boldsymbol{e}_r + \frac{\partial^2 u}{\partial \theta^2}\boldsymbol{e}_r) \tag{C.34}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_\theta v) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \boldsymbol{e}_\theta v}{\partial \theta} = \frac{1}{r^2}(-2\frac{\partial v}{\partial \theta}\boldsymbol{e}_r - v\boldsymbol{e}_\theta + \frac{\partial^2 v}{\partial \theta^2}\boldsymbol{e}_\theta) \tag{C.35}$$

Moreover, the curl can be established as

$$\nabla \times \boldsymbol{u} = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right) \times (\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.36}$$

$$= \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \frac{1}{r}\partial_\theta & \partial_z \\ u & v & w \end{vmatrix} + \frac{1}{r}\boldsymbol{e}_\theta \times \frac{\partial(v\boldsymbol{e}_\theta)}{\partial \theta} \tag{C.37}$$

$$= (\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z})\boldsymbol{e}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\boldsymbol{e}_\theta + (\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta})\boldsymbol{e}_z \tag{C.38}$$

$$= (\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z})\boldsymbol{e}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\boldsymbol{e}_\theta + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta})\boldsymbol{e}_z \tag{C.39}$$

$$= \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix} \tag{C.40}$$

We note that to the vertical vorticity ω_z, the shear vorticity $\partial_r v$ and the curvature vorticity v/r have equal contributions. The relation (C.40) is generalizable and will be shown in section C.3.

Examples.

1. Rigid body rotation with angular velocity Ω and $u_\theta = v = \Omega r$. Vorticity $\omega_z = 2\Omega$ but there is no vortical motion.
2. Potential point vortex with $u_\theta = v = \Gamma/2\pi r$. Vorticity $\omega_z = 0$ according to (C.39).

We need more objective criterion for identifying vortices (cf. A.3.2).

C.1.2 Navier–Stokes in cylindrical coordinate

The Navier–Stokes equation in cylindrical coordinate reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{C.41}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{C.42}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{C.43}$$

Q.E.D.

Additionally, the transport equation of a passive scalar (say ρ)

$$\frac{\partial \rho}{\partial t}+\boldsymbol{u}\cdot\nabla\rho=\nu\nabla^2\rho \tag{C.44}$$

can be cast in cylinderical coordinate as

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\frac{v}{r}\frac{\partial \rho}{\partial \theta}+w\frac{\partial \rho}{\partial z}=\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \rho}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \rho}{\partial \theta^2}+\frac{\partial^2 \rho}{\partial z^2}\right] \tag{C.45}$$

C.2 Spherical coordinate

Consider the transformation

$$(x,y,z)\rightarrow(r,\phi,\theta) \tag{C.46}$$

where

$$x=r\sin\phi\cos\theta \tag{C.47}$$
$$y=r\sin\phi\sin\theta \tag{C.48}$$
$$z=r\cos\phi \tag{C.49}$$

or

$$r=\sqrt{x^2+y^2+z^2} \tag{C.50}$$
$$\phi=\arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \tag{C.51}$$
$$\theta=\arctan\left(\frac{y}{x}\right) \tag{C.52}$$

or thought of as from cylindrical with

$$z=r\cos\phi \tag{C.53}$$
$$r'=r\sin\phi \tag{C.54}$$
$$x=r'\sin\theta \tag{C.55}$$
$$y=r'\cos\theta \tag{C.56}$$

Here θ is the azimuthal angle with x-axis on the equatorial plane and ϕ is the polar angle with z-axis (North), for the convenience of going from cylindrical to polar and backwards.

We have the corresponding relation between unit vectors

$$\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix}=\begin{bmatrix}\sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta\\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta\\ \cos\phi & -\sin\phi & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\phi\\ \boldsymbol{e}_\theta\end{bmatrix} \tag{C.57}$$

and

$$\begin{bmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\phi\\ \boldsymbol{e}_\theta\end{bmatrix}=\begin{bmatrix}\sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi\\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi\\ -\sin\theta & \cos\theta & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_x\\ \boldsymbol{e}_y\\ \boldsymbol{e}_z\end{bmatrix}, \tag{C.58}$$

which can be proven graphically. We note that the grid transformation matrix is unitary and has $\det() = 1$ (rotation matrix).

C.2.1 From cylindrical to spherical

We have the transformation

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.59}$$

that can be factorized as

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.60}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.61}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{r'} \\ \boldsymbol{e}_{\theta'} \\ \boldsymbol{e}_{z'} \end{bmatrix} \tag{C.62}$$

with

$$\begin{bmatrix} \boldsymbol{e}_{r'} \\ \boldsymbol{e}_{\theta'} \\ \boldsymbol{e}_{z'} \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix}. \tag{C.63}$$

C.3 General curvilinear coordinates

Consider the coordinate transformations

$$q_i = q_i(x_1, x_2, x_3),\ x_i = x_i(q_1, q_2, q_3) \tag{C.64}$$

where (x_1, x_2, x_3) is the standard Cartesian coordinates and q_i are mutually independent.

C.3.1 Length, area, and volume

Consider the change of the vector

$$\boldsymbol{x} = x_1\boldsymbol{e}_{x_1} + x_2\boldsymbol{e}_{x_2} + x_3\boldsymbol{e}_{x_3} \tag{C.65}$$

$$= q_1\boldsymbol{h}_1 + q_2\boldsymbol{h}_2 + q_3\boldsymbol{h}_3 \tag{C.66}$$

where $\boldsymbol{x} = \boldsymbol{x}(x_i(q_j))$ as

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{e}_{x_1}\mathrm{d}x_1 + \boldsymbol{e}_{x_2}\mathrm{d}x_2 + \boldsymbol{e}_{x_3}\mathrm{d}x_3 \tag{C.67}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \boldsymbol{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \boldsymbol{x}}{\partial q_3}\mathrm{d}q_3 \tag{C.68}$$

and

$$\boldsymbol{h}_i = \frac{\partial \boldsymbol{x}}{\partial q_i}. \tag{C.69}$$

We note that $\boldsymbol{h}_i$ is the change of $\boldsymbol{x}$ with only changing q_i, so it does define direction of coordinate lines of q_i. We denote with $\hat{(\cdot)}$ unit vectors and note that $\boldsymbol{h}_i$ are not necessary unit vectors.

Now consider the length of $\mathrm{d}\boldsymbol{x}$:

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} && \text{(C.70)}\\
&= \frac{\partial \boldsymbol{x}}{\partial q_j}\mathrm{d}q_j \cdot \frac{\partial \boldsymbol{x}}{\partial q_k}\mathrm{d}q_k && \text{(C.71)}\\
&= \frac{\partial x_i}{\partial q_j}\mathrm{d}q_j \frac{\partial x_i}{\partial q_k}\mathrm{d}q_k && \text{(C.72)}\\
&= g_{jk}\mathrm{d}q_j\mathrm{d}q_k && \text{(C.73)}
\end{aligned}$$

with

$$g_{ij} = \frac{\partial x_l}{\partial q_i}\frac{\partial x_l}{\partial q_j} \tag{C.74}$$

being the metric tensor. When q_i are orthogonal coordinates,

$$\frac{\partial \boldsymbol{x}}{\partial q_i} \cdot \frac{\partial \boldsymbol{x}}{\partial q_j} = \delta_{ij} \tag{C.75}$$

and g_{ij} only has diagonal elements and

$$\begin{aligned}
\mathrm{d}s^2 &= g_{11}(\mathrm{d}q_1)^2 + g_{22}(\mathrm{d}q_3)^2 + g_{33}(\mathrm{d}q_3)^2 && \text{(C.76)}\\
&= h_1^2(\mathrm{d}q_1)^2 + h_2^2(\mathrm{d}q_3)^2 + h_3^2(\mathrm{d}q_3)^2 && \text{(C.77)}
\end{aligned}$$

Define the Lamé parameters as

$$\begin{aligned}
h_1 &= \sqrt{g_{11}} = \sqrt{\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2} = |\boldsymbol{h}_1| && \text{(C.78)}\\
h_2 &= \sqrt{g_{22}} = \sqrt{\left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2} = |\boldsymbol{h}_2| && \text{(C.79)}\\
h_3 &= \sqrt{g_{33}} = \sqrt{\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2} = |\boldsymbol{h}_3| && \text{(C.80)}
\end{aligned}$$

and unit vectors in q_i directions as

$$\boldsymbol{h}_i = \frac{\boldsymbol{h}_i}{|\boldsymbol{h}_i|} = \frac{\boldsymbol{h}_i}{h_i}. \tag{C.81}$$

We note that the Lamé parameters can depend on the coordinates as

$$h_i = h_i(q_1, q_2, q_3). \tag{C.82}$$

The increment can be rewritten as

$$\begin{aligned}
\mathrm{d}\boldsymbol{x} &= h_1\mathrm{d}q_1\boldsymbol{h}_1 + h_2\mathrm{d}q_2\boldsymbol{h}_2 + h_3\mathrm{d}q_3\boldsymbol{h}_3 && \text{(C.83)}\\
&= \mathrm{d}s_1\boldsymbol{h}_1 + \mathrm{d}s_2\boldsymbol{h}_2 + \mathrm{d}s_3\boldsymbol{h}_3 && \text{(C.84)}
\end{aligned}$$

with

$$\mathrm{d}s_i \tag{C.85}$$

being the projection of $\mathrm{d}\boldsymbol{x}$ on each coordinate.

Now consider the surface and volume of infinitesimal elements. The (directed) areas of surface elements are

$$\mathrm{d}\sigma_i = \boldsymbol{h}_i \cdot (h_j \mathrm{d}q_j \boldsymbol{h}_j \times h_k \mathrm{d}q_k \boldsymbol{h}_k) = h_j h_k \mathrm{d}q_j \mathrm{d}q_k \tag{C.86}$$

or

$$\mathrm{d}\sigma_1 = h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2 \tag{C.87}$$
$$\mathrm{d}\sigma_2 = h_1 h_3 \mathrm{d}q_1 \mathrm{d}q_3 \tag{C.88}$$
$$\mathrm{d}\sigma_3 = h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2 \tag{C.89}$$

The volume element (e.g. in volume integrals) spanned by the vector $\mathrm{d}\boldsymbol{x}$ is

$$\mathrm{d}V = (h_1 \mathrm{d}q_1 \boldsymbol{h}_1) \cdot (h_2 \mathrm{d}q_2 \boldsymbol{h}_2 \times h_3 \mathrm{d}q_3 \boldsymbol{h}_3) \tag{C.90}$$
$$= h_1 \mathrm{d}q_1 h_2 \mathrm{d}q_2 h_3 \mathrm{d}q_3 (\boldsymbol{h}_1) \cdot (\boldsymbol{h}_2 \times \boldsymbol{h}_3) \tag{C.91}$$
$$= h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3 \tag{C.92}$$

when $\boldsymbol{h}_i$ mutually orthogonal.

Example.

For cylindrical coordinate, by definition,

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1 \tag{C.93}$$

$$h_2 = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r \tag{C.94}$$

$$h_3 = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1 \tag{C.95}$$

C.3.2 Jacobian

Now we consider the Jacobian of the backward transformation

$$(q_1, q_2, q_3) \to (x_1, x_2, x_3) \tag{C.96}$$

which reads

$$\boldsymbol{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{bmatrix} \tag{C.97}$$

and the Jacobian determinant (with $\exists\, \boldsymbol{J}^{-1}$)

$$J = \det(\boldsymbol{J}) = \det(\boldsymbol{J}^{\mathrm{T}}) \tag{C.98}$$
$$= \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} \tag{C.99}$$

$$= \left(\frac{\partial x_1}{\partial q_1}\boldsymbol{x}_1 + \frac{\partial x_2}{\partial q_1}\boldsymbol{x}_2 + \frac{\partial x_3}{\partial q_1}\boldsymbol{x}_3\right) \cdot \begin{vmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} \tag{C.100}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial q_2} \times \frac{\partial \boldsymbol{x}}{\partial q_3}\right) \tag{C.101}$$

$$= \boldsymbol{h}_1 \cdot (\boldsymbol{h}_2 \times \boldsymbol{h}_3) \tag{C.102}$$

$$= h_1 h_2 h_3 \tag{C.103}$$

$$\neq 0 \tag{C.104}$$

Hence we have

$$\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = h_1h_2h_3\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3 = J\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3. \tag{C.105}$$

C.3.3 Three major calculus theorems

1. Gradient theorem:

$$\int_{\boldsymbol{l}:\boldsymbol{x}_1\to\boldsymbol{x}_2} (\nabla f) \cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1) \tag{C.106}$$

   The integral is independent of path since ∇f is potential (conservative, curl-free).

2. Divergence theorem:

$$\iint_{\Omega} (\nabla \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A} = \oint_{l=\partial\Omega} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} \tag{C.107}$$

   Implication: vorticity is circulation per unit area.

3. Curl theorem:

$$\iiint_V (\nabla \cdot \boldsymbol{u})\, \mathrm{d}V = \iint_{\Omega=\partial V} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A} \tag{C.108}$$

C.3.4 Differential operators in curvilinear coordinate systems

Next, let's consider differential operators in curvilinear coordinates. Consider a scalar $f = f(q_1, q_2, q_3)$ and its gradient ∇f. Starting from

$$\mathrm{d}f = \frac{\partial f}{\partial q_i}\mathrm{d}q_i, \tag{C.109}$$

due to the displacement $\boldsymbol{x}$. On the other hand,

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\boldsymbol{x} \tag{C.110}$$

$$= (\nabla f)_{q_i}\mathrm{d}s_i \tag{C.111}$$

$$= (\nabla f)_{q_i} h_i \mathrm{d}q_i \tag{C.112}$$

Compare (C.112) and (C.109) we have

$$(\nabla f)_{q_i} = \frac{1}{h_i}\frac{\partial f}{\partial q_i} \tag{C.113}$$

where

$$\nabla f = (\nabla f)_{q_i}\boldsymbol{h}_i \tag{C.114}$$

$$= \frac{1}{h_1}\frac{\partial f}{\partial q_1}\boldsymbol{h}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\boldsymbol{h}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\boldsymbol{h}_3 \tag{C.115}$$

Consider the divergence of a vector $\boldsymbol{u}$ in a coordinate-free form:

$$\nabla \cdot \boldsymbol{u} \triangleq \lim_{V\to 0} \frac{\oint_{\Omega=\partial V} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{\sigma}}{V} \tag{C.116}$$

$$= \frac{1}{V}\left(\frac{\partial(u_1 h_2 h_3 \mathrm{d}q_2 \mathrm{d}q_3)}{\partial q_1}\mathrm{d}q_1 + \frac{\partial(u_2 h_1 h_3 \mathrm{d}q_1 \mathrm{d}q_3)}{\partial q_2}\mathrm{d}q_2 + \frac{\partial(u_3 h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2)}{\partial q_3}\mathrm{d}q_3\right) \tag{C.117}$$

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(u_1 h_2 h_3)}{\partial q_1} + \frac{\partial(u_2 h_1 h_3)}{\partial q_2} + \frac{\partial(u_3 h_1 h_2)}{\partial q_3}\right) \tag{C.118}$$

Consider the curl of a vector $\boldsymbol{u}$ in a coordinate-free form, its compoment along $\boldsymbol{n}$ (normal of the surface $\boldsymbol{S} = S\boldsymbol{n}$) is

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} \triangleq \lim_{S\to 0} \frac{\int_{l=\partial S} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}}{S} \tag{C.119}$$

and (consider the area spanned by $\mathrm{d}s_2 = h_2 \mathrm{d}q_2$ and $\mathrm{d}s_3 = h_3 \mathrm{d}q_3$)

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_1 = \frac{\oint_l \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}}{\mathrm{d}\sigma_1} \tag{C.120}$$

$$= \frac{1}{h_2 h_3 \mathrm{d}q_2 \mathrm{d}q_3}[u_2 h_2 \mathrm{d}q_2 \tag{C.121}$$

$$- (u_2 h_2 + \frac{\partial u_2 h_2}{\partial q_3}\mathrm{d}q_3)\mathrm{d}q_2 \tag{C.122}$$

$$- u_3 h_3 \mathrm{d}q_3 \tag{C.123}$$

$$+ (u_3 h_3 + \frac{\partial u_3 h_3}{\partial q_2}\mathrm{d}q_2)\mathrm{d}q_3] \tag{C.124}$$

$$= \frac{1}{h_2 h_3}\left(\frac{\partial u_3 h_3}{\partial q_2} - \frac{\partial u_2 h_2}{\partial q_3}\right) \tag{C.125}$$

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_2 = \frac{1}{h_1 h_3}\left(\frac{\partial u_1 h_1}{\partial q_3} - \frac{\partial u_3 h_3}{\partial q_1}\right) \tag{C.126}$$

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_3 = \frac{1}{h_1 h_2}\left(\frac{\partial u_2 h_2}{\partial q_1} - \frac{\partial u_1 h_1}{\partial q_2}\right) \tag{C.127}$$

We note that the Lamé coefficient also changes as the coordinate changes.

In determinant form,

$$\nabla \times \boldsymbol{u} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \boldsymbol{h}_1 & h_2 \boldsymbol{h}_2 & h_3 \boldsymbol{h}_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ h_1 u_1 & h_2 u_2 & h_3 u_3 \end{vmatrix} \tag{C.128}$$

The Laplacian can be obtained by taking the divergence of ∇f as combining (C.115) and (C.118)

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right) \tag{C.129}$$

C.3.5 Derivatives of unit vectors

In general curvilinear coordinates, the directions of unit vectors could change with coordinate as well. We are basically concerned about

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \tag{C.130}$$

and we will establish that

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} // \boldsymbol{h}_j,\ i \neq j. \tag{C.131}$$

First we have

$$\boldsymbol{h}_i \cdot \frac{\partial \boldsymbol{h}_i}{\partial q_j} = \frac{\partial \boldsymbol{h}_i^2/2}{\partial q_j} = 0 \tag{C.132}$$

and hence

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \perp \boldsymbol{h}_i,\ i \neq j. \tag{C.133}$$

According to the orthogonality we have

$$\boldsymbol{h}_1 \cdot \boldsymbol{h}_2 = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2} = 0 \tag{C.134}$$

and

$$\begin{aligned}
0 = \frac{\partial}{\partial q_3}\left(\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2}\right) &= \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} + \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_3} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2} && \text{(C.135)}\\
&= \frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_1} + \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_3} && \text{(C.136)}\\
&= \frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} + \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_2} \cdot \frac{\partial \boldsymbol{x}}{\partial q_1} && \text{(C.137)}
\end{aligned}$$

then

$$\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} + \frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_1} + \frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} = 0 \tag{C.138}$$

and then

$$\begin{aligned}
\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} &= 0 && \text{(C.139)}\\
\frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_3} &= 0 && \text{(C.140)}\\
\frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} &= 0 && \text{(C.141)}
\end{aligned}$$

From

$$\begin{aligned}
0 &= \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial}{\partial q_2}\left(\frac{\partial \boldsymbol{x}}{\partial q_3}\right) && \text{(C.142)}\\
&= h_1 \boldsymbol{h}_1 \cdot \frac{\partial h_3 \boldsymbol{h}_3}{\partial q_2} && \text{(C.143)}\\
&= h_1 \boldsymbol{h}_1 \cdot \left(h_3 \frac{\partial \boldsymbol{h}_3}{\partial q_2} + \boldsymbol{h}_3 \frac{\partial h_3}{\partial q_2}\right) && \text{(C.144)}\\
&= h_1 h_3 \boldsymbol{h}_1 \cdot \frac{\partial \boldsymbol{h}_3}{\partial q_2} && \text{(C.145)}
\end{aligned}$$

we have (similarly)

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \perp \boldsymbol{h}_k,\ i \neq j \neq k \neq i. \tag{C.146}$$

Combining (C.133) and (C.146) we have

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} // \boldsymbol{h}_j,\ i \neq j, \tag{C.147}$$

and using

$$\frac{\partial^2 \boldsymbol{x}}{\partial q_i \partial q_j} = \frac{\partial^2 \boldsymbol{x}}{\partial q_j \partial q_i} \tag{C.148}$$

we have

$$\frac{\partial}{\partial q_j}\left(\frac{\partial \boldsymbol{x}}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\frac{\partial \boldsymbol{x}}{\partial q_j}\right) \tag{C.149}$$

$$\boldsymbol{h}_i \frac{\partial h_i}{\partial q_j} + h_i \frac{\partial \boldsymbol{h}_i}{\partial q_j} = \boldsymbol{h}_j \frac{\partial h_j}{\partial q_i} + h_j \frac{\partial \boldsymbol{h}_j}{\partial q_i} \tag{C.150}$$

with repeated indices not implying summation. Since $i \neq j$, $\boldsymbol{h}_i$ and $\boldsymbol{h}_j$ are linearly independent, we have

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} = \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \boldsymbol{h}_j. \tag{C.151}$$

Now we turn back and consider $\partial \boldsymbol{h}_i / \partial q_j$.

$$\frac{\partial \boldsymbol{h}_i}{\partial q_i} = \frac{\partial(\boldsymbol{h}_j \times \boldsymbol{h}_k)}{\partial q_i} \tag{C.152}$$

$$= \frac{\partial \boldsymbol{h}_j}{\partial q_i} \times \boldsymbol{h}_k + \boldsymbol{h}_j \times \frac{\partial \boldsymbol{h}_k}{\partial q_i} \tag{C.153}$$

$$= \frac{1}{h_j} \frac{\partial h_i}{\partial q_j} \boldsymbol{h}_i \times \boldsymbol{h}_k + \boldsymbol{h}_j \times \boldsymbol{h}_i \frac{1}{h_k} \frac{\partial h_i}{\partial q_k} \tag{C.154}$$

$$= -\left(\frac{1}{h_j} \frac{\partial h_i}{\partial q_j} \boldsymbol{h}_j + \frac{1}{h_k} \frac{\partial h_i}{\partial q_k} \boldsymbol{h}_k\right) \tag{C.155}$$

without repeated indices being summed over.

Using the relations (C.151) and (C.155), gradient, curl, divergence, Laplacian, as well as operators like $\nabla \boldsymbol{u}$ and $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ can be expressed.

Example: $\nabla \cdot \boldsymbol{u}$.
We have before

$$\nabla = \frac{\boldsymbol{h}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\boldsymbol{h}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\boldsymbol{h}_3}{h_3} \frac{\partial}{\partial q_3} \tag{C.156}$$

and now consider $\nabla \boldsymbol{u}$ with

$$\boldsymbol{u} = u_1 \boldsymbol{h}_1 + u_2 \boldsymbol{h}_2 + u_3 \boldsymbol{h}_3 \tag{C.157}$$

and we have

$$\nabla \cdot \boldsymbol{u} = \left(\frac{\boldsymbol{h}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\boldsymbol{h}_2}{h_2} \frac{\partial}{\partial q_2} \boldsymbol{h}_2 + \frac{\boldsymbol{h}_3}{h_3} \frac{\partial}{\partial q_3} \boldsymbol{h}_3\right) \cdot (u_1 \boldsymbol{h}_1 + u_2 \boldsymbol{h}_2 + u_3 \boldsymbol{h}_3) \tag{C.158}$$

$$= \frac{\boldsymbol{h}_1}{h_1} \frac{\partial}{\partial q_1} (u_1 \boldsymbol{h}_1 + u_2 \boldsymbol{h}_2 + u_3 \boldsymbol{h}_3) + \ldots \tag{C.159}$$

$$= \frac{1}{h_1} \left(\frac{\partial u_1}{\partial q_1} + \frac{u_2}{h_2} \frac{\partial h_1}{\partial q_2} + \frac{u_3}{h_3} \frac{\partial h_1}{\partial q_3}\right) \tag{C.160}$$

$$+ \frac{1}{h_2} \left(\frac{\partial u_2}{\partial q_2} + \frac{u_3}{h_3} \frac{\partial h_2}{\partial q_3} + \frac{u_1}{h_1} \frac{\partial h_2}{\partial q_1}\right) \tag{C.161}$$

$$+ \frac{1}{h_3} \left(\frac{\partial u_3}{\partial q_3} + \frac{u_1}{h_1} \frac{\partial h_3}{\partial q_1} + \frac{u_2}{h_2} \frac{\partial h_3}{\partial q_2}\right) \tag{C.162}$$

$$= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial u_1 h_2 h_3}{\partial q_1} + \frac{\partial u_2 h_1 h_3}{\partial q_2} + \frac{\partial u_3 h_1 h_2}{\partial q_3}\right) \tag{C.163}$$

References: Appendices in Batchelor (1967); Griffiths (2013), and text book of Wu (1982).

C.3.6 Examples

1. Cartesian. $(q_1, q_2, q_3) = (x_1, x_2, x_3)$
   Elements:

$$h_1 = h_2 = h_3 = 1 \tag{C.164}$$
$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{C.165}$$
$$\mathrm{d}V = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \tag{C.166}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial f}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial f}{\partial x_3}\boldsymbol{e}_3 \tag{C.167}$$
$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{C.168}$$
$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{C.169}$$
$$= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\boldsymbol{e}_x + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\boldsymbol{e}_y + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\boldsymbol{e}_z \tag{C.170}$$
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.171}$$

2. Cylindrical. $(q_1, q_2, q_3) = (r, \theta, z)$
   Elements:

$$h_1 = h_3 = 1, h_2 = r \tag{C.172}$$
$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2 \tag{C.173}$$
$$\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \tag{C.174}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial f}{\partial z}\boldsymbol{e}_z \tag{C.175}$$
$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\left(\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial (rw)}{\partial z}\right) \tag{C.176}$$
$$= \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.177}$$
$$\nabla \times \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix} \tag{C.178}$$
$$= (\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z})\boldsymbol{e}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\boldsymbol{e}_\theta + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta})\boldsymbol{e}_z \tag{C.179}$$
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.180}$$

3. Spherical. $(q_1, q_2, q_3) = (r, \phi, \theta)$, ϕ is the polar angle and θ is the azimuthal.
   Elements:

$$h_1 = 1, h_2 = r, h_3 = r\sin\phi \tag{C.181}$$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + r^2\sin^2\phi\mathrm{d}\theta^2 \tag{C.182}$$

$$\mathrm{d}V = r^2\sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta \tag{C.183}$$

Operators:

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta \tag{C.184}$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r^2\sin\phi}\left(\frac{\partial(r^2\sin\phi u)}{\partial r} + \frac{\partial(r\sin\phi v)}{\partial\phi} + \frac{\partial(rw)}{\partial\theta}\right) \tag{C.185}$$

$$= \frac{1}{r^2}\frac{\partial(r^2u)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(\sin\phi v)}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial w}{\partial\theta} \tag{C.186}$$

$$\nabla\times\boldsymbol{u} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\phi & r\sin\phi\boldsymbol{e}_\theta \\ \partial_r & \partial_\phi & \partial_\theta \\ u & rv & r\sin\phi w \end{vmatrix} \tag{C.187}$$

$$= \frac{1}{r\sin\phi}(\frac{\partial\sin\phi w}{\partial\phi} - \frac{\partial v}{\partial\theta})\boldsymbol{e}_r + (\frac{1}{r\sin\phi}\frac{\partial u}{\partial\theta} - \frac{1}{r}\frac{\partial rw}{\partial r})\boldsymbol{e}_\phi + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial\phi})\boldsymbol{e}_\theta \tag{C.188}$$

$$\nabla^2 f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2\sin\phi}\frac{\partial^2 f}{\partial\theta^2} \tag{C.189}$$

D Exact solutions of the Navier–Stokes

Ref. Drazin & Riley (2006).

D.1 Parallel flows

Plane Poiseuille, Coette, and their superposition.

Hagen-Poiseuille (circular) flow. Wiki.

D.2 Vortical flows

D.2.1 Lamb-Oseen similarity solution, Burgers vortex

We recall that a potential/irrotational vortex has a velocity profile as

$$u_\theta = \frac{\Gamma}{2\pi r} \tag{D.1}$$

where $\Gamma = \int_0^{2\pi} u_\theta r\,\mathrm{d}\theta$ is the circulation. Oseen considered viscous solution in the form of

$$u_r = 0,\, u_\theta = \frac{\Gamma}{2\pi r}g(r,t), \tag{D.2}$$

where $g(r,t)$ is a non-dimensional similarity function that combines the time-dependent spreading vortex core size $R(t)$ and time t, that collapses when the non-dimensional radius agree.

It is easy to see that the viscous length scale is $\sqrt{\nu t}$ in a diffusion process. So we take

$$g(r,t) = g(\hat{r}^2) = g\left(\frac{r^2}{4\nu t}\right) \tag{D.3}$$

where $\hat{r} = r/2\sqrt{\nu t}$ is the self-similar length scale and $\eta = \hat{r}^2$ is the similarity variable. The same similarity transform converts the heat equation to an ODE from which the heat kernel (D.13) can be solved.

The simplified azimuthal momentum equation is

$$\frac{\partial u_\theta}{\partial t} = \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right), \tag{D.4}$$

which represents a diffusion process, and it reduces to and ODE under the similarity transform (D.3):

$$g' + g'' = 0, \tag{D.5}$$

where $g'(\eta) = \partial_\eta g$. The general solution is

$$g(\eta) = c_1 + c_2 e^{-\eta}. \tag{D.6}$$

Given the boundary conditions

$$g(0) = 0,\ g(\infty) = 1, \tag{D.7}$$

we have

$$g(r,t) = g(\eta) = 1 - \exp\left(\frac{r^2}{4\nu t}\right) \tag{D.8}$$

and the velocity is

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right). \tag{D.9}$$

It approaches the limit of potential vortex as $r >> R = 2\sqrt{\nu t}$, with the diffusion speed being $u_d = R/t = 2\sqrt{\nu/t}$. Also, at the inviscid limit ($\nu \to \infty$), it is just the potential vortex solution (with $\omega_z = 0$).

The vorticity, is

$$\omega_z = \frac{1}{r}\frac{\partial r u_\theta}{\partial r} = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right). \tag{D.10}$$

It is straightforward to verify that ω_z satisfies the diffusion equation in cylinderical coordinates (with $\partial_\theta = 0$ and no advection terms)

$$\frac{\partial \omega_z}{\partial t} = \nu \nabla^2 \omega_z \tag{D.11}$$

$$= \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_z}{\partial r}\right), \tag{D.12}$$

and show that (D.10) is the just heat kernel (fundamental solution of the heat equation) with two spatial dimensions. In a d-dimensional space, the heat kernel is generally written as

$$K(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) = \frac{1}{(4\pi\nu(t-t_0))^{d/2}}\exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{x}_0|^2}{4\nu(t-t_0)}\right). \tag{D.13}$$

Similar to the Lamb–Oseen solution, the Burgers solution can be established as:

$$u_r = -\alpha r \tag{D.14}$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{\alpha r^2}{2\nu}\right)\right) \tag{D.15}$$

$$u_z = 2\alpha z \tag{D.16}$$

with an axisymmetric stagnation flow and α being the strain rate. It has a vortex stretching mechanism.

E Hyperbolic functions, error functions, Gaussians

E.1 Defining ODEs

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