A note on hydrostatics and geostrophy

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Abstract: In this note the validity and relevant scaling relations for the two axiomatic assumptions in geophysical flows – geostrophic and hydrostatic, are discussed. The first section will define symbols and notations and separate discussions on the two assumptions will follow.

CONTENTS

1. Background and notation definition

1.1. *The ambient pressure and the dymanic pressure*

The density is decomposed as

$$\rho(x,y,z,t) = \rho_0 + \bar{\rho}(z) + \rho'(x,y,z,t) \tag{1.1}$$

z-momentum for a steady state:

$$\frac{\partial p_0}{\partial z} = -\rho g \tag{1.2}$$

where p_0 is the ambient pressure as in

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t) \tag{1.3}$$

For a constantly and stably stratified fluid:

$$\frac{\partial \bar{\rho}(z)}{\partial z} = -\frac{\rho_0}{g} N^2 \tag{1.4}$$

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and hence

$$\bar{\rho}(z) = \rho_0 - \int_{z_0}^{z} N^2 \frac{\rho_0}{g}\,\mathrm{d}z = \rho_0(1 - N^2\frac{z}{g}) \tag{1.5}$$

Integration of Eq. (1.3) gives

$$p_0 = p_a - \rho_0 g(z - z_0) + \frac{\rho_0}{2g} N^2 (z - z_0)^2 \tag{1.6}$$

where the three parts contributing to the pressure are: the surface/reference pressure $p_a = p(z = z_0)$, the pressure due to depth increase, and the one due to density variation. Note that $z < z_0$.

1.2. *Bousinessq approximation*

Note that we denote here

$$\rho(z) = \rho_0 + \bar{\rho}(z) \tag{1.7}$$

instead of Eq. (1.1).

z-momentum in the inviscid governing equation:

$$\frac{Dw}{Dt} = -\frac{1}{\rho(z) + \rho'}\frac{\partial}{\partial z}(p_0 + p') - g \tag{1.8}$$

$$(\rho(z) + \overset{\text{small}}{\cancel{\rho'}})\frac{Dw}{Dt} = -\frac{\partial}{\partial z}(\overset{\text{ambient}}{\cancel{p_0}} + p') - (\overset{\text{ambient}}{\cancel{\rho(z)}} + \rho')g \tag{1.9}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b \tag{1.10}$$

where the buoyancy is defined as

$$b = -\frac{\rho' g}{\rho_0} \tag{1.11}$$

1.3. *Buoyancy equation*

From Eq. (1.11),

$$\frac{Db}{Dt} = -\frac{g}{\rho_0}\frac{D\rho'}{Dt} \tag{1.12}$$

Continuity:

$$\frac{D\rho}{Dt} = \frac{D\rho'}{Dt} + w\frac{\partial\bar{\rho}}{\partial z} = 0 \tag{1.13}$$

$$\frac{D\rho'}{Dt} = wN^2\frac{\rho_0}{g} \tag{1.14}$$

and hence

$$\frac{Db}{Dt} + wN^2 = 0 \tag{1.15}$$

1.4. *Hydrostatic*

Hydrostatic assumption is almost axiomatic in ocean models. It is valid in some situations. The sections below follows basically Marshall *et al.* (1997) but steps and notations might defer.

(i) fully hydrostatic:

$$\frac{\partial p_0}{\partial z} = -(\rho_0 + \bar{\rho}(z))g \tag{1.16}$$

$$\frac{\partial p'}{\partial z} = \rho_0 b = -\rho' g \tag{1.17}$$

Summing together Eqs. (1.16) and (1.17) we have

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}(p_0 + p') = -(\rho_0 + \bar{\rho} + \rho')g = -\rho g \tag{1.18}$$

which is the standard hydrostatic assumption.

(ii) non-hydrostatic:

$$\frac{\partial p_0}{\partial z} = -(\rho_0 + \bar{\rho}(z))g \tag{1.19}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b,\; b = -\frac{\rho' g}{\rho_0} \tag{1.20}$$

$$\tag{1.21}$$

Summing together Eqs. (1.19) and (1.20) we have

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{1.22}$$

which is again the full z-momentum equation.

2. Buoyancy scaling for hydrostatic flows

The buoyancy equation (1.15) gives

$$w \sim U\frac{b}{L}\frac{1}{N^2} \tag{2.1}$$

In Eq. (1.10) hydrostatic requires the order-of-magnitude relation

$$-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} \sim b \gg \frac{Dw}{Dt} \tag{2.2}$$

and hence

$$\frac{U^2}{L^2 N^2} \ll 1 \tag{2.3}$$

Here L and h are horizontal and vertical length scales, respectively.

The Froude and Richardson numbers are

$$Fr = \frac{U}{Nh} \tag{2.4}$$

$$Ri = \frac{N^2 h^2}{U^2} \tag{2.5}$$

and the condition for the hydrostatic limit is

$$n \triangleq \frac{U^2}{L^2 N^2} = \frac{\gamma^2}{Ri} \ll 1 \tag{2.6}$$

or equivalently

$$\gamma Fr \ll 1 \tag{2.7}$$

where

$$\gamma = \frac{h}{L} \tag{2.8}$$

is the aspect ratio.

Comments on conditions for hydrostatic flows:

(i) strong stratification / weak flow / low Fr

(ii) small aspect ratio, could be in terms of 'pancake' vortices or 'pancake' turbulence.

3. Quasi-hydrostatic

In Eq. (1.10) assume there is one 'hydrostatic' part of the hydrostatic pressure p_{HY} that balances the buoyancy as in Eq. (1.17) and the remainder p_{NH} – the non-hydrostatic pressure.

$$p' = p_{\mathrm{HY}} + p_{\mathrm{NH}} \tag{3.1}$$

Note that ambient pressure is also 'hydrostatic' but excluded in this part.

The ratio

$$n = \frac{p_{\mathrm{NH}}}{p_{\mathrm{HY}}} \tag{3.2}$$

is call the hydrostatic index, which measures quantitatively to which extent the hydrostatic assumption is valid and will be shown later to appear as a small parameter (in the case of quasi-hydrostatic flows) in the pertubation expansion of the governing equations.

From

$$-\frac{1}{\rho_0}\frac{\partial p_{\mathrm{HY}}}{\partial z} + b = 0 \tag{3.3}$$

we could have

$$p_{\mathrm{HY}} = \int_{z_0}^{z} \rho_0 b \,\mathrm{d}z = \int_{z_0}^{z} -\rho' g \,\mathrm{d}z \tag{3.4}$$

and then

$$p_{\mathrm{NH}} = p' - p_{\mathrm{HY}}. \tag{3.5}$$

The more efficient way might be using

$$n = \frac{\gamma^2}{Ri} \tag{3.6}$$

than using (3.2) which will be shown equivalent in the momentum equation scaling later.

Governing equations are

$$\frac{D\boldsymbol{u}_{\mathrm{h}}}{Dt} = -\frac{1}{\rho}\nabla_{\mathrm{h}}(p_0 + p_{\mathrm{HY}} + p_{\mathrm{NH}}) - f\hat{\boldsymbol{k}} \times \boldsymbol{u}_{\mathrm{h}} \tag{3.7}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p_{\mathrm{NH}}}{\partial z} \tag{3.8}$$

$$\frac{Db}{Dt} + N^2 w = 0 \tag{3.9}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{3.10}$$

where $\boldsymbol{u} = (u, v, w)$ and $\boldsymbol{u}_{\mathrm{h}} = (u, v)$.

4. Quasi-geostrophic

Following the section above, we separate the scalings for the hydrostatic and non-hydrostatic parts in pressure and exam the geostrophic condition.

Scales:

$$u, v : U \tag{4.1}$$
$$w : W \tag{4.2}$$
$$x, y : L \tag{4.3}$$
$$z : h \tag{4.4}$$
$$p_0, p_{\mathrm{HY}} : P_{\mathrm{HY}} \tag{4.5}$$
$$p_{\mathrm{Nh}} : P_{\mathrm{NH}} \tag{4.6}$$
$$f : F \tag{4.7}$$
$$b : g\rho'/\rho_0 \tag{4.8}$$
$$N : -\frac{g}{\rho_0}\frac{\Delta\rho}{h} \tag{4.9}$$

(1) Geostrophy scaling:

$$\frac{1}{\rho_0}\frac{P_{\mathrm{HY}}}{L} \sim FU \to \frac{P_{\mathrm{HY}}}{\rho_0 U^2} \sim \frac{FL}{U} = \frac{1}{Ro} \tag{4.10}$$

(2) Hydrostatic pressure scaling:

$$b = -\frac{\rho' g}{\rho_0} = \frac{1}{\rho_0}\frac{\partial P_{\mathrm{HY}}}{\partial z} \to P_{\mathrm{HY}} \sim \rho' g h \tag{4.11}$$

(3) Continuity scaling: (assumingly, and commented on later)

$$\frac{U}{L} \sim \frac{1}{Ro}\frac{W}{h} \tag{4.12}$$

(4) Buoyancy scaling:

$$\frac{Db}{Dt} = -N^2 w \to \frac{U}{L}\frac{\rho' g}{\rho_0} \sim w\frac{g}{\rho_0}\frac{\Delta\rho}{h} \to \frac{\rho'}{\Delta\rho} \sim \frac{WL}{Uh} \sim Ro \tag{4.13}$$

Combining Eqs. (4.10) – (4.13) we have

$$Ri Ro^2 \sim 1 \to Bu = \frac{Ro^2}{Fr^2} \sim 1 \tag{4.14}$$

which is the celebrated condition for (quasi-) **geostrophy**. Note that by using (4.11) we don't apply hydrostatic assumption but just use the hydrostatic pressure scaling.

Comment on continuity equation scaling:

(i) In shallow water (with rotation) systems, the continuity equation is converted into the equation for the surface water displacement height η, and the scaling in Eq. (4.12) works when the wave height scales as

$$\eta^* = \frac{g}{fUL}\eta, \tag{4.15}$$

according to Vallis (2017).

(ii) In general, we may naively expect

$$\frac{\partial u}{\partial x} \sim \frac{\partial w}{\partial z} \to \frac{U}{L} \sim \frac{W}{h} \tag{4.16}$$

but it doesn't work. Firstly, the Rossby number order zero expansion gives the geostrophic balance (it results from the momentum equation instead of the continuity equation!)

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0. \tag{4.17}$$

From

$$\frac{\partial w}{\partial z} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}), \tag{4.18}$$

we know $\frac{\partial w}{\partial z}$ scales with the horizontal velocity divergence, which is small according to Eq. (4.17), instead of scaling with $\frac{\partial u}{\partial x}$ or $\frac{\partial v}{\partial y}$. So

$$\frac{W}{h} \sim Ro\frac{U}{L} \tag{4.19}$$

could be a safe guess. But it is not guaranteed to work (Vallis 2017).

Comment on density fluctuation in geostrophic flow:

From Eq. (4.13) we learn that

$$\frac{\rho'}{\Delta\rho} \sim Ro \tag{4.20}$$

which means in order to have non-hydrostatic pressure, the density fluctuation should be smaller than what the upwelling of the background density can compensate, in case of geostrophy.

5. Non-dimensional governing equations

Assuming quasi-hydrostatic and quasi-geostrophic, i.e., the hydrostatic index n and Ro are both small, we can rewrite the governing equation into the following non-dimensional form with small parameters:

$$\frac{D\boldsymbol{u}_\mathrm{h}}{Dt} = -\frac{1}{Ro}\nabla_\mathrm{h}(p_0 + p_\mathrm{HY} + np_\mathrm{NH}) - \frac{1}{Ro}f\hat{\boldsymbol{k}} \times \boldsymbol{u}_\mathrm{h} \tag{5.1}$$

$$\frac{Dw}{Dt} = -\frac{\partial p_\mathrm{NH}}{\partial z} \tag{5.2}$$

$$\frac{Db}{Dt} + N^2 w = 0 \tag{5.3}$$

$$\nabla_\mathrm{h} \cdot \boldsymbol{u}_\mathrm{h} + Ro\frac{\partial w}{\partial z} = 0 \tag{5.4}$$

6. Hydrostatic index revisited

Scaling of hydrostatic pressure:

$$\frac{1}{\rho_0}\frac{\partial p_\mathrm{HY}}{\partial z} = b \triangleq -\frac{\rho' g}{\rho_0} \to P_\mathrm{HY} \sim \rho' g h \tag{6.1}$$

Scaling of non-hydrostatic pressure:

$$\frac{Dw}{Dt} \sim \frac{1}{\rho_0}\frac{\partial p_\mathrm{NH}}{\partial z} \to U\frac{W}{L} \sim \frac{1}{\rho_0}\frac{P_\mathrm{NH}}{h} \to P_\mathrm{NH} \sim \frac{UWh}{L}\rho_0 \tag{6.2}$$

6.1. *Geostrophic flow*

Hydrostatic index:

$$n \triangleq \frac{P_{\mathrm{NH}}}{P_{\mathrm{HY}}} = \frac{U^2}{L^2}\frac{h}{\rho'}\frac{\rho_0}{g} = \frac{U^2}{L^2N^2} = \gamma^2 Fr^2 \tag{6.3}$$

Note that the *Ro*-scaling (4.20) for fluctuating density ρ' is used and *Ro* finally cancelled out in both numerator and denominator. Note also the use of velocity scaling (4.19) which is also under geostrophic assunmption.

6.2. *Non-geostrophic flow*

The velocity scaling (4.19) falls back to

$$\frac{W}{h} \sim \frac{U}{L} \tag{6.4}$$

given *Ro* is no longer small. And the buoyancy scaling (4.13) follows as

$$\frac{\rho'}{\Delta\rho} \sim \frac{WL}{Uh} \sim \mathcal{O}(1) \tag{6.5}$$

and we resume

$$n \triangleq \frac{P_{\mathrm{NH}}}{P_{\mathrm{HY}}} = \frac{U^2}{L^2N^2} = \gamma^2 Fr^2 \tag{6.6}$$

Comment:

From Eqs. (6.3) and (6.6) we learn that the hydrostatic assumption is independent on geostrophy, as the opposite is also true.

And the Rossby and the Froude numbers can be regarded as any other relevant time scales devided by f or N, not necessarily the global/large time scale $U/L, W/h$ but could also be some other scales defined locally.

7. Stratification: how strong is enough?

This section follows basically the beautiful paper by Billant & Chomaz (2001), and rotation is neglected from now on. Questions remain whether the vertical convenction and horizontal convection are in the same order when we simply write the LHS as $\frac{D}{Dt}$ in previous sections.

7.1. *Momentum scaling and hydrostatic*

Governing equations for inviscid gravitating Boussinesq flow are

$$\frac{\partial \boldsymbol{u}_{\mathrm{h}}}{\partial t} + \boldsymbol{u}_{\mathrm{h}} \cdot \nabla_{\mathrm{h}} \boldsymbol{u}_{\mathrm{h}} + w\frac{\partial \boldsymbol{u}_{\mathrm{h}}}{\partial z} = -\frac{1}{\rho_0}\nabla_{\mathrm{h}}(p_0 + p') \tag{7.1}$$

$$\frac{\partial w}{\partial t} + \boldsymbol{u}_{\mathrm{h}} \cdot \nabla_{\mathrm{h}} w + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} \tag{7.2}$$

$$\nabla_{\mathrm{h}} \cdot \boldsymbol{u}_{\mathrm{h}} + \frac{\partial w}{\partial z} = 0 \tag{7.3}$$

$$\frac{\partial \rho'}{\partial t} + \boldsymbol{u}_{\mathrm{h}} \cdot \nabla_{\mathrm{h}} \rho' + w\frac{\partial \rho'}{\partial z} + \frac{\partial \bar{\rho}}{\partial z} w = 0 \tag{7.4}$$

Buckingham Π-theorem:

physical variables: U, ρ, g, L, h

physical dimensions: M, L, T

non-dimensional set: $Fr_{\mathrm{h}} = \frac{U}{NL}, \gamma = \frac{h}{L}$

In the context of homogeneously horizontal shear flow with **strong** vertical density stratification (Billant & Chomaz 2001) where no explicit vertical length scale is posted by the initial and boundary conditions, we have:

scales known to the problem: U, L, ρ_0, g

scales need to be determined: $W, P(p'), R(\rho'), h$

Scalings:

(1) Pressure scaling: from (7.1), assuming vertical convection is not grater than the horizontal one (which is a safe estimation), we have the pressure gradient that balances the advection

$$P = \rho_0 U^2 \tag{7.5}$$

which is standard. We would leave the continuity equation later.

(2) Density equation: from (7.4), analogously balancing the horizontal advection of ρ' and the vertical advection of $\frac{\partial \bar{\rho}}{\partial z}$,

$$\boldsymbol{u}_{\mathrm{h}} \cdot \nabla_{\mathrm{h}} \rho' \sim \frac{\partial \bar{\rho}}{\partial z} w \to \frac{U\rho'}{L} \sim \frac{\rho_0}{g} N^2 W \to W \sim Fr_{\mathrm{h}} \frac{\rho' g}{\rho_0 N} \tag{7.6}$$

where the horizontal Froude number is

$$Fr_{\mathrm{h}} = \frac{U}{LN} \tag{7.7}$$

(3) Vertical momentum: looking for density pertubation scaling to be applied in (7.6). We should compare the (horizontal) advection term and the buoyancy term.

$$\boldsymbol{u}_{\mathrm{h}} \cdot \nabla_{\mathrm{h}} w \sim \frac{U}{L} Fr_{\mathrm{h}} \frac{\rho' g}{\rho_0 N} \tag{7.8}$$

which means the whole LHS $\frac{Dw}{Dt}$ is order $\mathcal{O}(Fr_{\mathrm{h}}^2)$ compared to the buoyancy term. Hence the dominant balance for (7.2) is the hydrostatic balance –

$$\rho' \sim \frac{P}{gh} \tag{7.9}$$

Combining the previous three non-trivial scalings:

(i) Vertical velocity:

$$W \sim \frac{Fr_{\mathrm{h}} \rho' g}{\rho_0 N} \tag{7.10}$$

(ii) The ratio of vertical to horizontal advection:

$$\frac{WL}{hU} \sim \frac{Fr_{\mathrm{h}}^2}{\gamma^2} = Fr_{\mathrm{v}} \tag{7.11}$$

where the vertical Froude number is

$$Fr_{\mathrm{v}} = \frac{U}{Nh} \tag{7.12}$$

which will be shown later to be ordedr unity if the vertical length is properly chosen to be $h = U/N$. The meaning of the horizontal Froude number

$$Fr_{\mathrm{h}} = \frac{U}{NL} = \frac{U/N}{L} = \frac{h}{L} \tag{7.13}$$

The smaller Fr_{h} is, the more 'flattened' the flow is.

Again, the non-dimensional governing equations is written as:

$$\frac{\partial \boldsymbol{u}_\mathrm{h}}{\partial t} + \boldsymbol{u}_\mathrm{h} \cdot \nabla_\mathrm{h} \boldsymbol{u}_\mathrm{h} + \frac{Fr_\mathrm{h}^2}{\gamma^2} w \frac{\partial \boldsymbol{u}_\mathrm{h}}{\partial z} = -\nabla_\mathrm{h}(p_0 + p') \tag{7.14}$$

$$Fr_\mathrm{h}^2 (\frac{\partial w}{\partial t} + \boldsymbol{u}_\mathrm{h} \cdot \nabla_\mathrm{h} w + \frac{Fr_\mathrm{h}^2}{\gamma^2} w \frac{\partial w}{\partial z}) = -\frac{\partial p'}{\partial z} - \rho' \tag{7.15}$$

$$\nabla_\mathrm{h} \cdot \boldsymbol{u}_\mathrm{h} + \frac{Fr_\mathrm{h}^2}{\gamma^2} \frac{\partial w}{\partial z} = 0 \tag{7.16}$$

$$\frac{\partial \rho'}{\partial t} + \boldsymbol{u}_\mathrm{h} \cdot \nabla_\mathrm{h} \rho' + \frac{Fr_\mathrm{h}^2}{\gamma^2} w \frac{\partial \rho'}{\partial z} - w = 0 \tag{7.17}$$

From Eq. (7.15) we show again the hydrostatic assumption is valid given that Fr_h is small, and show that the non-hydrostatic pressure

$$P_\mathrm{NH} = nP_\mathrm{HY} = \frac{U^2}{L^2 N^2} P_\mathrm{HY} \tag{7.18}$$

is indeed order $\mathcal{O}(Fr_\mathrm{h}^2)$.

In the next subsection, we will show that the vertical length scale is $h = U/N$ and hence

$$Fr_\mathrm{v} = \frac{U}{Nh} = 1 \tag{7.19}$$

or

$$Fr_\mathrm{h}/\gamma = 1 \tag{7.20}$$

so the equations (7.14) – (7.17) simplify to

$$\frac{\partial \boldsymbol{u}_\mathrm{h}}{\partial t} + \boldsymbol{u}_\mathrm{h} \cdot \nabla_\mathrm{h} \boldsymbol{u}_\mathrm{h} + w \frac{\partial \boldsymbol{u}_\mathrm{h}}{\partial z} = -\nabla_\mathrm{h}(p_0 + p') \tag{7.21}$$

$$Fr_\mathrm{h}^2 (\frac{\partial w}{\partial t} + \boldsymbol{u}_\mathrm{h} \cdot \nabla_\mathrm{h} w + w \frac{\partial w}{\partial z}) = -\frac{\partial p'}{\partial z} - \rho' \tag{7.22}$$

$$\nabla_\mathrm{h} \cdot \boldsymbol{u}_\mathrm{h} + \frac{\partial w}{\partial z} = 0 \tag{7.23}$$

$$\frac{\partial \rho'}{\partial t} + \boldsymbol{u}_\mathrm{h} \cdot \nabla_\mathrm{h} \rho' + w \frac{\partial \rho'}{\partial z} - w = 0 \tag{7.24}$$

The physical implication is that, the vertical velocity is small but the vertical gradients are large (given a small h in the presence of strong stratification), so that the vertical transport is in balance with its horizontal counterparts. The flows described by Eqs. (7.14) – (7.17) are not 2-D.

The undetermined scalings are now

$$W \sim UFr_\mathrm{h} \tag{7.25}$$

$$\rho' \sim \frac{UN}{g} \rho_0 \tag{7.26}$$

$$z \sim U/N \tag{7.27}$$

$$t \sim L/U \tag{7.28}$$

Note the scales h, L above are local scales. Although it 'appears' to be true that locally hydrostatic is satisfied in strongly stratified flows, we still don't know if there are global length scales and whether the hydrostatic assumption can be applied in the whole domain. Generally, hydrostatic is local instead of global.

7.2. *Choice of vertical length scale*

Consider the strong stratification limit – $Fr_{\mathrm{h}} = 0$, Eqs. (7.14) – (7.17) will pocess group of invariance (Billant & Chomaz 2001) with the transform being

$$(x, y, z) = (x^*, y^*, \alpha z^*) \tag{7.29}$$

$$w = \alpha w^* \tag{7.30}$$

$$\rho' = \frac{1}{\alpha}\rho'^* \tag{7.31}$$

$$N = \frac{1}{\alpha}N^* \tag{7.32}$$

which implies that the horizontal scales remains unchange while the vertical scale and density fluctuation decreases or increases propotionally to the stratification strength N. Thereby $h \sim U/N$.

We note that the inclusion of Dw/Dt in (7.22) breaks this similarity. Also, viscous term breaks this similarity as well. Rotation doesn't.

References

Billant, Paul & Chomaz, Jean-Marc 2001 Self-similarity of strongly stratified inviscid flows. *Physics of fluids* **13** (6), 1645–1651.

Marshall, John, Hill, Chris, Perelman, Lev & Adcroft, Alistair 1997 Hydrostatic, quasi-hydrostatic, and nonhydrostatic ocean modeling. *Journal of Geophysical Research: Oceans* **102** (C3), 5733–5752.

Vallis, Geoffrey K 2017 *Atmospheric and oceanic fluid dynamics*. Cambridge University Press.