Principles of fluid flows

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Preface: This note, intended for being used as a quick reference, provides a collection of a wide range of equations in fluid mechanics. We try to provide enough details so that all derivations can be reproduced. We also try to use symbols and notations as consistently as possible, and it is unavoidable that this note is biased to the author's own taste. The logical order of the materials is from general to special, which is arguably counter-pedagogical. Thus, this note does not pretend to be an introductory one (on how this World works). This note is still under construction.

Contents

Chapter 1

Conservation laws

1.1 Generic conservation laws

Consider a control volume V enclosed by a surface A and a scalar ψ (per unit mass) carried by the fluid. The rate of change of the total scalar in V, in the absence of any source or sink, can be established as

$$\frac{d}{dt}\left(\int_V \psi(\boldsymbol{x},t)dm\right) = \frac{d}{dt}\left(\int_V \psi(\boldsymbol{x},t)\rho(\boldsymbol{x},t)dV\right) \tag{1.1}$$

$$= -\int_{A=\partial V} \rho\psi\boldsymbol{u}\cdot\boldsymbol{n}dA \tag{1.2}$$

$$= -\int_V \nabla\cdot(\rho\psi\boldsymbol{u})dV \tag{1.3}$$

Hence,

$$\int_V \left[\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u})\right] dV = 0. \tag{1.4}$$

Since the control volume is arbitrary and can be shrunk to infinitesimal, we have the differential form

$$\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u}) = 0. \tag{1.5}$$

Here, the scalar can be taken to be mass ($\psi = 1$), momentum ($\psi := \boldsymbol{u}$), or any other scalar ($\psi = X_i$, where X_i is the species concentration).

Example: Moving control volume.

In what follows, we will discuss several conservation laws. Ref. Chorin & Marsden (1990).

1.2 Mass conservation

Taking $\psi = 1$,

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right] dV = 0. \tag{1.6}$$

In differential form:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0. \tag{1.7}$$

More over, using the definition of material derivative, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho(\nabla \cdot \boldsymbol{u}) \tag{1.8}$$

$$= \frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{u}) \tag{1.9}$$

$$= 0. \tag{1.10}$$

1.2.1 Incompressibility, continuity

It can be seen from (1.9) that solenoidality of the velocity field ($\nabla \cdot \boldsymbol{u} = 0$) is a sufficient and necessary condition for imcompressibility ($D\rho/Dt = 0$, density doesn't change along the material). The physical meaning is as the following.

Consider a cubic finite volume $dV = dxdydz$, the volume change is

$$\delta(dV) = [(u + \partial_x u dx) - u]dydzdt + [(v + \partial_y v dy) - v]dxdzdt + [(w + \partial_z w dz) - w]dxdydt \tag{1.11}$$

$$= (\partial_x u + \partial_y v + \partial_x w)dxdydzdt. \tag{1.12}$$

We have the divergence of a fluid parcel

$$\nabla \cdot \boldsymbol{u} = \partial_x u + \partial_y v + \partial_x w = \frac{1}{dV}\frac{\delta(dV)}{\delta t}, \tag{1.13}$$

which is the volume change rate per unit volumn. The specific volume (not to be confused with viscosity) is defined as $\nu = 1/\rho$. We can also have

$$\nabla \cdot \boldsymbol{u} = \frac{1}{\nu}\frac{D\nu}{Dt}, \tag{1.14}$$

which is stating that the divergence of the velocity filed is the normalized rate of change of specific volume.

One might have already been satisfied with such statements. Actually, there are more to it. The compressibility (change of density) can be due to pressure or thermal effects, which will be discussed more in detail in Chapter 6.

Ref. Batchelor (1967).

1.3 Material derivative and the Reynolds transport theorem

1.3.1 Material derivative

Consider the total time derivative of a scalar ψ:

$$\frac{d}{dt}\psi(\boldsymbol{x}(t), t) = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt} = \frac{\partial \psi}{\partial t} + \boldsymbol{u} \cdot \nabla \psi. \tag{1.15}$$

We define

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \tag{1.16}$$

as the material derivative, with $\partial/\partial t$ being the local rate-of-change and $\boldsymbol{u} \cdot \nabla$ being the convective derivative.

This is actually a bridge between the Eulerian and Lagrangian description of fluids.

1.3.2 Reynolds transport theorem

1.3.3 A Lagrangian perspective

An Eulerian field is

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}(\varphi(\boldsymbol{x}_0,t),t) \tag{1.17}$$

where $\varphi : \boldsymbol{x}_0 \to \boldsymbol{x}$ is a smooth and invertible flow mapping that maps the initial location of the fluid parcel $\boldsymbol{x}_0$ at $t=0$ to $\boldsymbol{x}$ at t such that

$$\boldsymbol{x} = \varphi(\boldsymbol{x}_0,t). \tag{1.18}$$

1.4 Momentum conservation

Let $\psi = \boldsymbol{u}$ to be the quantity being transported. The change of momentum in V is equal to the momentum flux in the direction $\boldsymbol{n}$ and volumetric contribution from the external body forcing per unit mass $\boldsymbol{f}$ ($\boldsymbol{f} = \lim_{\Delta V\to 0} \Delta\boldsymbol{F}/\Delta m$):

$$\frac{d}{dt}\int_V \rho\boldsymbol{u}dV = -\int_A (\rho\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{n}dA + \int_A \boldsymbol{\sigma}\cdot\boldsymbol{n}dA + \int_V \rho\boldsymbol{f}dV \tag{1.19}$$

$$= -\int_V \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u})dV + \int_V (\nabla\cdot\boldsymbol{\sigma})dV + \int_V \rho\boldsymbol{f}dV, \tag{1.20}$$

where $\boldsymbol{\sigma} = \lim_{\Delta A\to 0} \Delta\boldsymbol{F}/\Delta\boldsymbol{A}$ is the stress tensor.

We have the integration from of the Euler equation:

$$\int_V \left[\partial_t(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u})\right] dV = \int_v [\nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{f}]dV \tag{1.21}$$

which is valid for a finite volume V. Hence, (the differential form)

$$\partial_t(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u}) = \nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{f}. \tag{1.22}$$

For example, if the body force is gravity, $\boldsymbol{f} = \boldsymbol{g}$. Or if the body force is Coriolis, $\boldsymbol{f} = \boldsymbol{u}\times\boldsymbol{f}_c$.

Using the continuity equation, the above equation converts to

$$\rho\ddot{\boldsymbol{x}} = \nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{f}, \tag{1.23}$$

which is a general form of momentum balance in continuum mechanics, where $-\rho\ddot{\boldsymbol{x}}$ is D'Alembert's force (with $\boldsymbol{x}$ being the displacement), $\boldsymbol{\sigma}$ is the stress tensor, and $\boldsymbol{f}$ is the body force (per unit volume).

Now we consider the sources of stresses,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_p + \boldsymbol{\tau}. \tag{1.24}$$

- Pressure. It is isotropic and the direction is $-\boldsymbol{n}$ so $\boldsymbol{\sigma}_p = -p\boldsymbol{I}$.
- Viscous stress. For a Newtonian fluid (see section 1.7),

$$\boldsymbol{\tau} = 2\mu\boldsymbol{S} + \lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{\delta} \tag{1.25}$$

$$= 2\mu\boldsymbol{S} - \frac{2}{3}\mu(\nabla\cdot\boldsymbol{u})\boldsymbol{\delta} \tag{1.26}$$

where δ_{ij} is the Kronecker delta. According to Stokes assumption, the bulk viscosity $\lambda = -2/3\mu$.

- For incompressible flows, $\boldsymbol{\tau} = 2\mu \boldsymbol{S}$ and

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{S}. \tag{1.27}$$

Hence, the momentum equation for incompressible flows is

$$\partial_t(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u}) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{f}. \tag{1.28}$$

We note that $\partial_j(2\mu S_{ij}) = \partial_j(\mu\partial_j u_i)$.

1.5 Energy conservation

1.5.1 Mechanical energy equation

Multiply the momentum equation

$$\frac{\partial\rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial\sigma_{ij}}{\partial x_j} + \rho f_i \tag{1.29}$$

with u_i and contract:

$$\frac{\partial\rho u_i^2/2}{\partial t} + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho u_i^2 u_j\right) = u_i\frac{\partial\sigma_{ij}}{\partial x_j} + \rho u_i f_i. \tag{1.30}$$

Here we use the product rule

$$u_i\frac{\partial\sigma_{ij}}{\partial x_j} = \frac{\partial u_i\sigma_{ij}}{\partial x_j} - \sigma_{ij}\frac{\partial u_i}{\partial x_j}, \tag{1.31}$$

and

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = \left(-p\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} \tag{1.32}$$

$$= -p\frac{\partial u_k}{\partial x_k} + 2\mu S_{ij}S_{ij} - \frac{2}{3}\mu\left(\frac{\partial u_k}{\partial x_k}\right)^2 \tag{1.33}$$

After rearrangements, we have the mechanical energy ($u_k^2/2$; per unit mass) equation:

$$\rho\frac{D(\rho u_k^2/2)}{Dt} = \frac{\partial(\rho u_k^2/2)}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i u_k^2/2) = \rho u_j f_j + \frac{\partial}{\partial x_j}(u_i\sigma_{ij}) + p\frac{\partial u_k}{\partial x_k} - \phi, \tag{1.34}$$

where

$$\phi = 2\mu S_{ij}S_{ij} - \frac{2}{3}\mu\left(\frac{\partial u_k}{\partial x_k}\right)^2 = 2\mu\left(S_{ij} - \frac{1}{3}\frac{\partial u_j}{\partial x_j}\delta_{ij}\right)^2, \tag{1.35}$$

the viscous dissipation, is always positive, transfering mechanical energy to internal energy. Note $p(\nabla\cdot\boldsymbol{u})$ is the volume expansion work for the mechanical energy equation (mechanical energy increases when $\nabla\cdot\boldsymbol{u} > 0$), and $-p(\nabla\cdot\boldsymbol{u})$ the compression heating (internal energy increases when $\nabla\cdot\boldsymbol{u} < 0$) for the internal energy.

1.5.2 Total energy equation

The total energy ($E = e + u_k^2/2$; per unit mass) conservation follows the thermodynamic first law (in the Lagrangian framework), that the change of total energy is equal to the work done to the system (control volume) plus heat added,

$$\frac{D}{Dt}\int_V \rho(e + u_i^2/2)\, dV = \int_V \rho u_i f_i \, dV + \int_A u_i \sigma_{ij}\, dA_j - \int_A q_i \, dA_i, \tag{1.36}$$

where the force acting on the surface of the CV is $\boldsymbol{\sigma} \cdot d\boldsymbol{A}$ and heat flux into the CV is $-\boldsymbol{q} \cdot \boldsymbol{n}$, $\boldsymbol{q} = -k\nabla T$ is the heat flux according to Fourier's law, and k is the thermal conductivity. Using the Reynolds transport theorem and the continuity equation,

$$\frac{D}{Dt}\int_V \rho(e + u_i^2/2)\, dV = \int_V \rho \frac{D}{Dt}(e + u_i^2/2)\, dV. \tag{1.37}$$

Finally, we have the total energy equation:

$$\rho\frac{D(e + u_k^2/2)}{Dt} = \frac{\partial \rho(e + u_k^2/2)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_j (e + u_k^2/2)\right) = \rho u_j f_j + \frac{\partial}{\partial x_j}(u_i \sigma_{ij}) - \frac{\partial q_j}{\partial x_j} \tag{1.38}$$

Here σ_{ij} is the stress tensor,

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{1.39}$$

and the stress work can be broken into the pressure work and the viscous stress work as

$$\frac{\partial}{\partial x_j}(u_i \sigma_{ij}) = -\frac{\partial (u_j p)}{\partial x_j} + \frac{\partial (u_i \tau_{ij})}{\partial x_j}. \tag{1.40}$$

We note that the potential energy is included in $\rho u_i f_j = -\rho u_i \partial_i \Phi = -\rho D\Phi/Dt$ ($\partial_t \Phi = 0$), where $\Phi = gz$ is the gravitational potential. Otherwise, the total energy is defined as $e + u_k^2/2 + gz$.

1.5.3 Internal energy equation

Subtract the mechanical energy equation from the total energy equation, we obtain the internal energy ($e = c_v T$; per unit mass) equation:

$$\rho\frac{De}{Dt} = \frac{\partial (\rho e)}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i e) = -\frac{\partial q_j}{\partial x_j} - p\frac{\partial u_j}{\partial x_j} + \phi, \tag{1.41}$$

with the meanings of each term already explained above.

1.5.4 Heat equation

Heat equation (under Boussinesq): when incompressible and the heating due to the viscous dissipation is neglected, the inernal energy equation turns to

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T, \tag{1.42}$$

or

$$\frac{DT}{Dt} = \alpha\nabla^2 T, \tag{1.43}$$

where $\alpha = k/(\rho c_v)$ is called thermal diffusivity.

1.5.5 Entropy equation

Following the Tds relation

$$Tds = de + pd\nu = de - \frac{p}{\rho^2}d\rho, \tag{1.44}$$

we have

$$\begin{aligned} \rho T\frac{DS}{Dt} &= \rho\frac{De}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} && (1.45)\\ &= -\frac{\partial q_j}{\partial x_j} - p\frac{\partial u_j}{\partial x_j} + \phi + \frac{p}{\rho}\rho\frac{\partial u_j}{\partial x_j} && (1.46)\\ &= -\frac{\partial q_j}{\partial x_j} + \phi && (1.47) \end{aligned}$$

Hence,

$$\begin{aligned} \rho\frac{DS}{Dt} &= -\frac{1}{T}\frac{\partial q_j}{\partial x_j} + \frac{\phi}{T} && (1.48)\\ &= -\frac{\partial}{\partial x_j}\left(\frac{q_i}{T}\right) + \frac{k}{T^2}\left(\frac{\partial T}{\partial x_j}\right)^2 + \frac{\phi}{T} && (1.49) \end{aligned}$$

where in the RHS the first term represents reversible entropy transfer since it can take both signs, and the last two terms are always positive (requiring $k, \nu > 0$) and represent the irreversible entropy production.

1.6 Bernoulli equation

Assumptions:

- Inviscid.
- Barotropic.
- Potential force.
- Steady.

Consider the inviscid Euler equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{F} \tag{1.50}$$

with a conservative force

$$\boldsymbol{F} = -\nabla\Phi. \tag{1.51}$$

Using Eq. (A.20) we have

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\frac{\boldsymbol{u}^2}{2} - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = \nabla\frac{\boldsymbol{u}^2}{2} - \boldsymbol{u}\times\boldsymbol{\omega} \tag{1.52}$$

We defined

$$\boldsymbol{L} = \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{\omega} \tag{1.53}$$

which is called the Lamb vector. The Euler equation becomes

$$\nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{p}{\rho} \right) - \nabla \Phi \tag{1.54}$$

and hence

$$\nabla \left(\frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi \right) = \boldsymbol{L} \tag{1.55}$$

which is called the Lamb-Gromyko equation. Define

$$H = \frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi \tag{1.56}$$

we have

$$\nabla H = \boldsymbol{L}. \tag{1.57}$$

If the flow is irrotational, i.e., $\boldsymbol{L} = 0$ and $H = \text{const.}$, we recover a special version (isentropic) of Bernuolli's theorem. It is a combination of momentum and energy conservation. When we take $\Phi = gz$ to be the gravitational potential, $\boldsymbol{F} = -\nabla\Phi = (0, 0, -g)$, the conservation of H implies

$$\frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + gz = \text{const.}, \tag{1.58}$$

which is useful for pipe flows when neglecting friction loss.

The Bernoulli equation can be interpreted as energy conservation as well. The total energy equation (1.38), after some rearrangements, turns to

$$\frac{\partial \rho (e + u_k^2/2)}{\partial t} + \frac{\partial}{\partial x_j} \left(\rho u_j \left(e + u_k^2/2 + \frac{p}{\rho} \right) \right) = \rho f_j u_j + \frac{\partial}{\partial x_j} (u_i \tau_{ij}) - \frac{\partial q_j}{\partial x_j}. \tag{1.59}$$

In steady flows, $\partial_t = 0$. The external force is gravity and $u_i f_i = -u_i \partial \Phi / \partial x_i = -u_i \partial (gz)/\partial x_i$. We have

$$\rho u_j \frac{\partial}{\partial x_j} \left(e + u_k^2/2 + \frac{p}{\rho} + gz \right) = 0 \tag{1.60}$$

in the absence of viscosity and external heating, implying

$$e + \frac{|\boldsymbol{u}|^2}{2} + \frac{p}{\rho} + gz = h + \frac{|\boldsymbol{u}|^2}{2} + gz = \text{const.} \tag{1.61}$$

along the streamlines (its gradient is perpendicular to $\nabla \boldsymbol{u}$), where we used $h = e + p/\rho$. This enthalpy/energy version of Bernoulli will be more useful in compressible flows.

1.7 Constitutive relations

A note on angular momentum conservation.

A difference between solid and fluid mechanics is that, in solid mechanics, stress is proportional to strain, while in fluid mechanics, stress is proportional to strain rate.

Some comments about stress–strain relation in the solids.

1.8 Pressure Poisson

Take the divergence of the follow equation for incompressible flows ($\nabla \cdot \boldsymbol{u} = 0$):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} \tag{1.62}$$

we have

$$-\frac{1}{\rho} \nabla^2 p = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u} \nabla + \boldsymbol{u} \cdot \nabla (\nabla \cdot \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u} \nabla \tag{1.63}$$

i.e.

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_i} = -\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \tag{1.64}$$

In incompressible CFD, the continuity equation ($\nabla \cdot \boldsymbol{u} = 0$) is responsible for solving for the pressure as each of the three momentum equations are responsible for one velocity component. Note that there is no time in the continuity/Poisson equations, putting challenges on obtaining time-accurate, divergence-free velocity fields. See fractional-step/projection methods.

Chapter 2

Vortex dynamics

Vorticity is defined as the curl of the velocity field:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{2.1}$$

2.1 Kelvin's theorem

Ref. Kundu and Cohen.

The circulation along a closed line is defined as

$$\Gamma = \oint_l \boldsymbol{u} \cdot d\boldsymbol{x} = \iint_A (\nabla \times \boldsymbol{u}) \cdot d\boldsymbol{A}, \tag{2.2}$$

according to Stokes' theorem.

Consider the momentum equation

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}. \tag{2.3}$$

The material (Lagrangian) derivative of Γ is

$$\begin{aligned}
\frac{D\Gamma}{Dt} &= \frac{D}{Dt}\left(\oint_l \boldsymbol{u} \cdot d\boldsymbol{x}\right) && (2.4)\\
&= \oint_l \frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{x} + \oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} && (2.5)\\
&= -\oint_l \left(\frac{1}{\rho}\nabla p\right) \cdot d\boldsymbol{x} + \oint_l (\nabla \cdot \boldsymbol{\sigma}) \cdot d\boldsymbol{x} + \oint_l \boldsymbol{f} \cdot d\boldsymbol{x} + \oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} && (2.6)
\end{aligned}$$

Assumptions:

1. Inviscid fluid: $\nabla \cdot \boldsymbol{\sigma} = 0$.

2. Conservative body force: $\boldsymbol{f} = -\nabla\Phi, \nabla \times \boldsymbol{f} = 0$ (it is the gradient of a potential field). Conservative means the potential difference does not depend on the path:

$$\int_A^B \boldsymbol{f} \cdot d\boldsymbol{x} = -\int_A^B \nabla\Phi \cdot d\boldsymbol{x} = -\int_A^B d\Phi = \Phi_A - \Phi_B. \tag{2.7}$$

For a close path, $A = B$ hence the integral is zero.

3. Barotropic flow: $\nabla\rho \times \nabla p = 0$.

Moreover, by

$$\boldsymbol{u} + d\boldsymbol{u} = \frac{D}{Dt}(\boldsymbol{x} + d\boldsymbol{x}) = \frac{D\boldsymbol{x}}{Dt} + \frac{D(d\boldsymbol{x})}{Dt}, \tag{2.8}$$

we have

$$\frac{D(d\boldsymbol{x})}{Dt} = d\boldsymbol{u} = d\boldsymbol{x} \cdot \nabla\boldsymbol{u} \tag{2.9}$$

so the last term in (2.6) is

$$\oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} = \oint_l \boldsymbol{u} \cdot d\boldsymbol{u} = \oint_l d(\boldsymbol{u}^2) = 0. \tag{2.10}$$

Then we are able to prove all RHS terms in (2.6) are zero hence

$$\frac{D\Gamma}{Dt} = 0 \tag{2.11}$$

along any arbitrary closed curves.

2.1.1 Helmholtz's theorems

2.2 Vorticity transport equation

By Eq. (A.11) we know

$$\nabla \cdot \boldsymbol{\omega} = 0 \tag{2.12}$$

i.e., the continuity of vorticity.

The incompressible Navier–Stokes equation in vector form:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f} \tag{2.13}$$

Using Eq. (1.52) we have

$$\frac{\partial\boldsymbol{u}}{\partial t} + \nabla\frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F} \tag{2.14}$$

Using the (A.21) and take the curl of Eq. (2.14), where $\boldsymbol{H} = \boldsymbol{u} \times \boldsymbol{\omega}$ is the Lamb vector,

$$\begin{aligned}
\text{LHS} &= \nabla \times (\frac{\partial\boldsymbol{u}}{\partial t} + \nabla\frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega}) && (2.15)\\
&= \frac{\partial\boldsymbol{\omega}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) && (2.16)\\
&= \frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}) - \boldsymbol{u}(\nabla \cdot \boldsymbol{\omega}) && (2.17)\\
&= \frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} && (2.18)\\
\text{RHS} &= \nabla \times (-\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}) && (2.19)
\end{aligned}$$

$$= \nu \nabla^2 \boldsymbol{\omega} + \nabla \times \boldsymbol{f} + \frac{1}{\rho^2} \nabla \rho \times \nabla p \tag{2.20}$$

Equaling both sides we obtain

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t}}_{\substack{\text{rate of}\\\text{change}}} + \underbrace{\boldsymbol{u} \cdot \nabla \boldsymbol{\omega}}_{\text{advection}} = \underbrace{\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}}_{\substack{\text{vortex}\\\text{stretching}}} + \underbrace{\nu \nabla^2 \boldsymbol{\omega}}_{\substack{\text{viscous}\\\text{diffusion}}} + \underbrace{\nabla \times \boldsymbol{f}}_{\substack{\text{external torque in}\\\text{a non-conservative field}}} + \underbrace{\frac{1}{\rho^2} \nabla \rho \times \nabla p}_{\substack{\text{baroclinic}\\\text{torque}}} \tag{2.21}$$

Again, the stretching term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ comes from the non-linear advection/inertial term in N-S. It is important in the energy cascade in turbulence. Also from (3.18), $\boldsymbol{\omega} \cdot \boldsymbol{\Omega} = \boldsymbol{0}$, we have

$$\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \boldsymbol{\omega} \cdot \boldsymbol{S}, \tag{2.22}$$

i.e., vortex stretching happens as a result of its interaction with the strain rate. Hence, the alignment of vorticity with the eigenvectors of the strain rate is important. It was found by Ashurst *et al.* (1987) that, in turbulence, vorticity tend to align to the eigenvector corresponding to the second eigenvalue of $\boldsymbol{S}$, which tend to be positive (indicating positive stretching production). This is called the kinematic alignment effect.

2.2.1 Enstrophy transport equation

Define enstrophy as:

$$\mathcal{E} \triangleq \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\omega} = \frac{1}{2} \omega_i \omega_i \tag{2.23}$$

Re-write (2.21) into tensor notation we have

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2} + \epsilon_{ijk} \frac{\partial f_k}{\partial x_j} + \frac{1}{\rho^2} \epsilon_{ijk} \frac{\partial \rho}{\partial x_j} \frac{\partial p}{\partial x_k} \tag{2.24}$$

$\omega_i \times$ (2.24) we have

$$\frac{\partial}{\partial t} (\frac{1}{2} \omega_i \omega_i) + u_j \frac{\partial}{\partial x_j} (\frac{1}{2} \omega_i \omega_i) = \omega_i \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2}{\partial x_j^2} (\frac{1}{2} \omega_i \omega_i) - \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} \tag{2.25}$$

$$+ \epsilon_{ijk} \omega_i \frac{\partial F_k}{\partial x_j} + \frac{1}{\rho^2} \epsilon_{ijk} \omega_i \frac{\partial \rho}{\partial x_j} \frac{\partial p}{\partial x_k} \tag{2.26}$$

Note that ϵ_{ijk} is the Levi-Civita symbol, not to be confused with the turbulent kinetic energy rate ε or Reynolds stresses dissipation rate ε_{ij}.

Re-write back into vector form:

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{u} \cdot \nabla \mathcal{E} = \boldsymbol{\omega\omega} : \nabla \boldsymbol{u} + \nu \nabla^2 \mathcal{E} \underbrace{- \nu \nabla \boldsymbol{\omega} : \nabla \boldsymbol{\omega}}_{\substack{\text{viscous}\\\text{dissipation}}} + \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{F}) + \boldsymbol{\omega} \cdot \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{2.27}$$

Note that we are assuming an incompressible flow, hence $\nabla \cdot \boldsymbol{u}$ related terms are not appearing in Eq. (2.27). A new mechanism compared to (2.21) is the viscous dissipation of enstrophy. This term is always negative and removing enstrophy. Since $\omega_i \omega_j$ is a symmetric tensor and Ω_{ij} is an antisymmetric tennsor, $\omega_i \omega_j \Omega_{ij} = 0$ and we can simplify

$$\boldsymbol{\omega\omega} : \nabla \boldsymbol{u} = \boldsymbol{\omega\omega} : \boldsymbol{S}. \tag{2.28}$$

2.3 Vorticity–streamfunction formulation

According to the Helmholtz decomposition theorem, the velocity field $\boldsymbol{u}$ can be decomposed to the addition of a potential and a curl field as

$$\boldsymbol{u} = \nabla\phi + \nabla \times \boldsymbol{A}, \tag{2.29}$$

with the first part being irrotational and the second solenoidal. The two potentials ϕ and $\boldsymbol{A}$ are called the scalar and vector potentials.

In two-dimensional flows, the vector potential reduces to $\boldsymbol{A} = (0, 0, \psi)$, where ψ is the streamfunction, defined as

$$\psi(x, y) = \int_0^y u(0, \xi)\, d\xi - \int_0^x v(\zeta, y)\, d\zeta + C, \tag{2.30}$$

is constant along the streamlines. A reference point would need to be specified. And it is noted that between two streamlines the volume flow rate is a constant (physical meaning of streamfunctions). For equally spaced streamlines, the locations where they are denser have a faster velocity.

In general, for a divergence-free field ($\nabla \cdot \boldsymbol{u} = 0$) there exist a vector potential. The equations to solve are, $\boldsymbol{u} = \nabla \times \boldsymbol{A}$, or component-wise

$$u = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \tag{2.31}$$

$$v = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \tag{2.32}$$

$$w = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \tag{2.33}$$

whose solution is far from unique. We note that $\nabla \times (\boldsymbol{A} + \nabla f) = \nabla \times \boldsymbol{A}$, for any arbitrary scalar potential ∇f. It can be chosen to $\nabla f = (0, 0, -A_3)$ by integrating $f = -\int A_3\, dz$ such that the above equations are simplifed to

$$u = -\frac{\partial A_2}{\partial z} \tag{2.34}$$

$$v = \frac{\partial A_1}{\partial z} \tag{2.35}$$

$$w = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \tag{2.36}$$

which are easier to solve.

2.3.1 In two dimensions

As a strategy to solve the N–S, it is common to transform (u, v, p) to (ω, ψ), equivalent to taking the curl of the N–S to eliminate the pressure. Under such transformation, 2D N–S equations are converted to

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \tag{2.37}$$

$$\omega = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \tag{2.38}$$

where

$$u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x}. \tag{2.39}$$

The common procedures in a Fourier spectral method (in a doubly periodic domain) solving the above equations include time-stepping of (2.37), with the streamfunction calculated from the inversion of (2.38) and used to obtained the velocities through (2.39). Such a formulation can be similarly obtain for axisymmetric flows – see section D.1.5. The elliptic character of (2.38) and the parabolic character of (2.37) jointly reflect the mixed elliptic-parabolic nature of the Navier–Stokes. We also note that the elliptic nature is closely associated with incompressibility.

The absence of evolution equation for the pressure is reflected in the Poisson equation (2.38). If the pressure field is desired, it can be recovered by solving the pressure Poisson:

$$\left(\frac{\partial^2 p}{\partial^2 x} + \frac{\partial^2 p}{\partial^2 y}\right) = -\rho\frac{\partial u_\beta}{\partial x_\alpha}\frac{\partial u_\alpha}{\partial x_\beta} \tag{2.40}$$

$$= -\rho\left[\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2\right] \tag{2.41}$$

$$= -2\rho\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) \tag{2.42}$$

$$= 2\rho\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right] \tag{2.43}$$

where the last simplification uses the continuity to reach an expression that requires fewer operation counts. Since the equation is second-order in space, we need two BC's in each direction. At solid walls, we have the no-slip condition $u = v = 0$ such that $\partial_x u = \partial_x v = 0$ at the wall. The N–S equation at the wall reads

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2} = \mu\frac{\partial\omega}{\partial y}, \tag{2.44}$$

where in a finite-difference scheme the pressure can be computed from a one-sided FDA of $\partial\omega/\partial y$, given pressure should be known at least for one point on the wall.

The convenience that this method offers comes at the expense of difficulties of imposing the boundary conditions. For streamfunction it is relatively easy – it should be a constant on solid boundaries. For vorticity, neither its value nor its first derivative is known before the computation. Special treatments are required. For example, at the wall of $y = 0$, the Poisson equation of ψ reduces to

$$\omega = -\frac{\partial^2\psi}{\partial y^2} \tag{2.45}$$

since $\partial_x v = 0$.

2.3.2 In three dimensions (omega y-Laplacian v)

When deriving the vorticity equation, we took the curl of the Lamb-type equation (2.14), resulting in

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times \boldsymbol{H} = \nu \nabla^2 \boldsymbol{\omega}, \tag{2.46}$$

where $\boldsymbol{H} = \boldsymbol{u} \times \boldsymbol{\omega}$ is the Lamb vector. The components of the vortcity equation are

$$\partial_t \omega_x - (\partial_y H_3 - \partial_z H_2) = \nu \nabla^2 \omega_x \tag{2.47}$$

$$\partial_t \omega_y - (\partial_z H_1 - \partial_x H_3) = \nu \nabla^2 \omega_y \tag{2.48}$$

$$\partial_t \omega_z - (\partial_x H_2 - \partial_y H_1) = \nu \nabla^2 \omega_z \tag{2.49}$$

We take the second equation to be one part of the ω_y–$\triangle v$ formulation. Next, we establish a relation:

$$\begin{aligned} \partial_x \omega_z - \partial_z \omega_x &= \partial_x(\partial_x v - \partial_y u) - \partial_z(\partial_y w - \partial_z v) && (2.50)\\ &= \partial_{xx} v - \partial_y(\partial_x u + \partial_z w) + \partial_{zz} v && (2.51)\\ &= (\partial_{xx} + \partial_{yy} + \partial_{zz}) v && (2.52)\\ &= \nabla^2 v && (2.53) \end{aligned}$$

Similarly, we take the curl of the vorticity equation (the second curl), $\partial_x \omega_z - \partial_z \omega_x$, which yields the omega y-Laplacian v equations:

$$\partial_t(\nabla^2 v) - (\partial_{xx} + \partial_{zz}) H_2 + \partial_y(\partial_x H_1 + \partial_z H_3) = \nu \nabla^2(\nabla^2 v) \tag{2.54}$$

$$\partial_t \omega_y - (\partial_z H_1 - \partial_x H_3) = \nu \nabla^2 \omega_y \tag{2.55}$$

which are second- and fourth-order equations. The fourth order equation requires four boundary conditions at $y = 0, 2h$ for a channel flow setting. Naturally, $v = 0$ at the walls are two. The no-slip conditions, $u = w = 0$ at the walls, leads to $\partial_x u = \partial_z w = 0$ at the walls. Hence, we can impose $\partial_y v = 0$ at both walls. These are the equations solved in Kim *et al.* (1987), KMM87, where the way to recover the primitive variables (u, v, w, p) was also described. We first obtain v from solving $\nabla^2 v$, and then u, w are obtained from

$$\partial_x u + \partial_z w = -\partial_y v \tag{2.56}$$

$$\partial_z u - \partial_x w = \omega_y \tag{2.57}$$

where we note that x, z are Fourier directions so it should be straightforward. Then pressure (see KMM87).

We note that this formulation is common in incompressible CFD and stability problems, due to the advantage of going pressure-less – reduced computational cost.

stability (algebraic growth paper cf.)

2.4 Potential flow

In potential flows, $\boldsymbol{A} = \boldsymbol{0}$ in (2.29) and the velocity field can be represented as the gradient of the scalar potential. The continuity condition, $\nabla \cdot \boldsymbol{u} = 0$, leads to

$$\nabla^2 \phi = 0, \tag{2.58}$$

i.e., the scalar potential satisfies the Laplacian equation. Since the Laplacian equation is linear, superposition is enabled and basic solutions can be used as building blocks to construct more complicated flow fields.

Note that the Helmholtz decomposition is not necessarily unique. For example, in 2D potential flows, the velocity can be expressed by solely an irrotational or a solenoidal field:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{2.59}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{2.60}$$

where both the velocity potential and the streamfunction satisfy the Cauchy–Riemann (C–R) condition and are harmonic functions that solve Laplacian equation:

$$\nabla^2 \phi = 0 \tag{2.61}$$

$$\nabla^2 \psi = 0 \tag{2.62}$$

As a consequence,

$$\nabla \phi \cdot \nabla \psi = 0, \tag{2.63}$$

i.e., contours of velocity potential and streamfunctions are perpendicular.

In the language of complex analysis:

$$z = x + iy \tag{2.64}$$

$$\bar{z} = x - iy \tag{2.65}$$

and

$$x = \frac{1}{2}(z + \bar{z}) \tag{2.66}$$

$$y = \frac{1}{2i}(z - \bar{z}). \tag{2.67}$$

We have the derivatives

$$\partial_z = \partial_x \partial_z x + \partial_y \partial_z y = \frac{1}{2}(\partial_x - i\partial_y) \tag{2.68}$$

and similarly

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{2.69}$$

Denote the complex potential as $\Phi = \phi + i\psi$, we can show that

$$\frac{d\Phi}{dz} = \frac{1}{2}(\phi_x + i\psi_x) - \frac{i}{2}(\phi_y + i\psi_y) = u - iv, \tag{2.70}$$

which can be sometimes easier to find. We can also show (by the C–R equations) that

$$\frac{d\Phi}{d\bar{z}} = 0. \tag{2.71}$$

Such function Φ is called analytic/homomorphic function whose real and imaginary parts ϕ, ψ are called the harmonic conjugates which satiafy the C–R equation and the Laplacian.

2.4.1 2D potential flow in cylindrical coordinates

The continuity equation in two dimensions reads

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \theta} = 0. \tag{2.72}$$

We can define the streamfunction and scalar potential as

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta},\ v = -\frac{\partial \psi}{\partial r} \tag{2.73}$$

$$u = \frac{\partial \phi}{\partial r}, \quad v = \frac{1}{r}\frac{\partial \phi}{\partial \theta}. \tag{2.74}$$

In the limit of 2D axisymmetric flow, we have $\partial_\theta = 0$ and hence

$$\frac{\partial(ru)}{\partial r} = 0,\ u = u(r),\ v = v(r), \tag{2.75}$$

the integration of the former leads to

$$u = \frac{Q}{2\pi r} \tag{2.76}$$

where Q is constant volume flow rate. In an inviscid barotropic potential flow, the conservation of circulation implies $\Gamma = \int_0^{2\pi} vr\, d\theta = 2\pi r v = \text{const}$, and hence

$$v = \frac{\Gamma}{2\pi r}. \tag{2.77}$$

The corresponding streamfunction and scalar potential are

$$\psi = -\frac{\Gamma}{2\pi}\ln r + \frac{Q}{2\pi}\theta + C \tag{2.78}$$

$$\phi = \frac{Q}{2\pi}\ln r + \frac{\Gamma}{2\pi}\theta + C' \tag{2.79}$$

representing the superposition of two typical potential solutions: a line source/sink with

$$u = \frac{Q}{2\pi r},\ v = 0 \tag{2.80}$$

and a line vortex

$$u = 0,\ v = \frac{\Gamma}{2\pi r}. \tag{2.81}$$

The vorticity reduces to

$$\omega_z = \frac{1}{r}\frac{\partial(rv)}{\partial r} = \frac{v}{r} + \frac{\partial v}{\partial r}, \tag{2.82}$$

with the first term called the curvature vorticity and the second called the shear vorticity.

2.5 Stokes flows

In Stokes flows (creeping flows at very low Re), the governing equation reduces to

$$0 = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}. \tag{2.83}$$

Taking the divergence, curl, and Laplacian respectively, we find that

$$\nabla^2 p = 0 \tag{2.84}$$

$$\nabla^2 \boldsymbol{\omega} = \mathbf{0} \tag{2.85}$$

$$\nabla^2 \nabla^2 \boldsymbol{u} = \mathbf{0} \tag{2.86}$$

$$\nabla^2 \nabla^2 \psi = 0 \,(\text{2D}) \tag{2.87}$$

that all flow variables are harmonic or biharmonic functions.

2.6 Beltrami flows

Recall in (1.53) that the Lamb vector is defined as $\boldsymbol{H} = \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{\omega}$. The Beltrami flows satisfy

$$\boldsymbol{H} = \mathbf{0}, \tag{2.88}$$

i.e., streamlines parallel to vortex lines. Moreover, the generalized Beltrami flows satisfy

$$\nabla \times \boldsymbol{H} = \mathbf{0} \tag{2.89}$$

such that the second term in (2.16) is zero and the vorticity equation (2.21), in the absence of external torque and baroclinicity, reduces to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \nabla^2 \boldsymbol{\omega}. \tag{2.90}$$

Hence, there is no vorticity generation mechanism in Beltrami and generalized Beltrami flows. Although (2.90) is easy to solve, the boundary conditions for vorticity are usually hard to prescribe.

In two dimensions, we also have

$$\omega = -\nabla^2 \psi, \tag{2.91}$$

and the generalized Beltrami flow condition (2.89) translates to

$$\frac{\partial \omega_z}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \omega_z}{\partial y}\frac{\partial \psi}{\partial x} = 0, \tag{2.92}$$

i.e.,

$$\nabla \omega_z \,//\, \nabla \psi, \tag{2.93}$$

implying that the two rows are linearly dependent and $\omega_z = f(\psi, t)$ that vorticity is a constant along the streamline.

The equation that ψ satisfies is

$$\frac{\partial}{\partial t}(\nabla^2 \psi) = \nu \nabla^2 \nabla^2 \psi, \tag{2.94}$$

with some found exact solutions provided below.

2.6.1 Examples

The ABC (Arnold–Beltrami–Childress) flow

$$u = A\sin z + C\cos y \tag{2.95}$$
$$v = B\sin x + A\cos z \tag{2.96}$$
$$w = C\sin y + B\cos x \tag{2.97}$$
$$\tag{2.98}$$

has $\boldsymbol{\omega} = \boldsymbol{u}$. It is a Beltrami flow in the narrow sense.

Taylor's decaying vortex:

$$\psi(x,y,t) = a\cos(mx)\cos(ny)\exp[-\nu(m^2+n^2)t], \tag{2.99}$$

with the wavenumbers m, n being free parameters. It is often used for testing numerical algorithms.

Kelvin's cat eye vortex:

$$\psi(x,y,t) = a\cosh(mx)\cos(ny)\exp[\nu(m^2-n^2)t], \tag{2.100}$$

with the wavenumbers m, n being free parameters.

2.7 Point vortex models

Hamiltonian system, single, dipole, and more. Biot-Savart's law.

2.8 Lamb–Oseen similarity solution, Burgers vortex

We recall that a potential/irrotational vortex has a velocity profile as

$$u_\theta = \frac{\Gamma}{2\pi r} \tag{2.101}$$

where $\Gamma = \int_0^{2\pi} u_\theta r \, \mathrm{d}\theta$ is the circulation. The vorticity is

$$\omega_z = \frac{1}{r}\frac{\partial\,(ru_\theta)}{\partial r} = 0 \tag{2.102}$$

hence the potential vortex is *irrotational*.

Oseen considered a viscous solution in the form of

$$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r} g(r,t), \tag{2.103}$$

where $g(r,t)$ is a non-dimensional similarity function that combines the time-dependent spreading vortex core size $R(t)$ and time t, that collapses when the non-dimensional radius agree.

It is easy to see that the viscous length scale is $\sqrt{\nu t}$ in a diffusion process. So we take

$$g(r,t) = g(\hat{r}^2) = g\left(\frac{r^2}{4\nu t}\right) \tag{2.104}$$

where $\hat{r} = r/2\sqrt{\nu t}$ is the self-similar length scale and $\eta = \hat{r}^2$ is the similarity variable. The same similarity transform converts the heat equation to an ODE from which the heat kernel (2.114) can be solved.

The simplified azimuthal momentum equation is

$$\frac{\partial u_\theta}{\partial t} = \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right), \tag{2.105}$$

which represents a diffusion process, and it reduces to and ODE under the similarity transform (2.104):

$$g' + g'' = 0, \tag{2.106}$$

where $g'(\eta) = \partial_\eta g$. The general solution is

$$g(\eta) = c_1 + c_2 e^{-\eta}. \tag{2.107}$$

Given the boundary conditions

$$g(0) = 0,\ g(\infty) = 1, \tag{2.108}$$

we have

$$g(r,t) = g(\eta) = 1 - \exp\left(\frac{r^2}{4\nu t}\right) \tag{2.109}$$

and the velocity is

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right). \tag{2.110}$$

It approaches the limit of potential vortex as $r >> R = 2\sqrt{\nu t}$, with the diffusion speed being $u_d = R/t = 2\sqrt{\nu/t}$. Also, at the inviscid limit ($\nu \to \infty$), it is just the potential vortex solution (with $\omega_z = 0$).

The vorticity, is

$$\omega_z = \frac{1}{r}\frac{\partial(r u_\theta)}{\partial r} = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right). \tag{2.111}$$

It is straightforward to verify that ω_z satisfies the diffusion equation in cylinderical coordinates (with $\partial_\theta = 0$ and no advection term since $u_r = 0$; see (D.76))

$$\frac{\partial \omega_z}{\partial t} = \nu \nabla^2 \omega_z \tag{2.112}$$

$$= \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_z}{\partial r}\right), \tag{2.113}$$

and show that (2.111) is the just heat kernel (fundamental solution of the heat equation) with two spatial dimensions. In a d-dimensional space, the heat kernel is generally written as

$$K(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) = \frac{1}{(4\pi\nu(t-t_0))^{d/2}}\exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{x}_0|^2}{4\nu(t-t_0)}\right). \tag{2.114}$$

It is easy to verify that the Lamb vector is $\boldsymbol{L} = L(r)\boldsymbol{e}_r$ so $\nabla \times \boldsymbol{L} = 0$ and it is verified that the Lamb-Oseen solution is generalized Beltrami.

Similar to the Lamb–Oseen solution, the Burgers solution can be established as:

$$u_r = -\alpha r \tag{2.115}$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{\alpha r^2}{2\nu}\right)\right) \tag{2.116}$$

$$u_z = 2\alpha z \tag{2.117}$$

with an axisymmetric stagnation flow and α being the strain rate. It has a vortex stretching mechanism.

2.9 Heat diffusion: a similarity solution

Consider one-dimensional heat equation:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right), \tag{2.118}$$

when the thermal conductivity k is a constant in space, it reduces to

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \tag{2.119}$$

where $\alpha = k/\rho c_p$ is the heat diffusivity. Eqn. (2.119) is also the governing equation for general diffusion processes, for example, of a scalar or chemical species.

Consider the following initial-boundary conditions:

$$T(x, t < 0) = T_s \tag{2.120}$$

$$T(x, t) = T_0,\ T(\infty, t) = T_s, \tag{2.121}$$

corresponding to an infinity long rod with initial temperature T_s subject to an abrupt input at $t = 0$ that brings $T(0, t) = T_0$.

Non-dimensionalization:

$$\theta = \frac{T - T_0}{T_s - T_0} \tag{2.122}$$

such that the PDE is

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}, \tag{2.123}$$

and the I/BCs are

$$\theta(x, t < 0) = 1 \tag{2.124}$$

$$\theta(0, t) = 0,\ \theta(\infty, t) = 1. \tag{2.125}$$

Consider the non-dimensional similarity variable

$$\eta = \frac{x}{2\sqrt{\alpha t}} \tag{2.126}$$

and $\theta(x,t) = f(\eta)$. Eqn. (2.123) can be conveted to an ODE as

$$f'' + 2\eta f' = 0, \tag{2.127}$$

subject to boundary conditions of

$$f(0) = 0,\ f(\infty) = 1. \tag{2.128}$$

Direct integration of

$$\frac{f''}{f'} = -2\eta \tag{2.129}$$

yields

$$\ln f' = -\eta^2 + \ln C_1 \tag{2.130}$$

and

$$f' = C_1 e^{-\eta^2}. \tag{2.131}$$

Integrating again we have

$$f(\eta) = f(\eta) - f(0) = C_1 \int_0^\eta e^{-t^2}\, dt \tag{2.132}$$

where the error function is defined as

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-t^2}\, dt \tag{2.133}$$

and

$$\mathrm{erf}(0) = 0,\ \mathrm{erf}(\infty) = 1. \tag{2.134}$$

Hence, we have $C_1 = 2/\sqrt{\pi}$ and

$$f(\eta) = \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-t^2}\, dt. \tag{2.135}$$

The solution to the original equation reads

$$T(x,t) = (T_s - T_0)\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) + T_0. \tag{2.136}$$

Moreover, the complementary error function is defined as

$$\mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-t^2}\, dt \tag{2.137}$$

and reaches values of

$$\mathrm{erfc}(0) = 1,\ \mathrm{erfc}(\infty) = 0. \tag{2.138}$$

Chapter 3

Velocity gradient tensor, its decomposition and dynamics

Following the previous section, we continue to consider the local flow structures represented by the velocity gradients.

The velocity gradient tensor $\boldsymbol{u}\nabla$ is

$$\boldsymbol{u}\nabla = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial w} \end{bmatrix} \tag{3.1}$$

and in entity notation

$$(\boldsymbol{u}\nabla)_{ij} = \frac{\partial u_i}{\partial x_j}, \tag{3.2}$$

and we note that is a Jacobian such that

$$\delta\boldsymbol{u} = (\boldsymbol{u}\nabla)\cdot\delta\boldsymbol{x} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial w} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}. \tag{3.3}$$

For an arbitrary tensor $\boldsymbol{A}$, it is always possible to decompose it into symmetric and antisymmetric (skew-symmetric) parts:

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{\Omega} \tag{3.4}$$

where

$$\boldsymbol{S} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) \tag{3.5}$$

$$\boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T) \tag{3.6}$$

such that

$$\boldsymbol{S}^T = \boldsymbol{S},\ \boldsymbol{\Omega}^T = -\boldsymbol{\Omega}. \tag{3.7}$$

For the velocity gradient tensor, $\boldsymbol{S} = 1/2(\boldsymbol{u}\nabla + \nabla\boldsymbol{u})$ is called the rate-of-strain tensor and $\boldsymbol{\Omega} = 1/2(\boldsymbol{u}\nabla - \nabla\boldsymbol{u})$ is called the rotation tensor.

3.1 Pseudo-vector and associated antisymmetric rotation tensor

Vorticity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{3.8}$$

$$= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{3.9}$$

$$= \begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} \tag{3.10}$$

is a pseudo-vector ($\omega_i = \epsilon_{ijk}\partial_j u_k$) whose sign depends on the coordinate system (the order of i, j, k; left-hand or right-hand; cyclic or anticyclic), and is related to the antisymmetric part of velocity gradient tensor $\boldsymbol{u}\nabla$ (the rotation rate tensor $\boldsymbol{\Omega}$):

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{3.11}$$

or

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & -\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & -\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ -\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) & \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix} \tag{3.12}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3.13}$$

$$= \begin{bmatrix} 0 & -\omega_z^* & \omega_y^* \\ \omega_z^* & 0 & -\omega_x^* \\ -\omega_y^* & \omega_x^* & 0 \end{bmatrix} \tag{3.14}$$

where $\boldsymbol{\omega}^*$ is the angular velocity and $\boldsymbol{\omega} = 2\boldsymbol{\omega}^*$ (vorticity is twice of the angular velocity of the local solid-body rotation motion).

Each antisymmetric tensor $\boldsymbol{\Omega}$ can be represented by a pseudo-vector $\boldsymbol{\omega}^*$ (since it just has three independent elements), such that

$$\Omega_{ij} = -\epsilon_{ijk}\omega_k^* \tag{3.15}$$

$$\omega_k^* = -\frac{1}{2}\epsilon_{ijk}\Omega_{ij} \tag{3.16}$$

and according to (3.15)-(3.16) the inner product of the tensor $\boldsymbol{\Omega}$ with and arbitrary vector $\boldsymbol{a}$ can be written as

$$\boldsymbol{\Omega} \cdot \boldsymbol{a} = \boldsymbol{\omega}^* \times \boldsymbol{a}. \tag{3.17}$$

It is easy to verify (3.15) by definition and (3.16) using (A.8). As a corollary, we have

$$\boldsymbol{\Omega} \cdot \boldsymbol{\omega}^{\boldsymbol{*}} = \boldsymbol{\omega}^* \times \boldsymbol{\omega}^* = \mathbf{0}. \tag{3.18}$$

We can also find that

$$\|\boldsymbol{\Omega}\|^2 = \Omega_{ij}\Omega_{ij} = 2\delta_{kl}\omega_k^*\omega_l^* = \frac{1}{2}\boldsymbol{\omega}^2, \tag{3.19}$$

i.e., the norm of the rotation tensor is just the enstrophy.

Note the vorticity is $\boldsymbol{\omega} = 2\boldsymbol{\omega}^*$. We will further look into the eigenvalue decomposition and principle directions of $\boldsymbol{S}$ to further understand how it describes the geometry of the flow.

3.2 Strain rate tensor

$$\boldsymbol{S} = \begin{bmatrix} \epsilon_1 & \frac{1}{2}\gamma_3 & \frac{1}{2}\gamma_2 \\ \frac{1}{2}\gamma_3 & \epsilon_2 & \frac{1}{2}\gamma_1 \\ \frac{1}{2}\gamma_2 & \frac{1}{2}\gamma_1 & \epsilon_3 \end{bmatrix} \tag{3.20}$$

where the normal strain rate is

$$\epsilon_i = \frac{\partial u_{(i)}}{\partial x_{(i)}}, \tag{3.21}$$

indices inside parentheses don't imply summation, and the shear strain rate is

$$\gamma_i = \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right), \; i \neq j \neq k. \tag{3.22}$$

Additionally, it is easy to verify with index notation that

$$\boldsymbol{S} : \boldsymbol{\Omega} = 0, \tag{3.23}$$

where (:) denotes tensor inner-product, which induces the Frobenius norm. But the kinematic alignment between the principal directions of $\boldsymbol{S}$ and the vorticity vector is still an important question in turbulence (Ashurst *et al.*, 1987).

Examples.

1. Consider a pure stretching motion, $\epsilon_1 = \partial_x u \neq 0$ only.

$$\frac{d(\delta x)}{dt} = u = \epsilon_1 \delta x \tag{3.24}$$

hence

$$\epsilon_1 = \frac{1}{\delta x}\frac{d(\delta x)}{dt} \tag{3.25}$$

is the expansion rate (per unit time and per unit length).

2. Consider a pure rotation, $\gamma_1 \neq 0$ only. According to

$$\gamma_1 = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{3.26}$$

and

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \tag{3.27}$$

we have

$$v = \frac{1}{2}\gamma_1 \delta z,\ w = \frac{1}{2}\gamma_1 \delta y \tag{3.28}$$

Define α_{23} as the angle between two line elements that are initially aligned with x_2 and x_3, we have (assumingi infinitesimal angle of rotaion during δt)

$$\frac{d\alpha_{23}}{dt} = \frac{w\delta t/\delta y - v\delta t/\delta z}{\delta t} = (\frac{1}{2}\gamma_1 - (-\frac{1}{2}\gamma_1)) = \gamma_1. \tag{3.29}$$

Hence, γ_i can be understood as the rate of change of two perpendicular elements in the plane normal to $\boldsymbol{e}_i$.

Motivation: during the deformation, can we find a set of axes the mutual angle between each pair is not change? That leads to the eigenvalues/principal strains and principal directions of the strain rate tensor.

3.3 Velocity field decomposition examples

1. Consider a plane constant-rate pure solid body rotation with the position vector being

$$\boldsymbol{r} = \cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y, \tag{3.30}$$

and we have

$$\begin{aligned}
\boldsymbol{u} &= \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} && (3.31)\\
&= \dot\theta(-\sin\theta \boldsymbol{e}_x + \cos\theta \boldsymbol{e}_y) && (3.32)\\
&= \begin{bmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \boldsymbol{e}_x \\ \sin\theta \boldsymbol{e}_y \end{bmatrix} && (3.33)\\
&= \boldsymbol{\Omega} \cdot \boldsymbol{r}. && (3.34)
\end{aligned}$$

With $\boldsymbol{\omega}^* = \dot\theta \boldsymbol{e}_z$ being the angular velocity, we have

$$\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{\Omega} \cdot \boldsymbol{r} = \boldsymbol{\omega}^* \times \boldsymbol{r} \tag{3.35}$$

where $\boldsymbol{\omega}^*$ is the angular velocity.

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix}, \tag{3.36}$$

In general, the decomposition of velocity gradient tensor can be applied to yield:

$$\begin{aligned}\delta \boldsymbol{u} &= \boldsymbol{u}\nabla \cdot \delta \boldsymbol{x} && (3.37)\\ &= \boldsymbol{S}\cdot\delta\boldsymbol{x} + \boldsymbol{\Omega}\cdot\delta\boldsymbol{x} && (3.38)\\ &= \boldsymbol{S}\cdot\delta\boldsymbol{x} + \boldsymbol{\omega}^*\times\delta\boldsymbol{x} && (3.39)\\ &= \nabla\varphi + \boldsymbol{\omega}^*\times\delta\boldsymbol{x} && (3.40)\end{aligned}$$

where the potential is

$$\begin{aligned}\varphi &= \frac{1}{2}(\epsilon_1\delta x^2 + \epsilon_2\delta y^2 + \epsilon_3\delta z^2 + \gamma_1\delta y\delta z + \gamma_2\delta z\delta x + \gamma_3\delta x\delta y) && (3.41)\\ &= \frac{1}{2}\delta\boldsymbol{x}\cdot\boldsymbol{S}\cdot\delta\boldsymbol{x} && (3.42)\end{aligned}$$

such that

$$\nabla\varphi = \boldsymbol{S}\cdot\delta\boldsymbol{x}. \tag{3.43}$$

According to the Helmholtz decomposition theorem, a vector field can be decomposed into the sum of an irrotational field (gradient field/curl-free) and a solenoidal field (curl field/divergence free). Here we provide a construction for the velocity field.

2. Consider plain shear flow with $U(y) = ay$ and

$$\boldsymbol{u}\nabla = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a/2 \\ a/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a/2 \\ -a/2 & 0 \end{bmatrix} = \boldsymbol{S} + \boldsymbol{\Omega}. \tag{3.44}$$

The flow is liearly decomposed into a pure deformation/strain motion with $\partial_y u = \partial_x v = a/2$ and a rotation motion with $\partial_y u = -\partial_x v = a/2$.

3.4 Dynamics of the velocity gradient tensor

3.4.1 Dynamics of A and its powers

Let

$$\boldsymbol{A} = \boldsymbol{u}\nabla = \left[\frac{\partial u_i}{\partial x_j}\right]. \tag{3.45}$$

Taking the gradient of Navier–Stokes we have

$$\frac{\partial \boldsymbol{A}}{\partial t} + u_k\frac{\partial \boldsymbol{A}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} + \nu\frac{\partial^2 \boldsymbol{A}}{\partial x_k^2} - \boldsymbol{A}\cdot\boldsymbol{A} \tag{3.46}$$

Contracting the indices in (3.46) we have

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_k^2} = \mathrm{tr}(\boldsymbol{A}^2) = -u_{i,j}u_{j,i} \tag{3.47}$$

and i.e.,

$$\nabla^2 p = 2\rho Q, \tag{3.48}$$

where

$$Q = \frac{1}{2}[(\mathrm{tr}\,\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)] \tag{3.49}$$

is the second invariant of $\boldsymbol{A}$ (see Appendix A.3). Hence, people argue that the sign of Q implies the existence of local pressure minimum and so forth the existence of vortices Hunt *et al.* (1988); Jeong & Hussain (1995).

The next question is the dynamics of its eigenvalues, invariants, and powers. We immediately deal with the latter since $\boldsymbol{A}^2$ appears in the equation of $\boldsymbol{A}$.

Re-write (3.46) as

$$\mathcal{L}(A_{ik}) = 0,\ \mathcal{L}(A_{kj}) = 0 \tag{3.50}$$

and multiply with A_{kj} and A_{ik} we have

$$\frac{\partial \boldsymbol{A}^2}{\partial t} + u_k \frac{\partial \boldsymbol{A}^2}{\partial x_k} = -\frac{1}{\rho}\left(\boldsymbol{A}\cdot\nabla(\nabla p) + \nabla(\nabla p)\cdot\boldsymbol{A}\right) + \nu\frac{\partial^2 \boldsymbol{A}^2}{\partial x_k^2} - 2\boldsymbol{A}^3. \tag{3.51}$$

Here $\nabla(\nabla p)$ is the pressure Hessian. We can see that, similar to the Reynolds average closure problem, the transport equations for $\boldsymbol{A}$ are never closed due to the appearance of even-higher powers.

Contracting the indices in (3.51) and assuming incompressibility ($\mathrm{tr}\,\boldsymbol{A} = 0$) we have

$$\frac{\partial Q}{\partial t} + u_k \frac{\partial Q}{\partial x_k} = \frac{1}{\rho}(\boldsymbol{A} : \nabla(\nabla p)) + \nu \frac{\partial^2 Q}{\partial x_k^2} + \mathrm{tr}(\boldsymbol{A}^3), \tag{3.52}$$

while we note that $-2Q = A_{ik}A_{ki}$.

3.4.2 Dynamics of S, $\boldsymbol{\Omega}$, and the eigenvalues of $\mathbf{S^2 + \Omega^2}$

$(3.46)^T \cdot \boldsymbol{A} + (3.46) \cdot \boldsymbol{A}^T$ we have

$$\frac{\partial S_{ij}}{\partial t} + u_k \frac{\partial S_{ij}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} + \nu \frac{\partial^2 S_{ij}}{\partial x_k^2} - (S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj}) \tag{3.53}$$

Consider a balance between the last term in the RHS and pressure Hessain,

$$\nabla(\nabla p) = -\rho(\boldsymbol{\Omega}^2 + \boldsymbol{S}^2), \tag{3.54}$$

which can be, similarly to (3.48) (which can also be obtained by contracting (3.53)), be used to search for pressure minimum (Jeong & Hussain, 1995).

Assume $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\boldsymbol{\Omega}^2 + \boldsymbol{S}^2$. Since

$$Q = -\frac{1}{2}\mathrm{tr}(\boldsymbol{A}^2) = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\Omega}^2 + \boldsymbol{S}^2) = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3), \tag{3.55}$$

a positive Q corresponds to $\lambda_1 + \lambda_2 + \lambda_3 < 0$. Then assume $-\lambda_3 > -\lambda_2 > -\lambda_1$, we have $\lambda_3 < 0, \lambda_1 > 0$. The requirement for p to have a (plane) local minimum (where $\nabla p = 0$ and a sub-matrix of $\nabla(\nabla p)$ is positive definite) is that the second eigenvalue $\lambda < 0$. Hence, a negative

λ_2 is a sufficient condition for a pressure minimum that defines a vortex core (Jeong & Hussain, 1995).

On the other hand, the equation for $\boldsymbol{\Omega}$,

$$\frac{\partial \Omega_{ij}}{\partial t} + u_k \frac{\partial \Omega_{ij}}{\partial x_k} = \nu \frac{\partial^2 \Omega_{ij}}{\partial x_k^2} - (S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj}), \tag{3.56}$$

does not involve pressure directly.

3.4.3 Dynamics of the invariant space

In Euler equations (neglecting viscosity), (3.46) is written as

$$\frac{\partial \boldsymbol{A}}{\partial t} + u_k \frac{\partial \boldsymbol{A}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - \boldsymbol{A}\cdot\boldsymbol{A} \tag{3.57}$$

Given

$$-\nabla^2 p = \mathrm{tr}(\boldsymbol{A}) = A_{ik}A_{ki}, \tag{3.58}$$

we have

$$\begin{aligned}
\mathrm{RHS} &= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - A_{ik}A_{kj} && (3.59)\\
&= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - A_{ik}A_{kj} + \frac{1}{3}\left(\frac{\partial^2 p}{\partial x_k x_k} + A_{mk}A_{km}\right)\delta_{ij} && (3.60)\\
&= -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x_i x_j} - \frac{1}{3}\frac{\partial^2 p}{\partial x_k x_k}\delta_{ij}\right) - \left(A_{ik}A_{kj} - \frac{1}{3}A_{mk}A_{km}\delta_{ij}\right) && (3.61)
\end{aligned}$$

Assume the heterogeneous part of the pressure is negligible (at least homogeneous in the small scales), we have

$$\frac{d\boldsymbol{A}}{dt} + \left(A_{ik}A_{kj} - \frac{1}{3}A_{mk}A_{km}\delta_{ij}\right) = 0. \tag{3.62}$$

Multiply (3.62) with A_{ji}, given

$$A_{ji}\frac{dA_{ij}}{dt} = \frac{1}{2}\frac{d}{dt}(A_{ij}A_{ji}) = -\frac{dQ}{dt}, \tag{3.63}$$

and $Q = -1/2\,\mathrm{tr}(\boldsymbol{A}), R = 1/3\,\mathrm{tr}(\boldsymbol{A}^3)$, we have

$$\frac{dQ}{dt} = 3R. \tag{3.64}$$

Multiply (3.62) with $A_{jk}A_{ki}$, given

$$A_{jk}A_{ki}\frac{dA_{ij}}{dt} = \frac{1}{3}\frac{d}{dt}(A_{ij}A_{jk}A_{ki}) = \frac{dR}{dt} \tag{3.65}$$

we have

$$\frac{dR}{dt} + \mathrm{tr}(\boldsymbol{A}^4) - \frac{1}{3}[\mathrm{tr}(\boldsymbol{A})^2]^2 = 0 \tag{3.66}$$

From Appendix A.3 we know (in incompressible flows)

$$P = \operatorname{tr}(\boldsymbol{A}) = 0 \tag{3.67}$$

$$Q = \frac{1}{2}[\operatorname{tr}(\boldsymbol{A})^2 - \operatorname{tr}(\boldsymbol{A}^2)] = -\frac{1}{2}\operatorname{tr}(\boldsymbol{A}^2) \tag{3.68}$$

$$R = \det(\boldsymbol{A}) = \frac{1}{6}\Big(\operatorname{tr}(\boldsymbol{A})^3 - 3\operatorname{tr}(\boldsymbol{A})\operatorname{tr}(\boldsymbol{A}^2) + 2\operatorname{tr}(\boldsymbol{A}^3)\Big) = \frac{1}{3}\operatorname{tr}(\boldsymbol{A}^3) \tag{3.69}$$

where the last relation from the Newton's identity.

According to Cayley–Hamilton theory, matrix $\boldsymbol{A}$ satisfies its characteristic polynomial as

$$\boldsymbol{A}^3 - P\boldsymbol{A}^2 + Q\boldsymbol{A} - R\boldsymbol{I} = 0 \tag{3.70}$$

so

$$\boldsymbol{A}^4 = \boldsymbol{A}\boldsymbol{A}^3 = -Q\boldsymbol{A}^2 + R\boldsymbol{A} \tag{3.71}$$

and

$$\operatorname{tr}(\boldsymbol{A}^4) = -Q\operatorname{tr}(\boldsymbol{A}^2) = \frac{1}{2}\operatorname{tr}(\boldsymbol{A}^2)^2. \tag{3.72}$$

Finally, we have

$$\frac{dR}{dt} = -\frac{2}{3}Q^2. \tag{3.73}$$

Equations (3.64)-(3.73) form a autonomous system, which can be further integrated:

$$dt = \frac{dQ}{3R} = \frac{dR}{-2Q/3} \tag{3.74}$$

$$Q^3 + \frac{27}{4}R^2 = \text{const.} \tag{3.75}$$

We note that $\triangle = Q^3 + 27/4R^2$ is just the discriminant of $\boldsymbol{A}$. The approximate inviscid dynamics is just for the discriminant to conserve along the path. But from (3.62) on the characters of the N–S is less seen.

Ref. Meneveau (2011).

3.5 Lagrangian representations

Cauchy-Green tensor etc.

Chapter 4

Laminar wall flows

When the flow is parallel, there is subsitential simplification can be done to the N–S, allowing analytical solutions.

4.1 Plane Poiseuille–Couette

Consider 2D parallel flow at steady state between two planes at $y = 0, 2h$ (h is the channel half height) and the boundary conditions are

$$u(x, 0) = v(x, 0) = 0, \tag{4.1}$$
$$u(x, 2h) = U,\ v(x, 2h) = 0. \tag{4.2}$$

The 2D N–S equation reads:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.3}$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4.4}$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{4.5}$$

With streamwise invariance, we have $\partial_x u = \partial_x v = 0$. With continuity we also have $\partial_y v = 0$, leading to $v(x, y) = v(x, 0) = 0$. Hence the second derivatives $\partial_{xx} u = \partial_{xx} v = \partial_{yy} v = 0$ as well. The momentum equations are reduced to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{4.6}$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{4.7}$$

At the wall, (4.6) implies

$$\frac{1}{\rho}\frac{\partial p_w}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{4.8}$$

where the RHS is independent of x (fully developed). Combined with (4.7), we have

$$\frac{\partial p}{\partial x} = \frac{\partial p_w}{\partial x} \triangleq \frac{\partial P}{\partial x}, \tag{4.9}$$

i.e., the pressure gradient is a constant. Then we can solve for the velocity with:

$$\nu \frac{d^2 u}{dy^2} = \frac{1}{\rho} \frac{dP}{dx}. \tag{4.10}$$

Given the two boundary conditions (4.2), the general velocity profile for Poiseuille–Couette flows can be written as

$$u(y) = U \frac{y}{2h} - \frac{2h^2}{\rho\nu} \frac{dP}{dx} \left[\frac{y}{2h} - \left(\frac{y}{2h} \right)^2 \right] \tag{4.11}$$

and it reduces to the Poiseuille solution with $U = 0$

$$u(y) = -\frac{2h^2}{\rho\nu} \frac{dP}{dx} \left[\frac{y}{2h} - \left(\frac{y}{2h} \right)^2 \right] \tag{4.12}$$

and the Couette solution with $dP/dx = 0$

$$u(y) = U \frac{y}{2h}. \tag{4.13}$$

We note the both the Poiseuille and Couette, or the Poiseuille–Couette profiles are Re-independent (but the friction coefficients will be Re-dependent) and have no inflection point. The mechanism for turbulence generation has to be transient. For the balance in a turbulent channel see section 5.6.1.

4.2 Laminar Poiseuille channel

Now lets consider the momentum balance in a control volume of $[0, L_x] \times [0, 2h]$. In plane Poiseuille, it can be shown that the pressure gradient equals twice the wall shear stress (simply because there are two walls; a half-channel balance within $[0, L_x] \times [0, h]$ can be established too, with the centerplane being stress-free) as

$$-2h \frac{dP}{dx} = 2\tau_w = 2\mu \frac{\partial u}{\partial y}. \tag{4.14}$$

That being said, the wall shear stress is balanced by the constant pressure gradient. On one side of the wall,

$$-\frac{dP}{dx} = \frac{\tau_w}{h}, \tag{4.15}$$

which along with (4.14), don't depend on the shape of the profile.

The centerline velocity is

$$U_0 = u(h) = -\frac{h^2}{2\rho\nu} \frac{dP}{dx} \tag{4.16}$$

and the bulk velocity is

$$\bar{U} = \frac{1}{2h} \int_0^{2h} u(y)\, dy = \frac{1}{h} \int_0^{h} u(y)\, dy = -\frac{h^2}{3\rho\nu} \frac{dP}{dx} = \frac{2}{3} U_0 \tag{4.17}$$

and the centerline and bulk Reynolds numbers are

$$Re_0 = U_0 h/\nu, \tag{4.18}$$

$$Re = 2\bar{U}h/\nu = 4/3Re_0. \tag{4.19}$$

The friction velocity is

$$u_\tau = \sqrt{\tau_w/\rho} = \sqrt{-\frac{h}{\rho}\frac{dP}{dx}} = \sqrt{\frac{2\nu U_0}{h}} = \sqrt{\frac{3\nu\bar{U}}{h}}, \tag{4.20}$$

and the ratios are

$$\frac{u_\tau}{U_0} = \sqrt{\frac{2}{Re_0}} = \sqrt{\frac{8}{3Re}}, \tag{4.21}$$

$$\frac{u_\tau}{\bar{U}} = \sqrt{\frac{6}{Re}}. \tag{4.22}$$

The critical bulk Reynolds number that the channel can stay laminar is $Re \approxeq 1350$, corresponding to

$$\frac{u_\tau}{U_0} = 0.0444, \tag{4.23}$$

leading to a friction Reynolds number of

$$Re_\tau = \frac{u_\tau h}{\nu} = 45, \tag{4.24}$$

which is the theoretical minimum for turbulent channels. The relation between Re_τ and Re in laminar channels is

$$Re_\tau = \sqrt{\frac{3Re}{2}}. \tag{4.25}$$

The Reynolds numbers in the turbulent channel of Kim *et al.* (1987) are about $Re = 5600$ and $Re_\tau = 180$.

Define the skin friction coefficient as

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} \tag{4.26}$$

it can be shown that

$$c_f = \frac{16}{3Re}, \tag{4.27}$$

which is called the laminar friction law for channels. Equally useful is the skin friction coefficient based on the bulk velocity,

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{U}^2}, \tag{4.28}$$

with $U_0 = 3\bar{U}/2$ and

$$C_f = \frac{12}{Re}. \tag{4.29}$$

4.3 Hagen–Poiseuille (circular pipe) flow

Hagen–Poiseuille flow has been quite important in experiments, due to the fact that a circular pipe is easy to build and the implications on pipelines. In a circular pipe, the laminar Hagen–Poiseuille flow is governed by

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{4.30}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{4.31}$$

and the solution is (a parabola)

$$u(r) = -\frac{1}{4\mu}\frac{dP}{dx}(R^2 - r^2) \tag{4.32}$$

where the constant pressure gradient $dP/dx < 0$ and R is the pipe radius. It can be shown that the centerline velocity is

$$U_0 = u(0) = -\frac{1}{4\mu}\frac{dP}{dx}R^2 \tag{4.33}$$

and the bulk velocity is

$$\bar{U} = \frac{1}{\pi R^2}\int_0^R u(r) 2\pi r \, dr = -\frac{1}{8\mu}\frac{dP}{dx}R^2 = \frac{1}{2}U_0. \tag{4.34}$$

It defines the bulk Reynolds number $Re = \rho\bar{U}D/\mu$. The streamwise pressure gradient is a constant for the same reason as in Poiseuille channels. The volume flow rate

$$Q = \pi R^2 \bar{U} = -\frac{\pi R^4}{8\mu}\frac{dP}{dx} \tag{4.35}$$

is proportional to the pressure gradient.

The shear stress,

$$\tau = -\mu\frac{du}{dr} = -\frac{r}{2}\frac{dP}{dx} \tag{4.36}$$

has a linear profile that vanishes at the centerline and peaks at the wall, which leads to the balance between the wall shear stress and the pressure gradient

$$-\frac{dP}{dx} = \frac{2\tau_w}{R}, \tag{4.37}$$

which can also be obtained from a control volume analysis. The latter is more general as it does not depend on the details of the velocity profile (whether it is laminar or turbulent). We also note that the shear stress is independent on viscosity/density.

The skin friction coefficient is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{U}^2} = \frac{16}{Re}, \tag{4.38}$$

where the bulk Reynolds number is $Re = \rho\bar{u}D/\mu$, and it can be expressed as a form easier to measure in experiments as the (Darcy) friction factor,

$$f = \frac{\Delta p D}{\frac{1}{2}\rho\bar{U}^2 L} = 4C_f = \frac{64}{Re} \tag{4.39}$$

where D is the pipe diameter, L is the pipe length and Δp is the pressure drop in between. Eqn. (4.39) is called the laminar friction law. It also sets that the pressure drop is prortional to the bulk velocity,

$$\Delta p/L \propto \bar{U}, \tag{4.40}$$

where in turbulent flows there is

$$\Delta p/L \propto \bar{U}^{1.75}. \tag{4.41}$$

The corresponding non-dimensional relations, which can alternatively be obtained by Π-group analysis and calibrated/measured from data, are

$$\frac{D\Delta p}{\frac{1}{2}\rho\bar{U}^2 L} = \phi\left(\frac{\rho\bar{U}D}{\mu}\right) = \frac{64}{Re} \tag{4.42}$$

and

$$\frac{\rho D^3\Delta p}{L\mu^2} = \phi\left(\frac{\rho\bar{U}D}{\mu}\right) \approx 0.155\left(\frac{\rho\bar{U}D}{\mu}\right)^{1.75} \tag{4.43}$$

for laminar and turbulent pipes. We notice that the pressure drop (and hence wall shear stress) depends weakly on μ in turbulent pipes ($\Delta p \sim \mu^{0.25}$) than in laminar pipes ($\Delta p \sim \mu$). The relation (4.43) was first discovered by Blasius by fitting experimental data with insights from non-dimensional groups.

The friction Reynolds number is related to the bulk Reynolds number in laminar pipes as

$$Re_\tau = \frac{\rho u_\tau R}{\mu} = \sqrt{2Re} \tag{4.44}$$

and the critical Reynolds number above which the flow cannot remain laminar is around $Re_{\mathrm{cri}} \approx 2300$ White (2008), corresponding to $Re_\tau \approx 68$.

4.4 The boundary layer theory

The boundary layer theory played central roles in the development of 20th century's fluid mechanics and aerodyanmics. Consider a uniform freestream flowing over a flat plat. The effect of viscosity, reflected by the no-slip boundary condition, is confined in a thin layer near the wall that scales with $\sqrt{\nu x/U_0}$ and the external flow is potential. This decomposition of flow regions enables separation of treatments of the lift, which can be computed from the potential theory, and of the drag, which can be obtained from another set of viscous-dependent and outer-flow-dependent boundary-layer equations. The effect of the boundary-layer solution to the potential flow can be approximated as a 'displacement' of the streamlines and the potential solution can be corrected accordingly. The boundary-layer solution, known as the Blasius solution, is self-similar. Similar self-similar solutions are found in corner flow (Falkner–Skan) and plane jets, wakes, and mixing layers, whose governing equations are of boundary-layer type.

4.5 Boundary layer equations

Here the freestreem pressure and velocity are $p_0(x), U_0(x)$ and they are related by the Bernoulli theorem (since the outer flow is inviscid), $p_0 + 1/2\rho U_0^2 = $ constant, hence

$$-\frac{dp_0}{dx} = \rho U_0 \frac{dU_0}{dx} \tag{4.45}$$

and the boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.46}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_0}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{4.47}$$

$$= U_0\frac{dU_0}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{4.48}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4.49}$$

which can be easily reached by considering the relevant scales (see Schlichting & Gersten (2016) if needed). And we note that since there is no vertical pressure gradient, the streamwise pressure gradient inside the boundary layer is the same as that of the freestreem (determined by the Bernoulli equations). The viscous term is written as the divergence of the shear stress which takes

$$\tau = \mu\frac{\partial u}{\partial y} \tag{4.50}$$

in laminar flows and

$$\tau = \mu\frac{\partial u}{\partial y} - \rho\overline{u'v'} \tag{4.51}$$

in turbulent flows.

4.5.1 von Kármán momentum integral

The von Kármán integrals provides an approach to estimate critical parameters of the boundary layer, such as the boundary layer thickness, the wall shear stress, and the shape factor, without needing to have the solution analytically. In fact, even some velicity profiles that are quite approximate can lead to results close to the Blasius solution.

$(u - U_0)\times$ (4.46) + (4.48) we have

$$\frac{\partial}{\partial x}[u(u - U_0)] + \frac{\partial}{\partial y}[v(u - U_0)] = (U_0 - u)\frac{dU_0}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}. \tag{4.52}$$

Integrate in y from 0 to ∞ we have

$$\frac{\partial}{\partial x}\left[\int_0^\infty u(U_0 - u)\,dy\right] + \frac{dU_0}{dx}\int_0^\infty (U_0 - u)\,dy = \frac{\tau_w}{\rho} \tag{4.53}$$

So far, we have not assumed whether (u, v) are the laminar velocities or the turbulent mean velocities. The nominal, displacement, and the momentum thicknesses are defined as

$$u(\delta_{99}) = 0.99U_0 \tag{4.54}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_0}\right) dy \tag{4.55}$$

$$\theta = \int_0^\infty \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy \tag{4.56}$$

and (4.53) can be turned into the von Kármán momentum integral

$$\frac{1}{2}c_f = \frac{\tau_w}{\rho U_0^2} = \frac{d\theta}{dx} + \frac{(\delta^* + 2\theta)}{U_0}\frac{dU_0}{dx} \tag{4.57}$$

$$= \frac{d\theta}{dx} + \frac{(H+2)\theta}{U_0}\frac{dU_0}{dx} \tag{4.58}$$

where $H = \delta^*/\theta$ is the shape factor and is around 2.6 and 1.3 for laminar and turbulent boundary layers, respectively. By assuming velocity profiles $u/U_0 = f(y\delta_{99})$, the above parameters can be solved.

Historically, von Kármán (1921) performed the momentum integral with a control-volume analysis of the boundary layer (mass and momentum balances). Assume a ZPGBL, such that the freestreem is a constant. Select a control volume of $(0,0), (0,h), (x,\delta), (x,0)$ where a streamline passes $(0,h)$ and (x,δ) (no mass penetrating the streamlines). The length of the plate into the paper is b.

The momentum conservation is

$$-D = \int_0^h b\rho U_0^2\, dy - \int_0^\delta b\rho u^2(y)\ dy \tag{4.59}$$

so

$$D = \rho b U_0^2 h - \rho b \int_0^\delta u^2(y)\, dy. \tag{4.60}$$

The mass conservation is (since the curve passing $(0,h)$ and (x,δ) is a streamline)

$$\int_0^h b\rho U_0\, dy = \int_0^\delta b\rho u(y)\ dy \tag{4.61}$$

which is used to simplify (4.60) to

$$D = \rho b U_0^2 \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy = \rho b U_0^2 \theta, \tag{4.62}$$

or in dimensionless form

$$C_D = \frac{D}{\frac{1}{2}\rho U_0^2 bL} = \frac{2\theta(L)}{L}. \tag{4.63}$$

We can see that the momentum integral relates the force on the solid body to the momentum loss in the boundary layer. What is also shown is that the momentum thickness $\theta(x)$ is a measure of the total drag up to x. Such ideas and definition of momentum thickness are also applicable in wakes, where the drag can be computed the same way using momentum integrals. The difference is that in wakes, θ stays as a constant past the body.

Note that the BL starts at $(0,0)$ so the total dray until $(x,0)$ is

$$D = \int_0^x b\tau_w\, dx \tag{4.64}$$

(τ_w is force per unit area) and hence

$$\tau_w = \frac{1}{b}\frac{dD}{dx} = \rho U_0^2 \frac{d\theta}{dx} \tag{4.65}$$

and the friction coefficient is

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} = 2\frac{d\theta}{dx}. \tag{4.66}$$

This is the same as (4.58) without external pressure gradients. Its implication is that, the momentum thickness represents the loss of momentum (deficit) due to the drag/shear stress acting on the bottom wall.

The equations (4.55)-(4.61) can also be interpreted as

$$U_0\delta^* = \int_0^\delta (U_0 - u)\,dy = U_0\delta - \int_0^\delta u\,dy \tag{4.67}$$

$$U_0 h = \int_0^h U_0\,dy = \int_0^\delta u\,dy = U_0(\delta - \delta^*) \tag{4.68}$$

$$\delta_* = \int_0^h \left(1 - \frac{u}{U_0}\right) dy = \delta - h \tag{4.69}$$

Hence, δ^* is interpreted as the displacement of the stremelines (or the bulk of fluid with velocity U_0 and height h) by the boundary layer. The last equality in (4.68) can be re-written as

$$\int_0^H u\,dy = U_0(H - \delta^*) \tag{4.70}$$

where H is an arbitrary height above δ_{99}. Consider two locations x_1, x_2 and a control volume in between with height H. The mass conservation

$$\int_0^H u_1\,dy = \int_0^H u_2\,dy + V_0(x)(x_2 - x_1) \tag{4.71}$$

where $H > \delta_1^*, \delta_2^*$ and V_0 is the vertical velocity of the external flow. Taking (4.70), it can be converted to

$$U_0(H - \delta_1^*) = U(H - \delta_2^*) + V_0(x_2 - x_1), \tag{4.72}$$

that being said,

$$\frac{V_0}{U_0} = \frac{d\delta^*}{dx}, \tag{4.73}$$

the rate at which the displacement thickness grows measures the displacement velocity.

It is now possible to consider the flow near the 'nominal' edge of the boundary layer. The line of $\delta_{99}(x)$ follows $\sqrt{x}$ according to the Blasius solution (see the next section) and its direction is $(1, d\delta_{99}/dx)$. At the edge of the boundary layer (where δ_{99} is), the direction of the velocity is $\boldsymbol{U}_0 = (U_0, V_0) = U_0(1, d\delta^*/dx)$. It can be seen from the Blasius solution that at each location (x), $\delta_{99} \approx 3\delta^*$. That being said, there is flow normal to $y = \delta_{99}(x)$ into the 'boundary layer', which is not a streamline!

Similar to (4.68) that

$$\int_0^\delta u\,dy = U_0(\delta - \delta^*), \tag{4.74}$$

we can also write down

$$\int_0^\delta u^2\,dy = U_0^2(\delta - \delta^* - \theta), \tag{4.75}$$

which interprets θ as the extra displacement (in addition to δ^*) so compensate for the momentum flux so that it matches that of the potential flow. The (x-)momentum conservation in again the control volume of $[x_1, x_2] \times [0, H]$ gives

$$\rho b \int_0^H u_1^2\, dy = \rho b \int_0^H u_2^2\, dy + \rho b \int_{x_1}^{x_2} V_0 U_0\, dx + F_D \tag{4.76}$$

where $F_D = D_2 - D_1$ is the drag force between x_1, x_2 while D counts from the leading edge. With

$$\int_{x_1}^{x_2} V_0 dx = U_0(\delta_2^* - \delta_1^*), \tag{4.77}$$

we have

$$U_0(H - \delta_1^* - \theta_1) = U_0^2(H - \delta_2^* - \theta_2) + U_0^2(\delta_2^* - \delta_1^*) + \frac{F_D}{\rho b} \tag{4.78}$$

and

$$\theta_2 - \theta_1 = \frac{F_D}{\rho b U_0^2}. \tag{4.79}$$

That being said, the change of momentum thickness reflects the drag acting on the fluid during its travel. This is an equivalant statement to (4.65).

NB. The control volume selection of $[x_1, x_2] \times [0, H]$ instead of a CV starts at the origin bypasses the issues with the transient period and takes advantages of the self-similar solution being estabilished and the intial conditions being forgotton.

4.5.2 The Blasius similarity solution

The references are Schlichting & Gersten (2016); Kundu *et al.* (2015); White (2008), with the definition of $\delta(x)$ different by a factor of $\sqrt{2}$. Here we will follow the notations in Kundu *et al.* (2015); White (2008).

The laminar zero-pressure-gradient boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.80}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{4.81}$$

The idea of self-similar solutions is that the velocity profile $u(y)$ at different x will be the same under some proper transformation/normalization of u and y. The scale for u is apparently U_∞, while the scale for y is δ. From the viscous scaling $v \sim \nu/\delta$ ($\delta \sim \sqrt{\nu T}, T \sim x/U_\infty$) and the scaling of the continuity equation $v/\delta \sim U_\infty/x$ we have

$$\delta^2 \sim \nu T \sim \frac{\nu x}{U_\infty} \tag{4.82}$$

and we accodringly define the boundary layer length scale as

$$\delta_\nu(x) = \sqrt{\frac{\nu x}{U_\infty}} \tag{4.83}$$

such that the similarity transformation is

$$\eta = \frac{y}{\delta_\nu(x)} = y\sqrt{\frac{U_\infty}{\nu x}} \tag{4.84}$$

and

$$\frac{u}{U_\infty} = f(\eta), \tag{4.85}$$

where $f(\eta)$ is the similarity function and η is the similarity coordinate. And we note

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{\eta}{x} \tag{4.86}$$

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}} \tag{4.87}$$

We note that the streamfunction ψ depends on ν, U_∞, x, y and dimensionally

$$\psi(x,y) = U_\infty \delta_\nu(x) f(\eta) = \sqrt{\nu U_\infty x} f(\eta) \tag{4.88}$$

and hence

$$u = \frac{\partial \psi}{\partial y} = U_\infty f' \tag{4.89}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(\eta f' - f) \tag{4.90}$$

The derivatives are

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2x} f''\eta \tag{4.91}$$

$$\frac{\partial u}{\partial y} = U_\infty f''\sqrt{\frac{U_\infty}{\nu x}} \tag{4.92}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} f''' \tag{4.93}$$

and then

$$u\frac{\partial u}{\partial x} = -\frac{U_\infty^2}{2x} f' f''\eta \tag{4.94}$$

$$v\frac{\partial u}{\partial y} = \frac{U_\infty^2}{2x} f''(\eta f' - f) \tag{4.95}$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{x} f''' \tag{4.96}$$

and finally we have the ODE

$$f''' + \frac{1}{2} f f'' = 0 \tag{4.97}$$

with the boundary conditions being

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1, \tag{4.98}$$

corresponding to

$$v(y=0)=0,\ u(y=0)=0,\ u(y=\infty)=U_\infty. \tag{4.99}$$

Note that $v(y=0)=0$ requires $(\eta f' - f)|_{y=0}=0$, meaning $f(0)=0$.

We also note that in the convention of Schlichting & Gersten (2016), $\delta_\nu = \sqrt{2\nu x/U_\infty}$ and the Blasius equation reads

$$f''' + ff'' = 0, \tag{4.100}$$

with the boundary conditions being the same. The choice of $\sqrt{2}$ is arbitrary and only for equation-simplification reasons.

It is common to use a Runge-Kutta shooting method to solve (4.98). The solution is called the Blasius solution, which is 'almost' an exact solution since you can know the functional value at any point to any precision that you want. The resulting relations are (approximately)

$$\frac{\delta_{99}}{x} = \frac{4.91}{\sqrt{Re_x}} \tag{4.101}$$

$$\frac{\delta^*}{x} = \frac{1.721}{\sqrt{Re_x}} \tag{4.102}$$

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}} \tag{4.103}$$

$$Re_\theta = 0.664\sqrt{Re_x} \tag{4.104}$$

$$H = \frac{\delta^*}{\theta} = 2.59 \tag{4.105}$$

$$\tau_w = \rho U_0^2 \frac{d\theta}{dx} = \frac{0.332\rho U_0^2}{\sqrt{Re_x}} \tag{4.106}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{0.664^2}{Re_\theta} \tag{4.107}$$

$$C_{Df} = \frac{D}{\frac{1}{2}\rho U_0^2 bL} = \frac{\int_0^L b\tau_w\, dx}{\frac{1}{2}\rho U_0^2 bL} = \frac{1.328}{\sqrt{Re_L}} \tag{4.108}$$

We can see that $\theta < \delta^* < \delta_{99}$.

4.5.3 Falkner–Skan Equations

The Falkner–Skan solution to the boundary-layer corner flow is another class of exact solutions of Blasius-kind. The external flow is $U_e = ax^n$ and the streamfunction and similarity variable are

$$\psi(x,y) = \sqrt{\nu U_e(x) x} f(\eta) = \sqrt{\nu a} x^{\frac{n+1}{2}} f(\eta) \tag{4.109}$$

$$\eta = y\sqrt{\frac{U_e}{\nu x}} = \sqrt{\frac{a}{\nu}} y x^{\frac{n-1}{2}} \tag{4.110}$$

The velocities are

$$u = \frac{\partial \psi}{\partial y} = ax^n f' \tag{4.111}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\sqrt{\nu a}}{2} x^{\frac{n-1}{2}} [(n+1)f + (n-1)\eta f']. \tag{4.112}$$

We check that when $n = 0$ (Blasius BL), $v = 1/2\sqrt{U_e \nu / x}(\eta f' - f)$ is what we are familiar with.

Let's do some preparation work,

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{a}{\nu}} x^{\frac{n-1}{2}} \tag{4.113}$$

$$\frac{\partial \eta}{\partial x} = \frac{n-1}{2} \frac{\eta}{x}, \tag{4.114}$$

and

$$\frac{\partial u}{\partial x} = nax^{n-1} f' + ax^n f'' \frac{n-1}{2} \frac{\eta}{x} \tag{4.115}$$

$$\frac{\partial u}{\partial y} = ax^n f'' \sqrt{\frac{a}{\nu}} x^{\frac{n-1}{2}} \tag{4.116}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{a^2}{\nu} x^{2n-1} f''' \tag{4.117}$$

$$U_e \frac{dU_e}{dx} = na^2 x^{2n-1}. \tag{4.118}$$

The x-momentum equation

$$uu_x + vu_y = U_e U_{e,x} + \nu u_{yy} \tag{4.119}$$

then converts to

$$na^2 x^{2n-1} f' f' - \frac{n+1}{2} a^2 x^{2n-1} f f'' = na^2 x^{2n-1} + a^2 x^{2n-1} f''' \tag{4.120}$$

which is the Falkner–Skan equation

$$f''' + \frac{n+1}{2} f f'' - n f'^2 + n = 0. \tag{4.121}$$

The BCs are

$$f(0) = 0;\ f'(0) = 0;\ f(+\infty) = 1. \tag{4.122}$$

When $n = 0$ it reduces to the Blasius equation $f''' + 1/2 f f'' = 0$.

4.5.4 von Kármán swirling disk flow

Also shooting method.

4.5.5 Laminar planar mixing layer

Governing equation (BL equations)

4.5.6 Laminar planar jet and wake

Momentum integrals. Momentum thickness. Lift coefficients.

4.5.7 Lift and drag

Consider forces on a 2D cylinder in a uniform freestream. The angle $0 < \theta < 2\pi$ counts clockwise from the front stagnation point. On each surface element at an angle of θ with the horizon, there is a wall shear stress τ_w tangential to the surface and a pressure p normal to it. The lift and drag are

$$D = F_x = \int p \cos\theta \, dA + \int \tau_w \sin\theta \, dA \tag{4.123}$$

$$L = F_y = -\int p \sin\theta \, dA + \int \tau_w \cos\theta \, dA \tag{4.124}$$

where the drag can be decomposed into two parts, the friction drag and the pressure (form) drag

$$D_f = \int \tau_w \sin\theta \, dA, \; D_p = \int p \cos\theta \, dA. \tag{4.125}$$

Non-dimensionally,

$$C_D = \frac{D}{\frac{1}{2}\rho U_0^2 A} = \frac{D_f + D_p}{\frac{1}{2}\rho U_0^2 A} = C_{Df} + C_{Dp} \tag{4.126}$$

where

$$C_{Dp} = \frac{D_p}{\frac{1}{2}\rho U_0^2 A} = \frac{\int p \cos\theta \, dA}{\frac{1}{2}\rho U_0^2 A} = \frac{\int \frac{p - p_0}{1/2\rho U_0^2} \cos\theta \, dA}{A} = \frac{\int C_p \cos\theta \, dA}{A} \tag{4.127}$$

where $C_p = (p - p_0)/(1/2\rho U_0^2)$ is the pressure coefficient.

Chapter 5

Turbulent flows

5.1 Mean flow and fluctuations

5.1.1 Reynolds average

We denote time average as $\overline{(\cdot)}$, space or ensemble average as $\langle\cdot\rangle$, and sometimes use these notations interchangeably given that they are equivalent under the ergodicity assumption. The properties proved for one definition are expected to hold for another. Although Reynolds decomposition and RANS modelings are not an accurate way of computing turbulence, they consist the foundation of our understanding of turbulence.

Below we give briefly some properties of Reynolds averaging:

(i) (Definition) The time average of a physical variable A is

$$\overline{A} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T A\,\mathrm{d}t \tag{5.1}$$

In practice, the limit is often neglected and the average window is assumed to be long enough.

(ii) (Definition) The fluctuation of a physical variable A is

$$A' \triangleq A - \overline{A} \tag{5.2}$$

(iii) (Proposition) The average of fluctuation is zero.

$$\overline{A'} = \overline{A - \overline{A}} = \overline{A} - \overline{A} = 0 \tag{5.3}$$

5.1.2 Continuity and momentum

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{5.4}$$

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i}{\partial x_j}) \tag{5.5}$$

Taking the average of Eq. (5.4) we have

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} = 0 \tag{5.6}$$

where

$$\frac{\partial \overline{u}_i}{\partial x_i} = \frac{\partial}{\partial x_i}(\frac{1}{T}\int_0^T u_i \,\mathrm{d}t) = \frac{1}{T}\int_0^T (\frac{\partial u_i}{\partial x_i})\,\mathrm{d}t = \frac{1}{T}\int_0^T 0 \,\mathrm{d}t = 0 \tag{5.7}$$

Hence we have the continuity for fluctuating velocity

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{5.8}$$

Taking the average of Eq. (5.5) we have

$$\mathrm{LHS} = (\frac{1}{T}\int_0^T \mathrm{d}t) * [\frac{\partial}{\partial t}(\overline{u}_i + u_i') + (\overline{u}_j + u_j')\frac{\partial}{\partial x_j}(\bar{u}_i + u_i')] \tag{5.9}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{\frac{\partial u_i'}{\partial t}} + \overline{\overline{u}_j \frac{\partial}{\partial x_j}\overline{u}_i} + \overline{\overline{u}_j \frac{\partial}{\partial x_j} u_i'} + \overline{u_j' \frac{\partial}{\partial x_j}\overline{u}_i} + \overline{u_j' \frac{\partial}{\partial x_j} u_i'} \tag{5.10}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\overline{u}_i + \overline{u_j' \frac{\partial}{\partial x_j} u_i'} \tag{5.11}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}\overline{u}_i + \overline{\frac{\partial}{\partial x_j}(u_j' u_i')} - \overline{u_i' \frac{\partial u_j'}{\partial x_j}} = \frac{\partial}{\partial x_j}\overline{u_j' u_i'} \tag{5.12}$$

$$= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u_j' u_i'} \tag{5.13}$$

$$\mathrm{RHS} = (\frac{1}{T}\int_0^T \mathrm{d}t)[-\frac{1}{\rho}\frac{\partial}{\partial x_i}(\overline{p} + p') + \frac{\partial}{\partial x_j}[\nu \frac{\partial}{\partial x_j}(\overline{u}_i + u_i')] \tag{5.14}$$

$$= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \overline{u}_i}{\partial x_j}) \tag{5.15}$$

Equaling both sides yields:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{5.16}$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i' u_j'}) \tag{5.17}$$

where the cross-correlation term having dimension of shear stress

$$\tau_{\mathrm{Rey}} = -\overline{u_i' u_j'} \tag{5.18}$$

is called the Reynolds stress term. It is a rank 2 tensor. It comes from the Reynolds averaging of the non-linear advection term on the LHS of Navier–Stokes, and it distinguishes turbulent flows from laminar ones. It represents the momentum transport due to turbulent motions, in anology to the molecular diffusion.

5.1.3 Transport equation of the fluctuating velocity

Denote the material derivative based on the mean flow advection as

$$\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}_k \frac{\partial}{\partial x_k} \tag{5.19}$$

and subtract the Reynolds equation from N-S equation

$$\frac{\bar{D}u_i'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u_i'}{\partial x_j} + \overline{u_i'u_j'} - u_i'u_j') - u_j'\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.20}$$

The last term shows the mean-flow stretching of the fluctuation, which is a generation mechanism be shown later related to the shear production of turbulent kinetic energy. In the latter section of 8 we will see the same form of perturbation equations will be used for hydrodynamic stability analysis.

5.1.4 Mean-flow and turbulent kinetic energy

The total kinetic energy of the flow can be devided into the mean kinetic energy (MKE) and the turbulent kinetic energy (TKE)

$$\begin{aligned}
K_{\text{tot}} &= \frac{1}{2}\overline{u_i u_i} && (5.21)\\
&= \overline{\frac{1}{2}(\overline{u}_i + u_i')(\overline{u}_i + u_i')} && (5.22)\\
&= \overline{\frac{1}{2}\overline{u}_i\overline{u}_i + \overline{u}_i u_i' + \frac{1}{2}u_i'u_i'} && (5.23)\\
&= \frac{1}{2}\overline{u}_i\overline{u}_i + \frac{1}{2}\overline{u_i'u_i'} && (5.24)\\
&= K + k && (5.25)
\end{aligned}$$

We will show how these two parts are related dynamically.

5.1.5 MKE equation

Multiply the Reynolds equation (5.17) by $\overline{u}_i$ we have

$$\text{LHS} = \overline{u}_i\frac{\bar{D}\overline{u}_i}{Dt} = \frac{\bar{D}K}{Dt} \tag{5.26}$$

$$\begin{aligned}
\overline{u}_i(-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}) &= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_i}{\partial x_i} + \frac{1}{\rho}\frac{\partial \overline{u}_i}{\partial x_i} && (5.27)\\
&= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_i}{\partial x_i} && (5.28)\\
&= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_j}{\partial x_j} && (5.29)
\end{aligned}$$

$$\overline{u}_i\frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}) = \frac{\partial}{\partial x_j}[\overline{u}_i(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'})] - \frac{\partial \overline{u}_i}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}) \tag{5.30}$$

$$= -\nu\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_i}{\partial x_j}\overline{u_i'u_j'} + \frac{\partial}{\partial x_j}(\nu\frac{\partial K}{\partial x_j}) - \frac{\partial \overline{u}_i\,\overline{u_i'u_j'}}{\partial x_j} \tag{5.31}$$

Equaling both sides we have

$$\frac{\bar{D}K}{Dt} = \frac{\partial}{\partial x_j}(\underbrace{-\frac{1}{\rho}\overline{p}\,\overline{u}_j}_{\substack{\text{pressure}\\\text{distortion}}}\underbrace{+\nu\frac{\partial K}{\partial x_j}}_{\substack{\text{molecular}\\\text{diffusion}}}\underbrace{-\overline{u}_i\,\overline{u_i'u_j'}}_{\substack{\text{turbulent}\\\text{diffusion}}})\underbrace{-\frac{1}{2}P_{kk}}_{\substack{\text{production}\\\text{of TKE}}}\underbrace{-\nu\left(\frac{\partial \overline{u}_i}{\partial x_j}\right)^2}_{\text{dissipation}} \tag{5.32}$$

where the term

$$P_{kk} = -2\overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.33}$$

is the production term of the turbulent kinetic energy, and, on the other hand, is the sink in MKE.

5.1.6 TKE equation

Similarly, multiply (5.20) by u_i' and then take the average

$$\text{LHS} = \overline{u_i'(\frac{\partial u_i'}{\partial t} + \overline{u_k}\frac{\partial u_i'}{\partial x_k})} \tag{5.34}$$

$$= \frac{\partial \frac{1}{2}\overline{u_i'u_i'}}{\partial t} + \overline{u_k}\frac{\partial \frac{1}{2}\overline{u_i'u_i'}}{\partial x_k} \tag{5.35}$$

$$= \frac{\bar{D}k}{Dt} \tag{5.36}$$

$$\overline{u_i'(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{1}{\rho}\frac{\partial\overline{p'u_i'}}{\partial x_i'} \tag{5.37}$$

$$= -\frac{1}{\rho}\frac{\partial\overline{p'u_k'}}{\partial x_k'} \tag{5.38}$$

$$\overline{u_i'(\frac{\partial}{\partial x_k}\nu\frac{\partial u_i'}{\partial x_k})} = \overline{\frac{\partial}{\partial x_k}(\nu u_i'\frac{\partial u_i'}{\partial x_k}) - \nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} \tag{5.39}$$

$$= \overline{\frac{\partial}{\partial x_k}(\nu\frac{\partial \frac{1}{2}u_i'u_i'}{\partial x_k}) - \nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} \tag{5.40}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial k}{\partial x_k}) - \nu\overline{(\frac{\partial u_i'}{\partial x_k})^2} \tag{5.41}$$

$$\overline{u_i'(\frac{\partial}{\partial x_k}\overline{u_i'u_k'})} = 0 \tag{5.42}$$

$$-\overline{u_i'(\frac{\partial}{\partial x_k}u_i'u_k')} = -\frac{1}{2}\frac{\partial\overline{u_i'u_i'u_k'}}{\partial x_k} \tag{5.43}$$

$$= -\frac{1}{2}\frac{\partial\overline{u_i'u_i'u_k'}}{\partial x_k} \tag{5.44}$$

$$\overline{u_i'(-u_k'\frac{\partial \overline{u}_i}{\partial x_k})} = -\overline{u_i'u_k'}\frac{\partial \overline{u}_i}{\partial x_k} \tag{5.45}$$

Equaling both sides we have

$$\frac{\bar{D}k}{Dt} = \frac{\partial}{\partial x_k}(\underbrace{\nu\frac{\partial k}{\partial x_k}}_{\substack{\text{molecular}\\ \text{diffusion}}} - \underbrace{\frac{1}{2}\overline{u_i'u_i'u_k'}}_{\substack{\text{turbulent}\\ \text{diffusion}}} - \underbrace{\frac{1}{\rho}\overline{p'u_k'}}_{\substack{\text{pressure}\\ \text{distortion}}}) + \underbrace{\frac{1}{2}P_{kk}}_{\substack{\text{production}\\ \text{of TKE}}} - \underbrace{\nu\overline{\left(\frac{\partial u_i'}{\partial x_k}\right)\left(\frac{\partial u_i'}{\partial x_k}\right)}}_{\text{(pseudo-) dissipation}} \tag{5.46}$$

Comments:

(1) The turbulent kinetic energy production term

$$P = \frac{1}{2}P_{kk} = -\overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.47}$$

can be expressed in tensor notation as

$$P = \boldsymbol{\tau}_{\text{Rey}} : \nabla\overline{\boldsymbol{u}} = \boldsymbol{\tau}_{\text{Rey}} : \bar{\boldsymbol{S}} \tag{5.48}$$

where the inner product represents the projection of the velocity fluctuation correlation on the mean shear/strain rate. The convention of $P_{kk}/2$ is to be consistent with the notations in (5.72).

(2) The (pseudo-) dissipation term

$$\tilde{\varepsilon} = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial u_i'}{\partial x_j}\right)} = \nu\overline{S_{ij}'S_{ij}'} + \nu\overline{\Omega_{ij}'\Omega_{ij}'} \tag{5.49}$$

is always positive, representing the dissipation mechanism of turbulence kinetic energy. We can also see that (perturbation) enstrophy is directly linked to the dissipation rate of TKE/total KE. We note that the relations

$$\boldsymbol{S} : \boldsymbol{\Omega} = 0 \tag{5.50}$$

and

$$\nabla\boldsymbol{u} : \nabla\boldsymbol{u} = \boldsymbol{S} : \boldsymbol{S} + \boldsymbol{\Omega} : \boldsymbol{\Omega} \tag{5.51}$$

also carries for the perturbation quantities. Also note $\|\boldsymbol{\Omega}\|^2 = \|\boldsymbol{\omega}\|^2/2$ – the perturbation enstrophy is largely related to turbulent dissipation.

(3) Note that eqn. (5.46) can alternatively be written as

$$\frac{\bar{D}k}{Dt} = \frac{\partial}{\partial x_k}(\underbrace{2\nu\overline{u_j's_{ij}'}}_{\substack{\text{molecular}\\ \text{diffusion}}} - \underbrace{\frac{1}{2}\overline{u_i'u_i'u_k'}}_{\substack{\text{turbulent}\\ \text{diffusion}}} - \underbrace{\frac{1}{\rho}\overline{p'u_k'}}_{\substack{\text{pressure}\\ \text{distortion}}}) + \underbrace{\frac{1}{2}P_{kk}}_{\substack{\text{production}\\ \text{of TKE}}} - \underbrace{2\nu\overline{s_{ij}'s_{ij}'}}_{\text{dissipation}} \tag{5.52}$$

with the viscous term in the original equation being $\tau_{ij,j} = 2\nu s_{ij,j}$ instead of a Laplacian, and the relation between dissipation and pseudo-dissipation being

$$\varepsilon = \tilde{\varepsilon} + \nu\frac{\partial^2\overline{u_i'u_j'}}{\partial x_i\partial x_j}, \tag{5.53}$$

which is small at vanishing nu or can be absorbed into the transport term otherwise.

5.1.7 Reynolds stress transport equation

The velocity fluctuation transport equation is

$$\frac{\bar{D}u_i'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i'}{\partial x_j} + \overline{u_i'u_j'} - u_i'u_j') - u_j'\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.54}$$

Or if we exchange the two subscripts we obtain:

$$\frac{\bar{D}u_j'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \frac{\partial}{\partial x_i}(\nu\frac{\partial u_j'}{\partial x_i} + \overline{u_i'u_j'} - u_i'u_j') - u_i'\frac{\partial \overline{u}_j}{\partial x_i} \tag{5.55}$$

$u_j' \times (19) + u_i' \times (20)$ and take the time average:

$$\text{LHS} = \frac{\bar{D}\overline{u_i'u_j'}}{Dt} \tag{5.56}$$

$$\text{RHS}_1 = -\frac{1}{\rho}[-2\overline{p's_{ij}} + \frac{\partial}{\partial x_i}(\overline{p'u_j'}) + \frac{\partial}{\partial x_j}(\overline{p'u_i'})] \tag{5.57}$$

$$\text{RHS}_2 = \overline{u_j'\frac{\partial}{\partial x_k}(\nu\frac{\partial u_i'}{\partial x_k}) + u_i'\frac{\partial}{\partial x_k}(\nu\frac{\partial u_j'}{\partial x_k})} \tag{5.58}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u_i'u_j'}}{\partial x_k}) - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{5.59}$$

$$\text{RHS}_3 = \overline{u_j'\frac{\partial}{\partial x_k}\overline{u_i'u_k'} + u_i'\frac{\partial}{\partial x_k}\overline{u_j'u_k'}} \tag{5.60}$$

$$= 0 \tag{5.61}$$

$$\text{RHS}_4 = -\overline{u_j'\frac{\partial}{\partial x_k}(u_i'u_k') + u_i'\frac{\partial}{\partial x_k}(u_j'u_k')} \tag{5.62}$$

$$= -\overline{u_j'u_k'\frac{\partial}{\partial x_k}(u_i') + u_i'u_k'\frac{\partial}{\partial x_k}(u_j') + u_i'u_j'\frac{\partial}{\partial x_k}(u_k')} \tag{5.63}$$

$$(Continuity, \frac{\partial u_k'}{\partial x_k} = 0, is\,used\,twice\,here.) \tag{5.64}$$

$$= -\frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'} \tag{5.65}$$

$$\text{RHS}_5 = \overline{-u_k'u_j'\frac{\partial \overline{u}_i}{\partial x_k} - u_k'u_i'\frac{\partial \overline{u}_j}{\partial x_k}} \tag{5.66}$$

$$= -\overline{u_k'u_j'}\frac{\partial \overline{u}_i}{\partial x_k} - \overline{u_k'u_i'}\frac{\partial \overline{u}_j}{\partial x_k} \tag{5.67}$$

$$\tag{5.68}$$

By equalizing both sides we obtain

$$\frac{\bar{D}\overline{u_i'u_j'}}{Dt} = \frac{2}{\rho}\overline{p's_{ij}} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p'u_j'})\delta_{ik} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p'u_i'})\delta_{jk} + \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u_i'u_j'}}{\partial x_k}) \tag{5.69}$$

$$- 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} - \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'} - \overline{u_k'u_j'}\frac{\partial \overline{u}_i}{\partial x_k} - \overline{u_k'u_i'}\frac{\partial \overline{u}_j}{\partial x_k} \tag{5.70}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial \overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'u_j'u_k'} - \frac{1}{\rho}\overline{p'u_i'}\delta_{jk} - \frac{1}{\rho}\overline{p'u_j'}\delta_{ik}) \tag{5.71}$$

$$- (\overline{u_k'u_j'}\frac{\partial \overline{u}_i}{\partial x_k} + \overline{u_k'u_i'}\frac{\partial \overline{u}_j}{\partial x_k}) + \frac{2}{\rho}\overline{p's_{ij}} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{5.72}$$

Equaling both sides we have

$$\frac{\bar{D}\overline{u_i'u_j'}}{Dt} = d_{ij} + P_{ij} + \Phi_{ij} - \varepsilon_{ij} \tag{5.73}$$

where

$$d_{ij} = \frac{\partial}{\partial x_k}\left(\nu\frac{\partial \overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'u_j'u_k'} - \frac{1}{\rho}\overline{p'u_i'}\delta_{jk} - \frac{1}{\rho}\overline{p'u_j'}\delta_{ik}\right) \tag{5.74}$$

$$P_{ij} = -\overline{u_k'u_j'}\frac{\partial \overline{u}_i}{\partial x_k} - \overline{u_k'u_i'}\frac{\partial \overline{u}_j}{\partial x_k} \tag{5.75}$$

$$\Phi_{ij} = \frac{2}{\rho}\overline{p's_{ij}'} = \frac{1}{\rho}\overline{p'\left(\frac{\partial x_i'}{\partial x_j} + \frac{\partial x_j'}{\partial x_i}\right)} \tag{5.76}$$

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{5.77}$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{5.78}$$

Comments:

(1) The left hand side term $\frac{\bar{D}\overline{u_i'u_j'}}{Dt}$ is the rate of change of the Reynolds stress along the particle line.

(2) The term d_{ij} is the diffusion term in the equation, appearing in the form of gradient. It includes viscous term, Reynolds stress term and pressure-velocity fluctuation coupling term. The diffusion is resulted by the spatial non-uniformity of these properties.

(3) The term P_{ij} is the generation term of Reynolds stress, showed in the form of the product of Reynolds stress and the mean flow strain rate.

(4) The term Φ_{ij} is the redistribution term. We note that the contraction of Reynolds stress transport equation is the transport equation for turbulence kinetic energy. And the contraction of Φ_{ij} is $\Phi_{ii} = \frac{2}{\rho}\overline{p's_{ii}'} = 0$ as continuity holds. So the term contributes nothing to the growth of turbulent kinetic energy. It just takes the kinetic energy from one component of fluid motion to another component.

(5) The term ε_{ij}, whose contraction is positive forever, representing the dissipation mechanism of kinetic energy.

5.1.8 Dissipation rate transport equation

The dissipation term in Reynolds stresses transport equation is defined as

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_j'}{\partial x_p}} \tag{5.79}$$

Multiply equation (5.20) by $2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}$ and take the time derivative we have:

$$\text{LHS} = 2\nu\frac{\bar{D}}{Dt}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_p}} = \frac{\bar{D}\varepsilon}{Dt} + 2\nu\frac{\partial \overline{u}_k}{\partial x_p}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} \tag{5.80}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{2\nu}{\rho}\frac{\partial}{\partial x_k}(\overline{\frac{\partial u_k'}{\partial x_p}\frac{\partial p'}{\partial x_p}}) \tag{5.81}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}(\nu\frac{\partial u_i'}{\partial x_k}))} = \frac{\partial}{\partial x_k}(\nu\frac{\partial \varepsilon}{\partial x_k}) - 2\overline{(\nu\frac{\partial^2 u_i'}{\partial x_p\partial x_k})^2} \tag{5.82}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}\overline{u_i'u_k'})} = 0 \tag{5.83}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k} - u_i'u_k')} = -2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} + \frac{\partial}{\partial x_k}\overline{u_k'\nu(\frac{\partial u_i'}{\partial x_p})^2} \tag{5.84}$$

$$= -2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} + \frac{\partial}{\partial x_k}\overline{u_k'\varepsilon'} \tag{5.85}$$

$$\overline{2\nu\frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-u_k'\frac{\partial \overline{u}_i}{\partial x_k})} = -2\nu\frac{\partial \overline{u}_i}{\partial x_k}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}} - 2\nu\frac{\partial^2 \overline{u}_i}{\partial x_k\partial x_p}\overline{u_k'\frac{\partial u_i'}{\partial x_p}} \tag{5.86}$$

By equalizing both side we yield the transport equation for turbulence dissipation rate

$$\frac{\bar{D}\varepsilon}{Dt} = \frac{\partial}{\partial x_k}\left(-\frac{2\nu}{\rho}\overline{\frac{\partial u_k}{\partial x_p}\frac{\partial p}{\partial x_p}} + \nu\frac{\partial \varepsilon}{\partial x_k} - \overline{u_k'\varepsilon'}\right) - 2\nu\frac{\partial \overline{u}_i}{\partial x_k}\left(\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}} + \overline{\frac{\partial u_p'}{\partial x_k}\frac{\partial u_p'}{\partial x_i}}\right) \tag{5.87}$$

$$- 2\nu\overline{u_k'\frac{\partial u_i'}{\partial x_p}}\frac{\partial^2 \overline{u}_i}{\partial x_p\partial x_k} - 2\nu\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} - 2\overline{\left(\nu\frac{\partial u_i'}{\partial^2 x_p\partial x_k}\right)^2} \tag{5.88}$$

The final equation of the equation agrees with that given in the turbulence book by Shi (1994). This should be the most complicated RANS equation we attempt here. We can see that the RANS second moment equations are never closed and a closure is needed (Chou, 1945).

5.1.9 Scalar flux, its mean and variance transport equations

Similar to Eq. (5.20) we have the transport equation for the mean and fluctuation of a passive scalar c:

$$\frac{\bar{D}\bar{c}}{\bar{D}t} = \frac{\partial}{\partial x_j}(\nu\frac{\partial \bar{c}}{\partial x_j} - \overline{c'u_j'}) \tag{5.89}$$

and

$$\frac{\bar{D}c'}{Dt} = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial c'}{\partial x_j} + \overline{c'u'_j} - c'u'_j) - u'_j\frac{\partial \overline{c}}{\partial x_j} \tag{5.90}$$

where Γ is the molecular diffusion coefficient of c.

Take $c'\times$ (5.20) $+\, u'_i\times$ (5.90) and apply the average

$$\text{LHS} = \frac{\bar{D}\overline{c'u'_i}}{\bar{D}t} \tag{5.91}$$

$$\text{RHS}_1 = -\frac{1}{\rho}\overline{c'\frac{\partial p'}{\partial x_i}} = -\frac{1}{\rho}(\frac{\partial}{\partial x_j}\overline{p'c'}\delta_{ij} - \overline{p'\frac{\partial c'}{\partial x_i}}) \tag{5.92}$$

$$\text{RHS}_2 = \frac{\partial}{\partial x_j}(\Gamma\overline{u'_i\frac{\partial c'}{\partial x_j}} + \nu\overline{c'\frac{\partial u'_i}{\partial x_j}}) - (\nu+\Gamma)\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.93}$$

$$\text{RHS}_3 = -\frac{\partial}{\partial x_j}(\overline{c'u'_iu'_j}) \tag{5.94}$$

$$\text{RHS}_4 = -\overline{c'u'_j}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_iu'_j}\frac{\partial \overline{c}}{\partial x_j} \tag{5.95}$$

then we obtain the transport equation for scalar flux

$$\frac{\bar{D}\overline{c'u'_i}}{\bar{D}t} = d_{jc} + P_{jc} + \Phi_{jc} - \varepsilon_{jc} \tag{5.96}$$

where

$$d_{ic} = \frac{\partial}{\partial x_j}(\Gamma\overline{u'_i\frac{\partial c'}{\partial x_j}} + \nu\overline{c'\frac{\partial u'_i}{\partial x_j}} - \frac{1}{\rho}\overline{p'c'}\delta_{ij} - \overline{c'u'_iu'_j}) \tag{5.97}$$

$$P_{ic} = -\overline{c'u'_j}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_iu'_j}\frac{\partial \overline{c}}{\partial x_j} \tag{5.98}$$

$$\Phi_{ic} = \frac{1}{\rho}\overline{p'\frac{\partial c'}{\partial x_i}} \tag{5.99}$$

$$\varepsilon_{ic} = (\nu+\Gamma)\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.100}$$

Comments:

(1) Gradient diffusion: velocity-fluctuation scalar-diffusion correlation, momentum-diffusion scalar-fluctation ccorelation, pressure diffusion, turbulence diffusion.

(2) Production: scalar flux interacting with mean shear, turbulent flux (Reynolds stresses) interacting with mean scalar gradient.

(3) Re-distribution.

(4) Dissipation.

Define scalar mean and variance/fluctuation energy as

$$K_c = \frac{1}{2}\bar{c}^2 \tag{5.101}$$

$$k_c = \frac{1}{2}\overline{c'c'} \tag{5.102}$$

$c' \times$ (5.90) and apply the average

$$\text{LHS} = \frac{\bar{D}k_c}{\bar{D}t} \tag{5.103}$$

$$\text{RHS}_1 = \frac{\partial}{\partial x_j}\Gamma\frac{\partial k_c}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.104}$$

$$\text{RHS}_2 = -\frac{1}{2}\frac{\partial}{\partial x_j}\overline{c'c'u_j} \tag{5.105}$$

$$\text{RHS}_3 = -\overline{c'u'_j}\frac{\partial \bar{c}}{\partial x_j} \tag{5.106}$$

then we obtain the transport equation for scalar fluctuation energy

$$\frac{\bar{D}k_c}{\bar{D}t} = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial}{\partial x_j}k_c - \frac{1}{2}\overline{c'c'u'_j}) - \overline{c'u'_j}\frac{\partial \bar{c}}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.107}$$

For active scalar (for example, density which appears in the momentum equation as buoyancy force), see section 6.7.

5.1.10 Poisson equation for mean and fluctuation pressure

The Reynolds average equation is

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}) \tag{5.108}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} \tag{5.109}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2\overline{p} - \frac{\partial^2\,\overline{u'_i u'_j}}{\partial x_i \partial x_j} \tag{5.110}$$

Poisson equation for mean pressure:

$$-\frac{1}{\rho}\nabla^2\overline{p} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial^2\,\overline{u'_i u'_j}}{\partial x_i \partial x_j} \tag{5.111}$$

The velocity fluctuation transport equation is

$$\frac{\bar{D}u'_i}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u'_i}{\partial x_j} + \overline{u'_i u'_j} - u'_i u'_j) - u'_j\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.112}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} \tag{5.113}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2 p' - \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j} - \frac{\partial u'_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} - \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} \tag{5.114}$$

Poisson equation for fluctuation pressure:

$$\frac{1}{\rho}\nabla^2 p' = -\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} - \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} - \frac{\partial u'_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} + \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j} \tag{5.115}$$

$$= -\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} + \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j} - 2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} \tag{5.116}$$

$$= -2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} - \frac{\partial^2}{\partial x_i x_j}(u'_i u'_j - \overline{u'_i u'_j}) \tag{5.117}$$

where the last equation is due to the fact that

$$\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} = \frac{\partial^2 u'_i u'_j}{\partial x_i x_j}. \tag{5.118}$$

According to the source terms, the pressure can be decomposed into the rapid, slow, and harmonic components

$$p' = p^{(r)} + p^{(s)} + p^{(h)}, \tag{5.119}$$

such that

$$\frac{1}{\rho}\nabla^2 p^{(r)} = -2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} \tag{5.120}$$

$$\frac{1}{\rho}\nabla^2 p^{(s)} = -\frac{\partial^2}{\partial x_i x_j}(u'_i u'_j - \overline{u'_i u'_j}) \tag{5.121}$$

$$\nabla^2 p^{(h)} = 0 \tag{5.122}$$

and modelled separately.

5.1.11 Turbulent vorticity and enstrophy

Similarly, vorticity can be decomposed into the mean and the pertubation. We give the equation of pertuabtion vorticity without derivation:

$$\frac{\bar{D}\omega'_i}{\bar{D}t} = \omega'_j \overline{S}_{ij} + \overline{\omega_j} S'_{ij} + \omega'_j S'_{ij} - \overline{\omega'_j S'_{ij}} - u'_j \frac{\partial \overline{\omega}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u'_j \omega'_i} - u'_j \omega'_i) + \nu \frac{\partial^2 \omega'_i}{\partial x_j^2} \tag{5.123}$$

where $\overline{S}_{ij}$ and S'_{ij} are the mean and the fluctuation shear, respectively.

We define the fluctuating enstrophy as

$$\mathscr{E} = \frac{1}{2}\overline{\omega'_i \omega'_i} \tag{5.124}$$

$\omega_i' \times$ (5.123) and take the time average

$$\text{LHS} = \frac{\bar{D}\mathscr{E}}{\bar{D}t} \tag{5.125}$$

$$\text{RHS}_1 = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega}_j\overline{\omega_i' S_{ij}'} + \overline{\omega_i'\omega_j' S_{ij}'} \tag{5.126}$$

$$\text{RHS}_2 = -\overline{\omega_i' u_j'}\frac{\partial \overline{\omega_i}}{\partial x_j} \tag{5.127}$$

$$\text{RHS}_3 = -\frac{1}{2}\frac{\partial}{\partial x_j}(\overline{u_j'\omega_i'\omega_i'}) \tag{5.128}$$

$$\text{RHS}_4 = \nu\overline{\frac{\partial^2 \mathscr{E}}{\partial x_j^2}} - \overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} \tag{5.129}$$

Equaling both sides we obtain

$$\frac{\bar{D}\mathscr{E}}{\bar{D}t} = P_{\mathscr{E}} + D_{\mathscr{E}} - \varepsilon_{\mathscr{E}} \tag{5.130}$$

$$P_{\mathscr{E}} = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega_j}\overline{\omega_i' S_{ij}'} + \overline{\omega_i'\omega_j' S_{ij}'} - \overline{\omega_i' u_j'}\frac{\partial \overline{\omega_i}}{\partial x_j} \tag{5.131}$$

$$D_{\mathscr{E}} = \frac{\partial}{\partial x_j}(\nu\frac{\partial \mathscr{E}}{\partial x_j} - \frac{1}{2}\overline{u_j'\omega_i'\omega_i'}) \tag{5.132}$$

$$\varepsilon_{\mathscr{E}} = \nu\overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} \tag{5.133}$$

Comment: The energy balance process of fluctuation enstrophy obeys four principle processes in nature (Kolmogorov):

$$\text{rate of change} = \text{production} + \text{diffusion} + \text{dissipation}$$

Moreover, in instability problems, it is convenience to consider the linearized inviscid perturbation vorticity equation, which reads

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j'\bar{S}_{ij} + \bar{\omega}_j S_{ij}' - u_j'\frac{\partial \bar{\omega}_i}{\partial x_j}, \tag{5.134}$$

since perturbation enstrophy is related to turbulent dissipation (see 5.51). It can be obtained either by linearizing the vorticity equation or taking the curl of the linearized momentum equations. It has a version in rotating frames in (6.109).

5.1.12 Spectral TKE equation

Staring with the fluctuation equations

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{5.135}$$

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial u_i'}{\partial x_j} - u_i'u_j' + \overline{u_i'u_j'}\right) \tag{5.136}$$

we take the Fourier expansion of the fluctuation in the homogeneous directions

$$\boldsymbol{u}'(x,y,z,t) = \sum_{k_x}\sum_{k_z}\hat{\boldsymbol{u}}(k_x,y,k_z,t)e^{i(k_x x+k_z z)} \tag{5.137}$$

$$p'(x,y,z,t) = \sum_{k_x}\sum_{k_z}\hat{p}(k_x,y,k_z,t)e^{i(k_x x+k_z z)} \tag{5.138}$$

where we assume that there is at least one spatial inhomogeneous direction; otherwise, see section 5.3 for the transport equation of the correlation function.

The energy spectrum is

$$\tilde{e}(k_x,y,k_z,t) = \frac{1}{2}\hat{u}_i\hat{u}_i^* = \frac{1}{2}|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2 \tag{5.139}$$

and its evolution equation (for each wavenumber pair (k_x,k_z)) can be derived by taking the Fourier transform of the fluctuation equations (5.135)–(5.136) in x,z,

$$ik_x\hat{u} + \partial_y\hat{v} + ik_z\hat{w} = 0 \tag{5.140}$$

$$\frac{\partial\hat{u}_i}{\partial t} + \bar{u}_j\widehat{\frac{\partial u_i'}{\partial x_j}} + \hat{u}_j\frac{\partial\overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\widehat{\frac{\partial p'}{\partial x_i}} + \widehat{\frac{\partial}{\partial x_j}\left(-u_i'u_j'\right)} + \nu\widehat{\frac{\partial^2 u_i'}{\partial x_j^2}} \tag{5.141}$$

where $\partial/\partial x_j = (ik_x,\partial_y,ik_z)$. We note that only the nonlinear term (RHS2) involves scale interaction. Note that the Reynolds stress $\overline{u_i'u_j'}$ involves no k_x,k_z dependence and its FT is Dirac delta (meaning that this term only appears in the 0-0 mode – the standard TKE equation – when integrated over the Fourier space).

Multiply (5.141) with $\hat{u}_i^*(k_x,y,k_z)$ and denote $\boldsymbol{k}=(k_x,y,k_z)$, we have the evolution equation of the Fourier modes:

$$\frac{\partial\tilde{e}}{\partial t} = \Re\left\{-\hat{u}_i^*\hat{u}_j\frac{\partial\overline{u}_i}{\partial x_j}\right\} + \Re\left\{-\frac{1}{\rho}\frac{\partial\hat{p}\hat{v}^*}{\partial y}\right\} + \Re\left\{-\hat{u}_i^*\widehat{\frac{\partial u_i'u_j'}{\partial x_j}}\right\} + \nu\frac{\partial^2\tilde{e}}{\partial y^2} - 2\nu(k_x^2+k_z^2)\tilde{e} - \nu\frac{\partial\hat{u}_i^*}{\partial y}\frac{\partial\hat{u}_i}{\partial y} \tag{5.142}$$

where the scale-dependent spectral TKE terms are

$$\hat{P}(k_x,y,k_z) = \Re\left\{-\hat{u}_j(\boldsymbol{k})\hat{u}_i^*(\boldsymbol{k})\frac{\partial\overline{u}_i}{\partial x_j}\right\} = \Re\left\{-\hat{u}^*\hat{v}\frac{\partial\overline{u}}{\partial y}\right\} \tag{5.143}$$

$$\hat{T}_p(k_x,y,k_z) = \Re\left\{-\frac{1}{\rho}\frac{\partial\hat{p}(\boldsymbol{k})\hat{v}^*(\boldsymbol{k})}{\partial y}\right\} \tag{5.144}$$

$$\hat{T}_{nl}(k_x,y,k_z) = \Re\left\{-\hat{u}_i^*(\boldsymbol{k})\widehat{\frac{\partial u_i'(\boldsymbol{k}_1)u_j'(\boldsymbol{k}_2)}{\partial x_j}}(\boldsymbol{k})\right\} = \Re\left\{-\hat{u}_i^*(\boldsymbol{k})\widehat{\frac{\partial u_i'(\boldsymbol{k}_1)v'(\boldsymbol{k}_2)}{\partial y}}(\boldsymbol{k})\right\} \tag{5.145}$$

$$= \Re\left\{-\hat{u}_i^*(\boldsymbol{k})\widehat{u_j'\frac{\partial u_i'}{\partial x_j}}(\boldsymbol{k})\right\} \tag{5.146}$$

$$= \Re\left\{-\frac{\partial}{\partial y}[\hat{u}_i^*(\boldsymbol{k})(\widehat{u_i'(\boldsymbol{k}_1)v'(\boldsymbol{k}_2)})] + \widehat{u_i'(\boldsymbol{k}_1)v'(\boldsymbol{k}_2)}\frac{\partial\hat{u}_i^*}{\partial y}(\boldsymbol{k})\right\} \tag{5.147}$$

$$\hat{D}(k_x,y,k_z) = \nu\frac{\partial^2\tilde{e}(\boldsymbol{k})}{\partial y^2} \tag{5.148}$$

$$\hat{\varepsilon}(k_x, y, k_z) = -2\nu(k_x^2 + k_z^2)\tilde{e}(\boldsymbol{k}) - \nu \frac{\partial \hat{u}_i^*(\boldsymbol{k})}{\partial y} \frac{\partial \hat{u}_i(\boldsymbol{k})}{\partial y} \tag{5.149}$$

are the spectral density of production, pressure fluctuation transport, (nonlinear) turbulent transport, viscous diffusion, and (pseudo-)dissipation at each wavenumber and wall distance, with the last equation in (5.143) giving its representation in a turbulent channel flow with the only non-zero shear being $\partial_y \bar{u}$. These equations are consistent with Mizuno (2016); Cho *et al.* (2018). We note that there are different ways of writing $\hat{T}_{nl}$; note $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}$. Typically, the spectral TKE equation will be averaged in time in practice.

We note that the production term is linear, in the sense that it does not involve scale interactions. Since $\partial_y \bar{u}(y)$ has no spatial dependence in the Fourier direction, it is immune to the FT. The FT of $\overline{uv}$ is the co-spectrum $\Phi_{uv}(\boldsymbol{k}) = \Re\{\hat{u}^*(\boldsymbol{k})\hat{v}(\boldsymbol{k})\}$. As a result, $\hat{P}$ is the production spectrum whose integration over the entire Fourier space recovers $P = -\overline{uv}\partial_y \bar{u}$. The diffusion of spectral density happens only in the inhomogeneous y-direction while dissipation occurs in all spatial directions. The nonlinear term $\hat{T}_{nl}$, is the only term that involves different wavenumbers (in the FT of $\widehat{u_i'(\boldsymbol{k}_1)u_j'(\boldsymbol{k}_2)}$) and hence scale interactions. The other terms are hence linear/scale-local (e.g., it is said that the production is linear). In statistically stationary flows, $d\tilde{e}/dt = 0$.

Some steps of the derivation are provided below:

$$\frac{\partial \tilde{e}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\hat{u}_i \hat{u}_i^*) = \frac{1}{2}\left(\hat{u}_i^* \frac{\partial \hat{u}_i}{\partial t} + \hat{u}_i \frac{\partial \hat{u}_i^*}{\partial t}\right) = \Re\{\hat{u}_i^* \frac{\partial \hat{u}_i}{\partial t}\} \tag{5.150}$$

$$\hat{u}_i^* \bar{u}_j \widehat{\frac{\partial u_i'}{\partial x_j}} = \hat{u}_i^*(ik_x \bar{u} + \bar{v}\partial_y + ik_z \bar{w})\hat{u}_i,\ \Re\{\cdot\} = 0,\ \bar{v} = 0 \tag{5.151}$$

$$\hat{u}_i^* \widehat{\frac{\partial p'}{\partial x_i}} = (\hat{u}^*, \hat{v}^*, \hat{w}^*) \cdot (ik_x \hat{p}, \partial_y \hat{p}, ik_z \hat{p}) \tag{5.152}$$

$$= \hat{v}^* \partial_y \hat{p} + \hat{p}(ik_x \hat{u}^* + ik_z \hat{w}^*) \tag{5.153}$$

$$= \hat{v}^* \partial_y \hat{p} + \hat{p}\partial_y \hat{v}^* \tag{5.154}$$

$$= \partial_y(\hat{p}\hat{v}^*) \tag{5.155}$$

$$\overline{\widehat{u_i' u_j'}} = \widehat{\overline{u_i' u_j'}} = 0 \tag{5.156}$$

$$\Re\left\{-\hat{u}_i^* \frac{\partial \widehat{u_i' u_j'}}{\partial x_j}\right\} = \Re\left\{-\hat{u}_i^* \widehat{u_j' \frac{\partial u_i'}{\partial x_j}}\right\} \text{ (another way of writing } \hat{T}_{nl}) \tag{5.157}$$

$$\widehat{u_i' u_j'}(\boldsymbol{k}) = \int_{-\infty}^{+\infty} \hat{u}_j(\boldsymbol{k}')\hat{u}_i(\boldsymbol{k} - \boldsymbol{k}')\, \mathrm{d}\boldsymbol{k}' \tag{5.158}$$

$$\nu \hat{u}_i^* \widehat{\frac{\partial^2 u_i'}{\partial x_j^2}} = \nu \hat{u}_i^* [-(k_x^2 + k_z^2) + \partial_y^2]\hat{u}_i \tag{5.159}$$

$$= -\nu(k_x^2 + k_z^2)\hat{u}_i \hat{u}_i^* + \nu \hat{u}_i^* \partial_y(\partial_y \hat{u}_i) \tag{5.160}$$

$$= -\nu(k_x^2 + k_z^2)\hat{u}_i \hat{u}_i^* + \nu \partial_y\left(\hat{u}_i^* \partial_y \hat{u}_i\right) - \nu \partial_y \hat{u}_i^* \partial_y \hat{u}_i \tag{5.161}$$

$$\Re\left\{\nu \hat{u}_i^* \widehat{\frac{\partial^2 u_i'}{\partial x_j^2}}\right\} = -\nu(k_x^2 + k_z^2)\hat{u}_i \hat{u}_i^* + \nu \partial_y^2(\hat{u}_i^* \hat{u}_i/2) - \nu \partial_y \hat{u}_i^* \partial_y \hat{u}_i \tag{5.162}$$

$$\Re\{\hat{u}_i^* \partial_y \hat{u}_i\} = \Re\{\partial_y[\partial_y(\hat{u}_i^* \hat{u}_i)] - \partial_y[\hat{u}_i \partial_y \hat{u}_i^*]\} = \frac{1}{2}\Re\{\partial_y^2 \hat{u}_i^* \hat{u}_i\} = \frac{1}{2}\partial_y^2 \hat{u}_i^* \hat{u}_i \tag{5.163}$$

where (5.151) implies that the mean convection does not change the spectrum. We can commute

$$\widehat{\frac{\partial a}{\partial x_j}} = \frac{\partial \hat{a}}{\partial x_j} \tag{5.164}$$

as long as $\partial \hat{a}/\partial x_j$ is interpreted as $(ik_x, \partial_y, ik_z)\hat{a}$. The co-spectrum of two variables u' and v' is

$$\Phi_{uv}(\boldsymbol{k}) = \Re\{\hat{u}^*(\boldsymbol{k})\hat{v}(\boldsymbol{k})\} \tag{5.165}$$

whose physical-space dual is the two-point correlation

$$R_{uv}(\boldsymbol{x}, \boldsymbol{x}') = \overline{u'(\boldsymbol{x})v'(\boldsymbol{x}')} \tag{5.166}$$

and whose integration over the wavenumber space is the cross-correlation $R_{uv}(0) = \overline{u'v'}$ (when $\boldsymbol{x}' = \boldsymbol{x}$).

Integration of (5.142) over the entire wavenumber space will reduce to the standard TKE equation (with x, z being homogeneous directions):

$$0 = -\overline{u'v'}\frac{\partial \overline{u}}{\partial y} - \frac{1}{\rho}\frac{\partial \overline{p'v'}}{\partial y} - \frac{\partial \overline{ev'}}{\partial y} + \nu\frac{\partial^2 \bar{e}}{\partial y^2} - \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} \tag{5.167}$$

where $e = 1/2u_i'u_i'$ and simplifications for a channel geometry has been made. We note that the integral of the nonlinear interscale transfer term over all wavenumbers is zero

$$\int_{\boldsymbol{k}} \hat{T}_{nl}(\boldsymbol{k}, y)\,\mathrm{d}\boldsymbol{k} = 0, \tag{5.168}$$

hence it does not make any contributions to the TKE budget. The spatial integral any spatial transport terms is also zero, which has some implications on the importance of linear mechanisms in perturbation amplification (see (8.27)).

5.1.13 Error evolution equation

5.2 LES equations

5.2.1 Filtering

Assume an unfiltered quantity $f(x)$ contains all scales and denote the filtered quantity is

$$\tilde{f}(x) = \int g(x, x')f(x')\,\mathrm{d}x', \tag{5.169}$$

and typically in homogeneous directions as the convolution of the function $f(x)$ with the kernel $g(x)$:

$$\tilde{f}(x) = \int g(x - x')f(x')\,\mathrm{d}x'. \tag{5.170}$$

There are many choices for the kernel.

- The box filter. It is given by the following function

$$g(x) = \begin{cases} 1/2\Delta & |x| \leq \Delta \\ 0 & \text{otherwise} \end{cases} \tag{5.171}$$

whose Fourier pair is

$$2\pi\hat{g}(k) = \frac{\sin(k\Delta)}{k\Delta}. \tag{5.172}$$

This filter corresponds to discretized measurement or computation in physical space, where filtering is usually not explicitly performed but implied by the truncation of Taylor's series. For example, a second-order approximation of $f'(x)$ at x_i is

$$f'(x_i) \approx \frac{f(x_i+\Delta) - f(x_i-\Delta)}{2\Delta} = \frac{d}{dx}\left(\frac{1}{2\Delta}\int_{x_i-\Delta}^{x_i+\Delta} f(x')\,\mathrm{d}x'\right) = \frac{d\tilde{f}}{dx} \tag{5.173}$$

where the RHS is exactly $d\tilde{f}/dx$ with the convolution kernel being the box filter function.

- The spectral cut-off (sharp Fourier) filter. The filtering kernel

$$g(x) = \frac{\sin(2\pi x/\Delta)}{\pi x} \tag{5.174}$$

corresponds to

$$2\pi\hat{g}(k) = \begin{cases} 1 & |k| \leq 2\pi/\Delta \\ 0 & \text{otherwise} \end{cases} \tag{5.175}$$

whose effect it to kill all modes with $|\boldsymbol{k}| > \boldsymbol{k}_{\max} = 2\pi/\Delta$.

- Gaussian filter. The Gaussian kernel

$$g(x) = \sqrt{\frac{6}{\pi\Delta^2}}e^{-6x^2/\Delta^2} \tag{5.176}$$

has a Fourier counterpart that is also Gaussian

$$2\pi\hat{g}(k) = e^{-(k\Delta)^2/24}. \tag{5.177}$$

Hence it is Gaussian in both physical and Fourier space.

We note that the Fourier counterpart of convolution (filtering) is multiplication in Fourier space, hence the filtered spectrum,

$$\hat{\tilde{f}} = 2\pi\hat{g}\hat{f}, \tag{5.178}$$

is spectral content of the signal f modified by filtering (multiplied by the transfer function $2\pi\hat{g}$). The co-spectrum of two arbitrary variables u, v

$$C_{uv}(k) = \Re\{\hat{u}\hat{v}^*\}, \tag{5.179}$$

which is the Fourier transform pair of

$$R_{uv}(x, x') = \langle u(x)v(x')\rangle, \tag{5.180}$$

will be modified to

$$C_{\tilde{u}\tilde{v}}(k) = 4\pi^2|\hat{g}(k)|^2 C_{uv}. \tag{5.181}$$

Relation to spatial average.

5.2.2 Filtered equations of motion

Adopt the large-small scale decomposition

$$u_i(x_i,t) = \tilde{u}_i(x_i,t) + u_i''(x_i,t), \tag{5.182}$$

where $\tilde{u}_i$ is the filtered velocity and u_i'' is the residual obtained by subtracting $\tilde{u}_i$ from u_i, and note that typically (unless a sharp spectral filter is used)

$$\widetilde{u''} = \widetilde{u - \tilde{u}} = \tilde{u} - \tilde{\tilde{u}} \neq 0, \tag{5.183}$$

unlike $\bar{u'} = 0$ in Reynolds averaging.

Assume filtering and spatial/temporal derivatives commute (which is typically the case with uniform filters), we have the filtered N–S equations

$$\frac{\partial \tilde{u}_i}{\partial x_j} = 0 \tag{5.184}$$

$$\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial \widetilde{u_i u_j}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} + f_i \tag{5.185}$$

where the filtered total stress can be decomposed into

$$\widetilde{u_i u_j} = \tilde{u}_i \tilde{u}_j + (\widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j) \tag{5.186}$$

and the second part is called the subgrid-scale stress (generalized small-scale co-variance),

$$\tau_{ij}^{\mathrm{SGS}} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j, \tag{5.187}$$

analogous to the Reynolds stress tensor $(-\overline{u_i' u_j'})$

$$\tau_{ij}^{\mathrm{Rey}} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j = \overline{u_i' u_j'}. \tag{5.188}$$

The momentum equation that the filtered velocity satisfies is

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} - \frac{\partial \tau_{ij}^{\mathrm{SGS}}}{\partial x_j} + \frac{\partial}{\partial x_j}(2\nu \tilde{S}_{ij}) + \tilde{f}_i \tag{5.189}$$

where $\tilde{S}_{ij} = 1/2(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i)$. Note that τ_{ij}^{SGS} is a large-scale quantity and the idea that the effect of small scales to the large scales can be represented by the large scales only leads to LES modellings.

The subgrid scale stress can be decomposed as (Leonard, 1975)

$$\tau_{ij}^{\mathrm{SGS}} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j = L_{ij} + C_{ij} + R_{ij} \tag{5.190}$$

where

$$L_{ij} = \widetilde{\tilde{u}_i \tilde{u}_j} - \tilde{u}_i \tilde{u}_j \tag{5.191}$$

$$C_{ij} = \widetilde{\tilde{u}_i u_j''} + \widetilde{u_i'' \tilde{u}_j} \tag{5.192}$$

$$R_{ij} = \widetilde{u_i'' u_j''} \tag{5.193}$$

representing the contributions of large-scale self-interaction, cross-scale interaction, and the small-scale interaction to the SGS stress. However, the above decomposition is not Galilean-invariant.

More generally, the generalized second and third moments (Germano *et al.*, 1991),

$$\tau^l(a,b) = \widetilde{ab} - \tilde{a}\tilde{b} \tag{5.194}$$

and

$$\tau^l(a,b,c) = \widetilde{abc} - \tilde{a}\tau^l(b,c) - \tilde{b}\tau^l(a,c) - \tilde{c}\tau^l(a,c) - \tilde{a}\tilde{b}\tilde{c} \tag{5.195}$$

can be used, where the superscript l implies these operations are with respect to the filter scale l (Johnson, 2021).

5.2.3 LES energy equations

In terms of energy, the (filtered) total energy equals to the energy of the filtered scales and of the sub-filter scale

$$\frac{1}{2}\widetilde{u_i u_i} = \frac{1}{2}\tilde{u}_i\tilde{u}_i + \frac{1}{2}(\widetilde{u_i u_i} - \tilde{u}_i\tilde{u}_i) = \frac{1}{2}\tilde{u}_i\tilde{u}_i + \frac{1}{2}\tau_{ii}^{\mathrm{SGS}}. \tag{5.196}$$

Multiply the filtered momentum equation with $\tilde{u}_i$ and note that

$$-\tilde{u}_i\frac{\partial\tau_{ij}^{\mathrm{SGS}}}{\partial x_j} + \tilde{u}_i\frac{\partial}{\partial x_j}(2\nu\tilde{S}_{ij}) = -\frac{\partial}{\partial x_j}(\tilde{u}_i\tau_{ij}^{\mathrm{SGS}}) + \tilde{S}_{ij}\tau_{ij}^{\mathrm{SGS}} + \frac{\partial}{\partial x_j}(\tilde{u}_i 2\nu\tilde{S}_{ij}) - 2\nu\tilde{S}_{ij}\tilde{S}_{ij} \tag{5.197}$$

we have the large-scale energy equation

$$\left(\frac{\partial}{\partial t} + \tilde{u}_j\frac{\partial}{\partial x_j}\right)\frac{1}{2}\tilde{u}_i\tilde{u}_i = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\tilde{p}\tilde{u}_j + \tilde{u}_i\tau_{ij}^{\mathrm{SGS}} - 2\nu\tilde{u}_i\tilde{S}_{ij}\right) - \Sigma - 2\nu\tilde{S}_{ij}\tilde{S}_{ij} + \tilde{u}_i\tilde{f}_i \tag{5.198}$$

where (positive value of) $\Sigma = -\tau_{ij}^{\mathrm{SGS}}\tilde{S}_{ij}$ represents the rate at which energy transfered between the large scale and the small scales. We note that the above equation can also be written as

$$\left(\frac{\partial}{\partial t} + \tilde{u}_j\frac{\partial}{\partial x_j}\right)\frac{1}{2}\tilde{u}_i\tilde{u}_i = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\tilde{p}\tilde{u}_j - 2\nu\tilde{u}_i\tilde{S}_{ij}\right) - \tilde{u}_i\frac{\partial\tau_{ij}^{\mathrm{SGS}}}{\partial x_j} - 2\nu\tilde{S}_{ij}\tilde{S}_{ij} + \tilde{u}_i\tilde{f}_i \tag{5.199}$$

where the inter-scale transfer term is expressed as $\Psi = \tilde{u}_i\partial_j\tau_{ij}^{\mathrm{SGS}}$. Although both are positive when there is direct energy cascade and they are related as $\Sigma = \Psi - \partial_j(\tilde{u}_i\tau_{ij}^{\mathrm{SGS}})$ and hence share the same spatial average, their pointwise values can be a lot different.

The (filtered) total energy equation can be obtained by multiplying the momentum equation with u_i and then filtering:

$$\frac{\partial}{\partial t}\frac{\widetilde{u_i u_i}}{2} + \frac{\partial}{\partial x_j}\frac{\widetilde{u_i u_i u_j}}{2} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\widetilde{pu_j} - 2\nu\widetilde{u_i S_{ij}}\right) - 2\nu\widetilde{S_{ij}S_{ij}} + \widetilde{u_i f_i} \tag{5.200}$$

and the equation for small-scale energy can be obtained by subtracting the large-scale energy equation (5.198) from above equation:

$$\frac{\partial\tau_{ii}^{\mathrm{SGS}}/2}{\partial t} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}(\widetilde{pu_j} - \tilde{p}\tilde{u}_j) - 2\nu(\widetilde{u_i S_{ij}} - \tilde{u}_i\tilde{S}_{ij})\right) + \Sigma - 2\nu(\widetilde{S_{ij}S_{ij}} - \tilde{S}_{ij}\tilde{S}_{ij}) + (\widetilde{u_i f_i} - \tilde{u}_i\tilde{f}_i) \tag{5.201}$$

$$- \frac{\partial}{\partial x_j}\left(\frac{1}{2}\widetilde{u_i u_i u_j} - \frac{1}{2}\tilde{u}_i\tilde{u}_i\tilde{u}_j - \tilde{u}_i\tau_{ij}^{\mathrm{SGS}}\right) \tag{5.202}$$

where the last term (5.202) can also be written with the generalized moments as

$$-\frac{\partial \phi_j}{\partial x_j},\ \phi_i = \frac{1}{2}\tau_{ii}^{\mathrm{SGS}}\tilde{u}_j + \frac{1}{2}\tau(u_i, u_i, u_j). \tag{5.203}$$

or the entire equation with the generalized moments as (Johnson, 2021)

$$\frac{\partial \tau^l(u_i, u_i)/2}{\partial t} = -\frac{\partial}{\partial x_j}\left(\tau^l(p, u_j) - 2\nu\tau^l(u_i, \tilde{S}_{ij}) + \frac{1}{2}\tau^l(u_i, u_i)\tilde{u}_j + \frac{1}{2}\tau^l(u_i, u_i, u_j)\right) \tag{5.204}$$

$$- \tau^l(u_i, u_j)\tilde{S}_{ij} - 2\nu\tau^l(S_{ij}, S_{ij}) + \tau^l(u_i, f_i) \tag{5.205}$$

It can easily be seen that positive value of $\Sigma = -\tau_{ij}^{\mathrm{SGS}}\tilde{S}_{ij}$ represents direct energy cascade across scale l. Since $\tilde{S}_{ij}$ is symmetric, Σ can be evaluated as

$$\Sigma = -\check{\tau}_{ij}^{\mathrm{SGS}}\tilde{S}_{ij},\ \check{\tau}_{ij}^{\mathrm{SGS}} = \tau_{ij}^{\mathrm{SGS}} - \frac{1}{3}\tau_{kk}^{\mathrm{SGS}}\delta_{ij}. \tag{5.206}$$

Similarly, the equation for the small-scale (sub-filter scale) motions, often overlooked, can be derived as

$$\frac{\partial u_i''}{\partial x_j} = 0 \tag{5.207}$$

$$\frac{\partial u_i''}{\partial t} + \tilde{u}_j\frac{\partial u_i''}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p''}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij}^{\mathrm{SGS}} - \tilde{u}_i u_j'' - u_i'' u_j'') + \frac{\partial}{\partial x_j}(2\nu S_{ij}'') + f_i'' \tag{5.208}$$

and its energy equation can be obtained by multiplying (5.208) with u_i'':

$$\left(\frac{\partial}{\partial t} + \tilde{u}_j\frac{\partial}{\partial x_j}\right)\frac{1}{2}u_i''u_i'' = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}p''u_j'' - u_i''\tau_{ij}^{\mathrm{SGS}} + \frac{1}{2}u_i''u_i''u_j'' - 2\nu u_i'' S_{ij}''\right) \tag{5.209}$$

$$- u_i''u_j''\frac{\partial \tilde{u}_i}{\partial x_j} + \Sigma'' - 2\nu S_{ij}''S_{ij}'' + u_i'' f_i'' \tag{5.210}$$

where $\Sigma'' = -\tau_{ij}^{\mathrm{SGS}}S_{ij}''$ and both itself and $-u_i''u_j''\partial_j\tilde{u}_i$ are responsible for inter-scale energy transfer, from the small scales' perspective. It can also be alternatively written as

$$\left(\frac{\partial}{\partial t} + \tilde{u}_j\frac{\partial}{\partial x_j}\right)\frac{1}{2}u_i''u_i'' = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}p''u_j'' + \frac{1}{2}u_i''u_i''u_j'' - 2\nu u_i'' S_{ij}''\right) \tag{5.211}$$

$$- u_i''u_j''\tilde{S}_{ij} + u_j''\frac{\partial \tau_{ij}^{\mathrm{SGS}}}{\partial x_j} - 2\nu S_{ij}''S_{ij}'' + u_i'' f_i'' \tag{5.212}$$

where $T = -u_i''u_j''\tilde{S}_{ij} + u_j''\partial_j\tau_{ij}^{\mathrm{SGS}}$ is interpreted as the transfer term (Cardesa & Lozano-Durán, 2019).

Moreover, since $\widetilde{u''} \neq 0$ in general, there is a part of energy residing in the correlated/cross term such that

$$\frac{1}{2}u_i u_i = \frac{1}{2}(\tilde{u}_i + u_i'')(\tilde{u}_i + u_i'') = \frac{1}{2}\tilde{u}_i\tilde{u}_i + \frac{1}{2}u_i''u_i'' + \tilde{u}_i u_i''. \tag{5.213}$$

The equation of the cross-scale-energy can be similarly derived as

$$\left(\frac{\partial}{\partial t}+\tilde{u}_j\frac{\partial}{\partial x_j}\right)\tilde{u}_iu''_i=-\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\tilde{p}u''_i+p''\tilde{u}_i-(\tilde{u}_j-u''_j)\tau^{\mathrm{SGS}}_{ij}+\frac{1}{2}\tilde{u}_i\tilde{u}_iu''_j-2\nu\tilde{u}_iS''_{ij}-2\nu u''_i\tilde{S}_{ij}\right) \tag{5.214}$$

$$-\tilde{u}_i\frac{\partial u''_iu''_j}{\partial x_j}+(\Sigma-\Sigma'')-4\nu\tilde{S}_{ij}S''_{ij} \tag{5.215}$$

When all three energy equations are added together, we can see the following

- Σ, Σ'' don't contribute to the total energy $1/2u_iu_i$ but just redistributes among components.
- The total dissipation equals to

$$-2\nu\tilde{S}_{ij}\tilde{S}_{ij}-2\nu S''_{ij}S''_{ij}-4\nu\tilde{S}_{ij}S''_{ij}=-2\nu S_{ij}S_{ij} \tag{5.216}$$

5.2.4 Smagorinsky and Dynamic models

In an eddy-viscosity formulation, $\tau^{\mathrm{SGS}}_{ij}-1/3\tau^{\mathrm{SGS}}_{kk}\delta_{ij}=-2\nu_T\tilde{S}_{ij}$, the SGS 'dissipation' $\Sigma=2\nu_T\tilde{S}_{ij}\tilde{S}_{ij}$ is always positive. But in reality, there is typically backscatter (negative Σ). Typical eddy-viscodity models include the Smagorinsky model, $\nu_T=L_c\sqrt{2\tilde{S}_{ij}\tilde{S}_{ij}}$, where $L_c=\min(\kappa d, C_s\sqrt[3]{\delta_x\Delta_y\Delta_z})$ is the characteristic length scale, κ is the Karman constant, and d is the distance from the wall. The model coefficient can be a constant (in static Smagorinsky) or dependent on the large-scale flow (in dynamics Smagorinsky).

Germano *et al.* (1991)

5.2.5 Triple decomposition TKE equation

Similar to the Reynolds decomposition, the flow variables can be triply decomposed as

$$[\boldsymbol{u}(\boldsymbol{x},t),p(\boldsymbol{x},t)]=[\bar{\boldsymbol{u}}(\boldsymbol{x})+\bar{p}(\boldsymbol{x})]+[\boldsymbol{u}'(\boldsymbol{x},t),p'(\boldsymbol{x},t)] \tag{5.217}$$

$$=[\bar{\boldsymbol{u}}(\boldsymbol{x})+\bar{p}(\boldsymbol{x})]+[\tilde{\boldsymbol{u}}(\boldsymbol{x},t)+\tilde{p}(\boldsymbol{x},t)]+[\boldsymbol{u}''(\boldsymbol{x},t),p''(\boldsymbol{x},t)] \tag{5.218}$$

where tilded variables denote the so-called 'coherent' part of the fluctuation and double-primed variables are the residual. The 'coherent' part could be conditionally averaged unsteady flow (conditioned to spatial or temporal reference phase), called phase-relevant (Hussain & Reynolds, 1970), or by other definitions. The governing equation for the kinetic energy associated with each part can be similarly established.

Let us start from the decomposition of the kinetic energy into the mean (MKE), the large-scale/coherent (LKE), and the small-scale/residual (SKE) parts:

$$k=\frac{1}{2}\overline{u_iu_i}\triangleq k_M+k_L+k_S \tag{5.219}$$

$$=\frac{1}{2}\bar{u}_i\bar{u}_i+\frac{1}{2}\overline{\tilde{u}_i\tilde{u}_i}+\frac{1}{2}\overline{u''_iu''_i}, \tag{5.220}$$

where the TKE $k_T=k_L+k_S$. In order to eliminate cross kinetic energy (CKE), it requires that

$$\overline{\bar{u}_i\tilde{u}_i}=\overline{\bar{u}_iu''_i}=\overline{\tilde{u}_iu''_i}=0. \tag{5.221}$$

That being said, the three parts of the flow need to be mutually uncorrelated for successful triple decomposition. Think of the case of turbulence superposed on a perturbation wave in Hussain & Reynolds (1970), the turbulent motion and the 'wave' motion are uncorrelated. But we will see this is not strictly necessary since the CKE equation can be derived and its role is like a bridge between LKE and SKE. The ease of this strict requirement will enable more filter choices.

Here we consider a triple decomposition roughly through cut-off filtering in Fourier space where the average $\overline{(\cdot)}$ denotes averaging over all homogeneous directions and ensembles (the zero Fourier mode), and the filtering is performed in all Fourier directions.

Denote the filtering operation as

$$\widetilde{u'_i} = \tilde{u}_i,\ u''_i = u'_i - \tilde{u}_i \tag{5.222}$$

and in the case of a sharp Fourier filter (not all filters have these properties),

$$\tilde{\tilde{u}}_i = \tilde{u}_i,\ \widetilde{u''_i} = 0 \tag{5.223}$$

and also

$$\overline{\tilde{u}_i} = \overline{u''_i} = 0,\ \tilde{\bar{u}}_i = \bar{u}_i, \tag{5.224}$$

The triple decomposition here can generally be into three non-overlapping scales: mean, large, and small scales. The first one need not necessarily be the mean flow.

By taking the mean and the Fourier cutoff of the continuity equation we have (up to commutation error)

$$\frac{\partial \bar{u}_i}{\partial x_i} = \frac{\partial u'_i}{\partial x_i} = \frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial u''_i}{\partial x_i} = 0. \tag{5.225}$$

The equation for the mean field is almost the same as the RANS equation except that the fluctuations are decomposed as

$$\overline{u'_i u'_j} = \overline{(\tilde{u}_i + u''_i)(\tilde{u}_j + u''_j)} = \overline{\tilde{u}_i \tilde{u}_j} + \overline{u''_i u''_j}. \tag{5.226}$$

The RANS equation can be re-written in an LES-consistent form as

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial}{\partial x_j}(\overline{u_i u_j} - \bar{u}_i \bar{u}_j) + \frac{\partial}{\partial x_j}(2\nu \bar{S}_{ij}) + \bar{f}_i, \tag{5.227}$$

or, in a more familiar form

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial}{\partial x_j}(\overline{u'_i u'_j}) + \frac{\partial}{\partial x_j}(2\nu \bar{S}_{ij}) + \bar{f}_i, \tag{5.228}$$

where the Reynolds stress can be interpreted as the sub-filter residual stress where $\overline{(\cdot)}$ is the first filtering operation:

$$\tau_{ij}^{\mathrm{Rey}} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j = \overline{u'_i u'_j}. \tag{5.229}$$

The fluctuation equation can be obtained by subtracting the RANS equation from the total momentum equation:

$$\frac{\partial u'_i}{\partial t} + \bar{u}_j \frac{\partial u'_i}{\partial x_j} + u'_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} - \frac{\partial}{\partial x_j}\left(u'_i u'_j - \overline{u'_i u'_j}\right) + \frac{\partial}{\partial x_j}(2\nu S'_{ij}) + f'_i. \tag{5.230}$$

The large-scale fluctuation equation can be obtained by filtering the fluctuation equation (denote $\tilde{u}_i = \widetilde{(u'_i)}$ as the filtered velocity and $u''_i = u'_i - \tilde{u}_i$ as its residual; also assume $\bar{\tilde{u}}_i = \bar{u}_i$), leading to

$$\frac{\partial \tilde{u}_i}{\partial t} + \bar{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} - \frac{\partial}{\partial x_j}\left(\widetilde{u'_i u'_j} - \overline{u'_i u'_j}\right) + \frac{\partial}{\partial x_j}(2\nu \tilde{S}_{ij}) + \tilde{f}_i \tag{5.231}$$

It can also be written as

$$\frac{\partial \tilde{u}_i}{\partial t} + \bar{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} - \frac{\partial}{\partial x_j}\left((\widetilde{u'_i u'_j} - \tilde{u}_i\tilde{u}_j) - \overline{u'_i u'_j}\right) + \frac{\partial}{\partial x_j}(2\nu \tilde{S}_{ij}) + \tilde{f}_i \tag{5.232}$$

for the needs of deriving the energy equation.

If $\widetilde{(\cdot)}$ is the phase average used in Hussain & Reynolds (1970), due to

$$\widetilde{u'_i u'_j} = \tilde{u}_i\tilde{u}_j + \widetilde{u''_i u''_j} \tag{5.233}$$

we will recover the large-scale equation there. However, with the filtering operator,

$$\widetilde{u'_i u'_j} = \widetilde{\tilde{u}_i\tilde{u}_j} + \widetilde{u''_i u''_j} + \widetilde{u''_j \tilde{u}_j} + \widetilde{\tilde{u}_i u''_j}, \tag{5.234}$$

there will be additional terms.

The small-scale equation is obtained by subtracting the large-scale equation from the fluctuation equation:

$$\frac{\partial u''_i}{\partial t} + \bar{u}_j \frac{\partial u''_i}{\partial x_j} + u''_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p''}{\partial x_i} - \frac{\partial}{\partial x_j}(u'_i u'_j - \widetilde{u'_i u'_j}) + \frac{\partial}{\partial x_j}(2\nu S''_{ij}) + f''_i \tag{5.235}$$

$$\frac{\partial u''_i}{\partial t} + \bar{u}_j \frac{\partial u''_i}{\partial x_j} + u''_j \frac{\partial \bar{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial u''_i}{\partial x_j} + u''_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p''}{\partial x_i} - \frac{\partial}{\partial x_j}(u''_i u''_j + \tilde{u}_i\tilde{u}_j - \widetilde{u'_i u'_j}) + \frac{\partial}{\partial x_j}(2\nu S''_{ij}) + f''_i \tag{5.236}$$

$$\frac{\partial u''_i}{\partial t} + \bar{u}_j \frac{\partial u''_i}{\partial x_j} + u''_j \frac{\partial \bar{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial u''_i}{\partial x_j} + u''_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p''}{\partial x_i} - \frac{\partial}{\partial x_j}(u''_i u''_j - \tau_{ij}^{\mathrm{SGS}}) + \frac{\partial}{\partial x_j}(2\nu S''_{ij}) + f''_i \tag{5.237}$$

$$\tau_{ij}^{\mathrm{SGS}} = \widetilde{u'_i u'_j} - \tilde{u}_i\tilde{u}_j \tag{5.238}$$

which is still consistent with the small-scale equation in Hussain & Reynolds (1970) if (5.233) were right. It also takes the same form as the u'_i equation whose respective large-scale is $\bar{u}_i$ – the stresses include the self-interaction stresses and the sub-filter stresses (that couples the large and the small scales).

Multiply respective equations with respective velocity components, the scale-divided kinetic energy equations (MKE, TKE, LKE, and SKE) can be derived as

$$\left(\frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j}\right)\frac{\bar{u}_i\bar{u}_i}{2} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\bar{p}\bar{u}_j + \bar{u}_i\overline{u'_i u'_j} - 2\nu\bar{u}_i\bar{S}_{ij}\right) + \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - 2\nu\bar{S}_{ij}\bar{S}_{ij} + \bar{u}_i\bar{f}_i$$

$$\left(\frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j}\right)\frac{\overline{u'_i u'_i}}{2} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{p' u'_j} + \frac{1}{2}\overline{u'_i u'_i u'_j} - 2\nu\overline{u'_i S'_{ij}}\right) - \overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - 2\nu\overline{S'_{ij}S'_{ij}} + \overline{f'_i u'_i}$$

$$
\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)\frac{\overline{\tilde{u}_i\tilde{u}_i}}{2} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{\tilde{p}\tilde{u}_j}+\frac{1}{2}\overline{\tilde{u}_i\tilde{u}_i\tilde{u}_j}+\overline{\tilde{u}_i(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)}-2\nu\overline{\tilde{u}_i\tilde{S}_{ij}}\right)-2\nu\tilde{S}_{ij}\tilde{S}_{ij}+\tilde{u}_i\tilde{f}_i
$$

$$
-\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial\bar{u}_i}{\partial x_j}+\overline{(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)\frac{\partial\tilde{u}_i}{\partial x_j}}
$$

$$
\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)\frac{\overline{u''_iu''_i}}{2} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{p''u''_j}+\frac{1}{2}\overline{(\tilde{u}_j+u''_j)u''_iu''_i}-\overline{u''_i(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)}-2\nu\overline{u''_iS''_{ij}}\right)
$$

$$
-\overline{u''_iu''_j}\frac{\partial\bar{u}_i}{\partial x_j}-\overline{u''_iu''_j\frac{\partial\tilde{u}_i}{\partial x_j}}-\overline{(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)\frac{\partial u''_i}{\partial x_j}}-2\nu\overline{S''_{ij}S''_{ij}}+\overline{u''_if''_i}
$$

Additionally, even though in the present formulation $\overline{u''_i\tilde{u}_i}=0$ due to the fact that the large scale and small scale fluctuations are uncorrelated, its transport equation can still be useful serving as a bridge in the large-small interaction / transfer. Also, it will be useful when filters other than Fourier sharp filter is used.

The cross kinetic energy (CKE) is:

$$
\left(\frac{\partial}{\partial t}+\bar{u}_j\frac{\partial}{\partial x_j}\right)\frac{\overline{u''_i\tilde{u}_i}}{2}+\overline{(u''_i\tilde{u}_j+u''_j\tilde{u}_i)}\frac{\partial\bar{u}_i}{\partial x_j}
$$

$$
= -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{(u''_j\tilde{p}+\tilde{u}_jp'')}+\overline{\tilde{u}_iu''_iu''_j}+\overline{u''_i\tilde{u}_i\tilde{u}_j}+\frac{1}{2}\overline{\tilde{u}_i\tilde{u}_iu''_j}+\overline{(u''_i-\tilde{u}_i)(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)}-2\nu\overline{(u''_i\tilde{S}_{ij}+\tilde{u}_iS''_{ij})}\right)
$$

$$
+\overline{u''_iu''_j\frac{\partial\tilde{u}_i}{\partial x_j}}+\overline{(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)\frac{\partial u''_i}{\partial x_j}}-\overline{(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)\frac{\partial\tilde{u}_i}{\partial x_j}}-4\nu\overline{S''_{ij}\tilde{S}_{ij}}+\overline{(u''_i\tilde{f}_i+\tilde{u}_if''_i)}
$$

and in the case of $\overline{\tilde{a}b''}=\overline{a''\tilde{b}}=0$ for any a', b', it simplifies to the following auxiliary equation

$$
0 = -\frac{\partial}{\partial x_j}(\overline{\tilde{u}_iu''_iu''_j}+\overline{u''_i\tilde{u}_i\tilde{u}_j}+\frac{1}{2}\overline{\tilde{u}_i\tilde{u}_iu''_j}+\overline{(u''_i-\tilde{u}_i)(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)}) \tag{5.239}
$$

$$
+\overline{u''_iu''_j\frac{\partial\tilde{u}_i}{\partial x_j}}+\overline{(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)\frac{\partial u''_i}{\partial x_j}}-\overline{(\widetilde{u'_iu'_j}-\tilde{u}_i\tilde{u}_j)\frac{\partial\tilde{u}_i}{\partial x_j}} \tag{5.240}
$$

Comments:

- Since we are in a RANS framework, the material derivative is associated the mean flow (sitting on a reference frame moving with the mean).

- In the MKE equation,

$$
-\frac{\partial}{\partial x_j}(\bar{u}_i\overline{u'_iu'_j}-2\nu\bar{u}_i\bar{S}_{ij}) = \frac{\partial}{\partial x_j}[(-\tau_{ij}^{\mathrm{Rey}}+\tau_{ij}^{\mathrm{vis}})\bar{u}_i] \tag{5.241}
$$

  represents the turbulent and molecular diffusion. Both are spatial transports. Note that for consistency with SGS stresses, the Reynolds stress ($\tau_{ij}^{\mathrm{Rey}}=\overline{u'_iu'_j}$) has an opposite sign to its conventional definition. We also note $\overline{(\cdot)}$ does not necessarily need to be the Reynolds average, but can be any filtering operation that has the same correlation/commulation properties with $\widetilde{(\cdot)}$.

- If we add LKE, SKE and CKE together, we will exactly recover TKE. The budgets are closed and the triple decomposition is complete. With some algebra it can be shown that

$$\frac{1}{2}\overline{u_i'u_i'u_j'} = \frac{1}{2}\overline{\tilde{u}_i\tilde{u}_i\tilde{u}_j} + \frac{1}{2}\overline{u_i''u_i''u_j''} + \frac{1}{2}\overline{\tilde{u}_j u_i''u_i''} + \overline{u_i''\tilde{u}_i\tilde{u}_j} + \overline{\tilde{u}_i u_i''u_j''} + \frac{1}{2}\overline{\tilde{u}_i\tilde{u}_i u_j''}, \tag{5.242}$$

relating the fluctuation-related spatial transport terms (∂_j) in those equations.

- The sink in MKE equation is $-\bar{\Sigma}$, where

$$\bar{\Sigma} = -\overline{u_i'u_j'}\frac{\partial \bar{u}}{\partial x_j} = -\tau_{ij}^{\text{Rey}}\bar{S}_{ij} \tag{5.243}$$

is the source in the TKE equation and can be splitted into

$$\bar{\Sigma} = -\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} = -\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial \bar{u}_i}{\partial x_j} - \overline{u_i''u_j''}\frac{\partial \bar{u}_i}{\partial x_j} = \tilde{\Sigma} + \Sigma'' \tag{5.244}$$

$$\tilde{\Sigma} = -\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial \bar{u}_i}{\partial x_j} \tag{5.245}$$

$$\Sigma'' = -\overline{u_i''u_j''}\frac{\partial \bar{u}_i}{\partial x_j} \tag{5.246}$$

with positive $\bar{\Sigma}$ indicating energy being extracted from the mean flow supplied to the LS and SS fluctuations (sources in LKE and SKE equations).

- The inter-scale transfer term, from the large scales' perspective is $-\Sigma^{\text{SGS}}$, where

$$\Sigma^{\text{SGS}} = -\overline{(\widetilde{u_i'u_j'} - \tilde{u}_i\tilde{u}_j)\frac{\partial \tilde{u}_i}{\partial x_j}} = -\overline{\tau_{ij}^{\text{SGS}}\tilde{S}_{ij}}, \tag{5.247}$$

with positive Σ^{SGS} indicating forward energy cascade. However, from the small scales' perspective, the transfer term is

$$-\overline{u_i''u_j''\frac{\partial \tilde{u}_i}{\partial x_j}} - \overline{(\widetilde{u_i'u_j'} - \tilde{u}_i\tilde{u}_j)\frac{\partial u_i''}{\partial x_j}}, \tag{5.248}$$

which can be shown equal to Σ^{SGS} from the auxiliary equation (CKE). Even though the CKE is eventually zero statistically, the cross-correlation between large and small scale represents the way that energy is transferred: the energy is drained from the LKE, transferred to the CKE, and then from CKE to SKE. CKE is like a catylist who doesn't change at the end, but serves a bridging role. That being said, the LS and SS are coupled during some periods of their lifes.

5.3 Theory of homogeneous isotropic turbulence

5.3.1 Correlation function and spectrum

See THU lecture notes;

5.3.2 The Kármán–Horwarth equations

5.3.3 Structure function

5.4 Scales of turbulent motions

5.4.1 The Kolmogorov scale

The Kolmogorov scales are:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, u_\eta = (\varepsilon\nu)^{1/4}, \tau_\eta = (\nu/\varepsilon)^{1/2}. \tag{5.249}$$

Note the scales depend only on viscosity and dissipation rate (hence the expressions above can be derived from a dimensional analysis). A resulting Reynolds number

$$Re_\eta = \frac{u_\eta \eta}{\nu} = 1 \tag{5.250}$$

indicates that viscous effect is active at the Kolmogorov scale. Also,

$$\varepsilon = \nu \left(\frac{u_\eta}{\eta}\right)^2 \tag{5.251}$$

indicates that u_η and η are the scales of dissipative eddies. The inviscid scaling for dissipation also gives that

$$\varepsilon \sim \frac{u_L^3}{L} \tag{5.252}$$

where L is the energy-containing scale and u_L is the characteristic velocity. At the end, the scale separation is related to the (large-scale) Reynolds number as

$$\frac{L}{\eta} = \frac{L}{[\nu^3/(u_L^3/L)]^{1/4}} = \left(\frac{Lu_L}{\nu}\right)^{3/4} = Re_L^{3/4} \tag{5.253}$$

and similarly

$$\frac{u_L}{u_\eta} = Re_L^{1/4} \tag{5.254}$$

$$\frac{T_L}{\tau_\eta} = Re_L^{1/2} \tag{5.255}$$

5.4.2 The Taylor microscale

In isotropic turbulence, the velocity gradient takes the form of

$$\left\langle \left(\frac{\partial u_i}{\partial x_j}\right)^2 \right\rangle = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle \tag{5.256}$$

(see exercise 5.28 in Pope (2001)) and dissipation can be expressed as

$$\varepsilon = \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\rangle = 15\nu\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle. \tag{5.257}$$

Practically, the Taylor length scale can be computed as

$$\lambda_g = \sqrt{\frac{15\nu u'^2}{\varepsilon}} \tag{5.258}$$

via measuring u' (the rms velocity) and ε. Then the Taylor Reynolds number is

$$Re_\lambda = \frac{u'\lambda_g}{\nu}, \tag{5.259}$$

which is commonly used to characterize isotropic turbulence. It can be related to the large scale Reynolds number,

$$Re = \frac{ul}{\nu} \tag{5.260}$$

through the inviscid scaling of dissipation ($\varepsilon = u^3/l$) as

$$Re_\lambda = (15Re)^{1/2}. \tag{5.261}$$

The Taylor length scale can also be estimated from correlation functions. The transverse Taylor length scale is defined as

$$\lambda_g = \left[-\frac{1}{2}g''(0)\right]^{-1/2} \tag{5.262}$$

which follows a polynomial approximation of the transverse correlation function

$$p(r) = g(0) + g'(0)r + \frac{1}{2}g''(0)r^2 \tag{5.263}$$

and

$$g(r) = \frac{\langle u_2(\boldsymbol{x} + r\boldsymbol{e}_1)u_2(\boldsymbol{x})\rangle}{\langle u_2^2\rangle} \tag{5.264}$$

(think the correlation between two points of opposite locations in a vortex). It is known that $g(0) = 1, g'(0) = 0$ and $g''(0) < 0$. Then

$$p(r) = 1 + \frac{1}{2}g''(0)r^2, \tag{5.265}$$

which is a parabola that intersects the horizontal axis at

$$r = \lambda_g = \left[-\frac{1}{2}g''(0)\right]^{-1/2}, \tag{5.266}$$

which can be interpreted as an approximate 'de-correlation' length. Think two points having a small correlation coefficient when they are more than an eddy size apart (statistical eddy). It can also be shown (see Pope (2001)) $\lambda_f = \sqrt{2}\lambda_g$ where $\lambda_f = [-\frac{1}{2}f''(0)]^{-1/2}$ and $f(r) = \langle u_1(\boldsymbol{x} + r\boldsymbol{e}_1)u_2(\boldsymbol{x})\rangle\langle u_1^2\rangle$. Using λ_g, the dissipation is given by

$$\varepsilon = 15\nu\frac{u'^2}{\lambda_g}. \tag{5.267}$$

5.4.3 Other useful scales

The Corrsin scale (below which the flow doesn't feel the shear and turbulence in different shear flows are similar):

$$L_C = \left(\frac{\varepsilon}{S^3}\right)^{1/2}, \tag{5.268}$$

where S is the background shear. Consider the time scale comparison between dissipation and shear distortion:

$$\frac{T_\varepsilon}{T_S} = \frac{k/\varepsilon}{1/S} = \frac{(SL)^2/\varepsilon}{1/S} = S^3 L/\varepsilon = \frac{L^2}{L_C^2}. \tag{5.269}$$

Here L is the length scale of energy containing eddies. When $L > L_C$ or $T_\varepsilon > T_S$, the eddies will be distroyed by shear before being dissipated.

An analog to the Corrsin scale in stratified turbulence is the Ozmidov scale:

$$L_O = \left(\frac{\varepsilon}{N^3}\right)^{1/2} \tag{5.270}$$

where S is replaced by N and a similar timescale comparison can be made. The Kolmogorov scale is

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \tag{5.271}$$

the ratio between these two gives $L_O/\eta = Re_b^{3/4}$. The buoyancy Reynolds is:

$$Re_b = \frac{\varepsilon}{\nu N^2}. \tag{5.272}$$

It can be regarded as a measure of the separation between scales unaffected by stratification and the dissipation range, and hence an indicator of the extent of small-scale isotropy.

Taking L_O as the vertical length scale for overturns and $q_h = \sqrt{(\langle u'^2 + v'^2 \rangle)}/2$ to be the velocity scale, the Froude number is unity hence the scales greater than L_O are influenced by stratification.

5.5 Turbulent free shear flows

5.5.1 Momentum integral

Similarity solutions (turbulent). Pope (2001).

5.5.2 Similarity solutions

The characteristic velocity and length scales are U_s and δ_s, respectively.

The example of plane jet is the easiest to understand and derive so we are the most detailed in that case and more loosely on the others. The same principles and machinery apply to all cases.

Flow type	U_s	δ_s	$U_s \propto x^m$	$\delta_s \propto x^n$	$f(\eta)$
Round jet	$\bar{u}(x, y=0)$	$r_{1/2}$	-1	1	$1/(1+a\eta^2)^2$
Plane jet	$\bar{u}(x, r=0)$	$y_{1/2}$	$-1/2$	1	$\mathrm{sech}^2(\ln(1+\sqrt{2})\,\eta)$
Round wake	$U_\infty - \bar{u}(x, y=0)$	$r_{1/2}$	$-2/3$	$1/3$	$\exp(-\ln 2\,\eta^2)$
Plane wake	$U_\infty - \bar{u}(x, r=0)$	$y_{1/2}$	$-1/2$	$1/2$	$\exp(-\ln 2\,\eta^2)$
Plane mixing layer	$U_2 - U_1$	$y_{0.9} - y_{0.1}$	0	1	$1/2\,\mathrm{erf}(\eta/\sigma\sqrt{2})$

Table 5.1: Self-similar solution table.

5.5.3 Round jet

Characteristic scales:
The centerline velocity is

$$U_s(x) = \bar{u}(x, r=0) \tag{5.273}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = \bar{u}(x, r_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{5.274}$$

Momentum integral constraint:
The boundary layer equation in cylindrical coordinates reads

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{r}\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{5.275}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial (r\bar{v})}{\partial r} = 0 \tag{5.276}$$

by $r\bar{u}$ and add it to (5.275) multiplied by r we obtain

$$\frac{\partial (r\bar{u}\bar{u})}{\partial x} + \frac{\partial (r\bar{u}\bar{v})}{\partial r} = -\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{5.277}$$

Integrate (5.277) in r we obtain

$$\int_0^\infty \frac{\partial (r\bar{u}\bar{u})}{\partial r}\,\mathrm{d}r + r\bar{u}\bar{v}|_0^\infty = -r\overline{u'v'}|_0^\infty \tag{5.278}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^\infty r\bar{u}^2\,\mathrm{d}r\right) = 0 \tag{5.279}$$

which implies the momentum flux

$$\dot{J}(x) = \int_0^\infty \rho\bar{u}^2 2\pi r\,\mathrm{d}r = J_0 \tag{5.280}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.
Self-similar assumptions:

$$\bar{u} = U_s(x)f(\eta),\ \overline{u'v'} = U_s^2(x)g(\eta),\ \eta = \frac{r}{r_{1/2}} \tag{5.281}$$

where $\eta = r/\delta_s(x)$ with $\delta_s = r_{1/2}$. Substitute (5.281) into (5.280) we have

$$\dot{J}(x) = (2\pi\rho)(U_s^2\delta_s^2)\left(\int_0^\infty \eta f^2(\eta)\,\mathrm{d}\eta\right) \tag{5.282}$$

to be a constant and implying

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2\delta_s^2) = 0 \tag{5.283}$$

and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{5.284}$$

Using the continuity equation we have

$$\bar{v} = -\frac{1}{r}\int_0^r \frac{\partial(r\bar{u})}{\partial x}\,\mathrm{d}y = U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{\eta}\int_0^\eta f\eta\,\mathrm{d}\eta\right). \tag{5.285}$$

Next we establish the constant spread rate of the round jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left[f^2\eta + ff'\eta + \left(\frac{f}{\eta} + f'\right)\int_0^\eta f\eta\,\mathrm{d}\eta\right] = g + g'\eta \tag{5.286}$$

and then $\mathrm{d}\delta_s/\mathrm{d}x$ has to be a constant. Combining with momentum integral restriction we have

$$\delta_s \propto x,\ U_s \propto x^{-1}. \tag{5.287}$$

We note that $\bar{v}$ is positive near the centerline and it switches sign from positive to negative as r reaches a certain value. This is the entrainment of ambient fluid. The entrainment is can be shown by the mass flux that

$$\dot{M}(x) = \int_0^\infty \rho\bar{u}2\pi r\,\mathrm{d}r = 2\pi\rho(U_s\delta_s^2)\left(\int_0^\infty \eta f(\eta)\,\mathrm{d}\eta\right) \sim x, \tag{5.288}$$

i.e., the mass flux increases as x.

5.5.4 Plane jet

Characteristic scales:
The centerline velocity is

$$U_s(x) = \bar{u}(x, y=0) \tag{5.289}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = \bar{u}(x, y_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{5.290}$$

Momentum integral constraint:
The boundary layer equation for the mean velocity simplifies to

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{5.291}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{5.292}$$

by $\bar{u}$ and add it to (5.291) we obtain

$$\frac{\partial \bar{u}\bar{u}}{\partial x} + \frac{\partial \bar{u}\bar{v}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{5.293}$$

Integrate (5.293) in y we obtain

$$\int_{-\infty}^{\infty} \frac{\partial \bar{u}\bar{u}}{\partial x}\,\mathrm{d}y + \bar{u}\bar{v}|_{-\infty}^{\infty} = -\overline{u'v'}|_{-\infty}^{\infty} \tag{5.294}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty} \bar{u}^2\,\mathrm{d}y\right) = 0 \tag{5.295}$$

which implies the momentum flux

$$\dot{J}(x) = \int_{-\infty}^{\infty} \rho\bar{u}^2\,\mathrm{d}y = J_0 \tag{5.296}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.
Self-similar assumptions:

$$\bar{u} = U_s(x)f(\eta),\ \overline{u'v'} = U_s^2(x)g(\eta),\ \eta = \frac{y}{y_{1/2}} \tag{5.297}$$

where $\eta = y/\delta_s(x)$ and we have

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{\delta_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{5.298}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta_s} \tag{5.299}$$

Substitute (5.297) into (5.296) we have

$$\dot{J}(x) = (U_s^2\delta_s)\left(\int_{-\infty}^{\infty} f^2(\eta)\,\mathrm{d}\eta\right) = J_0 \tag{5.300}$$

is a constant. So it must be

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2\delta_s) = 0 \tag{5.301}$$

which gives the momentum flux constraint in terms of characteristic variables, and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{5.302}$$

Using the continuity equation we have

$$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x}\,\mathrm{d}y = U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{2}\int_0^\eta f\,\mathrm{d}\eta\right) \tag{5.303}$$

Next we establish the constant spread rate of the plane jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(f^2 + f'\int_0^\eta f\,\mathrm{d}\eta) = g' \tag{5.304}$$

and then

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x} = \frac{2g'}{f^2 + f'\int_0^\eta f\,\mathrm{d}\eta} = C \tag{5.305}$$

with the LHS only depend on x and RHS only depend on η. Then both sides have to be constant. Combining (5.305) and (5.301) we have

$$\delta_s \propto x,\, U_s \propto x^{-1/2}. \tag{5.306}$$

Entrainment. The mass flux is

$$\dot{M}(x) = (U_s\delta_s)\left(\int_{-\infty}^{\infty} f(\eta)\,\mathrm{d}\eta\right) \sim x^{1/2}. \tag{5.307}$$

5.5.5 Round wake

Characteristic scales:
The centerline velocity deficit is

$$U_0(x) = U_\infty - \bar{u}(x, r=0) = U_d(x,0) \tag{5.308}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = U_\infty - \bar{u}(x, r_{1/2}(x)) = \frac{1}{2}U_0(x). \tag{5.309}$$

Momentum integral constraint:
Here we start from the simplified (see plane wake) momentum equation

$$U_\infty\frac{\partial(\bar{u} - U_\infty)}{\partial x} = -\frac{1}{r}\frac{\partial(r\overline{u'v'})}{\partial r} \tag{5.310}$$

and the corresponding far-wake simplified momentum deficit flux conservation

$$\dot{J}(x) = \int_0^\infty \rho\bar{u}(U_\infty - \bar{u})2\pi r\,\mathrm{d}r \approx \int_0^\infty \rho U_\infty(U_\infty - \bar{u})2\pi r\,\mathrm{d}r = J_0. \tag{5.311}$$

Note that we have already replaced $\bar{u}$ with U_∞, assuming (or by order of magnitude analysis) $\bar{u}/U_\infty$ is small and the convection velocity is U_∞.

Self-similar assumptions:

$$U_\infty - \bar{u} = U_s(x)f(\eta),\ \overline{u'v'} = U_s^2(x)g(\eta),\ \eta = \frac{r}{r_{1/2}}, \tag{5.312}$$

We have

$$\dot{J}(x) = (U_s\delta_s^2)(2\pi\rho U_\infty)\int_0^\eta f\,\mathrm{d}\eta \tag{5.313}$$

is a constant and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s\delta_s^2) = 0. \tag{5.314}$$

Consider the momentum equation, the other constraint reads

$$-\frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(2f + f'\eta)\eta = (g'\eta + g) \tag{5.315}$$

We define the spread rate as

$$S = \frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}, \tag{5.316}$$

and then

$$-S(2f\eta + f'\eta^2) = (g\eta)'. \tag{5.317}$$

S, as a function of x, has to be a constant.

Include the boundary conditions and after integration we get

$$g = -S\eta f \tag{5.318}$$

same as in plane wakes. Combining (5.314) and (5.316) we have

$$\delta_s \propto x^{1/3},\, U_s \propto x^{-2/3}. \tag{5.319}$$

Eddy-viscosity model:
Invoke the eddy viscosity assumption ($\overline{u'v'} = U_s^2 g \sim \nu_T\partial_y U$, $\nu_T = \hat{\nu}_T r_{1/2}U_s$), we have

$$g = \hat{\nu}_T f', \tag{5.320}$$

and by assuming $\hat{\nu}_T$ is a constant (which is quite good for the core of the wake/jet), it can be solved that

$$f(\eta) = Ce^{-\frac{S}{2\hat{\nu}_T}\eta^2}. \tag{5.321}$$

The boundary conditions are $f(0) = 1, f(1) = 1/2$. Finally, the self-similar function is determined to be

$$f(\eta) = e^{-\ln 2\,\eta^2} \tag{5.322}$$

with $S = 2\ln 2\,\hat{\nu}_T$ (constant spreading rate).

5.5.6 Plane wake

Characteristic scales:
The centerline velocity deficit is

$$U_s(x) = U_\infty - \bar{u}(x, y = 0) = U_d(x, 0) \tag{5.323}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = U_\infty - \bar{u}(x, y_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{5.324}$$

Momentum integral constraint:
The boundary layer equation:

$$\bar{u}\frac{\partial(\bar{u} - U_\infty)}{\partial x} + \bar{v}\frac{\partial(\bar{u} - U_\infty)}{\partial y} = \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{5.325}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{5.326}$$

by $\bar{u} - U_\infty$ and add it to (5.325) we obtain

$$\frac{\partial \bar{u}(\bar{u} - U_\infty)}{\partial x} + \frac{\partial \bar{v}(\bar{u} - U_\infty)}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}. \tag{5.327}$$

Integrate (5.293) in y we obtain

$$\int_{-\infty}^{\infty} \frac{\partial \bar{u}(\bar{u} - U_\infty)}{\partial x}\,\mathrm{d}y + \bar{v}(\bar{u} - U_\infty)|_{-\infty}^{\infty} = -\overline{u'v'}|_{-\infty}^{\infty} \tag{5.328}$$

and since $\overline{u'v'}$ and $\bar{u} - U_\infty$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty} \bar{u}(\bar{u} - U_\infty)\,\mathrm{d}y\right) = 0 \tag{5.329}$$

which implies the momentum deficit flux

$$\dot{J}(x) = \int_{-\infty}^{\infty} \rho\bar{u}(U_\infty - \bar{u})\,\mathrm{d}y \tag{5.330}$$

is conserved (we note that we haven't assumed far wake or $U_s \sim \bar{u} - U_\infty \ll U_\infty$ yet).
Self-similar assumptions:

$$U_\infty - \bar{u} = U_s(x)f(\eta),\ \overline{u'v'} = U_s^2(x)g(\eta),\ \eta = \frac{y}{y_{1/2}} \tag{5.331}$$

Substitute (5.331) into (5.330), and assume the far wake is reached ($U_s/U_\infty \ll 1$) we have

$$\dot{J}(x) = \int_{-\infty}^{\infty} \rho(U_\infty - U_s f)U_s f \delta_s\,\mathrm{d}\eta \tag{5.332}$$

$$= U_\infty^2 \int_{-\infty}^{\infty} \rho(1 - \frac{U_s f}{U_\infty}) \frac{U_s}{U_\infty} f \delta_s \, \mathrm{d}\eta \tag{5.333}$$

$$= \rho U_\infty U_s \delta_s \int_{-\infty}^{\infty} f \, \mathrm{d}\eta \tag{5.334}$$

is a constant. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s \delta_s) = 0. \tag{5.335}$$

Using the continuity equation we have

$$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x} \, \mathrm{d}y = -U_s \frac{\mathrm{d}\delta_s}{\mathrm{d}x} f\eta. \tag{5.336}$$

Note the negative speed ($\partial_x \bar{u} > 0$) corresponds to wake entrainment (of high momentum into low momentum region).

Now we consider another constraint. Since in the far wake, the velocity deficit $U_s/U_\infty \ll 1$, we have the simplification of the momentum equation as

$$\frac{\partial \bar{u}(\bar{u} - U_\infty)}{\partial x} + \frac{\partial \bar{v}(\bar{u} - U_\infty)}{\partial y} = U_\infty \frac{\partial \bar{u}}{\partial x} = -\frac{\partial \overline{u'v'}}{\partial y} \tag{5.337}$$

where

$$\bar{u}(\bar{u} - U_\infty) = (U_\infty - U_s f)(-U_s f) = U_\infty^2 (1 - \frac{U_s f}{U_\infty})(-\frac{U_s}{U_\infty} f) = -U_s U_\infty f = U_\infty(\bar{u} - U_\infty). \tag{5.338}$$

And the scale for $\partial \bar{u}(\bar{u} - U_\infty)/\partial x$ is

$$\frac{U_\infty U_s}{L_x} \tag{5.339}$$

while the scale for $\partial \bar{v}(\bar{u} - U_\infty)/\partial y$ (from (5.336)) is

$$\frac{U_s}{\delta_s} \left(U_s \frac{\delta_s}{L_x} \right). \tag{5.340}$$

Define the spread rate as

$$S = \frac{U_\infty}{U_s} \frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{5.341}$$

Take $\bar{v}$ into the simplified momentum equation we have

$$(f + f'\eta) \frac{U_\infty}{U_s} \frac{\mathrm{d}\delta_s}{\mathrm{d}x} = -g' \tag{5.342}$$

with S depends only on x and the rest on η hence S has to be a constant. Then (5.342) can be rewritten as

$$g' + S(f + f'\eta) = 0 \tag{5.343}$$

which is to say

$$(g + S\eta f)' = 0. \tag{5.344}$$

Integrate from $\eta = 0$ to η and note that $g(0) = 0$, we have

$$g = -S\eta f. \tag{5.345}$$

Combining two conditions (5.335) and (5.341) we have

$$\delta_s \propto x^{1/2},\, U_s \propto x^{-1/2}. \tag{5.346}$$

Eddy-viscosity model:
Invoke the constant eddy viscosity assumption,

$$g = \hat{\nu}_T f', \tag{5.347}$$

and the boundary conditions $f(0) = 1, f(1) = 1/2$ it can be solved that the self-similar function is determined to be

$$f(\eta) = e^{-\ln 2\, \eta^2} \tag{5.348}$$

with $S = 2\ln 2\, \hat{\nu}_T$. We note that this self-similar function is the same as in asymmetry wakes.

5.5.7 Plane mixing layer

Characteristic scales:
The two velocities are $U_2 > U_1$ with U_2 on the top. The mean convection velocity is

$$U_c = \frac{1}{2}(U_1 + U_2) \tag{5.349}$$

and the characteristic velocity scale is

$$U_s = U_2 - U_1. \tag{5.350}$$

The characteristic length is the mixing layer width,

$$\delta_s(x) = y_{0.9} - y_{0.1} \tag{5.351}$$

with cross-stream location $y_\alpha(x)$ such that

$$\bar{u}(x, y_\alpha(x)) = U_1 + \alpha U_s. \tag{5.352}$$

a reference position is

$$\hat{y} = \frac{1}{2}(y_{0.1} + y_{0.9}) \tag{5.353}$$

such that the self-similar variable is defined as

$$\eta = \frac{y - \hat{y}}{\delta_s(x)} \tag{5.354}$$

5.6 Turbulent wall flows

5.6.1 Turbulent channel flow

The mean flow equations are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{5.355}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'v'}}{\partial y} \tag{5.356}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - \frac{\partial \overline{v'^2}}{\partial y} \tag{5.357}$$

Integrate (5.357) in y from 0 to y we have

$$\frac{\bar{p}}{\rho} + \overline{v'^2} = \frac{p_w}{\rho} \tag{5.358}$$

where p_w is the mean wall pressure. Hence, we have

$$\frac{\partial \bar{p}}{\partial x} = \frac{\partial p_w}{\partial x}, \tag{5.359}$$

implying that the pressure drop over the same distance is the same for different heights, even $\partial \bar{p}/\partial y \neq 0$. The wall shear stress balance (4.15) is still valid (control volume from 0 to h) as

$$-\frac{\partial p_w}{\partial x} = \frac{\tau_w}{h}. \tag{5.360}$$

Substitute (5.358) and (5.360) back to (5.356) we have

$$-\frac{\tau_w}{h} = \nu\frac{\partial^2 \bar{u}}{\partial^2 y} - \frac{\partial \overline{u'v'}}{\partial y} \tag{5.361}$$

the $\int_y^h dy$ of which leads to the channel basin equation

$$\tau(y) = \tau_w\left(1 - \frac{y}{h}\right) = \rho u_\tau^2 = \mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'} \tag{5.362}$$

which is stating that, the total stress (viscous shear stress and Reynolds stress) decreases linearly toward the centerline. Near the wall, the viscous shear stress dominates and it decreases as the distance from the wall increases. From data, one can plot the total stress profile (which is quite linear) and figure out u_τ.

FIK;

Chapter 6

Geophysical fluid dynamics

6.1 Basics

6.1.1 Non-inertial frames, centrifugal and Coriolis forces

6.1.2 Absolute velocity

In an inertial frame,

$$\boldsymbol{u}_a = \boldsymbol{u}_{(r)} + \boldsymbol{\Omega} \times \boldsymbol{r}, \tag{6.1}$$

where $\boldsymbol{u}_{(r)}$ is the relative (to the non-inertial frame) velocity. Especially, in a cylindrical frame with a plane-normal rotation,

$$u_{\theta,a} = u_\theta + \Omega r. \tag{6.2}$$

6.1.3 Inertial oscillations: buoyancy and Coriolis frequencies

6.2 Boussinesq approximation

6.3 Balanced flows

6.3.1 Hydrostatic and geostrophic balances

In balanced flow, which is usually the case to the first order, there is a background horizontal pressure gradient that balances the Coriolis forces due to horizontal motions and a vertical pressure gradient that balances the background unperturbed density:

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x} + fV \tag{6.3}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y} - fU \tag{6.4}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial z} - \frac{\rho^* g}{\rho_0} \tag{6.5}$$

with

$$p_g = p - p_0,\ \rho^* = \rho - \rho_0 - \rho_b(z) \tag{6.6}$$

and the background balance

$$0 = -\frac{\partial p_0}{\partial z} - (\rho_0 + \rho_b)g \tag{6.7}$$

already subtracted. Note that the Boussinesq and hydrostatic approximations are already applied.

The above equations in vector form:

$$\boldsymbol{f}_{\rm c} \times \boldsymbol{U} = -\frac{1}{\rho_0}\nabla p_g + \frac{\rho^*}{\rho_0}\boldsymbol{g}. \tag{6.8}$$

We have

$$\boldsymbol{U} = (U, V, 0) = -\frac{1}{\rho_0 f}(\frac{\partial p_g}{\partial y}, -\frac{\partial p_g}{\partial x}, 0). \tag{6.9}$$

And we have

$$\nabla_{\rm h} \cdot \boldsymbol{U} = 0. \tag{6.10}$$

In the world geostrophic, geo means Coriolis and strophic means cyclone/anticyclone or low-/high-pressure systems.

6.3.2 Thermal wind relations

In hydrostatic Boussinesq flow. Taking the vertical gradient of (6.8) and using the hydrostatic balance, we have

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial x} + f\frac{\partial V}{\partial z} \tag{6.11}$$

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial y} - f\frac{\partial U}{\partial z} \tag{6.12}$$

and hence

$$(\frac{\partial U}{\partial z}, \frac{\partial V}{\partial z}) = \frac{g}{\rho_0 f}(\frac{\partial \rho^*}{\partial y}, -\frac{\partial \rho^*}{\partial x}), \tag{6.13}$$

or in vector form,

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f} \times \nabla \rho^* \tag{6.14}$$

In a more general case. Without introducing the hydrostatic balance and Boussinesq approximation, we write

$$\boldsymbol{f}_c \times \boldsymbol{U} = \frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{6.15}$$

Taking its curl:

$$\text{LHS} = \nabla \times (\boldsymbol{f}_c \times \boldsymbol{U}) = -\boldsymbol{f}_c \cdot \nabla \boldsymbol{U} = -f\frac{\partial \boldsymbol{U}}{\partial z} \tag{6.16}$$

$$\text{RHS} = -\nabla \times (\frac{1}{\rho}\nabla p) + \nabla \times \boldsymbol{g} = -\frac{1}{\rho^2}(\nabla p \times \nabla \rho) \tag{6.17}$$

Re-introduce hydrostatic ($\partial_z p = -\rho g$) and Boussinesq, we have

$$\nabla p \approx \frac{\partial p}{\partial z}\boldsymbol{e}_z = -\rho g \boldsymbol{e}_z \tag{6.18}$$

and

$$-\frac{1}{\rho^2}(\nabla p \times \nabla \rho) = \frac{\rho g}{\rho_0^2}(\boldsymbol{e}_z \times \nabla \rho^*) \tag{6.19}$$

$$= -\frac{\boldsymbol{g}}{\rho_0} \times \nabla \rho^* \tag{6.20}$$

hence we recover

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f} \times \nabla \rho^*. \tag{6.21}$$

Note:

6.3.3 Cyclostrophic wind relations

In analogy to the thermal wind (Coriolis balances the horizontal pressure gradient), we have similarly the cyclostrophic wind (centrifugal balances the radial pressure gradient):

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_0}\frac{\partial p^*}{\partial r} \tag{6.22}$$

$$\frac{\partial p^*}{\partial z} = -\frac{\rho^* g}{\rho_0} \tag{6.23}$$

$$\frac{\partial \rho^*}{\partial r} = -\frac{\rho_0}{g}\frac{\partial}{\partial z}\left(\frac{u_\theta^2}{r}\right) \tag{6.24}$$

where the mean vertical wind shear is supported with a horizontal density gradient, which provides the centripetal accelaration ($a = u_\theta^2/r$). This has something to do with the Jet Stream. Cyclo means 'cyclone' or low-pressure system and strophic means 'turning'.

6.3.4 Example: Taylor-Proudman theory

Consider steady flow with negligible convective term (in geostrophic balance, $Ro \ll 1$):

$$0 = -\frac{1}{\rho}\nabla p + \boldsymbol{u} \times \boldsymbol{f}_\mathrm{c}. \tag{6.25}$$

Taking the curl of the above equation we have

$$0 = \nabla \times (\boldsymbol{u} \times \boldsymbol{f}_\mathrm{c}). \tag{6.26}$$

Using triple product rules (A.21), in an f-plane, we have

$$\boldsymbol{f}_\mathrm{c} \times \nabla \boldsymbol{u} = 0. \tag{6.27}$$

Assuming the rotation axis is normal to the plane, $\boldsymbol{f}_\mathrm{c} = f\boldsymbol{k}$, we have

$$\frac{\partial \boldsymbol{u}}{\partial z} = \boldsymbol{0}. \tag{6.28}$$

That implies very strong rotation negates vertical gradients.

Stratified Taylor column: consider the thermal wind balance

$$0 = -\frac{1}{\rho_0}\nabla p + \boldsymbol{u} \times \boldsymbol{f}_{\mathrm{c}} - \frac{\rho^* g}{\rho_0}\boldsymbol{k}. \tag{6.29}$$

Taking the curl we have

$$\boldsymbol{f}_{\mathrm{c}} \cdot \nabla \boldsymbol{u} = (\boldsymbol{i}\partial_y - \boldsymbol{j}\partial_x)\frac{\rho^* g}{\rho_0}, \tag{6.30}$$

i.e.,

$$[f_3\frac{\partial u}{\partial z}, f_3\frac{\partial v}{\partial z}, f_3\frac{\partial w}{\partial z}] = [\partial_y\frac{\rho^* g}{\rho_0}, -\partial_x\frac{\rho^* g}{\rho_0}, 0], \tag{6.31}$$

i.e.,

$$f_3\frac{\partial u}{\partial z} = \partial_y\frac{\rho^* g}{\rho_0} \tag{6.32}$$

$$f_3\frac{\partial v}{\partial z} = -\partial_x\frac{\rho^* g}{\rho_0} \tag{6.33}$$

$$\frac{\partial w}{\partial z} = 0 \tag{6.34}$$

which implies Q2D flow with

$$\nabla_H \cdot \boldsymbol{u} = \partial_x u + \partial_y v = -\partial_z w = 0. \tag{6.35}$$

The relations (6.32)-(6.33) are essentially thermal wind relations where the vertical wind shear is balanced by (the pressure gradient created by) horizontal density gradients.

6.3.5 Surface and bottom Ekman layers

In geostrophy, the dominant balance is the Coriolis force and the pressure gradient. Here in the laminar Ekman layer, we consider the steady solution resulting from a balance between viscous shear stress and the Coriolis force:

$$\begin{aligned} 0 &= fv + \nu\frac{\partial^2 u}{\partial z^2} \\ 0 &= -fu + \nu\frac{\partial^2 v}{\partial z^2} \end{aligned} \tag{6.36}$$

where the shear is only in the vertical direction due to horizontally uniform winds blowing over the surface (or uni-directional drag on the ocean bottom).

We denote

$$\boldsymbol{\tau}_a = (\tau_a^X, \tau_a^Y) = \left(\nu\frac{\partial u}{\partial z}\bigg|_{z=0}, \nu\frac{\partial v}{\partial z}\bigg|_{z=0}\right) \tag{6.37}$$

and

$$\mathcal{U}(z) = u(z) + iv(z). \tag{6.38}$$

Hence,

$$0 = f(u + iv) + i\nu\frac{\partial^2}{\partial z^2}(u + iv) \tag{6.39}$$

converts to

$$\frac{\partial^2 \mathcal{U}}{\partial z^2} = \frac{if}{\nu}\mathcal{U}. \tag{6.40}$$

The eigenvalues are

$$\lambda = \pm\sqrt{\frac{if}{\nu}} = \pm\frac{1+i}{\sqrt{2}}\sqrt{\frac{f}{\nu}} \tag{6.41}$$

Define the Ekman depth as

$$d_E = \sqrt{\frac{2\nu}{f}} \tag{6.42}$$

which increases as $\sqrt{\nu}$ and decreases as $\sqrt{1/f}$, and the general solution to (6.40) is

$$\mathcal{U}(z) = C_1 \exp\left[\frac{1+i}{\sqrt{2}}\sqrt{f/\nu z}\right] + C_2 \exp\left[-\frac{1+i}{\sqrt{2}}\sqrt{f/\nu z}\right]. \tag{6.43}$$

By knowing $\mathcal{U}(z=-\infty)$ we have $C_2 = 0$ and

$$\mathcal{U}(z) = \mathcal{U}_0 \exp\left[\frac{1+i}{\sqrt{2}}\sqrt{f/\nu z}\right] = \mathcal{U}_0 e^{z/d_E} e^{iz/d_E} \tag{6.44}$$

which tells us that as you go from the surface (z decreases from zero to $z < 0$), the magnitude of the velocity decreases and the velocity vector spirals CW (as you look down), in agreement with the direction of the Coriolis force.

Now we attempt to fix the constant $\mathcal{U}_0$. Given the complex velocity, the (directional) shear stress can be compactly written as

$$\boldsymbol{\tau}_a = \tau_a^X + i\tau_a^Y. \tag{6.45}$$

And the boundary condition is

$$\nu \left.\frac{\partial \mathcal{U}}{\partial z}\right|_{z=0} = \frac{\nu(1+i)\mathcal{U}_0}{d_E}, \tag{6.46}$$

leading to

$$\mathcal{U}_0 = \frac{\boldsymbol{\tau}_a d_E}{(1+i)\rho_0 \nu} \tag{6.47}$$

where

$$\frac{1}{1+i} = \frac{1}{\sqrt{2}} e^{-i\pi/4}, \tag{6.48}$$

which implies that the relative direction of the surface flow is 45 deg to the right (CW) of the wind stress.

Now we turn our attention to the Ekman transport over a finite depth of the mixed layer, $z_E \sim -200$ m, which is a few Ekman depths. The Ekman depth $d_E = \sqrt{2\nu_{\text{eff}}/f} \sim 45$ m for $f \sim 10^{-4} \sim 2\pi \times 24$ hr^{-1}. We note that $e^{-5} = 0.0067$ which might have been sufficient to assume the stress is weak enough to be neglected.

Vertically integrate (6.36) from $-z_E$ to 0, we have

$$\mathcal{U}_E = \int_{-z_E}^{0} u\, dz = \frac{\tau_a^Y}{\rho_0 f} \tag{6.49}$$

$$\mathcal{V}_E = \int_{-z_E}^{0} v\, dz = -\frac{\tau_a^X}{\rho_0 f} \tag{6.50}$$

or in a more compact way,

$$(\mathcal{U}_E, \mathcal{V}_E) = -\frac{1}{\rho_0 f}\boldsymbol{k} \times \boldsymbol{\tau}_a \tag{6.51}$$

where we note that the dimensions for $\mathcal{U}_E, \mathcal{V}_E$ are velocity times length. The direction of the transport, $(\mathcal{U}_E, \mathcal{V}_E)$, is $(\tau_a^Y, -\tau_a^X)$, which is perpendicular to and pointing to the right of the direction of the surface winds (and the wind forcing is balanced by the mean Coriolis force that is 90 deg to the right of the Ekman transport – opposite to the surface forcing). This does not mean that the surface forcing does no work, since the velocity on the surface is at a 45 deg angle to the winds.

We note several useful directions:

- The local velocity direction: $\mathcal{U} = u + iv = (u, v)$;
- The local shear direction: $\boldsymbol{\tau} = \tau^X + i\tau^Y = (\tau^X, \tau^Y) = (\nu\partial_z u, \nu\partial_z v)$;
- And the local Reynolds stress direction: $\boldsymbol{\tau}_{Rey} = (-\overline{u'w'}, -\overline{v'w'})$.

6.4 QGPV theory

The geostrophic balance in a stratified fluid is

$$\frac{Du}{Dt} = -p_x + fv \tag{6.52}$$

$$\frac{Dv}{Dt} = -p_y - fu \tag{6.53}$$

$$\frac{Dw}{Dt} = -p_z + b \tag{6.54}$$

$$\frac{Db}{Dt} + wN^2 = 0 \tag{6.55}$$

$$u_x + v_y + w_z = 0 \tag{6.56}$$

where the horizontal divergence-free condition naturally results from the geostrophy. The vertical divergence w_z is an order ($O(Ro)$) smaller and the (hydrostatically and geostrophically) balanced base flow is 2D.

With a streamfunction ψ, the above system can be written as

$$\begin{pmatrix} p \\ u \\ v \\ w \\ b \end{pmatrix} = \begin{pmatrix} f\psi \\ -\psi_y \\ \psi_x \\ 0 \\ f\psi_z \end{pmatrix} \tag{6.57}$$

where we have chosen the streamfunction $\psi = p/f$ to be proportional to the pressure (as an analog to the sea surface height). The idea of QG theory is to approximate (and then add back) the terms missing in the balanced flow ($Du_i/Dt, Db/Dt, w_z$, in blue) with the known relations (6.57). This technique is called recursion. For example,

$$\frac{Du}{Dt} = u_t + uu_x + vu_y = -\psi_{yt} + \psi_{yx}\psi_y - \psi_x\psi_{yy} = -\psi_{yt} - J(\psi, \psi_y) \tag{6.58}$$

where we define the Jacobian operator as

$$J(p, q) = p_x q_y - p_y q_x = \begin{vmatrix} p_x & p_y \\ q_x & q_y \end{vmatrix} \tag{6.59}$$

for convenience of expressing the convection terms. We have $J = 0$ when $p = q$. As a consequence,

$$\partial_y J(\psi, \psi_y) = J(\psi_y, \psi_y) + J(\psi, \psi_{yy}) = J(\psi, \psi_{yy}). \tag{6.60}$$

Also, J is a linear operator.

The (quasi-geostrophic) QG equations become

$$-\psi_{yt} - J(\psi, \psi_y) + p_x - fv = 0 \tag{6.61}$$
$$\psi_{xt} + J(\psi, \psi_x) + p_y + fu = 0 \tag{6.62}$$
$$p_z = b \tag{6.63}$$
$$f\psi_{zt} + J(\psi, f\psi_z) + wN^2 = 0 \tag{6.64}$$
$$u_x + v_y + w_z = 0 \tag{6.65}$$

By eliminating the pressure (cross-differentiating the first two momentum equations),

$$(\triangle\psi)_t + J(\psi, \triangle\psi) = fw_z, \tag{6.66}$$

where

$$\triangle\psi = (\partial_{xx} + \partial_{yy})\psi = \zeta \tag{6.67}$$

is the vertical (relative) vorticity. Taking the z-derivative of the buoyancy equation,

$$\frac{f^2}{N^2}\psi_{zzt} + J(\psi, \frac{f^2}{N^2}\psi_{zz}) + fw_z = 0, \tag{6.68}$$

that is,

$$(\mathcal{L}\psi)_t + J(\psi, \mathcal{L}\psi) + fw_z = 0, \tag{6.69}$$

where $\mathcal{L} = \partial_z(f^2/N^2\partial_z)$. Combine the vorticity and the buoyancy equations (6.66)&(6.69), we have

$$q_t + J(\psi, q) = 0, \tag{6.70}$$

where

$$q \triangleq \triangle\psi + \mathcal{L}\psi \tag{6.71}$$

is the potential vorticity (QGPV), which is conserved along the geostrophic motions as dictated by (6.70).

6.5 Governing equations of unbalanced motions

It is reasonable to assume directions of both system rotation and gravity are in $\boldsymbol{z}$.

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{6.72}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} - f\epsilon_{ij3}(u_j - U_j) = -\frac{1}{\rho_0}\frac{\partial p^*}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\rho^* g}{\rho_0}\delta_{i3}, \tag{6.73}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i}, \tag{6.74}$$

$$\tau_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ J_{\rho,i} = \kappa\frac{\partial \rho}{\partial x_i}. \tag{6.75}$$

In vector form,

$$\begin{gathered}\nabla\cdot\boldsymbol{u} = 0\\ \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\cdot(\boldsymbol{u}\boldsymbol{u}) + f\boldsymbol{e}_z\times(\boldsymbol{u}-\boldsymbol{U}) = -\frac{1}{\rho_0}\nabla p^* + \nabla\cdot\boldsymbol{\tau} - \frac{\rho^* g}{\rho_0}\boldsymbol{e}_z\\ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = \nabla\cdot\boldsymbol{J}_\rho\end{gathered} \tag{6.76}$$

where the stress and the scalar flux are

$$\boldsymbol{\tau} = \nu(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla),\ \boldsymbol{J}_\rho = \kappa\nabla\rho. \tag{6.77}$$

The total density ρ is decomposed into the reference density ρ_0, the background density $\rho_b(z)$, and the density perturbation ρ^* due to fluid motion,

$$\rho(x,y,z,t) = \rho_0 + \rho_b(z) + \rho^*(x,y,z,t). \tag{6.78}$$

The total pressure is written as

$$p(x,y,z,t) = p_0 + p_g(x,y) + p_a(z) + p^*(x,y,z,t), \tag{6.79}$$

where the reference pressure p_0 is a constant, the hydrostatic (ambient) pressure p_a has a vertical gradient that balances the ambient density ($\rho_a = \rho_0 + \rho_b(z)$), and the geostrophic pressure p_g has a transverse gradient that balances the Coriolis force due to the geostrophic wind $\boldsymbol{U}$. Only the dynamic pressure p^* appears in the momentum equation (6.73).

Instead of using ρ^*, it is also common to express the buoyancy term as

$$b = -\frac{\rho^* g}{\rho_0}, \tag{6.80}$$

and the 'total' buoyancy

$$\tilde{b} = b + \bar{b} = -\frac{(\rho^* + \bar{\rho}(z))g}{\rho_0}, \tag{6.81}$$

where the background linear stratification is $N^2 = \partial\bar{b}/\partial z$ and we have $\tilde{b} = b + N^2 z$ with the reference value $\bar{\rho}(z=0) = 0$.

Eqn, (6.74) can also be expressed as

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* u_i}{\partial x_i} + w\frac{\partial \bar{\rho}}{\partial z} = \kappa \frac{\partial^2 \rho^*}{\partial x_i^2}, \tag{6.82}$$

and hence we have the buoyancy equation

$$\frac{\partial b}{\partial t} + \frac{\partial b u_i}{\partial x_i} + wN^2 = \kappa \frac{\partial^2 b}{\partial x_i^2}, \tag{6.83}$$

and the equation for the total buoyancy is

$$\frac{\partial \tilde{b}}{\partial t} + \frac{\partial \tilde{b} u_i}{\partial x_i} = \kappa \frac{\partial^2 \tilde{b}}{\partial x_i^2}. \tag{6.84}$$

Roughly, the following independent non-dimensional parameters can be obtained from the comparison of relevant terms:

1. The Reynolds number $Re = UL/\nu$, where L is the horizontal length scale.
2. The Rossby number $Ro = U/fL$.
3. The Froude number $Fr = U/NL$.
4. The aspect ratio $\alpha = H/L$.
5. The Mach number $Ma = U/(L/T)$, where L/T is the group velocity. It is the ratio between the nonlinear term and the tendency term.

6.5.1 Incompressibility

Evan though there is a density transport due to the diffusion (due to the special role that ρ plays; this is not the mass conservation equation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i} \neq 0, \tag{6.85}$$

we could still establish incompressible condition

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{6.86}$$

with some additional assumtions.

First we review the integration form of the general conservation equation for an arbitraty scalar (per unit mass)

$$\begin{aligned}\frac{\partial}{\partial t}\left(\iiint_V \rho\psi \,\mathrm{d}V\right) &= -\iint_{\Omega=\partial V} (\rho \boldsymbol{u}\psi)\cdot \mathrm{d}\boldsymbol{A} - \iint_{\Omega=\partial V} \rho\kappa(-\nabla\psi)\cdot \mathrm{d}\boldsymbol{A} && (6.87)\\ &= -\iiint_V \nabla\cdot(\rho\boldsymbol{u}\psi)\,\mathrm{d}V - \iiint_V \nabla\cdot(\rho\kappa(-\nabla\psi))\,\mathrm{d}V && (6.88)\end{aligned}$$

and we have

$$\frac{\partial \rho\psi}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u}) = \nabla\cdot(\rho\kappa\nabla\psi). \tag{6.89}$$

It is in a general form of a conservational principle

$$\frac{\partial Q_v}{\partial t} + \nabla \cdot \boldsymbol{F} = 0, \tag{6.90}$$

where $\boldsymbol{F}$ is the flux and $\nabla \cdot \boldsymbol{F}$ is the transport term.

Taking $\psi = 1$ we recover the mass conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = \frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{u}) = 0. \tag{6.91}$$

Usually, density change along the material lines, is small enough such that $(1/\rho)D\rho/Dt \ll U/L$ and hence $\nabla \cdot \boldsymbol{u} \ll U/L$. That being said, being non-dimensionalised, the velocity field is solenoidal. We note in density-variable flows that

$$\nabla \cdot \boldsymbol{u} = 0 \tag{6.92}$$

is an approximation. See Batchelor (1967), section 3.2, for details as why this is valid.

6.5.2 Scalar transport equation

Taking $\psi = s$ (salinity or temperature) and assume diffusivity κ is constant, we have the scalar transport equation

$$s\frac{\partial \rho}{\partial t} + \rho\frac{\partial s}{\partial t} + u_j \rho \frac{\partial s}{\partial x_j} + u_j s \frac{\partial \rho}{\partial x_j} = \kappa \frac{\partial s}{\partial x_j}\frac{\partial \rho}{\partial x_j} + \rho\kappa \frac{\partial^2 s}{\partial x_j^2}, \tag{6.93}$$

taking into account

$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0 \tag{6.94}$$

we have

$$\rho\frac{\partial s}{\partial t} + u_j \rho \frac{\partial s}{\partial x_j} = \kappa \frac{\partial s}{\partial x_j}\frac{\partial \rho^*}{\partial x_j} + \rho\kappa \frac{\partial^2 s}{\partial x_j^2}. \tag{6.95}$$

We note that under Boussinesq assumption, $\rho^*/\rho = \rho^*/(\rho_0 + \rho*) \ll 1$, we have

$$\frac{\partial s}{\partial t} + u_j \frac{\partial s}{\partial x_j} = \kappa \frac{\partial^2 s}{\partial x_j^2}. \tag{6.96}$$

With some linear equations of state, such as

$$\rho = \rho_0 \left(1 - \frac{T - T_0}{T_0}\right), \tag{6.97}$$

we can relate s or T to ρ and get a scalar transport equation for ρ as

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = \kappa \frac{\partial^2 \rho}{\partial x_j^2}, \tag{6.98}$$

with the incompressibility being implied by $\nabla \cdot \boldsymbol{u} = 0$ from Eqn. (6.91). We note that Eqns. (6.91) and (6.98) correspond to two different physical principles.

Consider the gas EOS,

$$p = \rho RT \tag{6.99}$$

$$\ln p = ln(\rho) + \ln T + \ln R \tag{6.100}$$

$$\frac{\delta p}{p} = \frac{\delta \rho}{\rho} + \frac{\delta T}{T}. \tag{6.101}$$

In general, for $\rho = \rho(T, p)$,

$$\frac{\delta \rho}{\rho} = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p dT + \frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right|_T dT dp \tag{6.102}$$

$$= -\alpha dT + \chi_T dp \tag{6.103}$$

where

$$\alpha = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_p, \ \chi_T = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right|_T \tag{6.104}$$

are called isobaric thermal expansion coefficient and isothermal compressibility, respectively.

Assuming an isobaric process, we have

$$\frac{\rho^*}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = -\alpha(T - T_0) = -\alpha T^* \tag{6.105}$$

$$\frac{1}{\rho_0}\frac{\partial \bar{\rho}}{\partial z} = -\alpha \frac{\partial \bar{T}}{\partial z}, \tag{6.106}$$

and it follows that

$$b = -\frac{\rho^* g}{\rho_0} = \alpha T^* g \tag{6.107}$$

$$N^2 = -\frac{1}{\rho_0}\frac{\partial \bar{\rho}}{\partial z} = \alpha g \frac{\partial \bar{T}}{\partial z}. \tag{6.108}$$

6.6 GFD vorticity equations

6.6.1 Absolute vorticity equation

The 'absolute' vorticity, defined as $\boldsymbol{\omega}_a = \boldsymbol{\omega} + \boldsymbol{f}_c$, is the 'relative' vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ plus the 'planetary' vorticity $\boldsymbol{f}_c = 2\boldsymbol{\Omega}_c$ ($\Omega_c = \Omega \sin\phi$).

Similar to Eq. (2.21), we can derive the governing equation for $\boldsymbol{\omega}_a$ starting from

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}_c \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F} \tag{6.109}$$

without the hydrostatic part separated and/or Boussinesq assumed.

According to identity (A.21) we have, *in an f-plane*,

$$\nabla \times (\boldsymbol{f}_c \times \boldsymbol{u}) = -\boldsymbol{f}_c \cdot \nabla \boldsymbol{u}. \tag{6.110}$$

Similar to Eq. (2.14), by taking the curl of (6.109) and taking $\boldsymbol{F} = b\boldsymbol{e}_z$ we have the vorticity equation in a rotating frame

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}_a \cdot \nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega} + \nabla\times\boldsymbol{F} + \frac{1}{\rho^2}\nabla\rho\times\nabla p, \tag{6.111}$$

i.e., the absolute vorticity equation:

$$\frac{D\boldsymbol{\omega}_a}{Dt} = \frac{\partial \boldsymbol{\omega}_a}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega}_a = \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} \tag{6.112}$$

$$= \boldsymbol{\omega}_a\cdot\nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega} + \nabla\times\boldsymbol{F} + \frac{1}{\rho^2}\nabla\rho\times\nabla p. \tag{6.113}$$

In a Boussinesq fluid,

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}_a\cdot\nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega} + \nabla\times(b\boldsymbol{e}_z) + \frac{1}{\rho_0^2}\nabla\rho^*\times\nabla p^*, \tag{6.114}$$

where $\nabla\times(b\boldsymbol{e}_z) = \epsilon_{ij3}\partial_j b$.

Additionally, the linearized inviscid evolution equation for the perturbation vorticity, similar to (5.134), reads

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j'\bar{S}_{ij} + (\bar{\omega}_j + f\delta_{j3})S_{ij}' - u_j'\frac{\partial\bar{\omega}_i}{\partial x_j} + \frac{1}{2}\epsilon_{ij3}f\omega_j' + \epsilon_{ij3}\frac{\partial b'}{\partial x_j}, \tag{6.115}$$

which is convenient for instability considerations.

Example (Taylor-Proudman theorem, 6.3.4; another proof): Assume inviscid, barotropic fluid acted on by conservative force, and that the rotation rate $\boldsymbol{\Omega}_c = \boldsymbol{f}_c/2$ is much greater than other frequencies. Eqn. (6.113) becomes

$$0 = \boldsymbol{f}_c\cdot\nabla\boldsymbol{u} \tag{6.116}$$

and that completes the proof.

6.6.2 Potential vorticity equation; Ertal's theorem

Ref. Pedlosky (2013).

Assume a conserved scalar λ with a governing operator $D\lambda/Dt = S$ where S is a source term for λ. Consider

$$\frac{D}{Dt}\left(\frac{\partial\lambda}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial\lambda}{\partial t} + u_j\frac{\partial\lambda}{\partial x_j}\right) - \frac{\partial u_j}{\partial x_i}\frac{\partial\lambda}{\partial x_j}, \tag{6.117}$$

i.e.,

$$\frac{D}{Dt}(\nabla\lambda) = \nabla\left(\frac{D\lambda}{Dt}\right) - \nabla\boldsymbol{u}\cdot\nabla\lambda. \tag{6.118}$$

$\nabla\lambda\cdot$ (6.113) + $\boldsymbol{\omega}_a\cdot$ (6.118), with a magic that two opposite-sign $\boldsymbol{\omega}_a\cdot\nabla\boldsymbol{u}\cdot\nabla\lambda$ terms cancel, we have

$$\frac{D}{Dt}(\boldsymbol{\omega}_a\cdot\nabla\lambda) = \boldsymbol{\omega}_a\cdot\nabla S + \nu\nabla^2\boldsymbol{\omega}_a\cdot\nabla\lambda + (\nabla\times\boldsymbol{F})\cdot(\nabla\lambda) + \frac{1}{\rho^2}(\nabla\rho\times\nabla p)\cdot(\nabla\lambda) \tag{6.119}$$

Take $\lambda = \tilde{b}$ which is the total buoyancy, with its governing equation being (6.84), assuming conservative external force $\boldsymbol{F}$ and barotropic flow*, we have the potential vorticity (PV) equation:

$$\frac{D}{Dt}\left(\boldsymbol{\omega}_a\cdot\nabla\tilde{b}\right) = \nu\nabla^2\boldsymbol{\omega}_a\cdot\nabla\tilde{b} + \kappa[\nabla^2(\nabla\tilde{b})]\cdot\boldsymbol{\omega}_a, \tag{6.120}$$

where

$$\Pi = \boldsymbol{\omega}_a\cdot\nabla\tilde{b} \tag{6.121}$$

is called the potential vorticity, which is the component of the absolute vorticity perpendicular to the isosurface (or parallel to the gradient) of $\tilde{b}$. In the absence of dissipation,

$$\frac{D\Pi}{Dt} = 0, \tag{6.122}$$

i.e., PV is conserved along the streamlines. Eq. (6.120) is like a double-diffusion problem with one 'passive' scalar diffuse together with vorticity.

Ertel's theorem: under the follwing assumptions, PV conservation along fluid motion is satisfied (from Eq. (6.119)):

- λ is a conserved quantity that following fluid motion $S = 0$.
- Conservative external force: $\nabla \times \boldsymbol{F} = 0$.
- Either
  1. Baroclinicity absent ($\nabla\rho \times \nabla p = 0$)
  2. λ is only a thermodynamic function of p, ρ, i.e., $\lambda = \lambda(p, \rho)$ so that the last term vanishes when $\cdot(\nabla\lambda)$. For example, $\lambda = s$ (entropy).
- Diffusion-less/inviscid: $\nu = \kappa = 0$.

6.6.3 Relation of PV to Kelvin's theorem in a rotating frame

Similarly, Kelvin's circulation theorem (see (2.6)) in a rotating frame is

$$\frac{D\Gamma_a}{Dt} = \iint_A \frac{\nabla\rho \times \nabla p}{\rho^2} \cdot d\boldsymbol{A}, \tag{6.123}$$

where the absolute circulation is

$$\Gamma_a = \int_A \boldsymbol{\omega}_a \cdot d\boldsymbol{A} = \Gamma + \int_A \boldsymbol{f}_{\mathrm{c}} \cdot d\boldsymbol{A}. \tag{6.124}$$

When the surface A is specifically chosen to be on $\lambda = \lambda(\rho, p)) =$ constant (and A is enclosed by a contour l that stays on $\lambda =$ constant for all time), what follows is that $\nabla\lambda$ must be in the parameter plane spanned by $\nabla\rho$ and ∇p, hence $\nabla\lambda \cdot (\nabla\rho \times \nabla p) = 0$. Here, we note that the normal vector for A is $\boldsymbol{n} = \nabla\lambda/|\nabla\lambda|$ so $(\nabla\rho \times \nabla p) \cdot \boldsymbol{n} = 0$ on the entire plane and

$$\frac{D\Gamma_a}{Dt} = 0. \tag{6.125}$$

The choice of λ is just to choose a surface/contour (of $\lambda =$ constant) on which $\nabla\rho, \nabla p$ lie in the surface and the baroclinicity term makes zero contribution to the circulation in a baroclinic flow. We can regard PV conservation as a special statement of Kelvin's theorem.

Example: $\lambda(\rho, p) = \rho^2 + p^2$. Think of $\lambda =$ constant as a cylinderical surface.

Example: For a finite flat area A, $\Gamma_a \approxeq \omega_a A$ and its conservation implies angular momentum conservation, and the change of spacing of λ surfaces (associated with change of area) can change relative vorticity — stretching and compressing.

6.7 Turbulence equations for an active scalar

6.7.1 Mean flow equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{6.126}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} - f\epsilon_{ij3}(\bar{u}_j - U_j) = -\frac{1}{\rho_0}\frac{\partial \overline{p^*}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \overline{u_i' u_j'}) - \frac{\overline{\rho^*} g}{\rho_0}\delta_{i3} \tag{6.127}$$

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{u}_j \frac{\partial \bar{\rho}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial \bar{\rho}}{\partial x_j} - \overline{\rho' u_j'}\right), \tag{6.128}$$

We note that only vertical variation of density is considered:

$$\bar{\rho} = \overline{\rho^*} + \rho_b(z),\ \rho_b = \rho_0\left(1 - \frac{N^2}{g}z\right),\ \frac{\partial \rho_b}{\partial z} = -\frac{\rho_0 N^2}{g} \tag{6.129}$$

and

$$\rho' = \rho - \bar{\rho} = \rho^* - \overline{\rho^*} = \rho^{*\prime}. \tag{6.130}$$

The equation for $\overline{\rho^*}$ becomes

$$\frac{\partial \overline{\rho^*}}{\partial t} + \bar{u}_j \frac{\partial \overline{\rho^*}}{\partial x_j} - \bar{w}\frac{\rho_0 N^2}{g} = \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial \overline{\rho^*}}{\partial x_j} - \overline{\rho' u_j'}\right). \tag{6.131}$$

6.7.2 Fluctuation equations

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{6.132}$$

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} - f\epsilon_{ij3}u_j' = -\frac{1}{\rho_0}\frac{\partial p^{*\prime}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u_i'}{\partial x_j} + \overline{u_i' u_j'} - u_i' u_j') - u_j' \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\rho^{*\prime} g}{\rho_0}\delta_{i3} \tag{6.133}$$

$$\frac{\partial \rho^{*\prime}}{\partial t} + \bar{u}_j \frac{\partial \rho^{*\prime}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial \rho^{*\prime}}{\partial x_j} + \overline{\rho^{*\prime} u_j'} - \rho^{*\prime} u_j'\right) - u_j' \frac{\partial \bar{\rho}}{\partial x_i} \tag{6.134}$$

$$= \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial \rho^{*\prime}}{\partial x_j} + \overline{\rho^{*\prime} u_j'} - \rho^{*\prime} u_j'\right) - u_j' \frac{\partial \overline{\rho^*}}{\partial x_i} + w' \frac{\rho_0 N^2}{g} \tag{6.135}$$

We will see later the Coriolis term won't appear in the transport equations of MKE, TKE, and Reynolds stresses. Coriolis just bends the direction of the velocity. In other words, Coriolis does not have direct influence on turbulence budgets, but indirectly through the change of the mean flow. On the other hand, it does make a difference in the vorticity/enstrophy equation (see section **??**).

The mean and fluctuation density equations are from the scalar transport equations in Section 5.1.9.

6.7.3 MKE, MPE, TKE, TPE, and buoyancy flux equations

Define the mean and turbulent kinetic and potential energy as

$$K = \frac{1}{2}\bar{u}_i\bar{u}_i \tag{6.136}$$

$$K_\rho = \frac{1}{2N^2}\bar{b}^2 \tag{6.137}$$

and

$$k = \frac{1}{2}\overline{u_i'u_i'} \tag{6.138}$$

$$k_\rho = \frac{1}{2N^2}\overline{b'b'} \tag{6.139}$$

where the instantaneous, mean, and fluctuation buoyancy are (with the dimension being acceleration)

$$b = -\frac{\rho^* g}{\rho_0},\ \bar{b} = -\frac{\overline{\rho^*} g}{\rho_0},\ b' = -\frac{\rho^{*\prime} g}{\rho_0}, \tag{6.140}$$

such that k and k_ρ have the same dimension as kinetic energy. We note that

$$\rho^{*\prime} = \rho^* - \overline{\rho^*} = (\rho - \rho_b) - \overline{(\rho - \rho_b)} = \rho - \bar{\rho} = \rho'. \tag{6.141}$$

The **MKE equations** is (repeating (5.32) but with a mean transport) :

$$\frac{\partial K}{\partial t} + \bar{u}_j\frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho}\bar{p}\,\bar{u}_j + \nu\frac{\partial K}{\partial x_j} - \bar{u}_i\,\overline{u_i'u_j'}\right) + \overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \nu\frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j} + \bar{b}\bar{w} \tag{6.142}$$

The **MPE equations** is:

$$\frac{\partial K_\rho}{\partial t} + \bar{u}_j\frac{\partial K_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial K_\rho}{\partial x_j} - \frac{1}{N^2}\bar{b}\,\overline{b'u_j'}\right) + \frac{1}{N^2}\overline{b'u_j'}\frac{\partial \bar{b}}{\partial x_j} - \frac{\kappa}{N^2}\frac{\partial \bar{b}}{\partial x_j}\frac{\partial \bar{b}}{\partial x_j} - \bar{b}\bar{w} \tag{6.143}$$

We note that the mean buoyancy flux $\bar{b}\bar{w}$ moves energy from MKE to MPE and the turbulent buoyancy flux participates in the production $\overline{b'u_j'}\partial\bar{b}/\partial x_j$, when $\partial\bar{b}/\partial x_j$ is negative, is a sink in the MPE equation and is a source in the TPE equation.

The **TKE equations** is:

$$\frac{\partial k}{\partial t} + \bar{u}_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_k}(\underbrace{\nu\frac{\partial k}{\partial x_k}}_{\substack{\text{molecular}\\ \text{diffusion}}} + \underbrace{\frac{1}{2}\overline{u_i'u_i'u_k'}}_{\substack{\text{turbulent}\\ \text{diffusion}}} \underbrace{- \frac{1}{\rho_0}\overline{p'u_k'})}_{\substack{\text{pressure}\\ \text{distortion}}} \underbrace{-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}}_{\text{production } P} - \underbrace{\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}}_{\text{dissipation } \varepsilon} + \underbrace{\overline{b'w'}}_{\substack{\text{buoyancy}\\ \text{flux } B}} \tag{6.144}$$

$$= \nabla\cdot\boldsymbol{T} + P - \varepsilon + B \tag{6.145}$$

where the turbulent buoyancy flux is

$$B = -\frac{g}{\rho_0}\overline{\rho^{*\prime}w'} = \overline{b'w'}. \tag{6.146}$$

The **TPE equation** is:

$$\frac{\partial k_\rho}{\partial t} + \bar{u}_j\frac{\partial k_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}(\kappa\frac{\partial k_\rho}{\partial x_j} - \frac{1}{2N^2}\overline{b'b'u_j'}) - \frac{1}{N^2}\overline{b'u_j'}\frac{\partial \bar{b}}{\partial x_j} - \frac{\kappa}{N^2}\overline{\frac{\partial b'}{\partial x_j}\frac{\partial b'}{\partial x_j}} - \overline{b'w'} \tag{6.147}$$

We can see that the turbulent buoyancy flux B (negative, think $-\overline{u_i'u_j'}$) moves energy from TKE to TPE (similar to the mean buoyancy flux that moves energy from MKE to MPE) and also participates in TPE's own gradient production through $\overline{b'u_j'}\partial\bar{b}/\partial z$ that moves energy from MPE and TPE. We also note that if we sum over MKE, MPE, TKE, TPE equations the production and buoyancy flux terms will have net zero contribution to the total energy. We can also find the role of buoyancu flux bw by multiplying the w equation with b and the b equation with w in (7.1).

The **buoyancy flux equation** is:

$$\frac{\partial\overline{b'u_i'}}{\partial t}+\bar{u}_j\frac{\partial\overline{b'u_i'}}{\partial x_j}=d_{b,i}+P_{b,i}+\Phi_{b,i}-\varepsilon_{b,i}+B_i \tag{6.148}$$

where

$$d_{b,i}=\frac{\partial}{\partial x_j}(\kappa\overline{u_i'\frac{\partial b'}{\partial x_j}}+\nu\overline{b'\frac{\partial u_i'}{\partial x_j}}-\frac{1}{\rho_0}\overline{p^{*\prime}b'}\delta_{ij}-\overline{b'u_i'u_j'}) \tag{6.149}$$

$$P_{b,i}=-\overline{b'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}-\overline{u_i'u_j'}\frac{\partial\bar{b}}{\partial x_j} \tag{6.150}$$

$$\Phi_{b,i}=-\frac{1}{\rho_0}\overline{p^{*\prime}\frac{\partial b'}{\partial x_i}} \tag{6.151}$$

$$\varepsilon_{b,i}=(\nu+\kappa)\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{6.152}$$

$$B_i=-\overline{u_i'w'}N^2+\overline{b'b'}\delta_{i3} \tag{6.153}$$

We note that we have re-written

$$-\frac{1}{\rho_0}\overline{b'\frac{\partial p^{*\prime}}{\partial x_i}}=-\frac{1}{\rho_0}\left(\frac{\partial}{\partial x_j}\overline{p^{*\prime}b'}\delta_{ij}-\overline{p^{*\prime}\frac{\partial b'}{\partial x_i}}\right), \tag{6.154}$$

and the new term (6.153) did not appear when we derive the passive scalar flux equation.

During $\partial\bar{\rho}/\partial x_3\neq 0$ we are mostly concerned about $\overline{b'w'}$, which is typically nagative (think about a displaced parcel argument). Its equation is ($i=3$)

$$\frac{\partial\overline{b'w'}}{\partial t}+\bar{u}_j\frac{\partial\overline{b'w'}}{\partial x_j}=\frac{\partial}{\partial x_j}(\kappa\overline{w'\frac{\partial b'}{\partial x_j}}+\nu\overline{b'\frac{\partial w'}{\partial x_j}}-\frac{1}{\rho_0}\overline{p^{*\prime}b'}\delta_{j3}-\overline{b'w'u_j'}) \tag{6.155}$$

$$-\overline{b'u_j'}\frac{\partial\overline{w}}{\partial x_j}-\overline{w'u_j'}\frac{\partial\bar{b}}{\partial x_j}-\overline{w'w'}N^2+\overline{b'b'} \tag{6.156}$$

$$-\frac{1}{\rho_0}\overline{p^{*\prime}\frac{\partial b'}{\partial x_3}}+(\nu+\kappa)\overline{\frac{\partial w'}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{6.157}$$

Two additional terms (with respect to scalar flux equation, for $\overline{u_i'\phi'}$, where ϕ' is a passive scalar) appear:

$$-\overline{w'w'}N^2+\overline{b'b'} \tag{6.158}$$

that mark the difference made by an active scalar. They would both be zero if the density is not coupled to buoyancy (simply by setting the gravity g to zero). When $g\neq 0$, density-velocity coupling plays a role in the dynamics of $\overline{b'w'}$. The effect of buoyancy is restoring (making $B = \overline{b'w'}$ less negative) and the effect of vertical perturbation increases the magnitude of B in a stably stratified medium ($N^2 > 0$).

If we neglect the transport terms, inhomogeneity (think homogeneous stratified turbulence; we have eliminated $\partial_z \bar{w}$, $\partial_z \overline{\rho^*}$ but not $\partial_z \rho_b$), and pressure fluctuations, we can reach a simple set of autonomous equations

$$\frac{\partial(\overline{w'w'}/2)}{\partial t} = \overline{b'w'} \tag{6.159}$$

$$\frac{\partial(\overline{b'b'}/2N^2)}{\partial t} = -\overline{b'w'} \tag{6.160}$$

$$\frac{\partial\overline{b'w'}}{\partial t} = -\overline{w'w'}N^2 + \overline{b'b'} \tag{6.161}$$

which can be written as a linear system that has eigenvalues $\lambda = \pm \mathrm{i}\sqrt{2}N$, $\lambda = 0$ (when $N^2 > 0$, stable stratification), where the purely imaginary EVs can be seen from the reducing the system to

$$\partial_{tt}\overline{b'w'} = -2N^2\overline{b'w'}. \tag{6.162}$$

This implies that the transfer mechanism between TKE, TPE, and buoyancy flux is energy-conserving. The phase relations can be seen from the general solution

$$\overline{b'w'} = \cos(\sqrt{2}Nt) \tag{6.163}$$

$$\overline{w'w'}/2 = \frac{1}{\sqrt{2}N}\sin(\sqrt{2}Nt) \tag{6.164}$$

$$\overline{b'b'}/2N^2 = -\frac{1}{\sqrt{2}N}\sin(\sqrt{2}Nt) \tag{6.165}$$

6.8 Miscellaneous

coriolis frequency;

6.8.1 Derivation of Coriolis force

6.8.2 Boussinesq approximation

Chapter 7

Waves in GFD

7.1 Internal waves: governing equations and dispersion relation

The governing equations for waves can be simplified from (6.76), with the following assumptions:

1. The Ekman number is small (neglect viscosity).
2. The fluid is Boussinesq.
3. The 'Mach' number is small (neglect the nonlinear term/only effects to the first order are retained).
4. The aspect ratio is $\alpha = H/L = W/U \ll 1$.

The set of simplified equations read (with $\rho_0 = 1$)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + fv \\
\frac{\partial v}{\partial t} &= -\frac{\partial p}{\partial y} - fu \\
\frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} + b \\
\frac{\partial b}{\partial t} &= -wN^2 \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned} \tag{7.1}$$

and by eliminating pressure and cross-differentiation, an equation for w can be obtained as

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + f^2\frac{\partial^2 w}{\partial z^2} + N^2(z)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{7.2}$$

The detailed steps include: (1) eliminate pressure from the momentum equations and obtain a system of three pressure-less equations; (2) cross-differentiate to get the forms of $\partial_{xx}b$ and use the buoyancy equation to replace b with w; we note that we assume $N(z)$ is only a function of the vertical coordinate so its z-dependence can be retained; (3) ∂_{yz} the u-equation and ∂_{xz} the v-equation, to elinimate triple-cross-difference terms as $\partial_{txy}u$; (4) organization.

A 'trivial' solution to (7.2) is simply the geostrophic balance:

$$w = 0,\ fv = p_x,\ fu = -p_y,\ b = p_z. \tag{7.3}$$

Assume non-zero solution in the form of normal modes

$$[u, v, w, p, b] = [\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{b}] \exp[i(kx + ly + mz - \omega t)] \tag{7.4}$$

where the wavenumber vector is

$$\boldsymbol{k} = (k, l, m), \tag{7.5}$$

the waves that the equation (7.2) describes should satisfy the dispersion relation

$$\omega^2 = \frac{f^2 m^2 + (k^2 + l^2)N^2}{k^2 + l^2 + m^2} = f^2 \sin^2\phi + N^2 \cos^2\phi \tag{7.6}$$

where ϕ is the angle between the wave-vector $\boldsymbol{k}$ and the horizontal plane (k, l). It can be seen that the feasible frequencies lie in the range of

$$f < \omega < N \tag{7.7}$$

for the general case in the ocean that $N/f \gg 1$.

From (7.6) we can get

$$\frac{N^2 - \omega^2}{\omega^2 - f^2} = \frac{(N^2 - f^2)\sin^2\phi}{(N^2 - f^2)\cos^2\phi} = \tan^2\phi = \frac{1}{\alpha^2}, \tag{7.8}$$

where the vertical-to-horizontal aspect ratio is defined as

$$\alpha = \frac{k^2 + l^2}{m^2} = \cot\phi, \tag{7.9}$$

which is close to zero at the hydrostatic limit ($m^2 \ll k^2 + l^2$). In such case, $N \gg \omega$.

We can also see that the continuity equation translates to

$$\boldsymbol{k} \cdot \hat{\boldsymbol{u}} = 0, \tag{7.10}$$

that being said, the wavenumber vector is perpendicular to the particle motions.

The dispersion relation also implies that the frequency only depends on the direction of the wavevector, but not its magnitude. All wavenumbers on a cone of constant ϕ have the same frequency, and the direction of the group velocity, $c_p = d\omega/dk$, is perpendicular to the cone surface. A trivial solution to the equation (7.2) is simply $w = 0$, which is corresponding to a geostrophic+hydrostatic balance.

Examples. (1) For a non-rotationg fluid, $\omega^2 = N^2 \cos^2\phi$, and the portion of stratification that the motion feels is proportional to the projection of its motion in the vertical direction. Oscillation that feels the full effect of stratification is vertical, and correspond to the largest possible frequency, a horizontal wavenumber, and $\phi = 0$. (2) For $\phi = \pi/2$, the wavevector is perpendicular and there is only inertial oscillations.

7.1.1 General solutions

Assume $N = N_0$ is a constant and without lost of generality the coordinate can be re-oriented such that $\partial_y = 0$ and $\partial_x u + \partial_z w = 0$. The rest of the flow quantities can be recovered from the solution of $\hat{w}$ as

$$\hat{u} = -\frac{m}{k}\hat{w} \tag{7.11}$$

$$\hat{v} = i\frac{fm}{\omega k}\hat{w} = -i\frac{f}{\omega}\hat{u} \tag{7.12}$$

$$\hat{p} = -\left(\frac{N^2 - \omega^2}{m\omega}\right)\hat{w} = \left(\frac{N^2 - \omega^2}{k\omega}\right)\hat{u} \tag{7.13}$$

$$\hat{b} = -i\frac{N^2}{\omega}\hat{w},\ \hat{p} = -i\left(\frac{N^2 - \omega^2}{mN^2}\right)\hat{b} \tag{7.14}$$

The above relations have a number of implications:

1. $\hat{u}$ and $\hat{w}$ are in phase.
2. $\hat{u}$ and $\hat{v}$ are out of phase (think inertial oscillations).
3. Pressure and vertical motion are in phase, but both are out of phase with buoyancy.
4. (TBD) the vertical structures (CW or CCW) of the waves.

7.1.2 Hydrostatic approximation

With the hydrostatic approximation is equivalent to

$$k^2 + l^2 \ll m^2, \tag{7.15}$$

with which the dispersion relation (7.6) reduces to

$$\omega^2 = f^2 + \frac{k^2 + l^2}{m^2}N^2. \tag{7.16}$$

7.1.3 Near-inertial waves

The near-inertialness,

$$\frac{\omega - f}{f} \ll 1, \tag{7.17}$$

implies that

$$\frac{\omega^2 - f^2}{f^2} = \frac{(\omega - f)^2 - 2f^2 + 2\omega f}{f^2} \approx \frac{2(\omega - f)}{f}. \tag{7.18}$$

While (7.6) is re-organized to

$$\frac{\omega^2 - f^2}{f^2} = \left(\frac{N^2}{f^2} - 1\right)\frac{k^2 + l^2}{k^2 + l^2 + m^2}, \tag{7.19}$$

which can only be made very small when the second factor is small, that said,

$$k^2 + l^2 \ll m^2, \tag{7.20}$$

which is the hydrostatic relation so (7.16) applies. With that, the dispersion relation for near-inertial waves is

$$\omega = f + \frac{N^2}{2f}\frac{k^2 + l^2}{m^2}. \tag{7.21}$$

In fact, near-inertial waves are super-hydrostatic. Taking an appropriate orientation of the coordinate frame ($l = 0$,$\partial_y = 0$), we can express the near-inertialness indicator

$$\frac{\omega - f}{f} = \frac{N^2 k^2}{f^2 m^2} = \frac{Bu}{2} \ll 1, \tag{7.22}$$

and the group velocities

$$c_{gv} = \frac{d\omega}{dm} = -\frac{N^2 k^2}{f m^3} = -Bu\frac{f}{m}. \tag{7.23}$$

$$c_{gh} = \frac{d\omega}{dk} = \frac{N^2 k}{f m^2} = Bu\frac{f}{k}. \tag{7.24}$$

Both c_{gv} and c_{gh} are small.

7.1.4 Vertical modes

If we ease the assumption (7.4) to allow a vertical dependence of the modes, since N is typically a function of z, i.e.,

$$w(x, y, z, t) = a(z)\exp[i(kx + ly - \omega t)], \tag{7.25}$$

by substituting into (7.2) we have

$$\frac{d^2 a}{dz^2} + \left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right)(k^2 + l^2)a = 0, \tag{7.26}$$

with rigid lid BC's $w(0) = w(-h) = 0$ that implies $a(0) = a(-h) = 0$. This is a typical Sturm–Liouville eigenvalue problem. Moreover, the derivative $a' = \partial_z a$ also satisfies another Sturm–Liouville equation. It will be shown later that the set of eigenmodes $\{a_m\}$ are mutually orthogonal under the inner-product xxx, where m is now the mode index.

The solution of (7.26) can be done for each specified freuquency: regard $k^2 + l^2$ as the eigenvalue, solve (7.26) with given $N(z)$ and ω. Under hydrostatic, it can be shown from $\alpha \approx 1$ that $N \gg \omega$. Assume the hydrostatic dispersion relation (for each mode m)

$$\omega_m^2 \approx f^2 + c_m^2(k^2 + l^2) \tag{7.27}$$

we have

$$\frac{d^2 a}{dz^2} + \frac{N^2}{c^2}a = 0, \tag{7.28}$$

where the eigenvalues $\{c_m^{-2}\}$ are the wave speed of each mode m and corresponding Rossby radius of deformation is $\lambda_m = c_m/f$. We note that f is absent in (7.28) and the eigenmodes it defines.

7.2 Inertial and buoyancy oscillations

The two most fundamental pedagogical oscillations in GFD are the inertial and buoyancy oscillations. The inertial oscillation results from the pressure-less version of the horizontal momentum equations (7.1),

$$u_t - fv = 0 \tag{7.29}$$
$$v_t + fu = 0. \tag{7.30}$$

It can be arranged into

$$(u + iv)_t + if(u + iv) = 0 \tag{7.31}$$

which has the solution

$$u + iv = e^{-ift}[u(0) + iv(0)], \tag{7.32}$$

which is rotating clockwise in Northern Hemisphere at an angular frequency f – the Coriolis frequency. Similarly, the pressure-less vertical and buoyancy equations give

$$w_{tt} = -N^2 w, \tag{7.33}$$

leading to the solutions

$$w = \begin{cases} w(0)e^{iNt}, \ N^2 > 0 \\ w(0)e^{\sqrt{-N^2}t}, \ N^2 < 0 \end{cases} \tag{7.34}$$

with the first solution corresponding to buoyancy oscillation in stably stratified fluid and the second corresponding to a gravitationally unstable mode under unstable stratification.

7.3 Shallow water waves

Assumptions:

- Inviscid
- Homogeneous fluid
- Horizontal scale $\gg$ vertical scale ($L \gg H$)
- Small amplitude ($\eta \ll H$)

Scaling analysis. Continuity:

$$\frac{W}{H} \sim \frac{U}{L}, \tag{7.35}$$

momentum:

$$\frac{\partial u}{\partial t} \sim \frac{U}{T} \tag{7.36}$$
$$u\frac{\partial u}{\partial x} \sim v\frac{\partial u}{\partial y} \sim U\frac{U}{L} \tag{7.37}$$
$$w\frac{\partial u}{\partial z} \sim U\frac{W}{H} \tag{7.38}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} \sim \frac{U^2}{L} \tag{7.39}$$

$$\frac{\partial w}{\partial t} \sim \frac{W}{T} \tag{7.40}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} \sim \frac{U^2}{H} \tag{7.41}$$

The incompressibility implies that the speed of motion is much smaller than some sort of wave speed:

$$\frac{U}{L/T} = \frac{\text{inertial}}{\text{acceleration}} \ll 1, \tag{7.42}$$

and the aspect ratio implies

$$\frac{\partial w/\partial t}{-(1/\rho_0)\partial p/\partial z} = \left(\frac{H}{L}\right)^2 \ll 1. \tag{7.43}$$

Hence, we have neglected the non-linear terms in horizontal momentum equations and the vertical momentum equation reduces to a hydrostatic balance. The shallow water equations (SWE) read

$$u_t = -\frac{1}{\rho_0}p_x \tag{7.44}$$

$$v_t = -\frac{1}{\rho_0}p_y \tag{7.45}$$

$$0 = -\frac{1}{\rho_0}p_z - g, \tag{7.46}$$

$$u_x + v_y = 0 \tag{7.47}$$

The hydrostatic pressure balance leads to

$$p(z) = p(H+\eta) + \int_{H+\eta}^{z} -\rho g dz = p_0 + \rho g(H+\eta-z), \tag{7.48}$$

where η is the surface height displacement and H is the depth ($z = 0$ at the bottom). The horizontal pressure gradients are given as $p_x = \rho g \eta_x, p_y = \rho g \eta_y$. The intuition is that when the surface is elevated it also gives the fluid below a higher hydrostatic pressure.

Now let's derive an equation for the surface motions.

$$\frac{\partial \eta}{\partial t} \sim \frac{D\eta}{Dt} = w(H+\eta) = w(H+\eta) - w(0) \tag{7.49}$$

$$= \int_0^{H+\eta} \frac{\partial w}{\partial z} dz = \int_0^{H+\eta} -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz \tag{7.50}$$

$$\approx -(H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{7.51}$$

$$\approx -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{7.52}$$

where we have applied the small amplitude assumption and the slab assumption.

Finally, the SWE become

$$u_t - fv + g\eta_x = 0 \tag{7.53}$$

$$v_t + fu + g\eta_y = 0 \tag{7.54}$$
$$\eta_t + H(u_x + v_y) = 0 \tag{7.55}$$

where the lst equation can be written as

$$(g\eta)_t + c^2(u_x + v_y) = 0 \tag{7.56}$$

where $c^2 = gH$ is the phase/group speed and $p = g\eta$ is the hydrostatic pressure. Shallow water waves are non-dispersive. Cross-differentiate the SWE, then we reach

$$\eta_{tt} - c^2(\eta_{xx} + \eta_{yy}) + Hf\zeta = 0, \tag{7.57}$$

where the relative vorticity $\zeta = v_x - u_y$. The curl of the first two equations of the SWE gives

$$\zeta_t + f(u_x + v_y) = 0. \tag{7.58}$$

Combined with the η_t equation (continuity), we have

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{f} - \frac{\eta}{H}\right) = \frac{\partial q}{\partial t} = 0, \tag{7.59}$$

where $q = \zeta/f - \eta/H$ is the potential vorticity (reduction in SWE). The conservation implies

$$\zeta = \frac{\eta f}{H} + fq_0. \tag{7.60}$$

Substitute this into (7.57) we have (the homogeneous part of the equation):

$$\eta_{tt} - c^2(\eta_{xx} + \eta_{yy}) + f^2\eta = 0. \tag{7.61}$$

Assuming normal modes $\eta = \eta_0 \exp[i(kx + ly - \omega t)]$ with oriented coordinates ($l = 0$), we have the dispersion relation

$$\omega^2 = c^2(k^2 + l^2) + f^2. \tag{7.62}$$

- Large $k^2 + l^2$ (fast waves): $\omega \approx \sqrt{gH}k$. Phase speed $c_p = \omega/k = \sqrt{gH}$. Group speed $c_g = d\omega/dk = \sqrt{gH} = c_p$. They don't feel the rotation of the Earth. We note that the phase is

  $$\Phi = kx + ly - \omega t = \sqrt{k^2 + l^2}(kx + ly - \omega t)/\sqrt{k^2 + l^2} = |\boldsymbol{k}|(\hat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_p t).$$

  Example. (Tsunami speed) Assume $H = 4$ km is the averaged ocean depth and $L = 8000$ km is the width of the ocean basin. With the SWE assumption $H \ll L$ satisfied, $c = \sqrt{gH}$ gives ≈ 200 m/s or 720 km/h of propagation speed.

- Small $k^2 + l^2$ (slow waves, small Ro): $\omega \approx f$. Near-inertial waves.

7.3.1 Travelling wave solutions

Consider non-rotating (or fast) waves. The formal solution to the 1D SWE is

$$\eta = G(x - ct) + F(x + ct) \tag{7.63}$$

with both G and F satisfying ($c^2 = Hg$)

$$\eta_{tt} - c^2\eta_{xx} = (\partial_t + c\partial_x)(\partial_t - c\partial_x)\eta = 0. \tag{7.64}$$

Moreover,

$$u = \int \frac{\partial u}{\partial t} dt = \int -g\frac{\partial \eta}{\partial x} = \frac{g}{c}[G(x - ct) - F(x + ct)]. \tag{7.65}$$

Without the loss of generality, we consider right-propagating waves with $F = 0$. Consider a single wavenumber,

$$\eta = \Re\{A \exp[i(kx - \omega t)]\} = A\cos(kx - \omega t), \tag{7.66}$$

where $\omega = ck$ and c is the phase speed. We have

$$u = \frac{gk}{\omega} A\cos(kx - \omega t), \tag{7.67}$$

which is in phase with η. We check *a posteriori* that in order to make the 'Mach' number very small, we should have

$$\frac{u}{c_p} = \frac{gkA/\omega}{\omega/k} = \frac{gk^2/\omega}{\omega^2} A \ll 1, \tag{7.68}$$

where A is the amplitude.

Here

$$\phi = kx - \omega t = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t \tag{7.69}$$

is called the phase. For 2D waves,

$$\phi = kx + ly - \omega t = \sqrt{k^2 + l^2}\left(\frac{kx + ly}{\sqrt{k^2 + l^2}} - \frac{\omega t}{\sqrt{k^2 + l^2}}\right) = |\boldsymbol{k}|\left(\frac{kx + ly}{|\boldsymbol{k}|} - |c_p|t\right) \tag{7.70}$$

and the phase speed is

$$|c_p| = \frac{\omega}{|\boldsymbol{k}|}. \tag{7.71}$$

The phase 'velocity', by definition,

$$\boldsymbol{c}_p = \frac{\omega}{|\boldsymbol{k}|}\left(\frac{k}{|\boldsymbol{k}|}, \frac{l}{|\boldsymbol{k}|}\right) // \boldsymbol{k}, \tag{7.72}$$

is parallel to the wavevector

$$\boldsymbol{k} = (k, l), \tag{7.73}$$

which is perpendicular to the constant phase lines

$$\phi_0 = kx + ly - \omega t_0 = C \to y = -\frac{k}{l}x + \frac{\phi_0}{l} // \left(1, -\frac{k}{l}\right). \tag{7.74}$$

7.4 Deep water waves

We relax the assumption $H \ll L$ in SWE and allow vertical motions (the fluid is no longer moving like slabs) and reach the deep water equations (DWE)

$$u_t = -\frac{1}{\rho_0} p_x \tag{7.75}$$

$$v_t = -\frac{1}{\rho_0} p_y \tag{7.76}$$

$$w_t = -\frac{1}{\rho_0} p_z \tag{7.77}$$

$$u_x + v_y + w_z = 0 \tag{7.78}$$

where the pressure is its deviation from the background hydrostatic balance

$$-\frac{1}{\rho_0}\bar{p}_z = g \to \tilde{p}(x, y, z, t) = p(x, y, z, t) - \rho_0 g z + p_0. \tag{7.79}$$

We note that $H \ll L$ is essential in eliminating the w-equation in the SWE.

Chapter 8

Hydrodynamic stability

8.1 Linearized Navier–Stokes

Consider the incompressible N-S equations

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{8.2}$$

and the decomposition of velocity and pressure into the base and perturbation states:

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}' \tag{8.3}$$

$$p = P + p' \tag{8.4}$$

We note that the base state also satisfies N-S:

$$\nabla \cdot \boldsymbol{U} = 0 \tag{8.5}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \boldsymbol{U} \tag{8.6}$$

hence by plugging in the decomposition to (8.1)-(8.2) we have the perturbation equation:

$$\nabla \cdot \boldsymbol{u}' = 0 \tag{8.7}$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{U} + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' = -\frac{1}{\rho}\nabla p' + \nu \nabla^2 \boldsymbol{u}' \tag{8.8}$$

And we note that the boundary conditions that the perturbation $\boldsymbol{u}', p'$ satisfy is homogeneous, such that $\boldsymbol{U}$ and p_b satisfy the same BC's as $\boldsymbol{u}$ and p in the original equation.

In linear stability, with the assumption that

$$O(\boldsymbol{u}') = \epsilon O(\boldsymbol{U}), \tag{8.9}$$

we neglect the nonlinear term $\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'$ and the primes, and have the linearised perturbation equation

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.10}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} - \nu \nabla^2 \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho} \nabla p \tag{8.11}$$

or if we define the linear operator as

$$\mathcal{L}_{\boldsymbol{U}} = \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{8.12}$$

there is

$$\mathcal{L}_{\boldsymbol{U}} \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho} \nabla p. \tag{8.13}$$

That being said, the linear mechanism is that the fluctuations extract energy from the mean flow, to the leading order effect, instead of interacting with themselvess.

The linearised equations (8.10)-(8.11), if written in matrix form (Arratia, 2011), is

$$\mathcal{L}_{\mathrm{NS}} \boldsymbol{q} = \begin{bmatrix} \mathcal{L}_{\boldsymbol{U}} + \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial V}{\partial x} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial W}{\partial z} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = 0, \tag{8.14}$$

where $\boldsymbol{q} = [u, v, w, p]^T$. This is in the KKT form that will be described below, where we will see that the same mathematical properties of the operators will be shared in both stability analysis and CFD.

On the other hand, it is sometimes also convenient to define the RHS operator as

$$\mathcal{A}_{\boldsymbol{U}} \boldsymbol{u} = -\boldsymbol{U} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} \tag{8.15}$$

such that

$$\frac{\partial \boldsymbol{u}}{\partial t} = \mathcal{A}_{\boldsymbol{U}} \boldsymbol{u}. \tag{8.16}$$

In practice, the pressure gradient can be neglected when calculating the operator $\mathcal{A}_{\boldsymbol{U}}$ and the resulting pressure is projected onto a divergence-free space (along with satisfying the boundary conditions in each numerical iteration/time-step), under the framework of projection methods. In general, in theoretical analysis or numerical methods where the pressure p is elimenated (by procedures similar to taking the curl of the N–S), the equations are to some extent equivalent to the vorticity equation.

8.1.1 The role of pressure

A separate short note on the pressure playing the role of Lagrangian multiplier in incompressible system. Consider the Stokes flow (actually that can be the linearized equations as described above)

$$-\nabla^2 \boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{8.17}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.18}$$

and in the matrix form

$$\begin{bmatrix} -\nabla^2 & \nabla \\ -\nabla\cdot & O \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} \tag{8.19}$$

and its discrete version

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{8.20}$$

which is a saddle point problem or a KKT (Karush-Kuhn-Tucker) system (Benzi *et al.*, 2005). The Stokes equations can be interpreted as a constrained optimization problem (section 3.15.5 of Gresho & Sani (1998))

$$\min\ J(\boldsymbol{u}) = \frac{1}{2}\int \|\nabla \boldsymbol{u}\|_2^2 dV - \int \boldsymbol{f}\cdot\boldsymbol{u} dV \tag{8.21}$$

$$\text{subject to } \nabla\cdot\boldsymbol{u} = 0 \tag{8.22}$$

where the variable p, introduced to satisfy an additional constraint, plays the role of a Lagrangian multiplier (Gresho & Sani, 1987). We note that the adjoint of the gradient operator is the (negative) divergence operator

$$(\nabla)^\dagger = -\nabla\cdot(\,) \tag{8.23}$$

We now establish this fact. Consider a scalar f and a vector $\boldsymbol{F}$. Consider

$$\int_V \nabla\cdot(f\boldsymbol{F})\,dV = \int_V f(\nabla\cdot\boldsymbol{F})\,dV + \int_V \nabla f\cdot\boldsymbol{F}\,dV = \iint_{\Omega=\partial V} f\boldsymbol{F}\cdot d\boldsymbol{A} \tag{8.24}$$

and if the boundary integral vanishes we have

$$(\nabla f, \boldsymbol{F}) = \int_V (\nabla f\cdot\boldsymbol{F})\,dV = -\int_V f(\nabla\cdot\boldsymbol{F})\,dV = (f, -\nabla\cdot\boldsymbol{F}) = (f, (\nabla)^\dagger(\boldsymbol{F})) \tag{8.25}$$

and hence

$$(\nabla)^\dagger = -\nabla\cdot(\,) \tag{8.26}$$

8.1.2 The role of linear mechanisms, the Reynolds–Orr equation

Integrate the TKE equation space and note the integration of the spatial transport terms vanish at the boundaries, we have the Reynolds–Orr equation, whose general form is

$$\frac{d}{dt}\int_\Omega \frac{1}{2}u_i'u_i'\,\mathrm{d}V = \int_\Omega -u_i'u_j'\frac{\partial U_i}{\partial x_j}\,\mathrm{d}V - \int_\Omega \nu\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\,\mathrm{d}V. \tag{8.27}$$

. Here U_j is the laminar base flow in a stability problem, which is itself a solution to the NSE and u_i' are the respectively defined perturbations. The implication is that the total production balances the total dissipation, and the spatial integral of the nonlinear transport term is zero. An equivalant statement in Fourier space is that the sum of the spectral transport term $\hat{T}_n l$ over all wavenumbers is zero. Recall the derivation (5.142). Also, the production term is linear. That being said, the mechanisms responsible for the global energy amplification have to be linear.

8.2 Parallel shear flows

Here, we start from a more generalized framework with a vertical density profile and a buoyancy equation (linearized N–S under Boussinesq) similar to in Section 6.7.2. We will also restrict our attention on one-dimensional horizontal shear $U(y)$ and vertical stratification ($Ri_g = N^2L^2/U^2$) and rotation ($f = 2\Omega$) where the scales for $(\boldsymbol{u}, \boldsymbol{x}, t, p, \rho)$ are $(U, L, L/U, \rho_0 U^2, (\rho_0/g)N^2 L)$.

The non-dimensional linearized perturbation equations read

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} - 2\Omega v &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + 2\Omega u &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} - Ri_b\rho + \frac{1}{Re}\nabla^2 w \\
\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} - w &= \frac{1}{RePr}\nabla^2 \rho
\end{aligned} \tag{8.28}$$

where the primes for perturbational variables are omitted.

With the normal-mode *ansatz*

$$(\boldsymbol{u}, p, \rho) = (\hat{\boldsymbol{u}}, \hat{p}, \hat{\rho}) \exp(ik_1 x + ik_3 z + \sigma t) \tag{8.29}$$

where $\sigma = -ik_1 c = -i\omega$ is the growth rate and $c = \omega/k$ is the wave speed, the equations (8.28) can be turned into the normal-mode equations

$$ik_1\hat{u} + D\hat{v} + ik_3\hat{w} = 0 \tag{8.30}$$

$$(\sigma + ik_1 U - \nu\triangle)\hat{u} + (DU - 2\Omega)\hat{v} = -ik_1\hat{p} \tag{8.31}$$

$$(\sigma + ik_1 U - \nu\triangle)\hat{v} + 2\Omega\hat{u} = -D\hat{p} \tag{8.32}$$

$$(\sigma + ik_1 U - \nu\triangle)\hat{w} = -ik_3\hat{p} - Ri_b\hat{\rho} \tag{8.33}$$

$$(\sigma + ik_1 U - \kappa\triangle)\hat{\rho} = \hat{w} \tag{8.34}$$

where we denote

$$D = \partial_y \tag{8.35}$$

$$\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz} \tag{8.36}$$

$$k^2 = k_1^2 + k_3^2 \tag{8.37}$$

$$\triangle = D^2 - k^2 \tag{8.38}$$

$$\nu = \frac{1}{Re} \tag{8.39}$$

$$\kappa = \frac{1}{RePr} \tag{8.40}$$

following the convention in Arobone & Sarkar (2012).

In a compact form, the normal-mode equations can be expressed as

$$\boldsymbol{A}\begin{bmatrix}\hat{u}\\ \hat{v}\\ \hat{\rho}\end{bmatrix} = \sigma\boldsymbol{B}\begin{bmatrix}\hat{u}\\ \hat{v}\\ \hat{\rho}\end{bmatrix} \tag{8.41}$$

where

$$\boldsymbol{A} = \begin{bmatrix} -2\Omega k^2 & ik_1(U\triangle - D^2U + 2\Omega D) - \nu\triangle^2 & -ik_3 Ri_b D \\ ik_1(DU + UD - 2\Omega) - \nu\triangle D & D^2U + k_1^2U + DUD - 2\Omega D + \nu ik_1\triangle & 0 \\ -ik_1 & -D & k_1k_3U + \kappa ik_3\triangle \end{bmatrix} \tag{8.42}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & -\triangle & 0 \\ -D & ik_1 & 0 \\ 0 & 0 & ik_3 \end{bmatrix} \tag{8.43}$$

as a generalized eigenvalue problem in terms of σ (Arobone & Sarkar, 2012). It is noted that both w and p are eliminated and this technique of deriving a pressure-less system is quite common in stability analysis and CFD. The cost is, however, differential equations of higher orders that require more boundary conditions, which are hopefully simple, in this parallel flow case.

The first row is effectively the Laplacian-v ($-\triangle\hat{v}$) equation, obtained by $\partial_y(\partial_z M_3\&C) + \partial_{zz}M_2 + \partial_x(\partial_x M_2 - \partial_y M_1)$, or effectively $\partial_x\omega_z - \partial_z\omega_x = \partial_x(\partial_x M_2 - \partial_y M_1) - \partial_z(\partial_y M_3 - \partial_z M_2)$ (taking the curl twice). Here $M_i (i = 1, 2, 3)$ denote the i-th momentum equation and C denotes the continuity equation, and & means substitute.

$\partial_z M_3\&C$:

$$(\sigma + ik_1U - \nu\triangle)(ik_1\hat{u} + D\hat{v}) = -k_3^2\hat{p} + ik_3Ri_b\hat{\rho} \tag{8.44}$$

$\partial_y(\partial_z M_3\&C)$:

$$ik_1(\sigma + ik_1U - \nu\triangle)D\hat{u} - k_1^2(DU)\hat{u} + (\sigma + ik_1U - \nu\triangle)D^2\hat{v} + ik_1(DU)D\hat{v} = -k_3^2D\hat{p} + ik_3Ri_bD\hat{\rho} \tag{8.45}$$

$\partial_{zz}M_2$:

$$-k_3^2(\sigma + ik_1U - \nu\triangle)\hat{v} - 2k_3^2\Omega\hat{u} = Dk_3^2\hat{p} \tag{8.46}$$

$-\partial_x(\partial_y M_1)$:

$$-ik_1(\sigma + ik_1U - \nu\triangle)D\hat{u} + k_1^2(DU)\hat{u} - ik_1(DU - 2\Omega)D\hat{v} - ik_1(D^2U)\hat{v} = -k_1^2D\hat{p} \tag{8.47}$$

$\partial_x(\partial_x M_2)$:

$$-k_1^2(\sigma + ik_1U - \nu\triangle)\hat{v} - 2k_1^2\Omega\hat{u} = k_1^2D\hat{p} \tag{8.48}$$

$\partial_y(\partial_z M_3\&C) + \partial_{zz}M_2 + \partial_x(\partial_x M_2 - \partial_y M_1)$:

$$\text{RHS} = ik_3Ri_bD\hat{\rho} \tag{8.49}$$

$$\text{LHS}_1 = -(\sigma + ik_1U - \nu\triangle)(k_1^2 + k_3^2)\hat{v} - 2(k_1^2 + k_3^2)\Omega\hat{u} \tag{8.50}$$

$$= -(\sigma + ik_1U - \nu\triangle)k^2\hat{v} - 2k^2\Omega\hat{u} \tag{8.51}$$

$$\mathrm{LHS}_2 = ik_1(DUD - DUD + 2\Omega D - D^2U)\hat{u} = ik_1(2\Omega D - D^2U)\hat{v} \tag{8.52}$$

$$\mathrm{LHS}_3 = \sigma D^2\hat{v} + ik_1UD^2\hat{v} - \nu\triangle D^2\hat{v} \tag{8.53}$$

Rearrange:

$$\mathrm{RHS} = -\sigma\triangle\hat{v} \tag{8.54}$$

$$\mathrm{LHS} = -2k^2\Omega\hat{u} + ik_1(2\Omega D - D^2U + U\triangle)\hat{v} - \nu\triangle^2\hat{v} - ik_3Ri_bD\hat{\rho} \tag{8.55}$$

The second row is effectively an equation for ω_z ($=ik_1\hat{v} - D\hat{u}$), obtained by by cross differentiating u, v momentum equations. Pay attention to stuff like the chain rule: $D((DU)\hat{v}) = D^2U\hat{v} + (DU)D\hat{v}$. The third row is from combining the continuity and the scalar equations. The first two equations represents the time-evolution ($\sigma \sim \partial_t$) of Laplacian-v and ω_z (which is more suitable than ω_y for there is stratification/rotation). It is foruth-order in $\hat{v}$ since it contains $\nu\triangle(\triangle\hat{v})$.

It will be easy to reduce from the general case derived here to the specific cases described later.

8.2.1 Orr–Sommerfield equations

In what follows, we consider homogeneous parallel shear flows with $U(y) \neq 0, \Omega = 0, Ri_g = 0$. Per Squire's theorem (next section), it is always possible to re-orient the coordinate $x - z$ such that any each unstable 3D perturbation always corresponds to a more unstable 2D one. Hence, it is sufficient to study 2D perturbations with $k_3 = 0, \hat{w} = 0$.

The 2D perturbation equations are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\ \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \end{aligned} \tag{8.56}$$

and by introducing the streamfunction and normal modes

$$u = \psi_y,\ v = -\psi_x,\ (u, v, \psi) = (\hat{u}, \hat{v}, \hat{\phi})\exp[ik(x - ct)] \tag{8.57}$$

we have to the Orr–Sommerfield equation

$$(U - c)(\hat{\phi}_{yy} - k^2\hat{\phi}) - U_{yy}\hat{\phi} = \frac{1}{ikRe}(\hat{\phi}_{yyyy} - 2k^2\hat{\phi}_{yy} + k^4\phi) \tag{8.58}$$

representing the viscous instability of shear flows.

We note that the normal mode perturbations (8.57) are in the form of plane waves, where $c_r = \omega/k$ is the phase speed and $\sigma_{2D} = kc_i$ is the temporal growth rate. The expression of the perturbations ($\in \mathbb{R}$) are to be understood as, for example,

$$w = \Re\{\hat{w}[\cos(kx - kc_rt) - i\sin(kx - kc_rt)]\}e^{c_ikt} \tag{8.59}$$

$$= \{\Re(\hat{w})[\cos(kx - kc_r t)] - \Im(\hat{w})[\sin(kx - kc_r t)]\}e^{kc_i t}. \tag{8.60}$$

We also note that since we have taken the curl, the ODE is thrid order in velocity and fourth order in streamfunction. The boundary conditions for two no-slip walls are $\hat{\phi}(0) = \hat{\phi}_y(0) = 0$, $\hat{\phi}(1) = \hat{\phi}_y(1) = 0$. We will revisit the O–S equation in a later section on transient growth.

8.2.2 Rayleigh's inflection point criterion

The inviscid limit of (8.58) is the Rayleigh equation

$$(U - c)\left(\frac{d^2}{dy^2} - k^2\right)\hat{\phi} - \frac{d^2U}{dy^2}\hat{\phi} = 0 \tag{8.61}$$

with two required BC's being no-penetration, $\hat{\phi}(0) = \hat{\phi}(1) = 0$. It can be shown by taking the complex conjugate of this equation that if c and $\hat{\phi}$ are an eigen pair, so as c^* and $\hat{\phi}^*$. We also note that the temporal growth rate is $\exp(c_i t)$ ($c = c_r + ic_i$). Hence, for each unstable mode there is a corresponding stable mode (and v.v.), and for each (neutrally) stable mode c must be real.

Multiply the following form of (8.61) with $\hat{\phi}^*$ and integrate both sides of

$$\hat{\phi}_{yy} - k^2\hat{\phi} - \frac{U_{yy}}{U - c}\hat{\phi} = 0 \tag{8.62}$$

within $0 < y < 1$ (assume $c_i \neq 0$ so $U - c \neq 0$), as a common technique in Sturm–Liouville problems, we have

$$\int |\hat{\phi}_y|^2\,dy + k^2 \int |\hat{\phi}|^2\,dy + \int \frac{U_{yy}}{U - c}|\hat{\phi}|^2\,dy = 0. \tag{8.63}$$

Since the first two terms are real, the imaginary part of the third term must be zero, that said,

$$\Im\left(\int \frac{U - c^*}{U - c^*}\frac{U_{yy}}{U - c}|\hat{\phi}|^2\,dy\right) = c_i \int \frac{U_{yy}}{|U - c|^2}|\hat{\phi}|^2\,dy = 0, \tag{8.64}$$

which is valid when either $c_i = 0$ (stable) or the integral being zero when U_{yy} changes sign (has an inflection point where $U_{yy} = 0$) within the domain.

For neutrally stable modes $c_i = 0$, the highest derivative drops out from (8.61) and a critical layer is form near $U = c$. Actually, we can obtain a more sufficient condition (a stronger necessary condition that has a narrower range) than the inflection point criterion. The real part of (8.63) is

$$\int \frac{U_{yy}(U - c_r)}{|U - c|^2}|\hat{\phi}|^2\,dy = -\int (|\hat{\phi}_y|^2 + k^2|\hat{\phi}|^2)\,dy \tag{8.65}$$

and for nonzero c_i (8.64) reads

$$\int \frac{U_{yy}}{|U - c|^2}|\hat{\phi}|^2\,dy = 0. \tag{8.66}$$

Assuming the velocity profile is such that Rayleigh's criterion is satisfied and there exist a unique inflection point y_s, where $U(y_s) = U_s$ and $U''(y_s)$, multiply the above equation with $(c_r - U_s)$ and add it to (8.65) we have

$$\int \frac{U_{yy}(U - U_s)}{|U - c|^2}|\hat{\phi}|^2\,dy = -\int (|\hat{\phi}_y|^2 + k^2|\hat{\phi}|^2)\,dy \leq 0 \tag{8.67}$$

We note that both U_{yy} and $U - U_s$ changes sign at y_s so the product cannot change sign there. The only way for the above inequality to hold is for the product to be negative somewhere in the flow, i.e.,

$$U_{yy}(U - U_s) \leq 0, \tag{8.68}$$

with the equality only achieved at y_s. This is referred to as the Fjørtoft's criterion.

Example: shear layer with a velocity profile of a hyperbolic tangent

$$U(y) = U_0 \tanh(y/L), \tag{8.69}$$

where the velocity scale is $U_0 = (U_1 - U_2)/2$ and L is the half-width of the layer. We can see that the shear is the largest at the centerline:

$$S^* = U'(y)L/U_0 = \mathrm{sech}^2(y/L), \tag{8.70}$$

which achives a non-dimensional value of unity at $y = 0$ and decays gradually outwards. The second derivative is

$$U''(y) = -\frac{2U_0}{L^2}\mathrm{sech}^2(y/L)\tanh(y/L) \tag{8.71}$$

and it has one inflection point at $y = 0$ such that $U''(0) = 0$. The instability of flows with inflection point leads to rolled-up blobs, such as the Kelvin–Helmholtz billow in shear layers.

8.2.3 Squire's transformation and theorem

In order to justify the 2D analysis in the previous sections, we consider the inviscid system described by (8.56), but subject to 3D normal mode perturbations:

$$(u, v, w, p) = (\hat{u}, \hat{v}, \hat{w}, \hat{p})\exp[i(k_1 x + k_3 z - k_1 ct)] \tag{8.72}$$

which is a plane wave with phase velocity $c_p = k_1 c_r/(k_1^1 + k_2^2)^{1/2}$ (the phase, $\varphi = \boldsymbol{k}\cdot\boldsymbol{x} - c_p kt$ is written in terms of $k_1 ct$ instead of $k = (k_1^1 + k_2^2)^{1/2}$ for convenience), propagating along the (k_1, k_3) direction, oblique to the base flow.

We have

$$ik_1\hat{u} + \hat{v}_y + ik_3\hat{w} = 0 \tag{8.73}$$

$$ik_1(U - c)\hat{u} + \hat{v}U_y = -ik_1\hat{p} + \frac{1}{Re}[\hat{u}_{yy} - (k_1^2 + k_3^2)\hat{u}] \tag{8.74}$$

$$ik_1(U - c)\hat{v} = -\hat{p}_y + \frac{1}{Re}[\hat{v}_{yy} - (k_1^2 + k_3^2)\hat{v}] \tag{8.75}$$

$$ik_1(U - c)\hat{w} = -ik_3\hat{p} + \frac{1}{Re}[\hat{v}_{yy} - (k_1^2 + k_3^2)\hat{v}] \tag{8.76}$$

Under the Squire transformation,

$$\bar{k} = (k_1^1 + k_2^2)^{1/2},\ \bar{k}\bar{u} = k_1\hat{u} + k_3\hat{w},\ \bar{p}/\bar{k} = \hat{p}/k_1, \tag{8.77}$$

$$\bar{c} = c,\ \bar{v} = \hat{v},\ \bar{k}\overline{Re} = k_1 Re \tag{8.78}$$

the first and third momentum equations can be combined to yield

$$i\bar{k}\bar{u} + \bar{v}_y = 0 \tag{8.79}$$

$$i\bar{k}(U-c)\bar{u} + \bar{v}U_y = -i\bar{k}\bar{p} + \frac{1}{\overline{Re}}[\bar{u}_{yy} - \bar{k}^2\bar{u}] \tag{8.80}$$

$$i\bar{k}(U-c)\bar{v} = -\bar{p}_y + \frac{1}{\overline{Re}}[\bar{v}_{yy} - \bar{k}^2\bar{v}] \tag{8.81}$$

which look exactly the same to the normal mode equations in 2D resulting from (8.56) (not shown). However, the growth rate is $\sigma_{2D} = \bar{k}c_i > k_1 c_i = \sigma_{2D}$. That being said, for each 3D perturbation, there exist an equivalent 2D perturbation that has a higher growth rate. Also, since $\overline{Re} < Re$, for each critical Reynolds number in 3D, there is a lower corresponding critical Reynolds number for 2D perturbations.

Hence, we may say that for parallel shear flows the 2D perturbations are more unstable than 3D ones and the previous discussions on the O–S and the Rayleigh equations in 2D are sufficient.

8.2.4 Taylor–Goldstein equation

Similar to (8.28), the dimensional, inviscid, linearized perturbation equations for vertical shear flow $(U(z), 0, 0)$ can be expressed as

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} \\
\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} - w\frac{N^2\rho_0}{g} &= 0
\end{aligned} \tag{8.82}$$

where the perturbations are assumed to be 2D (according to Squire's theorem) in the form of (u, v, w, p, ρ).

Introducing the streamfunction

$$u = \psi_z,\; w = -\psi_x \tag{8.83}$$

and the normal modes

$$(\rho, p, \psi) = (\hat{\rho}(z), \hat{p}(z), \hat{\psi}(z)) \exp[ik(x - ct)] \tag{8.84}$$

the equations (8.82) are converted into

$$\begin{aligned}
(U-c)\hat{\psi}_z - U_x\hat{\psi} &= -\frac{1}{\rho_0}\hat{p} \\
(U-c)k^2\hat{\psi} &= -\frac{1}{\rho_0}\hat{p}_z - \frac{g}{\rho_0}\hat{\rho} \\
(U-c)\hat{\rho} + \frac{N^2\rho_0}{g}\hat{\psi} &= 0
\end{aligned} \tag{8.85}$$

and by eliminating pressure we have the Taylor–Goldstein equation

$$(U-c)\left(\frac{d^2}{dz^2} - k^2\right)\hat{\psi} + \left(\frac{N^2}{U-c} - \frac{d^2U}{dz^2}\right)\hat{\psi} = 0, \tag{8.86}$$

which is just the Rayleigh equation (8.61) with an additional term from density stratification.

Since

$$\overline{\frac{1}{U-c}} = \overline{\left(\frac{U-c^*}{|U-c|^2}\right)} = \frac{U-c}{|U-c|^2} = \frac{1}{U-c^*}, \tag{8.87}$$

it can be seen that $(c, \hat{\psi})$ is an eigen pair, so is $(c^*, \hat{\psi})$. Hence, for each growing mode, there is a corresponding decaying mode. The no-penetration conditions $w(0) = w(1)$ requires that $\hat{\psi}(0) = \hat{\psi}(1) = 0$ at the walls. We also note that since $w = -\psi_x$, the normal modes of w are related to those of ψ as $\hat{w} = -ik\hat{\psi}$ so the TGE can also be cast as

$$(U-c)\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} + \left(\frac{N^2}{U-c} - \frac{d^2U}{dz^2}\right)\hat{w} = 0. \tag{8.88}$$

If there is no background flow, the T–G that describes the instability triggered by perturbations only reads

$$\frac{d^2\hat{\psi}}{dz^2} + k^2\left(\frac{N^2}{\Omega^2} - 1\right)\hat{\psi} = 0, \tag{8.89}$$

where $\Omega^2 = c^2k^2$ is the angular frequency. Here $N = N(z)$ is an arbitraty density profile.

The T–G equation (8.86) and the Rayleigh equation (8.61) are both second-order ODEs in the Sturm–Liouville form (8.196). They are eigenvalue problems with c or Ω as the eigenvalue and k as a parameter, and are **not** self-adjoint but can be transformed to self-adjoint (refs) via the Howard transformations (ref). What viscosity does on the normality?

8.2.5 Howard's semicircle theorem

Takeing the Howard's transformation (non-singular if $c \neq U$)

$$F = \frac{\hat{\psi}}{U-c} \tag{8.90}$$

the T–G equation is transformed into

$$\frac{d}{dz}[(U-c)^2F_z] - k^2(U-c)^2F + N^2F. \tag{8.91}$$

Multiply by F^*, integrate in the vertical domain, and use the homogeneous boundary conditions, we get

$$\int (U-c)^2Q\,dz = \int N^2|F|^2\,dz \tag{8.92}$$

where

$$Q = |F_z|^2 + k^2|F|^2 \tag{8.93}$$

is real positive. Separating the real and imaginary parts of (8.92), we have

$$\int [(U-c_r)^2 - c_i^2]Q\,dz = \int N^2|F|^2\,dz \tag{8.94}$$

$$c_i \int (U-c_r)Q\,dz = 0. \tag{8.95}$$

Since we require $c_i \neq 0$ for instability, the second equation is essentially

$$\int (U - c_r)Q\, dz = 0, \tag{8.96}$$

which can be satisfied only if $U - c_r$ changes sign in the domain, i.e.,

$$U_{\min} < c_r < U_{\max}. \tag{8.97}$$

That implies the propagation direction of the unstable waves is the same as the base flow (assume $U_{\min} > 0$) and provides a bound for their phase speed – could not be faster or slower than the base flow.

Combining (8.94) and (8.95) we have

$$\int [U^2 - c_r^2 - c_i^2]Q\, dz \geq 0, \tag{8.98}$$

also naturally we have

$$\int (U - U\text{min})(U - U_{\max})Q\, dz \leq 0 \rightarrow \int [U_{\max}U_{\min} + U^2 - U(U_{\max} + U_{\min})]Q\, dz \leq 0. \tag{8.99}$$

Combining (8.98) and (8.99) and realizing (8.96) we finally have

$$\int [U_{\max}U_{\min} + c_r^2 + c_i^2 - c_r(U_{\max} + U_{\min})]Q\, dz \leq 0. \tag{8.100}$$

That implies a semicircle ($c_i > 0$)

$$[c_r - \frac{1}{2}(U_{\max} + U_{\min})^2] + c_i^2 \leq [\frac{1}{2}(U_{\max} - U_{\min})]^2 \tag{8.101}$$

which limits the possible phase speed (c_r) of the unstable waves and their growth rate ($\sigma = kc_i$). This is a necessary condition. Also, the maximum growth rate is limited by (not necessarily reached)

$$kc_i \leq \frac{k(U_{\max} - U_{\min})}{2}. \tag{8.102}$$

In the numerical computation of the complex eigenvalue $c(k)$, the above bounds efficiently narrow the searching range. We also note that $N = 0$ will only zero out the RHS of (8.92) and (8.98) but not that of (8.99), so the semicircle theorem is unaffected and is valid for both unstratified and stratified shear flows.

8.2.6 Miles–Howard sufficient condition

Under the transformation (Howard, 1961)

$$\phi = \frac{\hat{\psi}}{\sqrt{U - c}} \tag{8.103}$$

the T–G equation can be converted to

$$\frac{d}{dz}[(U - c)\phi_z] - \left[k^2(U - c) + \frac{1}{2}U_{zz} + \frac{1}{U - c}\left(\frac{1}{4}U_z^2 - N^2\right)\right]\phi = 0. \tag{8.104}$$

Multiply the equation above with ϕ^*, integrate in z, and use the homogeneous boundary conditions, we get

$$\int \frac{1}{U-c}(N^2 - 1/4U_z^2)\,dz = \int (U-c)(|\phi_z|^2 + k^2|\phi|^2)\,dz + \int \frac{1}{2}\int U_{zz}|\phi|^2\,dz \tag{8.105}$$

and notice that

$$\frac{1}{U-c} = \frac{U-c_r+ic_i}{|U-c|^2} \tag{8.106}$$

we have (separating real and imaginary parts):

$$\int \frac{U-c_r}{|U-c|^2}(N^2 - 1/4U_z^2)\,dz = \int (U-c_r)(|\phi_z|^2 + k^2|\phi|^2)\,dz + \int \frac{1}{2}\int U_{zz}|\phi|^2\,dz \tag{8.107}$$

$$\int \frac{c_i}{|U-c|^2}(N^2 - 1/4U_z^2)\,dz = -c_i \int (|\phi_z|^2 + k^2|\phi|^2)\,dz. \tag{8.108}$$

Since the integrant in the second equation is positive, the necessary condition for instability ($c_i \neq 0$) is for $N^2 - 1/4U_z^2$ to be negative at least in some part of the domain, i.e., at least one point in the domain has

$$Ri_g = \frac{N^2}{U_z^2} < \frac{1}{4}. \tag{8.109}$$

This adverse condition is generally referred to as the Miles–Howard sufficient condition: if the flow has everywhere $Ri_g > 1/4$, inviscid linear stability of the normal modes is guaranteed.

8.2.7 Transient growth

Here we revisit the Orr-Sommerfield equation and the Squire equation, in an ω_y-$\nabla^2 v$ formulation. Assume the base flow is $U(y)$ and the perturbations are (u, v, w). The linearized perturbation equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8.110}$$

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + U'v = -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \tag{8.111}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \tag{8.112}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w \tag{8.113}$$

and the linearized fluctuation pressure equation is

$$\nabla^2 p = -2U'\frac{\partial v}{\partial x}. \tag{8.114}$$

Define

$$\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{8.115}$$

$$\phi = \nabla^2 v \tag{8.116}$$

and their governing equations can be derived and will be of central roles in stability of parallel shear flows.

Take the divergence of the v-equation and substitute in pressure, we have the Orr–Sommerfield ($\nabla^2 v$) equation; and take the curl of the momentum equations we get the Squire (ω_y) equation:

$$\frac{\partial}{\partial t}(\nabla^2 v) + U\frac{\partial}{\partial x}(\nabla^2 v) = U''\frac{\partial v}{\partial x} + \frac{1}{Re}\nabla^2(\nabla^2 v) \tag{8.117}$$

$$\frac{\partial \omega_y}{\partial t} + U\frac{\partial \omega_y}{\partial x} = -U'\frac{\partial v}{\partial z} + \frac{1}{Re}\nabla^2 \omega_y \tag{8.118}$$

noting that the simplification of the second-order operator is

$$\nabla^2\left(U\frac{\partial v}{\partial x}\right) = U''\frac{\partial v}{\partial x} + 2U'\frac{\partial^2 v}{\partial x \partial y} + U\frac{\partial}{\partial x}(\nabla^2 v). \tag{8.119}$$

These operations are called Orr–Sommerfield transform, also commonly used in numerics (see section 2.3.2).

The v-equation is autonomous, although it is coupled to the second equation, in turn driven by v (seemingly one-way coupling) through the continuity constraint. These two equations are called the Orr–Sommerfield and the Squire equations *in physical space*, respectively. Since they are a fourth-order and a second order equation, we need six boundary conditions in total, being

$$v = \partial_y v = 0,\, \omega_y = 0 \tag{8.120}$$

at both walls, where the $\partial_y v$ condition comes from continuity.

The production terms in these two equations,

$$U''\frac{\partial v}{\partial x},\, -U'\frac{\partial v}{\partial z}, \tag{8.121}$$

represent the Orr and the lift-up mechanisms, that are two of the most important linear amplification mechanisms, especially in flows without inflection points, and they are through the interaction between the perturbations and the mean flow.

The OSE and SE in Fourier space, vector form, and non-normality.

Non-self-adjointness: The Rayleigh equation is not self-adjoint; but it is after the Howard transformation. So as the Taylor–Goldstein.

8.2.8 Axisymmetric parallel shear flows

It is useful to consider the perturbation equations in cylindrical coordinates (x, r, θ) for applications in fully developed pipe flow and quasi-parallel axisymmetric jets and wakes. The base flow is $(U(r), 0, 0, P)$ and the perturbations are (u_x, u_r, u_θ, p) where the primes are dropped. Axisymmetry base flow implies $\partial_\theta U = 0$ and quasi-parallel implies $\partial_x U = 0$.

The perturbation equations are

$$\frac{\partial u_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \tag{8.122}$$

$$\frac{\partial u_x}{\partial t} + U\frac{\partial u_x}{\partial x} + U' u_r = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u_x \tag{8.123}$$

$$\frac{\partial u_r}{\partial t} + U\frac{\partial u_r}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{Re}\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right) \tag{8.124}$$

$$\frac{\partial u_\theta}{\partial t} + U\frac{\partial u_\theta}{\partial x} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{1}{Re}\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) \tag{8.125}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{8.126}$$

8.3 Centrifugal and rotational instabilities

8.3.1 Rayleigh's criterion

Consider cylindrical coordinates (x, r, θ).

The θ-momentum equation reduces to

$$\frac{\partial u_\theta}{\partial t} + u_x\frac{\partial u_\theta}{\partial x} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} = 0 \tag{8.127}$$

when inviscid, axisymmetric ($\partial_\theta = 0$). It can be re-written as

$$\frac{1}{r}\left[\frac{\partial}{\partial t}(ru_\theta) + u_x\frac{\partial}{\partial x}(ru_\theta) + u_r\frac{\partial}{\partial r}(ru_\theta)\right] = 0, \tag{8.128}$$

or simply

$$\frac{D(ru_\theta)}{Dt} = \frac{DL}{Dt} = 0, \tag{8.129}$$

which is the conservation of angular momentum,

$$L = ru_\theta, \tag{8.130}$$

in the inciscid limit.

The inviscid r-momentum equation reads

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}. \tag{8.131}$$

Consider an axisymmetric base flow $(0, 0, U_\theta(r), P)$, satisfying the balance

$$\rho\frac{U_\theta^2}{r} = \frac{\partial P}{\partial r}. \tag{8.132}$$

When a ring of fluid at r_1 is displaced to $r_2 > r_1$ due to a radial perturbation $u_r > 0$, angular momentum is conserved

$$r_1 U_\theta(r_1) = r_2 u_\theta^* \tag{8.133}$$

and the derivative of the radial velocity is

$$\frac{Du_r}{Dt} = \frac{(u_\theta^*)^2}{r_2} - \frac{1}{\rho}\left.\frac{\partial P}{\partial r}\right|_{r=r_2} \tag{8.134}$$

$$= \frac{(r_1/r_2)^2 U_\theta^2(r_1)}{r_2} - \frac{U_\theta^2(r_2)}{r_2} \tag{8.135}$$

$$= \frac{1}{r_2^3}[r_1^2 U_\theta^2(r_1) - r_2^2 U_\theta^2(r_2)] \tag{8.136}$$

and the flow is unstable if at one point

$$r_1^2 U_\theta^2(r_1) > r_2^2 U_\theta^2(r_2) \tag{8.137}$$

or stable if everywhere

$$r_1^2 U_\theta^2(r_1) < r_2^2 U_\theta^2(r_2). \tag{8.138}$$

The condition for instability is that the absolute magnitude of the angular momentum decreases radially,

$$\frac{1}{r^3}\frac{dL^2}{dr} < 0. \tag{8.139}$$

This is commonly referred to as the Rayleigh's criterion and it is a necessary and sufficient condition for the instability of inviscid columnar vortices subject to 3-D axisymmetric perturbations (Drazin, 2002). Since the flow is axisymmetric, $\omega_z = \partial_r u_\theta + u_\theta/r - \partial_\theta u_r = \partial_r u_\theta + u_\theta/r$, the stability condition (8.139) is equivalent to

$$\frac{U_\theta}{r}\left(\frac{U_\theta}{r} + \frac{\partial U_\theta}{\partial r}\right) = \frac{U_\theta \omega_z}{r} < 0 \tag{8.140}$$

(velocity and vorticity are of opposite signs) everywhere.

8.3.2 Axisymmetric circular flows: normal mode analysis

The above conclusion can also be reached in a standard normal mode analysis.

Consider cylindrical coordinates (x, r, θ), base flow $(0, 0, U_\theta(r), P)$ and perturbations $(u_x', u_r', u_\theta', p')$. The governing equations for axisymmetric perturbations $(\partial_\theta = 0)$ $(\partial_\theta = 0, u_\theta = 0)$ are

$$\frac{\partial u_x'}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta') = 0 \tag{8.141}$$

$$\frac{\partial u_x'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \tag{8.142}$$

$$\frac{\partial u_r'}{\partial t} - 2\frac{U_\theta u_\theta'}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} \tag{8.143}$$

$$\frac{\partial u_\theta'}{\partial t} + u_r'\left(\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\right) = 0 \tag{8.144}$$

The last equation can be expressed as

$$\frac{\partial u_\theta'}{\partial t} = -\frac{u_r'}{r}\frac{\partial}{\partial r}(r u_\theta'). \tag{8.145}$$

8.3.3 Inertial/Coriolis instability

The Rayleigh's criterion can be generalized in a rotating reference frame (non-inertial; with rotation rate Ω) by replacing the angular momentum L with the absolute angular momentum

$$L_a = r(U_\theta + \Omega r) = r(U_\theta + \frac{f}{2}r) \tag{8.146}$$

where $U_\theta + \Omega r$ is interpreted as the absolute velocity. It is conserved along material lines

$$\frac{DL_a}{Dt} = 0. \tag{8.147}$$

The condition for instability

$$\frac{1}{r^3}\frac{dL_a^2}{dr} < 0 \tag{8.148}$$

translates to

$$\chi = \left(\frac{2U_\theta}{r} + f\right)(\omega_z + f) < 0, \tag{8.149}$$

which is called the generalized Rayleigh criterion (Kloosterziel & Van Heijst, 1991). Equation (8.149) can be interpreted as that if the absolute velocity and the absolute vorticity are of opposite signs, the flow is unstable.

It can be estabilished similarly to (8.139) as follows. The θ-momentum equation translates to the absolute angular momentum conservation. The inviscid r-momentum equation reads

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + fu_\theta. \tag{8.150}$$

Consider an axisymmetric base flow $(0, 0, U_\theta(r), P)$, satisfying the geostrophic balance (r momentum balance)

$$-\frac{U_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + fU_\theta. \tag{8.151}$$

When a ring of fluid at r_1 is displaced to $r_2 > r_1$ due to a radial perturbation $u_r > 0$, the absolute angular momentum conservation leads to

$$r_1 U_\theta(r_1) + \frac{1}{2}fr_1^2 = r_2 u_\theta^* + \frac{1}{2}fr_2^2 \tag{8.152}$$

and the derivative of the radial velocity is

$$\frac{Du_r}{Dt} = \frac{(u_\theta^*)^2}{r_2} - \frac{1}{\rho}\left.\frac{\partial P}{\partial r}\right|_{r=r_2} + fu_\theta^* \tag{8.153}$$

$$= \frac{(u_\theta^*)^2 - U_\theta^2(r_2)}{r_2} + f[u_\theta^* - U_\theta(r_2)] \tag{8.154}$$

$$= \frac{1}{r_2^3}[(r_1^2U_\theta^2(r_1) - r_2^2U_\theta^2(r_2)) + (\frac{1}{4}f^2r_1^4 - \frac{1}{4}f^2r_2^4) + (fU_\theta(r_1)r_1^3 - fU_\theta(r_2)r_2^3)] \tag{8.155}$$

$$= \frac{1}{r_2^3}[L_a^2(r_1) - L_a^2(r_2)] \tag{8.156}$$

and the flow is unstable if at one point

$$\frac{1}{r_3}\frac{dL_a^2}{dr} < 0 \tag{8.157}$$

or stable if everywhere

$$\frac{1}{r_3}\frac{dL_a^2}{dr} > 0. \tag{8.158}$$

Taking $r \to \infty$ (translation = rotation around infinity), we recover the absolute vorticity criterion

$$f(\omega_z + f) < 0 \tag{8.159}$$

for anticyclonic parallel shear flows (Holton, 1972), where ω_z also reduces to $\omega_z = \partial_r u_\theta + u_\theta/r = \partial_r u_\theta = -S$. This marks the similarity between the inertial instability in 2D parallel and axisymmetric base flows. The structures resulting from these two instabilities are quasi-streamwise/azimuthal vortices, respectively. However, 8.159 implies that cyclonic shear layer (f and ω_z same sign) is always stable, but both cyclonic and anticyclic axisymmetric vortices can be unstable.

8.4 Some classical instabilities

8.4.1 Kelvin–Helmholtz and Rayleigh–Taylor instabilities

The KHI and RTI can be analysed together as special cases of a general problem of a vortex sheet separating two streams of fluids with velocity U_1, U_2 and density ρ_1, ρ_2. Here 1 denotes the bottom fluid and 2 denotes the top fluid.

The vortex sheet is an interface $z = \zeta(x, y, t)$ and other than that the flow is inviscid/irrotational – $\nabla^2\phi_1 = 0$, $\phi_1 = U_1 x$; $\nabla^2\phi_2 = 0$, $\phi_2 = U_2 x$, where ϕ is the velocity potential such that $\boldsymbol{u} = \nabla\phi$. The momentum equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\frac{\boldsymbol{u}^2}{2}\right) - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\boldsymbol{u} - g\boldsymbol{e}_z \tag{8.160}$$

can then be converted into

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{(\nabla\phi)^2}{2} + \frac{p}{\rho} + gz\right) = 0, \tag{8.161}$$

neglecting viscosity. Hence, in each region, the governing equations are

$$p_1 = \rho_1\left[C_1 - \frac{\partial\phi_1}{\partial t} - \frac{(\nabla\phi_1)^2}{2} - gz\right] \tag{8.162}$$

$$p_2 = \rho_2\left[C_2 - \frac{\partial\phi_2}{\partial t} - \frac{(\nabla\phi_2)^2}{2} - gz\right] \tag{8.163}$$

Then we consider the boundary conditions. First, the kinematic boundary condition is that a material surface ($f(x, y, z, t) = z - \zeta(x, y, t) = 0$) remains a material surface, so $Df/Dt = \partial f/\partial t + \boldsymbol{u} \cdot \nabla f = 0$, or

$$-\zeta_t - \phi_x\zeta_x - \phi_y\zeta_y + \phi_z = 0, \tag{8.164}$$

which holds for both sides of the interface at $z = \zeta$:

$$\phi_{1,z} = \zeta_t + \phi_{1,x}\zeta_x + \phi_{1,y}\zeta_y \tag{8.165}$$

$$\phi_{2,z} = \zeta_t + \phi_{2,x}\zeta_x + \phi_{2,y}\zeta_y \tag{8.166}$$

Second, the dynamic boundary condition concerns the force balance of the interface. Since the flow is inviscid, at $z = \zeta$ the pressure on both sides should be equal:

$$\rho_1\left[C_1 - \frac{\partial\phi_1}{\partial t} - \frac{(\nabla\phi_1)^2}{2} - gz\right] = \rho_2\left[C_2 - \frac{\partial\phi_2}{\partial t} - \frac{(\nabla\phi_2)^2}{2} - gz\right]. \tag{8.167}$$

We want to be systematic on the procedures and will follow the standard steps of base flow – linearization – normal modes.

Base flow.

- Outside the vortex sheet: $\nabla^2\Phi_1 = \nabla^2\Phi_2 = 0$ and $\Phi_1(z=-\infty) = U_1 x, \Phi_2(z=+\infty) = U_2 x$.
- Kinematic BC: $\zeta = 0$.
- Dynamic BC: $\rho_1[C_1 - U_1^2/2] = \rho_2[C_2 - U_2^2/2]$.

Linearization/governing equations for perturbations. Where $\phi_i = \Phi_i + \phi_i'$.

- Outside the vortex sheet: $\nabla^2\phi_1' = \nabla^2\phi_2' = 0$ and $\phi_1'(z=-\infty) = \phi_2'(z=+\infty) = 0$.
- Linearized kinematic BC at $z = \zeta$ or, approximately, at $z = 0$:

$$\phi_{1,z}' = \zeta_t + U_1\zeta_x \tag{8.168}$$
$$\phi_{2,z}' = \zeta_t + U_2\zeta_x \tag{8.169}$$

- Linearized dynamic BC at $z = \zeta$ or, approximately, at $z = 0$:

$$\rho_1\left[C_1 - \frac{\partial\phi_1}{\partial t} - \frac{(\nabla\phi_1)^2}{2} - gz\right] = \rho_2\left[C_2 - \frac{\partial\phi_2}{\partial t} - \frac{(\nabla\phi_2)^2}{2} - gz\right]. \tag{8.170}$$

where

$$(\nabla\phi_1)^2 = (\nabla(U_1 x + \phi_1'))^2 = (U_1 + \phi_{1,x}', \phi_{1,y}', \phi_{1,z}')^2 = U_1^2 + 2U_i\phi_{1,x}' + (\nabla\phi_1')^2 \approx U_1^2 + 2U_1\phi_{1,x}' \tag{8.171}$$

and similar for 2. The replacement of $z = \zeta$ with $z = 0$ is justifies as

$$\phi_{1,x}'(x,y,\zeta,t) = \phi_{1,x}'(x,y,0,t) + \partial_z(\phi_{1,x}')(x,y,0,t)\zeta + \partial_{zz}(\phi_{1,x}')(x,y,0,t)\frac{\zeta^2}{2} + \ldots \approx \phi_{1,x}'(x,y,0,t). \tag{8.172}$$

Using (8.167), we have

$$\rho_1\left[\frac{\partial\phi_1'}{\partial t} + U_1\frac{\partial\phi_1'}{\partial x} + gz\right] = \rho_2\left[\frac{\partial\phi_2'}{\partial t} + U_2\frac{\partial\phi_2'}{\partial x} + gz\right] \tag{8.173}$$

- Normal modes.

$$\zeta = \hat{\zeta}(z)e^{st+i(kx+ly)} \tag{8.174}$$
$$\phi_1 = \hat{\phi}_1(z)e^{st+i(kx+ly)} \tag{8.175}$$
$$\phi_2 = \hat{\phi}_2(z)e^{st+i(kx+ly)} \tag{8.176}$$

Substitute into the governing equations $\nabla^2\phi_1' = \nabla\phi_2' = 0$, we have

$$\partial_{zz}\hat{\phi}_1 = (k^2 + l^2)\hat{\phi}_1 \tag{8.177}$$
$$\partial_{zz}\hat{\phi}_2 = (k^2 + l^2)\hat{\phi}_2 \tag{8.178}$$

leading to

$$\hat{\phi}_1(z) = A_1 e^{\tilde{k}z} + B_1 e^{-\tilde{k}z} \tag{8.179}$$

$$\hat{\phi}_2(z) = A_2 e^{\tilde{k}z} + B_2 e^{-\tilde{k}z} \tag{8.180}$$

where $\tilde{k}^2 = k^2 + l^2$. Given the respective vanishing conditions at $\pm\infty$, we have

$$\hat{\phi}_1(z) = A_1 e^{\tilde{k}z},\ \hat{\phi}_2(z) = B_2 e^{-\tilde{k}z}. \tag{8.181}$$

Then the kinematic BCs are used to determine the constants:

$$A_1 = \frac{s + ikU_1}{\tilde{k}}\hat{\zeta} \tag{8.182}$$

$$B_2 = -\frac{s + ikU_2}{\tilde{k}}\hat{\zeta} \tag{8.183}$$

The dynamic BC at $z = \zeta = 0$ leads to the dispersion relation

$$\rho_1[A_1(s + ikU_1) + g] = \rho_2[B_2(s + ikU_2) + g] \tag{8.184}$$

i.e.

$$\rho_1\left[\frac{(s + ikU_1)^2}{\tilde{k}} + g\right] = \rho_2\left[-\frac{(s + ikU_2)^2}{\tilde{k}} + g\right] \tag{8.185}$$

which is quadratic in s and has the solution

$$s = -ik\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k^2\rho_1\rho_2(U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{\tilde{k}g(\rho_1 - \rho_2)}{\rho_1 + \rho_2}\right]^{1/2}. \tag{8.186}$$

We can see that for large k the first term in the square root which is positive will be larger than the second, negative term hence all large wavenumbers are unstable. Also, when there is no density difference and only pure shear, any wavenumber is unstable.

There are several special cases that have classical implications.

- The KH instability. Assume $\rho_1 = \rho_2$. The growth rate is

$$s = -ik\frac{U_1 + U_2}{2} \pm \frac{k(U_1 - U_2)}{2}. \tag{8.187}$$

  The KH waves travel at $c = \omega/k = \Re\{is\}/k = (U_1+U_2)/2$, the average speed of two streams and the growth rate is positive for any wavenumber and proportional to the wavenumber. This is generally not true for typical finite-amplitude perturbations where large wavelengths are suppressed by the vertical shear length scale (δ) and small ones suppressed by viscosity/capillarity. The largest inviscid growth rate is achived at $k_{\max} \sim 2\pi/\delta$ with frequency $\sim 2\pi(U/\delta)$ proportional to the shear.

- The RT instability. This can be due to density inversion (bottom light) or due to accelerations (such as in supernovae). Assume $U_1 = U_2 = 0$. The growth rate is

$$s = \pm\left[\frac{\tilde{k}g'(\rho_2 - \rho_1)}{\rho_1 + \rho_2}\right]^{1/2}. \tag{8.188}$$

  where $g' = g + a$ and a is the acceleration of the system/frame.

- Gravity waves. Assume $U_1 = U_2 = 0$ and $\rho_2 < \rho_1$. The growth rate is

$$s = \pm i \left[\frac{\tilde{k} g(\rho_1 - \rho_2)}{\rho_1 + \rho_2} \right]^{1/2}, \tag{8.189}$$

which gives purely oscillatory motions, which interprets wave motions as a special case of stability. The eigenmodes (z-modes) decay exponentially away from the density interface or they are called trapped modes (in the language of quantum mechanics). When $\rho_2 = 0$ (such as the air-water interface), $s = \pm i\sqrt{\tilde{k}g}$, $\omega = is = \pm\sqrt{\tilde{k}g}$ and the wavespeed is

$$c = \omega/\tilde{k} = \pm\sqrt{g/\tilde{k}}. \tag{8.190}$$

8.4.2 Taylor–Couette instability

8.4.3 Rayleigh–Bénard instability

Consider the flow driven by two parallel plates with a temperature difference (hot bottom). The governing equations are (the Boussinesq equations)

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.191}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla \left(\frac{p + \rho_0 g z}{\rho_0} \right) + \nu \nabla^2 \boldsymbol{u} + \alpha g (T - T_0) \boldsymbol{e}_z \tag{8.192}$$

$$\frac{\partial \boldsymbol{u}}{\partial T} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T \tag{8.193}$$

8.4.4 The Lorenz system

Its derivation, linearization, and bifurcations.

8.4.5 Adjoint matrices, operators, and equations

For a complex matrix $A \in \mathbb{C}^{N\times N}$, define an inner product

$$(u, v)_A = (Au, v) \tag{8.194}$$

where $u, v \in \mathbb{C}^N$ and $(u, v) = v^H u$, $(\cdot)^H$ is the Hermitian transpose. Define the adjoint matrix of A as $A^\dagger$ such that

$$(Au, v) = (u, A^\dagger v). \tag{8.195}$$

We note that if A is Hermitian ($A^H = A$), (8.195) is valid. Such matrix A is also called self-adjoint ($A = A^\dagger$). For operators defined on domains like $\mathbb{C}^N$, Hermitian and self-adjointness imply each other and we don't distinguish these two in what follows.

Consider the following standard Sturm–Liouville eigenvalue problem:

$$\frac{d}{dx} \left[p(x) \frac{d\phi}{dx} \right] + q(x)\phi = -\lambda \sigma(x) \phi, \tag{8.196}$$

where $p(x), w(x)$ are positive, and $\lambda, \phi(x)$ are the eigenvalue and corresponding eigenfunction of the problem. The boundary conditions are

$$\alpha_1 \phi(a) + \alpha_2 \frac{d\phi}{dx}(a) = 0 \tag{8.197}$$

$$\beta_1\phi(b)+\beta_2\frac{d\phi}{dx}(b)=0 \tag{8.198}$$

$$\tag{8.199}$$

with $\alpha_1^2+\alpha_2^2\neq 0, \beta_1^2+\beta_2^2\neq 0$.

The LHS operator is defined as

$$\mathcal{L}(y)=\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y \tag{8.200}$$

and the S–L eigenvalue problem is

$$\mathcal{L}(\phi)+\lambda\sigma(x)\phi=0. \tag{8.201}$$

The Lagrange identity is

$$u\mathcal{L}(v)-v\mathcal{L}(u)=\frac{d}{dx}\left[p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\right] \tag{8.202}$$

and the Green's formula is

$$\int_a^b[u\mathcal{L}(v)-v\mathcal{L}(u)]dx=p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_a^b \tag{8.203}$$

When u, v satisfy the same set of boundary conditions (either homogeneous or periodic), we have the self-adjointness, i.e.,

$$\int_a^b[u\mathcal{L}(v)-v\mathcal{L}(u)]dx=0 \tag{8.204}$$

or

$$\int_a^b u\mathcal{L}(v)dx=\int_a^b v\mathcal{L}(u)dx. \tag{8.205}$$

$$=\int_a^b v\mathcal{L}^\dagger(u)dx. \tag{8.206}$$

and we note the definition of adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$ is that

$$(u,\mathcal{L}(v))=(v,\mathcal{L}^\dagger(u)), \tag{8.207}$$

with the inner product defined based on spatial integral and the adjoint is dependent on the inner product.

Examples.

1. The Laplacian operator.

$$\mathcal{L}=\nabla^2. \tag{8.208}$$

The multidimensional variation of (8.203), with $\mathcal{L}=\nabla^2$, is

$$\iiint[u\mathcal{L}(v)-v\mathcal{L}(u)]dV=\iiint\nabla\cdot[u\nabla v-v\nabla u]dV=\iint(u\nabla v-v\nabla u)\cdot d\boldsymbol{A} \tag{8.209}$$

and if u, v satisfy the same homogeneous BC,

$$\iiint[u\mathcal{L}(v)-v\mathcal{L}(u)]dV=\iint(u\nabla v-v\nabla u)\cdot d\boldsymbol{A}=0, \tag{8.210}$$

∇^2 is self-adjoint.

2. The wave equation.

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\nabla^2. \tag{8.211}$$

The Green's formula

$$\int_{t_i}^{t_f} \iiint [u\mathcal{L}(v) - v\mathcal{L}(u)] dV dt \tag{8.212}$$

$$= \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right) dV \Big|_{t_i}^{t_f} - c^2 \int_{t_i}^{t_f} \left(\iint (u\nabla v - v\nabla u) \cdot d\boldsymbol{A} \right) dt \tag{8.213}$$

And we note that the $\mathcal{L} = \partial_{tt}$ operator alone is self-adjoint if the boundary terms vanish, by

$$\int_{t_i}^{t_f} [u\mathcal{L}(v) - v\mathcal{L}(u)] dt = \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t} \right) \Big|_{t_i}^{t_f}. \tag{8.214}$$

3. Heat equation. Given above, we only consider the temporal derivative $\mathcal{L} = \partial_t$ here. Since

$$\int_{t_i}^{t_f} \left(u\frac{dv}{dt} + v\frac{du}{dt} \right) dt = uv|_{t_i}^{t_f} \tag{8.215}$$

vanishes if the RHS is zero,

$$\int_{t_i}^{t_f} u\mathcal{L}v\, dt = \int_{t_i}^{t_f} -v\mathcal{L}u\, dt \tag{8.216}$$

and the adjoint operator is $\mathcal{L}^\dagger = -\partial_t$. And we note that for the boundary term to vanish, we usually only have one BC for u in a first order problem such as $u(a) = 0$ and need to introduce an adjoint BC as $v(b) = 0$.

8.4.6 Non-self-adjointness and non-normality

A normal matrix $L \in \mathbb{C}^{N\times N}$ is defined as

$$L^H L = LL^H \tag{8.217}$$

and it is unitarily diagonalizable ($L = U\Lambda U^H$). The eigenvectors of L span an orthogonal basis of $\mathbb{C}^N$. More specifically, the eigenvectors corresponding to different eigenvalues are orthogonal, and even for degenerate eigenvalues an orthogonal basis can be found. A normal matrix is Hermitian if and only if all its eigenvalues are real.

The normality of a linear operator $\mathcal{L}$ is defined as

$$\mathcal{L}^\dagger\mathcal{L} = \mathcal{L}\mathcal{L}^\dagger. \tag{8.218}$$

The eigenmodes of $\mathcal{L}$ are normal to each other. A self-adjoint operator is hence (an example of) normal operators and a non-normal operator must be non-self-adjoint.

For two eigenmodes $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$, where $\boldsymbol{\Phi}_i = e^{\lambda_i t}\boldsymbol{\phi}_i$, and $\lambda_i, \boldsymbol{\phi}_i$ are the eigenpair. There difference/cancellation $\boldsymbol{f} = \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2$ decays if both decays and $(\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2) = 0$. That said, if the real part of each eigenvalue is negative, the energy of the perturbation will decay. However, for non-self-adjoint operators, there could be a transient growth of the cancellation $\boldsymbol{f}$ (Schmid, 2007), where the decay of individual eigenmodes does not imply the transient decay of the total energy. The idea of optimal perturbations it to find such a transient mode that grows most within a certain period of time.

8.4.7 Adjoint of the linearised N-S equations (Op's)

Op's: optimal linear perturbation solved as an optimal control problem in an optimization formulation constrained by the PDEs with linear operators.

Similar to (8.10)-(8.11), the linearised perturbation equation with buoyancy is

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.219}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho_0}\nabla p + \nu \nabla^2 \boldsymbol{u} + b\boldsymbol{e}_z \tag{8.220}$$

$$\frac{\partial b}{\partial t} + \boldsymbol{U} \cdot \nabla b + \boldsymbol{u} \cdot \nabla B = \kappa \nabla^2 b \tag{8.221}$$

where $b = B + b'$ is buoyancy and the primes are dropped from the equations above.

Assume the adjoints of $(\boldsymbol{u}, p, b)$ are $(\boldsymbol{v}, q, \varphi)$, multiplying each term in (8.219) by (v_1, v_2, v_3), (8.220) by $\boldsymbol{v}$, and (8.221) by φ, we can derive the adjoint equations of (8.219)-(8.221) as

$$\nabla \cdot \boldsymbol{v} = 0 \tag{8.222}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{U} + \varphi \nabla B - \frac{1}{\rho_0}\nabla q - \nu \nabla^2 \boldsymbol{v} \tag{8.223}$$

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{U} \cdot \nabla \varphi = -v_3 - \kappa \nabla^2 \varphi \tag{8.224}$$

We note the cross contribution terms $\varphi \nabla B$ and $-v_3$. The same set of equations can also be derived from a Lagrangian multiplier approach (a more morden method, but now classic), with the total perturbation energy being the Lagrangian and the set of governing equations along with BC's being the constraints enforced as multipliers. Such Lagrangian is in the form of energy gain as (Arratia, 2011; Luchini & Bottaro, 2014; Kaminski *et al.*, 2014)

$$\mathscr{L}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T)\rangle + \langle b(T), b(T)\rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0\rangle + \langle b_0, b_0\rangle / N^2} \tag{8.225}$$

$$- [\frac{\partial u_i}{\partial t} + u_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j^2} - b\delta_{i3}, v_i] \tag{8.226}$$

$$- [\frac{\partial b}{\partial t} + u_j \frac{\partial B}{\partial x_j} + U_j \frac{\partial b}{\partial x_j} - \kappa \frac{\partial^2 b}{\partial x_j^2}, \varphi] - [\frac{\partial u_i}{\partial x_i}, q] \tag{8.227}$$

$$- \langle u_i(0) - u_{0i}, v_{0i}\rangle - \langle b(0) - b_0, \varphi_0\rangle \tag{8.228}$$

constrained by the equations through the multipliers $(\boldsymbol{v}, q, \varphi)$ and similarly the BC's. The inner products

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \int_V \boldsymbol{u} \cdot \boldsymbol{v} dV \tag{8.229}$$

$$[\boldsymbol{u}, \boldsymbol{v}] = \int_0^T \langle \boldsymbol{u}, \boldsymbol{v}\rangle dt \tag{8.230}$$

are defined as respective spatial and spatio-temporal integrations.

In the phraseology of optimal control (with PDE constraints, Ref. section 5 of Manzoni *et al.* (2021)):

- Target functional (finite time transient gain):

$$\mathscr{G}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T)\rangle + \langle b(T), b(T)\rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0\rangle + \langle b_0, b_0\rangle / N^2}. \tag{8.231}$$

- State variables: $(\boldsymbol{u}, b)$.
- State equations (PDE constraints): (8.219)-(8.221). We note that (adjoint) pressure only appears as a Lagrangian multiplier that enforces the continuity.
- Control variables: $\boldsymbol{u}(0), b(0)$ and hence $\boldsymbol{u}(\boldsymbol{x}, t), b(\boldsymbol{x}, t)$.
- Admissible constraints on controls: none for now.

Taking the variation of (8.228) w.r.t.:

- The multipliers $(\boldsymbol{v}, q, \varphi)$: we recover the 'direct' equations (8.219)-(8.221).
- The terms $\boldsymbol{v}_0, \varphi_0$: we obtain the definition of IC's $\boldsymbol{u}_0, b_0$.
- The 'direct' variables $(\boldsymbol{u}, p, b)$: we obtain the adjoint equations (8.222)-(8.224). This step will be shown in detail.

Other than deriving from a Lagrangian perspective, the adjoint can also be derived using (multiple) integrations by parts. Starting from (8.14), i.e.,

$$\mathcal{L}_{\mathrm{NS}}\boldsymbol{q} = 0, \tag{8.232}$$

with the direct and adjoint variables being $\boldsymbol{q} = (\boldsymbol{u}, p)$ and $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$, we look for the adjoint $\mathcal{L}^{\dagger}_{\mathrm{NS}}$ such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] - [\mathcal{L}^{\dagger}_{\mathrm{NS}}\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) \tag{8.233}$$

and the boundary conditions that make the RHS boundary terms vanish.

Where the inner product $[\cdot, \cdot]$ is that same as in (8.230) such that

$$[\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \int_T \int_V (\boldsymbol{v} \cdot \boldsymbol{u} + qp) dt dV. \tag{8.234}$$

The weak form of (8.14) is

$$[\mathbb{1}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V (\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + \partial_i p + u_j \partial_j U_i) dt dV = 0. \tag{8.235}$$

We note that (8.235) should also be valid on any arbitrary test function $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$ for (8.14) to hold, such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V v_i(\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + u_j \partial_j U_i) + q \partial_i p \, dt dV = 0. \tag{8.236}$$

By integration by parts we have

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p \partial_i q \, dt dV + \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) \tag{8.237}$$

$$= \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p\partial_i q \, dtdV \tag{8.238}$$

$$= [\mathcal{L}_{\mathrm{NS}}^{\dagger} \boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] \tag{8.239}$$

that defines the adjoint operator of $\mathcal{L}_{\boldsymbol{U}} = \partial_t + \boldsymbol{U} \cdot \nabla - \nu \nabla^2$:

$$\mathcal{L}_{\boldsymbol{U}}^{\dagger} = -\partial_t - \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{8.240}$$

and the adjoint equation

$$\partial_t v_i + U_j \partial_j v_i + \nu \partial_j^2 v_i = v_j \partial_j U_i - \partial_i q \tag{8.241}$$

and we note that by advancing forward in time, the viscous term is injecting energy into the system. Using the transfrom $\tau = -t$ we have

$$\partial_\tau v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i = -v_j \partial_j U_i + \partial_i q. \tag{8.242}$$

In another form, the adjoint equation (without density) can be expressed similar to (8.16) as

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\mathcal{A}_{\boldsymbol{U}}^{\dagger} \boldsymbol{v} \tag{8.243}$$

where $\mathcal{A}_{\boldsymbol{U}}^{\dagger}$ is the adjoint operator of $\mathcal{A}_{\boldsymbol{U}}$ in (8.15)

$$\mathcal{A}_{\boldsymbol{U}}^{\dagger} \boldsymbol{v} = \boldsymbol{U} \cdot \nabla \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{U} + \frac{1}{\rho_0} \nabla q + \nu \nabla^2 \boldsymbol{v}. \tag{8.244}$$

Chapter 9

Compressible flows, gas dynamics

9.1 Compressibility

Density change due to the pressure and the thermal effects (EOS). Batchelor (1967) Ref.

9.2 Thermodynamics

9.2.1 Useful relations

The equation of state, internal energy, enthalpy, and specific heats are:

$$p = \rho R T \tag{9.1}$$

$$e = c_v T \tag{9.2}$$

$$h = c_p T \tag{9.3}$$

$$c_p = \frac{\gamma R}{\gamma - 1},\ c_v = \frac{R}{\gamma - 1},\ c_p - c_v = R,\ \gamma = \frac{c_p}{c_v} \tag{9.4}$$

Here R is the gas constant and it is realted to the universal constant Λ as

$$R = \frac{\Lambda}{M} \tag{9.5}$$

where M is the molecular weight (kg/kmol or g/mol). For example, taking $\Lambda = 8.314\,\mathrm{J/(mol \cdot K)} = 8314\,\mathrm{J/(kmol \cdot K)}$ and the molecular weight of air $M = 28.97\,\mathrm{kg/kmol}$, we have $R = 287 \cdot \mathrm{J/(kg \cdot K)} = 287\,\mathrm{m^2/(s^2 \cdot K)}$. The specific heat ratio of air is $\gamma = 1.4$, which does not change too much with T. Consequently, $c_v = 718\,\mathrm{m^2/(s^2 \cdot K)}$ and $c_p = 1005\,\mathrm{m^2/(s^2 \cdot K)}$.

Example. Adiabatic expansion of air (temperature lapse rate).

9.2.2 First and second laws of thermodynamics

The thermodynamics first law states that the increase of the internal energy of the system equals heat added (positive δQ for added to the system) minus work done (positive δW for work done by the system),

$$\delta e = \delta Q - \delta W, \tag{9.6}$$

where the work done by isobaric (constant pressure) expansion and contraction in a frictionless system is $pd\nu$ (positive $d\nu$ when expanding and doing work on the surroundings). Hence, for an infinitesimal reversible process

$$de = dQ - pd\nu, \tag{9.7}$$

or

$$dQ = de + pd\nu \tag{9.8}$$

We define enthalpy as (recall Bernoulli, (1.56))

$$h = e + p\nu = e + \frac{p}{\rho}, \tag{9.9}$$

and then we have also

$$dQ = dh + \nu dp. \tag{9.10}$$

The entropy change is

$$Tds = dQ. \tag{9.11}$$

Using $dQ = de + pd\nu$ from the first law and the definition of h,

$$Tds = de + pd\nu \tag{9.12}$$

$$Tds = dh - \nu dp \tag{9.13}$$

where

$$dh = c_p dT,\ de = c_v dT \tag{9.14}$$

and

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p,\ c_v = \left(\frac{\partial e}{\partial T}\right)_\nu. \tag{9.15}$$

In a constant volume process, $-pd\nu = 0$ and we have

$$\left(\frac{\partial Q}{\partial T}\right)_\nu = \left(\frac{\partial e}{\partial T}\right)_\nu = c_v \tag{9.16}$$

and similarly, in a constant pressure process,

$$\left(\frac{\partial Q}{\partial T}\right)_p = \left(\frac{\partial h}{\partial T}\right)_p = c_p \tag{9.17}$$

9.3 Conservation laws

Apart from deriving the Euler equation by neglecting the viscous term in the Navier–Stokes, we present another way here that is more to the taste of gas dynamics. (AFD notes/Shu).

In compressible flows, it is typical to write the gooverning equations in the following conservative form, especially when doing CFD. It is also noted that the density change can no longer be neglected and it will result in the change of pressure, which is a thermodynamic state variable now.

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0. \tag{9.18}$$

Momentum conservation:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho f_i, \tag{9.19}$$

where the viscous stresses are

$$\tau_{ij} = \lambda \frac{\partial u_k}{\partial x_k}\delta_{ij} + 2\mu S_{ij} \tag{9.20}$$

$$= 2\mu\left(S_{ij} - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) \tag{9.21}$$

and $\lambda = -2/3\mu$ according to the Stokes assumption.

Total energy $E = e + u_k^2/2$ conservation:

$$\frac{\partial \rho(e + u_k^2/2)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j(e + u_k^2/2)) = \rho f_j u_j - \frac{\partial(u_j p)}{\partial x_j} + \frac{\partial(u_i \tau_{ij})}{\partial x_j} + \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right). \tag{9.22}$$

In a vecterized form where the unknown variable is $U = [\rho, \rho u, \rho v, \rho w, \rho E]^T$ instead of the primitive variables ρ, u, v, w, e, T, p, the conservation laws can be expressed as

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho E\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ \rho u^2 + p - \tau_{xx}\\ \rho uv - \tau_{yx}\\ \rho uw - \tau_{zx}\\ (E+p)u - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} + q_x\end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix}\rho v\\ \rho uv - \tau_{yx}\\ \rho v^2 + p - \tau_{yy}\\ \rho vw - \tau_{yz}\\ (E+p)v - u\tau_{yx} - v\tau_{yy} - w\tau_{yz} + q_y\end{bmatrix} \tag{9.23}$$

$$+ \frac{\partial}{\partial z}\begin{bmatrix}\rho w\\ \rho uw - \tau_{zx}\\ \rho vw - \tau_{zy}\\ \rho w^2 + p - \tau_{zz}\\ (E+p)w - u\tau_{zx} - v\tau_{zy} - w\tau_{zz} + q_z\end{bmatrix} = \begin{bmatrix}0\\ \rho f_x\\ \rho f_y\\ \rho f_z\\ \rho(uf_x + vf_y + wf_x)\end{bmatrix} \tag{9.24}$$

where $\boldsymbol{\tau} =$ and $\boldsymbol{q} = -k\nabla T$. The general consevation law can be written as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = J, \tag{9.25}$$

where F, G, H are called the fluxes. In order to close the system of five equations and nine unknowns (including μ, k), we need to use the EOS $p = \rho RT$, the calorically perfect gas relation $e = c_v T$, constant Prandtl number ($Pr = \nu/\alpha$, $\alpha = k/c_p\rho$), and the Sutherland's formula ($\mu = \mu(T)$). One of the difference between incompressible and compressible flows is that the energy equation is coupled in compressible flows and is to be solved together with the momentum equations. Another difference is that pressure is a thermodynamic variable.

9.3.1 Speed of sound

Through a control-volume analysis (White, 2008; Kundu *et al.*, 2015), it can be estabilished that the sound speed of a compressible fluid is

$$a = \sqrt{\frac{\partial p}{\partial \rho}} \tag{9.26}$$

and the correct way to calculate this derivative, since both p and ρ are thermodynamic state variables, is adiabatically/isentropically. The isentropic process refers to the propagation of sound waves – the change of fluid parcels as the sound wave (which is a weak wave) passes by, can be described as a isentropic process. It doesn't mean the following derivation can only be valid in isentropic flows. Besides, when the wave is strong, such as a shock wave, the entropy production can no longer be neglected.

Nevertheless,

$$\left.\frac{\partial p}{\partial \rho}\right|_s = \frac{\partial}{\partial \rho}(C\rho^\gamma) = \gamma \frac{p}{\rho} = \gamma RT. \tag{9.27}$$

We also note that

$$\left.\frac{\partial p}{\partial \rho}\right|_T = \frac{\partial}{\partial \rho}(\rho RT) = RT \neq \left.\frac{\partial p}{\partial \rho}\right|_s. \tag{9.28}$$

Hence, the speed of sound is

$$a = \sqrt{\gamma RT}. \tag{9.29}$$

Its value for air in the standard atmosphere at room temperature (288 K/sea level) is about 340.3 m/s (1225 km/h). At 10000 m (223K for standard atmosphere), sound speed is about 299.5 m/s (1078 km/h). In incompressible flows, $\delta\rho \approx 0$ following material, so the sound speed (or the propagation speed of pressure signals) is infinity.

9.4 Adiabatic isentropic flows

In shock-less regions away from the walls, for example, the potential flow region, where heating and entropy production due to shocks or boundary layer friction are absent, the flow can be treated as isentropic and the problem is greatly simplified.

The integration of the Tds relations from state 1 to 2 lead to

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{9.30}$$

$$= c_v \ln \frac{T_2}{T_2} - R \ln \frac{\rho_2}{\rho_1} \tag{9.31}$$

and the isentropic condition implies $s_2 = s_1$ and

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} = \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}, \; \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}. \tag{9.32}$$

These are called the isentropic relations. The last condition is equivalent to

$$\frac{p}{\rho^\gamma} = \text{const}. \tag{9.33}$$

An equation of state like so, $p = p(\rho) = C(s)\rho^\gamma$ is called a barotropic EOS, where the constant $C(s)$ is a function of entropy.

The energy conservation (in terms of Bernoulli, (1.61); along the streamlines) for one-dimensional flow is

$$h_0 = h + \frac{V^2}{2} = c_p T + \frac{V^2}{2} = c_p T_0 = \text{const.} \tag{9.34}$$

The total temperature

$$T_0 = T + \frac{V^2}{2c_p} \tag{9.35}$$

is conserved, following the energy (total enthalpy) conservation. It contains a static part and a dynamic part.

Provided the definition of the Mach number

$$Ma = \frac{V}{a} = \frac{V}{\sqrt{\gamma R T}}, \tag{9.36}$$

the total temperature conservation can be written as

$$T_0 = T\left(1 + \frac{\gamma - 1}{2} Ma^2\right), \tag{9.37}$$

which relates static temperature and the Mach number. This relation requires adiabatic (total temperature/enthalpy conservation) but not necessarily isentropic. Subsequently,

$$a_0 = a\left(1 + \frac{\gamma - 1}{2} Ma^2\right)^{1/2} \tag{9.38}$$

since $a = \sqrt{\gamma R T}$.

With the isentropic relations (γ-relations),

$$p_0 = p\left(1 + \frac{\gamma - 1}{2} Ma^2\right)^{\gamma/(\gamma-1)} \tag{9.39}$$

$$\rho_0 = \rho\left(1 + \frac{\gamma - 1}{2} Ma^2\right)^{1/(\gamma-1)} \tag{9.40}$$

mass conservation ($\rho_1 V_1 A_1 = \rho_2 V_2 A_2$) and ideal gas law, the area ratios for a converging-diverging nozzle is

$$\frac{A_2}{A_1} = \frac{Ma_1}{Ma_2}\left(\frac{1 + \frac{\gamma-1}{2} Ma_2^2}{1 + \frac{\gamma-1}{2} Ma_1^2}\right)^{(k+1)/2(k-1)}. \tag{9.41}$$

Taking $A_1 = A^*$ to be the critical/sonic condition (which might not need to exist in the flow; reference value) at the throat of the nozzle, we obtain the isentropic area-Mach number relation:

$$\frac{A}{A^*} = \frac{1}{Ma}\left(\frac{1 + \frac{\gamma-1}{2} Ma^2}{\frac{\gamma+1}{2}}\right)^{(k+1)/2(k-1)}. \tag{9.42}$$

9.4.1 Area effects

The energy equation

$$h + \frac{V^2}{2} = \text{const.} \tag{9.43}$$

can be taken differential

$$dh + VdV = 0 \tag{9.44}$$

and turn into a momentum equation. In isentropic flows,

$$0 = Tds = dh - \frac{dp}{\rho} \tag{9.45}$$

hence

$$dh = \frac{dp}{\rho} \tag{9.46}$$

and

$$\frac{dp}{\rho} + VdV = 0, \tag{9.47}$$

where $dp = a^2 d\rho$. The continuity equation, $\rho V \rho = \text{const.}$ is differentiated to

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0 \tag{9.48}$$

and hence

$$\frac{dV}{V} = \frac{1}{Ma^2 - 1}\frac{dA}{A}. \tag{9.49}$$

We can see the opposite behavior of the flow when $Ma < 1$ and $Ma < 1$.

9.5 Shock waves, discontinuities, and jump conditions

Shock waves are ideally infinitely thin (order of molecular mean-free-path), discontinuous structures in compressible flows. Across the shock, p, ρ, T, s increase, Ma, p_0 decrease, and T_0, h_0 conserve. The pre- and post-shock variables satisfy normal shock relations that will be derived below. Here we only deal with normal shock waves where the flow is perpendicular to the shock. For oblique shock waves the normal component of the flow satisfies the same relation.

Assume a shock wave moving from right to left. When the reference frame is fixed in the shock, the flow goes from left (left= 1) to right (right= 2).

The conservation laws (for a thin CV across the shock) and thermodynamic relations are:

- Mass conservation:

$$\rho_1 V_1 = \rho_2 V_2. \tag{9.50}$$

- Momentum conservation:

$$\rho_1 V_1^2 - \rho_2 V_2^2 = p_2 - p_1. \tag{9.51}$$

- Energy (total enthalpy/total temperature) conservation:

$$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2 = h_0. \tag{9.52}$$

- The enthalpy is
$$h = c_p T \tag{9.53}$$
and c_p and k are constants.
- (Perfect gas law):
$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2}. \tag{9.54}$$

9.5.1 Jump conditions

We have 5 equations for 5 unknowns, p, V, ρ, h, T. By eliminating density (including multiplying the momentum equation with $1/\rho_1 + 1/\rho_2$), we reach the Rankine–Hugoniot relation

$$h_2 - h_1 = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(p_2 - p_1), \tag{9.55}$$

in which only thermodynamic properties are contained and the EOS is not invoked.

In more general cases; Integral form of the equations; jump conditions. Conservation form, weak form, weak continuity, test functions.

9.5.2 Rankine–Hugoniot relations / Mach number relations for perfect gas

In a perfect gas, $h = c_p T = \gamma/(\gamma-1)p/\rho$, the density ratio can be related to the pressure ratio as

$$\frac{\rho_2}{\rho_1} = \frac{1 + \beta\frac{p_2}{p_1}}{\beta + \frac{p_2}{p_1}}, \tag{9.56}$$

where $\beta = (\gamma+1)/(\gamma-1)$. Furthermore, the downstream solution are only functions of upstream Mach number Ma_1 and γ, as the Rankine–Hugoniot relations (Mach number relations):

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(Ma_1^2 - 1) \tag{9.57}$$
$$\frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)Ma_1^2}{(\gamma-1)Ma_1^2 + 2} \tag{9.58}$$
$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{\gamma Ma_1^2 + 1}{Ma_1^2}(Ma_1^2 - 1) \tag{9.59}$$
$$Ma_2^2 = \frac{(\gamma-1)Ma_1^2 + 2}{2\gamma Ma_1^2 - (\gamma-1)} \tag{9.60}$$
$$\frac{s_2 - s_1}{c_v} = \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right] \tag{9.61}$$

where in order to reach second to last equation the momentum equation in terms of Mach numbers

$$\frac{p_2}{p_1} = \frac{1 + \gamma Ma_1^2}{1 + \gamma Ma_2^2} \tag{9.62}$$

is used.

It can be shown that when $Ma_1^2 - 1$ is small,

$$\frac{s_2 - s_1}{c_v} \approx \frac{2\gamma(\gamma-1)}{3(\gamma+1)^2}(Ma_1^2 - 1)^3, \tag{9.63}$$

which should be positive according to the second law of thermodynamics. Hence, we must have $Ma_1 > 1$ for the upstream flow. Subsequently,

$$
\begin{align}
& p_2 > p_1 \tag{9.64}\\
& \rho_2 > \rho_1 \tag{9.65}\\
& T_2 > T_1 \tag{9.66}\\
& s_2 > s_1 \tag{9.67}\\
& Ma_2 < 1 < Ma_1 \tag{9.68}\\
& V_2 < V_1 \tag{9.69}\\
& \tag{9.70}
\end{align}
$$

post-shock.

9.6 Euler equation

9.6.1 Characteristics

Consider 1D Euler equation. We denote the state vector as $U = [\rho, \rho u, E]$ (total energy $E = \rho e + 1/2\rho u^2$), and the 1D conservation law is

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \tag{9.71}$$

where the flux vector is

$$F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}. \tag{9.72}$$

It can be written as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0, \tag{9.73}$$

where $A = \partial F/\partial U$. If A is diagonalizable, $A = R\Lambda R^{-1}$, a change to the characteristic variable $v = R^{-1}u$ will diagonalize the system (9.73) to

$$v_t + \Lambda v_x = 0, \tag{9.74}$$

which are decoupled, and such a system is called hyperbolic. The directions Λ are called the characteristics.

With the equation of state $e = c_v T = (p/\rho)/(\gamma-1)$, $E = p/(\gamma-1) + 1/2\rho u^2$, the Jacobian of the flux vector is

$$F'(U) = \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & (\gamma-1) \\ \frac{1}{2}(\gamma-1)u^3 - u(E+p)/\rho & (E+p)/\rho - (\gamma-1)u^2 & \gamma u \end{bmatrix}, \tag{9.75}$$

which has three real eigenvalues

$$\lambda_1 = u - a,\ \lambda_2 = u,\ \lambda_3 = u + a, \tag{9.76}$$

where

$$a = \sqrt{\gamma p/\rho} \tag{9.77}$$

is just the speed of sound.

9.6.2 Riemann invariants

The 1D Euler equation is

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = -\rho\frac{\partial u}{\partial x} \tag{9.78}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{9.79}$$

The sound speed is

$$a^2 = \frac{\partial p}{\partial \rho} = \gamma\frac{p}{\rho} \tag{9.80}$$

with an isentropic EOS $p = C\rho^\gamma$. It can be shown that

$$2a\,da = (\gamma - 1)a^2\frac{d\rho}{\rho} \tag{9.81}$$

and hence

$$\frac{\partial \rho}{\partial t} = \frac{2\rho}{a(\gamma - 1)}\frac{\partial a}{\partial t} \tag{9.82}$$

$$\frac{\partial \rho}{\partial x} = \frac{2\rho}{a(\gamma - 1)}\frac{\partial a}{\partial x} \tag{9.83}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{2a}{\gamma - 1}\frac{\partial a}{\partial x} \tag{9.84}$$

The momentum equation converts to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -a\frac{\partial}{\partial x}\left(\frac{2a}{\gamma - 1}\right) \tag{9.85}$$

and the continuity equation to

$$\frac{\partial}{\partial t}\left(\frac{2a}{\gamma - 1}\right) + u\frac{\partial}{\partial x}\left(\frac{2a}{\gamma - 1}\right) = -a\frac{\partial u}{\partial x} \tag{9.86}$$

The above two equations can be combined to

$$\frac{\partial}{\partial t}\left(u + \frac{2a}{\gamma - 1}\right) + u\frac{\partial}{\partial x}\left(u + \frac{2a}{\gamma - 1}\right) = -a\left(u + \frac{2a}{\gamma - 1}\right) \tag{9.87}$$

$$\frac{\partial}{\partial t}\left(u - \frac{2a}{\gamma - 1}\right) + u\frac{\partial}{\partial x}\left(u - \frac{2a}{\gamma - 1}\right) = a\left(u - \frac{2a}{\gamma - 1}\right) \tag{9.88}$$

i.e.

$$\left(\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right)\left(u + \frac{2a}{\gamma-1}\right) = 0 \tag{9.89}$$

$$\left(\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right)\left(u - \frac{2a}{\gamma-1}\right) = 0 \tag{9.90}$$

where the two Riemann invariants

$$R_{\pm} = u \pm \frac{2a}{\gamma-1} \tag{9.91}$$

are conserved in isentropic flows along the two characteristics

$$\frac{dx}{dt} = u \pm a, \tag{9.92}$$

respectively. Once the two information-carrying invariants are solved at any (x, t), the velocity and sound speed can be solved as

$$u = \frac{1}{2}(R_+ + R_-) \tag{9.93}$$

$$a = \frac{\gamma-1}{2}(R_+ - R_-) \tag{9.94}$$

9.6.3 Small perturbation linearized equations; acoustic equation

Linearize the Euler equation around the base state $\rho_0, p_0, \boldsymbol{u}_0$ and decompose the variables as

$$\rho = \rho_0 + \rho' \tag{9.95}$$

$$p = p_0 + p' \tag{9.96}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}' = \boldsymbol{u}' \tag{9.97}$$

where we have chosen a uniform ρ_0 such that $\nabla\rho_0 = \mathbf{0}$ and $\boldsymbol{u}_0 = \mathbf{0}$ (moving with the base flow). The linearized perturbation equations are

$$\frac{\partial\rho}{\partial t} + \rho_0(\nabla\cdot\boldsymbol{u}) = 0 \tag{9.98}$$

$$\frac{\partial\boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\frac{dp}{d\rho}\nabla\rho \tag{9.99}$$

where the barotropic EOS, $p = p(\rho)$, is used and primes are dropped without causing confusions. Combining these two equations and eliminate $\boldsymbol{u}$, we have the acoustic wave equation

$$\frac{\partial^2\rho}{\partial t^2} = a^2\nabla^2\rho, \tag{9.100}$$

where $a = dp/d\rho$ is the (phase) speed of sound.

Alternatively, the density equation can be written with pressure ($\rho_t = (d\rho/dp)p_t = p_t/a^2$) as

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho_0}\right) + a^2(\nabla\cdot\boldsymbol{u}) = 0, \tag{9.101}$$

or ($\rho_0 = 1$)

$$p_t + a^2(u_x + v_y + w_z) = 0 \tag{9.102}$$

and

$$p_{tt} = a^2\nabla^2 p. \tag{9.103}$$

9.7 Kinetic theory, microscopic view of fluid and flow properties

pressure (molecules bouncing back from a flat wall), viscosity, etc.

kinetic theory. Cf. Frank Shu.

AFD notes; the temperature lapse rate example;

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Appendix A

Vectors, tensors, and their calculus

A.1 Levi-Civita symbol

A.1.1 Determinant representation

The matrix determinants can be expressed in terms of the Levi-Civita symbol. Assume A is a matrix

$$\det(A) = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{A.1}$$

where

$$\boldsymbol{a}_1 = (a_{11}, a_{12}, a_{13})^\top,\ \boldsymbol{a}_2 = (a_{21}, a_{22}, a_{23})^\top,\ \boldsymbol{a}_3 = (a_{31}, a_{32}, a_{33})^\top \tag{A.2}$$

Therefore the Levi-Civita symbol can be expressed as

$$\epsilon_{ijk} = \det(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k) = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) \tag{A.3}$$

Similarly, the outer product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \epsilon_{ijk} a_j b_k \boldsymbol{e}_i \tag{A.4}$$

Example: $\boldsymbol{\omega}$.

A.1.2 Epsilon identity

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{A.5}$$

$$= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{jl}(\delta_{in}\delta_{km} - \delta_{im}\delta_{kn}) + \delta_{kl}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \tag{A.6}$$

$$= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} \tag{A.7}$$

A.1.3 Contracted epsilon identity

Let $i = l$ and notice $\delta_{ii} = 3$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{A.8}$$

Futhur let $k = m$

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{A.9}$$

Futhermore

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{A.10}$$

A.2 Vector identities

Assume λ is a scalar and $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$,$\boldsymbol{d}$ are vectors in $\mathbb{R}^3$. The identities below might be useful in fluids, some of which have geometric implecations.

$$\nabla \cdot (\nabla \times \boldsymbol{b}) = 0 \tag{A.11}$$
$$\nabla \times (\nabla \boldsymbol{b}) = 0 \tag{A.12}$$
$$\nabla \cdot (\lambda \boldsymbol{b}) = \nabla\lambda \cdot \boldsymbol{b} + \lambda(\nabla \cdot \boldsymbol{b}) \tag{A.13}$$
$$\nabla \times (\lambda \boldsymbol{b}) = \lambda(\nabla \times \boldsymbol{b}) - \boldsymbol{b} \times \nabla\lambda \tag{A.14}$$
$$\nabla(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla\boldsymbol{a} + \boldsymbol{a} \cdot \nabla\boldsymbol{b} + \boldsymbol{b} \times (\nabla \times \boldsymbol{a}) + \boldsymbol{a} \times (\nabla \times \boldsymbol{b}) \tag{A.15}$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \tag{A.16}$$
$$\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla \times \boldsymbol{a}) \cdot \boldsymbol{b} - (\nabla \times \boldsymbol{b}) \cdot \boldsymbol{a} \tag{A.17}$$
$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}) \tag{A.18}$$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) \tag{A.19}$$
$$\boldsymbol{b} \times (\nabla \times \boldsymbol{b}) = \nabla(\frac{1}{2}\boldsymbol{b} \cdot \boldsymbol{b}) - \boldsymbol{b} \cdot \nabla\boldsymbol{b} \tag{A.20}$$
$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) \tag{A.21}$$
$$\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2\boldsymbol{a} \tag{A.22}$$

Their proofs are left as exercises.
Comments:

(1) Eq. (A.11): A curl field is solenoidal (divergence-free).

(2) Eq. (A.12): A gradient field is irrotational (curl-free).

(3) Eq. (A.21): $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$, so its curl is in the space spaned by $\boldsymbol{a}$ and $\boldsymbol{b}$.

(4) Eq. (A.19): $\boldsymbol{a} \times (\cdot)$ is perpendicular to $\boldsymbol{a}$ and $(\cdot) \times (\boldsymbol{b} \times \boldsymbol{c})$ is in the space spaned by $\boldsymbol{b}$ and $\boldsymbol{c}$. This two facts in combinition gives the bases of $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

(5) Eq. (A.16): This is the volume spaned by $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, and the identity is basically the invariance of a determinant with respect to row/column permutation.

(6) Eq. (A.18): By letting $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$ and noticing the inner product with itself is non-negative, we re-discover the Cauchy-Schwartz inequality.

(7) From (A.15) it can be immediately seen that

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla(\boldsymbol{u}^2/2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}). \tag{A.23}$$

A.3 Tensor eigenvalues, invariants, and its application in fluids

Consider a tensor $\boldsymbol{A}$ in Cartesian coordinate

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{A.24}$$

Its eigenvalues are roots of the characteristic polynomial

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{A.25}$$

with the three coefficients being the three principle invariants of $\boldsymbol{A}$

$$\begin{aligned}
I_1 &= a_{11} + a_{22} + a_{33} && \text{(A.26)} \\
&= \operatorname{tr}(\boldsymbol{A}) && \text{(A.27)} \\
&= a_{ii} && \text{(A.28)} \\
I_2 &= a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31} && \text{(A.29)} \\
&= \frac{\operatorname{tr}(\boldsymbol{A})^2 - \operatorname{tr}(\boldsymbol{A}^2)}{2} && \text{(A.30)} \\
&= \frac{1}{2}((a_{ii})^2 - a_{ij}a_{ji}) && \text{(A.31)} \\
I_3 &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) && \text{(A.32)} \\
&= \det(\boldsymbol{A}) && \text{(A.33)}
\end{aligned}$$

in both element-wise and coordinate-independent expression.

Now we consider the factorization of the characteristic polynomial as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\lambda - \lambda_1\lambda_2\lambda_3 = 0, \tag{A.34}$$

and obtain the Vieta's theorem for cubic equations as

$$\begin{aligned}
I_1 &= \lambda_1 + \lambda_2 + \lambda_3 && \text{(A.35)} \\
I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 && \text{(A.36)} \\
I_3 &= \lambda_1\lambda_2\lambda_3 && \text{(A.37)}
\end{aligned}$$

which are the three principle invariants of tensor $\boldsymbol{A}$.

Additionally, there are more invariants (although not independent) of $\boldsymbol{A}$, such as the main invariants

$$J_1 = \lambda_1 + \lambda_2 + \lambda_3 = I_1 = \mathrm{tr}(\boldsymbol{A}) \tag{A.38}$$

$$J_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1^2 - 2I_2 = \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{A}) \tag{A.39}$$

$$J_1 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = I_1^3 - 3I_1I_2 + 3I_3 = \mathrm{tr}(\boldsymbol{A}\cdot\boldsymbol{A}\cdot\boldsymbol{A}) \tag{A.40}$$

which are the coefficients of the characteristic polynomial of the deviatoric part of $\boldsymbol{A}$:

$$\boldsymbol{A} - \frac{\mathrm{tr}(\boldsymbol{A})}{3}\boldsymbol{I}, \tag{A.41}$$

which is traceless and has eigenvalues

$$\lambda_i - \frac{1}{3}. \tag{A.42}$$

A.3.1 Discriminant of a cubic equation

Consider

$$ax^3 + bx^2 + cx + d = 0, \tag{A.43}$$

its determinant is

$$\Delta = (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2 \tag{A.44}$$

$$= 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \tag{A.45}$$

with x_1, x_2, x_3 being the three roots.

1. $\Delta > 0$: Three distinct real roots.
2. $\Delta = 0$: All roots are real with at least two identical.
3. $\Delta < 0$: One real and a pair of complex conjugate roots (proof: assume complex roots are $x \pm iy$).

Proof. The Vieta's theorem for (A.43) and the invariant relations can be used to simplify (A.43) to obtain (A.45).

Note: Eq. (A.45) can also be obtained as follows (with some reasons/meanings in algebraic geometry). Consider a cubic equation in canonical form

$$f(x, w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = 0. \tag{A.46}$$

The Hessain matrix is

$$H(f) = \begin{bmatrix} 6Ax + 6Bw & 6Bx + 6Cw \\ 6Bx + 6Cw & 6Cx + 6Dw \end{bmatrix} \tag{A.47}$$

and the Hessain

$$\det(H) = 36[(AC - B^2)x^2 + (AD - BC)xw + (BD - C^2)w^2] \tag{A.48}$$

$$= 18[x, w] \begin{bmatrix} 2(AC - B^2) & (AD - BC) \\ (AD - BC) & 2(BD - C^2) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}, \tag{A.49}$$

in quadratic form. Define the Hessain

$$\boldsymbol{H} = \begin{bmatrix} 2(AC - B^2) & (AD - BC) \\ (AD - BC) & 2(BD - C^2) \end{bmatrix} \tag{A.50}$$

The discriminant of the cubic is just the determinant of the Hessain $\boldsymbol{H}$:

$$\Delta = \det(\boldsymbol{H}) = -A^2D^2 + 6ABCD - 4AC^3 - 4B^3D + 3B^2C^2, \tag{A.51}$$

and $\Delta > 0$ for three real roots, $\Delta = 0$ for double or triple real root, and $\Delta < 0$ for single real root.

A.3.2 Application to the velocity gradient tensor

The cubic curve theory, especially root finding, has relation to the characteristic polynomial of the velocity gradient tensor and so forth the local flow geometry (basically the number of real/complex eigenvalues).

The characteristic polynomial for $\boldsymbol{u}\nabla$ is

$$\lambda^3 - P\lambda^2 + Q\lambda - R = 0, \tag{A.52}$$

where $(P, Q, R) = (I_1, I_2, I_3)$ are the three invariants. In incompressible flow, $P = u_{i,i} = 0$, and the equation above degenerates to

$$\lambda^3 + Q\lambda - R = 0. \tag{A.53}$$

It is in the so-called 'depressed' form (comparatively, an elliptic curve is called in the Weierstrass form if it satisfies the Weierstrass equation $y^2 = x^3 + ax + b$)

Positive second invariant $Q = -1/2u_{i,j}u_{j,i} = 1/2(\|\boldsymbol{\Omega}\|^2 - \|\boldsymbol{S}\|^2)$ is used for identifying vortical motions, or 'eddies' Hunt *et al.* (1988); Jeong & Hussain (1995). Consider the Poisson equation,

$$\frac{1}{\rho}\nabla^2 p = -u_{i,j}u_{j,i}, \tag{A.54}$$

we have positive Q corresponding to a local pressure minimum.

The discriminant for depressed cubic equation

$$x^3 + px + q = 0 \tag{A.55}$$

reduces to

$$\Delta = -4p^3 - 27q^2. \tag{A.56}$$

So we have the discriminant for the gradient of a solenoidal field (with renormalized coefficients; note the flipped sign)

$$\triangle = \left(\frac{1}{3}Q\right)^3 + \left(\frac{1}{2}R\right)^2 \tag{A.57}$$

and if $\triangle > 0$ there will be complex eigenvalues (in complex conjugate pair according to the algebra basic theorem) and so-defined vortical motions. Hence, we can see that the invariants of the velocity gradient tensor is largely related to the local geometry (Chong *et al.*, 1990) of the flow.

A.3.3 Application to the strain rate tensor

We note that both the rate-of-strain tensor $\boldsymbol{S}$ and the Reynolds stress tensor $-\overline{u_i' u_j'}$ are real symmetric, hence they have three real eigenvalues and three orthogonal eigenvectors (principle axes), or, in another word, they are unitarily similar to a diagonal matrix. The rate of strain tensor is diagonal and all strains are normal strains in the principle coordinate.

This part is largely related to eigenvalue decomposition (see section B).

Example. Consider a strain rate tensor

$$\boldsymbol{S} = \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix}. \tag{A.58}$$

Its eigenvalues are $\lambda_1 = -\gamma/2$, $\lambda_2 = \gamma/2$ and associated eigenvectors are $\boldsymbol{x}_1 = 1/\sqrt{2}[1, -1]^T$ and $\boldsymbol{x}_2 = 1/\sqrt{2}[1, 1]^T$. The two principle directions are 45 deg (stretching, the direction that receives the most amplification) and -45 deg (compressing). The similarity transform is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} -\gamma/2 & 0 \\ 0 & \gamma/2 \end{bmatrix} \tag{A.59}$$

where

$$\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2] \tag{A.60}$$

is a unitary matrix such that $\boldsymbol{AX} = \boldsymbol{X\Lambda}$.

Take $\gamma = \partial_y U$, this is an importan example of plane shear flow. Think of the deformation of a rectangle to a diamond as it moves with the shear.

A.3.4 Application to the Reynolds stress tensor, the invariant map, and the Lumley triangle,

Consider the anisotropic (deviatoric) tensor of Reynolds stress

$$a_{ij} = \frac{\overline{u_i' u_j'}}{2k} - \frac{1}{3}\delta_{ij} \tag{A.61}$$

and its three principle invariants

$$I = \sigma_1 + \sigma_2 + \sigma_3 \tag{A.62}$$

$$II = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \tag{A.63}$$

$$III = \sigma_1\sigma_2\sigma_3 \tag{A.64}$$

along with its three eigenvalues

$$\sigma_1,\ \sigma_2,\ \sigma_3. \tag{A.65}$$

Since a_{ij} is a deviator, it is traceless and

$$I = a_{ii} = 0. \tag{A.66}$$

Consider turbulence. and has zero determinant

$$\det\left(\frac{\overline{u_i'u_j'}}{2k}\right) = (\sigma_1+\frac{1}{3})(\sigma_2+\frac{1}{3})(\sigma_3+\frac{1}{3}) \tag{A.67}$$

$$= \sigma_1\sigma_2\sigma_3 + \frac{1}{3}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) + \frac{1}{9}(\sigma_1+\sigma_2+\sigma_3) + \frac{1}{27}, \tag{A.68}$$

and we define

$$F = 27III + 9II + 1 \tag{A.69}$$

since $I = 0$.

1. Two-dimensional turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ can be diagonalized to

   $$diag(a, k-a, 0)$$

   and has zero determinant (there's a direction that has no turbulence). $F = 0$.

2. Three-dimensional isotropic turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ is

   $$diag(k/3, k/3, k/3)$$

   and we have $F = 1$.

3. Axisymmetric turbulence. Similarly, the characteristic polynomial of a_{ij} is in Weierstrass form and the condition for repeated eigenvalues (same energy in two principle directions) is

   $$\triangle = \left(\frac{1}{3}II\right)^3 + \left(\frac{1}{2}III\right)^2 = 0 \tag{A.70}$$

   and hence

   $$III = \pm 2\left(-\frac{II}{3}\right)^3, \tag{A.71}$$

   corresponding to the negative/left (pancake) and positive/right (cigar) limit curves of the Lumley triangle (Lumley & Newman, 1977; Choi & Lumley, 2001).

Appendix B

Matrix Analysis

B.1 Unitary matrix

Unitary transformations preserve inner products (and hence length and angle). Rotation and reflection.

B.2 Singular value decomposition and eigenvalue decomposition

also bilateral relations between SVD/Schur and polar decomposition.

B.3 Conformal mapping

Analytic functions, homomorphism, and conformal mapping.

Angle preserving, Symplectic (area preserving), implications of the latter on numerical schemes.

Appendix C

Fourier analysis

Fourier transform is a linear transformation to a space in which the function can be represented by a set of complete basis where the physics might be better understood. Fourier basis functions (sines and cosines) are the eigenfunction of the Laplacian operator, the matrix of which can be diagonalized by the Fourier transform, and the corresponding eigenvalues are frequencies.

C.1 Fourier series

Assume a function $f(t) \in \mathcal{L}_2([-\pi, \pi])$ (hence $\|f(t)\|$ in (C.4) is bnounded) is periodic on the interval $[-\pi, \pi]$. The Fourier expansion of $f(t)$ in terms of sin's and cos's is

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty}(A_k \cos kt + B_k \sin kt), \tag{C.1}$$

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt \,\mathrm{d}t = \frac{1}{\|\cos kt\|^2}(f(t), \cos kt) \tag{C.2}$$

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt \,\mathrm{d}t = \frac{1}{\|\sin kt\|^2}(f(t), \sin kt) \tag{C.3}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathcal{L}_2$ space (which is a Hilbert space) and the norm $\|\cdot\|$ is defined as

$$\|f(t)\| = (f(t), f(t))^{1/2} \tag{C.4}$$

and the inner-product therein as

$$(f(t), g(t)) = \int_{-\pi}^{\pi} f(t)\overline{g}(t)\,\mathrm{d}t, \tag{C.5}$$

where $\overline{(\cdot)}$ is the complex conjugate.

If the function is periodic in $[0, L]$, after domain transformation we will get

$$f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty}\left(A_k \cos\left(\frac{2\pi k}{L}t\right) + B_k \sin\left(\frac{2\pi k}{L}t\right)\right), \tag{C.6}$$

$$A_k = \frac{2}{L}\int_{-\pi}^{\pi} f(t)\cos\left(\frac{2\pi k}{L}t\right)\,\mathrm{d}t \tag{C.7}$$

$$B_k = \frac{2}{L}\int_{-\pi}^{\pi} f(t)\sin\left(\frac{2\pi k}{L}t\right)\,\mathrm{d}t, \tag{C.8}$$

and we note that the normalizing factor just comes from the norms of the trigonometric functions on prescribed intervals and the pre-factor before t is to make the sin's and cos's periodic on this interval.

Now we can think of combining the sin's and cos's into a form similar to $e^{\mathrm{i}kt}$ by leveraging

$$e^{\mathrm{i}kx} = \cos kt + \mathrm{i}\sin kt = \psi_k. \tag{C.9}$$

This will give us the complex Fourier series. Assume $f(t) \in \mathbb{C}$. The series (C.1) on domain $[-\pi, \pi]$ can be re-organized as

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}kt} = \sum_{k=-\infty}^{\infty} (\alpha_k + i\beta_k)(\cos kt + \mathrm{i}\sin kt). \tag{C.10}$$

If $f(t) \in \mathbb{R}$, and given $\cos kt = (e^{\mathrm{i}kx} + e^{-\mathrm{i}kx})/2$ and $\sin kt = (e^{\mathrm{i}kx} - e^{-\mathrm{i}kx})/2i$ we have

$$c_k = \frac{\alpha_k + i\beta_k}{2},\; c_{-k} = \frac{\alpha_k - i\beta_k}{2}, \tag{C.11}$$

and the symmetry

$$c_k = \bar{c}_{-k}. \tag{C.12}$$

The relation of the coefficients c_k to those α_k, β_k in (C.1) can be obtained by expanding. For example, $k = 0$ and

$$\frac{A_0}{2} = c_0, \tag{C.13}$$

$k = \pm 1$ and

$$(\alpha_1 + i\beta_1)(\cos t + \mathrm{i}\sin t) + (\alpha_{-1} + i\beta_{-1})(\cos t - \mathrm{i}\sin t) = A_1\cos t + B_1\sin t, \tag{C.14}$$

where we have four conditions and four unknowns.

Assume a function $f(t)$ on the interval $[-L, L]$ is continuous and square-integrable. Its Fourier expansion that decompose $f(t)$ into infinite sum of sin's and cos's is

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}k\pi t/L} = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}\omega_k t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} (f(t), \psi_k)\psi_k \tag{C.15}$$

$$c_k = \frac{1}{2\pi}(f(t), \psi_k) = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-\mathrm{i}k\pi t/L}\,\mathrm{d}t, \tag{C.16}$$

where the basis functions are called $\{\psi_k\}$ and the corresponding expansion coefficients on this basis are $\{c_k\}$. The angular frequency is

$$\omega_k = \frac{k\pi}{L} = k\Delta\omega, \tag{C.17}$$

that has a dimension of inverse t, where $\Delta\omega$ is the frequency resolution.

C.2 Fourier transform

In the Fourier series, the corresponding function on $[-L, L]$ is the periodic extension of $f(t)$. Assume we don't want to have this periodic extension and take the limit of $L \to \infty$. Then we get the Fourier transform as:

$$\begin{aligned}
f(t) &= \lim_{\Delta\omega \to \infty} \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}\omega_k t} && \text{(C.18)}\\
&= \lim_{\Delta\omega \to \infty} \sum_{k=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} f(\xi) e^{-\mathrm{i}k\pi\xi/L} \,\mathrm{d}\xi\right) e^{\mathrm{i}\omega_k t} && \text{(C.19)}\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi) e^{-\mathrm{i}k\pi\xi/L} \,\mathrm{d}\xi\right) e^{\mathrm{i}\omega t} \,\mathrm{d}\omega && \text{(C.20)}\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{\mathrm{i}\omega t} \,\mathrm{d}\omega && \text{(C.21)}\\
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(\xi) e^{-\mathrm{i}\omega\xi} \,\mathrm{d}\xi && \text{(C.22)}
\end{aligned}$$

where

$$\begin{aligned}
\hat{f}(\omega) &= \mathcal{F}(f(t)) && \text{(C.23)}\\
f(t) &= \mathcal{F}^{-1}(\hat{f}(\omega)) && \text{(C.24)}
\end{aligned}$$

are the direct and inverse Fourier transforms. We call

$$f(t) \leftrightarrow \hat{f}(\omega) \tag{C.25}$$

a Fourier transform pair. Here the spectral density $(1/2\pi)\hat{f}(\omega)\mathrm{d}\omega$ is the analogy of the amplitude c_k. The (C.21) is can also be written as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-\mathrm{i}\omega(\xi - t)} \,\mathrm{d}\xi \,\mathrm{d}\omega \tag{C.26}$$

and (C.22) called the Fourier inversion theorem. In many cases, for convenience, (for example, Fourier expansion in spectral methods) the factor of $1/2\pi$ is often placed in front of the forward FT (C.22). Another choice is placing a factor of $1/\sqrt{2\pi}$ in front of both.

C.2.1 Orthogonality, completeness, and linear independence

Up to now, we have probably reached a point where the properties of the Fourier basis should be discussed. The bases are orthogonal and form an infinite-dimensional function space.

First we take a look at $\{\psi_k\}$ where $\psi_k = e^{\mathrm{i}kx} \in [-\pi, \pi]$. We have

$$(\psi_j, \psi_k) = \int_{-\pi}^{\pi} e^{\mathrm{i}jx} e^{-\mathrm{i}kx} \,\mathrm{d}t = \int_{-\pi}^{\pi} e^{\mathrm{i}(j-k)t} \,\mathrm{d}t = \frac{1}{\mathrm{i}(j-k)} e^{\mathrm{i}(j-k)t}|_{-\pi}^{\pi} = \begin{cases} 0, & j \neq k \\ 2\pi, & j = k \end{cases}. \tag{C.27}$$

C.2.2 Derivatives

It is easy to see that

$$f'(t) = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{\mathrm{i}\omega t}\,\mathrm{d}\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{i}\omega\hat{f}(\omega)e^{\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{C.28}$$

and hence

$$f'(t) \leftrightarrow \mathrm{i}\omega\hat{f}(\omega) \tag{C.29}$$

are a Fourier pair.

C.2.3 The inversion theorem; duality

As preparation, we introduce the flip operator

$$\mathcal{R}f(t) = f(-t) \tag{C.30}$$

and it can be shown that if $\hat{f}(\omega) = \mathcal{F}[f(t)]$

$$\mathcal{F}\mathcal{R}f(t) = \mathcal{F}f(-t) = \int_{-\infty}^{\infty} f(-\xi)e^{-\mathrm{i}\omega\xi}\,\mathrm{d}\xi = \int_{-\infty}^{\infty} f(\xi)e^{\mathrm{i}\omega\xi}\,\mathrm{d}\xi = \hat{f}(-\omega) = \mathcal{R}\mathcal{F}f(t) \tag{C.31}$$

$$= \int_{-\infty}^{\infty} f(t)e^{\mathrm{i}\omega t}\,\mathrm{d}t = 2\pi\mathcal{F}^{-1}[f(t)] \tag{C.32}$$

that being said,

$$2\pi\mathcal{F}^{-1}f = \mathcal{F}\mathcal{R}f = \mathcal{R}\mathcal{F}f \tag{C.33}$$

and it can be deduced that

$$f(-t) = \mathcal{R}(t) = \mathcal{F}^{-1}\mathcal{R}\hat{f}(\omega) = \mathcal{F}^{-1}\hat{f}(-\omega) = \frac{1}{2\pi}\mathcal{F}\mathcal{R}\hat{f}(-\omega) = \frac{1}{2\pi}\mathcal{F}\hat{f}(\omega) \tag{C.34}$$

that the duality

$$\hat{f}(\omega) = \mathcal{F}[f(t)] \leftrightarrow f(-t) = \frac{1}{2\pi}\mathcal{F}[\hat{f}(\omega)] \tag{C.35}$$

is true. Thai being said, if you know a function g is the Fourier transform of another function f, you can obtain the Fourier transform of g in terms of f.

C.2.4 Convolution theorem

The convolution of two functions f and g is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,\mathrm{d}\tau = (g * f)(t), \tag{C.36}$$

and we will show how it is related to the Fourier transform of the product of two signals.

The convolution theorem is that

$$\mathcal{F}((f * g)(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau e^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{C.37}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau e^{-\mathrm{i}\omega\tau}e^{-\mathrm{i}\omega(t-\tau)}\,\mathrm{d}t \tag{C.38}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g(t-\tau)e^{-\mathrm{i}\omega(t-\tau)}\,\mathrm{d}(t-\tau)\int_{-\infty}^{\infty} f(\tau)e^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau && \text{(C.39)}\\
&= \hat{f}(\omega)\hat{g}(\omega) && \text{(C.40)}\\
&= \mathcal{F}(f(t))\mathcal{F}(g(t)) && \text{(C.41)}
\end{aligned}
$$

the Fourier transform of the convolution of two functions is the product of the Fourier transforms of these two functions. This provides another way of computing the convolution of two functions.

$$
\begin{array}{ccc}
f & \xrightarrow{*g} & (f*g)(t)\\
\mathcal{F}\big\downarrow & & \big\downarrow\mathcal{F}^{-1}\\
\hat{f}(\omega) & \xrightarrow[\times\hat{g}(\omega)]{} & \hat{f}(\omega)\hat{g}(\omega)
\end{array}
$$

We will see that the other way around is also true. Assume

$$
\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\,\mathrm{d}t && \text{(C.42)}\\
\hat{g}(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-\mathrm{i}\omega t}\,\mathrm{d}t && \text{(C.43)}
\end{aligned}
$$

and the Fourier transform of the product of two functions is

$$
\begin{aligned}
\mathcal{F}[f(t)g(t)] &= \int_{-\infty}^{\infty} f(t)g(t)e^{-\mathrm{i}\omega t}\,\mathrm{d}t && \text{(C.44)}\\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\alpha)e^{\mathrm{i}\alpha t}\,\mathrm{d}\alpha\right)g(t)e^{-\mathrm{i}\omega t}\,\mathrm{d}t && \text{(C.45)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\alpha)\left(\int_{-\infty}^{\infty} g(t)e^{-\mathrm{i}(\omega-\alpha)t}\,\mathrm{d}t\right)\mathrm{d}\alpha && \text{(C.46)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\alpha)\hat{g}(\omega-\alpha)\,\mathrm{d}\alpha && \text{(C.47)}\\
&= \frac{1}{2\pi}\hat{f}*\hat{g} && \text{(C.48)}
\end{aligned}
$$

that is the convolution of the Fourier transforms of the two functions.

C.2.5 Correlation functions and spectra; Parseval's theorem

Define the cross-correlation of two complex-value functions u and v as

$$
\begin{aligned}
C_{uv}(\tau) &= \int_{-\infty}^{\infty} u^*(t)v(t+\tau)\,\mathrm{d}t && \text{(C.49)}\\
&= \int_{-\infty}^{\infty} u^*(t-\tau)v(t)\,\mathrm{d}t && \text{(C.50)}\\
&= C_{vu}^*(-\tau) && \text{(C.51)}
\end{aligned}
$$

Its Fourier counterpart is the cross-spectrum

$$
\Phi_{uv}(\omega) = \mathcal{F}[C_{uv}(\tau)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^*(t)v(t+\tau)e^{-\mathrm{i}\omega\tau}\,\mathrm{d}t\mathrm{d}\tau \qquad \text{(C.52)}
$$

$$= 2\pi \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(t) e^{\mathrm{i}\omega t} \, \mathrm{d}t \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} v(t+\tau) e^{-\mathrm{i}\omega(t+\tau)} \, \mathrm{d}\tau \right) \tag{C.53}$$

$$= 2\pi \hat{u}^*(\omega) \hat{v}(\omega) \tag{C.54}$$

whose real part is called the co-spectrum of uv. The phase relation between u, v is $\phi = \tan^{-1}(\Im(\Phi_{uv})/\Re(\Phi_{uv}))$. Note we have chosen to put the $\frac{1}{2\pi}$ factor in the forward FT here. In the special case of $u = v$,

$$\Phi_{uu}(\omega) = 2\pi \hat{u}^*(\omega)\hat{u}(\omega) = 2\pi |\hat{u}(\omega)|^2 \tag{C.55}$$

is the energy spectrum of u, which is the Fourier transform of the two-point auto-correlation C_{uu} (Wiener–Khinchin theorem). We note that $\Phi_{uu}(\omega)$ is real while Φ_{uv} is complex.

From the inverse FT,

$$C_{uv}(\tau) = \int_{-\infty}^{\infty} \Phi_{uv}(\omega) e^{\mathrm{i}\omega\tau} \, \mathrm{d}\omega = 2\pi \int_{-\infty}^{\infty} \hat{u}^*(\omega)\hat{v}(\omega) e^{\mathrm{i}\omega\tau} \, \mathrm{d}\omega. \tag{C.56}$$

Taking $\tau = 0$ we have

$$\int_{-\infty}^{\infty} u^*(t) v(t) \, \mathrm{d}t = C_{uv}(0) = 2\pi \int_{-\infty}^{\infty} \hat{u}^*(\omega)\hat{v}(\omega) \, \mathrm{d}\omega \tag{C.57}$$

or simply, the correlation between u, v being the average/integral in physical/Fourier space:

$$\langle u(t) v(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(t) v(t) \, \mathrm{d}t = \int_{-\infty}^{\infty} \hat{u}^*(\omega)\hat{v}(\omega) \, \mathrm{d}\omega. \tag{C.58}$$

With $u = v$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |u(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} |\hat{u}(\omega)|^2 \, \mathrm{d}\omega. \tag{C.59}$$

This is called the Parseval's theorem. In the context of DFT, it becomes

$$\frac{1}{N} \sum_{j=0}^{N-1} u_j^* u_j = \sum_{k=0}^{N-1} \hat{u}_k^* \hat{u}_k. \tag{C.60}$$

C.2.6 Symmetries of real FT

A real function $f(t) \in \mathbb{R}$ being Fourier transformed can be decomposed into odd and even parts:

$$f(t) = \frac{1}{2}(f(t) + f(-t)) + \frac{1}{2}(f(t) - f(-t)) \tag{C.61}$$

$$= f_e(t) + f_o(t) \tag{C.62}$$

where

$$f_e(t) = f_e(-t), \; f_o(t) = -f_o(-t). \tag{C.63}$$

The Fourier transform can be so decomposed as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(\xi) e^{-\mathrm{i}\omega\xi} d\xi \tag{C.64}$$

$$= \int_{-\infty}^{\infty} (f_e(\xi) + f_o(\xi))(\cos(\omega\xi) - \mathrm{i}\sin(\omega\xi))d\xi \tag{C.65}$$

$$= \int_{-\infty}^{\infty} f_e(\xi)\cos(\omega\xi)d\xi - i\int_{-\infty}^{\infty} f_o(\xi)\sin(\omega\xi)d\xi \tag{C.66}$$

$$= \Re\{\hat{f}(\omega)\} + \Im\{\hat{f}(\omega)\} \tag{C.67}$$

$$= \hat{f}_e(\omega) + \hat{f}_o(\omega) \tag{C.68}$$

where we note that the $\int_{-\infty}^{\infty}$ of the product of an even and an odd function is zero.

This has several implications:

- The FT of an even function is real and even ($\hat{f}(\omega) = \hat{f}(-\omega)$). The FT of an odd function is imaginary and odd ($\hat{f}(\omega) = -\hat{f}(-\omega)$).

- The real part of the FT is the FT of the even part of the signal. The imaginary part of the FT is the FT of the odd part of the signal.

These properties can be summarized into

$$\hat{f}(\omega) = \hat{f}_e(\omega) + \hat{f}_o(\omega) \tag{C.69}$$

$$= \hat{f}_e(-\omega) - \hat{f}_o(-\omega) \tag{C.70}$$

$$= \overline{\hat{f}_e(-\omega) + \hat{f}_o(-\omega)} \tag{C.71}$$

$$= \overline{\hat{f}(-\omega)} \tag{C.72}$$

which is called the Hermitian symmetry of the real FT. It can also be seen from replacing ω with $-\omega$ in (C.64). This Hermitian symmetry leads to $\hat{f}(0) = \hat{f}^*(0) \in \mathbb{R}$. In other words, the ω mode and the $-\omega$ mode have equal amplitude but opposite phases:

$$\hat{f}(\omega) = |\hat{f}(\omega)|e^{\mathrm{i}\phi(\omega)} \tag{C.73}$$

$$\hat{f}(-\omega) = |\hat{f}(-\omega)|e^{\mathrm{i}\phi(-\omega)} = \overline{\hat{f}(\omega)} = |\hat{f}(\omega)|e^{-\mathrm{i}\phi(\omega)} \rightarrow \phi(-\omega) = -\phi(\omega) \tag{C.74}$$

C.2.7 Fourier transform in multiple dimensions

FT can be straightforwardly extended to higher dimensions as

$$f(x, y, z) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(k_x, k_y, k_z)e^{\mathrm{i}k_x x}e^{\mathrm{i}k_y y}e^{\mathrm{i}k_z z}\,\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z \tag{C.75}$$

$$\hat{f}(k_x, k_y, k_z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z)e^{-\mathrm{i}k_x x}e^{-\mathrm{i}k_y y}e^{-\mathrm{i}k_z z}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{C.76}$$

The Hermitian symmetry of real FT ($f \in \mathbb{R}$) is

$$\hat{f}(k_x, k_y, k_z) = \overline{\hat{f}(-k_x, -k_y, -k_z)}. \tag{C.77}$$

C.2.8 Examples

- Dirac delta and unity.

$$\mathcal{F}[\delta(t-t_0)] = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-\mathrm{i}\omega t}\,\mathrm{d}t = e^{-\mathrm{i}\omega t_0} = \cos(\omega t_0) - \mathrm{i}\sin(\omega t_0), \tag{C.78}$$

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-\mathrm{i}\omega t}\,\mathrm{d}t = 1, \tag{C.79}$$

$$\delta(-t) = \delta(t) = \mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1\cdot e^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{C.80}$$

$$2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t}\,\mathrm{d}t = \mathcal{F}[1]. \tag{C.81}$$

that the Dirac $\delta(t)$ and unity are a Fourier transform pair. The Fourier dual of a constant signal is $\mathcal{F}(C) = C\cdot 2\pi\delta(\omega)$.

- Sine and cosine. First we establish

$$\mathcal{F}[e^{\mathrm{i}\alpha t}] = \int_{-\infty}^{\infty} e^{\mathrm{i}\alpha t}e^{-\mathrm{i}\omega t}\,\mathrm{d}t = 2\pi\delta(\omega-\alpha). \tag{C.82}$$

With

$$\cos(\alpha x) = \frac{e^{\mathrm{i}\alpha x}+e^{-\mathrm{i}\alpha x}}{2} \tag{C.83}$$

$$\sin(\alpha x) = \frac{e^{\mathrm{i}\alpha x}-e^{-\mathrm{i}\alpha x}}{2i} \tag{C.84}$$

we have

$$\mathcal{F}[\cos(\alpha x)] = \pi[\delta(\omega+\alpha)+\delta(\omega-\alpha)] \tag{C.85}$$

$$\mathcal{F}[\sin(\alpha x)] = i\pi[\delta(\omega+\alpha)-\delta(\omega-\alpha)] \tag{C.86}$$

that being said, the frequency content of these functions are only at isolated frequency(ies) even though they are widespread in space. On the other hand, the FT of highly spatially localized function $\delta(t-t_0)$ is widespread in Fourier space (see (C.78)).

- Box and sinc. Consider a box function with unit area

$$g(t) = \begin{cases} 1/2\Delta & |t|\le\Delta \\ 0 & \text{otherwise} \end{cases} \tag{C.87}$$

whose Fourier pair is

$$\hat{g}(\omega) = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} e^{-\mathrm{i}\omega t}\,\mathrm{d}t \tag{C.88}$$

$$= \frac{1}{2\Delta}\frac{1}{-i\omega}(e^{-\mathrm{i}\omega\Delta}-e^{\mathrm{i}\omega\Delta}) \tag{C.89}$$

$$= \frac{\sin(\omega\Delta)}{\omega\Delta} = \mathrm{sinc}(\omega\Delta). \tag{C.90}$$

We note the sinc (cardinal sine) function has a finite integration (computed using the Laplace transform) over the real domain

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi \tag{C.91}$$

and its normalized version is $\mathrm{sinc}_\pi x = \sin(\pi x)/\pi x$ whose integral over the real axis is unity. Thus, spatially compact function square box has an oscillatory spectral counterpart.

On the other hand, spatially oscillatory function

$$h(t) = \frac{\sin(2\pi t/\Delta)}{\pi t} \tag{C.92}$$

corresponds to a spectrally sharp counterpart

$$\hat{h}(\omega) = \begin{cases} 1, & |\omega| \le 2\pi/\Delta \\ 0, & \text{otherwise} \end{cases} \tag{C.93}$$

which can be shown from the inverse transform of the sinc function

$$\left.\begin{matrix} 1/2\Delta, & |t| \le \Delta \\ 0, & \text{otherwise} \end{matrix}\right\} = g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega\Delta)}{\omega\Delta} e^{\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{C.94}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin(\Delta t)}{\Delta t} e^{\mathrm{i}\omega t}\,\mathrm{d}t \tag{C.95}$$

equivalently

$$\left.\begin{matrix} 1, & |\alpha| \le 2\pi/\Delta \\ 0, & \text{otherwise} \end{matrix}\right\} = g(\alpha) = \int_{-\infty}^{\infty} \frac{\sin(2\pi\tau/\Delta)}{\pi\tau} e^{-\mathrm{i}\alpha\tau}\,\mathrm{d}\tau \tag{C.96}$$

with COVs $\alpha = -2\pi\omega/\Delta^2$ and $\tau = \Delta^2 t/2\pi$. Taking $\hat{h}(\omega) = g(\alpha)$ completes the proof.

- Gaussian. The Fourier transform of a normalized Gaussian function $f(t) = \sqrt{a/\pi}e^{-at^2}$ is

$$\mathcal{F}[f(t)] = \sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} e^{-at^2} e^{-\mathrm{i}\omega t}\,\mathrm{d}t = \sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} \exp\left[-a\left(t + \frac{\mathrm{i}\omega}{2a}\right)^2 - \frac{\omega^2}{4a}\right]\mathrm{d}t \tag{C.97}$$

$$= \sqrt{\frac{a}{\pi}} e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} \exp\left[-a\left(t + \frac{\mathrm{i}\omega}{2a}\right)^2\right]\mathrm{d}t = e^{-\frac{\omega^2}{4a}} \tag{C.98}$$

where we have used the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-at^2}\,\mathrm{d}t = \sqrt{\frac{\pi}{a}}, \tag{C.99}$$

provided

$$\left(\int_{-\infty}^{\infty} e^{-at^2}\,\mathrm{d}t\right)^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\,\mathrm{d}y\right) \tag{C.100}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y \tag{C.101}$$

$$= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} e^{-ar^2} r \, \mathrm{d}r \tag{C.102}$$

$$= \frac{\pi}{a}. \tag{C.103}$$

Take $a = 6/\Delta^2$ in a typical choice of filtering,

$$f(t) = \frac{6}{\pi\Delta^2} e^{-\frac{6t^2}{\Delta^2}} \tag{C.104}$$

has a Fourier transform pair

$$\hat{f}(\omega) = e^{-\frac{(\omega\Delta)^2}{24}}. \tag{C.105}$$

Note that the FT of Gaussian is still Gaussian. Take $a = 1/2\sigma$ we recover the standard Gaussian distribution

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}. \tag{C.106}$$

Gaussian is compact in both physical and Fourier space.

C.3 Discrete Fourier transform

Consider the discretization of physical domain (sampling, denoted by a subscript 's') first. Assume the sampling period is $L_s = L_0/N$ (or sampling frequency $\omega_s = 2\pi/L_s$) and the sampled function f_s is given by

$$f_s(t) = \sum_{j=-\infty}^{\infty} f(t)\delta(t - jL_s), \tag{C.107}$$

where $\delta(t)$ is the Dirac delta and

$$\int_{-\infty}^{\infty} \delta(t) \, \mathrm{d}t = 1; \tag{C.108}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) \, \mathrm{d}t = f(t_0). \tag{C.109}$$

Given (C.22), we have

$$\hat{f}_s(\omega) = \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f(t)\delta(t - jL_s) e^{-\mathrm{i}\omega t} \, \mathrm{d}t \tag{C.110}$$

$$= \sum_{j=-\infty}^{\infty} f(jL_s) e^{-\mathrm{i}\omega j L_s}. \tag{C.111}$$

Now we also need to discretize the frequency domain. Assume there are $N + 1$ samples with indices $0, 1, ..., N$, $f_N = f_0$ and the sampling period being L_s and the length of sampling, or the period of extension is $L_0 = NL_s$. The lowest frequency that can be resolved is $\omega_0 = 2\pi/L_0$ and the discrete frequencies are $\omega_k = k\omega_0 = 2\pi k/L_0$. Replacing ω in the spectral density function $\hat{f}_s(\omega)$ we get

$$\hat{f}_s(k\omega_0) = \int_{-L_0/2}^{L_0/2} \sum_{j=0}^{N-1} f(jL_s) e^{-\mathrm{i}k\omega_0 j L_s} \, \mathrm{d}t \tag{C.112}$$

$$= \sum_{j=0}^{N-1} f(-\frac{L_0}{2} + jL_s)e^{-\mathrm{i}kj\frac{2\pi}{N}}, \tag{C.113}$$

denoted as

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j e^{-\mathrm{i}2\pi kj/N}. \tag{C.114}$$

We also note that, as commonly used, the k-th frequency $\tilde{f}_k$ (the tilde distinguishes 'f'requency from 'f'unction) corresponding to the spectral density (Fourier coefficient) $\hat{f}_k$ is given as

$$\tilde{f}_k = \frac{\tilde{\omega}_k}{2\pi} = \frac{k\tilde{\omega}_0}{2\pi} = \frac{k}{2\pi}\frac{2\pi}{L_0} = \frac{k}{2\pi}\frac{2\pi}{NL_s} = \frac{k}{N}\tilde{f}_s = k\Delta\tilde{f}, \tag{C.115}$$

where $\tilde{f}_s = 1/L_s$ is the sampling frequency and $\Delta\tilde{f} = \tilde{f}_s/N = 1/L_0$ is the frequency resolution (the smallest/lowest resolved frequency). The angular frequency / wavenumber resolution is $\Delta\omega = 2\pi\Delta\tilde{f} = 2\pi/L_0$. Here tildes denotes frequency (not to be confused with the function f). When plotting the power spectral density, $\tilde{f}_k$ will be the discrete frequencies on the horizontal axis (you just multiply k/N, $k = 0, 1, ..., N-1$ with $\tilde{f}_s = 1/L_s$), and the Fourier coefficient $\hat{f}_k$ will be the vertical axis.

In a general case, assume on the interval $[t_0, t_{N-1}]$ the function f is periodic and take the values

$$f_i = f(t_i),\ 0 \le i \le N-1, \tag{C.116}$$

at N equispaced, collocation points $t_0, t_1, ..., t_{N-1}$ which are typically the measurement data, where

$$t_i = \frac{L_0}{N}i,\ i = 0, 1, ..., N-1. \tag{C.117}$$

The discrete Fourier transform (DFT) and its inverse are

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j e^{-\mathrm{i}2\pi jk/N} = \sum_{j=0}^{N-1} f_j (\omega_N^k)^j \tag{C.118}$$

$$f_j = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}_k e^{\mathrm{i}2\pi jk/N} = \frac{1}{N}\sum_{k=0}^{N-1} \hat{f}_k (\bar{\omega}_N^j)^k \tag{C.119}$$

where

$$\omega_N = e^{-\mathrm{i}2\pi/N} \in \mathbb{C} \tag{C.120}$$

or its complex conjugate $\bar{\omega}_N = e^{\mathrm{i}2\pi/N}$ is the primitive N-th complex root of unity since $(\omega_N)^N = (\bar{\omega}_N)^N = 1$. The FT linearly maps the data in physical space to spectral space or vice versa as

$$(f_0, f_1, ..., f_{N-1}) \leftrightarrow (\hat{f}_0, \hat{f}_1, ..., \hat{f}_{N-1}), \tag{C.121}$$

where both sides are linear combinations of one another and the weights are given by appropriate powers of ω_N.

Note the normalizing factor $1/N$ is similar to the factor of $1/2\pi$ in the inverse Fourier transform (C.21) that preserves the energy after applying the forward and inverse FTs. It can also be seen that the zero mode in (C.119) corresponds to the mean

$$\langle f \rangle = \frac{1}{N}\hat{f}_0 = \frac{1}{N}\sum_{j=0}^{N-1} f_j. \tag{C.122}$$

C.3.1 Discrete orthogonality

We prove that for each k in (C.118), basis functions $\{(\omega_N^k)^j\}$ are orthogonal. That also being said, the columns in the DFT matrix (C.152) form an orthogonal basis.

Define the inner product of two arbitrary powers i, j as

$$[(\omega_N^k)^i, (\omega_N^k)^j] = \sum_{k=0}^{N-1} \overline{(\omega_N^k)^i}(\omega_N^k)^j = \sum_{k=0}^{N-1} e^{\mathrm{i}2\pi k(i-j)/N}. \tag{C.123}$$

Due to the geometric sum formula

$$\sum_{k=0}^{N-1} \alpha^k = \frac{1-\alpha^N}{1-\alpha}, \tag{C.124}$$

and $1-\omega_N^N = 0$, we have

$$[(\omega_N^k)^i, (\omega_N^k)^j] = \begin{cases} 0, & i \neq j \\ N, & i = j \end{cases}. \tag{C.125}$$

Hence, it is also proved that the DFT matrix is full-rank.

C.3.2 Conjugate symmetry of the real DFT

Since $\hat{f}_k = \hat{f}_{-k}^*$ (here * is the complex conjugate), similar to (C.72), we have

$$\hat{f}_{N-k} = \hat{f}_{-k} = \hat{f}_k^* \tag{C.126}$$

if $f(t) \in \mathbb{R}$. This symmetry implies redundancy in the coefficients in real DFT that we only need to store half of the elements (N is even)

$$k = \{0, 1, ..., \frac{N}{2} - 1\}, \tag{C.127}$$

and the rest can be recovered by complex conjugation. Frequencies $k > N/2 - 1$ are aliased back to $k - N$. For example, $k = 0$ in (C.126) correspond to

$$\hat{f}_0 = \hat{f}_{N/2} \in \mathbb{R}, \tag{C.128}$$

as the DC component.

C.3.3 Frequency ordering and the Nyquist frequency

It is observed that according to the 2π periodicity of $e^{-2\pi i k j/N}$, the DFT (C.118) can be shifted to

$$\hat{f}_k = \sum_{j=-N/2}^{N/2-1} f_j e^{-\mathrm{i}2\pi k j/N}, \tag{C.129}$$

with the same number of terms. Accordingly, the discrete wavenumbers can be arranged as

$$k = \{0, 1, 2, ..., \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1, ..., N-2, N-1\} \tag{C.130}$$

$$= \{0, 1, 2, ..., \frac{N}{2} - 1, -\frac{N}{2}, -\frac{N}{2} + 1, ..., -2, -1\}, \tag{C.131}$$

according to the conjugate symmetry of (C.118) and (C.129) that

$$\hat{f}_{N-k} = \hat{f}_{-k}, \tag{C.132}$$

where the second half in (C.130) are interpreted as negative frequencies.

It can be seen that with (C.131), the k-th power of $(\bar{\omega}_N^k)^j$ that you multiply with / corresponds to $\hat{f}_k$ in the Fourier expansion (C.119) becomes

$$\{\bar{\omega}_N^k\} = \exp\left(\mathrm{i}2\pi\{0, 1, 2, ..., \frac{N}{2} - 1, -\frac{N}{2}, -\frac{N}{2} + 1, ..., -2, -1\}/N\right) = \exp(\mathrm{i}\{\tilde{\omega}_k^*\}/N), \tag{C.133}$$

where the corresponding discrete, non-dimensional (in order for the exponential to make sense), angular frequencies are

$$\tilde{\omega}_k^* = 2\pi k,\ k = \{0, 1, 2, ..., \frac{N}{2} - 1, -\frac{N}{2}, -\frac{N}{2} + 1, ..., -2, -1\}, \tag{C.134}$$

such that $\{\tilde{\omega}_k^*\}/N$ uniformly distribute within $[-\pi, \pi)$ on the unit circle in a counter-clockwise order, starting from 0 and having a jump from π to $-\pi$ when $k = N/2$. When multiplied by the frequency scale $1/L_0 = \Delta\tilde{f}$, the angular frequency becomes dimensional:

$$\tilde{\omega}_k = 2\pi\tilde{f}_k = \frac{2\pi k}{L_0},\ k = \{0, 1, 2, ..., \frac{N}{2} - 1, -\frac{N}{2}, -\frac{N}{2} + 1, ..., -2, -1\}, \tag{C.135}$$

or

$$\tilde{f}_k = \frac{k}{L_0} = k\Delta\tilde{f}. \tag{C.136}$$

It is easy to see that the highest frequency is $\tilde{f}_{N/2-1} = (N-2)/(2L_0)$. Given the interval size $L_s = L_0/N = 1/\tilde{f}_s$, $\tilde{f}_{N/2-1} \approx 1/2\tilde{f}_s$ given N is typically large. That being said, the resolvable frequency is at most nearly half of the sampling frequency. The frequency $\tilde{f}_s/2$ is called the Nyquist frequency and the frequencies above this frequency are subject to aliasing (folding). For example, $\{0.6, 1.4, 1.6\}\tilde{f}_s$ will all be aliased to $0.4\tilde{f}_s$, due to the symmetry (about $N/2$ or $\tilde{f}_s/2$) and the N-periodicity of (C.135). In other words, since Fourier modes contain real and complex parts, with the same amount of information from N points in physical space we can only obtain half the number of independent Fourier modes, consistent with the conjugate symmetry (C.126).

C.3.4 Aliasing and de-aliasing

It is common to involve new frequencies above the Nyquist frequency and hence aliasing errors during the multiplications in the physical space, especially in pseudo-spectral treatments of nonlinear PDEs.

For example, consider the quadratic nonlinear term f^2 and its Fourier transform. We first denote

$$[\mathcal{F}_N(f)]_k = \hat{f}_k = \sum_{j=-N/2}^{N/2-1} f_j e^{-\mathrm{i}2\pi kj/N} \tag{C.137}$$

$$[\mathcal{F}_N^{-1}(f)]_j = f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{\mathrm{i}2\pi jk/N} \tag{C.138}$$

as the DFTs with N points. The pseudo-spectral treatment of f^2 involves an inverse transform of $\hat{f}$, a multiplication in physical space, and a direct transform to Fourier space. This way, the cost of convolution in Fourier space as in a normal FT of a product in physical space can be avoided.

We have:

$$\begin{aligned}
[\mathcal{F}_N(f^2)]_k &= [\mathcal{F}_N([\mathcal{F}_N^{-1}(\hat{f})]^2)]_k && \text{(C.139)}\\
&= \sum_{j=-N/2}^{N/2-1} [\mathcal{F}_N^{-1}(\hat{f})]_j^2 e^{-\mathrm{i}2\pi kj/N} && \text{(C.140)}\\
&= \sum_{j=-N/2}^{N/2-1} \left(\frac{1}{N^2} \sum_{m=-N/2}^{N/2-1} \sum_{l=-N/2}^{N/2-1} \hat{f}_m \hat{f}_l e^{\mathrm{i}2\pi j(m+l)/N} \right) e^{-\mathrm{i}2\pi kj/N} && \text{(C.141)}\\
&= \frac{1}{N^2} \sum_{m=-N/2}^{N/2-1} \sum_{l=-N/2}^{N/2-1} \hat{f}_m \hat{f}_l \sum_{j=-N/2}^{N/2-1} e^{\mathrm{i}2\pi j(m+l-k)/N} && \text{(C.142)}\\
&= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} \hat{f}_{k-l} \hat{f}_l && \text{(C.143)}
\end{aligned}$$

where the discrete orthogonality condition is used in the simplification to the last equation. Aliasing error occurs as $k-l$ exceeds the frequency range, as can be seen from the decomposition of the following cyclic convolution

$$\begin{aligned}
[\mathcal{F}_N(f^2)]_k &= \frac{1}{N} \sum_{l=-N/2}^{N/2-1} \hat{f}_{k-l} \hat{f}_l && \text{(C.144)}\\
&= \frac{1}{N} \sum_{-N/2 \le l, k-l \le N/2-1} \hat{f}_{k-l} \hat{f}_l + \frac{1}{N} \sum_{\substack{-N/2 \le l \le N/2-1 \\ k-l > N/2-1 \text{ or } k-l < -N/2}} \hat{f}_{k-l} \hat{f}_l && \text{(C.145)}
\end{aligned}$$

where the second term involves aliasing errors.

The idea of the zero-padding method is to choose to perform the DFT in a large space, with K points, with the goal being keeping $[-\frac{N}{2}, \frac{N}{2}-1]$ aliasing-free. In such a case,

- Wavenumbers with $k-l > K/2-1$ will be aliased to $k-l-K$, requiring $k-l-K < -N/2$ for all possible $k, l \in [-\frac{N}{2}, \frac{N}{2}-1]$. The maximum of $k-l$ is $N-1$, achieved when $k = N/2-1, l = -N/2$. Hence, $K > 3N/2-1$.

- Wavenumbers with $k-l < -K/2$ will be aliased to $k-l+K$, requiring $k-l+K > N/2-1$ for all $k, l \in [-\frac{N}{2}, \frac{N}{2}-1]$. The minimum of $k-l$ is $-N+1$, when $k = -N/2, l = N/2-1$. Hence, $K > 3N/2-2$.

- In all, a choice of $K = 3N/2$ is typically made.

- The computational procedures are: (1) zeros are padded to extend the wavenumber range and $\mathcal{F}_K^{-1}(\hat{f})$ is computed; (2) it is squared in physical space and Fourier transformed to $\mathcal{F}_K([\mathcal{F}_K^{-1}(\hat{f})]^2)$; (3) 1/3 of the wavenumbers between $N/2$ and $K/2$ are dropped and the FT is scaled back to $\mathcal{F}_N$ by multiplying K/N.

C.3.5 MATLAB `fft()`, and `FFTW`

We note that the order (C.131),

$$k = \{0, 1, 2, ..., \frac{N}{2} - 1, -\frac{N}{2}, -\frac{N}{2} + 1, ..., -2, -1\},$$

is used in `FFTW3` (Frigo & Johnson, 2020) and in MATLAB `fft()`. In MATLAB, `fftshift()` can be used to shift (C.131) to

$$k = \{-\frac{N}{2}, -\frac{N}{2} + 1, ..., -1, 0, 1, ..., \frac{N}{2} - 2, \frac{N}{2} - 1\} \tag{C.146}$$

such that it is symmetric about zero and only the positive semi-axis of $\tilde{\omega}_k$ (assuming N is even) needs to be plotted (see (C.135)).

C.3.6 Examples

- Consider the azimuthal decomposition of $f(\theta)$ into its Fourier modes. The IFT is

$$f_k = \sum_{m=0}^{N-1} \hat{f}_m e^{\mathrm{i}2\pi mk/N} = \sum_{m=0}^{N-1} \hat{f}_m e^{\mathrm{i}(m2\pi k/N)} = \sum_{m=0}^{N-1} \hat{f}_m e^{\mathrm{i}(m\theta_k)} \tag{C.147}$$

  where we have absorbed $1/N$ into $\hat{f}_k$, integers $m = 0, 1, ..., N-1$ are the mode indices, and $\theta_k = (2\pi/N)k$, $k = 0, 1, ..., N-1$ are the discrete azimuth angles (collocation points). Domain $L_0 = 2\pi$.

- The DFTs in multiple dimensions are

$$\hat{f}_{k_1,k_2,k_3} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{l=0}^{L-1} f_{m,n,l} e^{-\mathrm{i}2\pi(k_1 m/M + k_2 n/N + k_3 l/L)} \tag{C.148}$$

$$f_{m,n,l} = \frac{1}{MNL} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{N-1} \sum_{k_3=0}^{L-1} \hat{f}_{k_1,k_2,k_3} e^{\mathrm{i}2\pi(k_1 m/M + k_2 n/N + k_3 l/L)} \tag{C.149}$$

  and it is seen that the mean (zero mode) is computed as

$$\langle f \rangle = \frac{1}{MNL} \hat{f}_{0,0,0} = \frac{1}{MNL} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{l=0}^{L-1} f_{m,n,l}. \tag{C.150}$$

  The conjugate symmetry condition for real DFT ($f \in \mathbb{R}$) is

$$\overline{\hat{f}}(k_1, k_2, k_3) = \hat{f}(-k_1, -k_2, -k_3). \tag{C.151}$$

C.4 The Fast Fourier Transform

C.4.1 The matrix form of the DFT

The DFT (C.118) is often written in matrix form as (following a slightly different normalizing convention that equally places a factor $1/\sqrt{N}$ in direct and inverse FT)

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_N^{(N-1)} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix}, \tag{C.152}$$

or

$$\hat{\boldsymbol{f}} = \boldsymbol{F}\boldsymbol{f},\ \hat{f}_k = F_{kj} f_j,\ 0 \le k, j \le N-1 \tag{C.153}$$

where

$$\boldsymbol{F} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_N^{(N-1)} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)^2} \end{bmatrix},\ F_{kj} = \omega_N^{kj},\ 0 \le k, j \le N-1 \tag{C.154}$$

is the Fourier transformation matrix. In terms of a special case of a Vandermonde matrix,

$$\boldsymbol{F} = \frac{1}{\sqrt{N}} \boldsymbol{V}(1, \omega_N, \omega_N^2, \dots, \omega_N^{(N-1)}), \tag{C.155}$$

where the factor $1/\sqrt{N}$ is to make $\boldsymbol{F}$ unitary.

The DFT matrix in (C.152) is fully dense due to Fourier transform's global property. And it is a Vandermonde matrix that has many nice properties and such as symmetry ($\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}$) and unitary ($\boldsymbol{F}\boldsymbol{F}^{\mathrm{H}} = \boldsymbol{I}$).

As a result, the inverse Fourier transform can be directly written down as

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w_N & w_N^2 & \dots & w_N^{N-1} \\ 1 & w_N^2 & w_N^4 & \dots & w_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w_N^{(N-1)} & w_N^{2(N-1)} & \dots & w_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix}, \tag{C.156}$$

or

$$\boldsymbol{f} = \boldsymbol{F}^{-1}\hat{\boldsymbol{f}} \tag{C.157}$$

where

$$\boldsymbol{F}^{-1} = \boldsymbol{F}^{\mathrm{H}} = \overline{\boldsymbol{F}} = \frac{1}{\sqrt{N}}\boldsymbol{V}(1, w_N, w_N^2, ..., w_N^{(N-1)}) \tag{C.158}$$

with $w_N = \bar{\omega}_n = e^{\mathrm{i}2\pi/N}$ being also the primitive N-th root of unity.

In the language of linear transformations, left-multiplication of $\boldsymbol{F}$ transforms the rows of the data (rows of column vectors say $\boldsymbol{f}$), such that

$$[\hat{\boldsymbol{f}}(x_1), \hat{\boldsymbol{f}}(x_2), ..., \hat{\boldsymbol{f}}(x_m)] = [\boldsymbol{F}\boldsymbol{f}(x_1), \boldsymbol{F}\boldsymbol{f}(x_2), ..., \boldsymbol{F}\boldsymbol{f}(x_m)] = \boldsymbol{F}[\boldsymbol{f}(x_1), \boldsymbol{f}(x_2), ..., \boldsymbol{f}(x_m)], \tag{C.159}$$

such as when the input has a spatial dependence on x. It is typically more convenient to write down the transpose of the transforms using a right multiplication (transforming the columns of $\boldsymbol{Q}$),

$$\hat{\boldsymbol{Q}} = [\hat{\boldsymbol{q}}_0, \hat{\boldsymbol{q}}_1, ..., \hat{\boldsymbol{q}}_{N-1}] = [\boldsymbol{q}_0, \boldsymbol{q}_1, ..., \boldsymbol{q}_{N-1}]\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{F} \tag{C.160}$$

and

$$\boldsymbol{Q} = \hat{\boldsymbol{Q}}\overline{\boldsymbol{F}} \tag{C.161}$$

where $\boldsymbol{q} \in \mathbb{C}^m$, and the Fourier mode $\hat{\boldsymbol{q}}_k$ at the kth discrete frequency ($k = 0, 1, ...N-1$) is a linear combination of $\{\boldsymbol{q}_l\}_{l=0}^{N-1}$ weighted by

$$(1, \omega_N^k, \omega_N^{2k}, ..., \omega_N^{k(N-1)}),\ \omega_N^k = e^{-\mathrm{i}2\pi k/N} = e^{-i\tilde{\omega}_k^*}$$

where $\tilde{\omega}_k^* = 2\pi k/N = 2\pi\tilde{f}_k L_s$ is the non-dimensionalized kth (angular) frequency. and the right-multiplication with $\boldsymbol{F}$ transforms the columns of $\boldsymbol{Q}$ (each column is a row vector; note $\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}$).

C.4.2 Vandermonde matrix

A square $(n+1) \times (n+1)$ Vandermonde matrix is

$$\boldsymbol{V} = \boldsymbol{V}(t_0, x_1, ..., x_n) = \begin{bmatrix} 1 & t_0 & t_0^2 & ... & t_0^n \\ 1 & x_1 & x_1^2 & ... & x_1^n \\ 1 & x_2 & x_2^2 & ... & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & ... & x_n^n \end{bmatrix} \tag{C.162}$$

whose determinant (called Vandermonde determinant) is

$$\det(\boldsymbol{V}) = \prod_{0 \le i \le j \le n} (x_i - x_j) \tag{C.163}$$

is non-zero if and only if x_i are distinct.

The Vandermonde matrix can come from the Lagrange interpolation problem where an nth order polynomial

$$p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_{n-1}t^{n-1} + a_n t^n \tag{C.164}$$

is used to represent the function that passes $n+1$ given points: $(t_0, y_0), (x_1, y_1), ..., (x_{n-1}, y_{n-1}), (x_n, y_n)$. The matrix form of the given problem is

$$\begin{bmatrix} 1 & t_0 & t_0^2 & ... & t_0^n \\ 1 & x_1 & x_1^2 & ... & x_1^n \\ 1 & x_2 & x_2^2 & ... & x_2^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & ... & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \tag{C.165}$$

When the number of parameters in the polynomial (order +1) matches the number of given points and the points are distinct, the above linear system can uniquely be solved for a_i ($\boldsymbol{a} = \boldsymbol{V}^{-1}\boldsymbol{y}$).

When the Fourier matrix (ignoring the normalization factor) is interpreted as a Vandermonde matrix, it is

$$\boldsymbol{F} = \boldsymbol{V}(\omega_N^0, \omega_N, ..., \omega_N^{(N-1)}) \tag{C.166}$$

C.4.3 An example

When $N = 4$, the forward Fourier transform is

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{C.167}$$

where the DFT matrix can be factorized as

$$\boldsymbol{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix} \tag{C.168}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\mathrm{i} \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & \mathrm{i} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{C.169}$$

$$= \boldsymbol{B}_4 \boldsymbol{F}_{2*2} \boldsymbol{P}_4 \tag{C.170}$$

where $\boldsymbol{P}_4$ is an odd-even sorting, permutation matrix, $\boldsymbol{F}_{2*2}$ contains two $N = 2$ DFT matrices

$$\boldsymbol{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{C.171}$$

and $\boldsymbol{B}_4$ is the butterfly matrix. We note that

$$\boldsymbol{F}_{2*2}\boldsymbol{P}_4 \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \\ f_1 + f_3 \\ f_1 - f_3 \end{bmatrix} \tag{C.172}$$

Here we can already see the reuse of a size $N/2$ DFT matrix and the sign of divide-and-conquer. These ideas will be shown more clearly in the following section.

Finally, we spell out the matrix multiplication

$$\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix} = \boldsymbol{B}_4 \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \\ f_1 + f_3 \\ f_1 - f_3 \end{bmatrix} = \begin{bmatrix} (f_0 + f_2) + 1 * (f_1 + f_3) \\ (f_0 - f_2) - \mathrm{i} * (f_1 + f_3) \\ (f_0 + f_2) - 1 * (f_1 + f_3) \\ (f_0 - f_2) + \mathrm{i} * (f_1 + f_3) \end{bmatrix} = \begin{bmatrix} \hat{f}_0^{\text{even}} + \omega_4^0 \hat{f}_0^{\text{odd}} \\ \hat{f}_1^{\text{even}} + \omega_4^1 \hat{f}_1^{\text{odd}} \\ \hat{f}_0^{\text{even}} - \omega_4^0 \hat{f}_0^{\text{odd}} \\ \hat{f}_1^{\text{even}} - \omega_4^1 \hat{f}_1^{\text{odd}} \end{bmatrix} \tag{C.173}$$

where

$$\begin{bmatrix} \hat{f}_0^{\text{even}} \\ \hat{f}_1^{\text{even}} \end{bmatrix} = \boldsymbol{F}_2 \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix} \tag{C.174}$$

and

$$\begin{bmatrix} \hat{f}_0^{\text{odd}} \\ \hat{f}_1^{\text{odd}} \end{bmatrix} = \boldsymbol{F}_2 \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix} \tag{C.175}$$

are the outputs of two size-$N/2$ DFTs of the even and odd elements of the input. This is a special case of (C.185)-(C.186). We can again see that $\hat{f}_k$ are nothing but linear combinations of f_j.

C.4.4 The idea of the FFT

First, we consider the time-complexity of/the operation counts for the DFT. Either the direct and the inverse DFT involves matrix multiplication of $\boldsymbol{F} \in \mathbb{C}^{N\times N}$ and a vector $\boldsymbol{f} \in \mathbb{R}^N$, requiring N^2 multiplications and $N(N-1)$ additions, totaling $O(N^2)$. The optimal operation count that can be achieved is $O(\frac{N}{2}\log_2 N)$, found by Cooley & Tukey (1965). We note that $\log_2 N$ grows almost like a constant when N is large, since it is beaten by any positive power of N. Hence, this scaling is also called the fast scaling. Below are the main ideas of the Cooley & Tukey (1965) FFT.

An observation can be made about the DFT:

$$\hat{f}_k = \sum_{j=0}^{N-1} F_{kj} f_j = \sum_{j=0}^{N-1} \omega_N^{kj} f_j, \tag{C.176}$$

where

$$\omega_N = e^{\mathrm{i}2\pi/N}. \tag{C.177}$$

When N is an even number, the above sum can be split into odd and even terms as

$$\hat{f}_k = \sum_{j=0}^{N/2-1} \omega_N^{k(2j)} f_{2j} + \sum_{j=0}^{N/2-1} \omega_N^{k(2j+1)} f_{2j+1} \tag{C.178}$$

and providing that

$$\omega_N^{2kj} = e^{\mathrm{i}2\pi 2kj/N} = e^{\mathrm{i}2\pi kj/(N/2)} = \omega_{N/2}^{kj}, \tag{C.179}$$

we have

$$\hat{f}_k = \sum_{j=0}^{N/2-1} \omega_{N/2}^{kj} f_{2j} + \omega_N^k \sum_{j=0}^{N/2-1} \omega_{N/2}^{kj} f_{2j+1}, \tag{C.180}$$

which is exactly the sum of two DFTs of size $N/2$, of the even and odd elements of the input f_j, respectively. It is naturally to think of a recursion that divides the problem into smaller and smaller, easier and easier to solve ones.

We note that the same formula (C.180) remains for $k = 0, 1, ..., N/2 - 1$, but some considerations are needed for $k = N/2, N/2 + 1, ..., N - 1$ since we want to constrain the weights $\omega_{N/2}^{kj}$ in $0 \leq k, j \leq N/2 - 1$. We can reuse (C.180) by a change of variable ($k = k' + N/2$, with $k' = 0, 1, ..., N/2 - 1$), such that, for $k = N/2, N/2 + 1, ..., N - 1$,

$$\hat{f}_k = \sum_{j=0}^{N/2-1} \omega_{N/2}^{(k'+N/2)j} f_{2j} + \omega_N^{k'+N/2} \sum_{j=0}^{N/2-1} \omega_{N/2}^{(k'+N/2)j} f_{2j+1}, \tag{C.181}$$

with another observation that

$$\omega_{N/2}^{(k'+N/2)j} = e^{-\mathrm{i}2\pi(k'+N/2)j/(N/2)} = e^{-\mathrm{i}2\pi k'j/(N/2)} e^{-\mathrm{i}2\pi j} = e^{-\mathrm{i}2\pi k'j/(N/2)} = \omega_{N/2}^{k'j} \tag{C.182}$$

since $j = 0, 1, ..., N/2 - 1$. Similarly, we also have

$$\omega_N^{k'+N/2} = -\omega_N^{k'j}. \tag{C.183}$$

Thus, the above equation (C.181) can be simplified as

$$\hat{f}_k = \sum_{j=0}^{N/2-1} \omega_{N/2}^{k'j} f_{2j} - \omega_N^{k'} \sum_{j=0}^{N/2-1} \omega_{N/2}^{k'j} f_{2j+1}. \tag{C.184}$$

In summary, a formal representation is

$$\hat{f}_k^{\text{upper}} = \hat{f}_k^{\text{odd}} + \omega_N^k \hat{f}_k^{\text{even}} \, (0 \leq k \leq N/2 - 1) \tag{C.185}$$

$$\hat{f}_{k'+N/2}^{\text{lower}} = \hat{f}_{k'}^{\text{odd}} - \omega_N^{k'} \hat{f}_{k'}^{\text{even}} \, (N/2 \leq k \leq N - 1, \, 0 \leq k' \leq N/2 - 1) \tag{C.186}$$

and such a structure, when written in matrix form, is call butterfly.

C.4.5 Miscellaneous

Note that matrix multiplication is not the only thing to do. If we just need to compute $\hat{\boldsymbol{f}}$ at several frequencies of interest, we can incrementally update (C.118) on-the-fly with the newest available snapshot multiplied with an appropriate weight. The time complexity is not favorable, partly compensated by fewer computed frequencies, but more importantly there is memory saving on the other hand, if there are large snapshots being FT'ed in one of the dimensions. This method is called streaming FFT.

Appendix D

Coordinate systems and their transformations

D.1 Cylindrical coordinates

Consider the cylindrical transformation

$$(x, y) \to (r, \theta) \tag{D.1}$$

where

$$x = r\cos\theta \tag{D.2}$$
$$y = r\sin\theta \tag{D.3}$$

or

$$r = \sqrt{x^2 + y^2} \tag{D.4}$$
$$\theta = \text{actan}\left(\frac{y}{x}\right) \tag{D.5}$$

we have the corresponding relation between unit vectors

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{D.6}$$

and

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix}, \tag{D.7}$$

which can also be proven graphically. The transformation matrices in (D.6) and (D.7) are inverse to each other, and both are unitary. That being said, they are rotation matrices with $\det() = 1$.

The Jacobian of the forward transformation $(r, \theta) = F(x, y)$ is

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix} \tag{D.8}$$

and of the backward transformation $(x, y) = G(r, \theta)$ is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \tag{D.9}$$

and it can be seen with a bit of algebra that these two are the inverse of each other.

We note that the directions of the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ depend on space, i.e.,

$$\frac{\partial \boldsymbol{e}_r}{\partial r} = \frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0 \tag{D.10}$$

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = -\sin\theta \boldsymbol{e}_x + \cos\theta \boldsymbol{e}_y = \boldsymbol{e}_\theta \tag{D.11}$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\cos\theta \boldsymbol{e}_x - \sin\theta \boldsymbol{e}_y = -\boldsymbol{e}_r \tag{D.12}$$

which can also be seen graphically. These relations are crucial to later derivations.

Consider the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{D.13}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{D.14}$$

here we proof the cylindrical coordinate representation of $\nabla \cdot \boldsymbol{u} = 0$. Assume $\boldsymbol{u} = u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \rightarrow \begin{cases} u = \dot{x} = \frac{\partial}{\partial t}(r\cos\theta) = u_r\cos\theta - u_\theta\sin\theta \\ v = \dot{y} = \frac{\partial}{\partial t}(r\sin\theta) = u_r\sin\theta + u_\theta\cos\theta \end{cases} \tag{D.15}$$

We have prepared

$$\begin{cases} \frac{\partial r}{\partial x} = \frac{x}{r} \\ \frac{\partial r}{\partial y} = \frac{y}{r} \\ \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \\ \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \end{cases} . \tag{D.16}$$

Starting from

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{D.17}$$

we use the chain rule to replace all coordinates and velocity components with their cylindrical counterparts, leading to (with a bit of algebra)

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \tag{D.18}$$

D.1.1 Operators in cylindrical coordinates

For a scalar function, say $f(x, y) = f(r, \theta)$, the gradient operator can be expressed as

$$\nabla = \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial}{\partial z} \tag{D.19}$$

$$= \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\right)(\cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta) + \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\right)(\sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta) + \boldsymbol{e}_z\frac{\partial}{\partial z} \tag{D.20}$$

$$= \boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z} \tag{D.21}$$

The factor $r\partial\theta$ can be interpreted as infinitesimal length element in θ direction.

The Laplace operator

$$\nabla^2 = \nabla\cdot\nabla = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right) \tag{D.22}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \boldsymbol{e}_\theta\cdot\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\right)\right] \tag{D.23}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{D.24}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{D.25}$$

Now consider a vector

$$\boldsymbol{u} = \boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w \tag{D.26}$$

and its derivatives.

Its divergence is

$$\nabla\cdot\boldsymbol{u} = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right)\cdot(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{D.27}$$

$$= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{D.28}$$

$$= \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{D.29}$$

The convection term

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{D.30}$$

$$= \left(u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right)\boldsymbol{e}_r \tag{D.31}$$

$$+ \left(u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right)\boldsymbol{e}_\theta \tag{D.32}$$

$$+ \left(u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\right)\boldsymbol{e}_z \tag{D.33}$$

Now we deal with $\nabla^2\boldsymbol{u}$.

$$\nabla^2\boldsymbol{u} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{D.34}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{u}}{\partial r}\right) + \frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v) \tag{D.35}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right)\boldsymbol{e}_r \tag{D.36}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right)\boldsymbol{e}_\theta \tag{D.37}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)\boldsymbol{e}_z \tag{D.38}$$

with

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_r u) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \boldsymbol{e}_r u}{\partial \theta} = \frac{1}{r^2}(2\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta - u\boldsymbol{e}_r + \frac{\partial^2 u}{\partial \theta^2}\boldsymbol{e}_r) \tag{D.39}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_\theta v) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \boldsymbol{e}_\theta v}{\partial \theta} = \frac{1}{r^2}(-2\frac{\partial v}{\partial \theta}\boldsymbol{e}_r - v\boldsymbol{e}_\theta + \frac{\partial^2 v}{\partial \theta^2}\boldsymbol{e}_\theta) \tag{D.40}$$

Moreover, the curl can be established (in a compact form) as

$$\nabla \times \boldsymbol{u} = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right) \times (\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{D.41}$$

$$= \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \frac{1}{r}\partial_\theta & \partial_z \\ u & v & w \end{vmatrix} + \frac{1}{r}\boldsymbol{e}_\theta \times \frac{\partial(v\boldsymbol{e}_\theta)}{\partial \theta} \tag{D.42}$$

$$= \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\boldsymbol{e}_\theta + \left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\boldsymbol{e}_z \tag{D.43}$$

$$= \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta}\right)\boldsymbol{e}_z \tag{D.44}$$

$$= \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix}. \tag{D.45}$$

Examples.

1. Axisymmetric flows ($u_\theta \neq 0$). We have the $\omega_z = 1/r\partial_r(ru_\theta) = u_\theta/r + \partial_r u_\theta$. The first term is called the curvature vorticity and the second called the shear vorticity.

2. Rigid body rotation with angular velocity Ω and $u_\theta = v = \Omega r$. Vorticity $\omega_z = 2\Omega$ (with equal contributions from curvature and shear) but there is no vortical motion.

3. Potential point vortex with $u_\theta = v = \Gamma/2\pi r$. Vorticity $\omega_z = 0$ according to (D.44) – there is no vorticity.

Now we turn our attention to velocity gradient and strain-rate tensors. In Cartesion coordinates,

$$\nabla\boldsymbol{u} = (\partial_i\boldsymbol{e}_i)(u_j\boldsymbol{e}_j) = \partial_i u_j \boldsymbol{e}_i\boldsymbol{e}_j. \tag{D.46}$$

In cylindrical coordinates,

$$\nabla\boldsymbol{u} = (\boldsymbol{e}_r\partial_r + \boldsymbol{e}_\theta 1/r\partial_\theta + \boldsymbol{e}_z\partial_z)(\boldsymbol{e}_r u_r + \boldsymbol{e}_\theta u_\theta + \boldsymbol{e}_z u_z) \tag{D.47}$$

$$= (\boldsymbol{e}_r\boldsymbol{e}_r\partial_r u + \boldsymbol{e}_r\boldsymbol{e}_\theta\partial_r v + \boldsymbol{e}_r\boldsymbol{e}_z\partial_r w) \tag{D.48}$$

$$+ [(\boldsymbol{e}_\theta\boldsymbol{e}_r 1/r\partial_\theta u + \boldsymbol{e}_\theta\boldsymbol{e}_\theta u/r) + (\boldsymbol{e}_\theta\boldsymbol{e}_\theta 1/r\partial_\theta v - \boldsymbol{e}_\theta\boldsymbol{e}_r v/r) + \boldsymbol{e}_\theta\boldsymbol{e}_z 1/r\partial_\theta w] \tag{D.49}$$

$$+ (\boldsymbol{e}_z\boldsymbol{e}_r\partial_z u + \boldsymbol{e}_z\boldsymbol{e}_\theta\partial_z v + \boldsymbol{e}_z\boldsymbol{e}_z\partial_z w) \tag{D.50}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{1}{r}\frac{\partial w}{\partial \theta} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{D.51}$$

and the rate-of-strain tensor is

$$\boldsymbol{S} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \\ ... & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}\right) \\ ... & ... & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ ... & S_{\theta\theta} & S_{\theta z} \\ ... & ... & S_{zz} \end{bmatrix} \tag{D.52}$$

D.1.2 Navier–Stokes in cylindrical coordinates

With the preparation in the previous section, we are now able to write down the Navier–Stokes equations in cylindrical coordinates as

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{D.53}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{D.54}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{D.55}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{D.56}$$

Q.E.D.

D.1.3 Scalar equation in cylindrical coordinates

Additionally, the transport equation of a passive scalar (say ρ)

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho = \nu\nabla^2\rho \tag{D.57}$$

can be cast in cylinderical coordinate as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{v}{r}\frac{\partial \rho}{\partial \theta} + w\frac{\partial \rho}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \rho}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial^2 \rho}{\partial z^2}\right] \tag{D.58}$$

D.1.4 Vorticity equation in cylindrical coordinates

The divergence-free condition for vorticity reads

$$\nabla \cdot \boldsymbol{\omega} = \frac{1}{r}\frac{\partial (r\omega_r)}{\partial r} + \frac{1}{r}\frac{\partial \omega_\theta}{\partial \theta} + \frac{\partial \omega_z}{\partial z} = 0. \tag{D.59}$$

The convective term is

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)(\boldsymbol{e}_r\omega_r + \boldsymbol{e}_\theta\omega_\theta + \boldsymbol{e}_z\omega_z) \tag{D.60}$$

$$= \left(u\frac{\partial \omega_r}{\partial r} + \frac{v}{r}\frac{\partial \omega_r}{\partial \theta} + w\frac{\partial \omega_r}{\partial z} - \frac{v\omega_\theta}{r}\right)\boldsymbol{e}_r \tag{D.61}$$

$$+ \left(u\frac{\partial \omega_\theta}{\partial r} + \frac{v}{r}\frac{\partial \omega_\theta}{\partial \theta} + w\frac{\partial \omega_\theta}{\partial z} + \frac{v\omega_r}{r}\right)\boldsymbol{e}_\theta \tag{D.62}$$

$$+ \left(u\frac{\partial \omega_z}{\partial r} + \frac{v}{r}\frac{\partial \omega_z}{\partial \theta} + w\frac{\partial \omega_z}{\partial z}\right)\boldsymbol{e}_z. \tag{D.63}$$

The stretching term is

$$(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \left(\omega_r\frac{\partial}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial}{\partial \theta} + \omega_z\frac{\partial}{\partial z}\right)(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{D.64}$$

$$= \left(\omega_r\frac{\partial u}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial u}{\partial \theta} + \omega_z\frac{\partial u}{\partial z} - \frac{v\omega_\theta}{r}\right)\boldsymbol{e}_r \tag{D.65}$$

$$+ \left(\omega_r\frac{\partial v}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial v}{\partial \theta} + \omega_z\frac{\partial v}{\partial z} + \frac{u\omega_\theta}{r}\right)\boldsymbol{e}_\theta \tag{D.66}$$

$$+ \left(\omega_r\frac{\partial w}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial w}{\partial \theta} + \omega_z\frac{\partial w}{\partial z}\right)\boldsymbol{e}_z. \tag{D.67}$$

The diffusion term is

$$\nabla^2\boldsymbol{\omega} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \omega_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial \omega_\theta}{\partial \theta} - \frac{\omega_r}{r^2} + \frac{\partial^2 \omega_r}{\partial z^2}\right)\boldsymbol{e}_r \tag{D.68}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \omega_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial \omega_r}{\partial \theta} - \frac{\omega_\theta}{r^2} + \frac{\partial^2 \omega_\theta}{\partial z^2}\right)\boldsymbol{e}_\theta \tag{D.69}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \omega_z}{\partial \theta^2} + \frac{\partial^2 \omega_z}{\partial z^2}\right)\boldsymbol{e}_z \tag{D.70}$$

Hence, the transport equation of vorticity, (2.21), in the absense of non-conservative external force and baroclinic torque, is cast in cylindrical coordinates as

$$\frac{\partial \omega_r}{\partial t} + u\frac{\partial \omega_r}{\partial r} + \frac{v}{r}\frac{\partial \omega_r}{\partial \theta} + w\frac{\partial \omega_r}{\partial z} = \omega_r\frac{\partial u}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial u}{\partial \theta} + \omega_z\frac{\partial u}{\partial z} \tag{D.71}$$

$$+ \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \omega_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial \omega_\theta}{\partial \theta} - \frac{\omega_r}{r^2} + \frac{\partial^2 \omega_r}{\partial z^2}\right) \tag{D.72}$$

$$\frac{\partial \omega_\theta}{\partial t} + u\frac{\partial \omega_\theta}{\partial r} + \frac{v}{r}\frac{\partial \omega_\theta}{\partial \theta} + w\frac{\partial \omega_\theta}{\partial z} + \frac{v\omega_r}{r} = \omega_r\frac{\partial v}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial v}{\partial \theta} + \omega_z\frac{\partial v}{\partial z} + \frac{u\omega_\theta}{r} \tag{D.73}$$

$$
+ \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \omega_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial \omega_r}{\partial \theta} - \frac{\omega_\theta}{r^2} + \frac{\partial^2 \omega_\theta}{\partial z^2} \right) \tag{D.74}
$$

$$
\frac{\partial \omega_z}{\partial t} + u \frac{\partial \omega_z}{\partial r} + \frac{v}{r} \frac{\partial \omega_z}{\partial \theta} + w \frac{\partial \omega_z}{\partial z} = \omega_r \frac{\partial w}{\partial r} + \frac{\omega_\theta}{r} \frac{\partial w}{\partial \theta} + \omega_z \frac{\partial w}{\partial z} \tag{D.75}
$$

$$
+ \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \omega_z}{\partial \theta^2} + \frac{\partial^2 \omega_z}{\partial z^2} \right) \tag{D.76}
$$

D.1.5 Vorticity–streamfunction in 3D axisymmetric flow

With $\partial_\theta = 0$, the continuity equation simplifies to

$$
\frac{\partial (ru)}{\partial r} + \frac{\partial wr}{\partial z} = 0 \tag{D.77}
$$

such that an axisymmetric (Stokes) streamfunction can be defined as

$$
u = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \; w = \frac{1}{r} \frac{\partial \psi}{\partial r}. \tag{D.78}
$$

Further assuming swirl-free $v = 0$, the vorticity–streamfunction equations are

$$
\frac{\partial \omega_\theta}{\partial t} + u \frac{\partial \omega_\theta}{\partial r} + w \frac{\partial \omega_\theta}{\partial z} = \frac{u \omega_\theta}{r} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega_\theta}{\partial r} \right) - \frac{\omega_\theta}{r^2} + \frac{\partial^2 \omega_\theta}{\partial z^2} \right) \tag{D.79}
$$

$$
-r \omega_\theta = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} \tag{D.80}
$$

Also see Panton (2013), appendix 4 for a more complete list of streamfunctions in other coordinates.

D.2 Spherical coordinate

Consider the transformation

$$
(x, y, z) \to (r, \phi, \theta) \tag{D.81}
$$

where

$$
x = r \sin\phi \cos\theta \tag{D.82}
$$

$$
y = r \sin\phi \sin\theta \tag{D.83}
$$

$$
z = r \cos\phi \tag{D.84}
$$

or

$$
r = \sqrt{x^2 + y^2 + z^2} \tag{D.85}
$$

$$
\phi = \arctan \left(\frac{\sqrt{x^2 + y^2}}{z} \right) \tag{D.86}
$$

$$
\theta = \arctan \left(\frac{y}{x} \right) \tag{D.87}
$$

or thought of as from cylindrical with

$$z = r\cos\phi \tag{D.88}$$
$$r' = r\sin\phi \tag{D.89}$$
$$x = r'\sin\theta \tag{D.90}$$
$$y = r'\cos\theta \tag{D.91}$$

Here θ is the azimuthal angle with x-axis on the equatorial plane and ϕ is the polar angle with z-axis (North), for the convenience of going from cylindrical to polar and backwards.

We have the corresponding relation between unit vectors

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{D.92}$$

and

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}, \tag{D.93}$$

which can be proven graphically. We note that the grid transformation matrix is unitary and has $\det() = 1$ (rotation matrix).

D.2.1 From cylindrical to spherical

We have the transformation

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{D.94}$$

that can be factorized as

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{D.95}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{D.96}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{r'} \\ \boldsymbol{e}_{\theta'} \\ \boldsymbol{e}_{z'} \end{bmatrix} \tag{D.97}$$

with

$$\begin{bmatrix} \boldsymbol{e}_{r'} \\ \boldsymbol{e}_{\theta'} \\ \boldsymbol{e}_{z'} \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{r} \\ \boldsymbol{e}_{\phi} \\ \boldsymbol{e}_{\theta} \end{bmatrix}. \tag{D.98}$$

D.3 General curvilinear coordinates

Consider the coordinate transformations

$$q_i = q_i(x_1, x_2, x_3),\ x_i = x_i(q_1, q_2, q_3) \tag{D.99}$$

where (x_1, x_2, x_3) is the standard Cartesian coordinates and q_i are mutually independent.

D.3.1 Length, area, and volume

Consider the change of the vector

$$\boldsymbol{x} = x_1\boldsymbol{e}_{x_1} + x_2\boldsymbol{e}_{x_2} + x_3\boldsymbol{e}_{x_3} \tag{D.100}$$

$$= q_1\boldsymbol{h}_1 + q_2\boldsymbol{h}_2 + q_3\boldsymbol{h}_3 \tag{D.101}$$

where $\boldsymbol{x} = \boldsymbol{x}(x_i(q_j))$ as

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{e}_{x_1}\mathrm{d}x_1 + \boldsymbol{e}_{x_2}\mathrm{d}x_2 + \boldsymbol{e}_{x_3}\mathrm{d}x_3 \tag{D.102}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial \boldsymbol{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial \boldsymbol{x}}{\partial q_3}\mathrm{d}q_3 \tag{D.103}$$

and

$$\boldsymbol{h}_i = \frac{\partial \boldsymbol{x}}{\partial q_i}. \tag{D.104}$$

We note that $\boldsymbol{h}_i$ is the change of $\boldsymbol{x}$ with only changing q_i, so it does define direction of coordinate lines of q_i. We denote with $\hat{(\cdot)}$ unit vectors and note that $\boldsymbol{h}_i$ are not necessary unit vectors.

Now consider the length of $\mathrm{d}\boldsymbol{x}$:

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} \tag{D.105}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_j}\mathrm{d}q_j \cdot \frac{\partial \boldsymbol{x}}{\partial q_k}\mathrm{d}q_k \tag{D.106}$$

$$= \frac{\partial x_i}{\partial q_j}\mathrm{d}q_j \frac{\partial x_i}{\partial q_k}\mathrm{d}q_k \tag{D.107}$$

$$= g_{jk}\mathrm{d}q_j\mathrm{d}q_k \tag{D.108}$$

with

$$g_{ij} = \frac{\partial x_l}{\partial q_i}\frac{\partial x_l}{\partial q_j} \tag{D.109}$$

being the metric tensor. When q_i are orthogonal coordinates,

$$\frac{\partial \boldsymbol{x}}{\partial q_i} \cdot \frac{\partial \boldsymbol{x}}{\partial q_j} = \delta_{ij} \tag{D.110}$$

and g_{ij} only has diagonal elements and

$$\begin{aligned} \mathrm{d}s^2 &= g_{11}(\mathrm{d}q_1)^2 + g_{22}(\mathrm{d}q_3)^2 + g_{33}(\mathrm{d}q_3)^2 &\text{(D.111)}\\ &= h_1^2(\mathrm{d}q_1)^2 + h_2^2(\mathrm{d}q_3)^2 + h_3^2(\mathrm{d}q_3)^2 &\text{(D.112)}\end{aligned}$$

Define the Lamé parameters as

$$\begin{aligned} h_1 &= \sqrt{g_{11}} = \sqrt{\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2} = |\boldsymbol{h}_1| &\text{(D.113)}\\ h_2 &= \sqrt{g_{22}} = \sqrt{\left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2} = |\boldsymbol{h}_2| &\text{(D.114)}\\ h_3 &= \sqrt{g_{33}} = \sqrt{\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2} = |\boldsymbol{h}_3| &\text{(D.115)}\end{aligned}$$

and unit vectors in q_i directions as

$$\boldsymbol{h}_i = \frac{\boldsymbol{h}_i}{|\boldsymbol{h}_i|} = \frac{\boldsymbol{h}_i}{h_i}. \tag{D.116}$$

We note that the Lamé parameters can depend on the coordinates as

$$h_i = h_i(q_1, q_2, q_3). \tag{D.117}$$

The increment can be rewritten as

$$\begin{aligned} \mathrm{d}\boldsymbol{x} &= h_1\mathrm{d}q_1\boldsymbol{h}_1 + h_2\mathrm{d}q_2\boldsymbol{h}_2 + h_3\mathrm{d}q_3\boldsymbol{h}_3 &\text{(D.118)}\\ &= \mathrm{d}s_1\boldsymbol{h}_1 + \mathrm{d}s_2\boldsymbol{h}_2 + \mathrm{d}s_3\boldsymbol{h}_3 &\text{(D.119)}\end{aligned}$$

with

$$\mathrm{d}s_i \tag{D.120}$$

being the projection of d$\boldsymbol{x}$ on each coordinate.

Now consider the surface and volume of infinitesimal elements. The (directed) areas of surface elements are

$$\mathrm{d}\sigma_i = \boldsymbol{h}_i \cdot (h_j\mathrm{d}q_j\boldsymbol{h}_j \times h_k\mathrm{d}q_k\boldsymbol{h}_k) = h_jh_k\mathrm{d}q_j\mathrm{d}q_k \tag{D.121}$$

or

$$\begin{aligned} \mathrm{d}\sigma_1 &= h_1h_2\mathrm{d}q_1\mathrm{d}q_2 &\text{(D.122)}\\ \mathrm{d}\sigma_2 &= h_1h_3\mathrm{d}q_1\mathrm{d}q_3 &\text{(D.123)}\\ \mathrm{d}\sigma_3 &= h_1h_2\mathrm{d}q_1\mathrm{d}q_2 &\text{(D.124)}\end{aligned}$$

The volume element (e.g. in volume integrals) spanned by the vector $\mathrm{d}\boldsymbol{x}$ is

$$
\begin{aligned}
\mathrm{d}V &= (h_1 \mathrm{d}q_1 \boldsymbol{h}_1) \cdot (h_2 \mathrm{d}q_2 \boldsymbol{h}_2 \times h_3 \mathrm{d}q_3 \boldsymbol{h}_3) && \text{(D.125)} \\
&= h_1 \mathrm{d}q_1 h_2 \mathrm{d}q_2 h_3 \mathrm{d}q_3 (\boldsymbol{h}_1) \cdot (\boldsymbol{h}_2 \times \boldsymbol{h}_3) && \text{(D.126)} \\
&= h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3 && \text{(D.127)}
\end{aligned}
$$

when $\boldsymbol{h}_i$ mutually orthogonal.

Example.

For cylindrical coordinate, by definition,

$$
\begin{aligned}
h_1 &= \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1 && \text{(D.128)} \\
h_2 &= \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r && \text{(D.129)} \\
h_3 &= \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1 && \text{(D.130)}
\end{aligned}
$$

D.3.2 Jacobian

Now we consider the Jacobian of the backward transformation

$$
(q_1, q_2, q_3) \to (x_1, x_2, x_3) \qquad \text{(D.131)}
$$

which reads

$$
\boldsymbol{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{bmatrix} \qquad \text{(D.132)}
$$

and the Jacobian determinant (with $\exists\, \boldsymbol{J}^{-1}$)

$$
\begin{aligned}
J &= \det(\boldsymbol{J}) = \det(\boldsymbol{J}^{\mathrm{T}}) && \text{(D.133)} \\
&= \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} && \text{(D.134)} \\
&= \left(\frac{\partial x_1}{\partial q_1}\boldsymbol{x}_1 + \frac{\partial x_2}{\partial q_1}\boldsymbol{x}_2 + \frac{\partial x_3}{\partial q_1}\boldsymbol{x}_3\right) \cdot \begin{vmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} && \text{(D.135)} \\
&= \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial q_2} \times \frac{\partial \boldsymbol{x}}{\partial q_3}\right) && \text{(D.136)} \\
&= \boldsymbol{h}_1 \cdot (\boldsymbol{h}_2 \times \boldsymbol{h}_3) && \text{(D.137)}
\end{aligned}
$$

$$= h_1 h_2 h_3 \tag{D.138}$$
$$\neq 0 \tag{D.139}$$

Hence we have

$$\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = h_1h_2h_3\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3 = J\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3. \tag{D.140}$$

D.3.3 Three major calculus theorems

Here we review the three major theorems of calculus and in the next section we will extend them in general curvilinear coordinates.

1. Gradient theorem:
$$\int_{\boldsymbol{l}:\boldsymbol{x}_1\to\boldsymbol{x}_2} (\nabla f)\cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1) \tag{D.141}$$
The integral is independent of path since ∇f is potential (conservative, curl-free).

2. Divergence theorem:
$$\iint_\Omega (\nabla\times\boldsymbol{u})\cdot \mathrm{d}\boldsymbol{A} = \oint_{l=\partial\Omega} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l} \tag{D.142}$$
Implication: vorticity is circulation per unit area.

3. Curl theorem:
$$\iiint_V (\nabla\cdot\boldsymbol{u})\,\mathrm{d}V = \iint_{\Omega=\partial V} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{A} \tag{D.143}$$

D.3.4 Differential operators in curvilinear coordinate systems

Next, let's consider differential operators in curvilinear coordinates. Consider a scalar $f = f(q_1, q_2, q_3)$ and its gradient ∇f. Starting from

$$\mathrm{d}f = \frac{\partial f}{\partial q_i}\mathrm{d}q_i, \tag{D.144}$$

due to the displacement $\boldsymbol{x}$. On the other hand,

$$\mathrm{d}f = \nabla f\cdot \mathrm{d}\boldsymbol{x} \tag{D.145}$$
$$= (\nabla f)_{q_i}\mathrm{d}s_i \tag{D.146}$$
$$= (\nabla f)_{q_i} h_i \mathrm{d}q_i \tag{D.147}$$

Compare (D.147) and (D.144) we have

$$(\nabla f)_{q_i} = \frac{1}{h_i}\frac{\partial f}{\partial q_i} \tag{D.148}$$

where

$$\nabla f = (\nabla f)_{q_i}\boldsymbol{h}_i \tag{D.149}$$
$$= \frac{1}{h_1}\frac{\partial f}{\partial q_1}\boldsymbol{h}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\boldsymbol{h}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\boldsymbol{h}_3 \tag{D.150}$$

Consider the divergence of a vector $\boldsymbol{u}$ in a coordinate-free form:

$$\nabla \cdot \boldsymbol{u} \triangleq \lim_{V \to 0} \frac{\oint_{\Omega = \partial V} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{\sigma}}{V} \tag{D.151}$$

$$= \frac{1}{V}\left(\frac{\partial(u_1 h_2 h_3 \mathrm{d}q_2 \mathrm{d}q_3)}{\partial q_1}\mathrm{d}q_1 + \frac{\partial(u_2 h_1 h_3 \mathrm{d}q_1 \mathrm{d}q_3)}{\partial q_2}\mathrm{d}q_2 + \frac{\partial(u_3 h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2)}{\partial q_3}\mathrm{d}q_3\right) \tag{D.152}$$

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(u_1 h_2 h_3)}{\partial q_1} + \frac{\partial(u_2 h_1 h_3)}{\partial q_2} + \frac{\partial(u_3 h_1 h_2)}{\partial q_3}\right) \tag{D.153}$$

Consider the curl of a vector $\boldsymbol{u}$ in a coordinate-free form, its compoment along $\boldsymbol{n}$ (normal of the surface $\boldsymbol{S} = S\boldsymbol{n}$) is

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} \triangleq \lim_{S \to 0} \frac{\int_{l = \partial S} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}}{S} \tag{D.154}$$

and (consider the area spanned by $\mathrm{d}s_2 = h_2 \mathrm{d}q_2$ and $\mathrm{d}s_3 = h_3 \mathrm{d}q_3$)

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_1 = \frac{\oint_l \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}}{\mathrm{d}\sigma_1} \tag{D.155}$$

$$= \frac{1}{h_2 h_3 \mathrm{d}q_2 \mathrm{d}q_3}[u_2 h_2 \mathrm{d}q_2 \tag{D.156}$$

$$- (u_2 h_2 + \frac{\partial u_2 h_2}{\partial q_3}\mathrm{d}q_3)\mathrm{d}q_2 \tag{D.157}$$

$$- u_3 h_3 \mathrm{d}q_3 \tag{D.158}$$

$$+ (u_3 h_3 + \frac{\partial u_3 h_3}{\partial q_2}\mathrm{d}q_2)\mathrm{d}q_3] \tag{D.159}$$

$$= \frac{1}{h_2 h_3}\left(\frac{\partial u_3 h_3}{\partial q_2} - \frac{\partial u_2 h_2}{\partial q_3}\right) \tag{D.160}$$

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_2 = \frac{1}{h_1 h_3}\left(\frac{\partial u_1 h_1}{\partial q_3} - \frac{\partial u_3 h_3}{\partial q_1}\right) \tag{D.161}$$

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_3 = \frac{1}{h_1 h_2}\left(\frac{\partial u_2 h_2}{\partial q_1} - \frac{\partial u_1 h_1}{\partial q_2}\right) \tag{D.162}$$

We note that the Lamé coefficient also changes as the coordinate changes.

In determinant form,

$$\nabla \times \boldsymbol{u} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \boldsymbol{h}_1 & h_2 \boldsymbol{h}_2 & h_3 \boldsymbol{h}_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ h_1 u_1 & h_2 u_2 & h_3 u_3 \end{vmatrix} \tag{D.163}$$

The Laplacian can be obtained by taking the divergence of ∇f as combining (D.150) and (D.153)

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right) \tag{D.164}$$

D.3.5 Derivatives of unit vectors

In general curvilinear coordinates, the directions of unit vectors could change with coordinate as well. We are basically concerned about

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \tag{D.165}$$

and we will establish that

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} // \boldsymbol{h}_j,\ i \neq j. \tag{D.166}$$

First we have

$$\boldsymbol{h}_i \cdot \frac{\partial \boldsymbol{h}_i}{\partial q_j} = \frac{\partial \boldsymbol{h}_i^2/2}{\partial q_j} = 0 \tag{D.167}$$

and hence

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \perp \boldsymbol{h}_i,\ i \neq j. \tag{D.168}$$

According to the orthogonality we have

$$\boldsymbol{h}_1 \cdot \boldsymbol{h}_2 = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2} = 0 \tag{D.169}$$

and

$$0 = \frac{\partial}{\partial q_3}\left(\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2}\right) = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} + \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_3} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2} \tag{D.170}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_1} + \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_3} \tag{D.171}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} + \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_2} \cdot \frac{\partial \boldsymbol{x}}{\partial q_1} \tag{D.172}$$

then

$$\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} + \frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_1} + \frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} = 0 \tag{D.173}$$

and then

$$\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} = 0 \tag{D.174}$$

$$\frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_3} = 0 \tag{D.175}$$

$$\frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} = 0 \tag{D.176}$$

From

$$0 = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial}{\partial q_2}\left(\frac{\partial \boldsymbol{x}}{\partial q_3}\right) \tag{D.177}$$

$$= h_1 \boldsymbol{h}_1 \cdot \frac{\partial h_3 \boldsymbol{h}_3}{\partial q_2} \tag{D.178}$$

$$= h_1 \boldsymbol{h}_1 \cdot \left(h_3 \frac{\partial \boldsymbol{h}_3}{\partial q_2} + \boldsymbol{h}_3 \frac{\partial h_3}{\partial q_2} \right) \tag{D.179}$$

$$= h_1 h_3 \boldsymbol{h}_1 \cdot \frac{\partial \boldsymbol{h}_3}{\partial q_2} \tag{D.180}$$

we have (similarly)

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \perp \boldsymbol{h}_k,\ i \neq j \neq k \neq i. \tag{D.181}$$

Combining (D.168) and (D.181) we have

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} // \boldsymbol{h}_j,\ i \neq j, \tag{D.182}$$

and using

$$\frac{\partial^2 \boldsymbol{x}}{\partial q_i \partial q_j} = \frac{\partial^2 \boldsymbol{x}}{\partial q_j \partial q_i} \tag{D.183}$$

we have

$$\frac{\partial}{\partial q_j} \left(\frac{\partial \boldsymbol{x}}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left(\frac{\partial \boldsymbol{x}}{\partial q_j} \right) \tag{D.184}$$

$$\boldsymbol{h}_i \frac{\partial h_i}{\partial q_j} + h_i \frac{\partial \boldsymbol{h}_i}{\partial q_j} = \boldsymbol{h}_j \frac{\partial h_j}{\partial q_i} + h_j \frac{\partial \boldsymbol{h}_j}{\partial q_i} \tag{D.185}$$

with repeated indices not implying summation. Since $i \neq j$, $\boldsymbol{h}_i$ and $\boldsymbol{h}_j$ are linearly independent, we have

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} = \frac{1}{h_i} \frac{\partial h_j}{\partial q_i} \boldsymbol{h}_j. \tag{D.186}$$

Now we turn back and consider $\partial \boldsymbol{h}_i / \partial q_j$.

$$\frac{\partial \boldsymbol{h}_i}{\partial q_i} = \frac{\partial (\boldsymbol{h}_j \times \boldsymbol{h}_k)}{\partial q_i} \tag{D.187}$$

$$= \frac{\partial \boldsymbol{h}_j}{\partial q_i} \times \boldsymbol{h}_k + \boldsymbol{h}_j \times \frac{\partial \boldsymbol{h}_k}{\partial q_i} \tag{D.188}$$

$$= \frac{1}{h_j} \frac{\partial h_i}{\partial q_j} \boldsymbol{h}_i \times \boldsymbol{h}_k + \boldsymbol{h}_j \times \boldsymbol{h}_i \frac{1}{h_k} \frac{\partial h_i}{\partial q_k} \tag{D.189}$$

$$= -\left(\frac{1}{h_j} \frac{\partial h_i}{\partial q_j} \boldsymbol{h}_j + \frac{1}{h_k} \frac{\partial h_i}{\partial q_k} \boldsymbol{h}_k\right) \tag{D.190}$$

without repeated indices being summed over.

Using the relations (D.186) and (D.190), gradient, curl, divergence, Laplacian, as well as operators like $\nabla \boldsymbol{u}$ and $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ can be expressed.

Example: $\nabla \cdot \boldsymbol{u}$.

We have before

$$\nabla = \frac{\boldsymbol{h}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\boldsymbol{h}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\boldsymbol{h}_3}{h_3} \frac{\partial}{\partial q_3} \tag{D.191}$$

and now consider $\nabla \boldsymbol{u}$ with

$$\boldsymbol{u} = u_1 \boldsymbol{h}_1 + u_2 \boldsymbol{h}_2 + u_3 \boldsymbol{h}_3 \tag{D.192}$$

and we have

$$\nabla \cdot \boldsymbol{u} = \left(\frac{\boldsymbol{h}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\boldsymbol{h}_2}{h_2} \frac{\partial}{\partial q_2} \boldsymbol{h}_2 + \frac{\boldsymbol{h}_3}{h_3} \frac{\partial}{\partial q_3} \boldsymbol{h}_3 \right) \cdot (u_1 \boldsymbol{h}_1 + u_2 \boldsymbol{h}_2 + u_3 \boldsymbol{h}_3) \tag{D.193}$$

$$= \frac{\boldsymbol{h}_1}{h_1} \frac{\partial}{\partial q_1} (u_1 \boldsymbol{h}_1 + u_2 \boldsymbol{h}_2 + u_3 \boldsymbol{h}_3) + ... \tag{D.194}$$

$$= \frac{1}{h_1} \left(\frac{\partial u_1}{\partial q_1} + \frac{u_2}{h_2} \frac{\partial h_1}{\partial q_2} + \frac{u_3}{h_3} \frac{\partial h_1}{\partial q_3} \right) \tag{D.195}$$

$$+ \frac{1}{h_2} \left(\frac{\partial u_2}{\partial q_2} + \frac{u_3}{h_3} \frac{\partial h_2}{\partial q_3} + \frac{u_1}{h_1} \frac{\partial h_2}{\partial q_1} \right) \tag{D.196}$$

$$+ \frac{1}{h_3} \left(\frac{\partial u_3}{\partial q_3} + \frac{u_1}{h_1} \frac{\partial h_3}{\partial q_1} + \frac{u_2}{h_2} \frac{\partial h_3}{\partial q_2} \right) \tag{D.197}$$

$$= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial u_1 h_2 h_3}{\partial q_1} + \frac{\partial u_2 h_1 h_3}{\partial q_2} + \frac{\partial u_3 h_1 h_2}{\partial q_3} \right) \tag{D.198}$$

References: Appendices in Batchelor (1967); Griffiths (2013), and textbook of Wu (1982).

D.3.6 Examples

1. Cartesian. $(q_1, q_2, q_3) = (x_1, x_2, x_3)$
   Elements:

$$h_1 = h_2 = h_3 = 1 \tag{D.199}$$

$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{D.200}$$

$$\mathrm{d}V = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \tag{D.201}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial x_1} \boldsymbol{e}_1 + \frac{\partial f}{\partial x_2} \boldsymbol{e}_2 + \frac{\partial f}{\partial x_3} \boldsymbol{e}_3 \tag{D.202}$$

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{D.203}$$

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{D.204}$$

$$= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) \boldsymbol{e}_x + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) \boldsymbol{e}_y + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \boldsymbol{e}_z \tag{D.205}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{D.206}$$

2. Cylindrical. $(q_1, q_2, q_3) = (r, \theta, z)$
   Elements:

$$h_1 = h_3 = 1, h_2 = r \tag{D.207}$$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2 \tag{D.208}$$

$$\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \tag{D.209}$$

Operators:

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial f}{\partial z}\boldsymbol{e}_z \tag{D.210}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\left(\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial(rw)}{\partial z}\right) \tag{D.211}$$

$$= \frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{D.212}$$

$$\nabla \times \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix} \tag{D.213}$$

$$= (\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z})\boldsymbol{e}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\boldsymbol{e}_\theta + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta})\boldsymbol{e}_z \tag{D.214}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{D.215}$$

3. Spherical. $(q_1, q_2, q_3) = (r, \phi, \theta)$, ϕ is the polar angle and θ is the azimuthal.
Elements:

$$h_1 = 1, h_2 = r, h_3 = r\sin\phi \tag{D.216}$$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + r^2\sin^2\phi\mathrm{d}\theta^2 \tag{D.217}$$

$$\mathrm{d}V = r^2\sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta \tag{D.218}$$

Operators:

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta \tag{D.219}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r^2\sin\phi}\left(\frac{\partial(r^2\sin\phi u)}{\partial r} + \frac{\partial(r\sin\phi v)}{\partial \phi} + \frac{\partial(rw)}{\partial \theta}\right) \tag{D.220}$$

$$= \frac{1}{r^2}\frac{\partial(r^2 u)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(\sin\phi v)}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial w}{\partial \theta} \tag{D.221}$$

$$\nabla \times \boldsymbol{u} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\phi & r\sin\phi\boldsymbol{e}_\theta \\ \partial_r & \partial_\phi & \partial_\theta \\ u & rv & r\sin\phi w \end{vmatrix} \tag{D.222}$$

$$= \frac{1}{r\sin\phi}(\frac{\partial \sin\phi w}{\partial \phi} - \frac{\partial v}{\partial \theta})\boldsymbol{e}_r + (\frac{1}{r\sin\phi}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial rw}{\partial r})\boldsymbol{e}_\phi + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \phi})\boldsymbol{e}_\theta \tag{D.223}$$

$$\nabla^2 f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin\phi}\frac{\partial^2 f}{\partial \theta^2} \tag{D.224}$$

Appendix E

Special functions

E.1 Polynomials and Taylor's expansions

The Euler's identity

$$e^{i\pi} + 1 = 0, \tag{E.1}$$

is actualy a special case of the Euler's formula

$$e^{ix} = \cos x + i \sin x, \tag{E.2}$$

where $i = \sqrt{-1}$ is the imaginary unit.

It can be shown by Taylor expansions of $\cos x$ and $\sin x$ at $x = 0$ in Table E.1 and we have already observed some similarities to the Taylor expansion of an exponential

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ... \tag{E.3}$$

Taking the Laurent expansion of e^{ix} at $x = 0$ we have proven that

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} && \text{(E.4)} \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + ... && \text{(E.5)} \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) && \text{(E.6)} \\ &= \cos x + i \sin x. && \text{(E.7)} \end{aligned}$$

The Taylor expansion of an infinity continuously differentiable function $f \in \mathbb{C}$ around $x = x_0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f(n)(x_0)}{n!}(x - x_0)^n, \tag{E.8}$$

and the expansions of commonly used functions are listed in the Table E.1.

Note that C_n^α is the combination number representing the total number of possibilities of selecting n from α and B_k are the Bernoulli numbers.

In this section, we will also connect the functions to their defining ODEs – the ODEs that they satisfy – to reveal their physical implications.

function (convergence radius R)	expansion sum	power series
Geometric functions ($\|x\| < 1$)		
$\frac{1}{1-x}$	$= \sum_{\infty}^{n=0} x^n$	$= 1 + x + x^2 + x^3 + ...$
$\frac{1}{(1-x)^2}$	$= \sum_{\infty}^{n=1} nx^{n-1}$	$= 1 + 2x + 3x^2 + 4x^3 + ...$
Binomial functions ($\|x\| < 1$)	$C_n^\alpha = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$	$\alpha \in \mathbb{C}$
$(1+x)^\alpha$	$= \sum_{\infty}^{n=0} C_n^\alpha x^n$	
$(1+x)^{1/2}$		$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + ...$
$(1+x)^{-1/2}$		$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + ...$
Trigonometric functions ($R = \infty$)		
$\sin x$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$
$\cos x$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$
($\|x\| < \pi/2$), $\tan x$	$= \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1}$	$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + ...$
($\|x\| < \pi/2$), $\sec x$		$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + ...$
($\|x\| \leq 1$), $\arcsin x$		$= x + \frac{x^3}{6} + \frac{3x^5}{40} + ...$
($\|x\| \leq 1$), $\arccos x$		$= \frac{\pi}{2} - \arcsin x$
($\|x\| \leq 1, x \neq \pm i$), $\arctan x$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$
Hyperbolic functions ($R = \infty$)		
$\sinh x$	$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$
$\cosh x$	$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$
$\tanh x$	$= \sum_{n=1}^{\infty} \frac{B_{2n}4^n(4^n-1)}{(2n)!} x^{2n-1}$	$= x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + ...$
$\operatorname{arctanh} x$	$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$	$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ...$

Table E.1: Taylor (polynomial) expansion of some special functions. These expansions can be used to draw mental pictures of these functions. The similarity in the expansion of the hyperbolic functions to the corresponding trigonometric functions can be seen.

E.2 Trigonometric functions

E.2.1 Trigonometric relations

E.3 Hyperbolic functions

The hyperbolic functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{E.9}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{E.10}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{E.11}$$

$$\tag{E.12}$$

and their first derivatives are listed in Table E.2.

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x\,(= 1/\cosh^2 x)$

Table E.2: Derivatives of hyperbolic functions.

It is easy to verify that both $y = \sinh kx$ and $y = \cosh kx$ are solutions to the second-order ODE

$$y'' - k^2 y = 0. \tag{E.13}$$

Similar to

$$\sin^2 x + \cos^2 x = 1, \tag{E.14}$$

for hyperbolic functions we have

$$\cosh^2 x - \sinh^2 x = 1. \tag{E.15}$$

Examples.

Mixing layer:

$$U(y) = U_0 \tanh(y/L). \tag{E.16}$$

Jet:

$$U(y) = U_0 \text{sech}^2(y/L) = U_0/\cosh^2(y/L). \tag{E.17}$$

We will see this in section xx as analytical solutions to lam/turb profiles? First derivatives, second derivatives, inflection points.

E.4 Bessel functions

E.5 Chebyshev polynomials

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