Principles of fluid flows

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Preface: This note, intended for being used as a quick reference, provides a collection of a wide range of equations in fluid mechanics, ranging from basic governing equations seen in introductory textbooks to secondary conclusions left as an exercise in graduate textbooks, monographs, or research papers, the detailed derivations of which are typically omitted. The principles outlined are supposed to be underlying how this World works. We try to use symbols and notations as consistently as possible, and it is unavoidable that this note is biased to the author's own preference and taste. The logical order of the materials is from general to special, which is probably not pedagogical and this note is hence not an introductory note that describes how fluids flow. This note is still under construction.

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1 Conservation laws

1.1 Generic conservation laws

Consider a control volume V enclosed by a surface A and a scalar ψ (per unit mass) carried by the fluid. The rate of change of the total scalar in V, in the absence of any source or sink, can be established as

$$\begin{aligned}\frac{d}{dt}\left(\int_V \psi(\boldsymbol{x},t)dm\right) &= \frac{d}{dt}\left(\int_V \psi(\boldsymbol{x},t)\rho(\boldsymbol{x},t)dV\right) &(1.1)\\ &= -\int_{A=\partial V} \rho\psi\boldsymbol{u}\cdot\boldsymbol{n}dA &(1.2)\\ &= -\int_V \nabla\cdot(\rho\psi\boldsymbol{u})dV &(1.3)\end{aligned}$$

Hence,

$$\int_V \left[\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u})\right] dV = 0. \tag{1.4}$$

Since the control volume is arbitrary and can be shrunk to infinitesimal, we have the differential form

$$\frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u}) = 0. \tag{1.5}$$

Here, the scalar can be taken to be mass ($\psi = 1$), momentum ($\psi := \boldsymbol{u}$), or any other scalar ($\psi = X_i$, where X_i is the species concentration).

Example: Moving control volume.

In what follows, we will discuss several conservation laws. Ref. Chorin *et al.* (1990).

1.2 Mass conservation

Taking $\psi = 1$,

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right] dV = 0. \tag{1.6}$$

In differential form:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0. \tag{1.7}$$

More over, using the definition of material derivative, we have

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) &= \frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho + \rho(\nabla\cdot\boldsymbol{u}) &(1.8)\\ &= \frac{D\rho}{Dt} + \rho(\nabla\cdot\boldsymbol{u}) &(1.9)\\ &= 0. &(1.10)\end{aligned}$$

1.2.1 Incompressibility, continuity

It can be seen from (1.9) that solenoidality of the velocity field ($\nabla\cdot\boldsymbol{u} = 0$) is a sufficient and necessary condition for imcompressibility ($D\rho/Dt = 0$, density doesn't change along the material). The physical meaning is as the following.

Consider a cubic finite volume $dV = dxdydz$, the volume change is

$$\begin{aligned}\delta(dV) &= [(u + \partial_x u dx) - u]dydzdt + [(v + \partial_y v dy) - v]dxdzdt + [(w + \partial_z w dz) - w]dxdydt &(1.11)\\ &= (\partial_x u + \partial_y v + \partial_x w)dxdydzdt. &(1.12)\end{aligned}$$

We have the divergence of a fluid parcel

$$\nabla \cdot \boldsymbol{u} = \partial_x u + \partial_y v + \partial_x w = \frac{1}{dV}\frac{\delta(dV)}{\delta t}, \tag{1.13}$$

which is the volume change rate per unit volumn. The specific volume (not to be confused with viscosity) is defined as $\nu = 1/\rho$. We can also have

$$\nabla \cdot \boldsymbol{u} = \frac{1}{\nu}\frac{D\nu}{Dt}, \tag{1.14}$$

which is stating that the divergence of the velocity filed is the normalized rate of change of specific volume.

One might have already been satisfied with such statements. Actually, there is more to it. Note on incompressibility condition - Drho/Dt, and the volume change rate relation ,and the thermal effects (EOS).

Ref. Batchelor (1967).

1.3 Material derivative and the Reynolds transport theorem

1.3.1 Material derivative

Consider the total time derivative of a scalar ψ:

$$\frac{d}{dt}\psi(\boldsymbol{x}(t), t) = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt} = \frac{\partial \psi}{\partial t} + \boldsymbol{u} \cdot \nabla \psi. \tag{1.15}$$

We define

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla \tag{1.16}$$

as the material derivative, with $\partial/\partial t$ being the local rate-of-change and $\boldsymbol{u} \cdot \nabla$ being the convective derivative.

This is actually a bridge between the Eulerian and Lagrangian description of fluids.

1.3.2 Reynolds transport theorem

1.3.3 A Lagrangian perspective

An Eulerian field is

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\varphi(\boldsymbol{x}_0, t), t) \tag{1.17}$$

where $\varphi : \boldsymbol{x}_0 \to \boldsymbol{x}$ is a smooth and invertible flow mapping that maps the initial location of the fluid parcel $\boldsymbol{x}_0$ at $t = 0$ to $\boldsymbol{x}$ at t such that

$$\boldsymbol{x} = \varphi(\boldsymbol{x}_0, t). \tag{1.18}$$

1.4 Momentum conservation

Let $\psi = \boldsymbol{u}$ to be the quantity being transported. The change of momentum in V is equal to the momentum flux in the direction $\boldsymbol{n}$ and volumetric contribution from the external body forcing per unit mass $\boldsymbol{f}$ ($\boldsymbol{f} = \lim_{\Delta V \to 0} \Delta \boldsymbol{F}/\Delta m$):

$$\frac{d}{dt}\int_V \rho \boldsymbol{u} dV = -\int_A (\rho \boldsymbol{u})\boldsymbol{u} \cdot \boldsymbol{n} dA + \int_A \boldsymbol{\sigma} \cdot \boldsymbol{n} dA + \int_V \rho \boldsymbol{f} dV \tag{1.19}$$

$$= -\int_V \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) dV + \int_V (\nabla \cdot \boldsymbol{\sigma}) dV + \int_V \rho \boldsymbol{f} dV, \tag{1.20}$$

where $\boldsymbol{\sigma} = \lim_{\Delta A \to 0} \Delta \boldsymbol{F}/\Delta \boldsymbol{A}$ is the stress tensor.

We have the integration from of the Euler equation:

$$\int_V [\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u})]\, dV = \int_v [\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}] dV \tag{1.21}$$

which is valid for a finite volume V. Hence, (the differential form)

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = \nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f}. \tag{1.22}$$

For example, if the body force is gravity, $\boldsymbol{f} = \boldsymbol{g}$. Or if the body force is Coriolis, $\boldsymbol{f} = \boldsymbol{u} \times \boldsymbol{f}_c$.
In general, for continuum mechanics, the force balance can generally be written as

$$\rho \ddot{\boldsymbol{u}} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} \tag{1.23}$$

where $-\rho \ddot{\boldsymbol{u}}$ is D'Alembert's force (with $\boldsymbol{u}$ being the displacement instead), $\boldsymbol{\sigma}$ is the stress tensor, and $\boldsymbol{f}$ is the body force per unit volume.

Now we consider the sources of stresses.

- Pressure. It is direction is $-\boldsymbol{n}$ so $\boldsymbol{\sigma}_p = -p\boldsymbol{I}$.
- Viscous stress. For imcompressible Newtonian fluid (see section 1.7), $\boldsymbol{\sigma}_v = \boldsymbol{\tau} = 2\mu \boldsymbol{S}$.
- Hence,

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu \boldsymbol{S}. \tag{1.24}$$

Hence, the momentum conservation is

$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{f}. \tag{1.25}$$

We note that $\partial_j(2\mu S_{ij}) = \partial_j(\mu \partial_j u_i)$.

1.5 Energy conservation

1.6 Bernoulli equation

It is a combination of momentum and energy conservations. (?)

Assumptions:

- Inviscid.
- Barotropic.
- Potential force.
- Steady.

Consider the inviscid Euler equation:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \boldsymbol{F} \tag{1.26}$$

with a conservative force

$$\boldsymbol{F} = -\nabla \Phi. \tag{1.27}$$

Using Eq. (A.20) we have

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} \tag{1.28}$$

We defined

$$\boldsymbol{L} = \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{\omega} \tag{1.29}$$

which is called the Lamb vector.

The Euler equation becomes

$$\nabla \frac{\boldsymbol{u}^2}{2} - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{p}{\rho}\right) - \nabla \Phi \tag{1.30}$$

and hence

$$\nabla \left(\frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi\right) = \boldsymbol{L} \tag{1.31}$$

which is called the Lamb-Gromyko equation. Define

$$H = \frac{\boldsymbol{u}^2}{2} + \frac{p}{\rho} + \Phi \tag{1.32}$$

we have

$$\nabla H = \boldsymbol{L}. \tag{1.33}$$

If the flow is irrotational, i.e., $\boldsymbol{L} = 0$, we recover a special version (isentropic) of Bernuolli's theorem.

1.7 Constitutive relations

A note on angular momentum conservation.

A difference between solid and fluid mechanics is that, in solid mechanics, stress is proportional to strain, while in fluid mechanics, stress is proportional to strain rate.

Some comments about stress–strain relation in the solids.

1.8 Pressure Poisson

Take the divergence of the follow equation for incompressible flows ($\nabla \cdot \boldsymbol{u} = 0$):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} \tag{1.34}$$

we have

$$-\frac{1}{\rho} \nabla^2 p = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u} \nabla + \boldsymbol{u} \cdot \nabla (\nabla \cdot \boldsymbol{u}) = \nabla \boldsymbol{u} : \boldsymbol{u} \nabla \tag{1.35}$$

i.e.

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x_i \partial x_i} = -\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \tag{1.36}$$

In CFD, the continuity equation ($\nabla \cdot \boldsymbol{u} = 0$) is responsible for solving pressure for the above reason.

2 Vortex dynamics

Vorticity is defined as the curl of the velocity field:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{2.1}$$

2.1 Kelvin's theorem

Ref. Kundu and Cohen.

The circulation along a closed line is defined as

$$\Gamma = \oint_l \boldsymbol{u} \cdot d\boldsymbol{x} = \iint_A (\nabla \times \boldsymbol{u}) \cdot d\boldsymbol{A}, \tag{2.2}$$

according to Stokes' theorem.

Consider the momentum equation

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f}. \tag{2.3}$$

The material (Lagrangian) derivative of Γ is

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\left(\oint_l \boldsymbol{u} \cdot d\boldsymbol{x}\right) \tag{2.4}$$

$$= \oint_l \frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{x} + \oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} \tag{2.5}$$

$$= -\oint_l \left(\frac{1}{\rho}\nabla p\right) \cdot d\boldsymbol{x} + \oint_l (\nabla \cdot \boldsymbol{\sigma}) \cdot d\boldsymbol{x} + \oint_l \boldsymbol{f} \cdot d\boldsymbol{x} + \oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} \tag{2.6}$$

Assumptions:

1. Inviscid fluid: $\nabla \cdot \boldsymbol{\sigma} = 0$.
2. Conservative body force: $\boldsymbol{f} = -\nabla\Phi, \nabla \times \boldsymbol{f} = 0$ (it is the gradient of a potential field). Conservative means the potential difference does not depend on the path:

$$\int_A^B \boldsymbol{f} \cdot d\boldsymbol{x} = -\int_A^B \nabla\Phi \cdot d\boldsymbol{x} = -\int_A^B d\Phi = \Phi_A - \Phi_B. \tag{2.7}$$

   For a close path, $A = B$ hence the integral is zero.
3. Barotropic flow: $\nabla\rho \times \nabla p = 0$.

Moreover, by

$$\boldsymbol{u} + d\boldsymbol{u} = \frac{D}{Dt}(\boldsymbol{x} + d\boldsymbol{x}) = \frac{D\boldsymbol{x}}{Dt} + \frac{D(d\boldsymbol{x})}{Dt}, \tag{2.8}$$

we have

$$\frac{D(d\boldsymbol{x})}{Dt} = d\boldsymbol{u} = d\boldsymbol{x} \cdot \nabla\boldsymbol{u} \tag{2.9}$$

so the last term in (2.6) is

$$\oint_l \boldsymbol{u} \cdot \frac{D(d\boldsymbol{x})}{Dt} = \oint_l \boldsymbol{u} \cdot d\boldsymbol{u} = \oint_l d(\boldsymbol{u}^2) = 0. \tag{2.10}$$

Then we are able to prove all RHS terms in (2.6) are zero hence

$$\frac{D\Gamma}{Dt} = 0 \tag{2.11}$$

along any arbitrary closed curves.

2.1.1 Helmholtz's theorems

2.2 Vorticity transport equation

By Eq. (A.11) we know

$$\nabla \cdot \boldsymbol{\omega} = 0 \tag{2.12}$$

i.e., the continuity of vorticity.

The incompressible Navier–Stokes equation in vector form:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f} \tag{2.13}$$

Using Eq. (1.28) we have

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \frac{\boldsymbol{u}^2}{2} + \boldsymbol{\omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F} \tag{2.14}$$

Using the (A.21) and take the curl of Eq. (2.14)

$$\begin{aligned}
\text{LHS} &= \nabla \times \left(\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \frac{\boldsymbol{u}^2}{2} + \boldsymbol{\omega} \times \boldsymbol{u}\right) && (2.15)\\
&= \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) && (2.16)\\
&= \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}) - \boldsymbol{u}(\nabla \cdot \boldsymbol{\omega}) && (2.17)\\
&= \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} && (2.18)\\
\text{RHS} &= \nabla \times \left(-\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}\right) && (2.19)\\
&= \nu \nabla^2 \boldsymbol{\omega} + \nabla \times \boldsymbol{f} + \frac{1}{\rho^2}\nabla \rho \times \nabla p && (2.20)
\end{aligned}$$

Equaling both sides we obtain

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t}}_{\text{rate of change}} + \underbrace{\boldsymbol{u} \cdot \nabla \boldsymbol{\omega}}_{\text{advection}} = \underbrace{\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}}_{\text{vortex stretching}} + \underbrace{\nu \nabla^2 \boldsymbol{\omega}}_{\text{viscous diffusion}} + \underbrace{\nabla \times \boldsymbol{f}}_{\text{external torque in a non-conservative field}} + \underbrace{\frac{1}{\rho^2}\nabla \rho \times \nabla p}_{\text{baroclinic torque}} \tag{2.21}$$

Again, the stretching term $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$ comes from the non-linear advection/inertial term in N-S. It is important in the energy cascade in turbulence. Also from (3.18), $\boldsymbol{\omega} \cdot \boldsymbol{\Omega} = \boldsymbol{0}$, we have

$$\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = \boldsymbol{\omega} \cdot \boldsymbol{S}, \tag{2.22}$$

i.e., vortex stretching happens as a result of its interaction with the strain rate. Hence, the alignment of vorticity with the eigenvectors of the strain rate is important. It was found by Ashurst *et al.* (1987) that, in turbulence, vorticity tend to align to the eigenvector corresponding to the second eigenvalue of $\boldsymbol{S}$, which tend to be positive (indicating positive stretching production). This is called the kinematic alignment effect.

2.2.1 Enstrophy transport equation

Define enstrophy as:

$$\mathcal{E} \triangleq \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\omega} = \frac{1}{2}\omega_i \omega_i \tag{2.23}$$

Re-write (2.21) into tensor notation we have

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j^2} + \epsilon_{ijk} \frac{\partial f_k}{\partial x_j} + \frac{1}{\rho^2} \epsilon_{ijk} \frac{\partial \rho}{\partial x_j} \frac{\partial p}{\partial x_k} \tag{2.24}$$

$\omega_i \times$ (2.24) we have

$$\frac{\partial}{\partial t}(\frac{1}{2}\omega_i\omega_i) + u_j \frac{\partial}{\partial x_j}(\frac{1}{2}\omega_i\omega_i) = \omega_i\omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2}{\partial x_j^2}(\frac{1}{2}\omega_i\omega_i) - \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} \tag{2.25}$$

$$+ \epsilon_{ijk}\omega_i \frac{\partial F_k}{\partial x_j} + \frac{1}{\rho^2} \epsilon_{ijk}\omega_i \frac{\partial \rho}{\partial x_j} \frac{\partial p}{\partial x_k} \tag{2.26}$$

Note that ϵ_{ijk} is the Levi-Civita symbol, not to be confused with the turbulent kinetic energy rate ε or Reynolds stresses dissipation rate ε_{ij}.

Re-write back into vector form:

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{u} \cdot \nabla \mathcal{E} = \boldsymbol{\omega\omega} : \nabla \boldsymbol{u} + \nu \nabla^2 \mathcal{E} \underbrace{- \nu \nabla \boldsymbol{\omega} : \nabla \boldsymbol{\omega}}_{\substack{\text{viscous} \\ \text{dissipation}}} + \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{F}) + \boldsymbol{\omega} \cdot \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{2.27}$$

Note that we are assuming an incompressible flow, hence $\nabla \cdot \boldsymbol{u}$ related terms are not appearing in Eq. (2.27). A new mechanism compared to (2.21) is the viscous dissipation of enstrophy. This term is always negative.

2.3 Vorticity–streamfunction formulation

According to the Helmholtz decomposition theorem, the velocity field $\boldsymbol{u}$ can be decomposed to the addition of a potential and a curl field as

$$\boldsymbol{u} = \nabla \phi + \nabla \times \boldsymbol{A}, \tag{2.28}$$

with the first part being irrotational and the second solenoidal. The two potentials ϕ and $\boldsymbol{A}$ are called the scalar and vector potentials.

In two-dimensional flows, the vector potential reduces to $\boldsymbol{A} = (0, 0, \psi)$, where ψ is the streamfunction, defined as

$$\psi(x, y) = \int_0^y u(0, \xi)\, d\xi - \int_0^x v(\zeta, y)\, d\zeta + C, \tag{2.29}$$

is constant along the streamlines. A reference point would need to be specified. And it is noted that between two streamlines the volume flow rate is a constant (physical meaning of streamfunctions). For equally spaced streamlines, the locations where they are denser have a faster velocity.

In general, for a divergence-free field ($\nabla \cdot \boldsymbol{u} = 0$) there exist a vector potential. The equations to solve are, $\boldsymbol{u} = \nabla \times \boldsymbol{A}$, or component-wise

$$u = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \tag{2.30}$$

$$v = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \tag{2.31}$$

$$w = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \tag{2.32}$$

whose solution is far from unique. We note that $\nabla \times (\boldsymbol{A} + \nabla f) = \nabla \times \boldsymbol{A}$, for any arbitrary scalar potential ∇f. It can be chosen to $\nabla f = (0, 0, -A_3)$ by integrating $f = -\int A_3\, dz$ such that the above equations are simplified to

$$u = -\frac{\partial A_2}{\partial z} \tag{2.33}$$

$$v = \frac{\partial A_1}{\partial z} \tag{2.34}$$

$$w = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \tag{2.35}$$

which are easier to solve.

2.3.1 Vorticity–streamfunction in two dimensions

As a strategy to solve the N–S, it is common to transform (u, v, p) to (ω, ψ), equivalent to taking the curl of the N–S to eliminate the pressure. Under such transformation, 2D N–S equations are converted to

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \tag{2.36}$$

$$\omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \tag{2.37}$$

where

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x}. \tag{2.38}$$

The common procedures in a Fourier spectral method (in a doubly periodic domain) solving the above equations include time-stepping of (2.36), with the streamfunction calculated from the inversion of (2.37) and used to obtained the velocities through (2.38). Such a formulation can be similarly obtain for axisymmetric flows – see section C.1.5. The elliptic character of (2.37) and the parabolic character of (2.36) jointly reflect the mixed elliptic-parabolic nature of the Navier–Stokes. We also note that the elliptic nature is closely associated with incompressibility.

The absence of evolution equation for the pressure is reflected in the Poisson equation (2.37). If the pressure field is desired, it can be recovered by solving the pressure Poisson:

$$\left(\frac{\partial^2 p}{\partial^2 x} + \frac{\partial^2 p}{\partial^2 y} \right) = -\rho \frac{\partial u_\beta}{\partial x_\alpha} \frac{\partial u_\alpha}{\partial x_\beta} \tag{2.39}$$

$$= -\rho \left[\left(\frac{\partial u}{\partial x} \right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y} \right)^2 \right] \tag{2.40}$$

$$= -2\rho \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} \right) \tag{2.41}$$

$$= 2\rho \left[\frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \tag{2.42}$$

where the last simplification uses the continuity to reach an expression that requires fewer operation counts. Since the equation is second-order in space, we need two BC's in each direction. At solid walls, we have the no-slip condition $u = v = 0$ such that $\partial_x u = \partial_x v = 0$ at the wall. The N–S equation at the wall reads

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} = \mu \frac{\partial \omega}{\partial y}, \tag{2.43}$$

where in a finite-difference scheme the pressure can be computed from a one-sided FDA of $\partial\omega/\partial y$, given pressure should be known at least for one point on the wall.

The convenience that this method offers comes at the expense of difficulties of imposing the boundary conditions. For streamfunction it is relatively easy – it should be a constant on solid boundaries. For vorticity, neither its value nor its first derivative is known before the computation. Special treatments are required. For example, at the wall of $y = 0$, the Poisson equation of ψ reduces to

$$\omega = -\frac{\partial^2 \psi}{\partial y^2} \tag{2.44}$$

since $\partial_x v = 0$.

2.3.2 Point vortex models

Hamiltonian system, single, dipole, and more. Biot-Savart's law.

2.3.3 Vorticity–streamfunction in three dimensions

KMM; stability (algebraic growth paper cf.)

2.4 Potential flow

In potential flows, $\boldsymbol{A} = \mathbf{0}$ in (2.28) and the velocity field can be represented as the gradient of the scalar potential. The continuity condition, $\nabla \cdot \boldsymbol{u} = 0$, leads to

$$\nabla^2 \phi = 0, \tag{2.45}$$

i.e., the scalar potential satisfies the Laplacian equation. Since the Laplacian equation is linear, superposition is enabled and basic solutions can be used as building blocks to construct more complicated flow fields.

Note that the Helmholtz decomposition is not necessarily unique. For example, in 2D potential flows, the velocity can be expressed by solely an irrotational or a solenoidal field:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{2.46}$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{2.47}$$

where both the velocity potential and the streamfunction satisfy the Cauchy–Riemann (C–R) condition and are harmonic functions that solve Laplacian equation:

$$\nabla^2 \phi = 0 \tag{2.48}$$

$$\nabla^2 \psi = 0 \tag{2.49}$$

As a consequence,

$$\nabla \phi \cdot \nabla \psi = 0, \tag{2.50}$$

i.e., contours of velocity potential and streamfunctions are perpendicular.

In the language of complex analysis:

$$z = x + iy \tag{2.51}$$

$$\bar{z} = x - iy \tag{2.52}$$

and

$$x = \frac{1}{2}(z + \bar{z}) \tag{2.53}$$

$$y = \frac{1}{2i}(z - \bar{z}). \tag{2.54}$$

We have the derivatives

$$\partial_z = \partial_x \partial_z x + \partial_y \partial_z y = \frac{1}{2}(\partial_x - i\partial_y) \tag{2.55}$$

and similarly

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{2.56}$$

Denote the complex potential as $\Phi = \phi + i\psi$, we can show that

$$\frac{d\Phi}{dz} = \frac{1}{2}(\phi_x + i\psi_x) - \frac{i}{2}(\phi_y + i\psi_y) = u - iv, \tag{2.57}$$

which can be sometimes easier to find. We can also show (by the C–R equations) that

$$\frac{d\Phi}{d\bar{z}} = 0. \tag{2.58}$$

Such function Φ is called analytic/homomorphic function whose real and imaginary parts ϕ, ψ are called the harmonic conjugates which satiafy the C–R equation and the Laplacian.

2.4.1 2D potential flow in cylindrical coordinates

The continuity equation in two dimensions reads

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \theta} = 0. \tag{2.59}$$

We can define the streamfunction and scalar potential as

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta},\ v = -\frac{\partial \psi}{\partial r} \tag{2.60}$$

$$u = \frac{\partial \phi}{\partial r}, \quad v = \frac{1}{r}\frac{\partial \phi}{\partial \theta}. \tag{2.61}$$

In the limit of 2D axisymmetric flow, we have $\partial_\theta = 0$ and hence

$$\frac{\partial(ru)}{\partial r} = 0,\ u = u(r),\ v = v(r), \tag{2.62}$$

the integration of the former leads to

$$u = \frac{Q}{2\pi r} \tag{2.63}$$

where Q is constant volume flow rate. In an inviscid barotropic potential flow, the conservation of circulation implies $\Gamma = \int_0^{2\pi} vr\, d\theta = 2\pi r v = \text{const}$, and hence

$$v = \frac{\Gamma}{2\pi r}. \tag{2.64}$$

The corresponding streamfunction and scalar potential are

$$\psi = -\frac{\Gamma}{2\pi}\ln r + \frac{Q}{2\pi}\theta + C \tag{2.65}$$

$$\phi = \frac{Q}{2\pi}\ln r + \frac{\Gamma}{2\pi}\theta + C' \tag{2.66}$$

representing the superposition of two typical potential solutions: a line source/sink with

$$u = \frac{Q}{2\pi r},\ v = 0 \tag{2.67}$$

and a line vortex

$$u = 0,\ v = \frac{\Gamma}{2\pi r}. \tag{2.68}$$

The vorticity reduces to

$$\omega_z = \frac{1}{r}\frac{\partial(rv)}{\partial r} = \frac{v}{r} + \frac{\partial v}{\partial r}, \tag{2.69}$$

with the first term called the curvature vorticity and the second called the shear vorticity.

2.5 Stokes flows

In Stokes flows (creeping flows at very low Re), the governing equation reduces to

$$0 = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}. \tag{2.70}$$

Taking the divergence, curl, and Laplacian respectively, we find that

$$\nabla^2 p = 0 \tag{2.71}$$

$$\nabla^2 \boldsymbol{\omega} = \boldsymbol{0} \tag{2.72}$$

$$\nabla^2\nabla^2 \boldsymbol{u} = \boldsymbol{0} \tag{2.73}$$

$$\nabla^2\nabla^2 \psi = 0\,(2\text{D}) \tag{2.74}$$

that all flow variables are harmonic or biharmonic functions.

2.6 Beltrami flows

Recall in (1.29) that the Lamb vector is defined as $\boldsymbol{L} = \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{\omega}$. The Beltrami flows satisfy

$$\boldsymbol{L} = \boldsymbol{0}, \tag{2.75}$$

i.e., streamlines parallel to vortex lines. Moreover, the generalized Beltrami flows satisfy

$$\nabla \times \boldsymbol{L} = \boldsymbol{0} \tag{2.76}$$

such that the second term in (2.16) is zero and the vorticity equation (2.21), in the absence of external torque and baroclinicity, reduces to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \nabla^2 \boldsymbol{\omega}. \tag{2.77}$$

Hence, there is no vorticity generation mechanism in Beltrami flows. Although (2.77) is easy to solve, the boundary conditions for vorticity are usually hard to prescribe.

In two dimensions, we also have

$$\omega = -\nabla^2 \psi, \tag{2.78}$$

and the generalized Beltrami flow condition (2.76) translates to

$$\frac{\partial \omega_z}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \omega_z}{\partial y}\frac{\partial \psi}{\partial x} = 0, \tag{2.79}$$

i.e.,

$$\nabla \omega_z \,//\, \nabla \psi, \tag{2.80}$$

implying that the two rows are linearly dependent and $\omega_z = f(\psi, t)$ that vorticity is a constant along the streamline.

The equation that ψ satisfies is

$$\frac{\partial}{\partial t}(\nabla^2 \psi) = \nu \nabla^2\nabla^2 \psi, \tag{2.81}$$

with some found exact solutions provided below.

2.6.1 Examples

The ABC (Arnold–Beltrami–Childress) flow

$$u = A\sin z + C\cos y \tag{2.82}$$
$$v = B\sin x + A\cos z \tag{2.83}$$
$$w = C\sin y + B\cos x \tag{2.84}$$
$$\tag{2.85}$$

has $\boldsymbol{\omega} = \boldsymbol{u}$. It is a Beltrami flow in the narrow sense.

Taylor's decaying vortex:

$$\psi(x, y, t) = a\cos(mx)\cos(ny)\exp[-\nu(m^2 + n^2)t], \tag{2.86}$$

with the wavenumbers m, n being free parameters. It is often used for testing numerical algorithms.

Kelvin's cat eye vortex:

$$\psi(x, y, t) = a\cosh(mx)\cos(ny)\exp[\nu(m^2 - n^2)t], \tag{2.87}$$

with the wavenumbers m, n being free parameters.

2.7 Lamb–Oseen similarity solution, Burgers vortex

We recall that a potential/irrotational vortex has a velocity profile as

$$u_\theta = \frac{\Gamma}{2\pi r} \tag{2.88}$$

where $\Gamma = \int_0^{2\pi} u_\theta r \,\mathrm{d}\theta$ is the circulation. The vorticity is

$$\omega_z = \frac{1}{r}\frac{\partial\,(r u_\theta)}{\partial r} = 0 \tag{2.89}$$

hence the potential vortex is *irrotational*.

Oseen considered a viscous solution in the form of

$$u_r = 0,\; u_\theta = \frac{\Gamma}{2\pi r} g(r, t), \tag{2.90}$$

where $g(r, t)$ is a non-dimensional similarity function that combines the time-dependent spreading vortex core size $R(t)$ and time t, that collapses when the non-dimensional radius agree.

It is easy to see that the viscous length scale is $\sqrt{\nu t}$ in a diffusion process. So we take

$$g(r, t) = g(\hat{r}^2) = g\left(\frac{r^2}{4\nu t}\right) \tag{2.91}$$

where $\hat{r} = r/2\sqrt{\nu t}$ is the self-similar length scale and $\eta = \hat{r}^2$ is the similarity variable. The same similarity transform converts the heat equation to an ODE from which the heat kernel (2.101) can be solved.

The simplified azimuthal momentum equation is

$$\frac{\partial u_\theta}{\partial t} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right), \tag{2.92}$$

which represents a diffusion process, and it reduces to and ODE under the similarity transform (2.91):

$$g' + g'' = 0, \tag{2.93}$$

where $g'(\eta) = \partial_\eta g$. The general solution is

$$g(\eta) = c_1 + c_2 e^{-\eta}. \tag{2.94}$$

Given the boundary conditions

$$g(0) = 0,\ g(\infty) = 1, \tag{2.95}$$

we have

$$g(r,t) = g(\eta) = 1 - \exp\left(\frac{r^2}{4\nu t}\right) \tag{2.96}$$

and the velocity is

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right). \tag{2.97}$$

It approaches the limit of potential vortex as $r >> R = 2\sqrt{\nu t}$, with the diffusion speed being $u_d = R/t = 2\sqrt{\nu/t}$. Also, at the inviscid limit ($\nu \to \infty$), it is just the potential vortex solution (with $\omega_z = 0$).

The vorticity, is

$$\omega_z = \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} = \frac{\Gamma}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right). \tag{2.98}$$

It is straightforward to verify that ω_z satisfies the diffusion equation in cylinderical coordinates (with $\partial_\theta = 0$ and no advection term since $u_r = 0$; see (C.71))

$$\frac{\partial \omega_z}{\partial t} = \nu\nabla^2\omega_z \tag{2.99}$$

$$= \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega_z}{\partial r}\right), \tag{2.100}$$

and show that (2.98) is the just heat kernel (fundamental solution of the heat equation) with two spatial dimensions. In a d-dimensional space, the heat kernel is generally written as

$$K(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) = \frac{1}{(4\pi\nu(t-t_0))^{d/2}}\exp\left(-\frac{|\boldsymbol{x} - \boldsymbol{x}_0|^2}{4\nu(t-t_0)}\right). \tag{2.101}$$

It is easy to verify that the Lamb vector is $\boldsymbol{L} = L(r)\boldsymbol{e}_r$ so $\nabla \times \boldsymbol{L} = 0$ and it is verified that the Lamb-Oseen solution is generalized Beltrami.

Similar to the Lamb–Oseen solution, the Burgers solution can be established as:

$$u_r = -\alpha r \tag{2.102}$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{\alpha r^2}{2\nu}\right)\right) \tag{2.103}$$

$$u_z = 2\alpha z \tag{2.104}$$

with an axisymmetric stagnation flow and α being the strain rate. It has a vortex stretching mechanism.

3 Velocity gradient tensor, its decomposition and dynamics

Following the previous section, we continue to consider the local flow structures represented by the velocity gradients.

The velocity gradient tensor $\boldsymbol{u}\nabla$ is

$$\boldsymbol{u}\nabla = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial w} \end{bmatrix} \tag{3.1}$$

and in entity notation

$$(\boldsymbol{u}\nabla)_{ij} = \frac{\partial u_i}{\partial x_j}, \tag{3.2}$$

and we note that is a Jacobian such that

$$\delta\boldsymbol{u} = (\boldsymbol{u}\nabla)\cdot\delta\boldsymbol{x} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial w} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}. \tag{3.3}$$

For an arbitrary tensor $\boldsymbol{A}$, it is always possible to decompose it into symmetric and antisymmetric (skew-symmetric) parts:

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{\Omega} \tag{3.4}$$

where

$$\boldsymbol{S} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) \tag{3.5}$$

$$\boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T) \tag{3.6}$$

such that

$$\boldsymbol{S}^T = \boldsymbol{S},\ \boldsymbol{\Omega}^T = -\boldsymbol{\Omega}. \tag{3.7}$$

For the velocity gradient tensor, $\boldsymbol{S} = 1/2(\boldsymbol{u}\nabla + \nabla\boldsymbol{u})$ is called the rate-of-strain tensor and $\boldsymbol{\Omega} = 1/2(\boldsymbol{u}\nabla - \nabla\boldsymbol{u})$ is called the rotation tensor.

3.1 Pseudo-vector and associated antisymmetric rotation tensor

Vorticity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{3.8}$$

$$= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{3.9}$$

$$= \begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix} \tag{3.10}$$

is a pseudo-vector ($\omega_i = \epsilon_{ijk}\partial_j u_k$) whose sign depends on the coordinate system (the order of i, j, k; left-hand or right-hand; cyclic or anticyclic), and is related to the antisymmetric part of velocity gradient tensor $\boldsymbol{u}\nabla$ (the rotation rate tensor $\boldsymbol{\Omega}$):

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{3.11}$$

or

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & -\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & -\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ -\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) & \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix} \tag{3.12}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3.13}$$

$$= \begin{bmatrix} 0 & -\omega_z^* & \omega_y^* \\ \omega_z^* & 0 & -\omega_x^* \\ -\omega_y^* & \omega_x^* & 0 \end{bmatrix} \tag{3.14}$$

where $\boldsymbol{\omega}^*$ is the angular velocity and $\boldsymbol{\omega} = 2\boldsymbol{\omega}^*$ (vorticity is twice of the angular velocity of the local solid-body rotation motion).

Each antisymmetric tensor $\boldsymbol{\Omega}$ can be represented by a pseudo-vector $\boldsymbol{\omega}^*$ (since it just has three independent elements), such that

$$\Omega_{ij} = -\epsilon_{ijk}\omega_k^* \tag{3.15}$$

$$\omega_k^* = -\frac{1}{2}\epsilon_{ijk}\Omega_{ij} \tag{3.16}$$

and according to (3.15)-(3.16) the inner product of the tensor $\boldsymbol{\Omega}$ with and arbitrary vector $\boldsymbol{a}$ can be written as

$$\boldsymbol{\Omega} \cdot \boldsymbol{a} = \boldsymbol{\omega}^* \times \boldsymbol{a}. \tag{3.17}$$

It is easy to verify (3.15) by definition and (3.16) using (A.8). As a corollary, we have

$$\boldsymbol{\Omega} \cdot \boldsymbol{\omega}^{\boldsymbol{*}} = \boldsymbol{\omega}^* \times \boldsymbol{\omega}^* = \mathbf{0}. \tag{3.18}$$

We can also find that

$$\|\boldsymbol{\Omega}\|^2 = \Omega_{ij}\Omega_{ij} = 2\delta_{kl}\omega_k^*\omega_l^* = \frac{1}{2}\boldsymbol{\omega}^2, \tag{3.19}$$

i.e., the norm of the rotation tensor is just the enstrophy.

Note the vorticity is $\boldsymbol{\omega} = 2\boldsymbol{\omega}^*$. We will further look into the eigenvalue decomposition and principle directions of $\boldsymbol{S}$ to further understand how it describes the geometry of the flow.

3.2 Strain rate tensor

$$\boldsymbol{S} = \begin{bmatrix} \epsilon_1 & \frac{1}{2}\gamma_3 & \frac{1}{2}\gamma_2 \\ \frac{1}{2}\gamma_3 & \epsilon_2 & \frac{1}{2}\gamma_1 \\ \frac{1}{2}\gamma_2 & \frac{1}{2}\gamma_1 & \epsilon_3 \end{bmatrix} \tag{3.20}$$

where the normal strain rate is

$$\epsilon_i = \frac{\partial u_{(i)}}{\partial x_{(i)}}, \tag{3.21}$$

indices inside parentheses don't imply summation, and the shear strain rate is

$$\gamma_i = \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right),\ i \neq j \neq k. \tag{3.22}$$

Additionally, it is easy to verify with index notation that

$$\boldsymbol{S} : \boldsymbol{\Omega} = 0, \tag{3.23}$$

where (:) denotes tensor inner-product, which induces the Frobenius norm. But the kinematic alignment between the principal directions of $\boldsymbol{S}$ and the vorticity vector is still an important question in turbulence (Ashurst *et al.*, 1987).

Examples.

1. Consider a pure stretching motion, $\epsilon_1 = \partial_x u \neq 0$ only.

$$\frac{d(\delta x)}{dt} = u = \epsilon_1 \delta x \tag{3.24}$$

hence

$$\epsilon_1 = \frac{1}{\delta x}\frac{d(\delta x)}{dt} \tag{3.25}$$

is the expansion rate (per unit time and per unit length).

2. Consider a pure rotation, $\gamma_1 \neq 0$ only. According to

$$\gamma_1 = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{3.26}$$

and

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \tag{3.27}$$

we have

$$v = \frac{1}{2}\gamma_1 \delta z,\ w = \frac{1}{2}\gamma_1 \delta y \tag{3.28}$$

Define α_{23} as the angle between two line elements that are initially aligned with x_2 and x_3, we have (assumingi infinitesimal angle of rotaion during δt)

$$\frac{d\alpha_{23}}{dt} = \frac{w\delta t/\delta y - v\delta t/\delta z}{\delta t} = (\frac{1}{2}\gamma_1 - (-\frac{1}{2}\gamma_1)) = \gamma_1. \tag{3.29}$$

Hence, γ_i can be understood as the rate of change of two perpendicular elements in the plane normal to $\boldsymbol{e}_i$.

Motivation: during the deformation, can we find a set of axes the mutual angle between each pair is not change? That leads to the eigenvalues/principal strains and principal directions of the strain rate tensor.

3.3 Velocity field decomposition examples

1. Consider a plane constant-rate pure solid body rotation with the position vector being

$$\boldsymbol{r} = \cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y, \tag{3.30}$$

and we have

$$\begin{aligned} \boldsymbol{u} &= \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} && (3.31)\\ &= \dot\theta(-\sin\theta \boldsymbol{e}_x + \cos\theta \boldsymbol{e}_y) && (3.32)\\ &= \begin{bmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \boldsymbol{e}_x \\ \sin\theta \boldsymbol{e}_y \end{bmatrix} && (3.33)\\ &= \boldsymbol{\Omega} \cdot \boldsymbol{r}. && (3.34) \end{aligned}$$

With $\boldsymbol{\omega}^* = \dot\theta \boldsymbol{e}_z$ being the angular velocity, we have

$$\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{\Omega} \cdot \boldsymbol{r} = \boldsymbol{\omega}^* \times \boldsymbol{r} \tag{3.35}$$

where $\boldsymbol{\omega}^*$ is the angular velocity.

$$
\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix}, \tag{3.36}
$$

In general, the decomposition of velocity gradient tensor can be applied to yield:

$$
\begin{aligned}
\delta\boldsymbol{u} &= \boldsymbol{u}\nabla\cdot\delta\boldsymbol{x} && (3.37)\\
&= \boldsymbol{S}\cdot\delta\boldsymbol{x} + \boldsymbol{\Omega}\cdot\delta\boldsymbol{x} && (3.38)\\
&= \boldsymbol{S}\cdot\delta\boldsymbol{x} + \boldsymbol{\omega}^*\times\delta\boldsymbol{x} && (3.39)\\
&= \nabla\varphi + \boldsymbol{\omega}^*\times\delta\boldsymbol{x} && (3.40)
\end{aligned}
$$

where the potential is

$$
\begin{aligned}
\varphi &= \frac{1}{2}(\epsilon_1\delta x^2 + \epsilon_2\delta y^2 + \epsilon_3\delta z^2 + \gamma_1\delta y\delta z + \gamma_2\delta z\delta x + \gamma_3\delta x\delta y) && (3.41)\\
&= \frac{1}{2}\delta\boldsymbol{x}\cdot\boldsymbol{S}\cdot\delta\boldsymbol{x} && (3.42)
\end{aligned}
$$

such that

$$
\nabla\varphi = \boldsymbol{S}\cdot\delta\boldsymbol{x}. \tag{3.43}
$$

According to the Helmholtz decomposition theorem, a vector field can be decomposed into the sum of an irrotational field (gradient field/curl-free) and a solenoidal field (curl field/divergence free). Here we provide a construction for the velocity field.

2. Consider plain shear flow with $U(y) = ay$ and

$$
\boldsymbol{u}\nabla = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a/2 \\ a/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a/2 \\ -a/2 & 0 \end{bmatrix} = \boldsymbol{S} + \boldsymbol{\Omega}. \tag{3.44}
$$

The flow is liearly decomposed into a pure deformation/strain motion with $\partial_y u = \partial_x v = a/2$ and a rotation motion with $\partial_y u = -\partial_x v = a/2$.

3.4 Dynamics of the velocity gradient tensor

3.4.1 Dynamics of A and its powers

Let

$$
\boldsymbol{A} = \boldsymbol{u}\nabla = \left[\frac{\partial u_i}{\partial x_j}\right]. \tag{3.45}
$$

Taking the gradient of Navier–Stokes we have

$$
\frac{\partial\boldsymbol{A}}{\partial t} + u_k\frac{\partial\boldsymbol{A}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} + \nu\frac{\partial^2\boldsymbol{A}}{\partial x_k^2} - \boldsymbol{A}\cdot\boldsymbol{A} \tag{3.46}
$$

Contracting the indices in (3.46) we have

$$
-\frac{1}{\rho}\frac{\partial^2 p}{\partial x_k^2} = \mathrm{tr}(\boldsymbol{A}^2) = -u_{i,j}u_{j,i} \tag{3.47}
$$

and i.e.,

$$
\nabla^2 p = 2\rho Q, \tag{3.48}
$$

where

$$Q = \frac{1}{2}[(\mathrm{tr}\,\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)] \tag{3.49}$$

is the second invariant of $\boldsymbol{A}$ (see Appendix A.3). Hence, people argue that the sign of Q implies the existence of local pressure minimum and so forth the existence of vortices Hunt *et al.* (1988); Jeong & Hussain (1995).

The next question is the dynamics of its eigenvalues, invariants, and powers. We immediately deal with the latter since $\boldsymbol{A}^2$ appears in the equation of $\boldsymbol{A}$.

Re-write (3.46) as

$$\mathcal{L}(A_{ik}) = 0,\ \mathcal{L}(A_{kj}) = 0 \tag{3.50}$$

and multiply with A_{kj} and A_{ik} we have

$$\frac{\partial \boldsymbol{A}^2}{\partial t} + u_k \frac{\partial \boldsymbol{A}^2}{\partial x_k} = -\frac{1}{\rho}\left(\boldsymbol{A}\cdot\nabla(\nabla p) + \nabla(\nabla p)\cdot\boldsymbol{A}\right) + \nu\frac{\partial^2 \boldsymbol{A}^2}{\partial x_k^2} - 2\boldsymbol{A}^3. \tag{3.51}$$

Here $\nabla(\nabla p)$ is the pressure Hessian. We can see that, similar to the Reynolds average closure problem, the transport equations for $\boldsymbol{A}$ are never closed due to the appearance of even-higher powers.

Contracting the indices in (3.51) and assuming incompressibility ($\mathrm{tr}\,\boldsymbol{A} = 0$) we have

$$\frac{\partial Q}{\partial t} + u_k \frac{\partial Q}{\partial x_k} = \frac{1}{\rho}(\boldsymbol{A} : \nabla(\nabla p)) + \nu\frac{\partial^2 Q}{\partial x_k^2} + \mathrm{tr}(\boldsymbol{A}^3), \tag{3.52}$$

while we note that $-2Q = A_{ik}A_{ki}$.

3.4.2 Dynamics of S, Ω, and the eigenvalues of $\mathbf{S}^2 + \mathbf{\Omega}^2$

$(3.46)^T \cdot \boldsymbol{A} + (3.46)\cdot \boldsymbol{A}^T$ we have

$$\frac{\partial S_{ij}}{\partial t} + u_k\frac{\partial S_{ij}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} + \nu\frac{\partial^2 S_{ij}}{\partial x_k^2} - (S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj}) \tag{3.53}$$

Consider a balance between the last term in the RHS and pressure Hessain,

$$\nabla(\nabla p) = -\rho(\boldsymbol{\Omega}^2 + \boldsymbol{S}^2), \tag{3.54}$$

which can be, similarly to (3.48) (which can also be obtained by contracting (3.53)), be used to search for pressure minimum (Jeong & Hussain, 1995).

Assume $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\boldsymbol{\Omega}^2 + \boldsymbol{S}^2$. Since

$$Q = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{A}^2) = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\Omega}^2 + \boldsymbol{S}^2) = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3), \tag{3.55}$$

a positive Q corresponds to $\lambda_1 + \lambda_2 + \lambda_3 < 0$. Then assume $-\lambda_3 > -\lambda_2 > -\lambda_1$, we have $\lambda_3 < 0, \lambda_1 > 0$. The requirement for p to have a (plane) local minimum (where $\nabla p = 0$ and a sub-matrix of $\nabla(\nabla p)$ is positive definite) is that the second eigenvalue $\lambda < 0$. Hence, a negative λ_2 is a sufficient condition for a pressure minimum that defines a vortex core (Jeong & Hussain, 1995).

On the other hand, the equation for $\boldsymbol{\Omega}$,

$$\frac{\partial \Omega_{ij}}{\partial t} + u_k\frac{\partial \Omega_{ij}}{\partial x_k} = \nu\frac{\partial^2 \Omega_{ij}}{\partial x_k^2} - (S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj}), \tag{3.56}$$

does not involve pressure directly.

3.4.3 Dynamics of the invariant space

In Euler equations (neglecting viscosity), (3.46) is written as

$$\frac{\partial \boldsymbol{A}}{\partial t} + u_k \frac{\partial \boldsymbol{A}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - \boldsymbol{A}\cdot\boldsymbol{A} \tag{3.57}$$

Given

$$-\nabla^2 p = \mathrm{tr}(\boldsymbol{A}) = A_{ik}A_{ki}, \tag{3.58}$$

we have

$$\begin{aligned}
\mathrm{RHS} &= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - A_{ik}A_{kj} && (3.59)\\
&= -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i x_j} - A_{ik}A_{kj} + \frac{1}{3}\left(\frac{\partial^2 p}{\partial x_k x_k} + A_{mk}A_{km}\right)\delta_{ij} && (3.60)\\
&= -\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x_i x_j} - \frac{1}{3}\frac{\partial^2 p}{\partial x_k x_k}\delta_{ij}\right) - \left(A_{ik}A_{kj} - \frac{1}{3}A_{mk}A_{km}\delta_{ij}\right) && (3.61)
\end{aligned}$$

Assume the heterogeneous part of the pressure is negligible (at least homogeneous in the small scales), we have

$$\frac{d\boldsymbol{A}}{dt} + \left(A_{ik}A_{kj} - \frac{1}{3}A_{mk}A_{km}\delta_{ij}\right) = 0. \tag{3.62}$$

Multiply (3.62) with A_{ji}, given

$$A_{ji}\frac{dA_{ij}}{dt} = \frac{1}{2}\frac{d}{dt}(A_{ij}A_{ji}) = -\frac{dQ}{dt}, \tag{3.63}$$

and $Q = -1/2\,\mathrm{tr}(\boldsymbol{A}), R = 1/3\,\mathrm{tr}(\boldsymbol{A}^3)$, we have

$$\frac{dQ}{dt} = 3R. \tag{3.64}$$

Multiply (3.62) with $A_{jk}A_{ki}$, given

$$A_{jk}A_{ki}\frac{dA_{ij}}{dt} = \frac{1}{3}\frac{d}{dt}(A_{ij}A_{jk}A_{ki}) = \frac{dR}{dt} \tag{3.65}$$

we have

$$\frac{dR}{dt} + \mathrm{tr}(\boldsymbol{A}^4) - \frac{1}{3}[\mathrm{tr}(\boldsymbol{A})^2]^2 = 0 \tag{3.66}$$

From Appendix A.3 we know (in incompressible flows)

$$\begin{aligned}
P &= \mathrm{tr}(\boldsymbol{A}) = 0 && (3.67)\\
Q &= \frac{1}{2}[\mathrm{tr}(\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)] = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{A}^2) && (3.68)\\
R &= \det(\boldsymbol{A}) = \frac{1}{6}\left(\mathrm{tr}(\boldsymbol{A})^3 - 3\,\mathrm{tr}(\boldsymbol{A})\,\mathrm{tr}(\boldsymbol{A}^2) + 2\,\mathrm{tr}(\boldsymbol{A}^3)\right) = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{A}^3) && (3.69)
\end{aligned}$$

where the last relation from the Newton's identity.

According to Cayley–Hamilton theory, matrix $\boldsymbol{A}$ satisfies its characteristic polynomial as

$$\boldsymbol{A}^3 - P\boldsymbol{A}^2 + Q\boldsymbol{A} - R\boldsymbol{I} = 0 \tag{3.70}$$

so

$$\boldsymbol{A}^4 = \boldsymbol{A}\boldsymbol{A}^3 = -Q\boldsymbol{A}^2 + R\boldsymbol{A} \tag{3.71}$$

and

$$\mathrm{tr}(\boldsymbol{A}^4) = -Q\,\mathrm{tr}(\boldsymbol{A}^2) = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{A}^2)^2. \tag{3.72}$$

Finally, we have

$$\frac{dR}{dt} = -\frac{2}{3}Q^2. \tag{3.73}$$

Equations (3.64)-(3.73) form a autonomous system, which can be further integrated:

$$dt = \frac{dQ}{3R} = \frac{dR}{-2Q/3} \tag{3.74}$$

$$Q^3 + \frac{27}{4}R^2 = \mathrm{const}. \tag{3.75}$$

We note that $\triangle = Q^3 + 27/4R^2$ is just the discriminant of $\boldsymbol{A}$. The approximate inviscid dynamics is just for the discriminant to conserve along the path. But from (3.62) on the characters of the N–S is less seen.

Ref. Meneveau (2011).

3.5 Lagrangian representations

Cauchy-Green tensor etc.

4 Laminar wall flows

When the flow is parallel, there is subsitential simplification can be done to the N–S, allowing analytical solutions.

4.1 Plane Poiseuille–Couette

Consider 2D parallel flow at steady state between two planes at $y = 0, 2h$ (h is the channel half height) and the boundary conditions are

$$u(x, 0) = v(x, 0) = 0, \tag{4.1}$$

$$u(x, 2h) = U,\ v(x, 2h) = 0. \tag{4.2}$$

The 2D N–S equation reads:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4.4}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{4.5}$$

With streamwise invariance, we have $\partial_x u = \partial_x v = 0$. With continuity we also have $\partial_y v = 0$, leading to $v(x, y) = v(x, 0) = 0$. Hence the second derivatives $\partial_{xx} u = \partial_{xx} v = \partial_{yy} v = 0$ as well. The momentum equations are reduced to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{4.6}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{4.7}$$

At the wall, (4.6) implies

$$\frac{1}{\rho}\frac{\partial p_w}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{4.8}$$

where the RHS is independent of x (fully developed). Combined with (4.7), we have

$$\frac{\partial p}{\partial x} = \frac{\partial p_w}{\partial x} \triangleq \frac{\partial P}{\partial x}, \tag{4.9}$$

i.e., the pressure gradient is a constant. Then we can solve for the velocity with:

$$\nu\frac{d^2 u}{dy^2} = \frac{1}{\rho}\frac{dP}{dx}. \tag{4.10}$$

Given the two boundary conditions (4.2), the general velocity profile for Poiseuille–Couette flows can be written as

$$u(y) = U\frac{y}{2h} - \frac{2h^2}{\rho\nu}\frac{dP}{dx}\left[\frac{y}{2h} - \left(\frac{y}{2h}\right)^2\right] \tag{4.11}$$

and it reduces to the Poiseuille solution with $U = 0$

$$u(y) = -\frac{2h^2}{\rho\nu}\frac{dP}{dx}\left[\frac{y}{2h} - \left(\frac{y}{2h}\right)^2\right] \tag{4.12}$$

and the Couette solution with $dP/dx = 0$

$$u(y) = U\frac{y}{2h}. \tag{4.13}$$

We note the both the Poiseuille and Couette, or the Poiseuille–Couette profiles are Re-independent (but the friction coefficients will be Re-dependent) and have no inflection point. The mechanism for turbulence generation has to be transient. For the balance in a turbulent channel see section 5.7.1.

4.2 Laminar Poiseuille channel

Now lets consider the momentum balance in a control volume of $[0, L_x] \times [0, 2h]$. In plane Poiseuille, it can be shown that the pressure gradient equals twice the wall shear stress (simply because there are two walls; a half-channel balance within $[0, L_x] \times [0, h]$ can be established too, with the centerplane being stress-free) as

$$-2h\frac{dP}{dx} = 2\tau_w = 2\mu\frac{\partial u}{\partial y}. \tag{4.14}$$

That being said, the wall shear stress is balanced by the constant pressure gradient. On one side of the wall,

$$-\frac{dP}{dx} = \frac{\tau_w}{h}, \tag{4.15}$$

which along with (4.14), don't depend on the shape of the profile.

The centerline velocity is

$$U_0 = u(h) = -\frac{h^2}{2\rho\nu}\frac{dP}{dx} \tag{4.16}$$

and the bulk velocity is

$$\bar{U} = \frac{1}{2h}\int_0^{2h} u(y)\,dy = \frac{1}{h}\int_0^{h} u(y)\,dy = -\frac{h^2}{3\rho\nu}\frac{dP}{dx} = \frac{2}{3}U_0 \tag{4.17}$$

and the centerline and bulk Reynolds numbers are

$$Re_0 = U_0 h/\nu, \tag{4.18}$$

$$Re = 2\bar{U}h/\nu = 4/3 Re_0. \tag{4.19}$$

The friction velocity is

$$u_\tau = \sqrt{\tau_w/\rho} = \sqrt{-\frac{h}{\rho}\frac{dP}{dx}} = \sqrt{\frac{2\nu U_0}{h}} = \sqrt{\frac{3\nu\bar{U}}{h}}, \tag{4.20}$$

and the ratios are

$$\frac{u_\tau}{U_0} = \sqrt{\frac{2}{Re_0}} = \sqrt{\frac{8}{3Re}}, \tag{4.21}$$

$$\frac{u_\tau}{\bar{U}} = \sqrt{\frac{6}{Re}}. \tag{4.22}$$

The critical bulk Reynolds number that the channel can stay laminar is $Re \approxeq 1350$, corresponding to

$$\frac{u_\tau}{U_0} = 0.0444, \tag{4.23}$$

leading to a friction Reynolds number of

$$Re_\tau = \frac{u_\tau h}{\nu} = 45, \tag{4.24}$$

which is the theoretical minimum for turbulent channels. The relation between Re_τ and Re in laminar channels is

$$Re_\tau = \sqrt{\frac{3Re}{2}}. \tag{4.25}$$

The Reynolds numbers in the turbulent channel of Kim *et al.* (1987) are about $Re = 5600$ and $Re_\tau = 180$.

Define the skin friction coefficient as

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} \tag{4.26}$$

it can be shown that

$$c_f = \frac{16}{3Re}, \tag{4.27}$$

which is called the laminar friction law for channels. Equally useful is the skin friction coefficient based on the bulk velocity,

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{U}^2}, \tag{4.28}$$

with $U_0 = 3\bar{U}/2$ and

$$C_f = \frac{12}{Re}. \tag{4.29}$$

4.3 Hagen–Poiseuille (circular pipe) flow

Hagen–Poiseuille flow has been quite important in experiments, due to the fact that a circular pipe is easy to build and the implications on pipelines. In a circular pipe, the laminar Hagen–Poiseuille flow is governed by

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{4.30}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{4.31}$$

and the solution is (a parabola)

$$u(r) = -\frac{1}{4\mu}\frac{dP}{dx}(R^2 - r^2) \tag{4.32}$$

where the constant pressure gradient $dP/dx < 0$ and R is the pipe radius. It can be shown that the centerline velocity is

$$U_0 = u(0) = -\frac{1}{4\mu}\frac{dP}{dx}R^2 \tag{4.33}$$

and the bulk velocity is

$$\bar{U} = \frac{1}{\pi R^2}\int_0^R u(r)2\pi r\,dr = -\frac{1}{8\mu}\frac{dP}{dx}R^2 = \frac{1}{2}U_0. \tag{4.34}$$

It defines the bulk Reynolds number $Re = \rho\bar{U}D/\mu$. The streamwise pressure gradient is a constant for the same reason as in Poiseuille channels. The volume flow rate

$$Q = \pi R^2\bar{U} = -\frac{\pi R^4}{8\mu}\frac{dP}{dx} \tag{4.35}$$

is proportional to the pressure gradient.

The shear stress,

$$\tau = -\mu\frac{du}{dr} = -\frac{r}{2}\frac{dP}{dx} \tag{4.36}$$

has a linear profile that vanishes at the centerline and peaks at the wall, which leads to the balance between the wall shear stress and the pressure gradient

$$-\frac{dP}{dx} = \frac{2\tau_w}{R}, \tag{4.37}$$

which can also be obtained from a control volume analysis. The latter is more general as it does not depend on the details of the velocity profile (whether it is laminar or turbulent). We also note that the shear stress is independent on viscosity/density.

The skin friction coefficient is

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{U}^2} = \frac{16}{Re}, \tag{4.38}$$

where the bulk Reynolds number is $Re = \rho\bar{u}D/\mu$, and it can be expressed as a form easier to measure in experiments as the (Darcy) friction factor,

$$f = \frac{\Delta pD}{\frac{1}{2}\rho\bar{U}^2L} = 4C_f = \frac{64}{Re} \tag{4.39}$$

where D is the pipe diameter, L is the pipe length and Δp is the pressure drop in between. Eqn. (4.39) is called the laminar friction law. It also sets that the pressure drop is prortional to the bulk velocity,

$$\Delta p/L \propto \bar{U}, \tag{4.40}$$

where in turbulent flows there is

$$\Delta p/L \propto \bar{U}^{1.75}. \tag{4.41}$$

The corresponding non-dimensional relations, which can alternatively be obtained by Π-group analysis and calibrated/measured from data, are

$$\frac{D\Delta p}{\frac{1}{2}\rho\bar{U}^2L} = \phi\left(\frac{\rho\bar{U}D}{\mu}\right) = \frac{64}{Re} \tag{4.42}$$

and

$$\frac{\rho D^3\Delta p}{L\mu^2} = \phi\left(\frac{\rho\bar{U}D}{\mu}\right) \approx 0.155\left(\frac{\rho\bar{U}D}{\mu}\right)^{1.75} \tag{4.43}$$

for laminar and turbulent pipes. We notice that the pressure drop (and hence wall shear stress) depends weakly on μ in turbulent pipes ($\Delta p \sim \mu^{0.25}$) than in laminar pipes ($\Delta p \sim \mu$). The relation (4.43) was first discovered by Blasius by fitting experimental data with insights from non-dimensional groups.

The friction Reynolds number is related to the bulk Reynolds number in laminar pipes as

$$Re_\tau = \frac{\rho u_\tau R}{\mu} = \sqrt{2Re} \tag{4.44}$$

and the critical Reynolds number above which the flow cannot remain laminar is around $Re_{\text{cri}} \approx 2300$, corresponding to $Re_\tau \approx 68$.

4.4 The boundary layer theory

The freestreem pressure and velocity are $p_0(x), U_0(x)$, related to by the Bernoulli theorem (since it is inviscid), $p_0 + 1/2\rho U_0^2 = \text{constant}$, hence

$$-\frac{dp_0}{dx} = \rho U_0 \frac{dU_0}{dx} \tag{4.45}$$

and the boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.46}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_0}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} = U_0\frac{dU_0}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} \tag{4.47}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4.48}$$

which can be easily reached by considering the relevant scales. And we note that since there is no vertical pressure gradient, the streamwise pressure gradient inside the boundary layer is the same as that of the freestreem. The shear stress is written in a more general way such that

$$\tau = \mu\frac{\partial u}{\partial y} \tag{4.49}$$

in laminar flows and

$$\tau = \mu\frac{\partial u}{\partial y} - \rho\overline{u'v'} \tag{4.50}$$

in turbulent flows.

$(u - U_0)\times$ (4.46) + (4.47) we have

$$\frac{\partial}{\partial x}[u(u-U_0)] + \frac{\partial}{\partial y}[v(u-U_0)] = (U_0 - u)\frac{dU_0}{dx} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}. \tag{4.51}$$

Integrate in y from 0 to ∞ we have

$$\frac{\partial}{\partial x}\left[\int_0^\infty u(U_0-u)\,dy\right] + \frac{dU_0}{dx}\int_0^\infty (U_0-u)\,dy = \frac{\tau_w}{\rho} \tag{4.52}$$

So far, we have not assume whether (u, v) are the laminar velocities or the turbulent mean velocities. The nominal, displacement, and the momentum thicknesses are defined as

$$u(\delta_{99}) = 0.99U_0 \tag{4.53}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_0}\right)\,dy \tag{4.54}$$

$$\theta = \int_0^\infty \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy \tag{4.55}$$

and (4.52) can be turned into the von Kármán momentum integral

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} = 2\frac{d\theta}{dx} + \frac{(2\delta^* + 4\theta)}{U_0}\frac{dU_0}{dx}. \tag{4.56}$$

4.4.1 von Kármán integrals

Actually, the von Kármán (1921) momentum integral was performed with a control-volume analysis. Assume a ZPGBL, such that the freestreem is a constant. Select a control volume of $(0,0), (0,h), (L,\delta), (L,0)$ where a streamline passes $(0,h)$ and (L,δ) (no mass penetrating the streamlines). Here h and δ are sufficiently high such that $h > \delta_{99}(L)$ and $u(\delta) = U_0$, and literally the integration bounds in (4.58) can be chosen to be infinity since it is finite. The length of the plate into the paper is b.

The momentum conservation is

$$-D = \int_0^h b\rho U_0^2\, dy - \int_0^\delta b\rho u^2(y)\ dy \tag{4.57}$$

so

$$D = \rho b U_0^2 h - \rho b \int_0^\delta u^2(y)\, dy. \tag{4.58}$$

The mass conservation is (since the curve passing $(0,h)$ and (L,δ) is a streamline)

$$\int_0^h b\rho U_0\, dy = \int_0^\delta b\rho u(y)\ dy \tag{4.59}$$

which is used to simplify (4.58) to

$$D = \rho b U_0^2 \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy = \rho b U_0^2 \theta, \tag{4.60}$$

or in dimensionless form

$$C_D = \frac{D}{\frac{1}{2}\rho U_0^2 b L} = \frac{2\theta(L)}{L}. \tag{4.61}$$

We can see that the momentum integral relates the force on the solid body to the integral of the velocity profile. What is also shown is that the momentum thickness $\theta(x)$ is a measure of the total drag up to x. Such ideas and definition of momentum thickness are also applicable in wakes, where the drag can be computed the same way using momentum integrals.

Note that the BL starts at $(0,0)$ so the total dray until $(L,0)$ is

$$D = \int_0^x b\tau_w\, dx \tag{4.62}$$

and hence

$$\tau_w = \frac{1}{b}\frac{dD}{dx} = \rho U_0^2 \frac{d\theta}{dx} \tag{4.63}$$

and the friction coefficient is

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U_0^2} = 2\frac{d\theta}{dx}. \tag{4.64}$$

This is the same as (4.56) without external pressure gradients.

The equation (4.59) can also be interpreted as

$$U_0 h = \int_0^h u\,dy + \int_h^{h+\delta_*} U_0\,dy \tag{4.65}$$

$$\delta_* = \int_0^h (1 - \frac{u}{U_0})\,dy \tag{4.66}$$

since it is assumed that $h > \delta_{99}$. Here δ_* is the displacement of the stremelines by the boundary layer at $x = L$. We can easily see that δ_* is just the displacement thickness δ^* defined in (4.54).

4.4.2 The Blasius similarity solution

The references are Schlichting & Gersten (2016); Kundu *et al.* (2015), with the definition of $\delta(x)$ different by a factor of $\sqrt{2}$. Here we will follow the notations in Schlichting & Gersten (2016).

The laminar zero-pressure-gradient boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.67}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{4.68}$$

The idea of self-similar solutions is that the velocity profile $u(y)$ will be the same under some proper transformation/normalization of u and y. The scale for u is apparently U_∞, while the scale for y is δ. From the viscous scaling $v \sim \nu/\delta$ and the scaling of the continuity equation $v/\delta \sim U_\infty/x$ we have

$$\delta^2 \sim \frac{\nu x}{U_\infty} \tag{4.69}$$

and for the sake of simplification of the final result (ODE) we define

$$\delta(x) = \sqrt{\frac{2x\nu}{U_\infty}} \tag{4.70}$$

such that the similarity transformation is

$$\eta = \frac{y}{\delta(x)} \tag{4.71}$$

such that

$$\frac{u}{U_\infty} = f(\eta) \tag{4.72}$$

where $f(\eta)$ is the similarity function and η is the similarity coordinate.

We note that the streamfunction ψ depends on ν, U_∞, x, y and dimensionally

$$\psi(x, y) = U_\infty \delta(s) f(\eta) = \sqrt{2\nu U_\infty x} f(\eta) \tag{4.73}$$

and hence

$$u = U_\infty f' \tag{4.74}$$

$$v = \sqrt{\frac{U_\infty \nu}{2x}}(\eta f' - f) \tag{4.75}$$

The derivatives are

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2x} f''\eta \tag{4.76}$$

$$\frac{\partial u}{\partial y} = U_\infty f'' \sqrt{\frac{U_\infty}{2\nu x}} \tag{4.77}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{2\nu x} f''' \tag{4.78}$$

and then

$$u\frac{\partial u}{\partial x} = -\frac{U_\infty^2}{2x} f' f'' \eta \tag{4.79}$$

$$v\frac{\partial u}{\partial y} = \frac{U_\infty^2}{2x} f''(\eta f' - f) \tag{4.80}$$

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{2x} f''' \tag{4.81}$$

and finally we have the ODE

$$ff'' + f''' = 0 \tag{4.82}$$

with the boundary conditions being

$$f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1, \tag{4.83}$$

corresponding to

$$v(y = 0) = 0,\ u(y = 0) = 0,\ u(y = \infty) = U_\infty. \tag{4.84}$$

It is common to use a Runge-Kutta shooting method to solve (4.83).

4.4.3 Lift and drag

4.5 von Kármán swirling disk flow

Also shooting method.

5 Turbulent flows

5.1 Mean flow and fluctuations

5.1.1 Reynolds average

We denote time average as $\overline{(\cdot)}$, space or ensemble average as $\langle\cdot\rangle$, and sometimes use these notations interchangeably given that they are equivalent under the ergodicity assumption. The properties proved for one definition are expected to hold for another. Although Reynolds decomposition and RANS modelings are not an accurate way of computing turbulence, they consist the foundation of our understanding of turbulence.

Below we give briefly some properties of Reynolds averaging:

(i) (Definition) The time average of a physical variable A is

$$\overline{A} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T A\,\mathrm{d}t \tag{5.1}$$

In practice, the limit is often neglected and the average window is assumed to be long enough.

(ii) (Definition) The fluctuation of a physical variable A is

$$A' \triangleq A - \overline{A} \tag{5.2}$$

(iii) (Proposition) The average of fluctuation is zero.

$$\overline{A'} = \overline{A - \overline{A}} = \overline{A} - \overline{A} = 0 \tag{5.3}$$

5.1.2 Continuity and momentum

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{5.4}$$

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i}{\partial x_j}) \tag{5.5}$$

Taking the average of Eq. (5.4) we have

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} = 0 \tag{5.6}$$

where

$$\frac{\partial \overline{u}_i}{\partial x_i} = \frac{\partial}{\partial x_i}(\frac{1}{T}\int_0^T u_i\,\mathrm{d}t) = \frac{1}{T}\int_0^T (\frac{\partial u_i}{\partial x_i})\,\mathrm{d}t = \frac{1}{T}\int_0^T 0\,\mathrm{d}t = 0 \tag{5.7}$$

Hence we have the continuity for fluctuating velocity

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{5.8}$$

Taking the average of Eq. (5.5) we have

$$\begin{aligned}
\text{LHS} &= (\frac{1}{T}\int_0^T \mathrm{d}t) * [\frac{\partial}{\partial t}(\overline{u}_i + u_i') + (\overline{u}_j + u_j')\frac{\partial}{\partial x_j}(\bar{u}_i + u_i')] && (5.9)\\
&= \frac{\partial \overline{u}_i}{\partial t} + \overline{\frac{\partial u_i'}{\partial t}} + \overline{\overline{u}_j\frac{\partial}{\partial x_j}\overline{u}_i} + \overline{\overline{u}_j\frac{\partial}{\partial x_j}u_i'} + \overline{u_j'\frac{\partial}{\partial x_j}\overline{u}_i} + \overline{u_j'\frac{\partial}{\partial x_j}u_i'} && (5.10)\\
&= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial}{\partial x_j}\overline{u}_i + \overline{u_j'\frac{\partial}{\partial x_j}u_i'} && (5.11)\\
&= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial}{\partial x_j}\overline{u}_i + \overline{\frac{\partial}{\partial x_j}(u_j'u_i')} - \overline{u_i'\frac{\partial u_j'}{\partial x_j}} = \frac{\partial}{\partial x_j}\overline{u_j'u_i'} && (5.12)\\
&= \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u_j'u_i'} && (5.13)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= (\frac{1}{T}\int_0^T \mathrm{d}t)[-\frac{1}{\rho}\frac{\partial}{\partial x_i}(\overline{p} + p') + \frac{\partial}{\partial x_j}[\nu\frac{\partial}{\partial x_j}(\overline{u}_i + u_i')] && (5.14)\\
&= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j}) && (5.15)
\end{aligned}$$

Equaling both sides yields:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{5.16}$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i'u_j'}) \tag{5.17}$$

where the cross-correlation term having dimension of shear stress

$$\tau_{\text{Rey}} = -\overline{u_i'u_j'} \tag{5.18}$$

is called the Reynolds stress term. It is a rank 2 tensor. It comes from the Reynolds averaging of the non-linear advection term on the LHS of Navier–Stokes, and it distinguishes turbulent flows from laminar ones. It represents the momentum transport due to turbulent motions, in anology to the molecular diffusion.

5.1.3 Transport equation of the fluctuating velocity

Denote the material derivative based on the mean flow advection as

$$\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}_k \frac{\partial}{\partial x_k} \tag{5.19}$$

and subtract the Reynolds equation from N-S equation

$$\frac{\bar{D}u'_i}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial u'_i}{\partial x_j} + \overline{u'_i u'_j} - u'_i u'_j) - u'_j \frac{\partial \overline{u}_i}{\partial x_j} \tag{5.20}$$

The last term shows the mean-flow stretching of the fluctuation, which is a generation mechanism be shown later related to the shear production of turbulent kinetic energy. In the latter section of 8 we will see the same form of perturbation equations will be used for hydrodynamic stability analysis.

5.1.4 Mean-flow and turbulent kinetic energy

The total kinetic energy of the flow can be devided into the mean kinetic energy (MKE) and the turbulent kinetic energy (TKE)

$$\begin{aligned} K_{\text{tot}} &= \frac{1}{2}\overline{u_i u_i} && (5.21)\\ &= \overline{\frac{1}{2}(\overline{u}_i + u'_i)(\overline{u}_i + u'_i)} && (5.22)\\ &= \overline{\frac{1}{2}\overline{u}_i\overline{u}_i + \overline{u}_i u'_i + \frac{1}{2}u'_i u'_i} && (5.23)\\ &= \frac{1}{2}\overline{u}_i\overline{u}_i + \frac{1}{2}\overline{u'_i u'_i} && (5.24)\\ &= K + k && (5.25) \end{aligned}$$

We will show how these two parts are related dynamically.

5.1.5 MKE equation

Multiply the Reynolds equation (5.17) by $\overline{u}_i$ we have

$$\begin{aligned} \text{LHS} = \overline{u}_i \frac{\bar{D}\overline{u}_i}{Dt} &= \frac{\bar{D}K}{Dt} && (5.26)\\ \overline{u}_i(-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}) &= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_i}{\partial x_i} + \frac{1}{\rho}\frac{\partial \overline{u}_i}{\partial x_i} && (5.27)\\ &= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_i}{\partial x_i} && (5.28)\\ &= -\frac{1}{\rho}\frac{\partial \overline{p}\,\overline{u}_j}{\partial x_j} && (5.29)\\ \overline{u}_i \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}) &= \frac{\partial}{\partial x_j}[\overline{u}_i(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j})] - \frac{\partial \overline{u}_i}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_i u'_j}) && (5.30)\\ &= -\nu\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_i}{\partial x_j}\overline{u'_i u'_j} + \frac{\partial}{\partial x_j}(\nu\frac{\partial K}{\partial x_j}) - \frac{\partial \overline{u}_i\,\overline{u'_i u'_j}}{\partial x_j} && (5.31) \end{aligned}$$

Equaling both sides we have

$$\frac{\bar{D}K}{Dt} = \frac{\partial}{\partial x_j}(\underbrace{-\frac{1}{\rho}\overline{p}\,\overline{u}_j}_{\substack{\text{pressure}\\ \text{distortion}}} \underbrace{+\nu\frac{\partial K}{\partial x_j}}_{\substack{\text{molecular}\\ \text{diffusion}}} \underbrace{-\overline{u}_i\,\overline{u_i'u_j'}}_{\substack{\text{turbulent}\\ \text{diffusion}}}) \underbrace{-\frac{1}{2}P_{kk}}_{\substack{\text{production}\\ \text{of TKE}}} \underbrace{-\nu\left(\frac{\partial\overline{u}_i}{\partial x_j}\right)^2}_{\text{dissipation}} \tag{5.32}$$

where the term

$$P_{kk} = -2\overline{u_i'u_j'}\frac{\partial\overline{u}_i}{\partial x_j} \tag{5.33}$$

is the production term of the turbulent kinetic energy, and, on the other hand, is the sink in MKE.

5.1.6 TKE equation

Similarly, multiply (5.20) by u_i' and then take the average

$$\text{LHS} = \overline{u_i'(\frac{\partial u_i'}{\partial t} + \overline{u_k}\frac{\partial u_i'}{\partial x_k})} \tag{5.34}$$

$$= \frac{\partial\frac{1}{2}\overline{u_i'u_i'}}{\partial t} + \overline{u_k}\frac{\partial\frac{1}{2}\overline{u_i'u_i'}}{\partial x_k} \tag{5.35}$$

$$= \frac{\bar{D}k}{Dt} \tag{5.36}$$

$$\overline{u_i'(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{1}{\rho}\frac{\partial\overline{p'u_i'}}{\partial x_i'} \tag{5.37}$$

$$= -\frac{1}{\rho}\frac{\partial\overline{p'u_k'}}{\partial x_k'} \tag{5.38}$$

$$\overline{u_i'(\frac{\partial}{\partial x_k}\nu\frac{\partial u_i'}{\partial x_k})} = \overline{\frac{\partial}{\partial x_k}(\nu u_i'\frac{\partial u_i'}{\partial x_k}) - \nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} \tag{5.39}$$

$$= \overline{\frac{\partial}{\partial x_k}(\nu\frac{\partial\frac{1}{2}u_i'u_i'}{\partial x_k}) - \nu\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}} \tag{5.40}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial k}{\partial x_k}) - \nu\overline{(\frac{\partial u_i'}{\partial x_k})^2} \tag{5.41}$$

$$\overline{u_i'(\frac{\partial}{\partial x_k}\overline{u_i'u_k'})} = 0 \tag{5.42}$$

$$\overline{-u_i'(\frac{\partial}{\partial x_k}u_i'u_k')} = -\frac{1}{2}\frac{\partial\overline{u_i'u_i'u_k'}}{\partial x_k} \tag{5.43}$$

$$= -\frac{1}{2}\frac{\partial\overline{u_i'u_i'u_k'}}{\partial x_k} \tag{5.44}$$

$$\overline{u_i'(-u_k'\frac{\partial\overline{u}_i}{\partial x_k})} = -\overline{u_i'u_k'}\frac{\partial\overline{u}_i}{\partial x_k} \tag{5.45}$$

Equaling both sides we have

$$\frac{\bar{D}k}{Dt} = \frac{\partial}{\partial x_k}(\underbrace{\nu\frac{\partial k}{\partial x_k}}_{\substack{\text{molecular}\\ \text{diffusion}}} \underbrace{-\frac{1}{2}\overline{u_i'u_i'u_k'}}_{\substack{\text{turbulent}\\ \text{diffusion}}} \underbrace{-\frac{1}{\rho}\overline{p'u_k'}}_{\substack{\text{pressure}\\ \text{distortion}}}) + \underbrace{\frac{1}{2}P_{kk}}_{\substack{\text{production}\\ \text{of TKE}}} \underbrace{-\nu\overline{\left(\frac{\partial u_i'}{\partial x_k}\right)\left(\frac{\partial u_i'}{\partial x_k}\right)}}_{\text{(pseudo-) dissipation}} \tag{5.46}$$

Comments:

(1) The turbulent kinetic energy production term

$$P = \frac{1}{2}P_{kk} = -\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.47}$$

can be expressed in tensor notation as

$$P = \boldsymbol{\tau}_{\text{Rey}} : \nabla \overline{\boldsymbol{u}} = \boldsymbol{\tau}_{\text{Rey}} : \bar{\boldsymbol{S}} \tag{5.48}$$

where the inner product represents the projection of the velocity fluctuation correlation on the mean shear/strain rate. The convention of $P_{kk}/2$ is to be consistent with the notations in (5.72).

(2) The (pseudo-) dissipation term

$$\tilde{\varepsilon} = \nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)\left(\frac{\partial u_i'}{\partial x_j}\right)} = \nu\overline{S_{ij}'S_{ij}'} + \nu\overline{\Omega_{ij}'\Omega_{ij}'} \tag{5.49}$$

is always positive, representing the dissipation mechanism of turbulence kinetic energy. We can also see that (perturbation) enstrophy is directly linked to the dissipation rate of TKE/total KE. We note that the relations

$$\boldsymbol{S} : \boldsymbol{\Omega} = 0 \tag{5.50}$$

and

$$\nabla \boldsymbol{u} : \nabla \boldsymbol{u} = \boldsymbol{S} : \boldsymbol{S} + \boldsymbol{\Omega} : \boldsymbol{\Omega} \tag{5.51}$$

also carries for the perturbation quantities. Also note $\|\boldsymbol{\Omega}\|^2 = \|\boldsymbol{\omega}\|^2/2$ – the perturbation enstrophy is largely related to turbulent dissipation.

(3) Note that eqn. (5.46) can alternatively be written as

$$\frac{\bar{D}k}{Dt} = \frac{\partial}{\partial x_k}(\underbrace{2\nu\overline{u_j' s_{ij}'}}_{\substack{\text{molecular}\\ \text{diffusion}}} - \underbrace{\frac{1}{2}\overline{u_i' u_i' u_k'}}_{\substack{\text{turbulent}\\ \text{diffusion}}} \underbrace{-\frac{1}{\rho}\overline{p' u_k'}}_{\substack{\text{pressure}\\ \text{distortion}}}) + \underbrace{\frac{1}{2}P_{kk}}_{\substack{\text{production}\\ \text{of TKE}}} \underbrace{- 2\nu\overline{s_{ij}' s_{ij}'}}_{\text{dissipation}} \tag{5.52}$$

with the viscous term in the original equation being $\tau_{ij,j} = 2\nu s_{ij,j}$ instead of a Laplacian, and the relation between dissipation and pseudo-dissipation being

$$\varepsilon = \tilde{\varepsilon} + \nu\frac{\partial^2 \overline{u_i' u_j'}}{\partial x_i \partial x_j}, \tag{5.53}$$

which is small at vanishing nu or can be absorbed into the transport term otherwise.

5.1.7 Reynolds stress transport equation

The velocity fluctuation transport equation is

$$\frac{\bar{D}u_i'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u_i'}{\partial x_j} + \overline{u_i' u_j'} - u_i' u_j') - u_j'\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.54}$$

Or if we exchange the two subscripts we obtain:

$$\frac{\bar{D}u_j'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \frac{\partial}{\partial x_i}(\nu\frac{\partial u_j'}{\partial x_i} + \overline{u_i' u_j'} - u_i' u_j') - u_i'\frac{\partial \overline{u}_j}{\partial x_i} \tag{5.55}$$

$u_j' \times (19) + u_i' \times (20)$ and take the time average:

$$\text{LHS} = \frac{\bar{D}\overline{u_i'u_j'}}{Dt} \tag{5.56}$$

$$\text{RHS}_1 = -\frac{1}{\rho}[-2\overline{p's_{ij}} + \frac{\partial}{\partial x_i}(\overline{p'u_j'}) + \frac{\partial}{\partial x_j}(\overline{p'u_i'})] \tag{5.57}$$

$$\text{RHS}_2 = \overline{u_j'\frac{\partial}{\partial x_k}(\nu\frac{\partial u_i'}{\partial x_k}) + u_i'\frac{\partial}{\partial x_k}(\nu\frac{\partial u_j'}{\partial x_k})} \tag{5.58}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u_i'u_j'}}{\partial x_k}) - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{5.59}$$

$$\text{RHS}_3 = \overline{u_j'\frac{\partial}{\partial x_k}\overline{u_i'u_k'} + u_i'\frac{\partial}{\partial x_k}\overline{u_j'u_k'}} \tag{5.60}$$

$$= 0 \tag{5.61}$$

$$\text{RHS}_4 = -\overline{u_j'\frac{\partial}{\partial x_k}(u_i'u_k') + u_i'\frac{\partial}{\partial x_k}(u_j'u_k')} \tag{5.62}$$

$$= -\overline{u_j'u_k'\frac{\partial}{\partial x_k}(u_i') + u_i'u_k'\frac{\partial}{\partial x_k}(u_j') + u_i'u_j'\frac{\partial}{\partial x_k}(u_k')} \tag{5.63}$$

$$(Continuity, \frac{\partial u_k'}{\partial x_k} = 0, is\, used\, twice\, here.) \tag{5.64}$$

$$= -\frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'} \tag{5.65}$$

$$\text{RHS}_5 = \overline{-u_k'u_j'\frac{\partial\overline{u}_i}{\partial x_k} - u_k'u_i'\frac{\partial\overline{u}_j}{\partial x_k}} \tag{5.66}$$

$$= -\overline{u_k'u_j'}\frac{\partial\overline{u}_i}{\partial x_k} - \overline{u_k'u_i'}\frac{\partial\overline{u}_j}{\partial x_k} \tag{5.67}$$

$$\tag{5.68}$$

By equalizing both sides we obtain

$$\frac{\bar{D}\overline{u_i'u_j'}}{Dt} = \frac{2}{\rho}\overline{p's_{ij}} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p'u_j'})\delta_{ik} - \frac{1}{\rho}\frac{\partial}{\partial x_k}(\overline{p'u_i'})\delta_{jk} + \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u_i'u_j'}}{\partial x_k}) \tag{5.69}$$

$$- 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} - \frac{\partial}{\partial x_k}\overline{u_i'u_j'u_k'} - \overline{u_k'u_j'}\frac{\partial\overline{u}_i}{\partial x_k} - \overline{u_k'u_i'}\frac{\partial\overline{u}_j}{\partial x_k} \tag{5.70}$$

$$= \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'u_j'u_k'} - \frac{1}{\rho}\overline{p'u_i'}\delta_{jk} - \frac{1}{\rho}\overline{p'u_j'}\delta_{ik}) \tag{5.71}$$

$$- (\overline{u_k'u_j'}\frac{\partial\overline{u}_i}{\partial x_k} + \overline{u_k'u_i'}\frac{\partial\overline{u}_j}{\partial x_k}) + \frac{2}{\rho}\overline{p's_{ij}} - 2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{5.72}$$

Equaling both sides we have

$$\frac{\bar{D}\overline{u_i'u_j'}}{Dt} = d_{ij} + P_{ij} + \Phi_{ij} - \varepsilon_{ij} \tag{5.73}$$

where

$$d_{ij} = \frac{\partial}{\partial x_k}(\nu\frac{\partial\overline{u_i'u_j'}}{\partial x_k} - \overline{u_i'u_j'u_k'} - \frac{1}{\rho}\overline{p'u_i'}\delta_{jk} - \frac{1}{\rho}\overline{p'u_j'}\delta_{ik}) \tag{5.74}$$

$$P_{ij} = -\overline{u_k'u_j'}\frac{\partial\overline{u}_i}{\partial x_k} - \overline{u_k'u_i'}\frac{\partial\overline{u}_j}{\partial x_k} \tag{5.75}$$

$$\Phi_{ij} = \frac{2}{\rho}\overline{p' s_{ij}} \tag{5.76}$$

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{5.77}$$

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \tag{5.78}$$

Comments:

(1) The left hand side term $\frac{\bar{D}\overline{u_i'u_j'}}{Dt}$ is the rate of change of the Reynolds stress along the particle line.

(2) The term d_{ij} is the diffusion term in the equation, appearing in the form of gradient. It includes viscous term, Reynolds stress term and pressure-velocity fluctuation coupling term.The diffusion is resulted by the spatial non-uniformity of these property.

(3) The term P_{ij} is the generation term of Reynolds stress, showed in the form of the product of Reynolds stress and the mean flow strain rate.

(4) The term Φ_{ij} is the redistribution term. We note that the contraction of Reynolds stress transport equation is the transport equation for turbulence kinetic energy. And the contraction of Φ_{ij} is $\Phi_{ii} = \frac{2}{\rho}\overline{p' s_{ii}} = 0$ as continuity holds. So the term contributes nothing to the growth of turbulent kinetic energy. It just takes the kinetic energy from one component of fluid motion to another component.

(5) The term ε_{ij}, whose contraction is positive forever, representing the dissipation mechanism of kinetic energy.

5.1.8 Dissipation rate transport equation

The dissipation term in Reynolds stresses transport equation is defined as

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_j'}{\partial x_p}} \tag{5.79}$$

Multiply equation (5.20) by $2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}$ and take the time derivative we have:

$$\text{LHS} = 2\nu \frac{\bar{D}}{Dt}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_p}} = \frac{\bar{D}\varepsilon}{Dt} + 2\nu \frac{\partial \overline{u_k}}{\partial x_p}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} \tag{5.80}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-\frac{1}{\rho}\frac{\partial p'}{\partial x_i})} = -\frac{2\nu}{\rho}\frac{\partial}{\partial x_k}(\overline{\frac{\partial u_k'}{\partial x_p}\frac{\partial p'}{\partial x_p}}) \tag{5.81}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}(\nu \frac{\partial u_i'}{\partial x_k}))} = \frac{\partial}{\partial x_k}(\nu \frac{\partial \varepsilon}{\partial x_k}) - 2\overline{(\nu \frac{\partial^2 u_i'}{\partial x_p \partial x_k})^2} \tag{5.82}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k}\overline{u_i'u_k'})} = 0 \tag{5.83}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(\frac{\partial}{\partial x_k} - u_i'u_k')} = -2\nu \overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} + \frac{\partial}{\partial x_k}\overline{u_k'\nu(\frac{\partial u_i'}{\partial x_p})^2} \tag{5.84}$$

$$= -2\nu \overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}\frac{\partial u_i'}{\partial x_k}} + \frac{\partial}{\partial x_k}\overline{u_k'\varepsilon'} \tag{5.85}$$

$$\overline{2\nu \frac{\partial u_i'}{\partial x_p}\frac{\partial}{\partial x_p}(-u_k'\frac{\partial \overline{u}_i}{\partial x_k})} = -2\nu \frac{\partial \overline{u}_i}{\partial x_k}\overline{\frac{\partial u_i'}{\partial x_p}\frac{\partial u_k'}{\partial x_p}} - 2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_p}\overline{u_k'\frac{\partial u_i'}{\partial x_p}} \tag{5.86}$$

By equalizing both side we yield the transport equation for turbulence dissipation rate

$$\frac{\bar{D}\varepsilon}{Dt} = \frac{\partial}{\partial x_k}\left(-\frac{2\nu}{\rho}\overline{\frac{\partial u_k}{\partial x_p}\frac{\partial p}{\partial x_p}} + \nu\frac{\partial \varepsilon}{\partial x_k} - \overline{u'_k\varepsilon'}\right) - 2\nu\frac{\partial \overline{u}_i}{\partial x_k}\left(\overline{\frac{\partial u'_i}{\partial x_p}\frac{\partial u'_k}{\partial x_p}} + \overline{\frac{\partial u'_p}{\partial x_k}\frac{\partial u'_p}{\partial x_i}}\right) \tag{5.87}$$

$$- 2\nu\overline{u'_k\frac{\partial u'_i}{\partial x_p}}\frac{\partial^2 \overline{u}_i}{\partial x_p \partial x_k} - 2\nu\overline{\frac{\partial u'_i}{\partial x_p}\frac{\partial u'_k}{\partial x_p}\frac{\partial u'_i}{\partial x_k}} - 2\overline{\left(\nu\frac{\partial u'_i}{\partial^2 x_p \partial x_k}\right)^2} \tag{5.88}$$

The final equation of the equation agrees with that given in the turbulence book by Shi (1994). This should be the most complicated RANS equation we attempt here. We can see that the RANS second moment equations are never closed and a closure is needed (Chou, 1945).

5.1.9 Scalar flux, its mean and variance transport equations

Similar to Eq. (5.20) we have the transport equation for the mean and fluctuation of a passive scalar c:

$$\frac{\bar{D}\bar{c}}{\bar{D}t} = \frac{\partial}{\partial x_j}(\nu\frac{\partial \bar{c}}{\partial x_j} - \overline{c'u'_j}) \tag{5.89}$$

and

$$\frac{\bar{D}c'}{Dt} = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial c'}{\partial x_j} + \overline{c'u'_j} - c'u'_j) - u'_j\frac{\partial \bar{c}}{\partial x_j} \tag{5.90}$$

where Γ is the molecular diffusion coefficient of c.

Take $c'\times$ (5.20) $+u'_i\times$ (5.90) and apply the average

$$\text{LHS} = \frac{\bar{D}\overline{c'u'_i}}{\bar{D}t} \tag{5.91}$$

$$\text{RHS}_1 = -\frac{1}{\rho}\overline{c'\frac{\partial p'}{\partial x_i}} = -\frac{1}{\rho}(\frac{\partial}{\partial x_j}\overline{p'c'}\delta_{ij} - \overline{p'\frac{\partial c'}{\partial x_i}}) \tag{5.92}$$

$$\text{RHS}_2 = \frac{\partial}{\partial x_j}(\Gamma\overline{u'_i\frac{\partial c'}{\partial x_j}} + \nu\overline{c'\frac{\partial u'_i}{\partial x_j}}) - (\nu+\Gamma)\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.93}$$

$$\text{RHS}_3 = -\frac{\partial}{\partial x_j}(\overline{c'u'_iu'_j}) \tag{5.94}$$

$$\text{RHS}_4 = -\overline{c'u'_j}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_iu'_j}\frac{\partial \bar{c}}{\partial x_j} \tag{5.95}$$

then we obtain the transport equation for scalar flux

$$\frac{\bar{D}\overline{c'u'_i}}{\bar{D}t} = d_{jc} + P_{jc} + \Phi_{jc} - \varepsilon_{jc} \tag{5.96}$$

where

$$d_{ic} = \frac{\partial}{\partial x_j}(\Gamma\overline{u'_i\frac{\partial c'}{\partial x_j}} + \nu\overline{c'\frac{\partial u'_i}{\partial x_j}} - \frac{1}{\rho}\overline{p'c'}\delta_{ij} - \overline{c'u'_iu'_j}) \tag{5.97}$$

$$P_{ic} = -\overline{c'u'_j}\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_iu'_j}\frac{\partial \bar{c}}{\partial x_j} \tag{5.98}$$

$$\Phi_{ic} = \frac{1}{\rho}\overline{p'\frac{\partial c'}{\partial x_i}} \tag{5.99}$$

$$\varepsilon_{ic} = (\nu+\Gamma)\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.100}$$

Comments:

(1) Gradient diffusion: velocity-fluctuation scalar-diffusion correlation, momentum-diffusion scalar-fluctation ccorelation, pressure diffusion, turbulence diffusion.

(2) Production: scalar flux interacting with mean shear, turbulent flux (Reynolds stresses) interacting with mean scalar gradient.

(3) Re-distribution.

(4) Dissipation.

Define scalar mean and variance/fluctuation energy as

$$K_c = \frac{1}{2}\bar{c}^2 \tag{5.101}$$

$$k_c = \frac{1}{2}\overline{c'c'} \tag{5.102}$$

$c' \times$ (5.90) and apply the average

$$\text{LHS} = \frac{\bar{D}k_c}{\bar{D}t} \tag{5.103}$$

$$\text{RHS}_1 = \frac{\partial}{\partial x_j}\Gamma\frac{\partial k_c}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.104}$$

$$\text{RHS}_2 = -\frac{1}{2}\frac{\partial}{\partial x_j}\overline{c'c'u_j} \tag{5.105}$$

$$\text{RHS}_3 = -\overline{c'u'_j}\frac{\partial \bar{c}}{\partial x_j} \tag{5.106}$$

then we obtain the transport equation for scalar fluctuation energy

$$\frac{\bar{D}k_c}{\bar{D}t} = \frac{\partial}{\partial x_j}(\Gamma\frac{\partial}{\partial x_j}k_c - \frac{1}{2}\overline{c'c'u'_j}) - \overline{c'u'_j}\frac{\partial \bar{c}}{\partial x_j} - \Gamma\overline{\frac{\partial c'}{\partial x_j}\frac{\partial c'}{\partial x_j}} \tag{5.107}$$

For active scalar (for example, density which appears in the momentum equation as buoyancy force), see section 6.7.

5.1.10 Poisson equation for mean and fluctuation pressure

The Reynolds average equation is

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} - \overline{u'_iu'_j}) \tag{5.108}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} \tag{5.109}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2\overline{p} - \frac{\partial^2\,\overline{u'_iu'_j}}{\partial x_i\partial x_j} \tag{5.110}$$

Poisson equation for mean pressure:

$$-\frac{1}{\rho}\nabla^2\overline{p} = \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial^2\,\overline{u'_iu'_j}}{\partial x_i\partial x_j} \tag{5.111}$$

The velocity fluctuation transport equation is

$$\frac{\bar{D}u'_i}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial u'_i}{\partial x_j} + \overline{u'_i u'_j} - u'_i u'_j) - u'_j\frac{\partial \overline{u}_i}{\partial x_j} \tag{5.112}$$

Take the divergence of the equation:

$$\text{LHS} = \frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} \tag{5.113}$$

$$\text{RHS} = -\frac{1}{\rho}\nabla^2 p' - \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j} - \frac{\partial u'_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} - \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} \tag{5.114}$$

Poisson equation for fluctuation pressure:

$$\frac{1}{\rho}\nabla^2 p' = -\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} - \frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} - \frac{\partial u'_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} + \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j} \tag{5.115}$$

$$= -\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} + \frac{\partial^2 \overline{u'_i u'_j}}{\partial x_i \partial x_j} - 2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} \tag{5.116}$$

$$= -2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} - \frac{\partial^2}{\partial x_i x_j}(u'_i u'_j - \overline{u'_i u'_j}) \tag{5.117}$$

where the last equation is due to the fact that

$$\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} = \frac{\partial^2 u'_i u'_j}{\partial x_i x_j}. \tag{5.118}$$

According to the source terms, the pressure can be decomposed into the rapid, slow, and harmonic components

$$p' = p^{(r)} + p^{(s)} + p^{(h)}, \tag{5.119}$$

such that

$$\frac{1}{\rho}\nabla^2 p^{(r)} = -2\frac{\partial \overline{u}_j}{\partial x_i}\frac{\partial u'_i}{\partial x_j} \tag{5.120}$$

$$\frac{1}{\rho}\nabla^2 p^{(s)} = -\frac{\partial^2}{\partial x_i x_j}(u'_i u'_j - \overline{u'_i u'_j}) \tag{5.121}$$

$$\nabla^2 p^{(h)} = 0 \tag{5.122}$$

and modelled separately.

5.1.11 Turbulent vorticity and enstrophy

Similarly, vorticity can be decomposed into the mean and the pertubation. We give the equation of pertuabtion vorticity without derivation:

$$\frac{\bar{D}\omega'_i}{\bar{D}t} = \omega'_j\overline{S}_{ij} + \overline{\omega_j}S'_{ij} + \omega'_j S'_{ij} - \overline{\omega'_j S'_{ij}} - u'_j\frac{\partial \overline{\omega}_i}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u'_j\omega'_i} - u'_j\omega'_i) + \nu\frac{\partial^2 \omega'_i}{\partial x_j^2} \tag{5.123}$$

where $\overline{S}_{ij}$ and S'_{ij} are the mean and the fluctuation shear, respectively.

We define the fluctuating enstrophy as

$$\mathscr{E} = \frac{1}{2}\overline{\omega'_i\omega'_i} \tag{5.124}$$

$\omega_i' \times$ (5.123) and take the time average

$$\text{LHS} = \frac{\bar{D}\mathscr{E}}{\bar{D}t} \tag{5.125}$$

$$\text{RHS}_1 = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega}_j\overline{\omega_i' S_{ij}'} + \overline{\omega_i'\omega_j' S_{ij}'} \tag{5.126}$$

$$\text{RHS}_2 = -\overline{\omega_i' u_j'}\frac{\partial \overline{\omega_i}}{\partial x_j} \tag{5.127}$$

$$\text{RHS}_3 = -\frac{1}{2}\frac{\partial}{\partial x_j}(\overline{u_j'\omega_i'\omega_i'}) \tag{5.128}$$

$$\text{RHS}_4 = \nu\overline{\frac{\partial^2 \mathscr{E}}{\partial x_j^2}} - \overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} \tag{5.129}$$

Equaling both sides we obtain

$$\frac{\bar{D}\mathscr{E}}{\bar{D}t} = P_{\mathscr{E}} + D_{\mathscr{E}} - \varepsilon_{\mathscr{E}} \tag{5.130}$$

$$P_{\mathscr{E}} = \overline{\omega_i'\omega_j'}\,\overline{S_{ij}} + \overline{\omega_j}\overline{\omega_i' S_{ij}'} + \overline{\omega_i'\omega_j' S_{ij}'} - \overline{\omega_i' u_j'}\frac{\partial \overline{\omega_i}}{\partial x_j} \tag{5.131}$$

$$D_{\mathscr{E}} = \frac{\partial}{\partial x_j}(\nu\frac{\partial \mathscr{E}}{\partial x_j} - \frac{1}{2}\overline{u_j'\omega_i'\omega_i'}) \tag{5.132}$$

$$\varepsilon_{\mathscr{E}} = \nu\overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} \tag{5.133}$$

Comment: The energy balance process of fluctuation enstrophy obeys four principle processes in nature (Kolmogorov):

change rate = production + diffusion + dissipation

Moreover, in instability problems, it is convenience to consider the linearized inviscid perturbation vorticity equation, which reads

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j'\bar{S}_{ij} + \bar{\omega}_j S_{ij}' - u_j'\frac{\partial \bar{\omega}_i}{\partial x_j}, \tag{5.134}$$

since perturbation enstrophy is related to turbulent dissipation (see 5.51). It has a version in rotating frames in section 6.86.

5.2 Farve average in compressible flows

5.3 LES equations

5.4 Theory of homogeneous isotropic turbulence

K-H etc.

Taylor microscale (velocity gradient based on that and urms, dissipation) and decorrelation; scale separation v.s. Re;

5.5 Scales of turbulent motions

5.5.1 Kolmogorov scale

The Kolmogorov scales are:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \, u_\eta = (\epsilon\nu)^{1/4}, \tau_\eta = (\nu/\epsilon)^{1/2}. \tag{5.135}$$

Note the scales depend only on viscosity and dissipation rate (hence the expressions above can be derived from a dimensional analysis).

A resulting Reynolds number

$$Re_\eta = \frac{u_\eta \eta}{\nu} = 1 \tag{5.136}$$

indicates that viscous effect is active at the Kolmogorov scale.

5.5.2 Taylor microscale

Also the idea of using urms and lambdaf to estimate dissipation.

Taylor Reynolds number

5.5.3 Other useful scales

Corrsin scale (below which the flow doesn't feel the shear and turbulence in different shear flows are similar).

Ozmidov scale.

The Ozimidov scale is

$$L_O = \left(\frac{\epsilon}{N^3}\right)^{1/2} \tag{5.137}$$

and the Kolmogorov scale is

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \tag{5.138}$$

the ratio between these two gives $Re_b^{3/4}$.

Buoyancy Reynolds number:

$$Re_b = \frac{\varepsilon}{\nu N^2}. \tag{5.139}$$

It can be regarded as a measure of the separation between scales unaffected by stratification and the dissipation range, and hence an indicator of the extent of small-scale isotropy.

Taking L_O as the vertical length scale for overturns and $q_h = \sqrt{(\langle u'^2 + v'^2 \rangle)}/2$ to be the velocity scale, the Froude number is unity hence the scales greater than L_O are influenced by stratification.

5.6 Turbulent free shear flows

5.6.1 Momentum integral

Similarity solutions (turbulent). Pope (2001).

5.6.2 Similarity solutions

The characteristic velocity and length scales are U_s and δ_s, respectively.

The example of plane jet is the easiest to understand and derive so we are the most detailed in that case and more loosely on the others. The same principles and machinery apply to all cases.

5.6.3 Round jet

Characteristic scales:

The centerline velocity is

$$U_s(x) = \bar{u}(x, r = 0) \tag{5.140}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = \bar{u}(x, r_{1/2}(x)) = \frac{1}{2} U_s(x). \tag{5.141}$$

Flow type	U_s	δ_s	$U_s \propto x^m$	$\delta_s \propto x^n$	$f(\eta)$
Round jet	$\bar{u}(x, y=0)$	$r_{1/2}$	-1	1	$1/(1+a\eta^2)^2$
Plane jet	$\bar{u}(x, r=0)$	$y_{1/2}$	$-1/2$	1	$\operatorname{sech}^2(\ln(1+\sqrt{2})\,\eta)$
Round wake	$U_\infty - \bar{u}(x, y=0)$	$r_{1/2}$	$-2/3$	$1/3$	$\exp(-\ln 2\,\eta^2)$
Plane wake	$U_\infty - \bar{u}(x, r=0)$	$y_{1/2}$	$-1/2$	$1/2$	$\exp(-\ln 2\,\eta^2)$
Plane mixing layer	$U_2 - U_1$	$y_{0.9} - y_{0.1}$	0	1	$1/2\,\mathrm{erf}(\eta/\sigma\sqrt{2})$

Table 1: Self-similar solution table.

Momentum integral constraint:
The boundary layer equation in cylindrical coordinates reads

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{r}\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{5.142}$$

Multiply the continuity equation

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial (r\bar{v})}{\partial r} = 0 \tag{5.143}$$

by $r\bar{u}$ and add it to (5.142) multiplied by r we obtain

$$\frac{\partial (r\bar{u}\bar{u})}{\partial x} + \frac{\partial (r\bar{u}\bar{v})}{\partial r} = -\frac{\partial (r\overline{u'v'})}{\partial r}. \tag{5.144}$$

Integrate (5.144) in r we obtain

$$\int_0^\infty \frac{\partial (r\bar{u}\bar{u})}{\partial r}\,\mathrm{d}r + r\bar{u}\bar{v}|_0^\infty = -r\overline{u'v'}|_0^\infty \tag{5.145}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^\infty r\bar{u}^2\,\mathrm{d}r\right) = 0 \tag{5.146}$$

which implies the momentum flux

$$\dot{M}(x) = \int_0^\infty \rho\bar{u}^2 2\pi r\,\mathrm{d}r = J_0 \tag{5.147}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.
Self-similar assumptions:

$$\bar{u} = U_s(x)f(\eta),\ \overline{u'v'} = U_s^2(x)g(\eta) \tag{5.148}$$

where $\eta = r/\delta_s(x)$ with $\delta_s = r_{1/2}$. Substitute (5.148) into (5.147) we have

$$\dot{M}(x) = (2\pi\rho)(U_s^2\delta_s^2)\left(\int_0^\infty \eta f^2(\eta)\,\mathrm{d}\eta\right) \tag{5.149}$$

to be a constant and implying

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2\delta_s^2) = 0 \tag{5.150}$$

and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{5.151}$$

Using the continuity equation we have

$$\bar{v} = -\frac{1}{r}\int_0^r \frac{\partial(r\bar{u})}{\partial x}\,\mathrm{d}y = U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{\eta}\int_0^\eta f\eta\,\mathrm{d}\eta\right) \tag{5.152}$$

We note that $\bar{v}$ switch sign from positive to negative when r is greater than a certain value (entrainment).

Next we establish the constant spread rate of the round jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left[f^2\eta + ff'\eta + \left(\frac{f}{\eta} + f'\right)\int_0^\eta f\eta\,\mathrm{d}\eta\right] = g + g'\eta \tag{5.153}$$

and then $\mathrm{d}\delta_s/\mathrm{d}x$ has to be a constant. Combining with momentum integral restriction we have

$$\delta_s \propto x,\ U_s \propto x^{-1}. \tag{5.154}$$

5.6.4 Plane jet

Characteristic scales:
The centerline velocity is

$$U_s(x) = \bar{u}(x, y=0) \tag{5.155}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = \bar{u}(x, y_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{5.156}$$

Momentum integral constraint:
The boundary layer equation for the mean velocity simplifies to

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} = -\frac{\partial\overline{u'v'}}{\partial y}. \tag{5.157}$$

Multiply the continuity equation

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} = 0 \tag{5.158}$$

by $\bar{u}$ and add it to (5.157) we obtain

$$\frac{\partial\bar{u}\bar{u}}{\partial x} + \frac{\partial\bar{u}\bar{v}}{\partial y} = -\frac{\partial\overline{u'v'}}{\partial y}. \tag{5.159}$$

Integrate (5.159) in y we obtain

$$\int_{-\infty}^{\infty}\frac{\partial\bar{u}\bar{u}}{\partial x}\,\mathrm{d}y + \bar{u}\bar{v}|_{-\infty}^{\infty} = -\overline{u'v'}|_{-\infty}^{\infty} \tag{5.160}$$

and since $\overline{u'v'}$ and $\bar{u}$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty}\bar{u}^2\,\mathrm{d}y\right) = 0 \tag{5.161}$$

which implies the momentum flux

$$\dot{M}(x) = \int_{-\infty}^{\infty} \rho \bar{u}^2 \, \mathrm{d}y = J_0 \tag{5.162}$$

is conserved (as a result of both mass and momentum conservation), where J_0 is the jet exit strength.
Self-similar assumptions:

$$\bar{u} = U_s(x) f(\eta), \; \overline{u'v'} = U_s^2(x) g(\eta) \tag{5.163}$$

where $\eta = y/\delta_s(x)$ and we have

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{\delta_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{5.164}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta_s} \tag{5.165}$$

Substitute (5.163) into (5.162) we have

$$\dot{M}(x) = (U_s^2 \delta_s)\left(\int_{-\infty}^{\infty} f^2(\eta) \, \mathrm{d}\eta\right) \tag{5.166}$$

is a constant. So it must be

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s^2 \delta_s) = 0 \tag{5.167}$$

which gives the momentum flux constraint in terms of characteristic variables, and hence

$$\frac{\delta_s}{U_s}\frac{\mathrm{d}U_s}{\mathrm{d}x} = -\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} \tag{5.168}$$

Using the continuity equation we have

$$\bar{v} = -\int_0^y \frac{\partial \bar{u}}{\partial x} \, \mathrm{d}y = U_s \frac{\mathrm{d}\delta_s}{\mathrm{d}x}\left(\eta f - \frac{1}{2}\int_0^\eta f \, \mathrm{d}\eta\right) \tag{5.169}$$

Next we establish the constant spread rate of the plane jet (i.e. $\mathrm{d}\delta_s/\mathrm{d}x$ is a constant). Take $\bar{v}$ into the momentum equation we have

$$\frac{1}{2}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(f^2 + f' \int_0^\eta f \, \mathrm{d}\eta) = g' \tag{5.170}$$

and then

$$\frac{\mathrm{d}\delta_s}{\mathrm{d}x} = \frac{2g'}{f^2 + f' \int_0^\eta f \, \mathrm{d}\eta} = C \tag{5.171}$$

with the LHS only depend on x and RHS only depend on η. Then both sides have to be constant. Combining (5.171) and (5.167) we have

$$\delta_s \propto x, \, U_s \propto x^{-1/2}. \tag{5.172}$$

5.6.5 Round wake

Characteristic scales:
The centerline velocity deficit is

$$U_0(x) = U_\infty - \bar{u}(x, r = 0) = U_d(x, 0) \tag{5.173}$$

and the characteristic length is the half width, $\delta_s = r_{1/2}(x)$, such that

$$U_d(x, r_{1/2}) = U_\infty - \bar{u}(x, r_{1/2}(x)) = \frac{1}{2}U_0(x). \tag{5.174}$$

Momentum integral constraint:
Here we start from the simplified (see plane wake) momentum equation

$$U_\infty \frac{\partial \bar{u}}{\partial x} = -\frac{1}{r}\frac{\partial (r\overline{u'v'})}{\partial r} \tag{5.175}$$

and the momentum deficit flux conservation

$$\dot{M}(x) = \int_0^\infty \rho U_\infty (U_\infty - \bar{u}) 2\pi r \, \mathrm{d}r. \tag{5.176}$$

Note that we have already replaced the $\bar{u}$ with U_∞ assuming (or by order of magnitude analysis) the convection velocity is U_∞.
Self-similar assumptions:

$$U_\infty - \bar{u} = U_s(x)f(\eta), \; \overline{u'v'} = U_s^2(x)g(\eta) \tag{5.177}$$

We have

$$\dot{M}(x) = (U_s\delta_s^2)(2\pi\rho U_\infty)\int_0^\eta f \, \mathrm{d}\eta \tag{5.178}$$

is a constant and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s\delta_s^2) = 0. \tag{5.179}$$

Consider the momentum equation, the other constraint reads

$$-\frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}(2f + f'\eta)\eta = (g'\eta + g) \tag{5.180}$$

We define the spread rate as

$$S = \frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}, \tag{5.181}$$

it has to be a constant. Then

$$-S(2f\eta + f'\eta^2) = (g\eta)' \tag{5.182}$$

and including boundary conditions after integration we get

$$g = -S\eta f \tag{5.183}$$

same as in plane wakes. Combining (5.179) and (5.181) we have

$$\delta_s \propto x^{1/3}, \; U_s \propto x^{-2/3}. \tag{5.184}$$

5.6.6 Plane wake

Characteristic scales:
The centerline velocity deficit is

$$U_s(x) = U_\infty - \bar{u}(x, y = 0) = U_d(x, 0) \tag{5.185}$$

and the characteristic length is the half width, $\delta_s = y_{1/2}(x)$, such that

$$U_d(x, y_{1/2}) = U_\infty - \bar{u}(x, y_{1/2}(x)) = \frac{1}{2}U_s(x). \tag{5.186}$$

Momentum integral constraint:
The boundary layer equation:

$$\bar{u}\frac{\partial(\bar{u}-U_\infty)}{\partial x}+\bar{v}\frac{\partial(\bar{u}-U_\infty)}{\partial y}=\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}=-\frac{\partial\overline{u'v'}}{\partial y}. \tag{5.187}$$

Multiply the continuity equation

$$\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}=0 \tag{5.188}$$

by $\bar{u}-U_\infty$ and add it to (5.187) we obtain

$$\frac{\partial\bar{u}(\bar{u}-U_\infty)}{\partial x}+\frac{\partial\bar{v}(\bar{u}-U_\infty)}{\partial y}=-\frac{\partial\overline{u'v'}}{\partial y}. \tag{5.189}$$

Integrate (5.159) in y we obtain

$$\int_{-\infty}^{\infty}\frac{\partial\bar{u}(\bar{u}-U_\infty)}{\partial x}\,\mathrm{d}y+\bar{v}(\bar{u}-U_\infty)|_{-\infty}^{\infty}=-\overline{u'v'}|_{-\infty}^{\infty} \tag{5.190}$$

and since $\overline{u'v'}$ and $\bar{u}-U_\infty$ are zero at infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{-\infty}^{\infty}\bar{u}(\bar{u}-U_\infty)\,\mathrm{d}y\right)=0 \tag{5.191}$$

which implies the momentum deficit flux

$$\dot{M}(x)=\int_{-\infty}^{\infty}\rho\bar{u}(U_\infty-\bar{u})\,\mathrm{d}y \tag{5.192}$$

is conserved (we note that we haven't assumed far wake yet).
Self-similar assumptions:

$$U_\infty-\bar{u}=U_s(x)f(\eta),\ \overline{u'v'}=U_s^2(x)g(\eta) \tag{5.193}$$

Substitute (5.193) into (5.192), and assume the far wake is reached ($U_s/U_\infty\ll 1$) we have

$$\begin{aligned}\dot{M}(x)&=\int_{-\infty}^{\infty}\rho(U_\infty-U_sf)U_sf\delta_s\,\mathrm{d}\eta &(5.194)\\ &=U_\infty^2\int_{-\infty}^{\infty}\rho(1-\frac{U_sf}{U_\infty})\frac{U_s}{U_\infty}f\delta_s\,\mathrm{d}\eta &(5.195)\\ &=\rho U_\infty U_s\delta_s\int_{-\infty}^{\infty}f\,\mathrm{d}\eta &(5.196)\end{aligned}$$

is a constant. Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}(U_s\delta_s)=0. \tag{5.197}$$

Using the continuity equation we have

$$\bar{v}=-\int_0^y\frac{\partial\bar{u}}{\partial x}\,\mathrm{d}y=-U_s\frac{\mathrm{d}\delta_s}{\mathrm{d}x}f\eta. \tag{5.198}$$

Note the negative speed corresponding to wake entrainment (of high momentum into low momentum region).

Now we consider another constraint. Since in the far wake, the velocity deficit $U_s/U_\infty \ll 1$, we have the simplification of the momentum equation as

$$\frac{\partial \bar{u}(\bar{u}-U_\infty)}{\partial x} + \frac{\partial \bar{v}(\bar{u}-U_\infty)}{\partial y} = U_\infty \frac{\partial \bar{u}}{\partial x} = -\frac{\partial \overline{u'v'}}{\partial y} \tag{5.199}$$

where

$$\bar{u}(\bar{u}-U_\infty) = (U_\infty - U_s f)(-U_s f) = U_\infty^2(1-\frac{U_s f}{U_\infty})(-\frac{U_s}{U_\infty}f) = -U_s U_\infty f = U_\infty(\bar{u}-U_\infty). \tag{5.200}$$

And the scale for $\partial \bar{u}(\bar{u}-U_\infty)/\partial x$ is

$$\frac{U_\infty U_s}{L_x} \tag{5.201}$$

while the scale for $\partial \bar{v}(\bar{u}-U_\infty)/\partial y$ (from (5.198)) is

$$\frac{U_s}{\delta_s}\left(U_s \frac{\delta_s}{L_x}\right). \tag{5.202}$$

Define the spread rate as

$$S = \frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x}. \tag{5.203}$$

Take $\bar{v}$ into the simplified momentum equation we have

$$(f + f'\eta)\frac{U_\infty}{U_s}\frac{\mathrm{d}\delta_s}{\mathrm{d}x} = -g' \tag{5.204}$$

with S depends only on x and the rest on η hence S has to be a constant. Then (5.204) can be rewritten as

$$g' + S(f + f'\eta) = 0 \tag{5.205}$$

which is to say

$$(g + S\eta f)' = 0. \tag{5.206}$$

Integrate from $\eta = 0$ to η and note that $g(0) = 0$, we have

$$g = -S\eta f. \tag{5.207}$$

Combining two conditions (5.197) and (5.203) we have

$$\delta_s \propto x^{1/2},\ U_s \propto x^{-1/2}. \tag{5.208}$$

5.6.7 Plane mixing layer

Characteristic scales:
The two velocities are $U_2 > U_1$ with U_2 on the top. The mean convection velocity is

$$U_c = \frac{1}{2}(U_1 + U_2) \tag{5.209}$$

and the characteristic velocity scale is

$$U_s = U_2 - U_1. \tag{5.210}$$

The characteristic length is the mixing layer width,

$$\delta_s(x) = y_{0.9} - y_{0.1} \tag{5.211}$$

with cross-stream location $y_\alpha(x)$ such that

$$\bar{u}(x, y_\alpha(x)) = U_1 + \alpha U_s. \tag{5.212}$$

a reference position is

$$\hat{y} = \frac{1}{2}(y_{0.1} + y_{0.9}) \tag{5.213}$$

such that the self-similar variable is defined as

$$\eta = \frac{y - \hat{y}}{\delta_s(x)} \tag{5.214}$$

5.7 Turbulent wall flows

5.7.1 Turbulent channel flow

The mean flow equations are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{5.215}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'v'}}{\partial y} \tag{5.216}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - \frac{\partial \overline{v'^2}}{\partial y} \tag{5.217}$$

Integrate (5.217) in y from 0 to y we have

$$\frac{\bar{p}}{\rho} + \overline{v'^2} = \frac{p_w}{\rho} \tag{5.218}$$

where p_w is the mean wall pressure. Hence, we have

$$\frac{\partial \bar{p}}{\partial x} = \frac{\partial p_w}{\partial x}, \tag{5.219}$$

implying that the pressure drop over the same distance is the same for different heights, even $\partial \bar{p}/\partial y \neq 0$. The wall shear stress balance (4.15) is still valid (control volume from 0 to h) as

$$-\frac{\partial p_w}{\partial x} = \frac{\tau_w}{h}. \tag{5.220}$$

Substitute (5.218) and (5.220) back to (5.216) we have

$$-\frac{\tau_w}{h} = \nu \frac{\partial^2 \bar{u}}{\partial^2 y} - \frac{\partial \overline{u'v'}}{\partial y} \tag{5.221}$$

the $\int_y^h dy$ of which leads to the channel basin equation

$$\tau(y) = \tau_w \left(1 - \frac{y}{h}\right) = \rho u_\tau^2 = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} \tag{5.222}$$

which is stating that, the total stress (viscous shear stress and Reynolds stress) decreases linearly toward the centerline. Near the wall, the viscous shear stress dominates and it decreases as the distance from the wall increases. From data, one can plot the total stress profile (which is quite linear) and figure out u_τ.

FIK;

6 Geophysical fluid dynamics

6.1 Basics

6.1.1 Non-inertial frames, centrifugal and Coriolis forces

6.1.2 Absolute velocity

In an inertial frame,

$$\boldsymbol{u}_a = \boldsymbol{u}_{(r)} + \boldsymbol{\Omega} \times \boldsymbol{r}, \tag{6.1}$$

where $\boldsymbol{u}_{(r)}$ is the relative (to the non-inertial frame) velocity. Especially, in a cylindrical frame with a plane-normal rotation,

$$u_{\theta,a} = u_\theta + \Omega r. \tag{6.2}$$

6.1.3 Inertial oscillations: buoyancy and Coriolis frequencies

6.2 Boussinesq approximation

6.3 Balanced flows

6.3.1 Hydrostatic and geostrophic balances

In balanced flow, which is usually the case to the first order, there is a background horizontal pressure gradient that balances the Coriolis forces due to horizontal motions and a vertical pressure gradient that balances the background unperturbed density:

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial x} + fV \tag{6.3}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial y} - fU \tag{6.4}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_g}{\partial z} - \frac{\rho^* g}{\rho_0} \tag{6.5}$$

with

$$p_g = p - p_0,\ \rho^* = \rho - \rho_0 - \rho_b(z) \tag{6.6}$$

and the background balance

$$0 = -\frac{\partial p_0}{\partial z} - (\rho_0 + \rho_b)g \tag{6.7}$$

already subtracted. Note that the Boussinesq and hydrostatic approximations are already applied.

The above equations in vector form:

$$\boldsymbol{f}_{\mathrm{c}} \times \boldsymbol{U} = -\frac{1}{\rho_0}\nabla p_g + \frac{\rho^*}{\rho_0}\boldsymbol{g}. \tag{6.8}$$

We have

$$\boldsymbol{U} = (U, V, 0) = -\frac{1}{\rho_0 f}\left(\frac{\partial p_g}{\partial y}, -\frac{\partial p_g}{\partial x}, 0\right). \tag{6.9}$$

And we have

$$\nabla_{\mathrm{h}} \cdot \boldsymbol{U} = 0. \tag{6.10}$$

In the world geostrophic, geo means Coriolis and strophic means cyclone/anticyclone or low-/high-pressure systems.

6.3.2 Thermal wind relations

In hydrostatic Boussinesq flow. Taking the vertical gradient of (6.8) and using the hydrostatic balance, we have

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial x} + f\frac{\partial V}{\partial z} \tag{6.11}$$

$$0 = \frac{g}{\rho_0}\frac{\partial \rho^*}{\partial y} - f\frac{\partial U}{\partial z} \tag{6.12}$$

and hence

$$(\frac{\partial U}{\partial z}, \frac{\partial V}{\partial z}) = \frac{g}{\rho_0 f}(\frac{\partial \rho^*}{\partial y}, -\frac{\partial \rho^*}{\partial x}), \tag{6.13}$$

or in vector form,

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f} \times \nabla \rho^* \tag{6.14}$$

In a more general case. Without introducing the hydrostatic balance and Boussinesq approximation, we write

$$\boldsymbol{f}_c \times \boldsymbol{U} = \frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{6.15}$$

Taking its curl:

$$\text{LHS} = \nabla \times (\boldsymbol{f}_c \times \boldsymbol{U}) = -\boldsymbol{f}_c \cdot \nabla \boldsymbol{U} = -f\frac{\partial \boldsymbol{U}}{\partial z} \tag{6.16}$$

$$\text{RHS} = -\nabla \times (\frac{1}{\rho}\nabla p) + \nabla \times \boldsymbol{g} = -\frac{1}{\rho^2}(\nabla p \times \nabla \rho) \tag{6.17}$$

Re-introduce hydrostatic ($\partial_z p = -\rho g$) and Boussinesq, we have

$$\nabla p \approx \frac{\partial p}{\partial z}\boldsymbol{e}_z = -\rho g \boldsymbol{e}_z \tag{6.18}$$

and

$$-\frac{1}{\rho^2}(\nabla p \times \nabla \rho) = \frac{\rho g}{\rho_0^2}(\boldsymbol{e}_z \times \nabla \rho^*) \tag{6.19}$$

$$= -\frac{\boldsymbol{g}}{\rho_0} \times \nabla \rho^* \tag{6.20}$$

hence we recover

$$\frac{\partial \boldsymbol{U}}{\partial z} = \frac{\boldsymbol{g}}{\rho_0 f} \times \nabla \rho^*. \tag{6.21}$$

Note:

6.3.3 Cyclostrophic wind relations

In analogy to the thermal wind (Coriolis balances the horizontal pressure gradient), we have similarly the cyclostrophic wind (centrifugal balances the radial pressure gradient):

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_0}\frac{\partial p^*}{\partial r} \tag{6.22}$$

$$\frac{\partial p^*}{\partial z} = -\frac{\rho^* g}{\rho_0} \tag{6.23}$$

$$\frac{\partial \rho^*}{\partial r} = -\frac{\rho_0}{g}\frac{\partial}{\partial z}\left(\frac{u_\theta^2}{r}\right) \tag{6.24}$$

where the mean vertical wind shear is supported with a horizontal density gradient, which provides the centripetal accelaration ($a = u_\theta^2/r$). This has something to do with the Jet Stream. Cyclo means 'cyclone' or low-pressure system and strophic means 'turning'.

6.3.4 Example: Taylor-Proudman theory

Consider steady flow with negligible convective term (in geostrophic balance, $Ro \ll 1$):

$$0 = -\frac{1}{\rho}\nabla p + \boldsymbol{u} \times \boldsymbol{f}_{\mathrm{c}}. \tag{6.25}$$

Taking the curl of the above equation we have

$$0 = \nabla \times (\boldsymbol{u} \times \boldsymbol{f}_{\mathrm{c}}). \tag{6.26}$$

Using triple product rules (A.21), in an f-plane, we have

$$\boldsymbol{f}_{\mathrm{c}} \times \nabla \boldsymbol{u} = 0. \tag{6.27}$$

Assuming the rotation axis is normal to the plane, $\boldsymbol{f}_{\mathrm{c}} = f\boldsymbol{k}$, we have

$$\frac{\partial \boldsymbol{u}}{\partial z} = \mathbf{0}. \tag{6.28}$$

That implies very strong rotation negates vertical gradients.

Stratified Taylor column: consider the thermal wind balance

$$0 = -\frac{1}{\rho_0}\nabla p + \boldsymbol{u} \times \boldsymbol{f}_{\mathrm{c}} - \frac{\rho^* g}{\rho_0}\boldsymbol{k}. \tag{6.29}$$

Taking the curl we have

$$\boldsymbol{f}_{\mathrm{c}} \cdot \nabla \boldsymbol{u} = (\boldsymbol{i}\partial_y - \boldsymbol{j}\partial_x)\frac{\rho^* g}{\rho_0}, \tag{6.30}$$

i.e.,

$$[f_3\frac{\partial u}{\partial z}, f_3\frac{\partial v}{\partial z}, f_3\frac{\partial w}{\partial z}] = [\partial_y\frac{\rho^* g}{\rho_0}, -\partial_x\frac{\rho^* g}{\rho_0}, 0], \tag{6.31}$$

i.e.,

$$f_3\frac{\partial u}{\partial z} = \partial_y\frac{\rho^* g}{\rho_0} \tag{6.32}$$

$$f_3\frac{\partial v}{\partial z} = -\partial_x\frac{\rho^* g}{\rho_0} \tag{6.33}$$

$$\frac{\partial w}{\partial z} = 0 \tag{6.34}$$

which implies Q2D flow with

$$\nabla_H \cdot \boldsymbol{u} = \partial_x u + \partial_y v = -\partial_z w = 0. \tag{6.35}$$

The relations (6.32)-(6.33) are essentially thermal wind relations where the vertical wind shear is balanced by (the pressure gradient created by) horizontal density gradients.

6.3.5 Surface and bottom Ekman layers

In geostrophy, the dominant balance is the Coriolis force and the pressure gradient. Here in the laminar Ekman layer, we consider the steady solution resulting from a balance between viscous shear stress and the Coriolis force:

$$\begin{aligned} 0 &= fv + \nu\frac{\partial^2 u}{\partial z^2} \\ 0 &= -fu + \nu\frac{\partial^2 v}{\partial z^2} \end{aligned} \tag{6.36}$$

where the shear is only in the vertical direction due to horizontally uniform winds blowing over the surface (or uni-directional drag on the ocean bottom).

We denote

$$\boldsymbol{\tau}_a = (\tau_a^X, \tau_a^Y) = \left(\nu \frac{\partial u}{\partial z}\bigg|_{z=0}, \nu \frac{\partial v}{\partial z}\bigg|_{z=0} \right) \tag{6.37}$$

and

$$\mathcal{U}(z) = u(z) + iv(z). \tag{6.38}$$

Hence,

$$0 = f(u+iv) + i\nu \frac{\partial^2}{\partial z^2}(u+iv) \tag{6.39}$$

converts to

$$\frac{\partial^2 \mathcal{U}}{\partial z^2} = \frac{if}{\nu}\mathcal{U}. \tag{6.40}$$

The eigenvalues are

$$\lambda = \pm\sqrt{\frac{if}{\nu}} = \pm\frac{1+i}{\sqrt{2}}\sqrt{\frac{f}{\nu}} \tag{6.41}$$

Define the Ekman depth as

$$d_E = \sqrt{\frac{2\nu}{f}} \tag{6.42}$$

which increases as $\sqrt{\nu}$ and decreases as $\sqrt{1/f}$, and the general solution to (6.40) is

$$\mathcal{U}(z) = C_1 \exp\left[\frac{1+i}{\sqrt{2}}\sqrt{f/\nu}z\right] + C_2 \exp\left[-\frac{1+i}{\sqrt{2}}\sqrt{f/\nu}z\right]. \tag{6.43}$$

By knowing $\mathcal{U}(z=-\infty)$ we have $C_2 = 0$ and

$$\mathcal{U}(z) = \mathcal{U}_0 \exp\left[\frac{1+i}{\sqrt{2}}\sqrt{f/\nu}z\right] = \mathcal{U}_0 e^{z/d_E} e^{iz/d_E} \tag{6.44}$$

which tells us that as you go from the surface (z decreases from zero to $z < 0$), the magnitude of the velocity decreases and the velocity vector spirals CW (as you look down), in agreement with the direction of the Coriolis force.

Now we attempt to fix the constant $\mathcal{U}_0$. Given the complex velocity, the (directional) shear stress can be compactly written as

$$\boldsymbol{\tau}_a = \tau_a^X + i\tau_a^Y. \tag{6.45}$$

And the boundary condition is

$$\nu \left.\frac{\partial \mathcal{U}}{\partial z}\right|_{z=0} = \frac{\nu(1+i)\mathcal{U}_0}{d_E}, \tag{6.46}$$

leading to

$$\mathcal{U}_0 = \frac{\boldsymbol{\tau}_a d_E}{(1+i)\rho_0 \nu} \tag{6.47}$$

where

$$\frac{1}{1+i} = \frac{1}{\sqrt{2}}e^{-i\pi/4}, \tag{6.48}$$

which implies that the relative direction of the surface flow is 45 deg to the right (CW) of the wind stress.

Now we turn our attention to the Ekman transport over a finite depth of the mixed layer, $z_E \sim -200$ m, which is a few Ekman depths. The Ekman depth $d_E = \sqrt{2\nu_{\text{eff}}/f} \sim 45$ m for $f \sim 10^{-4} \sim 2\pi \times 24$

hr^{-1}. We note that $e^{-5} = 0.0067$ which might have been sufficient to assume the stress is weak enough to be neglected.

Vertically integrate (6.36) from $-z_E$ to 0, we have

$$\mathcal{U}_E = \int_{-z_E}^{0} u\,dz = \frac{\tau_a^Y}{\rho_0 f} \tag{6.49}$$

$$\mathcal{V}_E = \int_{-z_E}^{0} v\,dz = -\frac{\tau_a^X}{\rho_0 f} \tag{6.50}$$

or in a more compact way,

$$(\mathcal{U}_E, \mathcal{V}_E) = -\frac{1}{\rho_0 f}\boldsymbol{k} \times \boldsymbol{\tau}_a \tag{6.51}$$

where we note that the dimensions for $\mathcal{U}_E, \mathcal{V}_E$ are velocity times length. The direction of the transport, $(\mathcal{U}_E, \mathcal{V}_E)$, is $(\tau_a^Y, -\tau_a^X)$, which is perpendicular to and pointing to the right of the direction of the surface winds (and the wind forcing is balanced by the mean Coriolis force that is 90 deg to the right of the Ekman transport – opposite to the surface forcing). This does not mean that the surface forcing does no work, since the velocity on the surface is at a 45 deg angle to the winds.

We note several useful directions:

- The local velocity direction: $\mathcal{U} = u + iv = (u, v)$;
- The local shear direction: $\boldsymbol{\tau} = \tau^X + i\tau^Y = (\tau^X, \tau^Y) = (\nu\partial_z u, \nu\partial_z v)$;
- And the local Reynolds stress direction: $\boldsymbol{\tau}_{Rey} = (-\overline{u'w'}, -\overline{v'w'})$.

6.4 QGPV theory

6.5 Governing equations of unbalanced motions

It is reasonable to assume directions of both system rotation and gravity are in $\boldsymbol{z}$.

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{6.52}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} - f\epsilon_{ij3}(u_j - U_j) = -\frac{1}{\rho_0}\frac{\partial p^*}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\rho^* g}{\rho_0}\delta_{i3}, \tag{6.53}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i}, \tag{6.54}$$

$$\tau_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ J_{\rho,i} = \kappa\frac{\partial \rho}{\partial x_i}. \tag{6.55}$$

In vector form,

$$\begin{gathered}
\nabla \cdot \boldsymbol{u} = 0 \\
\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) + f\boldsymbol{e}_z \times (\boldsymbol{u} - \boldsymbol{U}) = -\frac{1}{\rho_0}\nabla p^* + \nabla \cdot \boldsymbol{\tau} - \frac{\rho^* g}{\rho_0}\boldsymbol{e}_z \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = \nabla \cdot \boldsymbol{J}_\rho
\end{gathered} \tag{6.56}$$

where the stress and the scalar flux are

$$\boldsymbol{\tau} = \nu(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla),\ \boldsymbol{J}_\rho = \kappa\nabla\rho. \tag{6.57}$$

The total density ρ is decomposed into the reference density ρ_0, the background density $\rho_b(z)$, and the density perturbation ρ^* due to fluid motion,

$$\rho(x, y, z, t) = \rho_0 + \rho_b(z) + \rho^*(x, y, z, t). \tag{6.58}$$

The total pressure is written as

$$p(x,y,z,t) = p_0 + p_g(x,y) + p_a(z) + p^*(x,y,z,t), \tag{6.59}$$

where the reference pressure p_0 is a constant, the hydrostatic (ambient) pressure p_a has a vertical gradient that balances the ambient density ($\rho_a = \rho_0 + \rho_b(z)$), and the geostrophic pressure p_g has a transverse gradient that balances the Coriolis force due to the geostrophic wind $\boldsymbol{U}$. Only the dynamic pressure p^* appears in the momentum equation (6.53).

Instead of using ρ^*, it is also common to express the buoyancy term as

$$b = -\frac{\rho^* g}{\rho_0}, \tag{6.60}$$

and the 'total' buoyancy

$$\tilde{b} = b + \bar{b} = -\frac{(\rho^* + \bar{\rho}(z))g}{\rho_0}, \tag{6.61}$$

where the background linear stratification is $N^2 = \partial\bar{b}/\partial z$ and we have $\tilde{b} = b + N^2 z$ with the reference value $\bar{\rho}(z=0) = 0$.

Eqn, (6.54) can also be expressed as

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial \rho^* u_i}{\partial x_i} + w\frac{\partial \bar{\rho}}{\partial z} = \kappa \frac{\partial^2 \rho^*}{\partial x_i^2}, \tag{6.62}$$

and hence we have the buoyancy equation

$$\frac{\partial b}{\partial t} + \frac{\partial b u_i}{\partial x_i} + wN^2 = \kappa \frac{\partial^2 b}{\partial x_i^2}, \tag{6.63}$$

and the equation for the total buoyancy is

$$\frac{\partial \tilde{b}}{\partial t} + \frac{\partial \tilde{b} u_i}{\partial x_i} = \kappa \frac{\partial^2 \tilde{b}}{\partial x_i^2}. \tag{6.64}$$

Roughly, the following independent non-dimensional parameters can be obtained from the comparison of relevant terms:

1. The Reynolds number $Re = UL/\nu$, where L is the horizontal length scale.
2. The Rossby number $Ro = U/fL$.
3. The Froude number $Fr = U/NL$.
4. The aspect ratio $\alpha = H/L$.
5. The Mach number $Ma = U/(L/T)$, where L/T is the group velocity. It is the ratio between the nonlinear term and the tendency term.

6.5.1 Incompressibility

Evan though there is a density transport due to the diffusion (due to the special role that ρ plays; this is not the mass conservation equation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \frac{\partial J_{\rho,i}}{\partial x_i} \neq 0, \tag{6.65}$$

we could still establish incompressible condition

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{6.66}$$

with some additional assumtions.

First we review the integration form of the general conservation equation for an arbitraty scalar (per unit mass)

$$\frac{\partial}{\partial t}\left(\iiint_V \rho\psi \,\mathrm{d}V\right) = -\iint_{\Omega=\partial V} (\rho\boldsymbol{u}\psi)\cdot \mathrm{d}\boldsymbol{A} - \iint_{\Omega=\partial V} \rho\kappa(-\nabla\psi)\cdot \mathrm{d}\boldsymbol{A} \tag{6.67}$$

$$= -\iiint_V \nabla\cdot(\rho\boldsymbol{u}\psi)\,\mathrm{d}V - \iiint_V \nabla\cdot(\rho\kappa(-\nabla\psi))\,\mathrm{d}V \tag{6.68}$$

and we have

$$\frac{\partial\rho\psi}{\partial t} + \nabla\cdot(\rho\psi\boldsymbol{u}) = \nabla\cdot(\rho\kappa\nabla\psi). \tag{6.69}$$

It is in a general form of a conservational principle

$$\frac{\partial Q_v}{\partial t} + \nabla\cdot\boldsymbol{F} = 0, \tag{6.70}$$

where $\boldsymbol{F}$ is the flux and $\nabla\cdot\boldsymbol{F}$ is the transport term.

Taking $\psi = 1$ we recover the mass conservation equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = \frac{D\rho}{Dt} + \rho(\nabla\cdot\boldsymbol{u}) = 0. \tag{6.71}$$

Usually, density change along the material lines, is small enough such that $(1/\rho)D\rho/Dt \ll U/L$ and hence $\nabla\cdot\boldsymbol{u} \ll U/L$. That being said, being non-dimensionalised, the velocity field is solenoidal. We note in density-variable flows that

$$\nabla\cdot\boldsymbol{u} = 0 \tag{6.72}$$

is an approximation. See Batchelor (1967), section 3.2, for details as why this is valid.

6.5.2 Scalar transport equation

Taking $\psi = s$ (salinity or temperature) and assume diffusivity κ is constant, we have the scalar transport equation

$$s\frac{\partial\rho}{\partial t} + \rho\frac{\partial s}{\partial t} + u_j\rho\frac{\partial s}{\partial x_j} + u_j s\frac{\partial\rho}{\partial x_j} = \kappa\frac{\partial s}{\partial x_j}\frac{\partial\rho}{\partial x_j} + \rho\kappa\frac{\partial^2 s}{\partial x_j^2}, \tag{6.73}$$

taking into account

$$\frac{\partial\rho}{\partial t} + u_j\frac{\partial\rho}{\partial x_j} = 0 \tag{6.74}$$

we have

$$\rho\frac{\partial s}{\partial t} + u_j\rho\frac{\partial s}{\partial x_j} = \kappa\frac{\partial s}{\partial x_j}\frac{\partial\rho^*}{\partial x_j} + \rho\kappa\frac{\partial^2 s}{\partial x_j^2}. \tag{6.75}$$

We note that under Boussinesq assumption, $\rho^*/\rho = \rho^*/(\rho_0 + \rho*) \ll 1$, we have

$$\frac{\partial s}{\partial t} + u_j\frac{\partial s}{\partial x_j} = \kappa\frac{\partial^2 s}{\partial x_j^2}. \tag{6.76}$$

With some linear equation of state, we can relate s or T to ρ and get a scalar transport equation for ρ as

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u_j\frac{\partial\rho}{\partial x_j} = \kappa\frac{\partial^2\rho}{\partial x_j^2}, \tag{6.77}$$

with the incompressibility being implied by $\nabla\cdot\boldsymbol{u} = 0$ from Eqn. (6.71). We note that Eqns. (6.71) and (6.77) correspond to two different physical principles.

E.g.,

$$p = \rho RT \tag{6.78}$$

$$\ln p = ln(\rho) + \ln T + \ln R \tag{6.79}$$

$$\frac{\delta p}{p} = \frac{\delta\rho}{\rho} + \frac{\delta T}{T}. \tag{6.80}$$

Assuming isobaric process we have

$$\frac{\partial\rho}{\partial z} \propto -\frac{\partial T}{\partial z} \tag{6.81}$$

and $b = -(g/\rho_0)\partial\rho^*/\partial z = \partial T^*/\partial z$. It is also possible to solve or interpret as the temperature equation (6.76) is being solved and density will be obtain using an equation-of-state such as

$$\frac{\rho-\rho_0}{\rho_0} = -\beta(T-T_0),\ \beta = -\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial T}\right)_p \tag{6.82}$$

where β is call the themal expansion coefficient. Then

$$\frac{\partial\rho}{\partial z} = -\beta\frac{\partial T}{\partial z} \tag{6.83}$$

and we can define

$$b = \frac{gT^*}{T_0} \tag{6.84}$$

$$N^2 = \frac{g}{T_0}\frac{\partial\bar{T}}{\partial z} \tag{6.85}$$

6.6 GFD vorticity equations

6.6.1 Absolute vorticity equation

The 'absolute' vorticity, defined as $\boldsymbol{\omega}_a = \boldsymbol{\omega} + \boldsymbol{f}_\mathrm{c}$, is the 'relative' vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ plus the 'planetary' vorticity $\boldsymbol{f}_\mathrm{c} = 2\boldsymbol{\Omega}_\mathrm{c}$ ($\Omega_\mathrm{c} = \Omega\sin\phi$).

Similar to Eq. (2.21), we can derive the governing equation for $\boldsymbol{\omega}_a$ starting from

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{f}_\mathrm{c}\times\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F} \tag{6.86}$$

without the hydrostatic part separated and/or Boussinesq assumed.

According to identity (A.21) we have, *in an f-plane*,

$$\nabla\times(\boldsymbol{f}_\mathrm{c}\times\boldsymbol{u}) = -\boldsymbol{f}_\mathrm{c}\cdot\nabla\boldsymbol{u}. \tag{6.87}$$

Similar to Eq. (2.14), by taking the curl of (6.86) and taking $\boldsymbol{F} = b\boldsymbol{e}_z$ we have the vorticity equation in a rotating frame

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}_a\cdot\nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}_a + \nabla\times\boldsymbol{F} + \frac{1}{\rho^2}\nabla\rho\times\nabla p, \tag{6.88}$$

i.e., the absolute vorticity equation:

$$\frac{D\boldsymbol{\omega}_a}{Dt} = \frac{\partial\boldsymbol{\omega}_a}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega}_a = \frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} \tag{6.89}$$

$$= \boldsymbol{\omega}_a\cdot\nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega} + \nabla\times\boldsymbol{F} + \frac{1}{\rho^2}\nabla\rho\times\nabla p. \tag{6.90}$$

In a Boussinesq fluid,

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega} + \nabla \times (b\boldsymbol{e}_z) + \frac{1}{\rho_0^2} \nabla \rho^* \times \nabla p^*, \tag{6.91}$$

where $\nabla \times (b\boldsymbol{e}_z) = \epsilon_{ij3}\partial_j b$.

Additionally, the linearized inviscid evolution equation for the perturbation vorticity, similar to (5.134), reads

$$\frac{\bar{D}\omega_i'}{\bar{D}t} = \omega_j' \bar{S}_{ij} + (\bar{\omega}_j + f\delta_{j3})S_{ij}' - u_j' \frac{\partial \bar{\omega}_i}{\partial x_j} + \frac{1}{2}\epsilon_{ij3} f \omega_j' + \epsilon_{ij3}\frac{\partial b'}{\partial x_j}, \tag{6.92}$$

which is convenient for instability considerations.

Example (Taylor-Proudman theorem, 6.3.4; another proof): Assume inviscid, barotropic fluid acted on by conservative force, and that the rotation rate $\boldsymbol{\Omega}_c = \boldsymbol{f}_c/2$ is much greater than other frequencies. Eqn. (6.90) becomes

$$0 = \boldsymbol{f}_c \cdot \nabla \boldsymbol{u} \tag{6.93}$$

and that completes the proof.

6.6.2 Potential vorticity equation; Ertal's theorem

Ref. Pedlosky (2013).

Assume a conserved scalar λ with a governing operator $D\lambda/Dt = S$ where S is a source term for λ. Consider

$$\frac{D}{Dt}\left(\frac{\partial \lambda}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial \lambda}{\partial t} + u_j \frac{\partial \lambda}{\partial x_j}\right) - \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j}, \tag{6.94}$$

i.e.,

$$\frac{D}{Dt}(\nabla \lambda) = \nabla\left(\frac{D\lambda}{Dt}\right) - \nabla \boldsymbol{u} \cdot \nabla \lambda. \tag{6.95}$$

$\nabla\lambda \cdot$ (6.90) + $\boldsymbol{\omega}_a \cdot$ (6.95), with a magic that two opposite-sign $\boldsymbol{\omega}_a \cdot \nabla \boldsymbol{u} \cdot \nabla\lambda$ terms cancel, we have

$$\frac{D}{Dt}(\boldsymbol{\omega}_a \cdot \nabla \lambda) = \boldsymbol{\omega}_a \cdot \nabla S + \nu \nabla^2 \boldsymbol{\omega}_a \cdot \nabla \lambda + (\nabla \times \boldsymbol{F}) \cdot (\nabla \lambda) + \frac{1}{\rho^2}(\nabla \rho \times \nabla p) \cdot (\nabla \lambda) \tag{6.96}$$

Take $\lambda = \tilde{b}$ which is the total buoyancy, with its governing equation being (6.64), assuming conservative external force $\boldsymbol{F}$ and barotropic flow*, we have the potential vorticity (PV) equation:

$$\frac{D}{Dt}\left(\boldsymbol{\omega}_a \cdot \nabla \tilde{b}\right) = \nu \nabla^2 \boldsymbol{\omega}_a \cdot \nabla \tilde{b} + \kappa [\nabla^2 (\nabla \tilde{b})] \cdot \boldsymbol{\omega}_a, \tag{6.97}$$

where

$$\Pi = \boldsymbol{\omega}_a \cdot \nabla \tilde{b} \tag{6.98}$$

is called the potential vorticity, which is the component of the absolute vorticity perpendicular to the isosurface (or parallel to the gradient) of $\tilde{b}$. In the absence of dissipation,

$$\frac{D\Pi}{Dt} = 0, \tag{6.99}$$

i.e., PV is conserved along the streamlines. Eq. (6.97) is like a double-diffusion problem with one 'passive' scalar diffuse together with vorticity.

Ertel's theorem: under the follwing assumptions, PV conservation along fluid motion is satisfied (from Eq. (6.96)):

- λ is a conserved quantity that following fluid motion $S = 0$.
- Conservative external force: $\nabla \times \boldsymbol{F} = 0$.

- Either
  1. Baroclinicity absent ($\nabla\rho \times \nabla p = 0$)
  2. λ is only a thermodynamic function of p, ρ, i.e., $\lambda = \lambda(p, \rho)$ so that the last term vanishes when $\cdot(\nabla\lambda)$. For example, $\lambda = s$ (entropy).
- Diffusion-less/inviscid: $\nu = \kappa = 0$.

6.6.3 Relation of PV to Kelvin's theorem in a rotating frame

Similarly, Kelvin's circulation theorem (see (2.6)) in a rotating frame is

$$\frac{D\Gamma_a}{Dt} = \iint_A \frac{\nabla\rho \times \nabla p}{\rho^2} \cdot d\boldsymbol{A}, \tag{6.100}$$

where the absolute circulation is

$$\Gamma_a = \int_A \boldsymbol{\omega}_a \cdot d\boldsymbol{A} = \Gamma + \int_A \boldsymbol{f}_{\mathrm{c}} \cdot d\boldsymbol{A}. \tag{6.101}$$

When the surface A is specifically chosen to be on $\lambda = \lambda(\rho, p)) =$ constant (and A is enclosed by a contour l that stays on $\lambda =$ constant for all time), what follows is that $\nabla\lambda$ must be in the parameter plane spanned by $\nabla\rho$ and ∇p, hence $\nabla\lambda \cdot (\nabla\rho \times \nabla p) = 0$. Here, we note that the normal vector for A is $\boldsymbol{n} = \nabla\lambda/|\nabla\lambda|$ so $(\nabla\rho \times \nabla p) \cdot \boldsymbol{n} = 0$ on the entire plane and

$$\frac{D\Gamma_a}{Dt} = 0. \tag{6.102}$$

The choice of λ is just to choose a surface/contour (of $\lambda =$ constant) on which $\nabla\rho, \nabla p$ lie in the surface and the baroclinicity term makes zero contribution to the circulation in a baroclinic flow. We can regard PV conservation as a special statement of Kelvin's theorem.

Example: $\lambda(\rho, p) = \rho^2 + p^2$. Think of $\lambda =$ constant as a cylinderical surface.

Example: For a finite flat area A, $\Gamma_a \approxeq \omega_a A$ and its conservation implies angular momentum conservation, and the change of spacing of λ surfaces (associated with change of area) can change relative vorticity — stretching and compressing.

6.7 Turbulence equations for an active scalar

6.7.1 Mean flow equations

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{6.103}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} - f\epsilon_{ij3}(\bar{u}_j - U_j) = -\frac{1}{\rho_0}\frac{\partial \overline{p^*}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \overline{u_i' u_j'}) - \frac{\overline{\rho^*} g}{\rho_0}\delta_{i3} \tag{6.104}$$

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{u}_j \frac{\partial \bar{\rho}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial \bar{\rho}}{\partial x_j} - \overline{\rho' u_j'}\right), \tag{6.105}$$

We note that

$$\rho' = \rho - \bar{\rho} = \rho^* - \overline{\rho^*} = \rho^{*\prime}. \tag{6.106}$$

6.7.2 Fluctuation equations

$$\frac{\partial u_i'}{\partial x_i} = 0 \tag{6.107}$$

$$\frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} - f\epsilon_{ij3}u_j' = -\frac{1}{\rho_0}\frac{\partial p^{*\prime}}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu\frac{\partial \overline{u}_i}{\partial x_j} + \overline{u_i'u_j'} - u_i'u_j') - u_j'\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\rho^{*\prime}g}{\rho_0}\delta_{i3} \tag{6.108}$$

$$\frac{\partial \rho^{*\prime}}{\partial t} + \bar{u}_j \frac{\partial \rho^{*\prime}}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\kappa\frac{\partial \rho^{*\prime}}{\partial x_j} + \overline{\rho^{*\prime}u_j'} - \rho^{*\prime}u_j'\right) - \rho^{*\prime}\frac{\partial \overline{p^*}}{\partial x_i} \tag{6.109}$$

We will see later the Coriolis term won't appear in the transport equations of MKE, TKE, and Reynolds stresses. Coriolis just bends the direction of the velocity. In other words, Coriolis does not have direct influence on turbulence budgets, but indirectly through the change of the mean flow. On the other hand, it does make a difference in the vorticity/enstrophy equation (see section 6.7.4).

6.7.3 MKE, MPE, TKE, TPE, and buoyancy flux equations

Define the mean and turbulent kinetic and potential energy as

$$K = \frac{1}{2}\bar{u}_i\bar{u}_i \tag{6.110}$$

$$K_\rho = \frac{1}{2}\bar{b}^2 \tag{6.111}$$

and

$$k = \frac{1}{2}\overline{u_i'u_i'} \tag{6.112}$$

$$k_\rho = \frac{1}{2}\overline{b'b'} \tag{6.113}$$

where the instantaneous, mean, and fluctuation buoyancy forces are

$$b = -\frac{\rho^* g}{\rho_0},\ \bar{b} = -\frac{\overline{\rho^*} g}{\rho_0},\ b' = -\frac{\rho^{*\prime} g}{\rho_0}, \tag{6.114}$$

such that k and k_ρ have the same dimension as the kinetic energy.

The **MKE equations** is (repeating (5.32)) :

$$\frac{\partial K}{\partial t} + \bar{u}_j\frac{\partial K}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho}\overline{p}\,\overline{u}_j + \nu\frac{\partial K}{\partial x_j} - \overline{u}_i\,\overline{u_i'u_j'}\right) + \overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu\frac{\partial \overline{u}_i}{\partial x_j}\frac{\partial \overline{u}_i}{\partial x_j} \tag{6.115}$$

The **MPE equations** is:

$$\frac{\partial K_\rho}{\partial t} + \bar{u}_j\frac{\partial K_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}(\kappa\frac{\partial K_\rho}{\partial x_j} - \bar{b}\,\overline{b'u_j'}) + \overline{b'u_j'}\frac{\partial \bar{b}}{\partial x_j} - \kappa\frac{\partial \bar{b}}{\partial x_j}\frac{\partial \bar{b}}{\partial x_j} \tag{6.116}$$

We note that the buoyancy flux $\overline{b'u_j'}\partial\bar{b}/\partial x_j$ is a sink in the MPE equation and is a source in the TPE equation.

The **TKE equations** is:

$$\frac{\partial k}{\partial t} + \bar{u}_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_k}(\underbrace{\nu\frac{\partial k}{\partial x_k}}_{\text{molecular diffusion}} + \underbrace{\frac{1}{2}\overline{u_i'u_i'u_k'}}_{\text{turbulent diffusion}} \underbrace{-\frac{1}{\rho_0}\overline{p'u_k'}}_{\text{pressure distortion}}) \underbrace{-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}}_{\text{production } P} - \underbrace{\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}}_{\text{dissipation } \varepsilon} + \underbrace{\overline{b'w'}}_{\text{buoyancy flux } B} \tag{6.117}$$

$$= \nabla\cdot\boldsymbol{T} + P - \varepsilon + B \tag{6.118}$$

where the turbulent buoyancy flux

$$B = -\frac{g}{\rho_0}\overline{\rho^{*\prime} w'} = \overline{b'w'} \tag{6.119}$$

consumes TKE and lead to the production of TPE.

The **TPE equation** is:

$$\frac{\partial k_\rho}{\partial t} + \bar{u}_j \frac{\partial k_\rho}{\partial x_j} = \frac{\partial}{\partial x_j}(\kappa \frac{\partial k_\rho}{\partial x_j} - \frac{1}{2}\overline{b'b'u_j'}) - \overline{b'u_j'}\frac{\partial \bar{b}}{\partial x_j} - \kappa \overline{\frac{\partial b'}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{6.120}$$

We can see that the turbulent buoyancy flux B (negative, think $-\overline{u_i'u_j'}$) works with the density distortion $\partial\bar{b}/\partial z$ to remove energy from TKE and MPE to produce TPE.

The **buoyancy flux equation** is:

$$\frac{\partial \overline{b'u_i'}}{\partial t} + \bar{u}_j \frac{\partial \overline{b'u_i'}}{\partial x_j} = d_{b,i} + P_{b,i} + \Phi_{b,i} - \varepsilon_{b,i} \tag{6.121}$$

where

$$d_{b,i} = \frac{\partial}{\partial x_j}(\overline{\kappa u_i' \frac{\partial b'}{\partial x_j}} + \overline{\nu b' \frac{\partial u_i'}{\partial x_j}} - \frac{1}{\rho_0}\overline{p'b'}\delta_{ij} - \overline{b'u_i'u_j'}) \tag{6.122}$$

$$P_{b,i} = -\overline{b'u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - \overline{u_i'u_j'}\frac{\partial \bar{b}}{\partial x_j} \tag{6.123}$$

$$\Phi_{b,i} = \frac{1}{\rho_0}\overline{p' \frac{\partial b'}{\partial x_i}} \tag{6.124}$$

$$\varepsilon_{b,i} = (\nu + \kappa)\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial b'}{\partial x_j}} \tag{6.125}$$

6.7.4 Perturbation vorticity and enstrophy equations

6.8 Miscellaneous

coriolis frequency;

6.8.1 Derivation of Coriolis force

6.8.2 Boussinesq approximation

7 Waves in GFD

7.1 Internal waves: governing equations and dispersion relation

The governing equations for waves can be simplified from (6.56), with the following assumptions:

1. The Ekman number is small (neglect viscosity).
2. The fluid is Boussinesq.
3. The 'Mach' number is small (neglect the nonlinear term/only effects to the first order are retained).
4. The aspect ratio is $\alpha = H/L = W/U \ll 1$.

The set of simplified equations read (with $\rho_0 = 1$)

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\partial p}{\partial x} + fv \\
\frac{\partial v}{\partial t} &= -\frac{\partial p}{\partial y} - fu \\
\frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} + b \\
\frac{\partial b}{\partial t} &= -wN^2 \\
\nabla \cdot \boldsymbol{u} &= 0
\end{aligned}
\tag{7.1}
$$

and by eliminating pressure and cross-differentiation, an equation for w can be obtained as

$$
\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + f^2\frac{\partial^2 w}{\partial z^2} + N^2(z)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{7.2}
$$

The detailed steps include: (1) eliminate pressure from the momentum equations and obtain a system of three pressure-less equations; (2) cross-differentiate to get the forms of $\partial_{xx}b$ and use the buoyancy equation to replace b with w; we note that we assume $N(z)$ is only a function of the vertical coordinate so its z-dependence can be retained; (3) ∂_{yz} the u-equation and ∂_{xz} the v-equation, to elinimate triple-cross-difference terms as $\partial_{txy}u$; (4) organization.

A 'trivial' solution to (7.2) is simply the geostrophic balance:

$$
w = 0,\ fv = p_x,\ fu = -p_y,\ b = p_z. \tag{7.3}
$$

Assume non-zero solution in the form of normal modes

$$
[u, v, w, p, b] = [\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{b}] \exp[i(kx + ly + mz - \omega t)] \tag{7.4}
$$

where the wavenumber vector is

$$
\boldsymbol{k} = (k, l, m), \tag{7.5}
$$

the waves that the equation (7.2) describes should satisfy the dispersion relation

$$
\omega^2 = \frac{f^2 m^2 + (k^2 + l^2)N^2}{k^2 + l^2 + m^2} = f^2 \sin^2\phi + N^2 \cos^2\phi \tag{7.6}
$$

where ϕ is the angle between the wave-vector $\boldsymbol{k}$ and the horizontal plane (k, l). It can be seen that the feasible frequencies lie in the range of

$$
f < \omega < N \tag{7.7}
$$

for the general case in the ocean that $N/f \gg 1$.

From (7.6) we can get

$$
\frac{N^2 - \omega^2}{\omega^2 - f^2} = \frac{(N^2 - f^2)\sin^2\phi}{(N^2 - f^2)\cos^2\phi} = \tan^2\phi = \frac{1}{\alpha^2}, \tag{7.8}
$$

where the vertical-to-horizontal aspect ratio is defined as

$$
\alpha = \frac{k^2 + l^2}{m^2} = \cot\phi, \tag{7.9}
$$

which is close to zero at the hydrostatic limit ($m^2 \ll k^2 + l^2$). In such case, $N \gg \omega$.

We can also see that the continuity equation translates to

$$
\boldsymbol{k} \cdot \hat{\boldsymbol{u}} = 0, \tag{7.10}
$$

that being said, the wavenumber vector is perpendicular to the particle motions.

The dispersion relation also implies that the frequency only depends on the direction of the wavevector, but not its magnitude. All wavenumbers on a cone of constant ϕ have the same frequency, and the direction of the group velocity, $c_p = d\omega/dk$, is perpendicular to the cone surface. A trivial solution to the equation (7.2) is simply $w = 0$, which is corresponding to a geostrophic+hydrostatic balance.

Examples. (1) For a non-rotationg fluid, $\omega^2 = N^2 \cos^2\phi$, and the portion of stratification that the motion feels is proportional to the projection of its motion in the vertical direction. Oscillation that feels the full effect of stratification is vertical, and correspond to the largest possible frequency, a horizontal wavenumber, and $\phi = 0$. (2) For $\phi = \pi/2$, the wavevector is perpendicular and there is only inertial oscillations.

7.1.1 General solutions

Assume $N = N_0$ is a constant and without lost of generality the coordinate can be re-oriented such that $\partial_y = 0$ and $\partial_x u + \partial_z w = 0$. The rest of the flow quantities can be recovered from the solution of $\hat{w}$ as

$$\hat{u} = -\frac{m}{k}\hat{w} \tag{7.11}$$

$$\hat{v} = i\frac{fm}{\omega k}\hat{w} = -i\frac{f}{\omega}\hat{u} \tag{7.12}$$

$$\hat{p} = -\left(\frac{N^2-\omega^2}{m\omega}\right)\hat{w} = \left(\frac{N^2-\omega^2}{k\omega}\right)\hat{u} \tag{7.13}$$

$$\hat{b} = -i\frac{N^2}{\omega}\hat{w},\ \hat{p} = -i\left(\frac{N^2-\omega^2}{mN^2}\right)\hat{b} \tag{7.14}$$

The above relations have a number of implications:

1. $\hat{u}$ and $\hat{w}$ are in phase.
2. $\hat{u}$ and $\hat{v}$ are out of phase (think inertial oscillations).
3. Pressure and vertical motion are in phase, but both are out of phase with buoyancy.
4. (TBD) the vertical structures (CW or CCW) of the waves.

7.1.2 Hydrostatic approximation

With the hydrostatic approximation is equivalent to

$$k^2 + l^2 \ll m^2, \tag{7.15}$$

with which the dispersion relation (7.6) reduces to

$$\omega^2 = f^2 + \frac{k^2+l^2}{m^2}N^2. \tag{7.16}$$

7.1.3 Near-inertial waves

The near-inertialness,

$$\frac{\omega - f}{f} \ll 1, \tag{7.17}$$

implies that

$$\frac{\omega^2 - f^2}{f^2} = \frac{(\omega-f)^2 - 2f^2 + 2\omega f}{f^2} \approx \frac{2(\omega - f)}{f}. \tag{7.18}$$

While (7.6) is re-organized to

$$\frac{\omega^2 - f^2}{f^2} = \left(\frac{N^2}{f^2} - 1\right) \frac{k^2 + l^2}{k^2 + l^2 + m^2}, \tag{7.19}$$

which can only be made very small when the second factor is small, that said,

$$k^2 + l^2 \ll m^2, \tag{7.20}$$

which is the hydrostatic relation so (7.16) applies. With that, the dispersion relation for near-inertial waves is

$$\omega = f + \frac{N^2}{2f} \frac{k^2 + l^2}{m^2}. \tag{7.21}$$

In fact, near-inertial waves are super-hydrostatic. Taking an appropriate orientation of the coordinate frame ($l = 0$,$\partial_y = 0$), we can express the near-inertialness indicator

$$\frac{\omega - f}{f} = \frac{N^2 k^2}{f^2 m^2} = \frac{Bu}{2} \ll 1, \tag{7.22}$$

and the group velocities

$$c_{gv} = \frac{d\omega}{dm} = -\frac{N^2 k^2}{f m^3} = -Bu\frac{f}{m}. \tag{7.23}$$

$$c_{gh} = \frac{d\omega}{dk} = \frac{N^2 k}{f m^2} = Bu\frac{f}{k}. \tag{7.24}$$

Both c_{gv} and c_{gh} are small.

7.1.4 Vertical modes

If we ease the assumption (7.4) to allow a vertical dependence of the modes, since N is typically a function of z, i.e.,

$$w(x, y, z, t) = a(z) \exp[i(kx + ly - \omega t)], \tag{7.25}$$

by substituting into (7.2) we have

$$\frac{d^2 a}{dz^2} + \left(\frac{N^2 - \omega^2}{\omega^2 - f^2}\right) (k^2 + l^2) a = 0, \tag{7.26}$$

with rigid lid BC's $w(0) = w(-h) = 0$ that implies $a(0) = a(-h) = 0$. This is a typical Sturm–Liouville eigenvalue problem. Moreover, the derivative $a' = \partial_z a$ also satisfies another Sturm–Liouville equation. It will be shown later that the set of eigenmodes $\{a_m\}$ are mutually orthogonal under the inner-product xxx, where m is now the mode index.

The solution of (7.26) can be done for each specified freuquency: regard $k^2 + l^2$ as the eigenvalue, solve (7.26) with given $N(z)$ and ω. Under hydrostatic, it can be shown from $\alpha \approx 1$ that $N \gg \omega$. Assume the hydrostatic dispersion relation (for each mode m)

$$\omega_m^2 \approx f^2 + c_m^2 (k^2 + l^2) \tag{7.27}$$

we have

$$\frac{d^2 a}{dz^2} + \frac{N^2}{c^2} a = 0, \tag{7.28}$$

where the eigenvalues $\{c_m^{-2}\}$ are the wave speed of each mode m and corresponding Rossby radius of deformation is $\lambda_m = c_m/f$. We note that f is absent in (7.28) and the eigenmodes it defines.

7.2 Inertial and buoyancy oscillations

The two most fundamental pedagogical oscillations in GFD are the inertial and buoyancy oscillations. The inertial oscillation results from the pressure-less version of the horizontal momentum equations (7.1),

$$u_t - fv = 0 \tag{7.29}$$
$$v_t + fu = 0. \tag{7.30}$$

It can be arranged into

$$(u + iv)_t + if(u + iv) = 0 \tag{7.31}$$

which has the solution

$$u + iv = e^{-ift}[u(0) + iv(0)], \tag{7.32}$$

which is rotating clockwise in Northern Hemisphere at an angular frequency f – the Coriolis frequency. Similarly, the pressure-less vertical and buoyancy equations give

$$w_{tt} = -N^2 w, \tag{7.33}$$

leading to the solutions

$$w = \begin{cases} w(0)e^{iNt},\ N^2 > 0 \\ w(0)e^{\sqrt{-N^2}t},\ N^2 < 0 \end{cases} \tag{7.34}$$

with the first solution corresponding to buoyancy oscillation in stably stratified fluid and the second corresponding to a gravitationally unstable mode under unstable stratification.

7.3 Shallow water waves

Assumptions:

- Inviscid
- Homogeneous fluid
- Horizontal scale ≫ vertical scale ($L \gg H$)
- Small amplitude ($\eta \ll H$)

Scaling analysis. Continuity:

$$\frac{W}{H} \sim \frac{U}{L}, \tag{7.35}$$

momentum:

$$\frac{\partial u}{\partial t} \sim \frac{U}{T} \tag{7.36}$$
$$u\frac{\partial u}{\partial x} \sim v\frac{\partial u}{\partial y} \sim U\frac{U}{L} \tag{7.37}$$
$$w\frac{\partial u}{\partial z} \sim U\frac{W}{H} \tag{7.38}$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial x} \sim \frac{U^2}{L} \tag{7.39}$$
$$\frac{\partial w}{\partial t} \sim \frac{W}{T} \tag{7.40}$$
$$-\frac{1}{\rho}\frac{\partial p}{\partial z} \sim \frac{U^2}{H} \tag{7.41}$$

The incompressibility implies that the speed of motion is much smaller than some sort of wave speed:

$$\frac{U}{L/T} = \frac{\text{inertial}}{\text{acceleration}} \ll 1, \tag{7.42}$$

and the aspect ratio implies

$$\frac{\partial w/\partial t}{-(1/\rho_0)\partial p/\partial z} = \left(\frac{H}{L}\right)^2 \ll 1. \tag{7.43}$$

Hence, we have neglected the non-linear terms in horizontal momentum equations and the vertical momentum equation reduces to a hydrostatic balance. The shallow water equations (SWE) read

$$u_t = -\frac{1}{\rho_0} p_x \tag{7.44}$$

$$v_t = -\frac{1}{\rho_0} p_y \tag{7.45}$$

$$0 = -\frac{1}{\rho_0} p_z - g, \tag{7.46}$$

$$u_x + v_y = 0 \tag{7.47}$$

The hydrostatic pressure balance leads to

$$p(z) = p(H+\eta) + \int_{H+\eta}^{z} -\rho g dz = p_0 + \rho g(H + \eta - z), \tag{7.48}$$

where η is the surface height displacement and H is the depth ($z = 0$ at the bottom). The horizontal pressure gradients are given as $p_x = \rho g \eta_x, p_y = \rho g \eta_y$. The intuition is that when the surface is elevated it also gives the fluid below a higher hydrostatic pressure.

Now let's derive an equation for the surface motions.

$$\frac{\partial \eta}{\partial t} \sim \frac{D\eta}{Dt} = w(H+\eta) = w(H+\eta) - w(0) \tag{7.49}$$

$$= \int_0^{H+\eta} \frac{\partial w}{\partial z} dz = \int_0^{H+\eta} -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz \tag{7.50}$$

$$\approx -(H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{7.51}$$

$$\approx -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{7.52}$$

where we have applied the small amplitude assumption and the slab assumption.

Finally, the SWE become

$$u_t - fv + g\eta_x = 0 \tag{7.53}$$

$$v_t + fu + g\eta_y = 0 \tag{7.54}$$

$$\eta_t + H(u_x + v_y) = 0 \tag{7.55}$$

where the lst equation can be written as

$$(g\eta)_t + c^2(u_x + v_y) = 0 \tag{7.56}$$

where $c^2 = gH$ is the phase/group speed and $p = g\eta$ is the hydrostatic pressure. Shallow water waves are non-dispersive. Cross-differentiate the SWE, then we reach

$$\eta_{tt} - c^2(\eta_{xx} + \eta_{yy}) + Hf\zeta = 0, \tag{7.57}$$

where the relative vorticity $\zeta = v_x - u_y$. The curl of the first two equations of the SWE gives

$$\zeta_t + f(u_x + v_y) = 0. \tag{7.58}$$

Combined with the η_t equation (continuity), we have

$$\frac{\partial}{\partial t}\left(\frac{\zeta}{f} - \frac{\eta}{H}\right) = \frac{\partial q}{\partial t} = 0, \tag{7.59}$$

where $q = \zeta/f - \eta/H$ is the potential vorticity (reduction in SWE). The conservation implies

$$\zeta = \frac{\eta f}{H} + f q_0. \tag{7.60}$$

Substitute this into (7.57) we have (the homogeneous part of the equation):

$$\eta_{tt} - c^2(\eta_{xx} + \eta_{yy}) + f^2\eta = 0. \tag{7.61}$$

Assuming normal modes $\eta = \eta_0 \exp[i(kx + ly - \omega t)]$ with oriented coordinates ($l = 0$), we have the dispersion relation

$$\omega^2 = c^2(k^2 + l^2) + f^2. \tag{7.62}$$

- Large $k^2 + l^2$ (fast waves): $\omega \approx \sqrt{gH}k$. Phase speed $c_p = \omega/k = \sqrt{gH}$. Group speed $c_g = d\omega/dk = \sqrt{gH} = c_p$. They don't feel the rotation of the Earth. We note that the phase is

  $$\Phi = kx + ly - \omega t = \sqrt{k^2 + l^2}(kx + ly - \omega t)/\sqrt{k^2 + l^2} = |\boldsymbol{k}|(\hat{\boldsymbol{k}} \cdot \boldsymbol{x} - c_p t).$$

  Example. (Tsunami speed) Assume $H = 4$ km is the averaged ocean depth and $L = 8000$ km is the width of the ocean basin. With the SWE assumption $H \ll L$ satisfied, $c = \sqrt{gH}$ gives ≈ 200 m/s or 720 km/h of propagation speed.

- Small $k^2 + l^2$ (slow waves, small Ro): $\omega \approx f$. Near-inertial waves.

7.3.1 Travelling wave solutions

Consider non-rotating (or fast) waves. The formal solution to the 1D SWE is

$$\eta = G(x - ct) + F(x + ct) \tag{7.63}$$

with both G and F satisfying ($c^2 = Hg$)

$$\eta_{tt} - c^2\eta_{xx} = (\partial_t + c\partial_x)(\partial_t - c\partial_x)\eta = 0. \tag{7.64}$$

Moreover,

$$u = \int \frac{\partial u}{\partial t} dt = \int -g\frac{\partial \eta}{\partial x} = \frac{g}{c}[G(x - ct) - F(x + ct)]. \tag{7.65}$$

Without the loss of generality, we consider right-propagating waves with $F = 0$. Consider a single wavenumber,

$$\eta = \Re\{A \exp[i(kx - \omega t)]\} = A\cos(kx - \omega t), \tag{7.66}$$

where $\omega = ck$ and c is the phase speed. We have

$$u = \frac{gk}{\omega} A\cos(kx - \omega t), \tag{7.67}$$

which is in phase with η. We check *a posteriori* that in order to make the 'Mach' number very small, we should have

$$\frac{u}{c_p} = \frac{gkA/\omega}{\omega/k} = \frac{gk^2/\omega}{\omega^2} A \ll 1, \tag{7.68}$$

where A is the amplitude.

Here

$$\phi = kx - \omega t = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t \tag{7.69}$$

is called the phase. For 2D waves,

$$\phi = kx + ly - \omega t = \sqrt{k^2 + l^2} \left(\frac{kx + ly}{\sqrt{k^2 + l^2}} - \frac{\omega t}{\sqrt{k^2 + l^2}} \right) = |\boldsymbol{k}| \left(\frac{kx + ly}{|\boldsymbol{k}|} - |c_p| t \right) \tag{7.70}$$

and the phase speed is

$$|c_p| = \frac{\omega}{|\boldsymbol{k}|}. \tag{7.71}$$

The phase 'velocity', by definition,

$$\boldsymbol{c}_p = \frac{\omega}{|\boldsymbol{k}|} \left(\frac{k}{|\boldsymbol{k}|}, \frac{l}{|\boldsymbol{k}|} \right) // \boldsymbol{k}, \tag{7.72}$$

is parallel to the wavevector

$$\boldsymbol{k} = (k, l), \tag{7.73}$$

which is perpendicular to the constant phase lines

$$\phi_0 = kx + ly - \omega t_0 = C \to y = -\frac{k}{l} x + \frac{\phi_0}{l} // \left(1, -\frac{k}{l} \right). \tag{7.74}$$

7.4 Deep water waves

We relax the assumption $H \ll L$ in SWE and allow vertical motions (the fluid is no longer moving like slabs) and reach the deep water equations (DWE)

$$u_t = -\frac{1}{\rho_0} p_x \tag{7.75}$$

$$v_t = -\frac{1}{\rho_0} p_y \tag{7.76}$$

$$w_t = -\frac{1}{\rho_0} p_z \tag{7.77}$$

$$u_x + v_y + w_z = 0 \tag{7.78}$$

where the pressure is its deviation from the background hydrostatic balance

$$-\frac{1}{\rho_0} \bar{p}_z = g \to \tilde{p}(x, y, z, t) = p(x, y, z, t) - \rho_0 g z + p_0. \tag{7.79}$$

We note that $H \ll L$ is essential in eliminating the w-equation in the SWE.

8 Hydrodynamic stability

8.1 Linearized Navier–Stokes

Consider the incompressible N-S equations

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} \tag{8.2}$$

and the decomposition of velocity and pressure into the base and perturbation states:

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}' \tag{8.3}$$
$$p = P + p' \tag{8.4}$$

We note that the base state also satisfies N-S:

$$\nabla \cdot \boldsymbol{U} = 0 \tag{8.5}$$
$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \boldsymbol{U} \tag{8.6}$$

hence by plugging in the decomposition to (8.1)-(8.2) we have the perturbation equation:

$$\nabla \cdot \boldsymbol{u}' = 0 \tag{8.7}$$
$$\frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \boldsymbol{U} + \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' = -\frac{1}{\rho}\nabla p' + \nu \nabla^2 \boldsymbol{u}' \tag{8.8}$$

And we note that the boundary conditions that the perturbation $\boldsymbol{u}', p'$ satisfy is homogeneous, such that $\boldsymbol{U}$ and p_b satisfy the same BC's as $\boldsymbol{u}$ and p in the original equation.

In linear stability, with the assumption that

$$O(\boldsymbol{u}') = \epsilon O(\boldsymbol{U}), \tag{8.9}$$

we neglect the nonlinear term $\boldsymbol{u}' \cdot \nabla \boldsymbol{u}'$ and the primes, and have the linearised perturbation equation

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.10}$$
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} - \nu \nabla^2 \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho}\nabla p \tag{8.11}$$

or if we define the linear operator as

$$\mathcal{L}_{\boldsymbol{U}} = \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{8.12}$$

there is

$$\mathcal{L}_{\boldsymbol{U}} \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho}\nabla p. \tag{8.13}$$

That being said, the linear mechanism is that the fluctuations extract energy from the mean flow, to the leading order effect, instead of interacting with themselvess.

The linearised equations (8.10)-(8.11), if written in matrix form (Arratia, 2011), is

$$\mathcal{L}_{\mathrm{NS}} \boldsymbol{q} = \begin{bmatrix} \mathcal{L}_{\boldsymbol{U}} + \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} & \frac{\partial}{\partial x} \\ \frac{\partial V}{\partial x} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \mathcal{L}_{\boldsymbol{U}} + \frac{\partial W}{\partial z} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = 0, \tag{8.14}$$

where $\boldsymbol{q} = [u, v, w, p]^T$. This is in the KKT form that will be described below, where we will see that the same mathematical properties of the operators will be shared in both stability analysis and CFD.

On the other hand, it is sometimes also convenient to define the RHS operator as

$$\mathcal{A}_{\boldsymbol{U}} \boldsymbol{u} = -\boldsymbol{U} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{8.15}$$

such that

$$\frac{\partial \boldsymbol{u}}{\partial t} = \mathcal{A}_{\boldsymbol{U}}\boldsymbol{u}. \tag{8.16}$$

In practice, the pressure gradient can be neglected when calculating the operator $\mathcal{A}_{\boldsymbol{U}}$ and the resulting pressure is projected onto a divergence-free space (along with satisfying the boundary conditions in each numerical iteration/time-step), under the framework of projection methods. In general, in theoretical analysis or numerical methods where the pressure p is elimenated (by procedures similar to taking the curl of the N–S), the equations are to some extent equivalent to the vorticity equation.

8.1.1 The role of pressure

A separate short note on the pressure being the Lagrangian multiplier in incompressible system. Consider the Stokes flow (actually that can be the linearised equations as described above)

$$-\nabla^2\boldsymbol{u} + \nabla p = \boldsymbol{f} \tag{8.17}$$

$$\nabla\cdot\boldsymbol{u} = 0 \tag{8.18}$$

and in the matrix form

$$\begin{bmatrix} -\nabla^2 & \nabla \\ \nabla\cdot & O \end{bmatrix}\begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} \tag{8.19}$$

and its discrete version

$$\begin{bmatrix} A & B^T \\ B & O \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{8.20}$$

which is a saddle point problem or a KKT (Karush-Kuhn-Tucker) system (Benzi *et al.*, 2005). The Stokes equations can be interpreted as a constained optimisation problem (section 3.15.5 of Gresho & Sani (1998))

$$\min\ J(\boldsymbol{u}) = \frac{1}{2}\int \|\nabla\boldsymbol{u}\|_2^2 dV - \int \boldsymbol{f}\cdot\boldsymbol{u}dV \tag{8.21}$$

$$\text{subject to } \nabla\cdot\boldsymbol{u} = 0 \tag{8.22}$$

where the variable p, introduced to satisfy an additional constraint, plays the role of a Lagrangian multiplier. We note that the adjoint of the gradient operator is the (negative) divergence operator

$$(\nabla)^\dagger = -\nabla\cdot \tag{8.23}$$

We not establish this fact. Consider a scalar f and a vector $\boldsymbol{F}$. consider

$$\int_V \nabla\cdot(f\boldsymbol{F})\,dV = \int_V f(\nabla\cdot\boldsymbol{F})\,dV + \int_V \nabla f\cdot\boldsymbol{F}\,dV = \iint_{\Omega=\partial V} f\boldsymbol{F}\cdot d\boldsymbol{A} \tag{8.24}$$

and if we the boundary integral vanishes we have

$$(\nabla f, \boldsymbol{F}) = \int_V (\nabla f\cdot\boldsymbol{F})\,dV = -\int_V f(\nabla\cdot\boldsymbol{F})\,dV = (f, -\nabla\cdot\boldsymbol{F}) = (f, (\nabla)^\dagger(\boldsymbol{F})) \tag{8.25}$$

and hence

$$(\nabla)^\dagger = -\nabla\cdot \tag{8.26}$$

8.2 Parallel shear flows

Here, we start from a more generalized framework with a vertical density profile and a buoyancy equation (linearized N–S under Boussinesq) similar to in section 6.7.2. We will also restrict our attention on one-dimensional horizontal shear $U(y)$ and vertical stratification ($Ri_g = N^2L^2/U^2$) and rotation ($f = 2\Omega$) where the scales for $(\boldsymbol{u}, \boldsymbol{x}, t, p, \rho)$ are $(U, L, L/U, \rho_0 U^2, (\rho_0/g)N^2L)$.

The non-dimensional linearized perturbation equations read

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} - 2\Omega v &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + 2\Omega u &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} - Ri_b\rho + \frac{1}{Re}\nabla^2 w \\
\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} - w &= \frac{1}{RePr}\nabla^2 \rho
\end{aligned} \tag{8.27}$$

where the primes for perturbational variables are omitted.

With the normal-mode *ansatz*

$$(\boldsymbol{u}, p, \rho) = (\bar{\boldsymbol{u}}, \hat{p}, \hat{\rho})\exp(ik_1x + ik_3z + \sigma t) \tag{8.28}$$

where $\sigma = -ik_1c = -i\omega$ is the growth rate and $c = \omega/k$ is the wave speed, the equations (8.27) can be turned into the normal-mode equations

$$ik_1\hat{u} + D\hat{v} + ik_3\hat{w} = 0 \tag{8.29}$$
$$(\sigma + ik_1U - \nu\triangle)\hat{u} + (DU - 2\Omega)\hat{v} = -ik_1\hat{p} \tag{8.30}$$
$$(\sigma + ik_1U - \nu\triangle)\hat{v} + 2\Omega\hat{u} = -iD\hat{p} \tag{8.31}$$
$$(\sigma + ik_1U - \nu\triangle)\hat{w} = -ik_3\hat{p} - Ri_b\rho \tag{8.32}$$
$$(\sigma + ik_1U - \kappa\triangle)\hat{\rho} = \hat{w} \tag{8.33}$$

where we denote

$$D = \partial_y \tag{8.34}$$
$$\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz} \tag{8.35}$$
$$k^2 = k_1^2 + k_3^2 \tag{8.36}$$
$$\triangle = D^2 - k^2 \tag{8.37}$$
$$\nu = \frac{1}{Re} \tag{8.38}$$
$$\kappa = \frac{1}{RePr} \tag{8.39}$$

following the convention in Arobone & Sarkar (2012).

In a compact form, the normal-mode equations can be expressed as

$$\boldsymbol{A}\begin{bmatrix} u \\ v \\ \rho \end{bmatrix} = \sigma\boldsymbol{B}\begin{bmatrix} u \\ v \\ \rho \end{bmatrix} \tag{8.40}$$

where

$$
\boldsymbol{A} = \begin{bmatrix} -2\Omega k^2 & ik_1(U\triangle - D^2U + 2\Omega D) + \nu\triangle & -ik_3 Ri_b D \\ ik_1(DU + UD - 2\Omega) - \nu\triangle D & D^2U + k_1^2 U + DUD - 2\Omega D + \nu ik_1 \triangle & 0 \\ -ik_1 & -D & k_1k_3U + \kappa ik_3\triangle \end{bmatrix} \tag{8.41}
$$

$$
\boldsymbol{B} = \begin{bmatrix} 0 & -\triangle & 0 \\ -D & ik_1 & 0 \\ 0 & 0 & ik_3 \end{bmatrix} \tag{8.42}
$$

as a generalized eigenvalue problem in terms of σ (Arobone & Sarkar, 2012). The first row is obtained by cross differentiating u, w momentum equations (equivalant of taking the curl of the N–S), the second row by cross differentiating u, v momentum equations (pay attention to stuff like the chain rule: $D((DU)\hat{v}) = D(DU)\hat{v} + (DU)D\hat{v})$, the third row by combining the continuity and the scalar equations.

8.2.1 Orr–Sommerfield equations

In what follows, we consider homogeneous parallel shear flows with $U(y) \neq 0, \Omega = 0, Ri_g = 0$. Per Squire's theorem (next section), it is always possible to re-orient the coordinate $x - z$ such that any each unstable 3D perturbation always corresponds to a more unstable 2D one. Hence, it is sufficient to study 2D perturbations with $k_3 = 0, \hat{w} = 0$.

The 2D perturbation equations are

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v
\end{aligned} \tag{8.43}
$$

and by introducing the streamfunction and normal modes

$$
u = \psi_y,\; v = -\psi_x,\; (u, v, \psi) = (\hat{u}, \hat{v}, \hat{\phi}) \exp[ik(x - ct)] \tag{8.44}
$$

we have to the Orr–Sommerfield equation

$$
(U - c)(\hat{\phi}_{yy} - k^2\hat{\phi}) - U_{yy}\hat{\phi} = \frac{1}{ikRe}(\hat{\phi}_{yyyy} - 2k^2\hat{\phi}_{yy} + k^4\phi) \tag{8.45}
$$

representing the viscous instability of shear flows.

We note that the normal mode perturbations (8.44) are in the form of plane waves, where $c_r = \omega/k$ is the phase speed and $\sigma_{2D} = kc_i$ is the temporal growth rate. The expression of the perturbations ($\in \mathbb{R}$) are to be understood as, for example,

$$
w = \Re\{\hat{w}[\cos(kx - kc_r t) - i\sin(kx - kc_r t)]\}e^{c_i kt} \tag{8.46}
$$

$$
= \{\Re(\hat{w})[\cos(kx - kc_r t)] - \Im(\hat{w})[\sin(kx - kc_r t)]\}e^{kc_i t}. \tag{8.47}
$$

We also note that since we have taken the curl, the ODE is thrid order in velocity and fourth order in streamfunction. The boundary conditions for two no-slip walls are $\hat{\phi}(0) = \hat{\phi}_y(0) = 0$, $\hat{\phi}(1) = \hat{\phi}_y(1) = 0$. We will revisit the O–S equation in a later section on transient growth.

8.2.2 Rayleigh's inflection point criterion

The inviscid limit of (8.45) is the Rayleigh equation

$$(U-c)\left(\frac{d^2}{dy^2}-k^2\right)\hat{\phi}-\frac{d^2U}{dy^2}\hat{\phi}=0 \tag{8.48}$$

with two required BC's being no-penetration, $\hat{\phi}(0)=\hat{\phi}(1)=0$. It can be shown by taking the complex conjugate of this equation that if c and $\hat{\phi}$ are an eigen pair, so as c^* and $\hat{\phi}^*$. We also note that the temporal growth rate is $\exp(c_i t)$ $(c=c_r+ic_i)$. Hence, for each unstable mode there is a corresponding stable mode (and v.v.), and for each (neutrally) stable mode c must be real.

Consider multiply the following form of (8.48) with $\hat{\phi}^*$ and integrate both sides of

$$\hat{\phi}_{yy}-k^2\hat{\phi}-\frac{U_{yy}}{U-c}\hat{\phi}=0 \tag{8.49}$$

within $0<y<1$ (assume $c_i\neq 0$ so $U-c\neq 0$), we have

$$\int|\hat{\phi}_y|^2\,dy+k^2\int|\hat{\phi}|^2\,dy+\int\frac{U_{yy}}{U-c}|\hat{\phi}|^2\,dy=0. \tag{8.50}$$

Since the first two terms are real, the imaginary part of the third term must be zero, that said,

$$\Im\left(\int\frac{U-c^*}{U-c^*}\frac{U_{yy}}{U-c}|\hat{\phi}|^2\,dy\right)=c_i\int\frac{U_{yy}}{|U-c|^2}|\hat{\phi}|^2\,dy=0, \tag{8.51}$$

which is valid when either $c_i=0$ (stable) or the integral being zero when U_{yy} changes sign (has an inflection point where $U_{yy}=0$) within the domain.

For neutrally stable modes $c_i=0$, the highest derivative drops out from (8.48) and a critical layer is form near $U=c$. Actually, we can obtain a more sufficient condition (a stronger necessary condition that has a narrower range) than the inflection point criterion. The real part of (8.50) is

$$\int\frac{U_{yy}(U-c_r)}{|U-c|^2}|\hat{\phi}|^2\,dy=-\int(|\hat{\phi}_y|^2+k^2|\hat{\phi}|^2)\,dy \tag{8.52}$$

and for nonzero c_i (8.51) reads

$$\int\frac{U_{yy}}{|U-c|^2}|\hat{\phi}|^2\,dy=0. \tag{8.53}$$

Assuming the velocity profile is such that Rayleigh's criterion is satisfied and there exist a unique inflection point y_s, where $U(y_s)=U_s$ and $U''(y_s)$, multiply the above equation with (c_r-U_s) and add it to (8.52) we have

$$\int\frac{U_{yy}(U-U_s)}{|U-c|^2}|\hat{\phi}|^2\,dy=-\int(|\hat{\phi}_y|^2+k^2|\hat{\phi}|^2)\,dy\leq 0 \tag{8.54}$$

We note that both U_{yy} and $U-U_s$ changes sign at y_s so the product cannot change sign there. The only way for the above inequality to hold is for the product to be negative somewhere in the flow, i.e.,

$$U_{yy}(U-U_s)\leq 0, \tag{8.55}$$

with the equality only achieved at y_s. This is referred to as the Fjørtoft's criterion.

Example: shear layer with a velocity profile of a hyperbolic tangent

$$U(y)=U_0\tanh(y/L), \tag{8.56}$$

where the velocity scale is $U_0=(U_1-U_2)/2$ and L is the half-width of the layer. We can see that the shear is the largest at the centerline:

$$S^*=U'(y)L/U_0=\mathrm{sech}^2(y/L), \tag{8.57}$$

which achives a non-dimensional value of unity at $y = 0$ and decays gradually outwards. The second derivative is

$$U''(y) = -\frac{2U_0}{L^2}\operatorname{sech}^2(y/L)\tanh(y/L) \tag{8.58}$$

and it has one inflection point at $y = 0$ such that $U''(0) = 0$. The instability of flows with inflection point leads to rolled-up blobs, such as the Kelvin–Helmholtz billow in shear layers.

8.2.3 Squire's transformation and theorem

In order to justify the 2D analysis in the previous sections, we consider the inviscid system described by (8.43), but subject to 3D normal mode perturbations:

$$(u, v, w, p) = (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \exp[i(k_1 x + k_3 z - k_1 ct)] \tag{8.59}$$

which is a plane wave with phase velocity $c_p = k_1 c_r/(k_1^1 + k_2^2)^{1/2}$ (the phase, $\varphi = \boldsymbol{k}\cdot\boldsymbol{x} - c_p kt$ is written in terms of $k_1 ct$ instead of $k = (k_1^1 + k_2^2)^{1/2}$ for convenience), propagating along the (k_1, k_3) direction, oblique to the base flow.

We have

$$ik_1\hat{u} + \hat{v}_y + ik_3\hat{w} = 0 \tag{8.60}$$

$$ik_1(U - c)\hat{u} + \hat{v}U_y = -ik_1\hat{p} + \frac{1}{Re}[\hat{u}_{yy} - (k_1^2 + k_3^2)\hat{u}] \tag{8.61}$$

$$ik_1(U - c)\hat{v} = -\hat{p}_y + \frac{1}{Re}[\hat{v}_{yy} - (k_1^2 + k_3^2)\hat{v}] \tag{8.62}$$

$$ik_1(U - c)\hat{w} = -ik_3\hat{p} + \frac{1}{Re}[\hat{v}_{yy} - (k_1^2 + k_3^2)\hat{v}] \tag{8.63}$$

Under the Squire transformation,

$$\bar{k} = (k_1^1 + k_2^2)^{1/2},\ \bar{k}\bar{u} = k_1\hat{u} + k_3\hat{w},\ \bar{p}/\bar{k} = \hat{p}/k_1, \tag{8.64}$$

$$\bar{c} = c,\ \bar{v} = \hat{v},\ \bar{k}\overline{Re} = k_1 Re \tag{8.65}$$

the first and third momentum equations can be combined to yield

$$i\bar{k}\bar{u} + \bar{v}_y = 0 \tag{8.66}$$

$$i\bar{k}(U - c)\bar{u} + \bar{v}U_y = -i\bar{k}\bar{p} + \frac{1}{\overline{Re}}[\bar{u}_{yy} - \bar{k}^2\bar{u}] \tag{8.67}$$

$$i\bar{k}(U - c)\bar{v} = -\bar{p}_y + \frac{1}{\overline{Re}}[\bar{v}_{yy} - \bar{k}^2\bar{v}] \tag{8.68}$$

which look exactly the same to the normal mode equations in 2D resulting from (8.43) (not shown). However, the growth rate is $\sigma_{2D} = \bar{k}c_i > k_1 c_i = \sigma_{2D}$. That being said, for each 3D perturbation, there exist an equivalent 2D perturbation that has a higher growth rate. Also, since $\overline{Re} < Re$, for each critical Reynolds number in 3D, there is a lower corresponding critical Reynolds number for 2D perturbations.

Hence, we may say that for parallel shear flows the 2D perturbations are more unstable than 3D ones and the previous discussions on the O–S and the Rayleigh equations in 2D are sufficient.

8.2.4 Taylor–Goldstein equation

Similar to (8.27), the dimensional, inviscid, linearized perturbation equations for vertical shear flow $(U(z), 0, 0)$ can be expressed as

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} \\
\frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} - w\frac{N^2\rho_0}{g} &= 0
\end{aligned}
\tag{8.69}
$$

where the perturbations are assumed to be 2D (according to Squire's theorem) in the form of (u, v, w, p, ρ).

Introducing the streamfunction

$$
u = \psi_z,\ w = -\psi_x \tag{8.70}
$$

and the normal modes

$$
(\rho, p, \psi) = (\hat{\rho}(z), \hat{p}(z), \hat{\psi}(z)) \exp[ik(x - ct)] \tag{8.71}
$$

the equations (8.69) are converted into

$$
\begin{aligned}
(U - c)\hat{\psi}_z - U_x\hat{\psi} &= -\frac{1}{\rho_0}\hat{p} \\
(U - c)k^2\hat{\psi} &= -\frac{1}{\rho_0}\hat{p}_z - \frac{g}{\rho_0}\hat{\rho} \\
(U - c)\hat{\rho} + \frac{N^2\rho_0}{g}\hat{\psi} &= 0
\end{aligned}
\tag{8.72}
$$

and by eliminating pressure we have the Taylor–Goldstein equation

$$
(U - c)\left(\frac{d^2}{dz^2} - k^2\right)\hat{\psi} + \left(\frac{N^2}{U - c} - \frac{d^2U}{dz^2}\right)\hat{\psi} = 0, \tag{8.73}
$$

which is just the Rayleigh equation (8.48) with an additional term from density stratification.

Since

$$
\overline{\frac{1}{U - c}} = \overline{\left(\frac{U - c^*}{|U - c|^2}\right)} = \frac{U - c}{|U - c|^2} = \frac{1}{U - c^*}, \tag{8.74}
$$

it can be seen that $(c, \hat{\psi})$ is an eigen pair, so is $(c^*, \hat{\psi})$. Hence, for each growing mode, there is a corresponding decaying mode. The no-penetration conditions $w(0) = w(1)$ requires that $\hat{\psi}(0) = \hat{\psi}(1) = 0$ at the walls. We also note that since $w = -\psi_x$, the normal modes of w are related to those of ψ as $\hat{w} = -ik\hat{\psi}$ so the TGE can also be cast as

$$
(U - c)\left(\frac{d^2}{dz^2} - k^2\right)\hat{w} + \left(\frac{N^2}{U - c} - \frac{d^2U}{dz^2}\right)\hat{w} = 0. \tag{8.75}
$$

If there is no background flow, the T–G that describes the instability triggered by perturbations only reads

$$
\frac{d^2\hat{\psi}}{dz^2} + k^2\left(\frac{N^2}{\Omega^2} - 1\right)\hat{\psi} = 0, \tag{8.76}
$$

where $\Omega^2 = c^2k^2$ is the angular frequency. Here $N = N(z)$ is an arbitraty density profile.

The T–G equation (8.73) and the Rayleigh equation (8.48) are both second-order ODEs in the Sturm–Liouville form (8.108). They are eigenvalue problems with c or Ω as the eigenvalue and k as a parameter, and are **not** self-adjoint but can be transformed to self-adjoint (refs) via the Howard transformations (ref). What viscosity does on the normality?

8.2.5 Howard's semicircle theorem

Takeing the Howard's transformation (non-singular if $c \neq U$)

$$F = \frac{\hat{\psi}}{U - c} \tag{8.77}$$

the T–G equation is transformed into

$$\frac{d}{dz}[(U-c)^2 F_z] - k^2(U-c)^2 F + N^2 F. \tag{8.78}$$

Multiply by F^*, integrate in the vertical domain, and use the homogeneous boundary conditions, we get

$$\int (U-c)^2 Q \, dz = \int N^2 |F|^2 \, dz \tag{8.79}$$

where

$$Q = |F_z|^2 + k^2 |F|^2 \tag{8.80}$$

is real positive. Separating the real and imaginary parts of (8.79), we have

$$\int [(U-c_r)^2 - c_i^2] Q \, dz = \int N^2 |F|^2 \, dz \tag{8.81}$$

$$c_i \int (U - c_r) Q \, dz = 0. \tag{8.82}$$

Since we require $c_i \neq 0$ for instability, the second equation is essentially

$$\int (U - c_r) Q \, dz = 0, \tag{8.83}$$

which can be satisfied only if $U - c_r$ changes sign in the domain, i.e.,

$$U_{\min} < c_r < U_{\max}. \tag{8.84}$$

That implies the propagation direction of the unstable waves is the same as the base flow (assume $U_{\min} > 0$) and provides a bound for their phase speed – could not be faster or slower than the base flow.

Combining (8.81) and (8.82) we have

$$\int [U^2 - c_r^2 - c_i^2] Q \, dz \geq 0, \tag{8.85}$$

also naturally we have

$$\int (U - U\mathrm{min})(U - U_{\max}) Q \, dz \leq 0 \to \int [U_{\max} U_{\min} + U^2 - U(U_{\max} + U_{\min})] Q \, dz \leq 0. \tag{8.86}$$

Combining (8.85) and (8.86) and realizing (8.83) we finally have

$$\int [U_{\max} U_{\min} + c_r^2 + c_i^2 - c_r(U_{\max} + U_{\min})] Q \, dz \leq 0. \tag{8.87}$$

That implies a semicircle ($c_i > 0$)

$$[c_r - \frac{1}{2}(U_{\max} + U_{\min})^2] + c_i^2 \leq [\frac{1}{2}(U_{\max} - U_{\min})]^2 \tag{8.88}$$

which limits the possible phase speed (c_r) of the unstable waves and their growth rate ($\sigma = kc_i$). This is a necessary condition. Also, the maximum growth rate is limited by (not necessarily reached)

$$kc_i \leq \frac{k(U_{\max} - U_{\min})}{2}. \tag{8.89}$$

In the numerical computation of the complex eigenvalue $c(k)$, the above bounds efficiently narrow the searching range. We also note that $N = 0$ will only zero out the RHS of (8.79) and (8.85) but not that of (8.86), so the semicircle theorem is unaffected and is valid for both unstratified and stratified shear flows.

8.2.6 Miles–Howard sufficient condition

Under the transformation (Howard, 1961)

$$\phi = \frac{\hat{\psi}}{\sqrt{U-c}} \tag{8.90}$$

the T–G equation can be converted to

$$\frac{d}{dz}[(U-c)\phi_z] - \left[k^2(U-c) + \frac{1}{2}U_{zz} + \frac{1}{U-c}\left(\frac{1}{4}U_z^2 - N^2\right)\right]\phi = 0. \tag{8.91}$$

Multiply the equation above with ϕ^*, integrate in z, and use the homogeneous boundary conditions, we get

$$\int \frac{1}{U-c}(N^2 - 1/4U_z^2)\,dz = \int (U-c)(|\phi_z|^2 + k^2|\phi|^2)\,dz + \int \frac{1}{2}\int U_{zz}|\phi|^2\,dz \tag{8.92}$$

and notice that

$$\frac{1}{U-c} = \frac{U - c_r + ic_i}{|U-c|^2} \tag{8.93}$$

we have (separating real and imaginary parts):

$$\int \frac{U-c_r}{|U-c|^2}(N^2 - 1/4U_z^2)\,dz = \int (U-c_r)(|\phi_z|^2 + k^2|\phi|^2)\,dz + \int \frac{1}{2}\int U_{zz}|\phi|^2\,dz \tag{8.94}$$

$$\int \frac{c_i}{|U-c|^2}(N^2 - 1/4U_z^2)\,dz = -c_i \int (|\phi_z|^2 + k^2|\phi|^2)\,dz. \tag{8.95}$$

Since the integrant in the second equation is positive, the necessary condition for instability ($c_i \neq 0$) is for $N^2 - 1/4U_z^2$ to be negative at least in some part of the domain, i.e., at least one point in the domain has

$$Ri_g = \frac{N^2}{U_z^2} < \frac{1}{4}. \tag{8.96}$$

This adverse condition is generally referred to as the Miles–Howard sufficient condition: if the flow has everywhere $Ri_g > 1/4$, inviscid linear stability of the normal modes is guaranteed.

8.2.7 Non-self-adjointness

The Rayleigh equation is not self-adjoint; but it is after the Howard transformation. So as the Taylor–Goldstein.

8.3 Centrifugal and rotational instability

8.3.1 Rayleigh's criterion

In the previous sections, instability analysis was mainly equation-based. In this section, we will also show an energy-based argument that leads to the well-known Rayleigh's criterion.

The angular momentum is

$$L = ru_\theta \tag{8.97}$$

and the condition that the absolute magnitude of the angular momentum decreases radially,

$$\frac{1}{2r^3}\frac{dL^2}{dr} < 0, \tag{8.98}$$

corresponds to instability. It is commonly referred to as the Rayleigh's criterion and it is a necessary and sufficient condition for the instability of inviscid columnar vortices subject to 3-D axisymmetric perturbations (Drazin, 2002). Since the flow is axisymmetric, $\omega_z = \partial_r u_\theta + u_\theta/r - \partial_\theta u_r = \partial_r u_\theta + u_\theta/r$, the condition (8.98) is equivalant to

$$\frac{u_\theta}{r}\left(\frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) < 0. \tag{8.99}$$

Example: Taylor–Couette.

8.3.2 Inertial/Coriolis instability

The Rayleigh's criterion can be generalized in a rotating frame (with rotation rate $\varOmega$) by replacing the angular momentum L with the absolute angular momentum

$$L_a = r(u_\theta + \varOmega r) \tag{8.100}$$

where $u_\theta + \varOmega r$ is interpreted as the absolute velocity. The condition for instability

$$\frac{1}{2r^3}\frac{dL_a^2}{dr} < 0 \tag{8.101}$$

translates to

$$\chi = \left(\frac{2u_\theta}{r} + f\right)(\omega_z + f) < 0 \tag{8.102}$$

where

$$\frac{dL}{dr} = r\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + 2\Omega\right) = r(\omega_z + f). \tag{8.103}$$

Equation (8.102) can be understood as that the absolute velocity and the absolute vorticity are of opposite signs. It is called the generalized Rayleigh criterion (Kloosterziel & Van Heijst, 1991). We note that conservation of absolute angular momentum,

$$\frac{d}{dr}(ru_\theta + \varOmega r) \tag{8.104}$$

Taking $r \to \infty$ (translation = rotation around infinity), we recover the absolute vorticity criterion

$$f(\omega_z + f) < 0 \tag{8.105}$$

for anticyclonic parallel shear flows (Holton, 1972), where ω_z also reduces to $\omega_z = \partial_r u_\theta + u_\theta/r = \partial_r u_\theta = -S$. This marks the similarity between the inertial instability in 2D parallel and axisymmetric base flows. The structures resulting from these two instabilities are quasi-streamwise/azimuthal vortices, respectively.

8.3.3 Taylor–Couette instability

8.4 Non-normal instability

8.4.1 Transient growth

Also vorticity / laplacian v formulation of the O–S. See Caulfield, also Henningson.

8.4.2 Adjoint matrices, operators, and equations

For a complex matrix $A \in \mathbb{C}^{N\times N}$, define an inner product

$$(u, v)_A = (Au, v) \tag{8.106}$$

where $u, v \in \mathbb{C}^N$ and $(u, v) = v^H u$, $(\cdot)^H$ is the Hermitian transpose. Define the adjoint matrix of A as $A^\dagger$ such that

$$(Au, v) = (u, A^\dagger v). \tag{8.107}$$

We note that if A is Hermitian ($A^H = A$), (8.107) is valid. Such matrix A is also called self-adjoint ($A = A^\dagger$). For operators defined on domains like $\mathbb{C}^N$, Hermitian and self-adjointness imply each other and we don't distinguish these two in what follows.

Consider the following standard Sturm–Liouville eigenvalue problem:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi = -\lambda\sigma(x)\phi, \tag{8.108}$$

where $p(x), w(x)$ are positive, and $\lambda, \phi(x)$ are the eigenvalue and corresponding eigenfunction of the problem. The boundary conditions are

$$\alpha_1\phi(a) + \alpha_2\frac{d\phi}{dx}(a) = 0 \tag{8.109}$$

$$\beta_1\phi(b) + \beta_2\frac{d\phi}{dx}(b) = 0 \tag{8.110}$$

$$\tag{8.111}$$

with $\alpha_1^2 + \alpha_2^2 \neq 0, \beta_1^2 + \beta_2^2 \neq 0$.

The LHS operator is defined as

$$\mathcal{L}(y) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y \tag{8.112}$$

and the S–L eigenvalue problem is

$$\mathcal{L}(\phi) + \lambda\sigma(x)\phi = 0. \tag{8.113}$$

The Lagrange identity is

$$u\mathcal{L}(v) - v\mathcal{L}(u) = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right] \tag{8.114}$$

and the Green's formula is

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\bigg|_a^b \tag{8.115}$$

When u, v satisfy the same set of boundary conditions (either homogeneous or periodic), we have the self-adjointness, i.e.,

$$\int_a^b [u\mathcal{L}(v) - v\mathcal{L}(u)]dx = 0 \tag{8.116}$$

or

$$\int_a^b u\mathcal{L}(v)dx = \int_a^b v\mathcal{L}(u)dx. \tag{8.117}$$

$$= \int_a^b v\mathcal{L}^\dagger(u)dx. \tag{8.118}$$

and we note the definition of adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$ is that

$$(u, \mathcal{L}(v)) = (v, \mathcal{L}^\dagger(u)), \tag{8.119}$$

with the inner product defined based on spatial integral and the adjoint is dependent on the inner product.

Examples.

1. The Laplacian operator.
$$\mathcal{L} = \nabla^2. \tag{8.120}$$
The multidimensional variation of (8.115), with $\mathcal{L} = \nabla^2$, is
$$\iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV = \iiint \nabla \cdot [u\nabla v - v\nabla u]dV = \iint (u\nabla v - v\nabla u) \cdot d\boldsymbol{A} \tag{8.121}$$
and if u, v satisfy the same homogeneous BC,
$$\iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV = \iint (u\nabla v - v\nabla u) \cdot d\boldsymbol{A} = 0, \tag{8.122}$$
∇^2 is self-adjoint.

2. The wave equation.
$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\nabla^2. \tag{8.123}$$
The Green's formula
$$\int_{t_i}^{t_f} \iiint [u\mathcal{L}(v) - v\mathcal{L}(u)]dV dt \tag{8.124}$$
$$= \iiint \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) dV\bigg|_{t_i}^{t_f} - c^2 \int_{t_i}^{t_f} \left(\iint (u\nabla v - v\nabla u) \cdot d\boldsymbol{A}\right) dt \tag{8.125}$$
And we note that the $\mathcal{L} = \partial_{tt}$ operator alone is self-adjoint if the boundary terms vanish, by
$$\int_{t_i}^{t_f} [u\mathcal{L}(v) - v\mathcal{L}(u)]dt = \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\bigg|_{t_i}^{t_f}. \tag{8.126}$$

3. Heat equation. Given above, we only consider the temporal derivative $\mathcal{L} = \partial_t$ here. Since
$$\int_{t_i}^{t_f} \left(u\frac{dv}{dt} + v\frac{du}{dt}\right) dt = uv|_{t_i}^{t_f} \tag{8.127}$$
vanishes if the RHS is zero,
$$\int_{t_i}^{t_f} u\mathcal{L}v\, dt = \int_{t_i}^{t_f} -v\mathcal{L}u\, dt \tag{8.128}$$
and the adjoint operator is $\mathcal{L}^\dagger = -\partial_t$. And we note that for the boundary term to vanish, we usually only have one BC for u in a first order problem such as $u(a) = 0$ and need to introduce an adjoint BC as $v(b) = 0$.

8.4.3 Non-self-adjointness and non-normality

A normal matrix $L \in \mathbb{C}^{N\times N}$ is defined as
$$L^H L = LL^H \tag{8.129}$$
and it is unitarily diagonalizable ($L = U\Lambda U^H$). The eigenvectors of L span an orthogonal basis of $\mathbb{C}^N$. More specifically, the eigenvectors corresponding to different eigenvalues are orthogonal, and even for degenerate eigenvalues an orthogonal basis can be found. A normal matrix is Hermitian if and only if all its eigenvalues are real.

The normality of a linear operator $\mathcal{L}$ is defined as
$$\mathcal{L}^\dagger\mathcal{L} = \mathcal{L}\mathcal{L}^\dagger. \tag{8.130}$$

The eigenmodes of $\mathcal{L}$ are normal to each other. A self-adjoint operator is hence (an example of) normal operators and a non-normal operator must be non-self-adjoint.

For two eigenmodes $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$, where $\mathbf{\Phi}_i = e^{\lambda_i t}\boldsymbol{\phi}_i$, and $\lambda_i, \boldsymbol{\phi}_i$ are the eigenpair. There difference/-cancellation $\boldsymbol{f} = \mathbf{\Phi}_1 - \mathbf{\Phi}_2$ decays if both decays and $(\mathbf{\Phi}_1, \mathbf{\Phi}_2) = 0$. That said, if the real part of each eigenvalue is negative, the energy of the perturbation will decay. However, for non-self-adjoint operators, there could be a transient growth of the cancellation $\boldsymbol{f}$ (Schmid, 2007), where the decay of individual eigenmodes does not imply the transient decay of the total energy. The idea of optimal perturbations it to find such a transient mode that grows most within a certain period of time.

8.4.4 Adjoint of the linearised N-S equations (Op's)

Op's: optimal linear perturbation solved as an optimal control problem in an optimization formulation constrained by the PDEs with linear operators.

Similar to (8.10)-(8.11), the linearised perturbation equation with buoyancy is

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8.131}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \boldsymbol{U} - \frac{1}{\rho_0}\nabla p + \nu \nabla^2 \boldsymbol{u} + b\boldsymbol{e}_z \tag{8.132}$$

$$\frac{\partial b}{\partial t} + \boldsymbol{U} \cdot \nabla b + \boldsymbol{u} \cdot \nabla B = \kappa \nabla^2 b \tag{8.133}$$

where $b = B + b'$ is buoyancy and the primes are dropped from the equations above.

Assume the adjoints of $(\boldsymbol{u}, p, b)$ are $(\boldsymbol{v}, q, \varphi)$, multiplying each term in (8.131) by (v_1, v_2, v_3), (8.132) by $\boldsymbol{v}$, and (8.133) by φ, we can derive the adjoint equations of (8.131)-(8.133) as

$$\nabla \cdot \boldsymbol{v} = 0 \tag{8.134}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{v} = \boldsymbol{v} \cdot \nabla \boldsymbol{U} + \varphi \nabla B - \frac{1}{\rho_0}\nabla q - \nu \nabla^2 \boldsymbol{v} \tag{8.135}$$

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{U} \cdot \nabla \varphi = -v_3 - \kappa \nabla^2 \varphi \tag{8.136}$$

We note the cross contribution terms $\varphi \nabla B$ and $-v_3$. The same set of equations can also be derived from a Lagrangian multiplier approach (a more morden method, but now classic), with the total perturbation energy being the Lagrangian and the set of governing equations along with BC's being the constraints enforced as multipliers. Such Lagrangian is in the form of energy gain as (Arratia, 2011; Luchini & Bottaro, 2014; Kaminski *et al.*, 2014)

$$\mathscr{L}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T)\rangle + \langle b(T), b(T)\rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0\rangle + \langle b_0, b_0\rangle / N^2} \tag{8.137}$$

$$- [\frac{\partial u_i}{\partial t} + u_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \nu \frac{\partial^2 u_i}{\partial x_j^2} - b\delta_{i3}, v_i] \tag{8.138}$$

$$- [\frac{\partial b}{\partial t} + u_j \frac{\partial B}{\partial x_j} + U_j \frac{\partial b}{\partial x_j} - \kappa \frac{\partial^2 b}{\partial x_j^2}, \varphi] - [\frac{\partial u_i}{\partial x_i}, q] \tag{8.139}$$

$$- \langle u_i(0) - u_{0i}, v_{0i}\rangle - \langle b(0) - b_0, \varphi_0\rangle \tag{8.140}$$

constrained by the equations through the multipliers $(\boldsymbol{v}, q, \varphi)$ and similarly the BC's. The inner products

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \int_V \boldsymbol{u} \cdot \boldsymbol{v} dV \tag{8.141}$$

$$[\boldsymbol{u}, \boldsymbol{v}] = \int_0^T \langle \boldsymbol{u}, \boldsymbol{v}\rangle dt \tag{8.142}$$

are defined as respective spatial and spatio-temporal integrations.

In the phraseology of optimal control (with PDE constraints, Ref. section 5 of Manzoni *et al.* (2021)):

- Target functional (finite time transient gain):

$$\mathscr{G}(T) = \frac{\langle \boldsymbol{u}(T), \boldsymbol{u}(T)\rangle + \langle b(T), b(T)\rangle / N^2}{\langle \boldsymbol{u}_0, \boldsymbol{u}_0\rangle + \langle b_0, b_0\rangle / N^2}. \tag{8.143}$$

- State variables: $(\boldsymbol{u}, b)$.
- State equations (PDE constraints): (8.131)-(8.133). We note that (adjoint) pressure only appears as a Lagrangian multiplier that enforces the continuity.
- Control variables: $\boldsymbol{u}(0), b(0)$ and hence $\boldsymbol{u}(\boldsymbol{x}, t), b(\boldsymbol{x}, t)$.
- Admissible constraints on controls: none for now.

Taking the variation of (8.140) w.r.t.:

- The multipliers $(\boldsymbol{v}, q, \varphi)$: we recover the 'direct' equations (8.131)-(8.133).
- The terms $\boldsymbol{v}_0, \varphi_0$: we obtain the definition of IC's $\boldsymbol{u}_0, b_0$.
- The 'direct' variables $(\boldsymbol{u}, p, b)$: we obtain the adjoint equations (8.134)-(8.136). This step will be shown in detail.

Other than deriving from a Lagrangian perspective, the adjoint can also be derived using (multiple) integrations by parts. Starting from (8.14), i.e.,

$$\mathcal{L}_{\mathrm{NS}}\boldsymbol{q} = 0, \tag{8.144}$$

with the direct and adjoint variables being $\boldsymbol{q} = (\boldsymbol{u}, p)$ and $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$, we look for the adjoint $\mathcal{L}^{\dagger}_{\mathrm{NS}}$ such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] - [\mathcal{L}^{\dagger}_{\mathrm{NS}}\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) \tag{8.145}$$

and the boundary conditions that make the RHS boundary terms vanish.

Where the inner product $[\cdot, \cdot]$ is that same as in (8.142) such that

$$[\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] = \int_T \int_V (\boldsymbol{v} \cdot \boldsymbol{u} + qp) dt dV. \tag{8.146}$$

The weak form of (8.14) is

$$[\mathbb{1}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V (\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + \partial_i p + u_j \partial_j U_i) dt dV = 0. \tag{8.147}$$

We note that (8.147) should also be valid on any arbitrary test function $\boldsymbol{q}_{\mathrm{d}} = (\boldsymbol{v}, q)$ for (8.14) to hold, such that

$$[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] = \int_T \int_V v_i(\partial_t u_i + U_j \partial_j u_i - \nu \partial_j^2 u_i + u_j \partial_j U_i) + q \partial_i p \, dt dV = 0. \tag{8.148}$$

By integration by parts we have

$$\begin{aligned}
[\boldsymbol{q}_{\mathrm{d}}, \mathcal{L}_{\mathrm{NS}}\boldsymbol{q}] &= \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p \partial_i q \, dt dV + \mathrm{BT}(\boldsymbol{q}, \boldsymbol{q}_{\mathrm{d}}) && (8.149)\\
&= \int_T \int_V u_i(-\partial_t v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i + v_j \partial_j U_i) - p \partial_i q \, dt dV && (8.150)\\
&= [\mathcal{L}^{\dagger}_{\mathrm{NS}}\boldsymbol{q}_{\mathrm{d}}, \boldsymbol{q}] && (8.151)
\end{aligned}$$

that defines the adjoint operator of $\mathcal{L}_{\boldsymbol{U}} = \partial_t + \boldsymbol{U} \cdot \nabla - \nu \nabla^2$:

$$\mathcal{L}^{\dagger}_{\boldsymbol{U}} = -\partial_t - \boldsymbol{U} \cdot \nabla - \nu \nabla^2 \tag{8.152}$$

and the adjoint equation

$$\partial_t v_i + U_j \partial_j v_i + \nu \partial_j^2 v_i = v_j \partial_j U_i - \partial_i q \tag{8.153}$$

and we note that by advancing forward in time, the viscous term is injecting energy into the system. Using the transfrom $\tau = -t$ we have

$$\partial_\tau v_i - U_j \partial_j v_i - \nu \partial_j^2 v_i = -v_j \partial_j U_i + \partial_i q. \tag{8.154}$$

In another form, the adjoint equation (without density) can be expressed similar to (8.16) as

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\mathcal{A}_{\boldsymbol{U}}^\dagger \boldsymbol{v} \tag{8.155}$$

where $\mathcal{A}_{\boldsymbol{U}}^\dagger$ is the adjoint operator of $\mathcal{A}_{\boldsymbol{U}}$ in (8.15)

$$\mathcal{A}_{\boldsymbol{U}}^\dagger \boldsymbol{v} = \boldsymbol{U} \cdot \nabla \boldsymbol{v} - \boldsymbol{v} \cdot \nabla \boldsymbol{U} + \frac{1}{\rho_0} \nabla q + \nu \nabla^2 \boldsymbol{v}. \tag{8.156}$$

8.5 Some classical instabilities

8.5.1 Kelvin–Helmholtz instability

8.5.2 Rayleigh–Bénard instability

8.5.3 The Lorenz system

8.5.4 Rayleigh–Taylor instability

9 Compressible flows, gas dynamics

9.1 Conservation laws

Apart from deriving the Euler equation by neglecting the viscous term in the Navier–Stokes, we present another way here that is more to the taste of gas dynamics.

9.2 Thermodynamics

9.3 Kinetic theory, microscopic view of fluid and flow properties

pressure (molecules bouncing back from a flat wall), viscosity, etc.

kinetic theory. Cf. Frank Shu.

AFD notes; the temperature lapse rate example;

Incompressible and isentropic flows ($p/\rho^\gamma = C$ for perfect gas), are barotropic flows.

9.4 Small perturbation linearized equations, sound waves

Simple waves, wave equations, enthalpy, shock wakes, etc. Small perturbation linearized equations, aerodynamics.

9.5 Shock waves, discontinuities, and jump conditions

Conservation form, weak form, weak continuity, test functions.

A Vectors, tensors, and their calculus

Aris (1989) is a good reference.

A.1 Levi-Civita symbol

A.1.1 Determinant representation

The matrix determinants can be expressed in terms of the Levi-Civita symbol. Assume A is a matrix

$$\det(A) = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{A.1}$$

where

$$\boldsymbol{a}_1 = (a_{11}, a_{12}, a_{13})^\top,\ \boldsymbol{a}_2 = (a_{21}, a_{22}, a_{23})^\top,\ \boldsymbol{a}_3 = (a_{31}, a_{32}, a_{33})^\top \tag{A.2}$$

Therefore the Levi-Civita symbol can be expressed as

$$\epsilon_{ijk} = \det(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k) = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k) \tag{A.3}$$

Similarly, the outer product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \epsilon_{ijk} a_j b_k \boldsymbol{e}_i \tag{A.4}$$

Example: $\boldsymbol{\omega}$.

A.1.2 Epsilon identity

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{A.5}$$

$$= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{jl}(\delta_{in}\delta_{km} - \delta_{im}\delta_{kn}) + \delta_{kl}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \tag{A.6}$$

$$= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} \tag{A.7}$$

A.1.3 Contracted epsilon identity

Let $i = l$ and notice $\delta_{ii} = 3$

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{A.8}$$

Futhur let $k = m$

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{A.9}$$

Futhermore

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{A.10}$$

A.2 Vector identities

Assume λ is a scalar and $\boldsymbol{a}$,$\boldsymbol{b}$,$\boldsymbol{c}$,$\boldsymbol{d}$ are vectors in $\mathbb{R}^3$. The identities below might be useful in fluids, some of which have geometric implecations.

$$\nabla \cdot (\nabla \times \boldsymbol{b}) = 0 \tag{A.11}$$
$$\nabla \times (\nabla \boldsymbol{b}) = 0 \tag{A.12}$$
$$\nabla \cdot (\lambda \boldsymbol{b}) = \nabla \lambda \cdot \boldsymbol{b} + \lambda(\nabla \cdot \boldsymbol{b}) \tag{A.13}$$
$$\nabla \times (\lambda \boldsymbol{b}) = \lambda(\nabla \times \boldsymbol{b}) - \boldsymbol{b} \times \nabla \lambda \tag{A.14}$$
$$\nabla(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \boldsymbol{a} + \boldsymbol{a} \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \times (\nabla \times \boldsymbol{a}) + \boldsymbol{a} \times (\nabla \times \boldsymbol{b}) \tag{A.15}$$
$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \tag{A.16}$$
$$\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla \times \boldsymbol{a}) \cdot \boldsymbol{b} - (\nabla \times \boldsymbol{b}) \cdot \boldsymbol{a} \tag{A.17}$$
$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}) \tag{A.18}$$
$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) \tag{A.19}$$
$$\boldsymbol{b} \times (\nabla \times \boldsymbol{b}) = \nabla(\frac{1}{2}\boldsymbol{b} \cdot \boldsymbol{b}) - \boldsymbol{b} \cdot \nabla \boldsymbol{b} \tag{A.20}$$
$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) \tag{A.21}$$
$$\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2 \boldsymbol{a} \tag{A.22}$$

Their proofs are left as exercises.
Comments:

(1) Eq. (A.11): A curl field is solenoidal (divergence-free).

(2) Eq. (A.12): A gradient field is irrotational (curl-free).

(3) Eq. (A.21): $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$, so its curl is in the space spaned by $\boldsymbol{a}$ and $\boldsymbol{b}$.

(4) Eq. (A.19): $\boldsymbol{a} \times (\cdot)$ is perpendicular to $\boldsymbol{a}$ and $(\cdot) \times (\boldsymbol{b} \times \boldsymbol{c})$ is in the space spaned by $\boldsymbol{b}$ and $\boldsymbol{c}$. This two facts in combinition gives the bases of $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

(5) Eq. (A.16): This is the volume spaned by $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, and the identity is basically the invariance of a determinant with respect to row/column permutation.

(6) Eq. (A.18): By letting $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$ and noticing the inner product with itself is non-negative, we re-discover the Cauchy-Schwartz inequality.

(7) From (A.15) it can be immediately seen that

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla(\boldsymbol{u}^2/2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}). \tag{A.23}$$

A.3 Tensor eigenvalues, invariants, and its application in fluids

Consider a tensor $\boldsymbol{A}$ in Cartesian coordinate

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \tag{A.24}$$

Its eigenvalues are roots of the characteristic polynomial

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{vmatrix} = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{A.25}$$

with the three coefficients being the three principle invariants of $\boldsymbol{A}$

$$
\begin{aligned}
I_1 &= a_{11} + a_{22} + a_{33} && \text{(A.26)} \\
&= \mathrm{tr}(\boldsymbol{A}) && \text{(A.27)} \\
&= a_{ii} && \text{(A.28)} \\
I_2 &= a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31} && \text{(A.29)} \\
&= \frac{\mathrm{tr}(\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)}{2} && \text{(A.30)} \\
&= \frac{1}{2}((a_{ii})^2 - a_{ij}a_{ji}) && \text{(A.31)} \\
I_3 &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) && \text{(A.32)} \\
&= \det(\boldsymbol{A}) && \text{(A.33)}
\end{aligned}
$$

in both element-wise and coordinate-independent expression.

Now we consider the factorization of the characteristic polynomial as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\lambda - \lambda_1\lambda_2\lambda_3 = 0, \quad \text{(A.34)}$$

and obtain the Vieta's theorem for cubic equations as

$$
\begin{aligned}
I_1 &= \lambda_1 + \lambda_2 + \lambda_3 && \text{(A.35)} \\
I_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 && \text{(A.36)} \\
I_3 &= \lambda_1\lambda_2\lambda_3 && \text{(A.37)}
\end{aligned}
$$

which are the three principle invariants of tensor $\boldsymbol{A}$.

Additionally, there are more invariants (although not independent) of $\boldsymbol{A}$, such as the main invariants

$$
\begin{aligned}
J_1 &= \lambda_1 + \lambda_2 + \lambda_3 = I_1 = \mathrm{tr}(\boldsymbol{A}) && \text{(A.38)} \\
J_2 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1^2 - 2I_2 = \mathrm{tr}(\boldsymbol{A} \cdot \boldsymbol{A}) && \text{(A.39)} \\
J_1 &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = I_1^3 - 3I_1I_2 + 3I_3 = \mathrm{tr}(\boldsymbol{A} \cdot \boldsymbol{A} \cdot \boldsymbol{A}) && \text{(A.40)}
\end{aligned}
$$

which are the coefficients of the characteristic polynomial of the deviatoric part of $\boldsymbol{A}$:

$$\boldsymbol{A} - \frac{\mathrm{tr}(\boldsymbol{A})}{3}\boldsymbol{I}, \quad \text{(A.41)}$$

which is traceless and has eigenvalues

$$\lambda_i - \frac{1}{3}. \quad \text{(A.42)}$$

A.3.1 Discriminant of a cubic equation

Consider

$$ax^3 + bx^2 + cx + d = 0, \quad \text{(A.43)}$$

its determinant is

$$
\begin{aligned}
\Delta &= (x_1 - x_2)^2(x_2 - x_3)^2(x_3 - x_1)^2 && \text{(A.44)} \\
&= 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 && \text{(A.45)}
\end{aligned}
$$

with x_1, x_2, x_3 being the three roots.

1. $\Delta > 0$: Three distinct real roots.

2. $\Delta = 0$: All roots are real with at least two identical.

3. $\Delta < 0$: One real and a pair of complex conjugate roots (proof: assume complex roots are $x \pm iy$).

Proof. The Vieta's theorem for (A.43) and the invariant relations can be used to simplify (A.43) to obtain (A.45).

Note: Eq. (A.45) can also be obtained as follows (with some reasons/meanings in algebraic geometry). Consider a cubic equation in canonical form

$$f(x, w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = 0. \tag{A.46}$$

The Hessain matrix is

$$H(f) = \begin{bmatrix} 6Ax + 6Bw & 6Bx + 6Cw \\ 6Bx + 6Cw & 6Cx + 6Dw \end{bmatrix} \tag{A.47}$$

and the Hessain

$$\det(H) = 36[(AC - B^2)x^2 + (AD - BC)xw + (BD - C^2)w^2] \tag{A.48}$$

$$= 18[x, w] \begin{bmatrix} 2(AC - B^2) & (AD - BC) \\ (AD - BC) & 2(BD - C^2) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}, \tag{A.49}$$

in quadratic form. Define the Hessain

$$\boldsymbol{H} = \begin{bmatrix} 2(AC - B^2) & (AD - BC) \\ (AD - BC) & 2(BD - C^2) \end{bmatrix} \tag{A.50}$$

The discriminant of the cubic is just the determinant of the Hessain $\boldsymbol{H}$:

$$\Delta = \det(\boldsymbol{H}) = -A^2D^2 + 6ABCD - 4AC^3 - 4B^3D + 3B^2C^2, \tag{A.51}$$

and $\Delta > 0$ for three real roots, $\Delta = 0$ for double or triple real root, and $\Delta < 0$ for single real root.

A.3.2 Application to the velocity gradient tensor

The cubic curve theory, especially root finding, has relation to the characteristic polynomial of the velocity gradient tensor and so forth the local flow geometry (basically the number of real/complex eigenvalues).

The characteristic polynomial for $\boldsymbol{u}\nabla$ is

$$\lambda^3 - P\lambda^2 + Q\lambda - R = 0, \tag{A.52}$$

where $(P, Q, R) = (I_1, I_2, I_3)$ are the three invariants. In incompressible flow, $P = u_{i,i} = 0$, and the equation above degenerates to

$$\lambda^3 + Q\lambda - R = 0. \tag{A.53}$$

It is in the so-called 'depressed' form (comparatively, an elliptic curve is called in the Weierstrass form if it satisfies the Weierstrass equation $y^2 = x^3 + ax + b$)

Positive second invariant $Q = -1/2u_{i,j}u_{j,i} = 1/2(\|\boldsymbol{\Omega}\|^2 - \|\boldsymbol{S}\|^2)$ is used for identifying vortical motions, or 'eddies' Hunt *et al.* (1988); Jeong & Hussain (1995). Consider the Poisson equation,

$$\frac{1}{\rho}\nabla^2 p = -u_{i,j}u_{j,i}, \tag{A.54}$$

we have positive Q corresponding to a local pressure minimum.

The discriminant for depressed cubic equation

$$x^3 + px + q = 0 \tag{A.55}$$

reduces to

$$\Delta = -4p^3 - 27q^2. \tag{A.56}$$

So we have the discriminant for the gradient of a solenoidal field (with renormalized coefficients; note the flipped sign)

$$\triangle = \left(\frac{1}{3}Q\right)^3 + \left(\frac{1}{2}R\right)^2 \tag{A.57}$$

and if $\triangle > 0$ there will be complex eigenvalues (in complex conjugate pair according to the algebra basic theorem) and so-defined vortical motions. Hence, we can see that the invariants of the velocity gradient tensor is largely related to the local geometry (Chong *et al.*, 1990) of the flow.

A.3.3 Application to the strain rate tensor

We note that both the rate-of-strain tensor $\boldsymbol{S}$ and the Reynolds stress tensor $-\overline{u_i' u_j'}$ are real symmetric, hence they have three real eigenvalues and three orthogonal eigenvectors (principle axes), or, in another word, they are unitarily similar to a diagonal matrix. The rate of strain tensor is diagonal and all strains are normal strains in the principle coordinate.

This part is largely related to eigenvalue decomposition (see section B).

Example. Consider a strain rate tensor

$$\boldsymbol{S} = \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix}. \tag{A.58}$$

Its eigenvalues are $\lambda_1 = -\gamma/2$, $\lambda_2 = \gamma/2$ and associated eigenvectors are $\boldsymbol{x}_1 = 1/\sqrt{2}[1, -1]^T$ and $\boldsymbol{x}_2 = 1/\sqrt{2}[1, 1]^T$. The two principle directions are 45 deg (stretching, the direction that receives the most amplification) and −45 deg (compressing). The similarity transform is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \gamma/2 \\ \gamma/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} -\gamma/2 & 0 \\ 0 & \gamma/2 \end{bmatrix} \tag{A.59}$$

where

$$\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2] \tag{A.60}$$

is a unitary matrix such that $\boldsymbol{AX} = \boldsymbol{X\Lambda}$.

Take $\gamma = \partial_y U$, this is an importan example of plane shear flow. Think of the deformation of a rectangle to a diamond as it moves with the shear.

A.3.4 Application to the Reynolds stress tensor, the invariant map, and the Lumley triangle,

Consider the anisotropic (deviatoric) tensor of Reynolds stress

$$a_{ij} = \frac{\overline{u_i' u_j'}}{2k} - \frac{1}{3}\delta_{ij} \tag{A.61}$$

and its three principle invariants

$$I = \sigma_1 + \sigma_2 + \sigma_3 \tag{A.62}$$

$$II = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \tag{A.63}$$

$$III = \sigma_1\sigma_2\sigma_3 \tag{A.64}$$

along with its three eigenvalues

$$\sigma_1,\, \sigma_2,\, \sigma_3. \tag{A.65}$$

Since a_{ij} is a deviator, it is traceless and

$$I = a_{ii} = 0. \tag{A.66}$$

Consider turbulence. and has zero determinant

$$\det\left(\frac{\overline{u_i'u_j'}}{2k}\right) = (\sigma_1 + \frac{1}{3})(\sigma_2 + \frac{1}{3})(\sigma_3 + \frac{1}{3}) \tag{A.67}$$

$$= \sigma_1\sigma_2\sigma_3 + \frac{1}{3}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{27}, \tag{A.68}$$

and we define

$$F = 27III + 9II + 1 \tag{A.69}$$

since $I = 0$.

1. Two-dimensional turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ can be diagonalized to

   $$diag(a, k-a, 0)$$

   and has zero determinant (there's a direction that has no turbulence). $F = 0$.

2. Three-dimensional isotropic turbulence: the Reynolds stress tensor $\overline{u_i'u_j'}$ is

   $$diag(k/3, k/3, k/3)$$

   and we have $F = 1$.

3. Axisymmetric turbulence. Similarly, the characteristic polynomial of a_{ij} is in Weierstrass form and the condition for repeated eigenvalues (same energy in two principle directions) is

   $$\triangle = \left(\frac{1}{3}II\right)^3 + \left(\frac{1}{2}III\right)^2 = 0 \tag{A.70}$$

   and hence

   $$III = \pm 2\left(-\frac{II}{3}\right)^3, \tag{A.71}$$

   corresponding to the negative/left (pancake) and positive/right (cigar) limit curves of the Lumley triangle (Lumley & Newman, 1977; Choi & Lumley, 2001).

B Matrix Analysis

B.1 Unitary matrix

Unitary transformations preserve inner products (and hence length and angle). Rotation and reflection.

B.2 Singular value decomposition and eigenvalue decomposition

also bilateral relations between SVD/Schur and polar decomposition.

B.3 Conformal mapping

Analytic functions, homomorphism, and conformal mapping.

Angle preserving, Symplectic (area preserving), implications of the latter on numerical schemes.

B.4 Coordinate transformation

C Coordinate systems

C.1 Cylindrical coordinates

Consider the cylindrical transformation

$$(x, y) \to (r, \theta) \tag{C.1}$$

where

$$x = r\cos\theta \tag{C.2}$$

$$y = r\sin\theta \tag{C.3}$$

or

$$r = \sqrt{x^2 + y^2} \tag{C.4}$$

$$\theta = \mathrm{actan}\left(\frac{y}{x}\right) \tag{C.5}$$

we have the corresponding relation between unit vectors

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.6}$$

and

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{bmatrix}, \tag{C.7}$$

which can be proven graphically. We note that the grid transformation matrix is unitary and has $\det() = 1$ (rotation matrix).

The Jacobian of the forward transformation $(r, \theta) = F(x, y)$ is

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix} \tag{C.8}$$

We note that the directions of the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ depend on space, i.e.,

$$\frac{\partial \boldsymbol{e}_r}{\partial r} = \frac{\partial \boldsymbol{e}_\theta}{\partial r} = 0 \tag{C.9}$$

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = -\sin\theta \boldsymbol{e}_x + \cos\theta \boldsymbol{e}_y = \boldsymbol{e}_\theta \tag{C.10}$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\cos\theta \boldsymbol{e}_x - \sin\theta \boldsymbol{e}_y = -\boldsymbol{e}_r \tag{C.11}$$

which can also be seen graphically. These relations are crucial to later derivations.

Consider the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{C.12}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{C.13}$$

C.1.1 Operators in cylindrical coordinates

For a scalar function, say $f(x, y) = f(r, \theta)$, the gradient operator can be expressed as

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{C.14}$$

$$= \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\right)(\cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta) + \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\right)(\sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta) + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{C.15}$$

$$= \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z} \tag{C.16}$$

The factor $r\partial\theta$ can be interpreted as infinitesimal length element in θ direction.

The Laplace operator

$$\nabla^2 = \nabla \cdot \nabla = \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \cdot \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \tag{C.17}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \boldsymbol{e}_\theta \cdot \frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)\right] \tag{C.18}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{C.19}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{C.20}$$

Now consider a vector

$$\boldsymbol{u} = \boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w \tag{C.21}$$

and its derivatives.

Its divergence is

$$\nabla \cdot \boldsymbol{u} = \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z \frac{\partial}{\partial z}\right) \cdot (\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.22}$$

$$= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.23}$$

$$= \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.24}$$

The convection term

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.25}$$

$$= \left(u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right)\boldsymbol{e}_r \tag{C.26}$$

$$+ \left(u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right)\boldsymbol{e}_\theta \tag{C.27}$$

$$+ \left(u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\right)\boldsymbol{e}_z \tag{C.28}$$

Now we deal with $\nabla^2\boldsymbol{u}$.

$$\nabla^2\boldsymbol{u} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.29}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \boldsymbol{u}}{\partial r}\right) + \frac{\partial^2 \boldsymbol{u}}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_r u + \boldsymbol{e}_\theta v) \tag{C.30}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right)\boldsymbol{e}_r \tag{C.31}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right)\boldsymbol{e}_\theta \tag{C.32}$$

$$+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right)\boldsymbol{e}_z \tag{C.33}$$

with

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_r u) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \boldsymbol{e}_r u}{\partial \theta} = \frac{1}{r^2}(2\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta - u\boldsymbol{e}_r + \frac{\partial^2 u}{\partial \theta^2}\boldsymbol{e}_r) \tag{C.34}$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{e}_\theta v) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{\partial \boldsymbol{e}_\theta v}{\partial \theta} = \frac{1}{r^2}(-2\frac{\partial v}{\partial \theta}\boldsymbol{e}_r - v\boldsymbol{e}_\theta + \frac{\partial^2 v}{\partial \theta^2}\boldsymbol{e}_\theta) \tag{C.35}$$

Moreover, the curl can be established (in a compact form) as

$$\nabla \times \boldsymbol{u} = \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_z\frac{\partial}{\partial z}\right) \times (\boldsymbol{e}_r u + \boldsymbol{e}_\theta v + \boldsymbol{e}_z w) \tag{C.36}$$

$$= \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \frac{1}{r}\partial_\theta & \partial_z \\ u & v & w \end{vmatrix} + \frac{1}{r}\boldsymbol{e}_\theta \times \frac{\partial(v\boldsymbol{e}_\theta)}{\partial \theta} \tag{C.37}$$

$$= \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\boldsymbol{e}_\theta + \left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\boldsymbol{e}_z \tag{C.38}$$

$$= \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)\boldsymbol{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta}\right)\boldsymbol{e}_z \tag{C.39}$$

$$= \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix}. \tag{C.40}$$

Examples.

1. Axisymmetric flows ($u_\theta \neq 0$). We have the $\omega_z = 1/r\partial_r(ru_\theta) = u_\theta/r + \partial_r u_\theta$. The first term is called the curvature vorticity and the second called the shear vorticity.

2. Rigid body rotation with angular velocity Ω and $u_\theta = v = \Omega r$. Vorticity $\omega_z = 2\Omega$ (with equal contributions from curvature and shear) but there is no vortical motion.

3. Potential point vortex with $u_\theta = v = \Gamma/2\pi r$. Vorticity $\omega_z = 0$ according to (C.39) – there is no vorticity.

Now we turn our attention to velocity gradient and strain-rate tensors. In Cartesion coordinates,

$$\nabla \boldsymbol{u} = (\partial_i \boldsymbol{e}_i)(u_j \boldsymbol{e}_j) = \partial_i u_j \boldsymbol{e}_i \boldsymbol{e}_j. \tag{C.41}$$

In cylindrical coordinates,

$$\nabla \boldsymbol{u} = (\boldsymbol{e}_r\partial_r + \boldsymbol{e}_\theta 1/r\partial_\theta + \boldsymbol{e}_z\partial_z)(\boldsymbol{e}_r u_r + \boldsymbol{e}_\theta u_\theta + \boldsymbol{e}_z u_z) \tag{C.42}$$

$$= (\boldsymbol{e}_r\boldsymbol{e}_r\partial_r u + \boldsymbol{e}_r\boldsymbol{e}_\theta\partial_r v + \boldsymbol{e}_r\boldsymbol{e}_z\partial_r w) \tag{C.43}$$

$$+ [(\boldsymbol{e}_\theta\boldsymbol{e}_r 1/r\partial_\theta u + \boldsymbol{e}_\theta\boldsymbol{e}_\theta u/r) + (\boldsymbol{e}_\theta\boldsymbol{e}_\theta 1/r\partial_\theta v - \boldsymbol{e}_\theta\boldsymbol{e}_r v/r) + \boldsymbol{e}_\theta\boldsymbol{e}_z 1/r\partial_\theta w] \tag{C.44}$$

$$+ (\boldsymbol{e}_z\boldsymbol{e}_r\partial_z u + \boldsymbol{e}_z\boldsymbol{e}_\theta\partial_z v + \boldsymbol{e}_z\boldsymbol{e}_z\partial_z w) \tag{C.45}$$

$$= \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \\ \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{1}{r}\frac{\partial w}{\partial \theta} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{C.46}$$

and the rate-of-strain tensor is

$$\boldsymbol{S} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \\ ... & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}\right) \\ ... & ... & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ ... & S_{\theta\theta} & S_{\theta z} \\ ... & ... & S_{zz} \end{bmatrix} \tag{C.47}$$

C.1.2 Navier–Stokes in cylindrical coordinates

With the preparation in the previous section, we are now able to write down the Navier–Stokes equations in cylindrical coordinates as

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{C.48}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{C.49}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{C.50}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{C.51}$$

Q.E.D.

C.1.3 Scalar equation in cylindrical coordinates

Additionally, the transport equation of a passive scalar (say ρ)

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho = \nu\nabla^2\rho \tag{C.52}$$

can be cast in cylinderical coordinate as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{v}{r}\frac{\partial \rho}{\partial \theta} + w\frac{\partial \rho}{\partial z} = \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \rho}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \rho}{\partial \theta^2} + \frac{\partial^2 \rho}{\partial z^2}\right] \tag{C.53}$$

C.1.4 Vorticity equation in cylindrical coordinates

The divergence-free condition for vorticity reads

$$\nabla\cdot\boldsymbol{\omega} = \frac{1}{r}\frac{\partial (r\omega_r)}{\partial r} + \frac{1}{r}\frac{\partial \omega_\theta}{\partial \theta} + \frac{\partial \omega_z}{\partial z} = 0. \tag{C.54}$$

The convective term is

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = \left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right)(\boldsymbol{e}_r\omega_r + \boldsymbol{e}_\theta\omega_\theta + \boldsymbol{e}_z\omega_z) \tag{C.55}$$

$$= \left(u\frac{\partial \omega_r}{\partial r} + \frac{v}{r}\frac{\partial \omega_r}{\partial \theta} + w\frac{\partial \omega_r}{\partial z} - \frac{v\omega_\theta}{r}\right)\boldsymbol{e}_r \tag{C.56}$$

$$+ \left(u\frac{\partial \omega_\theta}{\partial r} + \frac{v}{r}\frac{\partial \omega_\theta}{\partial \theta} + w\frac{\partial \omega_\theta}{\partial z} + \frac{v\omega_r}{r}\right)\boldsymbol{e}_\theta \tag{C.57}$$

$$+\left(u\frac{\partial \omega_z}{\partial r}+\frac{v}{r}\frac{\partial \omega_z}{\partial \theta}+w\frac{\partial \omega_z}{\partial z}\right)\boldsymbol{e}_z. \tag{C.58}$$

The stretching term is

$$(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}=\left(\omega_r\frac{\partial}{\partial r}+\frac{\omega_\theta}{r}\frac{\partial}{\partial \theta}+\omega_z\frac{\partial}{\partial z}\right)(\boldsymbol{e}_r u+\boldsymbol{e}_\theta v+\boldsymbol{e}_z w) \tag{C.59}$$

$$=\left(\omega_r\frac{\partial u}{\partial r}+\frac{\omega_\theta}{r}\frac{\partial u}{\partial \theta}+\omega_z\frac{\partial u}{\partial z}-\frac{v\omega_\theta}{r}\right)\boldsymbol{e}_r \tag{C.60}$$

$$+\left(\omega_r\frac{\partial v}{\partial r}+\frac{\omega_\theta}{r}\frac{\partial v}{\partial \theta}+\omega_z\frac{\partial v}{\partial z}+\frac{u\omega_\theta}{r}\right)\boldsymbol{e}_\theta \tag{C.61}$$

$$+\left(\omega_r\frac{\partial w}{\partial r}+\frac{\omega_\theta}{r}\frac{\partial w}{\partial \theta}+\omega_z\frac{\partial w}{\partial z}\right)\boldsymbol{e}_z. \tag{C.62}$$

The diffusion term is

$$\nabla^2\boldsymbol{\omega}=\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\omega_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial \omega_\theta}{\partial\theta}-\frac{\omega_r}{r^2}+\frac{\partial^2\omega_r}{\partial z^2}\right)\boldsymbol{e}_r \tag{C.63}$$

$$+\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\omega_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial \omega_r}{\partial\theta}-\frac{\omega_\theta}{r^2}+\frac{\partial^2\omega_\theta}{\partial z^2}\right)\boldsymbol{e}_\theta \tag{C.64}$$

$$+\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\omega_z}{\partial\theta^2}+\frac{\partial^2\omega_z}{\partial z^2}\right)\boldsymbol{e}_z \tag{C.65}$$

Hence, the transport equation of vorticity, (2.21), in the absense of non-conservative external force and baroclinic torque, is cast in cylindrical coordinates as

$$\frac{\partial \omega_r}{\partial t}+u\frac{\partial \omega_r}{\partial r}+\frac{v}{r}\frac{\partial \omega_r}{\partial \theta}+w\frac{\partial \omega_r}{\partial z}=\omega_r\frac{\partial u}{\partial r}+\frac{\omega_\theta}{r}\frac{\partial u}{\partial \theta}+\omega_z\frac{\partial u}{\partial z} \tag{C.66}$$

$$+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\omega_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial \omega_\theta}{\partial\theta}-\frac{\omega_r}{r^2}+\frac{\partial^2\omega_r}{\partial z^2}\right) \tag{C.67}$$

$$\frac{\partial \omega_\theta}{\partial t}+u\frac{\partial \omega_\theta}{\partial r}+\frac{v}{r}\frac{\partial \omega_\theta}{\partial \theta}+w\frac{\partial \omega_\theta}{\partial z}+\frac{v\omega_r}{r}=\omega_r\frac{\partial v}{\partial r}+\frac{\omega_\theta}{r}\frac{\partial v}{\partial \theta}+\omega_z\frac{\partial v}{\partial z}+\frac{u\omega_\theta}{r} \tag{C.68}$$

$$+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\omega_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial \omega_r}{\partial\theta}-\frac{\omega_\theta}{r^2}+\frac{\partial^2\omega_\theta}{\partial z^2}\right) \tag{C.69}$$

$$\frac{\partial \omega_z}{\partial t}+u\frac{\partial \omega_z}{\partial r}+\frac{v}{r}\frac{\partial \omega_z}{\partial \theta}+w\frac{\partial \omega_z}{\partial z}=\omega_r\frac{\partial w}{\partial r}+\frac{\omega_\theta}{r}\frac{\partial w}{\partial \theta}+\omega_z\frac{\partial w}{\partial z} \tag{C.70}$$

$$+\nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\omega_z}{\partial\theta^2}+\frac{\partial^2\omega_z}{\partial z^2}\right) \tag{C.71}$$

C.1.5 Vorticity–streamfunction in 3D axisymmetric flow

With $\partial_\theta = 0$, the continuity equation simplifies to

$$\frac{\partial(ru)}{\partial r}+\frac{\partial wr}{\partial z}=0 \tag{C.72}$$

such that an axisymmetric (Stokes) streamfunction can be defined as

$$u=-\frac{1}{r}\frac{\partial\psi}{\partial z},\ w=\frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{C.73}$$

Further assuming swirl-free $v = 0$, the vorticity–streamfunction equations are

$$\frac{\partial \omega_\theta}{\partial t} + u\frac{\partial \omega_\theta}{\partial r} + w\frac{\partial \omega_\theta}{\partial z} = \frac{u\omega_\theta}{r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_\theta}{\partial r}\right) - \frac{\omega_\theta}{r^2} + \frac{\partial^2 \omega_\theta}{\partial z^2}\right) \tag{C.74}$$

$$-r\omega_\theta = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{C.75}$$

Also see Panton (2013), appendix 4 for a more complete list of streamfunctions in other coordinates.

C.2 Spherical coordinate

Consider the transformation

$$(x, y, z) \to (r, \phi, \theta) \tag{C.76}$$

where

$$x = r\sin\phi\cos\theta \tag{C.77}$$

$$y = r\sin\phi\sin\theta \tag{C.78}$$

$$z = r\cos\phi \tag{C.79}$$

or

$$r = \sqrt{x^2 + y^2 + z^2} \tag{C.80}$$

$$\phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \tag{C.81}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{C.82}$$

or thought of as from cylindrical with

$$z = r\cos\phi \tag{C.83}$$

$$r' = r\sin\phi \tag{C.84}$$

$$x = r'\sin\theta \tag{C.85}$$

$$y = r'\cos\theta \tag{C.86}$$

Here θ is the azimuthal angle with x-axis on the equatorial plane and ϕ is the polar angle with z-axis (North), for the convenience of going from cylindrical to polar and backwards.

We have the corresponding relation between unit vectors

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.87}$$

and

$$\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix}, \tag{C.88}$$

which can be proven graphically. We note that the grid transformation matrix is unitary and has $\det() = 1$ (rotation matrix).

C.2.1 From cylindrical to spherical

We have the transformation

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.89}$$

that can be factorized as

$$\begin{bmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \\ \boldsymbol{e}_z \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.90}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix} \tag{C.91}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{r'} \\ \boldsymbol{e}_{\theta'} \\ \boldsymbol{e}_{z'} \end{bmatrix} \tag{C.92}$$

with

$$\begin{bmatrix} \boldsymbol{e}_{r'} \\ \boldsymbol{e}_{\theta'} \\ \boldsymbol{e}_{z'} \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_\theta \end{bmatrix}. \tag{C.93}$$

C.3 General curvilinear coordinates

Consider the coordinate transformations

$$q_i = q_i(x_1, x_2, x_3),\ x_i = x_i(q_1, q_2, q_3) \tag{C.94}$$

where (x_1, x_2, x_3) is the standard Cartesian coordinates and q_i are mutually independent.

C.3.1 Length, area, and volume

Consider the change of the vector

$$\boldsymbol{x} = x_1\boldsymbol{e}_{x_1} + x_2\boldsymbol{e}_{x_2} + x_3\boldsymbol{e}_{x_3} \tag{C.95}$$

$$= q_1\boldsymbol{h}_1 + q_2\boldsymbol{h}_2 + q_3\boldsymbol{h}_3 \tag{C.96}$$

where $\boldsymbol{x} = \boldsymbol{x}(x_i(q_j))$ as

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{e}_{x_1}\mathrm{d}x_1 + \boldsymbol{e}_{x_2}\mathrm{d}x_2 + \boldsymbol{e}_{x_3}\mathrm{d}x_3 \tag{C.97}$$

$$= \frac{\partial\boldsymbol{x}}{\partial q_1}\mathrm{d}q_1 + \frac{\partial\boldsymbol{x}}{\partial q_2}\mathrm{d}q_2 + \frac{\partial\boldsymbol{x}}{\partial q_3}\mathrm{d}q_3 \tag{C.98}$$

and

$$\boldsymbol{h}_i = \frac{\partial\boldsymbol{x}}{\partial q_i}. \tag{C.99}$$

We note that $\boldsymbol{h}_i$ is the change of $\boldsymbol{x}$ with only changing q_i, so it does define direction of coordinate lines of q_i. We denote with $\hat{(\cdot)}$ unit vectors and note that $\boldsymbol{h}_i$ are not necessary unit vectors.

Now consider the length of $\mathrm{d}\boldsymbol{x}$:

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} && \text{(C.100)}\\
&= \frac{\partial \boldsymbol{x}}{\partial q_j}\mathrm{d}q_j \cdot \frac{\partial \boldsymbol{x}}{\partial q_k}\mathrm{d}q_k && \text{(C.101)}\\
&= \frac{\partial x_i}{\partial q_j}\mathrm{d}q_j \frac{\partial x_i}{\partial q_k}\mathrm{d}q_k && \text{(C.102)}\\
&= g_{jk}\mathrm{d}q_j\mathrm{d}q_k && \text{(C.103)}
\end{aligned}$$

with

$$g_{ij} = \frac{\partial x_l}{\partial q_i}\frac{\partial x_l}{\partial q_j} \tag{C.104}$$

being the metric tensor. When q_i are orthogonal coordinates,

$$\frac{\partial \boldsymbol{x}}{\partial q_i} \cdot \frac{\partial \boldsymbol{x}}{\partial q_j} = \delta_{ij} \tag{C.105}$$

and g_{ij} only has diagonal elements and

$$\begin{aligned}
\mathrm{d}s^2 &= g_{11}(\mathrm{d}q_1)^2 + g_{22}(\mathrm{d}q_3)^2 + g_{33}(\mathrm{d}q_3)^2 && \text{(C.106)}\\
&= h_1^2(\mathrm{d}q_1)^2 + h_2^2(\mathrm{d}q_3)^2 + h_3^2(\mathrm{d}q_3)^2 && \text{(C.107)}
\end{aligned}$$

Define the Lamé parameters as

$$\begin{aligned}
h_1 &= \sqrt{g_{11}} = \sqrt{\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial x_3}{\partial q_1}\right)^2} = |\boldsymbol{h}_1| && \text{(C.108)}\\
h_2 &= \sqrt{g_{22}} = \sqrt{\left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial x_3}{\partial q_2}\right)^2} = |\boldsymbol{h}_2| && \text{(C.109)}\\
h_3 &= \sqrt{g_{33}} = \sqrt{\left(\frac{\partial x_1}{\partial q_3}\right)^2 + \left(\frac{\partial x_2}{\partial q_3}\right)^2 + \left(\frac{\partial x_3}{\partial q_3}\right)^2} = |\boldsymbol{h}_3| && \text{(C.110)}
\end{aligned}$$

and unit vectors in q_i directions as

$$\boldsymbol{h}_i = \frac{\boldsymbol{h}_i}{|\boldsymbol{h}_i|} = \frac{\boldsymbol{h}_i}{h_i}. \tag{C.111}$$

We note that the Lamé parameters can depend on the coordinates as

$$h_i = h_i(q_1, q_2, q_3). \tag{C.112}$$

The increment can be rewritten as

$$\begin{aligned}
\mathrm{d}\boldsymbol{x} &= h_1\mathrm{d}q_1\boldsymbol{h}_1 + h_2\mathrm{d}q_2\boldsymbol{h}_2 + h_3\mathrm{d}q_3\boldsymbol{h}_3 && \text{(C.113)}\\
&= \mathrm{d}s_1\boldsymbol{h}_1 + \mathrm{d}s_2\boldsymbol{h}_2 + \mathrm{d}s_3\boldsymbol{h}_3 && \text{(C.114)}
\end{aligned}$$

with

$$\mathrm{d}s_i \tag{C.115}$$

being the projection of $\mathrm{d}\boldsymbol{x}$ on each coordinate.

Now consider the surface and volume of infinitesimal elements. The (directed) areas of surface elements are

$$\mathrm{d}\sigma_i = \boldsymbol{h}_i \cdot (h_j\mathrm{d}q_j\boldsymbol{h}_j \times h_k\mathrm{d}q_k\boldsymbol{h}_k) = h_jh_k\mathrm{d}q_j\mathrm{d}q_k \tag{C.116}$$

or

$$\mathrm{d}\sigma_1 = h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2 \tag{C.117}$$
$$\mathrm{d}\sigma_2 = h_1 h_3 \mathrm{d}q_1 \mathrm{d}q_3 \tag{C.118}$$
$$\mathrm{d}\sigma_3 = h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2 \tag{C.119}$$

The volume element (e.g. in volume integrals) spanned by the vector $\mathrm{d}\boldsymbol{x}$ is

$$\mathrm{d}V = (h_1 \mathrm{d}q_1 \boldsymbol{h}_1) \cdot (h_2 \mathrm{d}q_2 \boldsymbol{h}_2 \times h_3 \mathrm{d}q_3 \boldsymbol{h}_3) \tag{C.120}$$
$$= h_1 \mathrm{d}q_1 h_2 \mathrm{d}q_2 h_3 \mathrm{d}q_3 (\boldsymbol{h}_1) \cdot (\boldsymbol{h}_2 \times \boldsymbol{h}_3) \tag{C.121}$$
$$= h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3 \tag{C.122}$$

when $\boldsymbol{h}_i$ mutually orthogonal.

Example.

For cylindrical coordinate, by definition,

$$h_1 = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1 \tag{C.123}$$
$$h_2 = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r \tag{C.124}$$
$$h_3 = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = 1 \tag{C.125}$$

C.3.2 Jacobian

Now we consider the Jacobian of the backward transformation

$$(q_1, q_2, q_3) \to (x_1, x_2, x_3) \tag{C.126}$$

which reads

$$\boldsymbol{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(q_1, q_2, q_3)} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \frac{\partial x_1}{\partial q_3} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_2}{\partial q_3} \\ \frac{\partial x_3}{\partial q_1} & \frac{\partial x_3}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{bmatrix} \tag{C.127}$$

and the Jacobian determinant (with $\exists\, \boldsymbol{J}^{-1}$)

$$J = \det(\boldsymbol{J}) = \det(\boldsymbol{J}^{\mathrm{T}}) \tag{C.128}$$
$$= \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} \tag{C.129}$$
$$= \left(\frac{\partial x_1}{\partial q_1}\boldsymbol{x}_1 + \frac{\partial x_2}{\partial q_1}\boldsymbol{x}_2 + \frac{\partial x_3}{\partial q_1}\boldsymbol{x}_3\right) \cdot \begin{vmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_2} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_3} & \frac{\partial x_3}{\partial q_3} \end{vmatrix} \tag{C.130}$$
$$= \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial q_2} \times \frac{\partial \boldsymbol{x}}{\partial q_3}\right) \tag{C.131}$$

$$
\begin{aligned}
&= \boldsymbol{h}_1 \cdot (\boldsymbol{h}_2 \times \boldsymbol{h}_3) && \text{(C.132)}\\
&= h_1 h_2 h_3 && \text{(C.133)}\\
&\neq 0 && \text{(C.134)}
\end{aligned}
$$

Hence we have

$$
\mathrm{d}V = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = h_1 h_2 h_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3 = J \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3. \tag{C.135}
$$

C.3.3 Three major calculus theorems

1. Gradient theorem:

$$
\int_{\boldsymbol{l}:\boldsymbol{x}_1 \to \boldsymbol{x}_2} (\nabla f) \cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1) \tag{C.136}
$$

   The integral is independent of path since ∇f is potential (conservative, curl-free).

2. Divergence theorem:

$$
\iint_{\Omega} (\nabla \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{A} = \oint_{l=\partial\Omega} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} \tag{C.137}
$$

   Implication: vorticity is circulation per unit area.

3. Curl theorem:

$$
\iiint_V (\nabla \cdot \boldsymbol{u})\, \mathrm{d}V = \iint_{\Omega=\partial V} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A} \tag{C.138}
$$

C.3.4 Differential operators in curvilinear coordinate systems

Next, let's consider differential operators in curvilinear coordinates. Consider a scalar $f = f(q_1, q_2, q_3)$ and its gradient ∇f. Starting from

$$
\mathrm{d}f = \frac{\partial f}{\partial q_i} \mathrm{d}q_i, \tag{C.139}
$$

due to the displacement $\boldsymbol{x}$. On the other hand,

$$
\begin{aligned}
\mathrm{d}f &= \nabla f \cdot \mathrm{d}\boldsymbol{x} && \text{(C.140)}\\
&= (\nabla f)_{q_i} \mathrm{d}s_i && \text{(C.141)}\\
&= (\nabla f)_{q_i} h_i \mathrm{d}q_i && \text{(C.142)}
\end{aligned}
$$

Compare (C.142) and (C.139) we have

$$
(\nabla f)_{q_i} = \frac{1}{h_i} \frac{\partial f}{\partial q_i} \tag{C.143}
$$

where

$$
\begin{aligned}
\nabla f &= (\nabla f)_{q_i} \boldsymbol{h}_i && \text{(C.144)}\\
&= \frac{1}{h_1}\frac{\partial f}{\partial q_1}\boldsymbol{h}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\boldsymbol{h}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\boldsymbol{h}_3 && \text{(C.145)}
\end{aligned}
$$

Consider the divergence of a vector $\boldsymbol{u}$ in a coordinate-free form:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{u} &\triangleq \lim_{V \to 0} \frac{\oint_{\Omega=\partial V} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{\sigma}}{V} && \text{(C.146)}\\
&= \frac{1}{V}\left(\frac{\partial(u_1 h_2 h_3 \mathrm{d}q_2 \mathrm{d}q_3)}{\partial q_1}\mathrm{d}q_1 + \frac{\partial(u_2 h_1 h_3 \mathrm{d}q_1 \mathrm{d}q_3)}{\partial q_2}\mathrm{d}q_2 + \frac{\partial(u_3 h_1 h_2 \mathrm{d}q_1 \mathrm{d}q_2)}{\partial q_3}\mathrm{d}q_3 \right) && \text{(C.147)}
\end{aligned}
$$

$$= \frac{1}{h_1h_2h_3}\left(\frac{\partial(u_1h_2h_3)}{\partial q_1} + \frac{\partial(u_2h_1h_3)}{\partial q_2} + \frac{\partial(u_3h_1h_2)}{\partial q_3}\right) \tag{C.148}$$

Consider the curl of a vector $\boldsymbol{u}$ in a coordinate-free form, its compoment along $\boldsymbol{n}$ (normal of the surface $\boldsymbol{S} = S\boldsymbol{n}$) is

$$(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{n} \triangleq \lim_{S\to 0} \frac{\int_{l=\partial S} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}}{S} \tag{C.149}$$

and (consider the area spanned by $\mathrm{d}s_2 = h_2\mathrm{d}q_2$ and $\mathrm{d}s_3 = h_3\mathrm{d}q_3$)

$$\begin{aligned}
(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_1 &= \frac{\oint_l \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}}{\mathrm{d}\sigma_1} && \text{(C.150)}\\
&= \frac{1}{h_2h_3\mathrm{d}q_2\mathrm{d}q_3}[u_2h_2\mathrm{d}q_2 && \text{(C.151)}\\
&\quad - (u_2h_2 + \frac{\partial u_2h_2}{\partial q_3}\mathrm{d}q_3)\mathrm{d}q_2 && \text{(C.152)}\\
&\quad - u_3h_3\mathrm{d}q_3 && \text{(C.153)}\\
&\quad + (u_3h_3 + \frac{\partial u_3h_3}{\partial q_2}\mathrm{d}q_2)\mathrm{d}q_3] && \text{(C.154)}\\
&= \frac{1}{h_2h_3}\left(\frac{\partial u_3h_3}{\partial q_2} - \frac{\partial u_2h_2}{\partial q_3}\right) && \text{(C.155)}\\
(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_2 &= \frac{1}{h_1h_3}\left(\frac{\partial u_1h_1}{\partial q_3} - \frac{\partial u_3h_3}{\partial q_1}\right) && \text{(C.156)}\\
(\nabla \times \boldsymbol{u}) \cdot \boldsymbol{h}_3 &= \frac{1}{h_1h_2}\left(\frac{\partial u_2h_2}{\partial q_1} - \frac{\partial u_1h_1}{\partial q_2}\right) && \text{(C.157)}
\end{aligned}$$

We note that the Lamé coefficient also changes as the coordinate changes.

In determinant form,

$$\nabla \times \boldsymbol{u} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{h}_1 & h_2\boldsymbol{h}_2 & h_3\boldsymbol{h}_3 \\ \partial_{q_1} & \partial_{q_2} & \partial_{q_3} \\ h_1u_1 & h_2u_2 & h_3u_3 \end{vmatrix} \tag{C.158}$$

The Laplacian can be obtained by taking the divergence of ∇f as combining (C.145) and (C.148)

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right) \tag{C.159}$$

C.3.5 Derivatives of unit vectors

In general curvilinear coordinates, the directions of unit vectors could change with coordinate as well. We are basically concerned about

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \tag{C.160}$$

and we will establish that

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} // \boldsymbol{h}_j,\ i \neq j. \tag{C.161}$$

First we have

$$\boldsymbol{h}_i \cdot \frac{\partial \boldsymbol{h}_i}{\partial q_j} = \frac{\partial \boldsymbol{h}_i^2/2}{\partial q_j} = 0 \tag{C.162}$$

and hence

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \perp \boldsymbol{h}_i,\ i \neq j. \tag{C.163}$$

According to the orthogonality we have

$$\boldsymbol{h}_1 \cdot \boldsymbol{h}_2 = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2} = 0 \tag{C.164}$$

and

$$0 = \frac{\partial}{\partial q_3}\left(\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2}\right) = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} + \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_3} \cdot \frac{\partial \boldsymbol{x}}{\partial q_2} \tag{C.165}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_1} + \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_1} \cdot \frac{\partial \boldsymbol{x}}{\partial q_3} \tag{C.166}$$

$$= \frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} + \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_2} \cdot \frac{\partial \boldsymbol{x}}{\partial q_1} \tag{C.167}$$

then

$$\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} + \frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_3 \partial q_1} + \frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} = 0 \tag{C.168}$$

and then

$$\frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_2 \partial q_3} = 0 \tag{C.169}$$

$$\frac{\partial \boldsymbol{x}}{\partial q_2} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_3} = 0 \tag{C.170}$$

$$\frac{\partial \boldsymbol{x}}{\partial q_3} \cdot \frac{\partial^2 \boldsymbol{x}}{\partial q_1 \partial q_2} = 0 \tag{C.171}$$

From

$$0 = \frac{\partial \boldsymbol{x}}{\partial q_1} \cdot \frac{\partial}{\partial q_2}\left(\frac{\partial \boldsymbol{x}}{\partial q_3}\right) \tag{C.172}$$

$$= h_1 \boldsymbol{h}_1 \cdot \frac{\partial h_3 \boldsymbol{h}_3}{\partial q_2} \tag{C.173}$$

$$= h_1 \boldsymbol{h}_1 \cdot \left(h_3 \frac{\partial \boldsymbol{h}_3}{\partial q_2} + \boldsymbol{h}_3 \frac{\partial h_3}{\partial q_2}\right) \tag{C.174}$$

$$= h_1 h_3 \boldsymbol{h}_1 \cdot \frac{\partial \boldsymbol{h}_3}{\partial q_2} \tag{C.175}$$

we have (similarly)

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} \perp \boldsymbol{h}_k,\ i \neq j \neq k \neq i. \tag{C.176}$$

Combining (C.163) and (C.176) we have

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} // \boldsymbol{h}_j,\ i \neq j, \tag{C.177}$$

and using

$$\frac{\partial^2 \boldsymbol{x}}{\partial q_i \partial q_j} = \frac{\partial^2 \boldsymbol{x}}{\partial q_j \partial q_i} \tag{C.178}$$

we have

$$\frac{\partial}{\partial q_j}\left(\frac{\partial \boldsymbol{x}}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\left(\frac{\partial \boldsymbol{x}}{\partial q_j}\right) \tag{C.179}$$

$$\boldsymbol{h}_i\frac{\partial h_i}{\partial q_j} + h_i\frac{\partial \boldsymbol{h}_i}{\partial q_j} = \boldsymbol{h}_j\frac{\partial h_j}{\partial q_i} + h_j\frac{\partial \boldsymbol{h}_j}{\partial q_i} \tag{C.180}$$

with repeated indices not implying summation. Since $i \neq j$, $\boldsymbol{h}_i$ and $\boldsymbol{h}_j$ are linearly independent, we have

$$\frac{\partial \boldsymbol{h}_i}{\partial q_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial q_i}\boldsymbol{h}_j. \tag{C.181}$$

Now we turn back and consider $\partial \boldsymbol{h}_i/\partial q_j$.

$$\begin{aligned}
\frac{\partial \boldsymbol{h}_i}{\partial q_i} &= \frac{\partial(\boldsymbol{h}_j \times \boldsymbol{h}_k)}{\partial q_i} &\text{(C.182)}\\
&= \frac{\partial \boldsymbol{h}_j}{\partial q_i} \times \boldsymbol{h}_k + \boldsymbol{h}_j \times \frac{\partial \boldsymbol{h}_k}{\partial q_i} &\text{(C.183)}\\
&= \frac{1}{h_j}\frac{\partial h_i}{\partial q_j}\boldsymbol{h}_i \times \boldsymbol{h}_k + \boldsymbol{h}_j \times \boldsymbol{h}_i\frac{1}{h_k}\frac{\partial h_i}{\partial q_k} &\text{(C.184)}\\
&= -(\frac{1}{h_j}\frac{\partial h_i}{\partial q_j}\boldsymbol{h}_j + \frac{1}{h_k}\frac{\partial h_i}{\partial q_k}\boldsymbol{h}_k) &\text{(C.185)}
\end{aligned}$$

without repeated indices being summed over.

Using the relations (C.181) and (C.185), gradient, curl, divergence, Laplacian, as well as operators like $\nabla \boldsymbol{u}$ and $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ can be expressed.

Example: $\nabla \cdot \boldsymbol{u}$.

We have before

$$\nabla = \frac{\boldsymbol{h}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\boldsymbol{h}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\boldsymbol{h}_3}{h_3}\frac{\partial}{\partial q_3} \tag{C.186}$$

and now consider $\nabla \boldsymbol{u}$ with

$$\boldsymbol{u} = u_1\boldsymbol{h}_1 + u_2\boldsymbol{h}_2 + u_3\boldsymbol{h}_3 \tag{C.187}$$

and we have

$$\begin{aligned}
\nabla \cdot \boldsymbol{u} &= \left(\frac{\boldsymbol{h}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\boldsymbol{h}_2}{h_2}\frac{\partial}{\partial q_2}\boldsymbol{h}_2 + \frac{\boldsymbol{h}_3}{h_3}\frac{\partial}{\partial q_3}\boldsymbol{h}_3\right) \cdot (u_1\boldsymbol{h}_1 + u_2\boldsymbol{h}_2 + u_3\boldsymbol{h}_3) &\text{(C.188)}\\
&= \frac{\boldsymbol{h}_1}{h_1}\frac{\partial}{\partial q_1}(u_1\boldsymbol{h}_1 + u_2\boldsymbol{h}_2 + u_3\boldsymbol{h}_3) + ... &\text{(C.189)}\\
&= \frac{1}{h_1}\left(\frac{\partial u_1}{\partial q_1} + \frac{u_2}{h_2}\frac{\partial h_1}{\partial q_2} + \frac{u_3}{h_3}\frac{\partial h_1}{\partial q_3}\right) &\text{(C.190)}\\
&\quad + \frac{1}{h_2}\left(\frac{\partial u_2}{\partial q_2} + \frac{u_3}{h_3}\frac{\partial h_2}{\partial q_3} + \frac{u_1}{h_1}\frac{\partial h_2}{\partial q_1}\right) &\text{(C.191)}\\
&\quad + \frac{1}{h_3}\left(\frac{\partial u_3}{\partial q_3} + \frac{u_1}{h_1}\frac{\partial h_3}{\partial q_1} + \frac{u_2}{h_2}\frac{\partial h_3}{\partial q_2}\right) &\text{(C.192)}\\
&= \frac{1}{h_1h_2h_3}\left(\frac{\partial u_1h_2h_3}{\partial q_1} + \frac{\partial u_2h_1h_3}{\partial q_2} + \frac{\partial u_3h_1h_2}{\partial q_3}\right) &\text{(C.193)}
\end{aligned}$$

References: Appendices in Batchelor (1967); Griffiths (2013), and text book of Wu (1982).

C.3.6 Examples

1. Cartesian. $(q_1, q_2, q_3) = (x_1, x_2, x_3)$
   Elements:

$$h_1 = h_2 = h_3 = 1 \tag{C.194}$$
$$\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \tag{C.195}$$
$$\mathrm{d}V = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \tag{C.196}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial f}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial f}{\partial x_3}\boldsymbol{e}_3 \tag{C.197}$$
$$\nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{C.198}$$
$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} \tag{C.199}$$
$$= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\boldsymbol{e}_x + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\boldsymbol{e}_y + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\boldsymbol{e}_z \tag{C.200}$$
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.201}$$

2. Cylindrical. $(q_1, q_2, q_3) = (r, \theta, z)$
   Elements:

$$h_1 = h_3 = 1, h_2 = r \tag{C.202}$$
$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2 \tag{C.203}$$
$$\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \tag{C.204}$$

   Operators:

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial f}{\partial z}\boldsymbol{e}_z \tag{C.205}$$
$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\left(\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial \theta} + \frac{\partial (rw)}{\partial z}\right) \tag{C.206}$$
$$= \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \tag{C.207}$$
$$\nabla \times \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u & rv & w \end{vmatrix} \tag{C.208}$$
$$= (\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z})\boldsymbol{e}_r + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})\boldsymbol{e}_\theta + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial \theta})\boldsymbol{e}_z \tag{C.209}$$
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.210}$$

3. Spherical. $(q_1, q_2, q_3) = (r, \phi, \theta)$, ϕ is the polar angle and θ is the azimuthal.
   Elements:

$$h_1 = 1, h_2 = r, h_3 = r\sin\phi \tag{C.211}$$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + r^2\sin^2\phi\mathrm{d}\theta^2 \tag{C.212}$$
$$\mathrm{d}V = r^2\sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta \tag{C.213}$$

Operators:

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{1}{r\sin\phi}\frac{\partial f}{\partial\theta}\boldsymbol{e}_\theta \tag{C.214}$$
$$\nabla\cdot\boldsymbol{u} = \frac{1}{r^2\sin\phi}\left(\frac{\partial(r^2\sin\phi u)}{\partial r} + \frac{\partial(r\sin\phi v)}{\partial\phi} + \frac{\partial(rw)}{\partial\theta}\right) \tag{C.215}$$
$$= \frac{1}{r^2}\frac{\partial(r^2u)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(\sin\phi v)}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial w}{\partial\theta} \tag{C.216}$$
$$\nabla\times\boldsymbol{u} = \frac{1}{r^2\sin\phi}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\phi & r\sin\phi\boldsymbol{e}_\theta \\ \partial_r & \partial_\phi & \partial_\theta \\ u & rv & r\sin\phi w \end{vmatrix} \tag{C.217}$$
$$= \frac{1}{r\sin\phi}(\frac{\partial\sin\phi w}{\partial\phi} - \frac{\partial v}{\partial\theta})\boldsymbol{e}_r + (\frac{1}{r\sin\phi}\frac{\partial u}{\partial\theta} - \frac{1}{r}\frac{\partial rw}{\partial r})\boldsymbol{e}_\phi + \frac{1}{r}(\frac{\partial rv}{\partial r} - \frac{\partial u}{\partial\phi})\boldsymbol{e}_\theta \tag{C.218}$$
$$\nabla^2 f = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2\sin\phi}\frac{\partial^2 f}{\partial\theta^2} \tag{C.219}$$

D Special functions

The Euler's identity
$$e^{i\pi} + 1 = 0, \tag{D.1}$$
is actualy a special case of the Euler's formula
$$e^{ix} = \cos x + i\sin x, \tag{D.2}$$
where $i = \sqrt{-1}$ is the imaginary unit.

It can be shown by Taylor expansions of $\cos x$ and $\sin x$ at $x = 0$ in Table 2 and we have already observed some similarities to the Taylor expansion of an exponential
$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ... \tag{D.3}$$
Taking the Laurent expansion of e^{ix} at $x = 0$ we have proven that
$$e^{ix} = \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} \tag{D.4}$$
$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + ... \tag{D.5}$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) \tag{D.6}$$
$$= \cos x + i\sin x. \tag{D.7}$$

The Taylor expansion of an infinity continuously differentiable function $f \in \mathbb{C}$ around $x = x_0$ is
$$f(x) = \sum_{n=0}^{\infty}\frac{f(n)(x_0)}{n!}(x - x_0)^n, \tag{D.8}$$

function (convergence radius R)	expansion sum	power series
Geometric functions ($\lvert x\rvert < 1$)		
$\frac{1}{1-x}$	$=\sum_{\infty}^{n=0} x^n$	$= 1 + x + x^2 + x^3 + ...$
$\frac{1}{(1-x)^2}$	$=\sum_{\infty}^{n=1} nx^{n-1}$	$= 1 + 2x + 3x^2 + 4x^3 + ...$
Binomial functions ($\lvert x\rvert < 1$)	$C_n^\alpha = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$	$\alpha \in \mathbb{C}$
$(1+x)^\alpha$	$=\sum_{\infty}^{n=0} C_n^\alpha x^n$	
$(1+x)^{1/2}$		$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + ...$
$(1+x)^{-1/2}$		$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + ...$
Trigonometric functions ($R = \infty$)		
$\sin x$	$=\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$
$\cos x$	$=\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$
$(\lvert x\rvert < \pi/2)$, $\tan x$	$=\sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1}$	$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + ...$
$(\lvert x\rvert < \pi/2)$, $\sec x$		$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + ...$
$(\lvert x\rvert \le 1)$, $\arcsin x$		$= x + \frac{x^3}{6} + \frac{3x^5}{40} + ...$
$(\lvert x\rvert \le 1)$, $\arccos x$		$= \frac{\pi}{2} - \arcsin x$
$(\lvert x\rvert \le 1, x \neq \pm i)$, $\arctan x$	$=\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$
Hyperbolic functions ($R = \infty$)		
$\sinh x$	$=\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$
$\cosh x$	$=\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$
$\tanh x$	$=\sum_{n=1}^{\infty} \frac{B_{2n}4^n(4^n-1)}{(2n)!} x^{2n-1}$	$= x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + ...$
$\operatorname{arctanh} x$	$=\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$	$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ...$

Table 2: Taylor (polynomial) expansion of some special functions. These expansions can be used to draw mental pictures of these functions. The similarity in the expansion of the hyperbolic functions to the corresponding trigonometric functions can be seen.

and the expansions of commonly used functions are listed in the Table 2.

Note that C_n^α is the combination number representing the total number of possibilities of selecting n from α and B_k are the Bernoulli numbers.

In this section, we will also connect the functions to their defining ODEs – the ODEs that they satisfy – to reveal their physical implications.

The hyperbolic functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{D.9}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{D.10}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{D.11}$$

$$\tag{D.12}$$

and their first derivatives are listed in Table 3.

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\mathrm{sech}^2 x\,(= 1/\cosh^2 x)$

Table 3: Derivatives of hyperbolic functions.

It is easy to verify that both $y = \sinh kx$ and $y = \cosh kx$ are solutions to the second-order ODE

$$y'' - k^2 y = 0. \tag{D.13}$$

Examples.
Mixing layer:

$$U(y) = U_0 \tanh(y/L). \tag{D.14}$$

Jet:

$$U(y) = U_0 \mathrm{sech}^2(y/L) = U_0/\cosh^2(y/L). \tag{D.15}$$

We will see this in section xx as analytical solutions to lam/turb profiles?

E Other self-similar solutions of the Navier–Stokes

E.1 Transient heat conduction: similarity solution

Consider one-dimensional heat equation:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right), \tag{E.1}$$

when the thermal conductivity κ is a constant in space, it reduces to

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \tag{E.2}$$

where $\alpha = \kappa/\rho c_p$ is the heat diffusivity. Eqn. (E.2) is also the governing equation for general diffusion processes, for example, of a scalar or chemical species.

Consider the following initial-boundary conditions:

$$T(x, t < 0) = T_s \tag{E.3}$$

$$T(x, t) = T_0,\ T(\infty, t) = T_s, \tag{E.4}$$

corresponding to an infinity long rod with initial temperature T_s subject to an abrupt input at $t = 0$ that brings $T(0, t) = T_0$.

Non-dimensionalization:

$$\theta = \frac{T - T_0}{T_s - T_0} \tag{E.5}$$

such that the PDE is

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}, \tag{E.6}$$

and the I/BCs are

$$\theta(x, t < 0) = 1 \tag{E.7}$$

$$\theta(0, t) = 0,\ \theta(\infty, t) = 1. \tag{E.8}$$

Consider the non-dimensional similarity variable

$$\eta = \frac{x}{2\sqrt{\alpha t}} \tag{E.9}$$

and $\theta(x, t) = f(\eta)$. Eqn. (E.6) can be conveted to an ODE as

$$f'' + 2\eta f' = 0, \tag{E.10}$$

subject to boundary conditions of

$$f(0) = 0,\ f(\infty) = 1. \tag{E.11}$$

Direct integration of

$$\frac{f''}{f'} = -2\eta \tag{E.12}$$

yields

$$\ln f' = -\eta^2 + \ln C_1 \tag{E.13}$$

and

$$f' = C_1 e^{-\eta^2}. \tag{E.14}$$

Integrating again we have

$$f(\eta) = f(\eta) - f(0) = C_1 \int_0^\eta e^{-t^2}\, dt \tag{E.15}$$

where the error function is defined as

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-t^2}\, dt \tag{E.16}$$

and

$$\mathrm{erf}(0) = 0,\ \mathrm{erf}(\infty) = 1. \tag{E.17}$$

Hence, we have $C_1 = 2/\sqrt{\pi}$ and

$$f(\eta) = \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-t^2}\, dt. \tag{E.18}$$

The solution to the original equation reads

$$T(x, t) = (T_s - T_0)\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) + T_0. \tag{E.19}$$

Moreover, the complementary error function is defined as

$$\mathrm{erfc}(\eta) = 1 - \mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-t^2}\, dt \tag{E.20}$$

and reaches values of

$$\mathrm{erfc}(0) = 1,\ \mathrm{erfc}(\infty) = 0. \tag{E.21}$$

F Probability theory

moments, statistics, correlations, JPDFs, etc.

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