3.3 Chain Homotopy vs. Homotopy(extra)

同伦与链同伦的关系在前文中并没有讲清楚, 在此处进行着重讲解 (参考齐震宇, 代数拓扑 3 同伦诱导链同伦), 笔记暂时设计为英文, 穿插部分中文解释.

定义 3.3.1. (Chain Homotopy) Let C,C' be two chain complexes, $(f_0)_\#$ and $(f_1)_\#$ be chain maps.

$$\begin{tikzcd} C \arrow[r, shift left, "(f_0)_\#"] \arrow[r, shift right, "(f_1)_\#"'] & C' \end{tikzcd}$$

A chain homotopy from $(f_0)_\#$ to $(f_1)_\#$ consists of group homomorphisms $h_k : C_k \to C'_{k+1}$ so that $(f_1)_k - (f_0)_k = \partial'_{k+1} h_k + h_{k-1}\partial_k, k \in \mathbb{Z}$.

The diagram of chain homotopy is

$$\begin{tikzcd} & \cdots \arrow[r] & C_k \arrow[r, "\partial_k"] \arrow[dl, "h_k"'] \arrow[d, "(f_1)_k-(f_0)_k"] & C_{k-1} \arrow[r] \arrow[dl, "h_{k-1}"] & \cdots \\ \cdots \arrow[r] & C'_{k+1} \arrow[r, "\partial'_{k+1}"'] & C'_k \arrow[r] & \cdots & \end{tikzcd}$$

Notation : $f_0 \simeq^{h_\#} f_1$.

定义 3.3.2. (Homotopy) 即同伦定义

3.3.1 (Homotopy $\Rightarrow$ Chain Homotopy)

Let $f_0 \simeq f_1 : X \to Y$ in **Top**. we want to show that $(f_1)_\# \simeq (f_2)_\#$, notice that in **Top** we have:

$$\begin{tikzcd} X \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_0"'] \arrow[rr, bend left, "f_1"] \arrow[rr, bend right, "f_0"'] & X \times I \arrow[r, "h"] & Y \end{tikzcd}$$

Note that we may reduce the problem to the case $Y = X \times I$,$f_0 = i_0, f_1 = i_1$.

定理 3.3.1. It exist a family of natural transformations

$$h_k : S_k \to S_{k+1}(\cdot \times I), k \in \mathbb{Z}$$

so that $(i_1)_\# - (i_0)_\# = \partial'_{k+1} \circ h_k = h_{k-1} \circ \partial_k, \forall k \in \mathbb{Z}$.

$$\begin{tikzcd} \cdots \arrow[r] & S_{k+1} \arrow[r] \arrow[d] & S_k \arrow[r, "\partial_k"] \arrow[dl, "h_k"'] \arrow[d, "(i_1)_\#-(i_0)_\#"] & S_{k-1} \arrow[r] \arrow[dl, "h_{k-1}"'] \arrow[d] & \cdots \\ \cdots \arrow[r] & S_{k+1}(\cdot \times I) \arrow[r, "\partial'_{k+1}"] & S_k(\cdot \times I) \arrow[r] & S_{k-1}(\cdot \times I) \arrow[r] & \cdots \end{tikzcd}$$

In other words, for any topological space X, we need to construct $S_k(X) \xrightarrow{h_k(X)} S_{k+1}(X \times I), k \in \mathbb{Z}$, so that $i_1(X)_\# - i_0(X)_\# = \partial_{k+1}(X \times I) \circ h_k(X) + h_{k-1}(X) \circ \partial_k(X)$.

And

$$
\begin{tikzcd}
X \arrow[d, "f"] & S_k(X) \arrow[r, "h_k(X)"] \arrow[d, "f_\#"'] & S_{k+1}(X \times I) \arrow[d, "(f \times \mathrm{id}_I)_\#"] \\
Y & S_k(Y) \arrow[r, "h_k(Y)"'] & S_{k+1}(Y \times I)
\end{tikzcd}
$$

Proof. 1) If such h_k exists for every $k \in \mathbb{Z}$, it is determined by $h_k(\Delta_k)(\mathrm{id}_{\Delta_k})$(取 $S_k(X)$ 中的生成元 σ, 而 σ 可以完全由 $S_k(\Delta_k)$ 中的元素所决定, 不妨设这个元素为 id_{Δ_k}), thus the diagram below is commutative(the pink part is determined by σ)

$$
\begin{tikzcd}
\sigma : \Delta_k \to X \arrow[d, mapsto, "h_k(X)\sigma \text{ must be}"'] \arrow[dr, phantom, "\in"] & & & \mathrm{id}_{\Delta_k} \arrow[lll, mapsto] \arrow[d, mapsto] \arrow[dl, phantom, "\ni"] \\
 & S_k(X) \arrow[d, dashed, "h_k(X)"] & S_k(\Delta_k) \arrow[l, "\sigma_\#"] \arrow[d, dashed, "h_k(\Delta_k)"'] & \\
 & S_{k+1}(X \times I) \arrow[dl, phantom, "\in"] & S_{k+1}(\Delta_k \times I) \arrow[l, "(\sigma \times \mathrm{id}_I)_\#"'] \arrow[dr, phantom, "\ni"] & \\
(\sigma \times \mathrm{id})_\#(h_k(\Delta_k)\mathrm{id}_{\Delta_k}) & & & h_k(\Delta_k)\mathrm{id}_{\Delta_k} \arrow[lll, mapsto]
\end{tikzcd}
$$

Hence, if h_k exists, it is determined by its definitio, and its form is as shown in the diagram above. Thus we can use above step as the definition of h_k(事先选定好 h_k, 即选定了 $h_k(X)$ 作用在 σ 上的点, 接下来的问题就是如何选好 $(h_k(\Delta_k)\mathrm{id}_{\Delta_k})$ 使其满足定理所给条件).

2) On the other hand, if we have chosen an element $\xi_k \in S_{k+1}(\Delta_k \times I)$ to play a role of $h_k(\Delta_k)\mathrm{id}_{\Delta_k}$ and define

$$
\begin{tikzcd}
\bigoplus_{\sigma:\Delta_k \to X} \mathbb{Z}\sigma \arrow[d, phantom, "\parallel"] & & \\
S_k(X) \arrow[r, "h_k(X)"] \arrow[d, phantom, "\cup"] & S_{k+1}(X \times I) \arrow[d, phantom, "\cup"] & \\
\sigma \arrow[r, mapsto] & h_k(X)\sigma & := \quad (\sigma \times \mathrm{id})_\# \xi_k
\end{tikzcd}
$$

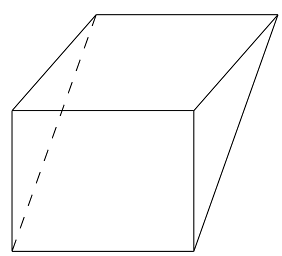
then we need to prove $h_k : S_k \to S_{k+1}(\cdot \times I)$ is a natural transformation(虽然前文说如果 h_k 是一个自然变换, 它就完全由上文决定, 但是并没有说明在这一步这样定义的 h_X

就是一个自然变换). Therefore, consider diagram below

$$
\begin{tikzcd}
 & & \mathrm{id}_{\Delta_k} \arrow[rrr, mapsto] \arrow[dll, mapsto] \arrow[d, phantom, sloped, "\in"] & & & \xi_k \arrow[d, phantom, sloped, "\in"] \arrow[dd, mapsto, bend left] \\
\sigma \arrow[ddd, mapsto] \arrow[dr, phantom, sloped, "\in"] & & S_k(\Delta_k) \arrow[rrr, dashed] \arrow[dl, "\sigma_\#"'] \arrow[ddl, "(f\times\sigma)_\#"] & & & S_{k+1}(\Delta_k \times I) \arrow[dll, "(\sigma\times\mathrm{id}_I)_\#"'] \arrow[ddll] \\
 & S_k(X) \arrow[rr, "h_k(X)"'] \arrow[d, "f_\#"'] & & S_{k+1}(X \times I) \arrow[d, "(f\times\mathrm{id}_I)_\#"'] \arrow[rr, phantom, "\ni"] & & (\sigma \times \mathrm{id}_I)_\# \xi_k \arrow[dd, mapsto] \\
 & S_k(Y) \arrow[rr, "h_k(Y)"'] & & S_{k+1}(Y \times I) \arrow[d, phantom, sloped, "\ni"] \arrow[drr, phantom, "\ni"] & & \\
f \circ \sigma \arrow[rrr, mapsto] \arrow[ur, phantom, sloped, "\in"] & & & ((f\circ\sigma) \times \mathrm{id}_I)_\# \xi_k & \overset{?}{=} & (f \times \mathrm{id}_I)_\# (\sigma \times \mathrm{id}_I)_\# \xi_k
\end{tikzcd}
$$

we need to check if $((f \circ \sigma) \times \mathrm{id}_I)_\# \xi_k = (f \times \mathrm{id}_I)_\# (\sigma \times \mathrm{id}_I)_\# \xi_k$, it determines whether the diagram commutative.

注 3.3.2. 此处, 可以将交换图表的主体想象为如下图所示的几何图形. 我们想证明的是



该几何图形的正面可以交换, 已有条件为其它几个面均可交换 (左右两侧的交换性来源于链映射 $\#$,).

3) Furthermore, if h_{k-1} and h_k are defined in the manner of 2), and $\xi_k = h_k(\Delta_k)\mathrm{id}_{\Delta_k}$ so that(假设是对的)

$$\partial_{k+1}(\Delta_k \times I) h_k(\Delta_k) \mathrm{id}(\Delta_k) = i_1(\Delta_k)_\# \mathrm{id}_{\Delta_k} - i_0(\Delta_k)_\# \mathrm{id}_{\Delta_k} - h_{k-1}(\Delta_k) \partial_k(\Delta_k) \mathrm{id}_{\Delta_k} \quad (*)$$

Consider the diagram

$$
\begin{tikzcd}
\mathrm{id}_{\Delta_k} \arrow[dddd, bend right] & \in & S_k(\Delta_k) \arrow[rr] \arrow[drr, "\sigma_\#"] \arrow[ddll, dashed] \arrow[ddr] & & S_{k-1}(\Delta_k) \arrow[drr, "\sigma_\#"] \arrow[ddl, dashed] & & \\
& & & & S_k(X) \arrow[rr, "\partial_k(X)"] \arrow[dddl, "h_k(X)"'] \arrow[dddrr, "i_1(X)_\# - i_0(X)_\#"] & & S_{k-1}(X) \arrow[ddd, "h_{k-1}(X)"] \\
S_{k+1}(\Delta_k \times I) \arrow[rrr] \arrow[ddrrr, "(\sigma\times \mathrm{id}_I)_\#"] & & & S_k(\Delta_k \times I) \arrow[ddrrr, "(\sigma\times \mathrm{id}_I)_\#"] & & & \\
\cup & & & & & & \\
\xi_k \arrow[ddrrr, mapsto] & & & S_{k+1}(X \times I) \arrow[rrr, "\partial_{k+1}(X\times I)"] & & & S_k(X \times I) \\
& & & \cup & & & \\
& & & h_k(X)\sigma & & &
\end{tikzcd}
$$

then

$$\partial_{k+1}(X \times I) \circ h_k(X)\sigma = \partial_{k+1}(X \times I)(\sigma \times \mathrm{id}_I)_\# h_k(\Delta_k)\mathrm{id}_{\Delta_k}$$

The formula ($*$) tell us

$$
\begin{aligned}
\partial_{k+1}(X \times I)(\sigma \times \mathrm{id}_I)_\# h_k(\Delta_k)\mathrm{id}_{\Delta_k} &= (\sigma \times \mathrm{id}_I)_\# \partial_{k+1}(\Delta_k \times I)\mathrm{id}_{\Delta_k} \\
&= (\sigma \times \mathrm{id}_I)_\# (i_1(\Delta_k)_\# - i_0(\Delta_k)_\# \mathrm{id}_{\Delta_k} - h_{k-1}(\Delta_k)\partial_k(\Delta_k)\mathrm{id}_{\Delta_k}) \\
&= (\iota_1(X)_\# - \iota_0(X)_\#)\sigma_\# \mathrm{id}_{\Delta_k} - h_{k-1}(X)\sigma_\# \partial_k(\Delta_k)\mathrm{id}_{\Delta_k} \\
&= \iota_1(X)_\# \sigma - \iota_0(X)_\# \sigma - h_{k-1}(X)\partial_k(X)\sigma
\end{aligned}
$$

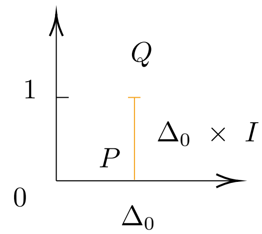
4) Use Mathematical Iinduction to find ξ_k which satisfies formula ($*$).
when $k = 0$,we have

$$
\begin{tikzcd}
& & \mathrm{id}_{\Delta_0} \arrow[ddddll, bend right, mapsto] & & \\
& & \in & & \\
& & S_0(\Delta_0) \arrow[r] \arrow[dl, dashed, "h_1(\Delta_0)"'] \arrow[d, "i_1(\Delta_0)_\# - i_0(\Delta_0)_\#"] & 0 \arrow[r] \arrow[dl, dashed, "0"] & \cdots \\
& S_1(\Delta_0 \times I) \arrow[r, "\partial_1(\Delta_0 \times I)"] & S_0(\Delta_0 \times I) \arrow[r] & 0 \arrow[r] & \cdots \\
& \in & & & \\
\xi_0 & & & &
\end{tikzcd}
$$

It means that we need to find a ξ_0 satisfies

$$\partial_1(\Delta_0 \times I) \circ \xi_0 = i_1(\Delta_0)_\# - i_0(\Delta_0)_\#(\mathrm{id}_{\Delta_0})$$

now we research $(i_1(\Delta_0) - i_0(\Delta_0))\mathrm{id}_{\Delta_0}$.Since Δ_0 is a point, we can picture that



Let $\iota_0(\Delta_0) = P, \iota_1(\Delta_0) = Q$. We have $(\iota_1(\Delta_0)_\# - \iota_0(\Delta_0)_\#)(\mathrm{id}_{\Delta_0}) = Q - P$, where Q and P both point. now we need $\xi_0 \in \Delta_0 \times I$ be a curve which boundary is Q and P. For example, one can take ξ_0 to be the 1- simplex $\xi_0 : (t_0, t_1) \mapsto (1, t_1)$
5) Suppose we have defined $h_l, l < k$(following 2)) so that

$$\forall X \in \mathrm{Ob}\,\mathbf{Top}, i_1(X)_\# - i_0(X)_\# = \partial_{l+1}(X \times I)h_l(X) + h_{l-1}(X)\partial_l(X), l < k$$

Now, consider the diagram

$$\begin{tikzcd}
 & \mathrm{id}_{\Delta_k} \arrow[dl, mapsto] & \in & S_k(\Delta_k) \arrow[r, "\partial_k(\Delta_k)"] \arrow[ddd, "i_1(\Delta_k)_\# - i_0(\Delta_k)_\#"'] & S_{k-1}(\Delta_k) \arrow[r, "\partial_{k-1}(\Delta_k)"] \arrow[ddd, "i_1(\Delta_k)_\# - i_0(\Delta_k)_\#"'] \arrow[dddl, "h_{k-1}(\Delta_k)"'] & S_{k-2}(\Delta_k) \arrow[r] \arrow[dddl, "h_{k-2}(\Delta_k)"'] & \cdots \\
\xi_k & & & & & & \\
\Cap & & & & & & \\
S_{k+1}(\Delta_k \times I) \arrow[rrr, "\partial_{k+1}(\Delta_k \times I)"'] & & & S_k(\Delta_k \times I) \arrow[r, "\partial_k(\Delta_k \times I)"'] & S_{k-1}(\Delta_k \times I) \arrow[r] & \cdots &
\end{tikzcd}$$

We need to find ξ_k which satisfies

$$\partial_{k+1}(\Delta_k \times I)(\xi_k) = i_1(\Delta_k)_\#\mathrm{id}_{\Delta_k} - i_0(\Delta_k)_\#\mathrm{id}_{\Delta_k} - h_{k-1}(\Delta_k) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k}$$

Since $\Delta_k \times I$ is a star-shaped set($H_k(X) = 0, k \geq 1$). We only need prove that $i_1(\Delta_k)_\#\mathrm{id}_{\Delta_k} - i_0(\Delta_k)_\#\mathrm{id}_{\Delta_k} - h_{k-1}(\Delta_k) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k} \in \ker(\partial_k(\Delta_k \times I))$.Since the diagram of chain complex is commutative.

$$\partial_k(\Delta_k \times I)(i_1(\Delta_k)_\#\mathrm{id}_{\Delta_k} - i_0(\Delta_k)_\#\mathrm{id}_{\Delta_k}) = (i_1(\Delta_k)_\# - i_0(\Delta_k)_\#) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k}$$

and from the assumption we can find

$$\partial_k(\Delta_k \times I)(h_{k-1}(\Delta_k) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k}) = ((i_1(\Delta_k)_\# - i_0(\Delta_k)_\#) - h_{k-2}(\Delta_k)\partial_{k-1}(\Delta_k)) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k}$$

Since $k - 1 < k$ and $\partial_{k-1}\partial_k = 0$we have

$$(i_1(\Delta_k)_\# - i_0(\Delta_k)_\#)_{k-1} = \partial_k(\Delta_k \times I) \circ h_{k-1}(\Delta_k) + h_{k-2}(\Delta_k) \circ \partial_{k-1}(\Delta_k)$$

and

$$h_{k-2}(\Delta_k)\partial_{k-1}(\Delta_k) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k} = 0$$

Thus

$$\partial_k(\Delta_k \times I)(h_{k-1}(\Delta_k) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k}) = (i_1(\Delta_k)_\# - i_0(\Delta_k)_\#) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k}$$

and hence,

$$i_1(\Delta_k)_\#\mathrm{id}_{\Delta_k} - i_0(\Delta_k)_\#\mathrm{id}_{\Delta_k} - h_{k-1}(\Delta_k) \circ \partial_k(\Delta_k)\mathrm{id}_{\Delta_k} = 0$$

□

3.3.2 Acyclic model theorem[8]

定义 3.3.3. (models and expressible functors) Let $\mathscr{C}$ be a category and $F : \mathscr{C} \to \mathbf{Ab}$ be a covariant functor.
Suppose we have chosen a set $\mathfrak{M} \subset \mathrm{Ob}(\mathscr{C})$ and $\mathcal{U} := \{U_M \subset F(M) : M \in \mathfrak{M}\}$ a family of subsets of $F(M)$, $M \in \mathfrak{M}$.

1) we can define a (covariant) functor

$$\begin{aligned} \tilde{F}^{\mathcal{U}} &: \mathscr{C} \to \mathbf{Ab} \\ \tilde{F}^{\mathcal{U}}(X) &:= \mathbb{Z}^{\oplus\{(\phi,m): M \xrightarrow[\mathscr{C}]{\phi} X : m \in U_M, M \in \mathfrak{M}\}} \end{aligned}$$

$$\begin{array}{ccccc} X \xrightarrow[\mathscr{C}]{f} Y & \longmapsto & \tilde{F}^{\mathcal{U}}(X) & \xrightarrow{\tilde{F}^{\mathcal{U}}(f)} & \tilde{F}^{\mathcal{U}}(Y) \\ & & & & \\ & & (\phi, m) & \longmapsto & (f \circ \phi, m) \end{array}$$

2) we have a natural transformation $\tilde{F}^{\mathcal{U}} \xrightarrow{\pi} F$.

$$\begin{array}{ccc} \tilde{F}^{\mathcal{U}}(X) & \xrightarrow{\pi(X)} & F(X) \\ (\phi, m) & \mapsto & F(\phi)m \end{array}$$

注 3.3.3. 考虑图表

$$\begin{array}{ccccccc} M & & F(M) & \supset & U_M & \ni & m \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle F(\phi)} & & & \swarrow & \\ X & & F(X) & \ni & F(\phi)\cdot m & & \end{array}$$

以得知前因后果.

Proof.

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \tilde{F}^{\mathcal{U}}(X) \arrow[r, "\pi_X"] \arrow[d, "\tilde{F}f"'] & F(X) \arrow[d, "Ff"] \\ \tilde{F}^{\mathcal{U}}(Y) \arrow[r, "\pi_Y"'] & F(Y) \end{tikzcd}$$

we need to show that $F(f) \circ \pi_X = \pi_Y \circ \tilde{F}f$. Since

$$\begin{aligned} F(f) \circ \pi(X) : \tilde{F}^{\mathcal{U}}(X) &\to F(Y) \\ (\phi, m) &\mapsto F(f)(F(\phi)m) \\ \pi_Y \circ \tilde{F}f : \tilde{F}^{\mathcal{U}}(X) &\to F(Y) \\ (\phi, m) &\mapsto F(f \circ \phi)m = F(f)(F(\phi)m) \end{aligned}$$

Thus π is a natural transformation. □

3) we say that F is $\mathcal{U}$-expressible if π is a natural equivalence.

例 3.3.4. Let $F = S_k : \textbf{Top} \to \textbf{Ab}$,and $\mathfrak{M} = \{\Delta_k\}$,$\mathcal{U} := \{\mathrm{id}_{\Delta_k}\} \subset S_k(\Delta_k)$.Thus $\tilde{F}^{\mathcal{U}}(X) = \mathbb{Z}^{\bigoplus\{(\sigma, \mathrm{id}_{\Delta_k}):\Delta_k \xrightarrow{\sigma} X\} = S_k(X)}$. It is easy to find that π is a natural equivalence, and then F is $\mathcal{U}$-expressible.

定理 3.3.5. (acyclic model theorem Ib)Let $\mathscr{C}$ be a category. Suppose we have covariant functors from $\mathscr{C}$ to **Ab** and natural transformations between them.as shown in below diagram

$$\begin{tikzcd} & K_k \arrow[r, "\partial_k"] \arrow[d, "f_k"] \arrow[dl, dashed, red] & K_{k-1} \arrow[r, "\partial_{k-1}"] \arrow[d, "f_{k-1}"] \arrow[dl, "h_{k-1}"'] & K_{k-2} \arrow[dl, "h_{k-2}"] \\ K'_{k+1} \arrow[r, "\partial'_{k+1}"'] & K'_k \arrow[r, "\partial'_k"'] & K'_{k-1} & \end{tikzcd}$$

where uppercase letters represent functors and arrow represent natural transformation between them so that $\partial_{k-1} \circ \partial_k = 0$,$\partial'_k \circ \partial'_{k+1} = 0$,$\partial'_k \circ f_k = f_{k-1} \circ \partial_k$(square commutative), and $f_{k-1} = \partial'_k \circ h_{k-1} + h_{k-2} \circ \partial_{k-1}$. If K_k is $\mathcal{U}$-expressible(for some $\mathcal{U}$ above,$\mathfrak{M} \subset \mathrm{Ob}(\mathscr{C})$, $U_M \in K_k(M), M \in \mathfrak{M} \Rightarrow \tilde{K}_k^{\mathcal{U}} \underset{\sim}{\xrightarrow{\pi}} K_k$) and if $\ker \partial'_k(M) = \mathrm{im}\, \partial'_{k+1}(M)$ for all $M \in \mathfrak{M}$, then it exists a natural transformation $h_k : K_k \to K_{k+1}$ so that $f_k = h_{k-1} \circ \partial_k + \partial'_{k+1} \circ h_k$.

Proof. 1) Define $h_k(M)u$ for $u \in U_M \subset K_k(M)$,$M \in \mathfrak{M}$, consider the diagram

$$
\begin{tikzcd}
 & u \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\
 & K_k(M) \arrow[r, "\partial_k(M)"] \arrow[d, "f_k(M)"] & K_{k-1}(M) \arrow[r, "\partial_{k-1}(M)"] \arrow[dl, "h_{k-1}(M)"'] \arrow[d, "f_{k-1}(M)"] & K_{k-2}(M) \arrow[dl, "h_{k-2}(M)"] \\
K'_{k+1}(M) \arrow[r, "\partial'_{k+1}(M)"'] & K'_k(M) \arrow[r, "\partial'_k(M)"'] \arrow[d, phantom, "\rotatebox{90}{$\ni$}"] & K'_{k-1}(M) & \\
 & f_k(M)u - h_{k-1}(M)\partial_k(M)u & &
\end{tikzcd}
$$

we need to prove $f_k(M)u - h_{k-1}(M)\partial_k(M)u \in K'_k(M)$.

$$
\begin{aligned}
& \partial'_k(M)(f_k(M)u - h_{k-1}(M)\partial_k(M)u) \\
= \;& f_{k-1}(M)\partial_k(M)u - (f_{k-1}(M)\partial_k(M)u - h_{k-2}(M)\partial_{k-1}(M)\partial_k(M)u) \\
= \;& 0
\end{aligned}
$$

Choose a $\xi_u \in K'_{k+1}(M)$ which satisfies

$$\partial'_{k+1}(M)\xi_u = f_k(M)u - h_{k-1}(M)\partial_k(M)u$$

2) Define a natural transformation $\tilde{K}_k^{\mathcal{U}} \xrightarrow{\tilde{h}_k} K'_{k+1}$ due to below diagram

$$
\begin{tikzcd}
 & \tilde{K}_k^{\mathcal{U}} \arrow[d, "\pi", "\sim"'] \arrow[ddl, "\tilde{h}_k"'] \\
 & K_k \arrow[dl, "\text{wish}"] \\
K'_{k+1} &
\end{tikzcd}
$$

we wish construct $h_k : K_k \to K'_{k+1}$ it is hard, but notice that K_k is $\mathcal{U}$-expressible thus we can get the diagram above, the way to determine $\tilde{h}_k(X)$ as shown in the diagram below

$$\begin{tikzcd}
& (\phi, u) \arrow[rrr, mapsto] \arrow[ddd, mapsto] \arrow[dr, phantom, sloped, "\in"] & & & K'_{k+1}(\phi)\xi_u \arrow[ddd] \arrow[dl, phantom, sloped, "\ni"] \\
X \arrow[d] & & \tilde{K}^{\mathcal{U}}_k(X) \arrow[r, "\tilde{h}_X"] \arrow[d, "\tilde{K}^{\mathcal{U}}_k(f)"] & K'_{k+1}(X) \arrow[d, "K'_{k+1}(f)"] & \\
Y & & \tilde{K}^{\mathcal{U}}_k(Y) \arrow[r, "\tilde{h}_Y"'] & K'_{k+1}(Y) & \\
& (f \circ \phi, u) \arrow[rrr, mapsto] \arrow[ur, phantom, sloped, "\in"] & & & K'_{k+1}(f \circ \phi)\xi_u \arrow[ul, phantom, sloped, "\ni"]
\end{tikzcd}$$

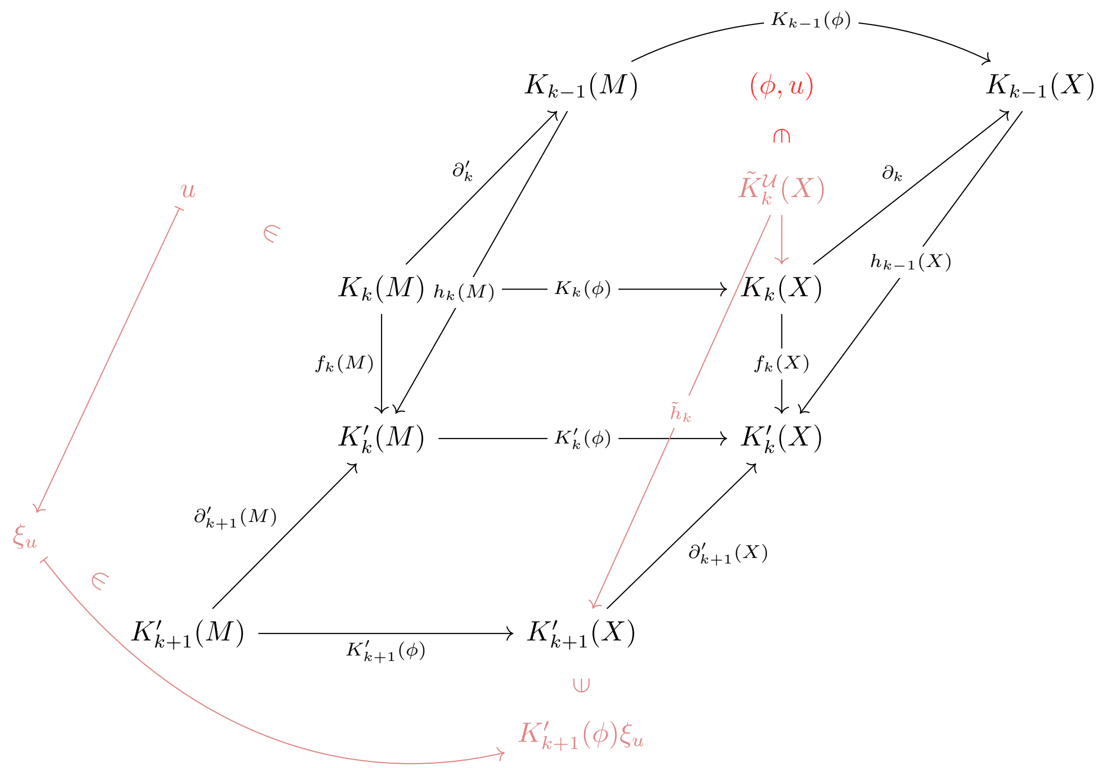
Thus we have

$$\begin{tikzcd}
\tilde{K}^{\mathcal{U}}_k(X) \arrow[r] & K'_{k+1}(X) \\
(\phi, u) \arrow[r, mapsto] & K'_{k+1}(\phi)\xi_u
\end{tikzcd}$$

3) We want to prove $f_k \circ \pi = \partial'_{k+1}\tilde{h}_k + h_{k-1} \circ \partial_k \circ \pi$ since

$$\begin{tikzcd}
& \tilde{K}^{\mathcal{U}}_k \arrow[d, "\sim"', "\pi"] \arrow[ddl, "\tilde{h}_k"'] & \\
& K_k \arrow[r, "\partial_k"] \arrow[d, "f_k"] & K_{k-1} \arrow[dl, "h_{k-1}"] \\
K'_{k+1} \arrow[r, "\partial'_{k+1}"'] & K'_k &
\end{tikzcd}$$

to prove this, we need to use below diagram



therefore

$$
\begin{aligned}
& f_k(X)\pi(X)(\phi, u) \\
= \;& f_k(X)K_k(\phi)u \\
= \;& K'_k(\phi)f_k(M)u \\
= \;& K'_k(\phi)(\partial'_k(M)\xi_u + h_{k-1}(M)\partial_{k-1}(M)u) \\
= \;& \partial'_{k+1}(X)K'_{k+1}(\phi)\xi_u + h_{k-1}(X)K_{k-1}(\phi)\partial_k(M)u \\
= \;& \partial'_{k+1}(X)\tilde{h}_k(X)(\phi, u) + h_{k-1}(X)\partial_k(X)K_k(\phi)u \\
= \;& \partial'_{k+1}(X)\tilde{h}_k(X)(\phi, u) + h_{k-1}(X)\partial_k(X)\pi(X)(\phi, u)
\end{aligned}
$$

□

注 3.3.6. We only need to assume that π has a (natural) right inverse(or say π is epi).

$$
\begin{tikzcd}
\tilde{K}_k^{\mathcal{U}} \arrow[r, shift left, "\pi"] & K_k \arrow[l, shift left, "\exists\tau"]
\end{tikzcd}
$$

so that

$$\pi \circ \tau = 1$$