NOTES IN THE ANALYTIC PRISMATIZATION OVER $\mathbb{Z}_p$

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Abstract. These are extended notes of a mini-series of lectures given during the spring of 2024 in Chicago University about the analytic prismatization over $\mathbb{Z}_p$.

Contents

1. Introduction

In this series of lectures we shall explain the desired/expected picture for the analytic prismatization over $\mathbb{Z}_p$ and to show the current progress we have made in achieving its construction. The analytic prismtization has emerged from the desired to understand new phenomenons in *analytic* p-adic Hodge theory, namely p-adic Hodge theory for rigid spaces, relating different cohomology theories and the theory of locally analytic representations of p-adic Lie groups (eg. [Pan22] and [RC22]). On the other hand, the works of Anschütz-Heuer-le Bras [JAB22, JAB23] about the relationship between perfect complexes of the Hodge-Tate stack of smooth formal schemes and v-bundles of their generic fibers conjectured the existence of an analytic Hodge-Tate stack X^{HT} naturally attached to a rigid space X.

More instructively, there are two concrete motivations that we expect the analytic prismatization will clarify.

(1) In analytic p-adic Hodge theory there are several cohomology theories and objects appearing from different sources. These gadgets are related via the so called *comparison theorems*. One of the main lessons that prismatic cohomology have left to us is that these comparison theorems can be substantially improved into geometric statements. Analytic prismatization ought to be the geometric object that verifies this expectation. To visualize the general picture, let us mention what kind of cohomology theories the analytic prismatization should encode (see also Section 1.4.4 and Section 6). Let C be a complete non-archimedean algebraically closed field of caracteristic 0 or p, and let X be a smooth rigid space over C.
   - For C of characteristic 0. We have proétale cohomology $R\Gamma_{\mathrm{proet}}(X, \mathbb{Q}_p)$. If X arises as the analytification of a variety over C, then its proétale cohomology coincides with the usual étale cohomology of the variety. However, for more rigid analytic objects such as the closed disc $\mathbb{D}_C = \mathrm{Spa}(C\langle T\rangle)$, the proétale cohomology seems to be patological, in particular it is not finite dimensional. For us, this

is not a bug in the theory but just evidence that a more complicate geometry is behind the scenes.

- For C of characteristic 0. De Rham cohomology of X, and more generally the theory of analytic D-modules, should play a fundamental role in the prismatization. From a cohomological point of view this is clear if one expects to have *étale to de Rham* comparison theorems for rigid varieties.
- More generally, still for C of characteristic 0, the work of le Bras and Vezzani [LBV23] on the de Rham-Fargues-Fontaine cohomology suggests that there should be a whole theory of analytic D-modules over the relative Fargues-Fontaine curve attached to X (or its diamond). This will be realized by the so called *de Rham-Fargues-Fontaine stack*.
- For C of characteristic 0. The cohomology $R\Gamma_{\mathrm{proet}}(X, \widehat{\mathscr{O}}_X)$ is also known as "Hodge-Tate cohomology". As its name suggests, it should be the same as the cohomology of a more refined object that we will call the *analytic Hodge-Tate stack*. Furthermore, features such as the geometric Sen operators of [Pan22] and [RC22] ought to be explained thanks to the gerbe structure on X^{HT} bounded by (a suitable version of) the tangent space of X.
- For C of characteristic p. Part of the the prismatization of X should encode a cohomology theory of X in characteristic 0. Classically, the cohomology that plays that role is rigid cohomology. We expect the prismatization to give a geometric improvement of rigid cohomology in terms of a *de Rham Fargues-Fontaine rigid cohomology.*
- Finally, for C of characteristic p, one should expect the existence of an *analytic de Rham stack* whose cohomology theory is given by the de Rham complex in the case of smooth rigid spaces, but that still encodes a convergence condition in the differential operators. Moreover, this de Rham cohomology should be the specialization of the rigid cohomology of the previous point to characteristic p. On the other hand, it should be a Frobenius twist of the specialization to characteristic p of the analytic Hodge-Tate stack of a smooth rigid variety in mixed characteristic.

(2) The second motivation comes from the geometrization of the local p-adic Langlands program. In [FS21] Fargues-Scholze wrote the foundations for the geometrization conjecture of Fargues on the classical local Langlands correspondence. The main players

in the geometrization are perfectoid spaces and the geometry of the Fargues-Fontaine curves attached to them. It is desirable to have an analogue of the geometrization conjecture of Fargues for the local p-adic Langlands program, compatible with the categorical conjectures of [MEH23]. In particular, one would expect such a formalism for the theory of locally analytic representations of p-adic Lie groups. One of the potential main applications of the theory of the analytic prismatization will be to provide a framework where this could be realized.

During these lectures we shall focus on the p-adic Hodge-theoretic motivations of the theory of the analytic prismatization discussed in (1). As a warm up, let us briefly recall some features of the p-adically complete prismatic cohomology ([BS22]) and the prismatization of p-adic formal schemes ([Dri22] and [BL22]).

1.1. p-complete prismatic sice.

One of the key concepts in the theory of prismatic cohomology is the notion of δ-ring.

Definition 1.1.1. Let A be a ring, a δ-structure on A is a map $\delta : A \to A$ satisfying the following conditions:

(1) $\delta(ab) = a^p\delta(a) + \delta(a)b^p + p\delta(a)\delta(b)$.
(2) $\delta(a+b) = \delta(a) + \delta(b) + \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} a^{p-k}b^k$.

A more conceptual description of a δ-ring structure is due to Joyal [Joy85a, Joy85b]. Let $W = \varprojlim_n W_n$ be the p-typical Witt vectors written as a limit of n-truncated Witt vectors.

Theorem 1.1.2 (Joyal)**.** *Let A be a ring, the following datums are naturally equivalent*

(1) *A δ-ring structure on A.*
(2) *A retraction of rings $A \to W_2(A) \to A$.*
(3) *An endomorphism of rings $\phi : A \to A$ together with a path of maps of animated rings*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & A \arrow[d] \\ A/^{\mathbb{L}}p \arrow[r, "\mathrm{Frob}"] & A/^{\mathbb{L}}p. \end{tikzcd}$$

As the Theorem 1.1.2 (3) suggests, a more natural definition of a δ-ring can be rephrased in terms of lifts of Frobenius in the world of animated rings. Because of this, and since in analytic geometry we will be force to work with animated rings, in the following discussion we shall jump from classical algebraic geometry to derived algebraic geometry by taking all rings to be animated.

We shall write Ani for the ∞-category of anima/spaces/∞-groupoids. Let Ring^{δ} be the ∞-category of δ-rings, i.e., animated rings A together with a lift of Frobenius $\phi : A \to A$. Let $\mathsf{Ring}^{\delta,\wedge p} \subset \mathsf{Ring}^{\delta}$ be the full subcategory of derived p-complete δ-rings. Given $(A,\delta) \in \mathsf{Ring}^{\delta,\wedge p}$ we let $\mathrm{Spf}(A,\delta)$ be the functor

$$\mathrm{Spf}(A,\delta) : \mathsf{Ring}^{\delta,p} \to \mathsf{Ani}$$

corepresented by the object (A,δ). We often drop any reference to the delta structure δ and simply write A for (A,δ).

Definition 1.1.3. We call $\mathrm{Func}(\mathsf{Ring}^{\wedge p},\mathsf{Ani})$ the category of p-adiccally formal (pre)stacks. Similarly, we let $\mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p},\mathsf{Ani})$ be the category of p-adically complete formal δ-(pre)stacks.

Remark 1.1.4. In [Mao21] it is explained how to construct the category of δ-rings from animation. Our point of view to define prismatic cohomology and prismatization will be closely related to the theory treated in *loc. cit.*

We now give the definition of prisms.

Definition 1.1.5. Let $A \in \mathsf{Ring}^{\delta,\wedge p}$, a prism structure on A is a (generalized) Cartier divisor $I \to A$ satisfying the following conditions:

(1) For the choice of any (local) generator d of I, the ring A is d-adically complete.
(2) We have $p \in (I,\phi(I))$ in $\pi_0(A)$. Equivalently, locally for the Zariski topology of A, there is a generator $d \in I$ such that $\delta(d) \in A^{\times}$.

A prism is a pair $I \to A$ with $A \in \mathsf{Ring}^{\delta,\wedge p}$ and I a prismatic structure on A. A morphism of prims is a morphism of pairs $(I \to A) \to (J \to B)$ such that $A \to B$ is a morphism of δ-rings. We let Prism be the ∞-category of prisms.

Given a map $(I \to A) \to (J \to B)$ of prisms, the rigidity lemma [BS22] implies that $J = I \otimes_A B$. However, as it is was highlight by Akhil Mathew during the talk, a base change of a prismatic structure $I \to A$ along a map of delta rings $A \to B$ is not necessarily a prismatic structure on B since B might not be I-complete. A way to solve this issue is to introduce the notion of "weakly distinguished element" of [Mao21].

Definition 1.1.6. Let A be a δ-ring. A weakly distinguished element is a map $A \xrightarrow{d} A$ of A-modules (depicted from an element $d \in \pi_0(A)$)

such that $\delta(d) \in \pi_0(A)$ is a unit in A/d. Let A be a p-complete δ-ring, a weakly prismatic structure on a is a Cartier divisor $I \to A$ such that, locally in the Zariski topology of A, I is generated by a weakly distinguished element.

We let $\mathrm{Div}^1 : \mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p}, \mathsf{Ani})$ be the prestack parametrizing the groupoid of weakly prismatic structures on p-complete δ-rings. We let $\mathsf{Prism}^{\mathrm{weak}} := \mathsf{Ring}^{\delta,\wedge p}/\mathrm{Div}^1$ be the category of weak prisms.

Remark 1.1.7. It is clear from the definition that weakly distinguished elements are compatible with base change. Moreover, [Mao21, Corollary 5.31] shows the notion of weakly prismatic structure is sensible since any generator is going to be weakly distinguished.

There is a natural functor from the category of prisms to the category of p-complete rings:

$$\begin{aligned} G\colon \mathsf{Prism}^{\mathrm{weak}} &\to \mathsf{Ring}^{\wedge p} \\ (I \to A) &\mapsto A/I. \end{aligned}$$

The functor G induces a map at the level of presheaves

$$\begin{aligned} \widetilde{G} : \mathrm{Func}(\mathsf{Ring}^{\wedge p}, \mathsf{Ani}) &\to \mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p}, \mathsf{Ani})_{/\mathrm{Div}^1} \\ X &\mapsto \widetilde{G}X(I \to A) = X(A/I). \end{aligned}$$

Remark 1.1.8. The functor $\widetilde{G}$ is a composition of the restriction functor

$$G_* : \mathrm{Func}(\mathsf{Ring}^{\wedge p}, \mathsf{Ani}) \to \mathrm{Func}(\mathsf{Prism}^{\mathrm{weak}}, \mathsf{Ani})$$

and the left Kan extension

$$K_! : \mathrm{Func}(\mathsf{Prism}^{\mathrm{weak}}, \mathsf{Ani}) \to \mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p}, \mathsf{Ani})_{/\mathrm{Div}^1}$$

induced by the inclusion $K : \mathsf{Prism}^{\mathrm{weak}} \subset \mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p}, \mathsf{Ani})_{/\mathrm{Div}^1}$.

We can give the following short definition of the prismatic site:

Definition 1.1.9. Let X be a p-adic formal scheme (or more generally an object in $\mathrm{Func}(\mathsf{Ring}^{\wedge p}, \mathsf{Ani})$). The absolute weak prismatic site of X is the slice category $X_{\Delta^{\mathrm{weak}}} := \mathsf{Ring}^{\delta,\wedge p}_{/G_*X}$.

1.2. **p-complete prismatization.** Instead of defining prismatic cohomology it is more convenient to directly define the prismatization functor. The key observation is that, by a theorem of Joyal, the forgetful functor

$$F : \mathsf{Ring}^{\delta,\wedge p} \to \mathsf{Ring}^{\wedge p}$$

has by right adjoint the Witt vector functor

$$W : \mathsf{Ring}^{\wedge p} \to \mathsf{Ring}^{\delta,\wedge p}. \tag{1.1}$$

The functor F induces a left Kan extension

$$F_! : \mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p}, \mathsf{Ani}) \to \mathrm{Func}(\mathsf{Ring}^{\wedge p}, \mathsf{Ani})$$

given by the unique colimit preserving functor such that

$$F_!\mathrm{Spf}(A, \delta) = \mathrm{Spf}(A).$$

Definition 1.2.1. Consider the functors

$$\begin{tikzcd} \mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p}, \mathsf{Ani})_{/\mathrm{Div}^1} \arrow[r, "F_!"] & \mathrm{Func}(\mathsf{Ring}^{\wedge p}, \mathsf{Ani})_{/F_!\mathrm{Div}^1} \\ \mathrm{Func}(\mathsf{Ring}^{\wedge p}, \mathsf{Ani}) \arrow[u, "\widetilde{G}"] & \end{tikzcd}$$

Given X a p-adic formal scheme (or more generally, an object in $\mathrm{Func}(\mathsf{Ring}^{\wedge p}, \mathsf{Ani})$) we define its weak prismatization $X^{\Delta^{\mathrm{weak}}}$ to be the formal stack over $F_!\mathrm{Div}^1$ given by

$$X^{\Delta^{\mathrm{weak}}} := F_!\widetilde{G}X.$$

Let us briefly describe the functor $X^{\Delta^{\mathrm{weak}}}$. By the right adjoint (1.1), for $T \in \mathrm{Func}(\mathsf{Ring}^{\delta,\wedge p}, \mathsf{Ani})$ and $R \in \mathsf{Ring}^{\wedge p}$, we have

$$F_!T(R) = T(W(R), \delta).$$

Therefore, the following holds:

- Let $R \in \mathsf{Ring}^{\wedge p}$, then $F_!\mathrm{Div}^1(R) = \mathrm{Div}^1(W(R))$ is the ∞-groupoid of weak prismatic structures $I \to W(R)$. Then, the prismatization $(\mathrm{Spf}\mathbb{Z}_p)^{\Delta}$ of [Dri22] and [BL22] is the substack

$$(\mathrm{Spf}\mathbb{Z}_p)^{\Delta} \subset (\mathrm{Spf}\mathbb{Z}_p)^{\Delta^{\mathrm{weak}}} = F_!\mathrm{Div}$$

  consisting on those weak prismatic structures $I \to W(R)$ such that $W(R)$ is I-adically complete.
- Let $R \in \mathsf{Ring}^{\wedge p}/\mathrm{Spf}(\mathbb{Z}_p)^{\Delta}$ with weak prismatic structure $I \to W(R)$. For X a p-adic formal scheme we have that

$$X^{\Delta^{\mathrm{weak}}}(R) = \widetilde{G}(X)(I \to W(R)) = X(W(R)/I).$$

  Then, if X^{Δ} denote the prismatization, we have a pullback diagram

$$\begin{tikzcd} X^{\Delta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X^{\Delta^{\mathrm{weak}}} \arrow[d] \\ (\mathrm{Spf}\mathbb{Z}_p)^{\Delta} \arrow[r] & (\mathrm{Spf}\mathbb{Z}_p)^{\Delta^{\mathrm{weak}}} = F_!\mathrm{Div}^1. \end{tikzcd}$$

Some formal properties of the functors $\widetilde{G}$ and F_* can be deduced: $\widetilde{G}$ commutes with limits and colimits of presheaves, and sends flat sheaves to flat sheaves. $F_!$ commutes with colimits and limits of presheaves, the sheafification of $F_!$ commutes with colimits and finite limits. In particular $X \mapsto X^{\mathbb{\Delta}}$ always commute with finite limits. Note however that a flat surjection $X \to Y$ of p-adic formal schemes does not necessarily produce a flat surjection $X^{\mathbb{\Delta}} \to Y^{\mathbb{\Delta}}$ at the level of prismatization (this amounts to prove non-trivial descent properties of the prismatization functor, eg. quasi-syntomic descent).

1.3. **Analytic prismatization: rough idea of its construction.** Let us now briefly describe the strategy in the construction of the analytic prismatization. We will follow similar steps as in the presentation of Sections 1.1 and 1.2.

1.3.1. *Idea of the construction: the rings modelling the geometry.* First, we need to decide what are going to be the rings that will model our geometry; in the case of p-adic formal schemes the rings were merely p-complete rings, since we want to see phenomena in rigid geometry we shall need to work with certain analytic rings admitting a pseudo-uniformizer. It turns out that Tate algebras can be generalized to bounded rings. Roughly speaking, a bounded ring is an analytic ring A admitting a pseudo-uniformizer π such that any function

$$f\colon A \to \mathbb{A}^1_A = \mathrm{AnSpec} A[T]$$

into the algebraic affine line factors through some open disc of radius $|\pi^{-n}|$

$$f\colon A \to \mathbb{D}^1_A(\pi^{-n}) \subset \mathbb{A}^1_A.$$

We let Ring^b denote the (∞-)category of bounded rings.

On the other hand, in the study of p-complete prismatic cohomology quasi-syntomic descent is a key tool used in several arguments. The key observation is that any algebra has a quasi-syntomic cover by a semiperfectoid ring (eg. by taking p-th roots of all elements). Then, in order to apply techniques of decent, it would be convenient to also have access to some analogue of "semiperfectoid rings" inside the category of bounded rings. A promising analogue to semiperfectoid rings is the category $\mathsf{NilPerf} \subset \mathsf{Ring}^b$ of "nil-perfectoid rings". Roughly speaking, a nil-perfectoid ring is a "nilpotent" thickening of a perfectoid ring, where now an element f is nilpotent if its norm is zero.

Example 1.3.1. A prototypical example of a bounded ring is a colimit of Banach $\mathbb{Q}_p$-algebras. So classical Tate algebras are bounded rings, but also the algebra of the "decompleted torus"

$$\varinjlim_n \mathbb{Q}_p\langle T^{\pm 1/p^n}\rangle$$

is a bounded ring.

Example 1.3.2. Let C be a complete algebraically closed field extension of $\mathbb{Q}_p$. Consider the ring

$$C\{T\}^\dagger := \varinjlim_n C\langle \frac{T}{p^n}\rangle.$$

Then $C\{T\}^\dagger$ is a nil-perfectoid ring. Indeed, the element T has norm 0 as $|T| \leq |p^n|$ for all n, and the "reduction" of $C\{T\}^\dagger$ is given by

$$C\{T\}^\dagger/(T) = C.$$

In any situation, let us write $\mathsf{Ring}^{\mathrm{nice}}$ for any of these categories, and we work with the principle that any reasonable analytic stack we want to study and produce will be an object in "nice analytic pre-stacks" $\mathsf{AnPStk}^{\mathrm{nice}} := \mathsf{Func}(\mathsf{Ring}^{\mathrm{nice}}, \mathsf{Ani})$.

1.3.2. *The idea of the construction: Fargues-Fontaine curves.* The next step in constructing the prismatization is to define an analogue of the category of δ-rings for p-complete prismatic cohomology. Moreover, what we actually need in order to perform the construction is the analogue of the category of δ-prestacks PStk^δ.

However, in the analytic prismatization, even the most basic objects will not be actual rings but analytic stacks. Moreover, we want some versions of Fargues-Fontaine curves of perfectoid rings to be the analogues of the perfect p-complete δ-rings. More precisely, given R a perfectoid ring in characteristic p with pseudo-uniformizer π, we would like the space

$$\mathcal{Y}_{[0,\infty),R} := \mathrm{Spa}\mathbb{A}_{\mathrm{inf}}(R)\backslash([\varpi] = 0),$$

together with its Frobenius automorphism, to be part of the analogues of δ-rings. For that reason, let us denote these key objects $\widetilde{\mathrm{FF}}$ and call them "generalized Fargues-Fontaine curves". We shall write $\mathsf{AnPStk}_{\mathrm{FF}} = \mathrm{PSh}(\widetilde{\mathrm{FF}}, \mathsf{Ani})$ to be its category of presheaves that we call "analytic Fargues-Fontaine stacks" [1].

Thus, we will have some forgetful map

$$F : \widetilde{\mathrm{FF}} \to \mathsf{AnPStk}^{\mathrm{nice}},$$

[1] It is not completely clear if the basic objects $\widetilde{\mathrm{FF}}$ always exist in the different versions of the prismatization. However, the whole category $\mathsf{AnPStk}_{\mathrm{FF}}$ of analytic Fargues-Fontaine stacks should always be present. In the case of characteristic zero we will see that $\widetilde{\mathrm{FF}}$ has a very simple description for the "perfect prismatization".

given by just regarding the generalized Fargues-Fontaine curves as analytic stacks. This functor will have a left Kan extension to analytic Fargues-Fontaine stacks

$$F_! : \mathsf{AnPStk}_{\mathrm{FF}} \to \mathsf{AnPStk}^{\mathrm{nice}}.$$

1.3.3. *The idea of the construction: prismatization.* A property that we will require for the objects in $\widetilde{\mathrm{FF}}$ is their underlying analytic stacks to be "nilpotent-thickenings" of usual Fargues-Fontaine curves (either in their $Y_{[0,\infty)}$ or $X_{\mathrm{FF}} = Y_{(0,\infty)}/\varphi^{\mathbb{Z}}$ constructions). The extra structure involving the category $\widetilde{\mathrm{FF}}$ is then attached to the Frobenius map.

In particular, we will be able to easily define a stack $\mathrm{Div}^1 \in \mathsf{AnPStk}_{\mathrm{FF}}$ whose points in $X \in \widetilde{\mathrm{FF}}$ are generalized Cartier divisors $D \subset X$ such that the pullback to the reduction $\overline{X} \subset X$ is a classical degree 1-divisor in a classical Fargues-Fontaine curve.

Having introduced all previous objects we can now construct the prismatization in complete analogy as for the p-complete version. Consider the slice category $\mathsf{AnPStk}_{\mathrm{FF},/\mathrm{Div}^1}$ of degree 1-divisors in de Rham Fargues-Fontaine stacks. The functor

$$\begin{aligned} G\colon \widetilde{\mathrm{FF}}_{/\mathrm{Div}^1} &\to \mathsf{AnPStk}^{\mathrm{nice}} \\ (D \subset X) &\mapsto D \end{aligned}$$

induces a functor

$$\widetilde{G} : \mathsf{AnPStk}^{\mathrm{nice}} \to \mathsf{AnPStk}_{\mathrm{FF}}.$$

We then have the diagram

$$\begin{tikzcd} \mathsf{AnPStk}_{\mathrm{FF},/\mathrm{Div}^1} \arrow[r, "F_!"] & \mathsf{AnPStk}^{\mathrm{nice}}_{/F_!\mathrm{Div}^1} \\ \mathsf{AnPStk}^{\mathrm{nice}} \arrow[u, "\widetilde{G}"] \arrow[ur, dashed] & \end{tikzcd}$$

Given $X \in \mathsf{AnPStk}^{\mathrm{nice}}$ the analytic prismatization functor will be then defined as

$$X^{\Delta,\mathrm{an}} := F_!\widetilde{G}X.$$

1.4. **How does the analytic prismatization look like?** In this short paragraph we will explain the qualitative picture of the analytic prismatization of analytic stacks (eg. rigid spaces) over $\mathbb{Z}_p$ with values in analytic stacks over $\mathbb{Q}_p$. Even though we expect to fill the picture in the future extending the coefficients to $\mathbb{Z}_p$, the technicalities to perform such a construction are much more subtle than for $\mathbb{Q}_p$ (and not yet in their final stage).

1.4.1. *Underlying topological space.* Let X be an analytic adic space over $\mathbb{Z}_p$ and let $\mathbb{X}^{\mathbb{\Delta}}$ be its analytic prismatization over $\mathbb{Z}_p$. The first question we can ask is what is the underlying topological space of $X^{\mathbb{\Delta}}$. This turns out to be very explicit:

Let $\mathsf{Perf}_{\mathbb{F}_p}$ be the category of perfectoid spaces in characteristic p. Let $\mathrm{Spd}(\mathbb{Z}_p)$ be the diamond whose value at $S \in \mathsf{Perf}_{\mathbb{F}_p}$ consists on all the untilts $S^\sharp$ of S. For X as before, we let $X^\diamondsuit$ be the diamond over $\mathrm{Spd}(\mathbb{Z}_p)$ whose values at S consists on pairs $(S^\sharp, \iota)$ where $S^\sharp$ is an untilt of S and ι an $S^\sharp$-point of X. Then, the underlying topological space of $X^\diamondsuit$ is isomorphic to $|X|$ by [Sch22, Lemma 15.6].

Proposition 1.4.1. *There is a natural homeomorphism*

$$|X^{\mathbb{\Delta}}| = |X^\diamondsuit \times \mathrm{Spd}(\mathbb{Z}_p)|.$$

Moreover, the natural Frobenius endomorphism $\varphi_X : X^{\mathbb{\Delta}} \to X^{\mathbb{\Delta}}$ induces the Frobenius $\varphi_{X^\diamondsuit}$ in its underlying topological space.

For example, if S is a perfectoid space then [SW20, Proposition 11.2.1] says that

$$S^\diamondsuit \times \mathrm{Spd}(\mathbb{Z}_p) = Y^\diamondsuit_{[0,\infty), S^\flat}$$

is the diamond associated to the $Y_{[0,\infty)}$-construction of S. In general, the underlying topological space of $X^{\mathbb{\Delta}}$ is the topological space of the relative $Y_{[0,\infty)}$-construction attached to its diamond $X^\diamondsuit$, obtained by descent using a presentation $X^\diamondsuit = S/R$ with S a perfectoid space and R a perfectoid equivalence relation.

Proposition (1.4.1) will produce a map of analytic stacks

$$X^{\mathbb{\Delta}} \to \mathcal{M}(X) = |X^\diamondsuit \times \mathrm{Spd}(\mathbb{Z}_p)|^{\mathrm{Hauss}}$$

where $\mathcal{M}(X)$ is the *Berkovich spectrum* of X, this is nothing but the maximal Hausdorff quotient of its adic spectrum by [Sch22, Proposition 13.9].

Let S be a perfectoid space over $\mathbb{Z}_p$ with fixed pseudo-uniformizer π. There is a map of compact Hausdorff spaces

$$|p|_\pi : \mathcal{M}(S) \to [0, 1)$$

given by the absolute value of $|p|$ for the fixed norm at each x making $|\pi|_x = 1/2$. The preimage $(0, 1)$ consists in the locus of S where p is a pseudo-uniformizer, i.e. its $\mathbb{Q}_p$-generic fiber. Let us now consider $Y_{[0,\infty),S^\flat}$ the relative curve attached to $S^\flat$. Pick $\pi^\flat$ a pseudo-uniformizer in $S^\flat$ with same valuation as π in each rank 1-point. Then $[\pi]$ induces a pseudo-uniformizer in $Y_{[0,\infty),S^\flat}$, and the function p gives rise a norm function

$$|p|_{[\pi^\flat]} : \mathcal{M}(Y_{[0,\infty),S^\flat}) \to [0,1)$$

as before. The map $|p|_{[\pi^\flat]}$ composed with

$$\frac{1}{\log_{1/2}(x)} : [0,1) \to [0,\infty)$$

is nothing but the usual radius map

$$\mathrm{rad} : \mathcal{M}(Y_{[0,\infty),S^\flat}) \to [0,\infty)$$

given by $\frac{1}{\log_{|\pi|}|p|}$.

In total, we have two maps

$$(|[p]|_{[\pi]}, |p|_{[\pi^\flat]}) : Y_{[0,\infty),S^\flat} \to [0,1) \times [0,1), \tag{1.2}$$

where $|[p]|_{[\pi]}$ is the composite of the map $|p|_\pi : \mathcal{M}(S) \to [0,1)$ with the projection

$$Y_{[0,\infty),S^\flat} \to \mathcal{M}(Y_{[0,\infty),S^\flat}) = \mathcal{M}(S^\lozenge \times \mathrm{Spd}(\mathbb{Z}_p)) \to \mathcal{M}(S).$$

Descending to the diamond $X^\lozenge$ and using Proposition 1.4.1 we get the following proposition

Proposition 1.4.2. *Let X be an adic space over $\mathbb{Z}_p$ with fixed pseudo-uniformizer π. There is a natural map of analytic stacks*

$$(|[p]|_{[\pi]}, |p|_{[\pi]}) : X^{\mathbb{\Delta}} \to [0,1) \times [0,1)$$

with the following features:

1. *The preimage of $(0,1) \times [0,1)$ is the locus where the geometry (i.e X) is $\mathbb{Q}_p$-generic.*
2. *The preimage of $[0,1) \times (0,1)$ is the locus where the coefficients are $\mathbb{Q}_p$-generic. We denote this locus by $X^{\mathbb{\Delta}_{(0,\infty)}}$ and call it the $\mathbb{Q}_p$-analytic prismatization of X.*
3. *The map $(|[p]|_{[\pi]}, |p|_{[\pi]})$ is φ-equivariant when $[0,1) \times [0,1)$ is endowed with the Frobenius map*

$$\begin{aligned} F : [0,1) \times [0,1) &\to [0,1) \times [0,1) \\ (x,y) &\mapsto (x, y^{1/p}). \end{aligned}$$

Remark 1.4.3. We use the notation $X^{\mathbb{\Delta}_{(0,\infty)}}$ for the $\mathbb{Q}_p$-analytic locus of the coefficients of the prismatization instead of $X^{\mathbb{\Delta}_{(0,1)}}$ in order to follow the more traditional $Y_{(0,\infty)}$-notation of relative Fargues-Fontaine curves. We apologize for the possible confusion this can create.

1.4.2. *The de Rham Fargues-Fontaine stack.* Let us now focus in the $\mathbb{Q}_p$-analytic prismatization of X. Generically, the analytic prismatization is nothing but an analytic de Rham stack:

Definition 1.4.4 (Pre-definition)**.** Let X be a (nice) analytic stack over $\mathbb{Q}_p$. The *analytic de Rham stack* of X is given by the quotient

$$X^{\mathrm{dR}} = X/(\Delta X \subset X \times X)^{\dagger}$$

where $(\Delta X \subset X \times X)^{\dagger} = \bigcap_{\Delta \subset U \subset X \times X}$ is the intersection of all the open neighbourhoods of the diagonal in $X \times X$.

There are different motivations for the introduction of the analytic de Rham stack. For us the main reason to consider this construction is because it provides a good theory of de Rham cohomology for spaces not having cotangent complex such as perfectoid spaces or relative Fargues-Fontaine curves. Indeed, the passage to the analytic de Rham stack is in some extend a "decompletion"

Example 1.4.5. Let $\mathbb{T} = \mathrm{AnSpec}(\mathbb{Q}_p\langle T^{\pm 1/p^{\infty}}\rangle)$ be the torus and $\mathbb{T}_{\infty} = \mathrm{AnSpec}(\mathbb{Q}_p\langle T^{\pm 1/p^{\infty}}\rangle)$ the pre-perfectoid torus. Let

$$\mathbb{T}_{\infty}^{\mathrm{sm}} = \mathrm{AnSpec}(\varinjlim_n \mathbb{Q}_p\langle T^{\pm 1/p^n}\rangle) = \varprojlim_n \mathbb{T}_n$$

be the *smooth* pre-perfectoid torus, with $\mathbb{T}_n = \mathrm{AnSpec}(\mathbb{Q}_p\langle T^{\pm 1/p^n}\rangle)$. Then there is a natural equivalence

$$\mathbb{T}_{\infty}^{\mathrm{dR}} \to \mathbb{T}_{\infty}^{\mathrm{sm},\mathrm{dR}}.$$

In particular, the de Rham cohomology of $\mathbb{T}_{\infty}$ is given by the colimit of de Rham cohomologies

$$R\Gamma_{\mathrm{dR}}(\mathbb{T}_{\infty}) = \varinjlim_n R\Gamma_{\mathrm{dR}}(\mathbb{T}_n).$$

Remark 1.4.6. To prove that $\mathbb{T}_{\infty}^{\mathrm{dR}} = \mathbb{T}_{\infty}^{\mathrm{sm},\mathrm{dR}}$ one needs the following important facts about the formation of the analytic de Rham stack:

(1) The formation of the analytic de Rham stack commutes with finite limits. In particular, if $X \to Y$, the Čech nerve of $X^{\mathrm{dR}} \to Y^{\mathrm{dR}}$ is the de Rham stack of the Čech nerve of $X \to Y$.
(2) If X is a smooth rigid variety (or more generally *formally smooth* in a suitable stronger sense, see [RC24, Definition 3.7.2]), then the natural map $X \to X^{\mathrm{dR}}$ is an epimorphism. In particular, it is for $\mathbb{T}_{\infty}^{\mathrm{sm}}$.
(3) If $X \to Y$ is a morphism of affine analytic stacks induced by a descendable map of algebras (as in [Mat16]), then it is an epimorphism. In particular, $\mathbb{T}_{\infty} \to \mathbb{T}_{\infty}^{\mathrm{sm}}$ is an epimorphism.

(4) Finally, if R is a semiperfectoid ring and R^{perf} is its perfectoidization, then $\mathrm{AnSpec}(R^{\mathrm{perf}}) \to \mathrm{AnSpec}(R)$ induces an equivalence of analytic de Rham stacks
$$\mathrm{AnSpec}(R^{\mathrm{perf}})^{\mathrm{dR}} = \mathrm{AnSpec}(R)^{\mathrm{dR}}.$$

Hence, consider the commutative square

$$\begin{tikzcd} \mathbb{T}_\infty \arrow[r] \arrow[d] & \mathbb{T}_\infty^{\mathrm{sm}} \arrow[d] \\ \mathbb{T}_\infty^{\mathrm{dR}} \arrow[r] & \mathbb{T}_\infty^{\mathrm{sm,dR}}. \end{tikzcd}$$

Then the right vertical arrow and the upper horizontal arrow are epimorphisms by (2) and (3). This implies that the bottom horizontal arrow is an epimorphism. To show that $\mathbb{T}_\infty^{\mathrm{dR}} \to \mathbb{T}_\infty^{\mathrm{sm,dR}}$ is an equivalence it suffices to show that it is an immersion. This is equivalent to proving that the diagonal map
$$\mathbb{T}_\infty^{\mathrm{dR}} \to \mathbb{T}_\infty^{\mathrm{dR}} \times_{\mathbb{T}_\infty^{\mathrm{sm,dR}}} \mathbb{T}_\infty^{\mathrm{dR}}$$
is an equivalence. But we have that
$$\mathbb{T}_\infty^{\mathrm{dR}} \times_{\mathbb{T}_\infty^{\mathrm{sm,dR}}} \mathbb{T}_\infty^{\mathrm{dR}} = (\mathbb{T}_\infty \times_{\mathbb{T}_\infty^{\mathrm{sm}}} \mathbb{T}_\infty)^{\mathrm{dR}},$$
by (1). A simple calculation shows that
$$(\mathbb{T}_\infty \times_{\mathbb{T}_\infty^{\mathrm{sm}}} \mathbb{T}_\infty)^{\mathrm{perf}} = \mathbb{T}_\infty$$
which implies our claim by (4).

We can then introduce the de Rham Fargues-Fontaine stack

Definition 1.4.7. Let S be a perfectoid ring in characteristic p, its non-perfected de Rham Fargues-Fontaine stack is given by
$$Y^{\mathrm{dR}}_{(0,\infty),S^\flat}.$$
Similarly, its (perfected) de Rham Fargues-Fontaine stack is given by
$$\mathrm{FF}^{\mathrm{dR}}_{S^\flat} = Y^{\mathrm{dR}}_{(0,\infty),S^\flat}/\varphi_S^{\mathbb{Z}}.$$

For a diamond $T = S/R$ given as a quotient of a perfectoid space S by a perfectoid equivalence relation R, we define its de Rham Fargues-Fontaine stacks $Y^{\mathrm{dR}}_{(0,\infty),T}$ and $\mathrm{FF}^{\mathrm{dR}}_S$ by descent, namely
$$\mathrm{FF}^{\mathrm{dR}}_T = \mathrm{FF}^{\mathrm{dR}}_S/\mathrm{FF}^{\mathrm{dR}}_R.$$

The formation of the de Rham Fargues-Fontaine stack satisfies strong descent properties for perfectoid spaces (almost as strong as v-descent). This justifies the well definiteness of the definition of the de Rham Fargues-Fontaine stack for (very) general diamonds.

Theorem 1.4.8 (Expected)**.**
(1) *Let A be a perfectoid ring weakly perfectly of finite type over a local field (see* [Man22, Definition 3.1.13]*). Let $Y_{(0,\infty),A^\flat}$ be its relative curve. Then the natural map*
$$Y_{(0,\infty),A^\flat} \to Y^{\mathrm{dR}}_{(0,\infty),A^\flat}$$
is a descendable map, in particular an epimorphism of analytic stacks.
(2) *Let A be as in (1), then there is a map $A \to B$ into a topologically countably generated strongly totally disconnected perfectoid ring B, such that $Y_{[0,\infty),B} \to Y_{[0,\infty),A}$ is a descendable map. Moreover, we can find B such that*
$$Y_{(0,\infty),B} \to Y^{\mathrm{dR}}_{(0,\infty),B}$$
is an epimorphism. In particular,
$$Y^{\mathrm{dR}}_{(0,\infty),B} \to Y^{\mathrm{dR}}_{(0,\infty),A}$$
is an epimorphism.
(3) *Let $f : A \to B$ be a map of strongly totally disconnected perfectoid rings that are topologically countably generated. Suppose that f is partially proper and a v-cover (eq. an* arc*-cover, eq. an epimorphism in Berkovich spaces). Then the natural map*
$$Y_{[0,\infty),B} \to Y_{[0,\infty),A}$$
is a descendable map, in particular an epimorphism of analytic stacks.

The $\mathbb{Q}_p$-analytic prismatization of a space X with trivial $\mathbb{Q}_p$-generic fiber is nothing but an analytic de Rham stack:

Theorem 1.4.9. *Let X be a (generalized) adic space over $\mathbb{Z}_p$ with pseudo-uniformizer π. Suppose that the map*
$$|p|_\pi : \mathcal{M}(X) \to [0,1)$$
lands in 0. Then there is a natural isomorphism
$$X^{\Delta_{(0,\infty)}} = Y^{\mathrm{dR}}_{(0,\infty),X^\diamond}.$$

1.4.3. *The perfect prismatization.* Let X be an adic space over $\mathbb{Q}_p$ and let $X^{\Delta_{(0,\infty)}}$ be its $\mathbb{Q}_p$-prismatization. Recall that we have a natural map of analytic stacks
$$X^{\Delta_{(0,\infty)}} \to \mathcal{M}(Y_{(0,\infty),X^\diamond}). \tag{1.3}$$
The space X induces a closed immersion of topological spaces
$$\mathcal{M}(X) \subset \mathcal{M}(Y_{(0,\infty),X^\diamond}).$$

The following theorem says that generically the prismatization is nothing but an analytic de Rham stack

Theorem 1.4.10. *Let $U \subset X^{\mathbb{A}_{(0,\infty)}}$ be the preimage of the open subspace $\mathcal{M}(Y_{(0,\infty),X^\diamond})\backslash\varphi_X^{\mathbb{N}}(\mathcal{M}(X))$. Then there is a natural isomorphism of analytic stacks*

$$U = (Y_{(0,\infty),X^\diamond}\backslash\varphi_X^{\mathbb{N}}(\mathcal{M}(X)))^{\mathrm{dR}}.$$

In other words, the analytic prismatization is generically (in characteristic 0) and analytic de Rham stack.

Finally, the locus given by the preimage of $\mathcal{M}(X)$ contains the information of the analytic Hodge-Tate stack.

Theorem 1.4.11. *Let $X^{\mathbb{B}_{\mathrm{dR}}^{+,\dagger}} \subset X^{\mathbb{A}_{(0,\infty)}}$ be the pre-image of $\mathcal{M}(X) \subset \mathcal{M}(Y_{(0,\infty),X^\flat})$ under the map (1.3). Then for all $n \geq 1$ the Frobenius map*

$$\varphi : \varphi_{X^{\mathbb{A}}}^{-n}(X^{\mathbb{B}_{\mathrm{dR}}^{+,\dagger}}) \xrightarrow{\sim} \varphi_{X^{\mathbb{A}}}^{-n+1}(X^{\mathbb{B}_{\mathrm{dR}}^{+,\dagger}})$$

is an equivalence. Moreover, there is a Cartier divisor

$$X^{\mathrm{HT}} \to X^{\mathbb{B}_{\mathrm{dR}}^{+,\dagger}}$$

called the analytic Hodge-Tate stack.

Combining Theorems 1.4.10 and 1.4.11 one can encode the analytic prismatization in the following two constructions.

Construction 1.4.12. Let X be an adic space over $\mathbb{Q}_p$. Recall that its de Rham-Fargues-Fontaine stack is given by

$$\mathrm{FF}_{X^\diamond}^{\mathrm{dR}} = (Y_{(0,\infty,X^\diamond)})^{\mathrm{dR}}/\varphi_X^{\mathbb{Z}}.$$

The *perfect prismatization* of X is defined as

$$X^{\mathbb{A}_{(0,\infty)},\mathrm{perf}} := \varprojlim_{\varphi_{X^{\mathbb{A}}}} X^{\mathbb{A}}.$$

We also write

$$X^{\mathbb{A}_{\mathrm{FF}}} := X^{\mathbb{A}_{(0,\infty)},\mathrm{perf}}/\varphi_{X^{\mathbb{A}}}^{\mathbb{Z}}.$$

Remark 1.4.13. Note that $X^{\mathbb{A}_{(0,\infty)},\mathrm{perf}}$ is the natural $\mathbb{Z}$-cover of $X^{\mathbb{A}_{\mathrm{FF}}}$ obtained by pulling back along the map of topological spaces

$$\mathcal{M}(X^{\mathbb{A}_{(0,\infty)},\mathrm{perf}}) = \mathcal{M}(Y_{(0,\infty),X^\diamond}) \to \mathcal{M}(\mathrm{FF}_{X^\diamond}) = \mathcal{M}(X^{\mathbb{A}_{\mathrm{FF}}}).$$

We often refer to both as the perfect prismatization of X.

Corollary 1.4.14. *Let $Y^{\mathrm{dR}}_{(1,\infty),X^\diamond}$ be the preimage of $(1,\infty)$ in the de Rham-Fargues Fontaine stack. Similarly, let $X^{\mathbb{\Delta}_{(0,p))},\mathrm{perf}}$ be the preimage of $(0,p)$ in the perfect prismatization. Then there is a natural isomorphism*

$$U = X^{\mathbb{\Delta}_{(1,p)},\mathrm{perf}} = Y^{\mathrm{dR}}_{(1,p),X^\diamond}$$

in the locus $(1,p)$, and we have a pushout diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Y^{\mathrm{dR}}_{(1,\infty),X^\diamond} \arrow[d] \\ X^{\mathbb{\Delta}_{(0,p)},\mathrm{perf}} \arrow[r] & X^{\mathbb{\Delta}_{(0,\infty)}} \end{tikzcd}$$

1.4.4. *First examples.* Let us finish this introduction with some examples of one of the incarnations of the analytic prismatization. In this version we will only care about characteristic zero phenomena, so that the category of "nice" rings live over $\mathbb{Q}_p$. We also simplify the examples by considering the "perfect prismatization" instead of the full prismatization (i.e. we made Frobenius invertible). In characteristic zero the perfect prismatization captures all essential features of the prismatization, and one can recover one from the other in an explicit way.

Example 1.4.15 (Banach Colmez spaces)**.** We said that the objects in $\widetilde{\mathrm{FF}}$ were some sort of generalizations of Fargues-Fontaine curves. In particular, given $\lambda \in \mathbb{Q}$, we will be able to construct a vector bundle $\mathscr{O}_X(\lambda)$ for any $X \in \widetilde{\mathrm{FF}}$. In this way we can define absolute Banach-Colmez spaces to be the cohomologies

$$\mathcal{BC}(\mathscr{O}(\lambda)) := R\Gamma(X, \mathscr{O}_X(\lambda)).$$

When restricted to usual Fargues-Fontaine curves attached to perfectoid rings these will be the same as the usual Banach-Colmez spaces. However, thanks to the additional "nilpotent-thickenings" that we are allowing in the theory, these Banach-Colmez spaces will have in general a non-trivial tangent space.

For instance, for $\lambda = 0$ and $\lambda = 1$, their realizations in analytic stacks will have the following form

- $F_! \mathcal{BC}(\mathscr{O}) = \mathbb{Q}_p^{la}$ is the analytic space attached to the locally analytic p-adic Lie manifold attached to $\mathbb{Q}_p$. Concretely, $\mathbb{Q}_p^{la} = \bigsqcup_n p^{-n}\mathbb{Z}_p^{la}$ and

$$\mathbb{Z}_p^{la} = \mathrm{AnSpec}(C^{la}(\mathbb{Z}_p, \mathbb{Q}_p))$$

is the analytic spectrum of the ring of locally analytic functions on $\mathbb{Z}_p$.

- $F_! \mathcal{BC}(\mathscr{O}(1))$ will be the "fundamental cover" of the generic fiber of $\widehat{\mathbb{G}}_m$. More precisely, let $\widehat{\mathbb{G}}_{m,\eta}$ be the open unit disc of radius 1 around $1 \in \mathbb{G}_m$, then

$$\mathcal{BC}(\mathscr{O}(1)) = \varprojlim_{\times p} \widehat{\mathbb{G}}_{m,\eta}$$

where the transition maps are p-powers, and the generic fiber is the open unit disc of radius one around $1 \in \mathbb{G}_m$. The limit is taken in the sense of analytic stacks, so the algebra of functions of $\mathcal{BC}(\mathscr{O}(1))$ is just the colimit of the algebra of functions of each finite level and not further completion is made.

Example 1.4.16 (Perfect prismatization of $\mathbb{Q}_p$)**.** By definition $(\mathrm{AnSpec}(\mathbb{Q}_p))^{⧍,\mathrm{an}} = F_! \mathrm{Div}^1$ and Div^1 parametrizes degree 1-divisors in the Fargues-Fontaine curve. Similarly as for perfectoid spaces, any degree 1-divisor will be locally isomorphic to $\mathscr{O}(1)$. One deduces that

$$\begin{aligned}(\mathrm{AnSpec}(\mathbb{Q}_p))^{⧍,\mathrm{an}} &= (\mathcal{BC}(\mathscr{O}(1))\backslash 0)/\mathcal{BC}(\mathscr{O})^{\times} \\ &= ((\varprojlim_{\times p} \widehat{\mathbb{G}}_{m,\eta})\backslash(1)_n)/\mathbb{Q}_p^{\times,la}\end{aligned}$$

where the action of $\mathbb{Q}_p^{\times,la} = \mathbb{Z}_p^{\times,la} \times p^{\mathbb{Z}}$ is such that for $(x_n)_n \in (\varprojlim_{\times p} \widehat{\mathbb{G}}_{m,\eta})\backslash(1)_n$ and $a \in \mathbb{Z}_p^{\times,la}$ we have

$$a \cdot (x_n)_n = (x_n^a)_n.$$

Similarly,

$$p \cdot (x_n)_n = (x_n^p)_n = (x_{n-1})_n.$$

Note that this is the same action one has in the definition of (ϕ, Γ)-modules in classical p-adic Hodge-theory. The stack $(\mathrm{AnSpec}(\mathbb{Q}_p))^{⧍}$ has a particular divisor, called the Hodge-Tate stack and denoted by $(\mathrm{AnSpec}(\mathbb{Q}_p))^{\mathrm{HT}}$, given by the image of the point $x = (\zeta_{p^n})$ in $(\mathrm{AnSpec}(\mathbb{Q}_p))^{⧍}$. The residue field of $\varprojlim_p \widehat{\mathbb{G}}_{m,\eta}$ at x is $\mathbb{Q}_p(\zeta_{p^\infty})$ and is stabilized by $\mathbb{Z}_p^{\times}$. Then, the analytic Hodge-Tate stack of $\mathbb{Q}_p$ has the following description:

$$(\mathrm{AnSpec}(\mathbb{Q}_p))^{\mathrm{HT}} = \mathrm{AnSpec}(\mathbb{Q}_p(\zeta_{p^\infty}))/\mathbb{Z}_p^{\times,la}.$$

Example 1.4.17 (Prismatization of $\mathbb{G}_a$)**.** The object $\mathbb{G}_a^{⧍,\mathrm{an}}$ lives over $(\mathrm{AnSpec}\mathbb{Q}_p)^{⧍,\mathrm{an}}$. By definition, the universal degree 1 divisor $d^{\mathrm{univ}} \in$

Div^1 in $\widetilde{\mathrm{FF}}$ sits in a fiber sequence

$$\mathscr{O} \xrightarrow{d^{\mathrm{univ}}} \mathscr{O}(1) \to G_* \mathbb{G}_a\{-1\},$$

where the $\{-1\}$ is a Breuil-Kisin twist. Taking the functor $F_!$ we get a fiber sequence of stacks over $(\mathrm{AnSpec}\mathbb{Q}_p)^{\Delta,\mathrm{an}}$

$$\mathbb{Q}_p^{la} \to (\varprojlim_{\times p} \widehat{\mathbb{G}}_{m,\eta}) \to \mathbb{G}_a^{\Delta,\mathrm{an}}\{-1\}. \tag{1.4}$$

Let us make the previous computation more explicit. Let C be a complete algebraically closed field in characteristic 0 and let $\theta\colon \mathrm{Spa}C \to X_{C^\flat}$ be the degree 1-divisor defined by C in the Fargues-Fontaine curve of $C^\flat$. This will produce a map

$$X_{C^\flat} \to (\mathrm{AnSpec}\mathbb{Q}_p)^{\Delta,\mathrm{an}}$$

and so we can talk about the relative analytic prismatization

$$\mathbb{G}_a^{\Delta/X_{C^\flat}} := X_{C^\flat} \times_{(\mathrm{AnSpec}\mathbb{Q}_p)^{\Delta,\mathrm{an}}} \mathbb{G}_a^{\Delta,\mathrm{an}}.$$

Then, (1.4) shows that we have an effective fiber sequence of analytic stacks over X_C

$$\mathbb{Q}_p^{la} \times_{\mathbb{Q}_p} X_C \to (\varprojlim_{\times p} \widehat{\mathbb{G}}_{m,\eta}) \times_{\mathbb{Q}_p} X_C \to \mathbb{G}_a^{\Delta/X_{C^\flat}}\{-1\}.$$

Then, if $[\epsilon] = (\xi_{p^n})_{n\in\mathbb{N}}$ is a fixed sequence of p-th roots of unit defining an element in $\mathbb{A}_{\inf}(C^\flat)$, we can take d^{univ} so that the action of $a \in \mathbb{Q}_p^{la} \times_{\mathbb{Q}_p} X_C$ on an element $(x_n)_n \in (\varprojlim_{\times p} \widehat{\mathbb{G}}_{m,\eta}) \times_{\mathbb{Q}_p} X_C$ is given by

$$a \cdot (x_n) = [\epsilon]^a (x_n).$$

So, if $a \in \mathbb{Z}_p^{la}$ we have just $a \cdot (x_n) = (\zeta_{p^n}^a x_n)$. We obtain that

$$\mathbb{G}_a^{\Delta/X_{C^\flat}}\{-1\} = (\varprojlim_{\times p} \widehat{\mathbb{G}}_{m,\eta}) \times_{\mathbb{Q}_p} X_C / (\mathbb{Q}_p^{la} \times_{\mathbb{Q}_p} X_C).$$

Example 1.4.18. Let us mention a last example of the perfect analytic prismatization in characteristic 0. Let us fix C as in Example 1.4.17 and let us consider the relative prismatization $\mathbb{G}_m^{\Delta/X_{C^\flat}}$ of $\mathbb{G}_m$. In this case, we will have an effective fiber sequence

$$\mathbb{Z}_p(1)^{la} \times_{\mathbb{Q}_p} X_C \to (\varprojlim_{p} \mathbb{G}_m) \times_{\mathbb{Q}_p} X_C \to \mathbb{G}_m^{\Delta/X_{C^\flat}},$$

where $\mathbb{Z}_p(1)$ is the Tate module of $\mathbb{G}_m$ seen as a p-adic Lie group. The action of the Tate module in the universal cover is then given by translations of $[(\zeta_{p^n})_{n\in\mathbb{N}}]$ where $(\zeta_{p^n})_{n\in\mathbb{N}} \in \mathbb{Z}_p(1)$ is a compatible family of p-power roots of unit. We obtain that

$$\mathbb{G}_m^{\Delta/X_{C^\flat}} = (\varprojlim_p \mathbb{G}_m) \times_{\mathbb{Q}_p} X_C/(\mathbb{Z}_p(1)^{la} \times_{\mathbb{Q}_p} X_C).$$

Finally, the pullback of $\mathbb{G}_m^{\Delta,X_{C^\flat}}$ by $\theta\colon \mathrm{Spa}C \to X_{C^\flat}$ defines the relative Hodge-Tate stack $\mathbb{G}_m^{\mathrm{HT}/C} \to \mathbb{G}_m^{\Delta,X_{C^\flat}}$ over C. Note that the image of a Teichmuller $[(\zeta_{p^n})_n]$ by θ is just 1. Therefore, we find that

$$\mathbb{G}_m^{\mathrm{HT}/C} = (\varprojlim_p \mathbb{G}_{m,C})/\mathbb{Z}_p(1)^{la}$$

where the action of $\mathbb{Z}_p(1)^{la}$ on $\varprojlim_p \mathbb{G}_{m,C}$ is actually smooth and corresponds to the natural Galois action over $\mathbb{G}_{m,C}$.

If we consider the absolute prismatization $X^{\Delta} \to \mathrm{AnSpec}(\mathbb{Q}_p)^{\Delta}$, the absolute Hodge-Tate stack of X will be given by the pullback

$$\begin{tikzcd} X^{\mathrm{HT}} \arrow[r] \arrow[d] & X^{\Delta} \arrow[d] \\ \mathrm{AnSpec}(\mathbb{Q}_p)^{\mathrm{HT}} \arrow[r] & \mathrm{AnSpec}(\mathbb{Q}_p)^{\Delta}. \end{tikzcd}$$

2. Solid Analytic geometry: six functor formalisms and analytic stacks

The analytic prismatization requires the foundations of condensed mathematics and analytic geometry. In this lecture we will briefly review the theory of analytic rings, the quasi-coherent six functor formalism and analytic stacks.

2.1. **Condensed mathematics and analytic rings.**

2.1.1. *Condensed sets.* Recall the definition of condensed set:

Definition 2.1.1. Let Prof be the category of light (eq. metrizable) profinite sets with covers generated by disjoint unions and surjective maps. The category of condensed sets Cond is the category of sheaves $T : \mathsf{Prof}^{\mathrm{op}} \to \mathsf{Sets}$. More generally, given an $(\infty$-)category $\mathscr{C}$ with all limits, the category of condensed $\mathscr{C}$-objects is defined to be the category of $\mathscr{C}$-valued (hyper-)sheaves

$$\mathsf{Cond}(\mathscr{C}) = \widehat{\mathrm{Sh}}(\mathsf{Prof}, \mathscr{C}).$$

In this way, a condensed group/ring/monoid is a sheaf on light profinite sets with values in the category of groups/rings/monoids. One of the objectives of condensed mathematics is to provide a better framework for performing algebraic constructions in topology. For example, the category of topological modules of a topological ring (which is not an abelian category) gets replaces by the category of condensed modules of a condensed ring (which is an abelian category). This improvement of the category of topological spaces is justified from the natural "condensification" functor

$$\underline{(-)}\colon \mathsf{Top} \to \mathsf{Cond}$$

mapping a topological space X to the condensed set $\underline{X}$ whose value at $S \in \mathsf{Prof}$ is $\underline{X}(S) = \mathrm{Map}^{\mathrm{cont}}(S, X)$. Then, [CS20, Proposition 1.2] shows that (metrizable) compact Hausdorff spaces are identified with qcqs condensed sets, and that we have a fully faithful embedding from metrazably compactly generated topological spaces into condensed sets. For us, the main use of condensed mathematics will be in the notion of analytic ring that we recall down below.

Example 2.1.2. Let $S = \varprojlim_n S_n$ be a light profinite set, we can describe explicitly its free condensed $\mathbb{Z}$-modules by the formula

$$\mathbb{Z}[S] = \bigcup_{k \in \mathbb{N}} \mathbb{Z}[S]_{\ell^0 \leq k}$$

where $\mathbb{Z}[S]_{\ell^0 \leq k}$ is the profinite set given by $\mathbb{Z}[S]_{\ell^0 \leq k} = \varprojlim_n \mathbb{Z}[S_n]_{\ell^0 \leq k}$ and for a finite set T, $\mathbb{Z}[T]_{\ell^0 \leq k} \subset \mathbb{Z}[T]$ is the subspace of sums $\sum_{t \in T} a_t \cdot [t]$ such that $\sum_t |a_t| \leq k$. (see [CS20, Proposition 2.1]).

Example 2.1.3 (Space of null sequences)**.** Let $\mathbb{N} \cup \{\infty\} = \varprojlim_n \{1, \ldots, n, \infty\}$ be the 1-point compactification of $\mathbb{N}$. The space of null sequences is the condensed abelian group

$$P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty).$$

Surprisingly the condensed module P is internally proyective in the category of light condensed abelian groups! This is a key feature that does not hold in arbitrary condensed abelian groups, and that is relevant to construct several examples of analytic rings.

The space P has also an algebra structure induced by the monoid structure on $\mathbb{N} \cup \{\infty\}$. We let $\mathbb{Z}[\hat{q}]$ denote P when considered as an algebra. We have inclusions

$$\mathbb{Z}[q] \subset \mathbb{Z}[\hat{q}] \subset \mathbb{Z}[[q]].$$

We can describe $\mathbb{Z}[\hat{q}]$ as a condensed set in the following way:

$$\mathbb{Z}[\hat{q}] = \bigcup_{k\in\mathbb{N}} \mathbb{Z}[\hat{q}]_{\leq k}$$

where

$$\mathbb{Z}[\hat{q}]_{\leq k} = \varprojlim_{n} (\mathbb{Z}[q]/q^{n+1})_{\leq k}$$

and $(\mathbb{Z}[q]/q^{n+1})_{\leq k}$ is the space of truncated polynomials $a_0+a_1q+\cdots+a_nq^n$ such that $\sum_{i=0}^{n}|a_i|\leq k$.

2.1.2. *Analytic rings.* The role of analytic rings in analytic geometry is analogue to the role of commutative rings in classical algebraic geometry. However, the datum of an analytic ring is stronger than the one of an usual ring. Given a commutative ring R its most fundamental invariant is the symetric monoidal category of R-modules $\mathsf{Mod}(R)$. An easy form of Tannaka duality implies that a morphism of rings $R\to S$ is the same datum as a colimit preserving symmetric monoidal functor

$$\mathsf{Mod}(R)\to\mathsf{Mod}(S).$$

A more robust version of Tannakian duality [Lur17, Corollary 4.8.5.21] says that a map of rings as before is the same datum as a colimit preserving morphism of symmetric monoidal functor of the derived ∞-categories of modules

$$\mathscr{D}(R)\to\mathscr{D}(S).$$

In conclusion, a ring R is completely determined by its derived ∞-category $\mathscr{D}(R)$ of modules[2].

Hence, an analytic ring A will be (essentially) determined by its ∞-category of "complete A-modules" $\mathscr{D}(A)$ (and up to animation constrains it will be equivalent to it). What is left to us to decide is which kind of categories $\mathscr{D}(A)$ will be allowed in the theory of analytic rings. Of course, we want to keep the intuition that an analytic ring has an underlying "topological ring" $A^{\triangleright}$ which should be quickly replaced with a condensed ring. Similarly, complete A-modules should come from an underlying condensed space that in addition should have the structure of module over $A^{\triangleright}$. On the other hand, for a module over $A^{\triangleright}$ being "A-complete" should be a condition and not extra structure. Hence $\mathscr{D}(A)$ ought to be a full subcategory $\mathscr{D}(A^{\triangleright})$ of all condensed $A^{\triangleright}$-modules. The previous desiderata leads to the following definition:

[2]This is true in general for $\mathbb{E}_\infty$-rings in spectra. The same statement fails for animated rings, the only constrain is that some symmetric power functors need to be tracked down.

Definition 2.1.4. An analytic ring is a pair $A = (A^{\triangleright}, \mathscr{D}(A))$ with $A^{\triangleright}$ a condensed (animated) ring, and $\mathscr{D}(A) \subset \mathscr{D}(A^{\triangleright})$ a full subcategory of condensed $A^{\triangleright}$-modules satisfying the following conditions:

(1) $\mathscr{D}(A)$ is stable under limits and colimits in $\mathscr{D}(A^{\triangleright})$ and there is a left adjoint of the inclusion $F : \mathscr{D}(A^{\triangleright}) \to \mathscr{D}(A)$.
(2) $\mathscr{D}(A)$ is linear over $\mathscr{D}(\mathsf{CondAb})$, i.e. for all $S \in \mathsf{Prof}$ and $M \in \mathscr{D}(A)$ we have
$$R\underline{\mathrm{Hom}}\mathbb{Z}(\mathbb{Z}[S], M) \in \mathscr{D}(A).$$
(3) The natural t-structure on $\mathscr{D}(A^{\triangleright})$ defines a t-structure on $\mathscr{D}(A)$. Equivalently, the localization functor $F : \mathscr{D}(A^{\triangleright}) \to \mathscr{D}(A) \subset \mathscr{D}(A^{\triangleright})$ preserves connective objects (i.e $\mathscr{D}_{\geq 0}(A^{\triangleright})$).
(4) The ring A is complete, i.e. $A^{\triangleright} \in \mathscr{D}(A)$.

Let us explain each of the conditions in the definition of an analytic ring: condition (1) is an obvious requirement that holds for any category of modules over a ring, namely, limits and colimits exist and can be taken in the category of abelian groups or spectra. The second condition is important if we want to have an internal Hom in $\mathscr{D}(A)$: it must be enriched in condensed sets. Condition (3) is a normalization used to stay in the world of derived algebraic geometry (i.e. animated rings) and not to jump into "spectral analytic geometry". Finally, condition (4) is reasonable if we want that the category $\mathscr{D}(A)$ "essentially" determines the ring $A^{\triangleright}$, namely, its underlying $\mathbb{E}_{\infty}$-ring ought to be the unit in $\mathscr{D}(A)$.

We have defined analytic rings, let us define morphisms between analytic rings.

Definition 2.1.5. A morphism of analytic rings $A \to B$ is a morphism of animated rings $A^{\triangleright} \to B^{\triangleright}$ such that the forgetful functor $\mathscr{D}(B^{\triangleright}) \to \mathscr{D}(A^{\triangleright})$ sends $\mathscr{D}(B)$ to $\mathscr{D}(A)$. We let AnRing denote the ∞-category of analytic rings.

In other words, a morphism of analytic rings $A \to B$ is just a morphism of underlying condensed rings $A^{\triangleright} \to B^{\triangleright}$ for which any complete B-module is also A-complete. The next theorem summarizes the main properties of the category of complete modules:

Theorem 2.1.6 ([CS20, Lecture XII])**.** *Let A be an analytic ring, then $\mathscr{D}(A)$ has a unique symmetric monoidal structure $\otimes_A$ making the localization functor $F : \mathscr{D}(A^{\triangleright}) \to \mathscr{D}(A)$ symmetric monoidal. We often denote $F = A \otimes_{A^{\triangleright}} -$. An object $M \in \mathscr{D}(A^{\triangleright})$ is A-complete if and only if each cohomology group $\pi_i(M)$ for $i \in \mathbb{Z}$ is A-complete.*

Let $A \to B$ be a morphism of analytic rings, then there is a natural (and unique up to contractible space of choices) commutative diagram of symmetric monoidal categories

$$\begin{tikzcd} \mathscr{D}(A^{\triangleright}) \arrow[r, "B^{\triangleright}\otimes_{A^{\triangleright}}"] \arrow[d, "A\otimes_{A^{\triangleright}}"] & \mathscr{D}(B^{\triangleright}) \arrow[d, "B\otimes_{B^{\triangleright}}"] \\ \mathscr{D}(A) \arrow[r, "B\otimes_A"] & \mathscr{D}(B). \end{tikzcd}$$

Other important properties of analytic rings are the following:

- The category AnRing admits all colimits and it is even a presentable ∞-category (for this it is important to be in the light setting of condensed mathematics). A colimit of analytic rings $B = \varinjlim_i A_i$ is computed as follows: first one takes the colimit of condensed animated rings $B^{',\triangleright} = \varinjlim_i A_i^{\triangleright}$, then one defines the category $\mathscr{D}(B) \subset \mathscr{D}(B^{',\triangleright})$ of complete B-modules by declaring a module B-complete if and only if it is A_i-complete for all i. The pair $(B^{',\triangleright}, \mathscr{D}(B))$ might not be an analytic ring because condition (4) in Definition 2.1.4 might fail. However, there is a (non-trivial) completion process that produces an analytic ring $B = (B^{\triangleright}, \mathscr{D}(B))$ from the "uncompleted analytic ring" $(B^{',\triangleright}, \mathscr{D}(B))$, see [Man22, Proposition 2.3.12]. Then, B is the colimit of the diagram $\{A_i\}_{i\in I}$.
- Let $A^{\triangleright}$ be an animated condensed ring. Then (uncompleted) analytic ring structures on $A^{\triangleright}$ are in bijection with (uncompleted) analytic ring structure on $\pi_0(A^{\triangleright})$. This is an incarnation that the category of complete modules only depend on the abelian category of its heart.
- Analytic ring structures are also invariant under nilpotent thickenings of condensed animated rings.
- Analytic rings can also be described using functors of measures: let $S \in \mathsf{Prof}$ be a light profinite set and let $A[S] := A \otimes_{A^{\triangleright}} A^{\triangleright}[S]$ be the free A-module generated by S. We can think of $A[S]$ as the space of "A-valued measures on S". The original definition of analytic ring ([CS20, Definition 12.1]) consisted in the datum of the objects $A[S]$ together with natural maps $S \to A[S]$ satisfying some conditions that guarantee the objects $A[S]$ to be the free objects generated by S in the category $\mathscr{D}(A)$.

2.1.3. *Examples of analytic rings.* Let us now discuss some basic examples of analytic rings that will be used in the next lectures.

Example 2.1.7. Let A be an analytic ring and let D be an animated $A^{\triangleright}$-algebra whose underling module is A-complete. Then we can define the induced analytic ring structure $D_{A/}$ by taking the condensed algebra to be D and declaring a condensed D-module to be $D_{A/}$-complete if and only if it is A-complete. By definition, $\mathscr{D}(D_{A/}) = \mathrm{Mod}_D(\mathscr{D}(A))$.

Example 2.1.8. The first analytic ring that was defined is the ring of solid integers $\mathbb{Z}_{\square}$. It has $\mathbb{Z}$ as underlying condensed ring and category of complete modules the derived category of solid abelian groups $\mathscr{D}(\mathsf{Solid})$. The free solid abelian group generated by a profinite set $S = \varprojlim_n S_n$ is

$$\mathbb{Z}_{\square}[S] = \varprojlim_n \mathbb{Z}[S_n].$$

Moreover, if S is infinite the free solid abelian group generated by S is isomorphic to $\prod_n \mathbb{Z}$, and it is a compact projective generator of Solid

An advantage of the light framework is that solid abelian groups are those abelian groups for which a null-sequence is summable. This can make precise in the following way: consider $\mathbb{Z}[\hat{q}]$ the algebra of null-sequences and let P denote the algebra when seen as a condensed abelian group. Given $M \in \mathsf{CondAb}$ a condensed abelian groups

$$\mathrm{Null}(M) := \underline{\mathrm{Hom}}(P, M)$$

is by definition the space of null-sequences of M. Multiplication by q in $\mathbb{Z}[\hat{q}]$ induces a shift in null-sequences: it maps $(a_0, a_1, \ldots)$ to $(a_1, a_2, \ldots)$. Thus, if $1-q^* : \mathrm{Null}(M) \to \mathrm{Null}(M)$ is an isomorphism, its inverse will send a null sequence $(a_0, a_1, \ldots)$ to the sequence $(\sum_{i\geq 0} a_i, \sum_{i\geq 1} a_i, \ldots)$.

This is precisely the definition of a solid abelian group: an object $M \in \mathsf{CondAb}$ is solid if and only if

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[\hat{q}]/(1-q), M) = 0.$$

Example 2.1.9. Let R be a finite type algebra over $\mathbb{Z}$. We can define the analytic ring $R_{\square}$ with underlying condensed ring R, and whose free complete module generated by a profinite $S = \varprojlim_n S_n$ is

$$R_{\square}[S] = \varprojlim_n R[S_n].$$

If S is infinite then these modules are isomorphic to $\prod_n R$ and it is a compact projective generator of the category of $R_{\square}$-complete modules. We will go back to this example in Section 3.

2.2. **Six functor formalism for quasi-coherent sheaves.** The theory of six functor formalisms in the sense of [Man22] (motivated by previous definitions of Liu-Zheng and Gaitsgory) is fundamental in the definition of analytic stacks. This is justified from the fact that analytic rings are essentially their category of complete modules, and localizing or gluing analytic rings is equivalent to localizing or gluing the categories. Six functors provide a very clean formalism to make these constructions systematically.

Let $\mathsf{Aff} := \mathsf{AnRing}^{\mathrm{op}}$ be the ∞-category of affine analytic stacks[3]. For an analytic ring A let $\mathrm{AnSpec}(A) \in \mathsf{Aff}$ be the object defined by A. Let us not give a summary of a definition of a six functor formalism (it is better to refer to [Sch23]), instead we construct directly the six functors for quasi-coherent sheaves on analytic rings by only asking for the following datum:

i. An "open immersion" $j : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ is such that the pullback functor $j^* : \mathscr{D}(A) \to \mathscr{D}(B)$ has a fully faithful left adjoint $j_! : \mathscr{D}(B) \to \mathscr{D}(A)$ satisfying projection formula: for $N \in \mathscr{D}(B)$ and $M \in \mathscr{D}(A)$ the natural map
$$j_!(N \otimes_B j^* M) \to j_! N \otimes_A M$$
is an equivalence.

ii. A "proper map" $p : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ is such that the pullback functor $p^* : \mathscr{D}(A) \to \mathscr{D}(B)$ has a colimit preserving right adjoint $p_* : \mathscr{D}(B) \to \mathscr{D}(A)$ ssatisfying projection formula: for $N \in \mathscr{D}(B)$ and $M \in \mathscr{D}(A)$ the natural map
$$p_* N \otimes_A M \to p_*(N \otimes_B p^* B)$$
is an equivalence.

iii. A map $f : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ is "!-able" if it admits a factorization $f = p \circ j$ with j an open immersion and p a proper map.

The previous datum are natural from the point of view of six functor formalisms, it turns out that we can explicitly describe the morphisms of (i)-(iii).

Proposition 2.2.1. *Let I, P and E denote the class of open, proper and !-able maps in Aff.*

(1) *A map $j : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ is in I if and only if the morphism $j^*\mathscr{D}(A) \to \mathscr{D}(B)$ is an open localization of presentably symmetric monoidal stable ∞-categories. In other words, j is an open localization if and only if there is an idempotent algebra object $D \in \mathscr{D}(A)$ such that we have a semi-orthogonal*

[3]This name causes a class with the already existing notion of affine stack in algebraic geometry, I apologize for this.

decomposition

$$\mathrm{Mod}_D(\mathscr{D}(A)) \subset \mathscr{D}(A) \xrightarrow{j^*} \mathscr{D}(B),$$

i.e. $\mathscr{D}(B) = \mathscr{D}(A)/\mathrm{Mod}_D(\mathscr{D}(A))$.

(2) *A map* $p : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ *is in* P *if and only if* B *has the induced analytic ring structure from* A. *More precisely,* p *is in* P *if and only if* $\mathscr{D}(B) = \mathrm{Mod}_{p_* B^{\triangleright}}(\mathscr{D}(A))$, *i.e. an* $B^{\triangleright}$*-module* M *is* B*-complete if and only if it is* A*-complete.*

(3) *A map* $f : \mathrm{AnSpec}B \to \mathrm{AnSpec}A$ *is* !*-able if and only if the map* $\mathrm{AnSpec}B \to \mathrm{AnSpec}B_{A/}$ *from the induced analytic ring structure on* $B^{\triangleright}$ *to* B *is an open immersion (this is some king of "canonical compactification" as in Huber's theory of affinoid rings).*

From Proposition 2.2.1 and the results of [Man22, Appendix A.5] one can construct a six functor formalism

$$\mathscr{D} : \mathrm{Corr}(\mathsf{Aff}, E) \to \mathrm{CAlg}(\mathsf{Pr}^{L,\mathrm{ex}})$$

extending the natural functor AnSpecA $\mapsto \mathscr{D}(A)$ where the maps admitting !-functors are precisely the !-able maps E.

Example 2.2.2. Let us give an example of an open immersion that will appear a lot in the theory of solid rings. Let $A = \mathbb{Z}[T]_{\mathbb{Z}_{\square}/}$ be the induced analytic ring structure on the polynomial algebra from the solid integers and let $B = \mathbb{Z}[T]_{\square}$ be the solid polynomial algebra of Example 2.1.9. The morphism of analytic rings $A \to B$ induces a map of analytic spectrums

$$j : \mathrm{AnSpec}B \to \mathrm{AnSpec}A.$$

The map j is an open immersion, namely, it is the complement of the idempotent A-algebra $\mathbb{Z}((T^{-1})) = \mathbb{Z}[[T^{-1}]][T]$ of Laurent power series in the variable T^{-1}. In other words,

$$\mathscr{D}(\mathbb{Z}[T]_{\square}) = \mathscr{D}(A)/\mathrm{Mod}_{\mathbb{Z}((T^{-1}))}(\mathscr{D}(A)).$$

One can compute explicitly the functor $j_!$ as follows: let $M \in \mathscr{D}(A)$, i.e. a solid abelian group endowed with a module structure over $\mathbb{Z}[T]$. Then

$$j_! j^* M = \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1] \otimes_A M.$$

2.3. **Analytic stacks.** Having constructed the six functor formalism for affine analytic stacks we can start the gluing process. First, let us define two notions of descent that are used to glue analytic rings.

Definition 2.3.1. Let $f : A \to B$ be a morphism of analytic rings and let $(B^{\otimes_A n+1})_{[n]\in\Delta}$ be its Čech nerve.

(1) We say that f satisfies $*$-descent if the natural map

$$\mathscr{D}(A) \to \varprojlim_{[n]\in\Delta}^{*} \mathscr{D}(B^{\otimes_A n+1}) \tag{2.1}$$

is an equivalence of categories, where the transition maps are given by pullback functors.

(2) We say that f satisfies !-descent if the natural map

$$\mathscr{D}(A) \to \varprojlim_{[n]\in\Delta}^{!} \mathscr{D}(B^{\otimes_A n+1}) \tag{2.2}$$

is an equivalence of categories, where the transition maps are given by upper !-functors.

Let us unravel what it means to have $*$ and !-descent at the level of modules. Let $f_n : \mathrm{AnSpec} B^{\otimes_A n+1} \to \mathrm{AnSpec} A$ be the natural map. Then the functor (2.1) has a right adjoint which sends a co-cartesian diagram $(M_n)_{[n]\in\Delta}$ with $M_n \in \mathscr{D}(B^{\otimes_A n+1})$ to

$$\varprojlim_{[n]\in\Delta} f_{n,*}M_n.$$

Thus, the fully-faithfulness of (2.1) is equivalent to: for any $N \in \mathscr{D}(A)$ the natural map

$$N \to \varprojlim_{[n]\in\Delta} f_{n,*}f_n^*N = \varprojlim_{[n]\in\Delta} (B^{\otimes_A n+1} \otimes_A N)$$

is an equivalence (this is usual descent).

On the other hand, the functor (2.2) has a left adjoint mapping a co-cartesian section $(M_n)_{[n]\in\Delta}$ to

$$\varinjlim_{[n]\in\Delta^{\mathrm{op}}} f_{n,!}M_n.$$

Thus, the fully-faithfulness of (2.2) is equivalent to: for any $N \in \mathscr{D}(A)$ the natural map

$$\varinjlim_{[n]\in\Delta^{\mathrm{op}}} f_{n,!}f_n^!N \to N$$

is an equivalence (this is also called co-descent).

A priori $*$ and !-decent properties look not related at all. Magically, thanks to the six functor formalism and adjunctions of $(\infty,2)$-categories, in the case of analytic rings one implies the other in a very strong sense:

Theorem 2.3.2 (Clausen and Scholze)**.** *Let $f : A \to B$ be a !-able map of analytic rings. Suppose that f has !-descent, then the following holds:*

(1) *f has $*$-descent.*
(2) *For all morphism $A \to C$ of analytic rings, the base change $f' : C \to B \otimes_A C$ has !-descent, in particular it also has $*$-descent.*

Remark 2.3.3. A much stronger statement holds: it turns out that the functor $A \mapsto \mathscr{D}(A)$ from analytic rings to $\mathrm{CAlg}(\mathsf{Pr}^L)$ commutes with colimits. For example, given a diagram $B \leftarrow A \to C$ of analytic rings, one has that

$$\mathscr{D}(B \otimes_A C) = \mathscr{D}(B) \otimes_{\mathscr{D}(A)} \mathscr{D}(C)$$

where the RHS is Lurie's tensor product in Pr^L. Thus we can talk about modules in Pr^L over $\mathscr{D}(A)$, which is denoted by $\mathsf{Pr}^L_{\mathscr{D}(A)}$ and given an algebraic way to define enrichments of (presentable) ∞-categories. Then, the following are equivalent for a !-able morphism of analytic rings $f : A \to B$:

(1) f has !-decent.
(2) The pullback $f^* : \mathsf{Pr}^L_{\mathscr{D}(A)} \to \mathsf{Pr}^L_{\mathscr{D}(B)}$ has $*$-descent.

Example 2.3.4. Let $f : A \to B$ be a morphism of classical (animated-)rings. Then we declare f to be proper the right adjoint $f_*\mathscr{D}(B) \to \mathscr{D}(A)$ satisfies projection formula. Then, f having !-descent is equivalent for f to be descendable in the sense of [Mat16]. More generally, let $f : A \to B$ be a morphism of analytic rings where B has the induced analytic structure. Then f has !-descent if and only if $B^{\triangleright} \in \mathscr{D}(A)$ is a descendable algebra.

With the previous preparations we can now define the category of analytic stacks.

Definition 2.3.5. A morphism of analytic rings $A \to B$ is said a !-cover if it is !-able and satisfies !-descent. The category of analytic stacks AnStk is the full subcategory of accessible pre-sheaves on anima of affine analytic stacks $\mathsf{AnStk} \subset \mathrm{PSh}(\mathsf{Aff}, \mathsf{Ani})$ satisfying descent for all hypercovers $(X_n)_{n \in \Delta^{\mathrm{op}}}$ of Y such that

$$\mathsf{Pr}^L_{\mathscr{D}(Y)} = \varprojlim_{[n] \in \Delta} \mathsf{Pr}^L_{\mathscr{D}(X_n)}.$$

Hence, roughly speaking, an analytic stack is a sheaf on affine analytic stacks with respect to the !-topology (and some additional hypercovers).

Add examples of analytic stacks

Example 2.3.6 (Schemes).

Example 2.3.7 (Complex analytic spaces).

Example 2.3.8 (Adic spaces)**.**

Example 2.3.9 (Betti stacks)**.**

3. Solid Analytic geometry: Bounded analytic rings

An analytic ring A is called "solid" if it lives over $\mathbb{Z}_\square$, we let $\mathsf{AnRing}_{\mathbb{Z}_\square}$ be the ∞-category of solid analytic rings. Since $\mathbb{Z}_\square$ is an analytic ring structure on the integers, being solid is actually a condition and not extra structure. For us, it will suffices to work with a smaller category $\mathsf{AffRing}_{\mathbb{Z}_\square} \subset \mathsf{AnRing}_{\mathbb{Z}_\square}$ of solid affinoid rings which is a generalization of a Huber ring $(A, A^+)_\square$. Following Clausen and Scholze we will define a stack of norms over which we can do "analytic geometry" in the classical sense, i.e. where we are given with a pseudo-uniformizer. Over the stack of norms we shall introduce the notion of "bounded ring" which is a generalization of Tate-algebras, i.e. "complete algebras with pseudo-uniformizer". We discuss a new character appearing in the theory of bounded rings; a new nil-radical of "norm zero elements" that we call the "†-nilradical". Finally, we briefly explain how bounded analytic rings can be used to define a very general notion of Tate (or analytic in the sense of Huber) adic space. All the previous objects will play a fundamental role in the definition of the analytic de Rham stack and the analytic prismatization.

3.1. **Solid affinoid rings.**

Recall the analytic rings from Example 2.1.9.

Definition 3.1.1. Let R be a finite type $\mathbb{Z}$-algebra. We let $R_\square$ be the analytic ring structure on R whose free complete module generated by a profinite set $S = \varprojlim_n S_n$ is given by

$$R_\square[S] = \varprojlim_n R[S_n].$$

We call $\mathbb{Z}[T]_\square$ the "solid polynomial algebra".

We have the following proposition.

Proposition 3.1.2 ([CS19])**.** *The following hold:*

(1) *We have $\mathbb{Z}[T_1, \ldots, T_n]_\square = \mathbb{Z}[T_1]_\square \otimes_{\mathbb{Z}_\square} \cdots \otimes_{\mathbb{Z}_\square} \mathbb{Z}[T_n]_\square$.*
(2) *Let R be an algebra of finite type over $\mathbb{Z}$ and $\mathbb{Z}[T_1, \ldots, T_n] \to R$ an integral map. Then $R_\square$ is the induced analytic ring structure from $\mathbb{Z}[T_1, \ldots, T_n]_\square$. More precisely, for S profinite we have*

$$R_\square[S] = R \otimes_{\mathbb{Z}[T_1,\ldots,T_n]} \mathbb{Z}[T_1, \ldots, T_n]_\square[S].$$

Part (1) of Proposition 3.1.2 says that in order to "solidify" a polynomial algebra it suffices to "solidify" each of its variables. Part (2) says that if $r \in R$ is integral over $a_1, \dots, a_n \in R$, and the last are solid, then r is also solid. This motivates the following definition which is a generalization of the "bounded elements" in an Huber pair.

Definition 3.1.3 ([RC24, Definition 2.6.1 (1)]). Let A be a solid analytic ring. Its subring of solid elements A^+ is the discrete ring

$$A^+ = \mathrm{Map}_{\mathsf{AnRing}}(\mathbb{Z}[T]_{\square}, A).$$

In other words, A^+ consists on all the elements $a \in \pi_0(A^{\triangleright})(*)$ such that any lift $\mathbb{Z}[T] \to A^{\triangleright}$ of animated rings mapping T to a localizes in a map of analytic rings $\mathbb{Z}[T]_{\square} \to A$.

If $A^{\triangleright}$ is a solid condensed ring we let $A^{\triangleright,+}$ be the solid elements of $A^{\triangleright}$ with the induced analytic ring structure from $\mathbb{Z}_{\square}$. In other words, $A^{\triangleright,+}$ consists on all the elements $a \in \pi_0(A^{\triangleright})(*)$ such that $A^{\triangleright}$ is a complete $\mathbb{Z}[a]_{\square}$-module.

Another natural construction in the theory of Huber pairs is the subspace of topologically nilpotent elements. Given (A, A^+) a classical complete Huber pair, the subspace $A^{\circ\circ}$ of topologically nilpotent elements consists on those $a \in A$ such that $(a^n)_n$ is a null-sequence. These are precisely the objects correpresented by the power series ring $\mathbb{Z}[[T]]$. We can then define topologically nilpotent elements for any solid analytic ring:

Definition 3.1.4 ([RC24, Definition 2.6.1 (2)]). Let A be a solid analytic ring, the subspace of topologically nilpotent elements is given by

$$A^{\circ\circ} = \mathrm{Map}_{\mathsf{AnRing}}(\mathbb{Z}[[T]], A).$$

Remark 3.1.5. Definition 3.1.3 is producing a discrete ring A^+, it is not clear whether there is a good way to promote A^+ to a non-discrete condensed ring. On the other hand, Definition 3.1.4 can be promoted to an actual solid $A^+_{\square}$-module by [RC24, Proposition 2.6.9].

It is natural to ask that for a solid ring $A^{\triangleright}$ what are the possible analytic strictures induced from solidifying some variables. This leads to the definition of solid rings attached to pairs:

Definition 3.1.6. Let $A^{\triangleright}$ be a solid animated ring and let $S \subset A^{\triangleright,+}$ be a subset. We let $(A, S)_{\square}$ be the analytic ring structure on $A^{\triangleright}$ making a module M complete if and only if it is $\mathbb{Z}[s]_{\square}$-complete for all $s \in S$.

A priori different pairs on a solid ring $A^{\triangleright}$ could produce the same analytic ring structure on $A^{\triangleright}$. Given a pair (A, S) there is a maximal $S \subset A^+(S) \subset A^+$ such that $(A, S)_{\square} = (A, A^+(S))_{\square}$:

Proposition 3.1.7. *Let $(A^{\triangleright}, S)$ be as in Definition 3.1.6 and let $A^+(S) := (A, S)^+_{\square}$. Then the natural map*

$$(A, S)_{\square} \to (A, A^+(S)))_{\square}$$

is an isomorphism. Moreover, $A^+(S)$ is an integrally closed subring containing $A^{\circ\circ}$.

Proof. The equivalence of analytic rings is clear by definition of $(A, S)_{\square}$ and A^+. The fact that $A^+(S)$ is integrally closed follows from Proposition 3.1.2 (2). Finally, A^+ contains $A^{\circ\circ}$ since we have a morphism of analytic rings $\mathbb{Z}[T]_{\square} \to \mathbb{Z}[[T]]$. $\square$

The previous discussion motivates the following definition:

Definition 3.1.8. A solid analytic ring A is solid affinoid if the natural map

$$(A^{\triangleright}, A^+)_{\square} \to A$$

is an equivalence. We let $\mathsf{AffRing}_{\mathbb{Z}_{\square}}$ be the ∞-category of solid affinoid rings.

Example 3.1.9. (1) Let A be a discrete ring, then $(A_{\mathbb{Z}_{\square}/})^+ = A$. Given $A^+ \subset A$ an integrally closed subring we can consider the discrete Huber pair (A, A^+) and its attached solid affinoid ring $(A, A^+)_{\square}$. Then, by [And21, Proposition 3.34] $A^+ = (A, A^+)^+_{\square}$ and so we get a fully faithful embedding of discrete Huber pairs into solid affinoid rings.

(2) More generally, let A be a complete Huber ring. Then $(A_{\mathbb{Z}_{\square}/})^+ = A^{\circ}$ is the ring of power bounded elements. Let $A^+ \subset A^{\circ}$ be an open and integrally closed subring and consider the solid affinoid ring $(A, A^+)_{\square}$. Then [And21, Proposition 3.34] implies that $A^+ = (A, A^+)^+_{\square}$ and so we have an embedding from complete Huber pairs into solid affinoid rings.

(3) We have $\mathbb{Z}_p \otimes_{\mathbb{Z}_{\square}} \mathbb{Z}[T]_{\square} = \mathbb{Z}_p\langle T\rangle_{\square} := (\mathbb{Z}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)_{\square}$. Then $\mathbb{Q}_p\langle T\rangle_{\square} := \mathbb{Q}_p \otimes_{\mathbb{Z}_{\square}} \mathbb{Z}[T]_{\square}$ is the solid affinoid ring attached to the affinoid Tate algebra in one variable over $\mathbb{Q}_p$.

(4) Let $A^{\triangleright} = \mathbb{Z}[[X]]$. Then $A^{\triangleright,+} = \mathbb{Z}[[X]](*)$. Moreover, since X is topologically nilpotent, the unique solid affinoid structure in $\mathbb{Z}[[X]]$ is the induced solid structure from $\mathbb{Z}_{\square}$: we have

$$(\mathbb{Z}[[X], \mathbb{Z})_{\square} = (\mathbb{Z}[[X]], \mathbb{Z}[[X]])_{\square}.$$

This defines a solid affinoid ring that still comes from the theory of Huber rings.

(5) Now let us take $A^{\triangleright} = \mathbb{Z}((X))$. Then $A^{\triangleright,+} = \mathbb{Z}[[X]]$ and so the unique solid affinoid structure in $A^{\triangleright}$ is the induced one from $\mathbb{Z}_{\square}$. The ring

$$(\mathbb{Z}((X)), \mathbb{Z})_{\square} = (\mathbb{Z}((X)), \mathbb{Z}[[X]])_{\square}$$

is an example of a solid affinoid ring that does not come from a classical completed Huber pair.

Finally, we endow $\mathsf{AffRing}$ with the !-topology and let

$$\mathsf{SolidAnStk} \subset \mathrm{PSh}(\mathsf{AffRing}_{\mathbb{Z}_{\square}}, \mathsf{Ani})$$

be the full subcategory of solid analytic stacks.

3.2. **Bounded rings and †-nilradical.** In order to do "honest" analytic geometry we need a pseudo-uniformizer. By definition, a pseudo-uniformizer should be a topologically nilpotent unit. In solid analytic rings, the universal ring parametrizing topologically nilpotent elements is the power series ring $\mathbb{Z}[[q]]$. Hence, the Laurent power series ring $\mathbb{Z}((q)) = \mathbb{Z}[[q]][q^{-1}]$ parametrizes pseudo-uniformisers. Let us then consider $\mathsf{AffRing}_{\mathbb{Z}((q))}$ be the category of solid affinoid rings over $\mathbb{Z}((q))$.

Another feature of Tate algebras or analytic Huber rings is that any function is bounded. In other words, if A is a Tate Huber ring with pseudo-uniformizer q, given $a \in A$ there is some $n \in \mathbb{N}$ such that $q^n a$ is a power bounded element. Equivalenly, there is some n such that the map $\mathbb{Z}((q))[T] \to A$ mapping T to a there is a unique extension to a map

$$\mathbb{Z}((q))\langle q^n T\rangle \to A,$$

where $\mathbb{Z}((q))\langle q^n T\rangle$ is the Tate algebra over $\mathbb{Z}((q))$ in the variable $q^n T$. Therefore, if we want to generalize "Tate Huber rings" we would like to work in a category of solid rings over $\mathbb{Z}((q))$ where a similar statement holds.

In order to realize this idea we need a version of Tate algebra generated by a profinite set:

Definition 3.2.1. For $S \in \mathsf{Prof}$ a light profinite set we let $\mathbb{Z}((q))\langle \mathbb{N}[S]\rangle$ be the "free Tate algebra generated by S". More precisely, $\mathbb{Z}((q))\langle \mathbb{N}[S]\rangle = \mathbb{Z}[[q]]\langle \mathbb{N}[S]\rangle[q^{-1}]$ where $\mathbb{Z}[[q]]\langle \mathbb{N}[S]\rangle$ is the q-adic completion of the free solid $\mathbb{Z}[[q]]$-algebra generated by S:

$$\mathbb{Z}[[q]]\langle \mathbb{N}[S]\rangle = (\mathrm{Sym}_{\mathbb{Z}[[q]]}\mathbb{Z}[[q]][S])^{\wedge q}.$$

It would be desirable that being a bounded function is a condition and not extra structure. This is guarantee thanks to the following proposition.

Proposition 3.2.2 ([RC24, Lemma 2.4.7])**.** *The* $\mathrm{Sym}_{\mathbb{Z}((q))}\mathbb{Z}((q))[S]$*-algebra* $\mathbb{Z}((q))\langle\mathbb{N}[S]\rangle$ *is idempotent.*

We can then define bounded solid rings:

Definition 3.2.3. Let A be a solid $\mathbb{Z}((q))$-algebra.

(1) The fullsubring $A^\circ \subset A$ of power-bounded elements is the solid $\mathbb{Z}[[q]]$-algebra whose values at $S \in \mathsf{Prof}$ are given by
$$A^\circ(S) := \mathrm{Map}_{\mathbb{Z}((q))}(\mathbb{Z}((q))\langle\mathbb{N}[S]\rangle, A).$$
(2) The subring $A^b \subset A$ of bounded elements of A is defined as $A^b = A^\circ[\frac{1}{q}]$.
(3) The †-nilradical $\mathrm{Nil}^\dagger(A)$ of thering A is defined to be the A^b-ideal of "elements of norm 0":
$$\mathrm{Nil}^\dagger(A) := \tau_{\geq 0} \varprojlim_{\times q} A^\circ.$$
In other words, $\mathrm{Nil}^\dagger(A) \subset A$ is the full subanima whose connected components are the "elements a" such that $q^n a$ is power bounded for all $n \in \mathbb{N}$.
(4) A solid ring A is said bounded if $A = A^b$. A solid affinoid ring A is bounded if $A^\triangleright$ is bounded. We let $\mathsf{Ring}^b_{\mathbb{Z}((q))}$ be the ∞-category of solid bounded $\mathbb{Z}((q))$-algebras and let $\mathsf{AffRing}^b_{\mathbb{Z}((q))}$ be the ∞-category of bounded affinoid rings over $\mathbb{Z}((q))$.

Example 3.2.4. Let us specialize to the map $\mathbb{Z}((q)) \to \mathbb{Q}_p$ mapping q to p. We can then talk about bounded rings over $\mathbb{Q}_p$.

(1) Any classical complete Tate $\mathbb{Q}_p$-algebra A is a bounded ring. Moreover, if A is a finite type Tate algebra then $\mathrm{Nil}^\dagger(A)$ is the subspace of spectral norm zero elements and so equal to its classical nilpotent radical.
(2) If $A = \mathbb{Q}_p[T]$ is a polynomial algebra then $A^b = \mathbb{Q}_p$ and so it is not a bounded ring.
(3) Let $A = \mathbb{Q}_p\{T\}^\dagger = \varinjlim_n \mathbb{Q}_p\langle\frac{T}{p^n}\rangle$ be the algebra of germs of analytic functions at 0 in the analytic affine line. Then $A = A^b$ is a bounded algebra and $\mathrm{Nil}^\dagger(A) = TA$. In particular its †-nilradical is not nilpotent.

The categories of bounded affinoid rings have the following permanence properties:

Proposition 3.2.5 ([RC24, Proposition 2.6.14])**.** *The category* $\mathsf{AffRing}^b_{\mathbb{Z}((q))} \subset \mathsf{AnRing}_{\mathbb{Z}((q))}$ *of bounded rings is stable under all colimits in analytic* *rings. Moreover,* $\mathsf{AffRing}^b_{\mathbb{Z}((q))}$ *is presentable and a family of generators is given by* $\mathbb{Z}((q))\langle T\rangle_\square$ *and* $\mathbb{Z}((q))\langle \mathbb{N}[S]\rangle$ *for* $S \in \mathsf{Prof}$.

We endow $\mathsf{AffRing}^b_{\mathbb{Z}((q))}$ with the !-topology and let

$$\mathsf{AnStk}^b_{\mathbb{Z}((q))} \subset \mathrm{PSh}(\mathsf{AffRing}^b_{\mathbb{Z}((q))}, \mathsf{Ani})$$

be the full subcategory of Tate-analytic stacks. We can formally define an "analytification functor" as follows:

Definition 3.2.6. Consider the embedding $G : \mathsf{AffRing}^b_{\mathbb{Z}((q))} \to \mathsf{AffRing}_{\mathbb{Z}((q))}$. The pullback or restriction functor

$$G^* : \mathrm{PSh}(\mathsf{AffRing}_{\mathbb{Z}((q))}, \mathsf{Ani}) \to \mathrm{PSh}(\mathsf{AffRing}^b_{\mathbb{Z}((q))}, \mathsf{Ani})$$

sends !-sheaves to !-sheaves and so restricts to the analytification functor

$$G^* : \mathsf{SolidAnStk}_{\mathbb{Z}((q))} \to \mathsf{TateAnStk}_{\mathbb{Z}((q))}.$$

Example 3.2.7. Let us restrict again to bounded rings over $\mathbb{Q}_p$.

(1) Consider $\mathbb{G}_a = \mathrm{AnSpec}\mathbb{Q}_p[T]$ the algebraic affine line over $\mathbb{Q}_p$. Its analytification $G^*\mathbb{G}_a$ sends a bounded algebra A to $A^{\triangleright}(*)$. Now, since A is bounded, $A^{\triangleright} = A^+[\frac{1}{q}]$ and so $\mathbb{G}_a$ represents the same functor as the analytic affine line $\mathbb{G}_a^{\mathrm{an}} = \bigcup_n \mathbb{D}_{\mathbb{Q}_p}(p^n)$ where $\mathbb{D}_{\mathbb{Q}_p}(p^n) = \mathrm{AnSpec}\mathbb{Q}\langle p^n T\rangle_\square$ is the affinoid disc of radius $|p^{-n}|$.

(2) Let X be a scheme over $\mathbb{Q}_p$, then there is a natural map $X^{\mathrm{an}} \to X$ from the analytification of X to X. If X is locally of finite type then this map is actually an open immersion in the sense of locales. If X is proper over X then $X^{\mathrm{an}} = X$ as analytic stacks; this is a geometric incarnation of GAGA.

(3) Let $\mathbb{G}_a^\dagger := \varprojlim_n \mathbb{D}(p^n)$ be the intersection of all discs around 0. Then $\mathbb{G}_a^\dagger$ is affinoid correpresented by the algebra $\mathbb{Q}_p\{T\}^\dagger$ and its values at a bounded algebra A are equal to the †-nilradical

$$\mathbb{G}_a^\dagger(A) = \mathrm{Nil}^\dagger(A)(*).$$

3.3. **The solid stack of norms.** In order to talk about Tate algebras of bounded rings it is not relevant to have a precise pseudo-uniformizer but just to have the "norm" defined by this pseudo-uniformizer. This was part of the motivation for Clausen and Scholze to introduce the gaseous "stack of norms" $\mathfrak{N}$. In the following we shall going to construct by hand the solidfication of the stack of norms using the algebra $\mathbb{Z}((X))$.

3.3.1. *Construction of the stack of norms.*

Definition 3.3.1. Let $S \in \mathsf{Prof}$ be a light profinite set. We let $\mathbb{Z}((q))\langle \mathbb{N}[S]\rangle^{\dagger}$ be the algebra of overconvergent functions of radius 1 generated by S, more precisely,

$$\mathbb{Z}((q))\langle \mathbb{N}[S]\rangle^{\dagger} = \varinjlim_{\epsilon \to 0^+} \mathbb{Z}((q))\langle \mathbb{N}[q^{\epsilon} S]\rangle.$$

Let A be a bounded ring over $\mathbb{Z}((q))$. A map $f : S \to A$ is of norm ≤ 1, written as $|f|_q \leq 1$, if it extends (necessarily uniquely) to a map $\mathbb{Z}((q))\langle \mathbb{N}[S]\rangle^{\dagger} \to A$. We say that f has norm $|f|_q < 1$ if there is $r > 0$ such that f extends to

$$\mathbb{Z}((q))\langle \mathbb{N}[q^{-r} S]\rangle^{\dagger} \to A.$$

Example 3.3.2. Let $\mathbb{A}^{\mathrm{an}}_{\mathbb{Z}((q))}$ be the analytic affine line over $\mathbb{Z}((q))$. Let $\overline{\mathbb{D}}^{\dagger}_{\mathbb{Z}((q))}$ be the overconvergent closed disc of radius 1 and $\mathring{\mathbb{D}}_{\mathbb{Z}((q))}$ the open unit disc. Then an element in a $\mathbb{Z}((q))$-ring $f \in A$ is of norm ≤ 1 if and only if we have a factorization

$$\begin{tikzcd} \mathrm{AnSpec} A \arrow[r, "f"] \arrow[dr, dashed] & \mathbb{A}^1_A \\ & \overline{\mathbb{D}}_A \arrow[u] \end{tikzcd}$$

Similarly, f has norm < 1 if and only if there is a factorization

$$\begin{tikzcd} \mathrm{AnSpec} A \arrow[r, "f"] \arrow[dr, dashed] & \mathbb{A}^1_A \\ & \mathring{\mathbb{D}}^{\dagger}_A \arrow[u] \end{tikzcd}$$

Definition 3.3.1 provides a notion of norm that is analogue to the one appearing in the theory of overconvergent versions of Berkovich geometry. We would like to make the "norm" $|-|_q$ independent of the variable q. Indeed, if A is a bounded ring over $\mathbb{Z}((q))$ and $a \in A$ is such that $|a|_q = 1$ (i.e. a is a unit and $|a|_q \leq 1$ and $|a^{-1}|_q \leq 1$), then $|-|_q = |-|_{aq}$ in the sense that the overconvergent algebras of radius 1 over A defined using the pseudo-uniformizers q and aq are naturally isomorphic. Using this idea we can make the following construction:

Construction 3.3.3. Let $X = \mathrm{AnSpec}\mathbb{Z}((q))$ and for $n \geq 0$ consider the $n+1$-th fold tensor product X^{n+1} over $\mathbb{Z}_{\square}$ with variables $q_0, \ldots, q_n$. Then, for each $i \in [0, n]$ we can fix the projection $\pi_i : X^{n+1} \to X$ and see X^{n+1} as a Tate stack over $\mathbb{Z}((q_i))$. The locus of X^{n+1} where

$0 < |q_j|_{q_i} < 1$ for all $j \neq i$ is an open substack of X^{n+1} that we will denote by $X^{n+1,\mathrm{an}}$. This locus is independent of i, and as $\mathbb{Z}((q_i))$-stack it is isomorphic to an n-th fold product of pointed open unit discs living over $\mathbb{Z}((q_i))$:

$$X^{n+1,\mathrm{an}} = \mathring{\mathbb{D}}^{\times,n}_{\mathbb{Z}((q_i))}$$

where the variables of the polydisc are precisely $\{q_j\}_{j\neq i}$.

Inside $X^{n+1,\mathrm{an}}$ we can look at the locus $X^{n+1,\dagger} \subset X^{n+1,\mathrm{an}}$ where $|q_j|_{q_i} = |q_i|_{q_i}$ for all j. The substack $X^{n+1,\dagger}$ is affinoid and independent of i, as $\mathbb{Z}((q_i))$-stack it is isomorphic to

$$X^{n+1,\dagger} = \mathrm{AnSpec}\mathbb{Z}((q_i))\langle T_j^{\pm 1} \colon j \neq i\rangle^{\dagger}$$

such that $q_j = T_j q_i$.

Then, the groupoid object $(X^{n+1})_{[n]\in\Delta^{\mathrm{op}}}$ stabilizes the subspaces $X^{n+1,\dagger}$ defining a groupoid object $(X^{n+1,\dagger})^{\mathrm{op}}_{[n]\in\Delta}$. The solid stack of norms is then the geometric realization

$$\mathfrak{N} = \varinjlim_{[n]\in\Delta^{\mathrm{op}}} X^{n+1,\dagger}.$$

Remark 3.3.4. The overconvergent algebras $\mathbb{Z}((q))\langle\mathbb{N}[S]\rangle^{\dagger}$ of radius 1, as well as the solid algebras $\mathbb{Z}((q))\langle T\rangle_{\square}$ can be extended to a descent datum over the diagram $(X^{n+1,\dagger})_{[n]\in\Delta^{\mathrm{op}}}$. Thus, they descent to (analytic) rings $\mathfrak{N}\langle\mathbb{N}[S]\rangle^{\dagger}$ and $\mathfrak{N}\langle T\rangle_{\square}$ over the stack of norms. In particular, we can define the categories $\mathsf{Ring}^b_{\mathfrak{N}}$ and $\mathsf{AffRing}^b_{\mathfrak{N}}$ of bounded (affinoid) rings over the stack of norms.

3.3.2. $\mathbb{R}_{>0}$ *action on the stack of norms.* The stack of norms can also be described in terms of a moduli problem. Prove statements in this section

Definition 3.3.5. Let A be a solid analytic ring. A norm on A is a map of analytic stacks

$$|T| : \mathbb{P}^1_A \to [0,\infty]$$

satisfying the following conditions[4]:

(1) We have a factorization

$$\begin{tikzcd} \mathrm{AnSpec}A \arrow[r, dashed] \arrow[d, "\infty"] & \infty \arrow[d] \\ \mathbb{P}^1_A \arrow[r, "|T|"] & {[0,\infty]} \end{tikzcd}$$

[4]Since the analytic stack attach to a condensed set is 0-truncated these are actual conditions and no extra structure.

(2) Let $\mathbb{A}^{\mathrm{an}}_A$ be the pullback $\mathbb{A}^1_A \times_{[0,\infty]} [0,\infty)$. By (1) we have an inclusion $\mathbb{A}^{\mathrm{an}}_A \subset \mathbb{A}^1_A$[5]. Then we have a factorization

$$\begin{tikzcd} \mathbb{A}^{\mathrm{an}}_A \times \mathbb{A}^{\mathrm{an}}_A \arrow[r] \arrow[d, dashed] & \mathbb{A}^1_A \times \mathbb{A}^1_A \arrow[d, "m"] \\ \mathbb{A}^{\mathrm{an}}_A \arrow[r] & \mathbb{A}^1_A \end{tikzcd}$$

where m is the multiplication map.

(3) The following diagrams commute

$$\begin{tikzcd} \mathbb{P}^1_A \arrow[r, "|T|"] \arrow[d, "T^{-1}"] & {[0,\infty]} \arrow[d, "(-)^{-1}"] \\ \mathbb{P}^1_A \arrow[r, "|T|"] & {[0,\infty]} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^{\mathrm{an}}_A \times \mathbb{A}^{\mathrm{an}}_A \arrow[r, "|T|\times|S|"] \arrow[d, "m"] & {[0,\infty) \times [0,\infty)} \arrow[d, "m"] \\ \mathbb{A}^{\mathrm{an}}_A \arrow[r, "T"] & {[0,\infty)} \end{tikzcd}$$

where T^{-1} is the antipode and m is the multiplication map.

(4) We have a factorization

$$\begin{tikzcd} \mathrm{AnSpec} A \otimes_{\mathbb{Z}} \mathbb{Z}[T]_{\square} \arrow[r, dashed] \arrow[d] & {[0,1]} \arrow[d] \\ \mathbb{P}^1_A \arrow[r, "|T|"] & {[0,\infty].} \end{tikzcd}$$

(5) The natural map of analytic stacks

$$\mathrm{AnSpec}(A \otimes_{\mathbb{Z}} \mathbb{Z}[[T]]) \times_{[0,1]} [0,1) \xrightarrow{\sim} \mathbb{P}^1_A \times_{[0,\infty]} [0,1)$$

is an equivalence.

Remark 3.3.6. Let us make some comments in the definition of a norm in a solid analytic ring.

- Condition (1) justs says that the norm of ∞ must be ∞. Together with the involutive property of (3) this is equivalent to the norm of 0 to be 0.
- Condition (2) says that the multiplication of two elements of finite norm has finite norm.
- Condition (3) says that the norm is multiplicative and involutive.

[5]This can be seen in the following way: it suffices to see that the category $\mathscr{D}(\infty \subset \mathbb{P}^1_A)$ obtained by the pullback of $\infty \in [0,\infty]$ contains the category of T^{-1}-torsion modules in $\mathscr{D}(\mathbb{P}^1_A)$. But by hypothesis it contains the skyscraper at sheaf at ∞, and so any extension of those modules.

- Condition (4) tells us that solid elements have norm ≤ 1. Since $\mathbb{Z}[T]_{\square}$ correpresents a ring, one deduces from this the ultrametric inequality on norms $|T+S| \leq \sup\{|T|, |S|\}$.
- Finally, condition (5) guarantees that the elements of norm < 1 are topologically nilpotent.

Theorem 3.3.7 (Clausen and Scholze)**.** *Let $\mathfrak{N}'$ be the pre-stack on solid ring mapping A to the ∞-groupoid*[6] *of norms on A. Then $\mathfrak{N}'$ is an analytic stack. Moreover, consider the map*

$$f : \mathrm{AnSpec}\mathbb{Z}((q)) \to \mathfrak{N}'$$

defining the norm on $\mathbb{Z}((q))$ such that $\mathbb{P}^1_{\mathbb{Z}((q))} \times_{[0,\infty]} [1, 1/2]$ is the analytic spectrum of the overconvergent algebra $\mathbb{Z}((q))\langle \frac{T}{q} \rangle^{\dagger}$ (i.e. making $|q| = 1/2$). Then f is surjective and induces an equivalence of analytic stacks

$$\mathfrak{N} \xrightarrow{\sim} \mathfrak{N}'.$$

Proof. □

Corollary 3.3.8. *The stack of norms $\mathfrak{N}$ has a natural action of the one parameter group $\mathbb{R}_{>0}$. Moreover, the surjective map*

$$\mathrm{AnSpec}\mathbb{Z}((q)) \to \mathfrak{N}/\mathbb{R}_{>0}$$

has by Čech nerve the groupoid $(X^{n+1,\mathrm{an}})_{[n]\in\Delta^{\mathrm{op}}}$ as in Construction 3.3.3.

Proof. □

Remark 3.3.9. The quotient stack $\mathfrak{N}/\mathbb{R}_{>0}$ behaves like the point $* = \mathrm{Spd}\mathbb{F}_p$ in the v-site of perfectoid spaces in characteristic p. Indeed, if S is an affinoid perfectoid space we have that

$$S \times \mathbb{F}_p((t^{1/p^{\infty}})) = \mathring{\mathbb{D}}_S^{\times,\mathrm{perf}}$$

is the perfectoid pointed open unit disc over S.

3.3.3. *Underlying Berkovich spaces.* Prove statements in this section

Now that we have a norm stack we can start defining Berkovich spaces for analytic stacks over $\mathfrak{N}$.Let A be a solid analytic ring over $\mathfrak{N}$. Given $a \in A^{\triangleright}(*)$ we have maps

$$|a| \colon \mathrm{AnSpec}A \xrightarrow{a} \mathbb{A}^1_A \subset \mathbb{P}^1_A \xrightarrow{|T|} [0, \infty].$$

Definition 3.3.10. Let A be as before and consider the map of analytic stacks

$$\mathrm{AnSpec}A \xrightarrow{\prod |a|} \prod_{a\in A^{\triangleright}(*)} [0, \infty].$$

[6]Since $[0, \infty]$ is 0-truncated this ∞-groupoid is just a set.

The Berkovich space $\mathcal{M}(A)$ is the coplement of the maximal compact open subspace with empty fiber. In other words, $\mathcal{M}(A)$ is the closed subspace of points x such that $\prod |a|^{-1}(x) \neq \emptyset$.

Definition 3.3.10 is justified by the following lemma.

Lemma 3.3.11. *Let S be a metrizable compact Hausdorff space seen as an analytic stack and let* $\mathrm{AnSpec} A \to S$ *be a map of analytic stacks. Let $s \in S$ be such that* $\mathrm{AnSpec} A \times_S s = \emptyset$*, then there is an open subspace $s \in U \subset S$ such that* $\mathrm{AnSpec} A \times_S U = \emptyset$.

Proof. □

In order to define Berkovich spaces for analytic stacks over $\mathfrak{N}$ we need the following result:

Lemma 3.3.12. *Let $A \to B$ be a !-cover of solid analytic rings over $\mathfrak{N}$. Then the map of Berkovich spaces $\mathcal{M}(B) \to \mathcal{M}(A)$ is surjective.*

Proof. Follows since !-covers are not empty. □

Proposition 3.3.13. *Let*

$$\mathcal{M} : \mathsf{AnRing}_{\mathfrak{N}}^{\mathrm{op}} \to \mathsf{AnStk}$$

be the functor sending $\mathrm{AnSpec} A$ *to its Berkovich space $\mathcal{M}(A)$ and consider its left Kan extension*

$$\widetilde{\mathcal{M}} : \mathrm{PSh}(\mathsf{AnRing}_{\mathfrak{N}}^{\mathrm{op}}, \mathsf{Ani}) \to \mathsf{AnStk}.$$

Then $\widetilde{\mathcal{M}}$ localizes to a functor of analytic stacks over $\mathfrak{N}$

$$\mathrm{PSh}(\mathsf{AnRing}_{\mathfrak{N}}^{\mathrm{op}}, \mathsf{Ani}) \to \mathsf{AnStk}_{/\mathfrak{N}} \xrightarrow{\mathcal{M}} \mathsf{AnStk}.$$

Proof. □

Fill examples of Berkovich spaces

Example 3.3.14 (Berkovich space of $\mathfrak{N}$)**.**

Example 3.3.15 (Berkovich space of an adic space)**.**

Introduce derived adic spaces

3.4. **Derived adic spaces.**

4. The analytic de Rham stack

The goal of this lecture is to construct the analytic de Rham stack over $\mathbb{Q}_p$ as in [RC24] and to extend it to a mixed characteristic analytic de Rham stack over the solid stack of norms.

4.1. **Simpson's algebraic de Rham stack.** As preparation for he forthcoming constructions let us recall how Simpson's algebraic de Rham stack is defined. Throughout this section we fix a field K of characteristic 0.

Definition 4.1.1. Let Alg_K be the category of K-algebra and let X be a variety over K. The algebraic de Rham stack of X is the functor

$$X^{\mathrm{dR,alg}} \colon \mathsf{Alg}_K \to \mathsf{Sets}$$
$$A \mapsto X(A^{\mathrm{red}})$$

where $A^{\mathrm{red}} = A/\mathrm{Nil}(A)$ is the reduction of A.

Algebraic de Rham stacks of smooth varieties encode the theory of algebraic D-modules in their theory of quasi-coherent sheaves. One can explicitly construct the algebraic de Rham stack of the affine line $\mathbb{G}_a = \mathrm{Spec} K[T]$:

$$\mathbb{G}_a^{\mathrm{dR,alg}} = \mathbb{G}_a/\widehat{\mathbb{G}}_a$$

where $\widehat{\mathbb{G}}_a$ is the formal completion of $\mathbb{G}_a$ at 0 (considered as an ind-scheme) acting by translations on $\mathbb{G}_a$. More generally, if X is a smooth scheme over K then the natural map

$$X \to X^{\mathrm{dR,alg}}$$

is an epimorphism and the Čech nerve is given by $(X^{n+1,\widehat{\Delta}})_{[n]\in\Delta^{\mathrm{op}}}$, where $X^{n+1,\widehat{\Delta}}$ is the formal completion of the diagonal $X^{\to}X^{n+1}$.

4.1.1. *Site theoretical construction.* We will present an equivalent construction of the de Rham stack in the same spirit of the definition of the prismatization functor in Section 1.2. For this, we need to introduce (a variant of) the infinitesimal site in terms of a suitable category of pairs as in [Mao21].

Definition 4.1.2. Let AniAlg_K be the ∞-category of animated K-algebras. A pair is an arrow $A \to B$ of animated algebras that is surjection on π_0. Given a pair $A \to B$ we let $I = \mathrm{fib}(A \to B)$ be its ideal. We say that a pair is locally nilpotent if the image of I in $\pi_0(A)$ is a locally nilpotent ideal. We let $\mathsf{Pair}_K \subset \mathrm{Func}(\Delta^1, \mathsf{AniAlg}_K)$ be the full subcategory of pairs and $\mathsf{Pair}_K^{\mathrm{lnil}} \subset \mathsf{Pair}_K$ be the full subcategory of locally nilpotent pairs.

Let us now consider the following functors:

- Let $G : \mathsf{Pair}_K^{\mathrm{lnil}} \to \mathsf{AniAlg}_K$ be the functor mapping a pair $A \to B$ to B. It induces a restriction functor

$$G_* : \mathrm{Func}(\mathsf{AniAlg}_K, \mathsf{Ani}) \to \mathrm{Func}(\mathsf{Pair}_K^{\mathrm{lnil}}, \mathsf{Ani}).$$

This functor carries the same information as (a slightly different version of) the infinitesimal site.

- Now let $F : \mathsf{Pair}_K^{\mathrm{lnil}} \to \mathsf{AniAlg}_K$ be the functor mapping $A \to B$ to A. It induces a left Kan extension

$$F_! : \mathrm{Func}(\mathsf{Pair}^{\mathrm{lnil}}, \mathsf{Ani}) \to \mathrm{Func}(\mathsf{AniAlg}_K, \mathsf{Ani})$$

which is the unique colimit preserving object mapping $\mathrm{Spec}(A \to B) \mapsto \mathrm{Spec}A$.

Definition 4.1.3. Consider the diagram

$$\begin{tikzcd} \mathrm{Func}(\mathsf{Pair}_K^{\mathrm{lnil}}, \mathsf{Ani}) \arrow[r, "F_!"] & \mathrm{Func}(\mathsf{AniAlg}_K, \mathsf{Ani}) \\ \mathrm{Func}(\mathsf{AniAlg}_K, \mathsf{Ani}). \arrow[u, "G_*"] & \end{tikzcd}$$

Then, given a prestack $X \in \mathrm{Func}(\mathsf{AniAlg}_K, \mathsf{Ani})$, its algebraic de Rham prestack is given by

$$X^{\mathrm{dR,alg}} := F_! G_* X.$$

To verify that both definitions of algebraic de Rham stacks agree we need the following easy lemma:

Lemma 4.1.4. *The functor $F : \mathsf{Pair}_K^{\mathrm{lnil}} \to \mathsf{AniAlg}_K$ mapping $A \to B$ to A has by right adjoint the functor*

$$R \mapsto (R \to R^{\mathrm{red}}).$$

Lemma 4.1.4 implies that for $Y \in \mathrm{Func}(\mathsf{Pair}^{lnil}, \mathsf{Ani})$ the functor $F_!$ can be computed explicitly as follows:

$$F_! Y(R) = Y(R \to R^{\mathrm{red}}).$$

Therefore, for $X \in \mathrm{Func}(\mathsf{AniAlg}_K, \mathsf{Ani})$ one has that

$$F_! G_* X(R) = G_* X(R \to R^{\mathrm{red}}) = X(R^{\mathrm{red}})$$

proving the claim.

4.2. Analytic de Rham stack over $\mathbb{Q}_p$.

Let us now recall the construction of the analytic de Rham stack over $\mathbb{Q}_p$. First, we need to introduce the notion of †-reduced rings.

Definition 4.2.1. A bounded ring A over the stack of norms $\mathfrak{N}$ is said †-reduced if and only if $\mathrm{Nil}^{\dagger}(A) = 0$. Given a bounded ring A we let $A^{\dagger-\mathrm{red}} := A/\mathrm{Nil}^{\dagger}(A)$ be its †-reduction.

The following proposition tells us that †-reduction have similar properties as the usual reduction.

Proposition 4.2.2 ([RC24]). *Let A be a bounded ring over $\mathfrak{N}$. The following hold:*

(1) $\mathrm{Nil}^{\dagger}(A^{\dagger-\mathrm{red}}) = 0$. *In particular, the $\dagger$-reduction of a bounded ring is $\dagger$-reduced.*
(2) *Let $f\colon \mathbb{Z}[T] \to A$ be a morphism of analytic rings. Then f extends to $\mathbb{Z}[T]_{\square}$ (resp. $\mathbb{Z}[[T]]$, resp. $\mathbb{Z}[T^{\pm 1}]$, resp. $\mathfrak{N}\langle T\rangle^{\dagger}$, resp.) if and only if the composite $\mathbb{Z}[T] \to A \to A^{\dagger-\mathrm{red}}$ does so.*

Remark 4.2.3. Proposition 4.2.2 (2) says that the algebras $\mathbb{Z}[T]$, $\mathbb{Z}[T^{\pm 1}]$, $\mathbb{Z}[T]_{\square}$, $\mathbb{Z}[[T]]$ and $\mathfrak{N}\langle T\rangle^{\dagger}$ are $\dagger$-formally smooth, see [RC24, Definition 3.7.2].

We can then define the analytic de Rham stack

Definition 4.2.4. Let $\mathsf{AffRing}_{\mathbb{Q}_p}$ be the ∞-category of bounded affinoid rings over $\mathbb{Q}_p$. Let $X \in \mathrm{Func}(\mathsf{AffRing}_{\mathbb{Q}_p}, \mathsf{Ani})$ be a Tate analytic prestack over $\mathbb{Q}_p$, its analytic de Rham prestack is the functor

$$X^{\mathrm{dR}}\colon \mathsf{AffRing}_{\mathbb{Q}_p} \to \mathsf{Ani}$$

given by

$$X^{\mathrm{dR}}(A) = X(A^{\dagger-\mathrm{red}}).$$

Example 4.2.5. Let us take $\mathbb{G}_a^{\mathrm{an}}$ be the analytic affine line over $\mathbb{Q}_p$. Then its de Rham stack is given by

$$\mathbb{G}_a^{\mathrm{an,dR}} = \mathbb{G}_a^{\mathrm{an}}/\mathbb{G}_a^{\dagger}$$

where $\mathbb{G}_a^{\dagger} = \mathrm{AnSpec}(\mathbb{Q}_p\{T\}^{\dagger})$ is the overconvergent neighborhood of 0 in $\mathbb{G}_a^{\dagger}$.

Example 4.2.6. Let X be a rigid variety over $\mathbb{Q}_p$, it turns out that the natural map $X \to X^{\mathrm{dR}}$ is an epimorphism. Its Čech nerve is given by $(X^{n+1,\dagger-\Delta})_{[n]\in\Delta^{\mathrm{op}}}$, where $X^{n+1,\dagger-\Delta} \subset X^{n+1}$ is the overconvergent neighborhood of the diagonal map $X \to X^{n+1}$.

Example 4.2.7. Let $Y \to X$ be an étale map for rigid spaces, then we have a cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y^{\mathrm{dR}} \arrow[r] & X^{\mathrm{dR}} \end{tikzcd}$$

This is a geometric way to say that étale maps of rigid spaces are actually (locally) formally $\dagger$-étale.

Example 4.2.8. Let $f : X \to Y$ be a smooth map of rigid spaces of relative dimension d, then $f^{\mathrm{dR}} : X^{\mathrm{dR}} \to Y^{\mathrm{dR}}$ is cohomologically smooth and

$$f^! 1_{Y^{\mathrm{dR}}} = 1_{X^{\mathrm{dR}}}[2d].$$

Example 4.2.9. We can attach analytic de Rham stacks to any analytic stack! In particular, we can do so for preperfectoid rings such as $A = \mathbb{Q}_{\shortmid}\langle T^{\pm 1/p^{\infty}}\rangle_{\square}>$ Indeed, if $\mathbb{T}_{\infty,\mathbb{Q}_p} = \mathrm{AnSpec} A$ and $\mathbb{T}_n = \mathrm{AnSpec}\mathbb{Q}_p\langle T^{\pm 1/p^n}\rangle_{\square}$, we have that

$$\mathbb{T}_{\infty}^{\mathrm{dR}} = \varprojlim_{n} \mathbb{T}_n^{\mathrm{dR}}.$$

In particular, the de Rham cohomology of the preperfectoid torus is the colimit of the de Rham cohomology of the finite level tori

$$R\Gamma_{\mathrm{dR}}(\mathbb{T}_{\infty}) = \varinjlim_{n} R\Gamma_{\mathrm{dR}}(\mathbb{T}_n).$$

4.2.1. *Motivation form Cartier duality.* One of the reasons for the introduction of the analytic de Rham stack over $\mathbb{Q}_p$ was to find a geometric incarnation of the theory of $\hat{D}$-modules of Ardakov and Wadsley [AW19, AW18] in condensed mathematics. The reason why one would like the de Rham stack of $\mathbb{G}_a^{\mathrm{an}}$ to be the quotient $\mathbb{G}_a^{\mathrm{an}}/\mathbb{G}_a^{\dagger}$ is thanks to Cartier duality.

Theorem 4.2.10 ([RC24, Theorem 4.3.13])**.** *There is a natural equivalence of stable ∞-categories*

$$\mathrm{FM} : \mathscr{D}(\mathbb{G}_a^{\mathrm{an}}) \xrightarrow{\sim} \mathscr{D}(B\mathbb{G}_a^{\dagger})$$

obtained from a Fourier-Moukai transform.

This tells us that an analytic D-module of $\mathbb{G}_a^{\mathrm{an}}$, i.e. an object in $\mathscr{D}(\mathbb{G}_a^{\mathrm{an}}/\mathbb{G}_a^{\dagger})$ should be given by a module over the Weyl algebra $\mathbb{Q}_p[T, \partial_T]$ such that both variables T and ∂_T localize into the analytic affine line. This is essentially the intuition behind a classical $\hat{D}$-module.

Remark 4.2.11. Usually the Cartier dual of a commutative stack X is defined to be $X^{\vee} := \underline{\mathrm{Hom}}(X, B\mathbb{G}_m)$. One of the most important aspects of Cartier duality is that, under some assumptions on X, the universal line bundle $\mathcal{L}$ living over $X \times X^{\vee}$ induces an equivalence of categories via a Fourier-Moukai transform

$$\mathrm{FM} : \mathscr{D}(X) \xrightarrow{\sim} \mathscr{D}(X^{\vee}).$$

It is not clear (yet) how to compute the stack

$$\underline{\mathrm{Hom}}(B\mathbb{G}_a^{\dagger}, B\mathbb{G}_m),$$

but it is not hard to construct the line bundle

$$\mathbb{G}_a^{\mathrm{an}} \times B\mathbb{G}_a^{\dagger} \to B\mathbb{G}_m$$

and show that the Fourier-Moukai transform is an isomorphism. A natural question is whether (under some conditions on the stacks) having a pairing

$$X \times Y \to B\mathbb{G}_m$$

whose Fourier-Moukai transform is an isomorphism (more specifically one gets an isomorphism in the category of kernels), then the natural map

$$Y \xrightarrow{\sim} X^{\vee}$$

is an isomorphism of analytic stacks.

From now on, we will refer to any version of "Cartier duality" to the situation when we have a pairing $X \times Y \to B\mathbb{G}_m$ whose Fourier-Moukai transform induces an equivalence of categories. Under strong Tannaka duality assumptions in the relevant analytic stacks, this should pin down Y naturally from X.

4.2.2. *Site theoretical construction.* Let us finish this section by defining the analytic de Rham stack via its "infinitesimal site".

Definition 4.2.12. The category of bounded pairs is the full subcategory of arrows $A \to B$ of bounded affinoid rings $\mathsf{PairAff}^b_{\mathbb{Q}_p} \subset \mathrm{Func}(\Delta^1, \mathrm{AffRing}^b_{\mathbb{Q}_p})$ such that

(1) $A^{\triangleright} \to B^{\triangleright}$ is a surjection on π_0.
(2) B has the induced analytic ring structure from A.

We say that a bounded pair $A \to B$ is $\dagger$-nilpotent if the ideal $I = \mathrm{fib}(A^{\triangleright} \to B^{\triangleright})$ lands in the $\dagger$-nilradical of A. We let $\mathsf{PairAff}^{b,\dagger}_{\mathbb{Q}_p} \subset \mathsf{PairAff}^b_{\mathbb{Q}_p}$ be the full subcategory of $\dagger$-nilpotent pairs.

Let us now consider the following functors

- $G : \mathsf{PairAff}^{b,\dagger}_{\mathbb{Q}_p} \to \mathsf{AffRing}^b_{\mathbb{Q}_p}$ mapping a $\dagger$-nilpotent pair $A \to B$ to B.
- $F : \mathsf{PairAff}^{b,\dagger}_{\mathbb{Q}_p} \to \mathsf{AffRing}^b_{\mathbb{Q}_p}$ mapping a $\dagger$-nilpotent pair $A \to B$ to A. This functor has by right adjoint the funcor mapping a bounded ring R to the pair $R \to R^{\dagger-\mathrm{red}}$.

Then, by defining the restriction G_* and the left Kan extension $F_!$ respectively, the analytic de Rham stack is given by

$$X^{\mathrm{dR}} = F_! G_* X.$$

4.3. **The de Rham Fargues-Fontaine stack.**

5. Analytic de Rham stack in mixed characteristic

5.1. **The solid de Rham stack.** In algebraic geometry we have both de Rham stacks in characteristic 0 and characteristic p. However, the ring stacks attached to $\mathbb{G}_a$ look very different. Indeed, one has

$$\mathbb{G}_{a,\mathbb{Q}}^{\mathrm{dR,alg}} = \mathbb{G}_{a,\mathbb{Q}}/\widehat{\mathbb{G}}_a^{\mathrm{formal}}$$

where $\widehat{\mathbb{G}}_a^{\mathrm{formal}}$ is the formal completion at zero seen as an ind-scheme, and

$$\mathbb{G}_{a,\mathbb{F}_p}^{\mathrm{dR,alg}} = \mathbb{G}_{a,\mathbb{F}_p}/\mathbb{G}_{a,\mathbb{F}_p}^{\sharp},$$

where $\mathbb{G}_{a,\mathbb{F}_p}^{\sharp}$ is the PD-envelope at 0 of $\mathbb{G}_a$.

A natural question is whether there is an interpolated version of the de Rham stack that recovers slightly different versions of the previous ones. This is indeed possible:

Theorem 5.1.1 (In progress, Aoki-R.-Zavyalov)**.** *There is a ring stack $\mathbb{G}_{a,\square}^{\mathrm{dR}}$ over $\mathbb{Z}_\square$ that induces a transmutation functor $X \mapsto X^{\mathrm{dR}}$ from schemes to solid analytic stacks. Explicitly, we have*

$$\mathbb{G}_{a,\square}^{\mathrm{dR}} = \mathbb{G}_{a,\square}/\widehat{\mathbb{G}}_a^{\sharp}$$

where $\mathbb{G}_{a,\square} = \mathrm{AnSpec}\mathbb{Z}[T]_\square$ is the solid affine line and $\widehat{\mathbb{G}}_a^{\sharp} = \mathrm{AnSpec}\mathbb{Z}[[T]]^{PD}$ is the "topologically nilpotent PD-completion at zero of $\mathbb{G}_a$". More precisely,

$$\mathbb{Z}[[T]]^{PD} = \prod_n (\mathbb{Z}\frac{T^n}{n!})$$

is the completion for the PD-filtration of $\mathbb{Z}[T]^{PD}$.

Again, the motivation to consider such solid de Rham stack comes from Cartier duality.

Theorem 5.1.2 (In progress, Aoki-R.-Zavyalov)**.** *There is a natural equivalence of stable ∞-categories*

$$\mathrm{FM} : \mathscr{D}(\mathbb{G}_{a,\square}) \xrightarrow{\sim} \mathscr{D}(B\widehat{\mathbb{G}}_a^{\sharp})$$

obtained from a Fourier-Moukai transform.

Remark 5.1.3. Note that the ring $\mathbb{Z}[[T]]^{PD}$ is the dual of $\mathbb{Z}[T]$ when endowed with its additive Hopf-algebra structure. This makes a guess of what the Cartier dual of $\mathbb{G}_{a,\square}$ should look like. Then the content of Theorem 5.1.2 is to construct the line bundle and prove that the Fourier-Moukai transform is an isomorphism.

In other words, an object in $\mathscr{D}(\mathbb{G}_{a,\square}^{\mathrm{dR}})$ will end up being a module over the Weyl algebra $\mathbb{Z}[T,\partial_T]$ whose restriction to both $\mathbb{Z}[T]$ and $\mathbb{Z}[\partial_T]$ is solid.

Example 5.1.4. When specializing in characteristic p, we have a map

$$\mathbb{G}_{a,\mathbb{F}_p,\square}^{\mathrm{dR}} \to \mathbb{G}_{a,\mathbb{F}_p}^{\mathrm{dR,alg}}$$

from the solid de Rham stack to the algebraic de Rham stack. The reason is that the map of PD-envelopes goes in the direction

$$\widehat{\mathbb{G}}_a^{\sharp} \to \mathbb{G}_a^{\sharp}.$$

Example 5.1.5. In characteristic zero, we can base change the solid de Rham stack to ultrasolid $\mathbb{Q}$. This is the analytic ring such that for a profinite set $S = \varprojlim_n S_n$ one has

$$\mathbb{Q}_{\square\square}[S] = \varprojlim_n \mathbb{Q}[S_n].$$

Then,

$$\widehat{\mathbb{G}}_a^{\sharp} \otimes_{\mathbb{Z}_\square} \mathbb{Q}_{\square\square} = \mathrm{AnSpec}(\mathbb{Q}[[T]])$$

is the analytic spectrum of a power series ring over $\mathbb{Q}$ with the ultrasolid structure. Let us write $\widehat{\mathbb{G}}_{a,\square} := \mathrm{AnSpec}(\mathbb{Q}[[T]])$. Then, we have a map of de Rham stacks in the other direction

$$\mathbb{G}_{a,\mathbb{Q}}^{\mathrm{dR,alg}} \to \mathbb{G}_{a,\mathbb{Q},\square}^{\mathrm{dR}}$$

since this time we have a map

$$\widehat{\mathbb{G}}_a^{\mathrm{formal}} \to \widehat{\mathbb{G}}_{a,\square}$$

from the formal completion to the spectrum of the power series ring.

5.1.1. *Cartier dual of the analytic affine line.* We would like to have some analogue of the solid de Rham stack with a more analytic flavour. In other words, we would like to have a globally defined analytic de Rham stack over the stack of norms $\mathfrak{N}$. Moreover, motivated from the Cartier duality Theorems 4.2.10 and 5.1.2, we would like that the de Rham stack of the affine line $\mathbb{G}_a^{\mathrm{an}}$ has the shape

$$\mathbb{G}_a^{\mathrm{an,dR}} = \mathbb{G}_a^{\mathrm{an}}/\mathbb{G}_a^{?},$$

where $\mathbb{G}_a^{?}$ is ought to be the Cartier dual of $\mathbb{G}_a^{\mathrm{an}}$. Combining Theorems 4.2.10 and 5.1.2 one arrives to the following guess:

Theorem 5.1.6 (Expected). *Let $\mathbb{G}_{a,\mathbb{Z}((q))}^{\dagger,\sharp}$ be the analytic stack over $\mathbb{Z}((q))$ given by the analytic spectrum of*

$$\mathbb{Z}((q))\{T\}^{\dagger,\mathrm{PD}} = \varinjlim_{r\to\infty} \mathbb{Z}((q))\langle \frac{T}{q^r} \rangle^{PD},$$

where $\mathbb{Z}((q))\langle \frac{T}{q^r}\rangle^{\mathrm{PD}} = \widehat{\bigoplus}_n \mathbb{Z}((q))\langle \frac{T^n}{q^{rn} n!}\rangle$ *is the q-adically complete PD-envelope of* $\mathbb{Z}((q))\langle \frac{T}{q^r}\rangle$ *at* 0*. Then there is an equivalence of categories*

$$\mathscr{D}(\mathbb{G}_{a,\mathbb{Z}((q))}^{\mathrm{an}}) \cong \mathscr{D}(B\mathbb{G}_{a,\mathbb{Z}((q))}^{\dagger,\sharp}) \tag{5.1}$$

induced by a Fourier-Moukai transform. Moreover, $\mathbb{G}_{a,\mathbb{Z}((q))}^{\dagger,\sharp}$ *and the equivalence* (5.1) *have a natural descent datum to the solid stack of norms* $\mathfrak{N}$*.*

Remark 5.1.7. In Theorem 5.1.6 we used Tate algebras to define $\mathbb{Z}((q))\{T\}^{\mathrm{PD}}$. Instead we could have done the following: let $\mathbb{Z}[[T]]^{\mathrm{PD}}$ be the topologically nilpotent PD ring of Theorem 5.1.1. Its base change to $\mathbb{Z}((q))$ produces the ring

$$\mathbb{Z}[[q]][[T]]^{\mathrm{PD}}[\frac{1}{q}]$$

where

$$\mathbb{Z}[[q]][[T]]^{\mathrm{PD}} = \prod_n (\mathbb{Z}[[q]]\frac{T^n}{n!}).$$

Then, one also has that

$$\mathbb{Z}((q))\{T\}^{\dagger,\mathrm{PD}} = \varinjlim_r \mathbb{Z}((q)) \otimes_{\mathbb{Z}_\square} \mathbb{Z}[[\frac{T}{q^r}]]^{\mathrm{PD}}.$$

Indeed, we have factorizations

$$\mathbb{Z}((q))\langle \frac{T}{q^r}\rangle^{\mathrm{PD}} \to \mathbb{Z}((q)) \otimes_{\mathbb{Z}_\square} \mathbb{Z}[[\frac{T}{q^r}]]^{\mathrm{PD}} \to \mathbb{Z}((q))\langle \frac{T}{q^{r+1}}\rangle^{\mathrm{PD}}.$$

5.2. **The global analytic (solid) de Rham stack.** Now that we have already made our guesses of the global definition of an analytic de Rham stack, we need to work a little bit in order to construct the ring stack $\mathbb{G}_a^{\mathrm{an,dR}} = \mathbb{G}_a^{\mathrm{an}}/\mathbb{G}_a^{\dagger,\sharp}$. Moreover, it would be much better if a framework such as the site theoretic construction of Section 4.2.2 is available. In the case of the algebraic de Rham stack in characteristic p this can be made thanks to the theory of animated PD-pairs of [Mao21]. Our next goal is to find an analogue theory of overconvergent PD-pairs over the stack of norms $\mathfrak{N}$. For simplicity, we will sketch the construction of that category only over $\mathbb{Z}((q))$, a further descent along $\mathrm{AnSpec}\mathbb{Z}((q)) \to \mathfrak{N}$ should be required.

To motivate the construction of overconvergent PD-pairs let us recall the following result of Mao

Theorem 5.2.1 ([Mao21])**.** *The category of (animated) pairs* Pair *has compact projective generators given by the pairs*

$$\mathbb{Z}[\underline{T}, \underline{S}] \to \mathbb{Z}[\underline{S}]$$

where $\underline{T}$ and $\underline{S}$ are finite sets of variables.

Remark 5.2.2. The pair $\mathbb{Z}[S] \to \mathbb{Z}[S]$ correpresents the functor mapping a pair $A \to B$ to A. On the other hand, the pair $\mathbb{Z}[T] \to \mathbb{Z}$ correpresents the ideal $I = \mathrm{fib}(A \to B)$.

The idea to define animated PD-pairs is by animation out of what the compact projective generators should be. In the 1-category of PD-pairs (A, I, γ), the PD-pair $(\mathbb{Z}[S], 0, \gamma)$ correpresents the underlying ring A. Similarly, the PD-pair $(\mathbb{Z}[T]^{\mathrm{PD}}, I^{[1]}, \gamma)$ (with $I^{[1]} = \ker(\mathbb{Z}[T]^{PD} \to \mathbb{Z})$) correpresents the underlying PD-ideal I. Then, one would like to define animated PD-pairs by taking the animation of the category generated by finite pushouts from these two PD-rings. The following lemma says that this is a reasonable procedure:

Lemma 5.2.3. *Let $\mathscr{C}^{\mathrm{PD}} \subset \mathrm{Pair}$ be the full subcategory of pairs given by*

$$\mathbb{Z}[\underline{T}]^{\mathrm{PD}} \otimes \mathbb{Z}[S] \to \mathbb{Z}[S].$$

Then $\mathbb{Z}[T]^{\mathrm{PD}} \to \mathbb{Z}$ correpresents the ideal $I = \mathrm{fib}(A \to B)$ in the category $\mathscr{C}$.

Definition 5.2.4. Let $\mathscr{C}^{\mathrm{PD}}$ be as in Lemma 5.2.3. The category of animated PD-pairs is given by the animation of $\mathscr{C}^{\mathrm{PD}}$

$$\mathsf{PDPair} := \mathrm{sInd}(\mathscr{C}^{\mathrm{PD}}).$$

We write $(A \to B, \gamma)$ for an object in PDPair and call γ its PD structure.

Remark 5.2.5. Features such as PD-envelopes are automatic from the definition: there is a natural forgetful functor

$$\mathsf{PDPair} \to \mathsf{Pair}$$

just by extending the immersion $\mathscr{C}^{\mathrm{PD}} \subset \mathsf{Pair}$ by sifted colimits. This functor preserves both limits and colimits, and its left adjoint is precisely the animated PD-envelope.

Remark 5.2.6. Consider the forgetful functor $\mathsf{PDPair} \to \mathsf{Ring}$ mapping an animated PD pair $(A \to B, \gamma)$ to A. This functor commutes with colimits and has by right adjoint G the functor mapping a ring R to

$$G(R) = (R \to R/\mathbb{G}_a^\sharp(R), \gamma).$$

Hence, the construction of the category of animated PD pairs is a way to prove that $\mathbb{G}_a/\mathbb{G}_a^\sharp$ is indeed a ring stack.

In the situation of the analytic ring stack, we want to construct a category $\mathsf{PDAff}^{b,\dagger}$ of $\dagger$-nilpotent PD-pairs over the solid stack of norms $\mathfrak{N}$ with the following desiderata:

0. The category $\mathsf{PDAff}^{b,\dagger}$ should be presentable (though not necessarily animated).
i. There should be a forgetful functor
$$\mathsf{PDAff}^{b,\dagger} \to \mathsf{PairAff}^{b,\dagger}$$
from †-nilpotent PD-pairs to †-nilpotent pairs. This functor should have both a left and right adjoint (eq. it should commute with limits and colimits).
ii. The pair $\mathbb{Z}((q))\{T\}^{\dagger,\sharp} \to \mathbb{Z}((q))$ should have a natural upgrade to an object in $\mathsf{PDAff}^{b,\dagger}_{/\mathbb{Z}((q))}$ correpresenting the points of the underlying ideal of the pair. Moreover, for each profinite set S we want algebras $\mathbb{Z}((q))\{\mathbb{N}[S]\}^{\dagger,\sharp}$ correpresenting the S-points of the ideal. We set $\mathbb{G}^{\dagger,\sharp}_{a,S,\mathbb{Z}((q))} = \mathrm{AnSpec}\mathbb{Z}((q))\{\mathbb{N}[S]\}^{\dagger,\sharp}$ and let $\mathbb{G}^{\dagger,\sharp}_{a,S}$ be its descend to the stack of norms $\mathfrak{N}$.
iii. The right adjoint of the functor $\mathsf{PDAff}^{b,\dagger} \to \mathsf{AffRing}^{b}$ mapping a †-nilpotent PD-pair $(A \to B, \gamma)$ to A has by right adjoint the functor mapping R to
$$(R \to R/\underline{\mathbb{G}_a}^{\dagger,\sharp}(R), \gamma),$$
where $\underline{\mathbb{G}_a}^{\dagger,\sharp}$ is the condensed stack (i.e. stack with values in condensed anima) given by the collection of analytic stacks $(\mathbb{G}^{\dagger,\sharp}_{a,S})_{S\in\mathsf{Prof}}$.

Suppose we have access to such category over the solid stack of norms $\mathfrak{N}$, then we can formally defined the global analytic de Rham stack as follows:

Construction 5.2.7. We define the functors
$$F : \mathsf{PDAff}^{b,\dagger} \to \mathsf{AffRing}^{b}$$
$$(A \to B, \gamma) \mapsto A$$
and
$$G : \mathsf{PDAff}^{b,\dagger} \to \mathsf{AffRing}^{b}$$
$$(A \to B, \gamma) \mapsto B.$$

Define the category of †-PD pre-stacks $\mathsf{PDPAnStk}^{b,\dagger} := \mathrm{Fub}(\mathsf{PDAff}^{b,\dagger}, \mathsf{Ani})$. And consider the diagram
$$\begin{array}{ccc} \mathsf{PDPAnStk}^{b,\dagger} & \xrightarrow{F_!} & \mathsf{TateAnPStk} \\ {\scriptstyle G_*}\big\uparrow & & \\ \mathsf{TateAnPStk} & & \end{array}$$
induced from the restriction G_* and the left Kan extension $F_!$.

For $X \in \mathsf{TateAnPStk}$ its analytic de Rham prestack is given by

$$X^{\mathrm{dR}} = F_! G_* X.$$

Remark 5.2.8. Thanks to the desiderata iii. we will formally have that

$$\mathbb{G}_a^{\mathrm{an,dR}} = \mathbb{G}_a^{\mathrm{an}} / \mathbb{G}_a^{\dagger,\sharp}$$

proving that it is indeed a ring stack. To verify that this construction is reasonable for general rigid spaces one needs to prove étale descent and a suitable overconvergent deformation condition for the rigid ball, see [RC24, Proposition 3.7.5]. Moreover, the existence of the left adjoint of desiderata i. says that one has a natural notion of †-PD envelopes in bounded rings that can be used to compute de Rham stacks of closed immersions.

5.3. **Construction of overconvergent PD-pairs.** In this section we shall sketch the construction of the category of overconvergent PD-pairs needed to define the global analytic de Rham stack over $\mathbb{Z}((q))$.

Construct the category of overconvergent PD pairs

6. Analytic prismatization

In this final lecture we construct the analytic prismatization of p-adic Tate analytic stacks. As a warm up we first will construct the $\mathbb{Q}_p$-base change of the analytic prismatization, it turns out that this version has a very simple description using some †-nilpotent thickenings of Fargues-Fontaine curves. We then discuss how the prismatization can be extended to the locus $|p| < 1$ of the stack of norms using different versions of Witt vectors.

6.1. **Nil-perfectoid rings.** An important tool in the p-complete theory of the prismatization is descent. The prismatization of a p-complete ring might in general not be connective. However, it is for the class of semiperfectoid rings, namely, quotients of perfectoid rings. Moreover, prismatic cohomology satisfies quasi-syntomic descent, and any p-complete animated ring has a quasi-syntomic cover given by a semiperfectoid ring. Thus, in order to define and to give some conceptual proofs of the prismatization or prismatic cohomology, one can just restrict to semi-perfectoid rings.

In the analytic prismatization, we know that the stacks we want to evaluate and want to produce ought to be Tate stacks over the stack of norms, in particular they will be modelled by bounded rings. However, not all the bounded rings are needed for the actual applications in analytic p-adic Hodge theory, so one can be a little flexible by taking suitable subcategories of bounded rings.

For us, the analytic prismatization will live over the locus $|p| < 1$ if the stack of norms $\mathfrak{N}$. This locus has as Berkovich space the interval $[0, 1)$ given by $|p|$. The locus $1 < |p| < 0$ of $\mathfrak{N}$ is isomorphic to $\mathrm{AnSpec}\mathbb{Q}_p \times (0, 1)$, so we essentially just see bounded rings over $\mathbb{Q}_p$. However, the locus $|p| = 0$ is more involved and non-archimedean fields $\mathbb{F}_p((q))$ map to them.

Let us now introduce the nil-perfectoid rings

Definition 6.1.1. A bounded ring over $\mathfrak{N}_{|p|<1}$ is nil-perfectoid if $A^{\dagger-\mathrm{red}}$ is a perfectoid Tate algebra. We let $\mathsf{NilPerf}$ denote the ∞-category of nil-perfectoid rings.

To justify the introduction of nil-perfectoid rings let us show that they generalize semiperfectoid rings.

Proposition 6.1.2. *Let B be a perfectoid ring and A any quotient of B as solid algebras. Then A is nil-perfectoid.*

Proof. Let $I = \ker(A \to B)$ and let $J \subset B$ be its closure as Banach space. Note that the elements in $J/I \subset A$ are $\dagger$-nilpotent, namely, given $r \in J$ we can write $r = \sum_n s_n$ with $s_n \in I$ converging to 0. But then the class of r in A is the same as the one of $\sum_{n \geq k} s_n$ for all $k \geq 0$, proving that $|r| \leq |s|_n$ for all n and so $|r| = 0$. A similar argument shows that, for any light profinite set S and any map $f : S \to J$ one has $|f| = 0$ (eg. one can approximate f by locally constant maps and then apply the one point case). Therefore, we can assume without loss of generality that A is a Banach ring, and so that it is a classical semiperfectoid ring. By [BS22, Theorem 7.4] the map $A \to A^{\mathrm{perf}}$ from A to its perfectoidization is a quotient, and that $\mathrm{Spa}A = \mathrm{Spa}A^{\mathrm{perf}}$ have the same adic spectrum given by the vanishing locus of the ideal I in $\mathrm{Spa}B$. Let $K = \ker(A \to A^{\mathrm{perf}})$, then for any $r \in K$ the locus where r is invertible is empty (as it is for A^{perf}). This implies that $|r| = 0$ and so that $K = \mathrm{Nil}^{\dagger}(A)$. ☐

Remark 6.1.3. It is plausible that any nil-perfectoid ring has a !-cover by a quotient of a perfectoid ring. In this way, the natural notion of semiperfectoid rings in bounded rings would also provide a basis for the !-topology.

As corollary we deduce that Banach algebras have descendable covers by nil-perfectoids.

Corollary 6.1.4. *Let A be a Banach algebra with a pseudo-uniformizer. Then A has a descendable cover by a semiperfectoid ring, in particular by a nil-perfectoid.*

Proof. We can consider a surjection

$$\mathbb{Z}((q))\langle T_{A_0}\rangle \to A$$

where $A_0 \subset A$ is a ring of definition. Then, after base change along the descendable cover

$$\mathbb{Z}((q))\langle T_{A_0}\rangle \to \mathbb{Z}((q))\langle T_{A_0}^{1/p^\infty}\rangle,$$

we get a semiperfectoid ring. $\square$

Talk about analytic stacks constructed from nil-perfectoid rings

6.2. **$\mathbb{Q}_p$-prisms and prismatization.** We can finally define the analytic prismatization in characteristic 0. Given S an affinoid perfectoid space with pseudo-uniformizer π we let Y_S denote the adic space

$$Y_S = \mathrm{AnSpec}(\mathbb{A}_{\mathrm{inf}}(S))\backslash V(p[\pi]).$$

We let φ_S denote the Frobenius automorphism of Y_S and $X_S = Y_S/\varphi_S^{\mathbb{Z}}$ be the Fargues-Fontaine curve of S. Finally, we let $|Y_S|$ and $|X_S|$ denote the underlying topological spaces.

The key definition is the one of $\mathbb{Q}_p$-generalized FF curve :

Definition 6.2.1 (generalized $\mathbb{Q}_p$-FF curves)**.** A generalized $\mathbb{Q}_p$-FF curve is a triple $(S, Y_S^\dagger, \varphi)$ consisting on an affinoid perfectoid space S in characteristic p, a derived adic space $Y_S^\dagger$ together with an endomorphism φ, and a map of φ-equivariant $\dagger$-nilpotent thickening

$$f : Y_S \to Y_S^\dagger.$$

A morphism of generalized $\mathbb{Q}_p$-FF curves $(S, Y_S^\dagger, \varphi_1) \to (T, Y_T^\dagger, \varphi_2)$ is a morphism of perfectoid spaces $S \to T$ and a commutative square of φ-spaces

$$\begin{tikzcd} Y_S \arrow[r] \arrow[d] & Y_S^\dagger \arrow[d] \\ Y_T \arrow[r] & Y_T^\dagger. \end{tikzcd}$$

We let $\mathsf{FF}_{\mathbb{Q}_p}$ denote the ∞-category of generalized $\mathbb{Q}_p$-Fargues-Fontaine curves. We often just write $Y_S^\dagger$ for the generalized $\mathbb{Q}_p$-FF curve.

The following is the perfect variant of an analytic prism

Definition 6.2.2 (Perfect generalized $\mathbb{Q}_p$-FF curves)**.** A perfect $\mathbb{Q}_p$-generalized FF curve is a pair $(S, X_S^\dagger)$ consisting on an affinoid perfectoid space S in characteristic p, a derived adic space $X_S^\dagger$, and a $\dagger$-nilpotent thickening.

A morphism of perfect $\mathbb{Q}_p$-generalized FF curves $(S, X_S^\dagger) \to (T, X_T^\dagger)$ is a map of perfectoid spaces $S \to T$ together with a commutative square

$$\begin{tikzcd} X_S \arrow[r] \arrow[d] & X_S^\dagger \arrow[d] \\ X_T \arrow[r] & X_T^\dagger. \end{tikzcd}$$

We let $\mathsf{FF}_{\mathbb{Q}_p}^{\mathrm{perf}}$ denote the ∞-category of perfect generalized $\mathbb{Q}_p$-Fargues-Fontaine curves We often write $X_S^\dagger$ for the generalized $\mathbb{Q}_p$-FF curve.

Remark 6.2.3. The idea behind a generalized $\mathbb{Q}_p$-Fargues-Fontaine is to encode deformations of nil-perfectoid rings to Fargues-Fontaine curves in the simplest way possible.

Remark 6.2.4. Let $\mathscr{C} \subset \mathsf{FF}_{\mathbb{Q}_p}$ be the full subcategory consisting on generalized $\mathbb{Q}_p$-Fargues-Fontaine curves such that φ is an isomorphism. Then the functor

$$\mathscr{C} \to \mathsf{FF}_{\mathbb{Q}_p}^{\mathrm{perf}} \colon (S, Y_S^\dagger, \varphi) \mapsto (S, Y_S^\dagger / \varphi^S)$$

is an equivalence of categories. An inverse is given by mapping $(S, X_S^\dagger)$ to the triple $(S, Y_S^\dagger, \varphi)$ where $Y_S^\dagger$ is the pullback

$$\begin{tikzcd} Y_S^\dagger \arrow[r] \arrow[d] & {|Y_S|} \arrow[d] \\ X_S^\dagger \arrow[r] & {|X_S|} \end{tikzcd}$$

and φ is induced from the Frobenius on $|Y_S|$. Hence, there is no ambiguity in the definition of a perfect generalized $\mathbb{Q}_p$-Fargues-Fontaine.

With the introduction of the categories of generalized Fargues-Fontaine curves we can define $\mathbb{Q}_p$-analytic prisms. Let us first introduce the definition of Div^1.

Definition 6.2.5 (The stacks Div^d)**.**
(1) Let $Y_S^\dagger$ be a generalized $\mathbb{Q}_p$-FF curve. A degree d-divisor in $Y_S^\dagger$ is a Cartier divisor $\mathcal{I} \to \mathscr{O}_{Y_S^\dagger}$ whose pullback to the reduction Y_S is a Cartier divisor of degree d. We let $\mathrm{Div}^d \in \mathrm{PSh}(\mathsf{FF}_{\mathbb{Q}_p}, \mathsf{Ani})$ be the prestack of ∞-groupoids of degree d Cartier divisors.
(2) Let $X_S^\dagger$ be a perfect generalized $\mathbb{Q}_p$-FF curve. A degree d-divisor in $X_S^\dagger$ is a Cartier divisor $\mathcal{I} \to \mathscr{O}_{X_S^\dagger}$ whose pullback to the reduction X_S is a Cartier divisor of degree d. We let

$\mathrm{Div}^{d,\varphi} \in \mathrm{PSh}(\mathsf{FF}^{\mathrm{perf}}_{\mathbb{Q}_p}, \mathsf{Ani})$ be the prestack of ∞-groupoids of degree d Cartier divisors.

Add definition of diamond functor!

With no more to introduce, we finally give the definition of the analytic prismatization over $\mathbb{Q}_p$. Let $\mathsf{NilPerf}_{\mathbb{Q}_p}$ be the category of nilperfectoid rings over $\mathbb{Q}_p$. Consider the functor

$$F : \mathsf{FF}_{\mathbb{Q}_p} \to \mathsf{AnPStk}^{\mathrm{nice}} = \mathrm{Func}(\mathsf{NilPerf}_{\mathbb{Q}_p}, \mathsf{Ani})$$

obtained by realizing a derived adic space as a functor in nil-perfectoid rings (see **??**). Also let

$$(-)^{\diamondsuit} : \mathsf{AnPStk}^{\mathrm{nice}} \to \mathrm{PSh}(\mathsf{FF}_{\mathbb{Q}_p}, \mathsf{Ani})_{/\mathrm{Div}^1}$$

be the functor that maps an analytic stack X on nil-perfectoid rings to the stack $X^{\diamond}$ whose values at $Y_S^{\dagger}$ consist on a degree 1-divisor $D \subset Y_S^{\dagger}$ and a map

$$D \to X.$$

Definition 6.2.6 (Analytic prismatization over $\mathbb{Q}_p$)**.** Consider the diagram

$$\begin{tikzcd} \mathrm{PSh}(\mathsf{FF}_{\mathbb{Q}_p}, \mathsf{Ani})_{/\mathrm{Div}^1} \arrow[r, "F_!"] & \mathsf{AnPStk}^{\mathrm{nice}}_{F_!\mathrm{Div}^1} \\ \mathsf{AnPStk}^{\mathrm{nice}} \arrow[u, "(-)^{\diamondsuit}"] & \end{tikzcd}$$

The analytic prismatization of $X \in \mathsf{AnPStk}^{\mathrm{nice}}$ over $\mathbb{Q}_p$ is given by

$$X^{\mathbb{\Delta}} := F_! X^{\diamondsuit}.$$

6.3. Geometry of the prismatization functor over $\mathbb{Q}_p$.

6.4. Analytic prismatization over $\mathbb{Z}_p$.

6.5. Geometry of the prismatization over $\mathbb{Z}_p$.

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