Formalization of Higher Categories

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Abstract

This book aims at introducing higher category theory in an axiomatic way: instead of building higher category theory on top of set theory, as is done in the framework of quasicategories, we will introduce higher category theory synthetically, with the aim of having access to its main features as quickly as possible: the Yoneda embedding, the straightening/unstraightening correspondence, the theory of Kan extensions, etcetera. We will then explore its consequences: the theory of presentable categories, topoi, stable categories, and the basics of K-theory. Our axiomatic approach will not only provide tools to comprehend the important aspect of higher categories as they are used in practice (derived algebraic geometry, homotopical algebra, etc.) but also in more general contexts (e.g. higher category theory internally in any higher topos) and in logic (dependent type theory).

Note by the author

This book has grown out of the author's lecture notes written for the lecture course *Formalization of Higher Categories*, taught by Denis-Charles Cisinski at the University of Regensburg during the winter term of 2023/2024 and the summer term of 2024. These lecture notes were initially written live during the lectures, and have later been carefully revised, expanded and reorganized. The intention is to turn these lecture notes into an actual book at some point in the future, and everything in this book should be considered joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde.

While the material contained in this book is directly based on Cisinski's course, we warn the reader that the organization of the material has completely changed, and that the axioms are presented in different order and different fashion. The presentation of the material in this book strongly reflects the author's personal perspective, which may at times be different from that of Cisinski, Nguyen and Walde.

The most substantial change is that we have chosen to formulate all the axioms from the very beginning in terms of *naive category theory*, rather than using the theory of tribes as was done in Cisinski's lectures. It will only be *after* our development of synthetic category theory that we discuss how to formalize the axioms within the theory of tribes; this will be the content of Part II of this book. This in particular allows the tribe to be a synthetic category itself.

The most recent version of this book can be found here. A previous version of this book, which was still written in terms of tribes, can be found here.

I thank Johannes Gloßner for various useful discussions. I thank Hayato Nasu for pointing out an incorrect statement in Section 14.4.

Warning: various parts of this book are very much unfinished and will be refined and expanded over the course of 2024 and 2025.

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Introduction

The goal of this book is to establish an axiomatic framework for higher category theory. This means that we will not give a *definition* of what a higher category is; rather, we will provide a list of *axioms* that impose the desired behavior of higher categories. These axioms will allow us to build up all the usual notions of higher category theory, similar to how in classical mathematics all mathematical objects are defined in terms of an axiomatic notion called *sets*. To emphasize the new role higher categories play in our theory, we will refer to them as *synthetic categories*.

Motivation

Let us start by motivating why we would want a formalization of higher category theory. There are many different answers to this question, which we can divide into two classes: philosophical and mathematical.

The *philosophical* reason is that it should provide a new foundation for mathematics and logic:

- Various existing foundations of mathematics (set theory, type theories) can be formulated using the language of category theory. Category theory provides natural tools to interpret and compare such formal systems:
  - We want to take the *plurality of foundations* seriously.
  - We would like the possibility of implementation in a proof assistant.
- A useful feature about category theory is that it can speak about itself: it is possible to talk about the *category of (small) categories*.
- By formalizing *higher* category theory, it is possible to get a richer concept of *equality* than in ordinary mathematics, abandoning the strict notion of equality that exists in set theory.

There are also various *mathematical* reasons for wanting a formalization of higher category theory:

- *Synthetic ∞-category theory*: it will make precise what people mean when they say they work 'independently of any model'.
- *Parametrized ∞-category theory*: it will describe the internal language of the ∞-category of sheaves of ∞-categories on any ∞-topos. This will in particular determine the kind of statements and proofs in higher category theory that can be extended to parametrized ∞-category theory (as for example developed by Barwick et al. [Bar+16] for the purpose of genuine equivariant homotopy theory or by Martini and Wolf [MW21] to describe category theory internal to an ∞-topos).
- *Naive ∞-category theory*: it will help to express "naive ∞-category theory" in the same way there is "naive set theory". We should be able to teach how to practice ∞-category theory without needing to go through all the technicalities. From our point of view, the technicalities of ∞-categories that arise in present-day treatments are a consequence of the choice of foundations, namely set theory, which is simply not suited to properly deal with higher categories.
- *Elementary ∞-category theory*: We will explore the scope of "elementary ∞-category theory". Everything will be done constructively, therefore expressing rather universal facts that include all the substantial parts of what we classically know after the work of Lurie and others.

Naive category theory

Our formalization of higher category theory rests upon a certain language of categories, functors and natural isomorphisms that we call *naive category theory*. Just like mathematicians can work with sets without making reference to any specific model for set theory, higher category theorists often write their articles in a 'model-independent' way that does not make reference to any specific model for higher quasicategories. We will now provide a rough overview of the language of naive category theory, leaving the details to Chapter 1.

Axiom A. We may speak of *synthetic categories* $C, D, E, \ldots$, of *functors* $f\colon C \to D$ between synthetic categories, of *natural isomorphisms* $\alpha\colon f \cong g$ between functors, of *isomorphisms*[1] $\alpha \cong \beta$ between natural isomorphisms, of *isomorphisms between isomorphisms* between natural isomorphisms, and so forth. One may compose functors, natural isomorphisms and 'higher' isomorphisms in the expected way.

A functor $f\colon C \to D$ is said to be an *equivalence* if there exists another functor $g\colon D \to C$ and natural isomorphisms $\mathrm{id}_C \cong gf$ and $\mathrm{id}_D \cong fg$.

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[1] In the literature, the word 'homotopy' or 'equivalence' is often used in cases where we use 'isomorphism'.

There are various ways to build new synthetic categories out of old ones. In each case, we will specify the behavior of these constructions by prescribing how to define functors into/out of them and how to define natural isomorphisms between functors into/out of them. To keep the theory simple, we omit the specification of all the expected higher coherences: we will see that we can develop the basic foundations without needing them.[2]

Axiom B. We demand the existence of the following synthetic categories:

(B.1) A *terminal category* $*$: for every C there is a functor $p_C\colon C \to *$, and for any two functors $f,g\colon C \to *$ there is a natural isomorphism $f \cong g$.

(B.2) An *initial category* $\emptyset$: for every C there is a functor $\emptyset \to C$, and for any two functors $f,g\colon \emptyset \to C$ there is a natural isomorphism $f \cong g$.

(B.3) For every two synthetic categories C and D a *product* $C \times D$ with functors $\mathrm{pr}_C\colon C \times D \to C$ and $\mathrm{pr}_D\colon C \times D \to D$: functors $E \to C \times D$ are specified by giving the two components $E \to C$ and $E \to D$, and natural isomorphisms between functors $E \to C \times D$ are specified by giving natural isomorphisms between the components.

(B.4) Similarly there is a *coproduct* $C \sqcup D$ with functors $i_C\colon C \sqcup D$ and $i_D\colon D \to C \sqcup D$: functors $C \sqcup D \to E$ are specified by giving the two components $C \to E$ and $D \to E$, and natural isomorphisms between functors $C \sqcup D \to E$ are specified by giving natural isomorphisms between the components.

(B.5) For synthetic categories C and D there is a *functor category* $\mathrm{Fun}(C,D)$. Every functor $f\colon E \times C \to D$ gives rise to a *curried functor* $f_c\colon E \to \mathrm{Fun}(C,D)$ and conversely every functor $g\colon E \to \mathrm{Fun}(C,D)$ gives rise to an *uncurried functor* $g^u\colon E \times C \to D$, satisfying $f \cong (f_c)^u$ and $g \cong (g^u)_c$. We may similarly curry/uncurry natural isomorphisms between functors. Currying is demanded to be functorial in E.

(B.6) For functors $f\colon C \to E$ and $g\colon D \to E$ there exists a *pullback* $C \times_E D$ with functors $\mathrm{pr}_C\colon C \times_E D \to C$ and $\mathrm{pr}_D\colon C \times_E D \to D$ and a natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$. Functors $T \to C \times_E D$ are specified by giving two components $t\colon T \to C$ and $s\colon T \to D$ together with a natural isomorphism $f \circ t \cong g \circ s$. One may similarly specify natural isomorphisms between two functors $T \to C \times_E D$.

We further demand that finite coproducts are *universal*, in the sense of Axiom B.7 and Axiom B.8.

We define an *absolute object* of a synthetic category C to be a functor $x\colon * \to C$. To speak of *morphisms*, we need another axiom:

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[2] The author is not sure whether this approach could lead to any problems down the road; while this doesn't seem to be the case, any insights suggesting otherwise are very welcome!

Axiom C. There is a synthetic category $[1]$, called the *walking morphism*, which comes equipped with objects $0 \in [1]$ and $1 \in [1]$.

We may then define a morphism in C to be a functor $f\colon [1] \to C$. We refer to the objects $f(0)$ and $f(1)$ of C as the *source* and *target* of f, respectively, and often write $f\colon x \to y$ to indicate that $f(0) = x$ and $f(1) = y$. Similarly, a *natural transformation* of functors $C \to D$ is defined as a morphism in $\mathrm{Fun}(C,D)$.

Every object $x\colon * \to C$ gives rise to an *identity morphism* $\mathrm{id}_x\colon x \to x$, defined as the composite $[1] \xrightarrow{p_{[1]}} * \xrightarrow{x} C$. The definition of *composition* of morphisms in C is a bit more subtle: it will require us to first introduce a notion of commutative triangle in C, and then demand that every two morphisms $f\colon x \to y$ and $g\colon y \to z$ can uniquely be extended to a commutative triangle. To do this, we introduce the notion of *diagram categories* in Section 1.3, based on the formalism of cubes, topes and extension types of Riehl and Shulman [RS17]. We may summarize it as follows:

Axiom D. We have the following structure:

(D.1) For every 'diagram shape' Φ and every synthetic category C there is a *diagram category* $\mathrm{Diag}(\Phi, C)$, which is functorial in C.

(D.2) For every map $\alpha\colon \Psi \to \Phi$ of diagram shapes there is a restriction functor $\alpha^*\colon \mathrm{Diag}(\Phi,C) \to \mathrm{Diag}(\Psi,C)$, satisfying $(\mathrm{id}_\Phi)^* \cong \mathrm{id}$ and $(\beta \circ \alpha)^* \cong \alpha^* \circ \beta^*$.

(D.3) Whenever $\alpha\colon \Psi \to \Phi$ is an isomorphism of diagram shapes, α^* is an equivalence.

(D.4) We have equivalences $\mathrm{Diag}(\Delta^0, C) \xrightarrow{\sim} C$ and $\mathrm{Diag}(\Delta^1, C) \xrightarrow{\sim} \mathrm{Fun}([1], C)$ which are compatible with evaluation at 0 and 1.

(D.5) For diagram shapes Φ and Ψ, there is an equivalence

$$\mathrm{Diag}(\Phi \times \Psi, C) \xrightarrow{\sim} \mathrm{Diag}(\Phi, \mathrm{Diag}(\Psi, C))$$

which is suitably functorial in C, Φ and Ψ.

(D.6) The canonical map $\mathrm{Diag}(\emptyset, C) \to *$ is an equivalence;

(D.7) Diagram categories satisfy excision: For subshapes $\Phi_1, \Phi_2 \subseteq \Phi$, the commutative square

$$\begin{tikzcd}
\mathrm{Diag}(\Phi_1 \cup \Phi_2, C) \arrow[r] \arrow[d] & \mathrm{Diag}(\Phi_1, C) \arrow[d] \\
\mathrm{Diag}(\Phi_2, C) \arrow[r] & \mathrm{Diag}(\Phi_1 \cap \Phi_2, C)
\end{tikzcd}$$

is a pullback square.

We can now formulate the Segal axiom:

Axiom E (Segal axiom)**.** For every synthetic category C, the restriction functor

$$\mathrm{Diag}(\Delta^2, C) \longrightarrow \mathrm{Diag}(\Lambda^2_1, C) \simeq \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C)$$

is an equivalence of synthetic categories.

The Segal axiom allows us to define the *composition* of morphisms in C. Given morphisms $f\colon x \longrightarrow y$ and $g\colon y \longrightarrow z$, the axiom provides an essentially unique diagram $\Delta^2 \longrightarrow C$ which on $[1] \cong \Delta^{\{0,1\}}$ restricts to f and on $[1] \cong \Delta^{\{1,2\}}$ restricts to g. The restriction of this diagram to the subdiagram $[1] \cong \Delta^{\{0,2\}}$ we obtain a morphism in C of the form $gf\colon x \longrightarrow z$, which we call the *composite of f and g*. One can prove that composition of morphisms is automatically unital and associative.

Having both identity morphisms and composition, we may now introduce the notion of *isomorphism* in C: a morphism $f\colon x \longrightarrow y$ in C is an isomorphism if it admits a left inverse $g\colon y \longrightarrow x$ (satisfying $gf \sim \mathrm{id}_x$) as well as a right inverse $h\colon y \longrightarrow x$ (satisfying $fh \sim \mathrm{id}_y$). Using diagram categories, we may in fact construct for every C a synthetic category $\mathrm{Iso}(C)$ whose objects are precisely the isomorphisms in C (equipped with the data of left and right inverses). The canonical example of an isomorphism in C is the identity morphism $\mathrm{id}_x\colon x \longrightarrow x$, in which case the left and right inverses are simply id_x itself. This construction provides a functor $i\colon C \longrightarrow \mathrm{Iso}(C)$, which we demand to be an equivalence:

Axiom F (Rezk axiom)**.** Every isomorphism in a synthetic category C is equivalent to an identity morphism. More precisely, the functor $i\colon C \longrightarrow \mathrm{Iso}(C)$ is an equivalence.

Axioms E and F are inspired by Rezk's notion of *complete Segal spaces*.

The axioms of synthetic category theory

Equipped with a basic language of categories, functors and natural isomorphisms, we can start formulating our axioms of synthetic category theory. As a first piece of structure, we require the existence of *groupoid cores* of synthetic categories. Using the above notion of isomorphisms in a synthetic category C, we say that C is a *groupoid* if all of its morphisms are isomorphisms.

Axiom G (Groupoid core axiom)**.** For every synthetic category C, there is a groupoid $C^{\simeq}$ called the *groupoid core of C*, equipped with a functor $\gamma_C\colon C^{\simeq} \longrightarrow C$. Every functor $f\colon X \longrightarrow C$ from a groupoid X factors through γ_C, and for functors $g, h\colon X \longrightarrow C^{\simeq}$ every natural isomorphism $\gamma_C \circ g \cong \gamma_C \circ h$ may be lifted to a natural isomorphism $g \cong h$.

For synthetic categories C and D, we write $\mathrm{Map}(C,D)$ for the groupoid $\mathrm{Fun}(C,D)^{\simeq}$.

Groupoids will play a fundamental role in our development of synthetic category theory, and we devote Chapter 2 to their study. Among other things, they determine the correct notion of *objects* in a synthetic category. In ordinary category theory, one frequently encounters statements that can be tested 'on objects'; think for example of the fact that a natural transformation is a natural isomorphism if and only if it is an objectwise isomorphism. To get the correct analogue of this notion in synthetic category theory, we must define an *object of a synthetic category C* to be a functor $x\colon \Gamma \to C$ from an *arbitrary* groupoid Γ.

Axiom H. For groupoids X and Y, the coproduct $X \sqcup Y$ is again a groupoid.

We also demand the existence of subcategories, localizations, fundamental groupoids and joins, discussed in detail in Chapter 4:

Axiom I (Subcategory axiom)**.** Consider a synthetic category C equipped with a *collection of morphisms $m\colon M \hookrightarrow \mathrm{Map}([1],C)$* of C which is *closed under composition* (see Definition 4.1.5 for a precise definition of these terms). Then there exists a synthetic category $\langle M\rangle_C$ equipped with a functor $i_M\colon \langle M\rangle_C \to C$ satisfying:

- The induced functor $(i_M)_*\colon \mathrm{Map}([1],\langle M\rangle_C) \to \mathrm{Map}([1],C)$ factors through M;
- The functor i_M is *universal* with respect to the previous property: every other functor $f\colon D \to C$ whose induced functor $f_*\colon \mathrm{Map}([1],D) \to \mathrm{Map}([1],C)$ factors through M admits an essentially unique factorization through $\langle M\rangle_C$.

Axiom J (Localization axiom)**.** For every synthetic category C and every collection of morphisms $W \hookrightarrow \mathrm{Map}([1],C)$, there exists a functor $l\colon C \to C[W^{-1}]$ exhibiting $C[W^{-1}]$ as a localization of C at the morphisms in W:

- The functor l sends the morphisms in W to isomorpisms $C[W^{-1}]$;
- The functor l is *universal* with respect to the previous property: every other functor $f\colon C \to D$ which sends morphisms in W to isomorphisms in D admits an essentially unique factorization through $C[W^{-1}]$.

In the case where W consists of *all* morphisms in C, we denote the localization by $\Pi_\infty(C)$ and call it the *fundamental groupoid of C*.

Axiom K (Fundamental groupoid axiom)**.** For every synthetic category C, the fundamental groupoid $\Pi_\infty(C)$ is a groupoid.

Axiom L (Join axiom)**.** For every pair of synthetic categories C and D, there exists a synthetic category $C \star D$ and a homotopy pushout square

$$\begin{tikzcd}
C \times D \sqcup C \times D \arrow[r] \arrow[d, "\langle \mathrm{pr}_C, \mathrm{pr}_D \rangle"'] & C \times [1] \times D \arrow[d] \\
C \sqcup D \arrow[r] & C \star D.
\end{tikzcd}$$

If we write $[0]$ for the terminal category $*$, we have $[1] \simeq [0] \star [0]$, and we may inductively define $[n+1] := [n] \star [0]$. One can show by induction that functors $[n] \to C$ corresponds to diagrams $\Delta^n \to C$.

Cartesian and cocartesian fibrations

An important concept in category theory is that of a cocartesian fibration: a functor $E \to C$ whose fibers $E(c)$ are in a certain 'covariantly functorial' in the object $c \in C$. In Chapter 5, we introduce a synthetic analogue of this definition: a functor $f\colon A \to B$ is called a cocartesian fibration if a certain 'directed evaluation map' $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$ admits a left adjoint section $\mathrm{lift}_0^f\colon A \vec{\times}_f B \to \mathrm{Fun}([1], A)$. Informally, this left adjoint section provides the cocartesian lifts $a \to \beta_!(a)$ in A for morphisms $\beta\colon b \to b'$ in B with specified lift a of b. One may dually define *cartesian fibrations*.

Axiom M (Exponentiability axiom)**.** Cartesian and cocartesian fibrations are *exponentiable* and exponentiation along (co)cartesian fibrations satisfies base change.

Axiom N (Functoriality of universals axiom)**.** Let $p\colon E \to C$ be a cocartesian (resp. cartesian) fibration. If the fiber $E(x)$ of p admits a terminal (resp. initial) object for every object x of C, then these objects assemble into a section $s\colon C \to E$ of p. Furthermore, this section comes equipped with a transformation $\eta\colon \mathrm{id}_E \to sp$ (resp. $\varepsilon\colon sp \to \mathrm{id}_E$) exhibiting it as a right (resp. left) adjoint of p.

Given a cocartesian fibration $p\colon U_\bullet \to U$, one can formulate what it means for p to satisfy *directed univalence*: roughly speaking, this means that for functors $f_0, f_1\colon C \to U$ with pullback fibrations $E_i := f_i^*(U_\bullet) \to C$, the groupoid of natural transformations $\alpha\colon f_0 \to f_1$ is equivalent to the groupoid of cocartesian functors $E_0 \to E_1$ over C. This axiom will in particular imply a form of *straightening/unstraightening*. A synthetic category U equipped with such a univalent cocartesian fibration p is called a *universe*. We say that U is *bicomplete* if it is closed under all the categorical operations introduced so far; see Definition 7.4.2 for a complete list.

Axiom O (Directed univalence axiom)**.** For every cocartesian fibration $q\colon E \to B$, there exists a bicomplete universe U such that B is U-small and q has U-small fibers.

We will usually fix a bicomplete universe $\pi_{\mathrm{univ}} \colon \mathrm{Cat}_{\bullet} \to \mathrm{Cat}$, work with synthetic categories that are small with respect to this universe, i.e. those categories C that can be obtained from an object c of Cat via the following pullback square:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_{\bullet} \arrow[d, "\pi_{\mathrm{univ}}"] \\
* \arrow[r, "c"] & \mathrm{Cat}.
\end{tikzcd}$$

Organization of the book

This version of the book is very much preliminary and is currently under the active process of being written and rewritten. The organization of the book may fluctuate substantially between different versions, and in particular the organizational overview presented here might no longer reflect the present state of the book. The book should be expected to continue to be updated throughout the course of 2024 and 2025.

We have divided the book into three parts:

(I) In Part I, we provide a synthetic development of (higher) category theory expressed in the basic language of *naive category theory* introduced before. In our exposition, we try to stick as closely as possible to the language employed by researchers working in the field of homotopy theory and higher category theory, avoiding foundational technicalities.

(II) In Part II, we provide a framework for constructing semantic interpretations of synthetic category theory, using Joyal's formalism of *tribes*. In other words: we use category theory to formalize itself. Examples of such semantic interpretations are the $(1, 1)$-category of quasicategories, the synthetic category Cat of small synthetic categories, and the category $\mathrm{Cat}(\mathcal{B})$ of categories internal to some (∞-)topos $\mathcal{B}$.

(III) In Part III, which has not yet been written, we will express our axioms of synthetic category theory within a version of type theory called *crisp simplicial type theory*, which is a combination of Shulman's *crisp type theory* (a.k.a. 'spatial' type theory) [Shu18] and Riehl-Shulman's *simplicial type theory* [RS17].

Part I

Synthetic category theory

1 The language of naive category theory

At the heart of mathematics lies the concept of a set—a collection of mathematical objects. In modern mathematics, sets serve as the building blocks for defining everything from numbers and relations to more complex structures like groups, rings, and topological spaces. Sets are manipulated using basic operations like forming unions, intersections, and complements, and are compared using functions between sets. It is fair to say that the language of set theory forms an essential toolkit for mathematicians.

Although formal axiomatizations of set theory exist, such as those by Zermelo and Fraenkel, they often diverge from the working mathematician's intuitive understanding of sets. For instance, in formal set theory, every natural number needs to be defined as a set, and in particular one is allowed to speak of the elements of a natural number. Similarly, every function between sets needs to be defined as a set. While this approach is formally valid, it can seem disconnected from everyday mathematical discourse.

In practice, mathematicians often use a more pragmatic approach known as *naive set theory*. In naive set theory, one works with sets based on intuitive understanding, without invoking formal logical or axiomatic implementations. For example, the set of natural numbers is simply the set containing the numbers $0, 1, 2$, etcetera, and functions between sets X and Y are simply 'specifications' that assign to every element of X an element of Y. The simplicity and accessibility of naive set theory make it relatively easy for students to learn it, and indeed in many introductory mathematics textbooks sets are used in the naive way without being rigorously introduced.

A similar trend is occurring in the field of higher category theory. In recent decades, higher category theory has seen rapid growth, in large part thanks to contributions from people like Joyal, Lurie, and others. Their work provides solid foundations for the field, and the language of higher categories is now extending beyond its origins in algebraic topology to fields like algebraic geometry and representation theory. But while the original works of Joyal and Lurie are based on explicit set-theoretical models for higher categories called 'quasicategories', more and more research papers are written in a 'model independent' way in which the usage of details about the specific choice of implementations is avoided. In other words, researchers are increasingly adopting a language of *naive category theory*, which enables a more intuitive approach to studying higher categorical structures.

We believe that naive category theory is easier to learn than the theory of quasicategories, just like naive set theory is easier to learn than formal set theory. With this in mind, the goal of Part I of this book is to provide a detailed treatment of the foundations of higher category theory written purely in the language of naive category theory, without reference to specific implementations. The goal of the present chapter is to give a careful introduction to this language: we introduce the relevant terms and explain the manipulations one is allowed to do. In particular, our approach is *synthetic* in nature rather than *analytic*: constructions of categories, like products or functor categories, are not *defined* in terms of set-level constructions, but are *axiomatized* to exist and satisfy the expected properties.

To keep the language as simple as possible, we have chosen to leave out a variety of 'higher coherences' that one would expect higher categorical constructions to have. For example, the terminal category $*$ is axiomatized to come equipped with a functor $p_C\colon C \to *$ for every other synthetic category C and with a natural isomorphism $f \cong g$ for every pair of functors $f, g\colon C \to *$. While the terminal category is supposed to also guarantee that any two such natural isomorphisms are isomorphic to each other, and more generally that any two 'higher' isomorphisms of the same level are isomorphic to each other, it turns out that these higher coherences are not needed to develop the foundational material, and omitting them makes the formalism much easier to work with. That being said, all relevant models for synthetic categories do satisfy these coherent versions of the axioms.

1.1 The external theory of synthetic categories

We start in this section by postulating the abstract properties we demand synthetic categories to have, ignoring for a moment their internal structure.

1.1.1 Categories, functors and natural isomorphisms

At the core of naive category theory are the notions of categories, functors and natural isomorphisms, which will be properly introduced now.

Axiom A.1. The following are the principal terms in the vocabulary of naive category theory:

(0) We may speak of *synthetic categories*, which we denote by symbols like C, D, E, etcetera;

(1) Given two synthetic categories C and D, we may speak of *functors* from C to D, which we denote by $f\colon C \to D$. Every synthetic category C has an *identity functor*

$\mathrm{id}_C\colon C \longrightarrow C$. Given two functors $f\colon C \longrightarrow D$ and $g\colon D \longrightarrow E$, there is a *composite functor* $g \circ f\colon C \longrightarrow E$.

(2) Given two functors $f, f'\colon C \longrightarrow D$, we may speak of *natural isomorphisms* $\alpha\colon f \cong f'$ between f and f'. We may sometimes display natural isomorphisms diagrammatically as follows:

$$\begin{tikzcd} C \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "f'"', ""{name=D}] & D. \arrow[from=U, to=D, Rightarrow, "\cong", "\alpha"'] \end{tikzcd}$$

Every functor f has an *identity isomorphism* $\mathrm{id}_f\colon f \cong f$. Every natural isomorphism α has an *inverse isomorphism* $\alpha^{-1}\colon f' \cong f$. Given two natural isomorphisms $\alpha\colon f \cong f'$ and $\beta\colon f' \cong f''$, there is a *(vertical) composite isomorphism* $\beta \circ \alpha\colon f \cong f''$.

To avoid cluttering of notation, we will frequently work with natural isomorphisms $f \cong f'$ that have not been given an explicit name.

(3+) Given two natural isomorphisms $\alpha, \alpha'\colon f \cong f'$, we may speak of *isomorphisms of natural isomorphisms* $\alpha \cong \alpha'$. For two of those, we may in turn speak of *isomorphisms of isomorphisms of natural isomorphisms*. This process can be repeated ad infinitum. For each of these 'iterated' notions of isomorphisms, we may speak of identity isomorphisms, of inverse isomorphisms, and of composite isomorphisms.

We may loosely think of a natural isomorphism $f \cong f'$ between two functors as expressing that f and f' are 'equal', and similarly for all the iterated versions of isomorphisms. The reason we allow for arbitrary iterations of isomorphisms is the following fundamental principle:

Fundamental principle of higher category theory: Whenever we would like to express that two things *are equal*, we should always remember *how* the two things are equal by specifying an isomorphism between them.

To illustrate the fundamental principle, let us see how it determines the correct notion of a 'commutative square' of functors between synthetic categories:

Definition 1.1.1. A *commutative square of functors*, often displayed by means of a diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\
C' \arrow[r, "f'"'] & D',
\end{tikzcd}$$

consists of four functors f, g, h and f' with sources and targets as displayed in the diagram, together with a specified natural isomorphism $\alpha\colon h \circ f \cong f' \circ g$. We may similarly define a *commutative square of natural isomorphisms*, etcetera.

We will often refer to the data of just the four functors f, g, h and f' as a *square*, and say that *the square commutes* whenever a natural isomorphism $\alpha\colon h \circ f \cong f' \circ g$ can be provided.

A *commutative triangle of functors*, displayed by means of a diagram

$$\begin{tikzcd}
& D \arrow[dr, "g"] & \\
C \arrow[ur, "f"] \arrow[rr, "h"'] & & E,
\end{tikzcd}$$

consists of three functors f, g and h with sources and targets as displayed in the diagram, together with a specified natural isomorphism $\alpha\colon h \cong g \circ f$. We may similarly define a *commutative triangle of natural isomorphisms*, etcetera.

Axiom A.2. The composition of functors is unital and associative up to isomorphism: for functors $f\colon C \longrightarrow D$, $g\colon D \longrightarrow E$ and $h\colon E \longrightarrow F$, there are natural isomorphisms

$$\mathrm{id}_D \circ f \cong f \cong f \circ \mathrm{id}_C \qquad \text{and} \qquad (h \circ g) \circ f \cong h \circ (g \circ f),$$

which we will leave nameless for the sake of notational simplicity. The composition of (iterated) isomorphisms is similarly unital and associative up to isomorphism, and for any natural isomorphism $\alpha\colon f \cong f'$ there are isomorphisms $\alpha^{-1} \circ \alpha \cong \mathrm{id}_f$ and $\alpha \circ \alpha^{-1} \cong \mathrm{id}_{f'}$.

Remark 1.1.2. From a higher categorical viewpoint, the data given in Axiom A.2 is not sufficient: one expects the associators and unitors to satisfy infinitely many higher coherences, including the pentagon and triangle diagrams. Since these higher coherences are hard to encode in our naive language, we will omit them from our axiomatization; it turns out that the resulting meta theory is still rich enough to express everything we need.

Axiom A.3. Composition of functors is compatible with natural isomorphisms: given a natural isomorphism $\alpha\colon f \cong f'$ of functors $C \longrightarrow D$ and a natural isomorphism $\beta\colon g \cong g'$ of functors $D \longrightarrow E$, we obtain a new natural isomorphism $\beta * \alpha\colon g \circ f \cong g' \circ f'$ of functors $C \longrightarrow E$, called the *horizontal composite*. Just like functor composition, horizontal composition of natural isomorphisms is also associative and unital up to isomorphism:

- For a natural isomorphism $\alpha\colon f \cong f'$ of functors $C \longrightarrow D$ there are commutative squares of natural isomorphisms

$$\begin{tikzcd}
f \circ \mathrm{id}_C \arrow[r, "\alpha * \mathrm{id}_{\mathrm{id}_C}", "\cong"'] \arrow[d, "\cong"'] & f' \circ \mathrm{id}_C \arrow[d, "\cong"] \\
f \arrow[r, "\alpha", "\cong"'] & f'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{id}_D \circ f \arrow[r, "\mathrm{id}_{\mathrm{id}_D} * \alpha", "\cong"'] \arrow[d, "\cong"'] & \mathrm{id}_D \circ f' \arrow[d, "\cong"] \\
f \arrow[r, "\alpha", "\cong"'] & f'.
\end{tikzcd}$$

- For natural isomorphisms $\alpha\colon f \cong f'$, $\beta\colon g \cong g'$ and $\gamma\colon h \cong h'$, the following square of natural isomorphisms commutes:

$$\begin{tikzcd}
(h \circ g) \circ f \arrow[r, "(\gamma * \beta) * \alpha", "\cong"'] \arrow[d, "\cong"'] & (h' \circ g') \circ f' \arrow[d, "\cong"] \\
h \circ (g \circ f) \arrow[r, "\gamma * (\beta * \alpha)", "\cong"'] & h' \circ (g' \circ f').
\end{tikzcd}$$

We further ask horizontal composition to be compatible with vertical composition: given another two natural isomorphisms $\alpha'\colon f' \cong f''$ and $\beta'\colon g' \cong g''$, there is an isomorphism

$$(\beta' * \alpha') \circ (\beta * \alpha) \cong (\beta' \circ \beta) * (\alpha' \circ \alpha)$$

of natural isomorphisms $g \circ f \cong g'' \circ f''$. The same requirements hold for horizontal compositions of iterated isomorphisms.

The resulting natural isomorphism may pictorially be displayed as follows:

$$\begin{tikzcd}[column sep=large]
C \arrow[r, bend left=60, "f"] \arrow[r, "f'" description] \arrow[r, bend right=60, "f''"'] \arrow[r, phantom, shift left=4, "\alpha \Downarrow \cong"] \arrow[r, phantom, shift right=4, "\alpha' \Downarrow \cong"] & D \arrow[r, bend left=60, "g"] \arrow[r, "g'" description] \arrow[r, bend right=60, "g''"'] \arrow[r, phantom, shift left=4, "\beta \Downarrow \cong"] \arrow[r, phantom, shift right=4, "\beta' \Downarrow \cong"] & E.
\end{tikzcd}$$

1.1.2 Equivalence of categories

An important notion in category theory is that of an *equivalence of categories*:

Definition 1.1.3 (Equivalence of categories)**.** A functor $f\colon C \to D$ is said to be an *equivalence* if there exists another functor $g\colon D \to C$ equipped with natural isomorphisms $\mathrm{id}_C \cong gf$ and $\mathrm{id}_D \cong fg$. We refer to g as the *inverse* of f. Equivalences will often be introduced by writing $f\colon C \xrightarrow{\sim} D$. We may write $C \simeq D$ to indicate that C and D are equivalent via an unnamed equivalence.

Note that any functor $f'\colon C \to D$ isomorphic to f is automatically also an equivalence with the same inverse g.

Lemma 1.1.4. *Let $f\colon C \to D$ be a functor and assume there are functors $g,h\colon D \to C$ equipped with natural isomorphisms $g \circ f \cong \mathrm{id}_C$ and $f \circ h \cong \mathrm{id}_D$. Then there is a natural isomorphism $g \cong h$, and f is an equivalence with inverse g.*

Proof. The desired isomorphism is given by composing the following chain of natural isomorphisms:

$$g \cong g \circ \mathrm{id}_D \cong g \circ (f \circ h) \cong (g \circ f) \circ h \cong \mathrm{id}_C \circ h \cong h.$$

It follows that f is an equivalence with inverse g, since there are natural isomorphisms $gf \cong \mathrm{id}_C$ and $fg \cong fh \cong \mathrm{id}_D$. □

Corollary 1.1.5. *Any two inverses g and h of an equivalence f are isomorphic.* □

Lemma 1.1.6. *If $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.*

Proof. We have $gg'f'f \cong g\,\mathrm{id}_D\,f \cong gf \cong \mathrm{id}_C$ and $f'fgg' \cong f'\,\mathrm{id}_D\,g' \cong f'g' \cong \mathrm{id}_E$. □

Lemma 1.1.7. *Equivalences of synthetic categories satisfy the 2-out-of-3 property: given functors $f\colon C \to D$ and $g\colon D \to E$, if two of the three functors f, g and gf are equivalences, then so is the third.*

Proof. If g and f are equivalences, then so is gf by Lemma 1.1.6. If f and gf are equivalences, then we may rewrite g as the composit $g \cong g \circ \mathrm{id}_D \cong g \circ f \circ f^{-1} = gf \circ f^{-1}$, so that g is the composite of two equivalences and hence itself an equivalence. A similar argument shows that f is an equivalence whenever g and gf are equivalences. □

Lemma 1.1.8. *Equivalences of synthetic categories satisfy the 2-out-of-6 property: given functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.*

Proof. Consider the functors $f' := f \circ (gf)^{-1} \colon E \to D$ and $h' := (hg)^{-1} \circ h \colon E \to D$. Then there are natural isomorphisms

$$g \circ f' = g \circ f \circ (gf)^{-1} = gf \circ (gf)^{-1} \cong \mathrm{id}_E$$

and

$$h' \circ g = (hg)^{-1} \circ h \circ g = (hg)^{-1} \circ hg \cong \mathrm{id}_D .$$

Hence it follows from Lemma 1.1.4 that g is an equivalence. In light of Lemma 1.1.7, we then get that also f and h are equivalences, and then so is hgf. □

Definition 1.1.9. We say that a functor $f \colon C \to D$ is a *retract* of a functor $f' \colon C' \to D'$ if there exists a commutative diagram of the form

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & C' \arrow[r, "h"] \arrow[d, "f'"] & C \arrow[d, "f"] \\
D \arrow[r, "k"] & D' \arrow[r, "j"] & D,
\end{tikzcd}$$

together with natural isomorphisms $\alpha \colon h \circ g \cong \mathrm{id}_C$ and $\beta \colon j \circ k \cong \mathrm{id}_D$ satisfying $f \circ \alpha \cong \beta \circ f$.

Lemma 1.1.10. *Assume that $f \colon C \to D$ is a retract of $f' \colon C' \to D'$. If f' is an equivalence, then so is f.*

Proof. Let $(f')^{-1} \colon D' \to C'$ be an inverse of f'. We claim that the composite $g := h \circ (f')^{-1} \circ k \colon D \to C$ is inverse to f. Indeed, we have natural isomorphisms

$$f \circ g = f \circ h \circ (f')^{-1} \circ k \cong j \circ f' \circ (f')^{-1} \circ k \cong k \circ j \cong \mathrm{id}_D$$

and

$$g \circ f = h \circ (f')^{-1} \circ k \circ f \cong h \circ (f')^{-1} \circ f' \circ g \cong h \circ g \cong \mathrm{id}_C .$$

This finishes the proof. □

1.1.3 Constructions of synthetic categories

On top of the language of synthetic categories, functors and natural isomorphisms, we need various basic categorical constructions, like functor categories and pullbacks. In each of these cases, we will only impose a ‘naive’ universal property that specifies the behavior of functors and natural isomorphisms but ignores all higher isomorphisms.

The terminal and initial categories

We start by introducing the terminal and initial synthetic categories.

Axiom B.1 (Terminal category)**.** There is a synthetic category $*$, called the *terminal category*. For every synthetic category C, there is a functor $p_C \colon C \to *$. For any two functors $f, f' \colon C \to *$, there is a natural isomorphism $f \cong f'$.

Axiom B.2 (Initial category)**.** The *initial category* is a synthetic category denoted $\emptyset$. For every synthetic category C, there is a functor $\emptyset \to C$. For any two functors $f, f' \colon \emptyset \to C$, there is a natural isomorphism $f \cong f'$.

Definition 1.1.11. An *(absolute) object* of a synthetic category C is a functor $x \colon * \to C$. We will frequently introduce objects of C by writing $x \in C$. Given two objects x and x' of C, an *isomorphism* between x and x' is a natural isomorphism $x \cong x'$ of functors $* \to C$.

Warning 1.1.12. In Section 3.2, we will introduce a more general notion of objects in a synthetic category. To disambiguate between the two notions, we will then use the phrase *absolute object* to refer to objects of C of the form $x \colon * \to C$.

Definition 1.1.13 (Contractible category)**.** A synthetic category C is called *contractible* if the functor $p_C \colon C \to *$ is an equivalence.

Products and coproducts of categories

We will now introduce various ways to construct new synthetic categories out of old ones, starting with *products* and *coproducts*.

Axiom B.3 (Product of categories)**.** The *product* of two synthetic categories C and D is a synthetic category $C \times D$ equipped with two functors $\mathrm{pr}_C \colon C \times D \to C$ and $\mathrm{pr}_D \colon C \times D \to D$ called the *projection functors*.

Given functors $f \colon E \to C$ and $g \colon E \to D$, we obtain a functor $(f, g) \colon E \to C \times D$ equipped with natural isomorphisms $\mathrm{pr}_C \circ (f, g) \cong f$ and $\mathrm{pr}_D \circ (f, g) \cong g$.

Given two functors $h, k \colon E \to C \times D$ and two natural isomorphisms $\alpha \colon \mathrm{pr}_C \circ h \cong \mathrm{pr}_C \circ k$ and $\beta \colon \mathrm{pr}_D \circ h \cong \mathrm{pr}_D \circ k$, we obtain a natural isomorphism $(\alpha, \beta) \colon h \cong k$ satisfying $\mathrm{pr}_C \circ (\alpha, \beta) \cong \alpha$ and $\mathrm{pr}_D \circ (\alpha, \beta) \cong \beta$.

Axiom B.4 (Coproduct of categories)**.** The *coproduct* of two synthetic categories C and D is a synthetic category $C \sqcup D$ equipped with two functors $i_C \colon C \to C \sqcup D$ and $i_D \colon D \to C \sqcup D$ called the *inclusion functors*. We will sometimes also refer to $C \sqcup D$ as the *disjoint union* of C and D.

Given functors $f \colon C \to E$ and $g \colon D \to E$, we obtain a functor $\langle f, g\rangle \colon C \sqcup D \to E$ equipped with natural isomorphisms $\langle f, g\rangle \circ i_C \cong f$ and $\langle f, g\rangle \circ i_D \cong g$.

Given two functors $h, k \colon C \sqcup D \to E$ and two natural isomorphisms $\alpha \colon h \circ i_C \cong k \circ i_C$ and $\beta \colon h \circ i_D \cong k \circ i_D$, we obtain a natural isomorphism $\langle \alpha, \beta\rangle \colon h \cong k$ satisfying $(\alpha, \beta) \circ i_C \cong \alpha$ and $(\alpha, \beta) \circ i_D \cong \beta$.

From these simple rules, we may deduce that products and coproducts are commutative, unital and associative. For commutativity of products, one argues as follows:

Lemma 1.1.14. *For synthetic categories C and D, the functor $(\mathrm{pr}_D, \mathrm{pr}_C) \colon C \times D \to D \times C$ is an equivalence, with inverse given by $(\mathrm{pr}_C, \mathrm{pr}_D) \colon D \times C \to C \times D$.*

Proof. We need to provide natural isomorphisms $(\mathrm{pr}_C, \mathrm{pr}_D) \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{id}_{C\times D}$ and $(\mathrm{pr}_D, \mathrm{pr}_C) \circ (\mathrm{pr}_C, \mathrm{pr}_D) \cong \mathrm{id}_{D\times C}$. By symmetry, it will suffice to produce the first natural isomorphism. For this, it will in turn suffice to produce natural isomorphisms after composing with the functors pr_C and pr_D. For the composition with pr_C we compute that

$$\mathrm{pr}_C \circ ((\mathrm{pr}_C, \mathrm{pr}_D) \circ (\mathrm{pr}_D, \mathrm{pr}_C)) \cong (\mathrm{pr}_C \circ (\mathrm{pr}_C, \mathrm{pr}_D)) \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C,$$

which is indeed isomorphic to $\mathrm{pr}_C \circ \mathrm{id}_{C\times D}$. The case for composition with pr_D is analogous. This finishes the proof. □

Unitality may be argued similarly:

Lemma 1.1.15. *For every synthetic category C, the projection functor $\mathrm{pr}_C \colon C \times * \to C$ is an equivalence, with inverse given by the functor $(\mathrm{id}_C, p_C) \colon C \to C \times *$.*

Proof. The composite $\mathrm{pr}_C \circ (\mathrm{id}_C, p_C) \colon C \to C$ is isomorphic to id_C by the defining property of (id_C, p_C). To show that the composite $(\mathrm{id}_C, p_C) \circ \mathrm{pr}_C \colon C \times * \to C \times *$ is isomorphic to $\mathrm{id}_{C\times *}$, it will suffice to do so after composing with pr_C and with pr_*. The case for pr_* is clear, since any two functors $C \times * \to *$ are isomorphic. In the case of pr_C we compute

$$\mathrm{pr}_C \circ ((\mathrm{id}_C, p_C) \circ \mathrm{pr}_C) \cong (\mathrm{pr}_C \circ (\mathrm{id}_C, p_C)) \circ \mathrm{pr}_C \cong \mathrm{id}_C \circ \mathrm{pr}_C \cong \mathrm{pr}_C \cong \mathrm{pr}_C \circ \mathrm{id}_{C\times *},$$

where the first isomorphism is associativity, the second is the relation proved in the first paragraph, and the third and fourth are unitality. This finishes the proof. □

Remark 1.1.16. In the previous proofs, we have been very explicit about the usage of associativity and naturality isomorphisms. In the remainder of this book, we will often leave the usage of such isomorphisms implicit in order to make the proofs more concise.

Exercise 1.1.17 (Exercise 1.6.1)**.** Formulate and prove the associativity of the product of synthetic categories.

Exercise 1.1.18 (Exercise 1.6.2)**.** Show that the coproduct of synthetic categories is associative, commutative and unital:

$$(C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Construction 1.1.19. The formation of products and coproducts of synthetic categories is functorial: given two functors $f\colon C \to C'$ and $g\colon D \to D'$, we obtain two new functors

$$f \times g := (f \circ \mathrm{pr}_C, g \circ \mathrm{pr}_D)\colon C \times D \to C' \times D'$$

and

$$f \sqcup g := \langle i_{C'} \circ f, i_{D'} \circ g \rangle\colon C \sqcup D \to C' \sqcup D'.$$

We leave it to the reader to verify that there are natural isomorphisms

$$\begin{aligned} \mathrm{id}_C \times \mathrm{id}_D &\cong \mathrm{id}_{C \times D} \quad \text{and} \quad (f' \times g') \circ (f \times g) \cong (f' \circ f) \times (g' \circ g), \\ \mathrm{id}_C \sqcup \mathrm{id}_D &\cong \mathrm{id}_{C \sqcup D} \quad \text{and} \quad (f' \sqcup g') \circ (f \sqcup g) \cong (f' \circ f) \sqcup (g' \circ g). \end{aligned}$$

Lemma 1.1.20. *For a synthetic category C, the projection map $C \times \emptyset \to \emptyset$ is an equivalence.*

Proof. This is immediate from Axiom B.7. □

Functor categories

We now introduce functor categories.

Axiom B.5 (Functor category)**.** The *functor category* between two synthetic categories C and D is a synthetic category denoted by $\mathrm{Fun}(C, D)$. Given another synthetic category E, we obtain for every functor $f\colon E \times C \to D$ a functor $f_c\colon E \to \mathrm{Fun}(C, D)$ obtained from f by *currying*. Conversely, there is for every functor $g\colon E \to \mathrm{Fun}(C, D)$ a functor $g^u\colon E \times C \to D$ obtained by *uncurrying*. These operations come equipped with natural isomorphisms $f \cong (f_c)^u$ and $g \cong (g^u)_c$, so that currying and uncurrying determine a one-to-one correspondence between functors $E \to \mathrm{Fun}(C, D)$ and functors $E \times C \to D$.

The operations of currying and uncurrying respect natural isomorphisms: given a natural isomorphism $\alpha\colon f \cong f'$ of functors $f\colon E\times C \to D$, we obtain a natural isomorphism $\alpha^c\colon f_c \cong f'_c$ between their curried functors. Conversely, an isomorphism $\beta\colon g \cong g'$ gives $\beta^u\colon g^u \cong (g')^u$. Again, we demand that[1] $(\alpha_c)^u \cong \alpha$ and $(\beta^u)_c \cong \beta$.

The operation of uncurrying is assumed to be functorial in E. More precisely, given functors $g\colon E \to \mathrm{Fun}(C,D)$ and $h\colon E' \to E$, then the uncurrying of the composite $g\circ h\colon E' \to \mathrm{Fun}(C,D)$ is required to be isomorphic to the composite $E'\times C \xrightarrow{h\times \mathrm{id}_C} E\times C \xrightarrow{g^u} D$. Similarly, given an isomorphism $\beta\colon g \cong g'$, the uncurrying of the isomorphism $\beta\circ h\colon g\circ h \cong g'\circ h$ is isomorphic to $\beta^u\circ(h\times\mathrm{id}_C)\colon g^u\circ(h\times\mathrm{id}_C) \cong (g')^u\circ(h\times\mathrm{id}_C)$.

By considering the case of the terminal category $E = *$, we observe that objects $* \to \mathrm{Fun}(C,D)$ of the functor category correspond to functors $C \simeq C\times * \to D$ from C to D, justifying the notation.

Construction 1.1.21 (Evaluation functor)**.** Given synthetic categories C and D, the *evaluation functor* $\mathrm{ev}\colon \mathrm{Fun}(C,D)\times C \to D$ is defined as $\mathrm{ev} := (\mathrm{id}_{\mathrm{Fun}(C,D)})^u$, i.e. as the functor obtained by uncurrying the identity functor $\mathrm{id}_{\mathrm{Fun}(C,D)}\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D)$. Given an object $x \in C$, we write $\mathrm{ev}_x\colon \mathrm{Fun}(C,D) \to D$ for the composite

$$\mathrm{ev}_x\colon \mathrm{Fun}(C,D) \simeq \mathrm{Fun}(C,D)\times * \xrightarrow{\mathrm{id}\times x} \mathrm{Fun}(C,D)\times C \xrightarrow{\mathrm{ev}} D,$$

and refer to it as the *evaluation functor at x*.

Lemma 1.1.22. *Given a functor $h\colon E \to \mathrm{Fun}(C,D)$, its uncurrying $h^u\colon E\times C \to D$ is isomorphic to the following composite functor:*

$$E\times C \xrightarrow{h\times\mathrm{id}_C} \mathrm{Fun}(C,D)\times C \xrightarrow{\mathrm{ev}} D.$$

Proof. This is immediate from the functoriality of uncurrying in E by writing h as the composite $\mathrm{id}_{\mathrm{Fun}(C,D)}\circ h$ and taking $g = \mathrm{id}_{\mathrm{Fun}(C,D)}$. □

The functoriality of the product naturally provides functoriality for the functor categories:

Construction 1.1.23 (Functoriality of functor categories)**.** Given a functor $g\colon D \to D'$, we define the functor $g\circ -\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D')$ as the currying of the composite

$$\mathrm{Fun}(C,D)\times C \xrightarrow{\mathrm{ev}} D \xrightarrow{g} D'.$$

---

[1] As in Axiom A.3, these isomorphisms should really be interpreted as commutative squares of natural isomorphisms.

Similarly, given a functor $f\colon C' \to C$, we define the functor $-\circ f\colon \operatorname{Fun}(C',D) \to \operatorname{Fun}(C,D)$ as the currying of the composite

$$\operatorname{Fun}(C',D) \times C \xrightarrow{\mathrm{id}\times f} \operatorname{Fun}(C',D) \times C' \xrightarrow{\mathrm{ev}} D.$$

In a completely analogous way, every natural isomorphism $\beta\colon g \cong g'$ of functors $D \to D'$ induces a natural isomorphism $(\alpha \circ -)\colon (g \circ -) \cong (g' \circ -)$ of functors $\operatorname{Fun}(C,D) \to \operatorname{Fun}(C,D')$, and similarly for the construction $-\circ f$.

Exercise 1.1.24 (Exercise 1.6.3)**.** Formulate and prove that the assignments $g \mapsto (g \circ -)$ and $f \mapsto (-\circ f)$ are functorial, in the sense that they respect identity functors and composition of functors.

Lemma 1.1.25. *For a synthetic category C and a functor $g\colon D \to D'$, the following square commutes:*

$$\begin{tikzcd}
\operatorname{Fun}(C,D) \times C \arrow[r, "\mathrm{ev}"] \arrow[d, "(g\circ -)\times \mathrm{id}_C"'] & D \arrow[d, "g"] \\
\operatorname{Fun}(C,D') \times C \arrow[r, "\mathrm{ev}"] & D'.
\end{tikzcd}$$

Proof. We need to show that the two composite functors $\operatorname{Fun}(C,D) \times C \to D'$ in the diagram are isomorphic. We may alternatively show that the two curried functors $\operatorname{Fun}(C,D) \to \operatorname{Fun}(C,D')$ are isomorphic. We claim that both of these functors are isomorphic to the functor

$$g \circ -\colon \operatorname{Fun}(C,D) \to \operatorname{Fun}(C,D'),$$

which would finish the proof. Indeed, for the top composite this holds by definition, while for the bottom this follows by applying Lemma 1.1.22 to the functor $h = (g \circ -)$. □

Lemma 1.1.26. *Let C be a synthetic category. Then the functor $p_{\operatorname{Fun}(C,*)}\colon \operatorname{Fun}(C,*) \to *$ is an equivalence.*

Proof. We will show that an inverse is given by the functor $(p_{*\times C})_c\colon * \to \operatorname{Fun}(C,*)$, obtained by uncurrying the functor $p_{*\times C}\colon *\times C \to *$. The composite $(p_{*\times C})_c \circ p_{\operatorname{Fun}(C,*)}\colon * \to *$ is automatically isomorphic to $\mathrm{id}_*\colon * \to *$, hence it remains to show that the composite $p_{\operatorname{Fun}(C,*)} \circ (p_{*\times C})_c\colon \operatorname{Fun}(C,*) \to \operatorname{Fun}(C,*)$ is isomorphic to $\mathrm{id}_{\operatorname{Fun}(C,*)}$. It will suffice to prove that the two uncurried functors $\operatorname{Fun}(C,*) \times C \to *$ are isomorphic. But this is again automatic. □

Lemma 1.1.27. *Let C, D and E be synthetic categories. Then the functor*

$$(\mathrm{pr}_D \circ -, \mathrm{pr}_E \circ -)\colon \operatorname{Fun}(C, D\times E) \to \operatorname{Fun}(C,D) \times \operatorname{Fun}(C,E)$$

is an equivalence.

Proof. For the purpose of this proof, let us abbreviate $\mathrm{Fun}(C,T)$ by T^C for every synthetic category T. We define a functor $h\colon D^C \times E^C \to (D\times E)^C$ by currying the following composite:

$$D^C \times E^C \times C \xrightarrow{(\mathrm{pr}_{D^C},\mathrm{pr}_C,\mathrm{pr}_{E^C},\mathrm{pr}_C)} D^C \times C \times E^C \times C \xrightarrow{\mathrm{ev}\times\mathrm{ev}} D \times E.$$

We start by showing that the composite $(\mathrm{pr}_D \circ -, \mathrm{pr}_E \circ -) \circ h$ is the identity. It will suffice to show this after projecting to both D^C and E^C, and by symmetry we may just do this for D^C. In other words, we must show that the composite

$$D^C \times E^C \xrightarrow{h} (D\times E)^C \xrightarrow{\mathrm{pr}_D \circ -} D^C$$

is isomorphic to pr_{D^C}. It will in turn suffice to show this after uncurrying. But there it holds since the uncurrying of both functors can be seen to be isomorphic to the composite

$$D^C \times E^C \times C \xrightarrow{(\mathrm{pr}_{D^C},\mathrm{pr}_C)} D^C \times C \xrightarrow{\mathrm{ev}} D.$$

We will now show that also the composite $h \circ (\mathrm{pr}_D \circ -, \mathrm{pr}_E \circ -)$ is isomorphic to the identity on $(D\times E)^C$. Since the target of these functors is $(D\times E)^C$, it will suffice to prove that the two uncurried functors $(D\times E)^C \times C \to D \times E$ are isomorphic. The target of these two functors is a product, hence it in turn suffices to show that the components of both functors are isomorphic. By symmetry, it again suffices to do this for the D-component. Using functoriality of uncurrying, and unwinding the constructions, we are reduced to showing that the following diagram commutes:

$$\begin{tikzcd}
(D\times E)^C \times C \arrow[r, "\mathrm{ev}"] \arrow[d, "(\mathrm{pr}_D \circ -)\times \mathrm{id}_C"'] & D\times E \arrow[d, "\mathrm{pr}_D"] \\
D^C \times C \arrow[r, "\mathrm{ev}"] & D.
\end{tikzcd}$$

But this is an instance of Lemma 1.1.25. □

Lemma 1.1.28. *For synthetic categories C, D and E, there is a preferred equivalence* $\mathrm{Fun}(C,\mathrm{Fun}(D,E)) \xrightarrow{\sim} \mathrm{Fun}(C\times D,E)$.

Proof. We define the preferred functor as the curried functor of the following composite:

$$\mathrm{Fun}(C,\mathrm{Fun}(D,E)) \times C \times D \xrightarrow{\mathrm{ev}\times \mathrm{id}_D} \mathrm{Fun}(D,E)\times D \xrightarrow{\mathrm{ev}} E.$$

To construct an inverse, it will by currying suffice to construct a functor $\mathrm{Fun}(C\times D,E)\times C \to \mathrm{Fun}(D,E)$, which we may take to be the curried functor of the following composite

$$\mathrm{Fun}(C\times D,E)\times C\times D \xrightarrow{\mathrm{ev}} E.$$

We leave it to the reader to verify that these two functors are inverse to each other. □

Lemma 1.1.29. *For a synthetic category C, the functor* $\mathrm{ev}_*\colon \mathrm{Fun}(*,C) \to C$ *is an equivalence.*

Proof. We leave it to the reader to verify that an inverse is given by currying the equivalence $C \times * \xrightarrow{\sim} C$. □

Lemma 1.1.30. *For a synthetic category C, the functor category* $\mathrm{Fun}(\emptyset, C)$ *is contractible, i.e.* $\mathrm{Fun}(\emptyset, C) \xrightarrow{\sim} *$.

Proof. For the inverse we pick the preferred functor $\emptyset \to C$. The composite $* \to \mathrm{Fun}(\emptyset, C) \to *$ is automatically isomorphic to the identity. For the composite $\mathrm{Fun}(\emptyset, C) \to * \to \mathrm{Fun}(\emptyset, C)$, we may show this after currying. But since $\mathrm{Fun}(\emptyset, C) \times \emptyset \xrightarrow{\sim} \emptyset$ by Lemma 1.1.20, this follows directly from the defining property of $\emptyset$. □

Pullbacks of categories

We introduce the concept of a pullback of synthetic categories.

Axiom B.6 (Pullback of categories)**.** Consider two functors $f\colon C \to E$ and $g\colon D \to E$. The *pullback* of f and g, also called the *fiber product of C and D over E*, is a synthetic category $C \times_E D$. It comes equipped with two functors $\mathrm{pr}_C\colon C \times_E D \to C$ and $\mathrm{pr}_D\colon C \times_E D \to D$ and a natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$, or in other words, a commutative square

$$\begin{tikzcd} C \times_E D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_D"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E. \end{tikzcd}$$

Given functors $t\colon T \to C$ and $s\colon T \to D$ equipped with a natural isomorphism $\alpha\colon f \circ t \cong g \circ s$, we obtain a functor $(t,s)\colon T \to C \times_E D$. This functor comes equipped with natural isomorphisms $\mathrm{pr}_C \circ (t,s) \cong t$ and $\mathrm{pr}_D \circ (t,s) \cong s$. Furthermore, the composite isomorphism

$$f \circ t \cong f \circ \mathrm{pr}_C \circ (t,s) \cong g \circ \mathrm{pr}_D \circ (t,s) \cong g \circ s$$

is isomorphic to α.

Consider now two functors $h, k\colon T \to C \times_E D$, and assume we are given two natural isomorphisms $\alpha\colon \mathrm{pr}_C \circ h \cong \mathrm{pr}_C \circ k$ and $\beta\colon \mathrm{pr}_D \circ h \cong \mathrm{pr}_D \circ k$ such that the following square of natural isomorphisms commutes:

$$\begin{tikzcd} f \circ \mathrm{pr}_C \circ h \arrow[r, "f\circ\alpha", "\cong"'] \arrow[d, "\cong"'] & f \circ \mathrm{pr}_C \circ k \arrow[d, "\cong"] \\ g \circ \mathrm{pr}_D \circ h \arrow[r, "g\circ\beta", "\cong"'] & g \circ \mathrm{pr}_D \circ k. \end{tikzcd}$$

Then we obtain a natural isomorphism $(\alpha,\beta)\colon h \cong k$ satisfying $\mathrm{pr}_C \circ (\alpha,\beta) \cong \alpha$ and $\mathrm{pr}_D \circ (\alpha,\beta) \cong \beta$, and inducing an isomorphic isomorphism of natural isomorphisms in the previous square.

Exercise 1.1.31 (Exercise 1.6.4)**.** Show that the fiber product is commutative, associative and unital: for functors $C \to E$, $D \to E$ and $B \to E$, there are preferred equivalences

$$C \times_E D \xrightarrow{\sim} D \times_E C, \qquad B \times_E (C \times_E D) \xrightarrow{\sim} (B \times_E C) \times_E D, \qquad C \times_E E \xrightarrow{\sim} C.$$

Construction 1.1.32. The construction of pullbacks of synthetic categories is functorial: given a commutative diagram

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "\varphi"'] & E \arrow[d, "\chi"] & D \arrow[l, "g"'] \arrow[d, "\psi"] \\
C' \arrow[r, "f'"'] & E' & D' \arrow[l, "g'"]
\end{tikzcd}$$

there is an induced functor

$$\varphi \times_\chi \psi := (\varphi \circ \mathrm{pr}_C, \psi \circ \mathrm{pr}_D)\colon C \times_E D \to C' \times_{E'} D'.$$

If each of the functors φ, ψ and χ are an equivalence, then so is $\varphi \times_\chi \psi$.

Corollary 1.1.33. *Equivalences are closed under pullback: if $g\colon D \to E$ is an equivalence and $f\colon C \to E$ is an arbitrary functor, then also $C \times_E D \to C$ is an equivalence.*

Proof. This is a special case of the last statement of the previous construction by taking $C' = C$, $E' = D' = E$, $\varphi = \mathrm{id}_C$, $\chi = \mathrm{id}_E$ and $\psi = g$. □

Definition 1.1.34 (Fiber)**.** Given a functor $f\colon C \to E$ and an object $e\colon * \to E$, we define the *fiber of f at x* as the fiber product of C and $*$ over E:

$$C_x := C \times_E *.$$

We will sometimes also write $C(x)$ for this fiber.

Lemma 1.1.35. *If $f\colon C \to E$ is an equivalence, then every fiber C_x is contractible.*

Proof. The map $C_x \to *$ is the base change of the equivalence $C \to E$ and hence is an equivalence by Corollary 1.1.33. □

Lemma 1.1.36. *Let T be a synthetic category and let $f\colon C \to E$ and $g\colon D \to E$ be functors. Then the functor*

$$(\mathrm{pr}_C \circ -, \mathrm{pr}_D \circ -)\colon \operatorname{Fun}(T, C \times_E D) \to \operatorname{Fun}(T,C) \times_{\operatorname{Fun}(T,E)} \operatorname{Fun}(T,D)$$

induced by the isomorphism $(f \circ -) \circ (\mathrm{pr}_C \circ -) \cong (g \circ -) \circ (\mathrm{pr}_D \circ -)$ is an equivalence.

Proof. We will again abbreviate $\operatorname{Fun}(T,-)$ by $(-)^T$. To construct an inverse to this functor, consider the two composites

$$(C^T \times_{E^T} D^T) \times T \xrightarrow{\mathrm{pr}_{C^T} \times \mathrm{id}_T} C^T \times T \xrightarrow{\mathrm{ev}} C$$

and

$$(C^T \times_{E^T} D^T) \times T \xrightarrow{\mathrm{pr}_{D^T} \times \mathrm{id}_T} D^T \times T \xrightarrow{\mathrm{ev}} D.$$

These two functors assemble into a functor to the pullback $C \times_E D$, due to the composite natural isomorphism exhibited by the following commutative diagram:

$$\begin{tikzcd}
 & & (C^T \times_{E^T} D^T) \times T \arrow[dll, "\mathrm{pr}_{C^T} \times \mathrm{id}_T"'] \arrow[drr, "\mathrm{pr}_{D^T} \times \mathrm{id}_T"] & & \\
\operatorname{Fun}(T,C) \times T \arrow[d, "\mathrm{ev}"'] \arrow[drr, "f\circ-"] & & & & \operatorname{Fun}(T,D) \times T \arrow[d, "\mathrm{ev}"] \arrow[dll, "g\circ-"'] \\
C \arrow[drr, "f"'] & & T^E \times T \arrow[d, "\mathrm{ev}"] & & D \arrow[dll, "g"] \\
 & & E. & &
\end{tikzcd}$$

By currying, we thus obtain a functor $h\colon C^T \times_{E^T} D^T \to (C \times_E D)^T$. We claim that this is the desired inverse. The proof is analogous to that of Lemma 1.1.27, with the added difficulty that one needs to make sure that the natural isomorphisms of functors into E are all compatible. We will leave the details as an exercise to the reader. □

Definition 1.1.37 (Pullback square)**.** A commutative square

$$\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, "s"'] & C \arrow[d, "f"] \\
D \arrow[r, "g"'] & E
\end{tikzcd}$$

with given isomorphism $\alpha\colon f \circ s \cong g \circ t$ is called a *pullback square* if the induced functor $(s,t)\colon T \to C \times_E D$ is an equivalence.

Lemma 1.1.38 (Pasting lemma for pullback squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\ D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3. \end{tikzcd}$$

(1) There is a preferred equivalence $D_1 \times_{D_3} C_3 \xrightarrow{\sim} D_1 \times_{D_2} (D_2 \times_{D_3} C_3)$*.*

(2) If the right-hand square is a pullback square, then the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

Proof. For part (1), we define the functor as $(\mathrm{pr}_{D_1}, (h_1 \circ \mathrm{pr}_{D_1}, \mathrm{pr}_{C_3}))$. We claim that an inverse is given by the functor

$$(\mathrm{pr}_{D_1}, \mathrm{pr}_{C_3} \circ \mathrm{pr}_{D_2 \times_{D_3} C_3}) \colon D_1 \times_{D_2} (D_2 \times_{D_3} C_3) \to D_1 \times_{D_3} C_3.$$

To show that the two composites are isomorphic to the respective identities, it suffices to construct these isomorphisms after projecting to the two components and verifying that these isomorphisms agree with composed with the functors to D_3. But after unwinding the definitions, this holds tautologically true by the very construction of the two functors in both directions.

Part (2) is an immediate consequence of the 2-out-of-3 property from Lemma 1.1.7 applied to the following commutative square:

$$\begin{tikzcd} C_2 \arrow[r] \arrow[d] & D_1 \times_{D_2} C_2 \arrow[d, "\simeq"] \\ D_1 \times_{D_3} C_3 \arrow[r, "\simeq"] & D_1 \times_{D_2} (D_2 \times_{D_3} C_3). \end{tikzcd}$$

□

Lemma 1.1.39. *Consider a commutative square*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\ E \arrow[r, "k"] & F \end{tikzcd}$$

and assume that h is an equivalence. Then g is an equivalence if and only if the square is a pullback square.

Proof. Since h is an equivalence, also the projection map $\mathrm{pr}_E \colon D \times_F E \to E$ is an equivalence by Corollary 1.1.33. Since the composite of the map $(f, g) \colon C \to D \times_F E$ with pr_E is isomorphic to g, the claim follows from th 2-out-of-3 property of equivalences, Lemma 1.1.7. □

Universality of coproducts

To get the expected behavior of coproducts, we need to impose they satisfy a certain 'universality' property.

Axiom B.7 (Universality of initial category)**.** Given a synthetic category C, any functor $C \to \emptyset$ is an equivalence.

Corollary 1.1.40. *For every functor $g\colon D \to E$, the commutative square*

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\
D \arrow[r, "g"] & E
\end{tikzcd}$$

is a pullback square.

Proof. We need to show that the induced map $\emptyset \to \times_E D$ is an equivalence. But this follows directly from Axiom B.7 by applying 2-out-of-3 to the maps $\emptyset \to \emptyset \times_E D \xrightarrow{\mathrm{pr}_\emptyset} \emptyset$. □

Axiom B.8 (Universality of coproducts)**.** Binary coproducts of categories are universal:

- Consider synthetic categories E_0 and E_1, and define $E := E_0 \sqcup E_1$. Then for every functor $f\colon C \to E$, the functor

$$\langle \mathrm{pr}_C, \mathrm{pr}_C \rangle \colon (C \times_E E_0) \sqcup (C \times_E E_1) \to C$$

  is an equivalence.

- For functors $f_0\colon C_0 \to E$, $f_1\colon C_1 \to E$ and $g\colon D \to E$, the commutative square

$$\begin{tikzcd}
(C_0 \times_E D) \sqcup (C_1 \times_E D) \arrow[r] \arrow[d] & C_0 \sqcup C_1 \arrow[d, "\langle f_0, f_1 \rangle"] \\
D \arrow[r, "g"] & E
\end{tikzcd}$$

  is a pullback square.

Pushouts of categories

The definition of pullbacks may be dualized as follows:

Definition 1.1.41 (Pushout square)**.** A commutative square of synthetic categories

$$\begin{tikzcd}
C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\
D' \arrow[r, "v"] & D
\end{tikzcd}$$

is called a *pushout square* (or *cocartesian*) if, for every synthetic category E, the induced square

$$\begin{tikzcd}
\mathrm{Fun}(D,E) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Fun}(D',C) \arrow[d, "(f')^*"] \\
\mathrm{Fun}(C,E) \arrow[r, "u^*"] & \mathrm{Fun}(C',E)
\end{tikzcd}$$

is a pullback square.

Lemma 1.1.42 (Pasting lemma for pullback squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\
D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3.
\end{tikzcd}$$

If the left-hand square is a pushout square, then the right-hand square is a pushout square if and only if the outer rectangle is a pushout square.

Proof. This follows immediately from Lemma 1.1.38. □

1.2 Embeddings of categories

We introduce the notion of *embeddings* of synthetic categories and prove basic properties about them. We will eventually see that embeddings precisely correspond to *subcategories*, see Proposition 6.4.1.

Definition 1.2.1 (Embedding)**.** A functor $f\colon C \to D$ is called an *embedding* if the diagonal $\Delta_f := (\mathrm{id}_C, \mathrm{id}_C)\colon C \to C \times_D C$ is an equivalence, or equivalently if the commutative diagram

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "f"] \\
C \arrow[r, "f"'] & D
\end{tikzcd}$$

is a pullback square.

Lemma 1.2.2. *A functor $f\colon C \to D$ is an embedding if and only if the commutative square*

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "(\mathrm{id}_C{,}\mathrm{id}_C)"'] & D \arrow[d, "(\mathrm{id}_D{,}\mathrm{id}_D)"] \\
C \times C \arrow[r, "f \times f"] & D \times D
\end{tikzcd}$$

is a pullback square.

Proof. This follows by applying the pasting lemma of pullback squares, Lemma 1.1.38, to the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "(\mathrm{id}_C{,}\mathrm{id}_C)"'] & C \arrow[r, "u"] \arrow[d, "(\mathrm{id}_C{,}u)"] & D \arrow[d, "(\mathrm{id}_D{,}\mathrm{id}_D)"] \\
C \times C \arrow[r, "1 \times u"] \arrow[d, "\mathrm{pr}_2"'] & C \times D \arrow[r, "u \times 1"] \arrow[d, "\mathrm{pr}_2"] & D \times D \\
C \arrow[r, "u"] & D.
\end{tikzcd}$$

□

Lemma 1.2.3. *Consider a pullback square*

$$\begin{tikzcd}
C' \arrow[r, "g"] \arrow[d, "u"'] & C \arrow[d, "v"] \\
D' \arrow[r, "f"] & C
\end{tikzcd}$$

of synthetic categories. If v is an embedding, then also u is an embedding.

Proof. This is left as an exercise for the reader, see Exercise 1.6.7 □

Lemma 1.2.4. *Let $f\colon C \to D$ and $g\colon D \to E$ be functors and assume that g is an embedding. Then the square*

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "f"'] & C \arrow[d, "gf"] \\
D \arrow[r, "g"] & E
\end{tikzcd}$$

is a pullback square.

Proof. This is immediate from the pasting law of pullback squares applied to the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "f"'] & C \arrow[d, "f"] \\
D \arrow[r, equal] \arrow[d, equal] & D \arrow[d, "g"] \\
D \arrow[r, "g"] & E.
\end{tikzcd}$$

Indeed, the bottom square is a pullback square by assumption on v, while the top square is clearly a pullback square. $\square$

Lemma 1.2.5. *Consider a commutative triangle*

$$\begin{tikzcd}
C \arrow[rr, "f"] \arrow[dr, "gf"'] & & D \arrow[dl, "g"] \\
& E &
\end{tikzcd}$$

of functors. If g is an embedding, then f is an embedding if and only if gf is an embedding.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, equal] & C \arrow[r, equal] \arrow[d, "u"] & C \arrow[d, "vu"] \\
C \arrow[r, "u"] & D \arrow[r, "v"] & E.
\end{tikzcd}$$

Since v is an embedding, the right square is a pullback square by Lemma 1.2.4. It follows from the pasting law of pullback squares, Lemma 1.1.38, that the left square is cartesian if and only if the outer rectangle is cartesian, proving the claim. $\square$

Corollary 1.2.6. *Consider a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"] & D
\end{tikzcd}$$

of functors such that v is an embedding. Then u is an embedding if and only if the induced map $(f,u)\colon A \to B \times_D C$ is an embedding.

Proof. The map $u\colon A \to C$ factors as the composite $A \xrightarrow{(f,u)} B \times_D C \xrightarrow{\mathrm{pr}_C} C$. The map pr_C is an embedding by Lemma 1.2.3, and hence the statement follows from Lemma 1.2.5. $\square$

Lemma 1.2.7. *Consider a commutative diagram*

$$\begin{tikzcd}
C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\
D' \arrow[r, "v"] \arrow[d, "g'"'] & D \arrow[d, "g"] \\
E' \arrow[r, "w"] & E.
\end{tikzcd}$$

If v and w are embeddings and the composite square is a pullback square, then also the top square is a pullback square.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
C' \arrow[r, equal] \arrow[d, "f'"'] & C' \arrow[r, "u"] \arrow[d, "(g'f',fu)"] & C \arrow[d, "f"] \\
D' \arrow[r, "(g',v)"] & E' \times_E D \arrow[r] \arrow[d] & D \arrow[d] \\
 & E' \arrow[r, "w"] & E.
\end{tikzcd}$$

Since w is an embedding, the map $E' \times_E D \to D$ is an embedding by Lemma 1.2.3. Since $v\colon D' \to D$ is also an embedding, it thus follows from Lemma 1.2.5 that the map $(g', v)\colon D' \to E' \times_E D$ is an embedding, and thus by Lemma 1.2.4 the top left square is a homotopy pullback square. Since the right bottom square and right outer rectangle are homotopy pullback squares, so is the top right square by the pasting law Lemma 1.1.38. Another instance of Lemma 1.1.38 shows that the outer rectangle is a homotopy pullback square, as desired. □

Lemma 1.2.8. *Let $f\colon C \to D$ be an embedding which admits a section $s\colon D \to c$. Then f is an equivalence.*

Proof. As $fs \cong \mathrm{id}_D$, it remains to show that $sf \cong \mathrm{id}_c$. Consider the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, dashed, "f"] \arrow[rr, equal, bend left] \arrow[d, "h"'] & C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "f"] \\
D \arrow[r, "s"] & C \arrow[r, "f"] & D.
\end{tikzcd}$$

Since the right square is a pullback square by assumption on f, there exists a map $h\colon A \to A$ making the diagram commute up to homotopy. It follows that $sf \cong h \cong \mathrm{id}_C$, finishing the proof. □

1.3 Morphisms and commutative diagrams

Our discussion so far has focused on the *external* behavior of synthetic categories. In this section, we will describe their *internal* structure by introducing the concepts of *morphisms* and *commutative diagrams* in a synthetic category C.

Axiom C. There is a synthetic category $[1]$, called the *directed interval* or the *walking morphism*. It comes equipped with two objects $0 \in [1]$ and $1 \in [1]$.

The synthetic category $[1]$ will take up the role of the 'free morphism' in a synthetic category, and it is frequently useful to pictorially visualize $[1]$ as follows:

$$0 \longrightarrow 1.$$

Definition 1.3.1 (Morphisms)**.** A *morphism* in a synthetic category C is defined as a functor $f\colon [1] \to C$. We refer to the synthetic category $\mathrm{Fun}([1],C)$ as the *arrow category* of C, whose objects are precisely the morphisms in C.

Every morphism $f\colon [1] \to C$ has a *source/domain* $f(0)\colon * \to C$ and a *target/codomain* $f(1)\colon * \to C$, obtained by precomposition with $0,1\colon * \to [1]$. If f is a morphism in C with domain $x = f(0)$ and codomain $y = f(1)$, we will often write $f\colon x \to y$ and say that f *is a morphism from* x *to* y.

We write $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}([1],C) \to C$ for the source and target functors, given by evaluation at 0 and 1, cf. Construction 1.1.21.

Definition 1.3.2 (Natural transformations)**.** If C and D are synthetic categories, we define a *natural transformation of functors* $C \to D$ to be a morphism in the functor category $\mathrm{Fun}(C,D)$. By (un)currying, this may equivalently be encoded as a functor $[1]\times C \to D$.

Of course, morphisms in a category do not exist in isolation: we may combine them to form *commutative diagrams*. To formalize the formation of diagram shapes and diagram categories, we will now introduce the language of *cubes* and *topes*, due to Riehl and Shulman [RS17]. While Riehl and Shulman provide a precise and formal type theoretical implementation of cubes and topes, we will state the definitions in natural language.

Definition 1.3.3 (Cubes and topes)**.** Given a natural number n, a *cube* of dimension n is an n-fold product $[1]^n = [1]\times\cdots\times[1]$ of the directed interval. A *tope* in dimension n is a first order formula in n variables $t_1,\ldots,t_n$ using the relation symbols $\leq$ and $=$ and constants 0 and 1.

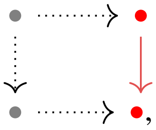
The relation symbols $\leq$ and $=$ satisfy the usual logical rules for a linear order, so that for example the tope $(t_i \leq t_j)\wedge(t_j \leq t_i)$ implies $t_i = t_j$. The constants 0 and 1 are the minimum and maximum for the relation $\leq$, in the sense that $0 \leq t$ and $t \leq 1$ are always assumed.

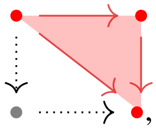
Example 1.3.4. The following are examples of topes in dimension 2:

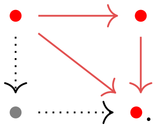
$$\varphi_1 = (1 \leq t_1), \qquad \varphi_2 = (t_2 \leq t_1), \qquad \varphi_3 = (t_2 = 0)\vee(t_1 = 1)\vee(t_1 = t_2).$$

Every tope φ of dimension n encodes a certain *diagram shape* Φ^n_φ, which may be thought of as the subshape of the n-dimensional cube $[1]^n$ consisting of the collection of n-tuples

$(t_1, \ldots, t_n)$ in $[1]$ that satisfy the formula φ. For example, the diagram shapes associated to the topes φ_1, φ_2 and φ_3 are the subshapes of $[1]^2$ marked in red as follows:

$\Phi^2_{\varphi_1} =$ 

$\Delta^2 := \Phi^2_{\varphi_2} =$ 

$\partial\Delta^2 := \Phi^2_{\varphi_3} =$ 

Here the horizontal direction indicates the t_1-coordinate, while the vertical direction indicates the t_2-coordinate. Note that the shapes Δ^2 and $\partial\Delta^2$ contain the same vertices and edges, but are nevertheless different since Δ^2 also contains the interior of the triangle, while $\partial\Delta^2$ only contains its boundary.

Axiom D.1 (Diagram category)**.** For a tope φ of dimension n and a synthetic category C, there is a *diagram category* $\mathrm{Diag}(\Phi^n_\varphi, C)$. We refer to objects of $\mathrm{Diag}(\Phi^n_\varphi, C)$ as *Φ^n_φ-shaped diagrams in C*.

The assignment $C \mapsto \mathrm{Diag}(\Phi^n_\varphi, C)$ is functorial in C: every functor $f \colon C \to D$ induces a functor

$$f \circ - \colon \mathrm{Diag}(\Phi^n_\varphi, C) \to \mathrm{Diag}(\Phi^n_\varphi, D),$$

and this process preserves identity functors and composition of functors.

Diagram categories are also functorial in the tope φ: if φ and ψ are two topes of dimension n such that the implication $\psi \to \varphi$ is satisfied, there is a restriction functor

$$(-)|_\psi \colon \mathrm{Diag}(\Phi^n_\varphi, C) \to \mathrm{Diag}(\Phi^n_\psi, C).$$

When $\varphi = \psi$ this functor is isomorphic to the identity, and when we have two implications $\gamma \to \psi \to \varphi$ we have an isomorphism $(-)|_\gamma \circ (-)|_\psi \cong (-)|_\psi$ of functors $\mathrm{Diag}(\Phi^n_\varphi, C) \to \mathrm{Diag}(\Phi^n_\gamma, C)$.

The functoriality of diagram categories in the cube-direction is a bit more subtle. Given two cubes $[1]^m$ and $[1]^n$, a *morphism of cubes from* $[1]^m$ *to* $[1]^n$ consists of n maps $[1]^m \to [1]$ that are of one of the following three forms:

- A projection map $\mathrm{pr}_j \colon [1]^m \to [1]$ to the j-th component for some $1 \leq j \leq m$;
- The constant map $\mathrm{const}_0 \colon [1]^m \to * \xrightarrow{0} [1]$ with value 0;
- The constant map $\mathrm{const}_1 \colon [1]^m \to * \xrightarrow{1} [1]$ with value 1.

Such a morphism is specified by choosing a function $\alpha\colon \{1,\dots,n\} \to \{\bot, 1,\dots,m,\top\}$, where the values $\bot$ and $\top$ are meant to refer to the constant maps const_0 and const_1, respectively. Given such function α, we denote the resulting morphism of cubes by $\alpha^*\colon [1]^m \to [1]^n$. Every tope φ of dimension n can then be turned into a tope $\alpha^*\varphi$ of dimension m via substitution: we define

$$\alpha^*\varphi := \varphi[t_{\alpha(1)}/t_1,\dots,t_{\alpha(n)}/t_n],$$

where the notation means that each variable t_i in φ gets substituted by the variable $t_{\alpha(i)}$ whenever $\alpha(i) \in \{1,\dots,m\}$, or gets substituted by the constant $t_\bot := 0$ or $t_\top := 1$ whenever $\alpha(i) \in \{\bot,\top\}$. We may think of the associated subshape $\Phi^m_{\alpha^*\varphi}$ of $[1]^m$ as the preimage under $\alpha^*\colon [1]^m \to [1]^n$ of the subshape Φ^n_φ of $[1]^n$. For example, the 2-dimensional tope φ_1 from Example 1.3.4 is obtained from the 1-dimensional tope $(1 \leq t_1)$ by substitution along the inclusion $\alpha\colon \{1\} \hookrightarrow \{1,2\} \hookrightarrow \{\bot,1,2,\top\}$.

Axiom D.2. Given a tope φ of dimension n and a function $\alpha\colon \{1,\dots,n\} \to \{\bot,1,\dots,m,\top\}$, we demand the existence of a *restriction functor*

$$\alpha^*\colon \mathrm{Diag}(\Phi^n_\varphi, C) \to \mathrm{Diag}(\Phi^m_{\alpha^*\varphi}, C).$$

If $n = m$ and $\alpha(i) = i$ for all i, then α^* is isomorphic to the identity functor. Given another map $\beta\colon \{1,\dots,m\} \to \{\bot,1,\dots,k,\top\}$, we obtain a composite $\beta\circ\alpha\colon \{1,\dots,n\} \to \{\bot,1,\dots,k,\top\}$ by setting $\beta(\bot) = \bot$ and $\beta(\top) = \top$, and we demand that $(\beta\circ\alpha)^*$ is isomorphic to $\alpha^*\circ\beta^*$.

Definition 1.3.5. Let ψ be an m-dimensional tope and let φ be an n-dimensional tope. A *morphism of topes* $\alpha\colon \psi \to \varphi$ is a morphism of cubes $\alpha\colon [1]^m \to [1]^n$ such that the implication $\psi \to \alpha^*(\varphi)$ is satisfied. We define the associated *restriction functor* as the composite

$$\alpha^*\colon \mathrm{Diag}(\Phi^n_\varphi, C) \xrightarrow{\alpha^*} \mathrm{Diag}(\Phi^m_{\alpha^*\varphi}, C) \xrightarrow{(-)|_\psi} \mathrm{Diag}(\Phi^m_\psi, C).$$

This construction is functorial in morphisms of topes.

We would next like to enforce that the diagram category $\mathrm{Diag}(\Phi^n_\varphi, C)$ really only depends on the underlying diagram shape of φ and not on the choice of embedding into some cube. We will do this by enforcing that certain ‘isomorphisms’ of topes induce equivalences at the level of diagram categories.

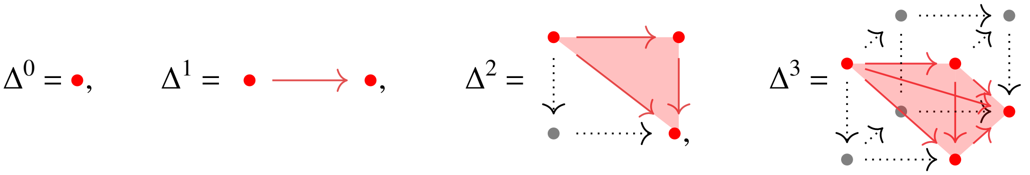
Definition 1.3.6. A morphism of topes $\alpha\colon \psi \to \varphi$ is called an *isomorphism* if there exists another morphism $\beta\colon \varphi \to \psi$ such that both implications $\alpha^*(\beta^*(\psi)) \to \psi$ and $\beta^*(\alpha^*(\varphi)) \to \varphi$ are satisfied.

Axiom D.3. Let $\alpha\colon \psi \to \varphi$ be an isomorphism of topes. Then the restriction functor $\alpha^*\colon \operatorname{Diag}(\Phi^n_\varphi, C) \to \operatorname{Diag}(\Phi^m_\psi, C)$ is an equivalence for every synthetic category C.

Definition 1.3.7 (Standard simplices)**.** We define the diagram shapes Δ^0, Δ^1, Δ^2 and Δ^3 as follows:

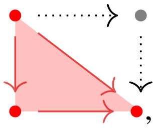
$$\Delta^0 := \Phi^0_\top, \qquad \Delta^1 := \Phi^1_\top, \qquad \Delta^2 := \Phi^2_{t_2\le t_1}, \qquad \Delta^3 := \Phi^3_{(t_2\le t_1)\wedge(t_3\le t_2)}.$$

Here $\top$ denotes the truth-value 'true', as a first order formula. We refer to the shape Δ^n as the *standard n-simplex* for $n = 0,1,2,3$. These shapes may be displayed as follows:



It will sometimes also be useful to consider the following 'alternative' 2-simplex:

$$\Delta^2_{\text{alt}} := \Phi^2_{t_1\le t_2} =$$



Since the swap map $[1]^2 \to [1]^2$ permuting the two factors transforms the standard 2-simplex Δ^2 into the alternative 2-simplex Δ^2_{alt}, there is for every synthetic category C a preferred equivalence

$$\operatorname{Diag}(\Delta^2, C) \xrightarrow{\sim} \operatorname{Diag}(\Delta^2_{\text{alt}}, C).$$

Definition 1.3.8 (Commutative triangle)**.** Given a synthetic category C, we define a *commutative triangle in C* to be a Δ^2-shaped diagram. The three implications

$$(t_1 = 1) \to (t_2 \le t_1), \qquad (t_1 = t_2) \to (t_2 \le t_1) \qquad \text{and} \qquad (t_2 = 0) \to (t_2 \le t_1)$$

provide three inclusions of diagram shapes $\Delta^1 \hookrightarrow \Delta^2$, and hence there are three restriction functors

$$\delta^2_0, \delta^2_1, \delta^2_2 : \operatorname{Diag}(\Delta^2, C) \to \operatorname{Diag}(\Delta^1, C).$$

We will frequently denote commutative triangles by

$$\sigma \quad = \quad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z, \end{tikzcd}$$

where $f = \delta^2_2(\sigma)$, $g = \delta^2_0(\sigma)$ and $h = \delta^2_1(\sigma)$. We will frequently leave commutative triangles unnamed, and only label their three edges. We will sometimes also use this notation for $\partial\Delta^2$-shaped diagrams that do *not* commute, which will then be mentioned explicitly.

Definition 1.3.9 (Commutative square)**.** We define the shape diagram $\Delta^1 \times \Delta^1$ as $\Phi^2_\top$, i.e. the full subshape of the 2-dimensional cube $[1]^2$. Given a synthetic category C, we define a *commutative triangle in* C to be a $(\Delta^1 \times \Delta^1)$-shaped diagram. We will often display commutative squares as follows:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w. \end{tikzcd}$$

Structural rules for diagram categories

To ensure that the diagram categories $\mathrm{Diag}(\Phi, C)$ have the expected behavior, we need to enforce various additional rules. First, we need to enforce that the Δ^0-shaped diagrams and Δ^1-shaped diagrams correspond to objects and morphisms, respectively:

Axiom D.4. For every synthetic category C there are equivalences

$$\mathrm{Diag}(\Delta^0, C) \xrightarrow{\sim} C \qquad \text{and} \qquad \mathrm{Diag}(\Delta^1, C) \xrightarrow{\sim} \mathrm{Fun}([1], C).$$

Moreover, the following two squares commute:

$$\begin{tikzcd} \mathrm{Diag}(\Delta^0, C) \arrow[d, "\simeq"'] & \mathrm{Diag}(\Delta^1, C) \arrow[l, "\mathrm{const}_0^*"'] \arrow[r, "\mathrm{const}_1^*"] \arrow[d, "\simeq"] & \mathrm{Diag}(\Delta^0, C) \arrow[d, "\simeq"] \\ C & \mathrm{Fun}([1], C) \arrow[l, "\mathrm{ev}_0"] \arrow[r, "\mathrm{ev}_1"'] & C. \end{tikzcd}$$

We further need to prescribe the behavior of diagrams indexed by a product of shapes:

Axiom D.5 (Product shapes)**.** Let φ and ψ be topes of dimensions n and m, respectively. We may define a tope $\varphi \times \psi$ of dimension $n+m$ as follows:

$$\varphi \times \psi := \varphi \wedge \psi[t_{n+1}/t_1, \ldots, t_{n+m}/t_m],$$

where the notation means that all instances of the variable t_i in ψ get replaced by the variable t_{n+i} for $1 \leq i \leq m$. This operation is compatible with implications of topes: if we have implications $\varphi \to \varphi'$ and $\psi \to \psi'$ then also $\varphi \times \psi \to \varphi' \times \psi'$. Similarly, given two morphisms of cubes $\alpha^* \colon [1]^n \to [1]^{n'}$ and $\beta^* \colon [1]^m \to [1]^{m'}$ we may form their product $(\alpha \times \beta)^* \colon [1]^{n+m} \to [1]^{n'+m'}$, and for topes φ and ψ of dimensions n' and m' we have

$$\alpha^* \varphi \times \beta^* \psi = (\alpha \times \beta)^* (\varphi \times \psi).$$

We now demand that there is for every synthetic category C an equivalence

$$\mathrm{Diag}(\Phi^n_\varphi, \mathrm{Diag}(\Phi^m_\psi, C)) \xrightarrow{\sim} \mathrm{Diag}(\Phi^{n+m}_{\varphi\times\psi}, C)$$

which is compatible with the functoriality in C from Axiom D.2 and the restriction functoriality in topes from Definition 1.3.5.

Lemma 1.3.10. *For every natural number n and every synthetic category C, there is an equivalence* $\mathrm{Diag}((\Delta^1)^n, C) \xrightarrow{\sim} \mathrm{Fun}([1]^n, C)$.

Proof. For $n = 0$ both sides are equivalent to C by Axiom D.4 and Lemma 1.1.29. For $n = 1$ this is the content of Axiom D.4. For $n \geq 2$, the claim follows by induction, since there are equivalences

$$\mathrm{Diag}((\Delta^1)^{n+1}, C) \overset{D.5}{\simeq} \mathrm{Diag}(\Delta^1, \mathrm{Diag}((\Delta^1)^n, C)) \simeq \mathrm{Fun}([1], \mathrm{Fun}([1]^n, C)) \overset{1.1.28}{\simeq} \mathrm{Fun}([1]^{n+1}, C).$$

This finishes the proof. □

Axiom D.6 (Empty diagram)**.** For every synthetic category C, the synthetic category $\mathrm{Diag}(\Phi^0_\perp, C)$ is contractible, i.e. the functor $\mathrm{Diag}(\Phi^0_\perp, C) \to *$ is an equivalence.

A final property that we expect from diagram categories is a form of ‘excision’: if a diagram shape can be written as a union of two subshapes, we may construct such diagrams by provide two diagrams indexed by the two subshapes and show they agree on their intersection. We may formalize this as follows:

Axiom D.7 (Excision)**.** Let φ and ψ be n-dimensional topes. Then for every synthetic category C, the commutative square

$$\begin{tikzcd}
\mathrm{Diag}(\Phi^n_{\varphi\vee\psi}, C) \arrow[r, "(-)|_{\varphi}"] \arrow[d, "(-)|_{\varphi}"'] & \mathrm{Diag}(\Phi^n_{\psi}, C) \arrow[d, "(-)|_{\varphi\wedge\psi}"] \\
\mathrm{Diag}(\Phi^n_{\varphi}, C) \arrow[r, "(-)|_{\varphi\wedge\psi}"'] & \mathrm{Diag}(\Phi^n_{\varphi\wedge\psi}, C)
\end{tikzcd}$$

is a pullback square.

The most important example of excision is the fact that a commutative square in C is given by two commutative triangles which agree along their diagonal:

Lemma 1.3.11. *For every synthetic category C, there is a canonical pullback square*

$$\begin{tikzcd}
\mathrm{Diag}(\Delta^1 \times \Delta^1, C) \arrow[r, "(-)|_{\Delta^2}"] \arrow[d, "(-)|_{\Delta^2_{\mathrm{alt}}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Diag}(\Delta^2, C) \arrow[d, "(-)|_{\Delta^1}"] \\
\mathrm{Diag}(\Delta^2, C) \simeq \mathrm{Diag}(\Delta^2_{\mathrm{alt}}, C) \arrow[r, "(-)|_{\Delta^1}"'] & \mathrm{Diag}(\Delta^1, C).
\end{tikzcd}$$

Proof. We apply excision to the formulas $\varphi = (t_2 \le t_1)$ and $\psi = (t_1 \le t_2)$. We have $\Phi^2_{\varphi} = \Delta^2$ and $\Phi^2_{\psi} = \Delta^2_{\mathrm{alt}}$. Since the formula $(t_2 \le t_1) \vee (t_1 \le t_2)$ is equivalent to $\top$, we get $\Phi^2_{(t_2\le t_1)\vee(t_1\le t_2)} = \Delta^1 \times \Delta^1$. Since $(t_2 \le t_1) \wedge (t_1 \le t_2)$ is equivalent to $(t_1 = t_2)$, we get $\Phi^2_{(t_2\le t_1)\wedge(t_1\le t_2)} = \Phi^2_{t_1=t_2}$, and by Axiom D.3 its diagram category is equivalent to $\mathrm{Diag}(\Delta^1, C)$. This finishes the proof. □

As a consequence of Lemma 1.3.11, we see that we may write commutative triangles in terms of partially degenerate commutative square:

Lemma 1.3.12. *For every synthetic category C, there is a canonical pullback square*

$$\begin{tikzcd}
\mathrm{Diag}(\Delta^2, C) \arrow[r] \arrow[d, "(\delta^2_1)^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Diag}(\Delta^1 \times \Delta^1, C) \arrow[d, "(-)|_{\Delta^2_{\mathrm{alt}}}"] & \mathrm{Diag}(\Delta^2, C) \arrow[l] \arrow[d, "(\delta^2_1)^*"] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathrm{Diag}(\Delta^1, C) \arrow[r, "s^2_1"] & \mathrm{Diag}(\Delta^2_{\mathrm{alt}}, C) & \mathrm{Diag}(\Delta^1, C). \arrow[l, "s^2_2"]
\end{tikzcd}$$

Proof. By symmetry it will suffice to produce the square on the left. This follows immediately from Lemma 1.3.11 and the pasting law of pullback squares, since the composite morphism of shapes $\Delta^1 \xrightarrow{\delta^2_1} \Delta^2_{\mathrm{alt}} \xrightarrow{s^2_1} \Delta^1$ is the identity. □

Corollary 1.3.13. *The restriction functor* $(-)|_{\Delta^2}\colon \mathrm{Diag}(\Delta^1 \times \Delta^1, C) \to \mathrm{Diag}(\Delta^2, C)$ *admits a section.* □

1.4 The Segal axiom

In this section, we will introduce the *Segal axiom*, which will allow us to talk about the *composition of morphisms* in a synthetic category.

Definition 1.4.1. We define the diagram shape Λ^2_1 as

$$\Lambda^2_1 := \Phi^2_{t_2=0\vee t_1=1} = \begin{tikzcd} \bullet \arrow[r] \arrow[d, dotted] \arrow[dr, dotted] & \bullet \arrow[d] \\ \bullet \arrow[r, dotted] & \bullet \end{tikzcd}.$$

By excision, there is for every synthetic category C a preferred equivalence

$$\mathrm{Diag}(\Lambda^2_1, C) \xrightarrow{\sim} \mathrm{Diag}(\Delta^1, C) \times_{\mathrm{Diag}(\Delta^0,C)} \mathrm{Diag}(\Delta^1, C) \xrightarrow{\sim} \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C).$$

Recall that the 2-simplex Δ^1 is defined as $\Phi^2_{t_2\leq t_1}$. Since the formula $t_2 = 0 \vee t_1 = 1$ implies $t_2 \leq t_1$, there is a canonical inclusion of diagram shapes $\Lambda^2_1 \hookrightarrow \Delta^2$.

Axiom E (Segal axiom)**.** For any synthetic category C, the restriction functor

$$(-)|_{\Lambda^2_1} \colon \mathrm{Diag}(\Delta^2, C) \to \mathrm{Diag}(\Lambda^2_1, C) \simeq \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C)$$

is an equivalence.

Informally speaking, the Segal axiom says that two composable morphisms in C admit an essentially unique composite. To make this precise, we will prove that for morphisms $f \colon x \to y$ and $g \colon y \to z$ in C, the synthetic category of commutative triangles

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & & z \end{tikzcd}$$

whose first leg is f and whose second leg is g is contractible.

Definition 1.4.2. Given two morphisms $f \colon x \to y$ and $g \colon y \to z$ in a synthetic category C, we define their *category of compositions* $\mathrm{Comp}_{g,f}$ as the following pullback:

$$\begin{tikzcd} \mathrm{Comp}_{g,f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Diag}(\Delta^2, C) \arrow[d] \\ * \arrow[r, "{(f,g)}"] & \mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C). \end{tikzcd}$$

Lemma 1.4.3. *For every pair of morphisms* (f,g) *in a synthetic category* C*, the synthetic category* $\mathrm{Comp}_{g,f}$ *is contractible.*

Proof. Since the right vertical map $\mathrm{Diag}(\Delta^2,C) \to \mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C)$ is an equivalence, this is an immediate consequence of Lemma 1.1.35. □

Due to the Segal axiom, synthetic categories behave like ordinary categories, in the sense that they come with notions of *identity morphisms* and *composition of morphisms*. We will now explain how to obtain this additional structure on a synthetic category.

Construction 1.4.4 (Identity morphisms)**.** For an object x of a synthetic category C, we define the *identity morphism* $\mathrm{id}_x\colon x \to x$ in C as the image of x under the map

$$p_{[1]}^{*}\colon C = \mathrm{Fun}(*,C) \to \mathrm{Fun}([1],C).$$

We may regard id_x as an object of the hom groupoid $C(x,x)$.

Definition 1.4.5 (Hom groupoid)**.** Let C be any synthetic category. Given objects x and y of C, we define the *hom groupoid* $C(x,y)$ via the following pullback square:

$$\begin{tikzcd}
C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
* \arrow[r, "(x,y)"] & C\times C.
\end{tikzcd}$$

Observe that the objects of $C(x,y)$ are triples (f,α,β), where $f\colon x' \to y'$ is a morphism in C and $\alpha\colon x \cong x'$ and $\beta\colon y \cong y'$ are isomorphisms in C. We will sometimes also use the alternative notation $\mathrm{Hom}_C(x,y)$ for $C(x,y)$.

Construction 1.4.6 (Composition functors)**.** Let x, y and z be objects of a synthetic category C. We define the synthetic category $C(x,y,z)$ via the following pullback square:

$$\begin{tikzcd}
C(x,y,z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Diag}(\Delta^2,C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1,\mathrm{ev}_2)"] \\
* \arrow[r, "(x,y,z)"] & C\times C\times C.
\end{tikzcd}$$

Consider now the zig-zag

$$\mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \xleftarrow{\sim} \mathrm{Diag}(\Delta^2,C) \xrightarrow{(\delta_1^2)^*} \mathrm{Fun}([1],C).$$

The first map is an equivalence by the Segal axiom, and hence it admits an inverse. We thus obtain a map

$$-\circ-\colon\ \mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \xrightarrow{\sim} \mathrm{Diag}(\Delta^2,C) \xrightarrow{(\delta_1^2)^*} \mathrm{Fun}([1],C).$$

Passing to fibers over (x,y,z) then produces a functor

$$-\circ-\colon C(x,y)\times C(y,z)\xrightarrow{\simeq} C(x,y,z)\to C(x,z),$$

which we call the *composition functor*.

We will now show that composition in a synthetic category is unital and associative.

Lemma 1.4.7 (Unitality, [RS17, Proposition 5.8]). *For every morphism $f\colon x\to y$ in a synthetic category C, there are isomorphisms*

$$\mathrm{id}_y\circ f\cong f\cong f\circ\mathrm{id}_x$$

in $\mathrm{Fun}([1],C)$. *Moreover, these isomorphisms are natural in f, in the sense that they form natural isomorphisms of functors* $\mathrm{Fun}([1],C)\to\mathrm{Fun}([1],C)$.

Proof. We will prove the first relation and leave the second to the reader. We have to prove that the composite

$$\mathrm{Fun}([1],C)\xrightarrow{\sim}\mathrm{Fun}([1],C)\times_C C\xrightarrow{1\times p^*_{[1]}}\mathrm{Fun}([1],C)\times_C\mathrm{Fun}([1])\xrightarrow{-\circ-}\mathrm{Fun}([1],C)$$

is isomorphic to the identity functor. But this is immediate from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r, "(s^2_1)^*"] \arrow[d, "\cong"'] \arrow[rr, bend left, equal] & \mathrm{Diag}(\Delta^2,C) \arrow[r, "(\delta^2_1)^*"] \arrow[d, "\simeq"] & \mathrm{Fun}([1],C) \arrow[d, equal] \\
\mathrm{Fun}([1],C)\times_C C \arrow[r, "1\times p^*_{[1]}"'] & \mathrm{Fun}([1],C)\times_C\mathrm{Fun}([1]) \arrow[r, "-\circ-"'] & \mathrm{Fun}([1],C).
\end{tikzcd}$$

□

Proposition 1.4.8 (Associativity, [RS17, Proposition 5.9]). *For composable morphisms $f\colon x\to y$, $g\colon y\to z$ and $h\colon z\to w$ in a synthetic category C, there is an isomorphism*

$$h\circ(g\circ f)\cong(h\circ g)\circ f$$

in $\mathrm{Fun}([1],C)$. *Moreover, this homotopy is natural in the triple (f,g,h), in the sense that it forms a natural isomorphism of functors*

$$-\circ(-\circ-)\cong(-\circ-)\circ-\colon\ \mathrm{Fun}([1],C)\times_C\mathrm{Fun}([1],C)\times_C\mathrm{Fun}([1],C)\to\mathrm{Fun}([1],C).$$

Proof. Consider first the forgetful functor

$$\mathrm{Diag}(\Delta^2,C)\times_{\mathrm{Fun}([1],C)}\mathrm{Diag}(\Delta^2,C)\to\mathrm{Fun}([1],C)\times_C\mathrm{Fun}([1],C)\times_C\mathrm{Fun}([1],C)$$

given by restriction along the following inclusion of diagram shapes:

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[d] & \\
 & (1,1) \arrow[r] & (1,2)
\end{tikzcd}
\quad \hookrightarrow \quad
\begin{tikzcd}
(0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \arrow[dr] & \\
 & (1,1) \arrow[r] & (1,2).
\end{tikzcd}$$

It follows immediately from the Segal axiom that this functor is an equivalence, and hence we may prove the claim after precomposing with this functor.

Consider next the forgetful functor

$$\mathrm{Fun}([1]\times[1],C)\times_{\mathrm{Fun}([1],C)}\mathrm{Fun}([1]\times[1],C)\to \mathrm{Diag}(\Delta^2,C)\times_{\mathrm{Fun}([1],C)}\mathrm{Diag}(\Delta^2,C)$$

given by restriction along the following inclusion of diagram shapes:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \arrow[dr] & \\
 & (1,1) \arrow[r] & (1,2)
\end{tikzcd}
\quad \hookrightarrow \quad
\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\
(1,0) \arrow[r] & (1,1) \arrow[r] & (1,2).
\end{tikzcd}$$

This restriction functor admits a section given informally by

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "g\circ f"'] & y \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
 & z \arrow[r, "h"'] & w
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, "g\circ f"] & y \arrow[r, "h\circ g"] \arrow[d, "g"] \arrow[dr, "h\circ g"] & w \arrow[d, equal] \\
x \arrow[r, "g\circ f"'] & z \arrow[r, "h"'] & w,
\end{tikzcd}$$

using the pullback description of $\mathrm{Fun}([1]\times[1],C)$ from Lemma 16.2.21. It will thus suffice to prove the claim after precomposing with this functor.

Consider thirdly the forgetful functor

$$\mathrm{Diag}(\Delta^2\times[1],C)\to \mathrm{Fun}([1]\times[1],C)\times_{\mathrm{Fun}([1],C)}\mathrm{Fun}([1]\times[1],C)$$

given by restriction along the following inclusion of diagram shapes:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\
(1,0) \arrow[r] & (1,1) \arrow[r] & (1,2)
\end{tikzcd}
\quad \hookrightarrow \quad
\begin{tikzcd}
 & (0,1) \arrow[dr] \arrow[dd] & \\
(0,0) \arrow[ur] \arrow[rr] \arrow[dd] & & (0,2) \arrow[dd] \\
 & (1,1) \arrow[dr] & \\
(1,0) \arrow[ur] \arrow[rr] & & (1,2).
\end{tikzcd}$$

This restriction functor is equivalent to the restriction functor

$$\mathrm{Diag}(\Delta^2,\mathrm{Fun}([1],C))\to \mathrm{Diag}(\Lambda^2_1,\mathrm{Fun}([1],C))$$

and hence it is an equivalence by the Segal axiom, Axiom E. We conclude once more that it suffices to prove the claim after precomposing with this functor. All in all, we must exhibit a natural isomorphism of the bottom square in the following diagram after composing with the left vertical map:

$$\begin{tikzcd}
\mathrm{Diag}(\Delta^2\times\Delta^1,C) \arrow[rrr] \arrow[drr] \arrow[dd] & & & \mathrm{Diag}(\Delta^2,C) \arrow[dr, "\delta_1^2"] \arrow[dd, "\simeq"] & \\
 & & \mathrm{Diag}(\Delta^2,C) \arrow[rr, "\delta_1^2"] \arrow[dd, "\simeq"] & & \mathrm{Fun}(\Delta^1,C) \arrow[dd, equal] \\
\mathrm{Fun}(\Delta^1,C)\times_C\mathrm{Fun}(\Delta^1,C)\times_C\mathrm{Fun}(\Delta^1,C) \arrow[rrr, "\mathrm{id}\times(-\circ-)"] \arrow[drr, "(-\circ-)\times\mathrm{id}"'] & & & \mathrm{Fun}(\Delta^1,C)\times_C\mathrm{Fun}(\Delta^1,C) \arrow[dr, "-\circ-"] & \\
 & & \mathrm{Fun}(\Delta^1,C)\times_C\mathrm{Fun}(\Delta^1,C) \arrow[rr, "-\circ-"] & & \mathrm{Fun}(\Delta^1,C)
\end{tikzcd}$$

The two unlabeled maps in the top square are given by restriction along the following two inclusions of diagram shapes:

$$\begin{tikzcd}
 & & & & (0,1) \arrow[dr] \arrow[dd] & & & & \\
(0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] & & (0,0) \arrow[ur] \arrow[rr] \arrow[dd] & & (0,2) \arrow[dd] & & (0,0) \arrow[d] \arrow[dr] & \\
 & (1,2) & \hookrightarrow & & (1,1) \arrow[dr] & & \hookleftarrow & (1,1) \arrow[r] & (1,2). \\
 & & & (1,0) \arrow[ur] \arrow[rr] & & (1,2) & & &
\end{tikzcd}$$

It is now clear from the definition of the composition functor that each of the four vertical squares commute up to homotopy, and since the top square clearly commutes this finishes the proof. □

Observe that another way of formulating Axiom E is that the commutative square

$$\begin{tikzcd}
\mathrm{Diag}(\Delta^2,C) \arrow[r, "(\delta_2^2)^*"] \arrow[d, "(\delta_0^2)^*"'] & \mathrm{Fun}(\Delta^1,C) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Fun}(\Delta^1,C) \arrow[r, "\mathrm{ev}_0"] & C
\end{tikzcd}$$

is a pullback square. This has the following corollary:

Corollary 1.4.9. *For a synthetic category C, there are pullback squares*

$$\begin{tikzcd}
\mathrm{Diag}(\Delta^2,C) \arrow[r] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}(\Delta^1\times\Delta^1,C) \arrow[d] & \textit{and} & \mathrm{Diag}(\Delta^2,C) \arrow[r] \arrow[d, "\mathrm{ev}_2"'] & \mathrm{Fun}(\Delta^1\times\Delta^1,C) \arrow[d] \\
C \arrow[r, "(p_{\Delta^1})^*"] & \mathrm{Fun}(\{0\}\times\Delta^1,C) & & C \arrow[r, "(p_{\Delta^1})^*"] & \mathrm{Fun}(\{1\}\times\Delta^1,C).
\end{tikzcd}$$

Proof. We will prove the claim for the left square and leave the analogous proof of the right square to the reader. To this end, we consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Diag}(\Delta^2, C) \arrow[r] \arrow[d, "(\delta_1^2)^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[d, "(-)|_{\Delta^2_{\mathrm{alt}}}"] & \\
\mathrm{Fun}(\Delta^1, C) \arrow[r, "s_1^2"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Diag}(\Delta^2_{\mathrm{alt}}, C) \arrow[r, "(\delta_0^2)^*"] \arrow[d, "(\delta_2^2)^*"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "\mathrm{ev}_0"] \\
C \arrow[r, "(p_{\Delta^1})^*"] & \mathrm{Fun}(\Delta^1, C) \arrow[r, "\mathrm{ev}_1"] & C.
\end{tikzcd}$$

Here the top square is the left pullback square from Lemma 1.3.12. The right square is the pullback square from the previous remark, and it follows from the pasting law of pullback squares that the left bottom square is also a pullback square. The composite rectangle on the left is then the desired pullback square. □

Remark 1.4.10. Informally, the previous corollary says that commutative triangles in C may equivalently be encoded in the following three ways:

(1) One might consider diagrams of the form $\Delta^2 \to C$, which we will display as

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "gf"] & & z.
\end{tikzcd}$$

(2) One might consider diagrams $\Delta^1 \times \Delta^1 \to C$ whose restriction to $\{0\} \times \Delta^1$ is constant, which we will display as

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, "g"] \\
x \arrow[r, "gf"] & z.
\end{tikzcd}$$

(3) One might consider diagrams $\Delta^1 \times \Delta^1 \to C$ whose restriction to $\Delta^1 \times \{1\}$ is constant, which we will display as

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "gf"'] & y \arrow[d, "g"] \\
z \arrow[r, equal] & z.
\end{tikzcd}$$

1.5 The Rezk axiom

The notions of identity morphisms and composition of morphisms in synthetic categories, provided by the Segal axiom, lead to the notion of an *isomorphism* in a synthetic category:

Definition 1.5.1 (Isomorphisms)**.** Consider a morphism $f\colon x \to y$ in a synthetic category C. We say that f is *invertible*, or that it is an *isomorphism*, if there are commutative triangles in C of the form

$$\begin{tikzcd}
y \arrow[r, "g"] \arrow[dr, "\mathrm{id}_y"'] & x \arrow[d, "f"] \\
& y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "\mathrm{id}_x"'] & y \arrow[d, "h"] \\
& x.
\end{tikzcd}$$

A priori, this use of the word 'isomorphism' clashes with that of Definition 1.1.11. For this reason, we will now introduce the *Rezk axiom*, which enforces that these two notions of isomorphisms agree with each other. We start by defining the synthetic category $\mathrm{Iso}(C)$ of isomorphisms in C.

Construction 1.5.2. Consider the following two subshapes Φ and Ψ of Δ^3:

$$\Phi \;=\; \begin{tikzcd}
& 1 \arrow[dd] \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & 3, \\
& 2 \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\Psi \;=\; \begin{tikzcd}
& 1 \arrow[dd, dashed] \arrow[dr] & \\
0 \arrow[ur, dashed] \arrow[dr] & & 3. \\
& 2 \arrow[ur, dashed] &
\end{tikzcd}$$

More precisely, Φ is the diagram shape determined by the 3-dimensional tope

$$\varphi = (t_3 = 0 \wedge t_2 \le t_1) \vee (t_1 = 1 \wedge t_3 \le t_2),$$

while Ψ is the diagram shape determined by the 3-dimensional tope

$$\psi = (t_3 = 0 \wedge t_2 = 0) \vee (t_1 = 1 \wedge t_2 = 1).$$

By excision, there are for every synthetic category C equivalences

$$\begin{aligned}
\mathrm{Diag}(\Phi, C) &\xrightarrow{\sim} \mathrm{Diag}(\Delta^2, C) \times_{\mathrm{Diag}(\Delta^1, C)} \mathrm{Diag}(\Delta^2, C),\\
\mathrm{Diag}(\Psi, C) &\xrightarrow{\sim} \mathrm{Diag}(\Delta^1, C) \times \mathrm{Diag}(\Delta^1, C).
\end{aligned}$$

Definition 1.5.3. Given a synthetic category C, we define the synthetic category $\mathrm{Iso}(C)$ via the following pullback square:

$$\begin{tikzcd}
\mathrm{Iso}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Diag}(\Phi, C) \arrow[d] \\
C \times C \arrow[r, "{(x,y)\mapsto(\mathrm{id}_x,\mathrm{id}_y)}"] & \mathrm{Fun}(\Delta^1, C) \times \mathrm{Fun}(\Delta^1, C).
\end{tikzcd}$$

Explicitly, a term of $\mathrm{Iso}(C)$ is a diagram in C of the form

$$\begin{tikzcd}
 & x \arrow[dd, "f"] \arrow[dr, "\mathrm{id}_x"] & \\
y \arrow[ur, "g"] \arrow[dr, "\mathrm{id}_y"'] & & x. \\
 & y \arrow[ur, "h"'] &
\end{tikzcd}$$

Restricting along the map $\Delta^{\{1,2\}} \to \Delta^3$ induces a functor

$$\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1], C).$$

Notice that by definition a morphism $f \in \mathrm{Fun}([1], C)$ is invertible in C if and only if it lifts to an object of $\mathrm{Iso}(C)$. This motivates the following definition:

Definition 1.5.4. Let $f \colon x \to y$ be a morphism in a synthetic category C. We define the category $\{f \text{ is invertible}\}$ via the following pullback diagram:

$$\begin{tikzcd}
\{f \text{ is invertible}\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(C) \arrow[d, "\pi_{\mathrm{Iso}}"] \\
* \arrow[r, "f"] & \mathrm{Fun}([1], C).
\end{tikzcd}$$

As one expects, the identity map $\mathrm{id}_x \colon x \to x$ is invertible in C for every object x of C:

Construction 1.5.5. We construct a functor $i \colon C \to \mathrm{Iso}(C)$ lifting the functor $1 \colon C \to \mathrm{Fun}([1], C) \colon x \mapsto \mathrm{id}_x$. The map $\Delta^3 \to \Delta^0$ of diagram shapes induces a functor $C \to \mathrm{Fun}(\Delta^3, C)$, which sends an object to the diagram informally displayed as follows:

$$\begin{tikzcd}
 & x \arrow[dd, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_x"] & \\
x \arrow[ur, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_x"'] \arrow[rr, dashed, "\mathrm{id}_x"] & & x. \\
 & x \arrow[ur, "\mathrm{id}_x"'] &
\end{tikzcd}$$

Since the restriction of this diagram to the edges $\Delta^{\{0,2\}}$ and $\Delta^{\{1,3\}}$ are the identity on x, this functor factors through $\mathrm{Iso}(C)$, producing the desired functor $i \colon C \to \mathrm{Iso}(C)$.

Note that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[dr, "x \mapsto \mathrm{id}_x"'] & \mathrm{Iso}(C) \arrow[r, "{(s,t)}"] \arrow[d, "\pi_{\mathrm{Iso}}"] & C \times C \\
 & \mathrm{Fun}([1], C). \arrow[ur, "{(s,t)}"'] &
\end{tikzcd}$$

Axiom F (Rezk axiom)**.** For any synthetic category C, the functor $i\colon C \to \mathrm{Iso}(C)$ is an equivalence.

The Rezk axiom is important, as it tells us that two objects x and y of C are isomorphic in the sense of Definition 1.1.11 if and only if there exists an invertible morphism $f\colon x \to y$ between them:

Corollary 1.5.6. *Let x and y be objects of C and assume there exists an invertible morphism $f\colon x \to y$. Then there exists an isomorphism $x \cong y$ in C.*

Proof. Since the functor $i\colon c \to \mathrm{Iso}(C)$ is an equivalence, there exists an object $z \in C$ together with an isomorphism $i(z) \cong f$ in $\mathrm{Iso}(C)$. Using the source and target maps $s,t\colon \mathrm{Iso}(C) \to C$, this isomorphism produces isomorphisms $z \cong x$ and $z \cong y$ in C. By inversion we obtain $x \cong z$ and by composition we obtain $x \cong y$, as desired. □

1.6 Exercises Chapter 1

Exercise 1.6.1. Show that the product of synthetic categories is associative: for synthetic categories C, D and E there is a preferred equivalence $C \times (D \times E) \xrightarrow{\sim} (C \times D) \times E$.

Exercise 1.6.2. Show that the coproduct of synthetic categories is associative, commutative and unital: for synthetic categories C, D and E there are preferred equivalences

$$(C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Exercise 1.6.3. Prove that the functors

$$g \circ -\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D') \qquad \text{and} \qquad - \circ f\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C',D)$$

constructed in Construction 1.1.23 respect identity functors and composition of functors:

$$\begin{aligned} (\mathrm{id}_D \circ -) &\cong \mathrm{id}_{\mathrm{Fun}(C,D)}, & g' \circ (g \circ -) &\cong (g' \circ g) \circ - \\ (- \circ \mathrm{id}_C) &\cong \mathrm{id}_{\mathrm{Fun}(C,D)}, & (- \circ f) \circ f' &\cong - \circ (f \circ f'). \end{aligned}$$

Exercise 1.6.4. Show that the fiber product of synthetic categories is commutative, associative and unital: for functors $C \to E$, $D \to E$ and $B \to E$, there are preferred equivalences

$$C \times_E D \xrightarrow{\sim} D \times_E C, \qquad B \times_E (C \times_E D) \xrightarrow{\sim} (B \times_E C) \times_E D, \qquad C \times_E E \xrightarrow{\sim} C.$$

Exercise 1.6.5. Fill in the missing details at the end of the proof of Lemma 1.1.36.

Exercise 1.6.6. Let $f\colon C \to D$ and $g\colon D \to E$ be two embeddings of synthetic categories. Show that the composite $gf\colon C \to E$ is also an embedding.

Exercise 1.6.7. Consider a pullback square

$$\begin{tikzcd}
C' \arrow[r, "g"] \arrow[d, "u"'] & C \arrow[d, "v"] \\
D' \arrow[r, "f"] & C
\end{tikzcd}$$

of synthetic categories and assume that v is an embedding. Show that also u is an embedding.

2 Groupoids

In this chapter, we discuss in detail the notion of a *groupoid*: a synthetic category in which every morphism is an isomorphism. Groupoids will play a fundamental role in our development of synthetic category theory: many statements about synthetic categories can ultimately be reduced to statements about groupoids. Every synthetic category C will come equipped with a maximal subgroupoid $C^{\simeq}$ called the *groupoid core* of C, which we may think of as remembering only the objects of C and forgetting its morphisms. The existence of the groupoid core is what will eventually allow us to check various properties about synthetic categories *objectwise*, in an appropriate sense.

2.1 Groupoids

A groupoid is a category in which every morphism is invertible. For the purpuse of synthetic category theory, the following alternative definition is somewhat more convenient:

Definition 2.1.1 (Groupoid)**.** A synthetic category X is called a *groupoid* if the functor

$$(p_{[1]})^* \colon X \xrightarrow{\sim} \mathrm{Fun}(*,X) \to \mathrm{Fun}([1],X)$$

induced by the map $p_{[1]} \colon [1] \to *$ is an equivalence.

Example 2.1.2. The terminal category $*$ is a groupoid: there is a commutative triangle

$$\begin{tikzcd}
& \mathrm{Fun}([1],*) \arrow[dr] & \\
* \arrow[ur, "(p_{[1]})^*"] \arrow[rr] & & *,
\end{tikzcd}$$

and since the bottom and right diagonal maps are equivalences (see Lemma 1.1.26) the claim follows by 2-out-of-3.

Example 2.1.3. If X and Y are groupoids, then also their product $X \times Y$ is a groupoid: this is an immediate consequence of Lemma 1.1.27.

The following proposition expresses the fact that groupoids are precisely those synthetic categories all of whose morphisms are invertible:

Proposition 2.1.4. *A synthetic category X is a groupoid if and only if the functor $\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(X) \to \mathrm{Fun}([1],X)$ admits a section: there exists a functor $s\colon \mathrm{Fun}([1],X) \to \mathrm{Iso}(X)$ together with a natural isomorphism $\pi_{\mathrm{Iso}} \circ s \cong \mathrm{id}_{\mathrm{Fun}([1],X)}$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& X \arrow[dl, "i"', "\sim"] \arrow[dr, "(p_{[1]})^*"] & \\
\mathrm{Iso}(X) \arrow[rr, "\pi_{\mathrm{Iso}}"] & & \mathrm{Fun}([1],X).
\end{tikzcd}$$

Since the map $(p_{[1]})^*$ admits a retraction given by $\mathrm{ev}_0\colon \mathrm{Fun}([1],X) \to X$ and i is an equivalence, also the map π_{Iso} admits a retraction. It thus follows from Lemma 1.1.4 that π_{Iso} admits a section if and only if it is an equivalence. But by 2-out-of-3 this is in turn equivalent to the functor $(p_{[1]})^*$ being an equivalence. This finishes the proof. □

Lemma 2.1.5. *Let $C \to D$ be an equivalence of synthetic categories. Then C is a groupoid if and only if D is a groupoid.*

Proof. This follows from the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, "\sim"] \arrow[d, "(p_{[1]})^*"'] & D \arrow[d, "(p_{[1]})^*"] \\
\mathrm{Fun}([1],C) \arrow[r, "\sim"] & \mathrm{Fun}([1],D).
\end{tikzcd}$$

Here the bottom functor is again an equivalence due to the functoriality of functor categories, Construction 1.1.23. □

Lemma 2.1.6. *Let X be a groupoid. Then for every synthetic category C, the functor category $\mathrm{Fun}(C,X)$ is a groupoid.*

Proof. This follows from the fact that the map $\mathrm{Fun}(C,X) \to \mathrm{Fun}([1],\mathrm{Fun}(C,X))$ is equivalent to the map $\mathrm{Fun}(C,X) \to \mathrm{Fun}(C,\mathrm{Fun}([1],X))$ induced by the equivalence $X \xrightarrow{\sim} \mathrm{Fun}([1],X)$. □

Lemma 2.1.7. *Groupoids are closed under retracts: if X is a groupoid and Y is a retract of X, then Y is also a groupoid.*

Proof. If Y is a retract of X, then the functor $Y \to \mathrm{Fun}([1],Y)$ is a retract of the functor $X \to \mathrm{Fun}([1],X)$.In particular, if the latter is an equivalence then so is the former by Lemma 1.1.10. □

2.2 Groupoid cores

A useful construction in category theory is the construction of the *groupoid core* of a category C: the largest groupoid contained in C. We will now introduce an analogue of this construction in the setting of synthetic category theory.

Axiom G (Groupoid core)**.** For every synthetic category C, there exists a groupoid $C^{\simeq}$ called the *groupoid core of* C, which comes equipped with a functor $\gamma_C\colon C^{\simeq} \longrightarrow C$. For every other groupoid X and every functor $f\colon X \longrightarrow C$, there exists a functor $\tilde{f}\colon X \longrightarrow C^{\simeq}$ and a natural isomorphism $f \cong \gamma_C \circ \tilde{f}$. Given two functors $g, h\colon X \longrightarrow C^{\simeq}$, there is for every natural isomorphism $\alpha\colon \gamma_C \circ g \cong \gamma_C \circ h$ of functors $X \longrightarrow C$ a natural isomorphism $\tilde{\alpha}\colon f \cong g$ of functors $X \longrightarrow C^{\simeq}$, together with an isomorphism of natural isomorphisms $\gamma_D \circ \tilde{\alpha} \cong \alpha$.

Applying the defining property of $C^{\simeq}$ to the terminal groupoid $X = *$ from Example 2.1.2, we see that there is a one-to-one correspondence between objects of $C^{\simeq}$ and objects of C: every object of C lifts to $C^{\simeq}$ and two objects of $C^{\simeq}$ are isomorphic if and only if their images in C are isomorphic. We may therefore think of $C^{\simeq}$ as the *groupoid of objects* of C.

The groupoid core $C^{\simeq}$ is automatically functorial in C:

Construction 2.2.1. Let $f\colon C \longrightarrow D$ be a functor. Then the composite $f \circ \gamma_C\colon C^{\simeq} \longrightarrow D$ is a functor from a groupoid into D, and thus we obtain a functor

$$f^{\simeq} := \widetilde{(f \circ \gamma_C)}\colon C^{\simeq} \longrightarrow D^{\simeq}$$

making the following diagram commute:

$$\begin{tikzcd}
C^{\simeq} \arrow[r, "f^{\simeq}"] \arrow[d, "\gamma_C"'] & D^{\simeq} \arrow[d, "\gamma_D"] \\
C \arrow[r, "f"'] & D.
\end{tikzcd}$$

Exercise 2.2.2 (Exercise 2.5.3)**.** Show that for every synthetic category C there is a preferred natural isomorphism $(\mathrm{id}_C)^{\simeq} \cong \mathrm{id}_{C^{\simeq}}$. Show that for functors $f\colon C \longrightarrow D$ and $g\colon D \longrightarrow E$ there is a preferred natural isomorphism $(g \circ f)^{\simeq} \cong g^{\simeq} \circ f^{\simeq}$.

We next prove various basic properties of the groupoid core construction $C \mapsto C^{\simeq}$.

[Note that we do not ask that γ_C is an embedding, as was done before! I feel like we shouldn't need it and that it will eventually follow from Proposition 4.4.5.]

Lemma 2.2.3. *Let X be a groupoid. Then the functor $\gamma_X\colon X^{\simeq} \to X$ is an equivalence.*

Proof. The identity map $\mathrm{id}_X\colon X \to X$ lifts to a map $\widetilde{\mathrm{id}_X}\colon X \to X^{\simeq}$ satisfying $\gamma_X \circ \widetilde{\mathrm{id}_X} \cong \mathrm{id}_X$. In particular, $\widetilde{\mathrm{id}_X}$ is right inverse to γ_X. To show that we also have $\widetilde{\mathrm{id}_X} \circ \gamma_X \cong \mathrm{id}_{X^{\simeq}}$, it will suffice to produce such a natural isomorphism after composing with $\gamma_X\colon X^{\simeq} \to X$. But then we have

$$\gamma_X \circ \widetilde{\mathrm{id}_X} \circ \gamma_X \cong \mathrm{id}_X \circ \gamma_X \cong \gamma_X \cong \gamma_X \circ \mathrm{id}_{X^{\simeq}},$$

finishing the proof. ◻

Lemma 2.2.4. *For synthetic categories C and D, the functor $(\mathrm{pr}_C^{\simeq}, \mathrm{pr}_D^{\simeq})\colon (C \times D)^{\simeq} \to C^{\simeq} \times D^{\simeq}$ is an equivalence.*

Proof. The source of the functor $\gamma_C \times \gamma_D\colon C^{\simeq} \times D^{\simeq} \to C \times D$ is a groupoid by Example 2.1.3, hence we obtain a lift $\widetilde{\gamma_C \times \gamma_D}\colon C^{\simeq} \times D^{\simeq} \to (C \times D)^{\simeq}$. We leave it to the reader to check that this provides an inverse to the functor $(\mathrm{pr}_C^{\simeq}, \mathrm{pr}_D^{\simeq})$. ◻

Similarly one may show:

Lemma 2.2.5. *For functors $C \to E$ and $D \to E$, the functor $(\mathrm{pr}_C^{\simeq}, \mathrm{pr}_D^{\simeq})\colon (C \times_E D)^{\simeq} \to C^{\simeq} \times_{E^{\simeq}} D^{\simeq}$ is an equivalence.* ◻

Since the walking morphism $[1]$ comes with two objects 0 and 1, we obtain a map $\langle 0,1 \rangle\colon * \sqcup * \to [1]^{\simeq}$. The following axiom expresses that every object of $[1]$ is isomorphic to either 0 or 1:

Axiom G.2. The map $\langle 0,1 \rangle\colon * \sqcup * \to [1]^{\simeq}$ admits a section $[1]^{\simeq} \to * \sqcup *$.

The reason we do not assume this map to be an equivalence is that we want groupoids to provide a semantic interpretation of synthetic category theory. In the theory of groupoids, the walking morphism would need to be the walking *isomorpism*, and this has (up to isomorphism) only a single object rather than two.

2.3 Mapping groupoids

An important example of the groupoid core is the following:

Definition 2.3.1 (Mapping groupoid)**.** For synthetic categories C and D, we define the *mapping groupoid* $\mathrm{Map}(C,D)$ of functors from C to D as the groupoid

$$\mathrm{Map}(C,D) := \mathrm{Fun}(C,D)^{\simeq}.$$

Combining the functoriality of the functor construction and that of the groupoid core construction, the mapping groupoids $\mathrm{Map}(C,D)$ are covariantly functorial in D and contravariantly functorial in C: for functors $f\colon C' \to C$ and $g\colon D \to D'$ we obtain functors

$$-\circ f\colon \mathrm{Map}(C,D) \to \mathrm{Map}(C',D) \qquad \text{and} \qquad g\circ -\colon \mathrm{Map}(C,D) \to \mathrm{Map}(C,D').$$

Lemma 2.3.2. *Let X be a groupoid and let D be a synthetic category. Then the functor*

$$\gamma_C \circ -\colon \mathrm{Map}(X, C^{\simeq}) \to \mathrm{Map}(X,C)$$

is an equivalence.

Proof. Consider the evaluation functor $\mathrm{ev}\colon \mathrm{Fun}(X,C)\times X \to C$. By the functoriality of the groupoid core, this induces a functor $\mathrm{ev}^{\simeq}\colon (\mathrm{Fun}(X,C)\times X)^{\simeq} \to C^{\simeq}$. The source of this functor is equivalent to $\mathrm{Map}(X,C)\times X$:

$$(\mathrm{Fun}(X,C)\times X)^{\simeq} \xrightarrow[\sim]{2.2.4} \mathrm{Fun}(X,C)^{\simeq}\times X^{\simeq} = \mathrm{Map}(X,C)\times X^{\simeq} \xrightarrow[\sim]{2.2.3} \mathrm{Map}(X,C)\times X.$$

By currying the resulting functor $\mathrm{ev}^{\simeq}\colon \mathrm{Map}(X,C)\times X \to C^{\simeq}$, we obtain a functor

$$(\mathrm{ev}^{\simeq})_c\colon \mathrm{Map}(X,C) \to \mathrm{Fun}(X,C^{\simeq}),$$

and since $\mathrm{Map}(X,C)$ this functor admits a lift $\widetilde{(\mathrm{ev}^{\simeq})_c}\colon \mathrm{Map}(X,C) \to \mathrm{Map}(X,C^{\simeq})$ to the groupoid core. We claim this functor is inverse to $\gamma_C \circ -$.

We start by showing that the composite $(\gamma_C\circ -)\circ \widetilde{(\mathrm{ev}^{\simeq})_c}\colon \mathrm{Map}(X,C) \to \mathrm{Map}(X,C)$ is naturally isomorphic to the identity. Since the source of this functor is a groupoid, it will suffice to produce such a natural isomorphism after composition with the functor $\gamma_{\mathrm{Fun}(X,C)}\colon \mathrm{Map}(X,C) \to \mathrm{Fun}(X,C)$. In other words, we must produce a natural isomorphism making the following square commutative:

$$\begin{tikzcd}
\mathrm{Map}(X,C) \arrow[r, "(\mathrm{ev}^{\simeq})_c"] \arrow[d, equal] & \mathrm{Fun}(X,C^{\simeq}) \arrow[d, "\gamma_C\circ -"] \\
\mathrm{Map}(X,C) \arrow[r, "\gamma_{\mathrm{Fun}(X,C)}"'] & \mathrm{Fun}(X,C).
\end{tikzcd}$$

By uncurrying, this is equivalent to producing a natural isomorphism for the following square:

$$\begin{tikzcd}
\mathrm{Map}(X,C)\times X \arrow[r, "\mathrm{ev}^{\simeq}"] \arrow[d, "\gamma_{\mathrm{Fun}(X,C)}\times \mathrm{id}_X"'] & C^{\simeq} \arrow[d, "\gamma_C"] \\
\mathrm{Fun}(X,C)\times X \arrow[r, "\mathrm{ev}"] & C.
\end{tikzcd}$$

But this square commutes by the very definition of $\mathrm{ev}^{\simeq}$.

Next, we will show that also the composite $\widetilde{(\mathrm{ev}^{\simeq})_c} \circ (\gamma_C \circ -)\colon \operatorname{Map}(X, C^{\simeq}) \to \operatorname{Map}(X, C^{\simeq})$ is isomorphic to the identity. Again we may do this after composing with $\gamma_{\operatorname{Fun}(X,C^{\simeq})}$, and by uncurrying we must show that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Map}(X, C^{\simeq}) \times X \arrow[r, "(\gamma_C \circ -)^{\simeq} \times X"] \arrow[d, "\gamma_{\operatorname{Fun}(X,C^{\simeq})} \times \mathrm{id}_X"'] & \operatorname{Map}(X, C) \times X \arrow[d, "\mathrm{ev}^{\simeq}"] \\
\operatorname{Fun}(X, C^{\simeq}) \times X \arrow[r, "\mathrm{ev}"] & C^{\simeq}.
\end{tikzcd}$$

Since $\operatorname{Map}(X, C^{\simeq}) \times X$ is a groupoid, it will in turn suffice to show that the two composites of the square become isomorphic after postcomposing with $\gamma_C\colon C^{\simeq} \to C$. But this may be exhibited by the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Map}(X, C^{\simeq}) \times X \arrow[r, "(\gamma_C \circ -)^{\simeq} \times X"] \arrow[d, "\gamma_{\operatorname{Fun}(X,C^{\simeq})} \times \mathrm{id}_X"'] & \operatorname{Map}(X, C) \times X \arrow[d, "\gamma_{\operatorname{Fun}(X,C)} \times \mathrm{id}_X"'] \arrow[dr, "\mathrm{ev}^{\simeq}"] & \\
\operatorname{Fun}(X, C^{\simeq}) \times X \arrow[r, "(\gamma_C \circ -) \times \mathrm{id}_X"] \arrow[dr, "\mathrm{ev}"'] & \operatorname{Fun}(X, C) \times X \arrow[dr, "\mathrm{ev}"] & C^{\simeq} \arrow[d, "\gamma_C"] \\
 & C^{\simeq} \arrow[r, "\gamma_C"] & C.
\end{tikzcd}$$

Here the left two squares commute by naturality, while the square on the right commutes by construction of $\mathrm{ev}^{\simeq}$. This finishes the proof. □

The mapping spaces $\operatorname{Map}(-,-)$ will play an important role in the development of the theory: frequently, we may check conditions at the level of mapping spaces. In the following, we will provide various examples of this phenomenon.

Proposition 2.3.3. *For a functor $f\colon C \to D$, the following are equivalent:*

(i) The functor f is an equivalence;

(ii) For any synthetic category E, the induced map

$$f_*\colon \operatorname{Map}(E, C) \to \operatorname{Map}(E, D)$$

is an equivalence.

(iii) For any synthetic category E, the induced map

$$f^*\colon \operatorname{Map}(D, E) \to \operatorname{Map}(C, E)$$

is an equivalence.

Proof. We will prove the equivalence between (i) and (ii); the proof of (i) $\iff$ (iii) is analogous. If f is an equivalence, it is immediate that f_* is an equivalence for every E. Conversely, assume that f_* is an equivalence for every E. By taking $E = D$, we obtain an equivalence $f_* \colon \operatorname{Map}(D,C) \xrightarrow{\sim} \operatorname{Map}(D,D)$, and hence there exists some functor $g \colon D \to C$ such that $f \circ g \cong \mathrm{id}_D$. It remains to show that also $g \circ f \cong \mathrm{id}_C$. To this end, we take $E = C$, so that we obtain an equivalence $f_* \colon \operatorname{Map}(C,C) \xrightarrow{\sim} \operatorname{Map}(C,D)$. To produce the desired isomorphism between $g \circ f$ and id_C, it will thus suffice to do so after postcomposing with f. But in that case we have the following string of isomorphisms:

$$f \circ (g \circ f) \cong (f \circ g) \circ f \cong \mathrm{id}_D \circ f \cong f \cong f \circ \mathrm{id}_C \,.$$

This finishes the proof. ◻

In the same way, one proves:

Proposition 2.3.4. *For a functor $f \colon X \to Y$ between groupoids, the following are equivalent:*

(i) The functor f is an equivalence;

(ii) For every groupoid Z, the functor $f_* \colon \operatorname{Map}(Z,X) \to \operatorname{Map}(Z,Y)$ is an equivalence;

(iii) For every groupoid Z, the functor $f^* \colon \operatorname{Map}(Y,Z) \to \operatorname{Map}(X,Z)$ is an equivalence. ◻

Corollary 2.3.5. *Consider a commutative square of synthetic categories*

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "v"] & D \end{tikzcd}$$

Then this square is a pullback square if and only if for every synthetic category E the induced square

$$\begin{tikzcd} \operatorname{Map}(E,C') \arrow[r, "u"] \arrow[d, "f'"'] & \operatorname{Map}(E,C) \arrow[d, "f"] \\ \operatorname{Map}(E,D') \arrow[r, "v"] & \operatorname{Map}(E,D) \end{tikzcd}$$

is a pullback square.

Proof. The assignment $C \mapsto \operatorname{Fun}(E,C)$ preserves pullback squares by Lemma 1.1.36, and the assignment $C \mapsto C^{\simeq}$ preserves pullbacks by Lemma 2.2.5. Hence the 'only if'-direction is immediate. For the 'if'-direction, assume that the square becomes a pullback square after applying $\operatorname{Map}(E,-)$ for every E. We have to show that the induced map $(f',u) \colon C' \to D' \times_D C$ is an equivalence. By Proposition 2.3.3, it will suffice to show that the map

$$(f'_*, r_*) \colon \operatorname{Map}(E,C') \to \operatorname{Map}(E, D' \times_D C) \xrightarrow{\sim} \operatorname{Map}(E,D') \times_{\operatorname{Map}(E,D)} \operatorname{Map}(E,C)$$

is an equivalence. But this holds by assumption. ◻

Corollary 2.3.6. *A functor $f\colon C \to D$ is an embedding if and only if for every synthetic category E the induced map $f_*\colon \mathrm{Map}(E,C) \to \mathrm{Map}(E,D)$ is an embedding.*

Proof. This follows immediately from Corollary 2.3.5 by considering the commutative diagram

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "f"] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

□

The above characterization of pullback squares admit a dual characterization for pushout squares:

Exercise 2.3.7 (Exercise 2.5.4)**.** Show that square of the form

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

is a pushout square if and only if for every synthetic category C the induced square

$$\begin{tikzcd}
\mathrm{Map}(B,C) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Map}(B',C) \arrow[d, "(f')^*"] \\
\mathrm{Map}(A,C) \arrow[r, "u^*"] & \mathrm{Map}(A',C)
\end{tikzcd}$$

is a pullback square.

2.4 The synthetic theory of groupoids

In this section, we show that the collection of groupoids satisfy all the postulates and axioms of synthetic category theory introduced so far.

Lemma 2.4.1. *Consider a pullback square*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

of synthetic categories. If X, Y and Y' are groupoids, then so is X'.

Proof. This is immediate from the fact that the assignment $C \mapsto \mathrm{Fun}([1],C)$ preserves pullbacks, see Lemma 1.1.36. □

Lemma 2.4.2. *The empty category* $\emptyset$ *is a groupoid.*

Proof. We need to show that the functor $\emptyset \to \mathrm{Fun}([1],\emptyset)$ is an equivalence, or equivalently that the map $\mathrm{ev}_0 \colon \mathrm{Fun}([1],\emptyset) \to \emptyset$ is an equivalence. But this is immediate from Axiom B.7. $\square$

Lemma 2.4.3. *Let C be a synthetic category and let X be a groupoid. Then also the functor category* $\mathrm{Fun}(C,X)$ *is a groupoid.*

Proof. We need to show that the functor $\mathrm{Fun}(C,X) \to \mathrm{Fun}([1],\mathrm{Fun}(C,X))$ is an equivalence. But under the equivalences

$$\mathrm{Fun}([1],\mathrm{Fun}(C,X)) \simeq \mathrm{Fun}([1]\times C,X) \simeq \mathrm{Fun}(C,\mathrm{Fun}([1],X)),$$

from Lemma 1.1.28, this functor corresponds to the one induced by the equivalence $X \xrightarrow{\sim} \mathrm{Fun}([1],X)$, hence it is an equivalence. $\square$

To guarantee that groupoids are also closed under disjoint unions, we need an additional axiom:

Axiom H. The synthetic category $* \sqcup *$ is a groupoid.

Corollary 2.4.4. *For every two synthetic categories C and D, the canonical map*

$$\mathrm{Fun}([1],C) \sqcup \mathrm{Fun}([1],D) \to \mathrm{Fun}([1],C \sqcup D)$$

is an equivalence.

Proof. When $C = * = D$, we have $\mathrm{Fun}([1],*) \cong *$ and the statement says that the map $* \sqcup * \to \mathrm{Fun}([1],* \sqcup *)$ is an equivalence. This is precisely the content of Axiom H. For arbitrary C and D, it follows from Lemma 1.1.36 that there are pullback squares

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C \sqcup D) \arrow[d] & \mathrm{Fun}([1],D) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathrm{Fun}([1],*) \arrow[r] & \mathrm{Fun}([1],* \sqcup *) & \mathrm{Fun}([1],*). \arrow[l]
\end{tikzcd}$$

The claim thus follows from Axiom B.8. $\square$

Corollary 2.4.5. *Let X and Y be groupoids. Then their disjoint union $X \sqcup Y$ is a groupoid.*

Proof. The map $X \sqcup Y \to \mathrm{Fun}([1], X \sqcup Y)$ factors as

$$X \sqcup Y \to \mathrm{Fun}([1], X) \sqcup \mathrm{Fun}([1], Y) \to \mathrm{Fun}([1], X \sqcup Y),$$

where the first map is the disjoint union of the two equivalences $X \xrightarrow{\sim} \mathrm{Fun}([1], X)$ and $Y \xrightarrow{\sim} \mathrm{Fun}([1], Y)$, while the second is the equivalence from Corollary 2.4.4. □

Corollary 2.4.6. *For synthetic categories C and D, the maps $C^{\simeq} \to (C \sqcup D)^{\simeq}$ and $D^{\simeq} \to (C \sqcup D)^{\simeq}$ induce an equivalence $C^{\simeq} \sqcup D^{\simeq} \xrightarrow{\sim} (C \sqcup D)^{\simeq}$.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
C^{\simeq} \sqcup D^{\simeq} \arrow[r] \arrow[d] & (C \sqcup D)^{\simeq} \arrow[d] \\
(*)^{\simeq} \sqcup (*)^{\simeq} \arrow[r, "\sim"] & (* \sqcup *)^{\simeq}.
\end{tikzcd}$$

The bottom map is an isomorphism by Axiom H, as both sides are equivalent to $* \sqcup *$. Combining Axiom B.8 with Lemma 2.2.5, we get that this square is a pullback square, and hence also the top map is an equivalence, as desired. □

2.5 Exercises Chapter 2

Exercise 2.5.1. Show that the synthetic category $*$ is a groupoid.

Exercise 2.5.2. Show that the product of two groupoids X and Y is again a groupoid.

Exercise 2.5.3. Show that for every synthetic category C there is a preferred natural isomorphism $(\mathrm{id}_C)^{\simeq} \cong \mathrm{id}_{C^{\simeq}}$ of functors $C^{\simeq} \to C^{\simeq}$. Show that for functors $f\colon C \to D$ and $g\colon D \to E$ there is a preferred natural isomorphism $(g \circ f)^{\simeq} \cong g^{\simeq} \circ f^{\simeq}$ of functors $C^{\simeq} \to E^{\simeq}$.

Exercise 2.5.4. Show that a commutative square in $\mathcal{E}$ of the form

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

is a pushout square if and only if for every synthetic category C the induced square

$$\begin{tikzcd}
\mathrm{Map}(B, C) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Map}(B', C) \arrow[d, "(f')^*"] \\
\mathrm{Map}(A, C) \arrow[r, "u^*"] & \mathrm{Map}(A', C)
\end{tikzcd}$$

is a pullback square.

3 Synthetic categories in contexts

At certain points in this book, we are working with ‘categories in context Γ’ for some groupoid Γ, or sometimes even for arbitrary synthetic categories Γ. In the language of tribes from Part II, this corresponds to working in the local tribe $\mathcal{E}(\Gamma)$. In the language of type theory from Part III, this corresponds to working in the (groupoidal/functorial) context Γ. As we haven’t yet completely established how we want to implement this notion of contexts within the language of naive category theory, we will omit any specifications of this for now, and add them in at some point in the future.

3.1 Contexts

To be written.

3.2 Objects in synthetic categories

In ordinary category theory, one frequently encounters statements that can be ‘checked on objects’: think for example of the fact that a natural transformation $\alpha\colon f \to g$ is a natural isomorphism if and only if the component $\alpha_x\colon f(x) \to g(x)$ is an isomorphism for every object x. To be able to formulate such statements in the setting of synthetic category theory, we need to somewhat expand our notion of ‘objects’ and allow objects in a given *context*:

Definition 3.2.1 (Objects in context Γ)**.** Let C be a synthetic category and let Γ be a groupoid. An *object of C in context* Γ is a functor $x\colon \Gamma \to C$.

We may fit this definition into the framework of ‘categories in context Γ’ introduced in the previous section.

Definition 3.2.2. For every synthetic category C, we let C_Γ denote the synthetic category in context Γ given by the projection functor $\mathrm{pr}_\Gamma\colon C \times \Gamma \to \Gamma$.

Observation 3.2.3. Working with synthetic categories in context Γ, an object of C_Γ in the sense of Definition 1.1.11 is given by a commutative triangle

$$\begin{tikzcd}
\Gamma \arrow[r, "(\mathrm{id}_\Gamma{,}x)"] \arrow[dr, equal] & \Gamma \times C \arrow[d, "\mathrm{pr}_1"] \\
 & \Gamma.
\end{tikzcd}$$

But the data of such a diagram is equivalent to that of a functor $x\colon \Gamma \to C$, justifying the terminology of Definition 3.2.1.

Convention 3.2.4. From now onward, we will use the phrase *object of C* for an object x in C in an *arbitrary* context Γ. When the context of x is $*$, we will speak of *absolute objects*. Although the groupoid Γ is allowed to be arbitrary, we may in practice always assume it is the terminal groupoid $*$ by replacing C by C_Γ and working with synthetic categories in context Γ instead. We will frequently reinforce the above philosophy in our notation and denote the context of an object x by $\{x\}$.

Definition 3.2.5 (Universal object)**.** For a synthetic category C, the map $\gamma_C\colon C^\simeq \to C$ is called the *universal object*. By adjunction any object $x\colon \{x\} \to C$ uniquely factors through $C^\simeq$. In practice, statements about *arbitrary* objects will often immediately reduce to statements about the *universal* object.

All of the definitions we have previously made for absolute objects have immediate generalizations for arbitrary objects. Let us highlight the two most important examples: fibers and hom groupoids.

Definition 3.2.6 (Fiber)**.** For an object x of C and a functor $p\colon E \to C$, we define the *fiber* $E(x)$ of p at x via the pullback square

$$\begin{tikzcd}
E(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
\{x\} \arrow[r, "x"] & C.
\end{tikzcd}$$

We will often treat $E(x)$ as a synthetic category in context $\{x\}$. We say that a property holds *fiberwise* if it holds for the fiber $E(x)$ for every object x of C.

Definition 3.2.7 (Hom groupoid)**.** Let C be any synthetic category. Given objects x and y of C, we define the *hom groupoid* $C(x,y)$ via the following pullback square:

$$\begin{tikzcd}
C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
\{(x,y)\} \arrow[r, "(x{,}y)"] & C \times C.
\end{tikzcd}$$

We will often treat $C(x,y)$ as a synthetic category in context $\{(x,y)\}$.

4 Constructions of synthetic categories

In this chapter, we introduce various fundamental categorical constructions in the setting of synthetic category theory: (full) subcategories, localizations, joins and slices.

4.1 Subcategory axiom

In ordinary category theory, if one is given a collection M of morphisms in a category C which is closed under composition, then one may form the subcategory of C whose morphisms are precisely the morphisms in M (and the objects in the subcategory are precisely the objects whose identity morphism is in M). In this section, we discuss the analogous situation in the context of synthetic category theory.

We start by definition the notion of a subcategory. For this, we need the following auxiliary construction:

Construction 4.1.1. For synthetic categories C, D and E, we construct a map

$$\mathrm{Map}(C,D) \to \mathrm{Map}(\mathrm{Map}(E,C),\mathrm{Map}(E,D))$$

which informally is given by sending a functor $f\colon C \to D$ to the functor $f_*\colon \mathrm{Map}(E,C) \to \mathrm{Map}(E,D)$ given by postcomposition with f. More precisely, consider the composition functor

$$-\circ-\colon \mathrm{Fun}(E,C)\times\mathrm{Fun}(C,D) \to \mathrm{Fun}(E,D),$$

defined as the functor adjoint to the composite

$$E\times\mathrm{Fun}(E,C)\times\mathrm{Fun}(C,D) \xrightarrow{\mathrm{ev}\times 1} C\times\mathrm{Fun}(C,D) \xrightarrow{\mathrm{ev}} D.$$

Applying the groupoid core construction $(-)^{\simeq}$, we thus obtain a map

$$\mathrm{Map}(E,C)\times\mathrm{Map}(C,D) \to \mathrm{Map}(E,D),$$

which by transposition defines a map

$$\mathrm{Map}(C,D) \to \mathrm{Fun}(\mathrm{Map}(E,C),\mathrm{Map}(E,D)).$$

Since $\mathrm{Map}(C,D)$ is a groupoid, this factors uniquely through the groupoid core of the target, producing the desired map

$$\mathrm{Map}(C,D) \to \mathrm{Map}(\mathrm{Map}(E,C),\mathrm{Map}(E,D)).$$

Definition 4.1.2 (Subcategory)**.** A functor $f\colon A \to C$ is called a *subcategory* if the following two conditions are satisfied:

(1) The map $f_*\colon \mathrm{Map}([1],A) \to \mathrm{Map}([1],C)$ is an embedding;

(2) For every synthetic category D, the induced square

$$\begin{tikzcd}
\mathrm{Map}(D,A) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}(D,C) \arrow[d] \\
\mathrm{Map}(\mathrm{Map}([1],D),\mathrm{Map}([1],A)) \arrow[r, "(f_*)_*"] & \mathrm{Map}(\mathrm{Map}([1],D),\mathrm{Map}([1],C))
\end{tikzcd}$$

is a pullback square. Here the two vertical maps are given by Construction 4.1.1.

In other words, if $f\colon A \to C$ is a subcategory, then a given functor $D \to C$ factors through A if and only if the associated map on morphisms $\mathrm{Map}([1],D) \to \mathrm{Map}([1],C)$ factors through $\mathrm{Map}([1],A)$.

Example 4.1.3. The identity functor $\mathrm{id}_C\colon C \to C$ and the empty category $\emptyset \to C$ are both seen to be subcategories.

Lemma 4.1.4. *Every subcategory $f\colon A \to C$ is an embedding.*

Proof. By Corollary 2.3.6, it will suffice to show that the induced map $\mathrm{Map}(D,A) \to \mathrm{Map}(D,C)$ is an embedding for every synthetic category D. But this follows from its defining property, as it is the base change of the map $(f_*)_*$ induced by the map $f_*\colon \mathrm{Map}([1],A) \to \mathrm{Map}([1],C)$ which was assumed to be an embedding. □

We would now like to construct subcategories of C out of the data of a collection of morphisms in C closed under composition. Let us start by defining this notion.

Definition 4.1.5 (Collection of morphisms)**.** Let C be a synthetic category. A *collection of morphisms in C* is a groupoid M equipped with an embedding

$$m\colon M \hookrightarrow \mathrm{Map}([1],C).$$

We say that *M is closed under composition* if the following two conditions are satisfied:

- If a morphism $f\colon x \to y$ in C is in M, then so are id_x and id_y. More concretely, there are commutative diagrams of the following form:

$$\begin{tikzcd}
M \arrow[d, hook, "m"'] & M \arrow[l, dashed] \arrow[r, dashed] \arrow[d, hook, "m"] & M \arrow[d, hook, "m"] \\
\mathrm{Map}([1],C) & \mathrm{Map}([1],C) \arrow[l, "f\mapsto \mathrm{id}_x"'] \arrow[r, "f\mapsto \mathrm{id}_y"] & \mathrm{Map}([1],C).
\end{tikzcd}$$

- For maps $f\colon x \to y$ and $g\colon y \to z$ in M, also the composition of f and g is in M. More concretely, there exists a dotted arrow in the following diagram:

$$\begin{tikzcd}
M \times_C M \arrow[rr, dashed] \arrow[d, hook] & & M \arrow[d, hook, "m"] \\
\mathrm{Map}([1],C) \times_C \mathrm{Map}([1],C) & \mathrm{Map}(\Delta^2, C) \arrow[l, hook', "\sim"'] \arrow[r, "(\delta^2_1)^*"] & \mathrm{Map}([1],C).
\end{tikzcd}$$

Example 4.1.6. Taking $M = \mathrm{Map}([1],C)$, it is clear that the identity map $M \to \mathrm{Map}([1],C)$ is a collection of morphisms of C closed under composition.

Example 4.1.7. Let $A \to C$ be a subcategory. Then the map $M_A := \mathrm{Map}([1],A) \to \mathrm{Map}([1],C)$ is a collection of morphisms in C closed under composition. This is immediate from the fact that the functor $A \to C$ preserves compositions and identities.

Also the collection of isomorphisms in C determines a collection of morphisms closed under composition:

Lemma 4.1.8. *For a synthetic category C, the map $\pi^{\simeq}_{\mathrm{Iso}}\colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Map}([1],C)$ is an embedding.*

Proof. We need to show that the diagonal $\Delta\colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Iso}(C)^{\simeq} \times_{\mathrm{Map}([1],C)} \mathrm{Iso}(C)^{\simeq}$ is an equivalence. Objects in the target may be described as pairs of diagrams

$$\left(
\begin{tikzcd}
 & x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "f"] & \\
y \arrow[ur, "g"] \arrow[dr, "\mathrm{id}_y"'] & & x \\
 & y \arrow[ur, "h"'] &
\end{tikzcd}
,
\begin{tikzcd}
 & x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "f"] & \\
y \arrow[ur, "\tilde{g}"] \arrow[dr, "\mathrm{id}_y"'] & & x \\
 & y \arrow[ur, "\tilde{h}"'] &
\end{tikzcd}
\right).$$

We claim that the first projection functor is an inverse of the diagonal. By definition, the composite $\mathrm{pr}_1 \circ \Delta$ is isomorphic to the identity, hence we need to provide an isomorphism $\Delta \circ \mathrm{pr}_1 \cong \mathrm{id}_{\mathrm{Iso}(C)^{\simeq} \times_{\mathrm{Map}([1],C)} \mathrm{Iso}(C)^{\simeq}}$. Since the first components of both functors agree by definition, we need to provide a natural isomorphism

$$\begin{tikzcd}
 & x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "f"] & \\
y \arrow[ur, "g"] \arrow[dr, "\mathrm{id}_y"'] & & x \\
 & y \arrow[ur, "h"'] &
\end{tikzcd}
\cong
\begin{tikzcd}
 & x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "f"] & \\
y \arrow[ur, "\tilde{g}"] \arrow[dr, "\mathrm{id}_y"'] & & x \\
 & y \arrow[ur, "\tilde{h}"'] &
\end{tikzcd}$$

in $\mathrm{Iso}(C)^{\simeq}$ whose image in $\mathrm{Map}([1], C)$ is the identity on f. By excision, it will suffice to provide isomorphisms for both of the two commutative triangles. Here we use the following isomorphisms:

$$g \cong hfg \cong h \cong hf\tilde{g} \cong \tilde{g} \qquad \text{and} \qquad h \cong hfg \cong g \cong \tilde{h}fg \cong \tilde{h}. \qquad \square$$

Lemma 4.1.9. *For a synthetic category C, the map $\pi^{\simeq}_{\mathrm{Iso}} \colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Map}([1], C)$ defines a collection of morphisms in C closed under composition.*

Proof. We saw the map is an embedding in the previous lemma. Since the Rezk axiom F gives an equivalence $C^{\simeq} \xrightarrow{\sim} \mathrm{Iso}(C)^{\simeq}$, it will thus suffice to show that the map $C^{\simeq} \to \mathrm{Map}([1], C)$ defines a collection of morphisms in C closed under composition. But for this map the first diagram becomes a tautology and the second diagram holds by the relation $\mathrm{id}_x \circ \mathrm{id}_x = \mathrm{id}_x$. $\square$

We may now state the subcategory axiom:

Axiom I (Subcategory axiom)**.** Let C be a synthetic category and let $M \hookrightarrow \mathrm{Map}([1], C)$ be a collection of morphisms in C closed under composition. Then there exists a synthetic category $\langle M \rangle_C$ equipped with a functor $i_M \colon \langle M \rangle_C \to C$ which exhibits $\langle M \rangle_C$ as a subcategory of C, and is equipped with an equivalence $\tilde{m} \colon \mathrm{Map}([1], \langle M \rangle_C) \xrightarrow{\sim} M$ fitting in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}([1], \langle M \rangle_C) \arrow[rr, "\tilde{m}", "\sim"'] \arrow[dr, hook, "(i_m)_*"'] & & M \arrow[dl, hook', "m"] \\
& \mathrm{Map}([1], C). &
\end{tikzcd}$$

Lemma 4.1.10. *Let $f \colon A \to C$ be a subcategory and let $M_A := \mathrm{Map}([1], A) \hookrightarrow \mathrm{Map}([1], C)$ denote the induced collection of morphisms in C. Then the map $i_{M_A} \colon \langle M_A \rangle_C \to C$ factors through an equivalence $\langle M_A \rangle_C \xrightarrow{\sim} A$.*

Proof. Since the functor $\langle M_A \rangle_C \to C$ induces the inclusion $\mathrm{Map}([1], A) \hookrightarrow \mathrm{Map}([1], C)$, the fact that A is a subcategory of C implies that this functor factors through A. It further follows that the induced map $\mathrm{Map}(D, \langle M_A \rangle_C) \to \mathrm{Map}(D, A)$ is an equivalence for every synthetic category C, since both sides sit in the same pullback square. The claim thus follows from the Proposition 2.3.3. $\square$

Corollary 4.1.11. *When $M = \mathrm{Map}([1], C)$ is the collection of all morphisms, the map $i_M \colon \langle M \rangle_C \to C$ is an equivalence.* $\square$

We will now explain the functoriality of the construction of the subcategory $\langle M\rangle_C$ in the pair (C, M).

Definition 4.1.12. Let C and D be synthetic categories, and assume that they come equipped with collections of morphisms $m\colon M \hookrightarrow \mathrm{Map}([1], C)$ and $n\colon N \hookrightarrow \mathrm{Map}([1], D)$. A functor $f\colon C \to D$ is said to *send morphisms in M to morphisms in N* if there exists a map $\tilde{f}\colon M \to N$ making the following diagram commute:

$$\begin{tikzcd}
M \arrow[r, dashed, "\tilde{f}"] \arrow[d, hook, swap, "m"] & N \arrow[d, hook, "n"] \\
\mathrm{Map}([1], C) \arrow[r, "f_*"] & \mathrm{Map}([1], D).
\end{tikzcd}$$

Since the map n is an embedding, the map $\tilde{f}\colon M \to N$ is unique up to natural isomorphism if it exists.

Construction 4.1.13. Let C and D be synthetic categories equipped with collections of morphisms M and N closed under composition, and let $f\colon C \to D$ be a functor which sends morphisms in M to morphisms in N. We construct a functor $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ making the following diagram commute:

$$\begin{tikzcd}
\langle M\rangle_C \arrow[r, dashed, "f|"] \arrow[d, swap, "i_M"] & \langle N\rangle_D \arrow[d, "i_N"] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

By the defining property of $\langle N\rangle_D$, it suffices to show that the the map $f \circ i_M\colon \langle M\rangle_C \to D$ sends all morphisms to morphisms in N. But this follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}([1], \langle M\rangle_C) \arrow[r, "\tilde{m}"] \arrow[dr, hook, swap, "(i_M)_*"] & M \arrow[r, "\tilde{f}"] \arrow[d, hook, "m"] & N \arrow[d, hook, "n"] \\
& \mathrm{Map}([1], C) \arrow[r, "f_*"] & \mathrm{Map}([1], D).
\end{tikzcd}$$

Remark 4.1.14 (Exercise 4.6.4)**.** The construction $f \mapsto f|$ satisfies the following properties:

(1) The map $f|$ is up to homotopy uniquely determined by f.

(2) Given a homotopy $f \sim f'$, we obtain an induced homotopy $f| \sim f'|$.

(3) The construction is compatible with composition: $(\mathrm{id}_C)| = \mathrm{id}_{\langle M\rangle_C}$ and $(g \circ f)| = g| \circ f|$.

(4) If the maps f and $\tilde{f}$ are equivalences, then also $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ is an equivalence.

4.2 Full subcategories

As a consequence of the subcategory axiom, we may in particular form *full* subcategories.

Construction 4.2.1. Given synthetic categories C and D, we construct a map $\mathrm{Map}(C,D) \to \mathrm{Map}(C^{\simeq}, D^{\simeq})$ as the adjunct of the following composite:

$$\mathrm{Map}(C,D) \times C^{\simeq} \cong (\mathrm{Fun}(C,D) \times C)^{\simeq} \xrightarrow{\mathrm{ev}^{\simeq}} D^{\simeq}.$$

Definition 4.2.2 (Full subcategory)**.** A functor $f\colon A \to C$ is called a *full subcategory* if the induced map $f^{\simeq}\colon A^{\simeq} \to C^{\simeq}$ is an embedding and for every synthetic category D, the commutative square

$$\begin{tikzcd}
\mathrm{Map}(D,A) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}(D,C) \arrow[d] \\
\mathrm{Map}(D^{\simeq},A^{\simeq}) \arrow[r, "(f^{\simeq})_*"] & \mathrm{Map}(D^{\simeq},C^{\simeq})
\end{tikzcd}$$

is a pullback square.

More informally, this says that a functor $D \to C$ factors through A if and only if the map $D^{\simeq} \to C^{\simeq}$ on objects factors through the embedding $A^{\simeq} \to C^{\simeq}$.

Corollary 4.2.3. *Every full subcategory is an embedding.*

Proof. The proof is similar to Lemma 4.1.4 and will be omitted. □

Definition 4.2.4 (Collection of objects)**.** Consider a synthetic category C. A *collection of objects* is a pair (Γ, j) consisting of a groupoid Γ equipped with an embedding $j\colon \Gamma \hookrightarrow C^{\simeq}$ which is an embedding. In this case, we define a groupoid M_{Γ} via the following pullback:

$$\begin{tikzcd}
M_{\Gamma} \arrow[r, hook, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\Gamma \times \Gamma \arrow[r, hook, "j \times j"] & C^{\simeq} \times C^{\simeq}.
\end{tikzcd}$$

Note that the map m is an embedding, as it is a base change of the embedding $j \times j$. It is also easy to verify that the resulting collection of morphisms M_{Γ} is closed under composition in C. This allows us to make the following definition:

Definition 4.2.5. Given an embedding $j\colon \Gamma \hookrightarrow C^{\simeq}$, we define

$$\langle \Gamma \subseteq C \rangle := \langle M_{\Gamma} \rangle_C,$$

and refer to it as the *full subcategory of C spanned by* Γ.

Proposition 4.2.6. *For any synthetic category D, there is a canonical pullback*

$$\begin{tikzcd}
\mathrm{Map}(D,\langle\Gamma\subseteq C\rangle) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(D,C) \arrow[d] \\
\mathrm{Map}(D^{\simeq},\Gamma) \arrow[r, "j_*"] & \mathrm{Map}(D^{\simeq},C^{\simeq}),
\end{tikzcd}$$

where the right vertical map is adjoint to the composite

$$\mathrm{Map}(C,D)\times C^{\simeq} \cong (\mathrm{Fun}(C,D)\times C)^{\simeq} \xrightarrow{\mathrm{ev}^{\simeq}} D^{\simeq}.$$

Proof. First consider the following diagram:

$$\begin{tikzcd}
\mathrm{Map}(D,\langle\Gamma\subseteq C\rangle) \arrow[r] \arrow[d] & \mathrm{Map}(D,C) \arrow[d] \\
\mathrm{Map}(\mathrm{Map}([1],D),M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(\mathrm{Map}([1],D),\mathrm{Map}([1],C)) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)_*"] \\
\mathrm{Map}(\mathrm{Map}([1],D),\Gamma\times\Gamma) \arrow[r] & \mathrm{Map}(\mathrm{Map}([1],D),C^{\simeq}\times C^{\simeq}).
\end{tikzcd}$$

The top square is a pullback by definition of $\langle\Gamma\subseteq C\rangle$, and the bottom square is a strict pullback by definition of M, and thus in particular a pullback square. In particular, the outer rectangle is a pullback.

Now we consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}(D,\langle\Gamma\subseteq C\rangle) \arrow[r] \arrow[d] & \mathrm{Map}(D,C) \arrow[d] \\
\mathrm{Map}(D^{\simeq},\Gamma) \arrow[r] \arrow[dd] & \mathrm{Map}(D^{\simeq},C^{\simeq}) \arrow[d] \\
 & \mathrm{Map}(D^{\simeq}\times D^{\simeq},C^{\simeq}\times C^{\simeq}) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)^*"] \\
\mathrm{Map}(\mathrm{Map}([1],D),\Gamma\times\Gamma) \arrow[r] & \mathrm{Map}(\mathrm{Map}([1],D),C^{\simeq}\times C^{\simeq}).
\end{tikzcd}$$

The bottom two horizontal maps are embeddings since they are induced by the embedding $j\colon\Gamma\to C^{\simeq}$. Since we showed before that the outer rectangle is a pullback square, it follows from Lemma 1.2.7 that the top square is a pullback square, as claimed. □

Corollary 4.2.7. *The map $\langle\Gamma\subseteq C\rangle\to C$ is a full subcategory and there is an equivalence*

$$\langle\Gamma\subseteq C\rangle^{\simeq} \xrightarrow{\sim} \Gamma.$$

Proof. We first prove the second statement. By applying Proposition 4.2.6 to $D = *$, we obtain a pullback square

$$\begin{tikzcd}
\langle \Gamma \subseteq C \rangle \arrow[r] \arrow[d] & C^{\simeq} \arrow[d] \\
\Gamma^{\simeq} \arrow[r, "j^*"] & C^{\simeq}.
\end{tikzcd}$$

As the right vertical map is the identity, we see that $\langle \Gamma \subseteq C \rangle \xrightarrow{\sim} \Gamma^{\simeq}$. But we have $\Gamma^{\simeq} \cong \Gamma$ since Γ is a groupoid. The first statement is now a direct consequence of Proposition 4.2.6. □

Lemma 4.2.8. *The inclusion $\langle C^{\simeq} \subseteq C \rangle \hookrightarrow C$ is an equivalence.*

Proof. Since the map $j\colon C^{\simeq} \to C^{\simeq}$ is the identity, it follows from Proposition 4.2.6 that for every synthetic category D the induced map

$$\mathrm{Map}(D, \langle C^{\simeq} \subseteq C \rangle) \to \mathrm{Map}(D, C)$$

is an equivalence. The claim thus follows from Proposition 2.3.3. □

Lemma 4.2.9. *Let C and D be synthetic categories, and let $j\colon \Gamma \hookrightarrow C$ and $k\colon \Lambda \hookrightarrow D$ be embeddings. Then there is a preferred equivalence*

$$\langle \Gamma \times \Lambda \subseteq C \times D \rangle \xrightarrow{\sim} \langle \Gamma \subseteq C \rangle \times \langle \Lambda \subseteq D \rangle.$$

Proof. This is an exercise in unwinding definitions and is left to the reader: see Exercise 4.6.3. □

Lemma 4.2.10. *Consider a functor $f\colon B \to C$. Given an embedding $\Gamma \hookrightarrow C^{\simeq}$, we define $\Lambda \hookrightarrow B^{\simeq}$ via the pullback*

$$\begin{tikzcd}
\Lambda \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^{\simeq} \arrow[d, "f^{\simeq}"] \\
\Gamma \arrow[r, hook] & C^{\simeq}.
\end{tikzcd}$$

Then the square

$$\begin{tikzcd}
\langle \Lambda \subseteq B \rangle \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
\langle \Gamma \subseteq C \rangle \arrow[r] & C
\end{tikzcd}$$

is a pullback square.

Proof. By Corollary 2.3.5, it will suffice to check that this square becomes a pullback square after applying the functor $\mathrm{Map}(D, -)$ for every synthetic category D. To this end, consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Map}(D,B) \arrow[rr] \arrow[dd] & & \mathrm{Map}(D^{\simeq},B^{\simeq}) \arrow[dd] \\
\mathrm{Map}(D,\langle\Lambda\subseteq B\rangle) \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & \mathrm{Map}(D^{\simeq},\Lambda) \arrow[ur] & \\
 & \mathrm{Map}(D,C) \arrow[rr] & & \mathrm{Map}(D^{\simeq},C^{\simeq}) \\
\mathrm{Map}(D,\langle\Gamma\subseteq C\rangle) \arrow[ur] \arrow[rr] & & \mathrm{Map}(D^{\simeq},\Gamma). \arrow[ur] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Here the top and bottom face of the cube are pullback squares by Proposition 4.2.6, while the right face is a square since $\mathrm{Map}(D^{\simeq},-)$ preserves pullback squares by Corollary 2.3.5. It follows from the pasting law of pullback squares that also the left face of the cube is a pullback square, finishing the proof. □

Lemma 4.2.11. *Let $f\colon B\to C$ be a functor exhibiting B as a full subcategory of C. Then f is an equivalence if and only if the induced map $f^{\simeq}\colon B^{\simeq}\to C^{\simeq}$ is an equivalence.*

Proof. This is clear from the definitions. □

4.3 Localization axiom

In ordinary category theory, if one is given a category C equipped with a collection of morphisms W of C, one may form the *localization* of C at W: the initial functor $C\to C[W^{-1}]$ out of C which sends all morphisms in W to isomorphisms. Our goal in this section is to define localizations in the context of synthetic category theory.

Construction 4.3.1. Let C be a synthetic category equipped with a collection of morphisms $w\colon W\hookrightarrow \mathrm{Map}([1],C)$, in the sense of Definition 4.1.5. We will construct for every synthetic category D a subcategory

$$\mathrm{Fun}^{W}(C,D)\subseteq\mathrm{Fun}(C,D)$$

In order to do this, we first define the groupoid $\mathrm{Map}^{W}(C,D)$ via the following pullback square:

$$\begin{tikzcd}
\mathrm{Map}^{W}(C,D) \arrow[r, hook, "\lambda^{W}_{C,D}"] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathrm{Map}(C,D) \arrow[d] \\
 & \mathrm{Map}(\mathrm{Map}([1],C),\mathrm{Map}([1],D)) \arrow[d, "w^{*}"] \\
\mathrm{Map}(W,\mathrm{Iso}(D)^{\simeq}) \arrow[r, hook, "(\pi_{\mathrm{Iso}})_{*}"] & \mathrm{Map}(W,\mathrm{Map}([1],D)),
\end{tikzcd}$$

where the top right vertical map is the map from Construction 4.1.1. The bottom map is induced by the embedding $\pi_{\mathrm{Iso}}^{\simeq}$ from Lemma 4.1.8 and hence is an embedding. It follows that also $\lambda_{C,D}^{W}$ is an embedding. We can now define

$$\mathrm{Fun}^{W}(C,D) := \langle \mathrm{Map}^{W}(C,D) \subseteq \mathrm{Fun}(C,D) \rangle.$$

In particular, the map $\lambda_{C,D}^{W}$ extends to a functor $\tilde{\lambda}_{C,D}^{W} \colon \mathrm{Fun}^{W}(C,D) \to \mathrm{Fun}(C,D)$. We may think of the category $\mathrm{Fun}^{W}(C,D)$ as the full subcategory of $\mathrm{Fun}(C,D)$ spanned by those functors sending the morphisms in W to isomorphisms in D.

Remark 4.3.2. The construction of $\mathrm{Fun}^{W}(C,D)$ is functorial in the pair (C,W): if C' is another synthetic category equipped with a collection of morphisms $w' \colon W' \hookrightarrow \mathrm{Map}([1],C')$ and $f \colon C \to C'$ is a functor which sends the morphisms of W to the morphisms of W' (in the sense of Definition 4.1.12), then for every synthetic category D there is an induced square

$$\begin{tikzcd}
\mathrm{Map}^{W'}(C',D) \arrow[r, "f^*"] \arrow[d, hook, "\lambda_{C',D}^{W'}"'] & \mathrm{Map}^{W}(C,D) \arrow[d, hook, "\lambda_{C,D}^{W}"] \\
\mathrm{Map}(C',D) \arrow[r, "f^*"] & \mathrm{Map}(C,D),
\end{tikzcd}$$

which in turn induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{W'}(C',D) \arrow[r, "f^*"] \arrow[d, "\tilde{\lambda}_{C',D}^{W'}"'] & \mathrm{Fun}^{W}(C,D) \arrow[d, "\tilde{\lambda}_{C,D}^{W}"] \\
\mathrm{Fun}(C',D) \arrow[r, "f^*"] & \mathrm{Fun}(C,D)
\end{tikzcd}$$

via Construction 4.1.13.

Before moving on to the localization axiom, we will establish some basic results that express that the categories $\mathrm{Fun}^{W}(C,D)$ behave as we expect. As a first result, we show that every functor $C \to D$ lies in $\mathrm{Fun}^{W}(C,D)$ whenever D is a groupoid:

Lemma 4.3.3. *Let C be a synthetic category equipped with a collection of morphisms $W \hookrightarrow \mathrm{Map}([1],C)$, and let Γ be a groupoid. Then the functor*

$$\mathrm{Fun}^{W}(C,\Gamma) \to \mathrm{Fun}(C,\Gamma)$$

is an equivalence.

Proof. The functor $\mathrm{Iso}(\Gamma)^{\simeq} \to \mathrm{Map}([1],\Gamma)$ is an equivalence, since Γ is a groupoid. It follows from its defining property that also the map $\mathrm{Map}^{W}(C,\Gamma) \to \mathrm{Map}(C,\Gamma)$ is an equivalence, and hence both sides define the same full subcategory of $\mathrm{Fun}(C,\Gamma)$. Since the full subcategory determined by $\mathrm{Map}(C,\Gamma)$ is just $\mathrm{Fun}(C,\Gamma)$ by Lemma 4.2.8, this finishes the proof.. □

The following lemma expresses the fact that any functor $C \to D$ takes isomorphisms in C to isomorphisms in D:

Proposition 4.3.4. *In the case of* $W := \mathrm{Iso}(C)^{\simeq} \hookrightarrow \mathrm{Map}([1], C)$*, the forgetful functor*

$$\mathrm{Fun}^{W}(C, D) \to \mathrm{Fun}(C, D)$$

is an equivalence.

Proof. We prove the claim by constructing an explicit inverse map $\mathrm{Fun}(C, D) \to \mathrm{Fun}^{\mathrm{Iso}(C)^{\simeq}}(C, D)$. As a first step, we define a map of the form

$$I\colon \mathrm{Map}(C, D) \to \mathrm{Map}(\mathrm{Iso}(C)^{\simeq}, \mathrm{Iso}(D)^{\simeq}).$$

By adjunction, it will suffice to construct a map of the form

$$\tilde{I}\colon \mathrm{Iso}(C) \times \mathrm{Fun}(C, D) \to \mathrm{Iso}(D).$$

The map $\tilde{I}$ is informally given by sending a pair (u, φ) consisting of an isomorphism $u\colon [1] \to C$ and a functor $\varphi\colon C \to D$ to the composite $\varphi \circ u\colon [1] \to D$. More formally we define it as the following composite:

$$\tilde{I}\colon \mathrm{Iso}(C) \times \mathrm{Fun}(C, D) \xrightarrow{1\times r} \mathrm{Iso}(C) \times \mathrm{Iso}(\mathrm{Fun}(C, D)) \simeq \mathrm{Iso}(C \times \mathrm{Fun}(C, D)) \xrightarrow{\mathrm{ev}} \mathrm{Iso}(D).$$

We claim that the map I fits in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}(C, D) \arrow[r, equal] \arrow[dd, "I"'] & \mathrm{Map}(C, D) \arrow[d] \\
 & \mathrm{Map}(\mathrm{Map}([1], C), \mathrm{Map}([1], D)) \arrow[d, "\pi_{\mathrm{Iso}}^{*}"] \\
\mathrm{Map}(\mathrm{Iso}(C)^{\simeq}, \mathrm{Iso}(D)^{\simeq}) \arrow[r, hook, "(\pi_{\mathrm{Iso}})_{*}"] & \mathrm{Map}(\mathrm{Iso}(C)^{\simeq}, \mathrm{Map}([1], D)).
\end{tikzcd}$$

Indeed, by transposition/currying and passing to groupoid cores, this follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}(C, D) \times \mathrm{Iso}(C) \arrow[r, equal] \arrow[d, "r\times 1"'] & \mathrm{Fun}(C, D) \times \mathrm{Iso}(C) \arrow[r, equal] \arrow[d, "p_{[1]}^{*}\times\pi_{\mathrm{Iso}}"'] & \mathrm{Fun}(C, D) \times \mathrm{Iso}(C) \arrow[d, "1\times\pi_{\mathrm{Iso}}"] \\
\mathrm{Iso}(\mathrm{Fun}(C, D)) \times \mathrm{Iso}(C) \arrow[r, "\pi_{\mathrm{Iso}}\times\pi_{\mathrm{Iso}}"] \arrow[d, equal] & \mathrm{Fun}([1], \mathrm{Fun}(C, D)) \times \mathrm{Fun}([1], C) \arrow[d, equal] & \mathrm{Fun}(C, D) \times \mathrm{Fun}([1], C) \arrow[l, "p_{[1]}^{*}\times 1"'] \arrow[dd, "-\circ-"] \\
\mathrm{Iso}(\mathrm{Fun}(C, D) \times C) \arrow[r, "\pi_{\mathrm{Iso}}"] \arrow[d, "\mathrm{ev}"'] & \mathrm{Fun}([1], \mathrm{Fun}(C, D) \times C) \arrow[d, "\mathrm{ev}"] & \\
\mathrm{Iso}(D) \arrow[r, "\pi_{\mathrm{Iso}}"'] & \mathrm{Fun}([1], D) \arrow[r, equal] & \mathrm{Fun}([1], D).
\end{tikzcd}$$

Since the bottom map $(\pi_{\mathrm{Iso}})_*$ in this commutative square is an embedding, it follows from Lemma 1.2.4 that the square is a pullback square. It follows that this square defines a homotopy equivalence

$$\mathrm{Map}(C,D) \xrightarrow{\sim} \mathrm{Map}^{\mathrm{Iso}(C)^{\simeq}}(C,D)$$

which is a section of the canonical map $\mathrm{Map}^{\mathrm{Iso}(C)^{\simeq}}(C,D) \to \mathrm{Map}(C,D)$. It follows that we get

$$\mathrm{Fun}^{\mathrm{Iso}(C)^{\simeq}}(C,D) = \langle \mathrm{Map}^{\mathrm{Iso}(C)^{\simeq}}(C,D) \subseteq \mathrm{Fun}(C,D)\rangle \simeq \langle \mathrm{Map}(C,D) \subseteq \mathrm{Fun}(C,D)\rangle \simeq \mathrm{Fun}(C,D)$$

as desired. □

Construction 4.3.5. Let $l\colon C \to L$ be a functor which sends all objects of W to isomorphism of L, in the sense of Definition 4.1.12. Then we see that for a synthetic category the induced map $l^*\colon \mathrm{Fun}(L,D) \to \mathrm{Fun}(C,D)$ factors through $\mathrm{Fun}^W(C,D)$, using the following composite:

$$\mathrm{Fun}(L,D) \simeq \mathrm{Fun}^{\mathrm{Iso}(L)^{\simeq}}(L,D) \xrightarrow{l^*} \mathrm{Fun}^W(C,D).$$

Here the map l^* is obtained via Remark 4.3.2.

Definition 4.3.6 (Localization)**.** We say that a functor $l\colon C \to L$ is a *localization of C at W* if for every synthetic category D, the functor

$$l^*\colon \mathrm{Fun}(L,D) \to \mathrm{Fun}^W(C,D)$$

is an equivalence.

If a localization exists, it is unique up to preferred equivalence. We will frequently denote it by $C[W^{-1}]$, and refer to the functor $l\colon C \to C[W^{-1}]$ as the *localization functor*.

Axiom J (Localization axiom)**.** For every collection of morphisms $w\colon W \to \mathrm{Map}([1],C)$ in C there exists a localization $l\colon C \to C[W^{-1}]$ of C at W.

Remark 4.3.7. The localization axiom essentially says that there exists a pushout square

$$\begin{tikzcd}
{[1]} \times W \arrow[r, "\tilde{w}"] \arrow[d, "\mathrm{pr}_2"'] & C \arrow[d] \\
W \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C[W^{-1}],
\end{tikzcd}$$

where the map $\tilde{w}$ is adjoint to the composite $W \xrightarrow{w} \mathrm{Map}([1],C) \to \mathrm{Fun}([1],C)$.

4.4 Fundamental groupoid axiom

The localization of a synthetic category C at *all* its morphisms deserves special notation and terminology:

Definition 4.4.1 (Fundamental groupoid)**.** In the case where $W := \mathrm{Map}([1], C)$ consists of all morphisms in C, we write

$$\Pi_\infty(C) := C[W^{-1}],$$

and we denote the localization functor by $\Pi_C \colon C \to \Pi_\infty(C)$. We refer to $\Pi_\infty(C)$ as the *fundamental groupoid* of C.

In other words, the fundamental groupoid is characterized by the property that for every other synthetic category C, the functor

$$\Pi_C^* \colon \mathrm{Fun}(\Pi_\infty(C), D) \to \mathrm{Fun}^{\mathrm{Map}([1],C)}(C, D)$$

is an equivalence.

Lemma 4.4.2. *If C is a synthetic category and Γ is a groupoid, then there is an equivalence*

$$\Pi_C^* \colon \mathrm{Fun}(\Pi_\infty(C), \Gamma) \xrightarrow{\sim} \mathrm{Fun}(C, \Gamma)$$

Proof. We may simply compose the equivalence $\mathrm{Fun}(\Pi_\infty(C), \Gamma) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Map}([1],C)}(C, \Gamma)$ with the equivalence $\mathrm{Fun}^{\mathrm{Map}([1],C)}(C, \Gamma) \xrightarrow{\sim} \mathrm{Fun}(C, \Gamma)$ from Lemma 4.3.3. □

Lemma 4.4.3. *Assume that X is a groupoid. Then the localization functor $\Pi_X \colon X \to \Pi_\infty(X)$ is an equivalence.*

Proof. Using that $\mathrm{Iso}(X) \simeq \mathrm{Fun}([1], X)$, we obtain by Proposition 4.3.4 equivalences

$$\mathrm{Fun}^{\mathrm{Map}([1],X)}(X, D) \simeq \mathrm{Fun}^{\mathrm{Iso}(X)^{\simeq}}(X, D) \simeq \mathrm{Fun}(X, D),$$

which shows that the identity map $X \to X$ exhibits X as a localization of X at all its morphisms. This gives the claim. □

As the name suggests, we expect the fundamental groupoid to be a groupoid. Perhaps somewhat surprisingly this does not seem to follow from the axioms, hence we will impose it:

Axiom K (Fundamental groupoid axiom)**.** For every synthetic category C, the synthetic category $\Pi_\infty(C)$ is a groupoid.

We can give a first simple calculation of a fundamental groupoid: that of the 1-simplex:

Lemma 4.4.4. *The functor $p_{[1]} \colon [1] \to *$ induces an equivalence*

$$\Pi_\infty([1]) \xrightarrow{\sim} *.$$

Proof. Since $\Pi_\infty([1])$ is a groupoid by Axiom K it suffices by the Yoneda lemma (just like the proof of Proposition 2.3.3) to show that for every groupoid Γ the induced map

$$\mathrm{Map}(*, \Gamma) \to \mathrm{Map}(\Pi_\infty([1]), \Gamma)$$

is an equivalence. By Lemma 4.4.2, we may identify this map (up to equivalence) with the map

$$p_{[1]}^* \colon \Gamma \to \mathrm{Map}([1], \Gamma).$$

But this map is an equivalence since Γ is a groupoid, finishing the proof. □

The main consequence of the above computation of $\Pi_\infty([1])$ is the fact that every synthetic category C is a full subcategory of its arrow category $\mathrm{Fun}([1], C)$:

Proposition 4.4.5. *Let C be a synthetic category. Then the functor $p_C^* \colon C \to \mathrm{Fun}([1], C)$ exhibits C as a full subcategory of $\mathrm{Fun}([1], C)$, in the sense of Definition 4.2.2.*

Proof. We need to show that there exists an embedding $j \colon \Gamma \to \mathrm{Fun}([1], C)^{\simeq}$ such that C is equivalent to $\langle \Gamma \subseteq \mathrm{Fun}([1], C) \rangle$. But by Lemma 4.4.4, the map $p_{[1]} \colon [1] \to *$ induces an equivalence

$$C \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Map}([1],[1])}([1], C),$$

which is a full subcategory of $\mathrm{Fun}([1], C)$ as desired. □

We obtain the following useful alternative characterization of groupoids:

Proposition 4.4.6. *Let X be a synthetic category. Then X is a groupoid if and only if the map*

$$X^{\simeq} = \mathrm{Map}(*, X) \xrightarrow{p_{[1]}^*} \mathrm{Map}([1], X)$$

is an equivalence.

Proof. We know that X is a groupoid if and only if the map $X \to \mathrm{Fun}([1], X)$ is an equivalence. Since this functor is a full subcategory by Proposition 4.4.5, it follows from Lemma 4.2.11 that we may check this on objects. This finishes the proof. □

4.5 Joins and slices

In this section, we introduce the *join* $C \star D$ of two synthetic categories C and D.

Axiom L (Join axiom)**.** For every groupoid Γ and synthetic categories C and D in context Γ, there is a synthetic category $C \star_\Gamma D$ in context Γ and a homotopy cocartesian square

$$\begin{tikzcd}
C \times_\Gamma (* \sqcup *)_\Gamma \times_\Gamma D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times_\Gamma [1]_\Gamma \times_\Gamma D \arrow[d] \\
C \sqcup D \arrow[r] & C \star_\Gamma D,
\end{tikzcd}$$

where the top map is induced by $\langle 0,1 \rangle \colon * \sqcup * \to [1]$ and the left vertical map is given by

$$C \times_\Gamma (* \sqcup *)_\Gamma \times_\Gamma D \xrightarrow{\sim} C \times_\Gamma D \sqcup C \times_\Gamma D \xrightarrow{\mathrm{pr}_C \sqcup \mathrm{pr}_D} C \sqcup D.$$

Remark 4.5.1. In other words, for every third synthetic category E in context Γ, the commutative square

$$\begin{tikzcd}
\mathrm{Fun}_\Gamma(C \star_\Gamma D, E) \arrow[r] \arrow[d] & \mathrm{Fun}_\Gamma(C \times_\Gamma [1]_\Gamma \times_\Gamma D, E) \arrow[d] \\
\mathrm{Fun}_\Gamma(C \sqcup D, E) \arrow[r] & \mathrm{Fun}(C \times_\Gamma \partial[1]_\Gamma \times_\Gamma D, E)
\end{tikzcd}$$

is a pullback square.

When $\Gamma = *$, we will simply write $C \star D$ for $C \star_* D$; this will be the case we will mostly work in.

Notation 4.5.2. For any functor $g \colon C \star D \to E$, we denote by

$$g_C \colon C \to E \qquad \text{and} \qquad g_D \colon D \to E$$

the composites of g with the inclusions $C \to C \star D$ and $D \to C \star D$, respectively.

Lemma 4.5.3 (Unitality of joins)**.** *For every synthetic category C, the maps $C \to \emptyset \star C$ and $C \to C \star \emptyset$ are equivalences.*

Proof. The synthetic category $\emptyset \star C$ sits in a pushout square

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\
C \arrow[r] & \emptyset \star C
\end{tikzcd}$$

and hence the bottom map is an equivalence. A similar argument applies to $C \star \emptyset$. □

Lemma 4.5.4 (Associativity of joins)**.** *For synthetic categories C, D and E, there is an equivalence*

$$(C \star D) \simeq E \simeq C \star (D \star E).$$

Proof. MISSING (and not completely figured out) □

The join operation can be used to describe functors into slice categories.

Construction 4.5.5 (Slice category)**.** Let $\psi\colon Y \to C$ be a functor. We construct the *slice* $C_{/\psi}$ via the following pullback square:

$$\begin{tikzcd}
C_{/\psi} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}([1] \times Y, C) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
C = C \times * \arrow[r, "(p_Y)^* \times \psi"] & \mathrm{Fun}(Y, C) \times \mathrm{Fun}(Y, C). &
\end{tikzcd}$$

Informally speaking, objects of $C_{/\psi}$ are pairs (x, α) consisting of an object x of C together with a natural transformation $\alpha\colon \mathrm{const}_x \to \psi$, where $\mathrm{const}_x\colon Y \to C$ is the constant function with values x.

Remark 4.5.6. We will mostly be interested in the case $Y = *$. In this case, we may identify functors $x\colon * \to C$ with (absolute) objects of C, and write $C_{/x}$ for the slice of C over x.

Lemma 4.5.7. *For every synthetic category X and every , there is a pullback square of the form*

$$\begin{tikzcd}
\mathrm{Fun}(X, C_{/\psi}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(X \star Y, C) \arrow[d] \\
\mathrm{Fun}(X, C) \arrow[r, "1 \times \psi"'] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C).
\end{tikzcd}$$

Proof. Since the assigment $C \mapsto \mathrm{Fun}(X, C)$ preserves pullbacks, applying it to the defining pullback square of $C_{/\psi}$ gives

$$\begin{tikzcd}
\mathrm{Fun}(X, C_{/\psi}) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}([1] \times Y, C) \arrow[d] \\
\mathrm{Fun}(X, C) \arrow[r, "1 \times \psi"] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C) \arrow[r, "\mathrm{pr}_X^* \times \mathrm{pr}_Y^*"] & \mathrm{Fun}(X \times Y, C) \times \mathrm{Fun}(X \times Y, C).
\end{tikzcd}$$

Because of the factorization of the bottom map, we may compute this pullback by first pulling back along the map $\mathrm{pr}_X^* \times \mathrm{pr}_Y^*$, giving a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(X, C_{/\psi}) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(X \sqcup Y, C) \times_{\mathrm{Fun}(X \times \partial[1] \times Y, C)} \mathrm{Fun}(X \times [1] \times Y, C) \arrow[d] \\
 & \mathrm{Fun}(X \sqcup Y, C) \arrow[d, "\cong"] \\
\mathrm{Fun}(X, C) = \mathrm{Fun}(X, C) \times * \arrow[r, "1 \times \psi"] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(Y, C).
\end{tikzcd}$$

Since the upper right corner is equivalent to $\mathrm{Fun}(X \star Y, C)$ by Remark 4.5.1, this finishes the proof. ◻

Remark 4.5.8. Informally speaking, the previous lemma says that a functor $X \to C_{/\psi}$ into the slice consists of a map $g\colon X \star Y \to C$ fitting in a commutative diagram as follows:

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr, "\psi"] & \\
X \star Y \arrow[rr, "g"'] & & C.
\end{tikzcd}$$

Completely analogously we obtain a dual slice construction:

Construction 4.5.9. For a functor $\varphi\colon X \to C$, we define the slice category $C_{\varphi/}$ via the following pullback square:

$$\begin{tikzcd}
C_{\varphi/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1] \times X, C) \arrow[d] \\
C = * \times C \arrow[r, "\varphi \times p_X^*"] & \mathrm{Fun}(X, C) \times \mathrm{Fun}(X, C).
\end{tikzcd}$$

We will now use the join construction to construct synthetic categories $[n]$ for every $n \geq 0$:

Definition 4.5.10. We define $[0] := *$. For $n \geq 1$, we inductively define

$$[n] := [n-1] \star [0].$$

Note that we do indeed have $[1] \simeq [0] \star [0]$ so that this definition makes sense for $n = 1$.

Proposition 4.5.11. *For every synthetic category C, there is a preferred equivalence*

$$\mathrm{Fun}([n], C) \xrightarrow{\sim} \mathrm{Diag}(\Delta^n, C).$$

Proof. We prove the claim by induction on n. For $n = 0$, both sides are canonically equivalent to C. So assume that the result has been proved for some n. Then by definition of the join $[n+1] = [n] \star *$, the category $\mathrm{Fun}([n+1], C)$ sits in the following pullback square:

$$\begin{tikzcd}
\mathrm{Fun}([n+1],C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([n]\times[1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\mathrm{Fun}([n],C)\times C \arrow[r] & \mathrm{Fun}([n],C)\times\mathrm{Fun}([n],C)
\end{tikzcd}$$

TO DO: Finish proof. □

4.6 Exercises Chapter 4

Exercise 4.6.1. Prove ??: For a synthetic category Γ, a synthetic category category D and a synthetic category E in context Γ, there is a pullback square of the form

$$\begin{tikzcd}
\mathrm{Fun}_\Gamma(D\times\Gamma,E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(D,E) \arrow[d, "q_*"] \\
\Gamma \arrow[r, "p_D^*"] & \mathrm{Fun}(D,\Gamma).
\end{tikzcd}$$

Hint: compare with Lemma 16.6.7.

Exercise 4.6.2. Show that for every groupoid Γ the forgetful functor $\mathcal{E}(\Gamma) \to \mathcal{E}$ preserves and detects homotopy (co)cartesian squares. Reformulate

Exercise 4.6.3. Prove Lemma 4.2.9: show that for embeddings $\Gamma \hookrightarrow C$ and $\Lambda \hookrightarrow D$ the projections $C \times D \to C$ and $C \times D \to D$ induce an equivalence

$$\langle \Gamma\times\Lambda \subseteq C\times D\rangle \xrightarrow{\sim} \langle \Gamma \subseteq C\rangle \times \langle \Lambda \subseteq D\rangle.$$

Exercise 4.6.4. Prove the assertions from Remark 4.1.14: given a functor $f\colon C \to D$ sending a collection of morphisms M in C to a collection of morphisms N in D, we have:

(1) The map $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ is uniquely up to homotopy determined by f.

(2) Given a homotopy $f \sim f'$, we obtain an induced homotopy $f| \sim f'|$.

(3) The construction is compatible with composition: we have $(\mathrm{id}_C)| = \mathrm{id}_{\langle M\rangle_C}$ and $(g\circ f)| = g|\circ f|$ if $g\colon D \to E$ is another functor preserving a collection of morphisms.

(4) If the functors $f\colon C \to D$ and $\tilde{f}\colon M \to N$ are equivalences, then also $f|\colon \langle M\rangle_C \to \langle N\rangle_D$ is an equivalence.

5 Cartesian and cocartesian fibrations

An important concept in category theory is that of a cocartesian fibration: a functor $E \to C$ whose fibers $E(c)$ are in a certain sense 'covariantly functorial' in the object $c \in C$. Similarly there is the dual concept of a cartesian fibration, for which the fibers are 'contravariantly functorial' in c. In this chapter, we will introduce synthetic analogues of these notions.

The material in this chapter is similar in spirit to the article [BW21] by Buchholtz and Weinberger, who give an extensive treatment of cartesian and cocartesian fibrations in the context of simplicial type theory.

5.1 Directed pullbacks

We begin by introducing the notion of the *directed pullback* of two functors.

Construction 5.1.1 (Directed pullback)**.** Consider two functors $f\colon A \to C$ and $g\colon B \to C$. We define the *directed pullback* $A \vec{\times}_{f,g} B$ *of* A *and* B *over* C via the pullback square

$$\begin{tikzcd}
A \vec{\times}_{f,g} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
A \times B \arrow[r, "f\times g"] & C \times C.
\end{tikzcd}$$

We write often abuse notation and write $A \vec{\times}_g B$ in the case where $C = A$ and $f = \mathrm{id}_A\colon A \to A$ and dually write $A \vec{\times}_f B$ when $C = B$ and $g = \mathrm{id}_B\colon B \to B$.

If $f'\colon A' \to C'$ and $g'\colon B' \to C'$ are two other functors and we are given functors $\alpha\colon A \to A'$, $\beta\colon B \to B'$ and $\gamma\colon C \to C'$ satisfying $f'\alpha = \gamma f$ and $g'\beta = \gamma g$, we denote the induced functor between directed pullbacks by

$$\alpha \vec{\times}_\gamma \beta\colon A \vec{\times}_{f,g} B \to A' \vec{\times}_{f',g'} B'.$$

Remark 5.1.2. An object of the directed pullback $A \vec{\times}_{f,g} B$ consists of a triple (a, b, γ) where a is an object of A, b is an object of B, and $\gamma\colon f(a) \to g(b)$ is a morphism in C.

Remark 5.1.3. Since the synthetic category $\mathrm{Iso}(C)$ is a path object of C by the Rezk axiom (Axiom F), we see that there is a pullback square

$$\begin{tikzcd}
A \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
A \times B \arrow[r, "f \times g"] & C \times C.
\end{tikzcd}$$

In particular the full subcategory $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1],C)$ induces a full subcategory $A \times_C B \to A \vec{\times}_C B$. Therefore, we may think of the pullback $A \times_C B$ as the full subcategory of $A \vec{\times}_C B$ spanned by those triples (a,b,γ) such that the map $\gamma \colon f(a) \xrightarrow{\sim} g(b)$ is an isomorphism in C.

Construction 5.1.4 (Directed evaluation maps)**.** Consider a functor $f \colon A \to B$. We define the *directed evaluation maps*

$$\vec{\mathrm{ev}}_0^f \colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B \qquad \text{and} \qquad \vec{\mathrm{ev}}_1^f \colon \mathrm{Fun}([1],A) \to B \vec{\times}_f A$$

via the following two commutative diagrams:

$$\begin{tikzcd}
\mathrm{Fun}([1],A) \arrow[r, dashed, "\vec{\mathrm{ev}}_0^f"] \arrow[rr, bend left, "f_*"] \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & A \vec{\times}_f B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],B) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
A \times A \arrow[r, "\mathrm{id}_A \times f"] & A \times B \arrow[r, "f \times \mathrm{id}_B"] & B \times B
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Fun}([1],A) \arrow[r, dashed, "\vec{\mathrm{ev}}_1^f"] \arrow[rr, bend left, "f_*"] \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & B \vec{\times}_f A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],B) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
A \times A \arrow[r, "f \times \mathrm{id}_A"] & B \times A \arrow[r, "\mathrm{id}_B \times f"] & B \times B.
\end{tikzcd}$$

Remark 5.1.5. When $B = *$, we have $A \vec{\times}_f B \xrightarrow{\sim} A \xrightarrow{\sim} B \vec{\times}_f A$ and the directed evaluation maps $\vec{\mathrm{ev}}_0^f$ and $\vec{\mathrm{ev}}_1^f$ simplify to the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}([1],A) \to A$.

For later use, we record various properties of the directed evaluation maps.

Lemma 5.1.6. *Assume that $f \colon A \to B$ is an equivalence. Then also $\vec{\mathrm{ev}}_0^f$ and $\vec{\mathrm{ev}}_1^f$ are equivalences.*

Proof. For $\vec{\mathrm{ev}}_0^f$, this follows from the 2-out-of-3 property applied to the following two maps:

$$\mathrm{Fun}([1],A) \xrightarrow{\vec{\mathrm{ev}}_0^f} A \vec{\times}_f B \to \mathrm{Fun}([1],B).$$

Indeed, the composite is induced by f and is hence an equivalence, and the second map is a base change of f and thus also an equivalence. The proof for $\vec{\mathrm{ev}}_1^f$ is similar. ◻

Lemma 5.1.7 (Base change)**.** *Consider a pullback square of the form*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B.
\end{tikzcd}$$

Then the commutative square

$$\begin{tikzcd}
\mathrm{Fun}([1],A') \arrow[r, "u_*"] \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"'] & \mathrm{Fun}([1],A) \arrow[d, "\vec{\mathrm{ev}}_0^{f}"] \\
A' \vec{\times}_{f'} B' \arrow[r, "v \vec{\times}_f u"] & A \vec{\times}_f B
\end{tikzcd}$$

is a pullback square. The dual result for $\vec{\mathrm{ev}}_1^{f}$ *and* $\vec{\mathrm{ev}}_1^{f'}$ *also holds.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
& \mathrm{Fun}([1],B') \arrow[rr] \arrow[dd, equal] & & \mathrm{Fun}([1],B) \arrow[dd, equal] \\
\mathrm{Fun}([1],A') \arrow[ur] \arrow[rr] \arrow[dd, "\vec{\mathrm{ev}}_0^{f'}"'] & & \mathrm{Fun}([1],A) \arrow[ur] \arrow[dd, "\vec{\mathrm{ev}}_0^{f}"] & \\
& \mathrm{Fun}([1],B') \arrow[rr] \arrow[dd, "{(s,t)}"] & & \mathrm{Fun}([1],B) \arrow[dd, "{(s,t)}"] \\
A' \vec{\times}_{f'} B' \arrow[ur] \arrow[rr] \arrow[dd] & & A \vec{\times}_f B \arrow[ur] \arrow[dd] & \\
& B' \times B' \arrow[rr] & & B \times B \\
A' \times B' \arrow[ur] \arrow[rr] & & A \times B. \arrow[ur] &
\end{tikzcd}$$

Observe that the bottom square is a pullback square, while the lower vertical squares on the left and right sides are pullback squares by definition of $A' \vec{\times}_{f'} B'$ and $A \vec{\times}_f B$. It follows by the pasting law of pullback squares that also the middle horizontal square is a pullback square. Since also the upper square at the back and the top square are pullback squares, it thus follows that the upper front square is a pullback square, as desired. □

Lemma 5.1.8 (Composition)**.** *For functors $f\colon A \to B$ and $g\colon B \to C$. Then there is a commutative diagram*

$$\begin{tikzcd}
& \mathrm{Fun}([1],A) \arrow[dl, "\vec{\mathrm{ev}}_0^f"'] \arrow[dr, "\vec{\mathrm{ev}}_0^{gf}"] & \\
A \vec{\times}_f B \arrow[rr] \arrow[d] \arrow[drr, phantom, very near start, "\lrcorner"] & & A \vec{\times}_{gf} C \arrow[d] \\
\mathrm{Fun}([1],B) \arrow[rr, "\vec{\mathrm{ev}}_0^g"] & & B \vec{\times}_g C,
\end{tikzcd}$$

where the bottom square is a pullback square.

Proof. It is easy to verify that the diagram commutes, hence it remains to show that the bottom square is a pullback square. To this end, consider the following commutative diagram:

$$\begin{tikzcd}
A \vec{\times}_f B \arrow[rr] \arrow[dr] \arrow[dd] & & A \vec{\times}_{gf} C \arrow[dr] \arrow[dd] & \\
& \mathrm{Fun}([1],B) \arrow[rr, "\vec{\mathrm{ev}}_0^g"] \arrow[dd] & & B \vec{\times}_g C \arrow[dd] \\
A \times B \arrow[rr, "1\times g"] \arrow[dr, "f\times 1"'] & & A \times C \arrow[dr, "f\times 1"] & \\
& B \times B \arrow[rr, "1\times g"] & & B \times C.
\end{tikzcd}$$

We want to show that the top square is a pullback square. The left square is a pullback square by definition and the bottom square is easily seen to be a pullback square, hence by the pasting law of pullback squares it will suffice to show that the right square is a pullback square. This follows from another instance of the pasting law, applied to the following commutative diagram:

$$\begin{tikzcd}
A \vec{\times}_{gf} C \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \vec{\times}_g C \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
A \times C \arrow[r, "f\times 1"] & B \times C \arrow[r, "g\times 1"] & C \times C.
\end{tikzcd}$$

Indeed, by the definitions of $A \vec{\times}_{gf} C$ and $B \vec{\times}_g C$ the right and outer squares are pullback squares and hence so is the left square. □

Lemma 5.1.9 (Retracts)**.** *Consider a retract diagram of functors:*

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\
B' \arrow[r] & B \arrow[r] & B'.
\end{tikzcd}$$

Then the map $\vec{\mathrm{ev}}_0^{f'}$ *is a retract of* $\vec{\mathrm{ev}}_0^{f}$*:*

$$\begin{tikzcd}
\mathrm{Fun}([1],A') \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"'] & \mathrm{Fun}([1],A) \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{f}"] & \mathrm{Fun}([1],A') \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"] \\
A' \vec{\times}_{f'} B' \arrow[r] & A \vec{\times}_f B \arrow[r] & A' \vec{\times}_{f'} B'.
\end{tikzcd}$$

Proof. This is an immediate consequence of the definitions. □

5.2 Left and right fibrations

We define the notions of *left* and *right fibrations*, and show that they admit a notion of *covariant/contravariant transport*.

Definition 5.2.1 (Left/right fibration)**.** A functor $f\colon A \to B$ is called a *left fibration* if the directed evaluation map $\vec{\mathrm{ev}}_0^{f}\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$ is an equivalence. Dually, we say that f is a *right fibration* if $\vec{\mathrm{ev}}_1^{f}\colon \mathrm{Fun}([1],A) \to B \vec{\times}_f A$ is an equivalence.

Lemma 5.2.2. *A functor* $f\colon A \to B$ *is a left fibration if and only if the commutative square*

$$\begin{tikzcd}
\mathrm{Fun}([1],A) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],B) \arrow[d, "\mathrm{ev}_0"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

is a pullback. A dual characterization for right fibrations is obtained by replacing $\vec{\mathrm{ev}}_0^{f}$ *by* $\vec{\mathrm{ev}}_1^{f}$ *and* ev_0 *by* ev_1.

Proof. This is immediate from the fact that the directed evaluation map $\vec{\mathrm{ev}}_0^{f}$ is defined as the unique dashed map filling the following diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],A) \arrow[dr, dashed, "\vec{\mathrm{ev}}_0^{f}"] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "\mathrm{ev}_0"'] & & \\
& A \vec{\times}_f B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],B) \arrow[d, "\mathrm{ev}_0"] \\
& A \arrow[r, "f"] & B.
\end{tikzcd}$$

The proof for right fibrations is dual. □

Corollary 5.2.3. *Let $f\colon A \to B$ be a left (resp. right) fibration. Then the commutative square*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "(p_{[1]})^*"'] & B \arrow[d, "(p_{[1]})^*"] \\
\mathrm{Fun}([1],A) \arrow[r, "f_*"] & \mathrm{Fun}([1],B)
\end{tikzcd}$$

is a pullback square.

Proof. We prove the claim for left fibrations; the claim for right fibrations is dual. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "(p_{[1]})^*"'] & B \arrow[d, "(p_{[1]})^*"] \\
\mathrm{Fun}([1],A) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],B) \arrow[d, "\mathrm{ev}_0"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

The outer square is clearly cartesian, and the bottom square is cartesian by Lemma 5.2.2. It thus follows from the pasting law of pullback squares that also the top square is cartesian. □

Left/right fibrations have unique covariant/contravariant lifts:

Definition 5.2.4 (Category of lifts). Let $f\colon A \to B$ be a functor. Consider an object (a,β) of $A \vec{\times}_f B$, given by an object a of A and a morphism $\beta\colon f(a) \to b'$ in B. We define the synthetic category $\mathrm{Lift}_0^f(a,\beta)$ of *covariant lifts* as the fiber of $\vec{\mathrm{ev}}_0^f$ over (a,β):

$$\begin{tikzcd}
\mathrm{Lift}_0^f(a,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],A) \arrow[d, "\vec{\mathrm{ev}}_0^f"] \\
\{(a,\beta)\} \arrow[r, "(a{,}\beta)"] & A \vec{\times}_f B.
\end{tikzcd}$$

An object of $\mathrm{Lift}_0^f(a,\beta)$ is called a *covariant lift of β starting in a* and consists of a morphism $\alpha\colon a \to a'$ in A whose source is a and whose image under f is β.

Dually, we define for every object (a',β) of $B \vec{\times}_f A$ the category $\mathrm{Lift}_1^f(a',\beta)$ of *contravariant lifts* as the fiber of $\vec{\mathrm{ev}}_1^f$ over (a',β):

$$\begin{tikzcd}
\mathrm{Lift}_1^f(a',\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],A) \arrow[d, "\vec{\mathrm{ev}}_1^f"] \\
\{(a',\beta)\} \arrow[r, "(a'{,}\beta)"] & B \vec{\times}_f A.
\end{tikzcd}$$

An object of $\mathrm{Lift}^f_1(a',\beta)$ is called a *contravariant lift of β ending in a'* and consists of a morphism $\alpha\colon a \to a'$ in A whose target is a and whose image under f is β.

Corollary 5.2.5. *If $f\colon A \to B$ is a left fibration, then the synthetic category* $\mathrm{Lift}^f_0(a,\beta)$ *is contractible for every $(a,\beta) \in A \vec{\times}_f B$. Dually, if f is a right fibration, then* $\mathrm{Lift}^f_1(a',\beta)$ *is contractible for all $(a',\beta) \in B \vec{\times}_f A$.*

In what follows, we will spell out various constructions in the case of left fibrations, and leave their dual constructions for right fibrations to the reader.

Notation 5.2.6 (Covariant transport)**.** Let $f\colon A \to B$ be a left fibration. We denote by

$$\mathrm{lift}^f_0\colon A \vec{\times}_f B \to \mathrm{Fun}([1],A)$$

the inverse of the functor $\vec{\mathrm{ev}}^f_0\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$. For an object (a,β) of $A \vec{\times}_f B$, we denote the morphism $\mathrm{lift}^f_0(a,\beta)$ in A by

$$\mathrm{lift}^f_0(a,\beta)\colon a \to \beta_!(a),$$

and refer to its target $\beta_!(a)$ as the *covariant transport of a along β*. Note that $\mathrm{lift}^f_0(a,\beta)$ is a covariant lift of β starting in a.

Dually, if f is instead a *right* fibration, we denote the inverse of $\vec{\mathrm{ev}}^f_1$ by lift^f_1. Its value at an object (a',β) of $B \vec{\times}_f A$ is denoted by $\mathrm{lift}^f_1(a',\beta)\colon \beta^*(a') \to a$ and we refer to $\beta^*(a')$ as the *contravariant transport of a' along β*.

We will now enhance the construction $a \mapsto \beta_!(a)$ to a functor $A(b) \to A(b')$:

Construction 5.2.7 (Covariant transport functor)**.** Let $f\colon A \to B$ be a left fibration, and let $\mathrm{lift}^f_0\colon A \vec{\times}_f B \to \mathrm{Fun}([1],A)$ be an inverse of $\vec{\mathrm{ev}}^f_0\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$. Given a morphism $\beta\colon b \to b'$ in B, we will construct a functor

$$\beta_!\colon A(b) \to A(b'),$$

called *covariant transport along β*. Note that the assignment $a \mapsto (a,\beta)$ extends to a functor $A(b) \to A \vec{\times}_f B$, defined as the unique dashed arrow making the following diagram commute:

$$\begin{tikzcd}
A(b) \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\lrcorner" very near start] & A \vec{\times}_f B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "f"] \\
\{\beta\} \arrow[r, "\beta"] & \mathrm{Fun}([1],B) \arrow[r, "\mathrm{ev}_0"] & B.
\end{tikzcd}$$

We may now consider the composite

$$A(b) \xrightarrow{a\mapsto(a,\beta)} A \vec{\times}_f B \xrightarrow{\mathrm{lift}^f_0} \mathrm{Fun}([1],A) \xrightarrow{\mathrm{ev}_1} A.$$

Since lift_0^f is a section of $\vec{\mathrm{ev}}_1$ we see that this composite factors uniquely through $A(b')$, finishing the construction of the functor $\beta_!\colon A(b) \to A(b')$.

Dually, given a right fibration $f\colon A \to B$, every morphism $\beta\colon b \to b'$ in B determines a functor $\beta^*\colon A(b') \to A(b)$ called *contravariant transport along β.*

Remark 5.2.8. The above construction can be equipped with additional naturality in β, in a sense we will now explain. Assume that b and b' are not just objects of B but are functors $b, b'\colon C \to B$ for some indexing category C. The fibers $A(b)$ and $A(b')$ are now replaced by the following pullbacks:

$$\begin{tikzcd}
A(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] & A(b') \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
C \arrow[r, "b"] & B & C. \arrow[l, "b'"']
\end{tikzcd}$$

Given a natural transformation $\beta\colon b \to b'$ of functors $C \to B$, we may replace the groupoid $\{\beta\}$ in the above construction by C to obtain a functor $\beta_!\colon A(b) \to A(b')$ over C: there is a commutative diagram

$$\begin{tikzcd}
A(b) \arrow[rr, "\beta_!"] \arrow[dr] & & A(b') \arrow[dl] \\
& C. &
\end{tikzcd}$$

For every object $c \in C$, passing to fibers over c recovers the map $\beta_!\colon A(b(c)) \to A(b'(c))$, so that this map over C assembles all these individual functors into one coherent map.

The next proposition shows that the covariant transport functors $\beta_!\colon A(b) \to A(b')$ are suitably 'functorial':

Proposition 5.2.9. *Let $f\colon A \to B$ be a left fibration.*

(1) For $\beta = \mathrm{id}_b\colon b \to b$, the functor $(\mathrm{id}_b)_!\colon A(b) \to A(b)$ is naturally isomorphic to the identity functor;

(2) For morphisms $\beta\colon b \to b'$ and $\beta'\colon b' \to b''$ in B, the composite functor

$$\beta'_! \circ \beta_!\colon A(b) \to A(b'')$$

is naturally isomorphic to $(\beta' \circ \beta)_!\colon A(b) \to A(b'')$.

The dual result for the contravariant transport functors of a right fibration is also valid.

Proof. For (1), consider the following commutative diagram:

$$\begin{tikzcd}
A(b) \arrow[r, "a\mapsto \mathrm{id}_a"] \arrow[d, "a\mapsto (a,\mathrm{id}_b)"'] \arrow[rr, bend left, equal] & \mathrm{Fun}([1],A(b)) \arrow[r, "\mathrm{ev}_1"] \arrow[d] & A(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{b\} \arrow[d] \\
A \vec{\times}_f B \arrow[r, "\mathrm{lift}_0^f"] & \mathrm{Fun}([1],A) \arrow[r, "\mathrm{ev}_1"] & A \arrow[r, "f"] & B.
\end{tikzcd}$$

To show that the left square commutes, it will suffice to show this after composing with the equivalence $\vec{\mathrm{ev}}\colon \mathrm{Fun}([1],A) \xrightarrow{\sim} A \vec{\times}_f B$, where it is clear. The middle square clearly commutes by naturality. By the universal property of the pullback $A(b) = A \times_B \{b\}$, it follows that the identity functor $\mathrm{id}_{A(b)}\colon A(b) \to A(b)$ must agree with $(\mathrm{id}_b)_!\colon A(b) \to A(b)$.

For (2), [To do!] We obtain for every object a of $A(b)$ a covariant lift

$$\mathrm{lift}_0^f(a,\beta)\colon a \to \beta_!(a)$$

of β starting in a, which in turn leads to a covariant lift

$$\mathrm{lift}_0^f(\beta_!(a),\beta')\colon \beta_!(a) \to \beta'_!(\beta_!(a)).$$

Since the composite of these two morphisms is a covariant lift of $\beta' \circ \beta$ starting in a, it follows that there is an isomorphism

$$\mathrm{lift}_0^f(\beta_!(a),\beta') \circ \mathrm{lift}_0^f(a,\beta) \cong \mathrm{lift}_0^f(a,\beta'\circ\beta),$$

and in particular we get $(\beta'\circ\beta)_!(a) \cong \beta'_!(\beta_!(a))$. Explain how to make this functorial in a! ☐

The covariant transport functors are compatible with maps of left fibrations:

Lemma 5.2.10. *Consider a morphism of left fibrations over B:*

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\
& B. &
\end{tikzcd}$$

Then there is for every morphism $\beta\colon b \to b'$ in B and every object a of $A(b)$ a natural isomorphism

$$\beta_!(\varphi_b(a)) \cong \varphi_{b'}(\beta_!(a))$$

of functors $A(b) \to A'(b')$.

Proof. This is clear from the observation that the morphism

$$\varphi(\mathrm{lift}^f_0(a,\beta))\colon \varphi_b(a) \to \varphi_{b'}(\beta_!(a))$$

in A' is a covariant lift of β starting in $\varphi_b(a)$, so that it must be isomorphic to the morphism $\mathrm{lift}^f_0(\varphi_b(a),\beta)\colon \varphi_b(a) \to \beta_!(\varphi_b(a))$. □

We now prove various preservation properties of left and right fibrations. We state some of the results only for left fibrations, and leave the dual statement for right fibrations to the reader.

Lemma 5.2.11 (Trivial fibrations)**.** *Every equivalence $f\colon A \xrightarrow{\sim} B$ is both a left and a right fibration.*

Proof. This is immediate from Lemma 5.1.6. □

Proposition 5.2.12 (Base change)**.** *Consider a pullback square of synthetic categories*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B.
\end{tikzcd}$$

(1) If f is a left fibration, then also f' is a left fibration;

(2) For a morphism $\beta\colon b \to b'$ in B', the following square commutes:

$$\begin{tikzcd}
A'(b) \arrow[r, "\beta_!"] \arrow[d, "u_b"', "\simeq"] & A'(b') \arrow[d, "u_{b'}", "\simeq"'] \\
A(v(b)) \arrow[r, "v(\beta)_!"] & A(v(b')).
\end{tikzcd}$$

Proof. Part (1) is immediate from Lemma 5.1.7, which says that the map $\vec{\mathrm{ev}}^{f'}_0$ is a base change of $\vec{\mathrm{ev}}^{f}_0$. This implies that the inverse map $\mathrm{lift}^{f'}_0$ is a base change of lift^f_0: given a morphism $\beta\colon b \to b'$ in B' and an object a of $A'(b)$, the covariant lift $\mathrm{lift}\,f'_0\colon a \to \beta_!(a)$ of β starting in a is sent under u to the covariant lift $\mathrm{lift}^f_0\colon u(a) \to v(\beta)_!(u(a))$ of $v(\beta)\colon v(b) \to v(b')$ starting in $u(a)$. This proves part (2). □

Proposition 5.2.13 (Composition)**.** *Consider a commutative diagram*

$$\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "gf"] & & C
\end{tikzcd}$$

of functors, and assume that g is a left fibration. Then f is a left fibration if and only if gf is a left fibration.

Proof. By Lemma 5.1.8, there is a commutative triangle

$$\begin{tikzcd}
& \mathrm{Fun}([1],A) \arrow[dl, "\vec{\mathrm{ev}}_0^f"'] \arrow[dr, "\vec{\mathrm{ev}}_0^{gf}"] & \\
B \vec{\times}_f A \arrow[rr] & & C \vec{\times}_{gf} A,
\end{tikzcd}$$

where the bottom map is a base change of the map $\vec{\mathrm{ev}}_0^g$ and hence is an equivalence as g is a left fibration. It follows by 2-out-of-3 that the map $\vec{\mathrm{ev}}_0^f$ is an equivalence if and only if $\vec{\mathrm{ev}}_0^{gf}$ is an equivalence. This finishes the proof. □

Lemma 5.2.14. *The classes of right and left fibrations are stable under retracts.*

Proof. This is immediate from Lemma 5.1.9. □

Proposition 5.2.15. *Let $p\colon C \to Y$ be a functor and assume that Y is a groupoid. Then the following conditions are equivalent:*

(1) The category C is a groupoid;

(2) The map p is a left fibration;

(3) The map p is a right fibration.

Proof. We prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r, "p_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],Y) \arrow[d, "\mathrm{ev}_0", "\sim"'] \\
C \arrow[r, "p"] & Y.
\end{tikzcd}$$

Since Y is a groupoid, the right vertical map is an equivalence. It follows that this square is a pullback square if and only if the left vertical map is an equivalence. But the former is by Lemma 5.2.2 equivalent to (2), while the latter is equivalent to (1). □

Corollary 5.2.16. *If $f\colon X \to Y$ is a left (resp. right) fibration and Y is a groupoid, then also X is a groupoid.* □

Corollary 5.2.17. *Let $f\colon C \to D$ be a left (resp. right) fibration. Then every fiber of f is a groupoid.*

Proof. For every object d of D, the map $C(d) \to \{d\}$ is again a left (resp. right) fibration, hence the claim follows from the previous corollary. □

5.3 Slice categories

In this section we provide two fundamental examples of left and right fibrations: the forgetful functors $C_{/x} \to C$ and $C_{x/} \to C$ for a synthetic category C and an object x of C.

Definition 5.3.1. Let C be a synthetic category and let x be an object of C. We define the *slice categories* $C_{/x}$ and $C_{x/}$ via the following two pullback diagrams:

$$\begin{tikzcd}
C_{/x} \arrow[r] \arrow[d, "p_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
C \arrow[r, "\mathrm{id}_C \times x"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times C \arrow[d, "\mathrm{pr}_2"] \\
* \arrow[r, "x"] & C.
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
C_{x/} \arrow[r] \arrow[d, "q_x"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
C \arrow[r, "x \times \mathrm{id}_C"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \times C \arrow[d, "\mathrm{pr}_1"] \\
* \arrow[r, "x"] & C.
\end{tikzcd}$$

Remark 5.3.2. The objects of the synthetic category $C_{/x}$ consists of pairs (y,u) where y is an object of C and $u\colon y \to x$ is a morphism in C. For $C_{x/}$, the second component is instead a morphism $v\colon x \to y$ in C.

Lemma 5.3.3. *Consider synthetic categories C and X, and let x be an object of C. Then there are equivalences*

$$\mathrm{Fun}(X, C_{/x}) \xrightarrow{\sim} \mathrm{Fun}(X,C)_{/x} \qquad \textit{and} \qquad \mathrm{Fun}(X, C_{x/}) \xrightarrow{\sim} \mathrm{Fun}(X,C)_{x/}$$

where on the right we write x for the object in $\mathrm{Fun}(X,C)$ *given by the composite* $X \to * \xrightarrow{x} C$.

Proof. Since $\mathrm{Fun}(X,-)$ preserves pullbacks, we have a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(X, C_{/x}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], \mathrm{Fun}(X,C)) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
\mathrm{Fun}(X,C) \arrow[r, "\mathrm{id}_{\mathrm{Fun}(X,C)} \times x"] & \mathrm{Fun}(X,C) \times \mathrm{Fun}(X,C),
\end{tikzcd}$$

showing that $\mathrm{Fun}(X, C_{/x}) \xrightarrow{\sim} \mathrm{Fun}(X,C)_{/x}$. The equivalence $\mathrm{Fun}(X, C_{x/}) \xrightarrow{\sim} \mathrm{Fun}(X,C)_{x/}$ is analogous. □

Theorem 5.3.4. *Let C be a synthetic category and let x be an object of C. Then the canonical functor $C_{/x} \to C$ is a right fibration. Dually, the map $C_{x/} \to C$ is a left fibration.*

Proof. By symmetry it will suffice to prove the claim for $C_{/x} \to C$. We have to show that the functor

$$\vec{\mathrm{ev}}_1^{p_x}\colon \mathrm{Fun}([1], C_{/x}) \to C \vec{\times}_{p_x} C_{/x}$$

is an equivalence. The source of this isofibration sits in a pullback square of the form

$$\begin{tikzcd}
\mathrm{Fun}([1],C_{/x}) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}([1],\mathrm{Fun}([1],C)) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
\mathrm{Fun}([1],C) \arrow[r, "1\times x"] & \mathrm{Fun}([1],C)\times C \arrow[r, "1\times p_{[1]}^*"] & \mathrm{Fun}([1],C)\times\mathrm{Fun}([1],C),
\end{tikzcd}$$

and as a consequence of Lemma 1.3.12 we thus obtain a pullback square

$$\begin{tikzcd}
\mathrm{Fun}([1],C_{/x}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2,C) \arrow[d, "\mathrm{ev}_2"] \\
* \arrow[r, "x"] & C.
\end{tikzcd}$$

On the other hand, the target $C \vec{\times}_{p_x} C_x$ is defined via the pullback square

$$\begin{tikzcd}
C \vec{\times}_{p_C} C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_1"] \\
C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "1\times x"] \\
\mathrm{Fun}([1],C) \arrow[r, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] & C\times C,
\end{tikzcd}$$

and consequently we obtain a pullback square

$$\begin{tikzcd}
C \vec{\times}_{p_C} C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Lambda^2_1,C) \arrow[d, "\mathrm{ev}_2"] \\
* \arrow[r, "x"] & C.
\end{tikzcd}$$

Under these two identifications, the map $\vec{\mathrm{ev}}_1^{p_x} \colon \mathrm{Fun}([1],C_{/x}) \to C \vec{\times}_{p_x} C_{/x}$ is obtained by restricting the map $\mathrm{Fun}(\Delta^2,C) \to \mathrm{Fun}(\Lambda^2_1,C)$. The fact that it is an equivalence thus follows from the Segal axiom E. □

Remark 5.3.5. In more down-to-earth terms, the above proof is capturing the equivalence between three ways of encoding maps in the slice $C_{/x}$:

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & z \arrow[d] & \overset{\textit{Lemma } 1.3.12}{\Longleftrightarrow} & & z \arrow[dr] & & \overset{\textit{Axiom E}}{\Longleftrightarrow} & & z \arrow[dr] & \\
x \arrow[r, equal] & x & & y \arrow[ur] \arrow[rr] & & x & & y \arrow[ur] & & x.
\end{tikzcd}$$

Corollary 5.3.6. *For two objects x and y in a synthetic category C, the hom groupoid $C(x,y)$ is a groupoid.*

Proof. Since $C(x,y)$ is the fiber over x of the right fibration $C_{/y} \to C$ from Theorem 5.3.4, this follows from Corollary 5.2.17. ☐

Observe that the fiber of the left fibration $C_{x/} \to C$ over an object y of C is the hom groupoid $C(x,y)$ from Definition 1.4.5. In particular, Construction 5.2.7 produces for every morphism $g\colon y \to z$ a *covariant transport functor*

$$g_!\colon C(x,y) \to C(x,z).$$

Proposition 5.3.7. *For every morphism $g\colon y \to z$, the functor $g_!\colon C(x,y) \to C(x,z)$ agrees with the composition functor*

$$g \circ -\colon C(x,y) \to C(x,z)$$

from Construction 1.4.6. Dually, the contravariant transport functor $g^*\colon C(z,x) \to C(y,x)$ associated to the right fibration $C_{/x} \to C$ agrees with the composition functor $- \circ g\colon C(z,x) \to C(y,x)$.

Proof. As we have seen in the proof of Theorem 5.3.4, the directed evaluation map $\vec{\mathrm{ev}}_0^{q_x}\colon \mathrm{Fun}([1],C_{x/}) \to C \vec{\times}_{q_x} C_{x/}$ is a base change of the map $\mathrm{Fun}(\Delta^2,C) \to \mathrm{Fun}(\Lambda^2_1,C)$, and hence the map $\mathrm{lift}_0^{q_x}$ is given by the inverse of this map. But this inverse precisely defines composition in C. In more detail, fixing the morphism $g\colon y \to z$, the covariant transport functor $g_!$ is given as the composite

$$C(x,y) \xrightarrow{f\mapsto(f,g)} C \vec{\times}_{q_x} C_{x/} \xrightarrow{\mathrm{lift}_0^{q_x}} \mathrm{Fun}([1],C_{x/}) \xrightarrow{\mathrm{ev}_1} C_{x/},$$

where one checks that this lands in the fiber $C(x,z)$ over z. Due to the above identification of the middle map, we may rewrite this as

$$C(x,y) \to \mathrm{Fun}([1],C) \xrightarrow{f\mapsto(f,g)} \mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \simeq \mathrm{Fun}(\Lambda^2_1,C) \xrightarrow{\sim} \mathrm{Fun}(\Delta^2,C) \xrightarrow{\mathrm{ev}_{\{1,2\}}} \mathrm{Fun}([1],C),$$

where again one checks that this lands in $C(x,z)$. But this is precisely the map $g \circ -\colon C(x,y) \to C(x,z)$. ☐

The previous results directly imply versions for the *relative slice categories* $C_{/d}$ for a functor $f\colon C \to D$ and an object d of D:

Definition 5.3.8 (Relative slice)**.** Given a functor $f\colon C \to D$ and an object d of D we define the *relative slices* $C_{/d}$ and $C_{d/}$ via the following pullback squares:

$$\begin{tikzcd} C_{/d} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_{/d} \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_{d/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_{d/} \arrow[d] \\ C \arrow[r, "f"] & D. \end{tikzcd}$$

Corollary 5.3.9. *For any functor $f\colon C \to D$ and any object d of D, the functor $C_{/d} \to C$ is a right fibration and the functor $C_{d/} \to C$ is a left fibration.*

Proof. This is immediate from Theorem 5.3.4 and Proposition 5.2.12. □

The above results for $C_{/\psi}$ dualize to analogous results for $C_{\varphi/}$, which we will leave to the reader to spell out.

Lemma 5.3.10. *For a functor $\psi\colon X \to C$, the forgetful functor $C_{/\psi} \to C$ is a right fibration. Dually, $C_{\psi/} \to C$ is a left fibration.*

Proof. We prove the first statement; the second statement is dual. Note that the cone category $C_{/\psi}$ sits in a pullback square of the form

$$\begin{tikzcd}
C_{/\psi} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(Y,C)_{/\psi} \arrow[d] \\
C \arrow[r, "p_Y^*"] & \mathrm{Fun}(Y,C).
\end{tikzcd}$$

Since the forgetful functor $\mathrm{Fun}(Y,C)_{/\psi} \to \mathrm{Fun}(Y,C)$ is a right fibration by Theorem 5.3.4, it thus follows from Proposition 5.2.12 that the map $C_{/\psi} \to C$ is also a right fibration, as desired. □

5.4 Adjunctions

The concept of an adjunction is fundamental in category theory, hence we will need an analogue in our setup of synthetic category theory. Given two functors $f\colon C \to D$ and $g\colon D \to C$ between ordinary categories, there are two equivalent ways to encode the data of an adjunction $f \dashv g$ between f and g:

(1) One may provide the data of a unit $\eta\colon \mathrm{id}_C \to gf$ and a counit $\varepsilon\colon fg \to \mathrm{id}_D$ satisfying the *triangle identities*;

(2) One may provide a natural isomorphism $D(fx,y) \xrightarrow{\sim} C(x,gy)$ for all x in C and y in D.

In this section, we will discuss the analogous situation in the context of directed type theory.

5.4.1 Units and counits

Our first definition of adjunctions will use units and counits.

Definition 5.4.1 (Adjunction)**.** Consider two functors $l\colon C \to D$ and $r\colon D \to C$. An *adjunction* $l \dashv r$ consists of natural transformations $\eta\colon \mathrm{id}_C \to rl$ and $\varepsilon\colon lr \to \mathrm{id}_D$ satisfying the *triangle identities*:

$$\begin{tikzcd}
 & lrl \arrow[dr, "\varepsilon l"] & \\
l \arrow[ur, "l\eta"] \arrow[rr, equal] & & l
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
 & rlr \arrow[dr, "r\varepsilon"] & \\
r \arrow[ur, "\eta r"] \arrow[rr, equal] & & r.
\end{tikzcd}$$

We call η the *unit* of the adjunction and ε the *counit*. We often write $(l \dashv r)\colon C \rightleftarrows D$ to indicate the data (l,r,η,ε) of an adjunction.

Warning 5.4.2. To obtain the correct *category of adjunctions*, Definition 5.4.1 needs to be modified as follows: an *adjunction* $l \vdash r$ consists of a natural transformation $\eta\colon \mathrm{id}_C \to rl$ and two natural transformations $\varepsilon, \varepsilon'\colon lr \to \mathrm{id}_D$ satisfying the triangle identities:

$$\begin{tikzcd}
 & lrl \arrow[dr, "\varepsilon l"] & \\
l \arrow[ur, "l\eta"] \arrow[rr, equal] & & l
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
 & rlr \arrow[dr, "r\varepsilon'"] & \\
r \arrow[ur, "\eta r"] \arrow[rr, equal] & & r.
\end{tikzcd}$$

One may then deduce a posteriori that the counits ε and ε' are equivalent to each other, much like the left and right inverses of an equivalence are canonically equivalent.

Since we will not have much interest in the category of adjunctions in this course, we will keep working with the somewhat simpler and more familiar definition of an adjunction given in Definition 5.4.1.

Proposition 5.4.3. *Let $(l \vdash r)\colon C \rightleftarrows D$ and $(l' \vdash r')\colon D \rightleftarrows E$ be adjunctions. Then the composite functors $l'l\colon C \to E$ and $rr'\colon E \to C$ constitute an adjunction $(l'l \vdash rr')\colon C \rightleftarrows E$.*

Proof. The corresponding unit and counit are given as the composites

$$\mathrm{id}_C \xrightarrow{\eta} rl \xrightarrow{r\eta' l} rr'l'l \qquad \text{and} \qquad l'lrr' \xrightarrow{l'\varepsilon r'} l'r' \xrightarrow{\varepsilon'} \mathrm{id}_E\,.$$

We leave the verification of the triangle identity to the reader, see Exercise 5.11.1 □

Proposition 5.4.4. *Let $(l \vdash r)\colon C \leftrightarrows D$ be an adjunction. Then for every synthetic category E we obtain adjunctions*

$$l_*\colon \mathrm{Fun}(E,C) \leftrightarrows \mathrm{Fun}(E,D) : r_* \qquad \textit{and} \qquad r^*\colon \mathrm{Fun}(D,E) \leftrightarrows \mathrm{Fun}(C,D) : l^*.$$

Proof. We construct the first adjunction and leave the second adjunction to the reader. To construct the unit $\eta^*\colon \mathrm{id}_{\mathrm{Fun}(E,C)} \to r^*l^*$, we may by currying equivalently produce a transformation between the two curried functors $E \times \mathrm{Fun}(E,C) \to C$, which we do by whiskering the transformation $\eta\colon \mathrm{id}_C \to rl$ with the evaluation functor $\mathrm{ev}\colon E \times \mathrm{Fun}(E,C) \to C$. The counit $\varepsilon^*\colon l^*r^* \to \mathrm{id}_{\mathrm{Fun}(E,D)}$ is constructed similarly. The triangle identities for η^* and ε^* follow directly from the triangle identities of η and ε after whiskering. ◻

Proposition 5.4.5. *Adjoints are unique: if* $(l \dashv r)\colon C \rightleftarrows D$ *and* $(l \dashv r')\colon C \rightleftarrows D$ *are adjunctions, there is a preferred natural isomorphism* $r \cong r'$.

Proof. Consider the following two composites:

$$r \xrightarrow{\eta' r} r'lr \xrightarrow{r'\varepsilon} r' \qquad \text{and} \qquad r' \xrightarrow{\eta r'} rlr' \xrightarrow{r\varepsilon'} r.$$

These two maps can be shown to be inverse to each other, finishing the proof. We leave the details to the reader, see Exercise 5.11.2 ◻

Proposition 5.4.6 (cf. [Uni13, Theorem 4.2.3])**.** *Let* $f\colon C \to D$ *be an equivalence with inverse* $g\colon D \to C$. *Then there exists an adjunction* $f \dashv g$ *between* f *and* g.

Proof. By assumption, there are natural isomorphisms $\eta\colon \mathrm{id}_C \xrightarrow{\sim} gf$ and $\varepsilon\colon fg \xrightarrow{\sim} \mathrm{id}_D$. We define a new transformation ε' as the composite

$$fg \xrightarrow{\varepsilon^{-1}fg} fgfg \xrightarrow{f\eta^{-1}g} fg \xrightarrow{\varepsilon} \mathrm{id}_D\,.$$

We claim that the transformations η and ε' satisfy the two triangle identities. To show that the composite $f \xrightarrow{f\eta} fgf \xrightarrow{\varepsilon' f} f$ is isomorphic to the identity on f, we consider the following commutative diagram:

$$\begin{tikzcd}
 & fgf \arrow[r, "\varepsilon^{-1}fgf"] \arrow[rrrd, bend left, "\varepsilon' f"] & fgfgf \arrow[dr, "f\eta^{-1}gf"] & & \\
f \arrow[ur, "f\eta"] \arrow[r, "\varepsilon^{-1}f"] \arrow[rrrr, equal, bend right] & fgf \arrow[ur, "fgf\eta"] \arrow[r, "f\eta^{-1}"] \arrow[rr, equal, bend right] & f \arrow[r, "f\eta"] & f \arrow[r, "\varepsilon f"] & f.
\end{tikzcd}$$

The top composite is $(\varepsilon' f) \circ (\varepsilon' f)$. The two faces of the diagram commute by naturality.

To show that the composite $g \xrightarrow{\eta g} gfg \xrightarrow{g\varepsilon'} g$ is isomorphic to the identity on g, consider the following commutative diagram:

$$\begin{tikzcd}
 & gfg \arrow[r, "g\varepsilon'"] \arrow[dr, swap, "\eta gfg"] & g \arrow[r, "\eta g"] & gfg \arrow[dr, "g\varepsilon'"] & \\
g \arrow[ur, "\eta g"] \arrow[dr, "\eta g"] & & gfgfg \arrow[ur, swap, "gfg\varepsilon'"] \arrow[dr, "g\varepsilon' fg"] & & g \\
 & gfg \arrow[ur, "gf\eta g"] \arrow[rr, equal] & & gfg. \arrow[ur, "g\varepsilon"] &
\end{tikzcd}$$

The three squares each commute by naturality and the bottom triangle commutes by the triangle identity established above. Therefore, the diagram expresses an isomorphism

$$(g\varepsilon') \circ (\eta g) \circ (g\varepsilon') \circ (\eta g) \cong (g\varepsilon') \circ (\eta g).$$

Since the transformations ηg and $g\varepsilon'$ are invertible, we may cancel them to obtain the desired isomorphism $(g\varepsilon') \circ (\eta g) \cong \mathrm{id}_g$. This finishes the proof. □

5.4.2 Adjunctions via hom groupoids

As explained in the introduction of this section, there is a second common way to define adjunctions $(l \dashv r)\colon C \rightleftarrows D$ between categories: rather than providing unit and counit transformations, one may also provide a natural isomorphism

$$D(lx, y) \xrightarrow{\sim} C(x, ry)$$

between hom groupoids for all x in C and y in D.

Proposition 5.4.7. *Let $l\colon C \to D$ and $r\colon D \to C$ be functors. Then the following three conditions are equivalent:*

(1) There exist a unit $\eta\colon \mathrm{id}_C \to rl$ and counit $\varepsilon\colon lr \to \mathrm{id}_D$ which exhibit r as a right adjoint of l;

(2) There exists an equivalence $\alpha\colon C \vec{\times}_l D \xrightarrow{\sim} C \vec{\times}_r D$ over $C \times D$.

Proof. We start by showing that (1) implies (2), so assume that the data (η, ε) of an adjunction has been given. For notational simplicity, we will show how to construct equivalences $\alpha_{x,y}\colon D(lx, y) \to C(x, ry)$ on fibers for all $x \in C$ and $y \in D$, and then argue afterwards that the construction can be upgraded to an equivalence over $C \times D$. We construct the map $\alpha_{x,y}\colon D(lx, y) \to C(x, ry)$ as the following composite:

$$\alpha_{x,y}\colon D(lx, y) \xrightarrow{r_*} C(rlx, ry) \xrightarrow{\eta_x^*} C(x, ry).$$

Here the first map is the map induced by r, while the second map is defined as the contravariant transport functor along the unit map $\eta_x\colon x \to rlx$ applied to the right fibration $C_{/ry} \to C$ from Theorem 5.3.4. Dually, we may construct a map $\beta_{x,y}\colon C(x,ry) \to D(lx,y)$ as the following composite:

$$\beta_{x,y}\colon C(x,ry) \xrightarrow{l_*} D(lx,lry) \xrightarrow{(\varepsilon_y)_!} D(lx,y).$$

Here the first map is induced by l and the second map is defined as the covariant transport functor along the counit map $\varepsilon_y\colon lry \to y$. W will now show that $\beta_{x,y}$ is an inverse to $\alpha_{x,y}$. We will show that $\beta_{x,y} \circ \alpha_{x,y} \sim \mathrm{id}_{D(lx,y)}$. To this end, consider the following diagram:

$$\begin{tikzcd}
D(lx,y) \arrow[r, "r_*"] \arrow[rr, bend left, "\alpha_{x,y}"] \arrow[dr, "(lr)_*"'] \arrow[ddr, bend right, "\varepsilon^*_{lx}"'] & C(rlx,ry) \arrow[r, "\eta^*_x"] \arrow[d, "l_*"'] \arrow[dd, bend left, "\beta_{rlx,y}"] & C(x,ry) \arrow[dd, "\beta_{x,y}"] \\
 & D(lrlx,lry) \arrow[d, "(\varepsilon_y)_!"'] & \\
 & D(lrlx,y) \arrow[r, "(l\eta_x)^*"'] & D(lx,y).
\end{tikzcd}$$

The composite along the top right is $\beta \circ \alpha$. We claim that the bottom composite is the identity functor. To see this, notice that by Proposition 5.2.9 the bottom composite is isomorphic to the functor $(\varepsilon_l x \circ l\eta_x)^*$. By the triangle identity, we may identify this with the functor $(\mathrm{id}_{lx})^*$. It thus follows from another application of Proposition 5.2.9 that this is indeed the identity on $D(lx,y)$.

We will now show that the diagram commutes. The square on the right commutes by applying (the dual of) Lemma 5.2.10 to the morphism $\beta\colon C_{/ry} \to C \vec{\times}_l D$ of right fibrations over C. Here we have used Proposition 5.2.12 to identify the bottom horizontal map. The top left triangle is easily seen to commute. It thus remains to see that the triangle on the bottom left commutes. Recall from Proposition 5.3.7 that the functors $(\varepsilon_y)_!$ and ε^*_{lx} are given by postcomposition with $\varepsilon_y\colon lrx \to y$ and precomposition with $\varepsilon_{lx}\colon lrlx \to lx$, respectively. The claim is thus equivalent to the claim that for every morphism $\varphi\colon lx \to y$, the following square commutes in D:

$$\begin{tikzcd}
lrlx \arrow[r, "lr(\varphi)"] \arrow[d, "\varepsilon_{lx}"'] & lry \arrow[d, "\varepsilon_y"] \\
lx \arrow[r, "\varphi"] & y.
\end{tikzcd}$$

But this is an instance of naturality of the transformation ε. We conclude that $\beta \circ \alpha \sim \mathrm{id}_{D(lx,y)}$. The proof that $\alpha \circ \beta \sim \mathrm{id}_{C(x,ry)}$ is entirely analogous and is left to the reader.

Although we have given equivalences $\alpha_{x,y}$ on fibers for individual objects $x \in C$ and $y \in D$, it is possible to carry out the above argument globally over $C \times D$, in the spirit of Remark 5.2.8. We leave the details to the reader. [Make sure the details actually work out!]

We now show that (2) implies (1). Consider an equivalence $\alpha\colon C \vec{\times}_l D \xrightarrow{\sim} C \vec{\times}_r D$ over $C \times D$. We define the unit $\eta\colon \operatorname{id}_C \to rl$ as the composite

$$C \to C \vec{\times}_l D \simeq C \vec{\times}_r D \to \operatorname{Fun}([1], C).$$

The fact that the source and target of this natural transformation are indeed id_C and rl, respectively, follows from the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "\Delta"'] & C \vec{\times}_l D \arrow[r, "\alpha"] \arrow[d] & C \vec{\times}_r D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Fun}([1], C) \arrow[d, "{(\operatorname{ev}_0, \operatorname{ev}_1)}"] \\
C \times C \arrow[r, "\operatorname{id}_C \times l"] & C \times D \arrow[r, equal] & C \times D \arrow[r, "\operatorname{id}_C \times r"] & C \times C.
\end{tikzcd}$$

Similarly, we define the counit $\varepsilon\colon lr \to \operatorname{id}_D$ as the composite

$$D \to C \vec{\times}_r D \simeq C \vec{\times}_l D \to \operatorname{Fun}([1], C).$$

We now need to verify the two triangle identities. By symmetry, it will suffice to show one of the triangle identities: we will show that the composite

$$r \xrightarrow{\eta r} rlr \xrightarrow{r\varepsilon} r$$

is isomorphic to the identity on r. For notational simplicity, we will show that this relation holds objectwise, i.e. that for every object y in D the composite $ry \xrightarrow{\eta_{ry}} rlr \xrightarrow{r\varepsilon_y} ry$ is the identity on ry. Since the same reasoning goes through for arbitrary functors $y\colon E \to D$, we may apply it in particular to the identity $y = \operatorname{id}_D\colon D \to D$ to get the claim. As a consequence of Lemma 5.2.10 we see that the following diagram commutes:

$$\begin{tikzcd}
D(lry, lry) \arrow[r, "\alpha_{lry,lry}"] \arrow[d, "(\varepsilon_y)_!"'] & C(ry, rlry) \arrow[d, "(r\varepsilon_y)_!"] \\
D(lry, y) \arrow[r, "\alpha_{lry,y}"] & D(ry, ry).
\end{tikzcd}$$

But by Proposition 5.3.7, the two covariant transport functors $(\varepsilon_y)_!$ and $(r\varepsilon_y)_!$ are given by postcomposition with ε_y and $r\varepsilon_y$ respectively. Since $\alpha_{lry,lry}(\operatorname{id}_{rly}) = \eta_{ry}$ and $\alpha_{lry,y}(\varepsilon_y) = \operatorname{id}_{ry}$ by definition of η and ε, plugging in the map $\operatorname{id}_{rly}\colon rly \to rly$ in the top left corner thus gives the desired relation

$$r\varepsilon_y \circ \eta_{ry} = (r\varepsilon_y)_!(\alpha_{lry,lry}(\operatorname{id}_{rly})) \sim \alpha_{lry,y}((\varepsilon_y)_!(\operatorname{id}_{rly})) = \alpha_{lry,y}(\varepsilon_y) = \operatorname{id}_{ry}\,.$$

This finishes the proof. □

5.5 Initial objects and left adjoint sections

In this section, we introduce the notion of an *initial object* of a synthetic category. More generally, we consider the notion of a *left adjoint section* of a functor, which we may think of as a fiberwise analogue of initial objects. Dually we obtain notions of *terminal objects* and *right adjoint sections*.

5.5.1 Left and right adjoint sections

Definition 5.5.1 (Left/right adjoint section)**.** Let $f\colon C \to D$ be a functor. A *left adjoint section* of f is a left adjoint $s\colon D \to C$ such that the unit transformation $\eta\colon \mathrm{id}_D \to fs$ is a natural isomorphism. Dually, a *right adjoint section* of f is a right adjoint $s\colon D \to C$ such that the counit transformation $\varepsilon\colon fs \to \mathrm{id}_D$ is a natural isomorphism.

In other words, f admits a left adjoint section if and only if it admits a section $s\colon D \to C$ (i.e. a functor satisfying $fs \cong \mathrm{id}_D$) which comes equipped with a natural transformation $\varepsilon\colon sp \to \mathrm{id}_C$ of functors $C \to C$ satisfying the following two conditions:

- The map ε is a transformation living over D: the induced map $f\varepsilon\colon f \cong fsf \to f$ is naturally isomorphic to the identity transformation.
- The map ε restricts to the identity on D: the map $\varepsilon s\colon s \cong sps \to s$ is naturally isomorphic to the identity transformation.

Definition 5.5.2 (Left/right reflector)**.** A functor $f\colon C \to D$ is called a *right reflector* if it admits a left adjoint section. We call f a *left reflector* if it admits a right adjoint section.

We will see in [ref] that a left/right adjoint section $s\colon D \to C$ of a functor $f\colon C \to D$ is automatically a full subcategory. We may then think of f as reflecting the category C into this subcategory D, explaining the previous terminology.

Lemma 5.5.3. *Every equivalence $C \xrightarrow{\sim} D$ admits both a left and a right adjoint section.*

Proof. This is immediate from Proposition 5.4.6. ◻

Lemma 5.5.4. *To do: fix the proof. For a synthetic category C, the target functor $\mathrm{ev}_1\colon \mathrm{Fun}([1],C) \to C$ admits a right adjoint section given by*

$$s = p_{[1]}^*\colon C \to \mathrm{Fun}([1],C).$$

Dually, the source functor $\mathrm{ev}_0\colon \mathrm{Fun}([1],C) \to C$ admits a left adjoint section, also given by $s = p_{[1]}^*$.

Proof. We prove the claim for ev_1; the claim for ev_0 is dual. Consider the following functor:

$$\mathrm{Fun}([1],C)\times[1]\to\mathrm{Fun}([1],C);\qquad (f,t)\mapsto(s\mapsto f(\max(t,s))).$$

Using **??**, we see that $f(\max(0,y))=f(y)$ and $f(\max(1,y))=f(1)$, so that this assignment defines a natural transformation

$$\eta\colon\ \mathrm{id}_{\mathrm{Fun}([1],C)}\to s\mathrm{ev}_1$$

of functors $\mathrm{Fun}([1],C)\to\mathrm{Fun}([1],C)$. It also follows from **??** that:

- The evaluation of η at 1 gives the identity of ev_1;
- Applying η to $s(c)$ gives the identity of $s(c)$.

This shows that η exhibits s as a right adjoint section of ev_1, finishing the proof. □

We can relax the condition on the transformation ε as follows:

Proposition 5.5.5. *Let $f\colon C\to D$ be a functor. Assume that there exists a section $s\colon D\to C$ of f and a natural transformation $\varepsilon\colon sp\to\mathrm{id}_C$ over D such that the transformation $\varepsilon s\colon s\cong sps\to s$ is a natural isomorphism. Then s is a left adjoint section of p.*

The analogous statement for right adjoint sections also holds.

Proof. Let $(\varepsilon s)^{-1}\colon s\to s$ be an inverse of $\varepsilon s\colon s\to s$. We define a new transformation $\varepsilon'\colon sp\to\mathrm{id}_C$ as the following composite:

$$sp\xrightarrow{(\varepsilon s)^{-1}p}sp\xrightarrow{\varepsilon}\mathrm{id}_C\,.$$

This is again a transformation over D, and it satsifies

$$\varepsilon's=(\varepsilon s)\circ((\varepsilon s)^{-1}ps)\cong(\varepsilon s)\circ(\varepsilon s)^{-1}\cong\mathrm{id}_C\,.$$

This shows that the map $\varepsilon'\colon sp\to\mathrm{id}_C$ exhibits s as a left adjoint section of p. □

Left and right reflectors are closed under a variety of operations.

Proposition 5.5.6 (Composition)**.** *Left (resp. right) reflectors are closed under composition.*

Proof. By Proposition 5.4.3 the composite functors form again an adjunction, and it is immediate from the descriptions of the unit and counit in the proof that the composite is again a left/right reflector. □

Proposition 5.5.7 (Functor categories)**.** *Let $p\colon C\to D$ be a right (resp. left) reflector. Then also the functor $p_*\colon$ $\mathrm{Fun}(A,C)\to\mathrm{Fun}(A,D)$ is a right (resp. left) reflector for any synthetic category A.*

Proof. This is immediate from Proposition 5.4.4. ☐

Proposition 5.5.8 (Base change)**.** *Left (resp. right) reflectors are stable under base change.*

Proof. We prove the claim for right reflectors; the claim for left reflectors is dual. Consider a pullback square

$$\begin{tikzcd}
C' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
D' \arrow[r, "v"] & D
\end{tikzcd}$$

where f is a right reflector. Let (s,ε) be a left adjoint section of f. By the universal property of the pullback, we obtain a section $s'\colon D' \to C'$ of f' satisfying $us' = sv$:

$$\begin{tikzcd}
D' \arrow[r, dashed, "s'"] \arrow[rr, bend left, "sv"] \arrow[dr, equal] & C' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
& D' \arrow[r, "v"] & D.
\end{tikzcd}$$

Similarly, we define the map $\varepsilon'\colon C' \times [1] \to C'$ over D' via the following commutative diagram:

$$\begin{tikzcd}
C' \times [1] \arrow[r, dashed, "\varepsilon'"] \arrow[rr, bend left, "\varepsilon\circ(u\times 1)"] \arrow[dr, "f'\circ \mathrm{pr}_1"'] & C' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
& D' \arrow[r, "v"] & D.
\end{tikzcd}$$

Note that ε' is indeed a transformation from $s'f'$ to $\mathrm{id}_{C'}$ over D'. Finally, we construct the desired commutative diagram $\sigma'\colon D' \times [2] \to C'$ via the following commutative diagram:

$$\begin{tikzcd}
D' \times [2] \arrow[r, dashed, "\sigma'"] \arrow[rr, bend left, "\sigma\circ(v\times 1)"] \arrow[dr, "\mathrm{pr}_1"'] & C' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
& D' \arrow[r, "v"] & D.
\end{tikzcd}$$

☐

Proposition 5.5.9 (Retracts)**.** *Left and right reflectors are stable under retracts: given a diagram*

$$\begin{tikzcd}
C \arrow[r] \arrow[rr, bend left, "\mathrm{id}_C"] \arrow[d, "f"'] & C' \arrow[r] \arrow[d, "f'"] & C \arrow[d, "f"] \\
D \arrow[r] \arrow[rr, bend right, "\mathrm{id}_D"'] & D' \arrow[r] & D,
\end{tikzcd}$$

if f' is a left/right reflector then so is f.

Proof. The proof is similar to the previous result, and we leave the proof as an exercise to the reader, see Exercise 5.11.3. □

5.5.2 Initial and terminal objects

We specialize the previous definitions to the cases of initial and terminal objects.

Definition 5.5.10 (Initial/terminal object)**.** Let C be a synthetic category. An *initial object* of C is a left adjoint section $x\colon * \to C$ of the functor $p_C\colon C \to *$. Dually, a *terminal object* of C is a right adjoint section of p_C.

The condition that x is initial admits various alternative formulations:

Lemma 5.5.11. *Let C be a synthetic category and let x be an object of C. Then the following conditions are equivalent:*

(1) The object x is initial;

(2) The forgetful functor $C_{x/} \to C$ is an equivalence;

(3) There exists a transformation $\varepsilon\colon \mathrm{const}_x \to \mathrm{id}_C$ and a natural isomorphism $\varepsilon_x \cong \mathrm{id}_x$ of functors $* \to C$. Here const_x is defined as the composite $C \xrightarrow{p_C} * \xrightarrow{x} C$.

(4) There exists a transformation $\varepsilon\colon \mathrm{const}_x \to \mathrm{id}_C$ such that the map $\varepsilon_x\colon x \to x$ is an isomorphism in C.

The dual statement for terminal objects is also valid.

Proof. The equivalence between (1) and (2) is a special case of Proposition 5.4.7, applied to $l = x\colon * \to C$ and $r = p_C\colon C \to *$. To see that (1) is equivalent to (3), note that there is only a single functor $C \times [1] \to *$, and hence the unit transformation $\eta\colon \mathrm{id}_* \to p_C \circ x$ is unique. One of the two triangle identities is automatic, and the remaining triangle identity amounts to condition (3). The equivalence between (3) and (4) is a special case of Proposition 5.5.5. □

Remark 5.5.12. If x is an initial object of a synthetic category C, then there exists for every other object y of C a morphism $\varepsilon_y\colon x \to y$. Moreover, every other morphism $f\colon x \to y$ in C is isomorphic to ε_y in $\mathrm{Fun}([1], C)$: by naturality of ε there is a commutative diagram in C of the form

$$\begin{tikzcd}
x \arrow[r, "\varepsilon_x"] \arrow[d, "\mathrm{id}_x"'] & x \arrow[d, "f"] \\
x \arrow[r, "\varepsilon_y"] & y,
\end{tikzcd}$$

and since ε_x is isomorphic to id_x by assumption this implies $\varepsilon_y \cong f$.

Lemma 5.5.13. *Let* $f\colon C \to D$ *be a right reflector with left adjoint section* $s\colon D \to C$*. For every object* d *of* D*, the object* $s(d) \in C(d)$ *is an initial object of the fiber* $C(d) := f^{-1}(d)$*. The dual statement for left reflectors and terminal objects also holds.*

Proof. It follows from Proposition 5.5.8 that the base change $f|_{C(d)}\colon C(d) \to \{d\}$ of f along $\{d\} \to D$ is again a right reflector with left adjoint section given by the restriction $s|_{\{d\}}\colon \{d\} \to C(d)$. By definition this means that $s(d)$ is an initial object of $C(d)$. ☐

We will now provide some basic examples of initial and terminal objects.

Lemma 5.5.14. *The synthetic category* $[1]$ *has both an initial object* $0 : [1]$ *and a terminal object* $1 : [1]$.

Proof. TO DO: update the proof! By symmetry it will suffice to show that 1 is terminal. Consider the map $\max\colon [1] \times [1] \to [1]$ defined in **??**. We may see this as a natural transformation $\eta\colon f_0 \to f_1$ between two morphisms $f_0, f_1\colon [1] \to [1]$, and we computed in **??** that $f_0 = \mathrm{id}_{[1]}$ and $f_1 = \mathrm{const}_1$. Furthermore, **??** also showed that the morphism $\eta_1\colon 1 \to 1$ in $[1]$ is the identity morphism on $1 : [1]$, which makes the required diagram trivially commute. This finishes the proof. ☐

Corollary 5.5.15. *Let* C *be a contractible category, i.e. the map* $p_C\colon C \to *$ *is an equivalence. Then* C *admits both a terminal and an initial object.*

Proof. If $x\colon * \to C$ is any homotopy inverse of p_C, then x is both initial and terminal: the groupoids $C(x,y)$ and $C(y,x)$ are contractible for every term y of C by contractibility of C. ☐

Lemma 5.5.16. *For an object* x *of a synthetic category* C*, the slice category* $C_{/x}$ *admits a terminal object* $(x, \mathrm{id}_x\colon x \to x)$*. Similarly,* $C_{x/}$ *admits an initial object* $(x, \mathrm{id}_x\colon x \to x)$.

Proof. We showed in Lemma 5.5.4 that the evaluation map $\mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$ admits a right adjoint section given by $p^*_{[1]}\colon C \to \mathrm{Fun}([1], C)$. Since $C_{/x}$ is the fiber over x of this map, the claim thus follows from Lemma 5.5.13. ☐

If one already knows the existence of a terminal object x, then other terminal objects are particularly easy to recognize:

Lemma 5.5.17. *Let* C *be a synthetic category and let* x *be a terminal object of* C*, as exhibited by the transformation* $\eta\colon \mathrm{id}_C \to \mathrm{const}_x$*. Let* y *be another object of* C*. Then* y *is a terminal object if and only if the morphism* $\eta_y\colon y \to x$ *is an isomorphism in* C*.*

The dual statement for initial objects also holds.

Proof. If η_y is an isomorphism, then the composite transformation

$$\mathrm{id}_C \xrightarrow{\eta} \mathrm{const}_x \xrightarrow{\mathrm{const}_{\eta_y^{-1}}} \mathrm{const}_y$$

exhibits y as a terminal object. Conversely, assume that y is a terminal object, as exhibited by the transformation $\eta'\colon \mathrm{id}_C \to \mathrm{const}_y$. We claim that the morphism $\eta'_x\colon x \to y$ is an inverse of $\eta_y\colon y \to x$ in C. Indeed, it follows from **??** that the composite $\eta_y \circ \eta'_x\colon x \to x$ is isomorphic to id_x and that the composite $\eta'_x \circ \eta_y\colon y \to y$ is isomorphic to id_y. □

Corollary 5.5.18. *All terminal objects of a synthetic category C are canonically isomorphic. Similarly, all initial objects of C are canonically isomorphic.* □

5.6 Cartesian and cocartesian fibrations

In Section 5.2 we introduced the notion of a *left fibration*: a functor $f\colon A \to B$ such that every morphism $\beta\colon b \to b'$ admits a unique covariant lift starting in a given lift a of b. The goal of this section is to introduce a variant of this notion in which the covariant lift is not necessarily unique but where there exists always an *initial* lift. Dually, we obtain a notion of *cartesian fibration*, in which there exists a *terminal* contravariant lift.

Definition 5.6.1 ((Co)cartesian fibration)**.** An isofibration $f\colon A \to B$ is called a *cocartesian fibration* the directed evaluation map $\vec{\mathrm{ev}}^f_0\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$ admits a left adjoint section $\mathrm{lift}^f_0\colon A \vec{\times}_f B \to \mathrm{Fun}([1],A)$.
Dually, f is called a *cartesian fibration* if the map $\vec{\mathrm{ev}}^f_1\colon \mathrm{Fun}([1],A) \to B \vec{\times}_f A$ admits a right adjoint section $\mathrm{lift}^f_1\colon B \vec{\times}_f A \to \mathrm{Fun}([1],A)$.

Remark 5.6.2. Throughout this section, we will frequently state results only for cocartesian fibrations and leave their dual formulation for cartesian fibrations to the reader.

Example 5.6.3. The following are the first few basic examples of (co)cartesian fibrations:

- Every left fibration is a cocartesian fibration: the map $\vec{\mathrm{ev}}^f_0$ is a trivial fibration and thus in particular admits a left adjoint section by Lemma 5.5.3. Dually, every right fibration is a cartesian fibration.

- In particular, every equivalence is both a cartesian fibration as well as a cocartesian fibration.

- For a synthetic category C, the map $p_C\colon C \to *$ is both a cartesian fibration and a cocartesian fibration. Indeed, since the map $\vec{\mathrm{ev}}^{p_C}_0\colon \mathrm{Fun}([1],C) \to C \vec{\times}_{p_C} *$ is isomorphic to $\mathrm{ev}_0\colon \mathrm{Fun}([1],C) \to C$ by Remark 5.1.5, it admits left adjoint section by Lemma 5.5.4, showing that p_C is a cocartesian fibration. The cartesian case is dual.

- For synthetic categories C and D, the projection maps $\mathrm{pr}_1\colon C \times D \to C$ and $\mathrm{pr}_2\colon C \times D \to D$ are both a cartesian fibration and a cocartesian fibration. This can again be argued directly, but also follows from the previous example and Proposition 5.6.13 below.

Cocartesian fibrations allow for a notion of *covariant transport* generalizing the situation for left fibration:

Notation 5.6.4 (Covariant transport)**.** For a term (a,β) of $A \vec{\times}_f B$, we denote the morphism $\mathrm{lift}_0^f(a,\beta)$ in A as

$$\mathrm{lift}_0^f(a,\beta)\colon a \to \beta_!(a)$$

and refer to its target $\beta_!(a)$ as the *covariant transport of a along β*. Observe that this map is a term of the category $\mathrm{Lift}_0^f(a,\beta)$ of covariant lifts of β starting in a, and that it is in fact an *initial object* of this category.

Just as in Construction 5.2.7, one can enhance the construction $a \mapsto \beta_!(a)$ to a functor

$$\beta_!\colon A(b) \to A(b').$$

called *covariant transport along β*.

We would like to show that the covariant transport functors satisfy the relations $(\mathrm{id}_b)_! = \mathrm{id}$ and $(\beta' \circ \beta)_! = \beta'_! \circ \beta_!$ as in Proposition 5.2.9. While the statement for left fibrations was immediate due to uniqueness of covariant lifts, the statement for cocartesian fibrations is slightly more involved and will need the following auxiliary notion of *cocartesian morphisms*:

Definition 5.6.5 (Cocartesian morphism)**.** Let $f\colon A \to B$ be a functor. A morphism $\alpha\colon a \to a'$ in A is called *f-cocartesian* (or just *cocartesian* if f is clear from context) if the commutative square

$$\begin{tikzcd}
A_{a'/} \arrow[r, "\alpha^*"] \arrow[d, "f"'] & A_{a/} \arrow[d, "f"] \\
B_{f(a')/} \arrow[r, "f(\alpha)^*"] & B_{f(a)/}
\end{tikzcd}$$

is a pullback square.

Corollary 5.6.6. *Let $f\colon A \to B$ be a left fibration. Then every morphism in A is f-cocartesian.*

Proof. We claim that both vertical maps in the commutative square in Definition 5.6.5 are equivalences, which in particular implies that the square is a pullback square. To see this, note that for an object x of A the map $f\colon A_{x/} \to B_{f(x)/}$ is obtained as the map induced on fibers over x by the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],A) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],B) \arrow[d, "\mathrm{ev}_0"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

But this square is a pullback square by Lemma 5.2.2, so the induced map on fibers is an equivalence. □

Lemma 5.6.7. *Let $f\colon A \to B$ be a cocartesian fibration.*

(1) For every object a in A, the identity morphism $\mathrm{id}_a\colon a \to a$ is f-cocartesian.

(2) For morphisms $\alpha\colon a \to a'$ and $\alpha'\colon a' \to a''$ in A such that α is f-cocartesian, we have that α' is f-cocartesian if and only if the composite $\alpha' \circ \alpha$ is f-cocartesian.

Proof. Part (1) is clear. For part (2), we apply the pasting law for pullback squares to the following commutative diagram:

$$\begin{tikzcd}
A_{a''/} \arrow[r, "\alpha',*"] \arrow[d, "f"'] & A_{a'/} \arrow[r, "\alpha^*"] \arrow[d, "f"'] & A_{a/} \arrow[d, "f"] \\
B_{f(a'')/} \arrow[r, "f(\alpha')^*"] & B_{f(a')/} \arrow[r, "f(\alpha)^*"] & B_{f(a)/}.
\end{tikzcd}$$

□

Since the functor lift_0^f is a left adjoint section of $\vec{\mathrm{ev}}_0^f$, it comes equipped with a natural transformation $\varepsilon\colon \mathrm{lift}_0^f \circ \vec{\mathrm{ev}}_0^f \to \mathrm{id}_{\mathrm{Fun}([1],A)}$. In particular, for a morphism $\alpha\colon a \to a'$ in A with underlying morphism $\beta := f(\alpha)\colon f(a) \to f(a')$ in B, we obtain a morphism $\varepsilon_\alpha\colon \mathrm{lift}_0^f(a,\beta) \to \alpha$ in $\mathrm{Fun}([1],A)$, corresponding to a commutative square in A of the form

$$\begin{tikzcd}
a \arrow[r, "\mathrm{lift}_0^f(a,\beta)"] \arrow[d, equal] & \beta_!(a) \arrow[d, "\varepsilon_\alpha"] \\
a \arrow[r, "\alpha"] & a'.
\end{tikzcd}$$

By Lemma 1.3.12, we may think of this square in A as a commutative triangle in A of the form

$$\begin{tikzcd}
& \beta_!(a) \arrow[dr, "\varepsilon_\alpha"] & \\
a \arrow[ur, "\mathrm{lift}_0^f(a,\beta)"] \arrow[rr, "\alpha"'] & & a'.
\end{tikzcd}$$

Lemma 5.6.8. *Let $f\colon A \to B$ be a cocartesian fibration. Then a morphism $\alpha\colon a \to a'$ is f-cocartesian if and only if the morphism $\varepsilon_\alpha\colon \mathrm{lift}_0^f(a,\beta) \to \alpha$ is an equivalence.*

Proof. TO DO! □

Corollary 5.6.9. *Let $f: A \to B$ be a cocartesian fibration.*

(1) For $\beta = \mathrm{id}_b: b \to b$, the functor $\beta_!: A(b) \to A(b)$ is isomorphic to the identity functor;

(2) For morphisms $\beta: b \to b'$ and $\beta': b' \to b''$ in B, the composite functor

$$\beta'_! \circ \beta_!: A(b) \to A(b'')$$

is isomorphic to $(\beta' \circ \beta)_!: A(b) \to A(b'')$.

Proof. TO DO!!!!! □

Covariant transport along natural transformations

TO DO: Rewrite this so that it no longer refers to tribes.

Recall from Construction 5.2.7 that a cocartesian fibration $p: C \to D$ admits *covariant transport*: for a morphism $\beta: x \to y$ in D, there is a preferred functor $\beta_!: C(x) \to C(y)$ between the fibers of p over x and y. Using the existence of functor categories, we may extend this construction to the case where x and y are functors out of some source category X and β is a natural transformation from x to y:

Construction 5.6.10 (Covariant transport along natural transformations)**.** Let $p: C \to D$ be a cocartesian fibration. Let X be a synthetic category, let $x, y: X \to D$ be functors and let $\beta: x \to y$ be a natural transformation. We will construct a *covariant transport functor*

$$\beta_!: C(x) \to C(y)$$

in the directed type theory $\mathcal{E}(X)$, where $C(x)$ and $C(y)$ are defined via the pullback squares

$$\begin{tikzcd} C(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\ X \arrow[r, "x"] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\ X \arrow[r, "y"] & D. \end{tikzcd}$$

In order to do this, we consider the auxiliary cocartesian fibration $\tilde{p}$ defined via the following pullback square:

$$\begin{tikzcd} \tilde{C} \arrow[r] \arrow[d, "\tilde{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\ X \times \mathrm{Fun}(X, D) \arrow[r, "\mathrm{ev}"] & D. \end{tikzcd}$$

We may regard $\tilde{p}$ as a morphism in the local tribe $\mathcal{E}(X)$ by equipping its target with the projection map $\mathrm{pr}_1: X \times \mathrm{Fun}(X, D) \to X$. It follows from **??** that $\tilde{p}$ is a cocartesian fibration in $\mathcal{E}(X)$, hence we may form its covariant transport in $\mathcal{E}(X)$. The maps x and y define two sections of the map $X \times \mathrm{Fun}(X,D) \to X$, and the natural transformation β defines a morphism between these, so that the covariant transport construction 5.2.7 (formed in the local tribe $\mathcal{E}(X)$) provides a functor $\beta_!\colon \tilde{C}(x) \to \tilde{C}(y)$ in $\mathcal{E}(X)$. But in light of the pullback diagram

$$\begin{tikzcd}
C(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{C}(x) \arrow[r] \arrow[d, "\tilde{p}"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\
X \arrow[r, "(\mathrm{id}_X{,}x)"] & X \times \mathrm{Fun}(X,D) \arrow[r, "\mathrm{ev}"] & D,
\end{tikzcd}$$

we see that $\tilde{C}(x)$ corresponds to the cocartesian fibration $C(x) \to X$, and similarly $\tilde{C}(y)$ corresponds to $C(y) \to X$. It follows that $\beta_!$ corresponds to a functor $\beta_!\colon C(x) \to C(y)$ in $\mathcal{E}(X)$, finishing the construction.

Closure properties of cartesian and cocartesian fibrations

We will now show that the classes of cartesian and cocartesian fibrations satisfy various closure properties.

Proposition 5.6.11. *Let $f\colon A \to B$ be a cocartesian fibration. Then for every synthetic category E, the functor $f_*\colon \mathrm{Fun}(E,A) \to \mathrm{Fun}(E,B)$ is a cocartesian fibration.*

Proof. Since $\mathrm{Fun}(E,-)$ preserves pullbacks, one observes that the directed evaluation map of f_* is given by applying $\mathrm{Fun}(E,-)$ to $\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1],A) \to A \vec{\times}_f B$. The claim is now an immediate consequence from the fact that $\mathrm{Fun}(E,-)$ preserves adjunctions, see Proposition 5.4.4. □

Proposition 5.6.12. *The classes of cartesian and cocartesian fibrations are stable under composition.*

Proof. We will only prove the claim about cocartesian fibrations, leaving the dual claim for cartesian fibrations to the reader. If $f\colon A \to B$ and $g\colon B \to C$ are two cocartesian fibrations, we have by Lemma 5.1.8 a commutative diagram

$$\begin{tikzcd}
& \mathrm{Fun}([1],A) \arrow[dl, "\vec{\mathrm{ev}}_0^f"'] \arrow[dr, "\vec{\mathrm{ev}}_0^{gf}"] & \\
A \vec{\times}_f B \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & A \vec{\times}_{gf} C \arrow[d] \\
\mathrm{Fun}([1],B) \arrow[rr, "\vec{\mathrm{ev}}_0^g"] & & B \vec{\times}_g C,
\end{tikzcd}$$

where the bottom square is a pullback square. As f and g are cocartesian fibrations, the maps $\vec{\mathrm{ev}}_0^g$ and $\vec{\mathrm{ev}}_0^f$ admit left adjoint sections, and it then follows from Proposition 5.5.6 and Proposition 5.5.8 that also $\vec{\mathrm{ev}}_0^{gf}$ admits a left adjoint section. □

Proposition 5.6.13. *The classes of cartesian and cocartesian fibrations are stable under base change.*

Proof. Again we only treat the case of cocartesian fibrations, leaving the dual statement for cartesian fibrations to the reader. Consider a pullback square of the form

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B,
\end{tikzcd}$$

and assume that f is a cocartesian fibration. By Lemma 5.1.7, there is a pullback square of the form

$$\begin{tikzcd}
\mathrm{Fun}([1], A') \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{f'}"'] & \mathrm{Fun}([1], A) \arrow[d, "\vec{\mathrm{ev}}_0^{f}"] \\
A' \vec{\times}_{f'} B' \arrow[r] & A \vec{\times}_f B,
\end{tikzcd}$$

and thus it follows from Proposition 5.5.8 that also $\vec{\mathrm{ev}}_0^{f'}$ admits a left adjoint section. □

Proposition 5.6.14. *The classes of cartesian and cocartesian fibrations are stable under retracts.*

Proof. This follows from a combination of Lemma 5.1.9 and Proposition 5.5.9. □

5.7 The source and target fibrations

TO DO: put back in the discussion of the maps min and max somewhere and fix the references.

In this section, we will provide two fundamental examples of (co)cartesian fibrations: the source and target maps $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$ for a synthetic category C. The fibers of these two isofibrations are the slice categories $C_{/x}$ and $C_{x/}$ constructed in Section 5.3 below, and the fact that ev_0 and ev_1 are cartesian and a cocartesian fibration, respectively, encodes the functoriality of these slices in x.

We start with an auxiliary result, for which we need to recall the maps $p_0, p_2 \colon [1] \times [1] \to \Delta^2$ and $j_0, j_2 \colon \Delta^2 \to [1] \times [1]$ from Constructions **??** and **??**, pictorially represented by the following diagrams:

$$p_0 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad p_2 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2, \end{tikzcd}$$

$$j_0 \quad = \quad \begin{tikzcd} (0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \\ & (1,1), \end{tikzcd} \qquad j_2 \quad = \quad \begin{tikzcd} (0,0) \arrow[d] \arrow[dr] & \\ (1,0) \arrow[r] & (1,1). \end{tikzcd}$$

Proposition 5.7.1. *The map j_0 is a left adjoint section of the map p_0. The map j_2 is right adjoint section of the map p_2.*

Proof. We prove the claim for j_2 and p_2 and leave the analogous case for j_0 and p_0 to the reader. For simplicity we write $j := j_2$ and $p := p_2$ for the remainder of this proof. We showed in **??** that j is a section of p. We will construct a natural transformation $\eta\colon \mathrm{id}_{[1]\times[1]} \to jp$, or equivalently a map

$$h\colon ([1]\times[1])\times[1] \to ([1]\times[1])$$

such that $h(x,y,0) = (x,y)$ and $h(x,y,1) = jp(x,y) = (x, \max(x,y))$, where the last equality holds by **??**. Formally, we may define h by

$$h(x,y,t) = (x, \max(\min(t,x), y)),$$

using the maps max and min from **??**. Indeed, when $t = 0$, we have $\min(t,x) = \min(0,x) = 0$, so that the second component is $\max(0,y) = y$. When $t = 1$, we have $\min(t,x) = \min(1,x) = x$ so that the second component is $\max(x,y)$.

Pictorially, we may display h as the following commutative cube in $[1]\times[1]$ (read from front to back):

$$\begin{tikzcd} & (0,0) \arrow[rr] \arrow[dd] & & (0,1) \arrow[dd, equal] \\ (0,0) \arrow[ur, equal] \arrow[rr] \arrow[dd] & & (0,1) \arrow[ur, equal] \arrow[dd] & \\ & (1,0) \arrow[rr] & & (1,1) \\ (1,0) \arrow[ur] \arrow[rr] & & (1,1). \arrow[ur, equal] & \end{tikzcd}$$

An argument similar to the one used in **??** shows that the composite ph is the identity transformation $\mathrm{id}_p\colon p \to p$. Similarly, one can show that the composite $h(j \times 1)$ is the identity transformation $\mathrm{id}_j\colon j \to j$. This finishes the proof. □

Construction 5.7.2. Given a natural transformation $H\colon [1]\times C\to D$ between two functors $f,g\colon C\to D$, and a synthetic category A, we construct a natural transformation

$$H^*\colon [1]\times\mathrm{Fun}(D,A)\to\mathrm{Fun}(C,A)$$

between the precomposition functors $f^*,g^*\colon \mathrm{Fun}(D,A)\to\mathrm{Fun}(C,A)$ as follows:

$$\mathrm{Fun}(D,A)\xrightarrow{H^*}\mathrm{Fun}([1]\times C,A)\cong\mathrm{Fun}([1],\mathrm{Fun}(C,A)).$$

It is clear that this construction sends identity transformations to identity transformations and compositions of transformations to compositions of transformations.

Theorem 5.7.3. *For any synthetic category C, the functor* $\mathrm{ev}_0\colon \mathrm{Fun}([1],C)\to C$ *is a cartesian fibration. Dually, the functor* $\mathrm{ev}_1\colon \mathrm{Fun}([1],C)\to C$ *is a cocartesian fibration.*

Proof. We show the statement for ev_0 and leave the dual case of ev_1 to the reader. We need to show that the map

$$\vec{\mathrm{ev}}_1^{\mathrm{ev}_0}\colon \mathrm{Fun}([1],\mathrm{Fun}([1],C))\to C\,\vec{\times}_{\mathrm{ev}_0}\,\mathrm{Fun}([1],C)$$

admits a right adjoint section. We may identify the source of this map with $\mathrm{Fun}([1]\times[1],C)$. The target sits by definition in a pullback square

$$\begin{tikzcd}
C\,\vec{\times}_{\mathrm{ev}_0}\,\mathrm{Fun}([1],C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Fun}([1],C) \arrow[r, "\mathrm{ev}_0"] & C,
\end{tikzcd}$$

and thus we may identify it with $\mathrm{Fun}(\Lambda^2_1,C)$. Unwinding definitions, we see that the map $\vec{\mathrm{ev}}_1^{\mathrm{ev}_0}$ factors as

$$\mathrm{Fun}([1]\times[1],C)\xrightarrow{j_2^*}\mathrm{Fun}(\Delta^2,C)\xrightarrow{\sim}\mathrm{Fun}(\mathrm{Sp}_2,C),$$

where the second map is a trivial fibration by the Segal axiom. It will thus suffice to show that j_2^* admits a right adjoint section.

We claim that p_2^* is a right adjoint section of j_2^*. Indeed, we have $j_2^*p_2^*=(p_2j_2)^*=(\mathrm{id})^*=\mathrm{id}$. The transformation $h\colon 1\to j_2p_2$ gives rise to a transformation $h^*\colon 1\to(j_2p_2)^*=p_2^*j_2^*$ via Construction 5.7.2. Furthermore, we have

$$j_2^*h^*=(hj_2)^*=\mathrm{id}_{j_2^*}\qquad\text{and}\qquad h^*p_2^*=(p_2h)^*=\mathrm{id}_{p_2^*}.$$

This finishes the proof. □

Remark 5.7.4. Pictorially, the functor $\vec{\mathrm{ev}}_1^{\mathrm{ev}_0}$ may be thought of as the following assignment:

$$\begin{tikzcd}
x \arrow[r, "u"] \arrow[dd, "\alpha"'] & y \arrow[dd, "\beta"] & & x \arrow[dd, "\alpha"'] \arrow[ddr, "u'\alpha"] & & & x \arrow[dd, "\alpha"'] & \\
& & \mapsto & & & \mapsto & & \\
x' \arrow[r, "u'"'] & y' & & x' \arrow[r, "u'"'] & y' & & x' \arrow[r, "u'"'] & y'.
\end{tikzcd}$$

Its right adjoint section is given informally by the following assignment:

$$\begin{tikzcd}
x \arrow[dd, "\alpha"'] & & & x \arrow[dd, "\alpha"'] \arrow[ddr, "u'\alpha"] & & & x \arrow[r, "u'\alpha"] \arrow[dd, "\alpha"'] & y' \arrow[dd, equal] \\
& & \mapsto & & & \mapsto & & \\
x' \arrow[r, "u'"'] & y' & & x' \arrow[r, "u'"'] & y' & & x' \arrow[r, "u'"'] & y'.
\end{tikzcd}$$

We will in fact need a slight generalization of Theorem 5.7.3. Given a functor $f\colon C \to D$, we may consider the directed pullbacks $C \vec{\times}_f D$ and $D \vec{\times}_f C$. They both come equipped with canonical maps to D, which we claim to be (co)cartesian fibrations:

Proposition 5.7.5. *For a functor $f\colon C \to D$, the projection $\mathrm{ev}_1\colon C \vec{\times}_f D \to D$ is a cocartesian fibration, while the projection $\mathrm{ev}_0\colon D \vec{\times}_f C \to D$ is a cartesian fibration.*

Proof. By symmetry, it suffices to prove the claim for $\mathrm{ev}_1\colon C \vec{\times}_f D \to D$. We thus need to show that the map

$$\vec{\mathrm{ev}}^0_{\mathrm{ev}_1}\colon \mathrm{Fun}([1], C \vec{\times}_f D) \to (C \vec{\times}_f D) \vec{\times}_\pi D$$

admits a left adjoint section. The proof is analogous to the proof of Theorem 5.7.3; to make the proof more readable we will explain what happens on objects and leave it to the reader to write a formal proof.

First observe that terms of the synthetic category $\mathrm{Fun}([1], C \vec{\times}_f D)$ are pairs (α, τ) where $\alpha\colon x \to x'$ is a morphism in C, and σ is a commutative square in D of the form

$$\begin{tikzcd}
f(x) \arrow[r, "u"] \arrow[d, "f(\alpha)"'] & y \arrow[d, "\beta"] \\
f(x') \arrow[r, "u'"] & y'.
\end{tikzcd}$$

On the other hand, terms of $(C \vec{\times}_f D) \vec{\times}_\pi D$ are of the form $(x, f(x) \xrightarrow{u} y \xrightarrow{v} z)$, which we might alternatively display as

$$\left(x, \begin{tikzcd}
f(x) \arrow[r, "u"] & y \arrow[d, "v"] \\
& z
\end{tikzcd}\right).$$

The map $\vec{\mathrm{ev}}^{0}_{\mathrm{ev}_1}$ is then given by the assignment

$$\left(\begin{tikzcd} x \arrow[d, "\alpha"'] \\ x' \end{tikzcd} , \begin{tikzcd} f(x) \arrow[r, "u"] \arrow[d, "f(\alpha)"'] & y \arrow[d, "\beta"] \\ f(x') \arrow[r, "u'"] & y' \end{tikzcd} \right) \quad \mapsto \quad \left(x, \begin{tikzcd} f(x) \arrow[r, "u"] & y \arrow[d, "\beta"] \\ & y' \end{tikzcd} \right)$$

A left adjoint section is then given by the assignment

$$\left(x, \begin{tikzcd} f(x) \arrow[r, "u"] & y \arrow[d, "v"] \\ & z \end{tikzcd} \right) \quad \mapsto \quad \left(\begin{tikzcd} x \arrow[d, equal] \\ x \end{tikzcd} , \begin{tikzcd} f(x) \arrow[r, "u"] \arrow[d, equal] & y \arrow[d, "v"] \\ f(x) \arrow[r, "vu"] & z \end{tikzcd} \right).$$

The precise definition of this functor and the fact that it is a left adjoint section proceed just like the (dual version of the) proof of Theorem 5.7.3, using the functors $j_0\colon \Delta^2 \to [1] \times [1]$ and $p_0\colon [1] \times [1] \to \Delta^2$. □

5.8 Functoriality of universals axiom

Let $f\colon C \to D$ be a functor. If f admits a left adjoint section $s\colon D \to C$, we saw in Lemma 5.5.13 that then every fiber $C(d)$ of f admits an initial object given by $s(d) \in C(d)$. If f is a *cartesian fibration*, we expect to be able to also go the other way: if every fiber $C(d)$ admits an initial object s_x, then these objects should assemble into a left adjoint section $s\colon D \to C$ of f. Informally, the functoriality of s is given as follows: given a morphism $\beta\colon d \to d'$ in D, its image $s_d \to s_{d'}$ under s corresponds to a morphism $s_d \to \beta^*(s_{d'})$ in the fiber $C(d)$, which is uniquely determined by the fact that s_d is an initial object of $C(d)$. We make this heuristic precise by introducing the following axiom:

Axiom M (Functoriality of universals axiom)**.** Let $p\colon E \to C$ be a cartesian (resp. cocartesian) fibration. Assume that for every object x in C, the fiber $E(x)$ of p at x admits an initial (resp. terminal) object s_x. Then there is a left adjoint (resp. right adjoint) section $s\colon C \to E$ of p equipped with an isomorphism $s_x \xrightarrow{\sim} s(x)$ in $E(x)$ for every object x of C.

Remark 5.8.1. Recall from Convention 3.2.4 that by 'object', we really mean 'an object $x\colon \{x\} \to C$ in an arbitrary groupoidal context $\{x\}$'. One easily observes that it suffices to check the condition for the *universal* object, i.e. the map $\gamma_C\colon C^{\simeq} \to C$. In this case, the fiber $E(x)$ is given by the left vertical map $\tilde{p}$ in the following pullback square:

$$\begin{tikzcd}
C^{\simeq} \times_C E \arrow[r] \arrow[d, "\tilde{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
C^{\simeq} \arrow[r, "\gamma_C"] & C.
\end{tikzcd}$$

Hence the following is an equivalent way of phrasing the axiom: If $\tilde{p}$ admits a left (resp. right) adjoint section $\tilde{s}\colon C^{\simeq} \to C^{\simeq} \times_C E$, then also p admits a left (resp. right) adjoint section $s\colon C \to E$ which the following diagram commute:

$$\begin{tikzcd}
C^{\simeq} \times_C E \arrow[r] & E \\
C^{\simeq} \arrow[r, "\gamma_C"] \arrow[u, "\tilde{s}"] & C. \arrow[u, "s"']
\end{tikzcd}$$

Axiom M will be of fundamental importance throughout our development of synthetic category theory: it allows us to reduce functorial statements to objectwise statements. In the remainder of this section, we will prove a wide range of examples of such phenomena.

5.8.1 Fiberwise criteria

We will start by showing that various proprties of (maps between) left/right fibrations may be tested at the level of fibers.

Proposition 5.8.2. *Let $f\colon E \to B$ be a left (resp. right) fibration. Assume that the fiber $E(b)$ of f at b is contractible for every object b of B. Then f is an equivalence.*

Proof. We will prove the claim for left fibrations; the claim for right fibrations is dual. For any object b of B, contractibility of $E(b)$ implies that there is an object $s(b)$ in $E(b)$ such that s_b is terminal. By Axiom M, it follows that the assignment $b \mapsto s_b$ is the restriction of a right adjoint section $s\colon B \to E$ of f. In particular, we have $fs = \mathrm{id}_B$ and a given transformation $\eta\colon \mathrm{id}_E \to sf$ such that $f\eta = \mathrm{id}_f$. In particular, the transformation $f\eta\colon f \to f$ is a natural isomorphism. But by combining Corollary 5.2.3 with the Rezk Axiom, we see that the commutative square

$$\begin{tikzcd}
\mathrm{Iso}(E) \arrow[r, "\pi_{\mathrm{Iso}}"] \arrow[d, "f_*"'] & \mathrm{Fun}([1],E) \arrow[d, "f_*"] \\
\mathrm{Iso}(B) \arrow[r, "\pi_{\mathrm{Iso}}"] & \mathrm{Fun}([1],C)
\end{tikzcd}$$

is a pullback square, and hence it follows that already the transformation $\eta\colon \mathrm{id}_E \to sf$ is a natural isomorphism. This shows that s is in fact an inverse to f, thus finishing the proof that f is an equivalence. □

Lemma 5.8.3. *Consider a commutative triangle*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C, & \end{tikzcd}$$

where p and q are left (resp. right) fibrations. Assume that the induced map $A(c) \to B(c)$ on fibers is an equivalence for every object c of C. Then f is an equivalence.

Proof. The functor f is also a left (resp. right) fibration by Proposition 5.2.13. By the previous proposition, it will thus suffice to show that the fibers of f are contractible. To this end, consider an object $b\colon \{b\} \to B$, and define $c := q \circ b\colon \{b\} \to C$. We then get a pullback diagram of the form

$$\begin{tikzcd} A(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A(c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \{b\} \arrow[r] & B(c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "q"] \\ & \{b\} \arrow[r, "c"] & C. \end{tikzcd}$$

Since the induced functor $A(c) \to B(c)$ is an equivalence by assumption , it follows that also its base change $A(b) \to \{b\}$ is an equivalence, showing that $A(b)$ is a contractible category in context $\{b\}$ as desired. □

Corollary 5.8.4. *Consider a commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

and assume that p and q are left (resp. right) fibrations. Then the square is a pullback square if and only if for every object x of C the induced map $f_x\colon A(x) \to B(f(x))$ on fibers is an equivalence.

Proof. Since left (resp. right) fibrations are closed under base change by Proposition 5.2.12, the functor $B \times_D C \to C$ is a right fibration. The square is a pullback square if and only if the map $(f, p)\colon A \to B \times_D C$ is an equivalence over C. By Lemma 5.8.3, we may check this condition fiberwise, where it precisely becomes the claim that the induced map $A(x) \to B(f(x))$ is an equivalence. □

Corollary 5.8.5. *Let $f\colon E \to C$ be a left (resp. right) fibration. Then the commutative square*

$$\begin{tikzcd}
E^{\simeq} \arrow[r, "\gamma_E"] \arrow[d, "f^{\simeq}"'] & E \arrow[d, "f"] \\
C^{\simeq} \arrow[r, "\gamma_C"] & C
\end{tikzcd}$$

is a pullback square.

Proof. Note that the functor $f^{\simeq}$ is again a left (resp. right) fibration by Proposition 5.2.15, since both its source and target are groupoids. By Corollary 5.8.4 it will thus suffice to show that γ_E induces equivalences on all fibers over $C^{\simeq}$. But given an object x of C, the induced map on fibers is $\gamma_{E_x}\colon (E_x)^{\simeq} \to E_x$, which is an equivalence since E_x is a groupoid by Corollary 5.2.17. □

Lemma 5.8.6. *For every synthetic category C, the commutative square*

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "\gamma_{\mathrm{Fun}([1],C)}"] \arrow[d, "(ev_0{,}ev_1)"'] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
C^{\simeq} \times C^{\simeq} \arrow[r, "\gamma_C \times \gamma_C"] & C \times C
\end{tikzcd}$$

is a pullback square.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r] \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"'] & C^{\simeq} \vec{\times}_{\gamma_C} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"] \\
C^{\simeq} \times C^{\simeq} \arrow[r, "\mathrm{id}_{C^{\simeq}} \times \gamma_C"] & C^{\simeq} \times C \arrow[r, "\gamma_C \times \mathrm{id}_C"] & C \times C.
\end{tikzcd}$$

By the pasting law of pullback squares, it will suffice to show that the left square is a pullback square. Using the first projection map $\mathrm{pr}_1\colon C^{\simeq} \times C \to C^{\simeq}$, we may regard this square as a diagram of synthetic categories in context $C^{\simeq}$, and it will suffice it is a pullback square in $\mathcal{E}(C^{\simeq})$. Since $C^{\simeq}$ is a groupoid, we may prove this claim after pulling back along an arbitrary map $x\colon \{x\} \to C^{\simeq}$ from some groupoid $\{x\}$. (This step is redundant since we may take $\{x\} = C^{\simeq}$, but the change in notation makes the proof psychologically easier to follow.) Observe that the resulting diagram of synthetic categories in context $\{x\}$ is given as follows:

$$\begin{tikzcd}
(C_{x/})^{\simeq} \arrow[r, "\gamma_{C_{x/}}"] \arrow[d] & C_{x/} \arrow[d] \\
C^{\simeq} \arrow[r, "\gamma_C"] & C.
\end{tikzcd}$$

But since the functor $C_{x/} \to C$ is a left fibration by Theorem 5.3.4, this square is a pullback square by Corollary 5.8.5. This finishes the proof. □

Recall that we defined in Definition 1.4.5 for any pair (x,y) of absolute objects in a synthetic category C the *hom groupoid* $C(x,y)$, which was generalized in Definition 3.2.7 to objects in arbitrary context. We now obtain the following alternative description of the hom groupoid:

Corollary 5.8.7. *For a synthetic category C and objects x and y of C, there is a pullback square*

$$\begin{tikzcd}
C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\{(x,y)\} \arrow[r, "(x,y)"] & C^{\simeq} \times C^{\simeq}.
\end{tikzcd}$$

Proof. The objects $x,y\colon \Gamma \to C$ uniquely factor through the groupoid core $C^{\simeq}$. Since $C(x,y)$ is defined as the fiber over Γ of the map $(\mathrm{ev}_0,\mathrm{ev}_1)\colon \mathrm{Fun}([1],C) \to C\times C$, it follows from Lemma 5.8.6 that it is equivalently the fiber of the map $(\mathrm{ev}_0,\mathrm{ev}_1)\colon \mathrm{Map}([1],C) \to C^{\simeq}\times C^{\simeq}$. □

Note that this gives an alternative proof of the claim in Corollary 5.3.6 that the hom groupoids $C(x,y)$ are indeed groupoids.

5.8.2 Objectwise criteria

Using the fiberwise criteria discussed above, we may reformulate various categorical notions in terms of objectwise properties.

Lemma 5.8.8. *Let x be an object of a synthetic category C. Then x is initial if and only if for every other object y of C the hom groupoid $C(x,y)$ is contractible. Dually, x is terminal if and only if $C(y,x)$ is contractible for every object y of C.*

Proof. By Lemma 5.5.11, the object x is initial if and only if the forgetful functor $C_{x/} \to C$ is an equivalence. By regarding both sides as left fibrations over C (invoking Theorem 5.3.4), it will by Lemma 5.8.3 suffice to show that the induced map on fibers is an equivalence. But for an object y of C, the induced map on fibers is the map $p_{C(x,y)}\colon C(x,y) \to *$, hence this is an equivalence if and only if $C(x,y)$ is contractible. □

Lemma 5.8.9. *Let $f\colon A \to B$ be a functor and let $\alpha\colon a \to a'$ be a morphism in A. Then α is f-cocartesian if and only if for every object x of A the commutative square*

$$\begin{tikzcd}
A(a',x) \arrow[r, "-\circ\alpha"] \arrow[d, "f"'] & A(a,x) \arrow[d, "f"] \\
B(f(a'),f(x)) \arrow[r, "-\circ f(\alpha)"] & B(f(a),f(x))
\end{tikzcd}$$

is a pullback square.

Proof. By definition, α is f-cocartesian if and only if the square

$$\begin{tikzcd}
A_{a'/} \arrow[r, "\alpha^*"] \arrow[d, "f"'] & A_{a/} \arrow[d, "f"] \\
B_{f(a')/} \arrow[r, "f(\alpha)^*"] & B_{f(a)/}
\end{tikzcd}$$

is a pullback square. Equivalently, the canonical map $A_{a'/} \to A_{a/} \times_{B_{f(a)/}} B_{f(a')/}$ is an equivalence. This map fits into a commutative triangle

$$\begin{tikzcd}
A_{a'/} \arrow[rr] \arrow[dr] & & A_{a/} \times_{B_{f(a)/}} B_{f(a')/} \arrow[dl] \\
& A &
\end{tikzcd}$$

of left fibrations over A, and thus by Lemma 5.8.3 it suffices to show that this map is an equivalence on fibers over x for every object x of A. But the induced map on fibers is the map $A(a',x) \to A(a,x) \times_{B(f(a'),f(x))} B(f(a),f(x))$ induced by the commutative square from the statement of the lemma, and hence it is an equivalence if and only if this square is a pullback square. □

Remark 5.8.10. In more informal terms, the previous result says that a morphism $\alpha\colon a \to a'$ is f-cocartesian if and only if for any morphism $v\colon a \to x$ in A and morphism $w\colon f(a') \to f(x)$ in B such that the triangle

$$\begin{tikzcd}
& f(a') \arrow[dr, "w"] & \\
f(a) \arrow[ur, "f(\alpha)"] \arrow[rr, "f(v)"] & & f(x)
\end{tikzcd}$$

commutes, there exists a unique morphism $\tilde{w}\colon a' \to x$ such that the triangle

$$\begin{tikzcd}
& a' \arrow[dr, "\tilde{w}"] & \\
a \arrow[ur, "\alpha"] \arrow[rr, "v"] & & x
\end{tikzcd}$$

commutes and is sent to the previous triangle under f.

Lemma 5.8.11. *Let $f\colon C \to D$ and $g\colon D \to C$ be functors and let $\varepsilon\colon fg \to \mathrm{id}_D$ be a natural transformation. Then ε exhibits f as a left adjoint of g if and only if for every object x of C and every object y of D the composite*

$$C(x,gy) \xrightarrow{f} D(fx,fgy) \xrightarrow{(\varepsilon_x)_*} D(fx,y)$$

is an equivalence of groupoids.

Proof. TO DO: reduce this to Proposition 5.4.7. □

5.8.3 Functoriality of universals for arbitrary functors

In our formulation of Axiom M, we assumed that the functor $f\colon C \to D$ was a cartesian fibration. We will now show that we may deduce from it a stronger version in which we do not require f to be a cartesian fibration.

Theorem 5.8.12. *Let $f\colon C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor f admits a right adjoint $g\colon D \to C$;

(2) The projection functor $p\colon C \vec{\times}_f D \to D$ admits a right adjoint section $s\colon D \to C \vec{\times}_f D$;

(3) For every object d in D, the relative slice $C_{/d} = C \times_D D_{/d}$ admits a terminal object s_d.

Dually, f admits a left adjoint if and only if the projection functor $D \vec{\times}_f C \to D \to D$ admits a left adjoint section, or equivalently if and only if the relative coslice $C_{d/}$ admits an initial object for every object d of D.

Remark 5.8.13. We will in fact proof the following somewhat more refined statement: assume that for every object d of D, the relative slice $C_{/d}$ admits a terminal object $s_d = (g_d, \varepsilon_d)$, where g_d is an object of C and $\varepsilon_d\colon f(g_d) \to d$ is a morphism in D. Then the right adjoint $g\colon D \to C$ of f may be chosen in such a way that for every object d of D the following two conditions are satisfied:

- There is an isomorphism $g_d \cong g(d)$;
- Under this isomorphism, the counit map $\varepsilon\colon f(g(d)) \to d$ of the adjunction corresponds to the given map $\varepsilon_d\colon f(g_d) \to d$.

Proof of Theorem 5.8.12. We will prove the equivalence between (1), (2) and (3), and leave the dual claim to the reader. Recall from Construction 5.1.1 the directed pullback $C \vec{\times}_f D$, defined via the following pullback diagram:

$$\begin{tikzcd}
C \vec{\times}_f D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], D) \arrow[d, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"] \\
C \times D \arrow[r, "f \times \mathrm{id}_D"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \times D \arrow[d, "\mathrm{pr}_1"] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

Objects of $C \vec{\times}_f D$ are pairs (x, u) where x is an object in C and u is an object in $\mathrm{Fun}([1], D)$ of the form $u\colon f(x) \to y$. There are two canonical projection maps

$$p\colon C \vec{\times}_f D \to D \qquad \text{and} \qquad q\colon C \vec{\times}_f D \to C$$

which are given on terms by $p(x, u\colon f(x) \to y) = y$ and $q(x, u) = x$. We have:

- The functor p is a cocartesian fibration by Proposition 5.7.5;
- The functor q is a base change of the evaluation map $\mathrm{ev}_0\colon \mathrm{Fun}([1],D) \to D$. Since ev_0 admits a left adjoint section given by $y \mapsto \mathrm{id}_y$ by Lemma 5.5.4, we get that also q admits a left adjoint section $i\colon C \to C \vec{\times}_f D$ given by $i(x) = (x, f(x) \xrightarrow{=} f(x))$.
- The composite
$$C \xrightarrow{i} C \vec{\times}_f D \xrightarrow{p} D$$
is equal to f.

For (1) $\implies$ (2), recall from Proposition 5.4.7 that the data of an adjunction $f \dashv g$ is equivalent to the data of an equivalence $C \vec{\times}_f D \simeq C \vec{\times}_g D$ over $C \times D$. We may therefore equivalently show that the projection $C \vec{\times}_g D \to D$ admits a right adjoint section. But this follows from Lemma 5.5.4, as this map is a base change of the evaluation functor $\mathrm{ev}_1\colon \mathrm{Fun}([1],C) \to C$.

For (2) $\implies$ (1), assume that p admits a right adjoint section $s\colon D \to C \vec{\times}_f D$. Since adjunctions compose by Proposition 5.4.3, we conclude that the composite $g := qs\colon D \to C$ is the desired right adjoint of $f = pi$:

$$\begin{tikzcd} C \arrow[r, shift left, "i"] \arrow[r, phantom, "\perp"] \arrow[rr, bend left, "f"] & C \vec{\times}_f D \arrow[l, shift left, "q"] \arrow[r, shift left, "p"] \arrow[r, phantom, "\perp"] & D. \arrow[l, shift left, "s"] \arrow[ll, bend left, "g"] \end{tikzcd}$$

The claim that (2) and (3) are equivalent is a direct consequence of the Functoriality of Universals Axiom M: p is a cocartesian fibration, and the fiber of p over an object d of D is precisely the relative slice $C_{/d}$.

For the the refined statement from Remark 5.8.13, we use that the terminal objects $s_d = (g_d, \varepsilon_d)$ of the relative slices $C_{/d}$ assemble into the right adjoint section $s\colon D \to C \vec{\times}_f D$, with counit $ps \cong \mathrm{id}_D$. The right adjoint g of f is then given by qs, with counit $fg \to \mathrm{id}_D$ given by the composite

$$fg(d) = piqs(d) \xrightarrow{p\varepsilon'_{s(d)}} ps(d) \cong d,$$

where the transformation $\varepsilon'\colon iq \to \mathrm{id}_{C \vec{\times}_f D}$ is the counit of the adjunction $i \dashv q$. After unwinding the definitions, one sees that this map is precisely the given map $\varepsilon_d\colon f(g_d) \to d$, as claimed. □

Proposition 5.8.14. *Let $f\colon C \to D$ be any functor. Assume that, for every object $d \in D$, there exists an object $g(d)$ in C together with a morphism $\varepsilon_d\colon f(g(d)) \to d$ (resp. $\eta_d\colon d \to f(g(b))$) such that for any object c in C the map*

$$C(c, g(d)) \xrightarrow{f} D(f(c), f(g(d))) \xrightarrow{\varepsilon_d \circ -} : D(f(c), d)$$

is an equivalence (resp. for every c in C the map

$$C(g(d),c) \xrightarrow{f} D(f(g(d)),f(c)) \xrightarrow{-\circ\eta_d} : D(d,f(c))$$

is an equivalence.) Then there is a right adjoint (resp. left adjoint) $g\colon D \to C$ of f such that $g(d)$ coincides on objects with the data above, and the counit (resp. unit) is on objects given by the maps ε_d (resp. η_d).

Proof. We will only prove the first case; the other case is dual. By Theorem 5.8.12 it suffices to check that, for each object d of D, the pair $(g(d),\varepsilon_d)$ is a final object of the relative slice $C_{/d} = C \times_D D_{/d}$. Let $(c,\alpha\colon f(c) \to d)$ be an object of $C_{/d}$. We need to show that the hom groupoid $\mathrm{Hom}_{C_{/d}}((c,\alpha),(g(d),\varepsilon_d))$ is contractible. By definition of $C_{/d}$ as a pullback, we may compute this hom groupoid via the following pullback square:

$$\begin{tikzcd}
\mathrm{Hom}_{C_{/d}}((c,\alpha),(g(d),\varepsilon_d)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{D_{/d}}((f(c),\alpha),(f(g(d)),\varepsilon_d)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
\mathrm{Hom}_C(c,g(d)) \arrow[r, "f"] & \mathrm{Hom}_D(f(c),f(g(d))) \arrow[r, "(\varepsilon_d)_*"] & \mathrm{Hom}_D(f(c),d).
\end{tikzcd}$$

But since the bottom composite is assumed to be an equivalence, it follows that the left-upper corner is an equivalence. This finishes the proof. □

Proposition 5.8.15. *Consider a pullback square*

$$\begin{tikzcd}
C' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\
D' \arrow[r, "v"] & D
\end{tikzcd}$$

such that p is a cocartesian fibration. If v admits a left adjoint, then also u admits a left adjoint.

Proof. By Theorem 5.8.12, the projection $D' \vec{\times}_v D \to D$ admits a left adjoint section. By that same theorem, it will suffice to show that the projection $C' \vec{\times}_u C \to C$ admits a left adjoint section. To this end, we consider the following commutative diagram:

$$\begin{tikzcd}
 & C' \vec{\times}_u C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\vec{\mathrm{ev}}_0^p"] \\
D \vec{\times}_v D' \arrow[d] & C \vec{\times}_{p,v} D' \arrow[l] \arrow[r, "\mathrm{id} \vec{\times} v"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & C \vec{\times}_p D \\
D & C. \arrow[l, "p"'] &
\end{tikzcd}$$

Since p is a cocartesian fibration, the functor $\vec{\mathrm{ev}}_0^p$ admits a left adjoint section. Since functors admitting left adjoint sections are closed under base change and composition, it remains to show that the two squares in the diagram are pullback squares. For the left bottom square, this follows by applying the pasting lemma of pullback squares to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],D) \arrow[d] & D \vec{\times}_v D' \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & C \vec{\times}_{p,v} D' \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
D \times D & D \times D' \arrow[l] \arrow[d] & C \times D' \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
 & D & C \arrow[l, "p"']
\end{tikzcd}$$

For the top right square, this is a consequence of the pasting law of pullback squares applied to the following commutative diagram:

$$\begin{tikzcd}
C \vec{\times}_u C' \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Fun}([1],C) \arrow[dr, "\mathrm{id}_C \times p'"] \arrow[dd] & & \\
 & C \vec{\times}_{p,v} D' \arrow[rr] \arrow[dd] & & C \vec{\times}_p D \arrow[r] \arrow[dd] & \mathrm{Fun}([1],D) \arrow[dd] \\
C \times C' \arrow[rr] \arrow[dr] & & C \times C \arrow[dr, "\mathrm{id}_C \times p"] & & \\
 & C \times D' \arrow[rr, "\mathrm{id}_C \times v"] & & C \times D \arrow[r] & D \times D
\end{tikzcd}$$

This finishes the proof. □

Let us finish the section with a recognition principle for (co)cartesian fibrations:

Proposition 5.8.16. *Let $f\colon C \to D$ be a functor. Assume that, for any object x of C and any morphism $v\colon y \to f(x)$ in D, there exists an f-cartesian lift $u\colon \tilde{y} \to x$ of v, meaning that $f(u) = v$. Then q is a cartesian fibration.*

Proof. TO DO: Update the notation and terminology in the proof.

Consider the directed evaluation map $\vec{\mathrm{ev}}_1^q\colon \mathrm{Fun}([1],X) \to A \vec{\times}_q X$. We have to show that it admits a right adjoint section. We will apply Theorem 5.8.12, so we have to prove that for every object (v,y) in $A \vec{\times}_q X$ the slice $\mathrm{Fun}([1],X)_{(v,y)}$ admits a terminal object. We proved in [ref] that the slice $\mathrm{Fun}([1],X)_{/u}$ admits a terminal object. Hence it will suffice to show that the functor

$$\begin{tikzcd}
\mathrm{Fun}([1],X)_{/u} \arrow[rr] \arrow[dr] & & \mathrm{Fun}([1],X)_{/(v,y)} \arrow[dl] \\
 & \mathrm{Fun}([1],X) &
\end{tikzcd}$$

is an equivalence over $\mathrm{Fun}([1],X)$. Since both functors are right fibrations, also the horizontal functor is a right fibration, and thus it will suffice to show that the fibers over any object in $\mathrm{Fun}([1],X)_{/(v,y)}$ are contractible. Such an object is given by a pair (α,τ) where $\alpha\colon y' \to y$ is a map fitting in a commutative square as follows:

$$\begin{tikzcd}
q(x') \arrow[r, "v'=q(u')"] \arrow[d, "w"'] & q(y') \arrow[d, "q(\alpha)"] \\
q(x) \arrow[r, "v"] & q(y).
\end{tikzcd}$$

But since u is q-cartesian, there exists a unique morphism $\tilde{w}$ making the following diagram commute:

$$\begin{tikzcd}
x' \arrow[r] \arrow[d, "\tilde{w}"] & y' \arrow[r] & y \\
x. \arrow[urr, "u"] & &
\end{tikzcd}$$

This proves that $\vec{\mathrm{ev}}_1^q$ admits a right adjoint $\sigma\colon A \vec{\times}_q X \to \mathrm{Fun}([1],X)$, with unit $\eta 1 \to \sigma \mathrm{ev}_q$ and counit $\vec{\mathrm{ev}}_1^q \sigma \to 1$ such that $\sigma(v,y)$ is the chosen q-cartesian lift u of v and such that ε restricts to the identity on objects. It follows that σ is a natural isomorphism, and hence [ref] σ determines a right adjoint section of ε. This shows that q is a cartesian fibration. □

5.9 Locally cocartesian fibrations

In this section, we introduce the notion of a locally (co)cartesian fibration.

Definition 5.9.1. Let $q\colon X \to A$ be an isofibration. A morphism $u\colon x \to y$ in X is called *locally q-cartesian* if, for any morphism $u'\colon x' \to y$ such that $q(u') = q(u)$, there is a unique morphism $v\colon x' \to x$ such that $q(v) = \mathrm{id}_{q(v)}$ there is a commutative triangle

$$\begin{tikzcd}
 & x \arrow[dr, "u"] & \\
x' \arrow[ur, "u'"] \arrow[rr, "u'"] & & y
\end{tikzcd}$$

which is sent to the degenerate commutative triangle in A under q. We say that q is a *locally cocartesian fibration* of every morphism $w\colon a \to b$ in A and any $x \in X(a)$ there exists a locally q-cocartesian lift $u\colon x \to y$ of w starting in x.

Equivalently, the functor $v \mapsto uv$ given by composing with u induces an equivalence

$$X_{q(u)}(x',x) \to X(x',y)_{q(u)},$$

where $X_{q(u)}$ is the fiber of q over $q(u)$ and $X(x',y)_{q(u)}$ is defined as the fiber

$$\begin{tikzcd}
X(x',y)_{q(u)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(x',y) \arrow[d, "q"] \\
* \arrow[r, "q(u)"] & A(q(x),q(y)).
\end{tikzcd}$$

Example 5.9.2. Any q-cartesian morphism is locally q-cartesian.

Example 5.9.3. Any isomorphism is q-cartesian, hence in particular locally q-cartesian.

Remark 5.9.4. For an isofibration $q\colon X \to [1]$, any locally q-cartesian morphism in X is q-cartesian. Indeed, a priori we need a condition for all maps in $[1]$, not just the canonical one. But since all other maps are identity maps, we are making a statement about the fibers, but the map $C \to *$ is a cocartesian fibration for every synthetic category C.

Exercise 5.9.5. Consider an isofibration $q\colon X \to A$. Prove that the following properties are equivalent:

(i) The isofibration q is a locally cocartesian fibration;

(ii) For any groupoid Γ and any functor $\Gamma \times [1] \to A$, the pullback of q along this map is a cartesian fibration

Proposition 5.9.6. *Assume that $q\colon X \to A$ is a locally cocartesian fibration. If locally q-cocartesian morphisms are closed under composition in X, then q is a cocartesian fibration.*

Proof. The proof is left as an exercise for the reader. □

5.10 Cocartesian functors

In Lemma 5.2.10, we saw that any functor between left fibrations is compatible with the covariant transport functors. The situation is different for cocartesian fibration: a functor between cocartesian fibrations is compatible with the covariant transport functors if and only if it preserves cocartesian morphisms. Functors satisfying this property are called *cocartesian functors* and will be made precise in this section.

Construction 5.10.1 (Beck-Chevalley transformation)**.** Let $p\colon A \to S$ and $q\colon B \to S$ be two cocartesian fibrations, and consider a commutative triangle

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& S. &
\end{tikzcd}$$

By definition of cocartesian fibrations, the map $\vec{\mathrm{ev}}_0^p\colon \mathrm{Fun}([1],A) \to A \vec{\times}_p S$ admits a left adjoint section lift_0^p, and similarly $\vec{\mathrm{ev}}_0^q$ admits a left adjoint section lift_0^q. In particular, we have natural isomorphisms $\vec{\mathrm{ev}}_0^p \,\mathrm{lift}_0^p \cong \mathrm{id}$ and $\vec{\mathrm{ev}}_0^q \,\mathrm{lift}_0^q\, \mathrm{id}\, \mathrm{id}$, and transformations $\varepsilon_p\colon \mathrm{lift}_0^p \,\vec{\mathrm{ev}}_0^p \to \mathrm{id}$ and $\varepsilon_q\colon \mathrm{lift}_0^q \,\vec{\mathrm{ev}}_0^q \to \mathrm{id}$. Observe that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}([1],A) \arrow[r, "f_*"] \arrow[d, "\vec{\mathrm{ev}}_0^p"'] & \mathrm{Fun}([1],B) \arrow[d, "\vec{\mathrm{ev}}_0^q"] \\
S \vec{\times}_p A \arrow[r, "\vec{f}"] & S \vec{\times}_q B.
\end{tikzcd}$$

We now define the *Beck-Chevalley transformation* $\mathrm{BC}_!\colon \mathrm{lift}_0^q \vec{f} \to f_* \,\mathrm{lift}_0^p$ as the following composite:

$$\mathrm{lift}_0^q \vec{f} \cong \mathrm{lift}_0^q \vec{f} \vec{\mathrm{ev}}_0^p \,\mathrm{lift}_0^p \cong \mathrm{lift}_0^q \,\vec{\mathrm{ev}}_0^q f_* \,\mathrm{lift}_0^p \xrightarrow{\varepsilon_q} f_* \,\mathrm{lift}_0^p .$$

Definition 5.10.2 (Cocartesian functor)**.** The functor f over S as above is said to be a *cocartesian functor over* S if the natural transformation $\mathrm{BC}_!\colon \mathrm{lift}_0^q \vec{f} \to f_* \,\mathrm{lift}_0^p$ is invertible. There is a dual notion of a *cartesian functor* between cartesian fibrations over S.

We can make the Beck-Chevalley transformation a bit more concrete as follows. For every morphism $v\colon x \to y$ in S and every term a of $A(x)$, the map $\mathrm{BC}_!(v)\colon \mathrm{lift}_0^q \vec{f}(a,v) \to f_* \,\mathrm{lift}_0^p(a,v)$ takes the form of a commutative square

$$\begin{tikzcd}
f(a) \arrow[r, equal] \arrow[d, "\mathrm{lift}_0^q(f(a),v)"'] & f(a) \arrow[d, "f(\mathrm{lift}_0^p(a,v))"] \\
v_!(f(a)) \arrow[r, "\overline{v}"] & f(v_!(a)).
\end{tikzcd}$$

In this sense, we see that a functor f is a cocartesian functor precisely if it commutes with the covariant transport functors $v_!$ for all morphisms v in S, see also Corollary 6.2.11.

Example 5.10.3. If p and q are left fibrations, then any functor f over S from A to B is a cocartesian functor over S: this is a direct consequence of the fact that the two adjunctions $\mathrm{lift}_0^p \dashv \vec{\mathrm{ev}}_0^p$ and $\mathrm{lift}_0^q \dashv \vec{\mathrm{ev}}_0^q$ are adjoint equivalences.

Example 5.10.4. For any isofibration $f\colon A \to B$, the diagram

$$\begin{tikzcd}
\mathrm{Fun}([1],A) \arrow[rr, "\vec{\mathrm{ev}}_1^f"] \arrow[dr, "\mathrm{ev}_1"'] & & B \vec{\times}_f A \arrow[dl] \\
& A &
\end{tikzcd}$$

is a cocartesian functor over A.

Proof. TO DO. The idea is to unfold the proof that these two functors to B are cocartesian fibrations and observe. $\square$

Proposition 5.10.5. *Let $p\colon A \to S$ and $q\colon B \to S$ be cocartesian fibrations and let $f\colon A \to B$ be a functor over S. If f is a cocartesian functor over S, then for every functor $T \to S$ the induced functor $f \times_S T\colon A \times_S T \to B \times_S T$ is a cocartesian functor over T.*

Proof. Let us abbreviate $A_T := A \times_S T$, $B_T := A \times_S T$ and $f_T := f \times_S T$. In light of (the proof of) Proposition 5.6.13, the functors $p_T\colon A_T \to T$ and $q_T\colon B_T \to T$ are again cocartesian fibrations, which left adjoint sections of the directed evaluation maps given via pullback:

$$\begin{tikzcd}
\mathrm{Fun}([1], A_T) \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{p_T}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], A) \arrow[d, "\vec{\mathrm{ev}}_0^{p}"] \\
A_T \vec{\times}_{p_T} T \arrow[r] \arrow[u, bend left, dashed, "\mathrm{lift}_0^{p_T}"] & A \vec{\times}_p S \arrow[u, bend left, "\mathrm{lift}_0^{p}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Fun}([1], B_T) \arrow[r] \arrow[d, "\vec{\mathrm{ev}}_0^{q_T}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], B) \arrow[d, "\vec{\mathrm{ev}}_0^{q}"] \\
B_T \vec{\times}_{q_T} T \arrow[r] \arrow[u, bend left, dashed, "\mathrm{lift}_0^{q_T}"] & B \vec{\times}_q S. \arrow[u, bend left, "\mathrm{lift}_0^{q}"]
\end{tikzcd}$$

We need to show that the Beck-Chevalley transformation $\mathrm{BC}_!\colon \mathrm{lift}_0^{q_T} \vec{f}_T \to (f_T)_* \mathrm{lift}_0^{p_T}$ is a natural isomorphism. Since this is a natural transformation of functors $T \vec{\times}_{p_T} A_T \to \mathrm{Fun}([1], B_T)$, it suffices to show this after application of $\vec{\mathrm{ev}}_0^{q_T}$ and $\mathrm{Fun}([1], B_T) \to \mathrm{Fun}([1], B)$. For the first, this is a consequence of the fact that the lifts form sections of the directed evaluation maps. For the second, this is a consequence of the assumption that f is a cocartesian functor. $\square$

5.11 Exercises Chapter 5

Exercise 5.11.1. Spell out the missing details in the proof of Proposition 5.4.3.

Exercise 5.11.2. Spell out the missing details in the proof of Proposition 5.4.5.

Exercise 5.11.3. Show that left and right reflectors are stable under retracts: given a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"] \arrow[rr, bend left, "\mathrm{id}_X"] & X' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\
Y \arrow[r] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Y' \arrow[r] & Y,
\end{tikzcd}$$

if p' is a left/right reflector then so is p.

Exercise 5.11.4. Let $p\colon A \to S$ and $q\colon B \to S$ be left fibrations over S. Show that any functor $f\colon A \to B$ over S is cocartesian over S.

6 The Fundamental Theorem of category theory

In ordinary category theory, it is a well-known fact that a functor $f\colon C \to D$ is an equivalence if and only if it is both fully faithful and essentially surjective. The goal of this chapter is to discuss an analogue of this result in synthetic category theory, and draw various important conclusions from it.

6.1 Fully faithful and essentially surjective functors

In this section, we introduce the notions of *fully faithful* and *essentially surjective* functors in the context of synthetic category theory.

Definition 6.1.1 (Essentially surjective functor)**.** We say that a functor $f\colon C \to D$ is *essentially surjective* if for every object d of D there exists an object c in C together with an isomorphism $f(c) \xrightarrow{\sim} d$ in D.

Warning 6.1.2. For emphasis, let us recall that the object d is allowed to live in an *arbitrary* context Γ, hence the statement should be read as: for every groupoid Γ and for every functor $d\colon \Gamma \to D$ there exists a functor $c\colon \Gamma \to C$ together with an isomorphism $f \circ c \cong d$ in $\mathrm{Fun}(\Gamma, D)$.

In particular, applying this to the universal object $\gamma_D\colon D^{\simeq} \to D$ we see that f is essentially surjective if and only if the induced map on groupoid cores $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ admits a section. We warn the reader that this is a stronger condition than the usual notion of 'essential surjectivity' in the higher categorical literature, in which the section is only asked to exist at the level of sets of isomorphism classes of objects.

Definition 6.1.3 (Fully faithful)**.** A functor $f\colon C \to D$ is said to be *fully faithful* if for all objects x and y of C the induced functor

$$f\colon C(x,y) \to D(f(x), f(y))$$

on hom groupoids is an equivalence.

Remark 6.1.4. The map $C(x,y) \to D(f(x),f(y))$ is obtained by passing to fibers over (x,y) in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}([1],D) \arrow[d] \\
C^{\simeq} \times C^{\simeq} \arrow[r, "f^{\simeq}\times f^{\simeq}"] & D^{\simeq} \times D^{\simeq},
\end{tikzcd} \tag{6.1}$$

where we use Corollary 5.8.7 to identify the fibers of the two vertical maps with $C(x,y)$ and $D(f(x),f(y))$, respectively. It follows that fully faithfulness of f is equivalent to the condition that the square (6.1) is a pullback square.

Lemma 6.1.5. *Every equivalence $u\colon C \xrightarrow{\sim} D$ is fully faithful.*

Proof. We leave this as an exercise to the reader, see Exercise 6.5.1 □

Lemma 6.1.6. *A composite of fully faithful functors is fully faithful.*

Proof. We leave this as an exercise to the reader, see Exercise 6.5.2 □

Lemma 6.1.7. *For a functor $f\colon C \to D$, the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) For every object x in C, the following commutative square is a pullback square:

$$\begin{tikzcd}
C_{/x} \arrow[r, "f_*"] \arrow[d] & D_{/f(x)} \arrow[d] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

(3) For every object x in C, the following commutative square is a pullback square:

$$\begin{tikzcd}
C_{x/} \arrow[r, "f_*"] \arrow[d] & D_{f(x)/} \arrow[d] \\
C \arrow[r, "f"] & D.
\end{tikzcd}$$

Proof. We will prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. The vertical two maps in (2) are both right fibrations by Theorem 5.3.4, hence by Corollary 5.8.4 the square is a pullback square if and only if it induces an equivalence on all fibers. But for an object y of C, the induced map on slices is the map $C(y,x) \to D(f(y),f(x))$, so this condition is satisfied if and only if f is fully faithful. □

Lemma 6.1.8. *Let X and Y be groupoids. Then a map $u\colon X \to Y$ is fully faithful if and only if it is an embedding.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[d, "\sim"'] & & Y \arrow[d, "\sim"] \\
\mathrm{Map}([1],X) \arrow[rr, "u_*"] \arrow[d] & & \mathrm{Map}([1],Y) \arrow[d] \\
X \times X \arrow[rr, "u \times u"] & & Y \times Y.
\end{tikzcd}$$

Since the top two vertical maps are equivalences, we see that the outer rectangle is a pullback square if and only if the bottom square is a pullback square. Since the former means that u is an embedding and the latter that u is fully faithful, this finishes the proof. □

6.2 Conservative functors

We introduce the notion of a *conservative* functor and prove some basic properties about them.

Definition 6.2.1 (Conservative functor)**.** A functor $f\colon C \to D$ is said to be *conservative* if the following condition is satisfied: for every pair of objects x and y of C and every morphism $u\colon x \to y$ in C, if the image $f(u)\colon f(x) \to f(y)$ of f is invertible in D then f is already invertible in C.

Conservative functors may be characterized in a variety of different ways:

Proposition 6.2.2. *Let $f\colon C \to D$ be any functor. The following conditions are equivalent:*

(i) The functor f is conservative.

(ii) The commutative square

$$\begin{tikzcd}
\mathrm{Iso}(C)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] \arrow[d, "f_*"'] & \mathrm{Map}([1],C) \arrow[d, "f_*"] \\
\mathrm{Iso}(D)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] & \mathrm{Map}([1],D)
\end{tikzcd}$$

is a pullback square.

(iii) The commutative square

$$\begin{tikzcd}
C^{\simeq} = \mathrm{Map}(*,C) \arrow[r, "(p_{[1]})^*"] \arrow[d] & \mathrm{Map}([1],C) \arrow[d, "f_*"] \\
D^{\simeq} = \mathrm{Map}(*,D) \arrow[r, "(p_{[1]})^*"] & \mathrm{Map}([1],D)
\end{tikzcd}$$

is a pullback square.

(iv) For any object d of D, the homotopy fiber of f at d is a groupoid in $\mathcal{E}(\{d\})$;

(v) For any groupoid Γ and any functor $d\colon \Gamma \to D$, the fiber $f^{-1}(d)$ is a groupoid.

(vi) A given commutative square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
Y \arrow[r] & D
\end{tikzcd}$$

with X and Y groupoids is a pullback square whenever the associated square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & C^{\simeq} \arrow[d, "f^{\simeq}"] \\
Y \arrow[r] & D^{\simeq}
\end{tikzcd}$$

is a pullback square.

(vii) The square

$$\begin{tikzcd}
C^{\simeq} \arrow[r] \arrow[d, "f^{\simeq}"'] & C \arrow[d, "f"] \\
D^{\simeq} \arrow[r] & D
\end{tikzcd}$$

is a pullback square.

Proof. We first show the equivalence between (i) and (ii). Recall from Lemma 4.1.9 that the map $(\pi_{\mathrm{Iso}})^{\simeq}\colon \mathrm{Iso}(C)^{\simeq} \to \mathrm{Map}([1], C)$ is an embedding for every synthetic category C. It thus follows from Lemma 1.2.5 that also the comparison map

$$\mathrm{Iso}(C)^{\simeq} \to \mathrm{Iso}(D)^{\simeq} \times_{\mathrm{Map}([1],D)} \mathrm{Map}([1], C)$$

is an embedding. As it is a morphism of groupoids, it thus follows from Proposition 15.5.9 that it is an equivalence if and only if it is surjective, in the sense of Definition 15.5.6. But since the objects in the target of this map are precisely those morphism $u\colon x \to y$ in C whose image in D is invertible, the surjectivity of this map is precisely the condition on f that every such morphism u in fact comes from an isomorphism in C, which is the definition of conservativity.

The equivalence between (ii) and (iii) is immediate from the equivalence $C^{\simeq} \xrightarrow{\sim} \mathrm{Iso}(C)^{\simeq}$ from the Rezk Axiom F. The equivalence between (iv) and (v) is immediate from **??**. To show that (v) implies (vi), consider the diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Y \times_D C \arrow[dl] \\
& Y &
\end{tikzcd}$$

of groupoids. By the hypothesis of (v), the synthetic category $Y \times_D C$ is a groupoid, hence $Y \times_D C \cong (Y \times_D C)^{\simeq} \cong Y \times_{D^{\simeq}} C^{\simeq}$ and the claim follows. The implication (vi) $\Longrightarrow$ (vii) is clear. To see that (vii) implies (iii), we use that the functor $\mathrm{Map}([1], -)$ preserves pullback squares, and apply the equivalence $C^{\simeq} \xrightarrow{\simeq} \mathrm{Map}([1], C^{\simeq})$ coming from the fact that $C^{\simeq}$ is a groupoid.

It thus remains to show that (iii) implies (v). Consider a pullback square

$$\begin{tikzcd}
X = f^{-1}(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
\Gamma \arrow[r, "b"] & D
\end{tikzcd}$$

with Γ a groupoid. We must show that X is a groupoid as well. To this end, consider the following commutative diagram:

$$\begin{tikzcd}
X^{\simeq} \arrow[rr] \arrow[dr] \arrow[dd] & & C^{\simeq} \arrow[dr] \arrow[dd] & \\
& \mathrm{Map}([1], X) \arrow[rr, crossing over] & & \mathrm{Map}([1], C) \arrow[dd] \\
\Gamma^{\simeq} \arrow[rr] \arrow[dr] & & D^{\simeq} \arrow[dr] & \\
& \mathrm{Map}([1], \Gamma) \arrow[rr] \arrow[from=uu, crossing over] & & \mathrm{Map}([1], D).
\end{tikzcd}$$

The back and front faces are pullback squares, and the right face is a pullback square by the hypothesis from (ii). We conclude from the pasting lemma of pullback squares that the left face

$$\begin{tikzcd}
X^{\simeq} \arrow[r] \arrow[d] & \mathrm{Map}([1], X) \arrow[d] \\
\Gamma^{\simeq} \arrow[r, "\simeq"] & \mathrm{Map}([1], \Gamma)
\end{tikzcd}$$

is a pullback square. Since the bottom map $\Gamma^{\simeq} \to \mathrm{Map}([1], \Gamma)$ is an equivalence, it follows that also the top map $X^{\simeq} \to \mathrm{Map}([1], X)$ is an equivalence, and thus we conclude from Proposition 4.4.6 that X is a groupoid as desired. □

Lemma 6.2.3. *Compositions of conservative functors are conservative.*

Proof. This is clear from the definition. □

Lemma 6.2.4. *Let $f \colon C \to D$ be a functor admitting a retraction $r \colon D \to C$, i.e. we have $rf \cong \mathrm{id}_C$. Then f is conservative.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Iso}(C)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] \arrow[d, "f_*"'] & \mathrm{Map}([1],C) \arrow[d, "f_*"] \\
\mathrm{Iso}(D)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] \arrow[d, "r_*"'] & \mathrm{Map}([1],D) \arrow[d, "r_*"] \\
\mathrm{Iso}(C)^{\simeq} \arrow[r, "(\pi_{\mathrm{Iso}})^{\simeq}"] & \mathrm{Map}([1],C).
\end{tikzcd}$$

Since the horizontal maps are embeddings and the outer square is clearly a pullback square, it follows from Lemma 1.2.7 that the top square is a pullback square. The claim thus follows from Proposition 6.2.2. □

Lemma 6.2.5. *Every left (resp. right) fibration is conservative.*

Proof. In light of Corollary 5.2.3, this follows immediately from characterization (iii) in the previous proposition. □

Proposition 6.2.6. *If a functor $f\colon C \longrightarrow D$ is fully faithful, then it is conservative.*

Proof. Consider a morphism $u\colon x \longrightarrow y$ in C whose image $f(u)$ in D is invertible. We will show that u admits a left inverse; the argument for the existence of a right inverse will be analogous. Since $f(u)$ is invertible in D, it admits a left inverse $v\colon f(y) \longrightarrow f(x)$, i.e. we have $vf(u) = \mathrm{id}_{f(x)}$. Since f is fully faithful, there exists a morphism $\tilde{v}\colon y \longrightarrow x$ in C satisfying $f(\tilde{v}) = v$. We claim that the relation $\tilde{v}u = \mathrm{id}_x$ is satisfied. Since f is fully faithful, it suffices to show that $f(\tilde{v}u) = f(\mathrm{id}_x)$. But since f preserves identities and compositions, this is equivalent to the relation $f(\tilde{v})f(u) = vf(u) = \mathrm{id}_{f(x)}$, which holds by assumption. □

Corollary 6.2.7. *Let $f\colon C \longrightarrow D$ be a fully faithful functor. Then the induced map $f^{\simeq}\colon C^{\simeq} \longrightarrow D^{\simeq}$ is an embedding.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
C^{\simeq} \arrow[r, "f^{\simeq}"] \arrow[d] & D^{\simeq} \arrow[d] \\
\mathrm{Map}([1],C) \arrow[r] \arrow[d] & \mathrm{Map}([1],D) \arrow[d] \\
C^{\simeq}\times C^{\simeq} \arrow[r, "f^{\simeq}\times f^{\simeq}"] & D^{\simeq}\times D^{\simeq}.
\end{tikzcd}$$

The bottom square is a pullback square by the assumption that f is fully faithful. The top square is a pullback square since the map $f\colon C \longrightarrow D$ is conservative by the previous proposition. It follows that the outer rectangle is a pullback square, which is precisely the condition that the functor $f^{\simeq}$ is an embedding. □

Corollary 6.2.8. *Let $f\colon C \to D$ be a fully faithful functor which is essentially surjective. Then the functor $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an equivalence.*

Proof. We have seen in Corollary 6.2.7 that the functor $f^{\simeq}$ is an embedding. Since it is also surjective by definition of essential surjectivity, it follows that $f^{\simeq}$ is an equivalence by Proposition 15.5.9. $\square$

We now come to the main result of this section: the fact that a natural transformation is a natural isomorphism if and only if it is so objectwise.

Theorem 6.2.9 (Objectwise criterion natural isomorphisms)**.** *For synthetic categories C and D, the canonical embedding $\gamma_C\colon C^{\simeq} \to C$ induces a conservative functor $\gamma_C^*\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C^{\simeq},D)$.*

In other words: given a natural transformation $\alpha\colon f \to g$ of functors $C \to D$, if the induced map $\alpha(x)\colon f(x) \to g(x)$ is an isomorphism in D for every object x of C, then α is a natural isomorphism.

Proof. By Proposition 6.2.2, we may show that the square

$$\begin{tikzcd}
\mathrm{Map}(C,D) \arrow[r] \arrow[d, "\gamma_C^*"'] & \mathrm{Map}([1],\mathrm{Fun}(C,D)) \arrow[d, "\gamma_C^*"] \\
\mathrm{Map}(C^{\simeq},D) \arrow[r] & \mathrm{Map}([1],\mathrm{Fun}(C^{\simeq},D))
\end{tikzcd}$$

is a pullback square. Note that this square is isomorphic to the square

$$\begin{tikzcd}
\mathrm{Map}(C,D) \arrow[r] \arrow[d] & \mathrm{Map}(C,\mathrm{Fun}([1],D)) \arrow[d] \\
\mathrm{Map}(C^{\simeq},D^{\simeq}) \arrow[r] & \mathrm{Map}(C^{\simeq},\mathrm{Map}([1],D)).
\end{tikzcd}$$

But this square is a pullback square since the functor $D \to \mathrm{Fun}([1],D)$ is a full subcategory by Proposition 4.4.5. $\square$

Corollary 6.2.10. *For every synthetic category C, the functor $(\mathrm{ev}_0,\mathrm{ev}_1)\colon \mathrm{Fun}([1],C) \to C \times C$ is conservative.*

Proof. We may factor this functor as a composite $\mathrm{Fun}([1],C) \to \mathrm{Fun}([1]^{\simeq},C) \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} C \times C$, so by Lemma 6.2.3 it suffices to show that both of them are conservative. The first is conservative by Theorem 6.2.9. For the second one, recall that we assumed in Axiom G.2 that the map $\langle 0,1\rangle\colon * \sqcup * \to [1]^{\simeq}$ admits a section $s\colon [1]^{\simeq} \to * \sqcup *$. It follows that the precomposition map $\mathrm{Fun}([1]^{\simeq},C) \to \mathrm{Fun}(* \sqcup *,C) = C \times C$ admits a retraction given by precomposition with s. In particular it is conservative by Lemma 6.2.4. $\square$

We may use this theorem to give a more concrete description of when a functor between cocartesian fibrations is a cocartesian functor.

Corollary 6.2.11. *Let $p\colon A \to S$ and $q\colon B \to S$ be two cocartesian fibrations, and consider a commutative triangle*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & S. & \end{tikzcd}$$

Then f is a cocartesian functor over S if and only if for every morphism $v\colon x \to y$ in S and every object a of $A(x)$, the map $\overline{v}\colon v_!(f(a)) \to f(v_!(a))$ is an isomorphism.

Proof. By definition, f is a cocartesian functor if and only if the natural transformation $\mathrm{BC}_!\colon \mathrm{lift}_0^q \vec{f} \to f_* \mathrm{lift}_0^p$ constructed in Construction 5.10.1 is a natural isomorphism. By Theorem 6.2.9, this may be tested pointwise: it is equivalent to asking that for every morphism $v\colon x \to y$ in S and every object a of $A(x)$, the morphism $\mathrm{BC}_!(a,v)\colon \mathrm{lift}_0^q \vec{f}(a,v) \to f_* \mathrm{lift}_0^p(a,v)$ in $\mathrm{Fun}([1],B)$ is an isomorphism. This morphism takes the form of a commutative square

$$\begin{tikzcd} f(a) \arrow[r, equal] \arrow[d, "\mathrm{lift}_0^q(f(a),v)"'] & f(a) \arrow[d, "f(\mathrm{lift}_0^p(a,v))"] \\ v_!(f(a)) \arrow[r, "\overline{v}"] & f(v_!(a)), \end{tikzcd}$$

and by Corollary 6.2.10 this is in turn equivalent to asking both horizontal maps to be isomorphisms. Since the top vertical morphism is an identity map and thus always an isomorphism, this finishes the proof. □

Proposition 6.2.12. *Let p and q as above and let $f\colon C \to D$ be a functor over S. Then the following conditions are equivalent:*

(1) The functor f is a (co)cartesian functor over S;

(2) For any functor $T \to S$, the induced functor $f_T\colon T \times_S C \to T \times_S D$ is a (co)cartesian functor over T;

(3) Condition (2) holds whenever T is of the form $T = [1] \times \Gamma$ for some groupoid Γ;

(4) Condition (2) holds for $T = [1] \times \mathrm{Map}([1],S)$ equipped with the evaluation map $[1] \times \mathrm{Map}([1],S) \to S$.

Proof. We showed that (1) implies (2) in Proposition 5.10.5. The implications (2) $\implies$ (3) $\implies$ (4) are clear. To see that (4) implies (3), consider a map $t\colon [1] \times \Gamma \to S$. We then obtain by currying a map $\tilde{t}\colon \Gamma \to \mathrm{Fun}([1],S)^{\simeq} = \mathrm{Map}([1],S)$, and the map t is given as the composite

$$[1]\times\Gamma \xrightarrow{1\times\tilde{t}} [1]\times\mathrm{Map}([1],S) \xrightarrow{\mathrm{ev}} S.$$

In particular, if the pullback of f to $[1]\times\mathrm{Map}([1],S)$ is a (co)cartesian functor, then its pullback to $[1]\times\Gamma$ will also be a (co)cartesian functor by Proposition 5.10.5, as desired.

It thus remains to show that (3) implies (1). By the previous corollary, it suffices to show that for every morphism $v\colon x \to y$ in S and every object c of $C(x)$, the map $\overline{v}\colon v_!(f(c)) \to f(v_!(c))$ is an isomorphism. But if x and y are objects in context Γ for some groupoid Γ, then v is a map of the form $[1]\times\Gamma \to S$. By assumption, the pullback f_T of f to $T = [1]\times\Gamma$ is a cocartesian functor of cocartesian fibrations over T. But the map $\overline{v}$ is precisely the Beck-Chevalley map induced by the canonical morphism in T (regarded as a category in context Γ) and hence it is an isomorphism. □

6.3 The Fundamental Theorem of category theory

The goal of this section is to prove the fundamental theorem of category theory: a functor is an equivalence if and only if it is both fully faithful and essentially surjective.

Theorem 6.3.1 (Fundamental Theorem of Category Theory). *A functor $f\colon C \to D$ is an equivalence if and only if it is fully faithful and essentially surjective.*

Proof. It is clear that every equivalence is fully faithful and essentially surjective. Conversely, let $f\colon C \to D$ be a functor which is fully faithful and essentially surjective. Our goal is to show that f is an equivalence.

We start by showing that f admits a right adjoint $g\colon D \to C$. By Theorem 5.8.12, it will suffice to show that the relative slice category $C_{/d} = C \times_D D_{/d}$ admits a terminal object for every object d of D. Since f is essentially surjective, we may assume that d is of the form $f(c)$ for some object c in C, and thus it suffices to show that $C_{/f(c)}$ admits a terminal object for every object c of C. But since f is fully faithful, it follows from Lemma 6.1.7 that the canonical map $C_{/c} \to C_{/f(c)} = C \times_D D_{/f(c)}$ is an equivalence, and hence it will suffice to show that the slice category $C_{/c}$ admits a terminal object for every object c of C. This was proved in Lemma 5.5.16.

Recall that the adjoint g we obtain is given by a composite of adjoints, as displayed as follows:

$$\begin{tikzcd}
C \arrow[r, shift left=1.5, "i"] \arrow[rr, bend left=30, "f"] & C \vec{\times}_f D \arrow[l, shift left=1.5, "q"] \arrow[l, phantom, "\perp"] \arrow[r, shift left=1.5, "p"] & D. \arrow[l, shift left=1.5, "s"] \arrow[l, phantom, "\perp"] \arrow[ll, bend left=30, "g"]
\end{tikzcd}$$

If we denote by $\eta\colon \mathrm{id}_{C \vec{\times}_f D} \to sp$ the unit of the adjunction $p \dashv s$ and by $\varepsilon\colon iq \to \mathrm{id}_{C \vec{\times}_f D}$ the counit of the adjunction $i \dashv q$, it will thus remain to show that the to natural transformations

$$fg = piqs \xrightarrow{p\varepsilon s} ps = \mathrm{id}_D$$

and

$$\mathrm{id}_C = qi \xrightarrow{q\eta i} qspi = gf$$

are natural isomorphisms.

To show that $p\varepsilon s$ is a natural isomorphism, consider an arbitrary type X, a term $x\colon X \to C$ of C and a morphism $u\colon f(x) \to y$ of D. Then the evaluation of the transformation $\varepsilon\colon iq \to \mathrm{id}_{C \vec{\times}_f D}$ at the object $(x, u\colon f(x) \to y)$ of $C \vec{\times}_f D$ is by the following morphism in $C \vec{\times}_f D$ (to be read from top to bottom):

$$\left(\begin{tikzcd} x \arrow[d, equal] \\ x \end{tikzcd}, \begin{tikzcd} f(x) \arrow[r, equal] \arrow[d, equal] & f(x) \arrow[d, "u"] \\ f(x) \arrow[r, "u"] & y \end{tikzcd} \right).$$

In particular, we see that the induced map $p\varepsilon_{(x,u)}\colon x \to x$ is the identity in D. This shows that the transformation $p\varepsilon s\colon fg \to \mathrm{id}_D$ is a natural isomorphism.

To show that $q\eta i$ is a natural isomorphism, it will by Theorem 6.2.9 suffice to do so on objects. To this end, we will start by giving an explicit description of the transformation $\eta\colon \mathrm{id}_{C \vec{\times}_f D} \to sp$ at an object $(x, u\colon f(x) \to y)$. By essential surjectivity of f, there exists an object x' in C with $f(x') \cong y$. By fully faithfulness, there is in turn a morphism $v\colon x \to x'$ in C such that $f(v)\colon f(x) \to f(x') \cong y$ is equal to u. Unwinding definitions, the section s is given on an object y of D by the pair $(x', f(x') \xrightarrow{\cong} y)$. The map $\eta_{(x,u)}\colon (x,u) \to (x', f(x') \xrightarrow{\cong} y)$ is given by the following morphism (to be read from top to bottom):

$$\left(\begin{tikzcd} x \arrow[d, "v"'] \\ x' \end{tikzcd}, \begin{tikzcd} f(x) \arrow[r, "u"] \arrow[d, "f(v)"] & y \arrow[d, equal] \\ f(x') \arrow[r, "\cong"] & y \end{tikzcd} \right).$$

In particular, the induced map $q\eta_{(x,u)}\colon y \to y$ is the identity as well. We conclude that $q\eta_i$ is an (objectwise) natural isomorphism, finishing the proof. □

Corollary 6.3.2. *A functor $f\colon C \to D$ is an equivalence if and only if the induced map*

$$f_*\colon \mathrm{Map}([1], C) \xrightarrow{\sim} \mathrm{Map}([1], D)$$

is an equivalence.

Proof. If f is an equivalence, it is clear that f_* is an equivalence. Conversely, assume that f_* is an equivalence. By the Fundamental Theorem, it will suffice to show that f is essentially surjective and fully faithful. Consider the following commutative diagram:

$$\begin{tikzcd}
C^{\simeq} \arrow[r, "p_{[1]}^*"] \arrow[d, "f^{\simeq}"'] & \mathrm{Map}([1],C) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "f_*"] & C^{\simeq} \arrow[d, "f^{\simeq}"] \\
D^{\simeq} \arrow[r, "p_{[1]}^*"] & \mathrm{Map}([1],D) \arrow[r, "\mathrm{ev}_0"] & D^{\simeq}.
\end{tikzcd}$$

It follows that $f^{\simeq}$ is a retract of f_*, and thus in particular it is an equivalence. In particular, f is essentially surjective. Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "f_*"] \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"'] & \mathrm{Map}([1],D) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"'] \\
C^{\simeq} \times C^{\simeq} \arrow[r, "f^{\simeq} \times f^{\simeq}"] & D^{\simeq} \times D^{\simeq}.
\end{tikzcd}$$

Since the top and bottom maps are equivalences, it follows that the square is a pullback square, and thus f is fully faithful by Remark 6.1.4. □

Exercise 6.3.3. Given types C and D, construct a canonical pullback square of the form

$$\begin{tikzcd}
\mathrm{Equiv}(C,D) \arrow[r] \arrow[d] & \mathrm{Equiv}(\mathrm{Map}([1],C),\mathrm{Map}([1],D)) \arrow[d] \\
\mathrm{Map}(C,D) \arrow[r] & \mathrm{Map}(\mathrm{Map}([1],C),\mathrm{Map}([1],D))
\end{tikzcd}$$

6.4 Applications

We will now discuss a variety of applications of the Fundamental Theorem. We start by showing that the various notions of 'subcategories' and 'full subcategories' we have previously introduced are compatible with each other.

Proposition 6.4.1. *Consider a functor $f\colon C \to D$. Then the following conditions are equivalent:*

(1) The functor f is an embedding, in the sense of Definition 1.2.1;

(2) The induced map $f_*\colon M_C := \mathrm{Map}([1],C) \to \mathrm{Map}([1],D)$ is an embedding and the induced map $C \to \langle M_C \rangle_D$ is an equivalence;

(3) The functor f exhibits C as a subcategory of D, in the sense of Definition 4.1.2.

Proof. It is clear that (2) implies (3), and we showed in Lemma 4.1.4 that (3) implies (1). To see that (1) implies (2), assume that f is an embedding. Then also the induced map $f_*\colon \mathrm{Map}([1],C) \to \mathrm{Map}([1],D)$ is an embedding. To show that the functor $C \to \langle M_C\rangle_D$ is an equivalence, it suffices by Corollary 6.3.2 to show that the induced functor $\mathrm{Map}([1],C) \to \mathrm{Map}([1],\langle M_C\rangle_D)$ is an equivalence. But this is clear since under the fixed identification $\mathrm{Map}([1],\langle M_C\rangle_D) \xrightarrow{\sim} M_C = \mathrm{Map}([1],C)$ this map corresponds to the identity map on $\mathrm{Map}([1],C)$. □

Proposition 6.4.2. *Consider a functor $f\colon C \to D$. Then the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) The functor $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an embedding and the canonical functor $C \to \langle C^{\simeq} \subseteq D\rangle$ is an equivalence;

(3) The functor f exhibits C as a full subcategory of D, in the sense of Definition 4.2.2.

Proof. First note that in all three cases the induced map $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an embedding, where in case (1) we use Corollary 6.2.7. Picking a section $s\colon [1]^{\simeq} \to * \sqcup *$ of the map $\langle 0,1\rangle\colon * \sqcup * \to [1]^{\simeq}$ as in Axiom G.2, we then obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}(([1])^{\simeq},C^{\simeq}) \arrow[r, "(f^{\simeq})_*"] \arrow[d, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"'] \arrow[dd, bend right=60, equal] & \mathrm{Map}(([1])^{\simeq},D^{\simeq}) \arrow[d, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"] \arrow[dd, bend left=60, equal] \\
C^{\simeq}\times C^{\simeq} \arrow[r, "f^{\simeq}\times f^{\simeq}"] \arrow[d, "s^*"'] & D^{\simeq}\times D^{\simeq} \arrow[d, "s^*"] \\
\mathrm{Map}(([1])^{\simeq},C^{\simeq}) \arrow[r, "(f^{\simeq})_*"] & \mathrm{Map}(([1])^{\simeq},D^{\simeq}).
\end{tikzcd}$$

Since the horizontal maps are embeddings and the outer square is a pullback square, it follows from Lemma 1.2.7 that also the top square is pullback square.

We will now prove that (1) implies (2), so assume that f is fully faithful. We already argued that $f^{\simeq}\colon C^{\simeq} \to D^{\simeq}$ is an embedding. To show that $C \to \langle C^{\simeq} \subseteq D\rangle$ is an equivalence, it suffices by Corollary 6.3.2 to show that the induced functor $\mathrm{Map}([1],C) \to \mathrm{Map}([1],\langle \Gamma \subseteq C\rangle)$ is an equivalence. By Proposition 4.2.6, this is equivalent to the condition that the top square in the following commutative diagram is a pullback square:

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}([1],D) \arrow[d] \\
\mathrm{Map}(([1])^{\simeq},C^{\simeq}) \arrow[r, "(f^{\simeq})_*"] \arrow[d, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(([1])^{\simeq},D^{\simeq}) \arrow[d, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"] \\
C^{\simeq}\times C^{\simeq} \arrow[r, "f^{\simeq}\times f^{\simeq}"] & D^{\simeq}\times D^{\simeq}.
\end{tikzcd}$$

Since we have seen before that the bottom square is a pullback square, this is equivalent to the outer square being a pullback square. But this precisely fully faithfulness of f.

The fact that (2) implies (3) was proved in Corollary 4.2.7.

Finally, condition (3) demands that for every synthetic category B the commutative square

$$\begin{tikzcd}
\mathrm{Map}(B,C) \arrow[r, "f_*"] \arrow[d] & \mathrm{Map}(B,D) \arrow[d] \\
\mathrm{Map}(B^{\simeq},C^{\simeq}) \arrow[r, "(f^{\simeq})_*"] & \mathrm{Map}(B^{\simeq},D^{\simeq})
\end{tikzcd}$$

is a pullback square, and as explained above this square for $B = [1]$ precisely says that f is fully faithful, giving (1). □

Next, we show that various properties we have introduced can be checked at the level of objects. To start with, recall that a functor $f\colon C \to D$ is a left fibration if and only if the commutative square

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1],D) \arrow[d, "\mathrm{ev}_0"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

is a pullback square, and dually for right fibrations. We may interpret this as saying that every morphism in B admits a unique lift to A with prespecified source, in a way which is fully functorial in the morphism in B. The next lemma says that it suffices to find these lifts objectwise:

Lemma 6.4.3. *Let $f\colon C \to D$ be a functor. Then f is a left (resp. right) fibration if and only if the square*

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "f_*"] \arrow[d, "\mathrm{ev}_i"'] & \mathrm{Map}([1],D) \arrow[d, "\mathrm{ev}_i"] \\
C^{\simeq} \arrow[r, "f^{\simeq}"] & D^{\simeq}
\end{tikzcd}$$

is a pullback for $i = 0$ (resp. $i = 1$).

Proof. By symmetry, it suffices to treat the case for left fibrations. One direction is clear by Lemma 5.2.2, so assume conversely that the above square is a pullback square. We need to show that the map

$$\vec{\mathrm{ev}}_0^f\colon \mathrm{Fun}([1],C) \to C \vec{\times}_f D$$

is an equivalence. By assumption, we know that the underlying map on groupoids

$$(\vec{\mathrm{ev}}_0^f)^{\simeq}\colon \mathrm{Map}([1],C) = \mathrm{Fun}([1],C)^{\simeq} \to C^{\simeq} \times_{D^{\simeq}} \mathrm{Fun}([1],D)^{\simeq} = (C \vec{\times}_f D)^{\simeq}$$

is an equivalence, and in particular $\vec{\mathrm{ev}}_0^f$ is essentially surjective. By the Fundamental Theorem 6.3.1, it will thus suffice to show that it is also fully faithful. To this end, consider objects $u\colon x_0 \to x_1$ and $v\colon y_0 \to y_1$ of $\mathrm{Fun}([1], C)$. We need to prove that the induced map

$$\mathrm{Hom}_{\mathrm{Fun}([1],C)}(u, v) \to \mathrm{Hom}_{C \vec{\times}_f D}((x_0, f(u)), (y_0, f(v)))$$

is an equivalence. Since it is a functor between groupoids, we may equivalently show that all its homotopy fibers are contractible. A map $(x_0, f(u)) \to (y_0, f(v))$ in $C \vec{\times}_f D$ is given by a morphism $\alpha\colon x_0 \to y_0$ in C and a morphism $\beta\colon f(x_1) \to f(y_1)$ in D together with a commutative diagram in D of the form

$$\begin{tikzcd}
f(x_0) \arrow[r, "f(u)"] \arrow[d, "f(\alpha)"'] & f(x_1) \arrow[d, "\beta"] \\
f(y_0) \arrow[r, "f(v)"] & f(y_1).
\end{tikzcd}$$

A lift of this data to a morphism $u \to v$ in $\mathrm{Fun}([1], C)$ consists of a morphism $\overline{\beta}\colon x_1 \to y_1$ and a commutative square in C of the form

$$\begin{tikzcd}
x_0 \arrow[r, "u"] \arrow[d, "\alpha"'] & x_1 \arrow[d, dashed, "\overline{\beta}"] \\
y_0 \arrow[r, "\alpha_1"] & y_1
\end{tikzcd}$$

such that f sends this commutative square to the above square in D. By invoking the Segal Axiom E, we thus have to show the existence of a unique dashed arrow in the following triangle on the left which gets sent to the triangle on the right:

$$\begin{tikzcd}
x_0 \arrow[rr, "u"] \arrow[dr, "\beta u"'] & & x_1 \arrow[dl, dashed, "\overline{\beta}"] & \mapsto & f(x_0) \arrow[rr, "f(u)"] \arrow[dr, "f(\beta u)"'] & & x_1 \arrow[dl, "\beta"] \\
& y_1 & & & & y_1.
\end{tikzcd}$$

By assumption on f there exists an essentially unique lift $\beta'\colon x_1 \to y'$ of β. But then the composite $\beta' u\colon x_0 \to y'$ is a lift of $f(\beta u)$, so by uniqueness it agrees with $\beta u\colon x_0 \to y_1$. In particular, $y' = y_1$ and β' is the desired unique lift $\overline{\beta}$. □

Our next goal is to provide a fiberwise criterion for a map between (co)cartesian fibrations to be an equivalence. In contract to the statement for left (resp. right) fibrations from Lemma 5.8.3, the statement for (co)cartesian fibrations requires that we restrict attention to *(co)cartesian functors*, a notion that was defined in Section 5.10.

Theorem 6.4.4. *A (co)cartesian functor between (co)cartesian fibrations is an equivalence if and only if it is an equivalence on fibers. More precisely, consider commutative triangle*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

such that p and q are cartesian (resp. cocartesian) fibrations and that f is a cartesian (resp. cocartesian) functor over S. If for every object s of S the induced functor $X(s) \to Y(s)$ is an equivalence, then f is an equivalence.

Proof. TO DO: clean up this proof.

Observe that the fibers of f are contractible. Indeed, for an object y of Y, let $s := q(y)$. Then there is a pullback square of the form:

$$\begin{tikzcd}
X(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \arrow[d, "y"] & \\
X(s) \arrow[r] \arrow[d] & Y(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \arrow[d, "s"] \\
X \arrow[r, "f"] & Y \arrow[r, "q"] & S.
\end{tikzcd}$$

In particular, it will suffice by Proposition 5.8.2 to show that f is a right (resp. left) fibration. For this, it in turn suffices by Lemma 6.4.3 to show that the map

$$(\vec{\mathrm{ev}}^0_f)^{\simeq}\colon \mathrm{Map}([1],X) \to (X \vec{\times}_f Y)^{\simeq}$$

is an equivalence.

To this end, let $c_p\colon X \vec{\times}_p S \to \mathrm{Fun}([1],X)$ be a left adjoint section of the functor $\vec{\mathrm{ev}}^0_p\colon \mathrm{Fun}([1],X) \to X \vec{\times}_p S$. The map $\varepsilon\colon c_p \vec{\mathrm{ev}}^0_p \to 1$ gives rise to cocartesian transports

$$\begin{tikzcd}
x \arrow[r, equal] \arrow[d, "v_x"'] & x \arrow[d, "\alpha"] \\
v_!(x) \arrow[r, "\overline{\alpha}"] & x'.
\end{tikzcd}$$

We may do similarly for q. The fact that f is cocartesian over S then translates into the condition that the map $\overline{f(v_x)}$ is an equivalence:

$$\begin{tikzcd}
f(x) \arrow[r, equal] \arrow[d, "v_{f(x)}"'] & f(x) \arrow[d, "f(v_x)"] \\
v_!(f(x)) \arrow[r, "\overline{f(v_x)}"] & f(v_!(x)).
\end{tikzcd}$$

Now, recall we need to show that $(\vec{\mathrm{ev}}^0_f)^\simeq \colon (X \vec{\times}_f Y) \to \mathrm{Map}([1], X)$ is an equivalence. An object of $(X \vec{\times}_f Y)^\simeq$ is a pair (x, u) with x an object of X and $u \colon f(x) \to y$ a morphism in Y (an object of $\mathrm{Fun}([1], Y)$). We may then consider $q(u) \colon p(x) = q(f(x)) \to q(y)$, which yields a map $q(u)_x \colon x \to q(u)_!(x)$.

Now, consider the square

$$\begin{tikzcd}
f(x) \arrow[r, equal] \arrow[d, "f(q(u)_x)"'] & f(x) \arrow[d, "u"] \\
f(q(u)_!(x)) \simeq q(u)_!(f(x)) \arrow[r, dashed, "\bar{u}"] & y.
\end{tikzcd}$$

Let us define $s := q(y) = p(q(u)_!(x))$. Then $\bar{u}$ is a morphism in the fiber Y_s. Because of the equivalence

$$f_s \colon X_s \xrightarrow{\sim} Y_s$$

there is a morphism $q(u)_!(x) \xrightarrow{\tilde{u}} x'$ such that $f(\tilde{u}) = \bar{u}$. In particular, we have $f(x') = y$. We then define the map $v(u) \colon x \to x'$ via the following commutative triangle:

$$\begin{tikzcd}
x \arrow[rr, "q(u)_x"] \arrow[dr, "v(u)"'] & & q(u)_!(x) \arrow[dl, "\tilde{u}"] \\
& x'. &
\end{tikzcd}$$

Applying f, this gives the triangle

$$\begin{tikzcd}
f(x) \arrow[rr, "f(q(u)_x)"] \arrow[dr, "u"'] & & f(q(u)_!(x)) \arrow[dl, "\bar{u}"] \\
& f(x'), &
\end{tikzcd}$$

which shows that $f(v(u)) = u$. The assignment $(x, u) \mapsto v(u)$ defines a morphism

$$\sigma \colon (X \vec{\times}_f Y)^\simeq \to \mathrm{Map}([1], X)$$

satisfying $\sigma(x, u) = f(u)$. (We may simply do the above construction in the local tribe over the groupoid $(X \vec{\times}_f Y)^\simeq$.) Observe that $\vec{\mathrm{ev}}^0_f \sigma = 1$ by construction. If $v \colon x \to x'$ is a map in X (an object of $\mathrm{Fun}([1], X)$) we get for $u = f(v)$ a factorization

$$\begin{tikzcd}
x \arrow[r, equal] \arrow[d, "q(u)_x"'] & x \arrow[d, "v"] \\
q(u)_!(x) \arrow[r] & x'.
\end{tikzcd}$$

Applying f, we get a diagram

$$\begin{tikzcd}
f(x) \arrow[r, equal] \arrow[d, "q(u)_{f(x)}"'] & f(x) \arrow[d, "u"] \\
q(u)_!(f(x)) \arrow[r] & f(x').
\end{tikzcd}$$

This implies that $\sigma(x,u) \simeq v$, because the map $q(u)_!(x) \to x'$ is a lift of the map $q(u)_!(f(x)) \to f(x)$ through the equivalence $f_s \colon X_s \xrightarrow{\sim} Y_s$ with $s = p(x')$. □

Corollary 6.4.5. *Let S be a type and consider a commutative square*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in $\mathcal{E}(S)$ in which the functors $T \to S$ are (co)cartesian fibrations for $T = X, Y, C, D$, and all four maps are (co)cartesian functors over S. Assume that for any object s of S, the induced square

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] & Y_s \arrow[d] \\
C_s \arrow[r] & D_s
\end{tikzcd}$$

is a pullback square. Then also the original square is a pullback square.

Proof. We choose a factorization of $Y \to D$ into

$$Y \overset{\sim}{\rightarrowtail} Y' \to D$$

into an anodyne map followed by an isofibration. Define X' by the following pullback square:

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\
C \arrow[r] & D.
\end{tikzcd}$$

Our goal is to show that the canonical map $X \to X'$ is an equivalence. Since this is a cocartesian functor over S between cocartesian fibrations, the map $X' \to S$ is a cocartesian fibration [provide a proof of this!] Furthermore, the functor $X \to X'$ is a cocartesian functor over S [provide a proof of this as well!] Thus, by Theorem 6.4.4 it will suffice to show that the induced map $X_s \to X'_s$ is an equivalence for every object s of S. But this follows from the assumption that the square we started with is a pullback square on fibers over s. □

Theorem 6.4.6. *Let $f \colon C \to D$ be any functor. Then the following conditions are equivalent:*

(1) The functor f is fully faithful;

(2) The functor f induces a pullback square

$$\begin{tikzcd}
\mathrm{Fun}([1], C) \arrow[r, "f_*"] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & \mathrm{Fun}([1], D) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
C \times C \arrow[r, "f \times f"] & D \times D.
\end{tikzcd}$$

(3) *For every synthetic category X, the induced functor $f_*\colon \mathrm{Fun}(X,C) \to \mathrm{Fun}(X,D)$ is fully faithful.*

Proof. For the equivalence between (2) and (3), notice that by Corollary 2.3.5 property (2) holds if and only if for every X the commutative square

$$\begin{tikzcd}
\mathrm{Map}(X,\mathrm{Fun}([1],C)) \arrow[r] \arrow[d] & \mathrm{Map}(X,\mathrm{Fun}([1],D)) \arrow[d] \\
\mathrm{Map}(X,C\times C) \arrow[r] & \mathrm{Map}(X,D\times D)
\end{tikzcd}$$

is a pullback square. Notice that this square is isomorphic to the following square:

$$\begin{tikzcd}
\mathrm{Map}([1],\mathrm{Fun}(X,C)) \arrow[r] \arrow[d] & \mathrm{Map}([1],\mathrm{Fun}(X,D)) \arrow[d] \\
\mathrm{Fun}(X,C)^{\simeq}\times\mathrm{Fun}(X,C)^{\simeq} \arrow[r] & \mathrm{Fun}(X,D)^{\simeq}\times\mathrm{Fun}(X,D)^{\simeq}.
\end{tikzcd}$$

But this square is a pullback square if and only if the functor $f_*\colon \mathrm{Fun}(X,C) \to \mathrm{Fun}(X,D)$ is fully faithful.

Now we show that (1) and (2) are equivalent. It is clear that (2) implies (1) by passing to groupoid cores. It thus remains to show that (1) implies (2). To this end, recall from Lemma 6.1.7 that for every object x of C the commutative square

$$\begin{tikzcd}
C_{/x} \arrow[r] \arrow[d] & D_{/f(x)} \arrow[d] \\
C \arrow[r, "f"] & D
\end{tikzcd} \tag{6.2}$$

is a pullback square. Now consider the following diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r] \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"'] & D\vec{\times}_f C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1],D) \arrow[d] \\
C\times C \arrow[r, "f\times 1"] & D\times C \arrow[r, "1\times f"] & D\times D.
\end{tikzcd}$$

We need to show that the outer rectangle is a pullback square. Since the right square is a pullback square, it suffices to show that the left square is a pullback square. We may regard this square as living in $\mathcal{E}(C)$ via the projection to the second factor, respectively via the evaluation map $\mathrm{ev}_1\colon \mathrm{Fun}([1],C) \to C$. Furthermore, all four maps in this square are cocartesian functors over C [argue this]. Hence by Corollary 6.4.5 it suffices to check that this is a a pullback square square fiberwise over a. But fiberwise we simply obtain the squares (6.2), which we have already argued are pullback squares. □

Corollary 6.4.7. *Let C be a synthetic category and let Γ be a groupoid equipped with an embedding $\Gamma \to C^{\simeq}$. Then for every other synthetic category X, the synthetic category* $\mathrm{Fun}(X, \langle \Gamma \subseteq C \rangle)$ *is equivalent to the full subcategory of* $\mathrm{Fun}(X,C)$ *spanned by those functors $F \colon X \to C$ satisfying the property that, for any object x of X, $F(X)$ is isomorphic to an object of C coming from Γ.*

Proof. The functor $\mathrm{Fun}(X, \langle \Gamma \subseteq C \rangle) \to \mathrm{Fun}(X,C)$ is fully faithful by Theorem 6.4.6. It thus follows from Proposition 6.4.2 that it induces an equivalence

$$\mathrm{Fun}(X, \langle \Gamma \subseteq C \rangle) \xrightarrow{\sim} \langle \mathrm{Map}(X, \langle \Gamma \subseteq C \rangle) \subseteq \mathrm{Fun}(X,C) \rangle,$$

and it remains to identify the groupoid $\mathrm{Map}(X, \langle \Gamma \subseteq C \rangle$ with the groupoid described in natural language in the statement of the corollary. But this follows from the following pullback square from Proposition 4.2.6:

$$\begin{tikzcd}
\mathrm{Map}(X, \langle \Gamma \subseteq C \rangle) \arrow[r] \arrow[d] & \mathrm{Map}(X,C) \arrow[d] \\
\mathrm{Map}(X^{\simeq}, \Gamma) \arrow[r] & \mathrm{Map}(X^{\simeq}, C^{\simeq}).
\end{tikzcd}$$

□

Remark 6.4.8. For every synthetic category E, we have a pullback square

$$\begin{tikzcd}
\mathrm{Fun}(E, \langle \Gamma \subseteq C \rangle) \arrow[r] \arrow[d] & \mathrm{Fun}(E,C) \arrow[d] \\
\mathrm{Fun}(E^{\simeq}, \langle \Gamma \subseteq C \rangle) \arrow[r] & \mathrm{Fun}(E^{\simeq}, C).
\end{tikzcd}$$

Indeed, we may show that the map from the left upper corner to the pullback is both essentially surjective and fully faithful. Essential surjectivity follows from the usual claim on mapping spaces. For fully faithfulness, notice that both the top map as well as the bottom map are fully faithful by [ref]. By the cancellation property of fully faithful functors [ref], it follows that also the map from $\mathrm{Fun}(E, \langle \Gamma \subseteq C \rangle)$ to the pullback of this square is also fully faithful. In light of the Fundamental Theorem of Category Theory, Theorem 6.3.1, this finishes the proof.

6.5 Exercises Chapter 6

Exercise 6.5.1. Show that every equivalence $u \colon C \xrightarrow{\sim} D$ is fully faithful.

Exercise 6.5.2. Show that a composite of fully faithful functors is fully faithful.

7 Universes and directed univalence

In this chapter, we will introduce a synthetic category Cat called the *universe of (small) categories*. It comes equipped with a cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_{\bullet} \to \mathrm{Cat}$ called the *universal (small) cocartesian fibration*. The conditions on Cat and π_{univ} will imply that all the constructions of synthetic category theory can be performed internally in the universe Cat. We will also construct the universe of groupoids Grpd (classically denoted by $\mathcal{S}$ and referred to as the *∞-category of 'spaces'*), together with its natural functors to/from Cat.

Warning: this chapter is in a somewhat unfinished state!

7.1 Exponentiability axiom and relative functor categories

Given a functor $p\colon C \to D$, we may associate to every functor $f\colon E \to D$ a pullback functor $p^*(f)\colon C \times_D E \to C$. In general, the assignment $f \mapsto p^*(f)$ does not admit a right adjoint:

Example 7.1.1. Given an integer $n \geq 0$, consider the synthetic category $[n]$, which we think of as the freely generated by the directed graph

$$0 \to 1 \to \cdots \to n.$$

We define the *simplex category* $\Delta \subseteq \mathrm{Cat}$ as the full subcategory whose objects are the categories $[n]$ for $n \geq 0$ and whose morphisms are the functors between them. In this case, a morphism $[m] \to [n]$ is exponentiable if and only if it is induced by a morphism of directed graphs.

Let us show for example that the map $\delta^2_1\colon [1] \to [2]$ given by $\delta^2_1(0) = 0$ and $\delta^2_1(1) = 2$ is not exponentiable. We will do this by showing that the pullback functor

$$(\delta^2_1)^*\colon \mathrm{Cat}_{/[2]} \to \mathrm{Cat}_{/[1]}$$

does not preserve colimits, and hence cannot have a right adjoint. To see this, consider the following pushout square in $\mathrm{Cat}_{/[2]}$:

$$\begin{tikzcd}
{[0]} \arrow[r, "\delta^1_2"] \arrow[d, "\delta^1_0"'] & {[1]} \arrow[d, "\delta^2_0"] \\
{[1]} \arrow[r, "\delta^2_2"'] \arrow[ur, phantom, "\ulcorner", very near end] & {[2]}.
\end{tikzcd}\tag{7.1}$$

If we pull this back along δ^2_1, then we get the square

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & {[0]} \arrow[d, "\delta^1_0"] \\
{[0]} \arrow[r, "\delta^1_1"'] & {[1]}.
\end{tikzcd}\tag{7.2}$$

But this is not a pushout square in $\mathrm{Cat}_{/[1]}$. Indeed, otherwise this would remain a pushout square after applying the left adjoint $\mathrm{Cat} \to \mathrm{Grpd}\colon A \mapsto A^{-1}A$ to the inclusion $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ (which inverts all maps in A), which would show that $* \sqcup * = *$, giving a contradiction.

Nevertheless, we *do* expect to have a right adjoint p_* whenever p is either a cartesian or cocartesian fibrations.

Definition 7.1.2. Let $f\colon E \to D$ and $g\colon E \to D$ be functors. We define the synthetic category $\mathrm{Fun}_D(E, E')$ of *functors $E' \to E$ over D* as the following pullback:

$$\begin{tikzcd}
\mathrm{Fun}_D(E,E') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(E,E') \arrow[d, "g_*"] \\
* \arrow[r, "f"] & \mathrm{Fun}(E,D).
\end{tikzcd}$$

Note that objects of $\mathrm{Fun}_D(E, E')$ are pairs (h, α) where $h\colon E \to E'$ is a functor and α is a natural isomorphism $f \cong g \circ h$. In other words, objects are commutative triangles of the form

$$\begin{tikzcd}
E \arrow[rr, "h"] \arrow[dr, "f"'] & & E' \arrow[dl, "g"] \\
& D. &
\end{tikzcd}$$

We write $\mathrm{Map}_D(E, E')$ for the groupoid core of $\mathrm{Fun}_D(E, E')$, which fits into a pullback square of the form

$$\begin{tikzcd}
\mathrm{Map}_D(E,E') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(E,E') \arrow[d, "g_*"] \\
* \arrow[r, "f"] & \mathrm{Map}(E,D).
\end{tikzcd}$$

Remark 7.1.3. Assume that $g\colon E \to D$ is a left (resp. right) fibration. Then also g_* is a left (resp. right) fibration, and in particular g_* is conservative. It follows that we have a pullback square of the form

$$\begin{tikzcd}
\mathrm{Map}(A,X) \arrow[r, hook] \arrow[d, "p_*"'] & \mathrm{Fun}(A,X) \arrow[d, "p_*"] \\
\mathrm{Map}(A,C) \arrow[r, hook] & \mathrm{Fun}(A,C),
\end{tikzcd}$$

and thus we have $\mathrm{Map}_D(E,E') \xrightarrow{\sim} \mathrm{Fun}_D(E,E')$.

Construction 7.1.4. Let $p\colon C \to D$ be a functor. We obtain an induced functor

$$p^*\colon \mathrm{Fun}_D(E,E') \to \mathrm{Fun}_{D'}(C \times_D E, C \times_D E')$$

by noting that the functor

$$\mathrm{Fun}_D(E,E') \to \mathrm{Fun}(E,E') \to \mathrm{Fun}(C \times_D E, C \times_D E')$$

lands in the correct fiber.

Axiom N.1 (Exponentiability Axiom)**.** Let $p\colon C \to D$ be a cartesian or cocartesian fibration.

- For every functor $f\colon E \to C$ there exists a functor $p_*(f)\colon p_*(E) \to D$ equipped with a map $\varepsilon\colon C \times_D p_*(E) \to E$ over C.
- For every functor $g\colon F \to D$, the functor

  $$\mathrm{Fun}_D(F, p_*(E)) \xrightarrow{p^*} \mathrm{Fun}_C(C \times_D F, C \times_D p_*(E)) \xrightarrow{\varepsilon\circ -} \mathrm{Fun}_C(C \times_D F, E)$$

  is an equivalence.

We might sometimes refer to $p_*(f)$ as the *dependent product* of f along p.

Given a functor $\varphi\colon F \to p_*(E)$ over D, we obtain a map

$$\varphi^u := \varepsilon \circ p^*(\varphi)\colon C \times_D F \to E$$

over C that we will refer to as the *relative uncurrying of* φ. The axiom says that conversely there is for every functor $\psi\colon C \times_D F \to E$ over D there is a *relative curried* functor $\psi_c\colon F \to p_*(E)$, satisfying

$$(\varphi^u)_c \cong \varphi \qquad \text{and} \qquad (\psi_c)^u \cong \psi.$$

In this way, we may think of the equivalence in the axiom as a *relative* version of currying/uncurrying.

We further want to impose that the assignments $f \mapsto p_*(f)$ satisfy base change.

Construction 7.1.5. Consider a pullback square

$$\begin{tikzcd} C' \arrow[r, "h"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "p"] \\ D' \arrow[r, "g"] & D \end{tikzcd}$$

of synthetic categories in which p and p' are cartesian or cocartesian fibrations. For every functor $f\colon E \to C$, we will construct a functor

$$\mathrm{BC}_*\colon D' \times_D p_*(E) \to p'_*(C' \times_C E)$$

over D' called the *Beck-Chevalley map*. By uncurrying, we may equivalently define BC_* by specifying a map $C' \times_{D'} (D' \times_D p_*(E)) \to C' \times_C E$, which we take to be the map

$$C' \times_{D'} (D' \times_D p_*(E)) \simeq C' \times_C (C \times_D p_*(E)) \xrightarrow{C' \times_C \varepsilon} C' \times_C E.$$

Axiom N.2. These assignments $f \mapsto p_*(f)$ satisfy *base change*: for every pullback square as in Construction 7.1.5, the functor $\mathrm{BC}_*\colon D' \times_D p_*(E) \to p'_*(C' \times_C E)$ is an equivalence.

Construction 7.1.6. Let $p\colon C \to D$ be a (co)cartesian fibration. Then every functor $\varphi\colon E \to F$ over C induces a map $p_*(\varphi)\colon p_*(E) \to p_*(F)$ over D. The construction of this map is completely analogous to the construction of the functoriality of functor categories in Construction 1.1.23 and the details are left to the reader.

Lemma 7.1.7. *Let $p\colon C \to D$ be a (co)cartesian fibration. Then the assignment $f \mapsto p_*(f)$ preserves pullbacks: given functors $E_1 \to E_3 \leftarrow E_2$ over C, there exists a preferred equivalence*

$$f_*(E_1) \times_{f_*(E_2)} f_*(E_3) \xrightarrow{\sim} f_*(E_1 \times_{E_3} E_2)$$

over D.

Proof. The preferred is constructed by adjunction, using that $f^*(-) = C \times_D -$ preserves pullbacks. The inverse is constructed using the functoriality of $f_*(-)$. It follows directly from universal properties that these two maps are mutual inverses. □

Corollary 7.1.8. *Let $p\colon C \to D$ be a (co)cartesian fibration. If $f\colon E \to C$ is an embedding, then also $p_*(f)\colon p_*(E) \to D$ is an embedding.* □

The exponentiability axiom allows us to construct certain 'fiberwise' functor categories, relative to a base category S:

Definition 7.1.9. Let $p\colon C \to S$ and $q\colon D \to S$ be two functors and assume that q is either a cartesian or a cocartesian fibration. We define the *relative functor category* $\underline{\mathrm{Fun}}_S(C,D) \to S$ as

$$\begin{tikzcd} \underline{\mathrm{Fun}}_S(C,D) \arrow[d] \\ S \end{tikzcd} \quad := \quad q_* \left(\begin{tikzcd} C \times_S D \arrow[d, "q^*(p)"] \\ C \end{tikzcd} \right).$$

It comes equipped with an evaluation map

$$\mathrm{ev}\colon \underline{\mathrm{Fun}}_S(C,D) \times_S C \to D$$

over S satisfying the property that for every other functor $E \to S$ the composite functor

$$\mathrm{Fun}_S(E, \underline{\mathrm{Fun}}_S(C,D)) \xrightarrow{- \times_S C} \mathrm{Fun}_S(E \times_S C, \underline{\mathrm{Fun}}_S(C,D) \times_S C) \xrightarrow{\mathrm{ev} \circ -} \mathrm{Fun}_S(E \times_S C, D).$$

is an equivalence. In other words, $\underline{\mathrm{Fun}}_S(C,D)$ is an 'internal hom' for synthetic categories over S.

Remark 7.1.10. Note that, for a synthetic category T, the data of a functor $T \to \underline{\mathrm{Fun}}_S(C,D)$ is equivalent to the data of a pair (t, f), where $t\colon T \to S$ is a functor and $f\colon T \times_S C \to T \times_S D$ is a functor over T.

Observation 7.1.11. For any functor $t\colon T \to S$ and categories C and D over S, there is a pullback square

$$\begin{tikzcd} \underline{\mathrm{Fun}}_T(T \times_S C, T \times_S D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}_S(C,D) \arrow[d] \\ T \arrow[r, "t"] & S. \end{tikzcd}$$

In particular, the fiber of $\underline{\mathrm{Fun}}_S(C,D) \to S$ over an object s of S is the functor category $\mathrm{Fun}(C_s, D_s)$, where C_s and D_s are the fibes of C and D over s, respectively.

Example 7.1.12. Assume that $S = \Gamma$ is a groupoid. By ?? any functor $q\colon D \to \Gamma$ is a cocartesian fibration, and hence the relative functor category $\underline{\mathrm{Fun}}_\Gamma(C,D)$ exists.

7.2 The subcategory of cocartesian functors

With the help of the relative functor categories, we may construct subcategories of cocartesian functors between cocartesian fibrations.

Construction 7.2.1. Let $p\colon C \to S$ and $q\colon D \to S$ be two cocartesian fibrations, and let $f\colon C \to D$ be a functor over S. We will construct a category $\mathrm{isCocart}(f)$ whose objects are proofs that f is a cocartesian functor.

Recall from Corollary 6.2.11 that a functor $f\colon C \to D$ over S is a cocartesian functor over S if and only if for every morphism $v\colon x \to y$ in S and for every object $c \in C(x)$ a certain canonical map $\mathrm{BC}_!(v,c)\colon v_! f(c) \to f(v_! c)$ is an isomorphism in D. This map is natural in the pair (v,x), in the sense that it is part of a natural transformation of functors $C \vec{\times}_p S \to D$. Precomposing with the directed evaluation functor $\vec{\mathrm{ev}}\colon \mathrm{Fun}([1],C) \to C \vec{\times}_p S$, we thus obtain a functor

$$\mathrm{BC}_!\colon \mathrm{Fun}([1],C) \to \mathrm{Fun}([1],D)$$

and f is a cocartesian functor over S if and only if $\mathrm{BC}_!$ factors through the subcategory $D \simeq \mathrm{Iso}(C) \hookrightarrow \mathrm{Fun}([1],D)$ of isomorphisms.

Consider now the following pullback square:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(D)^{\simeq} \arrow[d, hook, "\pi_{\mathrm{Iso}}"] \\
\mathrm{Map}([1],C) \arrow[r, "\mathrm{BC}_!"] & \mathrm{Map}([1],D).
\end{tikzcd}$$

Consider also the map $p = p_{\mathrm{Map}([1],C)}\colon \mathrm{Map}([1],C) \to *$. We then define

$$\mathrm{isCocart}(f) := p_*(P).$$

Lemma 7.2.2. *For every f as above, the functor* $\mathrm{isCocart}(f) \to *$ *is an embedding. In particular, if* $\mathrm{isCocart}(f)$ *admits an (absolute) object, then it is contractible.*

Proof. Note that $P \to \mathrm{Map}([1],C)$ is an embedding, since it is the base change of the embedding π_{Iso} from Lemma 4.1.8. It follows from Corollary 7.1.8 that also the map $\mathrm{isCocart}(f) \to *$ is an embedding. An absolute object of $\mathrm{isCocart}(f)$ determines a section of this embedding, which then by Lemma 1.2.8 implies that this embedding is an equivalence. □

Observation 7.2.3. Note that $\mathrm{isCocart}(f)$ admits an absolute object if and only if f is a cocartesian functor over S. Indeed, an object $* \to \mathrm{isCocart}(f)$ corresponds by adjunction to a section of the map $P \to \mathrm{Map}([1],C)$, which in turn is equivalent to a lift of $\mathrm{BC}_!$ along

π_{Iso}:

$$\begin{tikzcd}
& \mathrm{Iso}(D)^{\simeq} \arrow[d, hook, "\pi_{\mathrm{Iso}}"] \\
\mathrm{Map}([1],C) \arrow[ur, dashed] \arrow[r, "\mathrm{BC}_!"] & \mathrm{Map}([1],D).
\end{tikzcd}$$

Such a section precisely says that the map $\mathrm{BC}_!(\varphi)$ is an equivalence for every morphism φ in C, which is the condition for f to be a cocartesian functor.

Remark 7.2.4. The above constructions go through in the context of an arbitrary groupoid Γ: for a category S in context Γ, cocartesian fibration $C \to S$ and $D \to S$ in context Γ, and a functor $f\colon C \to D$ over S in context Γ, we obtain a category $\mathrm{isCocart}_\Gamma(f)$ in context Γ.

We continue to fix two cocartesian fibrations $p\colon C \to S$ and $q\colon D \to S$ over S. Recall that there exists an internal hom $\underline{\mathrm{Fun}}_S(C,D)$ of categories over S. Our next goal is to define a subcategory $\mathrm{CoCart}_S(C,D) \subseteq \underline{\mathrm{Fun}}_S(C,D)$ spanned by the cocartesian functors $C \to D$ over S. More precisely, we would like that for any other functor $T \to S$, a functor $T \to \underline{\mathrm{Fun}}_S(C,D)$ over S factors through $\mathrm{CoCart}_S(C,D)$ if and only if the associated functor $T \times_S C \to T \times_S D$ over T from Remark 7.1.10 is a cocartesian functor over T.

Definition 7.2.5. Consider a morphism of $\underline{\mathrm{Fun}}_S(C,D)$, i.e. an object of $\mathrm{Fun}([1],\underline{\mathrm{Fun}}_S(C,D))$. Such an object is given by a groupoid Γ together with a functor $[1] \times \Gamma \to S$ and a functor over $[1] \times \Gamma$ of the form

$$f\colon ([1] \times \Gamma) \times_S C \to ([1] \times \Gamma) \times_S D.$$

We say that the morphism is *cocartesian* if the associated map f is a cocartesian functor over $[1] \times \Gamma$. Equivalently, we may regard f as a functor over Γ and ask that it is a cocartesian functor over $[1]_\Gamma$ in context Γ. By Remark 7.2.4, this is classified by a category $\mathrm{isCocart}_\Gamma(f)$ in context Γ.

We will apply this to the *universal* morphism of $\underline{\mathrm{Fun}}_S(C,D)$:

Construction 7.2.6. Define $T := [1] \times \mathrm{Map}([1],\underline{\mathrm{Fun}}_S(X,Y))$. Then evaluation determines a map

$$\mathrm{ev}\colon T \to \underline{\mathrm{Fun}}_S(X,Y),$$

which corresponds to a functor $T \to S$ together with a functor

$$f_{\mathrm{univ}}\colon C_T \to D_T.$$

Applying Definition 7.2.5 to the case $\Gamma = \mathrm{Map}([1],\underline{\mathrm{Fun}}_S(X,Y))$, we thus obtain a category $\mathrm{isCocart}_\Gamma(f_{\mathrm{univ}})$ in context Γ, i.e. a functor

$$\mathrm{isCocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1],\underline{\mathrm{Fun}}_S(X,Y)).$$

Lemma 7.2.7. *The above functor is an embedding.*

Proof. This is a special case of Lemma 7.2.2, applied in context Γ. $\square$

Lemma 7.2.8. *The embedding* $M := \mathrm{isCocart}_\Gamma(f_{\mathrm{univ}}) \hookrightarrow \mathrm{Map}([1], \underline{\mathrm{Fun}}_S(X,Y))$ *forms a collection of morphisms in* $\underline{\mathrm{Fun}}_S(C,D)$ *which is closed under composition, in the sense of Definition 4.1.5.*

Proof. TO DO. $\square$

Definition 7.2.9. We define the (non-full) subcategory $\underline{\mathrm{CoCart}}_S(C,D) \subseteq \underline{\mathrm{Fun}}_S(C,D)$ as the subcategory spanned by M:

$$\underline{\mathrm{CoCart}}_S(C,D) := \langle M \rangle_{\underline{\mathrm{Fun}}_S(C,D)} \subseteq \underline{\mathrm{Fun}}_S(C,D).$$

Proposition 7.2.10. *Consider a functor $T \to S$ and let $\varphi \colon T \to \mathrm{Fun}_S(C,D)$ be a functor over S, corresponding to a functor $f \colon T \times_S C \to T \times_S D$ over T. Then f is cocartesian over T if and only if φ factors through the subcategory* $\underline{\mathrm{CoCart}}_S(C,D)$.

Proof. By definition of $\underline{\mathrm{CoCart}}_S(C,D)$ as a subcategory, φ factors through $\underline{\mathrm{CoCart}}_S(C,D)$ if and only if the induced map $\mathrm{Map}([1],T) \to \mathrm{Map}([1], \mathrm{Fun}_S(C,D)) = \mathrm{Map}_S([1] \times C, D)$ factors through M. This is in turn equivalent to the condition that for every Γ and every map $\alpha \colon \Gamma \to \mathrm{Map}_S([1],T)$, the object

$$\varphi(\alpha) \colon \Gamma \to \mathrm{Map}_S([1], \mathrm{Fun}_S(C,D))$$

factors through M. The map α corresponds to a map $\tilde{\alpha} \colon [1] \times \Gamma \to T$, and by definition of M this happens if and only if the induced functor

$$f_{\Gamma \times [1]} \colon (\Gamma \times [1]) \times_S C \to (\Gamma \times [1]) \times_S D$$

is a cocartesian functor over $\Gamma \times [1]$. Now, observe that this map is obtained from $f_T \colon T \times_S C \to T \times_S D$ by pulling back along $\tilde{\alpha} \colon \Gamma \times [1] \to T$, and thus it follows from Proposition 6.2.12 that the condition is equivalent to the condition that f_T is a cocartesian functor over T. This finishes the proof. $\square$

Corollary 7.2.11. *For every functor $T \to S$, there is a pullback square of the form*

$$\begin{tikzcd}
\underline{\mathrm{CoCart}}_T(T \times_S C, T \times_S D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_S(C,D) \arrow[d] \\
T \arrow[r, "t"] & S.
\end{tikzcd}$$

Proof. The pullback square from Observation 7.1.11 restricts a pullback square on subcategories of cocartesian fibrations due to Proposition 7.2.10. $\square$

7.3 Directed univalence and universes

In this section, we introduce a class of cocartesian fibrations called *directed univalent*. We start by defining what it means for a cocartesian fibration to be 'small', and by introducing various categories that classify cocartesian fibrations. Throughout this subsection, we fix a cocartesian fibration $\pi\colon U_\bullet \to U$.

Definition 7.3.1. We say that a cocartesian fibration $p\colon E \to S$ is *U-small* if it comes equipped with a pullback square of the form

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_\bullet \arrow[d, "\pi"] \\
S \arrow[r, "\ulcorner p \urcorner"] & U.
\end{tikzcd}$$

We say that the map $\ulcorner p \urcorner$ *classifies the fibration p*, or that *p is classified by the map $\ulcorner p \urcorner$*. We say that a synthetic category C is *U-small* if the functor $p_C\colon C \to *$ is U-small.

We may say that the category U *classifies* U-small cocartesian fibrations, with the fibration $\pi\colon U_\bullet \to U$ being the *universal U-small cocartesian fibration*. We may similarly think of the category $U \times U$ as the one classifying *pairs* of U-small cocartesian fibrations:

Observation 7.3.2. Let S be a synthetic category equipped with two U-small cocartesian fibrations $p_1\colon E_1 \to S$ and $p_2\colon E_2 \to S$, classified by functors $\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner\colon S \to U$. Then p_1 and p_2 may be obtained via pullback from the functors $\mathrm{id}\times\pi\colon U \times U_\bullet \to U \times U$ and $\pi \times \mathrm{id}\colon U_\bullet \times U \to U \times U$:

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U \times U_\bullet \arrow[r] \arrow[d, "\mathrm{id}\times\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_\bullet \arrow[d, "\pi"] \\
S \arrow[r, "(\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner)"'] & U \times U \arrow[r, "\mathrm{pr}_1"] & U
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E_2 \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_\bullet \times U \arrow[r] \arrow[d, "\pi\times\mathrm{id}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_\bullet \arrow[d, "\pi"] \\
S \arrow[r, "(\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner)"'] & U \times U \arrow[r, "\mathrm{pr}_2"] & U.
\end{tikzcd}$$

We may think of the pair of cocartesian fibrations $\mathrm{id}\times\pi\colon U \times U_\bullet \to U \times U$ and $\pi \times \mathrm{id}\colon U_\bullet \times U \to U \times U$ as the *universal pair of U-small cocartesian fibrations*.

Observation 7.3.3. Given the pair of cocartesian fibrations $\mathrm{id}\times\pi\colon U \times U_\bullet \to U \times U$ and $\pi \times \mathrm{id}\colon U_\bullet \times U \to U \times U$, Definition 7.1.9 provides a category $\underline{\mathrm{Fun}}_{U\times U}(U \times U_\bullet, U_\bullet \times U)$ together with a functor

$$\underline{\mathrm{Fun}}_{U\times U}(U \times U_\bullet, U_\bullet \times U) \to U \times U.$$

Given a synthetic category S, it follows from Remark 7.1.10 that the data of a functor $S \to \underline{\mathrm{Fun}}_{U\times U}(U \times U_\bullet, U_\bullet \times U)$ is the same as the data of a functor $(\ulcorner p_1 \urcorner, \ulcorner p_2 \urcorner)\colon S \to U \times U$ together with a functor $E_1 \to E_2$ over S between the cocartesian fibrations $p_1\colon E_1 \to S$ and $p_2\colon E_2 \to S$ classified by $\ulcorner p_1 \urcorner$ and $\ulcorner p_2 \urcorner$. In other words: the category $\underline{\mathrm{Fun}}_{U\times U}(U \times U_\bullet, U_\bullet \times U)$ classifies functors between two U-small cocartesian fibrations.

Observation 7.3.4. Consider the subcategory

$$\underline{\mathrm{CoCart}}_{U\times U}(U\times U_\bullet, U_\bullet\times U) \subseteq \underline{\mathrm{Fun}}_{U\times U}(U\times U_\bullet, U_\bullet\times U)$$

of cocartesian functors over $U\times U$, defined in Definition 7.2.9. By Proposition 7.2.10, a functor $S \to \underline{\mathrm{Fun}}_{U\times U}(U\times U_\bullet, U_\bullet\times U)$ factors through this subcategory if and only if the functor $E_1 \to E_2$ over S from Observation 7.3.3 is a cocartesian functor over S. In other words: the category $\underline{\mathrm{CoCart}}_{U\times U}(U\times U_\bullet, U_\bullet\times U)$ classifies *cocartesian* functors between two U-small cocartesian fibrations.

Remark 7.3.5. Recall from Observation 7.1.11 and Corollary 7.2.11 that the constructions $\underline{\mathrm{Fun}}$ and $\underline{\mathrm{CoCart}}$ are compatible with pullbacks. It thus follows that for any pair (p_1,p_2) of U-small cocartesian fibrations over S, classified by functors $\ulcorner p_1\urcorner, \ulcorner p_2\urcorner\colon S\to U$, there are pullback squares of the form

$$\begin{tikzcd}
\underline{\mathrm{Fun}}_S(E_1,E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Fun}}_{U\times U}(U\times U_\bullet, U_\bullet\times U) \arrow[d] \\
S \arrow[r, "(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)"] & U\times U
\end{tikzcd}$$

and

$$\begin{tikzcd}
\underline{\mathrm{CoCart}}_S(E_1,E_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_{U\times U}(U\times U_\bullet, U_\bullet\times U) \arrow[d] \\
S \arrow[r, "(\ulcorner p_1\urcorner, \ulcorner p_2\urcorner)"] & U\times U.
\end{tikzcd}$$

Proof. We already argued that the pair of p-fibrations q_1 and q_2 is classified by some map $(f_1,f_2)\colon B\to U\times U$. The given fiberwise homotopy equivalence u between q_1 and q_2 then provides a lift of the map $(f_1,f_2)\colon B\to U\times U$ lifts along the fibration $\overline{\pi}_{E_1,E_2}$, giving rise to a map $f\colon B\to\overline{\pi}_{E_1,E_2}$. It follows that there are pullback squares

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d, "q_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{E}_i \arrow[r] \arrow[d, "\overline{p}_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_i \arrow[d, "p_i"] \\
B \arrow[r, "f"] \arrow[rr, bend right, "(f_1,f_2)"'] & \overline{\pi}_{E_1,E_2} \arrow[r, "\overline{\pi}_{E_1,E_2}"] & U\times U,
\end{tikzcd}$$

and the fiberwise homotopy equivalence u is pulled back from the universal one u_{univ}. This finishes the proof. □

Using this remark, we may be more precise about the way that the category $\underline{\mathrm{CoCart}}_{U\times U}(U\times U_\bullet, U_\bullet\times U)$ 'classifies' cocartesian functors between two cocartesian fibrations.

Definition 7.3.6. Write $T := \underline{\mathrm{CoCart}}_{U\times U}(U \times U_\bullet, U_\bullet \times U)$ for short. Define cocartesian fibrations $\overline{\pi}_1 \colon \overline{U}_1 \to T$ and $\overline{\pi}_2 \colon \overline{U}_2 \to T$ via the following pullback squares:

$$\begin{tikzcd}
\overline{U}_1 \arrow[r] \arrow[d, "\overline{\pi}_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U \times U_\bullet \arrow[d, "\mathrm{id}\times\pi"] & \text{and} & \overline{U}_2 \arrow[r] \arrow[d, "\overline{\pi}_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & U_\bullet \times U \arrow[d, "\pi\times\mathrm{id}"] \\
T \arrow[r] & U \times U & & T \arrow[r] & U \times U.
\end{tikzcd}$$

Since the map $T \to U \times U$ lifts to a map $T \to \underline{\mathrm{CoCart}}_{U\times U}(U \times U_\bullet, U_\bullet \times U)$ (namely the identity map on T), it follows from Observation 7.3.4 that the cocartesian fibrations $\overline{\pi}_1$ and $\overline{\pi}_2$ come equipped with a cocartesian functor

$$\begin{tikzcd}
\overline{U}_1 \arrow[rr, "f_{\mathrm{univ}}"] \arrow[dr, "\overline{\pi}_1"'] & & \overline{U}_2 \arrow[dl, "\overline{\pi}_2"] \\
& T. &
\end{tikzcd}$$

Lemma 7.3.7. *Let $p_1 \colon E_1 \to S$ and $p_2 \colon E_2 \to S$ be two U-small cocartesian fibrations, and let $f \colon E_1 \to E_2$ be a cocartesian functor over S. Then there exists a functor $\lceil f \rceil \colon S \to T := \underline{\mathrm{CoCart}}_{U\times U}(U \times U_\bullet, U_\bullet \times U)$ such that f is the base change of f_{univ} along $\lceil f \rceil$:*

$$\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] & & \lceil f\rceil^*(\overline{U}_1) \arrow[rr, "\lceil f\rceil^*(f_{\mathrm{univ}})"] \arrow[dr, "\lceil f\rceil^*(\overline{\pi}_1)"'] & & S\lceil f\rceil^*(\overline{U}_2) \arrow[dl, "\lceil f\rceil^*(\overline{\pi}_2)"] \\
& S & \simeq & & & S. &
\end{tikzcd}$$

Proof. We saw in Observation 7.3.4 that the cocartesian functors between U-small cocartesian fibrations over a category S are classified by functors $S \to T = \underline{\mathrm{CoCart}}_{U\times U}(U \times U_\bullet, U_\bullet \times U)$. This process is functorial in S: given functors $\varphi \colon S \to S'$ and $\psi \colon S' \to T$, the cocartesian functor over S classified by the composite $\psi \circ \varphi \colon S \to T$ is obtained from the one over S' classified by ψ by pulling back along φ. Taking $S' = T$ now gives the claim. □

Our next goal will be to define what it means for a cocartesian fibration $\pi \colon U_\bullet \to U$ to be *directed univalent*. We first need an auxiliary construction. Recall that for a cocartesian fibration $W \to [1]$, the canonical morphism $v \colon 0 \to 1$ in $[1]$ determines a covariant transport functor $v_! \colon W_0 \to W_1$. We will now construct a version of covariant transport that works over a more general base category.

Construction 7.3.8. Let S be a synthetic category and let $p \colon W \to [1] \times S$ be a cocartesian fibration, and denote by $p_i \colon W_0 \to S$ for $i = 0, 1$ the base change of $p \colon W \to S$ along the inclusion $(i, \mathrm{id}) \colon S \to [1] \times S$. We will construct a cocartesian functor over $[1] \times S$ of the form

$$\begin{tikzcd}
{[1]} \times W_0 \arrow[rr, "l"] \arrow[dr, "\mathrm{id}_{[1]}\times p_0"'] & & W_1 \arrow[dl, "p"] \\
& {[1]} \times S. &
\end{tikzcd}$$

The functor l will be constructed as a dashed filler fitting in the following commutative diagram:

$$\begin{tikzcd} W_0 \arrow[r, "\iota_0"] \arrow[d, "{(0,\mathrm{id}_{W_0})}"'] & W \arrow[d, "p"] \\ {[1]\times W_0} \arrow[r, "\mathrm{id}_{[1]}\times p_0"'] \arrow[ur, dashed, "l"] & {[1]\times S.} \end{tikzcd}$$

Observe that such a choice of lift is equivalent to finding a filler for the following commutative diagram:

$$\begin{tikzcd} \{0\} \arrow[r, "\iota_0"] \arrow[d] & \mathrm{Fun}(W_0, W) \arrow[d, "p_*"] \\ {[1]} \arrow[r] \arrow[ur, dashed, "k"] & \mathrm{Fun}(W_0, [1]\times S), \end{tikzcd}$$

where the right vertical functor is given by postcomposition with p. In other words: we need to find a morphism in $\mathrm{Fun}(W_0, W)$ whose source is the inclusion $\iota_0\colon W_0 \to W$ and which lifts the morphism in $\mathrm{Fun}(W_0, [1]\times S)$ adjoint to $\mathrm{id}_{[1]}\times p_0\colon [1]\times W_0 \to [1]\times S$. Since p_* is a cocartesian fibration by Proposition 5.6.11, we may simply choose k to be a cocartesian lift which starts in ι_0.

Unwinding the proof of Proposition 5.6.11, we obtain an explicit description of l at the level of objects: given an object $(\varepsilon, w) \in [1]\times W_0$, write $s := \pi_0(w) \in S$ and let $u\colon 0 \to \varepsilon$ denote the morphism in $[1]$ obtained from the fact that $0 \in [1]$ is an initial object, see Lemma 5.5.14. Then we have

$$l(\varepsilon, w) = (u, \mathrm{id}_s)_!(w),$$

where the functor $(u, \mathrm{id}_s)_!\colon W_{(0,s)} \to W_{(\varepsilon,s)}$ denotes cocartesian transport along the morphism $(u, \mathrm{id}_s)\colon (0, s) \to (\varepsilon, s)$ in $[1]\times S$.

Lemma 7.3.9. *The functor* $l\colon [1]\times W_0 \to W$ *constructed in Construction 7.3.8 is a cocartesian functor over* $[1]\times S$.

Proof. Consider a morphism $\psi\colon (\varepsilon, s) \to (\varepsilon', s')$ in $[1]\times S$ and an object $(\varepsilon, w) \in \{\varepsilon\}\times W_s = ([1]\times W)_{(\varepsilon,s)}$. By Corollary 6.2.11, it will suffice to show that the canonical map

$$\overline{\psi}\colon \psi_!(l(\varepsilon, w)) \to l(\psi_!(\varepsilon, w))$$

is an isomorphism in W. Observe that we may factor ψ as a composite

$$(\varepsilon, s) \to (\varepsilon', s) \to (\varepsilon', s'),$$

and hence since the covariant transport functors compose, Corollary 5.6.9, it will suffice to show the claim in the two cases where either $\varepsilon = \varepsilon'$ or $s = s'$:

- Assume that $s = s'$ and $\psi = (v, \mathrm{id}_s)\colon (\varepsilon, s) \to (\varepsilon', s)$ for some morphism $v\colon \varepsilon \to \varepsilon'$ in $[1]$. Letting $u_\varepsilon\colon 0 \to \varepsilon$ and $u_{\varepsilon'}\colon 0 \to \varepsilon'$ denote the canonical maps in $[1]$, the map $\overline{\psi}$ then simplifies to the map

  $$(v, \mathrm{id}_s)_!(u_\varepsilon, \mathrm{id}_s)_!(w) \to (u_{\varepsilon'}, \mathrm{id}_s)(w).$$

  But since $u_{\varepsilon'}\colon 0 \to \varepsilon'$ is the composite of $u_\varepsilon\colon 0 \to \varepsilon$ and $v\colon \varepsilon \to \varepsilon'$, this map is an isomorphism by Corollary 5.6.9.

- Assume that $\varepsilon = \varepsilon'$ and that $\psi = (\mathrm{id}_\varepsilon, v)$ for some morphism $v\colon s \to s'$ in S. Since $(\varepsilon, w) = (u_\varepsilon, \mathrm{id}_s)_!(0, w)$, we may by the previous case reduce to $\varepsilon = 0$. The map $\overline{\psi}$ then simplifies to the map

  $$(\mathrm{id}_0, v)_!(u_0, \mathrm{id}_s)_!(0, w) \to (u_0, \mathrm{id}_s)_!(\mathrm{id}_0, v)_!(0, w),$$

  which is clearly the identity since $u_0\colon 0 \to 0$ is the identity and hence so is $(u_0, \mathrm{id}_s)_!$.

□

Definition 7.3.10 (Straightening)**.** Let $p\colon W \to [1] \times S$ be a cocartesian fibration, and denote by $p_i\colon W_0 \to S$ for $i = 0, 1$ the base change of $p\colon W \to S$ along the inclusion $(i, \mathrm{id})\colon S \to [1] \times S$. We define the *straightening* of p over $[1]$ to be the cocartesian functor over S of the form

$$\begin{tikzcd} W_0 \arrow[rr, "\mathrm{St}(p)"] \arrow[dr, "p_0"'] & & W_1 \arrow[dl, "p_1"] \\ & S, & \end{tikzcd}$$

obtained from the cocartesian functor $l\colon [1] \times W_0 \to W$ over $[1] \times S$ by forming the base change along $(1, \mathrm{id})\colon S \to [1] \times S$. Note that $\mathrm{St}(p)$ corresponds to a section

$$\sigma_p\colon S \to \underline{\mathrm{CoCart}}_S(W_0, W_1)$$

of the functor $\underline{\mathrm{CoCart}}_S(W_0, W_1) \to S$.

Remark 7.3.11. If $S' \to S$ is any functor and if $p\colon W \to [1] \times S$ is a cocartesian fibration, we may form the pullback

$$\begin{tikzcd} W' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "p"] \\ {[1] \times S'} \arrow[r] & {[1] \times S.} \end{tikzcd}$$

We obtain a pullback square

$$\begin{tikzcd} \underline{\mathrm{CoCart}}_S(W_0', W_1') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_S(W_0, W_1) \arrow[d] \\ {[1] \times S'} \arrow[r] & {[1] \times S.} \end{tikzcd}$$

There are now a priori two ways of constructing two ways of constructing a cocartesian functor $W_0' \to W_1'$ over S': we can either apply Definition 7.3.10 directly to p', or we first apply Definition 7.3.10 to p to get a cocartesian functor $W_0 \to W_1$ over S and then pull back along $S' \to S$. One can show that these two functors agree. In other words, the diagram

$$\begin{tikzcd}
\underline{\mathrm{CoCart}}_S(W_0', W_1') \arrow[r] & \underline{\mathrm{CoCart}}_S(W_0, W_1) \\
{[1] \times S'} \arrow[u, "\sigma_{p'}"] \arrow[r] & {[1] \times S} \arrow[u, "\sigma_p"']
\end{tikzcd}$$

is commutative.

The assignment of the straightening $f_p \colon W_0 \to W_1$ of a cocartesian fibration $p \colon W \to [1] \times S$ is *functorial in p* in a suitable sense. To make this precise, we will need to assume that p is a U-small cocartesian fibration, so that it fits in a pullback square of the form

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\
{[1] \times S} \arrow[r] & U.
\end{tikzcd}$$

But since functors $[1] \times S \to U$ correspond under (un)currying to functors $S \to \mathrm{Fun}([1], U)$, it will suffice to study the unstraightening in the *universal* case $S = \mathrm{Fun}([1], U)$. This is what we will do next.

Construction 7.3.12 (Universal straightening)**.** Let $\pi \colon U_\bullet \to U$ be a cocartesian fibration. We will construct a *straightening functor*

$$\begin{tikzcd}
\mathrm{Fun}([1], U) \arrow[rr, "\mathrm{St}_U"] \arrow[dr, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"'] & & \underline{\mathrm{CoCart}}_{U \times U}(U \times U_\bullet, U_\bullet \times U) \arrow[dl] \\
& U \times U. &
\end{tikzcd}$$

Consider the synthetic category $S := \mathrm{Fun}([1], U)$. This comes equipped with an evaluation functor $[1] \times S = [1] \times \mathrm{Fun}([1], U) \xrightarrow{\mathrm{ev}} U$, and we define the cocartesian fibration $p \colon W \to [1] \times S$ as the base change of π along ev:

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\
{[1] \times S} \arrow[r, "\mathrm{ev}"] & U.
\end{tikzcd}$$

The cocartesian fibrations $W_0 \to S$ and $W_1 \to S$, defined as before, are now obtained as base changes of π:

$$\begin{tikzcd}
W_i \arrow[r] \arrow[d, "p_i"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
\mathrm{Fun}([1], U) \arrow[r, "\mathrm{ev}_i"] & U.
\end{tikzcd}$$

It follows that we may alternatively write these as base changes of the universal pair of U-small cocartesian fibrations:

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d, "p_0"] \arrow[dr, phantom, "\lrcorner", very near start] & U \times U_\bullet \arrow[d, "\mathrm{id}_U \times \pi"] \\ \mathrm{Fun}([1],U) \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"] & U \times U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W_1 \arrow[r] \arrow[d, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & U \times U_\bullet \arrow[d, "\pi \times \mathrm{id}_U"] \\ \mathrm{Fun}([1],U) \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"] & U \times U. \end{tikzcd}$$

By Corollary 7.2.11, we thus obtain a pullback square

$$\begin{tikzcd} \underline{\mathrm{CoCart}}_{\mathrm{Fun}([1],U)}(W_0,W_1) \arrow[r] \arrow[d] & \underline{\mathrm{CoCart}}_{U \times U}(U \times U_\bullet, U_\bullet \times U) \arrow[d] \\ \mathrm{Fun}([1],U) \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \arrow[u, dashed, bend left, "\sigma_p"] \arrow[ur, dashed, "\mathrm{St}_U"] & U \times U. \end{tikzcd}$$

Now, Definition 7.3.10 provides a section σ_p of the left vertical functor, and this corresponds to the desired functor $\mathrm{St}_U \colon \mathrm{Fun}([1],U) \to \underline{\mathrm{CoCart}}_{U \times U}(U \times U_\bullet, U_\bullet \times U)$ over $U \times U$.

Definition 7.3.13 (Directed univalence)**.** A cocartesian fibration $\pi \colon U_\bullet \to U$ is called *directed univalent* if the induced functor $\mathrm{St}_U \colon \mathrm{Fun}([1],U) \to \underline{\mathrm{CoCart}}_{U \times U}(U \times U_\bullet, U_\bullet \times U)$ is an equivalence.

A *universe* is a synthetic category U equipped with a directed univalent cocartesian fibration $\pi \colon U_\bullet \to U$.

While there can a priori be many different universes, the collection of universes behaves like a partially ordered set:

Theorem 7.3.14. *Let U be a universe with directed univalent cocartesian fibration $\pi \colon U_\bullet \to U$, and let $i \colon V \to U$ be a functor. Then the following two conditions are equivalent:*

(1) The base change $V_\bullet := V \times_U U_\bullet \to V$ exhibits V as a universe;

(2) The functor i is fully faithful.

Proof. Consider the map $\pi' \colon V_\bullet \to V$ defined as the base change of π along i:

$$\begin{tikzcd} V_\bullet \arrow[r, "j"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\ V \arrow[r, "i"] & U. \end{tikzcd}$$

By taking the cartesian product of this square with a degenerate pullback square, we obtain two pullback squares

$$\begin{tikzcd} V \times V_\bullet \arrow[r, "i \times j"] \arrow[d, "\mathrm{id} \times \pi'"] \arrow[dr, phantom, "\lrcorner", very near start] & U \times U_\bullet \arrow[d, "\mathrm{id} \times \pi"] \\ V \times V \arrow[r, "i \times i"] & U \times U \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V_\bullet \times V \arrow[r, "j \times i"] \arrow[d, "\pi' \times \mathrm{id}"] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \times U \arrow[d, "\pi \times \mathrm{id}"] \\ V \times V \arrow[r, "i \times i"] & U \times U. \end{tikzcd}$$

As a consequence, we obtain from Corollary 7.2.11 a pullback square

$$\begin{tikzcd}
\underline{\mathrm{CoCart}}_{V\times V}(V\times V_\bullet, V_\bullet\times V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_{U\times U}(U\times U_\bullet, U_\bullet\times U) \arrow[d] \\
V\times V \arrow[r, "i\times i"] & U\times U.
\end{tikzcd}$$

Since the square

$$\begin{tikzcd}
{[1]\times\mathrm{Fun}([1],V)} \arrow[r, "\mathrm{ev}"] \arrow[d, "1\times i_*"'] & V \arrow[d, "i"] \\
{[1]\times\mathrm{Fun}([1],U)} \arrow[r, "\mathrm{ev}"] & U
\end{tikzcd}$$

commutes, we obtain pullback squares

$$\begin{tikzcd}
W_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U_\bullet \arrow[d, "\pi"] \\
{[1]\times\mathrm{Fun}([1],U)} \arrow[r, "1\times i_*"] & {[1]\times\mathrm{Fun}([1],U)} \arrow[r, "\mathrm{ev}"] & U.
\end{tikzcd}$$

Consider now the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],V) \arrow[r, "i_*"] \arrow[d, "\mathrm{St}_V"] & \mathrm{Fun}([1],U) \arrow[d, "\mathrm{St}_U"] \\
\underline{\mathrm{CoCart}}_{V\times V}(V\times V_\bullet, V_\bullet\times V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{CoCart}}_{U\times U}(U\times U_\bullet, U_\bullet\times U) \arrow[d] \\
V\times V \arrow[r, "i\times i"] & U\times U.
\end{tikzcd}$$

Since the bottom square is a pullback square, it follows from the pasting rule for pullback squares that i is fully faithful if and only if the top square is cartesian. Since the upper right vertical map St_U is an equivalence, this happens if and only if also the left vertical map St_V being an equivalence, i.e. if the cocartesian fibration $V_\bullet \to V$ is univalent. This finishes the proof. □

Lemma 7.3.15. *Let Γ be a groupoid and let U be a universe. Then the functor $\Gamma\times U\to\Gamma$ is a universe with respect to synthetic categories in context Γ.*

Proof. To do. □

7.3.1 Functoriality of straightening

The process of straightening associates to every cocartesian fibration over $[1]\times S$ a cocartesian functor over S. We will now argue that this assignment satisfies a weak form of functoriality.

Lemma 7.3.16. *Let $p_0 \colon W_0 \to S$ be a cocartesian fibration, and define $W := [1] \times W_0$ and $p := (\mathrm{id}_{[1]} \times p_0) \colon W \to [1] \times S$. Then there is a preferred equivalence $W_1 \simeq W_0$ over S and the straightening $\mathrm{St}(p) \colon W_0 \to W_0$ of p is isomorphic to the identity of W_0 over S.*

Proof. [To do, but should be clear from the definitions.] □

Lemma 7.3.17. *Consider a synthetic category S and a functor $p \colon [2] \times S \to \mathrm{Cat}$. For $0 \le i < j \le 2$, define the cocartesian fibration $p_{i,j} \colon W_{i,j} \to [1] \times S$ via the following pullback square:*

$$\begin{tikzcd}
W^{\{i,j\}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & W \arrow[d, "\pi"] \\
{[1] \times S} \arrow[r, "\{i,j\} \times \mathrm{id}_S"] & {[2] \times S.}
\end{tikzcd}$$

Then there is a natural isomorphism

$$\mathrm{St}(p_{0,2}) \cong \mathrm{St}(p_{1,2}) \circ \mathrm{St}(p_{0,1})$$

of cocartesian functors $W_0 \to W_2$ over S, where $p_i \colon W_i \to S$ is defined as the base change of p along the inclusion $(i, \mathrm{id}_S) \colon S \to [2] \times S$.

Proof. Add details to the proof.

Consider the functor $p = p_0 \colon [1] \times [1] \to [2]$ from [ref], satisfying

$$p(0,0) = p(0,1) = 0, \qquad p(1,0) = 1, \qquad p(1,1) = 2.$$

Define $\widetilde{W}$ and $W^{\{i,j\}}$ via the following two pullback squares:

$$\begin{tikzcd}
\widetilde{W} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & W \arrow[d, "\pi"] \\
{[1] \times [1] \times S} \arrow[r, "p \times \mathrm{id}_S"] & {[1] \times S}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
W^{\{i,j\}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & W \arrow[d, "\pi"] \\
{[1] \times S = \Delta^{\{i,j\}} \times S} \arrow[r] & {[2] \times S.}
\end{tikzcd}$$

The first diagram induces

$$\begin{tikzcd}
{[1] \times W_0} \arrow[rr] \arrow[dr] & & W^{\{1,2\}} \arrow[dl] \\
& {[1] \times S.} &
\end{tikzcd}$$

Similarly the second diagram induces

$$\begin{tikzcd}
W_i \arrow[rr, "\alpha_{i,j}"] \arrow[dr] & & W_j \arrow[dl] \\
& S. &
\end{tikzcd}$$

The fiber of the first map over 1 is precisely $\alpha_{0,2} \colon W_0 \to W_2$, which is the composition of $\alpha_{1,2}$ and $\alpha_{0,1}$. □

7.4 The universe of categories

Our next goal is to define the *universe of all (small) categories*, i.e. a synthetic category Cat whose objects classify (small) synthetic categories. We require that Cat is a universe, i.e. that it comes equipped with a directed univalent cocartesian fibration $\pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$. To ensure that Cat does indeed behave like a universe of categories, we will have to enforce that all the categorical constructions of synthetic categories we have introduced so far admit analogues internal to Cat.

Definition 7.4.1 (U-small fibers)**.** Let U be a universe and let $f \colon C \to D$ be a functor. We say that f *has U-small fibers* if for any U-small category T and any functor $T \to C$ the pullback $C \times_D T$ is U-small.

Definition 7.4.2 (Bicomplete universe)**.** We say that a universe U is *bicomplete* if the following conditions are satisfied:

(1) The composite of two U-small cocartesian fibrations is U-small;

(2) A cocartesian fibration is U-small if and only if it has U-small fibers;

(3) Every embedding $f \colon \Gamma \hookrightarrow \Gamma'$ between groupoids has U-small fibers;

(4) If C and D are U-small categories, then so is their join $C \star D$.

(5) A synthetic category C is U-small if and only if the groupoid $\mathrm{Map}([1], C)$ is U-small;

(6) Any retract of a U-small category is U-small;

(7) If C is U-small and $W \to \mathrm{Map}([1], C)$ is an embedding, then there exists a localization of C at W which is U-small;

(8) For any U-small cocartesian fibration $p \colon C \to D$ and a functor $E \to C$ with U-small fibers, also the functor $p_*(E) \to D$ has U-small fibers.

Axiom O (Directed Univalence Axiom)**.** For any cocartesian fibration $q \colon E \to B$, there exists a bicomplete universe U such that B is U-small and q has U-small fibers.

The closure properties of bicomplete universes automatically imply various other closure properties. In the next couple of results, U denotes a bicomplete universe.

Lemma 7.4.3. *The synthetic category $[n]$ is U-small for every natural number n.*

Proof. For $n = 0$, it suffices by (2) to show that the identity functor $* \to *$ has U-small fibers. But this is clear, since the pullback of the identity map along a map $T \to *$ is T. For $n \geq 1$, we have $[n] = [n-1] \star [0]$ and hence the claim follows inductively from (4). □

Lemma 7.4.4. *The product $p \times q \colon E \times F \to C \times D$ of two U-small cocartesian fibrations $p \colon E \to C$ and $q \colon F \to D$ is again U-small.*

Proof. We may factor $p \times q$ as a composite

$$E \times F \xrightarrow{p \times \mathrm{id}_F} C \times F \xrightarrow{\mathrm{id}_C \times q} C \times D.$$

The maps $p \times \mathrm{id}_F$ and $\mathrm{id}_C \times q$ are U-small, and hence so is their composite by (1). □

Lemma 7.4.5. *Let C be a U-small category and $p \colon E \to S$ a U-small cocartesian fibration. Then also $p_* \colon \mathrm{Fun}(C, E) \to \mathrm{Fun}(C, S)$ is a U-small cocartesian fibration.*

Proof. It is of the form $(p_C)_* p_C^*(p)$. □

Lemma 7.4.6. *If C and D are U-small categories, then also $\mathrm{Fun}(C, D)$ is U-small.*

Throughout the remainder of this section, we fix a bicomplete universe Cat and a universal cocartesian fibration

$$\pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \to \mathrm{Cat}.$$

We will say that a synthetic category C is *small* if it is Cat-small in the sense of Definition 7.3.1, i.e. if there exists an object $c \colon * \to \mathrm{Cat}$ and a pullback square

$$\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
* \arrow[r, "c"] & \mathrm{Cat}.
\end{tikzcd}$$

We will similarly speak of *small cocartesian fibrations*, etcetera.

Definition 7.4.7. Consider two small cocartesian fibrations $p_0 \colon E_0 \to S$ and $p_1 \colon E_1 \to S$:

$$\begin{tikzcd}
E_i \arrow[r, "\widetilde{F}_i"] \arrow[d, "p_i"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
S \arrow[r, "F_i"] & \mathrm{Cat}.
\end{tikzcd}$$

We define the *cocartesian mapping space over S* as

$$\mathrm{Map}_S^{\mathrm{CoCart}}(E_0, E_1) := (p_S)_* \left(\begin{array}{c} \underline{\mathrm{CoCart}}_S(E_0, E_1) \\ \downarrow \\ S \end{array} \right)^{\simeq}.$$

We similarly define

$$\mathrm{Map}_S(E_0,E_1) := (p_S)_* \left(\begin{array}{c} \underline{\mathrm{Fun}}_S(E_0,E_1) \\ \downarrow \\ S \end{array} \right)^{\simeq}.$$

Note that there is an embedding $\mathrm{Map}_S^{\mathrm{CoCart}}(E_0,E_1) \to \mathrm{Map}_S(E_0,E_1)$.

Lemma 7.4.8. *Given functors* $F_0, F_1 \colon S \to \mathrm{Cat}$*, the fiber of the functor*

$$\mathrm{Fun}(S, \mathrm{CoCart}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}\times\mathrm{Cat}_\bullet, \mathrm{Cat}_\bullet\times\mathrm{Cat})) \to \mathrm{Fun}(S,\mathrm{Cat})\times\mathrm{Fun}(S,\mathrm{Cat})$$

at the pair (F_0,F_1) *is* $\mathrm{Map}_S^{\mathrm{CoCart}}(E_0,E_1)$.

Proof. Consider a groupoid Γ. Then a map $\Gamma \to \mathrm{Map}_S^{\mathrm{CoCart}}(E_0,E_1)$ corresponds by definition to a cocartesian functor $f \colon \Gamma\times E_0 \to \Gamma\times E_1$ over $\Gamma\times S$.

On the other hand, a map from Γ to the fiber is given as a map $Y_0 \to Y_1$ over $\Gamma\times S$ together with a proof that it is a cocartesian functor over $\Gamma\times S$, where Y_0 and Y_1 are defined via the pullback squares

$$\begin{tikzcd} Y_i \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \Gamma\times S \arrow[r, "\mathrm{pr}"] & S \arrow[r, "F_0"] & \mathrm{Cat} \end{tikzcd}$$

But it is easy to see that Y_0 and Y_1 are equivalent to $\Gamma\times E_0$ and $\Gamma\times E_1$, respectively. □

Theorem 7.4.9. *For small cocartesian fibrations* $p_i \colon E_i \to S$ *classified by functors* $F_i \colon S \to \mathrm{Cat}$*, there is an equivalence of groupoids*

$$\mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Cat})}(F_0,F_1) \xrightarrow{\sim} \mathrm{Map}_S^{\mathrm{CoCart}}(E_0,E_1).$$

Proof. By definition, the hom groupoid $\mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Cat})}(F_0,F_1)$ is defined via the following pullback square:

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Fun}(S,\mathrm{Cat})}(F_0,F_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],\mathrm{Fun}(S,\mathrm{Cat})) \arrow[d] \\ * \arrow[r, "{(F_0,F_1)}"] & \mathrm{Fun}(S,\mathrm{Cat})\times\mathrm{Fun}(S,\mathrm{Cat}). \end{tikzcd}$$

By the previous lemma, it will thus suffice to show that the right vertical map is equivalent to the map in the lemma. But this follows directly from the following commutative diagram:

$$\begin{tikzcd} \mathrm{Fun}([1],\mathrm{Fun}(S,\mathrm{Cat})) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}(S,\mathrm{Fun}([1],\mathrm{Cat})) \arrow[r, "\mathrm{St}_*"] & \mathrm{Fun}(S,\mathrm{CoCart}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}\times\mathrm{Cat}_\bullet,\mathrm{Cat}_\bullet\times\mathrm{Cat})) \arrow[d] \\ \mathrm{Fun}(S,\mathrm{Cat})\times\mathrm{Fun}(S,\mathrm{Cat}) \arrow[rr, equal] & & \mathrm{Fun}(S,\mathrm{Cat})\times\mathrm{Fun}(S,\mathrm{Cat}), \end{tikzcd}$$

since the top two functors are equivalences. □

Remark 7.4.10. Given two cocartesian fibrations $p_i\colon X_i \to S$, every equivalence $X_0 \xrightarrow{\sim} X_1$ over S is automatically a cocartesian functor over S [ref], so the inclusion $\mathrm{Equiv}_S(X_0, X_1) \to \mathrm{Map}_S(X_0, X_1)$ factors through $\mathrm{Map}_S^{\mathrm{CoCart}}(X_0, X_1)$.

Consider now the following diagram:

$$\begin{tikzcd}
\mathrm{Iso}(\mathrm{Cat}) \arrow[r, dashed] \arrow[d] & \langle \mathrm{Equiv}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat} \times \mathrm{Cat}_\bullet) \rangle \arrow[d] \\
\mathrm{Fun}([1], \mathrm{Cat}) \arrow[r, "\mathrm{St}"] & \mathrm{CoCart}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat} \times \mathrm{Cat}_\bullet).
\end{tikzcd}$$

The dotted arrow is an equivalence because it is an equivalence of underlying subgroupoids between two full subcategories. It follows that the isofibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ is univalent in Voevodsky's sense.

Remark 7.4.11. Assume that an isofibration $p\colon X \to S$ is represented by two different functors $F, F'\colon S \to \mathrm{Cat}$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
S \arrow[r, "F"] & \mathrm{Cat}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
S \arrow[r, "F'"] & \mathrm{Cat}.
\end{tikzcd}$$

Then there is an equivalence $F \sim F'$. So the notion of smallness is really a property of an isofibration, there is not really data, at least up to homotopy.

Even better: three is a type of all such pullback squares and it is equivalent to $*$. As a result, we may define smallness up to equivalence.

7.5 Category theory internal to the universe

Last time we introduced the notion of a universe of categories, $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$. The goal of the next few lectures is to show that everything we have done so far can be internalized within the synthetic category Cat.

Cartesian products in Cat

If $p\colon X \to A$ and $q\colon Y \to B$ are small cocartesian fibrations, then also $p \times q$ is a small cocartesian fibration by Lemma 7.4.4. We thus obtain a functor $- \times -\colon \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat}$ fitting in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Cat}_\bullet \times \mathrm{Cat}_\bullet \arrow[r] \arrow[d, "\pi_{\mathrm{univ}} \times \pi_{\mathrm{univ}}"'] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Cat} \times \mathrm{Cat} \arrow[r, "\times"] & \mathrm{Cat}.
\end{tikzcd}$$

Functor categories

Now let A be a small type, i.e. sitting in a pullback square as follows:

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ * \arrow[r, "a"] & \mathrm{Cat}. \end{tikzcd}$$

Given any cocartesian fibration $p\colon X \to S$, we have a pullback square

$$\begin{tikzcd} \underline{\mathrm{Fun}}_S(A_S, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(A, X) \arrow[d, "p_*"] \\ S \arrow[r, "(p_A)^*"] & \mathrm{Fun}(A, S). \end{tikzcd}$$

We will also write $X^A := \underline{\mathrm{Fun}}_S(A_S, X)$ for short, and denote the map $X^A \to S$ by p^A. Recall that p_* is U-small by Lemma 7.4.5.

So:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] & \rightsquigarrow & X^A \arrow[r] \arrow[d, "p^A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ S \arrow[r, "F"] & \mathrm{Cat} & & S \arrow[r, "F^a"] & \mathrm{Cat}. \end{tikzcd}$$

For $p = \pi_{\mathrm{univ}}$, we get

$$\begin{tikzcd} \mathrm{Cat}_\bullet^A \arrow[r] \arrow[d, "\pi_{\mathrm{univ}}^A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\ \mathrm{Cat} \arrow[r, "(-)^a"] & \mathrm{Cat}. \end{tikzcd}$$

Observe that a map

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "p"'] & & X^A \arrow[dl, "p^A"] \\ & S & \end{tikzcd}$$

over S corresponds to a natural transformation $x \to x^a$ of funcors $S \to \mathrm{Cat}$. Applying this to $S = \mathrm{Cat}$ and $X = \mathrm{Cat}_\bullet$, we get a natural transformation $\mathrm{id}_{\mathrm{Cat}} \to (-)^a$ of functors $\mathrm{Cat} \to \mathrm{Cat}$.

7.6 The universe of groupoids

In the previous section, we constructed for every object $a \in \mathrm{Cat}$ a natural transformation $\mathrm{id}_{\mathrm{Cat}} \to (-)^a$ of functors $\mathrm{Cat} \to \mathrm{Cat}$. Taking a to be the walking morphism $[1]$, we in particular obtain a functor $\mathrm{Cat} \to \mathrm{Fun}([1], \mathrm{Cat})$ given on objects by sending $c \in \mathrm{Cat}$ to the functor $c \to c^{[1]}$. Since a synthetic category C is a groupoid if and only if the functor $C \to \mathrm{Fun}([1], C)$ is an equivalence, we are led to the following definition:

Definition 7.6.1 (The universe of groupoids)**.** We define the synthetic category Grpd via the pullback square

$$\begin{tikzcd}
\mathrm{Grpd} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(\mathrm{Cat})^{\simeq} \arrow[d, hook] \\
\mathrm{Cat} \arrow[r, "x \mapsto (x \to x^{[1]})"'] & \mathrm{Fun}([1], \mathrm{Cat}).
\end{tikzcd}$$

Since the right vertical map is fully faithful by [ref], we see that the resulting functor Grpd → Cat is fully faithful. We define the cocartesian fibration πuniv: $\mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ via the pullback square

$$\begin{tikzcd}
\mathrm{Grpd}_\bullet \arrow[r] \arrow[d, "\pi_{\mathrm{univ}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Grpd} \arrow[r] & \mathrm{Cat}.
\end{tikzcd}$$

Note that the cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ is directed univalent by Theorem 7.3.14, because it is classified by a fully faithful functor Grpd ↪ Cat.

Theorem 7.6.2. *The functor* $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ *is the universal small left fibration: given a small cocartesian fibration* $p\colon X \to S$ *classified by a functor* $F\colon S \to \mathrm{Cat}$, *we have that* p *is a left fibration if and only if* F *factors through the subcategory* Grpd *of* Cat.

Proof. Let $p\colon X \to S$ be a small cocartesian fibration classified by a functor F:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
S \arrow[r, "F"] & \mathrm{Cat}.
\end{tikzcd}$$

Since p is a cocartesian fibration, the directed evaluation map $\vec{\mathrm{ev}}_0^p$ admits a left adjoint section σ. We have to show that p is even a left fibration. It will suffice to show that every morphism $u\colon x_0 \to x_1$ in X is a p-cocartesian morphism. Let $v := p(u)\colon y_0 \to y_1$. Then u is p-cocartesian if and only if the canonical map $v_!(x_0) \to x_1$ is an isomorphism in X. Note that this map lives in the fiber over y_1, and hence we may assume without loss of generality that S is a groupoid.

For a groupoid S, consider the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "p"] & X^{[1]} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}([1], X) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
& S \arrow[r, "\simeq"] & \mathrm{Fun}([1], S) \arrow[r, "\simeq"] & S.
\end{tikzcd}$$

By directed univalence, the map $X \to X^{[1]}$ is an equivalence if and only if the map $F \to F^{[1]}$ is a natural isomorphism into Cat. But the latter precisely means that F factors through Grpd. □

Corollary 7.6.3. *A cocartesian fibration is a left fibration if and only if its fibers are groupoids.*

Proof. This follows from the previous proof. □

Lemma 7.6.4. *Let Γ be a groupoid. An isofibration $p\colon A \to \Gamma$ has small fibers if and only if for every small groupoid Γ' and any functor $\Gamma' \to \Gamma$ the pullback $A' := \Gamma' \times_\Gamma A$ is small.*

Proof. Let A be a type and consider an arbitrary functor $A \to \Gamma$. By one of the axioms of a bicomplete universe, there exists a fundamental groupoid $\Pi_\infty(A)$ of A which is again small. The map $A \to \Gamma$ will factor through $\Gamma' := \Pi_\infty(A)$ up to homotopy. **I**t follows [explain] that the pullback along $A \to \Gamma$ is equivalent to the fiber along the composite $A \to \Gamma' \to \Gamma$. But the latter is small by assumption, since Γ' is a small groupoid. □

This lemma allows us to think of the objects of Grpd as functors $\Gamma \to$ Grpd for a small groupoid Γ. For instance, given a cocartesian functor

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& A &
\end{tikzcd}$$

of cocartesian fibrations over A, with p and q small, then f is an equivalence if and only if the induced map $f_a\colon X_a \to Y_a$ for any map $a\colon \Gamma \to A$ from a *small* groupoid Γ is an equivalence.

7.7 Fiberwise maximal groupoids

Next, we would like to define fiberwise maximal groupoids for a cocartesian fibration $p\colon X \to A$. For every morphism $u\colon x_0 \to x_1$ in X with $v = p(u)$ in A, we get a commutative diagram

$$\begin{tikzcd}
& v_!(x_0) \arrow[dr, "\psi(u)"] & \\
x_0 \arrow[ur] \arrow[rr] & & x_1.
\end{tikzcd}$$

This defines a morphism

$$\psi\colon \operatorname{Map}([1], X) \to \operatorname{Map}([1], X)\colon u \mapsto \psi(u).$$

Observe that we have $p(\psi(u)) = \mathrm{id}_{p(x_0)}$. We now define a collection of maps in X via the following pullback square:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Iso}(X)^{\simeq} \arrow[d] \\
\mathrm{Map}([1], X) \arrow[r, "\psi"] & \mathrm{Map}([1], X).
\end{tikzcd}$$

Observe that a morphism is in M if and only if it is p-cocartesian. Since the p-cocartesian morphisms in X are closed under composition by Lemma 5.6.7, and thus we may make the following definition:

Definition 7.7.1. We define the *fiberwise maximal groupoid* of X over A as the subcategory

$$X^{\simeq/A} := \langle M \rangle_X \subseteq X.$$

Proposition 7.7.2. *The map $p' \colon X^{\simeq/A} \to A$ is a left fibration.*

Proof. By assumption, the functor $\vec{\mathrm{ev}}_0^p \colon \mathrm{Fun}([1], X) \to X \vec{\times}_p A$ admits a left adjoint section

$$\mathrm{lift}_0^p \colon X \vec{\times}_p A \to \mathrm{Fun}([1], X).$$

We claim that this restricts to a functor

$$X^{\simeq/A} \vec{\times}_{p'} A \to \mathrm{Fun}([1], X).$$

This is clear on objects, since $\mathrm{lift}_0^p(x, v) \colon x \to v_!(x)$ is p-cocartesian for all $(x, v) \in X \vec{\times}_p A$. It thus remains to prove the claim on morphisms. TO DO.

We deduce that p' is still a cocartesian fibration. To show that it is in fact a left fibration, it thus remains to show that its fibers are groupoids. But this is true by construction. □

Applying this result to $p = \pi_{\mathrm{univ}} \colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$, we in particular obtain a left fibration

$$\begin{tikzcd}
\mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \arrow[r] \arrow[dr] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
 & \mathrm{Cat}.
\end{tikzcd}$$

One observes that

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Cat}_\bullet^{\simeq/\mathrm{Cat}} \arrow[r] \arrow[d] & \mathrm{Grpd}_\bullet \arrow[d] \\
\mathrm{Cat} \arrow[r, "x \mapsto x^{\simeq}"'] & \mathrm{Grpd}.
\end{tikzcd}$$

Proposition 7.7.3. *Let $p\colon X \to A$ be a cocartesian fibration and let $f\colon B \to A$ be any functor. For any left fibration $q\colon Y \to B$, there is an equivalence*

$$\mathrm{Map}_B(Y, A \times_B X^{\simeq/A}) \xrightarrow{\sim} \mathrm{Map}_B^{\mathrm{CoCart}}(Y, B \times_A X).$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
(A \times_B X)^{\simeq/B} \arrow[r, "\simeq"] & A \times_B X^{\simeq/A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\simeq/A} \arrow[d] \\
 & B \arrow[r] & A.
\end{tikzcd}$$

Now use the universal property of the subcategory $\langle M \rangle_X$. ◻

Since the functor $\pi_{\mathrm{univ}}\colon \mathrm{Grpd}_\bullet \to \mathrm{Grpd}$ is a left fibration, it is in particular conservative. Hence, we get a pullback diagram

$$\begin{tikzcd}
\mathrm{Grpd}_\bullet^\simeq \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d] \\
\mathrm{Grpd}^\simeq \arrow[r] & \mathrm{Grpd} \arrow[r] & \mathrm{Cat}.
\end{tikzcd}$$

Theorem 7.7.4. *The functor $\mathrm{Grpd}_\bullet^\simeq \to \mathrm{Grpd}^\simeq$ is a univalent fibration, in the sense of [ref]. Therefore, groupoids form a semantic interpretation of univalent homotopy type theory.*

For the proof, we will need:

Lemma 7.7.5. *Let $X \to A$ and $Y \to A$ be left fibrations. Then the following two squares are pullbacks:*

$$\begin{tikzcd}
\underline{\mathrm{Equiv}}_{A^\simeq}(A^\simeq \times_A X, A^\simeq \times_A Y) \arrow[r] \arrow[d] & \underline{\mathrm{Equiv}}_A(X, Y) \arrow[d] \\
\underline{\mathrm{Fun}}_{A^\simeq}(A^\simeq \times_A X, A^\simeq \times_A Y) \arrow[r] \arrow[d] & \underline{\mathrm{Fun}}_A(X, Y) \arrow[d] \\
A^\simeq \arrow[r] & A.
\end{tikzcd}$$

Proof. The top square is always a pullback. The bottom square is a pullback square because equivalences between left fibrations can be detected fiberwise. ◻

Proof of the theorem. Consider the following commutative diagram: TO DO! ◻

7.8 Fiberwise localizations

Our next goal is to construct a functor $\pi_\infty\colon \mathrm{Cat} \to \mathrm{Grpd}$ which is given on objects by sending a synthetic category C to its fundamental groupoid Π_∞. The main ingredient will be to understand how to localize a cocartesian fibration fiberwise. We continue to fix a universe Cat of small categories, equipped with a universal cocartesian fibration $\pi_{\mathrm{univ}}\colon \mathrm{Cat}_\bullet \to \mathrm{Cat}$ with small fibers. Recall that we then defined the universal *left* fibration with small fibers as the pullback

$$\begin{tikzcd}
\mathrm{Grpd}_\bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\mathrm{Grpd} \arrow[r, hook, "i"] & \mathrm{Cat}.
\end{tikzcd}$$

Last time, we investigated the right adjoint to the inclusion i. This time, we would like to understand the *left* adjoint of i.

Theorem 7.8.1. *The inclusion functor* $i\colon \mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ *admits a left adjoint* $\pi_\infty\colon \mathrm{Cat} \to \mathrm{Grpd}$.

Proof. We will verify the conditions of Proposition 5.8.14. Let b be an object of Cat. We associate to it a synthetic category B defined via the following pullback square:

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\{b\} \arrow[r, "b"] & \mathrm{Cat}.
\end{tikzcd}$$

Let $\eta_B\colon B \to \Pi_\infty(B)$ be a functor exhibiting $\Pi_\infty(B)$ as a fundamental groupoid of B, i.e. as a localization of B at the collection of all of its morphisms. By directed univalence, there is an equivalence

$$\mathrm{Fun}([1], \mathrm{Cat}) \xrightarrow{\sim} \mathrm{CoCart}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}\times\mathrm{Cat}_\bullet, \mathrm{Cat}_\bullet\times\mathrm{Cat}).$$

Since the map $\eta_B\colon B \to \Pi_\infty(B)$ may be seen as a cocartesian functor between cocartesian fibrations over $*$, it corresponds to an object in the right-hand side, and hence there exists a morphism $\eta_b\colon b \to i\pi_\infty(b)$ corresponding to η_B.

We claim that this satisfies the assumptions of the proposition. Given an object a of Grpd, define A via the following pullback square:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{Grpd}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
\{a\} \arrow[r, "a"] & \mathrm{Grpd}.
\end{tikzcd}$$

The claim then follows by applying 2-out-of-3 to the following commutative square: [explain where this comes from]

$$\begin{tikzcd}
\mathrm{Grpd}(\pi_\infty(b), a) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Cat}(b, i(a)) \arrow[d, "\simeq"] \\
\mathrm{Map}(\Pi_\infty(B), A) \arrow[r, "\simeq"] & \mathrm{Map}(B, A).
\end{tikzcd}$$

This finishes the proof. □

So, we have thus obtained a functor $\pi_\infty \colon \mathrm{Cat} \to \mathrm{Grpd}$ which is left adjoint to $i \colon \mathrm{Grpd} \to \mathrm{Cat}$, together with a unit transformation $\mathrm{id}_{\mathrm{Cat}} \to i\pi_\infty$. As a result, we get for every synthetic category C an adjunction between the induced functors

$$\pi_\infty \colon \mathrm{Fun}(C, \mathrm{Cat}) \to \mathrm{Fun}(C, \mathrm{Grpd}) \qquad \text{and} \qquad i \colon \mathrm{Fun}(C, \mathrm{Grpd}) \hookrightarrow \mathrm{Fun}(C, \mathrm{Cat}).$$

The existence of the functor π_∞ allows us to form *fiberwise localizations* of (small) cocartesian fibrations. Consider a functor $F \colon C \to \mathrm{Cat}$ and let $p \colon X \to C$ be its associated cocartesian fibration:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
C \arrow[r, "F"] & \mathrm{Cat}.
\end{tikzcd}$$

We then define the cocartesian fibration $\Pi_\infty(X/C)$ via the following pullback diagram:

$$\begin{tikzcd}
\Pi_\infty(X/C) \arrow[rr] \arrow[d, "\Pi_\infty(p/C)"] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Grpd}_\bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d, "\pi_{\mathrm{univ}}"] \\
C \arrow[r, "F"] & \mathrm{Cat} \arrow[r, "\pi_\infty"] & \mathrm{Grpd} \arrow[r, hook, "i"] & \mathrm{Cat}.
\end{tikzcd}$$

It comes equipped with a canonical map from X over C:

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "p"'] & & \Pi_\infty(X/C) \arrow[dl, "\Pi_\infty(p/C)"] \\
& C. &
\end{tikzcd}$$

The map η_X can in fact be recovered as a pullback from the universal case. If we take $C = \mathrm{Cat}$ and $F = \mathrm{id}_{\mathrm{Cat}}$, we may apply the above construction to obtain a map

$$\begin{tikzcd}
\mathrm{Cat}_\bullet \arrow[rr, "\eta_{\mathrm{Cat}_\bullet}"] \arrow[dr, "\pi_{\mathrm{univ}}"'] & & \Pi_\infty(\mathrm{Cat}_\bullet/\mathrm{Cat}) \arrow[dl, "\Pi_\infty(\pi_{\mathrm{univ}}/\mathrm{Cat})"] \\
& \mathrm{Cat}. &
\end{tikzcd}$$

Then the map η_X is obtained from $\eta_{\mathrm{Cat}_\bullet}$ by forming the pullback along $F \colon C \to \mathrm{Cat}$. So in this sense, the formation of fiberwise localizations of cocartesian fibrations is completely compatible with pullback in C.

Comments on directed univalence

Let us write

$$\mathrm{cocart}_1 := \mathrm{cocart}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}\times\mathrm{Cat}_\bullet, \mathrm{Cat}_\bullet\times\mathrm{Cat}),$$

that is, cocart_1 is the synthetic category classifying cocartesian functors between cocartesian fibrations, cf. [ref]. It comes equipped with maps

$$s,t\colon\ \mathrm{cocart}_1 \xrightarrow{\sim} \mathrm{Fun}([1],\mathrm{Cat}) \to \mathrm{Cat}\times\mathrm{Cat}.$$

Definition 7.8.2. We define the synthetic category cocart_2 via the following pullback square:

$$\begin{tikzcd}
\mathrm{cocart}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{cocart}_1 \arrow[d, "s"] \\
\mathrm{cocart}_1 \arrow[r] & \mathrm{Cat}.
\end{tikzcd}$$

Note that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}([2],\mathrm{Cat}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}([1],\mathrm{Cat})\times_{\mathrm{Cat}}\mathrm{Fun}([1],\mathrm{Cat}) \arrow[d, "\simeq"] \\
\mathrm{cocart}_2 \arrow[r, "\simeq"] & \mathrm{cocart}_1\times_{\mathrm{Cat}}\mathrm{cocart}_1\,.
\end{tikzcd}$$

In other words: *commutative triangles in the universe* Cat *correspond to commutative triangles of functors in the sense of Definition 1.1.1.*

Similarly, using the pushout square

$$\begin{tikzcd}
{[1]} \arrow[r, "\delta^2_1"] \arrow[d, "\delta^2_1"'] & {[2]} \arrow[d] \\
{[2]} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & {[1]\times[1]},
\end{tikzcd}$$

we conclude that commutative squares in the universe correspond to commutative squares in the sense of Definition 1.1.1.

Fiberwise localization as a localization

Let $p\colon X \to A$ be a cocartesian fibration and let B be a synthetic category. We may consider the functor

$$\pi\colon\ \mathrm{Fun}_A(X, A\times B) \to A.$$

Given another functor $q\colon C \to A$, a map $f\colon C \to \mathrm{Fun}_A(X, A\times B)$ over A corresponds to a map $\varphi\colon C\times_A X \to C\times B$ over C:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[dr, "q"'] & \mathrm{Fun}_A(X,A\times B) \arrow[d, "\pi"] & & C\times_A X \arrow[r, "\varphi"] \arrow[dr] & C\times B \arrow[d, "\mathrm{pr}_1"] \\
& A & \Longleftrightarrow & & A.
\end{tikzcd}$$

Proposition 7.8.3. *The functor* $\pi\colon \operatorname{Fun}_A(X, A\times B)\to A$ *is a cartesian fibration with fiber over an object* a *of* A *given by* $\operatorname{Fun}(X_a, B)$.

Proof. Recall that for a map $\alpha\colon A'\to A$, we have a pullback square

$$\begin{tikzcd}
\operatorname{Fun}_{A'}(X', A'\times B) \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Fun}_A(X, A\times B) \arrow[d, "\pi"] \\
A' \arrow[r, "\alpha"] & A,
\end{tikzcd}$$

where we write $X' := A'\times_A X\to A'$. In particular, it follows directly that the fibers of π are given by $\operatorname{Fun}(X_a, B)$, and it remains to show that π is a cartesian fibration.

We start by showing that π is a locally cartesian fibration. Let $v\colon a\to b$ be a morphism in A. Let $F\colon X_b\to B$ be an object of the fiber of π over b. We need to show that there exists a locally π-cartesian fibration of v whose codomain is F.

Writing $v\colon [1]\to A$, consider the pullback Y defined as follows:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
{[1]} \arrow[r, "v"] & A.
\end{tikzcd}$$

It follows from the above observation that also the square

$$\begin{tikzcd}
\operatorname{Fun}_{[1]}(Y, [1]\times B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Fun}_A(X, A\times B) \arrow[d, "\pi"] \\
{[1]} \arrow[r, "v"] & A
\end{tikzcd}$$

is a pullback square. In this way, we have reduced to the case $A = [1]$ and where v is the universal morphism $v\colon 0\to 1$, i.e. we need to show that the map $\pi\colon \operatorname{Fun}_{[1]}(X, [1]\times B)\to [1]$ is a cartesian fibration, and it will suffice to find lifts of v with target given by some functor $F\colon X_1\to B$.

Consider the following lifting problem:

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{incl}"] \arrow[d, hook, "{(0,\mathrm{id}_{X_0})}"'] & X \arrow[d, "p"] \\
{[1]\times X_0} \arrow[r, "\mathrm{pr}_1"] \arrow[ur, dashed, "l"] & {[1].}
\end{tikzcd}$$

Since p is a cocartesian fibration, there exists a cocartesian lift $l\colon [1]\times X_0\to X$ over $[1]$, i.e. l is a cocartesian functor over $[1]$. Passing to fibers over $\{1\}$, we thus obtain a morphism

$v_! \colon X_0 \to X_1 = X_b$. We may now think of the commutative square

$$\begin{tikzcd} X_0 \arrow[r, "Fv_!"] \arrow[d, "v_!"'] & B \arrow[d, equal] \\ X_1 \arrow[r, "F"] & B \end{tikzcd}$$

as a commutative square in the universe, and thus as a morphism in $\mathrm{Fun}([1], \mathrm{Cat})$. Under this translation, it corresponds to a cocartesian functor between cocartesian fibrations over $[1]$ of the form

$$\begin{tikzcd} X \arrow[rr, "\tilde{F}"] \arrow[dr, "p"'] & & {[1] \times B} \arrow[dl, "\mathrm{pr}_1"] \\ & {[1].} & \end{tikzcd}$$

If we think of $F \colon X_1 \to B$ as contained in the fiber of $\mathrm{Fun}_{[1]}(X, [1] \times B)$ over 1 and of $F \circ v_! \colon X_0 \to B$ as contained in the fiber over 0, then this diagram determines a morphism $\tilde{u} \colon Fv_! \to F$ in $\mathrm{Fun}_{[1]}(X, [1] \times B)$ such that $\pi(\tilde{u}) = v$. To finish the proof, it will suffice to prove that $\tilde{u}$ is a π-cartesian morphism.

To do this, let us start by giving a very explicit description of those morphisms u in $\mathrm{Fun}_{[1]}(X, [1] \times B)$ satisfying $\pi(u) = v$. Notice that such a morphism is the same thing as a global section of π and thus corresponds by adjunction to a commutative diagram of the form

$$\begin{tikzcd} X \arrow[rr, "{(p,\varphi)}"] \arrow[dr, "p"'] & & {[1] \times B} \arrow[dl, "\mathrm{pr}_1"] \\ & {[1],} & \end{tikzcd}$$

or equivalently a map $\varphi \colon X \to B$. For $i = 0, 1$, we write $\varphi_i \colon X_i \to B$ fo the restriction of $(p, \varphi) \colon X \to [1] \times B$ to the fiber X_i over $\{i\} \hookrightarrow [1]$. Since p is a cocartesian fibration, the inclusion $i \colon X_1 \hookrightarrow X$ admits a left adjoint $\lambda \colon X \to X_1$, satisfying $\lambda i \cong \mathrm{id}_{X_1}$. For $x \in X_0$ we have $\lambda(x) = v_!(x)$. The unit $\eta \colon 1 \to i\lambda$ induces a map

$$\varphi\eta \colon \varphi \to \varphi i \lambda \cong \varphi_1 \lambda$$

such that its restriction to $X_0 \hookrightarrow X$ is the map $\tilde{\eta} \colon \varphi_0 \to \varphi_1 v_!$. In other words, we have a commutative diagram as follows:

$$\begin{tikzcd} X_0 \arrow[r, "\tilde{\eta}"] \arrow[d, "v_!"'] & \mathrm{Fun}([1], B) \arrow[d, "\mathrm{ev}_1"] \\ X_1 \arrow[r, "\varphi_1"] & B. \end{tikzcd}$$

Claim: One can recover φ (hence u) from just the data of $(\varphi_0, \varphi_1, \tilde{\eta}, \alpha)$, where α is the natural isomorphism $\mathrm{ev}_1 \circ \tilde{\eta} \cong \varphi_1 \circ v_!$ exhibiting the commutativity of the square above.

To prove this claim, let $q\colon W \to [1]$ be the cocartesian fibration corresponding to the functor

$$v_!\colon W_0 = \mathrm{Fun}([1], B) \xrightarrow{\mathrm{ev}_1} B = W_1.$$

The above square is then simply the same data as a functor over $[1]$ of the form

$$\begin{tikzcd}
X \arrow[rr, "\psi"] \arrow[dr, "p"'] & & W \arrow[dl, "q"] \\
& {[1].} &
\end{tikzcd}$$

(The map ψ corresponds to pairs $(\tilde{\eta}, \varphi_1)$ and the data of α corresponds to the commutativity of the triangle). [Warning: most likely I've misunderstood this part!]

We observe that q is a cartesian fibration as well. Indeed, for any object b in W with $q(b) = 1$, we may consider the morphism $\mathrm{id}_b \in \mathrm{Fun}([1], B)$ satisfying $\mathrm{ev}_1(\mathrm{id}_b) = b$. Since q is a cocartesian fibration, the isomorphism $\mathrm{ev}_1(\mathrm{id}_b) \cong b$ thus corresponds to a morphism $u\colon \mathrm{id}_b \to b$ in W lifting $v\colon 0 \to 1$. For any other morphism $u'\colon f \to b$ in W with $(f\colon x \to y)$ in $W_0^{\simeq} = \mathrm{Fun}([1], B)^{\simeq}$, we observe that $q(u') = v$, and hence there exists a q-cocartesian lift $f \to \eta$ of v and a morphism $\tilde{u}\colon \eta \to b$ in $B = W_1$. We get a commutative square

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "f"'] & b \arrow[d, equal] \\
y \arrow[r] & b,
\end{tikzcd}$$

or equivalently a map $\tilde{u}\colon f \to \mathrm{id}_b$ in $\mathrm{Fun}([1], B)$ together with a uniquely determined commutative triangle

$$\begin{tikzcd}
& \mathrm{id}_b \arrow[dr, "u"] & \\
f \arrow[ur, "\tilde{u}"] \arrow[rr, "u'"] & & b.
\end{tikzcd}$$

Exercise: show that u is a locally q-cartesian lift of v.

This implies that the inclusion $j\colon W_0 \hookrightarrow W$ admits a right adjoint ρ that is the identity on W_0 and sends $b \in W_1 = B$ to id_b. Consider the counit

$$\varepsilon\colon j\rho \to \mathrm{id}$$

that is the identity on W_0 and a map $\mathrm{id} \to b$ as above, for $b \in W_1 = B$.

Now, we may consider the following composite φ':

$$\varphi'\colon X \xrightarrow{\psi} W \xrightarrow{\rho} W_0 = \mathrm{Fun}([1], B) \xrightarrow{\mathrm{ev}_0} B.$$

We claim that $\varphi' \cong \varphi$. Note that φ' has some of the same constraints as φ: we have

$$\varphi'_0 = \varphi_0, \qquad \varphi'_1 = \varphi_1, \qquad \tilde{\eta}' = \tilde{\eta}\colon \varphi'_0 \to \varphi'_1 v_!.$$

We will finish the proof next time. □

New version of the result that Tashi presented:

Proposition 7.8.4. *Let $p\colon X \to A$ be a cocartesian fibration and let B be a synthetic category. Then the functor*

$$\pi\colon T := \underline{\mathrm{Fun}}_A(X, B \times A) \to A$$

is a cartesian fibration.

Proof. We will first show that π is locally cocartesian and then show that locally cartesian morphisms compose.

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([n],T) \arrow[r, "\pi_*"] \arrow[d] & A \vec{\times}_\pi T \arrow[r] \arrow[d] & T \arrow[d, "\pi"] \\
\mathrm{Fun}([1],A) \arrow[r, equal] & \mathrm{Fun}([1],A) \arrow[r, "\mathrm{ev}_1"] & A.
\end{tikzcd}$$

We want to show that the top left horizontal map has a right adjoint section $R\colon A \vec{\times}_\pi T \to \mathrm{Fun}([1],T)$. But given a morphism $u\colon a \to b$ in A, we may consider the pullback of this whole diagram along the inclusion $\{u\} \hookrightarrow \mathrm{Fun}([1],A)$. This gives the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],T)_u \arrow[r] \arrow[d] & T_b \arrow[r] \arrow[d] & T_b \arrow[d] \\
\{u\} \arrow[r, equal] & \{u\} \arrow[r] & \{b\}.
\end{tikzcd}$$

We claim that π is a *locally* cartesian fibration if and only if the top left horizontal map in this second diagram admits a right adjoint section $r\colon T_b \to \mathrm{Fun}([1],T)_u$. Indeed, the existence of r means that for every object $t \in T_b$ there exists a morphism $r(t)\colon t' \to t$ over u such that for every diagram

$$\begin{tikzcd}
s' \arrow[r] \arrow[d, dashed] & s \arrow[d] \\
t' \arrow[r, "r(t)"] & t
\end{tikzcd}$$

in T living over $u\colon a \to B$, there exists a dashed arrow filling the square. But this is equivalent to $r(t)$ being a locally cartesian lift of u.

Now, this was a general result that works for arbitrary $T \to A$, but for our specific T we compute what the map $\mathrm{Fun}([1],T)_u \to T_b$ is. In the source, we get $\mathrm{Fun}(X_u, B)$, where X_u

is defined via the following pullback diagram:

$$\begin{tikzcd}
X_u \arrow[rr] \arrow[dd] & & X \arrow[dd] & \\
& B\times[1] \arrow[rr, crossing over] \arrow[dl] & & B\times A \arrow[dl] \\
{[1]} \arrow[rr, "u"] & & A &
\end{tikzcd}$$

We also have $T_b \xrightarrow{\sim} \mathrm{Fun}(X_b,B)$. The map $\mathrm{Fun}(X_u,B) \to \mathrm{Fun}(X_b,B)$ is given by restriction along the inclusion $X_b \hookrightarrow X_u$. Our goal now is to find a right adjoint section of this functor. Recall that for an arbitrary right adjoint section $l \dashv s\colon C \leftrightarrows D$, the induced maps

$$s^*\colon \mathrm{Fun}(C,B) \leftrightarrows \mathrm{Fun}(D,B) \,:l^*$$

still form a right adjoint section (where now l^* is the section). So it will suffice to show that the inclusion $X_b \hookrightarrow X_u$ is the right adjoint section of some functor $X_u \to X_b$. This is a general fact about cocartesian fibrations. By definition, we have a pullback square as follows:

$$\begin{tikzcd}
X_b \arrow[r, hook] \arrow[d, "p_b"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_u \arrow[d, "p_u"] \\
\{1\} \arrow[r, hook] & {[1].}
\end{tikzcd}$$

Since $X \to A$ is a cocartesian fibration, so is the functor $X_u \to [1]$, and thus we have a left adjoint section

$$\mathrm{lift}_0^{p_u}\colon X_u \vec{\times}_{p_u} [1] \to \mathrm{Fun}([1],X_u)$$

of the directed evaluation map $\vec{\mathrm{ev}}_0^{p_u}\colon \mathrm{Fun}([1],X_u) \to X_u \vec{\times}_{p_u} [1]$. Consider now the following commutative diagram::

$$\begin{tikzcd}
X_b \arrow[r] \arrow[dr, hook] & X_u \vec{\times}_i X_b \arrow[r] \arrow[l, dashed, bend right] \arrow[d, shift left] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],X_u) \arrow[d, shift left, "\vec{\mathrm{ev}}_0^{p_u}"] \\
& X_u \arrow[r, hook] \arrow[u, dashed, shift left, "l"] & X_u \vec{\times}_{p_u} [1]. \arrow[u, dashed, shift left, "\mathrm{lift}_0^{p_u}"]
\end{tikzcd}$$

Since pullbacks of right reflectors have right reflectors, the left vertical map has again a left adjoint section. The top left horizontal map also admits a left adjoint, see [ref!]. It follows that the composite $X_b \hookrightarrow X_u$ admits a left adjoint $t\colon X_u \to X_b$. Since $X_b \hookrightarrow X_u$ is also fully faithful, it follows that it is a right adjoint section. This finishes the proof that $\pi\colon T \to A$ is a locally cartesian fibration.

Now, to show that the locally cartesian morphisms are closed under composition, we give an explicit description of the functor $t\colon X_u \to X_b$: on objects, it sends a pair (t,x) consisting of an object t of $[1]$ and an object x of $X_{u(t)}$ to the object $u(t \to 1)_!(x)$, given by the cocartesian transport of x along the morphism $u(t \to 1)$, where $t \to 1$ is the unique map in $[1]$ from t to 1.

To see that locally cartesian morphisms are closed under composition, we may equivalently show that the locally cartesian transport functors compose. The locally cartesian transport along $u\colon a \to b$ is given as the following composite:

$$u^*\colon \operatorname{Fun}(X_b, B) \xrightarrow{t^*} \operatorname{Fun}(X_u, B) \xrightarrow{\text{res}} \operatorname{Fun}(X_a, B).$$

The first functor sends $F\colon X_b \to B$ to the assignment

$$(t,x) \mapsto F(u(t \to 1)_!(x)).$$

Restricting to $t = 0$ thus shows that the composite sends F to the functor $x \mapsto F(u_!(x))$, which is the precomposition of F with $u_!\colon X_a \to X_b$.

Now, given two morphisms $u\colon a \to b$ and $v\colon b \to c$ in A, the composite

$$\operatorname{Fun}(X_c, B) \xrightarrow{v^*} \operatorname{Fun}(X_b, B) \xrightarrow{u^*} \operatorname{Fun}(X_a, B)$$

is given by precomposition with $v_! \circ u_!$, but since p is a cocartesian fibration this is $(v \circ u)_!\colon X_a \to X_c$. This finishes the proof. □

Remark 7.8.5. More generally, if $p\colon X \to A$ is merely *locally* cocartesian, $\pi\colon T \to A$ will be locally cartesian. Indeed, for a functor $A' \to A$ we have a pullback square

$$\begin{tikzcd}
\underline{\operatorname{Fun}}_{A'}(X \times_A A', B \times A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\operatorname{Fun}}_A(X, B \times A) \arrow[d] \\
A' \arrow[r] & A.
\end{tikzcd}$$

Taking $A' = [1]$ then shows that π being locally cartesian only depends on the pullbacks of $X \to A$ to $[1]$.

Theorem 7.8.6. *Let $p\colon X \to A$ be a cocartesian fibration. Then there exists a commutative triangle*

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "p"'] & \Pi_\infty(X/A) \arrow[d, "\Pi_\infty(p/A)"] \\
 & A
\end{tikzcd}$$

which the following properties:

(1) The functor $\Pi_\infty(p/A)$ is a left fibration;

(2) The functor η_X exhibits $\Pi_\infty(X/A)$ as the localization of X at the morphisms that are sent to identities in A;

(3) For any object a of A, the induced functor $X_a \to \Pi_\infty(X/A)_a$ induces an equivalence

$$\Pi_\infty(X_a) \xrightarrow{\sim} \Pi_\infty(X/A)_a.$$

(4) The formation of η_X and $\Pi_\infty(X/A)$ is stable under base change along any functor $A' \to A$.

Proof. We discussed how to construct such left fibrations $\Pi_\infty(p/A)$ and argued why they satisfy properties (1), (3) and (4). It thus remains to show that it also satisfies (2).

We have a fully faithful cartesian functor over A of the form

$$\begin{tikzcd}
\mathrm{Fun}_A(\Pi_\infty(X/A), A \times B) \arrow[rr] \arrow[dr] & & \mathrm{Fun}_A(X, A \times B) \arrow[dl] \\
& A, &
\end{tikzcd}$$

which is fiberwise given by the fully faithful functors

$$\mathrm{Fun}(\Pi_\infty(X_a), B) \hookrightarrow \mathrm{Fun}(X_a, B)$$

Since the two vertical maps are cartesian fibrations over A, it follows that $\mathrm{Fun}_A(\Pi_\infty(X/A), A \times B)$ is equivalent to the full subcategory of $\mathrm{Fun}_A(X, A \times B)$ spanned by those functors $\varphi\colon X \to B$ that invert all maps that are sent to identities in B. [To do: show that for cocartesian functors between cocartesian fibrations, we may detect fully faithfulness fiberwise, and we may also detect its essential image fiberwise.] Now, applying $(p_A)_*(-)$, we obtain a pullback diagram

$$\begin{tikzcd}
W \arrow[rr] \arrow[d] \arrow[drr, phantom, very near start, "\lrcorner"] & & \mathrm{Fun}([1], X) \arrow[d] \\
A \arrow[r, "\simeq"] & \mathrm{Iso}(A) \arrow[r, hook] & \mathrm{Fun}([1], A),
\end{tikzcd}$$

and we may summarize the above conclusion as describing an equivalence $\mathrm{Fun}(\Pi_\infty(X/A), B) \xrightarrow{\sim} \mathrm{Fun}^W(X, B)$. But this is precisely what we needed to show. □

Let us summarize what we have done. Essentially what we want to do is to write an arbitrary functor $f\colon X \to A$ as a composite of a cofinal functor $X \to Y$ followed by a left fibration $Y \to A$. The idea is to construct Y in two steps:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[dr, bend right, "f"'] & A \vec{\times}_f X \arrow[r, "\eta"] \arrow[d] & \Pi_\infty(A \vec{\times}_f X) =: Y \arrow[dl, bend left] \\
& A. &
\end{tikzcd}$$

The second top horizontal functor is a localization by the previous theorem, and hence it will be cofinal. The first top horizontal functor is a [left/right?] adjoint and hence also cofinal. Finally, the map $\Pi_\infty(A \vec{\times}_f X/A) \to A$ is a left fibration. This provides the desired factorization. These idea will be made precise in the next chapter.

8 Cofinality

Warning: this chapter is incomplete as it is currently being written!

8.1 Initial and terminal functors

Let C be a synthetic category and consider two synthetic categories over C given by functors $u\colon A \to C$ and $p\colon X \to C$. Recall from Definition 7.1.2 the groupoid $\mathrm{Map}_C(A, X)$ of functors $A \to X$ over C, defined via the pullback square

$$\begin{tikzcd} \mathrm{Map}_C(A,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(A,X) \arrow[d, "p_*"] \\ * \arrow[r, "u"] & \mathrm{Map}(A,C). \end{tikzcd}$$

Definition 8.1.1. A *covariant equivalence* (resp. a *contravariant equivalence*) over C is a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "u"'] & & B \arrow[dl, "v"] \\ & C & \end{tikzcd}$$

such that for any left (resp. right) fibration $p\colon X \to C$ the induced functor

$$- \circ f\colon \mathrm{Map}_C(B, X) \to \mathrm{Map}_C(A, X)$$

is an equivalence.

Definition 8.1.2 (Initial/terminal functors)**.** A functor $f\colon A \to B$ is called *initial* (or *limit-cofinal*) if for any synthetic category C and any functor $v\colon B \to C$, f is a covariant equivalence over C (with respect to the composite $A \xrightarrow{f} B \xrightarrow{v} C$). Dually, f is called *terminal* (or *colimit-cofinal*) if for any $v\colon B \to C$, f is a contravariant equivalence over C.

Lemma 8.1.3. *Consider a covariant (resp. contravariant) equivalence*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "u"'] & & B \arrow[dl, "v"] \\ & C & \end{tikzcd}$$

over C. Then for every functor $\varphi\colon C \to D$, the functor f is also a covariant (resp. contravariant) equivalence over D.

Proof. Let $X \to D$ be a left (resp. right) fibration. We obtain a commutative square as follows:

$$\begin{tikzcd}
\mathrm{Map}_D(B,X) \arrow[r, "-\circ f"] \arrow[d, "\simeq"'] & \mathrm{Map}_D(A,X) \arrow[d, "\simeq"] \\
\mathrm{Map}_C(B, C\times_D X) \arrow[r, "-\circ f"] & \mathrm{Map}_C(A, C\times_D X).
\end{tikzcd}$$

We need to show that the top map is an equivalence. But this follows from the fact that the bottom map is an equivalence, since the functor $C\times_D X \to C$ is again a left (resp. right) fibration. □

Proposition 8.1.4. *Let $f\colon A \to B$ be a functor. Then f is initial (resp. terminal) if and only if for every left (resp. right) fibration $X \to B$ the induced map*

$$-\circ f\colon \mathrm{Map}_B(B,X) \to \mathrm{Map}_B(A,X)$$

is an equivalence.

Proof. This is immediate from Lemma 8.1.3. □

We will now prove various basic properties of initial and terminal functors.

Proposition 8.1.5. *(1) Let $u\colon A \to B$ be an initial (resp. terminal) functor. Then for every synthetic category T, also the functor $u\times T\colon A\times T \to B\times T$ is initial (resp. terminal).*

(2) Consider two functors $u\colon A \to B$ and $v\colon B \to C$ and assume that u is initial (resp. terminal). Then v is initial (resp. terminal) if and only if vu is initial (resp. terminal).

Proof. For (1), consider a map $f\colon B\times T \to C$ and a left (resp. right) fibration $X \to Z$. We want to show that the induced functor

$$\mathrm{Map}_C(B\times T, X) \to -\circ(u\times T)\,\mathrm{Map}_C(A\times T, X)$$

is an equivalence. By uncurrying, the functor f corresponds to a functor $\tilde{f}\colon B \to \mathrm{Fun}(T,C)$, and the above functor is equivalent to the functor

$$\mathrm{Map}_{\mathrm{Fun}(T,C)}(B, \mathrm{Fun}(T,X)) \xrightarrow{-\circ u} \mathrm{Map}_{\mathrm{Fun}(T,C)}(A, \mathrm{Fun}(T,X)).$$

Since the functor $\mathrm{Fun}(T,X) \to \mathrm{Fun}(T,C)$ is still a left (resp. right) fibration, it follows from the assumption on u that this functor is an equivalence.

For (2), let $X \to C$ be a left (resp. right) fibration, and consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}_C(C,X) \arrow[rr, "v\circ -"] \arrow[dr, "vu\circ -"'] & & \mathrm{Map}_C(B,X) \arrow[dl, "u\circ -"] \\
& \mathrm{Map}_C(A,X). &
\end{tikzcd}$$

By assumption, the right diagonal map is an equivalence. It follows by 2-out-of-3 that the left diagonal map is an equivalence if and only if the top horizontal map is an equivalence. □

Proposition 8.1.6. *Let $f\colon A \to B$ be a localization, i.e. it factors as $A \to A[W^{-1}] \simeq B$ for some class of maps $W \hookrightarrow \mathrm{Map}([1],A)$. Then f is both initial and terminal.*

Proof. Recall that we denote by $\mathrm{Map}^W(A,T)$ the full subgroupoid of $\mathrm{Map}(A,T)$ on those maps that send all morphisms in W to equivalences in T. Consider a left (resp. right) fibration $p\colon X \to B$. By [ref], p is conservative, and hence for a functor $F\colon A \to X$ the functor F sends the morphisms in W to isomorphisms in X if and only if the composite $p \circ F\colon A \to B$ sends all morphisms in W to isomorphisms in B. In particular, we see that the right square in the following commutative diagram is a pullback square:

$$\begin{tikzcd}
\mathrm{Map}(B,X) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}^W(A,X) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(A,X) \arrow[d] \\
\mathrm{Map}(B,B) \arrow[r, "\simeq"] & \mathrm{Map}^W(A,B) \arrow[r, hook] & \mathrm{Map}(A,B).
\end{tikzcd}$$

The left square is also a pullback square since both horizontal functors are equivalences. By passing to fibers over the identity $\mathrm{id}_B\colon B \to B$, this shows that the induced functor

$$\mathrm{map}_B(B,X) \to \mathrm{Map}_B(A,X)$$

is an equivalence. This was what we needed to show. □

The following example of initial/terminal functors will generate a big class of other examples:

Example 8.1.7. The inclusion $\{1\} \hookrightarrow [1]$ is terminal. The inclusion $\{0\} \hookrightarrow [1]$ is initial.

Proof. Let us treat the case of $\{1\} \hookrightarrow [1]$, the other case is dual. Consider a map $[1] \to A$ and let $p\colon X \to A$ be a right fibration. By definition of what it means to be a right fibration, we have the following pullback square:

$$\begin{tikzcd}
\mathrm{Map}([1],X) \arrow[r] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Map}([1],X) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Map}(\{1\},X) \simeq X^{\simeq} \arrow[r] & \mathrm{Map}(\{1\},A) \simeq A^{\simeq}.
\end{tikzcd}$$

By passing to horizontal fibers over some morphism $f \in \mathrm{Map}([1], X)$, we see that the map

$$\mathrm{Map}_A([1], X) \to \mathrm{Map}_A(\{1\}, X)$$

is an equivalence. This was what we needed to show. □

Our next goal is to show that right adjoints are terminal and that left adjoints are initial. We start with the case for fully faithful functors.

Proposition 8.1.8. *Let $v\colon B \to A$ be a left reflector with right adjoint section $u\colon A \hookrightarrow B$. Then u is a terminal functor. Dually, a left adjoint section of v is an initial functor.*

Proof. We prove the first case. Consider a right fibration $p\colon X \to B$. The unit transformation $\eta\colon \mathrm{id} \to uv$ corresponds to a functor $\eta\colon [1] \times B \to B$. Consider now the following square:

$$\begin{tikzcd}
\mathrm{Map}_B([1] \times B, X) \arrow[r] \arrow[d] & \mathrm{Map}_B([1] \times A, X) \arrow[d] \\
\mathrm{Map}_B(\{1\} \times B, X) \arrow[r] & \mathrm{Map}_B(\{1\} \times A).
\end{tikzcd}$$

Since the inclusion $\{1\} \hookrightarrow [1]$ is terminal, so is $\{1\} \times A \to [1] \times A$, and hence both vertical functors are equivalences. It follows that the square is a pullback square, i.e. we have an equivalence

$$\mathrm{Map}_B([1] \times B, X) \xrightarrow{\sim} \mathrm{Map}_B([1] \times A, X) \times_{\mathrm{Map}_B(\{1\} \times A, X)} \mathrm{Map}_B(\{1\} \times B, X).$$

But the map $\mathrm{Map}_B(B, X) \to \mathrm{Map}_B(A, X)$ is a retract of this equivalence, hence it is an equivalence itself. This □

Proposition 8.1.9. *Any right adjoint is terminal. Any left adjoint is initial.*

Proof. By symmetry it suffices to do the right adjoint case, so let $f\colon A \to B$ be a right adjoint. We may factorize it as follows:

$$A \overset{i}{\hookrightarrow} A \vec{\times}_f B \xrightarrow{\pi} B,$$

where i is a fully faithful right adjoint and where π admits a left adjoint section. By Proposition 8.1.8, the functor i is terminal. It thus remains to show that π is terminal. Since we showed in Proposition 8.1.6 that localizations are terminal, this follows from the following lemma: □

Lemma 8.1.10. *Let $\pi\colon C \to D$ be a functor which admits a left (resp. right) adjoint section. Then π is a localization.*

Proof. Let $\lambda\colon D \to C$ be a left adjoint section of π, with natural isomorphism $\eta\colon \mathrm{id}_D \cong \pi\lambda$ and counit $\varepsilon\colon \lambda\pi \to \mathrm{id}_C$. For any synthetic category E, the induced functor

$$\pi^*\colon \mathrm{Fun}(D,E) \to \mathrm{Fun}(C,E)$$

is a left adjoint section of $\lambda^*\colon \mathrm{Fun}(C,E) \to \mathrm{Fun}(D,E)$, see Proposition 5.4.4. In particular, the functor π^* is fully faithful by [ref]. Let $W \hookrightarrow \mathrm{Map}([1],C)$ denote the collection of morphisms in C that are sent to isomorphisms in D under π:

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1],C) \arrow[d, "\pi_*"] \\
\mathrm{Iso}(D) \arrow[r, hook] & \mathrm{Map}([1],D).
\end{tikzcd}$$

Since the map $\pi(\varepsilon_x)\colon \pi(\lambda(\pi(x))) \to \pi(x)$ is an isomorphism by the triangle identity, it follows that the map $\varepsilon\colon \lambda(\pi(x)) \to x$ lies in W for any object x of C. In particular, for any functor $F\colon C \to E$ in $\mathrm{Fun}^W(C,E)$, the counit of the adjunction $\pi^* \dashv \lambda^*$ is a natural isomorphism: $\pi^*\lambda^*F \xrightarrow{\cong} F$. It follows that this adjunction restricts to an adjunction

$$\pi^*\colon \mathrm{Fun}(D,E) \leftrightarrows \mathrm{Fun}^W(C,X) : \lambda^*$$

in which both the unit and counit are an equivalence. We conclude that the induced functor

$$\pi^*\colon \mathrm{Fun}(D,E) \to \mathrm{Fun}^W(C,E)$$

is an equivalence, showing that π exhibits D as a localization of C at W as desired. □

[Remark: the following is a very important theorem: it is about the existence of colimits in the universe of groupoids, and about how to compute them.]

Theorem 8.1.11. *Any functor $f\colon C \to D$ admits a factorization of the form*

$$f\colon C \xrightarrow{i} C' \xrightarrow{f'} E \xrightarrow{p} D,$$

where

- *p is a left (resp. right) fibration;*
- *f' is a localization, and remains a localization after base change along an arbitrary functor $D' \to D$;*
- *i is a fully faithful left (resp. right) adjoint.*

Proof. We start by factoring f as

$$C \xrightarrow{i} C \vec{\times}_f D \xrightarrow{\pi} D,$$

where i is a fully faithful left (resp. right) adjoint and π is a cocartesian (resp. cartesian) fibration. We may now factor π as

$$C \vec{\times}_f D \xrightarrow{f'} \Pi_\infty(C \vec{\times}_f D/D) \xrightarrow{\Pi_\infty(\pi/D)} D.$$

We showed in [ref] that f' is a localization and that $\Pi_\infty(\pi/D)$ is a left fibration. Since this construction is compatible with base change along $D' \to D$ (see [ref]) we see that in fact any base change of f' along $D' \to D$ is still a localization. □

Corollary 8.1.12. *Any functor $f\colon C \to D$ admits a factorization of the form*

$$C \xrightarrow{j} X \xrightarrow{p} D,$$

where $j\colon C \to X$ is an initial (resp. terminal) functor, and $p\colon X \to D$ is a left (resp. right) fibration.

Remark 8.1.13. If C and D are small and f has small fibers, then all the functors in the above diagrams are again small. (For the factorization using $\Pi_\infty(-/D)$ we in fact only need small fibers, we don't even need source and target to be small.)

Theorem 8.1.14. *Let $f\colon C \to D$ be a functor which has small fibers.*

(1) Then the pullback functor

$$u^*\colon \mathrm{Fun}(D, \mathrm{Grpd}) \to \mathrm{Fun}(C, \mathrm{Grpd})$$

admits a left adjoint $u_!\colon \mathrm{Fun}(C, \mathrm{Grpd}) \to \mathrm{Fun}(D, \mathrm{Grpd})$.

(2) This left adjoint is given on objects as follows: for a functor $F\colon C \to \mathrm{Grpd}$, consider the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi"] \\ C \arrow[r, "F"] & \mathrm{Grpd}, \end{tikzcd}$$

and apply Corollary 8.1.12 to factor the composite $u \circ p\colon E \to D$ as a composite

$$E \xrightarrow{j} Y \xrightarrow{q} D,$$

where j is initial and q is a left fibration. Since u has small fibers, also q has small fibers, and thus there is a pullback square of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi"] \\ B \arrow[r, "u_!(F)"] & \mathrm{Grpd}. \end{tikzcd}$$

The map $(p, j)\colon E \to C \times_D Y$ then corresponds to a unit map $\eta_F\colon F \to u^* u_!(F)$. Then η_F exhibit $u_!(F)$ as a left adjoint object to F.

Proof. Consider a functor $G\colon D \to \mathrm{Grpd}$. We have to show that the canonical map

$$\mathrm{Hom}_{\mathrm{Fun}(D,\mathrm{Grpd})}(u_!(F), G) \to \mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{Grpd})}(F, u^*(G))$$

is an equivalence. Define the left fibration $r\colon Z \to D$ via the pullback square

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d, "\pi"] \\ D \arrow[r, "G"] & \mathrm{Grpd}. \end{tikzcd}$$

Using [ref], the above map then becomes equivalent to the map

$$\mathrm{Map}_D(Y, Z) \to \mathrm{Map}_C(E, C \times_D Z) \simeq \mathrm{Map}_D(E, Z)$$

induced by precomposition with $j\colon E \to Y$. But since j is initial and $r\colon Z \to D$ is a left fibration, this map is an equivalence, finishing the proof. □

Lemma 8.1.15. *Consider a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S, & \end{tikzcd}$$

where p and q are left (resp. right) fibrations. Then f is an equivalence if and only if it is a covariant (resp. contravariant) equivalence over S.

Proof. Left to the reader. □

Proposition 8.1.16. *Consider a commutative triangle*

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "p"'] & & D \arrow[dl, "q"] \\ & S, & \end{tikzcd}$$

where q is a left (resp. right) fibration. Then f is initial (resp. terminal) if and only if it is a covariant (resp. contravariant) equivalence over S.

Proof. It is clear that if f is initial, then it is a covariant equivalence over S. Conversely, assume that f is a covariant equivalence over S. We may factorize f as pi where i is initial and p is a left fibration. By the previous step we know that i is also a covariant equivalence over S, hence so is p by 2-out-of-3. It then follows from the previous lemma that p is an equivalence, hence $f \cong pi$ is initial. □

8.2 Proper and smooth functors

Definition 8.2.1 (Proper/smooth functors, Grothendieck)**.** A functor $p \colon E \to S$ is called *proper* (resp. *smooth*) if for any pullback diagram

$$\begin{tikzcd}
E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "p"] \\
S'' \arrow[r, "v'"] & S' \arrow[r, "v"] & S
\end{tikzcd}$$

in which v' is terminal (resp. initial), then so is u'.

Remark 8.2.2. Some remarks about this definition:

- We will see that a functor $u \colon A \to B$ is initial if and only if for every object b of B, the relative slice $A_{/b}$ is weakly contractible (meaning that its fundamental groupoid $\Pi_\infty(A_{/b})$ is contractible).
- We have also seen that a functor $u \colon A \to B$ is a left adjoint if and only if for any object b of B, the slice $A_{/b}$ admits a terminal object.
- More generally, every predicate on categories, we obtain a notion of "initiality", which has a corresponding notion of "smoothness".
- For the choice of left adjoints, we claim that this notion of smoothness corresponds to locally cocartesian fibrations.

Eventually, we will prove that for a synthetic category C with small limits and any functor $u \colon A \to B$ with A and B small, the restriction functor

$$u^* \colon \mathrm{Fun}(B,C) \to \mathrm{Fun}(A,C)$$

admits a right adjoint $u_* \colon \mathrm{Fun}(A,C) \to \mathrm{Fun}(B,C)$. Then u is proper if and only if for any C with small limits and any pullback square

$$\begin{tikzcd}
A' \arrow[r, "v"] \arrow[d, "u'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "u"] \\
B' \arrow[r, "w"] & B
\end{tikzcd}$$

the canonical map $w^*u_* \to u'_*v^*$ is a natural isomorphism, and in fact the same property holds for any base change of u.

Lemma 8.2.3. *The collection of proper (resp. smooth) functors is closed under composition and base change.*

Proof. It is clear that it is closed under base change. For compositions, let $p\colon X \to B$ and $q\colon B \to C$ be proper functor. For functors $C'' \to C' \to C$ with $C'' \to C'$ terminal, we consider

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
B'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "q"] \\
C'' \arrow[r] & C' \arrow[r] & C.
\end{tikzcd}$$

Since q is proper, also $B'' \to B'$ is terminal, but then by properness of p also $X'' \to X'$ is terminal. This finishes the proof. □

Theorem 8.2.4. *Let $p\colon E \to S$ be a functor, and assume that p satisfies the property that for any pullback diagram of the form*

$$\begin{tikzcd}
E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
S'' = \{\varepsilon\} \times S'' \arrow[r, hook, "v'"] & {[1]} \times S'' \arrow[r, "v"] & S
\end{tikzcd}$$

for $\varepsilon = 1$ (resp. $\varepsilon = 0$) the base change u' is terminal (resp. initial). Then p is proper (resp. smooth).

Proof. We prove the statement for proper functors; the statement for smooth functors is dual. Consider an arbitrary pullback diagram

$$\begin{tikzcd}
E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
S'' \arrow[r, "v'"] & S' \arrow[r, "v"] & S
\end{tikzcd}$$

in which v' is terminal. We need to show that also u' is terminal. To this end, we factor v' as

$$S'' = S''_0 \xrightarrow{j} S''_1 \xrightarrow{\lambda} B''_2 = \Pi_\infty(B''_1/B') \xrightarrow{q} B',$$

where j is a fully faithful right adjoint, λ is a localization and remains so after pullback, and q is a right fibration. Consider now the following commutative triangle:

$$\begin{tikzcd}
B'' \arrow[rr] \arrow[dr, "v'"'] & & B''_2 \arrow[dl, "q"] \\
& B'. &
\end{tikzcd}$$

The top map is a right adjoint and thus terminal. It follows that also q is terminal, and since it is also a right fibration it is an equivalence [ref]. Now consider the following pullback diagram:

$$\begin{tikzcd}
E'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E''_1 \arrow[r, "\lambda'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E''_2 \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
S'' \arrow[r, "j"] & S''_1 \arrow[r, "\lambda"] & S''_2 \arrow[r, "q", "\simeq"'] & S' \arrow[r, "v"] & S.
\end{tikzcd}$$

Since λ' is a localization, it is terminal by Proposition 8.1.6. It thus remains to show that the pullback of j along p is terminal. In other words: without loss of generality we may assume that v' is a fully faithful right adjoint. Let $w\colon S' \to S''$ be a left adjoint of v', so that we have maps $\varepsilon\colon wv' \cong \mathbb{1}_{S''}$ and $\eta\colon \mathbb{1}_{S'} \to v'w$. Now form the following pullback:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{p}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "p'"] \\
{[1] \times S'} \arrow[r, "\eta"] & S'.
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
\{1\} \times S'' \arrow[r, "v'"] \arrow[d, hook] & \{1\} \times S' \cong S' \arrow[d, hook] \arrow[dr, "w"] & \\
{[1] \times S''} \arrow[r, "\mathrm{id}_{[1]} \times v'"] \arrow[dr, "\mathrm{pr}_2"'] & {[1] \times S'} \arrow[dr, "\eta"] & S'' \arrow[d, "v'"] \\
& S'' \arrow[r, "v'"] & S'.
\end{tikzcd}$$

If we pull back this diagram along p', we obtain

$$\begin{tikzcd}
\{1\} \times X'' \arrow[r] \arrow[d, hook, "i"] & S' \times_{S''} X'' \arrow[d, "j"] & \\
{[1] \times X''} \arrow[r] & \tilde{X} \arrow[dr, "\tilde{\eta}"] & \\
& & X'.
\end{tikzcd}$$

The map i is clearly terminal, and j are terminal by the assumption since it is a pullback of the map $\{1\} \times S' \hookrightarrow [1] \times S'$:

$$\begin{tikzcd}
B' \times_{B''} X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "p'"] \\
\{1\} \times S' \arrow[r, hook] & {[1] \times S'} \arrow[r, "\eta"] & S'.
\end{tikzcd}$$

Now, to show that $u' \colon X'' \to X'$ is terminal, we have to show that or every right fibration $q \colon Z \to X'$ the induced map

$$\mathrm{Map}_{X'}(X', Z) \xrightarrow{(u')^*} \mathrm{Map}_{X'}(X'', Z)$$

is an equivalence. Observe that the above commutative square over X' induces a commutative square

$$\begin{tikzcd}
\mathrm{Map}_{X'}(\tilde{X}, Z) \arrow[r] \arrow[d, "j^*"'] & \mathrm{Map}_{X'}([1] \times X'', Z) \arrow[d, "i^*"] \\
\mathrm{Map}_{X'}(S' \times_{S''} X'') \arrow[r] & \mathrm{Map}_{X'}(\{1\} \times X'', Z)
\end{tikzcd}$$

where the two vertical maps are equivalences since i and j are terminal. It follows that we have an equivalence

$$\mathrm{Map}_{X'}(\tilde{X}, Z) \xrightarrow{\sim} \mathrm{Map}_{X'}(S' \times_{S''} X'', Z) \times_{\mathrm{Map}_{X'}(X'', Z)} \mathrm{Map}_{X'}([1] \times X'', Z).$$

Now we are done, since the map we wanted to show is an equivalence is a retract of this map. [Work this out!] □

Corollary 8.2.5. *Any cocartesian fibration is proper. Any cartesian fibration is smooth.*

Proof. Let $p \colon E \to S$ be a cocartesian fibration and consider a pullback square of the following form:

$$\begin{tikzcd}
E'' \arrow[r, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
S'' = \{\varepsilon\} \times S'' \arrow[r, hook, "v'"] & {[1] \times S''} \arrow[r, "v"] & S
\end{tikzcd}$$

Since p' is a cocartesian fibration and v' is a right adjoint, it follows from Proposition 5.8.15 that also u' is a right adjoint, and hence terminal by Proposition 8.1.9. The claim now follows from Theorem 8.2.4. □

Remark 8.2.6. The statement of the corollary in facts this holds true for *locally* (co)cartesian fibrations.

Corollary 8.2.7. *Any left fibration is proper. Any right fibration is smooth.*

Corollary 8.2.8. *Any projection of the form $C \times D \to C$ is both smooth and proper.*

8.3 Weak homotopy equivalences

We will now introduce the notion of a weak homotopy equivalence of synthetic categories.

Definition 8.3.1. A functor $f\colon A \to B$ is called a *weak homotopy equivalence* if it is a covariant equivalence over $*$.

Lemma 8.3.2. *For a functor $f\colon A \to B$, the following are equivalent:*

(1) The functor f is a weak homotopy equivalence;

(2) For every groupoid X, the induced map

$$\mathrm{Map}(B,X) \to \mathrm{Map}(A,X)$$

is an equivalence;

(3) The induced map $\Pi_\infty(f)\colon \Pi_\infty(A) \to \Pi_\infty(B)$ is an equivalence.

Proof. The equivalence between (1) and (2) is clear, since by Proposition 5.2.15 a functor $X \to *$ is a left fibration if and only if X is a groupoid. By Lemma 4.4.2, (2) is equivalent to the condition that f induces an equivalence

$$\mathrm{Map}(\Pi_\infty(B),X) \xrightarrow{\sim} \mathrm{Map}(\Pi_\infty(A),X)$$

for every groupoid X. This is equivalent to (3) by Proposition 2.3.4. □

Example 8.3.3. Any initial (or terminal) functor is a weak homotopy equivalence.

Lemma 8.3.4. *Consider a commutative triangle of the form*

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "u"'] & & X \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where B is initial and where p is a left fibration. Then, for any object b of B, the following two functors are weak homotopy equivalences:

$$A_{/b} \to X_{/b} \leftarrow X_b.$$

In fact, $A_{/b} \to X_{/b}$ is initial and $X_b \to X_{/b}$ is terminal.

Proof. Consider the following pullback diagram:

$$\begin{tikzcd}
X_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{/b} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
* \arrow[r, "{(b,\mathrm{id}_b)}"] & B_{/b} \arrow[r] & B.
\end{tikzcd}$$

Since p is a left fibration, it is proper, so the terminal functor $* \to B_{/b}$ pulls back to a terminal functor $X_b \to X_{/b}$. Consider now the following commutative diagram:

$$\begin{tikzcd}
A_{/b} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{/b} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_{/b} \arrow[d] \\
A \arrow[r, "i"] & X \arrow[r, "p"] & B.
\end{tikzcd}$$

Since the functor $B_{/b} \to B$ is a right fibration, it is in particular smooth, so the initial functor $i\colon A \to X$ pulls back to an initilal functor $A_{/b} \to X_{/b}$. ◻

Proposition 8.3.5. *Consider a commutative triangle*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "u"'] & & B \arrow[dl, "v"] \\
& C. &
\end{tikzcd}$$

Then f is a covariant equivalence if and only if for every object c of C the induced map

$$A_{/c} \to B_{/c}$$

is a weak homotopy equivalence.

Proof. The first step is to construct a commutative diagram of the form

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d] & & B \arrow[d] \\
A' \arrow[rr, "f'"] \arrow[dr, "u'"'] & & B' \arrow[dl, "v'"] \\
& C, &
\end{tikzcd}$$

where the two vertical maps are initial and the two diagonal maps u' and v' are left fibrations. In particular, the maps $A \to A'$ and $B \to B'$ are covariant equivalences, and we see that f is a covariant equivalence if and only if f' is a covariant equivalence. But since u' and v' are left fibrations, this is equivalent to saying that f is an equivalence. By [ref] this is in turn equivalent to the statement that the induced map $A'_c \to B'_c$ is an equivalence on all fibers. Again using the lemma, this translates into the statement that the induced map $A_{/c} \to B_{/c}$ is a weak equivalence. ◻

Corollary 8.3.6. *A functor $u\colon A \to B$ is initial if and only if for any object b of B the relative slice $A_{/b}$ is weakly contractible (meaning that $\Pi_\infty(A_{/b})$ is contractible.)*

Proof. The functor u is initial if and only if the diagram

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "u"'] & & B \arrow[dl, "\mathrm{id}_B"] \\
& B &
\end{tikzcd}$$

is a covariant equivalence. The statement now follows as the relative slice of the identity functor id_B is the point. □

Corollary 8.3.7 (Quillen's Theorem A)**.** *Let $f\colon A \to B$ be a functor such that the relative slice $A_{/b}$ is weakly contractible for any object b of B. Then f is a weak homotopy equivalence.*

Proof. This is immediate from the previous corollary. □

8.4 Proper and smooth base change

Let $f\colon A \to B$ be a functor between two small categories A and B. Recall from Theorem 8.1.14 that the restriction functor

$$f^*\colon \mathrm{Fun}(B, \mathrm{Grpd}) \to \mathrm{Fun}(A, \mathrm{Grpd})$$

admits a left adjoint $f_!$. In particular, taking $B = *$ shows that the synthetic category Grpd of small groupoids admits all small colimits.

Now consider a commutative square of small categories

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

and any functor $F\colon A \to \mathrm{Grpd}$. The goal of this section is to show that under suitable assumptions, the canonical Beck-Chevalley map

$$\mathrm{BC}_!\colon f'_! u^*(F) \to v^* f_!(F)$$

in $\mathrm{Fun}((B')^{\mathrm{op}}, \mathrm{Grpd})$ is an equivalence. This Beck-Chevalley map is defined as the composite

$$f'_! u^*(F) \xrightarrow{\eta} f'_! u^* f^* f_!(F) \simeq f'_! f'^* v^* f_!(F) \xrightarrow{\varepsilon} v^* f_!(F),$$

where $\eta\colon \mathrm{id}_A \to f^*f_!$ is the unit of the adjunction $f_! \dashv f^*$ and where $\varepsilon\colon f'_!f'^* \to \mathrm{id}_{B'}$ is the counit of the adjunction $f'_! \dashv f'^*$. Alternatively, it is the map adjoint to the map

$$u^*(F) \xrightarrow{\eta} u^*f^*f_!F \simeq f'^*v^*f_!(F).$$

Before continuing, it is useful to first give a more concrete description of the map $\mathrm{BC}_!$. Consider the following pullback square:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\
A \arrow[r, "F"] & \mathrm{Grpd},
\end{tikzcd}$$

where the right vertical map is the universal left fibration with small fibers. Choose a factorization of $f \circ p\colon X \to B$ as

$$X \xrightarrow{j} Y \xrightarrow{q} B,$$

where j is initial and q is a left fibration. In particular, q is given via a pullback square

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "p"'] & Y \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\
A \arrow[r, "f"] & B \arrow[r, dashed, "f_!(F)"] & \mathrm{Grpd}
\end{tikzcd}$$

for some functor $f_!(F)$. The commutative square on the left exhibits the unit $F \to f^*f_!F$. Pulling back along v and u, we obtain the commutative diagram

$$\begin{tikzcd}
A' \times_A X \arrow[r, "f' \times_f j"] \arrow[d, "u^*p"'] & B' \times_B Y \arrow[d, "v^*q"] \\
A' \arrow[r, "f'"] & B'.
\end{tikzcd}$$

We may factorize $f' \times_f j$ into

$$A' \times_A X \xrightarrow{j'} Y' \xrightarrow{r} B' \times_B Y,$$

where j' is initial and r is a left fibration. Then the composite

$$q' := v^*q \circ r\colon Y' \to B$$

is again a left fibration. All in all, we have constructed a commutative diagram as follows:

$$\begin{tikzcd}
X' \arrow[rr, "j'"] \arrow[dr] \arrow[dd, "u^*p"'] & & Y' \arrow[dr] \arrow[dd, "q'" near start] & \\
& X \arrow[rr, "j"] \arrow[dd, "p" near start] & & Y \arrow[dd, "q"] \\
A' \arrow[rr, "f'" near start] \arrow[dr, "u"'] & & B' \arrow[dr, "v"] & \\
& A \arrow[rr, "f"] & & B.
\end{tikzcd}$$

We have pullback squares

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\
B' \arrow[r, "f'_! u^*(F)"'] & \mathrm{Grpd}
\end{tikzcd}$$

and

$$\begin{tikzcd}
B' \times_B Y \arrow[r] \arrow[d, "v^*q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Grpd}_\bullet \arrow[d] \\
B' \arrow[r, "v"] \arrow[rr, bend right, "v^* f_!(F)"'] & B \arrow[r, "f_!(F)"] & \mathrm{Grpd}.
\end{tikzcd}$$

All in all, we see that the Beck-Chevalley map $\mathrm{BC}_! \colon f'_! u^*(F) \to v^* f_!(F)$ corresponds under directed univalence to the map $Y' \to B' \times_B Y$ over B'. We may now state the main theorem of this section:

Theorem 8.4.1 (Proper and smooth base change)**.** *Consider a pullback square of small synthetic categories*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B,
\end{tikzcd}$$

where either v is proper or f is smooth. Then for any functor $F \colon A \to \mathrm{Grpd}$, the Beck-Chevalley map $\mathrm{BC}_! \colon f'_! u^*(F) \to v^* f_!(F)$ is an equivalence.

Proof. We will prove this next time. □

9 Limits and colimits

This chapter still needs to be written.

10 Presentable categories and topoi

This chapter still needs to be written.

11 Stable categories

This chapter still needs to be written.

12 K-theory

This chapter still needs to be written.

Part II

Models for synthetic category theory

13 Introduction

In Part I of this book, we introduced synthetic category theory using the language of *naive category theory*. In our axiomatization, we simply took for granted that notions of synthetic categories, functors and natural isomorphisms exist, without specifying the semantic interpretation of synthetic categories. This provides a flexible formalism that may be applied to a wide range of situations, and the theorems proved using the language of naive category theory will carry over immediately to each of these situations.

The goal of Part II of this book is to provide a convenient formalism for constructing semantic interpretations of synthetic category theory, based on Joyal's formalism of *tribes*. The starting point is a given category $\mathcal{E}$, which remains fixed throughout all of Part II, and on which more and more axioms will be imposed. We want to think of objects of $\mathcal{E}$ as *synthetic categories* and of its morphisms as *functors*. The axioms that we will impose on $\mathcal{E}$ allow us to verify all the axioms of synthetic category theory introduced in Part I, and hence all the results we proved there will hold in $\mathcal{E}$ as well.

The main example for $\mathcal{E}$ will be Cat, the universe of (small) universes introduced in Section 7.4. Another example is Grpd, the universe of (small) groupoids (also known as the category of *spaces*, or ∞-*groupoids*, or *anima*). More generally, for any topos $\mathcal{B}$ the category $\mathrm{Cat}(\mathcal{B})$ of categories internal to $\mathcal{B}$ will be an example of a tribe satisfying the axioms.

In order to connect our theory to the classical approach to higher category theory, we will formulate our axioms on $\mathcal{E}$ in such a way that the category qCat of quasicategories also satisfies all the axioms of synthetic category theory. This in particular implies that all the results about synthetic categories proved in Part I must hold for quasicategories as well. Since qCat is a 1-category, the categorical pullbacks that exist in qCat cannot possibly model the pullback of synthetic categories, forcing us to introduce a notion of *homotopy pullback in* $\mathcal{E}$ instead. In order to make this possible, we require that $\mathcal{E}$ comes equipped with a designated class of morphisms called *isofibrations*, which we denote as $C \twoheadrightarrow D$. The isofibrations need to satisfy the following closure properties:

- Isofibrations are closed under composition and include all isomorphisms in $\mathcal{E}$;
- Base changes of isofibrations exist in $\mathcal{E}$ and are again isofibrations;

- There exists a terminal synthetic category $*$, and the unique map $C \to *$ is an isofibration for every synthetic category C.

In Joyal's terminology, the above properties are saying that $\mathcal{E}$ is a *clan*. The formalism of clans and its relation to dependent type theory is the topic of Chapter 14.

The notion of isofibration leads to a notion of *anodyne morphism*: those maps satisfying the left lifting property with respect to isofibrations. We demand that every functor factors as an anodyne morphism followed by an isofibration and that anodyne morphisms are closed under base change along isofibrations. In Joyal's terminology, this says that $\mathcal{E}$ is a *tribe*. These additional properties on $\mathcal{E}$ allow us to speak of *homotopies* between functors, which lead to the notion of *equivalence* of synthetic categories. The formalism of tribes and its relation to homotopy type theory is the topic of Chapter 15.

While the formalism of tribes suffices to formulate and verify all the *external* aspects of naive category theory, i.e. the axioms laid out in Section 1.1, it is unable to speak of the *internal* structure of synthetic categories. We will address this in Chapter 16 by introducing the notion of a *simplicial tribe*: a tribe $\mathcal{E}$ equipped with a functor $\Delta^{\bullet} \colon \Delta \to \mathcal{E}$ such that for every $[n] \in \Delta$ the functor $\Delta^n \times - \colon \mathcal{E} \to \mathcal{E}$ admits a right adjoint $\mathrm{Fun}(\Delta^n, -) \colon \mathcal{E} \to \mathcal{E}$. The object Δ^1 in particular provides the 'walking morphism' in $\mathcal{E}$ required by Axiom C. Furthermore, under some additional assumptions on $\mathcal{E}$ the functors $\mathrm{Fun}(\Delta^n, -)$ can be combined to give functors $\mathrm{Diag}(\Phi, -) \colon \mathcal{E} \to \mathcal{E}$ for every diagram shape Φ, satisfying all the requirements of diagram categories demanded in Section 1.3. If the objects of $\mathcal{E}$ further satisfy the Segal Axiom E and Rezk Axiom F, we say that $\mathcal{E}$ is a *simplicial type theory*, matching the terminology of Riehl and Shulman [RS17].

In Chapter 17, we explain how each of the axioms of synthetic category theory introduced in Part I may be formulated in terms of the tribe $\mathcal{E}$.

14 Clans and dependent type theory

In this chapter, we will give a detailed introduction to the formalism of *clans*, due to Joyal [Joy17]. This formalism provides a convenient way to axiomatize constructive dependent type theory using the language of category theory. All definitions and results in this chapter are due to Joyal.

14.1 Clans

We start with the notion of *clans*.

Definition 14.1.1 (Clan structure, Joyal [Joy17, Definition 1.1.1])**.** Let $\mathcal{E}$ be a category with a terminal object $*$. A *clan structure* on $\mathcal{E}$ is a class of maps $\mathcal{F}$, the elements of which are called *fibrations* and denoted by arrows of the form $\twoheadrightarrow$, satisfying the following conditions:

(1) Every isomorphism belongs to $\mathcal{F}$;

(2) The morphisms in $\mathcal{F}$ are closed under composition;

(3) The morphisms in $\mathcal{F}$ are closed under base change: for every pair of morphisms $p\colon X \twoheadrightarrow Y$ and $g\colon Y' \to Y$ such that p is in $\mathcal{F}$, there exists a pullback square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathcal{E}$, and p' is again in $\mathcal{F}$.

(4) For every object X in $\mathcal{E}$ the unique map $X \twoheadrightarrow *$ belongs to $\mathcal{F}$.

A *clan* is a category equipped with a clan structure.

Given a clan $\mathcal{E}$, we refer to its objects as *types*. The fibrations $p\colon A \twoheadrightarrow B$ should be thought of as *type families* $b : B \vdash A(b)$ type, where the type $A(b)$ is given as the *fiber* of the map p:

Definition 14.1.2 (Fiber)**.** For a fibration $p\colon A \twoheadrightarrow B$ and a map $b\colon * \to B$, we define the *fiber* $A(b)$ of p at b via the following pullback square:

$$\begin{tikzcd}
A(b) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, two heads, "p"] \\
* \arrow[r, "b"] & B.
\end{tikzcd}$$

Lemma 14.1.3. *Every clan $\mathcal{E}$ admits finite products. The cartesian product of two fibrations is again a fibration.*

Proof. As $\mathcal{E}$ admits a terminal object by assumption, it remains to show that it admits binary products. But a product $X \times Y$ of two types X and Y can be written as pullbacks of the form

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, two heads] \\
X \arrow[r] & *.
\end{tikzcd}$$

Since we assumed that the map $Y \twoheadrightarrow *$ is a fibration, this pullback exists in $\mathcal{E}$.

For the final claim, consider two fibrations $f\colon A \twoheadrightarrow B$ and $f'\colon A' \twoheadrightarrow B'$. Their product $f \times f'\colon A \times A' \to B \times B'$ may be written as the composite

$$A \times A' \xrightarrow{f \times 1_{A'}} B \times A' \xrightarrow{1_B \times f'} B \times B'.$$

Since $f \times 1_{A'}$ is a base change of f and $1_B \times f'$ is a base change of f', both of these morphisms are again fibrations, and thus so is their composite $f \times f'$. □

Remark 14.1.4. The previous lemma admits a converse: if $\mathcal{E}$ is a category which admits finite products, then it admits a *minimal* clan structure in which $\mathcal{F}$ is the class of those maps $p\colon X \to Y$ for which there exists a pullback square of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & F \arrow[d] \\
Y \arrow[r] & *.
\end{tikzcd}$$

(In other words, there is an isomorphism $X \cong Y \times F$ over Y for some object F in $\mathcal{E}$.)

There is a canonical notion of morphisms of clans:

Definition 14.1.5 (Morphisms of clans)**.** Given clans $\mathcal{E}$ and $\mathcal{E}'$, a functor $F\colon \mathcal{E} \to \mathcal{E}'$ is called a *morphism of clans* if it satisfies the following properties:

(1) The functor F preserves terminal objects;

(2) The functor F preserves fibrations: if $p\colon X \twoheadrightarrow Y$ is a morphism in $\mathcal{F}$, then $F(p)\colon F(X) \twoheadrightarrow F(Y)$ is a morphism in $\mathcal{F}'$;

(3) The functor F preserves base changes along fibrations: given a pullback square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

in $\mathcal{E}$ such that p is in $\mathcal{F}$, the square

$$\begin{tikzcd}
F(X') \arrow[r, "F(g')"] \arrow[d, two heads, "F(p')"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(X) \arrow[d, two heads, "F(p)"] \\
F(Y') \arrow[r, "F(g)"] & F(Y)
\end{tikzcd}$$

is a pullback square in $\mathcal{E}'$.

We say that F is an *equivalence of clans* if it is additionally an equivalence of categories.

Examples of clans

We will next discuss various examples of clans.

Example 14.1.6 (Sets)**.** The category Set admits the structure of a clan in which *all* maps are declared fibrations.

More generally, every category with finite limits admits the structure of a clan by declaring all maps to be fibrations.

Example 14.1.7 (Topological spaces)**.** Let Top denote the category of topological spaces with continuous maps as morphisms. Let $I = [0,1]$ denote the unit interval. For any space A, we denote the by $i_0\colon A \to A \times I$ the inclusion $a \mapsto (a,0)$. Recall that a continuous map $p\colon X \to Y$ is called a *Hurewicz fibration* if it satisfies the following lifting property: For any topological space A and any commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[d, "p"] \\
A \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & Y,
\end{tikzcd}$$

there exists a morphism $\tilde{h}\colon A \times I \to X$ satisfying $p\tilde{h} = h$ and $\tilde{h}i_0 = f$. Then the collection of Hurewicz fibrations defines a clan structure on the category Top.

Example 14.1.8 (Categories)**.** Let Cat_1 denote the 1-category of small 1-categories. A functor $p\colon C \to D$ is called an *isofibration* if for any isomorphism $v\colon d_0 \xrightarrow{\sim} d_1$ in D and any object c_0 in C with $p(c_0) = d_0$, there exists an isomorphism $u\colon a_0 \xrightarrow{\sim} a_1$ in C with $p(u) = v$. The category Cat_1 equipped with the class of isofibrations forms a clan.

Example 14.1.9 (Groupoids)**.** Similarly, the 1-category Grpd_1 of 1-groupoids becomes a clan when equipped with the class of isofibrations between 1-groupoids.

Example 14.1.10 (Kan complexes)**.** Recall that a map $f\colon X \to Y$ of simplicial sets is called a *Kan fibration* if it satisfies the right lifting property with respect to all horn inclusions $\Lambda^n_k \to \Delta^n$ for $0 \leq k \leq n$. We say that a simplicial set X is a *Kan complex* if the map $X \to *$ to the terminal simplicial set is a Kan fibration. Then the category Kan of Kan complexes becomes a clan when equipped with the class of Kan fibrations

Example 14.1.11 (Quasicategories)**.** Recall that a map $f\colon X \to Y$ of simplicial sets is called an *inner fibration* if it satisfying the right lifting property with respect to all *inner* horn inclusions $\Lambda^n_k \to \Delta^n$ for $0 < k < n$. We say that a simplicial set X is a *quasicategory* if the map $X \to *$ to the terminal simplicial set is an inner fibration. Inside a quasicategory one has a notion of invertible morphism, and we say that a map $f\colon X \to Y$ of quasicategories is an *isofibration* if it is an inner fibration and if every invertible morphism in Y lifts to an invertible morphism in X with given domain (resp. codomain).

The category qCat of quasicategories becomes a clan when equipped with the class of isofibrations.

Example 14.1.12 (Product clan)**.** If $\mathcal{E}$ and $\mathcal{E}'$ are two clans, then their product category $\mathcal{E} \times \mathcal{E}'$ is again a clan, in which a morphism (f, f') is defined to be a fibration if and only if f is a fibration in $\mathcal{E}$ and f' is a fibration in $\mathcal{E}'$.

14.2 Local clans: types in contexts

Given a type B in a clan $\mathcal{E}$, we may think of the fibrations $A \twoheadrightarrow B$ as type families

$$b : B \vdash A(b) \ \text{ type}.$$

The behavior of such type families closely parallels the behavior of individual types, and it will frequently be useful to talk about such type families as individual types:

Definition 14.2.1 (Types in contexts)**.** Given a type B in a clan $\mathcal{E}$, we say that a *type in context B* is a fibration $A \twoheadrightarrow B$. We denote by

$$\mathcal{E}(B) \subseteq \mathcal{E}_{/B}$$

the full subcategory spanned by the types in context B.

The claim that the behavior of types in context B parallels that of individual types is made precise by the following result:

Proposition 14.2.2. *Let B be a type in a clan $\mathcal{E}$. Consider the class of morphisms $\mathcal{F}_B$ in $\mathcal{E}(B)$ of the form*

$$\begin{tikzcd}
X \arrow[rr, two heads, "f"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\
& B, &
\end{tikzcd}$$

where the underlying morphism $f\colon X \twoheadrightarrow Y$ is a fibration. Then the class of morphisms $\mathcal{F}_B$ defines a clan structure on $\mathcal{E}(B)$.

Proof. The identity morphism of B is a terminal object of $\mathcal{E}(B)$. It is clear that every isomorphism belongs to $\mathcal{F}_B$ and that morphisms in $\mathcal{F}_B$ are stable under composition. To see that the morphisms in $\mathcal{F}_B$ are stable under base change, consider morphisms $p\colon X \twoheadrightarrow Y$ and $g\colon Y' \to Y$ over B such that p is in $\mathcal{F}_B$. We may form the pullback

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

in $\mathcal{E}$, and the map p' is in $\mathcal{F}$. Define a morphism $X' \twoheadrightarrow B$ as the composite

$$X' \xrightarrow{p'} Y' \xrightarrow{q'} B,$$

which is a composite of two fibrations and thus itself a fibration. We may thus regard the above diagram as a diagram in $\mathcal{E}(B)$, and we see that it is in fact a pullback square $\mathcal{E}(B)$. Finally, for every object $(X, p) \in \mathcal{E}(B)$, the unique map to the terminal object of $\mathcal{E}(B)$ is given by the diagram

$$\begin{tikzcd}
X \arrow[rr, two heads, "p"] \arrow[dr, two heads, "p"'] & & B \arrow[dl, equal] \\
& B &
\end{tikzcd}$$

and this is in $\mathcal{F}_B$. $\square$

Definition 14.2.3 (Local clan)**.** We refer to the clan $\mathcal{E}(B)$ as the *local clan* of $\mathcal{E}$ at B.

Remark 14.2.4. Let $F\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of clans. Then for every type $B \in \mathcal{E}$, the induced functor $F(B)\colon \mathcal{E}(B) \to \mathcal{E}(FB)$ on local clans is a morphism of clans; see Exercise 14.5.1.

Example 14.2.5 (Types in the empty context)**.** The terminal type $*$ is sometimes also referred to as the *empty context*. A type in the empty context is just a type: the functor $\mathcal{E} \to \mathcal{E}(*)$ given by $A \mapsto (A \twoheadrightarrow *)$ is an equivalence of clans.

Example 14.2.6 (Types in iterated contexts)**.** Consider a fibration $p\colon A \twoheadrightarrow B$, and regard it as an object (A, p) of the local clan $\mathcal{E}(B)$, i.e. as a type in context B. Then types in context (A, p) are the same as types in context A: the functor $\mathcal{E}(A) \to \mathcal{E}(B)(A, p)$ sending a fibration $q\colon E \twoheadrightarrow A$ in $\mathcal{E}$ to the fibration

$$\begin{tikzcd}
E \arrow[rr, two heads, "q"] \arrow[dr, two heads, "qp"'] & & A \arrow[dl, two heads, "p"] \\
& B &
\end{tikzcd}$$

in $\mathcal{E}(B)$ is an equivalence of clans.

The philosophy that a type A in context B is a collection of types $A(b)$ for every term b of B can be made concrete in the clan of sets:

Example 14.2.7. Consider the clan $\mathcal{E} = \text{Set}$ of sets from Example 14.1.6, in which every morphism is declared to be a fibration. For a set B, the local clan $\mathcal{E}(B)$ is simply the slice category $\text{Set}_{/B}$, whose objects are pairs (A, p) consisting of a set A and a function $p\colon A \to B$.

We claim that $\text{Set}_{/B}$ is equivalent to the clan Set^B whose objects are B-indexed collections of sets $(A_b)_{b\in B}$ and whose morphisms are B-indexed collections of functions; the clan structure on Set^B is provided by Example 14.1.6 since this category admits finite limits. A functor $\text{Set}_{/B} \to \text{Set}^B$ is given by sending a map $(p\colon A \to B)$ to the collection $(p^{-1}(b))_{b\in B}$ of all fibers of p. Conversely, a functor $\text{Set}^B \to \text{Set}_{/B}$ is given by sending a collection $(A_b)_{b\in B}$ to the disjoint union $\bigsqcup_{b\in B} A_b$ equipped with the map to B sending all elements of A_b to $b \in B$. We leave it to the reader to verify that these two functors are mutual inverses.

Given a morphism of types $f\colon A \to B$, every type family over B gives rise to a type family over A:

Construction 14.2.8 (Pullback functor)**.** Let $f\colon A \to B$ be a morphism in a clan $\mathcal{E}$. For every object $(X, p) \in \mathcal{E}(B)$, we may form the pullback $(A \times_B X, f^*(p))$, given by the following pullback diagram:

$$\begin{tikzcd}
A \times_B X \arrow[r, "p'"] \arrow[d, two heads, "f^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

Since the map $f^*(p)$ is again in $\mathcal{F}$ by assumption on $\mathcal{E}$, this determines a functor

$$\begin{aligned} f^*\colon \mathcal{E}(B) &\to \mathcal{E}(A) \\ (X, p) &\mapsto (A \times_B X, f^*(p)). \end{aligned}$$

We refer to f^* as the *pullback functor* associated to f.

Lemma 14.2.9. *Let $f\colon A \to B$ be a map in a clan $\mathcal{E}$. Then the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ is a morphism of clans.*

Proof. As the pullback of the identity map on B is the identity map on A, the functor f^* preserves terminal objects. It follows from the fact that fibrations are closed under base change that f^* preserves fibrations. Finally, the functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ preserves all pullbacks that exist in $\mathcal{E}(B)$, so in particular the base changes along fibrations. □

14.3 Dependent sums and products

In dependent type theory, two important type constructions one can perform on a type family $a : A \vdash E(a)$ are the *dependent sum* $\sum_{a:A} E(a)$ and the *dependent product* $\prod_{a:A} E(a)$. In this section, we will introduce the analogues of these constructions within the language of clans.

14.3.1 Dependent sums

We start with a discussion of dependent sums in a clan $\mathcal{E}$.

Definition 14.3.1 (Dependent sum)**.** Let $f\colon A \twoheadrightarrow B$ be a fibration in a clan $\mathcal{E}$. We define the *dependent sum functor*

$$f_!\colon \mathcal{E}(A) \to \mathcal{E}(B)$$

as the functor sending a fibration $p\colon E \twoheadrightarrow A$ to the composite $fp\colon E \twoheadrightarrow B$. This functor is sometimes also denoted by Σ_f in the literature. Note that the functor $f_!$ forms a left adjoint to the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$, see Exercise 14.5.4.

Notation 14.3.2. If $B = *$ is the terminal object, we will sometimes also denote the functor $f_!\colon \mathcal{E}(A) \to \mathcal{E}(*) = \mathcal{E}$ by

$$(E, p) \mapsto \sum_{a:A} E(a).$$

While this is just notation, it reinforces the philosophy that the total space E of a fibration $p\colon E \twoheadrightarrow A$ is the ‘sum’ of all the fibers $E(a)$ of p.

The word ‘sum’ may be understood as the type-theoretic interpretation of ‘disjoint union’:

Example 14.3.3 (Dependent sums in Set)**.** Let $\mathcal{E} = \mathrm{Set}$ be the clan of (small) sets. Recall from Example 14.2.7 that for a set B we may identify the local clan $\mathcal{E}(B) = \mathrm{Set}_{/B}$ with the clan Set^B of B-indexed collections of sets. Under this equivalence, the forgetful functor $\mathrm{Set}_{/B} \to \mathrm{Set}$ corresponds to the functor

$$\mathrm{Set}^B \to \mathrm{Set}\colon (A_b)_{b\in B} \mapsto \bigsqcup_{b\in B} A_b,$$

taking the disjoint union of a B-indexed collection of sets.

Dependent sums satisfy the following compatibility with base change:

Proposition 14.3.4 (Beck-Chevalley property for dependent sums)**.** *Consider a pullback square*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, two heads, "f'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, two heads, "f"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

in $\mathcal{E}$ such that f and f' are fibrations. Then there is a canonical natural isomorphism

$$f'_! u^* \xrightarrow{\cong} v^* f_!$$

of functors $\mathcal{E}(A) \to \mathcal{E}(B')$.

Proof. Consider a fibration $p\colon E \twoheadrightarrow A$, and consider the following pullback diagram:

$$\begin{tikzcd}
A' \times_A E \arrow[r, "f''"] \arrow[d, two heads, "u^*(p')"] \arrow[dr, phantom, very near start, "\lrcorner"] \arrow[dd, bend right=60, "f'_! u^*(p)"'] & E \arrow[d, two heads, "p"] \arrow[dd, bend left=60, "f_! p"] \\
A' \arrow[r, "u"] \arrow[d, two heads, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, two heads, "f"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

Since the outer diagram commutes, there is a canonical map $A' \times_A E \to B' \times_B E$ over B', or in other words a map $f'_! u^*(p) \to v^* f_!(p)$. Since the outer rectangle is also a pullback square by the pasting lemma of pullback squares, we see that this map is an isomorphism, finishing the proof. □

Dependent sum also satisfies the following 'projection formula':

Proposition 14.3.5 (Projection formula for dependent sums)**.** *Let $f\colon A \twoheadrightarrow B$ be a fibration. Then for every pair of fibrations $p\colon E \twoheadrightarrow A$ and $q\colon E' \twoheadrightarrow B$, the canonical map*

$$f_!(E \times_A f^*(E')) \to f_!(E) \times_B E'$$

is an isomorphism in $\mathcal{E}(B)$.

Proof. This map is the canonical map $E \times_A (A \times_B E') \xrightarrow{\cong} E \times_B E'$, which is indeed an isomorphism over B. □

14.3.2 Dependent products

We now discuss dependent products in a clan $\mathcal{E}$. In contrast to dependent sums, these do not automatically exist.

Definition 14.3.6 (Exponentiable fibrations)**.** Let $f\colon A \twoheadrightarrow B$ be a fibration in a clan $\mathcal{E}$. We say that f is *exponentiable* if for any fibration $p\colon E \twoheadrightarrow A$ there exists a fibration

$$f_*(E) \xrightarrow{f_*(p)} B$$

together with a map

$$\begin{tikzcd}
A \times_B f_*(E) \arrow[rr, "e(p)"] \arrow[dr, two heads, "f^* f_*(p)"'] & & E \arrow[dl, two heads, "p"] \\
& A &
\end{tikzcd}$$

in $\mathcal{E}(A)$ satisfying the following universal property: for any map $u\colon X \to B$, the composite

$$\mathrm{Hom}_{/B}(X, f_*(E)) \xrightarrow{f^*} \mathrm{Hom}_{/A}(A \times_B X, A \times_B f_*(E)) \xrightarrow{e(p)\circ -} \mathrm{Hom}_{/A}(A \times_B X, E)$$

is a bijection.

Remark 14.3.7. More explicitly, the condition of Definition 14.3.6 says that for every $u\colon X \to B$ and every commutative triangle

$$\begin{tikzcd}
A \times_B X \arrow[rr, "\psi"] \arrow[dr, "\mathrm{pr}_A"'] & & E \arrow[dl, two heads, "p"] \\
& A &
\end{tikzcd}$$

in $\mathcal{E}$, there exists a unique commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "u"'] & & f_*(E) \arrow[dl, two heads, "f_*(p)"] \\
& B &
\end{tikzcd}$$

such that the map ψ coincides with the composite

$$A \times_B X \xrightarrow{1_A \times_B \varphi} A \times_B f_*(E) \xrightarrow{e(p)} E.$$

Example 14.3.8 (Exercise 14.5.5)**.** Every isofibration between groupoids is exponentiable in the clan of groupoids.

Corollary 14.3.9. *Let $f\colon A \twoheadrightarrow B$ be an exponentiable fibration. Then the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ admits a right adjoint*

$$f_*\colon \mathcal{E}(A) \to \mathcal{E}(B)$$

given objectwise by $f_*(E,p) = (f_*(E), f_*(p))$.

Proof. This is immediate by taking the map $u\colon X \to B$ to be a fibration in Definition 14.3.6. □

Definition 14.3.10 (Dependent product)**.** For an exponentiable fibration $f\colon A \twoheadrightarrow B$, we refer to the functor $f_*\colon \mathcal{E}(A) \to \mathcal{E}(B)$ as the *dependent product along f*. This functor is sometimes also denoted by Π_f in the literature.

Notation 14.3.11. Let $B = *$ be the terminal object. In analogy with Notation 14.3.2, we will sometimes denote the functor $f_*\colon \mathcal{E}(A) \to \mathcal{E}$ by

$$(E,p) \mapsto \prod_{a:A} E(a).$$

It enforces the idea that the dependent product of a fibration p is the product of all its fibers $\mathcal{E}(a)$.

The following example explains the terminology 'dependent product' for the functor f_*:

Example 14.3.12 (Dependent products in Set)**.** Let $\mathcal{E} = \mathrm{Set}$ be the category of (small) sets, in which all morphisms are fibrations. We claim that every map of sets $f\colon X \to Y$ is exponentiable. First consider the case $Y = *$, in which case we have to determine the right adjoint to the functor $f^*\colon \mathcal{E} \to \mathcal{E}(X)$. As explained in Example 14.2.7, the category $\mathcal{E}(X) = \mathrm{Set}_{/X}$ is equivalent to Set^X via the assignment

$$\mathrm{Set}_{/X} \xrightarrow{\simeq} \mathrm{Set}^X\colon (p\colon E \to X) \mapsto (p^{-1}(x))_{x\in X}.$$

Under this equivalence, the functor $\mathrm{Set} \to \mathrm{Set}_{/X}$ sending Y to $\mathrm{pr}_X\colon Y \times X \to X$ corresponds to the diagonal functor $\mathrm{Set} \to \mathrm{Set}^X\colon Y \mapsto (Y)_{x\in X}$. It follows that its right adjoint is given by taking the X-indexed product:

$$\mathrm{Set}_{/X} \to \mathrm{Set}\colon (p\colon E \to X) \mapsto \prod_{x\in X} p^{-1}(x).$$

Note that an element of the set $\prod_{x\in X} p^{-1}(x)$ corresponds to a section $s\colon X \to E$ of p.

More generally, if $f\colon X \to Y$ is an arbitrary map in set, then the functor

$$f_*\colon \mathrm{Set}_{/X} \to \mathrm{Set}_{/Y}$$

sends a map $p\colon E \to X$ to the map $q\colon F \to Y$, where F is the set of pairs (y,σ) with $y \in Y$ and $\sigma\colon f^{-1}(y) \to E$ is a map satisfying $p(\sigma(x)) = x$ for all $x \in X$. Under the identifications $\mathrm{Set}_{/X} \simeq \mathrm{Set}^X$ and $\mathrm{Set}_{/Y} \simeq \mathrm{Set}^Y$, this is given by

$$(E_x)_{x\in X} \mapsto \left(\prod_{x\in f^{-1}(y)} E_x\right)_{y\in Y}.$$

Proposition 14.3.13 (Beck-Chevalley property of dependent products)**.** *Consider a pullback square*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] & B
\end{tikzcd}$$

in $\mathcal{E}$ in which f and f' are exponentiable fibrations. Then there is a canonical natural isomorphism

$$\mathrm{BC}_*\colon v^*f_* \xrightarrow{\cong} f'_*u^*$$

of functors $\mathcal{E}(A) \to \mathcal{E}(B')$.

Proof. For every pair of fibrations $p\colon X \twoheadrightarrow B'$ and $q\colon E \twoheadrightarrow A$, we have bijections

$$\mathrm{Hom}_{\mathcal{E}(B')}((X,p), v^*f_*E) \cong \mathrm{Hom}_{\mathcal{E}_{/B}}((X, v\circ p), f_*E) \cong \mathrm{Hom}_{\mathcal{E}_{/A}}(f^*(X, v\circ p), E)$$

and

$$\mathrm{Hom}_{\mathcal{E}(B')}((X,p), f'_*u^*E) \cong \mathrm{Hom}_{\mathcal{E}_{/A'}}((f'^*X, f'^*p), u^*E) \cong \mathrm{Hom}_{\mathcal{E}_{/A}}((f'^*X, u\circ f'^*(p)), E),$$

and thus it will suffice to show that the pullback of $p\colon X \to B'$ along $f'\colon A' \to B'$ agrees with the pullback of $v\circ p\colon X \to B$ along $f\colon A \to B$. But this follows just like in Proposition 14.3.4 from the pasting law of pullback squares:

$$\begin{tikzcd}
f'^*(X) \arrow[r, two heads, "f'^*(p)"] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & A' \arrow[r, "u"] \arrow[d, two heads, "f'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, two heads, "f"] \\
X \arrow[r, two heads, "p"] & B' \arrow[r, "v"] & B.
\end{tikzcd}$$

□

Corollary 14.3.14. *Let $u\colon A \twoheadrightarrow B$ be an exponentiable fibration which is also a monomorphism. Then the dependent product functor $u_*\colon \mathcal{E}(A) \to \mathcal{E}(B)$ is fully faithful.*

Proof. Since u is a monomorphism we have a pullback square of the form

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "u"] \\
A \arrow[r, "u"] & B.
\end{tikzcd}$$

It follows from Proposition 14.3.13 that the counit map $u^*u_* \to \mathrm{id}$ of the adjunction is a natural isomorphism, or equivalently that u_* is fully faithful. □

Lemma 14.3.15. *Consider a type $C \in \mathcal{E}$, and let $f\colon (A,p) \twoheadrightarrow (B,q)$ be a fibration in $\mathcal{E}(C)$. Assume that the underlying morphism of f in $\mathcal{E}$ is exponentiable in $\mathcal{E}$.*

(1) The morphism f is exponentiable in $\mathcal{E}(C)$.

(2) Under the canonical equivalences $\mathcal{E}(C)(A,p) \simeq \mathcal{E}(A)$ and $\mathcal{E}(C)(B,q) \simeq \mathcal{E}(B)$ from Example 14.2.6, the dependent product functor

$$f_*\colon \mathcal{E}(C)(A,p) \to \mathcal{E}(C)(B,p)$$

in $\mathcal{E}(C)$ corresponds to the dependent product functor $f_*\colon \mathcal{E}(A) \to \mathcal{E}(B)$ in $\mathcal{E}$.

Proof. This is left to the reader, see Exercise 14.5.6. □

14.3.3 Universally exponentiable fibrations

We have seen that for an exponentiable fibration $f\colon A \twoheadrightarrow B$, the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ admits a right adjoint. Furthermore, if f' is a base change of f which is also exponentiable, then the right adjoints f_* and f'_* are suitably compatible with each other as expressed by the Beck-Chevalley property from Proposition 14.3.13. In this subsection, we will see that these two conditions can be used as a characterization of exponentiability. Throughout, we fix a clan $\mathcal{E}$.

Definition 14.3.16 (Universally exponentiable fibration)**.** A fibration $f\colon A \twoheadrightarrow B$ is called *universally exponentiable* if for every pullback square

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r, "v"] & B \end{tikzcd}$$

in $\mathcal{E}$ the map f' is exponentiable.

Proposition 14.3.17 (cf. [EH23, Lemma 2.4.6])**.** *A fibration $f\colon A \twoheadrightarrow B$ is universally exponentiable if and only if the following two conditions are satisfied:*

(1) For every pullback square

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B' \arrow[r, "v"] & B \end{tikzcd}$$

in $\mathcal{E}$, the functor $f'^*\colon \mathcal{E}(B') \to \mathcal{E}(A')$ admits a right adjoint $f'\colon \mathcal{E}(A') \to \mathcal{E}(B')$;

(2) For every pullback diagram

$$\begin{tikzcd}
A'' \arrow[r, "u'"] \arrow[d, "f''"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B'' \arrow[r, "v'"] & B' \arrow[r, "v"] & B
\end{tikzcd}$$

in $\mathcal{E}$, the Beck-Chevalley transformation

$$\mathrm{BC}_* \colon v'^* f'_* \to f''_* u'^*$$

is a natural isomorphism of functors $\mathcal{E}(A') \to \mathcal{E}(B'')$.

Proof. If f is universally exponentiable, then parts (1) and (2) follow from Corollary 14.3.9 and Proposition 14.3.13, respectively. It thus remains to prove that (1) and (2) imply that f is universally exponentiable. In fact, since every base change of f automatically satisfied conditions (1) and (2) again, it suffices to show that f is exponentiable.

Consider a fibration $p \colon E \twoheadrightarrow A$, which we may regard as an object of $\mathcal{E}(A)$. Applying the right adjoint $f_* \colon \mathcal{E}(A) \to \mathcal{E}(B)$, we thus obtain a fibration $f_*(p) \colon f_*(E) \twoheadrightarrow B$, which comes equipped with a counit map $e(p) \colon A \times_B f_*(E) = f^* f_*(E) \to E$ in $\mathcal{E}(A)$. We need to show that this map satisfies the universal property from Definition 14.3.6, i.e. that for an arbitrary map $g \colon X \to B$ the composite

$$\mathrm{Hom}_{/B}(X, f_*(E)) \xrightarrow{f^*} \mathrm{Hom}_{/A}(A \times_B X, A \times_B f_*(E)) \xrightarrow{e(p)\circ -} \mathrm{Hom}_{/A}(A \times_B X, E)$$

is a bijection. Because $e(p)$ is the counit of the adjunction $f^* \dashv f_*$, we know that this is the case whenever g is a fibration. For the general case, we consider the following pullback square in $\mathcal{E}$:

$$\begin{tikzcd}
Y \arrow[r, "g'"] \arrow[d, two heads, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"] \\
X \arrow[r, "g"] & B.
\end{tikzcd}$$

The desired composite appears as the top composite in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{/B}(X, f_*(E)) \arrow[r, "f^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{/A}(Y, A \times_B f_*(E)) \arrow[r, "e(p)\circ -"] \arrow[d, "\cong"] & \mathrm{Hom}_{/A}(Y, E) \arrow[d, "\cong"] \\
\mathrm{Hom}_{/X}(X, g^* f_*(E)) \arrow[r, "f'^*"] \arrow[d, "\cong"'] & \mathrm{Hom}_{/Y}(Y, Y \times_X g^* f_*(E)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{/Y}(Y, g'^*(E)) \arrow[d, equal] \\
\mathrm{Hom}_{/X}(X, f'_* g'^*(E)) \arrow[r, "f'^*"] & \mathrm{Hom}_{/Y}(Y, Y \times_X f'_* g'^*(E)) \arrow[r, "e(p')\circ -"] & \mathrm{Hom}_{/Y}(Y, g'^*(E)).
\end{tikzcd}$$

Here $p' := g'^* p \colon g'^*(E) \to Y$. The top three vertical isomorphisms are the adjunction isomorphisms while the bottom two are induced by the Beck-Chevalley isomorphism

$g^* f_* \xrightarrow{\cong} f'_* g'^*$. It will thus suffice to show that the bottom composite is a bijection. But this follows from the fact that the fibration $f' \colon Y \twoheadrightarrow X$ is exponentiable and the identity map $\mathrm{id}_X \colon X \longrightarrow X$ is a fibration. □

14.3.4 Distributivity

Consider the category Set of (small) sets. This category admits coproducts and products, given by disjoint unions and cartesian products, respectively. Furthermore, products and disjoint unions satisfy a form of *distributivity*: given sets A, B and C, there is a canonical bijection

$$(A \sqcup B) \times C \xrightarrow{\cong} (A \times C) \sqcup (A \times C).$$

More generally, consider sets X and Y, and let $(F_{x,y})_{(x,y) \in X \times Y}$ be a collection of sets indexed by the product $X \times Y$. Then there is a canonical bijection

$$\prod_{x \in X} \coprod_{y \in Y} F_{x,y} \cong \coprod_{g \colon X \to Y} \prod_{x \in X} F_{x,g(x)},$$

where the second disjoint union is indexed by the set of functions $X \longrightarrow Y$. In this subsection, we will show that an analogous distributivity property holds for dependent sums and dependent products in a clan $\mathcal{E}$.

Proposition 14.3.18 (Distributivity)**.** *Let $\mathcal{E}$ be a clan and let $f \colon A \twoheadrightarrow B$ be a universally exponentiable fibration. For a fibration $p \colon E \twoheadrightarrow A$, consider the following commutative diagram in $\mathcal{E}$:*

$$\begin{tikzcd}
E & A \times_B f_*(E) \arrow[l, "e(p)"'] \arrow[r, two heads, "f'"] \arrow[d, two heads, "q'"] \arrow[dr, phantom, "\lrcorner", very near start] & f_*(E) \arrow[d, two heads, "q = f_*(p)"] \\
E \arrow[r, phantom, ""] & A \arrow[r, two heads, "f"] & B.
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[dr, two heads, "p"'] & A \times_B f_*(E) \arrow[l, "e(p)"'] \arrow[r, two heads, "f'"] \arrow[d, two heads, "q'"] & f_*(E) \arrow[d, two heads, "q = f_*(p)"] \\
& A \arrow[r, two heads, "f"] & B.
\end{tikzcd}$$

Then the composite

$$\mathcal{E}(E) \xrightarrow{p_!} \mathcal{E}(A) \xrightarrow{f_*} \mathcal{E}(B)$$

is canonically equivalent to the composite

$$\mathcal{E}(E) \xrightarrow{e(p)^*} \mathcal{E}(A \times_B f^*(E)) \xrightarrow{f'_*} \mathcal{E}(f_*(E)) \xrightarrow{q_!} \mathcal{E}(B).$$

The proof of Proposition 14.3.18 is somewhat long and technical. To improve the readability of this section, we have deferred its proof to the appendix, see Appendix B.

Let us make the distributivity relation more concrete in the special case where $B = *$, $A = X$ and $E = X \times Y$, where we take $f \colon X \twoheadrightarrow *$ and $p = \mathrm{pr}_X \colon X \times Y \longrightarrow X$. In this case, the type $f_*(E)$ is precisely the type of maps $X \longrightarrow Y$. Now, assume given any fibration $F \twoheadrightarrow X \times Y$, thought of as a family $\{F_{x,y}\}$ of types indexed by pairs (x,y) in $X \times Y$. Using the notation for dependent sums and dependent products from Notation 14.3.2 and Notation 14.3.11, the distributivity relation then takes the form

$$\prod_{x:X}\sum_{y:Y} F_{x,y} \cong \sum_{g\colon X\to Y}\prod_{x:X} F_{x,g(x)},$$

which looks exactly like the distributivity relation of disjoint unions and products of sets. More generally, if $p\colon E \twoheadrightarrow A$ is arbitrary and $F \twoheadrightarrow E$ is a fibration, the relation may be expressed as

$$\prod_{a:A}\sum_{e:E_a} F_e \cong \sum_{s\colon A\to E}\prod_{a:A} F_{s(a)},$$

where now the second sum is indexed by the type of sections $s\colon A \to E$ of p.

Corollary 14.3.19 (Joyal [Joy17, Proposition 2.4.16])**.** *For a universally exponentiable fibration $f\colon A \twoheadrightarrow B$ in $\mathcal{E}$, the dependent product functor*

$$f_*\colon \mathcal{E}(A) \to \mathcal{E}(B)$$

preserves fibrations. In particular, it is a morphism of clans.

Proof. Since the right adjoint f_* preserves all limits that exist in $\mathcal{E}(A)$, the second statement is immediate from the first statement, hence it remains to prove the first statement.

By definition of an exponentiable fibration, f_* sends fibrations in $\mathcal{E}(A)$ of the form

$$\begin{tikzcd}
F \arrow[rr, two heads, "p"] \arrow[dr, two heads, "p"'] & & A \arrow[dl, equal] \\
& A &
\end{tikzcd}$$

to fibrations. Observe that a general fibration

$$\begin{tikzcd}
F \arrow[rr, two heads, "g"] \arrow[dr, two heads, "pg"'] & & E \arrow[dl, two heads, "p"] \\
& A &
\end{tikzcd}$$

in $\mathcal{E}(A)$ can be written as the image of the map $g\colon (F,g) \to (E,1_E)$ under the functor $p_!\colon \mathcal{E}(E) \to \mathcal{E}(A)$. By Proposition 14.3.18, we have

$$f_*(g) = f_*p_!(g) \cong q_!f'_*e(p)^*(g) = q_!f'_*(g'),$$

where we use the notation from Proposition 14.3.18. Since g' is of the specific form mentioned above, f'_* sends it to a fibration in $\mathcal{E}(f_*(E))$, and then $p_!$ sends it to a fibration in $\mathcal{E}(B)$. This finishes the proof. □

14.4 Dependent type theory

We have seen that every clan $\mathcal{E}$ admits dependent sums. To also have sufficient dependent products, we make the following definition:

Definition 14.4.1 (Dependent type theory)**.** A clan $\mathcal{E}$ is called a *dependent type theory* if every fibration is exponentiable.

Warning 14.4.2. Joyal used the terminology *π-clan* for what we call a dependent type theory.

It follows that for a dependent type theory $\mathcal{E}$, every fibration $f\colon A \twoheadrightarrow B$ induces a pair of adjunctions

$$\begin{tikzcd}
\mathcal{E}(A) \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathcal{E}(B). \arrow[l, "f^*" description] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"]
\end{tikzcd}$$

Lemma 14.4.3. *Let $\mathcal{E}$ be a dependent type theory and let B be a type in $\mathcal{E}$. Then the local clan $\mathcal{E}(B)$ is a dependent type theory.*

Proof. We need to show that any fibration in $\mathcal{E}(B)$ is exponentiable. This is immediate from Lemma 14.3.15. □

There is a natural notion of a *morphism* of dependent type theories. Morally, a morphism of clans is a morphism of dependent type theories if it commutes with dependent sums and dependent products. Since the former is automatic, this leads to the following definition:

Definition 14.4.4. Let $\mathcal{E}$ and $\mathcal{E}'$ be dependent type theories. A functor $F\colon \mathcal{E} \to \mathcal{E}'$ is called a *morphism of dependent type theories* if it is a morphism of clans satisfying the following condition: for any fibration $f\colon A \twoheadrightarrow B$ in $\mathcal{E}$, the canonical *Beck-Chevalley map*

$$F_B f_* \to F(f)_* F_A$$

is a natural isomorphism.

Let us unwind this definition a little. For every type $B \in \mathcal{E}$, there is a functor

$$F_B\colon \mathcal{E}(B) \to \mathcal{E}'(F(B))\colon (E,p) \mapsto (F(E),F(p)).$$

We saw in Remark 14.2.4 that this is a morphism of clans. Now, given a fibration $f\colon A \twoheadrightarrow B$ in $\mathcal{E}$, we obtain a fibration $F(f)\colon F(A) \twoheadrightarrow F(B)$, and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E}(A) \arrow[r, "F_A"] \arrow[d, "f_!"'] & \mathcal{E}'(F(A)) \arrow[d, "F(f)_!"] \\
\mathcal{E}(B) \arrow[r, "F_B"] & \mathcal{E}'F(B).
\end{tikzcd}$$

In formulas, this is saying that there is a natural equivalence

$$F_B f_! \overset{\sim}{\Longrightarrow} F(f)_! F_A.$$

Now, since F is assumed to preserve pullbacks along fibrations, we may adjoin over $f_!$ and $F(f)_!$ to obtain a natural equivalence

$$F_A f^* \overset{\sim}{\Longrightarrow} F(f)^* F_B.$$

These two equivalences express that F commutes with both base change and dependent sums. In order for F to be a morphism of dependent type theories, we need F to also preserve dependent products. There exists a *Beck-Chevalley map* $F_B f_* \to F(f)_* F_A$, which is given as the following composite

$$F_B f_* \xrightarrow{\text{unit}} F(f)_* F(f)^* F_B f_* \cong F(f)_* F_A f^* f_* \xrightarrow{\text{counit}} F(f)_* F_A.$$

This is the map we ask to be a natural isomorphism in Definition 14.4.4. This condition is often referred to as the *Beck-Chevalley condition*.

Example 14.4.5. Consider a morphism $f \colon A \to B$ in $\mathcal{E}$. Then the pullback functor $f^* \colon \mathcal{E}(B) \to \mathcal{E}(A)$ is morphism of dependent type theories. Indeed, we proved in Lemma 14.2.9 that it is a morphism of clans, and the Beck-Chevalley condition was proved in Proposition 14.3.13.

14.5 Exercises Chapter 14

Exercise 14.5.1. Let $F \colon \mathcal{E} \to \mathcal{E}'$ be a morphism of clans. Show that for every type $B \in \mathcal{E}$, the induced functor $F(B) \colon \mathcal{E}(B) \to \mathcal{E}(FB)$ on local clans is a morphism of clans.

Exercise 14.5.2. Show that the category Top of topological spaces is a clan when we equip it with the Hurewicz fibrations from Example 14.1.7.

Exercise 14.5.3. Show that the 1-category Cat_1 of small 1-categories is a clan when we equip it with the isofibrations from Example 14.1.8. Show similarly that the 1-category Grpd_1 of 1-groupoids is a clan with the isofibrations as fibrations.

Exercise 14.5.4. Let $f \colon A \twoheadrightarrow B$ be a fibration in a clan $\mathcal{E}$. Show that the functor $f_! \colon \mathcal{E}(A) \to \mathcal{E}(B)$ from Definition 14.3.1 defines a left adjoint to the pullback functor $f^* \colon \mathcal{E}(B) \to \mathcal{E}(A)$.

Exercise 14.5.5. For an isofibration of groupoids $p \colon E \to B$, we let

$$\mathrm{Sect}(p) \subseteq \mathrm{Fun}(B, E)$$

denote the full subgroupoid spanned by the sections of p.

(a) Let B be a groupoid and let $f\colon B \to *$ denote the functor to the terminal groupoid. Show that f is exponentiable in the tribe Grpd and that, for any isofibration $p\colon E \twoheadrightarrow B$, the groupoid $f_*(E,p)$ is given by $\mathrm{Sect}(p)$.

(b) Show that any isofibration $f\colon B \twoheadrightarrow C$ of groupoids is exponentiable and give an explicit description of $f_*(E,p)$ for $(E,p) \in \mathrm{Grpd}(B)$.

Exercise 14.5.6. Consider a type C in a clan $\mathcal{E}$, and let $f\colon (A,p) \twoheadrightarrow (B,q)$ be a fibration in $\mathcal{E}(C)$. Assume that the underlying morphism of f in $\mathcal{E}$ is exponentiable in $\mathcal{E}$.

(1) The morphism f is exponentiable in $\mathcal{E}(C)$.

(2) Under the canonical equivalences $\mathcal{E}(C)(A,p) \simeq \mathcal{E}(A)$ and $\mathcal{E}(C)(B,q) \simeq \mathcal{E}(B)$, the dependent product functor

$$f_*\colon \mathcal{E}(C)(A,p) \to \mathcal{E}(C)(B,q)$$

corresponds to the dependent product functor $f_*\colon \mathcal{E}(A) \to \mathcal{E}(B)$.

15 Tribes and homotopy type theory

In the previous chapter, we introduced the notion of a *clan* and saw that it provides a categorical interpretation of dependent type theory. In this chapter, we will impose additional axioms on the clan, allowing for a natural notion of *homotopies* between morphisms of types. As a consequence, tribes provide a categorical framework for studying homotopy type theory.

15.1 Anodyne maps

In this section, we will introduce the notion of 'anodyne maps' in a clan, and prove various closure properties for anodyne maps.

Definition 15.1.1 (Anodyne maps)**.** Let $\mathcal{E}$ be a clan. A map $u\colon A \to B$ in $\mathcal{E}$ is called *anodyne* if it has the left lifting property with respect to fibrations. Explicitly, this means that for any commutative square

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, two heads, "p"] \\
B \arrow[r, "b"] & Y
\end{tikzcd}$$

with p in $\mathcal{F}$, there exists a morphism $l\colon B \to X$ satisfying the relations

$$pl = b \qquad \text{and} \qquad lu = a.$$

We will use the notation $u\colon A \overset{\sim}{\rightarrowtail} B$ to denote anodyne maps, or sometimes just $A \rightarrowtail B$ for simplicity.

Convention 15.1.2. We will often write the left lifting property above in the following more suggestive form:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, two heads, "p"] \\
B \arrow[r, "b"'] \arrow[ur, dashed, "l"] & Y.
\end{tikzcd}$$

Here the solid arrows are part of the given data, and the dashed arrow l is the morphism we are postulating should exist; the two conditions correspond to the commutativity of the two triangles in the diagram. We refer to such a solid square as a *lifting problem*, and to the map l as the *lift*.

Proposition 15.1.3. *The class of anodyne maps is closed under the following operations:*

(1) Any isomorphism is anodyne.

(2) The composition of anodyne maps is anodyne.

(3) Any retract of an anodyne map is anodyne.

Proof. For (1), it is clear that any isomorphism is anodyne. For (2), consider anodyne maps $u\colon A \overset{\sim}{\rightarrowtail} B$ and $v\colon B \overset{\sim}{\rightarrowtail} C$, and consider any solid commutative diagram as follows:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "u"'] & X \arrow[dd, two heads, "p"] \\
B \arrow[d, tail, "v"'] \arrow[ur, dashed, "l"] & \\
C \arrow[r, "c"'] \arrow[uur, dashed, "k"] & Y
\end{tikzcd}$$

in which the map p is a fibration. Since u is anodyne, we may find a map l making the top triangle and the bottom square commute. As v is anodone, we may find a map k making the middle and bottom right triangle commute. The map k then provides the desired lift, showing part (2). For (3), consider any retract diagram

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "u"'] \arrow[rr, equal, bend left] & A' \arrow[r, "r"] \arrow[d, tail, "u'"] & A \arrow[d, "u"] \\
B \arrow[r, "s'"] \arrow[rr, equal, bend right] & B' \arrow[r, "r'"] & B
\end{tikzcd}$$

and assume that the map u' is anodyne. Consider a commutative square as follows:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, two heads, "p"] \\
B \arrow[r, "b"'] & Y,
\end{tikzcd}$$

where p is a fibration. Consider now the following larger diagram obtained by pasting the previous two diagrams:

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "u"'] \arrow[rr, equal, bend left] & A' \arrow[r, "r"] \arrow[d, tail, "u'"] & A \arrow[r, "a"] & X \arrow[d, two heads, "p"] \\
B \arrow[r, "s'"] \arrow[rr, equal, bend right] & B' \arrow[r, dashed, "r'"] \arrow[rru, dashed, "l'"] & B \arrow[r, "b"'] & Y.
\end{tikzcd}$$

Since u' is anodyne, we may find a map $l'\colon B' \to X$ making the two triangles in the diagram commute. We claim that the composite $l := l's'\colon B \to X$ solves the original lifting problem. Indeed, we have

$$pl = pl's' = br's' = b \qquad \text{and} \qquad lu = l's'u = l'u's = ars = a.$$

This finishes the proof. □

Lemma 15.1.4. *Any anodyne map $u\colon A \overset{\sim}{\rightarrowtail} B$ admits a retraction $r\colon B \to A$, that is, a map satisfying $ru = \mathrm{id}_A$.*

Proof. Since the map $A \twoheadrightarrow *$ is assumed to be a fibration, we may find a map $r\colon B \to A$ filling the following square:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, tail, "u"'] & A \arrow[d, two heads] \\
B \arrow[r] \arrow[ru, dashed, "r"] & *.
\end{tikzcd}$$

In particular we have $ru = \mathrm{id}_A$ so that r is a retraction of u. □

Lemma 15.1.5. *Anodyne maps satisfy left-cancellation: if $u\colon A \to B$ and $v\colon B \to C$ are maps such that v and vu are anodyne, then also u is anodyne.*

Proof. By Lemma 15.1.4, the map v admits a retraction $r\colon C \to B$. We may consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u"'] & A \arrow[r, equal] \arrow[d, tail, "vu"] & A \arrow[d, "u"] \\
B \arrow[r, "v"] \arrow[rr, equal, bend right] & C \arrow[r, "r"] & B,
\end{tikzcd}$$

which exhibits u as a retract of the anodyne map vu in $\mathcal{E}$. Since anodyne maps are closed under retracts by part (3) of Proposition 15.1.3, this finishes the proof. □

Lemma 15.1.6 (cf. [Joy17, Lemma 3.1.4])**.** *A morphism $u\colon A \to B$ in a clan $\mathcal{E}$ is anodyne if and only if for every fibration $f\colon E \twoheadrightarrow B$ and every map $a\colon A \to E$ satisfying $fa = u$,* *there exists a section $s\colon B \to E$ of f satisfying $a = su$:*

$$\begin{tikzcd}
& E \arrow[d, two heads, "f"] \\
A \arrow[ur, "a"] \arrow[r, "u"'] & B. \arrow[u, dashed, bend right, "s"']
\end{tikzcd}$$

Proof. If u is anodyne, the desired section s can be obtained by finding a filler for the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "u"'] & E \arrow[d, two heads, "f"] \\
B \arrow[r, equal] \arrow[ur, dashed, "s"] & B.
\end{tikzcd}$$

Conversely, assume such section s exists for every f and a. To show that u is anodyne, consider a commutative square as follows:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "u"'] & X \arrow[d, two heads, "f"] \\
B \arrow[r, "b"] & Y.
\end{tikzcd}$$

Pulling back f along b, it will suffice to find a filler in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, tail, "u"'] & B \times_Y X \arrow[r] \arrow[d, two heads, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "f"] \\
B \arrow[r, equal] \arrow[ur, dashed, "s"] & B \arrow[r, "b"] & Y.
\end{tikzcd}$$

This filler exists by assumption on u. □

Corollary 15.1.7. *Consider an object B of a clan $\mathcal{E}$. Then a morphism in the local clan $\mathcal{E}(B)$ is anodyne if and only if the underlying morphism in $\mathcal{E}$ is anodyne.*

Proof. Consider a morphism

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\
& B &
\end{tikzcd}$$

in the local clan $\mathcal{E}(B)$. If the underlying morphism u in $\mathcal{E}$ is anodyne, it is straightforward to check that u is anodyne in the local clan $\mathcal{E}(B)$ as well. Consversely, assume that u is anodyne in the local clan $\mathcal{E}(B)$. We will show its underlying morphism in $\mathcal{E}$ is anodyne by checking the criterion from Lemma 15.1.6. Picking a fibration $f\colon E \twoheadrightarrow Y$ and a map $x\colon X \to E$ satisfying $fx = u$, we have to show that there exists a section $s : Y \to E$ of f satisfying $x = su$. Since the composite

$$qf\colon E \overset{f}{\twoheadrightarrow} Y \overset{q}{\twoheadrightarrow} B$$

is a fibration, we may regard E as an object in the local clan $\mathcal{E}(B)$. Observe that this turns the maps f and x into maps in $\mathcal{E}(B)$. Since f is a fibration in $\mathcal{E}$, it is by definition of $\mathcal{E}(B)$ also a fibration in $\mathcal{E}(B)$, so applying Lemma 15.1.6 to the clan $\mathcal{E}(B)$ shows that there exists a section $s\colon Y \to E$ of f in $\mathcal{E}(B)$ satisfying $x = su$. This produces the desired map s, finishing the proof. □

15.2 Tribes

Using the notion of anodyne morphisms, we now come to the definition of a tribe:

Definition 15.2.1 (Tribe, Joyal [Joy17, Definition 3.1.6]). A *tribe* is a clan $\mathcal{E}$ with the following properties:

(1) For every map $f\colon A \to B$ in $\mathcal{E}$, there exists a factorization

$$\begin{tikzcd}
A \arrow[r, tail, "i"'] \arrow[rr, bend left, two heads, "f"] & C \arrow[r, two heads, "p"'] & B
\end{tikzcd}$$

of f into an anodyne map $i\colon A \overset{\sim}{\rightarrowtail} C$ followed by a fibration $p\colon C \twoheadrightarrow B$;

(2) Anodyne maps are stable under base change along fibrations: for every pullback square

$$\begin{tikzcd}
X \arrow[r, "v"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, two heads, "q"] \\
A \arrow[r, tail, "u"] & B
\end{tikzcd}$$

in which q (and thus p) is a fibration and u is anodyne, also v is anodyne.

Remark 15.2.2. A tribe corresponds to a type theory in which the notion of equality between terms is defined (possibly constructively in the sense of Martin-Löf). We refer to Gambino and Garner [GG08] for details. Important is the fact that in a tribe any object X has a path object: we may factorize the diagonal $\Delta\colon X \to X \times X$ of X into an anodyne $r\colon X \overset{\sim}{\rightarrowtail} P(X)$ followed by a fibration $(d_0, d_1)\colon P(X) \twoheadrightarrow X \times X$; see Definition 15.3.1 below.

Example 15.2.3. Let $\mathcal{E}$ be a category with finite limits and let $\mathcal{F}$ consist of all maps in $\mathcal{E}$. Then the anodyne maps are precisely the isomorphisms in $\mathcal{E}$, and it is easy to see that this is a tribe.

Example 15.2.4. (Topological spaces, see Exercise 15.7.1) The clan Top of topological spaces and Hurewicz fibrations is a tribe; the anodyne maps are precisely the strong deformation retracts.

Example 15.2.5. (Small categories, see Exercise 15.7.2) The clan Cat_1 of small 1-categories and isofibrations is a tribe; the anodyne maps are precisely the equivalences of 1-categories which induce an injection on objects.

Lemma 15.2.6. *Let $\mathcal{E}$ be a tribe and let B be a type in $\mathcal{E}$. Then the local clan $\mathcal{E}(B)$ from Definition 14.2.3 is a tribe.*

Proof. From the characterization of the anodyne maps in $\mathcal{E}(B)$ from Corollary 15.1.7, it follows immediately that the anodyne maps in $\mathcal{E}(B)$ are closed under pullback along a fibration. It thus remains to show that any morphism $f\colon (X,p) \to (Y,q)$ in $\mathcal{E}(B)$ can be factorized as an anodyne map followed by a fibration. Since $\mathcal{E}$ is a tribe, we may factor the underlying map $f\colon X \to Y$ as

$$\begin{tikzcd} X \arrow[r, tail, "i"] & Z \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

where i is anodyne and g is a fibration. We may regard Z as an object in $\mathcal{E}(B)$ via the composite $qg\colon Z \twoheadrightarrow B$. But then this turns both maps i and g into morphisms in $\mathcal{E}(B)$. The map $g\colon (Z,qg) \to (Y,q)$ is a fibration in $\mathcal{E}(B)$ by definition, while the map $i\colon (X,p) \to (Z,qg)$ is anodyne in $\mathcal{E}(B)$ by Corollary 15.1.7. This finishes the proof. □

Definition 15.2.7 (Local tribe)**.** Let $\mathcal{E}$ be a tribe and let B be a type in $\mathcal{E}$. We refer to the tribe $\mathcal{E}(B)$ from Lemma 15.2.6 as the *local tribe* of $\mathcal{E}$ at B.

Proposition 15.2.8. *Let $\mathcal{E}$ be a tribe and let $f\colon A \to B$ be a morphism in $\mathcal{E}$. Then the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ preserves anodyne maps.*

Proof. Choose a factorization

$$A \overset{\sim}{\rightarrowtail} C \twoheadrightarrow B$$

of f as an anodyne map $u\colon A \overset{\sim}{\rightarrowtail} C$ followed by a fibration $g\colon C \twoheadrightarrow B$. The functor f^* is given as the composite

$$\mathcal{E}(B) \xrightarrow{g^*} \mathcal{E}(C) \xrightarrow{u^*} \mathcal{E}(A),$$

and thus it suffices to show that g^* and u^* both preserve anodyne maps. For g^* this is a straightforward consequence of the axiom that a pullback of an anodyne map along a fibration is again anodyne. To show that u^* preserves anodyne maps, consider an anodyne map

$$\begin{tikzcd} X \arrow[rr, tail, "v"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\ & C & \end{tikzcd}$$

in $\mathcal{E}(C)$, and form the following pullback diagram:

$$\begin{tikzcd}
A \times_C X \arrow[r, tail, "u_X"] \arrow[d, "1_A \times v"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "v"] \arrow[dd, bend left, two heads, "p"] \\
A \times_C Y \arrow[r, tail, "u_Y"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, "q"] \\
A \arrow[r, tail, "u"] & C.
\end{tikzcd}$$

The map v is anodyne by assumption, and the maps u_X and u_Y are anodyne as they are pullbacks of the anodyne map u under the fibrations q and p. It thus follows from part (2) of Proposition 15.1.3 that also the composite $vu_X \colon A \times_C X \to C$ is anodyne, and thus we deduce from Lemma 15.1.5 that $1_A \times v \colon A \times_C X \to A \times_C Y$ is anodyne. This finishes the proof. □

15.3 Homotopies and homotopy equivalences

In this section, we will see that tribes form a convenient context for axiomatic homotopy theory. Throughout the section, we fix a tribe $\mathcal{E}$.

15.3.1 Homotopies

We introduce the notion of homotopy in our tribe $\mathcal{E}$.

Definition 15.3.1 (Strict path object)**.** A *strict path object* of a type X consists of a fibration $(d_0, d_1) \colon P(X) \twoheadrightarrow X \times X$ and an anodyne map $r \colon X \overset{\sim}{\rightarrowtail} P(X)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\
X \arrow[ur, tail, "r"] \arrow[r, "\Delta"'] & X \times X.
\end{tikzcd}$$

Here $\Delta = (\mathrm{id}_X, \mathrm{id}_X) \colon X \to X \times X$ denotes the diagonal of X.

Observe that a strict path object $P(X)$ exists for every object X by applying axiom (1) of Definition 15.2.1 to the diagonal map $\Delta \colon X \to X \times X$.

Warning 15.3.2. Joyal refers to strict path objects simply as 'path objects', and this convention was adopted by Cisinski in his lectures. We add the adjective 'strict' since we will define a more general notion of path object in Definition 15.3.27 below.

Definition 15.3.3 (Homotopic maps)**.** Consider two types A and X in $\mathcal{E}$. We say that a map $f\colon A \to X$ is *homotopic* to another map $g\colon A \to X$ if there exists a strict path object

$$\begin{tikzcd} X \arrow[r, tail, "r"] & P(X) \arrow[r, two heads, "{(d_0,d_1)}"] & X \times X \end{tikzcd}$$

of X and a *homotopy* $\alpha\colon A \to P(X)$ satisfying $d_0 h = f$ and $d_1 h = g$:

$$\begin{tikzcd} & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\ A \arrow[ur, dashed, "\alpha"] \arrow[r, "{(f,g)}"'] & X \times X. \end{tikzcd}$$

In this case, we will say that α is a *homotopy from f to g*, and denote this by $\alpha\colon f \sim g$. We will frequently write $f \sim g$ if f is homotopic to g via some unnamed homotopy.

Lemma 15.3.4. *The notion of homotopy in Definition 15.3.3 does not depend on the chosen strict path object $P(X)$: if X admits another strict path object*

$$\begin{tikzcd} X \arrow[r, tail, "r'"] & P'(X) \arrow[r, two heads, "{(d'_0,d'_1)}"] & X \times X \end{tikzcd}$$

then there exists a homotopy $\alpha\colon A \to P(X)$ between f and g if and only if there exists a homotopy $\alpha'\colon A \to P'(X)$ between f and g.

Proof. By symmetry, it suffices to show one direction. So given a homotopy $\alpha\colon A \to P(X)$, we need to construct a homotopy $\alpha'\colon A \to P'(X)$. To this end, consider the following diagram

$$\begin{tikzcd} X \arrow[r, tail, "r'"] \arrow[d, tail, "r"'] & P'(X) \arrow[d, two heads, "{(d'_0,d'_1)}"] \\ P(X) \arrow[ur, dashed, "l"] \arrow[r, two heads, "{(d_0,d_1)}"'] & X \times X. \end{tikzcd}$$

As r is anodyne and (d'_0, d'_1) is a fibration, the dashed map l exists, making the triangles commute. We now claim that the composite $\alpha' := l\alpha\colon A \to P'(X)$ is the desired homotopy between f and g. Indeed, we have

$$(d'_0, d'_1)\alpha' = (d'_0, d'_1)l\alpha = (d_0, d_1)\alpha = (f, g). \qquad \square$$

We will now show that the homotopy relation is an equivalence relation.

Construction 15.3.5. Consider a type X equipped with a chosen strict path object $(P(X), r, d_0, d_1)$. We will construct maps

$$(-)^{-1}\colon P(X) \to P(X) \qquad \text{and} \qquad -\circ-\colon P(X) \times_X P(X) \to P(X)$$

called the *inversion* and *composition* maps, respectively. The inversion map is given by choosing a lift in the following lifting problem:

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, tail, "r"'] & P(X) \arrow[d, "{(d_0,d_1)}"] \\
P(X) \arrow[r, "{(d_1,d_0)}"'] \arrow[ur, dashed, "(-)^{-1}" description] & X \times X,
\end{tikzcd}$$

where we emphasize the order of d_0 and d_1 in the bottom horizontal map. For the concatenation map, let us first clarify that by $P(X) \times_X P(X)$ we mean the fiber product of $d_1 \colon P(X) \twoheadrightarrow X$ and $d_0 \colon P(X) \twoheadrightarrow X$. The concatenation map is then given by choosing a lift in the following lifting problem:

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, tail, "{(r,r)}"'] & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\
P(X) \times_X P(X) \arrow[r, "{(d_0 \circ \mathrm{pr}_1, d_1 \circ \mathrm{pr}_2)}"'] \arrow[ur, dashed, "- \circ -" description] & X \times X.
\end{tikzcd}$$

Construction 15.3.6. We construct identity homotopies, inverses of homotopies and compositions of homotopies::

- Given a map $f \colon A \to X$, we define the *identity homotopy* $\mathrm{id}_f \colon f \sim f$ as the composite $A \xrightarrow{f} X \overset{r}{\rightarrowtail} P(X)$;
- Given a homotopy $\alpha \colon f \sim g$, we define the *inverse homotopy* $\alpha^{-1} \colon g \sim f$ as the composite $A \xrightarrow{\alpha} P(X) \xrightarrow{(-)^{-1}} P(X)$.
- Given homotopies $\alpha \colon f \sim g$ and $\beta \colon g \sim h$, we define the *(vertically) composite homotopy* $\beta \circ \alpha \colon f \sim h$ as the composite $A \xrightarrow{(\alpha,\beta)} P(X) \times_X P(X) \xrightarrow{- \circ -} P(X)$.

Corollary 15.3.7. *The notion of homotopy is an equivalence relation on maps $A \to X$:*

(1) (Reflexivity) For every morphism $f \colon A \to X$ we have $f \sim f$;

(2) (Symmetry) For morphisms $f, g \colon A \to X$ we have $f \sim g$ if and only if $g \sim f$;

(3) (Transitivity) For morphisms $f, g, h \colon A \to X$, if $f \sim g$ and $g \sim h$ then also $f \sim h$.

Lemma 15.3.8. *The composition of homotopies is unital and associative, and admits inverses:*

(1) Given a homotopy $\alpha \colon f \sim g$, there are homotopies of maps $A \to P(X)$ of the form

$$\alpha \circ \mathrm{id}_f \sim \alpha \qquad \text{and} \qquad \mathrm{id}_g \circ \alpha \sim \alpha.$$

(2) Given homotopies $\alpha\colon f \sim g$, $\beta\colon g \sim h$ and $\gamma\colon h \sim i$, there is a homotopy

$$(\gamma \circ \beta) \circ \alpha \sim \gamma \circ (\beta \circ \alpha)$$

of maps $A \longrightarrow P(X)$.

(3) Given a homotopy $\alpha\colon f \sim g$, there are homotopies of maps $A \longrightarrow P(X)$ of the form

$$\alpha^{-1} \circ \alpha \sim \mathrm{id}_f \qquad \textit{and} \qquad \alpha \circ \alpha^{-1} \sim \mathrm{id}_g \,.$$

Proof. For part (1), consider lifts h and h' in the following two lifting problems:

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, rightarrowtail, "r"] & P(X) \arrow[r, "r"] & P(P(X)) \arrow[d, twoheadrightarrow, "{(d_0,d_1)}"] \\
P(X) \cong X \times_X P(X) \arrow[r, "{(r,\mathrm{id})}"'] \arrow[urr, dashed, "h"] & P(X) \times_X P(X) \arrow[r, "{(-\circ-,\mathrm{pr}_2)}"'] & P(X) \times P(X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, rightarrowtail, "r"] & P(X) \arrow[r, "r"] & P(P(X)) \arrow[d, twoheadrightarrow, "{(d_0,d_1)}"] \\
P(X) \cong P(X) \times_X X \arrow[r, "{(\mathrm{id},r)}"'] \arrow[urr, dashed, "h'"] & P(X) \times_X P(X) \arrow[r, "{(\mathrm{pr}_1,-\circ-)}"'] & P(X).
\end{tikzcd}$$

The desired homotopies $\mathrm{id}_g \circ \alpha \sim \alpha$ and $\alpha \circ \mathrm{id}_f \sim \alpha$ are then given by $h \circ \alpha\colon A \longrightarrow P(P(X))$ and $h' \circ \alpha\colon X \longrightarrow P(P(X))$, respectively.

For part (2), choose a lift h in the following lifting problem:

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, rightarrowtail, "{(r,r,r)}"'] & P(X) \arrow[r, "r"] & P(P(X)) \arrow[d, twoheadrightarrow, "{(d_0,d_1)}"] \\
P(X) \times_X P(X) \times_X P(X) \arrow[rr, "{((-\circ-)\circ-,-\circ(-\circ-))}"'] \arrow[urr, dashed, "h"] & & P(X) \times P(X).
\end{tikzcd}$$

The desired homotopy is $(\gamma \circ \beta) \circ \alpha \sim \gamma \circ (\beta \circ \alpha)$ then given by $h \circ (\alpha,\beta,\gamma)\colon A \longrightarrow P(P(X))$.

For part (3), we just do the case $\alpha^{-1} \circ \alpha \sim \mathrm{id}_f$; the other case is similar. Choose a lift h in the following lifting problem:

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, rightarrowtail, "r"] & P(X) \arrow[r, "r"] & P(P(X)) \arrow[d, twoheadrightarrow, "{(d_0,d_1)}"] \\
P(X) \arrow[rr, "{(\varphi,\psi)}"'] \arrow[urr, dashed, "h"] & & P(X) \times P(X),
\end{tikzcd}$$

where φ and ψ are given by

$$\varphi\colon P(X) \xrightarrow{((-)^{-1},\mathrm{id})} P(X) \times_X P(X) \xrightarrow{-\circ-} P(X) \qquad \text{and} \qquad \psi\colon P(X) \xrightarrow{d_0} X \xrightarrow{r} P(X),$$

respectively. The desired homotopy $\alpha^{-1} \circ \alpha \sim \mathrm{id}_f$ is then given by $h \circ \alpha\colon A \longrightarrow P(P(X))$. □

Definition 15.3.9 (Homotopy classes)**.** For types $A, X \in \mathcal{E}$, we write

$$[A,X] := \mathrm{Hom}_{\mathcal{E}}(A,X)/{\sim}$$

for the set of equivalence classes with respect to the homotopy relation $\sim$. We refer to its equivalence classes as *homotopy classes* of maps $A \to X$.

The homotopy relation $\sim$ is compatible with composition:

Lemma 15.3.10. *Let $f, g \colon A \to X$ be two maps and let $\alpha \colon f \sim g$ be a homotopy.*

(1) For any map $u \colon B \to A$, there is a homotopy $\alpha u \colon fu \sim gu$.

(2) For any map $v \colon X \to Y$, there is a homotopy $v\alpha \colon vf \sim vg$.

Proof. For part (1), we define αu simply as the composite $B \xrightarrow{u} A \xrightarrow{\alpha} P(X)$. Then αu is easily seen to be a homotopy between fu and gu. For part (2), we choose a lift $P(v) \colon P(X) \to P(Y)$ in following lifting problem:

$$\begin{tikzcd}
X \arrow[r, "v"] \arrow[d, tail, "r_X"'] & Y \arrow[r, tail, "r_Y"] & P(Y) \arrow[d, two heads, "{(d_0,d_1)}"] \\
P(X) \arrow[r, "{(d_0,d_1)}"'] \arrow[urr, dashed, "P(v)"] & X \times X \arrow[r, "v \times v"'] & Y \times Y.
\end{tikzcd}$$

We then define $v\alpha$ as the composite $A \xrightarrow{\alpha} P(X) \xrightarrow{P(v)} P(Y)$, which is easily seen to be a homotopy between vf and vg. □

Observation 15.3.11. Given a homotopy $\alpha \colon f \sim g$ and maps $u \colon B \to A$ and $v \colon X \to Y$, it is clear that the two homotopies $v(\alpha u) \colon vfu \sim vgu$ and $(v\alpha)u \colon vfu \sim vgu$ agree as maps $B \to P(Y)$.

Construction 15.3.12 (Horizontal composition of homotopies)**.** Given maps $f, f' \colon A \to X$ and $g, g' \colon X \to Y$ and maps homotopies $\alpha \colon f \sim f'$ and $\beta \colon g \sim g'$, we define their *horizontal composite $\beta * \alpha \colon g \circ f \sim g' \circ f'$* as the composite of $g\alpha \colon gf \sim gf'$ with $\beta f' \colon gf' \sim g'f'$.

Lemma 15.3.13. *Horizontal composition of homotopies satisfies the constraints concerning unitality, associativity and compatibility with vertical composition from Axiom A.3.*

Proof. The proofs of the unitality and associativity constraints are similar to previous proofs and are left to the reader. Since horizontal composition is defined in terms of vertical composition, their compatibility is an immediate consequence of Observation 15.3.11. □

Construction 15.3.14 (Homotopy category of a tribe)**.** We will construct the homotopy category $\mathrm{Ho}(\mathcal{E})$ of a tribe $\mathcal{E}$. The objects of $\mathrm{Ho}(\mathcal{E})$ are the types of $\mathcal{E}$. The morphisms of $\mathrm{Ho}(\mathcal{E})$ are homotopy classes of morphisms in $\mathcal{E}$: for $A, B \in \mathcal{E}$ we set

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(A, B) := [A, B] = \mathrm{Hom}_{\mathcal{E}}(A, B)/\!\sim .$$

It follows from Lemma 15.3.10 that this inherits a well-defined composition law from $\mathcal{E}$, thus defining a category $\mathrm{Ho}(\mathcal{E})$. The quotient maps

$$\mathrm{Hom}_{\mathcal{E}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}(A, B)/\!\sim = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(A, B)$$

define a functor $\mathcal{E} \to \mathrm{Ho}(\mathcal{E})$.

15.3.2 Homotopy equivalences

The notion of homotopy in a tribe $\mathcal{E}$ naturally leads to the notion of homotopy equivalence:

Definition 15.3.15 (Homotopy equivalence)**.** A map $u\colon A \to B$ is called a *homotopy equivalence* if there exists a map $v\colon B \to A$ with $vu \sim \mathrm{id}_A$ and $uv \sim \mathrm{id}_B$.

Remark 15.3.16 (See Exercise 15.7.4)**.** A map $u\colon A \to B$ is a homotopy equivalence if and only if its image in $\mathrm{Ho}(\mathcal{E})$ is an isomorphism.

Corollary 15.3.17. *The homotopy equivalences are stable under retracts and satisfy the 2-out-of-3 property.* □

An important class of examples of homotopy equivalences are the anodyne maps:

Proposition 15.3.18. *Any anydone map $u\colon A \xrightarrow{\sim} B$ is a homotopy equivalence.*

Proof. By Lemma 15.1.4, the map u admits a retraction $r\colon B \to A$, i.e. we have $ru = \mathrm{id}_A$. It remains to show that $ur \simeq \mathrm{id}_B$. Since $(u, u) = (uru, u)$, the following solid diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, tail, "u"'] & B \arrow[r, "r"] & P(B) \arrow[d, two heads, "{(d_0,d_1)}"] \\
B \arrow[rr, "{(ur,\mathrm{id}_B)}"'] \arrow[urr, dashed, "h"] & & B \times B.
\end{tikzcd}$$

We may thus find a filler $h\colon B \to P(B)$. But then h defines the desired homotopy $ur \simeq \mathrm{id}_B$. □

The axioms of a tribe guarantee that every morphism factors as an anodyne morphism followed by a fibration. The following construction shows that there is in fact a canonical choice for such a factorization:

Construction 15.3.19 (Mapping-path object)**.** Let $\mathcal{E}$ be a tribe and $f\colon A \to B$ a morphism in $\mathcal{E}$. We define the *mapping-path object*

$$\begin{tikzcd}
A \arrow[r, tail] & M(f) \arrow[r, two heads, "{(\delta_0,\delta_1)}"] & A \times B
\end{tikzcd}$$

of f via the pullback diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ddd, equal, bend right=60] & B \arrow[d, tail, "r"] \arrow[ddd, equal, bend left=60] \\
M(f) \arrow[r, "H"] \arrow[d, two heads, "{(\delta_0,\delta_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P(B) \arrow[d, two heads, "{(d_0,d_1)}"] \\
A \times B \arrow[r, "f \times 1_B"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \times B \arrow[d, "\mathrm{pr}_1"] \\
A \arrow[r, "f"] & B,
\end{tikzcd}$$

where $P(B)$ is a strict path-object for B.

Lemma 15.3.20. *The map $u\colon A \to M(f)$ is anodyne.*

Proof. We may regard $r\colon B \overset{\sim}{\rightarrowtail} P(B)$ as a map in $\mathcal{E}(B)$. Since r is anodyne, it follows from Corollary 15.1.7 that it also anodyne as a map of $\mathcal{E}(B)$. The map u is obtained by applying the functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ to v, and since f^* preserves anodynes by Proposition 15.2.8 we get that u is anodyne as a morphism in $\mathcal{E}(A)$. Another application of Corollary 15.1.7 shows that u is anodyne in $\mathcal{E}$. □

In particular, we see that the maps $u\colon A \overset{\sim}{\rightarrowtail} M(f)$ and $\delta_1\colon M(f) \twoheadrightarrow B$ provide a factorization of f into an anodyne followed by a fibration.

Remark 15.3.21. Note that H is a homotopy from $f\delta_0$ to δ_1. One can prove that $M(f)$ is in fact *universal* with this property: for any other type X equipped with maps $a\colon X \to A$ and $b\colon X \to B$ and a homotopy $h\colon X \to P(B)$ between fa and b, the homotopy h factors uniquely through H.

Corollary 15.3.22. *Let $f\colon A \to B$ be a homotopy equivalence in $\mathcal{E}$. Then there exists a commutative diagram*

$$\begin{tikzcd}
& A \arrow[dl, equal] \arrow[d, "\sim"] \arrow[dr, "f"] & \\
A & A' \arrow[l, two heads, "\sim"'] \arrow[r, two heads, "\sim"] & B
\end{tikzcd}$$

where the bottom two maps are fibrations which are homotopy equivalences.

Proof. Let $A' = M(f)$, and take the maps to A and B to be the maps δ_0 and δ_1, respectively. The map $u \colon A \longrightarrow M(f)$ is anodyne and thus an equivalence. It thus follows by 2-out-of-3 that also δ_0 and δ_1 are homotopy equivalences, as desired. □

The following lemma contains an important strictification result, showing in particular that a section up to homotopy of a fibration can be strictified to an honest section:

Lemma 15.3.23 (Straightening lemma)**.** *Let $p \colon X \twoheadrightarrow B$ be a fibration and let $f \colon A \longrightarrow B$ be any map. For any map $g \colon A \longrightarrow X$ such that $pg \sim f$, there exists a map $g' \colon A \longrightarrow X$ such that:*

(a) The map g' is homotopic to g;

(b) We have a strict equality $pg' = f$.

Proof. Let $h \colon A \longrightarrow P(B)$ be a homotopy from pg to f. We define the map $w \colon A \longrightarrow M(p)$ via the following diagram:

$$\begin{tikzcd}
A \arrow[dr, dashed, "w"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "{(g,f)}"'] & & \\
& M(p) \arrow[r, "H"] \arrow[d, tail, "{(\delta_0,\delta_1)}"] \arrow[dr, phantom, "\lrcorner", very near start] & P(B) \arrow[d, "{(d_0,d_1)}"] \\
& X \times B \arrow[r, "p \times 1_B"] & B \times B.
\end{tikzcd}$$

In particular, we have $Hw = h$, $\delta_0 w = g$ and $\delta_1 w = f$. We further consider a map $d \colon M(p) \longrightarrow X$ obtained by finding a lift in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, tail, "u"'] & X \arrow[d, two heads, "p"] \\
M(p) \arrow[r, "\delta_1"] \arrow[ur, dashed, "d"] & B,
\end{tikzcd}$$

so that $du = 1_X$ and $pd = \delta_1$. We then define $g' := dw \colon A \longrightarrow X$. Because of the relations $du = 1_X = \delta_0 u$ and the fact that u is a homotopy equivalence, we see that d and δ_0 are both homotopy inverses to u, and thus there exists a homotopy $d \sim \delta_0$. It thus follows that

$$g' = dw \sim \delta_0 w = g.$$

Finally, we see that

$$pg' = pdw = \delta_1 w = f,$$

finishing the proof. □

Definition 15.3.24 (Trivial fibration)**.** A *trivial fibration* is a fibration that is also a homotopy equivalence.

Corollary 15.3.25. *Any trivial fibration $p\colon X \twoheadrightarrow Y$ has a section.*

Proof. Since p is a trivial fibration, there exists a homotopy inverse $r\colon Y \longrightarrow X$ which in particular satisfies $pr \sim 1_Y$. By the Straightening Lemma, there thus exists $s\colon Y \longrightarrow X$ with $s \sim r$ and $ps = 1_Y$, showing that s is a section of p. □

15.3.3 Path objects

In Definition 15.3.1, we defined the notion of a *strict* path object of a type X in $\mathcal{E}$. In practice, it is sometimes convenient to work with a weaker notion of path object that still captures the correct notion of homotopy and homotopy equivalence.

Warning 15.3.26. In his course, Cisinski did not consider this weaker notion of path object; see also Warning 15.3.2.

Definition 15.3.27 (Path object)**.** A *path object* of a type X consists of a fibration $(d_0, d_1)\colon P(X) \twoheadrightarrow X \times X$ and a homotopy equivalence $r\colon X \xrightarrow{\sim} P(X)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\
X \arrow[ur, "r"] \arrow[r, "\Delta"'] & X \times X.
\end{tikzcd}$$

Given a path object $P(X)$ of X and two morphisms $f, g : A \longrightarrow X$, we say that a map $h\colon A \longrightarrow P(X)$ is a *homotopy between f and g* if the following diagram commutes:

$$\begin{tikzcd}
 & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\
A \arrow[ur, dashed, "h"] \arrow[r, "{(f,g)}"'] & X \times X.
\end{tikzcd}$$

Note that the only difference between path objects and strict path objects (Definition 15.3.1) is that the map r is only required to be a homotopy equivalence, not necessarily an anodyne map.

The next result shows that also non-strict path objects can be used to define homotopies between maps:

Proposition 15.3.28. *Let $P(X)$ be a path object of X. Then too morphisms $f, g\colon A \longrightarrow X$ are homotopic in the sense of Definition 15.3.3 if and only if there exists a homotopy between f and g of the form $h\colon A \longrightarrow P(X)$.*

Proof. Fix a *strict* path object

$$\begin{tikzcd} X \arrow[r, tail, "r'"] & P'(X) \arrow[r, two heads, "{(d'_0,d'_1)}"] & X \times X. \end{tikzcd}$$

We will show there exists a homotopy of the form $h\colon A \to P(X)$ if and only if there exists a homotopy of the form $h'\colon A \to P'(X)$. To this end, consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, tail, "r"] \arrow[d, tail, "r'"'] & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\ P'(X) \arrow[ur, dashed, "l"] \arrow[r, two heads, "{(d'_0,d'_1)}"'] & X \times X. \end{tikzcd}$$

As r' is anodyne and (d_0,d_1) is a fibration, the dashed map l exists, making both triangles commute. It follows at once that every homotopy between f and g of the form $h'\colon A \to P'(X)$ also provides a homotopy of the form $h\colon A \to P(X)$ by postcomposing with l. To show the converse, note that l is a homotopy equivalence, since r' and r are both homotopy equivalences. In particular, l admits a homotopy inverse $k\colon P(X) \to P'(X)$, and it follows that the diagram

$$\begin{tikzcd} & P'(X) \arrow[d, two heads, "{(d'_0,d'_1)}"] \\ P(X) \arrow[ur, "k"] \arrow[r, two heads, "{(d_0,d_1)}"'] & X \times X \end{tikzcd}$$

commutes up to homotopy. By the straightening lemma, Lemma 15.3.23, we may replace k up to homotopy by a map $k'\colon P(X) \to P'(X)$ which makes this diagram commute strictly in $\mathcal{E}$. But then every homotopy between f and g of the form $h\colon A \to P(X)$ defines a homotopy $h'\colon A \to P'(X)$ by postcomposing with k'. This finishes the proof. □

15.3.4 Morphisms of tribes

Since tribes are clans with a well-behaved notion of homotopy equivalences, there is a natural notion of morphisms between tribes:

Definition 15.3.29 (Morphism of tribes)**.** Let $\mathcal{E}$ and $\mathcal{E}'$ be tribes and let $F\colon \mathcal{E} \to \mathcal{E}'$ be a functor. We say that

(1) F is a *morphism of tribes* if it is a morphism of clans and preserves homotopy equivalences;

(2) F is a *strict morphism of tribes* if it is a morphism of clans and it preserves anodyne maps;

(3) F is a *(strict) equivalence of tribes* if F is an equivalence of categories and both F and its inverse are (strict) morphisms of tribes.

We show in Corollary 15.3.34 below that strict morphisms of tribes are indeed morphisms of tribes.

Warning 15.3.30. Joyal [Joy17] only considers strict morphisms of tribes, and simply refers to them as 'morphisms of tribes'; this convention was also used by Cisinski in his lectures. While in practice many morphisms of tribes that one encounters are indeed strict, we regard the more general definition as the preferred notion. It will play an important role in our discussion of *function extensionality* in **??**.

Lemma 15.3.31. *Let $\mathcal{E}$ be a tribe and let $f\colon A \to B$ be a morphism in f. Then the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ is a strict morphism of tribes.*

Proof. We proved in Lemma 14.2.9 that f^* is a morphism of clans, and we proved in Proposition 15.2.8 that it preserves anodyne morphisms. □

Morphisms of tribes have various alternative characterizations:

Proposition 15.3.32. *Let $\mathcal{E}$ and $\mathcal{E}'$ be tribes and let $F\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of clans. The following conditions are equivalent:*

(1) F is a morphism of tribes (i.e., F preserves homotopy equivalences);

(2) F sends anydone morphisms to homotopy equivalences;

(3) F sends strict path objects in $\mathcal{E}$ to path objects in $\mathcal{E}'$;

(4) For every type X in $\mathcal{E}$ there exists a path object $(P(X), r, d_0, d_1)$ of X such that the tuple $(F(P(X)), F(r), F(d_0), F(d_1))$ is a path object of $F(X)$ in $\mathcal{E}'$.

(5) Given two maps $f, g\colon A \to X$ in $\mathcal{E}$, if f and g are homotopic then also $F(f)$ and $F(g)$ are homotopic.

Proof. The implication (1) $\Longrightarrow$ (2) holds since every anodyne morphism is a homotopy equivalence by Proposition 15.3.18. For (2) $\Longrightarrow$ (3), consider a strict path object $X \overset{\sim}{\rightarrowtail} P(X) \twoheadrightarrow X \times X$ of X in $\mathcal{E}$. Given (2), the map $F(r)$ is a homotopy equivalence, and thus the induced diagram

$$F(X) \xrightarrow{F(r)} F(P(X)) \twoheadrightarrow F(X \times X) \simeq F(X) \times F(X)$$

is a path object for $F(X)$ in $\mathcal{E}'$ as F preserves fibrations. The implication (3) $\Longrightarrow$ (4) is immediate as every type admits a strict path object. The implication (4) $\Longrightarrow$ (5) is immediate from Proposition 15.3.28, and the implication (5) $\Longrightarrow$ (1) is clear from the definitions. □

Corollary 15.3.33. *A morphism of tribes $F\colon \mathcal{E} \to \mathcal{E}'$ induces a functor* $\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathcal{E}')$ *between homotopy categories.*

Proof. This is immediate from part (5) of Proposition 15.3.32. □

Corollary 15.3.34. *Every strict morphism of tribes is a morphism of tribes.*

Proof. If $F\colon \mathcal{E} \to \mathcal{E}'$ preserves anodynes, it in particular sends anodyne morphisms in $\mathcal{E}$ to homotopy equivalences in $\mathcal{E}'$, so the claim follows from Proposition 15.3.32. □

Lemma 15.3.35. *Consider tribes $\mathcal{E}$ and $\mathcal{E}'$ and asssume given an adjunction $F\colon E \rightleftarrows \mathcal{E}' :G$ such that both F and G are morphisms of tribes. Then F and G induce an adjunction*

$$\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathcal{E}) \rightleftarrows \mathrm{Ho}(\mathcal{E}') :\mathrm{Ho}(G)$$

on homotopy categories.

Proof. For types $X \in \mathcal{E}$ and $Y \in \mathcal{E}'$, there is a natural bijection

$$\mathrm{Hom}_{\mathcal{E}'}(FX, Y) \cong \mathrm{Hom}_{\mathcal{E}}(X, GY).$$

We have to show that two maps $f, g\colon FX \to Y$ are homotopic in $\mathcal{E}'$ if and only if their adjunct maps $\tilde{f}, \tilde{g}\colon X \to GY$ are homotopic in $\mathcal{E}$. For the "if"-direction, observe that we may express the map f in terms of $\tilde{f}$ as the following composite:

$$FX \xrightarrow{F\tilde{f}} FGY \xrightarrow{\varepsilon} Y,$$

where ε denotes the counit of the adjunction. Since F preserves homotopies, we see that $\tilde{f} \sim \tilde{g}$ implies $f \sim g$. The other direction is entirely dual, finishing the proof. □

Lemma 15.3.36. *Let $F\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of tribes. Then for every type $B \in \mathcal{E}$, the induced functor $F(B)\colon \mathcal{E}(B) \to \mathcal{E}'(FB)$ is a morphism of tribes. Furthermore, if F is strict then also $F(B)$ is strict.*

Proof. This is left as an exercise, see Exercise 14.5.1. □

15.3.5 Strong deformation retracts

The class of anodyne maps admit another characterization as those maps which are strong deformation retracts, in the following sense:

Definition 15.3.37. A *strong deformation retract* is a map $i\colon A \to B$ such that there is a map $q\colon B \to A$ such that $qi = 1_A$ as well as a homotopy $h\colon B \to P(B)$ with $d_0 = iq$ and $d_1 h = 1_B$, and such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[d, tail, "r"] \\
B \arrow[r, "h"] & P(B).
\end{tikzcd}$$

Remark 15.3.38. Informally, the last condition says that the h is a homotopy between iq and 1_B whose restriction to A is the constant homotopy on 1_A.

Proposition 15.3.39. *A morphism $i\colon A \to B$ in $\mathcal{E}$ is anodyne if and only if it is a strong deformation retract.*

Proof. The proof of Proposition 15.3.18 shows that any anodyne map is a strong deformation retract. Conversely, assume that $i\colon A \to B$ is a strong deformation retract. Consider the mapping path object of i:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, tail, "u"'] \arrow[dd, bend right=60, equal] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail, "r"] \arrow[dd, bend left=60, equal] \\
M(i) \arrow[r, "H"] \arrow[d, two heads, "\delta_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & P(B) \arrow[d, two heads, "d_0"] \\
A \arrow[r, "i"] & B.
\end{tikzcd}$$

Our next goal is to show that i is a retract of u. Define the map $s\colon B \to M(i)$ using the following diagram:

$$\begin{tikzcd}
B \arrow[dr, dashed, "s"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "q"'] & & \\
& M(i) \arrow[r, "H"] \arrow[d, two heads, "\delta_0"] \arrow[dr, phantom, "\lrcorner", very near start] & P(B) \arrow[d, two heads, "d_0"] \\
& A \arrow[r, "i"] & B.
\end{tikzcd}$$

Note that we have $si = u$, since u was defined as the unique map satisfying $\delta_0 u = 1_A$ and $Hu = ri$ and we also have $\delta_0 si = qi = 1_A$ and $Hsi = hi = ri$. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[r, equal] \arrow[d, tail, "u"] & A \arrow[d, "i"] \\
B \arrow[r, "s"] & M(i) \arrow[r, "\delta_1"] & B.
\end{tikzcd}$$

The left square commutes as $si = u$. The right square commutes as $\delta_1 u = i$, and the bottom composite is the identity. The diagram thus exhibits i as a retract of the anodyne map u, proving that i is anodyne by part (3) of Proposition 15.1.3. This finishes the proof. $\square$

15.3.6 Fiberwise homotopy equivalences

We continue to fix a tribe $\mathcal{E}$. For a type B in $\mathcal{E}$, the notions of homotopy and homotopy equivalence in particular make sense in the local tribe $\mathcal{E}(B)$ from Definition 15.2.7, leading to notions of *fiberwise homotopy* and *fiberwise homotopy equivalence*. The goal of this subsection is to explore these two notions and compare them with the notions homotopy and homotopy equivalences in $\mathcal{E}$.

Definition 15.3.40. Consider two fibrations $p\colon X \twoheadrightarrow B$ and $q\colon A \twoheadrightarrow B$, and let $f, g\colon A \to X$ be maps over B, i.e. we have $pf = g = pg$. We say that f and g are *fiberwise homotopic over B*, denoted by $f \sim_B G$, if they are homotopic as morphisms in the tribe $\mathcal{E}(B)$. Similarly, a morphism $f\colon A \to X$ over B is called a *fiberwise homotopy equivalence over B* if it is a homotopy equivalence in the tribe $\mathcal{E}(B)$.

Remark 15.3.41. By regarding a fibration $p\colon X \twoheadrightarrow B$ as an object of the local tribe $\mathcal{E}(B)$, we may form its path object inside $\mathcal{E}(B)$. We will denote such path object as follows:

$$\begin{tikzcd}
 & P_B(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\
X \arrow[ur, tail, "r"] \arrow[r, "\Delta"'] & X \times_B X.
\end{tikzcd}$$

Lemma 15.3.42. *For any map $f\colon A \to B$, the pullback functor*

$$f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$$

preserves fiberwise homotopy equivalences.

Proof. We proved in Proposition 15.2.8 that f^* is a morphism of tribes, and hence it sends homotopy equivalences in $\mathcal{E}(B)$ to homotopy equivalences in $\mathcal{E}(A)$. □

Proposition 15.3.43. *Let $f, g\colon A \to X$ be two morphisms in $\mathcal{E}(B)$. If f and g are fiberwise homotopic over B, then they are homotopic as morphisms in $\mathcal{E}$:*

$$f \sim_B g \implies f \sim g.$$

Proof. We may find a filler in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr] \arrow[d, tail] & & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\
P_B(X) \arrow[r] \arrow[urr, dashed] & X \times_B X \arrow[r] & X \times X.
\end{tikzcd}$$

Then any fiberwise homotopy $h\colon A \to P_B(X)$ between f and g induces a homotopy $A \to P(X)$ by composing with this map $P_B(X) \to P(X)$. □

Corollary 15.3.44. *Every fiberwise homotopy equivalence is a homotopy equivalence.* □

The following result gives a partial converse to Corollary 15.3.44:

Proposition 15.3.45. *Let $p\colon X \twoheadrightarrow B$ be a fibration. Then the map $p\colon (X,p) \to (B,1_B)$ in $\mathcal{E}(B)$ is a fiberwise homotopy equivalence over B if and only if $p\colon X \to B$ is a homotopy equivalence.*

Proof. If p is a fiberwise homotopy equivalence over B, it is a homotopy equivalence by Corollary 15.3.44. Conversely, assume that p is a homotopy equivalence. By Corollary 15.3.25, p admits a section $s\colon B \to X$. Since p is a homotopy equivalence, we have $sp \simeq 1_A$, so we may choose a homotopy $h\colon B \to P(B)$ exhibiting this, where $P(B)$ is a strict path object for B. Now consider the mapping-path object $M(s)$ of s:

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, tail, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "r"] \\
M(s) \arrow[r, "H"] \arrow[d, two heads, "{(\delta_0,\delta_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P(X) \arrow[d, two heads, "{(d_0,d_1)}"] \\
B\times X \arrow[r, "s\times 1_X"] & X\times X.
\end{tikzcd}$$

The commutativity of the bottom square expresses that we have equalities $d_0H = s\delta_0$ and $d_1H = \delta_1$ in $\mathcal{E}$. The map $u\colon B \overset{\sim}{\rightarrowtail} M(s)$ is defined as the unique map satisfying the equations

$$\delta_0 u = 1_B, \qquad \delta_1 u = s, \qquad Hu = rs.$$

We may now consider the following diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "p"'] & & \\
& M(s) \arrow[r, "H"] \arrow[d] & P(X) \arrow[d, two heads, "d_0"] \\
& B \arrow[r, "s"] & X.
\end{tikzcd}$$

The map φ is uniquely determined by the conditions

$$\delta_0\varphi = p, \qquad H\varphi = h.$$

In particular, we see that

$$\delta_1\varphi = d_1H\varphi = d_1h = 1_X,$$

and thus by uniqueness we have $\varphi i = u$.

Now choose a strict path object of X over B:

$$\begin{tikzcd}
& P_B(X) \arrow[d, two heads, "{(d_0',d_1')}"] \\
X \arrow[ur, tail, "r'"] \arrow[r, "\Delta"'] & X\times_B X.
\end{tikzcd}$$

We may now also construct the mapping path object of s in the tribe $\mathcal{E}(B)$:

$$\begin{tikzcd}
B \arrow[r, "s"] \arrow[d, tail, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "r'"] \\
M_B(s) \arrow[r, "H'"] \arrow[d, two heads, "{(\delta_0',\delta_1')}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_B(X) \arrow[d, two heads, "{(d_0',d_1')}"] \\
X \simeq B\times_B X \arrow[r, "s\times_B 1_X"] & X\times_B X.
\end{tikzcd}$$

Then the following solid diagram commutes, as both composites are s:

$$\begin{tikzcd}
B \arrow[r, "u'"] \arrow[d, tail, "u"'] & M_B(s) \arrow[d, two heads, "\delta_1"] \\
M(s) \arrow[r, "\delta_1"] \arrow[ur, dashed, "d"] & X.
\end{tikzcd}$$

We may thus find a filler d. But then we define $h' := H'd\varphi\colon X \to P_B(X)$. Since $\delta_1' = d_1'H'$ and $pd_1' = pd_0'$, we get

$$p\delta_1' = pd_1'H' = pd_0'H' = ps\delta_0' = \delta_0'.$$

Therefore we get

$$d_0'h' = d_0'H'd\varphi = s\delta_0'd\varphi = sp\delta_1'd\varphi = sp\delta_1\varphi = sp1_X = sp,$$

and similarly

$$d_1h' = d_1H'd\varphi = \delta_1 d\varphi = \delta_1\varphi = 1_X.$$

This shows that s is also a fiberwise homotopy inverse over B, finishing the proof. □

Corollary 15.3.46. *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\
& B &
\end{tikzcd}$$

in $\mathcal{E}(B)$. If f is a fibration, then f is a homotopy equivalence if and only if it is a fiberwise homotopy equivalence over B.

Proof. The "if"-direction is immediate from Corollary 15.3.44. For the "only if"-direction, it follows from Proposition 15.3.45 that f is fiberwise homotopy equivalence over Y. Applying Corollary 15.3.44 to the object (Y,q) in the tribe $\mathcal{E}(B)$, it follows that f is also a fiberwise homotopy equivalence over B, finishing the proof. □

Theorem 15.3.47. *For any type B in $\mathcal{E}$, the forgetful functor*

$$\mathcal{E}(B) \to \mathcal{E}\colon (X,p) \mapsto X$$

preserves and reflects homotopy equivalences.

Proof. Consider a map

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\
& B. &
\end{tikzcd}$$

We need to show that f is a homotopy equivalence if and only if f is a fiberwise homotopy equivalence. We already saw the “if”-direction in Corollary 15.3.44. To see the other direction, choose a factorization

$$\begin{tikzcd}
X \arrow[r, tail, "u"] & Z \arrow[r, two heads, "\pi"] & Y
\end{tikzcd}$$

of f into an anodyne followed by a fibration. We may regard Z as a fibration over B via the map $q\pi\colon Z \to B$, and thus regard it as an object of $\mathcal{E}(B)$. Doing this makes both maps u and π into maps in $\mathcal{E}(B)$. Since f is a homotopy equivalence and u is anodyne, also π is a homotopy equivalence. It thus follows from Corollary 15.3.46 that π is a fiberwise homotopy equivalence over B. It follows from Corollary 15.1.7 that u is anodyne in $\mathcal{E}(B)$, and thus is also a fiberwise homotopy equivalence by applying Proposition 15.3.18 to the tribe $\mathcal{E}(B)$. As $f = \pi u$, it follows that also f is a fiberwise homotopy equivalence over B, finishing the proof. □

Corollary 15.3.48. *Let $f\colon A \twoheadrightarrow B$ be any fibration. Then the functor*

$$f_!\colon \mathcal{E}(A) \to \mathcal{E}(B)\colon (X,p) \mapsto (X,fp)$$

preserves and reflects fiberwise homotopy equivalences.

Proof. Apply Theorem 15.3.47 to the tribe $\mathcal{E}(B)$ and use that $\mathcal{E}(B)(A,f) \simeq \mathcal{E}(A)$. □

Theorem 15.3.49. *Consider a pullback square*

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\
Y' \arrow[r, "v"] & Y.
\end{tikzcd}$$

(1) If p is a homotopy equivalence, then so is p'.

(2) If v is a homotopy equivalence, then so is u.

Proof. We start with part (1). Given that p is a homotopy equivalence, it follows from Proposition 15.3.45 that p is a homotopy equivalence over Y. Applying Lemma 15.3.42 to the functor

$$v^* \colon \mathcal{E}(Y) \to \mathcal{E}(Y'),$$

which is a morphism of tribes by Lemma 15.3.31, we get that p' is a fiberwise homotopy equivalence over Y', and thus in particular a homotopy equivalence.

For part (2), assume that v is a homotopy equivalence. We may factorize v as a composite

$$\begin{tikzcd} Y' \arrow[r, tail, "v'"] & Y'' \arrow[r, two heads, "r"] & Y, \end{tikzcd}$$

where v' is anodyne and r is a fibration. Since v and v' are homotopy equivalences, also r is a homotopy equivalence. Now consider the following pullback diagram:

$$\begin{tikzcd} X' \arrow[r, tail, "u'"] \arrow[d, two heads, "p'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X'' \arrow[r, "s"] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, two heads, "p"] \\ Y' \arrow[r, tail, "v'"] & Y'' \arrow[r, two heads, "r"] & Y. \end{tikzcd}$$

As v' is anodyne, also u' is anodyne, and thus in particular a homotopy equivalence. Similarly, as r is a homotopy equivalence, so is s by part (1). Since $u = su'$, it follows that also u is a homotopy equivalence, finishing the proof. □

15.4 Homotopy pullbacks

In this section, we introduce the notion of homotopy pullback squares in our fixed tribe $\mathcal{E}$. This leads to notions of *embeddings*, 0-*truncated types* and *n-truncated types* for all $n \geq -2$.

15.4.1 Homotopy pullback squares

Definition 15.4.1. A *homotopy pullback square* (or *homotopy cartesian square*) is a commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

in $\mathcal{E}$ such that there exists a factorization

$$\begin{tikzcd} Y \arrow[r, tail, "j"] & Z \arrow[r, two heads, "q"] & B \end{tikzcd}$$

of β into an anodyne map followed by a fibration such that the induced map

$$i = (\alpha, jg) \colon X \longrightarrow A \times_B Z$$

is a homotopy equivalence. Diagrammatically:

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, dashed, "\simeq"] \arrow[ddr, bend right, "\alpha"'] & & Y \arrow[d, tail, "j"] \\
& A \times_B Z \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, two heads, "q"] \\
& A \arrow[r, "f"] & B.
\end{tikzcd}$$

Proposition 15.4.2. *Consider a commutative square*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

in $\mathcal{E}$. Then the following conditions are equivalent:

(1) The square is homotopy cartesian.

(2) For any factorization of β into a homotopy equivalence $j \colon Y \longrightarrow Z$ followed by a fibration $q \colon Z \longrightarrow B$, the induced map

$$(\alpha, jg) \colon X \longrightarrow A \times_B Z$$

is a homotopy equivalence.

(3) The flipped square

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "g"'] & A \arrow[d, "f"] \\
Y \arrow[r, "\beta"] & B
\end{tikzcd}$$

is homotopy cartesian.

Proof. This is left as an exercise, see Exercise 15.7.5. □

Corollary 15.4.3. *Let*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, two heads, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, "\beta"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

be a pullback square in $\mathcal{E}$ such that β is fibration. Then this square is homotopy cartesian.

Proof. This is immediate from Proposition 15.4.2 by choosing the decomposition of β to be $Y \xlongequal{} Y \xrightarrow{\beta} B$. □

Proposition 15.4.4. *Let*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

be a commutative square in $\mathcal{E}$ such that f is a homotopy equivalence. Then this square is homotopy cartesian if and only if g is a homotopy equivalence.

Proof. This is left as an exercise, see Exercise 15.7.5. □

Proposition 15.4.5 (Pasting law homotopy pullback squares)**.** *Let*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
A \arrow[r] & B \arrow[r] & C
\end{tikzcd}$$

be a commutative diagram in $\mathcal{E}$ and assume that the right-hand square is homotopy cartesian. Then the left-hand square is homotopy cartesian if and only if the outer square is homotopy cartesian.

Proof. This is left as an exercise, see Exercise 15.7.5. □

Proposition 15.4.6. *For every morphism $f\colon A \longrightarrow B$, the commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "(1_A{,}f)"'] & B \arrow[d, "(1_B{,}1_B)"] \\
A \times B \arrow[r, "f\times 1"] & B \times B
\end{tikzcd}$$

is a homotopy pullback square.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "(1_A{,}f)"'] & B \arrow[d, "(1_B{,}1_B)"] \\
A \times B \arrow[r, "f\times 1"] \arrow[d, two heads, "\mathrm{pr}_1"'] & B \times B \arrow[d, two heads, "\mathrm{pr}_1"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

It follows from Corollary 15.4.3 that both the bottom square and the outer rectangle are homotopy pullback squares. It thus follows from Proposition 15.4.5 that also the top square is a homotopy pullback square, as desired. □

15.4.2 Homotopy pullbacks

We define the *homotopy pullback* of two morphisms in $\mathcal{E}$ with common target, and prove some basic properties of this construction.

Construction 15.4.7 (Homotopy pullback)**.** Given morphisms $f\colon A \to B$ and $v\colon Y \to B$, the *homotopy pullback* $A \times_B^h Y$ is defined via the following pullback diagram:

$$\begin{tikzcd}
A \times_B^h Y \arrow[r] \arrow[d, two heads, "\mathrm{pr}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & M(\beta) \arrow[d, two heads, "\delta_1"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

It comes equipped with canonical maps $\mathrm{pr}_A\colon A \times_B^h Y \twoheadrightarrow A$ and $\mathrm{pr}_Y\colon A \times_B^h Y \to Y$, where the latter is given by first projecting to $M(\beta)$ and then applying the map $\delta_0\colon M(\beta) \to Y$. The composite $A \times_B^h Y \to M(\beta) \to P(B)$ provides a canonical homotopy $f \circ \mathrm{pr}_A \sim v \circ \mathrm{pr}_B$.

Remark 15.4.8. Since the map $\delta_1\colon M(\beta) \to B$ is a fibration, this square is indeed a homotopy pullback square by Corollary 15.4.3.

Construction 15.4.9. Consider a commutative square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

in $\mathcal{E}$. We define the map

$$(u,g)\colon X \to A \times_B^h Y$$

as the unique dashed arrow which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[d, dashed, "(u{,}g)"] \arrow[dd, bend right=60, "u"'] & & Y \arrow[d, tail, "u"'] \arrow[dd, bend left=60, "\beta"] \\
A \times_B^h Y \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & M(\beta) \arrow[d, two heads, "\delta_1"'] \\
A \arrow[rr, "f"] & & B.
\end{tikzcd}$$

Notice that we have $u = \mathrm{pr}_A \circ (u,g)$ and $g = \mathrm{pr}_Y \circ (u,g)$.

More generally, assume that the above square does not commute strictly in $\mathcal{E}$ but only up to a specified homotopy $\alpha\colon v \circ g \sim f \circ u$. We may then define maps $(g,\alpha)\colon X \to M(\beta)$ and

$(u,g)\colon X \to A \times^h_B Y$ as the unique dashed arrows making the following diagrams commute:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\alpha"] \arrow[dr, dashed, "{(g,\alpha)}"] \arrow[ddr, bend right, "g"'] & & \\
& M(\beta) \arrow[r] \arrow[d, two heads, "\delta_0"] \arrow[dr, phantom, "\lrcorner", very near start] & P(B) \arrow[d, two heads, "d_0"] \\
& Y \arrow[r, "\beta"] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[drr, bend left, "{(g,\alpha)}"] \arrow[dr, dashed, "{(u,g)}"] \arrow[ddr, bend right, "u"'] & & \\
& A \times^h_B Y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & M(\beta) \arrow[d, two heads, "\delta_1"] \\
& A \arrow[r, "f"] & B.
\end{tikzcd}$$

We still have $\mathrm{pr}_A \circ (u,g) = u$ and $\mathrm{pr}_Y \circ (u,g) = g$.

Lemma 15.4.10. *A commutative square as in Construction 15.4.9 is a homotopy pullback square if and only if the map $(\alpha,g)\colon X \to A \times^h_B Y$ is a homotopy equivalence.*

Proof. This is an immediate consequence of part (2) of Proposition 15.4.2. □

Lemma 15.4.11. *Consider morphisms $f\colon A \to B$ and $\beta\colon Y \to B$. Then there is a homotopy equivalence*

$$A \times^h_B Y \simeq Y \times^h_B A.$$

Proof. Consider the two mapping path objects

$$A \overset{u_f}{\rightarrowtail} M(f) \overset{\delta_{1,f}}{\twoheadrightarrow} B$$

and

$$Y \overset{u_\beta}{\rightarrowtail} M(\beta) \overset{\delta_{1,\beta}}{\twoheadrightarrow} B$$

for the maps f and β. It will suffice to show that the maps u_f and u_β induce homotopy equivalences

$$A \times^h_B Y = A \times_B M(\beta) \xrightarrow{\simeq} M(f) \times_B M(\beta) \xleftarrow{\simeq} M(f) \times_B Y \cong Y \times_B M(f) = Y \times^h_B A.$$

Indeed this follows from part (2) of Theorem 15.3.49: the map $A \times_B M(\beta) \to M(f) \times_B M(\beta)$ is a base change of the equivalence $u_f\colon A \xrightarrow{\sim} M(f)$ along the fibration $M(f) \times_B M(\beta) \twoheadrightarrow M(f)$, and similarly the map $M(f) \times_B Y \to M(f) \times_B M(\beta)$ is a base change of u_β along a fibration. □

15.4.3 Embeddings

We introduce the notion of an *embedding* of types and prove some basic properties about embeddings.

Definition 15.4.12 (Embedding)**.** A morphism $u\colon A \to B$ in $\mathcal{E}$ is called an *embedding* if the commutative square

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "u"] \\
A \arrow[r, "u"] & B
\end{tikzcd}$$

is a homotopy pullback square.

Lemma 15.4.13. *Every homotopy equivalence is an embedding.*

Proof. This is an immediate consequence of Proposition 15.4.4. □

Lemma 15.4.14. *Let $u\colon A \twoheadrightarrow B$ be an fibration which is also a monomorphism in $\mathcal{E}$. Then u is an embedding.*

Proof. As u is a monomorphism, the above square is a pullback square in $\mathcal{E}$, and since u is a fibration it follows from Corollary 15.4.3 that it is also a homotopy pullback square. □

The following equivalent characterization of embeddings is frequently useful:

Lemma 15.4.15. *A morphism $u\colon A \to B$ is an embedding if and only if the following commutative square is homotopy cartesian:*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "{(1_A,1_A)}"'] & B \arrow[d, "{(1_B,1_B)}"] \\
A \times A \arrow[r, "u\times u"] & B \times B
\end{tikzcd}$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "{(1_A,1_A)}"'] & A \arrow[r, "u"] \arrow[d, "{(1_A,u)}"] & B \arrow[d, "{(1_B,1_B)}"] \\
A \times A \arrow[r, "1\times u"] \arrow[d, two heads, "\mathrm{pr}_2"'] & A \times B \arrow[r, "u\times 1"] \arrow[d, two heads, "\mathrm{pr}_2"] & B \times B \\
A \arrow[r, "u"] & B.
\end{tikzcd}$$

The upper right square is a homotopy pullback square by Proposition 15.4.6, while the bottom left square is a homotopy pullback square by Corollary 15.4.3. By applying the pasting law of homotopy pullback squares (Proposition 15.4.5) twice, it follows that the left outer rectangle is a homotopy pullback square if and only if the upper outer rectangle is a homotopy pullback square. This finishes the proof. □

Lemma 15.4.16. *Consider a homotopy pullback square*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

in $\mathcal{E}$. If v is an embedding, then also u is an embedding.

Proof. This is left as an exercise for the reader, see Exercise 1.6.7 ◻

Lemma 15.4.17. *Let $u\colon A \to B$ and $v\colon B \to C$ be morphisms in $\mathcal{E}$ and assume that v is an embedding. Then the square*

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u"'] & A \arrow[d, "vu"] \\
B \arrow[r, "v"] & C
\end{tikzcd}$$

is a homotopy pullback square.

Proof. This is immediate from the pasting law of homotopy pullback squares applied to the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u"'] & A \arrow[d, "u"] \\
B \arrow[r, equal] \arrow[d, equal] & B \arrow[d, "v"] \\
B \arrow[r, "v"] & C.
\end{tikzcd}$$

Indeed, the bottom square is a homotopy pullback square by assumption on v, while the top square is a homotopy pullback square by Corollary 15.4.3. ◻

Lemma 15.4.18. *Consider a commutative triangle*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "vu"'] & & B \arrow[dl, "v"] \\
& C &
\end{tikzcd}$$

in $\mathcal{E}$. If v is an embedding, then u is an embedding if and only if vu is an embedding.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, equal] \arrow[d, "u"] & A \arrow[d, "vu"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C.
\end{tikzcd}$$

Since v is an embedding, the right square is a homotopy pullback square by Lemma 1.2.4. It follows from the pasting law of homotopy pullback squares, Proposition 15.4.5, that the left square is homotopy cartesian if and only if the outer rectangle is homotopy cartesian, proving the claim. □

Corollary 15.4.19. *Consider a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, two heads, "f"'] & B \arrow[d, two heads, "g"] \\
C \arrow[r, "v"] & D
\end{tikzcd}$$

where f and g are fibrations and v is an embedding. Then u is an embedding if and only if the induced map $(u,f)\colon A \to B \times_D C$ is an embedding.

Proof. The map (u,f) fits into the following commutative diagram:

$$\begin{tikzcd}
A \arrow[d, "{(u,f)}"'] \arrow[dr, "u"] & \\
B \times_C D \arrow[r, "\mathrm{pr}_B"] \arrow[d, two heads, "\mathrm{pr}_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "g"] \\
C \arrow[r, "v"] & B.
\end{tikzcd}$$

Since v is an embedding, it follows from Lemma 1.2.3 that also pr_B is an embedding. The result thus follows from Lemma 1.2.5. □

Lemma 15.4.20. *Consider a commutative diagram*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "v"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\
C' \arrow[r, "w"] & C.
\end{tikzcd}$$

If v and w are embeddings and the composite square is homotopy cartesian, then also the top square is homotopy cartesian.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
A' \arrow[r, equal] \arrow[d, "f'"'] & A' \arrow[r, "u"] \arrow[d, "{(g'f',fu)}"] & A \arrow[d, "f"] \\
B'(g',v) \arrow[r] & C' \times^h_C B \arrow[r] \arrow[d] & B \arrow[d] \\
& C' \arrow[r, "w"] & C.
\end{tikzcd}$$

Since w is an embedding, the map $C' \times^h_C B \to B$ is an embedding by Lemma 1.2.3. Since $v\colon B' \to B$ is also an embedding, it thus follows from Lemma 1.2.5 that the map $(g', v)\colon B' \to C' \times^h_C B$ is an embedding, and thus by Lemma 1.2.4 the top left square is a homotopy pullback square. Since the right bottom square and right outer rectangle are homotopy pullback squares, so is the top right square by the pasting law Proposition 15.4.5. Another instance of Proposition 15.4.5 shows that the outer rectangle is a homotopy pullback square, as desired. ◻

15.4.4 Truncated types

In this subsection, we discuss the notion of *n-truncated types* for every $n \geq -2$.

Definition 15.4.21 (Contractible types)**.** A type A is called *contractible* if the map $p_A\colon A \to *$ is a homotopy equivalence.

Definition 15.4.22 (Proposition)**.** A type A is called a *proposition* if the map $p_A\colon A \to *$ is an embedding.

Definition 15.4.23 (0-truncated type)**.** A type A is called *0-truncated* if the diagonal map $\Delta\colon A \to A \times A$ is an embedding.

These three notions are three special cases of the notion of *n-truncated types*:

Definition 15.4.24 (n-truncated type)**.** For an integer $n \geq -2$, we inductively define what it means for a morphism $f\colon A \to B$ of types to be *n-truncated*:

- When $n = -2$, we say f is (-2)-truncated if and only if it is a homotopy equivalence;
- When $n > -2$, we say f is n-truncated if and only if the map $(1_A, 1_A)\colon A \to A \times^h_B A$ to the homotopy pullback of f along itself is $(n-1)$-truncated.

A type A is called *n-truncated* if the map $A \to *$ is n-truncated.

Exercise 15.4.25 (Exercise 15.7.13)**.** Show that the product of two n-truncated types is again n-truncated.

The condition for a morphism of types to be n-truncated can be formulated in a way that does not reference homotopy pullbacks:

Lemma 15.4.26. *Let $f\colon A \to B$ be a morphism of types and let $n \geq -1$.*

(1) If f is a fibration, then f is n-truncated if and only if the diagonal map $\Delta\colon A \to A \times_B A$ is $(n-1)$-truncated.

(2) For arbitrary f, f is n-truncated if and only if it can be factored into an anodyne map $i\colon A \to E$ followed by an n-truncated fibration $p\colon E \twoheadrightarrow B$.

Proof. Part (1) is immediate from Corollary 15.4.3, which implies that $A \times_B A$ is a homotopy pullback of f along itself. Part (2) is immediate from part (1) and the fact that i is a homotopy equivalence. □

15.5 Terms of types

As was explained in Chapter 19, dependent type theory does not only speak of *types* but also of *terms of types*. In this section, we will establish the terminology regarding terms of types.

Definition 15.5.1 (Term in context)**.** Let $X \in \mathcal{E}$ be a type. For another type Γ, we define a *term of X in context* Γ to be a morphism of types $x\colon \Gamma \to X$. If Γ is the terminal type $*$, we say that x is an *absolute term*, or a *term in the absolute context.*

Observe that a term x of X in context Γ may be identified with a section of the projection map $\mathrm{pr}_1\colon \Gamma \times X \twoheadrightarrow \Gamma$:

$$\begin{tikzcd}
\Gamma \arrow[r, "(1_\Gamma{,}x)"] \arrow[dr, equal] & \Gamma \times X \arrow[d, two heads, "\mathrm{pr}_1"] \\
& \Gamma.
\end{tikzcd}$$

We may regard the projection map pr_1 as an object X_Γ of $\mathcal{E}(\Gamma)$, and as such it is the incarnation of X as a type 'in context Γ'. Hence a term of X in context Γ is simply an absolute term of the type X_Γ, explaining our terminology. The fact that we may reinterpret every term as an absolute term of another type also motivates the following convention:

Convention 15.5.2. Given a type X and a term x, we will often denote the context of x by $\{x\}$, even though it is allowed to be an arbitrary type.

In practice, statements about terms can usually be reduced to the universal term:

Definition 15.5.3 (Universal term)**.** For a type X, we refer to the identity map $1_X\colon X \to X$ as the *universal term* of X. It is a term in context X.

In practice, statements about terms are frequently tautologically true by considering the universal term. However, they often allow for useful shifts in perspective, especially when combined with Convention 15.5.2. As a first example, let us explain how it provides another characterization of the 'fiberwise homotopy equivalences' from Definition 15.3.40.

Definition 15.5.4 (Fiber)**.** Let $p \colon P \twoheadrightarrow X$ be a fibration in $\mathcal{E}$ and let x be a term of X. We define the type $P(x)$ via the following pullback square in $\mathcal{E}$:

$$\begin{tikzcd}
P(x) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, two heads, "p"] \\
\{x\} \arrow[r] & X.
\end{tikzcd}$$

We may regard $P(x)$ as an object of the local tribe $\mathcal{E}(\{x\})$, and this resulting object will be called the *fiber of p over x*.

Lemma 15.5.5. *Let $\Gamma \in \mathcal{E}$ be a type and consider a morphism in $\mathcal{E}(B)$ of the form*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\
& B. &
\end{tikzcd}$$

Then f is a fiberwise homotopy equivalence if and only if for every term b of B the induced map $f(b) \colon X(b) \to Y(b)$ on fibers is a homotopy equivalence in the tribe $\mathcal{E}(b)$.

Proof. One implication is clear by applying the assumption to the universal term $b = 1_B \colon B \to B$. The other implication follows directly from the fact that the pullback functor $b^* \colon \mathcal{E}(B) \to \mathcal{E}(\{b\})$ preserves homotopy equivalences by Lemma 15.3.42. □

As a second example, it allows us to talk constructively about 'surjective maps of types' without needing to worry about the axiom of choice:

Definition 15.5.6 (Surjection of types)**.** A morphism of types $f \colon A \to B$ is called a *surjection* if for every term b of B there exists a term a of A such that $f(a) \sim b$.

Lemma 15.5.7. *A morphism of types $f \colon A \to B$ is a surjection if and only if it admits a section up to homotopy, i.e. a map $s \colon B \to A$ equipped with a homotopy $fs \sim 1_B$.*

Proof. First assume that f is a surjection. Taking b to be the universal term $b = 1_B \colon B \to B$, we get by assumption a term $a \colon B \to A$ such that there is a homotopy $f(a) \simeq b$ as terms of B_B in the local tribe $\mathcal{E}(B)$. But this precisely the condition that the composite $f \circ a \colon B \to B$ is homotopic to the identity of B, so that a is the desired section up to homotopy.

Conversely, assume that $s \colon B \to A$ is a section up to homotopy. For every term $b \colon \Gamma \to B$ of B, we may define a as the composite $a := s(b) = s \circ b \colon \Gamma \to A$ so that we have $f(a) = f \circ s \circ b \simeq \mathrm{id}_B \circ b = b$. This shows that f is a surjection. □

We will end this section by showing that a morphism of types is a homotopy equivalence if and only if it is a surjective embedding.

Lemma 15.5.8. *Let $u\colon A \to B$ be an embedding which admits a section $s\colon B \to A$. Then u is an equivalence.*

Proof. As $us = 1_B$, it remains to show that $su \sim 1_A$. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, dashed, "f"] \arrow[rr, bend left, equal] \arrow[d, "u"'] & A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "u"] \\
B \arrow[r, "s"] & A \arrow[r, "u"] & B.
\end{tikzcd}$$

Since the right square is homotopy cartesian by assumption on u, there exists a map $f\colon A \to A$ making the diagram commute up to homotopy. It follows that $su \sim f \sim 1_A$, finishing the proof. □

Proposition 15.5.9. *A morphism of types $f\colon A \to B$ is a homotopy equivalence if and only if it is both an embedding and a surjection.*

Proof. If f is a homotopy equivalence, it is clear that f is both an embedding and a surjection.

Conversely, assume that f is an embedding and a surjection. Let section $s\colon B \to A$ be a section up to homotopy, i.e. we have $fs \sim 1_B$. We may factor f as a composite $A \overset{v}{\rightarrowtail} C \overset{g}{\twoheadrightarrow} B$ where v is anodyne and g is a fibration. Since v is a homotopy equivalence, it is in particular an embedding by Lemma 15.4.13, and hence by Lemma 1.2.5 we see that g is an embedding. The map g again admits a section up to homotopy, and thus by the Straightening Lemma 15.3.23 we see that g admits a strict section. It thus follows from Lemma 1.2.8 that g is a homotopy equivalence, and consequently f is a homotopy equivalence. □

15.6 Homotopy type theory

So far, we have merely focused on the homotopical aspects of tribes. In the remainder of this chapter, we study the interaction of homotopy theory with dependent type theory, a field known under the name *homotopy type theory*.

Definition 15.6.1 (Homotopy type theory)**.** A *homotopy type theory* (or *Martin-Löf dependent dependent type theory*) is a clan $\mathcal{E}$ which is both a tribe and a dependent type theory.

If $\mathcal{E}$ and $\mathcal{E}'$ are homotopy type theories, we say that a morphism of clans $F\colon \mathcal{E} \to \mathcal{E}'$ is a *morphism of homotopy type theories* if it is both a morphism of tribes and a morphism of dependent type theories.

Example 15.6.2 (Exercise 15.7.19)**.** The following are examples of homotopy type theories:

- The category $\mathcal{E} = \mathrm{Set}$ of sets, in which all maps are fibrations;
- The category $\mathcal{E} = \mathrm{Grpd}$ of groupoids, where fibrations are isofibrations;
- The category $\mathcal{E} = \mathrm{Kan}$ of Kan complexes, in which the fibrations are the Kan fibrations;
- The category $\mathcal{E} = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$ of presheaves on any small category C.

Lemma 15.6.3. *Let $\mathcal{E}$ be a homotopy type theory. Then for every object $B \in \mathcal{E}$, the local clan $\mathcal{E}(B)$ is a homotopy type theory.*

Proof. This is immediate from Lemma 15.2.6 and Lemma 14.4.3. □

Lemma 15.6.4. *Let $f\colon A \to B$ be a morphism in a homotopy type theory $\mathcal{E}$. Then the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ is a morphism of homotopy type theories.*

Proof. This is immediate from Lemma 15.3.31 and Example 14.4.5. □

Example 15.6.5. Taking $B = *$, we get that the functor

$$\mathcal{E} \to \mathcal{E}(A)\colon X \mapsto (X \times A, \mathrm{pr}_A)$$

is a morphism of homotopy type theories. Similarly, for any term $a : A$ given by a morphism of the form $a\colon * \to A$, we obtain a morphism of homotopy type theories

$$a^*\colon \mathcal{E}(A) \to \mathcal{E}.$$

15.7 Exercises Chapter 15

Exercise 15.7.1. Consider the clan Top of topological spaces and Hurewicz fibrations from Example 14.1.7. Recall that a continuous map $i\colon A \to X$ is called a *strong deformation retract* if there exist continuous maps $q\colon X \to A$ and $h\colon I \times X \to X$ satisfying

(i) $qi = 1_A$,

(ii) $h(0,x) = x$ and $h(1,x) = iq(x)$ for all $x \in X$, and

(iii) $h(t,i(a)) = i(a)$ for all $a \in A$ and $t \in I$.

Prove the following:

(a) The anodyne maps in the clan Top are precisely the strong deformation retracts.

(b) For a topological space X, the path space $P(X) = \mathrm{Map}([0,1], X)$, equipped with the compact-open topology, is a path object for X.

(c) The clan Top is a tribe.

Exercise 15.7.2. Consider the clan Cat_1 of small 1-categories and isofibrations from Example 14.1.8.

(a) Prove that the anodyne maps in the clan Cat_1 are precisely the equivalences of 1-categories $i\colon A \to B$ which are injective on objects.

(b) Let C be a small 1-category and let $P(C)$ denote the full subcategory of the arrow category $\mathrm{Arr}(C)$ spanned by the isomorphisms. Show that $P(C)$ is a path object for C.

(c) Prove that the clan Cat_1 is a tribe.

Exercise 15.7.3. Let $F\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of tribes. Show that for every type $B \in \mathcal{E}$, the induced functor $F(B)\colon \mathcal{E}(B) \to \mathcal{E}(F(B))$ is a morphism of tribes. Show that $F(B)$ is a *strict* morphism of tribes whenever F is.

Exercise 15.7.4. Show that a map $u\colon A \to B$ in a tribe $\mathcal{E}$ is a homotopy equivalence if and only if its image in the homotopy category $\mathrm{Ho}(\mathcal{E})$ is an isomorphism.

Homotopy pullbacks

Exercise 15.7.5. In the following, let $\mathcal{E}$ be a tribe. Consider a commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd} \tag{15.1}$$

in $\mathcal{E}$ and let $B \overset{\sim}{\rightarrowtail} B' \twoheadrightarrow D$ be a factorization of $B \to D$ into an anodyne map followed by a fibration. Note that the pullback $B' \times_D C$ exists in $\mathcal{E}$. We say that the square (15.1) is *homotopy cartesian* (or a *homotopy pullback square*) if the morphism

$$A \to B' \times_D C$$

induced by the composites $A \to B \to B'$ and $A \to C$, is a homotopy equivalence:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, dashed, "\simeq"] & & B \arrow[d, tail] \\
& B' \times_D C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[d, two heads] \\
& C \arrow[r] & D.
\end{tikzcd}$$

(a) Show that the definition given above does not depend on the choice of the factorization of $B \to D$.

(b) Prove that a square (15.1) is homotopy cartesian if and only if the flipped square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

is homotopy cartesian.

(c) Show that any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] \\ C \arrow[r] & D \end{tikzcd}$$

in which the vertical arrows are homotopy equivalences, is a homotopy pullback square.

(d) Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & X \arrow[d] \\ C \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

be a commutative diagram in $\mathcal{E}$ and assume that the right-hand square is homotopy cartesian. Show that the left-hand square is homotopy cartesian if and only if the outer square is homotopy cartesian.

Exercise 15.7.6. Show that a morphism $f\colon A \to B$ in a tribe $\mathcal{E}$ is a homotopy equivalence if and only if the pullback functor

$$f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$$

induces an equivalence of categories on homotopy categories:

$$\mathrm{Ho}(f^*)\colon \mathrm{Ho}(\mathcal{E}(B)) \xrightarrow{\simeq} \mathrm{Ho}(\mathcal{E}(A)).$$

Exercise 15.7.7. Let $u\colon A \to B$ and $v\colon B \to C$ be two embeddings. Show that the composite $vu\colon A \to B$ is also an embedding.

Exercise 15.7.8. Consider a homotopy pullback square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

in $\mathcal{E}$ and assume that v is an embedding. Show that also u is an embedding.

Truncated types

Exercise 15.7.9. Show that a morphism is (-1)-truncated if and only if it is an embedding.

Exercise 15.7.10. Let A be a type.

(1) Show that A is contractible if and only if it is (-2)-truncated;

(2) Show that A is a proposition if and only if it is (-1)-truncated;

(3) Show that A is 0-truncated in the sense of Definition 15.4.23 if and only if it is 0-truncated in the sense of Definition 15.4.24.

Exercise 15.7.11. Show that the n-truncated morphisms are closed under composition.

Exercise 15.7.12. Show that the n-truncated morphisms are closed under homotopy pullback: given a homotopy pullback square

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
A \arrow[r, "g"] & B,
\end{tikzcd}$$

if β is n-truncated, then also α is n-truncated.

Exercise 15.7.13. Show that n-truncated morphisms are closed under product: if $f\colon A \to B$ and $f'\colon A' \to B'$ are n-truncated, then so is $f \times f'\colon A \times A' \to B \times B'$.

Exercise 15.7.14. Show that n-truncated morphisms are closed under retracts: given a retract diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\
B \arrow[r] & B' \arrow[r] & B
\end{tikzcd}$$

in which the horizontal composites are the identity, if f' is n-truncated then also f is n-truncated.

Exercise 15.7.15. Consider morphisms $f\colon A \to B$ and $g\colon B \to C$. Assume that gf is n-truncated and that g is $(n+1)$-truncated. Show that also f is n-truncated.

Exercise 15.7.16. Recall that a topological space X is n-truncated if and only if the homotopy groups $\pi_i(X)$ are trivial for $i \geq n+1$. In this exercise, we will prove the analogous statement for types.

For $n \geq 0$, we inductively define types X^{B^n}, $X^{S^{n-1}}$ and maps

$$\begin{tikzcd}
X \arrow[r, tail, "i_n"] & X^{B^n} \arrow[r, two heads, "\pi_n"] & X^{S^{n-1}}.
\end{tikzcd}$$

When $n = 0$, we set $X^{B^0} := X$ and $X^{S^{-1}} := *$, and define i_0 and π_n as

$$\begin{tikzcd} X \arrow[r, tail, "1_X"] & X \arrow[r, two heads, "p_X"] & *. \end{tikzcd}$$

Assuming that we have finished the construction for n, we define X^{S^n} by forming the pullback square

$$\begin{tikzcd} X^{S^n} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X^{B^n} \arrow[d, two heads, "\pi_n"] \\ X \arrow[r, "\pi_n \circ i_n"] & X^{S^{n-1}}. \end{tikzcd}$$

The maps $1_X \colon X \to X$ and $i_n \colon X \to X^{B^n}$ define a map $(1_X, i_n) \colon X \to X^{S^n}$, which we may factor into an anodyne followed by a fibration:

$$\begin{tikzcd} X \arrow[r, tail, "i_{n+1}"] & X^{B^{n+1}} \arrow[r, two heads, "\pi_{n+1}"] & X^{S^n}. \end{tikzcd}$$

Show that the type X is n-truncated if and only if the map $X \to X^{S^{n+1}}$ is a homotopy equivalence.

Homotopy type theory

Exercise 15.7.17. Let $\mathcal{E} = \text{Grpd}$ denote the tribe of groupoids with isofibrations as class of fibrations. Show that for every isofibration $p \colon E \to B$ the dependent product functor $p_* \colon \text{Grpd}(E) \to \text{Grpd}(B)$, constructed in Exercise 14.5.5, is a morphism of tribes.

Exercise 15.7.18. Establish a correspondence between isofibrations $p \colon E \twoheadrightarrow B$ of groupoids and pseudofunctors $B \to \text{Grpd}$. rcise 2.

Exercise 15.7.19. Show that the following are examples of homotopy type theories:

- The category $\mathcal{E} = \text{Set}$ of sets, in which all maps are fibrations;
- The category $\mathcal{E} = \text{Grpd}$ of groupoids, where fibrations are isofibrations;
- The category $\mathcal{E} = \text{Kan}$ of Kan complexes, in which the fibrations are the Kan fibrations;
- The category $\mathcal{E} = \text{Fun}(C^{\text{op}}, \text{Set})$ of presheaves on any small category C.

Exercise 15.7.20. Decide whether the map $\{1\} \hookrightarrow \{0, 1\}$ is a univalent fibration in the tribe Set of sets.

16 Simplicial type theory

In the previous two chapters, we have seen that the formalism of tribes provides a convenient framework for talking about abstract homotopy theory. In this chapter, we will introduce an enhancement of the notion of tribes that allows us to talk about abstract *category* theory. The additional structure we put on a tribe is directly inspired by the formalism of *simplicial type theory* introduced by Riehl and Shulman [RS17]. The main new ingredient of simplicial type theory is the introduction of a type Δ^1, called the *1-simplex*, which allows to talk about *morphisms* in a type C. The morphisms in C can be assembled into a type $\mathrm{Fun}(\Delta^1, C)$, called the *arrow category*. More generally we will require the existence of various *diagram categories* $\mathrm{Diag}(\Phi, C)$, allowing us to talk about commutative diagrams in a type. Using this structure, we then introduce two final axioms, the *Segal axiom* and the *Rezk axiom*, which ensure that compositions of morphisms are well-defined and that types behave like higher categories.

Convention. Since our eventual goal is to do synthetic category theory, we will from now on adopt the following terminology:

- Types in a tribe $\mathcal{E}$ will be called *synthetic categories*;
- Morphisms $F\colon C \to D$ of types will be called *functors*;
- Fibrations in $\mathcal{E}$ will be called *isofibrations*;
- The terminal object of $\mathcal{E}$ is denoted by Δ^0;
- We will refer to homotopy equivalences between types as *equivalences*.

Warning. The axioms used in this chapter are substantially different from the axioms used during Cisinski's lectures: whereas we directly assume the existence of the simplices Δ^n as part of the structure, Cisinski defined these inductively in terms of a certain *join operation*. We discuss this approach in Appendix A.

16.1 Simplicial tribes

The main feature which distinguishes categories from sets or groupoids is the existence of *non-invertible morphisms*. The first step towards describing categories using type theory is therefore to introduce a notion of morphisms in a type. More generally we would like to talk about *compositions of morphisms* in a type. We will encode this data with the help of *simplices*.

Definition 16.1.1 (Simplex category)**.** For an integer $k \geq 0$, let $[k]$ denote the partially ordered set (poset) $\{0 \leq 1 \leq \cdots \leq k\}$, regarded as a category. We define the *simplex category* Δ as the full subcategory of the category Cat spanned by the posets $[k]$ for $k \geq 0$.

Definition 16.1.2 (Simplicial tribe)**.** A *simplicial tribe* is a tribe $\mathcal{E}$ equipped with a functor $\Delta^\bullet \colon \Delta \to \mathcal{E}$ such that for every $[n] \in \Delta$ the functor $\Delta^n \times - \colon \mathcal{E} \to \mathcal{E}$ admits a right adjoint $\mathrm{Fun}(\Delta^n, -) \colon \mathcal{E} \to \mathcal{E}$ which is a morphism of tribes.

Remark 16.1.3. In practice, we will only need the first four simplices Δ^0, Δ^1, Δ^2 and Δ^3.

In the remainder of this chapter, we will fix a simplicial tribe $\mathcal{E}$. We shall next discuss how the simplicial structure leads to notions of morphisms, natural transformations and commutative triangles.

Definition 16.1.4 (Morphisms)**.** A *morphism* in a synthetic category C is defined as a functor $f \colon \Delta^1 \to C$. We refer to the synthetic category $\mathrm{Fun}(\Delta^1, C)$ as the *arrow category* of C, whose terms are precisely the morphisms in C.

The two inclusions $0, 1 \colon [0] \to [1] = \{0 \leq 1\}$ induce two maps $0, 1 \colon \Delta^0 \to \Delta^1$, and thus every morphism $f \colon \Delta^1 \to C$ has a *source/domain* $f(0) \colon \Delta^0 \to C$ and a *target/codomain* $f(1) \colon \Delta^0 \to C$. If f is a morphism in C with domain $x = f(0)$ and codomain $y = f(1)$, we will often write $f \colon x \to y$ and say that *f is a morphism from x to y*.

We write $\mathrm{ev}_0, \mathrm{ev}_1 \colon \mathrm{Fun}(\Delta^1, C) \to C$ for the source and target functors, given by evaluation at 0 and 1.

Definition 16.1.5 (Natural transformations)**.** More generally, if C and D are synthetic categories, we define a *natural transformation of functors $C \to D$* to be a functor $\alpha \colon C \to \mathrm{Fun}(\Delta^1, D)$. We define its *source* as $\mathrm{ev}_0 \circ \alpha \colon C \to D$ and its *target* as $\mathrm{ev}_0 \circ \alpha \colon C \to D$. Given two functors $f, g \colon C \to D$, a natural transformation whose source is f and whose target is g will also be denoted as $\alpha \colon f \Rightarrow g$. For $C = \Delta^0$ we recover the notion of morphisms discussed before.

Definition 16.1.6 (Commutative triangle)**.** Given a synthetic category C, we define a *commutative triangle in C* to be a functor $\sigma \colon \Delta^2 \to C$.

Observe that there are precisely three injective maps of posets $[1] \to [2]$ given by omitting one of the three elements in $[2]$:

$$\delta_0^2 \colon \{1 \le 2\} \hookrightarrow \{0 \le 1 \le 2\} \qquad \delta_1^2 \colon \{0 \le 2\} \hookrightarrow \{0 \le 1 \le 2\} \qquad \delta_2^2 \colon \{0 \le 1\} \hookrightarrow \{0 \le 1 \le 2\}.$$

We thus obtain three maps of simplices $\delta_0^2, \delta_1^2, \delta_2^2 \colon \Delta^1 \to \Delta^2$, and hence every commutative triangle σ in C has three underlying morphisms. We will frequently denote commutative triangles by

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & & z,
\end{tikzcd}$$

where $f = \sigma \circ \delta_2^2$, $g = \sigma \circ \delta_0^2$ and $h = \sigma \circ \delta_1^2$.

Definition 16.1.7 (Commutative square)**.** Given a synthetic category C, we define a *commutative square in C* to be a functor $\sigma \colon \Delta^1 \times \Delta^1 \to C$. We will frequently denote commutative triangles by

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\
z \arrow[r, "k"'] & w.
\end{tikzcd}$$

16.2 Diagram categories

In the previous section, we saw how the existence of the simplices Δ^n leads to various commutative diagrams in synthetic categories. However, the behavior of such diagrams is currently somewhat nontransparent: for example, every commutative square gives rise to two commutative triangles which agree along their diagonal, but it is not clear to what extent the original commutative square is determined by this data:

$$\begin{tikzcd}
x \arrow[d, "g"'] \arrow[dr, "l"] & & & x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "l"] & y \arrow[d, "h"] & & x \arrow[r, "f"] \arrow[dr, "l"'] & y \arrow[d, "h"] \\
z \arrow[r, "k"'] & w & \arrow[r, phantom, "\mapsfrom"] & z \arrow[r, "k"'] & w & \arrow[r, phantom, "\mapsto"] & & w.
\end{tikzcd}$$

In this section, we will introduce a more flexible formalism of diagram categories, which is directly inspired by the type theoretic approach of Riehl and Shulman [RS17] and in particular by their notion of *extension types*.

16.2.1 Diagram shapes

In this subsection, we introduce the collection of diagram shapes we want to allow for our diagram categories. Following Riehl and Shulman, we will take the diagram shapes to be suitable subshapes of cubes.

Definition 16.2.1 (Cube category)**.** We recall the definition of the *cube category* Cube:

For $n \geq 0$, consider the poset $[1]^n$ defined as the n-fold cartesian product of the poset $[1] = \{0 \leq 1\}$. Concretely, it consists of n-tuples $(x_1, \ldots, x_n)$ with $x_i \in \{0, 1\}$, and we have $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if and only if $x_i \leq y_i$ for every $i = 1, \ldots, n$. We refer to $[1]^n$ as the *n-dimensional cube*.

For $n \geq 0$, a morphism of posets $[1]^n \to [1]$ is called a *morphism of cubes* if it is of one of the following two forms:

- A projection map $\mathrm{pr}_i \colon [1]^n \to [1]$ onto the i-th component, for some $1 \leq i \leq n$;
- A constant map $\mathrm{const}_\varepsilon \colon [1]^n \to [1]$ with value $\varepsilon \in \{0, 1\}$.

For general $n, m \geq 0$, a map of posets $f \colon [1]^n \to [1]^m$ is called a *morphism of cubes* if each of its m components $f_j \colon [1]^n \to [1]$ is a morphism of cubes. Note that the composite of a morphism of cubes is again a morphism of cubes.

We write Cube for the subcategory of the category of posets whose objects are the cubes $[1]^n$ and whose morphisms are the morphisms of cubes.

We will often draw the cubes by using arrows instead of the relation symbol $\leq$, as in the following four examples:

$$\bullet$$

$$\begin{tikzcd} 0 \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[d] \\ (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

$$\begin{tikzcd} & (0,0,1) \arrow[rr] \arrow[dd] & & (0,1,1) \arrow[dd] \\ (0,0,0) \arrow[ur] \arrow[rr] \arrow[dd] & & (0,1,0) \arrow[ur] \arrow[dd] & \\ & (1,0,1) \arrow[rr] & & (1,1,1) \\ (1,0,0) \arrow[ur] \arrow[rr] & & (1,1,0) \arrow[ur] & \end{tikzcd}$$

Construction 16.2.2. The assignment $[1]^n \mapsto (\Delta^1)^n$ defines a functor Cube $\to \mathcal{E}$. To see this, it remains to associate a functor $(\Delta^1)^n \to (\Delta^1)^m$ to every morphism of cubes $f\colon [1]^n \to [1]^m$. Since maps into cartesian products are defined componentwise, we may reduce to the case $m = 1$. We are then in one of two cases:

- If $f = \mathrm{pr}_i\colon [1]^n \to [1]$ is the i-projection map for some $1 \le i \le n$, we take the induced functor $(\Delta^1)^n \to \Delta^1$ to also be the i-th projection map.
- If $f = \mathrm{const}_\varepsilon\colon [1]^n \to [1]$ is the constant map with value $\varepsilon \in \{0, 1\}$, we take the induced functor to be the composite

  $$(\Delta^1)^n \to \Delta^0 \xrightarrow{\varepsilon} \Delta^1,$$

  where the first map is the unique map to the terminal object.

Definition 16.2.3 (Simplex of a cube)**.** Consider a cube $[1]^n$ for some integer $n \ge 0$. Given another integer $k \ge 0$, we define a *k-simplex in* $[1]^n$ as a morphism of posets $\sigma\colon [k] \to [1]^n$, where $[k]$ denotes the poset $\{0 \le 1 \le \cdots \le k-1 \le k\}$. We say that σ is *non-degenerate* if it is injective. Given another simplex $\tau\colon [l] \to [1]^n$, we call τ a *subsimplex of σ* if there exists (not necessarily injective) map of posets $\alpha\colon [l] \to [k]$ such that $\tau = \sigma \circ \alpha$. In this case, we write $\tau \le \sigma$.

More explicitly, we may describe a k-simplex of $[1]^n$ as a $(k+1)$-tuple $\sigma = (v_0, v_1, \ldots, v_k)$ of vertices $v_i \in [1]^n$ satisfying the relation $v_{i-1} \le v_i$ for all $i = 1, \ldots, k$. With this notation, σ is non-degenerate if we have $v_{i-1} \ne v_i$ for all i. Another simplex $\tau = (w_0, \ldots, w_l)$ is a subsimplex of σ if we have $w_i \in \sigma$ for all $i = 0, \ldots, l$.

Definition 16.2.4 (Diagram shape)**.** A *diagram shape* is a pair (n, Φ), where $n \ge 0$ is a natural number and Σ is a collection of simplices of $[1]^n$ which is closed under subsimplices: if $\sigma \in \Phi$ and $\tau \le \sigma$ then also $\tau \in \Phi$. We will often abuse notation and denote a diagram shape just by Φ, leaving the dimension n implicit.

Given diagram shapes (n, Φ) and (n, Ψ) of the same dimension n, we say that Ψ is a *subshape* of Φ if $\sigma \in \Psi$ implies $\sigma \in \Phi$. We will also write $\Psi \subseteq \Phi$ in this situation.

Given diagram shapes of different dimensions, a *morphism of diagram shapes* from (n, Φ) to (m, Ψ) is a morphism of cubes $f\colon [1]^n \to [1]^m$ satisfying the condition that for every k-simplex $\sigma\colon [k] \to [1]^n$ in Φ, the k-simplex $f \circ \sigma\colon [k] \to [1]^m$ is in Ψ. Observe that in the case $f = \mathrm{id}_{[1]^n}$ this reduces to the definition of a subshape.

Since the morphisms of diagram shapes are clearly closed under composition, this defines a category Shape. It comes equipped with a forgetful functor Shape $\to$ Cube that assigns to a diagram shape the underlying cube in which it was embedded.

Let us give some examples of diagram shapes.

Example 16.2.5 (The n-cube)**.** For $n \geq 0$ we define the n-cube $(\Delta^1)^n$ as the subshape of $[1]^n$ consisting of *all* simplices. This construction defines a fully faithful functor

$$\mathrm{Cube} \hookrightarrow \mathrm{Shape}\,.$$

Example 16.2.6 (The standard n-simplex)**.** For $n \geq 0$, we define the *standard n-simplex* Δ^n as follows

$$\Delta^n := \{\sigma \mid \text{ for all } v = (x_1, \ldots, x_n) \in \sigma \text{ we have } x_n \leq x_{n-1} \leq \cdots \leq x_2 \leq x_1\}.$$

Here we follow Riehl-Shulman in the perhaps surprising reversal of order in the coordinates, which guarantees that the coordinate x_i parametrizes the i-th arrow in the spine of the simplex, cf. [RS17, Remark 3.3].

The first three standard simplices may be displayed as follows:

$$\begin{tikzcd}
 & & & & (0,0) \arrow[r] \arrow[dr] & (1,0) \arrow[d] \arrow[dl, phantom, "//"] \\
\bullet & & 0 \arrow[r] & 1 & (0,1) & (1,1). \\
\Delta^0 & & \Delta^1 & & \Delta^2 &
\end{tikzcd}$$

Lemma 16.2.7. *The assignment* $[n] \mapsto \Delta^n$ *extends to a fully faithful functor* $\Delta \hookrightarrow \mathrm{Shape}$.

Proof. Consider a morphism of posets $\alpha \colon [n] \to [m]$. We define map of posets $[1]^\alpha \colon [1]^n \to [1]^m$ as $[1]^\alpha(x_1, \ldots, x_n) := (y_1, \ldots, y_m)$, where y_j for $1 \leq j \leq m$ is defined as follows:

$$y_j := \begin{cases} 0 & j \leq \alpha(0) \\ x_i & \alpha(i-1) < j \leq \alpha(i) \text{ for some (necessarily unique) } 1 \leq i \leq n \\ 1 & j > \alpha(i) \text{ for all} i. \end{cases}$$

We leave it to the reader to verify that this is defines a functor $\Delta \to \mathrm{Shape}$ and that it is fully faithful. ◻

Example 16.2.8. Some other diagram shapes we will be using are the following:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[dr] & (1,0) & (0,0) \arrow[r] & (1,0) \arrow[d] & (0,0) \arrow[dr] & (1,0) \arrow[d] & (0,0) \arrow[r] \arrow[dr] & (1,0) \arrow[d] \\
(0,1) & (1,1), & (0,1) & (1,1), & (0,1) & (1,1), & (0,1) & (1,1). \\
\Lambda^2_0 & & \Lambda^2_1 & & \Lambda^2_2 & & \partial\Delta^2 &
\end{tikzcd}$$

In our definition of diagram shape, we have chosen a fixed embedding into some cube, and in particular a single shape might be embedded in several different ways into various cubes. We will now introduce a notion of *equivalence of shapes* which allows us to identify these various embeddings.

Definition 16.2.9 (Equivalence of shapes)**.** A morphism of shapes $f\colon (n,\Phi) \to (m,\Psi)$ is called an *equivalence of shapes* if for every non-degenerate simplex $\sigma\colon [k] \to [1]^n$ in Φ there is a unique simplex $\tau\colon [k] \to [1]^m$ in Ψ satisfying $\sigma = f \circ \tau$

Example 16.2.10 (Alternative 2-simplex)**.** It will sometimes be useful to consider the following alternative Δ^2_{alt} to the standard 2-simplex:

$$\Delta^2_{\mathrm{alt}} \quad = \quad \begin{tikzcd} (0,0) \arrow[d] \arrow[dr] & (1,0) \\ (0,1) \arrow[r] \arrow[ur, Rightarrow, shorten=4mm] & (1,1). \end{tikzcd}$$

Observe that it is equivalent to the standard 2-simplex Δ^2 via the swap-map $[1]^2 \to [1]^2\colon (x,y) \mapsto (y,x)$.

For future use, we will now record various constructions on diagram shapes:

Definition 16.2.11 (Intersection)**.** Let (n,Φ) be a diagram shape and let (n,Ψ) and (n,Ψ') be two subshapes. We define their *intersection* $\Psi \cap \Psi'$ as the subshape of Φ consisting of all simplices which are both in Ψ and in Ψ'.

Definition 16.2.12 (Product shape)**.** Let (n,Φ) and (m,Ψ) be diagram shapes. We define the *product shape* $(n+m, \Phi\times\Psi)$ as the shape consisting of those simplices $\sigma = (\sigma_1,\sigma_2)\colon [k] \to [1]^{n+m} \cong [1]^n \times [1]^m$ such that the first component $\sigma_1\colon [k] \to [1]^n$ is in Φ and the second component $\sigma_2\colon [k] \to [1]^m$ is in Ψ.

This construction defines a functor $-\times-\colon \mathrm{Shape}\times\mathrm{Shape} \to \mathrm{Shape}$.

16.2.2 Diagram categories

We will impose the existence of *diagram categories* $\mathrm{Diag}(\Phi, C)$ for every diagram shape Φ and every synthetic category C.

Construction 16.2.13 (Diagram category)**.** Let (n,Φ) be a diagram shape, and consider the poset Φ^{nondeg} of non-degenerate simplices in Φ, where we write $\tau \leq \sigma$ if τ is a subsimplex of σ in the sense of Definition 16.2.3. We then obtain a functor $\Phi^{\mathrm{nondeg}} \to \Delta$ which sends a k-simplex $\sigma_k\colon [k] \to [1]^n$ in Φ to the poset $[k]$. In particular, we obtain for every synthetic category C a functor

$$(\Phi^{\mathrm{nondeg}})^{\mathrm{op}} \to \mathcal{E}\colon \sigma_k \mapsto \mathrm{Fun}(\Delta^k, C).$$

We say that the simplicial tribe $\mathcal{E}$ *admits* Φ*-indexed diagram categories* if for every synthetic category C the above functor admits a limit. In this case we denote the limit by

$$\mathrm{Diag}(\Phi, C) := \lim_{\sigma_k \in (\Phi^{\mathrm{nondeg}})^{\mathrm{op}}} \mathrm{Fun}(\Delta^k, C)$$

and refer to it as the Φ*-indexed diagram category of* C. We further refer to its terms as Φ*-indexed diagrams in* C.

Remark 16.2.14. Denote by Δ/Φ the category of *simplices of* Φ, whose objects are (not necessarily degenerate) simplices $\sigma\colon [k] \to [1]^n$ in Φ, and whose morphisms from σ to $\tau\colon [l] \to [1]^n$ are maps $\alpha\colon [k] \to [l]$ such that $\sigma = \tau \circ \alpha$. Then Φ^{nondeg} is a full subcategory of Δ/Φ, and the inclusion $\Phi^{\mathrm{nondeg}} \hookrightarrow \Delta/\Phi$ is cofinal since every simplex uniquely factors through a degenerate simplex. Hence we could alternatively have defined

$$\mathrm{Diag}(\Phi, C) := \lim_{\sigma_k \in (\Delta/\Phi)^{\mathrm{op}}} \mathrm{Fun}(\Delta^k, C).$$

We will momentarily axiomatize the existence of diagram categories. First we need some more auxiliary constructions.

Construction 16.2.15 (Induced functors)**.** Let Φ be a diagram shape and assume that $\mathcal{E}$ admits Φ-indexed diagram categories. For every functor $p\colon C \to D$, we directly obtain an induced functor

$$p_*\colon \mathrm{Diag}(\Phi, C) \to \mathrm{Diag}(\Phi, D)$$

by taking the limit of the functors $p_*\colon \mathrm{Fun}(\Delta^k, C) \to \mathrm{Fun}(\Delta^k, D)$ over all $\sigma_k \in \Phi^{\mathrm{nondeg}}$. This construction determines a functor

$$\mathrm{Diag}(\Phi, -)\colon \mathcal{E} \to \mathcal{E}.$$

Construction 16.2.16 (Restriction functors)**.** Let $f\colon \Psi \to \Phi$ be a morphism of diagram shapes, and assume that $\mathcal{E}$ admits both Φ-indexed and Ψ-indexed diagram categories. We will construct for every synthetic category C a *restriction functor*

$$f^*\colon \mathrm{Diag}(\Phi, C) \to \mathrm{Diag}(\Psi, C).$$

Since $\mathrm{Diag}(\Psi, C)$ is defined as a limit of functor categories $\mathrm{Fun}(\Delta^k, C)$ over all non-degenerate simplices $\sigma_k\colon [k] \to [1]^m$ of Ψ, it will suffice to construct compatible collections of functors $\mathrm{Diag}(\Phi, C) \to \mathrm{Fun}(\Delta^k, C)$ for all $\sigma_k \in \Psi^{\mathrm{nondeg}}$. The map $f\colon \Psi \to \Phi$ sends the simplex $\sigma_k\colon [k] \to [1]^m$ to a simplex $f \circ \sigma\colon [k] \to [1]^n$ of Φ. Although this simplex might be degenerate, there is a unique non-degenerate simplex $\tau_l\colon [l] \to [1]^n$ of Φ with the same image as $f \circ \sigma$. In particular, there is a unique morphism $\alpha\colon [k] \to [l]$ of posets satisfying $f \circ \sigma = \tau \circ \alpha\colon [k] \to [1]^n$. The definition of $\mathrm{Diag}(\Phi, C)$ as a limit over all non-degenerate simplices of Φ now produces a projection functor $\mathrm{Diag}(\Phi, C) \to \mathrm{Fun}(\Delta^l, C)$ and thus we may define the required functor $\mathrm{Diag}(\Phi, C) \to \mathrm{Fun}(\Delta^k, C)$ as the composite

$$\mathrm{Diag}(\Phi, C) \to \mathrm{Fun}(\Delta^l, C) \xrightarrow{\alpha^*} \mathrm{Fun}(\Delta^k, C).$$

If $\sigma_{k'}$ is a subsimplex of σ_k in Ψ, the image $\tau_{l'}$ is a subsimplex of τ_l in Φ, and it is easy to verify that the functor $\mathrm{Diag}(\Phi, C) \to \mathrm{Fun}(\Delta^{k'}, C)$ agrees with the composite $\mathrm{Diag}(\Phi, C) \to$

$\mathrm{Fun}(\Delta^k, C) \to \mathrm{Fun}(\Delta^{k'}, C)$. In particular, these maps assemble into the desired restriction functor $f^*\colon \mathrm{Diag}(\Phi, C) \to \mathrm{Diag}(\Psi, C)$. We leave it to the reader to verify that we have $(\mathrm{id})^* = \mathrm{id}\colon \mathrm{Diag}(\Phi, C) \to \mathrm{Diag}(\Phi, C)$ and that $(g \circ f)^* = f^* \circ g^*\colon \mathrm{Diag}(\Xi, C) \to \mathrm{Diag}(\Phi, C)$ for a second morphism of diagram shapes $g\colon \Psi \to \Xi$.

Observe that this restriction functor is fully functorial in C: it defines a natural transformation

$$f^*\colon \mathrm{Diag}(\Phi, -) \to \mathrm{Diag}(\Psi, -)$$

of functors $\mathcal{E} \to \mathcal{E}$. If $\mathcal{E}$ admits Φ-indexed diagram categories for all diagrams Φ, then this construction defines a functor

$$\mathrm{Diag}(-,-)\colon \mathrm{Shape}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{E}.$$

Notation 16.2.17. If Ψ is a subshape of a diagram shape Φ, we will denote the restriction functor by

$$(-)|_{\Psi}\colon \mathrm{Diag}(\Phi, C) \to \mathrm{Diag}(\Psi, C).$$

Construction 16.2.18 (Comparison functors)**.** Consider two simplices Δ^n and Δ^m, and assume that $\mathcal{E}$ admits Φ-indexed diagram categories for $\Phi = \Delta^n \times \Delta^m$. We construct for every synthetic category C a *comparison functor*

$$\mathrm{Fun}(\Delta^n \times \Delta^m, C) \to \mathrm{Diag}(\Delta^n \times \Delta^m, C).$$

As in the previous construction, it suffices to construct a functor to $\mathrm{Fun}(\Delta^k, C)$ for every non-degenerate simplex σ_k of $\Delta^n \times \Delta^m$. Every such simplex corresponds to an injective map $\alpha\colon [k] \to [n] \times [m]$ of posets, whose components we will denote by $\alpha_1\colon [k] \to [n]$ and $\alpha_2\colon [k] \to [m]$. We thus obtain two functors $\alpha_1\colon \Delta^k \to \Delta^n$ and $\alpha_2\colon \Delta^k \to \Delta^m$, which we may combine into a functor $\alpha = (\alpha_1, \alpha_2)\colon \Delta^k \to \Delta^n \times \Delta^m$. The desired functor may now be given by

$$\alpha^*\colon \mathrm{Fun}(\Delta^n \times \Delta^m, C) \to \mathrm{Fun}(\Delta^k, C).$$

It is easy to verify that this map is compatible with passing to a subsimplex $\sigma_{k'} \leq \sigma_k$ and thus produce the desired functor to $\mathrm{Diag}(\Delta^n \times \Delta^m, C)$.

We are now ready to axiomatize the existence of diagram categories:

Tribe Axiom A (Diagram category axiom)**.** For every diagram shape Φ, the simplicial tribe $\mathcal{E}$ admits Φ-indexed diagram categories satisfying the following properties:

(A.1) For every subshape Ψ of a diagram shape Φ and any synthetic category C, the restriction functor $(-)|_{\Psi}\colon \operatorname{Diag}(\Phi, C) \to \operatorname{Diag}(\Psi, C)$ is an isofibration. More generally, if $p\colon C \twoheadrightarrow D$ is an isofibration, then the functor

$$\operatorname{Diag}(\Phi, C) \to \operatorname{Diag}(\Phi, D) \times_{\operatorname{Diag}(\Psi, D)} \operatorname{Diag}(\Psi, C)$$

induced by the commutative square

$$\begin{tikzcd}
\operatorname{Diag}(\Phi, C) \arrow[r, "|_{\Psi}"] \arrow[d, two heads, "p_*"'] & \operatorname{Diag}(\Psi, C) \arrow[d, two heads, "p_*"] \\
\operatorname{Diag}(\Phi, D) \arrow[r, "|_{\Psi}"] & \operatorname{Diag}(\Psi, D)
\end{tikzcd}$$

is an isofibration;

(A.2) For every pair of simplices Δ^n and Δ^m and every synthetic category C, the comparison functor

$$\operatorname{Fun}(\Delta^n \times \Delta^m, C) \to \operatorname{Diag}(\Delta^n \times \Delta^m, C)$$

is an isomorphism.

In the remainder of this section, we will establish various useful properties of diagram categories, like shape invariance, homotopy invariance and the excision property. Let us start with the former, which says that diagram categories are independent of the way their diagram shape is embedded into a cube:

Lemma 16.2.19 (Shape invariance)**.** *Given a synthetic category C and an equivalence of diagram shapes $f\colon (m, \Psi) \to (n, \Phi)$ in the sense of Definition 16.2.9, the restriction functor $f^*\colon \operatorname{Diag}(\Phi, C) \to \operatorname{Diag}(\Psi, C)$ is an isomorphism for every synthetic category C.*

Proof. Since f is an equivalence of shapes, it induces an isomorphism of posets $\Psi^{\text{nondeg}} \xrightarrow{\cong} \Phi^{\text{nondeg}}$, and hence the result is immediate from the definition of diagram categories. □

The following proposition tells us that we may construct diagrams in steps by breaking it up into smaller pieces:

Proposition 16.2.20 (Excision)**.** *For every diagram shape (n, Φ) equipped with subshapes Ψ and Ψ' satisfying $\Psi \cup \Psi' = \Phi$, the following square is a pullback square for all C:*

$$\begin{tikzcd}
\mathrm{Diag}(\Phi, C) \arrow[r, two heads, "(-)|_{\Psi}"] \arrow[d, two heads, "(-)|_{\Psi}"'] & \mathrm{Diag}(\Psi, C) \arrow[d, two heads, "(-)|_{\Psi\cap\Psi'}"] \\
\mathrm{Diag}(\Psi', C) \arrow[r, two heads, "(-)|_{\Psi\cap\Psi'}"] & \mathrm{Diag}(\Psi \cap \Psi', C).
\end{tikzcd}$$

Proof. Since Ψ and Ψ' are subshapes of Φ, their posets Ψ^{nondeg} and Ψ'^{nondeg} of non-degenerate simplices are subposets of Φ^{nondeg}, and their intersection is precisely the poset $(\Psi \cap \Psi')^{\mathrm{nondeg}}$ of non-degenerate simplices in their intersection. Since the union of Ψ^{nondeg} and $(\Psi')^{\mathrm{nondeg}}$ is all of Φ^{nondeg}, it immediately follows that a limit over Φ^{nondeg} can be computed by taking the limits over the subposets Ψ^{nondeg} and $(\Psi')^{\mathrm{nondeg}}$ and taking the pullback over the limit indexed by their intersection. But this is precisely what we had to prove. □

The most important example of excision is the fact that a commutative square in C is given by two commutative triangles which agree along their diagonal:

Lemma 16.2.21. *For every synthetic category C, there is a canonical pullback square*

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[r, two heads, "(-)|_{\Delta^2}"] \arrow[d, two heads, "(-)|_{\Delta^2_{\mathrm{alt}}}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}(\Delta^2, C) \arrow[d, two heads, "(\delta^2_1)^*"] \\
\mathrm{Fun}(\Delta^2_{\mathrm{alt}}, C) \arrow[r, two heads, "(\delta^2_1)^*"] & \mathrm{Fun}(\Delta^1, C).
\end{tikzcd}$$

Proof. This is an immediate consequence of Proposition 16.2.20, since the subshapes Δ^2 and Δ^2_{alt} of $\Delta^1 \times \Delta^1$ together cover the whole square and their intersection is (equivalent to) the 1-simplex Δ^1. □

Remark 16.2.22. Since the shape Δ^2 is equivalent to the shape Δ^2_{alt} by Example 16.2.10, we see that $\mathrm{Fun}(\Delta^2_{\mathrm{alt}}, C)$ is isomorphic to $\mathrm{Fun}(\Delta^2, C)$ for every C.

As a consequence of Lemma 16.2.21, we see that we may write commutative triangles in terms of partially degenerate commutative square:

Lemma 16.2.23. *For every synthetic category C, there is a canonical pullback square*

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^2, C) \arrow[r] \arrow[d, two heads, "(\delta^2_1)^*"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[d, two heads, "(-)|_{\Delta^2_{\mathrm{alt}}}"] & \mathrm{Fun}(\Delta^2, C) \arrow[l] \arrow[d, two heads, "(\delta^2_1)^*"] \arrow[dl, phantom, very near start, "\llcorner"] \\
\mathrm{Fun}(\Delta^1, C) \arrow[r, "s^2_1"] & \mathrm{Fun}(\Delta^2_{\mathrm{alt}}, C) & \mathrm{Fun}(\Delta^1, C). \arrow[l, "s^2_2"]
\end{tikzcd}$$

Proof. By symmetry it will suffice to produce the square on the left. This follows immediately from Lemma 16.2.21 and the pasting law of pullback squares, since the composite morphism of shapes $\Delta^1 \xrightarrow{\delta_1^2} \Delta^2_{\mathrm{alt}} \xrightarrow{s_1^2} \Delta^1$ is the identity. □

Corollary 16.2.24. *The restriction functor* $(-)|_{\Delta^2} \colon \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \to \mathrm{Diag}(\Delta^2, C)$ *admits a section.* □

Lemma 16.2.25 (Empty diagrams)**.** *Let* $\Phi = \emptyset$ *be the empty diagram shape, consisting of zero simplices. Then for every synthetic category* C *the canonical map* $\mathrm{Diag}(\emptyset, C) \to \Delta^0$ *is an isomorphism.*

Proof. The poset $\emptyset^{\mathrm{nondeg}}$ is empty, and a limit over an empty diagram is isomorphic to the terminal object. □

By part (A.1) of the diagram category axiom, the restriction functors $(-)|_\Psi \colon \mathrm{Diag}(\Phi, C) \to \mathrm{Diag}(\Psi, C)$ are isofibrations for every subshape Ψ of a shape Φ, and hence we may in particular consider the fibers of this functor. These fibers precisely correspond to the *extension types* in the simplicial type theory of [RS17]. As a special case we obtain hom groupoids:

Definition 16.2.26 (Hom groupoid)**.** Let C be any synthetic category. Given terms x and y of C, we define the *hom groupoid* $C(x,y)$ via the following pullback square:

$$\begin{tikzcd}
C(x,y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, C) \arrow[d, two heads, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
\{(x,y)\} \arrow[r, "(x,y)"] & C \times C.
\end{tikzcd}$$

Observe that the terms of $C(x,y)$ are precisely the morphisms in C of the form $f \colon x \to y$. We will sometimes also use the alternative notation $\mathrm{Hom}_C(x,y)$ for $C(x,y)$.

We shall now prove the 'homotopy invariance' of the diagram functors $\mathrm{Diag}(\Phi, -) \colon \mathcal{E} \to \mathcal{E}$.

Proposition 16.2.27 (Homotopy invariance)**.** *For every diagram shape* (n, Φ)*, the functor* $\mathrm{Diag}(\Phi, -) \colon \mathcal{E} \to \mathcal{E}$ *is a morphism of tribes.*

Proof. We prove the claim by induction on the number of non-degenerate simplices of Φ. If Φ has no simplices, then $\mathrm{Diag}(\Phi, -)$ is by Lemma 16.2.25 the constant functor on the terminal object, which is clearly a morphism of tribes. Hence assume that Φ admits $m \geq 1$ non-degenerate simplices and that we have proved the claim for shapes with fewer than m non-degenerate simplices. Choose a non-degenerate simplex σ_k of highest possible dimension k. Then the collection of simplices $\Phi \setminus \{\sigma_k\}$ of $[1]^n$ is still a diagram shape, and hence is a subshape of Φ. Write $\Delta^k_\sigma \subseteq \Phi$ for the subshape consisting of all subsimplices of σ_k and write $\partial\Delta^k_\sigma := \Delta^k_\sigma \setminus \{\sigma_k\}$ for the intersection with $\Phi \setminus \{\sigma_k\}$. By excision, we thus have for every synthetic category C an isomorphism

$$\mathrm{Diag}(\Phi, C) \xrightarrow{\cong} \mathrm{Diag}(\Phi \setminus \{\sigma_k\}, C) \times_{\mathrm{Diag}(\partial\Delta^k_\sigma, C)} \mathrm{Diag}(\Delta^k_\sigma, C).$$

By the induction hypothesis, the functors $\mathrm{Diag}(\Phi \setminus \{\sigma_k\}, -)$ and $\mathrm{Diag}(\partial\Delta^k_\sigma, -)$ are morphisms of tribes as their indexing diagrams have strictly less simplices than Φ. Furthermore, the functor $\mathrm{Diag}(\Delta^k_\sigma, -)$ is isomorphic to $\mathrm{Fun}(\Delta^k, -)$ by Lemma 16.2.19, and thus it is a morphism of tribes by the very definition of a simplicial tribe. By part (A.1) from Tribe Axiom A, the induced functors between them are all isofibrations, and it follows that the pullback of these three functors is again a morphism of tribes. This finishes the proof. □

Proposition 16.2.28 (Iterated diagrams)**.** *For all diagram shapes Φ and Ψ and every synthetic category C, there is a preferred natural isomorphism*

$$\mathrm{Diag}(\Phi \times \Psi, C) \xrightarrow{\cong} \mathrm{Diag}(\Phi, \mathrm{Diag}(\Psi, C)).$$

Proof. We start by constructing a comparison map. By definition of the right-hand side, it will suffice to construct for every non-degenerate simplex σ_k of Φ and every non-degenerate simplex τ_l of Ψ a functor

$$\mathrm{Diag}(\Phi \times \Psi, C) \to \mathrm{Fun}(\Delta^k, \mathrm{Fun}(\Delta^l, C)).$$

The maps σ_k and τ_l determine a subshape inclusion $\Delta^k \times \Delta^l \subseteq \Phi \times \Psi$, and hence we may define this map as the composite

$$\mathrm{Diag}(\Phi \times \Psi, C) \to \mathrm{Diag}(\Delta^k \times \Delta^l, C) \cong \mathrm{Fun}(\Delta^k \times \Delta^l, C) \cong \mathrm{Fun}(\Delta^k, \mathrm{Fun}(\Delta^l, C)),$$

where the first isomorphism is the one from part (A.2) of Tribe Axiom A and the second isomorphism is given by currying. These maps are seen to be compatible and thus determine the desired functor.

We may now show that this functor is an isomorphism using a double induction on the number of non-degenerate simplices of Φ and Ψ, similar to the proof of Proposition 16.2.27. We will leave the details to the interested reader, see Exercise 16.7.3. □

Construction 16.2.29 (cf. [RS17, Proposition 3.5])**.** We construct maps $\max, \min \colon \Delta^1 \times \Delta^1 \to \Delta^1$, encoding the 'maximum' and 'minimum' operations on Δ^1. By Lemma 16.2.21, it suffices to construct these maps on the subshapes Δ^2 and Δ^2_{alt} and show they agree on their intersection. But on the restrictions to Δ^2 and Δ^2_{alt} we may simply define them using the two projection maps $\mathrm{pr}_1, \mathrm{pr}_2 \colon [1]^2 \to [1]$:

$$\max|_{\Delta^2} := \mathrm{pr}_1, \qquad \max|_{\Delta^2_{\mathrm{alt}}} := \mathrm{pr}_2, \qquad \min|_{\Delta^2} := \mathrm{pr}_2, \qquad \min|_{\Delta^2_{\mathrm{alt}}} := \mathrm{pr}_1.$$

Since the two projection maps agree on the image of the diagonal $\Delta \colon [1] \to [1]^2$, these partial definitions of max and min agree on the intersection of their domains and so glue to functors $\Delta^1 \times \Delta^1 \to \Delta^1$.

16.3 The Segal axiom

In this section we introduce and discuss the *Segal axiom*:

Tribe Axiom B (Segal axiom)**.** For any synthetic category C, the restriction functor

$$(-)|_{\Lambda^2_1} \colon \mathrm{Fun}(\Delta^2, C) \twoheadrightarrow \mathrm{Diag}(\Lambda^2_1, C) \cong \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C)$$

is an equivalence.

The Segal axiom roughly speaking says that two composable morphisms in C admit an essentially unique composite: for morphisms $f \colon x \longrightarrow y$ and $g \colon y \longrightarrow z$ in C, there is an essentially unique commutative triangle

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & & z
\end{tikzcd}$$

whose first leg is f and whose second leg is g. We will now make this precise:

Definition 16.3.1. Given two morphisms $f \colon x \longrightarrow y$ and $g \colon y \longrightarrow z$ in a synthetic category C, we define their *category of compositions* $\mathrm{Comp}_{g,f}$ as the following pullback:

$$\begin{tikzcd}
\mathrm{Comp}_{g,f} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^2, C) \arrow[d, two heads] \\
\{(f,g)\} \arrow[r, "{(f,g)}"] & \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C).
\end{tikzcd}$$

Lemma 16.3.2. *The Segal axiom holds if and only if for every pair of morphisms* (f,g) *in an arbitrary context* $\{(f,g)\}$ *the synthetic category* $\mathrm{Comp}_{g,f}$ *is contractible.*

Proof. This is an immediate consequence of the general fact [ref] that an isofibration is an equivalence if and only if its fibers over all terms of its base are contractible. □

Remark 16.3.3. Another way of formulating Tribe Axiom B is that the commutative square

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^2, C) \arrow[r, two heads, "(\delta^2_2)^*"] \arrow[d, two heads, "(\delta^2_0)^*"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, two heads, "\mathrm{ev}_1"] \\
\mathrm{Fun}(\Delta^1, C) \arrow[r, two heads, "\mathrm{ev}_0"] & C
\end{tikzcd}$$

is a homotopy pullback square.

Corollary 16.3.4. *For a synthetic category C, there are homotopy pullback squares*

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^2, C) \arrow[r] \arrow[d, two heads, "\mathrm{ev}_0"'] & \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[d, two heads] \\
C \arrow[r, "(p_{\Delta^1})^*"] & \mathrm{Fun}(\{0\} \times \Delta^1, C)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{Fun}(\Delta^2, C) \arrow[r] \arrow[d, two heads, "\mathrm{ev}_2"'] & \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[d, two heads] \\
C \arrow[r, "(p_{\Delta^1})^*"] & \mathrm{Fun}(\{1\} \times \Delta^1, C).
\end{tikzcd}$$

Proof. We will prove the claim for the left square and leave the analogous proof of the right square to the reader. To this end, we consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^2, C) \arrow[r] \arrow[d, two heads, "(\delta^2_1)^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \arrow[d, two heads, "(-)|_{\Delta^2_{\mathrm{alt}}}"] & \\
\mathrm{Fun}(\Delta^1, C) \arrow[r, "s^2_1"] \arrow[d, two heads, "\mathrm{ev}_0"'] & \mathrm{Fun}(\Delta^2_{\mathrm{alt}}, C) \arrow[r, "(\delta^2_0)^*"] \arrow[d, two heads, "(\delta^2_2)^*"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, two heads, "\mathrm{ev}_0"] \\
C \arrow[r, "(p_{\Delta^1})^*"] & \mathrm{Fun}(\Delta^1, C) \arrow[r, "\mathrm{ev}_1"] & C.
\end{tikzcd}$$

Here the top square is the left pullback square from Lemma 1.3.12. The right square is the homotopy pullback square from the previous remark, and it follows from the pasting law of homotopy pullback squares that the left bottom square is also a homotopy pullback square. The composite rectangle on the left is then the desired homotopy pullback square. □

Remark 16.3.5. Informally, the previous corollary says that commutative triangles in C may equivalently be encoded in the following three ways:

(1) One might consider diagrams of the form $\Delta^2 \to C$, which we will display as

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "gf"] & & z.
\end{tikzcd}$$

(2) One might consider diagrams $\Delta^1 \times \Delta^1 \to C$ whose restriction to $\{0\} \times \Delta^1$ is constant, which we will display as

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, "g"] \\
x \arrow[r, "gf"] & z.
\end{tikzcd}$$

(3) One might consider diagrams $\Delta^1 \times \Delta^1 \to C$ whose restriction to $\Delta^1 \times \{1\}$ is constant, which we will display as

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "gf"'] & y \arrow[d, "g"] \\
z \arrow[r, equal] & z.
\end{tikzcd}$$

16.4 Identities and composition

Due to the Segal axiom, synthetic categories behave like ordinary categories, in the sense that they come with notions of *identity morphisms* and *composition of morphisms*. In this section, we will explain how to obtain this additional structure on a synthetic category. We will also explain how to associate a *homotopy category* $\mathrm{ho}(C)$ to every synthetic category C.

Construction 16.4.1 (Identity morphisms)**.** For an object x of a synthetic category C, we define the *identity morphism* $\mathrm{id}_x\colon x \to x$ in C as the image of x under the map

$$p_{\Delta^1}^*\colon C = \mathrm{Fun}(\Delta^0, C) \to \mathrm{Fun}(\Delta^1, C).$$

Note that id_x is an object of the hom groupoid $C(x,x)$.

Construction 16.4.2 (Composition functors)**.** Let x, y and z be objects of a synthetic category C. We define the synthetic category $C(x,y,z)$ via the following pullback square:

$$\begin{tikzcd}
C(x,y,z) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^2, C) \arrow[d, two heads, "(\mathrm{ev}_0, \mathrm{ev}_1, \mathrm{ev}_2)"] \\
\Delta^0 \arrow[r, "(x,y,z)"] & C \times C \times C.
\end{tikzcd}$$

Consider now the zig-zag

$$\mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \xleftarrow{\sim} \mathrm{Fun}(\Delta^2, C) \xrightarrow{(\delta_1^2)^*} \mathrm{Fun}(\Delta^1, C).$$

The first map is an equivalence by the Segal axiom, and hence it admits a section (see Corollary 15.3.25). We thus obtain a map

$$-\circ-\colon\ \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \xrightarrow{\sim} \mathrm{Fun}(\Delta^2, C) \xrightarrow{(\delta_1^2)^*} \mathrm{Fun}(\Delta^1, C).$$

Passing to fibers over (x,y,z) then produces a functor

$$-\circ-\colon C(x,y) \times C(y,z) \xrightarrow{\simeq} C(x,y,z) \to C(x,z),$$

which we call the *composition functor*.

Construction 16.4.3 (Composition of natural transformations)**.** In an analogous way, the Segal axiom leads to the construction of compositions of natural transformations. Let $f,g,h\colon D \to C$ be functors and let $\alpha\colon f \to g$ and $\beta\colon g \to h$ be natural transformations. Then this data determines a functor

$$(\alpha,\beta)\colon D \to \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C),$$

and hence we may compose it with the above map

$$- \circ - \colon \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \xrightarrow{\sim} \mathrm{Fun}(\Delta^2, C) \xrightarrow{(\delta^2_1)^*} \mathrm{Fun}(\Delta^1, C)$$

we obtain the *composite transformation*

$$\beta \circ \alpha \colon D \to \mathrm{Fun}(\Delta^1, C).$$

We will now show that composition in a synthetic category is unital and associative.

Lemma 16.4.4 (Unitality, [RS17, Proposition 5.8])**.** *For every morphism $f \colon x \to y$ in a synthetic category C, there are homotopies*

$$\mathrm{id}_y \circ f \sim f \sim f \circ \mathrm{id}_x .$$

Moreover, these homotopies are natural in f, in the sense that they form homotopies of functors $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\Delta^1, C)$.

Proof. We will prove the first relation and leave the second to the reader. We have to prove that the composite

$$\mathrm{Fun}(\Delta^1, C) \cong \mathrm{Fun}(\Delta^1, C) \times_C C \xrightarrow{1 \times p^*_{\Delta^1}} \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1) \xrightarrow{- \circ -} \mathrm{Fun}(\Delta^1, C)$$

is homotopic to the identity functor. But this is immediate from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, C) \arrow[r, "(s^2_1)^*"] \arrow[rr, bend left, equal] \arrow[d, "\cong"'] & \mathrm{Fun}(\Delta^2, C) \arrow[r, "(\delta^2_1)^*"] \arrow[d, "\simeq"] & \mathrm{Fun}(\Delta^1, C) \arrow[d, equal] \\
\mathrm{Fun}(\Delta^1, C) \times_C C \arrow[r, "1 \times p^*_{\Delta^1}"'] & \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1) \arrow[r, "- \circ -"'] & \mathrm{Fun}(\Delta^1, C).
\end{tikzcd}$$

□

Proposition 16.4.5 (Associativity, [RS17, Proposition 5.9])**.** *For composable morphisms $f \colon x \to y$, $g \colon y \to z$ and $h \colon z \to w$ in a synthetic category C, there is a homotopy*

$$h \circ (g \circ f) \sim (h \circ g) \sim f.$$

Moreover, this homotopy is natural in the triple (f, g, h), in the sense that it forms a homotopy of functors

$$- \circ (- \circ -) \sim (- \circ -) \circ - \colon \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\Delta^1, C).$$

Proof. Consider first the forgetful functor

$$\mathrm{Fun}(\Delta^2, C) \times_{\mathrm{Fun}(\Delta^1, C)} \mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C)$$

given by restriction along the following inclusion of diagram shapes:

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[d] & \\
& (1,1) \arrow[r] & (1,2)
\end{tikzcd}
\hookrightarrow
\begin{tikzcd}
(0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \arrow[dr] & \\
& (1,1) \arrow[r] & (1,2).
\end{tikzcd}$$

It follows immediately from the Segal axiom that this functor is an equivalence, and hence we may prove the claim after precomposing with this functor.

Consider next the forgetful functor

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, C) \times_{\mathrm{Fun}(\Delta^1, C)} \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \to \mathrm{Fun}(\Delta^2, C) \times_{\mathrm{Fun}(\Delta^1, C)} \mathrm{Fun}(\Delta^2, C)$$

given by restriction along the following inclusion of diagram shapes:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \arrow[dr] & \\
& (1,1) \arrow[r] & (1,2)
\end{tikzcd}
\hookrightarrow
\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\
(1,0) \arrow[r] & (1,1) \arrow[r] & (1,2).
\end{tikzcd}$$

This restriction functor admits a section given informally by

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, swap, "g\circ f"] & y \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
& z \arrow[r, swap, "h"] & w
\end{tikzcd}
\mapsto
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, "g\circ f"] & y \arrow[r, "h\circ g"] \arrow[d, "g"] \arrow[dr, "h\circ g"] & w \arrow[d, equal] \\
x \arrow[r, swap, "g\circ f"] & z \arrow[r, swap, "h"] & w,
\end{tikzcd}$$

using the pullback description of $\mathrm{Fun}(\Delta^1 \times \Delta^1, C)$ from Lemma 16.2.21. It will thus suffice to prove the claim after precomposing with this functor.

Consider thirdly the forgetful functor

$$\mathrm{Fun}(\Delta^2 \times \Delta^1, C) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, C) \times_{\mathrm{Fun}(\Delta^1, C)} \mathrm{Fun}(\Delta^1 \times \Delta^1, C)$$

given by restriction along the following inclusion of diagram shapes:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\
(1,0) \arrow[r] & (1,1) \arrow[r] & (1,2)
\end{tikzcd}
\hookrightarrow
\begin{tikzcd}
& (0,1) \arrow[dr] \arrow[dd] & \\
(0,0) \arrow[ur] \arrow[rr] \arrow[dd] & & (0,2) \arrow[dd] \\
& (1,1) \arrow[dr] & \\
(1,0) \arrow[ur] \arrow[rr] & & (1,2)
\end{tikzcd}$$

This restriction functor is isomorphic to the restriction functor

$$\mathrm{Fun}(\Delta^2, \mathrm{Fun}(\Delta^1, C)) \to \mathrm{Fun}(\Lambda^2_1, \mathrm{Fun}(\Delta^1, C))$$

and hence it is an equivalence by the Segal axiom, Tribe Axiom B. We conclude once more that it suffices to prove the claim after precomposing with this functor. All in all, we must exhibit a homotopy of the bottom square in the following diagram after composing with the left vertical map:

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^2 \times \Delta^1, C) \arrow[rrr] \arrow[drr] \arrow[dd] & & & \mathrm{Fun}(\Delta^2, C) \arrow[dd, "\simeq"] \arrow[dr, "\delta^2_1"] & \\
& & \mathrm{Fun}(\Delta^2, C) \arrow[rr, "\delta^2_1"] \arrow[dd, "\simeq"] & & \mathrm{Fun}(\Delta^1, C) \arrow[dd, equal] \\
\mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[rrr, "\mathrm{id} \times (- \circ -)"] \arrow[drr, "(- \circ -) \times \mathrm{id}"'] & & & \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[dr, "- \circ -"] & \\
& & \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[rr, "- \circ -"] & & \mathrm{Fun}(\Delta^1, C)
\end{tikzcd}$$

The two unlabeled maps in the top square are given by restriction along the following two inclusions of diagram shapes:

$$\begin{tikzcd}
 & & & & (0,1) \arrow[d] \arrow[dr] & & & & \\
(0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] & \hookrightarrow & (0,0) \arrow[ur] \arrow[rr] \arrow[dd] & {} & (0,2) \arrow[dd] & \hookleftarrow & (0,0) \arrow[d] \arrow[dr] & \\
 & (1,2) & & & (1,1) \arrow[dr] & & & (1,1) \arrow[r] & (1,2). \\
 & & & (1,0) \arrow[ur] \arrow[rr] & & (1,2) & & &
\end{tikzcd}$$

It is now clear from the definition of the composition functor that each of the four vertical squares commute up to homotopy, and since the top square clearly commutes this finishes the proof. □

Remark 16.4.6. The analogous statement for composition of natural transformations also holds true, see Exercise 16.7.6.

16.4.1 The homotopy category of a synthetic category

The above constructions allow us to define the homotopy category $\mathrm{ho}(C)$ of a synthetic category C.

Construction 16.4.7 (Path components)**.** Let C be a synthetic category. We define the *set of path components of C* as the set

$$\pi_0(C) := \mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(\Delta^0, C).$$

This defines a functor $\pi_0(-)\colon \mathcal{E} \to \mathrm{Set}$.

Lemma 16.4.8. *For synthetic categories C and D, the canonical map*

$$\pi_0(C \times D) \to \pi_0(C) \times \pi_0(D)$$

is a bijection.

Proof. This is clear from unwinding definitions. □

Construction 16.4.9 (Homotopy category)**.** For a synthetic category C, define the ordinary category $\mathrm{ho}(C)$ as follows:

- The objects of $\mathrm{ho}(C)$ are the objects of C, i.e. functors $x\colon \Delta^0 \to C$ in $\mathcal{E}$;
- Given objects x, y of C, the set of morphisms from x to y in $\mathrm{ho}(C)$ is the set of path components of the hom groupoid $C(x,y)$:

  $$\mathrm{Hom}_{\mathrm{ho}(C)}(x,y) := \pi_0 C(x,y).$$

  In other words, the set of morphisms in $\mathrm{ho}(C)$ from x to y is the set of homotopy classes of morphisms from x to y in C;
- The identity morphism $\mathrm{id}_x\colon x \to x$ in $\mathrm{ho}(C)$ is the homotopy class of the identity morphism $\mathrm{id}_x\colon x \to x$ in C;
- Given objects x, y, z of C, composition in $\mathrm{ho}(C)$ is induced by the composition functors for C:

  $$\mathrm{Hom}_{\mathrm{ho}(C)}(x,y) \times \mathrm{Hom}_{\mathrm{ho}(y,z)} \cong \pi_0(C(x,y) \times C(y,z)) \to \pi_0(C(x,z)) = \mathrm{Hom}_{\mathrm{ho}(C)}(x,z).$$

It follows from Lemma 1.4.7 and Proposition 1.4.8 that composition in $\mathrm{ho}(C)$ is associative and unital, so that $\mathrm{ho}(C)$ is indeed a category.

Construction 16.4.10. Let $f\colon C \to D$ be a functor between synthetic categories. We will construct a functor $\mathrm{ho}(f)\colon \mathrm{ho}(C) \to \mathrm{ho}(D)$ between their homotopy categories:

- On objects, $\mathrm{ho}(f)$ sends an object $x\colon \Delta^0 \to C$ to the composite $f(x)\colon \Delta^0 \to C \xrightarrow{f} D$.
- On morphisms, $\mathrm{ho}(f)$ sends a morphism in $\mathrm{ho}(C)$ represented by a morphism $\Delta^1 \to C$ in C to the composite $\Delta^1 \to C \xrightarrow{f} D$. As composition with f preserves homotopies by Lemma 15.3.10, this is well-defined.
- This construction preserves identity morphisms and composition because of the commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d] & D \arrow[d] \\
\mathrm{Fun}(\Delta^1, C) \arrow[r, "f_*"] & \mathrm{Fun}(\Delta^1, D)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^2, C) \arrow[r, "f_*"] \arrow[d, "(\delta^2_1)^*"'] & \mathrm{Fun}(\Delta^2, D) \arrow[d, "(\delta^2_1)^*"] \\
\mathrm{Fun}(\Delta^1, C) \arrow[r, "f_*"] & \mathrm{Fun}(\Delta^1, D).
\end{tikzcd}$$

16.5 The Rezk axiom

We will now introduce the Rezk axiom, which roughly speaking says that any isomorphism in a synthetic category is in fact an *equality*. Together with the Segal axiom, this allows us to think of synthetic categories semantically as some kind of *complete Segal types*.

Definition 16.5.1 (Isomorphisms)**.** Consider a morphism $f\colon x \to y$ in a synthetic category C. We say that f is *invertible*, or that it is an *isomorphism*, if there are commutative triangles in C of the form

$$\begin{tikzcd} y \arrow[r, "g"] \arrow[dr, "\mathrm{id}_y"'] & x \arrow[d, "f"] \\ & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "\mathrm{id}_x"'] & y \arrow[d, "h"] \\ & x. \end{tikzcd}$$

Our next goal is to define the type $\mathrm{Iso}(C)$ of isomorphisms in C.

Construction 16.5.2. Consider the following two subshapes Φ and Ψ of Δ^3:

$$\Phi \;=\; \begin{tikzcd} & 1 \arrow[dd] \arrow[dr] & \\ 0 \arrow[ur] \arrow[dr] & & 3, \\ & 2 \arrow[ur] & \end{tikzcd} \qquad \Psi \;=\; \begin{tikzcd} & 1 \arrow[dd, dashed] \arrow[dr] & \\ 0 \arrow[ur, dashed] \arrow[dr] & & 3. \\ & 2 \arrow[ur, dashed] & \end{tikzcd}$$

For every synthetic category C there are isomorphisms

$$\begin{aligned} \mathrm{Diag}(\Phi, C) &\cong \mathrm{Fun}(\Delta^2, C) \times_{\mathrm{Fun}(\Delta^1, C)} \mathrm{Fun}(\Delta^2, C), \\ \mathrm{Diag}(\Psi, C) &\cong \mathrm{Fun}(\Delta^1, C) \times \mathrm{Fun}(\Delta^1, C). \end{aligned}$$

The inclusion of Ψ into Φ induces an isofibration $\mathrm{Diag}(\Phi, C) \twoheadrightarrow \mathrm{Diag}(\Psi, C)$ by Tribe Axiom A.

Definition 16.5.3. Given a synthetic category C, we define the synthetic category $\mathrm{Iso}(C)$ via the following pullback square:

$$\begin{tikzcd} \mathrm{Iso}(C) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Diag}(\Phi, C) \arrow[d, two heads] \\ C \times C \arrow[r, "{(x,y) \mapsto (\mathrm{id}_x, \mathrm{id}_y)}"] & \mathrm{Fun}(\Delta^1, C) \times \mathrm{Fun}(\Delta^1, C). \end{tikzcd}$$

So a term of $\mathrm{Iso}(C)$ is a diagram in C of the form

$$\begin{tikzcd}
& x \arrow[dd, "f"] \arrow[dr, "\mathrm{id}_x"] & \\
y \arrow[ur, "g"] \arrow[dr, "\mathrm{id}_y"'] & & x. \\
& y \arrow[ur, "h"'] &
\end{tikzcd}$$

Restricting along the map $\Delta^{\{1,2\}} \longrightarrow \Delta^3$ induces a functor

$$\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \longrightarrow \mathrm{Fun}(\Delta^1, C).$$

Proposition 16.5.4. *The functor* $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \longrightarrow \mathrm{Fun}(\Delta^1, C)$ *is an isofibration. More generally, for every isofibration* $f \colon X \twoheadrightarrow Y$*, the induced map*

$$(f_*, \pi_{\mathrm{Iso}}) \colon \mathrm{Iso}(X) \longrightarrow \mathrm{Iso}(Y) \times_{\mathrm{Fun}(\Delta^1, Y)} \mathrm{Fun}(\Delta^1, X)$$

is an isofibration.

Proof. We prove the first claim, and leave the analogous proof of the second claim to the reader. Consider subshape $\widetilde{\Phi}$ of Δ^3 given as follows:

$$\begin{tikzcd}
& & 1 \arrow[dd] \arrow[dr] & \\
\widetilde{\Phi} \arrow[r, phantom, "="] & 0 \arrow[ur, dashed] \arrow[dr] & & 3. \\
& & 2 \arrow[ur, dashed] &
\end{tikzcd}$$

Note that $\Psi \subseteq \widetilde{\Phi} \subseteq \Phi$. It follows from Tribe Axiom A that the restriction map $\mathrm{Diag}(\Phi, C) \twoheadrightarrow \mathrm{Diag}(\widetilde{\Phi}, C)$ is an isofibration. Note that this isofibration lives over $\mathrm{Diag}(\Psi, C) = \mathrm{Fun}(\Delta^1, C) \times \mathrm{Fun}(\Delta^1, C)$, and that its pullback along the map $C \times C \longrightarrow \mathrm{Fun}(\Delta^1, C) \times \mathrm{Fun}(\Delta^1, C)$ is precisely the functor $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \longrightarrow \mathrm{Fun}(\Delta^1, C)$. It follows that this functor is an isofibration as well, finishing the proof. □

Notice that by definition a morphism $f \in \mathrm{Fun}(\Delta^1, C)$ is invertible in C if and only if it lifts to an object of $\mathrm{Iso}(C)$. This motivates the following definition:

Definition 16.5.5. Let $f \colon x \longrightarrow y$ be a morphism in a synthetic category C. We define the category $\{f \text{ is invertible}\}$ via the following pullback diagram:

$$\begin{tikzcd}
\{f \text{ is invertible}\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(C) \arrow[d, two heads, "\pi_{\mathrm{Iso}}"] \\
\Delta^0 \arrow[r, "f"] & \mathrm{Fun}(\Delta^1, C).
\end{tikzcd}$$

As one expects, the identity map $\mathrm{id}_x \colon x \to x$ is an isomorphism in C for every object x of C:

Construction 16.5.6. We construct a functor $i \colon C \to \mathrm{Iso}(C)$ lifting the functor $1 \colon C \to \mathrm{Fun}(\Delta^1, C) \colon x \mapsto \mathrm{id}_x$. Precomposition with $\Delta^3 \to \Delta^0$ defines a functor $C \to \mathrm{Fun}(\Delta^3, C)$, which sends an object to the diagram informally displayed as follows:

$$\begin{tikzcd}
 & x \arrow[rd, "\mathrm{id}_x"] \arrow[dd, "\mathrm{id}_x"] & \\
x \arrow[ru, "\mathrm{id}_x"] \arrow[rr, dashed, "\mathrm{id}_x"] \arrow[rd, "\mathrm{id}_x"'] & & x. \\
 & x \arrow[ru, "\mathrm{id}_x"'] &
\end{tikzcd}$$

Since the restriction of this diagram to the edges $\Delta^{\{0,2\}}$ and $\Delta^{\{1,3\}}$ are the identity on x, this functor factors through $\mathrm{Iso}(C)$, producing the desired functor $i \colon C \to \mathrm{Iso}(C)$.

Note that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[rd, "x \mapsto \mathrm{id}_x"'] & \mathrm{Iso}(C) \arrow[r, two heads, "{(s,t)}"] \arrow[d, two heads, "\pi_{\mathrm{Iso}}"] & C \times C \\
 & \mathrm{Fun}(\Delta^1, C). \arrow[ru, two heads, "{(s,t)}"'] &
\end{tikzcd}$$

Tribe Axiom C (Rezk axiom)**.** For any synthetic category C, the functor $i \colon C \to \mathrm{Iso}(C)$ is an equivalence. In particular, $\mathrm{Iso}(C)$ is a path object for C.

The Rezk axiom expresses the condition that two isomorphic objects in C are in fact *equal*. To make this precise, we construct the category of isomorphisms between two objects:

Definition 16.5.7. Consider two objects x and y of a synthetic category C. We define the category $\{x \cong y\}$ of isomorphisms between x and y via the following pullback diagram:

$$\begin{tikzcd}
\{x \cong y\} \arrow[r] \arrow[d] \arrow[rd, phantom, "\lrcorner", very near start] & \mathrm{Iso}(C) \arrow[d, two heads, "\pi_{\mathrm{Iso}}"] \\
\Delta^0 \arrow[r, "{(x,y)}"] & C \times C.
\end{tikzcd}$$

Corollary 16.5.8. *There is a fiberwise equivalence* $P(C) \xrightarrow{\sim} \mathrm{Iso}(C)$ *over* $C \times C$. *In particular, there is an equivalence*

$$\{x = y\} \xrightarrow{\sim} \{x \cong y\}$$

for all objects x and y in C.

Proof. We may find a filler h in the following commutative diagram:

$$\begin{tikzcd}
C \arrow[r, "i\ \sim"] \arrow[d, tail, "r", "\sim"'] & \mathrm{Iso}(C) \arrow[d, two heads] \\
P(C) \arrow[r, two heads] \arrow[ur, dashed, "h"] & C \times C.
\end{tikzcd}$$

It follows from 2-out-of-3 that h is an equivalence, and thus a fiberwise equivalence by Theorem 15.3.47. The last claim is immediate by passing to fibers over $(x, y) \in C \times C$. □

16.6 Simplicial type theory

We combine the four axioms on our tribe into a unified definition:

Definition 16.6.1. A simplicial tribe $\mathcal{E}$ is called a *simplicial type theory* if it satisfies Axioms A - C. If $\mathcal{E}'$ is another simplicial tribe, we say that a *morphism of simplicial type theories* $F\colon \mathcal{E} \to \mathcal{E}'$ is a morphism of tribes which preserves the functor categories $\mathrm{Fun}(\Delta^n, -)$, in the sense that it comes equipped with a natural isomorphism making the following square commute:

$$\begin{tikzcd}
\Delta^{\mathrm{op}} \times \mathcal{E} \arrow[r, "\mathrm{Fun}(-,-)"] \arrow[d, "\mathrm{id} \times F"'] & \mathcal{E} \arrow[d, "F"] \\
\Delta^{\mathrm{op}} \times \mathcal{E}' \arrow[r, "\mathrm{Fun}(-,-)"'] & \mathcal{E}'.
\end{tikzcd}$$

Observe that a morphism of simplicial type theories automatically preserves all the categorical structure available in the synthetic categories: morphisms, natural transformations, identity morphisms, compositions, commutative triangles, commutative squares, etcetera.

Lemma 16.6.2. *Every morphism of simplicial type theories $F\colon \mathcal{E} \to \mathcal{E}'$ preserves diagram categories: for every diagram shape Φ and for every synthetic category $C \in \mathcal{E}$ there is a preferred natural isomorphism*

$$F(\mathrm{Diag}(\Phi, C)) \xrightarrow{\cong} \mathrm{Diag}(\Phi, F(C)).$$

Proof. We first construct a canonical comparison map. For every non-degenerate simplex σ_k of Φ, we may consider the composite

$$F(\mathrm{Diag}(\Phi, C)) \to F(\mathrm{Fun}(\Delta^k, C)) \cong \mathrm{Fun}(\Delta^k, F(C)),$$

where the isomorphism is the one given as part of the structure of F. These restrictions are compatible under passing to subcomplexes and thus define a functor to $\mathrm{Diag}(\Phi, F(C))$. The fact that this is an isomorphism can be proved by induction on the number of simplices of the diagram shape. We leave the details to the interested reader. □

We obtain the following immediate corollary:

Corollary 16.6.3. *Every morphism of simplicial type theories $F\colon \mathcal{E} \to \mathcal{E}'$ preserves isomorphism categories: for every synthetic category $C \in \mathcal{E}$ there is a natural isomorphism*

$$F(\mathrm{Iso}(C)) \xrightarrow{\cong} \mathrm{Iso}(F(C)). \qquad \square$$

Remark 16.6.4. The arguments of the previous lemma and corollary do not use the fact that F was assumed to preserve equivalences, and in particular also hold true if F is merely assumed to be a morphism of *clans* which preserves diagram categories. Since the isomorphism categories $\mathrm{Iso}(C)$ serve at path objects in the tribe $\mathcal{E}$, due to the Rezk Axiom C, Corollary 16.6.3 directly implies that every such functor is automatically a morphism of tribes, and hence a morphism of simplicial type theories.

For every simplicial type theory $\mathcal{E}$, each of the local tribes $\mathcal{E}(B)$ admits a natural structure of a simplicial type theory again.

Definition 16.6.5. For a synthetic categories C and B, we define the object $C_B \in \mathcal{E}(B)$ as the projection $\mathrm{pr}_1\colon C \times B \twoheadrightarrow B$. This defines a functor $(-)_B\colon \mathcal{E} \to \mathcal{E}(B)$.

Remark 16.6.6. Note that $(-)_B$ agrees with the functor $p_B^*\colon C \to C(B)$ given by pullback along the map $p_B\colon B \to \Delta^0$. In particular it is a morphism of tribes.

Lemma 16.6.7. *The composite*

$$\Delta_B^\bullet\colon \Delta \xrightarrow{\Delta^\bullet} \mathcal{E} \xrightarrow{(-)_B} \mathcal{E}(B)$$

equips $\mathcal{E}(B)$ with the structure of a simplicial tribe.

Proof. We have to show that for every $[n] \in \Delta$ the functor $\Delta_B^n \times_B -\colon \mathcal{E}(B) \to \mathcal{E}(B)$ admits a right adjoint $\mathrm{Fun}_B(\Delta_B^1, -)\colon \mathcal{E}(B) \to \mathcal{E}(B)$ which is a morphism of tribes. For an isofibration $p\colon C \twoheadrightarrow B$, we define the isofibration $\mathrm{Fun}_B(\Delta_B^1, C) \twoheadrightarrow C$ via the following pullback square:

$$\begin{tikzcd}
\mathrm{Fun}_B(\Delta_B^n, C) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^n, C) \arrow[d, two heads, "p_*"] \\
B \arrow[r, "p_{\Delta^n}^*"] & \mathrm{Fun}(\Delta^n, B).
\end{tikzcd}$$

Let $q\colon D \twoheadrightarrow B$ be another isofibration. We have to show that functors $D \to \mathrm{Fun}_B(\Delta_B^n, C)$ over B are in natural bijection with functors $\Delta_B^n \times_B D \to C$ over B. By construction, the former correspond to functors $D \to \mathrm{Fun}(\Delta^1, C)$ fitting in the following commutative diagram:

$$\begin{tikzcd}
D \arrow[r, dashed] \arrow[d, two heads, "q"'] & \mathrm{Fun}(\Delta^n, C) \arrow[d, two heads, "p_*"] \\
B \arrow[r, "p_{\Delta^1}^*"] & \mathrm{Fun}(\Delta^n, B).
\end{tikzcd}$$

By adjunction, such morphisms correspond to morphisms $D \times \Delta^1 \to C$ fitting in the following commutative diagram:

$$\begin{tikzcd}
D \times \Delta^n \arrow[r, dashed] \arrow[d, two heads, "\mathrm{pr}_1"'] & C \arrow[d, two heads, "p"] \\
D \arrow[r, "q"] & B.
\end{tikzcd}$$

But these are precisely the morphisms $D \times_B \Delta^n_B \to C$ over B, as desired. We conclude that the construction $C \mapsto \mathrm{Fun}_B(\Delta^1_B, C)$ defines a right adjoint to $\Delta^n_B \times_B -$. From the fact that $\mathrm{Fun}(\Delta^n, -)\colon \mathcal{E} \to \mathcal{E}$ is a morphism of tribes it immediately follows that also $Fun_B(\Delta^1_B, -)\colon \mathcal{E}(B) \to \mathcal{E}(B)$ is a morphism of tribes. This finishes the proof. □

We now show the existence of diagram categories in $\mathcal{E}(B)$.

Lemma 16.6.8. *The local tribe $\mathcal{E}(B)$ admits diagram categories: for a diagram shape Φ and an isofibration $C \twoheadrightarrow B$, a diagram category in $\mathcal{E}(B)$ is given by the following pullback square:*

$$\begin{tikzcd}
\mathrm{Diag}_B(\Phi, C) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Diag}(\Phi, C) \arrow[d, two heads, "p_*"] \\
B \arrow[r, "p^*_\Phi"] & \mathrm{Diag}(\Phi, B).
\end{tikzcd}$$

Proof. This follows from Lemma 16.6.7 using an easy induction on the number of non-degenerate simplices of Φ. We leave the details to the reader. □

Proposition 16.6.9. *The local tribe $\mathcal{E}(B)$ is again a simplicial type theory.*

Proof. We just showed that $\mathcal{E}(B)$ admits diagram categories, and using their explicit descriptions one may directly deduce conditions (A.1) and (A.2) of Tribe Axiom A from the analogous conditions for diagram categories in $\mathcal{E}$. For the Segal axiom B and the Rezk axiom C, we need to show that for every isofibration $C \twoheadrightarrow B$, the maps

$$\mathrm{Fun}_B(\Delta^2, C) \twoheadrightarrow \mathrm{Diag}_B(\Lambda^2_1, C) \qquad \text{and} \qquad C \to \mathrm{Iso}_B(C)$$

are equivalences in $\mathcal{E}(B)$. Again this follows at once from the Segal axiom and Rezk axiom in $\mathcal{E}$ and the pullback description of the diagram categories in $\mathcal{E}(B)$. We leave the details as an exercise to the reader. □

Lemma 16.6.10. *For every functor $f\colon A \to B$, the pullback functor $f^*\colon \mathcal{E}(B) \to \mathcal{E}(A)$ is a morphism of simplicial type theories.*

Proof. By identifying $\mathcal{E}(A)$ with $\mathcal{E}(B)(A)$, we may assume that $B = \Delta^0$, hence we must show that the functor $(-)_A\colon \mathcal{E} \to \mathcal{E}(A)$ is a morphism of simplicial type theories. We showed in Lemma 15.3.31 that it is a morphism of tribes. Furthermore, the proof of

Lemma 16.6.7 shows that for every synthetic category C, there is a canonical pullback square

$$\begin{tikzcd}
\operatorname{Fun}_A(\Delta^n_A, C_A) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & \operatorname{Fun}(\Delta^n, C \times A) \arrow[d, two heads, "(\mathrm{pr}_A)_*"] \\
A \arrow[r, "(p_{\Delta^n})^*"] & \operatorname{Fun}(\Delta^n, A).
\end{tikzcd}$$

As a consequence we obtain a natural isomorphism $\operatorname{Fun}_A(\Delta^n_A, C_A) \cong \operatorname{Fun}(\Delta^n, C) \times A$, and under this isomorphism the map $\operatorname{Fun}_A(\Delta^n_A, C_A) \twoheadrightarrow A$ corresponds to the projection map $\operatorname{Fun}(\Delta^n, C) \times A \twoheadrightarrow A$. We conclude that there is a natural isomorphism $\operatorname{Fun}_A(\Delta^n_A, C_A) \cong \operatorname{Fun}(\Delta^n, C)_A$, as desired. □

Lemma 16.6.11. *Let $f \colon C \twoheadrightarrow D$ be a universally exponentiable fibration. Then the dependent product functor $f_* \colon \mathcal{E}(C) \to \mathcal{E}(D)$ is a morphism of simplicial type theories. In particular, it is a morphism of tribes.*

Proof. We have seen that f_* is a morphism of clans in Corollary 14.3.19. By Remark 16.6.4 it thus remains to show that f_* preserves the functor categories $\operatorname{Fun}(\Delta^n, -)$. By passing to left adjoints, this is equivalent to the claim that there are natural isomorphisms $f^*(\Delta^n_D \times_D E) \cong \Delta^n_C \times_C f^*(E)$ for $E \in \mathcal{E}(D)$. But this is clear from the fact that f^* preserves products and that we have $f^*(\Delta^n_D) = f^*(\Delta^n \times D) = \Delta^n \times C = \Delta^n_C$. □

16.7 Exercises Chapter 16

Exercise 16.7.1. Show that the functor $\Delta \to \mathrm{Shape}$ constructed in the proof of Lemma 16.2.7 is indeed well-defined.

Exercise 16.7.2. Let $\mathcal{E}$ be a simplicial type theory satisfying the following weaker version of (A.1) from Tribe Axiom A: for every subshape Ψ of a diagram shape Φ and every isofibration $p \colon C \twoheadrightarrow D$, the following implication holds:

> If $\mathcal{E}$ admits Φ-indexed and Ψ-indexed diagram categories, then the functor $\operatorname{Diag}(\Phi, C) \to \operatorname{Diag}(\Phi, D) \times_{\operatorname{Diag}(\Psi, D)} \operatorname{Diag}(\Psi, C)$ is an isofibration.

Show that $\mathcal{E}$ admits Φ-indexed diagram categories for every diagram shape Φ. *Hint:* proceed by induction on the number of non-degenerate sipmlices in Φ.

Exercise 16.7.3. Finish the proof of Proposition 16.2.28: show that for diagram shapes Φ and Ψ, the functor $\operatorname{Diag}(\Phi \times \Psi, C) \to \operatorname{Diag}(\Phi, \operatorname{Diag}(\Psi, C))$ constructed in the proof of Proposition 16.2.28 is indeed an isomorphism for every synthetic category C.

Hint: proceed by induction on the number of non-degenerate simplices of Φ and Ψ, and use that a decomposition of either Φ or Ψ into a union of smaller subshapes also induces a decomposition of their product $\Phi \times \Psi$.

Exercise 16.7.4. Finish the proof of Lemma 16.6.8.

Exercise 16.7.5. Verify the assertions made at the end of **??**.

Exercise 16.7.6. Show that the composition of natural transformations between functors is both unital and associative:

- For natural transformations $\alpha\colon f \to g$, $\beta\colon g \to h$ and $\gamma\colon h \to k$ of functors from C to D, show that there exists a homotopy

  $$(\gamma \circ \beta) \circ \alpha \sim \gamma \circ (\beta \circ \alpha)$$

  of functors $C \to \mathrm{Fun}(\Delta^1, D)$;
- For a natural transformation $\alpha\colon f \to g$ of functors from C to D, show that there are homotopies

  $$\mathrm{id}_g \circ \alpha \sim \alpha \sim \alpha \circ \mathrm{id}_f$$

  of functors $C \to \mathrm{Fun}(\Delta^1, D)$.

Exercise 16.7.7. Provide the missing details in the proof of Lemma 16.4.8.

Exercise 16.7.8. Provide a proof of the second assertion in Proposition 16.5.4.

17 Synthetic category theory

In Part I of this book, we introduced the language of “naive category theory” and used it to provide a synthetic development of category theory. The goal of the present chapter is to show how the axioms from Part I may be interpreted internal to a fixed simplicial type theory $\mathcal{E}$.

Warning: This chapter is currently under active construction and essential parts are still missing. Our goal is to give for every every axiom of synthetic category theory introduced in Part I either a proof that it is a consequence of the structure imposed on a simplicial type theory, or a formulation of an axiom within the language of tribes which implies the corresponding axiom of synthetic category theory. (In most cases we may simply use the formulation from the old version of the book.)

17.1 Naive category theory

We start by showing that all the ingredients of the language of naive category theory introduced in Chapter 1 admit interpretations internal to the simplicial type theory $\mathcal{E}$.

Definition 17.1.1 (Categories, functors and natural isomorphisms)**.** A *synthetic category* is an object of $\mathcal{E}$. A *functor* is a morphism in $\mathcal{E}$. Given two functors $f, g\colon C \to D$, a *natural isomorphism* $\alpha\colon f \cong g$ is a homotopy between f and g, i.e. a map $\alpha\colon C \to P(D)$ such that $f = d_0 \circ \alpha$ and $g = d_1 \circ \alpha$. An *isomorphism* $\alpha \cong \beta$ between two homotopies $\alpha, \beta\colon C \to P(D)$ is a homotopy between α and β, i.e. a map $H\colon C \to P(P(D))$ such that $\alpha = d_0 \circ H$ and $\beta = d_1 \circ H$. By iterating this process one obtains iterated notions of isomorphisms between natural isomorphisms.

Lemma 17.1.2. *The above definitions come with preferred notions of identities, composites and inverses that satisfy the properties of Axiom A.1, Axiom A.2 and Axiom A.3.*

Proof. The composition of functors is simply given by composition of maps in $\mathcal{E}$, hence is automatically associative and unital. The identities, compositions and inverses of (higher) natural isomorphisms is given by the analogous constructions for homotopies in the tribe

$\mathcal{E}$, see Construction 15.3.6. It was shown in Lemma 15.3.8 that this satisfies unitality and associativity, and that the constructed isomorphism $\alpha^{-1}\colon g \cong f$ is indeed a two-sided inverse to α. Similarly, horizontal composition of (higher) isomorphisms is defined as horizontal composition of homotopies in the tribe $\mathcal{E}$, see Construction 15.3.12. Unitality, associativity and compatibility with vertical composition were verified in Lemma 15.3.13. □

Lemma 17.1.3. *(1) The terminal object $*$ of $\mathcal{E}$ satisfies the conditions of the terminal category from Axiom B.1;*

(2) For synthetic categories C and D, the cartesian product $C \times D$ satisfies the conditions of the product category from Axiom B.3;

(3) For functors $f\colon C \to E$ and $g\colon D \to E$, the homotopy pullback $C \times^h_E D$ from Construction 15.4.7 satisfies the condition of the pullback from Axiom B.6.

Proof. For part (1), it is clear that every synthetic category C comes with a map $p_C\colon C \to *$. Moreover, given two maps $f, g\colon C \to *$ we have $f = g$ and thus in particular $f \cong g$.

For part (2), we equip the product with the two projection maps $\mathrm{pr}_C\colon C \times D \to C$ and $\mathrm{pr}_D\colon C \times D \to D$. Given maps $f\colon E \to C$ and $g\colon E \to D$, we obtain a map $(f,g)\colon E \to C \times D$. Moreover, given two maps $(f,g), (f',g')\colon E \to C \times D$ with natural isomorphisms $\alpha\colon f \cong f'$ and $\beta\colon g \cong g'$, we obtain a canonical map $(\alpha,\beta)\colon E \to P(C) \times P(D)$, and since there is an equivalence $P(C \times D) \simeq P(C) \times P(D)$ over $(C \times D) \times (C \times D)$ this shows that $(f,g) \cong (f',g')$ as desired.

For part (3), we equip the homotopy pullback with the two maps $\mathrm{pr}_C\colon C \times^h_E D \to C$ and $\mathrm{pr}_D\colon C \times^h_E D \to D$ and the canonical natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$ from Construction 15.4.7. Given two functors $t\colon T \to C$ and $s\colon T \to D$ equipped with a natural isomorphism $\alpha\colon f \circ t \cong g \circ s$, Construction 15.4.9 provides a functor $(t,s)\colon T \to C \times^h_D D$. We leave it to the reader to check that α can indeed be recovered as the natural isomorphism induced from $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$.

Moreover, given functors $h, k\colon T \to C \times^h_E D$ equipped with natural isomorphisms $\alpha\colon \mathrm{pr}_C \circ h \cong \mathrm{pr}_C \circ k$ and $\beta\colon \mathrm{pr}_D \circ h \cong \mathrm{pr}_D \circ k$ such that the square of natural isomorphisms displayed in Axiom B.6 commutes, we obtain by Construction 15.4.9 a map $T \to P(A) \times^h_{P(B)} P(Y)$ that recovers this data. Since there is an equivalence $P(A) \times^h_{P(B)} P(Y) \simeq P(A \times^h_B Y)$ that commutes with the projection maps, this finishes the proof. □

Coproducts of synthetic categories

In order to have an initial category $\emptyset$ and coproducts of categories $C \sqcup D$, we need to assume the following axiom on our simplicial type theory $\mathcal{E}$:

Tribe Axiom D (Disjoint unions)**.** The category $\mathcal{E}$ admits finite coproducts which are *universal*: for every finite set I and any family $(C_i)_{i\in I}$ of synthetic categories with coproduct $C = \bigsqcup_{i\in I} C_i$, the pullback functor

$$\mathcal{E}(C) \to \prod_{i\in I} \mathcal{E}(C_i) \colon E \mapsto (E \times_C C_i)_{i\in I}$$

is an equivalence of tribes, in the sense of Definition 15.3.29.

Note that the inverse to the pullback functor is necessarily given by the coproduct functor

$$\prod_{i\in I} \mathcal{E}(C_i) \to \mathcal{E}(\bigsqcup_{i\in I} C_i) \colon (E_i \twoheadrightarrow C_i)_{i\in I} \mapsto (\bigsqcup_{i\in I} E_i \twoheadrightarrow \bigsqcup_{i\in I} C_i),$$

and thus the above axiom in particular demands this to be a morphism of tribes. Note that the pullback functor $\mathcal{E}(C) \simeq \prod_{i\in I} \mathcal{E}(C_i)$ is automatically a morphism of tribes, since we proved in Lemma 15.3.31 that each of the functors $E \mapsto E \times_C C_i$ is a morphism of tribes.

Corollary 17.1.4. *Fibrations and anodyne morphisms are closed under finite disjoint unions.*

Proof. We have to show that the coproduct functor $\mathcal{E}^n \to \mathcal{E} \colon (E_i)_{i\in I} \mapsto \bigsqcup_{i\in I} E_i$ preserves anodynes and fibrations. Applying the equivalence from Tribe Axiom D to the synthetic categories $C_i = *$, we see that this functor is equivalent to the composite

$$\prod_{i\in I} \mathcal{E} = \prod_{i\in I} \mathcal{E}(*) \xrightarrow{\simeq} \mathcal{E}(\bigsqcup_{i\in I} *) \to \mathcal{E},$$

where the middle equivalence is the inverse of the equivalence from Tribe Axiom D and the last map is the forgetful functor. Since both of these functors preserve anodynes and fibrations, this finishes the proof. □

Lemma 17.1.5. *The category $\mathcal{E}$ admits an initial object $\emptyset$. Furthermore, any functor $C \to \emptyset$ is an isomorphism.*

Proof. Since $\mathcal{E}$ admits a coproduct over the empty set $J = \emptyset$, we obtain an initial object. By assumption we get $\mathcal{E}(\emptyset) \simeq \prod_{j\in\emptyset} \mathcal{E}(C_j) = *$. In particular, every object in $\mathcal{E}(\emptyset)$ is equivalent to the terminal object, showing that every *fibration* $C \twoheadrightarrow \emptyset$ is an isomorphism.

Consider now an anodyne map $u \colon C \overset{\sim}{\rightarrowtail} \emptyset$. By Lemma 15.1.4, it admits a retraction $r \colon \emptyset \to C$, meaning that $ru = 1_C$. But we also have $ur = 1_\emptyset$ since $\emptyset$ is an initial object of $\mathcal{E}$. It follows that u is an isomorphism.

Finally, given an arbitrary functor $C \to \emptyset$, we may factor it as an anodyne morphism $u\colon C \to D$ followed by a fibration $p\colon D \twoheadrightarrow \emptyset$. In the first paragraph we showed that p must be an isomorphism, so that in particular D is an initial object. The second paragraph then shows that also u is an isomorphism. This finishes the proof. □

Lemma 17.1.6. *Every functor $\emptyset \to C$ is a fibration.*

Proof. We may factor this map as an anodyne map followed by a fibration. Thus it suffices to show that any anodyne map $u\colon \emptyset \overset{\sim}{\rightarrowtail} C$ is an isomorphism. By Lemma 15.1.4, u admits a retraction $r\colon C \to \emptyset$, which is an isomorphism by Lemma 17.1.5. We conclude that also u is an isomorphism, finishing the proof. □

Corollary 17.1.7. *The initial object $\emptyset$ satisfies the conditions of Axiom B.2 and Axiom B.7.*

Proof. It is clear that every synthetic category C admits a functor $\emptyset \to C$ and that every two such functors are (equal hence) naturally isomorphic. Any functor $C \to \emptyset$ is an isomorphism by Lemma 17.1.5, and thus an equivalence. □

Lemma 17.1.8. *For synthetic categories $C, D \in \mathcal{E}$, the functors $C \to C \sqcup D$ an $D \to C \sqcup D$ are both a fibration and a monomorphism.*

Proof. By symmetry it suffices to prove this for $C \to C \sqcup D$. We may write this as the disjoint union of the identity map $\mathrm{id}_C\colon C \to C$ and the unique map $\emptyset \to D$. Since the latter is a fibration by Lemma 17.1.6, and fibrations are closed under coproducts by Corollary 17.1.4, it follows that $C \to C \sqcup D$ is a fibration. It follows from Tribe Axiom D that the pullback of the map $C \to C \sqcup D$ along itself is simply the identity on C, showing that this map is a monomorphism. □

Lemma 17.1.9. *The coproduct of categories satisfies the conditions of Axiom B.4 and Axiom B.8.*

Proof. Let C and D be synthetic categories. Given functors $f\colon C \to E$ and $g\colon D \to E$, we obtain a functor $\langle f, g\rangle\colon C \sqcup D \to E$. Furthermore, if $\alpha\colon f \cong f'$ and $\beta\colon g \cong g'$ are natural isomorphisms, then the the map $\langle \alpha, \beta\rangle\colon C \sqcup D \to P(E)$ is a natural isomorphism $\langle \alpha, \beta\rangle\colon \langle f, g\rangle \cong \langle f', g'\rangle$.

Given synthetic categories E_0 and E_1, we have an equivalence $\mathcal{E}(E_0 \sqcup E_1) \xrightarrow{\sim} \mathcal{E}(E_0) \times \mathcal{E}(E_1)$ given by pullback along the inclusions of E_0 and E_1 into $E_0 \sqcup E_1$ and with inverse given by taking disjoint union. It in particular follows that:

- For an isofibration

□

Lemma 17.1.10. *If $\mathcal{E}$ is a tribe satisfying Tribe Axiom D, then also the local tribe $\mathcal{E}(\Gamma)$ satisfies Tribe Axiom D for every type Γ.*

Proof. Given fibrations $p_i \colon X_i \to \Gamma$ indexed by some finite set I, a coproduct in $\mathcal{E}(\Gamma)$ is given by the composite

$$\bigsqcup_{i\in I} X_i \twoheadrightarrow \bigsqcup_{i\in I} \Gamma \xrightarrow{\nabla} \Gamma.$$

Note that the last map is a fibration as it is isomorphic to the projection map $(\bigsqcup_{i\in I}\Delta^0)\times\Gamma \twoheadrightarrow \Gamma$. These finite sums satisfy universality because the functor

$$\mathcal{E}(\Gamma)(\bigsqcup_{i\in I} X_i) \to \prod_{i\in I} \mathcal{E}(\Gamma)(X_i)$$

is identified with the functor $\mathcal{E}(\bigsqcup_{i\in I} X_i) \to \prod_{i\in I}\mathcal{E}(X_i)$, and is therefore an equivalence of tribes. □

Corollary 17.1.11. *For a fibration $f \colon C \twoheadrightarrow D$, the restriction functor $f^* \colon \mathcal{E}(D) \to \mathcal{E}(C)$ preserves finite coproducts.*

Proof. Let I be a finite set and let $g_i \colon Y_i \twoheadrightarrow D$ be an I-indexed family of fibrations over D. We have to show that the canonical map

$$\sqcup_{i\in I} f^*(Y_i) \to f^*(\sqcup_{i\in I} Y_i)$$

is an isomorphism. Notice that both sides are equipped with a fibration to $X_1 \sqcup X_2$, and thus by universality of coproducts it suffices to show that this map becomes an isomorphism after pullback along the inclusions $X_1 \to X_1 \sqcup X_2$ and $X_2 \to X_1 \sqcup X_2$, which is clear. □

Functor categories

The final missing ingredient is the existence of functor categories

Tribe Axiom E (Functor categories)**.** The category $\mathcal{E}$ is cartesian closed: for every synthetic category C, the functor $C\times - \colon \mathcal{E} \to \mathcal{E}$ admits a right adjoint $\mathrm{Fun}(C,-) \colon \mathcal{E} \to \mathcal{E}$.

We refer to the synthetic category $\mathrm{Fun}(C,D)$ as the *functor category*.

Lemma 17.1.12. *For every synthetic category C, the functor $p_C \colon C \twoheadrightarrow *$ is a universally exponentiable fibration, in the sense of Definition 14.3.16.*

Proof. We will apply the criterion from Proposition 14.3.17. For a synthetic category D, consider the projection map $\mathrm{pr}_D\colon C\times D\to D$. The functor $\mathrm{pr}_D^*\colon \mathcal{E}(D)\to\mathcal{E}(C\times D)$ sends an isofibration $E\twoheadrightarrow D$ to the product $C\times E\twoheadrightarrow C\times D$. A right adjoint $(\mathrm{pr}_D)_*\colon \mathcal{E}(C\times D)\to\mathcal{E}(D)$ to this functor is given by sending a isofibration $p\colon F\twoheadrightarrow C\times D$ to the isofibration $\mathrm{Fun}_D(C\times D,F)\twoheadrightarrow D$ given by the following pullback square:

$$\begin{tikzcd}
\mathrm{Fun}_D(C\times D,F) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(C,F) \arrow[d, two heads, "p_*"] \\
D \arrow[r] & \mathrm{Fun}(C,C\times D),
\end{tikzcd}$$

where the bottom map is obtained by currying the identity on $C\times D$. For the Beck-Chevalley condition, we have to show that for a functor $f\colon D\to D'$ and an isofibration $E\twoheadrightarrow C\times D'$, the canonical map

$$D\times_{D'}\mathrm{Fun}_{D'}(C\times D',F)\to\mathrm{Fun}_D(C\times D,(C\times D')\times_{C\times D}F)$$

is an isomorphism. But this is clear from the construction, using that the right adjoint $\mathrm{Fun}(C,-)\colon\mathcal{E}\to\mathcal{E}$ preserves pullbacks along isofibrations. □

Corollary 17.1.13. *For every synthetic category C, the functor* $\mathrm{Fun}(C,-)\colon\mathcal{E}\to\mathcal{E}$ *is a morphism of simplicial type theories.*

Proof. In light of the previous lemma, this is a special case of Lemma 16.6.11. □

Lemma 17.1.14. *For synthetic categories C and D, the functor category* $\mathrm{Fun}(C,D)$ *satisfies the conditions from Axiom B.5.*

Proof. By adjunction we have for every other synthetic category E an equivalence $\mathrm{Hom}_{\mathcal{E}}(E\times C,D)\simeq\mathrm{Hom}_{\mathcal{E}}(E,\mathrm{Fun}(C,D))$. In particular, every functor $f\colon E\times C\to D$ determines a functor $f_c\colon E\to\mathrm{Fun}(C,D)$, every functor $g\colon E\to\mathrm{Fun}(C,D)$ determines $g^u\colon E\times C\to D$, and these satisfy $(f_c)^u=f$ and $(g^u)_c=g$. Given a natural isomorphism $\alpha\colon f\cong f'$ of functors $E\times C\to D$, we may curry to obtain a functor $\alpha\colon E\to\mathrm{Fun}(C,P(D))$. But by combining the previous corollary with Proposition 15.3.32, we see that $\mathrm{Fun}(C,P(D))$ is a path object for $\mathrm{Fun}(C,D)$ and hence provides the desired isomorphism $\alpha_c\colon f_c\cong f'_c$. Conversely, if $\beta\colon E\to\mathrm{Fun}(C,P(D))$ is a natural isomorphism $g\cong g'$ of functors $C\to\mathrm{Fun}(C,D)$, then uncurrying determines an isomorphism $\beta^u\colon g^u\cong(g')^u$. Again we clearly have $(\alpha_c)^u=\alpha$ and $(\beta^u)_c=\beta$. Finally the functoriality in E is clear. □

Diagram categories

We will now show that the diagram categories in $\mathcal{E}$ constructed in Section 16.2.2 satisfy all the axioms of diagram categories postulated in Section 1.3.

First, notice that the 1-simplex $[1] := \Delta^1$ comes equipped with maps $0, 1\colon \Delta^0 \to \Delta^1$, and hence satisfies the conditions of Axiom C.

Proposition 17.1.15. *The diagram categories* $\mathrm{Diag}(\Phi, C)$ *in* $\mathcal{E}$ *satisfy the conditions of Axioms D.1, D.2, D.3, D.4, D.5, D.6 and D.7.*

Proof. The diagram category $\mathrm{Diag}(\Phi, C)$ was defined in Construction 16.2.13 and assumed to exist in Tribe Axiom A. Its functoriality in C was constructed in Construction 16.2.15 and the functoriality in Φ in Construction 16.2.16. The property from Axiom D.3 that an isomorphisms of topes induces an equivalence of diagram categories is a consequence of Lemma 16.2.19. It is immediate from the definition that there are isomorphisms $\mathrm{Diag}(\Delta^0, C) \cong C$ and $\mathrm{Diag}(\Delta^1, C) \cong \mathrm{Fun}(\Delta^1, C)$ that are compatible with the inclusions $0, 1\colon \Delta^0 \to \Delta^1$, giving Axiom D.4. The content of Axiom D.5 is covered by Proposition 16.2.28, that of Axiom D.6 by Lemma 16.2.25, and that of Axiom D.7 by Proposition 16.2.20. □

Summary

Let us summarize our progress in the following proposition:

Proposition 17.1.16. *Let* $\mathcal{E}$ *be a simplicial type theory satisfying the Tribe Axioms D and E. Then the underlying theory of synthetic categories in* $\mathcal{E}$ *satisfies all axioms A - F from Chapter 1.*

17.2 Groupoids

We now introduce an axiom on $\mathcal{E}$ that guarantees the existence of groupoid cores $C^{\simeq}$ for all synthetic categories C.

Tribe Axiom F (Groupoid Core Axiom)**.** There is a morphism of tribes

$$(-)^{\simeq}\colon \mathcal{E} \to \mathcal{E}$$

equipped with a natural transformation $\gamma\colon (-)^{\simeq} \to \mathrm{id}_{\mathcal{E}}$ satisfying the following conditions:

(i) The map $\gamma_C\colon C^{\simeq} \to C$ is both a monomorphism and an isofibration for every synthetic category C;

(ii) The synthetic category $C^{\simeq}$ is a groupoid for every synthetic category C;

(iii) The map $\gamma_X \colon X^{\simeq} \to X$ is an equivalence for every groupoid X.

We refer to the category $C^{\simeq}$ as the *groupoid core* of C.

Lemma 17.2.1. *For every synthetic category C, the functor $\gamma_C \colon C^{\simeq} \to C$ is an embedding.*

Proof. This is immediate from Lemma 15.4.14. □

Lemma 17.2.2. *For every groupoid X, the map $\gamma_X \colon X^{\simeq} \to X$ is an isomorphism.*

Proof. As it is an equivalence and an isofibration, it admits a section $s \colon X \to X^{\simeq}$ by Corollary 15.3.25. As $\gamma_X s = \mathrm{id}_X$, it thus remains to show that also $s\gamma_X = \mathrm{id}_{X^{\simeq}}$. Since γ_X is a monomorphism, it suffices to check this after postcomposing with γ_X, where it follows from the relation $\gamma_X s = \mathrm{id}_X$. □

Proposition 17.2.3. *The functor $(-)^{\simeq} \colon \mathcal{E} \to \mathrm{Grpd}(\mathcal{E})$ is right adjoint to the inclusion $\mathrm{Grpd}(\mathcal{E}) \hookrightarrow \mathcal{E}$, with counit given by the transformation $\gamma_C \colon C^{\simeq} \to C$.*

Proof. We have to show that for every synthetic category C and every groupoid X, composition with $\gamma_C \colon C^{\simeq} \to C$ determines a bijection of sets

$$\mathrm{Hom}_{\mathrm{Grpd}(\mathcal{E})}(X, C^{\simeq}) = \mathrm{Hom}_{\mathcal{E}}(X, C^{\simeq}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{E}}(X, C).$$

This map is injective since γ_C is a monomorphism in $\mathcal{E}$. To see it is surjective, consider any map $f \colon X \to C$. By naturality of γ, there is a commutative diagram

$$\begin{tikzcd}
X^{\simeq} \arrow[r, "f^{\simeq}"] \arrow[d, "\gamma_X"', "\cong"] & C^{\simeq} \arrow[d, "\gamma_C"] \\
X \arrow[r, "f"] & C.
\end{tikzcd}$$

Since γ_X is an isomorphism by Lemma 17.2.2, it follows that f is in the image of the above map, finishing the proof. □

Corollary 17.2.4. *The functor $(-)^{\simeq} \colon \mathrm{Ho}(\mathcal{E}) \to \mathrm{Ho}(\mathrm{Grpd}(\mathcal{E}))$ is a right adjoint to the inclusion $\mathrm{Ho}(\mathrm{Grpd}(\mathcal{E})) \hookrightarrow \mathrm{Ho}(\mathcal{E})$: For every category C and every groupoid X, there is a bijection*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(X, C) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{E})}(X, C^{\simeq}).$$

Proof. Since both the inclusion $\mathrm{Grpd}(\mathcal{E}) \hookrightarrow \mathcal{E}$ and its right adjoint $(-)^{\simeq} \colon \mathcal{E} \to \mathrm{Grpd}(\mathcal{E})$ are morphisms of tribes, this is immediate from Lemma 15.3.35. □

Corollary 17.2.5. *For a synthetic category C, there is a 1-to-1 correspondence between the objects of C and the objects of its groupoid core $C^{\simeq}$.*

Proof. Apply Proposition 17.2.3 to the groupoid $X = \Delta^0$. □

If Γ is a groupoid, then the local tribe $\mathcal{E}(\Gamma)$ will again satisfy Tribe Axiom F.

Lemma 17.2.6. *Let $p\colon X \twoheadrightarrow \Gamma$ be an isofibration over a groupoid Γ. Then (X, p) is a groupoid in $\mathcal{E}(\Gamma)$ if and only if X is a groupoid in $\mathcal{E}$.*

Proof. The proof is identical to that of **??**. □

Construction 17.2.7 (Maximal subgroupoid in $\mathcal{E}(\Gamma)$)**.** We construct a functor

$$(-)^{\simeq}\colon \mathcal{E}(\Gamma) \to \mathcal{E}(\Gamma).$$

Given an isofibration $p\colon C \twoheadrightarrow \Gamma$, we define $(C, p)^{\simeq}$ as the fibration

$$\begin{tikzcd} C^{\simeq} \arrow[r, two heads, "p^{\simeq}"] & \Gamma^{\simeq} \arrow[r, "\gamma_\Gamma", "\cong"'] & \Gamma. \end{tikzcd}$$

It is clear that this is functorial in the pair (C, p). We equip this construction with a natural transformation to the identity functor on $\mathcal{E}(\Gamma)$ by using the naturality squares

$$\begin{tikzcd} C^{\simeq} \arrow[r, "\gamma_X"] \arrow[d, "p^{\simeq}"'] & C \arrow[d, two heads, "p"] \\ \Gamma^{\simeq} \arrow[r, "\gamma_\Gamma"] & \Gamma. \end{tikzcd}$$

Lemma 17.2.8. *The functor $(-)^{\simeq}\colon \mathcal{E}(\Gamma) \to \mathcal{E}(\Gamma)$ equipped with the transformation $(-)^{\simeq} \to \mathrm{id}$ from Construction 17.2.7 satisfies the properties (i)-(iii) from Tribe Axiom F.*

Proof. Properties (i) and (iv) follow immediately from the analogous properties in $\mathcal{E}$, while (ii) and (iii) are direct consequences of Lemma 17.2.6. □

Lemma 17.2.9. *For every functor $f\colon \Lambda \to \Gamma$ between groupoids, the pullback functor $f^*\colon \mathcal{E}(\Gamma) \to \mathcal{E}(\Lambda)$ preserves groupoid cores.*

Proof. This is immediate from the construction of the groupoid cores in $\mathcal{E}(\Gamma)$ and $\mathcal{E}(\Lambda)$ together with the assumption that $(-)^{\simeq}\colon \mathcal{E} \to \mathcal{E}$ preserves pullbacks along fibrations. □

We will now show that the category $\mathrm{Grpd}(\mathcal{E})$ of groupoids in $\mathcal{E}$ admits a natural structure of a tribe.

Lemma 17.2.10. *Consider a pullback square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, two heads, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{E}$ such that f and f' are isofibrations. If X, Y and Y' are groupoids, then so is X'.

Proof. The proof is identical to that of Lemma 2.4.1. □

Corollary 17.2.11. *The category* $\mathrm{Grpd}(\mathcal{E})$*, equipped with the isofibrations in $\mathcal{E}$ between groupoids, is a clan.*

Proof. This follows from the fact that $\mathrm{Grpd}(\mathcal{E})$ contains the terminal object of $\mathcal{E}$ and is closed under base change along isofibrations by Lemma 17.2.10. □

We will now show that the clan $\mathrm{Grpd}(\mathcal{E})$ is in fact a tribe.

Lemma 17.2.12. *A morphism $i\colon X \to Y$ of groupoids is anodyne in the clan* $\mathrm{Grpd}(\mathcal{E})$ *if and only if it is anodyne in $\mathcal{E}$.*

Proof. Since isofibrations in $\mathrm{Grpd}(\mathcal{E})$ are in particular isofibrations in $\mathcal{E}$, it is clear that if i is anodyne in $\mathcal{E}$ then it is also anodyne in $\mathrm{Grpd}(\mathcal{E})$. For the converse, we use that for any isofibration $f\colon C \twoheadrightarrow D$ in $\mathcal{E}$, there is by Proposition 17.2.3 a one-to-one correspondence between the following two lifting problems:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & C \arrow[d, two heads, "f"] \\ Y \arrow[r] \arrow[ur, dashed] & D \end{tikzcd} \qquad \Longleftrightarrow \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & C^{\simeq} \arrow[d, two heads, "f^{\simeq}"] \\ Y \arrow[r] \arrow[ur, dashed] & D^{\simeq}. \end{tikzcd}$$

Since $f^{\simeq}$ is an isofibration in $\mathrm{Grpd}(\mathcal{E})$, the right square admits a lift, which implies that also the left square admits a lift. □

Proposition 17.2.13. *The clan* $\mathrm{Grpd}(\mathcal{E})$ *from Corollary 17.2.11 is a tribe. Furthermore, the functor* $(-)^{\simeq}\colon \mathcal{E} \to \mathrm{Grpd}(\mathcal{E})$ *is a morphism of tribes.*

Proof. Since a morphism in $\mathrm{Grpd}(\mathcal{E})$ is anodyne if and only if it is anodyne in $\mathcal{E}$ by the previous corollary, we see that anodynes are closed under pullback along fibrations in $\mathrm{Grpd}(\mathcal{E})$. For the required anodyne-isofibration factorizations, consider a morphism $f\colon X \to Y$ between groupoids. Since $\mathcal{E}$ is a tribe, we may find a factorization

$$\begin{tikzcd} X \arrow[r, tail, "i"] & C \arrow[r, two heads, "p"] & Y \end{tikzcd}$$

in $\mathcal{E}$ into an anodyne and an isofibration. Since i is in particular an equivalence, it follows from Lemma 2.1.5 that C is also a groupoid, and hence this gives the desired factorization of f in $\mathrm{Grpd}(\mathcal{E})$. This shows that $\mathrm{Grpd}(\mathcal{E})$ is a tribe. Furthermore, since the functor $(-)^{\simeq}\colon \mathcal{E} \to \mathcal{E}$ is a morphism of tribes by assumption and lands in $\mathrm{Grpd}(\mathcal{E})$, it is clear that the resulting functor $(-)^{\simeq}\colon \mathcal{E} \to \mathrm{Grpd}(\mathcal{E})$ is a morphism of tribes. $\square$

17.3 Constructions of synthetic categories

The tribe axioms for subcategories, localizations, fundamental groupoids and joins are essentially identical to those from Part I:

Tribe Axiom G (Subcategory axiom)**.** Let C be a synthetic category and let $M \twoheadrightarrow \mathrm{Map}(\Delta^1, C)$ be a collection of morphisms in C closed under composition. Then there exists a synthetic category $\langle M\rangle_C$ equipped with an isofibration $i_M\colon \langle M\rangle_C \twoheadrightarrow C$ which exhibits $\langle M\rangle_C$ as a subcategory of C, and is equipped with an equivalence $\tilde{m}\colon \mathrm{Map}(\Delta^1, \langle M\rangle_C) \xrightarrow{\sim} M$ fitting in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}(\Delta^1, \langle M\rangle_C) \arrow[rr, "\tilde{m}", "\sim"'] \arrow[dr, "(i_m)_*"'] & & M \arrow[dl, two heads, "m"] \\
& \mathrm{Map}(\Delta^1, C). &
\end{tikzcd}$$

Note that the only difference with Axiom I is that we ask the inclusion $i_M\colon \langle M\rangle_C \to C$ to be an isofibration.

Tribe Axiom H (Localization axiom)**.** For every collection of morphisms $w\colon W \to \mathrm{Map}(\Delta^1, C)$ in C there exists a localization $l\colon C \to C[W^{-1}]$ of C at W.

Tribe Axiom I (Fundamental groupoid axiom)**.** For every synthetic category C, the synthetic category $\Pi_\infty(C)$ is a groupoid.

Tribe Axiom J (Join axiom)**.** For every groupoid Γ and isofibrations $C \twoheadrightarrow \Gamma$ and $D \twoheadrightarrow \Gamma$, there is a synthetic category $C \star_\Gamma D \in \mathcal{E}(\Gamma)$ and a homotopy pushout square

$$\begin{tikzcd}
C \times_\Gamma \partial\Delta^1_\Gamma \times_\Gamma D \arrow[r] \arrow[d] & C \times_\Gamma \Delta^1_\Gamma \times_\Gamma D \arrow[d] \\
C \sqcup D \arrow[r] & C \star_\Gamma D
\end{tikzcd}$$

in the local tribe $\mathcal{E}(\Gamma)$.

17.4 Cartesian and cocartesian fibrations

In Chapter 5, we provided a synthetic development of the theory of cartesian and cocartesian fibrations. While this discussion goes through verbatim within the tribe $\mathcal{E}$, it is often convenient to work with (co)cartesian fibrations $p\colon C \twoheadrightarrow D$ that are additionally *isofibrations*. In this section, we shall introduce various ‘strictified’ versions of definitions from Chapter 5 and prove strictification results.

Directed evaluation maps

Recall from Construction 5.1.1 and Construction 5.1.4 the definition of the directed evaluation maps $\vec{\mathrm{ev}}^f_0\colon \mathrm{Fun}([1], A) \to A \vec{\times}_f B$ and $\vec{\mathrm{ev}}^f_1\colon \mathrm{Fun}([1], A) \to B \vec{\times}_f A$ for every functor $f\colon A \to B$.

Lemma 17.4.1. *If $f\colon A \twoheadrightarrow B$ is an isofibration, than so are the maps $\vec{\mathrm{ev}}^f_0\colon \mathrm{Fun}(\Delta^1, A) \to A \vec{\times}_f B$ and $\vec{\mathrm{ev}}^f_1\colon \mathrm{Fun}(\Delta^1, A) \twoheadrightarrow B \vec{\times}_f A$.*

Proof. The map $\vec{\mathrm{ev}}^f_0$ factors as a composite

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, A) \arrow[r, two heads] & \mathrm{Fun}(\Delta^1, B) \times_{B \times B} A \times A \arrow[r, two heads] & A \vec{\times}_f B,
\end{tikzcd}$$

where the first map is an isofibration by the gluing axiom and the second map is an isofibration as it is a base change of f:

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, B) \times_{B \times B} A \times A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \vec{\times}_f B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, B) \arrow[d, two heads] \\
A \times A \arrow[r, "1_A \times f"] \arrow[d, "\mathrm{pr}_2"] \arrow[dr, phantom, "\lrcorner", very near start] & A \times B \arrow[r, "f \times 1_B"] \arrow[d, "\mathrm{pr}_2"] & B \times B \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

An analogous proof shows that $\vec{\mathrm{ev}}_1^f \colon \mathrm{Fun}(\Delta^1, A) \to B \vec{\times}_f A$ is an isofibration. ☐

Lemma 17.4.2 (Trivial fibrations)**.** *Assume that $f \colon A \twoheadrightarrow B$ is a trivial fibration. Then also $\vec{\mathrm{ev}}_0^f$ and $\vec{\mathrm{ev}}_1^f$ are trivial fibrations.*

Proof. This is immediate from Lemma 5.1.6. ☐

Lemma 17.4.3 (Base change)**.** *Consider a pullback square of the form*

$$\begin{tikzcd}
A' \arrow[r, "u"] \arrow[d, two heads, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "f"] \\
B' \arrow[r, "v"] & B,
\end{tikzcd}$$

where f and f' are isofibrations. Then the commutative square

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, A') \arrow[r, "u_*"] \arrow[d, two heads, "\vec{\mathrm{ev}}_0^{f'}"'] & \mathrm{Fun}(\Delta^1, A) \arrow[d, two heads, "\vec{\mathrm{ev}}_0^f"] \\
A' \vec{\times}_{f'} B' \arrow[r, "v \vec{\times}_f u"] & A \vec{\times}_f B
\end{tikzcd}$$

is a pullback square. The dual result for $\vec{\mathrm{ev}}_1^f$ and $\vec{\mathrm{ev}}_1^{f'}$ also holds.

Proof. The proof is identical to that of Lemma 5.1.7. (We warn the reader that the statement in Lemma 17.4.3 is about *strict pullbacks* in $\mathcal{E}$, while Lemma 5.1.7 only says something about the *homotopy pullbacks* in $\mathcal{E}$.) ☐

Lemma 17.4.4 (Local tribes)**.** *Let B be a synthetic category and consider an isofibration*

$$\begin{tikzcd}
C \arrow[rr, two heads, "f"] \arrow[dr, two heads, "p"'] & & D \arrow[dl, two heads, "q"] \\
& B &
\end{tikzcd}$$

in $\mathcal{E}(B)$. Then there exists a pullback square

$$\begin{tikzcd}
\mathrm{Fun}_B(\Delta^1_B, C) \arrow[r] \arrow[d, two heads, "\vec{\mathrm{ev}}_{0,B}^f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, C) \arrow[d, two heads, "\vec{\mathrm{ev}}_0^f"] \\
C \vec{\times}_{f,B} D \arrow[r] & C \vec{\times}_f D,
\end{tikzcd}$$

where we denote by $\vec{\mathrm{ev}}_{0,B}^f$ the (underlying map in $\mathcal{E}$ of the) directed evaluation map of f in the local tribe $\mathcal{E}(B)$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}_B(\Delta^1_B, C) \arrow[r] \arrow[d, two heads, "\vec{\mathrm{ev}}^f_{0,B}"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, two heads, "\vec{\mathrm{ev}}^f_0"] \\
C \vec{\times}_{f,B} D \arrow[r] \arrow[d, two heads] & C \vec{\times}_f D \arrow[d, two heads] \\
B \arrow[r] & \mathrm{Fun}(\Delta^1, B).
\end{tikzcd}$$

In light of the description of the arrow category $\mathrm{Fun}_B(\Delta^1_B, C)$ in $\mathcal{E}(B)$ from Lemma 16.6.7, the outer rectangle is a pullback square, hence by the pasting law of pullback squares it will suffice to show that also the bottom square is a pullback square. This follows from an iterated use of the pasting law of pullback squares applied to the following large commutative diagram in $\mathcal{E}$:

$$\begin{tikzcd}
& C \vec{\times}_f D \arrow[rrr] \arrow[ddr] \arrow[ddd, two heads] & & & \mathrm{Fun}(\Delta^1, D) \arrow[ddd, two heads] \arrow[ddr] & \\
C \vec{\times}_{f,B} D \arrow[ur] \arrow[rrr] \arrow[ddd, two heads] & & & \mathrm{Fun}_B(\Delta^1_B, D) \arrow[ur] \arrow[ddd, two heads] & & \\
& & C \times D \arrow[rrr] \arrow[ddd, two heads] & & & D \times D \arrow[ddd, two heads] \\
& C \vec{\times}_B D \arrow[ur] \arrow[rrr] \arrow[ddd, two heads] & & & D \vec{\times}_B D \arrow[ur] \arrow[ddd, two heads] & \\
C \times_B D \arrow[ur] \arrow[rrr] \arrow[ddd, two heads] & & & D \times_B D \arrow[ur] \arrow[ddd, two heads] & & \\
& & B \times B \arrow[rrr, equal] & & & B \times B \\
& \mathrm{Fun}(\Delta^1, B) \arrow[ur] \arrow[rrr, equal] & & & \mathrm{Fun}(\Delta^1, B) \arrow[ur] & \\
B \arrow[ur] \arrow[rrr, equal] & & & B. \arrow[ur] & &
\end{tikzcd}$$

Indeed, in the bottom half of the diagram, all left and right faces are pullback squares, and since the bottom faces are trivially pullback squares it follows that the middle two faces are pullback squares. On the top half, the slanted back face is a pullback square by definition, hence so is the back face of the top front cube. As the front face is a pullback by definition, it thus remains to show that its right face is a pullback square. But this also follows from the pasting lemma as the composite vertical face on the right is a pullback square. □

Initial and terminal objects

Let $\mathcal{E}$ be an arbitrary simplicial type theory (i.e. we only require $\mathcal{E}$ to satisfy the Tribe Axioms A - C).

18 Examples of synthetic category theories

This chapter still needs to be written.

TO DO:

- Show that the tribe of quasicategories satisfies all the axioms. In cases where the quasicategorical definition does not immediately agree with the synthetic one, show that they are indeed equivalent, or provide a reference.
- Show that the universe Cat of small synthetic categories, introduced in Section 7.4, satisfies all the axioms.
- More generally, show that for a topos $\mathcal{B}$ the category $\mathrm{Cat}(\mathcal{B})$ of categories internal to $\mathcal{B}$ satisfies all the axioms.

Part III

Type theoretical implementation

19 Introduction to type theory

In Part I of this book, we introduced synthetic category theory using the language of *naive category theory*. In Part II, we saw how the formalism of *tribes* gives a nice way to provide models for higher categories that satisfy all the axioms of synthetic category theory. A reader interested in foundations might object at this point that this approach is circular: the formalism of tribes makes use of the language of category theory, and hence we are using category theory to formalize category theory. In this chapter, we will address this point by providing foundations for higher category theory that are independent of category theory.

The traditional way in the literature to provide such foundations is via the theory of *quasi-categories*, which in turn rests on set theory. While this can be done, the language of set theory is simply not very adequate to model homotopical notions, for reasons we explain in Section 19.1. Instead, we choose to formalize higher category theory within the formalism of *dependent type theory*, building on work of Riehl and Shulman [RS17] and Buchholtz and Weinberger [BW21]. In this chapter, we will provide a leisurely overview of dependent type theory.

The current version of the book does not yet contain a treatment of our axioms of synthetic category theory formulated within dependent type theory. We hope to implement this at some point in the future, but this may take a while!

19.1 A deficiency of set theory

In this section we say some words about the usual way mathematics is formalized using *set theory*, and indicate the main deficiency of this choice for the purpose of higher category theory.

The language of set theory consists of letters/symbols for sets, like

$$a, b, c, \dots,$$
$$A, B, C, \dots,$$

and two "relation symbols"

$$= \qquad \text{and} \qquad \in .$$

In a sense, it is really only the membership symbol $\in$ that is axiomatized, as the notion of membership fully determines the notion of equality: we have $A = B$ if and only if the following holds:

Given any set x, we have $x \in A$ if and only if $x \in B$.

In particular, a key feature of set theory is that equality of sets is a *property*: two sets are equal when they have the same elements, and otherwise they are not equal; it does not make sense to ask *how* two sets are equal. This strict notion of equality has the advantage that set theory relies on very few basic principles. However, for the purposes of this book we regard this 'feature' as a huge *deficiency*: the language of set theory does not natively allow us to make identifications which are not of the form "having the same elements". This is inconvenient, since in mathematical practice there is a wide variety of more general identifications:

- In algebra, two isomorphic algebraic structures may be regarded as 'the same' for all intents and purposes. For example, given a group isomorphism $G \xrightarrow{\sim} H$, statements proved about the group G should be true about the group H as well;
- In category theory, two isomorphic objects of a category C are often thought of as 'the same': having an isomorphism $X \xrightarrow{\sim} Y$ between two objects implies that all categorical properties about X also hold for Y;
- In homotopy theory, two homotopy equivalent topological spaces are 'homotopically equal': given a homotopy equivalence $X \xrightarrow{\sim} Y$, any statement about the homotopy type of X will be true about the homotopy type of Y as well;
- In homological algebra, establishing a chain homotopy equivalence $C_\bullet \xrightarrow{\sim} D_\bullet$ implies that all homological properties of the chain complex $C_\bullet$ are also true for $D_\bullet$.
- In category theory, if one is given an equivalence of categories $C \xrightarrow{\sim} D$, then every categorical property of the category C should also hold for D.
- And so on and so forth.

From these examples, it becomes clear that equality is in its core a logical concept which has nothing to do with set theory: if two mathematical objects X and Y in a certain mathematical context are 'equal', we should have that for every 'well-formed' predicate P in that context, if $P(X)$ holds then also $P(Y)$ holds:

$$\text{If } X \text{ and } Y \text{ are equal, then } P(X) \text{ implies } P(Y). \tag{$*$}$$

Note that with this principle of equality, the question of *how* two mathematical objects are equal is perfectly sensible: to identify two objects, one needs to remember the data of the isomorphism/homotopy equivalence/equivalence/etcetera.

The language of set theory is not particularly well-suited for dealing with the above principle of equality: if two structures are not literally *equal as sets*, then there will be predicates that distinguish these two structures. For example, the one-element group $\{1\}$ is isomorphic to the one-element group $\{0\}$, but the statement "The group G contains the element 0" holds for $\{0\}$ but not for $\{1\}$. As a result, developing mathematics using the foundations of set theory can be subtle, as one always needs to ensure that the statements one proves respect the relevant notion of equality in that context. While various mathematical fields like *homological algebra*, *homotopy theory* and *higher category theory* have successfully found ways to deal with this problem, these fields are often considered to have a steep learning curve, resulting from the large disconnect between the internal language of these subjects and the foundations on which they are built.

19.2 Informal presentation of dependent type theory

The goal of this book is to reduce the gap between the foundations of higher category theory and the way it is used in mathematical practice. This will in particular mean that we need to discard set theory as the basis of mathematics. Instead, we will work within *dependent type theory*, a formalism that allows for richer notions of equality. In this section, we will provide an informal overview of this theory.

Type theory was introduced by Bertrand Russel as a way to avoid paradoxes in set theory (in its naive form — Zermelo proposed his axioms for set theory in its modern form at the same time, 1908, but the definitive form of set theory, after the contributions of Fraenkel and Skolem, came only in 1921/22). A first attempt to define a logic based on type theory was made by Bertrand Russel and Alfred North Whitehead in their book *Principia Mathematica* [WR10] from the 1910's. Another formulation of type theory as a rigorous formal system in its own right is Alonzo Church's book [Chu32] on *λ-calculus* from the 1930's.

In the last part of the 20th century, Per Martin-Löf [Mar82] developed a predicative modification of Church's type theory. This is usually called *dependent type theory*, also known under the names *constructive type theory*, *intuistionistic type theory* or *Martin-Löf type theory*. In dependent type theory, there are two basic concepts:

- *Types*, usually denoted by capital letters: $A, B, C, \dots, X, Y, \dots$.
- *Terms*, usually denoted by small letters: $a, b, c, \dots, x, y, \dots$.

Type theory has a specific language to talk about types and terms. A statement one can make in type theory is called a *judgment*. There are only four kinds of basic judgments:

(i) A is a (well-formed) **type**:

$$A \ \text{ type.}$$

(ii) A and B are **judgmentally equal types**:

$$A \equiv B \ \text{ type.}$$

(iii) a is a (well-formed) **term** of type A:

$$a : A.$$

(iv) a and b are **judgmentally equal terms** of type A:

$$a \equiv b : A.$$

The phrase 'judgmentally equal' should be interpreted as saying that these terms/types are the same by their very definition. For example, if we have a function $f\colon \mathbb{N} \to \mathbb{N}$ defined by the formula $f(x) = x^2$, and if $n : \mathbb{N}$ is a natural number, then the equality $f(n) \equiv n^2 : \mathbb{N}$ is a judgmental equality.

One frequently wants to make judgments that only make sense in the context of some other variables. To make this precise, we define a *context* as a list of variable declarations of the form

$$x_1 : A_1,\ x_2 : A_2,\ \dots\ ,\ x_n : A_n. \tag{19.1}$$

The adjective 'dependent' in 'dependent type theory' means that all four judgements (i)-(iv) can be made relative to any context Γ, so that general judgments will have the following form:

$$\Gamma \vdash A \ \text{ type}, \qquad \Gamma \vdash A \equiv B \ \text{ type}, \qquad \Gamma \vdash a : A, \qquad \Gamma \vdash a \equiv b : A.$$

For example, the first judgment, $\Gamma \vdash A$ type, means that A *is a (well-formed) type in context* Γ, and similarly for the other three judgments. In more down-to-earth terms, if Γ is of the form (19.1) then it is saying: "for every term x_1 of A_1, every term x_2 of A_2, …, and every term x_n of A_n, we are given a type A". One may also write $A(x_1,\dots,x_n)$ instead of A to make the dependencies on the previous variables explicit.

We should emphasize that even a context itself can be formed dependently: for a context of the form (19.1), the k-th type A_k for $1 \le k \le n$ may be dependent on the first $k-1$ variables $x_1, \dots, x_{k-1}$. This means that before using this context we should already have established the judgment

$$x_1 : A_1,\ \dots\ ,\ x_{k-1} : A_{k-1} \vdash A_k \ \text{ type}$$

for all $1 \leq k \leq n$.

There exists a version of Martin-Löf type theory in which the contexts are considered as independent objects in their own right. This is known as *substitution calculus*, and was developed by Martin-Löf [Mar], see [CCD21] for more details. In this case, one has additional judgments that allow one to introduce contexts and to substitute variables between contexts; for example, one then also has the following two judgments:

$$\Gamma \ \text{context} \qquad\qquad \Gamma \equiv \Gamma' \ \text{context.}$$

In this book, we will not work with substitution calculus but rather treat contexts on a meta-level as done above.

In dependent type theory, there are specific *inference rules* that prescribe how one can construct new types out of old ones or how to produce and use terms in them. Rather than explaining the general formalism behind these rules, we will merely illustrate them with two examples.

Example 19.2.1 (Product types)**.** Given two types A and B in context Γ, there exists a *product type* $A \times B$ in context Γ. This is expressed formally using the *formation rule*:

$$\frac{\Gamma \vdash A \ \text{type} \qquad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A \times B \ \text{type}}$$

The notation with the horizontal line means that whenever one has already established the judgments above the line, one may deduce the judgment below the line.

To produce a term (a,b) of this product type, one needs to provide a term a of A and a term b of B; This is expessed by the *introduction rule*:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a,b) : A \times B}$$

Conversely, given a term x of $A \times B$, there are two *elimination rules* which allow us to form the A-component and the B-component of x:

$$\frac{\Gamma \vdash x : A \times B}{\Gamma \vdash \mathrm{pr}_1(x) : A} \qquad\qquad \frac{\Gamma \vdash x : A \times B}{\Gamma \vdash \mathrm{pr}_2(x) : B}$$

Finally, there are three *computation rules* which tell us how the introduction and elimination rules interact. These rules guarantee that terms of $A \times B$ really behave like pairs of terms of A and terms of B:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pr}_1(a,b) \equiv a : A} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{pr}_2(a,b) \equiv b : B} \qquad \frac{\Gamma \vdash x : A \times B}{\Gamma \vdash x \equiv (\mathrm{pr}_1(x), \mathrm{pr}_2(x)) : A \times B}$$

The first two rules express that the first and second component of the term (a,b) are a and b, while the third rule expresses that every term x of $A \times B$ can be recovered from its two components.

Note that this definition of the product type is essentially defining $A \times B$ by its universal property of the cartesian product. This is in stark contrast to the definition in set theory of products of *sets*, which relies on the somewhat arbitrary choice to define a tuple (a,b) of two elements a and b as the set $\{\{a,b\},\{a\}\}$. This is a general pattern in type theory: types are constructed using type constructors that express their universal property rather than being defined via a somewhat arbitrary choice of implementation as one does in set theory.

Here is another example of the construction of a type:

Example 19.2.2 (Function types)**.** Given two types A and B in context Γ, there is a type $A \to B$ called the *function type*:

$$\frac{\Gamma \vdash A \ \text{type} \qquad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A \to B \ \text{type}}$$

Producing a term of the function type amounts to assigning to every term $a : A$ a term $b(a) : B$. In type theory, the resulting term in the function type is usually denoted by $\lambda a.b(a)$, where the lambda indicates that the expression should be read as the function $a \mapsto b(a)$. Formally, we have an introduction rule of the following form:

$$\frac{\Gamma,\ a : A \vdash b(a) : B}{\Gamma \vdash \lambda a.b(a) : A \to B}$$

The elemination rule in this context says that any term $f : A \to B$ can be evaluated at any term $a : A$:

$$\frac{\Gamma \vdash f : A \to B}{\Gamma,\ a : A \vdash f(a) : A}$$

Finally, we have two computation rules which express that any term of the function type $A \to B$ is uniquely specified by its values $f(a)$:

$$\frac{\Gamma,\ a : A \vdash b(a) : B}{\Gamma,\ a' : A \vdash (\lambda a.b(a))(a') \equiv b(a') : B} \qquad \frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f \equiv \lambda a.f(a) : A \to B}$$

Again, note how this formal definition of functions in type theory is very close to the way we use functions in practice: a function $f\colon A \to B$ is something which assigns values $f(a) : B$ to all terms $a : A$. In contrast, the definition in set theory needs to make the arguably arbitrary choice to record a function via its graph as a subset of $A \times B$.

19.3 Equality in dependent type theory

In Section 19.1, we described a flexible concept of equality in which two objects X and Y in a given mathematical context are regarded as ‘equal’ whenever every statement P about these objects holds for X if and only if it holds for Y. We also indicated why set theory does not natively support such flexible notions of equality. In this section, we will see how equality is implemented in dependent type theory, and how this can be used to make precise the above heuristic.

A fundamental difference between set theory and type theory is the way in which it treats *propositions*. While set theory is built on top of first-order logic, type theory is itself a mathematical logic with its own inference rules. This leads to the following principle, known as the *Curry-Howard correspondence*:

“Propositions are types.”

In practice, this principle means that any propositional statement about types will be implemented as a type itself. For such a propositional type P, we think of its terms $p : P$ as *proofs that P holds*. The various type formers in type theory can now be reinterpreted as logical operators. For example:

- Given propositional types P and Q, the product type $P \times Q$ is a propositional type representing the *logical conjunction* ‘P and Q’: providing a proof of $P \times Q$ amounts to providing both a proof p of P as well as a proof q of Q.
- Given propositional types P and Q, the function type $P \to Q$ is a propositional type representing the *logical implication* ‘P implies Q’: providing a proof of $P \to Q$ amounts to providing a proof $f(p)$ of Q for every proof p of P.

In the spirit of the Curry-Howard correspondence, also the notion of *equality* should be implemented as a type: given a type A and two terms $a : A$ and $b : A$, there is a type $a =_A b$, often abbreviated to $a = b$, which is called the *identity type of a and b*. It is introduced via the following formation rule:

$$\frac{\Gamma \vdash A \ \text{ type}}{\Gamma, a : A, b : A \vdash a = b \ \text{ type}}$$

We want to think of the terms of $a = b$ as *identifications* of a and b, or as *proofs* that a and b are equal. The idea is that there can be several ‘proofs’ that a and b are equal, and the type $a = b$ encodes all the complexity of the logical relations between them.

One crucial property of equality that we need to demand is that every term a of a type A is equal to itself in a preferred way, leading to the following introduction rule:

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{id}_a : a = a \ \text{ type}}$$

We can think of the term id_a as the *identity* of a. In type theory, this term is often denoted by refl_a since it expresses reflexivity of equality, but the notation id_a fits better with our aim of developing synthetic category theory.

Just like the examples of product types $A \times B$ and function types $A \to B$, the identity types $a = b$ come with *elimination* and *computation* rules which directly axiomatize their desired behavior. For identity types, this desired behavior is that equal terms cannot be distinguished by any type family: whenever we are given a type family $P(x)$ dependent on the terms x of a type A, then any identification $f : a = b$ between two terms a and b of A should produce a function $f_* \colon P(a) \to P(b)$, informally saying that any proof of $P(a)$ also provides a proof of $P(b)$. In fact, it turns out to be better to allow the type family P to be dependent also on the identification f, so that we obtain the following elimination rule:

$$\frac{\Gamma, a : A, b : A, f : a = b \vdash P(a, b, f) \ \text{ type}}{\Gamma, a : A, b : A, f : a = b \vdash f_* : P(a, a, \mathrm{id}_a) \to P(a, b, f)}$$

This rule is often referred to as *path induction* by type theorists, in which case f_* is denoted as path-ind$_f$. Finally, we have the computation rule which demands that the function $(\mathrm{id}_a)_* \colon P(a, a, \mathrm{id}_a) \to P(a, a, \mathrm{id}_a)$ is the identity:

$$\frac{\Gamma, a : A, b : A, f : a = b \vdash P(a, b, f) \ \text{ type}}{\Gamma, a : A, p : P(a, a, \mathrm{id}_a) \vdash (\mathrm{id}_a)_*(p) \equiv p : P(a, a, \mathrm{id}_a)}$$

The reader will probably have observed that the above axiomatization of equality is asymmetric: an identification $f \colon a = b$ gives a function $f_* \colon P(a) \to P(b)$ but not a priori function in the opposite direction. The reason for this asymmetry is that asking for the opposite direction is redundant: applying path induction to the type family $P'(a, b, f) := (b = a)$, we see that every identification $f : a = b$ determines a function $f_* \colon (a = a) \to (b = a)$ and thus gives rise to an 'inverse identification' $\overline{f} : b = a$ defined as $\overline{f} :\equiv f_*(\mathrm{id}_a)$. This provides a function $\overline{f}_* \colon P(b) \to P(a)$ in the opposite direction, which can be shown to be inverse to f_*.

19.4 Types as categories

In the previous three sections, we saw how dependent type theory addresses the deficiency of set theory regarding its notion of equality. In this section, we will address another deficiency: the fact that set theory cannot natively speak of itself.

In mathematical practice, sets do not just exist in isolation but can be compared via *functions* between sets. The notion of a function is a fundamental aspect of mathematics, and a large chunk of mathematical theorems refer to them in one way or another. If sets are equipped with additional structure, the relevant functions one wants to consider are those that preserve this structure in a suitable way, in which case we refer to them as *morphisms*. The philosophy that the morphisms between mathematical objects are just as important as (or arguably even more important than) the objects themselves is what leads one to the formalism of *category theory*.

Unfortunately, we now hit a problem if we would like to 'do set theory internal to set theory': a set is merely a collection of mathematical objects which lacks a notion of 'morphism' between these objects. In particular, there is no object in set theory which by itself reflects the true nature of sets: the collection of all (small) sets merely forms a set, and it does not let us talk about functions between sets. In this sense, we might say that 'set theory cannot speak of itself'.

In this book, we propose that the fundamental mathematical notion forming the basis of mathematics should be that of a *category* rather than that of a *set*: instead of just having collections of mathematical objects, we should be working with collections of mathematical objects that come equipped with an inherent notion of *morphisms* between these objects. This philosophy is already bread and butter for many mathematicians working in fields like algebraic topology, homological algebra, algebraic geometry, functional analysis and representation theory, since a large part of the theorems in these fields consists of constructing and studying adjunctions and equivalences between various categories. We would like to work with foundations of mathematics which natively incorporate this philosophy, again reducing the gap between the foundations and the mathematical practice.

Combining the above philosophy with our desire to work with dependent type theory, the starting point of this book is to interpret *types as categories*. This in particular means that we are in need of an appropriate extension of dependent type theory, since ordinary type theory does not natively allow us to speak of morphisms in a type. At the time of writing, this is an active research area in the field of (homotopy) type theory which is known under the collective name of *directed type theory*, see for example [LH11; War13; Nuy15; Nor19]. Other proposed solutions for dealing with higher categorical structures are *opetopic type theory* [BD98; AFS21] and *displayed type theory* [KS24]. Finally, there is an enhancement of homotopy type theory called *simplicial type theory*, introduced by Riehl and Shulman [RS17], which we pick as the foundational framework in which to formalize synthetic category theory. We will give an introduction to simplicial type theory in Chapter 16.

19.5 Categorical semantics of type theory

In the previous sections we explained why dependent type theory provides an appropriate choice of foundations for an axiomatic development of higher category theory. It is now time for a confession: in the bulk of this book, we will not actually work with directed type theory directly, but instead use a *categorical interpretation* of type theory due to Joyal [Joy17]. This means that we fix a category $\mathcal{E}$ whose objects we refer to as *types* or *synthetic categories*, that we formulate all axioms and type constructions in terms of categorical properties and constructions in $\mathcal{E}$.

Before explaining this approach in more detail, let us explain the two main reasons for this choice:

- *Familiarity*: A disadvantage of the existing approaches of formalizing higher category theory using type theory is that the terminology and notation from type theory are unfamiliar to many mathematicians. Since one of the goals in this book is to provide an accessible introduction to higher category theory, we would like to avoid as much as possible the cognitive hurdle resulting from a switch to a different foundational system. By working with a categorical interpretation of type theory, we hope to take away most of this hurdle and keep the focus instead on the actual axioms of synthetic category theory we put forward.
- *Flexibility*: At the time of writing, it is not yet completely clear to the authors what is the most suitable type theory for formalizing higher category theory. By first developing synthetic category theory in terms of a categorical interpretation of type theory, we can ensure that our theory focuses on the categorical semantics , leaving us the flexibility of choosing axioms that most directly reflect the mathematical practice. While we will indicate an approach for a genuine type theoretic formalization in Part III of this book[1], we expect that the axioms we propose admit a variety of different implementations in terms of type theory, and for this reason it seems useful to separate the semantics from the type theoretical implementation.

In the remainder of this section, we give an overview of the translation between type theory and its categorical semantics. As mentioned before, we fix a (small) category $\mathcal{E}$ with a terminal object $*$, whose objects we refer to as *types*, and whose categorical behavior reflects the desired behavior of the type theory we are axiomatizing. The four basic judgments in type theory admit the following translations to the categorical setup:

$$\begin{array}{ccc}
A \ \text{type} & \iff & A \text{ is an object of } \mathcal{E} \\
A \equiv B \ \text{type} & \iff & A \text{ and } B \text{ are the same object of } \mathcal{E} \\
a : A & \iff & a \text{ is a morphism } * \longrightarrow A \text{ in } \mathcal{E} \\
a \equiv b : A & \iff & a \text{ and } b \text{ are the same morphism } * \longrightarrow A
\end{array}$$

Here the words “the same” should really be interpreted in the strictest possible sense.

In order to deal with *dependent* type theory, we should be able to make sense of these four judgments within a given context Γ. To every context Γ we will assign an object of $\mathcal{E}$, which we will abusively again denote by Γ. We assume that we are given a subcategory

$$\mathcal{E}(\Gamma) \subseteq \mathcal{E}_{/\Gamma}$$

---

[1] This part still needs to be written

of the slice category of $\mathcal{E}$ over Γ whose objects we think of as ‘types in context Γ’. We will draw the objects of this category $\mathcal{E}(\Gamma)$ as arrows $p\colon A \twoheadrightarrow \Gamma$ and refer to them as *fibrations in $\mathcal{E}$*. The four basic judgments in context Γ can now be translated as follows:

$$\begin{array}{ccc}
\Gamma \vdash A \ \text{type} & \Longleftrightarrow & \text{a fibration } p\colon A \twoheadrightarrow \Gamma \text{ in } \mathcal{E} \\
\Gamma \vdash A \equiv B \ \text{type} & \Longleftrightarrow & A \text{ and } B \text{ are the same object of } \mathcal{E}(\Gamma) \\
\Gamma \vdash a : A & \Longleftrightarrow & \text{a section } a\colon \Gamma \to A \text{ of } p\colon A \twoheadrightarrow \Gamma \\
\Gamma \vdash a \equiv b : A & \Longleftrightarrow & a \text{ and } b \text{ are the same section}
\end{array}$$

In particular, a context of the form

$$x_1 : A_1,\ x_2 : A_2,\ \ldots,\ x_n : A_n$$

should be interpreted as a choice of fibrations $p_i\colon A_i \twoheadrightarrow A_{i-1}$ in $\mathcal{E}$ for all $1 \le i \le n$, where we set $A_0 = *$. The object in $\mathcal{E}$ assigned to this context is simply the object $A_n \in \mathcal{E}$.

The fibrations in $\mathcal{E}$ need to satisfy various properties, like closure under composition and base change. In ensuring that the category $\mathcal{E}$ can reflect all the type constructions one can do in (homotopy) type theory, [Joy17] came up with the suitable properties on $\mathcal{E}$, leading to the notions of *clans* and *tribes*. We will give a detailed introduction to the theory of clans and tribes in Chapters 14 and 15 below. In Chapter 16 we indicate how to enhance Joyal’s framework to capture the simplicial type theory of Riehl and Shulman.

Let us finish this chapter by explaining how various type theoretic constructions are reflected within the category $\mathcal{E}$:

- Given two types A and B in $\mathcal{E}$, their *product type* $A \times B$ is simply their cartesian product in $\mathcal{E}$;
- Given two types A and B in $\mathcal{E}$, their *function type* $A \to B$ is the *exponential object* B^A, defined by the universal property that maps $C \to B^A$ correspond to maps $C \times A \to B$.
- Given a type A, the *identity types* $a = b$ for $a : A$ and $b : A$ form a type family over $A \times A$ and are hence implemented via the choice of a fibration $\mathrm{Equ}_A \twoheadrightarrow A \times A$. The identities $\mathrm{id}_a\colon a = a$ are implemented via the choice of a morphism $i\colon A \to \mathrm{Equ}_A$ making the following diagram commute:

$$\begin{tikzcd}
 & \mathrm{Equ}_A \arrow[d, two heads] \\
A \arrow[ur, dashed, "i"] \arrow[r, "\Delta=(1_A,1_A)"'] & A \times A.
\end{tikzcd}$$

While the interpretations of $A \times B$ and $A \to B$ are clear, the interpretation of the type Equ_A is perhaps more mysterious. Let us illustrate it via three examples:

- If one thinks of types as sets, then the fibration $\mathrm{Equ}_A \twoheadrightarrow A \times A$ is simply the diagonal map $A \to A \times A$, and the notion of equality recovers the usual (strict) notion of equality of elements of a set;

- If one thinks of types as homotopy types, then the map $\mathrm{Equ}_A \twoheadrightarrow A \times A$ is given by the path space $P(A) := \mathrm{Map}([0,1], A) \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} A \times A$. The associated notion of equality is *equality up to homotopy*, and the the type $a = b$ is the homotopy type of paths in A from a to b.

- If one thinks of types as (higher) categories, then Equ_A is the full subcategory of the arrow category $\mathrm{Fun}([1], A)$ spanned by the isomorphisms. The associated notion of equality is *equality up to isomorphism*, and the $a = b$ is the (higher) groupoid of isomorphisms from a to b.

20 Crisp type theory

This chapter still needs to be written.

21 Simplicial type theory

This chapter still needs to be written.

22 Synthetic category theory

This chapter still needs to be written.

Part IV

Appendix

A Old approach to simplices and joins

In this appendix, we record Cisinski's approach to the construction of simplices and to the join operation $C \star D$. **Note:** the writing style of this chapter has not been modified since its removal from the body of the text, which may lead to various incorrect cross-references.

A.1 The 1-simplex

We introduce a synthetic category $\Delta^1 \in \mathcal{E}$, called the *1-simplex* or the *directed interval*, which allows us to speak of *morphisms* in a synthetic category $C \in \mathcal{E}$.

Notation A.1. Recall that we denote the terminal object of $\mathcal{E}$ by Δ^0. We also write

$$\partial\Delta^1 := \Delta^0 \sqcup \Delta^0,$$

and denote the two structure maps of the coproduct by

$$0\colon \Delta^0 \to \partial\Delta^1 \qquad \text{and} \qquad 1\colon \Delta^0 \to \partial\Delta^1.$$

Axiom P.1 (Existence of the 1-simplex)**.** There is a synthetic category $\Delta^1 \in \mathcal{E}$ equipped with a map $i\colon \partial\Delta^1 \to \Delta^1$ such that:

(1) The map i is both an isofibration and a monomorphism;

(2) The map i is universally exponentiable, in the sense of Definition 14.3.16;

(3) The object Δ^1 is 0-truncated (i.e., the diagonal map $(1_{\Delta^1}, 1_{\Delta^1})\colon \Delta^1 \to \Delta^1 \times \Delta^1$ is an embedding).

We refer to Δ^1 as the *1-simplex*.

Remark A.2. Since i is both an isofibration and a monomorphism, it is also an embedding by Lemma 15.4.14.

Notation A.3. We define maps $\delta^1_0, \delta^1_1 \colon \Delta^0 \to \Delta^1$ as the following composites:

$$\begin{tikzcd}
\Delta^0 \arrow[dr, two heads, "0"'] \arrow[drr, bend left, two heads, "\delta^1_1"] & & \\
& \partial\Delta^1 \arrow[r, two heads, "i"] & \Delta^1. \\
\Delta^0 \arrow[ur, two heads, "1"] \arrow[urr, bend right, two heads, "\delta^1_0"'] & &
\end{tikzcd}$$

By Lemma 17.1.8, the maps δ^1_0 and δ^1_1 are both isofibrations and monomorphisms. Furthermore, they are disjoint, in the sense that there is a pullback square

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^0 \arrow[d, "\delta^1_0"] \\
\Delta^0 \arrow[r, "\delta^1_1"] & \Delta^1.
\end{tikzcd}$$

Terminology A.4. We introduce the following terminology:

- An *object* of a synthetic category C is a functor $x \colon \Delta^0 \to C$;
- A *morphism* of C is a functor $f \colon \Delta^1 \to C$.
- We write $f(0) \colon \Delta^0 \to C$ for the composite $f \circ \delta^1_1$ and refer to it as the *source* or *domain* of f. We similarly define $f(1) := f \circ \delta^1_0$, which is called the *target* or *codomain* of f.
- Given objects x and y of C, we write

  $$f \colon x \to y$$

  to denote a morphism f in C satisfying $x = f(0)$ and $y = f(1)$.
- Given two functors $\varphi, \psi \colon C \to D$, a *natural transformation* $\alpha \colon \varphi \to \psi$ is a morphism $\alpha \colon C \times \Delta^1 \to D$ such that the following diagram commutes:

$$\begin{tikzcd}
C = C \times \Delta^{\{0\}} \arrow[dr, "1_C \times \delta^1_1"'] \arrow[drr, bend left, "\varphi"] & & \\
& C \times \Delta^1 \arrow[r, "\alpha"] & D. \\
C = C \times \Delta^{\{0\}} \arrow[ur, "1_C \times \delta^1_0"] \arrow[urr, bend right, "\psi"'] & &
\end{tikzcd}$$

- Given a natural transformation $\alpha\colon \varphi \to \psi$ of functors $C \to D$ and a morphism $f\colon x \to y$ in C, the *naturality square* for α at f is the composite

  $$\Delta^1 \times \Delta^1 \xrightarrow{f \times 1} C \times \Delta^1 \xrightarrow{\alpha} D.$$

  We will often informally display this situation using the following diagram:

  $$\begin{tikzcd}
  \varphi(x) \arrow[r, "\alpha_x"] \arrow[d, "\varphi(f)"'] & \psi(x) \arrow[d, "\psi(f)"] \\
  \varphi(y) \arrow[r, "\alpha_y"] & \psi(y).
  \end{tikzcd}$$

A.2 The join construction

The exponentiability of the map $i\colon \partial\Delta^1 \to \Delta^1$ allows us to define a *join construction* $-\star-\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, equipping $\mathcal{E}$ with a monoidal structure.

Construction A.5 (Join)**.** We will define the *join* $A \star B$ of two synthetic categories A and B. By the sum axiom, the map $p_A \sqcup p_B\colon A \sqcup B \twoheadrightarrow \Delta^0 \sqcup \Delta^0 = \partial\Delta^1$ is a fibration. Since the map $i\colon \partial\Delta^1 \to \Delta^1$ is assumed to be exponentiable, we obtain a functor

$$i_*\colon \mathcal{E}(\partial\Delta^1) \to \mathcal{E}(\Delta^1).$$

We define the *join* $A \star B$ as

$$\begin{tikzcd}
A \star B \arrow[d, two heads, "p_A \star p_B"'] & := & i_*(A \sqcup B) \arrow[d, two heads, "i_*(p_A \sqcup p_B)"] \\
\Delta^1 & & \Delta^1.
\end{tikzcd}$$

Note that we have $\Delta^0 \star \Delta^0 = \Delta^1$ since i_* preserves the terminal object.

Lemma A.6. *For synthetic categories A and B, there exists a preferred pullback square*

$$\begin{tikzcd}
A \sqcup B \arrow[r, two heads, "i_{A,B}"] \arrow[d, two heads, "p_A \sqcup p_B"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \star B \arrow[d, "p_A \star p_B"] \\
\partial\Delta^1 \arrow[r, two heads, "i"] & \Delta^1.
\end{tikzcd}$$

Proof. The map i induces a triple of adjunctions

$$\begin{tikzcd}
\mathcal{E}(\partial\Delta^1) \arrow[r, shift left=6, "i_!"] \arrow[r, shift right=6, "i_*"'] & \mathcal{E}(\Delta^1). \arrow[l, "i^*" description] \arrow[l, phantom, shift left=3, "\perp"] \arrow[l, phantom, shift right=3, "\perp"]
\end{tikzcd}$$

Since i is a monomorphism, we see that for every pair $A \sqcup B \twoheadrightarrow \partial\Delta^1$, the square

$$\begin{tikzcd}
A \sqcup B \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \sqcup B \arrow[d] \\
\partial\Delta^1 \arrow[r, "i"] & \Delta^1
\end{tikzcd}$$

is a pullback square. It follows that the unit map $A \sqcup B \to i^* i_!(A \sqcup B)$ is an isomorphism, and thus the functor $i_!$ is fully faithful. By adjunction, it also follows that also the functor $i_* \colon \mathcal{E}(\partial\Delta^1) \to \mathcal{E}(\Delta^1)$ is fully faithful. In particular, the counit $i^*(A \star B) = i^* i_*(A \sqcup B) \to A \sqcup B$ is an isomorphism. This proves the claim. $\square$

We can now give a concrete description of the universal property of the join construction. Given a map $X \to \Delta^1$, we define types X_0 and X_1 via the following pullback squares:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] & X_1 \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\Delta^0 \arrow[r, "\delta^1_1"] & \Delta^1 & \Delta^0 . \arrow[l, "\delta^1_0"']
\end{tikzcd}$$

Lemma A.7. *Consider synthetic categories A and B and consider an isofibration $X \twoheadrightarrow \Delta^1$. Then a functor $\varphi \colon X \to A \star B$ over Δ^1 is the same data as a pair of functors $\varphi_0 \colon X_0 \to A$ and $\varphi_1 \colon X_1 \to B$.*

Proof. By the universal property of $A \star B = i_*(A \sqcup B)$, we have

$$\mathrm{Hom}_{\mathcal{E}(\Delta^1)}(X, A \star B) \cong \mathrm{Hom}_{\mathcal{E}(\partial\Delta^1)}(X \times_{\Delta^1} \partial\Delta^1, A \sqcup B) \cong \mathrm{Hom}_{\mathcal{E}}(X_0, A) \times \mathrm{Hom}_{\mathcal{E}}(X_1, B),$$

where we use that $\mathcal{E}(\partial\Delta^1) \simeq \mathcal{E} \times \mathcal{E}$ by Tribe Axiom D. $\square$

We may summarize the above situation with the following diagram:

$$\begin{tikzcd}
X_0 \sqcup X_1 \arrow[rr, "\varphi_1 \sqcup \varphi_2"] \arrow[d] \arrow[dr] & & A \sqcup B \arrow[dl] \arrow[d] \\
X \arrow[r, dash] \arrow[dr, "f"'] & \partial\Delta^1 \arrow[r] \arrow[d, "i"] & A \star B \arrow[dl, "p_A \star p_B"] \\
& \Delta^1 . &
\end{tikzcd}$$

A.2.1 Unitality and associativity of joins

We will now introduce various subaxioms of Axiom P.1 that guarantee that the join construction $A \star B$ defines a monoidal structure on $\mathcal{E}$.

Axiom P.2 (Unitality of joins)**.** The maps

$$i_{\emptyset,\Delta^0} \colon \Delta^0 = \emptyset \sqcup \Delta^0 \to \emptyset \star \Delta^0 \qquad \text{and} \qquad i_{\Delta^0,\emptyset} \colon \Delta^0 = \Delta^0 \sqcup \emptyset \to \Delta^0 \star \emptyset$$

are isomorphisms.

Lemma A.8 (Unitality of joins)**.** *For every synthetic category A, the maps*

$$i_{\emptyset,A} \colon A = \emptyset \sqcup A \to \emptyset \star A \qquad \textit{and} \qquad i_{A,\emptyset} \colon A = A \sqcup \emptyset \to A \star \emptyset$$

are isomorphisms.

Proof. This is immediate from the following pullback diagrams:

$$\begin{tikzcd}
\emptyset \sqcup A \arrow[r, "i_{\emptyset,A}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \star A \arrow[d] \\
\emptyset \sqcup \Delta^0 \arrow[r, "i_{\emptyset,\Delta^0}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \star \Delta^0 \arrow[d] \\
\Delta^0 \sqcup \Delta^0 \arrow[r, "i"] & \Delta^1
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
A \sqcup \emptyset \arrow[r, "i_{A,\emptyset}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \star \emptyset \arrow[d] \\
\Delta^0 \sqcup \emptyset \arrow[r, "i_{\Delta^0,\emptyset}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^0 \star \emptyset \arrow[d] \\
\Delta^0 \sqcup \Delta^0 \arrow[r, "i"] & \Delta^1.
\end{tikzcd}$$

□

Axiom P.3 (Associativity of joins)**.** There is an isomorphism

$$a \colon \Delta^0 \star \Delta^1 \xrightarrow{\cong} \Delta^1 \star \Delta^0$$

such that the following diagram commutes:

$$\begin{tikzcd}
\Delta^0 \sqcup (\Delta^0 \sqcup \Delta^0) \arrow[rr, "\cong"] \arrow[d, "i_{\Delta^0,\Delta^0 \sqcup \Delta^0}"'] & & (\Delta^0 \sqcup \Delta^0) \sqcup \Delta^0 \arrow[d, "i_{\Delta^0 \sqcup \Delta^0,\Delta^0}"] \\
\Delta^0 \star (\Delta^0 \sqcup \Delta^0) \arrow[d, "1 \star i"'] & & (\Delta^0 \sqcup \Delta^0) \star \Delta^0 \arrow[d, "i \star 1"] \\
\Delta^0 \star \Delta^1 \arrow[rr, "a"] & & \Delta^1 \star \Delta^0.
\end{tikzcd}$$

Construction A.9. Given synthetic categories A, B and C, we define an isomorphism

$$a_{A,B,C} \colon A \star (B \star C) \xrightarrow{\cong} (A \star B) \star C.$$

Consider the following diagram:

$$\begin{tikzcd}
A \sqcup (B \sqcup C) \arrow[r, "\cong"] \arrow[d, "1_A \sqcup i_{B,C}"'] & (A \sqcup B) \sqcup C \arrow[d, "i_{A,B} \sqcup 1_C"] \\
A \sqcup (B \star C) \arrow[d, "i_{A,B\star C}"'] & (A \star B) \sqcup C \arrow[d, "i_{A\star B,C}"] \\
A \star (B \star C) \arrow[r, dashed, "a_{A,B,C}"] \arrow[d] & (A \star B) \star C \arrow[d] \\
\Delta^0 \star \Delta^1 \arrow[r, "a"] & \Delta^1 \star \Delta^0,
\end{tikzcd}$$

Note that the outer diagram commutes, using the commutativity of the diagram from Axiom P.3. It thus follows from a double use of the universal property of $(A \star B) \star C$ from Lemma A.7 that there exists a unique dashed arrow $a_{A,B,C}$ making the diagram commute. One may construct a map $a^{-1}_{A,B,C} \colon (A \star B) \star C \to A \star (B \star C)$ in an analogous fashion, and it follows by iterated use of the universal property of the join constrution that these maps are inverses to each other, showing that $a_{A,B,C}$ is an isomorphism.

Axiom P.4 (Monoidal coherence axiom)**.** The following two diagrams commute:

$$\begin{tikzcd}
& \Delta^0 \star \Delta^0 \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\Delta^0 \star (\emptyset \star \Delta^0) \arrow[rr, "a_{\Delta^0,\emptyset,\Delta^0}"'] & & (\Delta^0 \star \emptyset) \star \Delta^0
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \Delta^1 \star \Delta^1 \arrow[dl, "a_{\Delta^1,\Delta^0,\Delta^0}"'] & \\
(\Delta^1 \star \Delta^0) \star \Delta^0 & & \Delta^0 \star (\Delta^0 \star \Delta^1) \arrow[ul, "a_{\Delta^0,\Delta^0,\Delta^1}"'] \arrow[d, "1\star a"] \\
(\Delta^0 \star \Delta^1) \star \Delta^0 \arrow[u, "a\star 1"] & & \Delta^0 \star (\Delta^1 \star \Delta^0). \arrow[ll, "a_{\Delta^0,\Delta^1,\Delta^0}"]
\end{tikzcd}$$

Proposition A.10. *The join functor* $- \star - \colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ *defines a monoidal structure on* $\mathcal{E}$ *with monoidal unit given by* $\emptyset$*.*

Proof. We provided the unitality isomorphisms in Lemma A.8 and the associativity isomorphisms in Construction A.9. It remains to verify the triangle axiom and the pentagon axiom. For the triangle axiom, we have to show that for all synthetic categories A and B, the following diagram commutes:

$$\begin{tikzcd}
& A\star B \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
A\star(\emptyset\star B) \arrow[rr, "a_{A,\emptyset,B}"'] & & (A\star\emptyset)\star B
\end{tikzcd}$$

This diagram lives over the commutative triangle from Axiom P.4. By the universal property of $\Delta^0\star\Delta^0$, it suffices to show that the diagram commutes when pulled back along the map $\Delta^0\sqcup\Delta^0\to\Delta^0\star\Delta^0$. But there it becomes the diagram

$$\begin{tikzcd}
& A\sqcup B \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
A\sqcup(\emptyset\sqcup B) \arrow[rr, "\cong"'] & & (A\sqcup\emptyset)\sqcup B,
\end{tikzcd}$$

where the isomorphisms are always the canonical isomorphisms. This diagram clearly commutes.

For the pentagon axiom, we have to show that for all synthetic categories A, B, C and D, the diagram

$$\begin{tikzcd}
& (A\star B)\star(C\star D) \arrow[dl, "a_{A\star B,C,D}"'] & \\
((A\star B)\star C)\star D & & A\star(B\star(C\star D)) \arrow[ul, "a_{A,B,C\star D}"'] \arrow[d, "1_A\star a_{B,C,D}"] \\
(A\star(B\star C))\star D \arrow[u, "a_{A,B,C}\star 1_D"] & & A\star((B\star C)\star D) \arrow[ll, "a_{A,B\star C,D}"]
\end{tikzcd}$$

commutes. The proof is analogous to the proof of the triangle axiom and is left to the reader. □

Using the monoidal structure, we can define iterated joins:

Definition A.11 (Iterated joins)**.** Given synthetic categories $A_1,\dots,A_n$ for $n\geq 3$, we inductively define the *iterated join construction* as

$$A_1\star A_2\star\cdots\star A_n := (A_1\star\cdots\star A_{n-1})\star A_n.$$

When $n=0$, we define this expression to mean $\emptyset$. When $n=1$ and $n=2$, these simply mean A_1 and $A_1\star A_2$, respectively.

Observation A.12 (Associativity and unitality for iterated joins)**.** Assume given for every $1 \leq i \leq k$ a finite collection of objects $A_1^{(i)}, \ldots, A_{n_i}^{(i)}$. By the triangle and pentagon axioms for monoidal categories, it follows that there is a preferred isomorphism

$$\begin{aligned}(A_1^{(1)} \star \cdots \star A_{n_1}^{(1)}) \star (A_1^{(2)} \star \cdots \star A_{n_2}^{(2)}) \star \cdots \star (A_1^{(k)} \star \cdots \star A_{n_k}^{(k)}) \\ A_1^{(1)} \star \cdots \star A_{n_1}^{(1)} \star A_1^{(2)} \star \cdots \star A_{n_2}^{(2)} \star \cdots \star A_1^{(k)} \star \cdots \star A_{n_k}^{(k)}.\end{aligned}$$

We will henceforth always use these preferred isomorphisms to rebracket any iterated join into the standard bracketing from Definition A.11.

A.3 Higher simplices

We now define the higher simplices Δ^n for all $n \geq -1$ and equip them with the familiar simplicial structure.

Definition A.13 (Higher simplices)**.** Let $S = \{s_0 < \cdots < s_n\}$ be a finite ordered set. For every $s \in S$, we define synthetic categories

$$\Delta^{\{s\}} := \Delta^0 \qquad \text{and} \qquad \Delta^S := \Delta^{\{s_0\}} \star \cdots \star \Delta^{\{s_n\}}.$$

For a natural number $n \geq 0$, we define

$$\Delta^n := \Delta^{\{0<1<\cdots<n\}}.$$

We also set

$$\Delta^{-1} := \Delta^{\emptyset} := \emptyset.$$

Observation A.14. For finite ordered sets $S_1, \ldots, S_m$, $m \geq 1$, we obtain by Observation A.12 a preferred isomorphism

$$\Delta^{S_1} \star \cdots \star \Delta^{S_m} \xrightarrow{\cong} \Delta^{S_1 \sqcup \cdots \sqcup S_m},$$

where the disjoint union $S \sqcup T$ is always given the ordering in which $s < t$ whenever $s \in S$ and $t \in T$.

Construction A.15 (Functoriality of higher simplices)**.** We show that the assignment $S \mapsto \Delta^S$ is natural in the finite ordered set S: for every morphism $\varphi \colon S \to T$ of finite ordered sets, we define the functor

$$\Delta^{\varphi} \colon \Delta^S \to \Delta^T$$

as the following composite:

$$\Delta^S \cong \Delta^{\varphi^{-1}(t_1)} \star \cdots \star \Delta^{\varphi^{-1}(t_m)} \to \Delta^{\{t_1\}} \star \cdots \star \Delta^{\{t_m\}} = \Delta^T,$$

where $T = \{t_1 < \cdots < t_m\}$, and where the first isomorphism is the preferred isomorphism from the previous observation. It is clear from the construction that this is completely functorial, so that we get a functor

$$\Delta^\bullet \colon \Delta_+ \to \mathcal{E},$$

where Δ_+ denotes the *extended simplex category*: the skeleton of the category of finite ordered sets spanned by the objects $[n] := \{0 < 1 < \cdots < n\}$ for $n \geq -1$.

Example A.16 (Simplicial structure maps)**.** Particularly prominent maps between simplices are maps of the form

$$d_i^n \colon \Delta^{n-1} \to \Delta^n \qquad \text{and} \qquad s_i^n \colon \Delta^{n+1} \to \Delta^n$$

for $0 \leq i \leq n$, called the *face maps* and *degeneracy maps*, respectively. The map d_i^n is given by 'skipping the i-th vertex':

$$d_i^n \colon \Delta^{n-1} \cong \Delta^{\{0\}} \star \cdots \star \Delta^{\{i-1\}} \star \emptyset \star \Delta^{\{i+1\}} \star \Delta^{\{n\}} \to \Delta^{\{0\}} \star \cdots \star \Delta^{\{n\}} = \Delta^n.$$

The map s_i^n is given by 'repeating the i-th vertex':

$$s_i^n \colon \Delta^{n+1} \cong \Delta^{\{0\}} \star \cdots \star \Delta^{\{i-1\}} \star \Delta^1 \star \Delta^{\{i+1\}} \star \Delta^{\{n\}} \to \Delta^{\{0\}} \star \cdots \star \Delta^{\{n\}} = \Delta^n.$$

We may sometimes also write d_i and s_i instead, leaving the degree n implicit.

Exercise A.17 (??). Show that the maps d_i^n and s_i^n satisfy the simplicial identities:

$$\begin{aligned} d_j^{n+1} \circ d_i^n &= d_i^{n+1} \circ d_{j-1}^n && \text{if } i < j, \\ s_j^{n-1} \circ s_i^n &= s_{i-1}^{n-1} \circ s_j^n && \text{if } i > j, \\ s_j^n \circ d_i^{n-1} &= \begin{cases} d_i^n \circ s_{j-1}^{n+1} & \text{if } i < j, \\ \mathrm{id} & \text{if } i \in \{j, j+1\}, \\ d_{i-1}^n \circ s_j^{n+1} & \text{if } i > j+1. \end{cases} \end{aligned}$$

A.4 Lattice structure

Using the functoriality of the assignment $S \mapsto \Delta^S$, it is possible to define maps

$$\begin{aligned} \max \colon \Delta^1 \times \Delta^1 \to \Delta^1 \\ \min \colon \Delta^1 \times \Delta^1 \to \Delta^1 \end{aligned}$$

that behave like the 'maximum' and 'minimum' functors on the partial order $\{0 \leq 1\}$, in the sense that they satisfy the equations

$$\begin{aligned} \max(x,0) &= x = \max(0,x), & \max(x,1) &= 1 = \max(1,x), \\ \min(x,0) &= 0 = \min(0,x), & \min(x,1) &= x = \min(1,x). \end{aligned}$$

We will use the following auxiliary construction:

Construction A.18. We construct a map $\pi\colon \Delta^1 \times \Delta^1 \to \Delta^3$ which informally encodes the commutative square in Δ^3 given by the following diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 1 \arrow[r] & 3. \end{tikzcd}$$

To do this, observe that for every isofibration $X \twoheadrightarrow \Delta^1$ whose fibers over $0 : \Delta^1$ and $1 : \Delta^1$ are denoted by X_0 and X_1, the identity on $X_0 \sqcup X_1$ corresponds by adjunction to a map

$$X \to X_1 \star X_0$$

over Δ^1. Given a synthetic category C, we may take $X := C \times \Delta^1$ to obtain a commutative diagram

$$\begin{tikzcd} C \times \Delta^1 \arrow[rr, "\tilde{p}_C"] \arrow[dr, "\mathrm{pr}_2"'] & & C \star C \arrow[dl, "p_C \star p_C"] \\ & \Delta^1. & \end{tikzcd}$$

Letting $C = \Delta^1$ we thus get a map

$$\pi := \tilde{p}_{\Delta^1} : \Delta^1 \times \Delta^1 \to \Delta^1 \star \Delta^1 = \Delta^3$$

over Δ^1.

Lemma A.19. *The following four diagrams commute:*

$$\begin{tikzcd} \Delta^1 \times \{0\} \arrow[r] \arrow[d, "\cong"'] & \Delta^1 \times \Delta^1 \arrow[d, "\pi"] \\ \Delta^{\{0,1\}} \arrow[r] & \Delta^3, \end{tikzcd} \qquad \begin{tikzcd} \Delta^1 \times \{1\} \arrow[r] \arrow[d, "\cong"'] & \Delta^1 \times \Delta^1 \arrow[d, "\pi"] \\ \Delta^{\{2,3\}} \arrow[r] & \Delta^3, \end{tikzcd}$$

$$\begin{tikzcd} \{0\} \times \Delta^1 \arrow[r] \arrow[d, "\cong"'] & \Delta^1 \times \Delta^1 \arrow[d, "\pi"] \\ \Delta^{\{0,2\}} \arrow[r] & \Delta^3, \end{tikzcd} \qquad \begin{tikzcd} \{1\} \times \Delta^1 \arrow[r] \arrow[d, "\cong"'] & \Delta^1 \times \Delta^1 \arrow[d, "\pi"] \\ \Delta^{\{1,3\}} \arrow[r] & \Delta^3. \end{tikzcd}$$

Proof. The first two diagrams are immediate from the fact that the maps $\Delta^1 \times \{i\} \to \Delta^1 \times \Delta^1$ factors through $\Delta^1 \sqcup \Delta^1$, on which π corresponds to the explicit map

$$\Delta^1 \sqcup \Delta^1 \cong \Delta^{\{0,1\}} \sqcup \Delta^{\{2,3\}} \to \Delta^3.$$

The last two diagrams are an immediate consequence from the naturality in C of the construction of the map $\tilde{p}_C \colon C \times \Delta^1 \to C \star C$ used to construct π. □

Using the map π, we can now construct the maximum and minimum operations on Δ^1:

Construction A.20. We define the map $\max \colon \Delta^1 \times \Delta^1 \to \Delta^1$ as the composite

$$\max \colon \Delta^1 \times \Delta^1 \xrightarrow{\pi} \Delta^3 = \Delta^0 \star \Delta^2 \xrightarrow{1_{\Delta^0} \star p_{\Delta^2}} \Delta^0 \star \Delta^0 = \Delta^1.$$

Similarly, we define the map $\min \colon \Delta^1 \times \Delta^1 \to \Delta^1$ as the composite

$$\min \colon \Delta^1 \times \Delta^1 \xrightarrow{\pi} \Delta^3 = \Delta^2 \star \Delta^0 \xrightarrow{p_{\Delta^2} \star 1_{\Delta^0}} \Delta^0 \star \Delta^0 = \Delta^1.$$

Lemma A.21. *The maps* max *and* min *satisfy the equations*

$$\begin{aligned} \max(x,0) &= x = \max(0,x), & \max(x,1) &= 1 = \max(1,x), \\ \min(x,0) &= 0 = \min(0,x), & \min(x,1) &= x = \min(1,x), \end{aligned}$$

in the sense that the following diagrams commute:

$$\begin{tikzcd}
\Delta^1 \times \{0\} \arrow[r] \arrow[dr, equal] & \Delta^1 \times \Delta^1 \arrow[d, "\max"] & \{0\} \times \Delta^1 \arrow[l] \arrow[dl, equal] \\
& \Delta^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^1 \times \{1\} \arrow[r] \arrow[d] & \Delta^1 \times \Delta^1 \arrow[d, "\max"] & \{1\} \times \Delta^1 \arrow[l] \arrow[d] \\
\{1\} \arrow[r] & \Delta^1 & \{1\} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^1 \times \{0\} \arrow[r] \arrow[d] & \Delta^1 \times \Delta^1 \arrow[d, "\min"] & \{1\} \times \Delta^1 \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & \Delta^1 & \{0\} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^1 \times \{1\} \arrow[r] \arrow[dr, equal] & \Delta^1 \times \Delta^1 \arrow[d, "\min"] & \{1\} \times \Delta^1 \arrow[l] \arrow[dl, equal] \\
& \Delta^1. &
\end{tikzcd}$$

Proof. In light of the fact that the simplices Δ^n satisfy the simplicial identities, cf. Construction A.15, the claim is an immediate consequence of Lemma A.19 by unwinding definitions. □

B Proof of distributivity

In Chapter 14, we introduced the notions of dependent sums and dependent products in a clan $\mathcal{E}$. We also formulated in Section 14.3.4 the *distributivity property* of dependent sums and dependent products. The goal of this appendix is to give a proof of distributivity.

Warning B.1. The notion of distributivity was not discussed during Cisinski's lectures.

Proposition B.2 (Distributivity, Proposition 14.3.18)**.** *Let $\mathcal{E}$ be a clan and let $f\colon A \twoheadrightarrow B$ be a fibration such that every base change of f is exponentiable. For a fibration $p\colon E \twoheadrightarrow A$, consider the following commutative diagram in $\mathcal{E}$:*

$$\begin{tikzcd}
E \arrow[drr, two heads, "p"'] & & A \times_B f_*(E) \arrow[ll, "e(p)"'] \arrow[rr, two heads, "f'"] \arrow[d, two heads, "q'"] \arrow[drr, phantom, "\lrcorner", very near start] & & f_*(E) \arrow[d, two heads, "q=f_*(p)"] \\
& & A \arrow[rr, two heads, "f"] & & B.
\end{tikzcd}$$

Then the composite

$$\mathcal{E}(E) \xrightarrow{p_!} \mathcal{E}(A) \xrightarrow{f_*} \mathcal{E}(B)$$

is canonically equivalent to the composite

$$\mathcal{E}(E) \xrightarrow{e(p)^*} \mathcal{E}(A \times_B f^*(E)) \xrightarrow{f'_*} \mathcal{E}(f_*(E)) \xrightarrow{q_!} \mathcal{E}(B).$$

Proof. Given a fibration $g\colon F \twoheadrightarrow E$, we show that the fibrations $f_*p_!(g)$ and $q_!f'_*e(p)^*(g)$ are canonically isomorphic. Consider the composite

$$p_!(g) = pg\colon F \twoheadrightarrow A.$$

Since f is exponentiable by assumption, there exists a fibration

$$f_*(pg)\colon f_*(F) \to B$$

equipped with a morphism $e(pg)\colon A \times_B f_*(F) \to F$ over A, satisfying the property that for all morphisms $u\colon X \to B$ the induced map

$$\mathrm{Hom}_{/B}(X, f_*(F)) \to \mathrm{Hom}_{/A}(A \times_B X, F)$$

is a bijection. By applying the functor $f_*\colon \mathcal{E}(A) \to \mathcal{E}(B)$ to the map $g\colon (F, gp) \to (E, p)$ in $\mathcal{E}(A)$ to obtain a map

$$f_*(g)\colon f_*(F) \to f_*(E)$$

over B. By naturality of the counit map $e(p)\colon A \times_B f_*(E) \to E$, we have a commutative diagram as follows:

$$\begin{tikzcd}
A \times_B f_*(F) \arrow[r, "e(pg)"] \arrow[d, "A\times_B f_*(g)"'] & F \arrow[d, "g"] \\
A \times_B f_*(E) \arrow[r, "e(p)"] & E.
\end{tikzcd}$$

If we define the object F' via the pullback square

$$\begin{tikzcd}
F' \arrow[r, "e(p)'"] \arrow[d, two heads, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[d, two heads, "g"] \\
A \times_B f_*(E) \arrow[r, "e(p)"] & E,
\end{tikzcd}$$

then we obtain a unique map $e(g')\colon A \times_B f_*(F) \to F'$ satisfying

$$g' \circ e(g') = A \times_B f_*(g) \qquad \text{and} \qquad e(p)' \circ e(g') = e(pg).$$

We may summarize the above constructions in the following large commutative diagram:

$$\begin{tikzcd}
 & & A \times_B f_*(F) \arrow[r, "f''"] \arrow[dl, "e(g')"'] \arrow[dd, "A\times_B f_*(g)"] \arrow[ddr, phantom, "\lrcorner", very near start] & f_*(F) \arrow[dd, "f_*(g)"] \arrow[dddd, bend left, two heads, "f_*(pg)"] \\
 & F' \arrow[dl, "e(p)'"'] \arrow[dr, two heads, "g'"] & & \\
F \arrow[dr, two heads, "g"'] & & A \times_B f_*(E) \arrow[r, "f'"] \arrow[dl, "e(p)"'] \arrow[dd, two heads, "A\times_B f_*(p)"] \arrow[ddr, phantom, "\lrcorner", very near start] & f_*(E) \arrow[dd, two heads, "f_*(p)"] \\
 & E \arrow[dr, two heads, "p"'] & & \\
 & & A \arrow[r, "f"] & B.
\end{tikzcd}$$

It will suffice to show that the map $f_*(g)\colon F_*(F) \to f_*(E)$ equipped with the map

$$\begin{tikzcd}
A \times_B f_*(E) \arrow[rr, "e(g')"] \arrow[dr, "A\times_B f_*(g)"'] & & F' \arrow[dl, two heads, "g'"] \\
 & A \times_B f_*(E) &
\end{tikzcd}$$

over $A \times_B f_*(E)$ is a dependent product of g' along f', or in other words, that we have $f_*(g) = f'_*(g')$. Indeed, if we have this then we get

$$f_* p_!(g) = f_*(pg) = f_*(p) \circ f_*(g) = f_*(p) \circ f'_*(g') = q_! f'_* e(p)^*(g).$$

To prove that $f_*(g)$ has the universal property of $f'_*(g')$, consider a map $u\colon X \longrightarrow f_*(E)$ and a map $\psi\colon A\times_B X \longrightarrow F'$ over $A\times_B f_*(E)$:

$$\begin{tikzcd}
A\times_B X \arrow[rr, "\psi"] \arrow[dr, "A\times_B u"'] & & F' \arrow[dl, ->>, "g'"] \\
& A\times_B f_*(E). &
\end{tikzcd}$$

We need to show that there exists a unique map

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "u"'] & & f_*(F) \arrow[dl, ->>, "f_*(g)"] \\
& f_*(E) &
\end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
A\times_B X \arrow[rr, "A\times_B\varphi"] \arrow[dr, "\psi"'] & & A\times_B f_*(F) \arrow[dl, ->>, "e(g')"] \\
& F' &
\end{tikzcd}$$

commutes. To this end, consider the map

$$\psi' := e(p')'\circ\varphi\colon A\times_B X \xrightarrow{\psi} F' \xrightarrow{e(p)'} F.$$

This is a map over E and thus in particular a map over A:

$$\begin{tikzcd}
A\times_B X \arrow[rr, "\psi"] \arrow[dr, "A\times_B(f_*(p)\circ u)"'] & & F \arrow[dl, "pg"] \\
& A. &
\end{tikzcd}$$

By the universal property of $f_*(pg)\colon f_*(F) \twoheadrightarrow B$, there exists a unique map

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "f_*(p)u"'] & & f_*(F) \arrow[dl, ->>, "f_*(p)f_*(g)=f_*(pg)"] \\
& B &
\end{tikzcd}$$

such that the diagram

$$\begin{tikzcd}
A\times_B X \arrow[rr, "A\times_B\varphi"] \arrow[dr, "e(p)'\psi=\psi'"'] & & A\times_B f_*(F) \arrow[dl, ->>, "e(pg)=e(p)'e(g')"] \\
& F &
\end{tikzcd} \tag{$*$}$$

commutes. We claim that φ is the desired map we are looking for. First we show that the triangle

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "u"'] & & f_*(F) \arrow[dl, two heads, "f_*(g)"] \\
& f_*(E) &
\end{tikzcd}$$

commutes. By the universal property of $f_*(E)$, we may equivalently show that the composite

$$A \times_B X \xrightarrow{A \times_B u} A \times_B f_*(E) \xrightarrow{e(p)} E$$

agrees with the composite

$$A \times_B X \xrightarrow{A \times_B \varphi} A \times_B f_*(F) \xrightarrow{A \times_B f_*(g)} A \times_B f_*(E) \xrightarrow{e(p)} E.$$

This follows from the following commutative diagram:

$$\begin{tikzcd}
& A \times_B X \arrow[dl, bend right, "A \times_B u"'] \arrow[d, "\psi"'] \arrow[dr, "A \times_B \varphi"] & \\
& F' \arrow[dl, "g'"'] \arrow[d, two heads, "e(p)'" description] & A \times_B f_*(F) \arrow[l, two heads, "e(g')" description] \arrow[dl, "e(pg)" description] \arrow[d, "A \times_B f_*(g)"] \\
A \times_B f_*(E) \arrow[dr, "e(p)"'] & F \arrow[d, "g"] & A \times_B f_*(E) \arrow[dl, "e(p)"] \\
& E &
\end{tikzcd}$$

Next we show that the diagram

$$\begin{tikzcd}
A \times_B X \arrow[rr, "A \times_B \varphi"] \arrow[dr, "\psi"'] & & A \times_B f_*(F) \arrow[dl, two heads, "e(g')"] \\
& F' &
\end{tikzcd}$$

commutes. By the definition of F' as a pullback, it suffices to show this after composition with both of the maps $e(p)' \colon F' \to F$ and $g' \colon F' \to A \times_B f_*(E)$. For $e(p)'$, this is immediate from the defining property $(*)$ of φ. For g', this follows from the following commutative diagram:

$$\begin{tikzcd}
A \times_B X \arrow[rr, "A \times_B \varphi"] \arrow[d, "\psi"'] \arrow[ddr, "A \times_B u"] & & A \times_B f_*(F) \arrow[d, "e(g')"] \arrow[ddl, "A \times_B f_*(g)"'] \\
F' \arrow[dr, "g'"'] & & F' \arrow[dl, "g'"] \\
& A \times_B f_*(E), &
\end{tikzcd}$$

where the inner triangle commutes by what we showed before. This shows the existence of φ.

The above proof also makes clear that the map φ is unique. This finishes the proof. $\square$

Bibliography

[AFS21] Antoine Allioux, Eric Finster, and Matthieu Sozeau. *Types are Internal ∞-Groupoids*. 2021. arXiv: 2105.00024 `[cs.LO]`.

[Bar+16] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. "Parametrized higher category theory and higher algebra: Exposé I - Elements of parametrized higher category theory". *arXiv preprint arXiv:1608.03657* (2016).

[BD98] John C. Baez and James Dolan. "Higher-dimensional algebra. III: n-categories and the algebra of opetopes". *Adv. Math.* 135.2 (1998), pp. 145–206.

[BW21] Ulrik Buchholtz and Jonathan Weinberger. "Synthetic fibered $(\infty, 1)$-category theory". *arXiv preprint arXiv:2105.01724* (2021).

[CCD21] Simon Castellan, Pierre Clairambault, and Peter Dybjer. "Categories with families: unityped, simply typed, and dependently typed". *Joachim Lambek: the interplay of mathematics, logic, and linguistics*. Cham: Springer, 2021, pp. 135–180.

[Chu32] A. Church. "A set of postulates for the foundation of logic." *Ann. Math. (2)* 33 (1932), pp. 346–366.

[EH23] Elden Elmanto and Rune Haugseng. "On distributivity in higher algebra. I: The universal property of bispans". *Compos. Math.* 159.11 (2023), pp. 2326–2415.

[GG08] Nicola Gambino and Richard Garner. "The identity type weak factorisation system". *Theor. Comput. Sci.* 409.1 (2008), pp. 94–109.

[Joy17] André Joyal. "Notes on clans and tribes". *arXiv preprint arXiv:1710.10238* (2017).

[Kel64] G. M. Kelly. "On MacLane's conditions for coherence of natural associativities, commutativities, etc". *J. Algebra* 1 (1964), pp. 397–402.

[KS24] Astra Kolomatskaia and Michael Shulman. *Displayed Type Theory and Semi-Simplicial Types*. 2024. arXiv: 2311.18781 `[math.CT]`.

[LH11] Daniel R. Licata and Robert Harper. "2-dimensional directed type theory". *Proceedings of the 27th conference on the mathematical foundations of programming semantics (MFPS XXVII), Pittsburgh, PA, USA, May 25–28, 2011*. Amsterdam: Elsevier, 2011, pp. 263–289.

[Mar] Per Martin-Löf. *Substitution calculus*. Notes from a lecture given in Göteborg, November 1992.

[Mar82] Per Martin-Löf. *Constructive mathematics and computer programming*. Logic, methodology and philosophy of science VI, Proc. 6th int. Congr., Hannover 1979, Stud. Logic Found. Math. 104, 153-175 (1982). 1982.

[MW21] Louis Martini and Sebastian Wolf. "Limits and colimits in internal higher category theory". *arXiv preprint arXiv:2111.14495* (2021).

[Nor19] Paige Randall North. "Towards a directed homotopy type theory". *Proceedings of the 35th conference on the mathematical foundations of programming semantics, MFPS XXXV, London, UK, June 4–7, 2019*. Amsterdam: Elsevier, 2019, pp. 223–239.

[Nuy15] Andreas Nuyts. *Towards a directed homotopy type theory based on 4 kinds of variance*. 2015.

[Rij21] Egbert Rijke. *Introduction to Homotopy Type Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2021.

[RS17] Emily Riehl and Michael Shulman. "A type theory for synthetic ∞-categories". *High. Struct.* 1.1 (2017), pp. 147–224.

[Shu18] Michael Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". *Math. Struct. Comput. Sci.* 28.6 (2018), pp. 856–941.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[War13] Michael Warren. *Directed type theory*. Video lecture. 2013.

[WR10] A. N. Whitehead and B. Russell. *Principia Mathematica. Vol. I.* Cambridge: University Press. xv, 666 S. (1910). 1910. URL: www.hti.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=AAT3201.