Introduction to stable homotopy theory

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Introduction

Stable homotopy theory started off as a branch of algebraic topology that emerged in the mid-20th century as mathematicians sought to understand the behavior of homotopy groups in high dimensions. The key insight was the *Freudenthal suspension theorem* [Fre37]: if X is a CW-complex with one 0-cell and no k-cells for $1 \leq k \leq n-1$, then the canonical map $X \to \Omega\Sigma X$ from X to the loop space of its suspension induces an isomorphism

$$\pi_k(X) \xrightarrow{\sim} \pi_k(\Omega\Sigma X) = \pi_{k+1}(\Sigma X)$$

on k-th homotopy groups for all $0 \leq k \leq 2n-2$ (and a surjection for $k = 2n-1$). Since ΣX is another CW-complex which now has no k-cells for $1 \leq k \leq n$, we may apply this theorem another time to ΣX, and proceeding in this fashion we see that the sequence

$$\pi_k(X) \to \pi_k(\Omega\Sigma X) \to \pi_k(\Omega^2\Sigma^2 X) \to \dots$$

eventually stabilizes for every natural number k. The limiting group in this sequence is known as the *k-th stable homotopy group* of X and is denoted by $\pi_k^{\mathrm{st}}(X)$. These stable homotopy groups, while difficult to compute, contain deep geometric and topological information.

Given pointed CW-complexes X and Y, we may more generally consider the set $[X,Y]_*^{\mathrm{st}}$ of (homotopy classes of) *stable maps* from X to Y, defined as the colimit of the sequence

$$[X,Y]_* \to [X,\Omega\Sigma Y]_* \to [X,\Omega^2\Sigma^2 Y]_* \to \dots;$$

note that we have $\pi_k^{\mathrm{st}}(X) = [S^k, X]_*^{\mathrm{st}}$ as a special case. Finite pointed CW-complexes and stable maps assemble into a category known as the *Spanier-Whitehead category* [SW53].

The introduction of generalized (co)homology theories by Eilenberg and Steenrod [ES52] reinforced the centrality of the notion of stability in homotopy theory: the stability isomorphism $E_k(X) \cong E_{k+1}(\Sigma X)$ was taken as one of the core axioms. At the same time, it also made clear that the Spanier-Whitehead category was not quite the right category to consider: not every generalized cohomology theory can be represented by an object from this category. Several solutions were proposed, for example by Boardman, Lima, Kan and Whitehead. It eventually lead to Adams' definition of the 'stable homotopy category'

[Ada74], which contains the Spanier-Whitehead category as a full subcategory. Its objects are known as *spectra*:[1]

Definition. A *spectrum* is a sequence of pointed spaces $X = \{X_n\}_{n\geq 0}$ together with homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for all $n \geq 0$, where Ω denotes the loop space functor. A spectrum X is called *connective* if the homotopy groups $\pi_k(X_n)$ vanish for $k < n$.

Spectra are absolutely fundamental in stable homotopy theory, and their role is best compared with the role of abelian groups in commutative algebra. Indeed, most of the familiar operations on abelian groups admit 'homotopical' reflections at the level of spectra:

- Given two spectra X and Y, we may form their *direct sum* $X \oplus Y$, which is both the product and the coproduct in the category of spectra. In particular, the category of spectra is *additive*;
- Given two spectra X and Y, we may form their *tensor product* $X \otimes Y$. This gives rise to the notion of a *ring spectrum*: a spectrum R equipped with a multiplication operation $m\colon R \otimes R \to R$ that is 'coherently' associative and unital;
- We may talk about *modules* over a ring spectrum. Familiar notions for modules over (commutative) rings, like the properties of being projective, perfect, or flat, admit generalizations to the setting of ring spectra;
- Given a spectrum X and a prime p, we may form its *p-localization* $X_{(p)}$ and its *p-completion* $X^{\wedge}_p$, which have similar behavior as their analogus for abelian groups;
- And so on and so forth...

The techniques and concepts of stable homotopy theory, while rooted in algebraic topology, have found significant applications in numerous neighboring fields of mathematics, like (derived) algebraic geometry, algebraic K-theory, geometric topology and representation theory.

Higher category theory

To fully appreciate the pervasive role of (stable) homotopy theory in modern mathematics, we must first internalize what I call the 'fundamental principle of homotopy theory':

> **Fundamental Principle of Homotopy Theory:** When expressing that two entities are equal, we must always specify *how* they are equal by providing a homotopy/isomorphism between them.

[1] In the older literature these are often called 'Ω-spectra'.

This principle is already familiar to most mathematicians when applied to mathematical *objects*: for instance, when we want to express that two abelian groups A and B are 'the same', we naturally construct an isomorphism $A \cong B$ between them. In homotopy theory, we extend this principle to *morphisms* between objects; for example, two continuous maps $f, g\colon X \to Y$ are considered 'equal' when equipped with a homotopy $f \sim g$ between them. More broadly, in homotopy theory, the notion of 'sameness' is not a *property* (a binary yes/no question) but rather *structure* that must be explicitly provided (a homotopy, isomorphism, etc.). This shift in perspective originated in algebraic topology, but has by now become fundamental in many neighboring fields of mathematics.

While this principle appears straightforward, it rapidly gives rise to intricate 'coherence problems'. Consider, for instance, the task of defining a homotopical version of an abelian group A. Following the Fundamental Principle, expressing the associativity of addition $+\colon A \times A \to A$ requires providing a homotopy between the two maps $A \times A \times A \to A$ given by $(a,b,c) \mapsto a+(b+c)$ and $(a,b,c) \mapsto (a+b)+c$. Once such a homotopy is given, one can construct two distinct homotopies between the maps $(a,b,c,d) \mapsto a+(b+(c+d))$ and $(a,b,c,d) \mapsto ((a+b)+c)+d$. Requiring these to coincide necessitates the introduction of *homotopies between homotopies*. This process continues indefinitely, leading to an infinite hierarchy of higher homotopies, as laid out for instance by Stasheff [Sta63].

The modern solution to such coherence problems lies in the theory of ∞-*categories*. Conceptually, an ∞-category behaves like an ordinary category, but one in which we take the Fundamental Principle to its logical conclusion. Like an ordinary category, an ∞-category has objects X and morphisms $f\colon X \to Y$, but it also allows for homotopies $f \sim g$ between morphisms f and g, as well as homotopies between these homotopies, and so on ad infinitum. Homotopical versions of familiar categorical constructions are conveniently formulated within this framework: for example, we may define for any ∞-category $\mathcal{C}$ with finite products a new ∞-category $\mathrm{CGrp}(\mathcal{C})$ of *commutative groups* in $\mathcal{C}$, where all the required higher coherences are automatically built into the objects.

As we will see in the course, there are ∞-categories $\mathcal{S}$ and Sp whose objects are spaces[2] and spectra, respectively, and whose 'higher homotopies' are the expected ones. We will discuss the following theorem, which provides crucial insight into the nature of stable homotopy theory:

Recognition Principle for Infinite Loop Spaces (Boardman–Vogt, May and Segal)**.** *There is a fully faithful functor of ∞-categories*

$$\mathrm{CGrp}(\mathcal{S}) \hookrightarrow \mathrm{Sp}$$

[2]For reasons that will be explained, we will call these objects 'animae' in the course, and write An rather than $\mathcal{S}$ for the resulting ∞-category.

whose image consists of the connective spectra.[3]

In other words, we may think of a connective spectrum as a "homotopy coherent" analogue of an abelian group. This result is part of a much broader phenomenon: many algebraic structures have natural "homotopy coherent" analogues in the world of spectra (think for example of the ring spectra mentioned before).

While coherence problems in homotopy theory can be (and historically have been) handled without ∞-categories, the ∞-categorical framework provides a natural setting where the Fundamental Principle is built into the foundations. For this reason, a significant portion of this course will be devoted to developing the essential aspects of ∞-category theory, with a focus on its applications to stable homotopy theory. Our approach will emphasize conceptual understanding over technical details, always guided by the Fundamental Principle.

Content of the course

The precise content of the course has not yet completely been established and will depend on how quickly we progress throughout the semester. I currently have the following topics in mind:

(1) Spectra, and their fundamental role through Brown representability.

(2) The language of ∞-categories, building on the Fundamental Principle of Homotopy Theory discussed above.

(3) Stable ∞-categories, with the ∞-category Sp of spectra as our main example.

(4) Algebraic structures in homotopy theory, including commutative monoids, symmetric monoidal ∞-categories and (perhaps?) ∞-operads. We will discuss May's recognition principle for infinite loop spaces and the group completion theorem.

(5) Vector bundles and K-theory.

(6) Thom spectra, and its relation to classical differential topology.

(7) Localization and completion techniques in stable homotopy theory, including Bousfield localizations and arithmetic completion. Time permitting, we will give a glimpse of chromatic homotopy theory.

(8) Duality in stable homotopy theory, culminating in Atiyah duality.

[3] The difference between spectra and connective spectra is similar to the difference between arbitrary chain complexes and non-negatively graded chain complexes in homological algebra.

This admittedly long list of topics will most likely be too much to cover in a one-semester course. Depending on our progress, we may adjust the emphasis on various topics, or even skip some of them entirely.

Acknowledgments

In writing these lecture notes, I am taking inspiration from various sources on (stable) homotopy theory. Especially influential are the following three sets of lecture notes:

- The lecture notes *Introduction to stable homotopy theory* by Denis Nardin [Nar21];
- The lecture notes by Ferdinand Wagner from a lecture course by Fabian Hebestreit on *algebraic and Hermitian K-theory* [HW21];
- The lecture notes by Jack Davies on *stable and chromatic homotopy theory* [Dav24].

Classical developments of the theory of spectra may be found for example in the textbooks by Switzer [Swi75] and Adams [Ada74]; see also Barnes and Roitzheim [BR20] for a recent account of spectra via model categories.

The book *Higher algebra* by Jacob Lurie [Lur17] has been highly influential for the modern approach to stable homotopy theory via ∞-categories, and is still one of the major references in the field.

The approach to ∞-categories taken in this course is based on joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde [Cis+24]; see here for details.

Various words of thanks are in order. I thank Marc Hoyois for pointing out various mathematical mistakes in the notes. I thank the participants of the course for their feedback, which has led to various useful clarifications and additions to the notes. I thank Tim Henke for various suggestions improving the exposition of these notes. Finally I wish to thank Denis Nardin, Fabian Hebestreit and Jack Davies for the excellent sets of lecture notes that have formed the basis for these notes; the lectures of Fabian were particularly influential to me since they were where I learned most of the basics on ∞-category theory.

In the process of writing these notes, I have now and then made use of Claude 1.5 Sonnet to find reformulations that best convey what I want to express.

1 Spectra and Brown representability

In this chapter, we will introduce the most fundamental objects in stable homotopy theory: spectra.

1.1 Homotopy pullbacks and homotopy pushouts

We start with some recollections regarding homotopy pullbacks and homotopy pushouts.

Definition 1.1.1. Consider maps $f\colon X \to Z$ and $g\colon Y \to Z$ of topological spaces. We define their *homotopy pullback* $X \times_Z^h Y$ as

$$\begin{aligned} X \times_Z^h Y &:= X \times_Z Z^{[0,1]} \times_Z Y \\ &= \{(x,p,y) \in X \times Z^{[0,1]} \times Y \mid p(0) = f(x), p(1) = g(y)\}. \end{aligned}$$

We denote by $\mathrm{pr}_X\colon X \times_Z^h Y \to X$ and $\mathrm{pr}_Y\colon X \times_Z^h Y \to Y$ the two projection maps, and we denote by $H\colon (X \times_Z^h Y) \times [0,1] \to Z$ the homotopy given by $((x,p,y),t) \mapsto p(t)$. Note that H satisfies $H_0 = f \circ \mathrm{pr}_X$ and $H_1 = g \circ \mathrm{pr}_Y$, so that it defines a homotopy between the two outer composites in the following square:

$$\begin{tikzcd} X \times_Z^h Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] \arrow[ur, phantom, "\overset{H}{\underset{\sim}{\Nearrow}}"] & Z. \end{tikzcd}$$

In fact, observe that this square is in a sense *universal* with this data: for an arbitrary space T, a map $T \to X \times_Z^h Y$ is the same as a pair of maps $t_X : T \to X$ and $t_Y : T \to Y$ together with a homotopy $f \circ t_X \sim g \circ t_Y$. We will discuss the resulting universal property of homotopy pullbacks in more detail in Chapter 2, see Section 2.3.2.

Example 1.1.2. Given a map $f\colon X \to Y$, the *homotopy fiber* of f at a point $y \in Y$ is the homotopy pullback of f along the map $y\colon * \to Y$.

Example 1.1.3. The *loop space* ΩX of a pointed space (X,x) is defined as the homotopy pullback of the map $x\colon * \to X$ along itself:

$$\Omega X = \Omega(X,x) := \{p\colon [0,1] \to X \mid p(0) = p(1) = x\}.$$

We regard ΩX again as a pointed space with basepoint given by the constant loop at x. Given a pointed map $f\colon (X,x) \to (Y,y)$, we obtain an induced map $\Omega f\colon \Omega X \to \Omega Y$ given by $\Omega f(p) := f \circ p$. This defines a functor

$$\Omega\colon \mathrm{Top}_* \to \mathrm{Top}_*,$$

where Top_* denotes the category of pointed topological spaces and pointed maps.

Definition 1.1.4. Consider maps $f\colon Z \to X$ and $g\colon Z \to Y$ of topological spaces. We define their *homotopy pushout* $X \sqcup^h_Z Y$ as

$$\begin{aligned} X \sqcup^h_Z Y &:= X \sqcup_Z (Z \times [0,1]) \sqcup_Z Y \\ &= (X \sqcup Z \times [0,1] \sqcup Y)/\simeq, \end{aligned}$$

where the equivalence relation $\simeq$ is generated by identifying $(z,0) \in Z \times [0,1]$ with $f(z) \in X$ and $(z,1) \in Z \times [0,1]$ with $g(z) \in Y$ for all $z \in Z$. We denote by $\iota_X\colon X \hookrightarrow X \sqcup^h_Z Y$, $x \mapsto x$ and $\iota_Y\colon Y \hookrightarrow X \sqcup^h_Z Y$, $y \mapsto y$ the two canonical inclusions, and we denote by $H\colon Z \times [0,1] \to X \sqcup^h_Z Y$ the homotopy given by $(z,t) \mapsto (z,t)$. Note that H satisfies $H_0 = \iota_X \circ f$ and $H_1 = \iota_Y \circ g$, so that it defines a homotopy between the two outer composites in the following square:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \\ X \arrow[r, "\iota_X"'] \arrow[ur, phantom, "\overset{H}{\underset{\sim}{\Nearrow}}"] & X \sqcup^h_Z Y. \end{tikzcd}$$

Again, note that this square is *universal* with this data: for another topological space T, a map $X \sqcup^h_Z Y \to T$ is precisely the data of maps $t_X\colon X \to T$ and $t_Y\colon Y \to T$ together with a homotopy $t_X \circ f \sim t_Y \circ g$.

Definition 1.1.5. Given a map $f\colon X \to Y$ of topological spaces, we define the *homotopy cofiber* $C(f)$ as

$$C(f) = Y \sqcup^h_X * = Y \sqcup_X (X \times [0,1])/X \times \{1\}.$$

Given a map $g\colon Y \to Z$ equipped with a *nullhomotopy* $g \circ f \sim \mathrm{const}_z$, i.e. a map $H\colon X \times [0,1] \to Z$ with $H_0 = gf$ and $H_1 = \mathrm{const}_z$, we obtain a canonical map

$$C(f) \to Z, \qquad y \mapsto g(y), \qquad (x,t) \mapsto H(x,t).$$

We say that the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a *cofiber sequence* if this map $C(f) \to Z$ is a homotopy equivalence.

Example 1.1.6. When $f = \mathrm{id}_X\colon X \to X$ is the identity, the cofiber $C(\mathrm{id}_X)$ is known as the *cone of* X and is denoted $C(X)$:

$$C(X) = (X \times [0,1])/X \times \{1\}.$$

Note that $C(f) = Y \sqcup_X C(X)$.

Remark 1.1.7. If (X,x) is a pointed topological space, we often want to consider the *reduced cone*

$$\tilde{C}(X) := C(X)/\{x\} \times [0,1].$$

Similarly, if $f\colon X \to Y$ is a pointed map, we may consider the *reduced cofiber* $\tilde{C}(f) = C(f) = Y \sqcup_X \tilde{C}(X)$. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of *pointed* spaces is called a *cofiber sequence in* CW_* if the composite gf comes with a null-homotopy $H\colon gf \sim \mathrm{const}_*$ such that the induced map $\tilde{C}(f) \to Z$ is a homotopy equivalence.

Example 1.1.8. The *suspension* SX of a topological space X is the homotopy cofiber of $X \to *$:

$$SX := * \sqcup_X (X \times [0,1]) \sqcup_X *.$$

The *reduced suspension* ΣX of a pointed topological space (X,x) is the quotient

$$\Sigma X := SX/(\{x\} \times [0,1]).$$

Note that the latter defines a functor $\Sigma\colon \mathrm{Top}_* \to \mathrm{Top}_*$ given on a pointed map $f\colon (X,x) \to (Y,y)$ by $(\Sigma f)[(x,t)] := [(f(x),t)]$.

Lemma 1.1.9. *There is an adjunction* $\Sigma\colon \mathrm{Top}_* \rightleftarrows \mathrm{Top}_* : \Omega$.

Proof. Consider pointed spaces (X,x) and (Y,y). The data of a pointed map $\Sigma X \to Y$ is that of a map $H\colon X \times [0,1] \to Y$ such that $H_0 = H_1 = \mathrm{const}_y$ and $H_t(x) = y$ for all $t \in [0,1]$. This in turn corresponds to a pointed map $\tilde{H}\colon X \to \Omega(Y,y) \subseteq \mathrm{Map}([0,1],Y)$, as desired. □

Recall that the *homotopy category* hTop_* is the category whose objects are the pointed topological spaces (X,x), and whose morphisms are the (pointed) homotopy classes of pointed maps $(X,x) \to (Y,y)$:

$$\mathrm{Hom}_{\mathrm{hTop}_*}((X,x),(Y,y)) = [X,Y]_*.$$

Corollary 1.1.10. *There is an adjunction* $\Sigma\colon \mathrm{hTop}_* \rightleftarrows \mathrm{hTop}_* : \Omega$.

Proof. It remains to show that two pointed maps $f,g\colon \Sigma X \to Y$ are pointed homotopic if and only if their adjoint maps $\tilde{f},\tilde{g}\colon X \to \Omega Y$ are, which is clear from chasing through the previous proof. □

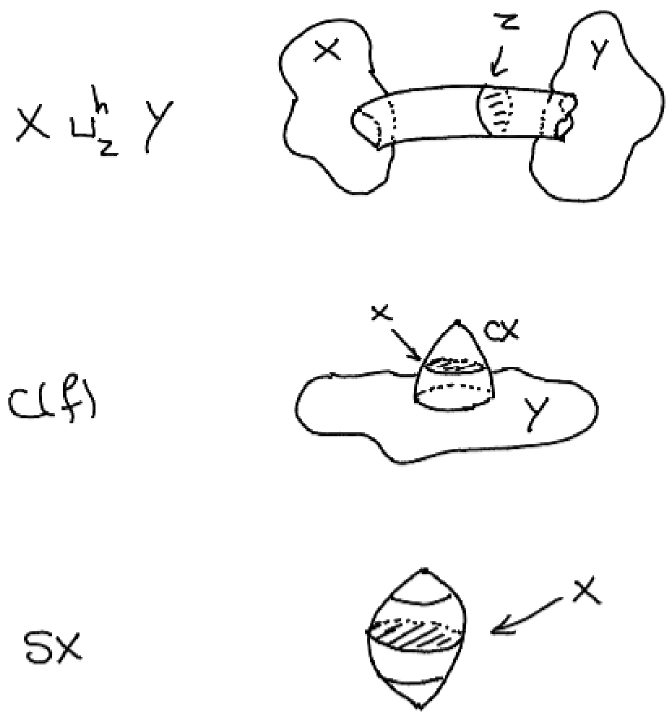


Figure 1.1: Illustration of the homotopy pushout, cofiber and suspension.

Remark 1.1.11. Recall that the k-th homotopy group of a pointed topological space (X,x) is defined as

$$\pi_k(X,x) := [S^k, X]_*.$$

The previous corollary thus in particular provides canonical isomorphisms

$$\pi_k(\Omega X) = [S^k, \Omega X]_* \cong [\Sigma S^k, X]_* \cong [S^{k+1}, X]_* = \pi_{k+1}(X).$$

1.2 Spaces versus topological spaces

In homotopy theory, we are interested in topological spaces up to weak homotopy equivalence, i.e. we regard two topolological spaces as 'the same' if there is a zig-zag of weak homotopy equivalences between them. Recall from your previous topology courses the following two theorems:

Theorem 1.2.1 (CW-approximation)**.** *For every topological space X there exists a CW-complex and a weak homotopy equivalence $Z \xrightarrow{\sim} X$.*

Theorem 1.2.2 (Whitehead theorem)**.** *Let X and Y be CW-complexes. Then every weak homotopy equivalence $X \xrightarrow{\sim} Y$ is already an actual homotopy equivalence.*

Because of these two theorems, we may safely work with CW-complexes up to homotopy equivalence. We will now introduce some terminology which emphasizes this perspective.

Convention 1.2.3. We define a *space* to be a topological space which is homotopy equivalent to a CW-complex. Similarly we define a *pointed space* to be a topological space which is pointed homotopy equivalent to a pointed CW-complex. We denote by

$$\mathrm{h}\mathcal{S} \subseteq \mathrm{hTop} \qquad \text{and} \qquad \mathrm{h}\mathcal{S}_* \subseteq \mathrm{hTop}_*$$

the full subcategories of (pointed) spaces. Since any homotopy equivalence $X \simeq Y$ becomes an isomorphism in hTop, the category $\mathrm{h}\mathcal{S}$ is equivalent to the homotopy category hCW of CW-complexes, and similarly $\mathrm{h}\mathcal{S}_* \simeq \mathrm{hCW}_*$.

Although this terminology may be quite confusing at first, it is very useful to distinguish the two different usages of topological spaces in homotopy theory:

- When we say 'topological space', we are talking about a specific set X equipped with a topology. In particular, it makes sense to speak of the set of points of X or the open subsets of X.
- Whenever we use the word 'space' without the adjective 'topological', we do not allow ourselves to say anything that is not invariant under homotopy equivalences. For example, we may speak of the homology or homotopy groups of a space, but we do not allow ourselves to speak about the 'set of points' of a space, or its 'open subsets', or its 'dimension': all of these will change if we replace the space X by a homotopy equivalent one.

When using the second perspective, we are in a sense forgetting the actual topological structure of the topological space, and we only remember its purely homotopical aspects, also known as its underlying *anima*.[1] See also Remarks 2.6.2 and 2.6.28 below.

Let us now check that all the relevant constructions on topological spaces preserve homotopy equivalences.

Exercise 1.2.4. Consider a diagram of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Z \arrow[d, "\gamma"] & Y \arrow[l, "g"'] \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Z' & Y' \arrow[l, "g'"'] \end{tikzcd}$$

which commutes up to homotopy, in the sense that we are given homotopies $H\colon \gamma \circ f \sim f' \circ \alpha$ and $K\colon \gamma \circ g \sim g' \circ \beta$.

[1] The word 'anima' means 'soul' in Latin.

- Construct a map $\alpha \times_\gamma \beta \colon X \times^h_Z Y \to X' \times^h_{Z'} Y'$. (This map will depend on the choices of homotopies H and K, though this is not reflected in our notation.)
- Show that $\alpha \times_\gamma \beta$ is a homotopy equivalence whenever α, β and γ are homotopy equivalences.

Exercise 1.2.5. Formulate and prove the analogous statement for homotopy pushouts.

Exercise 1.2.6. Show that if $f\colon Z \to X$ and $g\colon Z \to Y$ are maps of spaces, then the homotopy pushhout $X \sqcup^h_Z Y$ is again a space.
Hint: use Exercise 1.2.5 and the cellular approximation theorem.

The dual statement for homotopy *pullbacks* is also true but highly non-trivial:

Theorem 1.2.7 (Mather [Mat76], Milnor [Mil59])**.** *If $f\colon X \to Z$ and $g\colon Y \to Z$ are maps of spaces, then the homotopy pullback $X \times^h_Z Y$ is again a space.*

As a special case of the above exercise and theorem, we see that for a pointed space X the reduced suspension ΣX and the loop space ΩX are again pointed spaces. In particular, the adjunction from Corollary 1.1.10 restricts to an adjunction

$$\Sigma\colon \mathrm{h}\mathcal{S}_* \rightleftarrows \mathrm{h}\mathcal{S}_* \colon \Omega.$$

1.3 Cohomology theories and Brown representability

Recall from the introduction the following definition:

Definition 1.3.1. A *spectrum* X is a sequence of pointed spaces $X = (X_n)_{n\geq 0}$ together with pointed homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for all $n \geq 0$.
Given another spectrum $Y = (Y_n)_{n\geq 0}$, a *morphism of spectra* $f\colon X \to Y$ is a sequence of pointed maps $(f_n\colon X_n \to Y_n)_{n\geq 0}$ together with homotopies H_n making all the following diagrams commute up to homotopy:

$$\begin{tikzcd} X_n \arrow[r, "\simeq"] \arrow[d, "f_n"'] & \Omega X_{n+1} \arrow[d, "\Omega f_{n+1}"] \arrow[dl, Rightarrow, "\sim"', "H_n"] \\ Y_n \arrow[r, "\simeq"'] & \Omega Y_{n+1}. \end{tikzcd}$$

Example 1.3.2 (Eilenberg-MacLane spectra)**.** Let A be an abelian group. We define the *Eilenberg-MacLane spectrum* HA of A as the spectrum given in degree n by the Eilenberg-MacLane space

$$(HA)_n := K(A,n),$$

characterized up to homotopy equivalence by the property that the only non-zero homotopy group of $K(A,n)$ is $\pi_n(K(A,n)) \cong A$. Since the loop space operation shifts homotopy groups, cf. Remark 1.1.11, we obtain equivalences $\Omega K(A,n+1) \simeq K(A,n)$ turning this into a spectrum.

As we will now discuss, the notion of a spectrum is intimately tied to that of a (generalized) cohomology theory. Recall that a *graded abelian group* is a collection $(A_n)_{n\in\mathbb{Z}}$ of abelian groups, indexed by the integers. A *morphism of graded abelian groups* from $(A_n)_{n\in\mathbb{Z}}$ to $(B_n)_{n\in\mathbb{Z}}$ is a collection of maps $f_n\colon A_n \to B_n$. We let $\mathrm{Ab}^{\mathbb{Z}}$ denote the resulting category of graded abelian groups.

Definition 1.3.3. A *(generalized) cohomology theory* is a pair (E^*, ∂) consisting of:

- A functor $E^*\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$;
- A natural isomorphism
$$\partial\colon E^*(-) \xrightarrow{\cong} E^{*+1}(\Sigma(-))$$
of functors $\mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$, called the *suspension isomorphism*.

This pair is required to satisfy the following two conditions:

- (Wedge axiom) For every small collection of pointed spaces $\{X_i\}_{i\in I}$, the inclusions $\iota_i\colon X_i \hookrightarrow \bigvee_{i\in I} X_i$ induce an isomorphism
$$(\iota_i^*)_{i\in I}\colon E^*\left(\bigvee_{i\in I} X_i\right) \xrightarrow{\cong} \prod_{i\in I} E^*(X_i).$$
In particular, for $I = \emptyset$ we have $E^*(*) \cong 0$.
- (Exactness) For any cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of pointed spaces, the sequence
$$E^*(Z) \xrightarrow{g^*} E^*(Y) \xrightarrow{f^*} E^*(X)$$
is exact.

Example 1.3.4. For an abelian group A, we have a functor $H^*(-;A)\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{N}}$ given by the *singular cohomology with coefficients in A*. By defining $H^n(X;A) = 0$ for $n < 0$, this defines a generalized cohomology theory.

Later in the course we will see other cohomology theories, like complex K-theory and cobordism.

Remark 1.3.5. Given a cofiber sequence $X \to Y \to Z$, the cofiber of $Y \to Z$ is equivalent to $\Sigma(X)$, and the cofiber of $Z \to \Sigma(X)$ is $\Sigma(Y)$. Proceeding this way, we get a chain of cofiber sequences known as the *Puppe sequence*:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma(X) \xrightarrow{\Sigma(f)} \Sigma(Y) \xrightarrow{\Sigma(g)} \Sigma(Z) \to \Sigma^2(X) \to \dots .$$

Using that $E^n(\Sigma X) \cong E^{n-1}(X)$, we thus obtain a long exact sequence

$$\dots \xrightarrow{g^*} E^{n-1}(Y) \xrightarrow{f^*} E^{n-1}(X) \to E^n(Z) \xrightarrow{g^*} E^n(Y) \xrightarrow{f^*} E^n(X) \to E^{n+1}(Z) \xrightarrow{g^*} E^{n+1}(Y) \xrightarrow{f^*} \dots .$$

Remark 1.3.6. Every cohomology theory E^* in the above sense defines a cohomology theory on CW-pairs (X, A) by setting

$$E^*(X, A) := E^*(X/A),$$

where we regard the quotient space X/A as a pointed space with basepoint A/A. We then recover the long exact sequence of a pair by applying the previous remark to the cofiber sequence

$$A_+ \to X_+ \to X/A.$$

Every spectrum gives rise to a cohomology theory on pointed spaces:

Construction 1.3.7. Let $E = (E_n)_{n\geq 0}$ be a spectrum. We define a functor $E^*\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$ as follows:

$$E^k(X) := \begin{cases} [X, E_k]_* & k \geq 0 \\ [\Sigma^{-k}X, E_0]_* & k < 0. \end{cases}$$

Given the homotopy equivalences $E_k \simeq \Omega E_{k+1} \simeq \Omega^2 E_{k+2}$, these sets of homotopy classes admit canonical stuctures of abelian groups via concatenation of loops. We further define the suspension isomorphism ∂ for $k \geq 0$ as

$$\delta\colon E^k(X) = [X, E_k]_* \cong [X, \Omega E_{k+1}]_* \cong [\Sigma X, E_{k+1}]_* = E^{k+1}(\Sigma X),$$

where all these identifications are natural in X. For $k < 0$ we take $\delta\colon E^k(X) \xrightarrow{\cong} E^{k+1}(\Sigma X)$ to be the identity map of $[\Sigma^{-k}X, E_0]_* = [\Sigma^{-(k+1)}\Sigma X, E_0]_*$.

Proposition 1.3.8. *Let E be a spectrum. Then the pair (E^*, ∂) constructed above defines a cohomology theory.*

Proof. First note that for two homotopic maps $f \sim g\colon X \to Y$, the induced maps $f^*, g^*\colon E^k(Y) \to E^k(X)$ are equal, so that the functor $E^k\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}$ is well-defined. Given pointed spaces $\{X_i\}_{i\in I}$, pointed maps from $\bigvee_{i\in I} X_i$ to E_k correspond to collections of pointed maps $X_i \to E_k$, and since this process preserves pointed homotopy classes we obtain a bijection

$$E^k(\bigvee_{i\in I} X_i) = [\bigvee_{i\in I} X_i, E_k]_* \cong \prod_{i\in I} [X_i, E_k]_* = \prod_{i\in I} E^k(X_i);$$

the argument for $k < 0$ is analogous, using that Σ^{-k} preserves wedge sums. Finally, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence of pointed spaces, we claim that for any space E the sequence

$$[Z, E_k]_* \xrightarrow{g^*} [Y, E_k]_* \xrightarrow{f^*} [X, E_k]_*$$

is exact: the image of g^* agrees with the preimage $(f^*)^{-1}(\mathrm{const}_*)$. Since $g \circ f$ is nullhomotopic, it is clear that $\mathrm{im}(g^*) \subseteq (f^*)^{-1}(\mathrm{const}_*)$. Conversely, let $h\colon Y \to E$ be a pointed map such that the composite $h \circ f\colon X \to E$ is nullhomotopic. Choosing a nullhomotopy then defines an extension of h to the mapping cone $C(f) \to E$, showing that h is in the image of the restriction map $[C(f), E]_* \to [Y, E]_*$. But by assumption we have $Z \simeq C(f)$ and the claim follows. □

The main goal of this section is to prove a converse of Proposition 1.3.8: using the Brown representability theorem, stated below, we will show that every cohomology comes from some spectrum E. Let us start by introducing some terminology.

Definition 1.3.9. Consider a functor $F\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Set}$.

(1) We say that F is *representable* if it is of the form[2] $[-, Z]_*\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Set}$ for some pointed space Z. (By the Yoneda lemma the object Z is then uniquely determined up to isomorphism as an object of $\mathrm{h}\mathcal{S}_*$.)

(2) We say that F satisfies the *Wedge Axiom* if for every small collection of pointed spaces $\{X_i\}_{i \in I}$ the map

$$F\left(\bigvee_{i \in I} X_i\right) \xrightarrow{\cong} \prod_{i \in I} F(X_i)$$

induced by the inclusions $X_i \hookrightarrow \bigvee_{i \in I} X_i$ is a bijection (equivalently: the functor F preserves arbitrary products);

(3) We say that F satisfies *Mayer-Vietoris* if for every homotopy pushout square

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "l"'] & A \arrow[d, "i"] \\ B \arrow[r, "j"'] & X \end{tikzcd}$$

of connected pointed spaces, the induced map

$$(i^*, j^*)\colon F(X) \to F(A) \times_{F(C)} F(B)$$

is surjective.

[2]By 'of the form' we mean 'naturally isomorphic to a functor of the form'.

Remark 1.3.10. If F satisfies the Wedge-Axiom, then $F(\Sigma X)$ is automatically a group for every X, and $F(\Sigma^2 X)$ is even an abelian group. To see this, first observe that ΣX is a *cogroup* in $\mathrm{h}\mathcal{S}_*$: there are pointed maps

$$\Sigma X \to \mathrm{pt}, \qquad \mathrm{pinch}\colon \Sigma X \to \Sigma X \vee \Sigma X$$

where the first map is the unique map to the point and where the second map is the *pinch map* defined by identifying $\Sigma X \vee \Sigma X$ with the quotient of $X \times [0,1]$ at the subspace $X \times \{0, \frac{1}{2}, 1\} \cup \{x_0\} \times [0,1]$. We may more abstractly describe this, using the definition of ΣX as a homotopy pushout, as the map

$$\Sigma X = \mathrm{pt} \sqcup^h_X \mathrm{pt} \simeq \mathrm{pt} \sqcup^h_X X \sqcup^h_X \mathrm{pt} \to \mathrm{pt} \sqcup^h_X \mathrm{pt} \sqcup^h_X \mathrm{pt} \simeq \Sigma X \vee \Sigma X$$

induced by the unique map $X \to \mathrm{pt}$. This map satisfies counitality and coassociativity, and it has a coinverse map $i\colon \Sigma X \to \Sigma X$ given by 'inverting the homotopy'. We leave the details as an exercise to the reader: see Exercise 7.7 and Remark 7.8.

Applying the contravariant functor F then turns the cogroup object ΣX into a group $F(\Sigma X)$: the unit, multiplication and inversion are given by the maps

$$\begin{aligned} \mathrm{pt} = F(\mathrm{pt}) &\to F(\Sigma X), \\ F(\Sigma X) \times F(\Sigma X) \simeq F(\Sigma X \vee \Sigma X) &\xrightarrow{F(\mathrm{pinch})} F(\Sigma X), \\ F(\Sigma X) &\xrightarrow{F(i)} F(\Sigma X). \end{aligned}$$

The space $\Sigma^2 X$ is a cogroup in two a priori distinct ways, but it follows from the famous Eckmann-Hilton argument that they agree in $\mathrm{h}\mathcal{S}_*$ and are in fact (co)commutative. It follows that $F(\Sigma^2 X)$ is even an abelian group for every space X.

With similar reasoning one observes that for any natural transformation $F \to F'$ the induced map $F(\Sigma X) \to F'(\Sigma X)$ is a group morphism for every pointed space X.

Remark 1.3.11. If F is given as a functor $F\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}$ into the category of *abelian groups*, then the Mayer-Vietoris property is equivalent to saying that the sequence

$$F(X) \xrightarrow{(i^*, j^*)} F(A) \times F(B) \xrightarrow{k^* - l^*} F(C)$$

is exact. If F also satisfies the Wedge Axiom, then one may (exercise!) use the cofiber sequence $A \vee B \to X \to \Sigma C$ to extend this to a long exact sequence

$$\cdots \to F(\Sigma A) \times F(\Sigma B) \xrightarrow{(\Sigma k)^* - (\Sigma l)^*} F(\Sigma C) \xrightarrow{\partial} F(X) \xrightarrow{(i^*, j^*)} F(A) \times F(B) \xrightarrow{k^* - l^*} F(C).$$

The following is the main theorem of this section:

Theorem 1.3.12 (Brown representability theorem)**.** *Let* $\mathrm{h}\mathcal{S}_*^{\geq 0}$ *denote the homotopy category of connected pointed spaces. Then a functor* $F\colon (\mathrm{h}\mathcal{S}_*^{\geq 0})^{\mathrm{op}} \to \mathrm{Set}$ *is representable if and only if it satisfies the Wedge Axiom and Mayer-Vietoris.*

Remark 1.3.13. Counterexamples to this theorem exist if one either drops the pointedness or the connectedness assumption.

We will prove this below, after some preparations. First note that for every connected pointed space Z and every element $\xi \in F(Z)$ we obtain a natural transformation

$$T_\xi\colon [X,Z]_* \to F(X), \qquad (f\colon X \to Z) \mapsto f^*\xi,$$

where we write $f^* := F(f)\colon F(Z) \to F(X)$. (In fact, by the Yoneda lemma every natural transformation $[-,Z]_* \to F(-)$ is of this form.) We say that an element $\xi \in F(Z)$ is *n-universal* for $n \in \mathbb{N}$ if the map

$$T_\xi\colon \pi_k(Z) = [S^k, Z]_* \to F(S^k)$$

is a bijection for $1 \leq k < n$ and a surjection for $k = n$. We will say that ξ is *universal* if ξ is n-universal for all $n \geq 0$. The core ingredient of the proof of Brown representability is the following statement:

Proposition 1.3.14. *Let* $F\colon (\mathrm{h}\mathcal{S}_*^{\geq 0})^{\mathrm{op}} \to \mathrm{Set}$ *be a functor satisfying the Wedge Axiom and Mayer-Vietoris. For every pointed space* X *and every element* $\eta \in F(X)$*, there exists a pointed space* Z_X*, a pointed map* $f_X\colon X \to Z_X$ *and a universal element* $\xi_X \in F(Z_X)$ *satisfying* $f_X^*(\xi_X) = \eta$*.*

Proof. Step 1: We start by producing a sequence

$$X =: Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} \dots$$

of pointed spaces and pointed maps, together with n-universal elements $\xi_n \in F(Z_n)$ satisfying $f_n^*(\xi_{n+1}) = \xi_n \in F(Z_n)$, where $\xi_0 = \eta \in F(X)$. We proceed by induction on n. For $n = 1$, we define

$$Z_1 := X \vee \bigvee_{m\geq 1} \bigvee_{\gamma \in F(S^m)} S^m.$$

By the Wedge Axiom there is a bijection

$$F(Z_1) \cong F(X) \times \prod_{m \geq 1} \prod_{\gamma \in F(S^m)} F(S^m),$$

and we let $\xi_1 \in F(Z_1)$ be the element whose first component is $\eta \in F(X)$ and whose component at $\gamma \in F(S^m)$ is $\gamma \in F(S^m)$. Note that for every $k \geq 1$ the map

$$T_{\xi_1}\colon [S^k, Z_1]_* \to F(S^k)$$

is surjective, and in particular ξ_1 is 0-universal.

Now for $n \geq 1$ assume that we have constructed Z_n together with an n-universal element $\xi_n \in F(Z_n)$; our task is to define Z_{n+1} and ξ_{n+1}. Recall from Remark 1.3.10 that $F(S^n)$ is a group, which is abelian for $n \geq 2$. Consider the set

$$K_n := \{[f] \in [S^n, Z_n] \mid f^*\xi_n = 0 \in F(S^n)\}.$$

We may then define the pointed space Z_{n+1} by attaching an $(n+1)$-cell to Z_n along the map $f\colon S^n \to Z_n$ for every element of $[f] \in K_n$, i.e. we consider the following homotopy pushout:

$$\begin{tikzcd}
\bigvee_{[f]\in K_n} S^n \arrow[r] \arrow[d] & Z_n \arrow[d, "f_n"] \\
* \simeq \bigvee_{[f]\in K_n} D^{n+1} \arrow[r] & Z_{n+1}.
\end{tikzcd}$$

By assumption we have $f^*(\xi) = 0 \in F(S^n)$ for all $[f] \in K_n$, and so by Mayer-Vietoris there exists an element $\xi_{n+1} \in F(Z_{n+1})$ satisfying $f_n^*\xi_{n+1} = \xi_n \in F(Z_n)$. We claim that ξ_{n+1} is $(n+1)$-universal, i.e. the map

$$[S^k, Z_{n+1}]_* \to F(S^k), \qquad f \mapsto f^*(\xi_{n+1})$$

is a bijection for all $1 \leq k \leq n$ and a surjection for $k = n+1$. We already know it is a surjection for all k. By the cellular approximation theorem we further see that the map $f_n\colon Z_n \to Z_{n+1}$ induces bijections $[S^k, Z_n]_* \xrightarrow{\sim} [S^k, Z_{n+1}]_*$ for $0 \leq k < n$, and since ξ_n is n-universal we deduce that also ξ_{n+1} is n-universal. It remains to see that the map

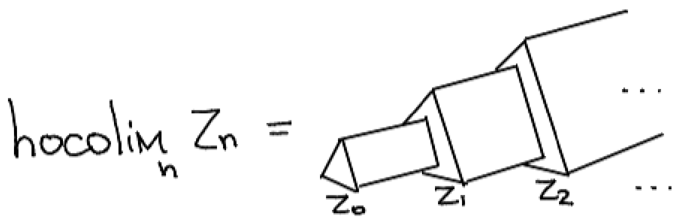
$$[S^n, Z_{n+1}]_* \to F(S^n), \qquad g \mapsto g^*\xi_{n+1}$$

is injective. By Remark 1.3.10 this is a group homomorphism, and so it suffices to show it has trivial kernel. Let $[g] \in [S^n, Z_{n+1}]_*$ be an element satisfying $g^*\xi_{n+1} = 0$. Again by the cellular approximation theorem we may assume that g factors as a map $S^n \xrightarrow{h} Z_n \xrightarrow{f_n} Z_{n+1}$. But then we have $h^*\xi_n = g^*\xi_{n+1} = 0$ hence $[h] \in K_n$. But then by construction the composite $g = f_n \circ h\colon S^n \to Z_n \to Z_{n+1}$ is null-homotopic, i.e. we have $[g] = 0 \in [S^n, Z_{n+1}]_*$. We conclude that ξ_{n+1} is $(n+1)$-universal. This finishes the induction, and completes the proof of Step 1.

Step 2: We will now define Z_X as the homotopy colimit of the sequence $\{Z_n\}$. Recall that the homotopy colimit is given by the *mapping telescope*

$$Z_X := \operatorname{hocolim}_n Z_n := \left(\bigvee_{n\in\mathbb{N}} Z_n \wedge [n, n+1]_+ \right) /\sim,$$

where the equivalence relation $\sim$ is given by $(x_n, n+1) \sim (f(x_n), n+1)$. Note that a map out of the mapping telescope into some other space T corresponds to a family of pointed maps $t_n\colon Z_n \to T$ together with pointed homotopies between $t_n\colon Z_n \to T$ and $t_{n+1} \circ f_n\colon Z_n \to T$.



One can show (exercise!) that the homotopy colimit may also be given by the following homotopy pushout:

$$\begin{tikzcd}
\bigvee_{n\geq 0} Z_n \arrow[r] \arrow[d] & \bigvee_{k\geq 0} Z_{2k} \arrow[d] \\
\bigvee_{k\geq 0} Z_{2k+1} \arrow[r] & Z = \operatorname{hocolim}_n Z_n.
\end{tikzcd}$$

Here the horizontal map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k}$ for $n = 2k$ even, and the map $f_{2k+1}\colon Z_{2k+1}Z_{2k+2}$ for $n = 2k+1$ odd. The vertical map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k+1}$ for $n = 2k+1$ odd, an the map $f_{2k}\colon Z_{2k} \to Z_{2k+1}$ for $n = 2k$ even. By the Mayer-Vietoris property, we then obtain an element $\xi \in F(Z)$ that restricts to $\xi_n \in F(Z_n)$ for all $n \geq 0$, hence in particular to $\eta \in F(X)$. We claim that ξ is universal. To see this, consider for $k \geq 0$ the following diagram:

$$\begin{tikzcd}
{[S^k, Z_0]_*} \arrow[r] \arrow[drrr, bend right, "T_{\xi_0}"'] & \dots \arrow[r] & {[S^k, Z_k]} \arrow[r] \arrow[dr, dashed, "T_{\xi_k}"] & {[S^k, Z_{k+1}]} \arrow[r, "\cong"] \arrow[d, dashed, "T_{\xi_{k+1}}"] & {[S^k, Z_{k+2}]} \arrow[r, "\cong"] \arrow[dl, dashed, "T_{\xi_{k+2}}"'] & \dots \arrow[r, "\cong"] & {[S^k, Z]} \arrow[dlll, bend left, "T_{\xi}"] \\
& & & F(S^k) & & &
\end{tikzcd}$$

By assumption, all the maps T_{ξ_n} are bijections for $n > k$. Also, since Z_{n+1} is obtained from Z_n by attaching $(n+1)$-cells, the top horizontal maps $[S^k, Z_n] \to [S^k, Z_{n+1}]$ are bijections for $n > k$. It follows that the map T_ξ is a bijection for every k, as desired. □

We are now ready to prove the Brown representability theorem:

Proof of Theorem 1.3.12. If $F \cong [-, Z]_*$ is representative, then the Wedge Axiom and Mayer-Vietoris are satisfied by the universal properties of the wedge and the homotopy pushout, just like in the proof of Proposition 1.3.8.

Conversely, assume that F satiesfies the Wedge Axiom and Mayer-Vietoris. To find the required pointed space Z, we take $X = \mathrm{pt}$, and we let $\eta := 0 \in F(\mathrm{pt})$ be the unique element. Applying Proposition 1.3.14 we obtain a pointed space $Z := Z_{\mathrm{pt}}$ and a universal element $\xi := \xi_{\mathrm{pt}}$. We have to show that for every pointed space X the induced map

$$T_\xi\colon [X, Z]_* \to F(X), \qquad (f\colon X \to Z) \mapsto f^*\xi$$

is a bijection.

Surjectivity: Pick an element $\eta \in F(X)$, and consider the space $\widetilde{X} := X \vee Z$. We let $\widetilde{\eta} \in F(\widetilde{X})$ be the element corresponding to the pair (η, ξ) under the bijection $F(\widetilde{X}) \cong F(X) \times F(Z)$ from the Wedge Axiom. Applying Proposition 1.3.14 to $(\widetilde{X}, \widetilde{\eta})$, we thus find a pointed space $\widetilde{Z}$, a pointed map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Z}$ and a universal element $\widetilde{\xi} \in F(\widetilde{Z})$ such that $\widetilde{f}^*(\widetilde{\xi}) = \widetilde{\eta}$. In particular, if we let $g\colon Z \to \widetilde{Z}$ denote the composite of $\widetilde{f}$ with the inclusion $Z \hookrightarrow X \vee Z = \widetilde{X}$ we see that $g^*(\widetilde{\xi}) = \xi$, and in particular the following diagram commutes for all $k \geq 0$:

$$\begin{tikzcd}
\pi_k(Z) = [S^k, Z]_* \arrow[rr, "g_*"] \arrow[dr, "T_\xi"', "\cong"] & & {[S^k, \widetilde{Z}]_* = \pi_k(\widetilde{Z})} \arrow[dl, "T_{\widetilde{\xi}}", "\cong"'] \\
& F(S^k). &
\end{tikzcd}$$

Since ξ and $\widetilde{\xi}$ are both universal, the bottom two maps are bijections. The top map is then also a bijection and it follows that the map $g\colon Z \to \widetilde{Z}$ is a weak homotopy equivalence. Since Z and $\widetilde{Z}$ are homotopy equivalent to CW-complexes this means that g is even a homotopy equivalence. It then follows from the commutative diagram

$$\begin{tikzcd}
{[X, Z]_*} \arrow[rr, "g_*", "\cong"'] \arrow[dr, "T_\xi"'] & & {[X, \widetilde{Z}]_*} \arrow[dl, "T_{\widetilde{\xi}}"] \\
& F(X) &
\end{tikzcd}$$

that the composite $X \to \widetilde{Z} \xrightarrow{\sim} Z$ is an element in $[X, Z]_*$ which is mapped to η by T_ξ.

Injectivity: Suppose that $f, g\colon X \to Z$ are two pointed maps satisfying $\eta := f^*(\xi) = g^*(\xi)$. Define the pointed space $\widetilde{X}$ as the following homotopy pushout:

$$\begin{tikzcd}
X \vee X \arrow[r, "{(f,g)}"] \arrow[d, "{(\mathrm{id}_X, \mathrm{id}_X)}"'] & Z \arrow[d] \\
X \arrow[r] & \widetilde{X} := X \sqcup^h_{X \vee X} Z.
\end{tikzcd}$$

By the Mayer-Vietoris property of F, there exists an element $\widetilde{\eta} \in F(\widetilde{X})$ which restricts to $\eta \in F(X)$ along the bottom map and to $\xi \in F(Z)$ along the right vertical map. Applying the claim to $(\widetilde{X}, \widetilde{\eta})$ we find a pointed space $\widetilde{Z}$ with a pointed map $\widetilde{X} \to \widetilde{Z}$ and a universal element $\widetilde{\xi} \in F(\widetilde{Z})$ which restricts to $\widetilde{\eta}$. Arguing as before, we see that the composite $Z \to \widetilde{Y} \to \widetilde{Z}$ is a homotopy equivalence. But this implies that there is a map $\widetilde{f}$ fitting in a homotopy commutative diagram as follows:

$$\begin{tikzcd}
X \vee X \arrow[r, "{(f,g)}"] \arrow[d, "{(\mathrm{id}_X, \mathrm{id}_X)}"'] & Z \\
X. \arrow[ur, dashed, "\widetilde{f}"'] &
\end{tikzcd}$$

We conclude that f and g are both homotopic to $\widetilde{f}$ and in particular that $[f]=[g]\in[X,Z]_*$, as desired. This finishes the proof of the theorem. $\square$

Having established Brown representability, we may now deduce that every cohomology theory is represented by a spectrum:

Proposition 1.3.15. *Let $h^*\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}}\to \mathrm{Ab}^{\mathbb{Z}}$ be a cohomology theory. Then there exists a spectrum E such that h^* is isomorphic to the cohomology theory E^* associated to E as in Construction 1.3.7.*

Proof. Step 1: We start by showing that for $n\geq 0$ the functor $h^n\colon (\mathrm{h}\mathcal{S}_*^{\geq 0})^{\mathrm{op}}\to \mathrm{Ab}$ satisfies the hypotheses of the Brown representability theorem. The Wedge Axiom holds by assumption. For the Mayer-Vietoris property, consider a homotopy pushout square

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "l"'] & A \arrow[d, "i"] \\ B \arrow[r, "j"'] & X \end{tikzcd}$$

By the pasting law of homotopy pushout squares (Exercise 2.2), the induced map on cofibers $C(k)\to C(j)$ is a homotopy equivalence. We thus obtain a diagram of long exact sequences

$$\begin{tikzcd} h^n(C(j)) \arrow[r, "q^*"] \arrow[d, "\cong"'] & h^n(X) \arrow[r, "j^*"] \arrow[d, "i^*"] & h^n(B) \arrow[r, "\partial"] \arrow[d, "l^*"] & h^{n+1}(C(j)) \arrow[d, "\cong"] \\ h^n(C(k)) \arrow[r, "p^*"] & h^n(A) \arrow[r, "k^*"] & h^n(C) \arrow[r, "\partial"] & h^{n+1}(C(k)). \end{tikzcd}$$

Consider elements $\alpha\in h^n(A)$ and $\beta\in h^n(B)$ such that $k^*(\alpha)=l^*(\beta)\in h^n(C)$. We must show that there exists an element $\gamma\in h^n(X)$ satisfying $\alpha=i^*(\gamma)$ and $\beta=j^*(\gamma)$. Because the bottom sequence is exact we get $\partial l^*\beta=\partial k^*\alpha=0$, and as the first vertical map is a bijection it follows that $\partial\beta=0$. We may thus find an element $\gamma_0\in h^n(Y')$ such that $\beta=j^*\gamma_0$. While γ_0 might not be mapped to α under i^*, we see that

$$k^*(\alpha-i^*\gamma_0)=k^*\alpha-k^*i^*\gamma_0=l^*\beta-l^*j^*\gamma_0=0,$$

and thus we see that $\alpha-i^*\gamma_0$ lies in the image of $p^*\colon h^n(C(f))\to h^n(Y)$. Since the left vertical map is a bijection, we may thus pick some $\gamma_1\in h^n(C(f))$ such that $\alpha-i^*\gamma_0=i^*q^*\gamma_1$. But then the element $\gamma:=\gamma_0+q^*\gamma_1$ satisfies

$$i^*(\gamma)=i^*\gamma_0+i^*q^*\gamma_1=\alpha \qquad \text{and} \qquad j^*(\gamma)=k^*\gamma_0+k^*q^*\gamma_1=\beta+0=\beta.$$

This finishes the proof that h^n satisfies Mayer-Vietoris.

Step 2: By Brown representability, we obtain pointed connected spaces Z_n together with natural isomorphisms

$$h^n(-) \cong [-, Z_n]_* \colon (\mathrm{h}\mathcal{S}_*^{\geq 0})^{\mathrm{op}} \to \mathrm{Set},$$

for all $n \geq 0$. Since ΣX is connected for all X, we then also obtain natural isomorphisms

$$h^n(X) \cong h^{n+1}(\Sigma X) \cong [\Sigma X, Z_{n+1}]_* \cong [X, \Omega Z_{n+1}] = [X, E_n]_*,$$

where we set $E_n := \Omega Z_{n+1}$. Since representing objects are unique up to isomorphism in $\mathrm{h}\mathcal{S}_*$, we then obtain pointed homotopy equivalences $E_n \simeq \Omega E_{n+1}$ which make the following diagram commute:

$$\begin{tikzcd} h^n(X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & h^{n+1}(\Sigma X) \arrow[d, "\cong"] \\ {[X, E_n]_*} \arrow[r, dashed, "\cong"'] & {[X, \Omega E_{n+1}]_*}. \end{tikzcd}$$

We conclude that the sequence $(E_n)_{n\in\mathbb{N}}$ defines a spectrum and that the cohomology theory h^* is naturally isomorphic to E^*. □

In a similar way we can show that every morphism of cohomology theories is represented by a morphism of spectra:

Lemma 1.3.16. *Let E and F be spectra and let $f^* \colon E^* \to F^*$ be a morphism of cohomology theories. Then there exists a morphism of spectra $f \colon E \to F$ that induces f^*.*

Proof. The map f^n has the form of a natural transformation $[-, E_n]_* \to [-, F_n]_*$, for $n \geq 0$. By the Yoneda lemma, it is given by postcomposition with a certain pointed map $f_n \colon E_n \to F_n$. In light of the commutative diagram

$$\begin{tikzcd} {[-, E_n]_*} \arrow[r, "\cong"] \arrow[d, "f_n \circ -"'] & {[-, \Omega E_{n+1}]_*} \arrow[d, "\Omega f_{n+1} \circ -"] \\ {[-, F_n]_*} \arrow[r, "\cong"] & {[-, \Omega F_{n+1}]_*} \end{tikzcd}$$

it follows that the diagram

$$\begin{tikzcd} E_n \arrow[r, "\sim"] \arrow[d, "f_n"'] & \Omega E_{n+1} \arrow[d, "\Omega f_{n+1}"] \\ F_n \arrow[r, "\sim"] & \Omega F_{n+1} \end{tikzcd}$$

commutes in $\mathrm{h}\mathcal{S}_*$. In particular we may pick a homotopy filling this diagram, turning the maps (f_n) into a morphism of spectra. □

Warning 1.3.17. There may be different maps of spectra which induce the same map at the level of cohomology theories! These are closely related to the famous *phantom maps*.

As announced in the introduction to this course, we would like to introduce various constructions of spectra that closely mimic their classical analogous for abelian groups, like direct sums and tensor products. In order to do this properly, we should work in the ∞-category Sp of spectra. We shall introduce this ∞-category in Chapter 3, after having introduced the general theory of ∞-categories in the next chapter.

2 Introduction to ∞-categories

The goal of this chapter is to get the reader up to speed with the theory of ∞-categories. The approach we will take is quite different from most other foundational sources on ∞-categories: instead of *defining* ∞-categories (e.g. in terms of simplicial sets), we will take ∞-categories as primitive objects whose behavior we have to *axiomatize* (just like how in ordinary mathematics one cannot *define* what a 'set' is).

2.1 The fundamental principle of homotopy theory

Before introducing ∞-categories, let me say a word about why I will not define what an ∞-category is in terms of more basic structures, like we would do for most other definitions in mathematics. The reason is that an ∞-category only becomes a natural concept to consider when one already has the mindset of a homotopy theorist, which requires a fundamental shift in perspective on the notion of *equality*:

> **Fundamental Principle of Homotopy Theory:** When expressing that two entities are equal, we must always specify *how* they are equal by providing a homotopy/isomorphism between them.

In other words, equality in homotopy theory is not a *property* (a binary yes/no question) but is additional *structure* that needs to be provided. This sits in stark contrast with the perspective of ordinary mathematics, in which equality is merely a property; e.g. two elements of a set are either equal or they are not equal.

Let us discuss various instances of this principle with which you are already familiar:

Example 2.1.1 (Algebraic topology)**.** Two continuous maps $f,g\colon X \to Y$ are considered 'homotopically the same' if we can provide a homotopy $H\colon X \times [0,1] \to Y$ between f and g. This leads to the following replacements of basic notions in algebraic topology:

- The notion of homeomorphism gets replaced by *homotopy equivalence*: instead of asking for an isomorphism of topological spaces, we look for continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ equipped with homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.

- Commutative diagrams get replaced by *homotopy coherent diagrams*. For instance, instead of requiring a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"] & W \end{tikzcd}$$

  to commute strictly, we provide a homotopy $H\colon h \circ f \sim k \circ g$. When dealing with larger diagrams, we need to specify homotopies between homotopies, leading to higher coherence data.

- The concept of uniqueness is replaced by *contractibility*: instead of asking for a unique solution, we require the space of all possible solutions to be contractible. For example, there is not a unique way to define the n-fold multiplication map $(\Omega X)^n \to \Omega X$ on the loop space of a pointed space, but the space of parameterizations of such maps is contractible.[1]

- Quotients are enhanced to *homotopy quotients*: for a group G acting on a space X, the strict quotient X/G identifies points in the same orbit, effectively collapsing each orbit to a single point. In contrast, the homotopy quotient $X /\!\!/ G$ adds paths between points in the same orbit, connecting x to gx for each group element g. In particular, points can become 'equal to themselves' in non-trivial ways.

Example 2.1.2 (Homological algebra)**.** The phenomenon of 'equality as structure' also appears prominently in homological algebra:

- A map $f\colon C_\bullet \to D_\bullet$ of chain complexes is called a *quasi-isomorphism* if it induces isomorphisms on all homology groups. Two chain complexes are considered 'the same' if they can be connected by a zigzag of quasi-isomorphisms.

- The *derived category* $D(A)$ of an abelian category A is formed by formally inverting all quasi-isomorphisms: we force quasi-isomorphic complexes to become isomorphic.

- A functor $F\colon A \to B$ between abelian categories may not preserve quasi-isomorphisms, and thus might not induce a well-defined functor between derived categories. The *derived functor* of F is essentially a way of modifying F to obtain a functor that does preserve this notion of sameness.

[1] More precisely, for each of the $n!$ possible orderings there is a contractible space of parametrizations for composing n loops in that given order.

- There is a more computational approach to these ideas using *chain homotopies* between two chain maps $f, g\colon C_\bullet \to D_\bullet$, leading to a notion of *chain homotopy equivalence*. For *projective* chain complexes, the notions of quasi-isomorphism and chain homotopy equivalence coincide.

- In nice abelian categories, every chain complex $C_\bullet$ admits a *projective resolution*: a quasi-isomorphism $P_\bullet \to C_\bullet$ where $P_\bullet$ is a complex of projective objects. This resolution is *unique up to chain homotopy equivalence*. For a right exact functor F, its left derived functor can be computed by evaluating F on projective resolutions; the uniqueness up to chain homotopy equivalence ensures this is well-defined.

Example 2.1.3 (Category theory)**.** Two objects X and Y in a category C are considered 'the same' when equipped with an isomorphism $f\colon X \xrightarrow{\sim} Y$. We need to keep track of these isomorphisms in our definitions:

- Two functors $F, G\colon C \to D$ are regarded 'the same' if we specify a natural isomorphism $\eta\colon F \overset{\cong}{\Longrightarrow} G$ between them: the chosen isomorphisms $\eta_X\colon F(X) \xrightarrow{\sim} G(X)$ are required to be compatible with the functors F and G.

- In the theory of monoidal categories, associativity and unitality are not strict equalities but rather natural isomorphisms: for objects X, Y, and Z, we have an associator isomorphism $\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ and unitor isomorphisms $\lambda_X\colon 1 \otimes X \xrightarrow{\sim} X$ and $\rho_X\colon X \otimes 1 \xrightarrow{\sim} X$. These must satisfy coherence conditions like the Mac Lane pentagon.

- Universal properties are typically defined up to unique isomorphism. For instance, a product of objects X and Y is unique only up to unique isomorphism.

We will get more used to the Fundamental Principle throughout the course, and from my point of view it is the most important aspect of ∞-category theory that one needs to learn. Although it is possible to develop ∞-category theory entirely within the classical framework of set theory (for example in terms of *quasicategories*, see Section 2.5 below) I think doing this is actually more confusing than helpful: it inherently goes against the Fundamental Principle. To understand what an ∞-category 'really' is, it is not very relevant how precisely ∞-categories can be implemented within ordinary mathematics, it is much more important to become fluent in the 'language of ∞-category theory'. For this reason, I have chosen to take a more axiomatic approach to ∞-category theory which aims to directly introduce such a language, building on recent joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde [Cis+24].

2.2 The language of ∞-category theory

As explained in the previous section, the theory of ∞-categories requires a language that embraces the fundamental principle of homotopy theory. In this section we will introduce such a language.

Remark 2.2.1. The language we introduce should be regarded as some kind of 'meta theory'. Eventually, after having set up enough theory, we will be able to formulate the existence of an *∞-category of ∞-categories*, denoted Cat_∞, and everything we say in the meta theory will hold internally to Cat_∞. (Usually the axioms imposed in the meta theory are only crude approximations of the 'true' universal properties satisfied inside Cat_∞, but they suffice to develop the theory.)

2.2.1 Categories, functors and natural isomorphisms

We start with the basic objects of this language:

Axiom A.1. (0) There exists mathematical structures called *∞-categories*, denoted C, D, E, etcetera.

(1) Given ∞-categories C and D, we may speak of *functors* $f\colon C \to D$ between them. Every ∞-category C has an *identity functor* $\mathrm{id}_C\colon C \to C$. Given two functors $f\colon C \to D$ and $g\colon D \to E$, there is a *composite functor* $g \circ f\colon C \to E$.

(2) Given two functors $f, f'\colon C \to D$, we may speak of *natural isomorphisms* $\alpha\colon f \cong f'$. We may sometimes display natural isomorphisms diagrammatically as follows:

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\alpha\Downarrow\cong"] & D. \end{tikzcd}$$

Every functor f has an *identity isomorphism* $\mathrm{id}_f\colon f \cong f$. Every natural isomorphism α has an *inverse isomorphism* $\alpha^{-1}\colon f' \cong f$. Given two natural isomorphisms $\alpha\colon f \cong f'$ and $\beta\colon f' \cong f''$, there is a *(vertical) composite isomorphism* $\beta \circ \alpha\colon f \cong f''$.

To avoid cluttering of notation, we will frequently work with natural isomorphisms $f \cong f'$ that have not been given an explicit name.

(3) Given two natural isomorphisms $\alpha, \alpha'\colon f \cong f'$, we may speak of *isomorphisms of natural isomorphisms* $\alpha \cong \alpha'$, also called 3-*isomorphisms* for short. We may again speak of identity 3-isomorphisms $\alpha \cong \alpha$, of composite 3-isomorphisms $\alpha \cong \alpha''$ and of inverse 3-isomorphisms $\alpha' \cong \alpha$.

(4+) Proceeding in this way, we demand for every natural n a notion of an *n-isomorphism* between two $(n-1)$-isomorphisms, and we may speak of identity n-isomorphisms, of inverse n-isomorphisms, and of composite n-isomorphisms.

We should think of a natural isomorphism $f \cong f'$ between two functors as expressing that f and f' are 'equal'. Similarly two n-isomorphisms are regarded as 'equal' if we are given an $(n+1)$-isomorphism between them.

Axiom A.2. The composition of functors is unital and associative up to isomorphism: for functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$, we have natural isomorphisms[2]

$$\lambda_f\colon \mathrm{id}_D \circ f \cong f, \qquad \rho_f\colon f \circ \mathrm{id}_C \cong f \qquad \text{and} \qquad \alpha_{f,g,h}\colon (h \circ g) \circ f \cong h \circ (g \circ f).$$

The composition of (higher) isomorphisms is similarly unital and associative up to isomorphism, and for any natural isomorphism $\alpha\colon f \cong f'$ there are preferred isomorphisms $\alpha^{-1} \circ \alpha \cong \mathrm{id}_f$ and $\alpha \circ \alpha^{-1} \cong \mathrm{id}_{f'}$.

Since we think of isomorpisms as 'equalities', we better make sure that all constructions we do preserve isomorphisms.

Axiom A.3. Composition of functors respects natural isomorphisms: given a natural isomorphism $\alpha\colon f \cong f'$ of functors $C \to D$ and a natural isomorphism $\beta\colon g \cong g'$ of functors $D \to E$, we obtain a new natural isomorphism $\beta * \alpha\colon g \circ f \cong g' \circ f'$ of functors $C \to E$, called the *horizontal composite*.

To be fully precise one needs to write down analogous rules for horizontal composition of n-isomorphisms for all n, and one needs to express their compatibility with vertical composition. We will not go into the details here and instead refer to [Cis+24, Section 1.1.2].

Definition 2.2.2. A *commutative square of functors*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd}$$

consists of four functors f, g, h and f' together with a specified natural isomorphism $\alpha\colon h \circ f \cong f' \circ g$. A *commutative triangle of functors*

$$\begin{tikzcd} & D \arrow[dr, "g"] & \\ C \arrow[ur, "f"] \arrow[rr, "h"'] & & E \end{tikzcd}$$

[2]We have given them names to make clear that the axiom provides preferred choices for such isomorphisms, but in practice we often don't write out the names for these isomorphisms each time.

consists of three functors f, g and h together with a specified natural isomorphism $\alpha\colon h \cong g \circ f$. We may similarly define a *commutative square/triangle of n-isomorphisms*.

Definition 2.2.3. A functor $f\colon C \to D$ is said to be an *equivalence* if there exists another functor $g\colon D \to C$ and natural isomorphisms $\mathrm{id}_C \cong gf$ and $\mathrm{id}_D \cong fg$. In this case we call g an *inverse* to f.

Exercise 2.2.4. Show that a functor f is an equivalence if and only if it admits both a section (i.e., a g such that $\mathrm{id}_D \cong fg$) and a retraction (i.e. an h such that $\mathrm{id}_C \cong hf$), and that in this case we have $g \cong h$.

Notation 2.2.5. It follows immediately from the previous exercise that if g and h are both inverses to f, then there is an isomorphism $g \cong h$. So up to isomorphism there is a unique inverse to f, which we will denote by $f^{-1}\colon D \to C$.

Exercise 2.2.6. Show that if $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.

Lemma 2.2.7. *Equivalences satisfy the 2-out-of-3 property: given functors $f\colon C \to D$ and $g\colon D \to E$, if two of the three functors f, g and gf are equivalences, then so is the third.*

Proof. If g and f are equivalences, then so is gf by Exercise 2.2.6. If f and gf are equivalences, then we may rewrite g as the composite $g \cong g \circ \mathrm{id}_D \cong g \circ f \circ f^{-1} = gf \circ f^{-1}$, so that g is the composite of two equivalences and hence itself an equivalence. A similar argument shows that f is an equivalence whenever g and gf are equivalences. □

Exercise 2.2.8. Show that equivalences of ∞-categories satisfy the 2-out-of-6 property: given functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.

2.2.2 Objects, morphisms and commutative triangles

Constructions from ordinary category may be imported to the setting of ∞-categories:[3]

Axiom C. Every ordinary category is an ∞-category. Every ordinary functor $f\colon C \to D$ is a functor of ∞-categories, and every functor arises in this way. Every ordinary natural isomorphism $\alpha\colon f \xrightarrow{\cong} f'$ is a natural isomorphism of functors of ∞-categories, and every natural isomorphism arises in this way. This process preserves composition of functors and natural isomorphisms.

[3]In a fully axiomatic approach, as we do in [Cis+24], one would not introduce such an axiom. In this course I wish to take a hybrid approach, since developing the full axiomatic theory would take too long.

The assumption that ordinary categories are ∞-categories provides a convenient way to define objects, morphisms and commutative triangles in an ∞-category C.

Notation 2.2.9. For $n \geq 0$, we write $[n]$ for the following partially ordered set:

$$[n] := \{0 \leq 1 \leq \cdots \leq n\}.$$

For $n > 1$ and $0 \leq i \leq n$, we define the *face map* $d_i\colon [n-1] \to [n]$ as the unique injective map of posets whose image leaves out i. Similarly we define the *degeneracy map* $s_i\colon [n+1] \to [n]$ as the unique surjective map of posets that hits i twice.

Recall that every partially ordered set $(P, \leq)$ can be turned into a category: given $p, q \in P$ we add a unique morphism $p \to q$ between them whenever $p \leq q$. The resulting categories $[0]$, $[1]$ and $[2]$ may be displayed as follows:

$$0, \qquad 0 \longrightarrow 1 \qquad \text{and} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2. \end{tikzcd}$$

The five face maps $d_0, d_1\colon [0] \to [1]$ and $d_0, d_1, d_2\colon [1] \to [2]$ and the relations between them may be displayd as follows:

$$\begin{tikzcd}
 & & 0 \arrow[dl, dashed, "d_0"'] \arrow[dr, dashed, "d_1"] & & \\
 & {\begin{smallmatrix} & 1 \\ \nearrow & \\ 0 & \end{smallmatrix}} \arrow[dr, dashed, "d_2"] & \cong & {\begin{smallmatrix} 0 & \\ & \searrow \\ & 1 \end{smallmatrix}} \arrow[dl, dashed, "d_0"'] & \\
 & \cong & {\begin{smallmatrix} & 1 & \\ \nearrow & & \searrow \\ 0 & \longrightarrow & 2 \end{smallmatrix}} & \cong & \\
 0 \arrow[uur, dashed, "d_1"] \arrow[rr, dashed, "d_1"] & & 0 \longrightarrow 1 \arrow[u, dashed, "d_1"'] & & 0. \arrow[uul, dashed, "d_0"'] \arrow[ll, dashed, "d_0"']
\end{tikzcd}$$

The two degeneracy maps $s_0, s_1\colon [2] \to [1]$ may be interpreted the following 'projections':

$$\begin{tikzcd}
0 \arrow[r] \arrow[dr] & 1 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] \arrow[dl] \\
 & 2 \arrow[rr, dashed, phantom] & {} \arrow[r, dashed, "s_0"] & 1 & {} \arrow[l, dashed, "s_1"'] \; 2 & 1. \arrow[l]
\end{tikzcd}$$

Definition 2.2.10. Let C be an ∞-category.

(1) An *object of* C is a functor $x\colon [0] \to C$;

(2) A *morphism of* C is a functor $f\colon [1] \to C$. We write $f(0)$ for its *source/domain* and $f(1)$ for its *target/codomain*, which are defined as the following composites:

$$f(0)\colon [0] \xrightarrow{d_1} [1] \xrightarrow{f} C \qquad \text{and} \qquad f(1)\colon [0] \xrightarrow{d_0} [1] \xrightarrow{f} C.$$

We will also write $f\colon x \to y$ to indicate that f comes with isomorphisms $x \cong f(0)$ and $y \cong f(1)$.

(3) A *commutative triangle in* C is a functor $\sigma\colon [2] \to C$. We will frequently denote commutative triangles by

$$\sigma \quad = \quad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z, \end{tikzcd}$$

where $f = \sigma \circ d_2$, $g = \sigma \circ d_0$ and $h = \sigma \circ d_1$. We will often leave commutative triangles unnamed, and only label their three edges.

Warning 2.2.11. While we may speak of *objects of* C, we may not speak of the *set of objects of* C! This is similar to how we may speak of points of a space X, but we may not speak of the *set of points*, cf. Convention 1.2.3.

Remark 2.2.12. We want to think of a commutative triangle σ as saying that h is a composite of f and g. Note that at this stage we do not yet say anything about the existence nor the uniqueness of composites; we will get back to this in Section 2.4.

Example 2.2.13. Given an object x of C, we define the *identity morphism* $\mathrm{id}_x\colon x \to x$ as the composite $[1] \to [0] \xrightarrow{x} C$. Since the composites

$$[0] \xrightarrow{d_1} [1] \xrightarrow{s_0} [0] \qquad \text{and} \qquad [0] \xrightarrow{d_0} [1] \xrightarrow{s_0} [0]$$

naturally isomorphic to the identity functor $\mathrm{id}_{[0]}$, we see that the source and target of id_x are indeed isomorphic to x.

Exercise 2.2.14. Construct for every morphism $f\colon x \to y$ commutative triangles in C of the form

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y. \end{tikzcd}$$

After having composition of morphisms available, the exercise will imply the expected relations $f \cong f \circ \mathrm{id}_x$ and $f \cong \mathrm{id}_y \circ f$, see Lemma 2.4.7.

2.3 Basic constructions of ∞-categories

There are various ways to build new ∞-categories out of old ones: we may form products, coproducts, pullbacks and functor categories.

2.3.1 Products and coproducts

Axiom B.1 (Terminal ∞-category)**.** The category $[0]$ is the *terminal ∞-category*: For every ∞-category C there is a functor $p_C\colon C \to [0]$, and for any two functors $f, f'\colon C \to [0]$ there is a natural isomorphism $f \cong f'$.

Notation 2.3.1. We will frequently also write $*$ for $[0]$.

Definition 2.3.2. Let C and D be ∞-categories and let x be an object of D. We define the *constant functor* $\mathrm{const}_x\colon C \to D$ as the composite $x \circ p_C\colon C \to D$:

$$\mathrm{const}_x : C \xrightarrow{p_C} [0] \xrightarrow{x} D.$$

Definition 2.3.3 (Contractible category)**.** An ∞-category C is called *contractible* if the functor $p_C\colon C \to *$ is an equivalence.

Remark 2.3.4. Observe that an ∞-category C is contractible if and only if there exists an absolute object $x\colon * \to C$ such that the composite $\mathrm{const}_x\colon C \to C$ is naturally isomorphic to the identity $\mathrm{id}_C\colon C \to C$. A natural isomorphism $H\colon\ \mathrm{const}_x \cong \mathrm{id}_C$ is called a *contraction of C onto x*.

Axiom B.2 (Initial ∞-category)**.** The empty category $\emptyset$ is the *initial ∞-category*: For every ∞-category C there is a functor $\emptyset \to C$, and for any two functors $f, f'\colon \emptyset \to C$ there is a natural isomorphism $f \cong f'$.

Axiom B.3 (Product of ∞-categories)**.** The *product* of two ∞-categories C and D is an ∞-category $C \times D$ equipped with two functors $\mathrm{pr}_C\colon C \times D \to C$ and $\mathrm{pr}_D\colon C \times D \to D$ called the *projection functors*. Given functors $f\colon T \to C$ and $g\colon T \to D$, we obtain a functor $(f,g)\colon T \to C \times D$ equipped with natural isomorphisms $\mathrm{pr}_C \circ (f,g) \cong f$ and $\mathrm{pr}_D \circ (f,g) \cong g$. Given two functors $h, k\colon T \to C \times D$ and two natural isomorphisms $\alpha\colon\ \mathrm{pr}_C \circ h \cong \mathrm{pr}_C \circ k$ and $\beta\colon\ \mathrm{pr}_D \circ h \cong \mathrm{pr}_D \circ k$, we obtain a natural isomorphism $(\alpha,\beta)\colon\ h \cong k$ satisfying $\mathrm{pr}_C \circ (\alpha,\beta) \cong \alpha$ and $\mathrm{pr}_D \circ (\alpha,\beta) \cong \beta$.

Remark 2.3.5. Taking $T = *$ we see that objects of $C \times D$ have the form (x,y), where x is an object of C and y is an object of D.

Axiom B.4 (Coproduct of categories)**.** The *coproduct* (or *disjoint union*) of two ∞-categories C and D is an ∞-category $C \sqcup D$ equipped with two functors $i_C : C \to C \sqcup D$ and $i_D : D \to C \sqcup D$ called the *inclusion functors*. Given functors $f : C \to E$ and $g : D \to E$, we obtain a functor $\langle f, g \rangle : C \sqcup D \to E$ equipped with natural isomorphisms $\langle f, g \rangle \circ i_C \cong f$ and $\langle f, g \rangle \circ i_D \cong g$. Given two functors $h, k : C \sqcup D \to E$ and two natural isomorphisms $\alpha : h \circ i_C \cong k \circ i_C$ and $\beta : h \circ i_D \cong k \circ i_D$, we obtain a natural isomorphism $\langle \alpha, \beta \rangle : h \cong k$ satisfying $(\alpha, \beta) \circ i_C \cong \alpha$ and $(\alpha, \beta) \circ i_D \cong \beta$.

From these simple rules, we may deduce that products and coproducts are commutative, unital and associative. For commutativity of products, one argues as follows:

Lemma 2.3.6. *For ∞-categories C and D, the functor $(\mathrm{pr}_D, \mathrm{pr}_C) : C \times D \to D \times C$ is an equivalence, with inverse given by $(\mathrm{pr}_C, \mathrm{pr}_D) : D \times C \to C \times D$.*

Proof. We need to provide natural isomorphisms $(\mathrm{pr}_C, \mathrm{pr}_D) \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{id}_{C \times D}$ and $(\mathrm{pr}_D, \mathrm{pr}_C) \circ (\mathrm{pr}_C, \mathrm{pr}_D) \cong \mathrm{id}_{D \times C}$. By symmetry, it will suffice to produce the first natural isomorphism. For this, it will in turn suffice to produce natural isomorphisms after composing with the functors pr_C and pr_D. For the composition with pr_C we compute that

$$\mathrm{pr}_C \circ ((\mathrm{pr}_C, \mathrm{pr}_D) \circ (\mathrm{pr}_D, \mathrm{pr}_C)) \cong (\mathrm{pr}_C \circ (\mathrm{pr}_C, \mathrm{pr}_D)) \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C \circ (\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C,$$

which is indeed isomorphic to $\mathrm{pr}_C \circ \mathrm{id}_{C \times D}$. The case for composition with pr_D is analogous. This finishes the proof. □

Unitality may be argued similarly:

Lemma 2.3.7. *For every ∞-category C, the projection functor $\mathrm{pr}_C : C \times * \to C$ is an equivalence, with inverse given by the functor $(\mathrm{id}_C, p_C) : C \to C \times *$.*

Proof. The composite $\mathrm{pr}_C \circ (\mathrm{id}_C, p_C) : C \to C$ is isomorphic to id_C by the defining property of (id_C, p_C). To show that the composite $(\mathrm{id}_C, p_C) \circ \mathrm{pr}_C : C \times * \to C \times *$ is isomorphic to $\mathrm{id}_{C \times *}$, it will suffice to do so after composing with pr_C and with pr_*. The case for pr_* is clear, since any two functors $C \times * \to *$ are isomorphic. In the case of pr_C we compute

$$\mathrm{pr}_C \circ ((\mathrm{id}_C, p_C) \circ \mathrm{pr}_C) \cong (\mathrm{pr}_C \circ (\mathrm{id}_C, p_C)) \circ \mathrm{pr}_C \cong \mathrm{id}_C \circ \mathrm{pr}_C \cong \mathrm{pr}_C \cong \mathrm{pr}_C \circ \mathrm{id}_{C \times *},$$

where the first isomorphism is associativity, the second is the relation proved in the first paragraph, and the third and fourth are unitality. This finishes the proof. □

Exercise 2.3.8. Formulate and prove the associativity of the product of ∞-categories.

Exercise 2.3.9. Show that the coproduct of ∞-categories is associative, commutative and unital:

$$(C \sqcup D) \sqcup E \xrightarrow{\sim} C \sqcup (D \sqcup E), \qquad C \sqcup D \xrightarrow{\sim} D \sqcup C, \qquad \emptyset \sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C \sqcup \emptyset.$$

Construction 2.3.10. The formation of products and coproducts of ∞-categories is functorial: given two functors $f\colon C \to C'$ and $g\colon D \to D'$, we obtain new functors

$$f \times g := (f \circ \mathrm{pr}_C, g \circ \mathrm{pr}_D)\colon C \times D \to C' \times D'$$
$$f \sqcup g := \langle i_{C'} \circ f, i_{D'} \circ g\rangle\colon C \sqcup D \to C' \sqcup D'.$$

We leave it to the reader to verify that there are natural isomorphisms

$$\begin{aligned} \mathrm{id}_C \times \mathrm{id}_D &\cong \mathrm{id}_{C\times D} \quad \text{and} \quad (f' \times g') \circ (f \times g) \cong (f' \circ f) \times (g' \circ g), \\ \mathrm{id}_C \sqcup \mathrm{id}_D &\cong \mathrm{id}_{C\sqcup D} \quad \text{and} \quad (f' \sqcup g') \circ (f \sqcup g) \cong (f' \circ f) \sqcup (g' \circ g). \end{aligned}$$

for functors $f\colon C \to C'$, $f'\colon C' \to C''$, $g\colon D \to D'$ and $g'\colon D' \to D''$.

2.3.2 Pullbacks of ∞-categories

We will now axiomatize the *pullback* of two functors $f\colon C \to E$ and $g\colon D \to E$ of ∞-categories. This will be similar to the axiom for the product $C \times D$, but it is more involved due to the fact that we need to record compatibilities with the structure maps to E.

Axiom B.5 (Pullbacks of ∞-categories)**.** Consider two functors $f\colon C \to E$ and $g\colon D \to E$. The *pullback* of f and g, also called the *fiber product of C and D over E*, is an ∞-category $C \times_E D$. It comes equipped with two functors $\mathrm{pr}_C\colon C \times_E D \to C$ and $\mathrm{pr}_D\colon C \times_E D \to D$ and a natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$, or in other words, a commutative square

$$\begin{tikzcd} C \times_E D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_D"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E. \end{tikzcd}$$

Given functors $t_C\colon T \to C$ and $t_D\colon T \to D$ equipped with a natural isomorphism $\alpha\colon f \circ t_C \cong g \circ t_D$, we obtain a functor $t = (t_C, t_D)\colon T \to C \times_E D$. This functor comes equipped with natural isomorphisms $\mathrm{pr}_C \circ t \cong t_C$ and $\mathrm{pr}_D \circ t \cong t_D$. Furthermore, the composite isomorphism

$$f \circ t_C \cong f \circ \mathrm{pr}_C \circ t \cong g \circ \mathrm{pr}_D \circ t \cong g \circ t_D$$

is isomorphic to α. We summarize this with the following diagram:

$$\begin{tikzcd} T \arrow[dr, dashed, "{(t_C,t_D)}"] \arrow[drr, bend left, "t_C"] \arrow[ddr, bend right, "t_D"'] & & \\ & C \times_E D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_D"'] & C \arrow[d, "f"] \\ & D \arrow[r, "g"] & E. \end{tikzcd}$$

Consider now two functors $t, t' \colon T \to C \times_E D$, and assume we are given two natural isomorphisms $\alpha \colon \operatorname{pr}_C \circ t \cong \operatorname{pr}_C \circ t'$ and $\beta \colon \operatorname{pr}_D \circ t \cong \operatorname{pr}_D \circ t'$ such that the following square of natural isomorphisms commutes:

$$\begin{tikzcd} f \circ \operatorname{pr}_C \circ t \arrow[r, "f \circ \alpha", "\cong"'] \arrow[d, "\cong"'] & f \circ \operatorname{pr}_C \circ t' \arrow[d, "\cong"] \\ g \circ \operatorname{pr}_D \circ t \arrow[r, "g \circ \beta", "\cong"'] & g \circ \operatorname{pr}_D \circ t'. \end{tikzcd}$$

Then we obtain a natural isomorphism $(\alpha, \beta) \colon t \cong t'$ satisfying $\operatorname{pr}_C \circ (\alpha, \beta) \cong \alpha$ and $\operatorname{pr}_D \circ (\alpha, \beta) \cong \beta$, and inducing an isomorphic isomorphism of natural isomorphisms in the previous square.

Remark 2.3.11. We refer to [Cis+24, Axiom B.5] for a precise formulation of the last condition.

Remark 2.3.12. While Axiom B.5 may look somewhat complicated, let us point out that it has the same form as that of Axioms B.1 and B.3:

- An ∞-category X is introduced, together with certain additional structure;
- For any other ∞-category T equipped with the same structure, we are given a functor $T \to X$ that is compatible with this structure;
- If we have *two* functors $T \to X$ that are compatible with this structure, then we are given a natural isomorphism between them that is compatible with this structure.

The Axioms B.2 and B.4 have a similar form, except that they talk about functors *out of* X.

Exercise 2.3.13. Show that the fiber product is commutative, associative and unital: for functors $C \to \Gamma$, $D \to \Gamma$ and $B \to \Gamma$, there are preferred equivalences

$$C \times_\Gamma D \xrightarrow{\sim} D \times_\Gamma C, \qquad B \times_\Gamma (C \times_\Gamma D) \xrightarrow{\sim} (B \times_\Gamma C) \times_\Gamma D, \qquad C \times_\Gamma \Gamma \xrightarrow{\sim} C.$$

Construction 2.3.14. The construction of pullbacks of ∞-categories is functorial: given a commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varphi"'] & \Gamma \arrow[d, "\chi"] & D \arrow[l, "g"'] \arrow[d, "\psi"] \\ C' \arrow[r, "f'"'] & \Gamma' & D' \arrow[l, "g'"] \end{tikzcd}$$

there is an induced functor

$$\varphi \times_\chi \psi := (\varphi \circ \operatorname{pr}_C, \psi \circ \operatorname{pr}_D) \colon C \times_\Gamma D \to C' \times_{\Gamma'} D'.$$

Exercise 2.3.15. In the above situation, show that if each of the functors φ, ψ and χ is an equivalence, then so is $\varphi \times_\chi \psi$. (This is essentially identical to Exercise 1.2 on Sheet 1.)

Corollary 2.3.16. *Equivalences are closed under pullback: if $g\colon D \to E$ is an equivalence and $f\colon C \to E$ is an arbitrary functor, then also $C \times_E D \to C$ is an equivalence.*

Proof. This is a special case of Exercise 2.3.15 by taking $C' = C$, $E' = D' = E$, $\varphi = \mathrm{id}_C$, $\chi = \mathrm{id}_E$ and $\psi = g$. □

Pullbacks of categories lead to the notion of a *fiber* of a functor:

Definition 2.3.17. Let $f\colon C \to D$ be a functor. For every object x of D, we define the *fiber* of f over x as the pullback

$$\begin{tikzcd} C_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ * \arrow[r, "x"] & D. \end{tikzcd}$$

Pullback squares

The definition of pullbacks of ∞-categories naturally leads to the notion of a pullback square.

Definition 2.3.18 (Pullback square)**.** A commutative square

$$\begin{tikzcd} T \arrow[r, "t"] \arrow[d, "s"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"'] & E \end{tikzcd}$$

is called a *pullback square* if the induced functor $(s,t)\colon T \to C \times_E D$ is an equivalence.

Lemma 2.3.19 (Pasting lemma for pullback squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\ D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3. \end{tikzcd}$$

(1) There is a preferred[4] equivalence $D_1 \times_{D_3} C_3 \xrightarrow{\sim} D_1 \times_{D_2} (D_2 \times_{D_3} C_3)$.

(2) If the right-hand square is a pullback square, then the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

[4]Whenever we use the word ‘preferred’, this is meant to indicate that the real content of the statement is not just that some equivalence/isomorphism exists, but that there is a specific choice of such an equivalence we are interested in, to be constructed in the proof.

Proof. For part (1), we define the functor as $(\mathrm{pr}_{D_1}, (h_1 \circ \mathrm{pr}_{D_1}, \mathrm{pr}_{C_3}))$. We claim that an inverse is given by the functor

$$(\mathrm{pr}_{D_1}, \mathrm{pr}_{C_3} \circ \mathrm{pr}_{D_2 \times_{D_3} C_3})\colon D_1 \times_{D_2} (D_2 \times_{D_3} C_3) \to D_1 \times_{D_3} C_3.$$

To show that the two composites are isomorphic to the respective identities, it suffices to construct these isomorphisms after projecting to the two components and verifying that these isomorphisms agree with composed with the functors to D_3. But after unwinding the definitions, this holds tautologically true by the very construction of the two functors in both directions. We leave the details to the reader.

Part (2) is an immediate consequence of the 2-out-of-3 property from Lemma 2.2.7 applied to the following commutative square:

$$\begin{tikzcd}
C_1 \arrow[r, "(f_1,g_1)"] \arrow[d, "(f_1,g_2g_1)"'] & D_1 \times_{D_2} C_2 \arrow[d, "\simeq"] \\
D_1 \times_{D_3} C_3 \arrow[r, "\simeq"] & D_1 \times_{D_2} (D_2 \times_{D_3} C_3),
\end{tikzcd}$$

where the right vertical map is an equivalence by applying Exercise 2.3.15 to the equivalence $C_2 \xrightarrow{\sim} D_2 \times_{D_3} C_3$. $\square$

Lemma 2.3.20. *Consider a commutative square*

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\
E \arrow[r, "k"] & F
\end{tikzcd}$$

and assume that h is an equivalence. Then g is an equivalence if and only if the square is a pullback square.

Proof. Since h is an equivalence, also the projection map $\mathrm{pr}_E\colon D \times_F E \to E$ is an equivalence by Corollary 2.3.16. Since the composite of the map $(f,g)\colon C \to D \times_F E$ with pr_E is isomorphic to g, the claim follows from the 2-out-of-3 property of equivalences, Lemma 2.2.7. $\square$

2.3.3 Universality of coproducts

So far the axioms for products and coproducts of ∞-categories are symmetric: all statements for products have dual analogues for coproducts. We will now introduce the *universality of coproducts* which breaks this symmetry:

Axiom B.2' (Universality of initial category)**.** The initial category is *strictly* initial: every functor $C \to \emptyset$ is an equivalence.

Axiom B.4' (Universality of coproducts)**.** (1) For functors $f\colon C' \to C$ and $g\colon D' \to D$, the commutative squares

$$\begin{tikzcd} C' \arrow[r, "i_{C'}"] \arrow[d, "f"'] & C' \sqcup D' \arrow[d, "f \sqcup g"] \\ C \arrow[r, "i_C"] & C \sqcup D \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} D' \arrow[r, "i_{D'}"] \arrow[d, "g"'] & C' \sqcup D' \arrow[d, "f \sqcup g"] \\ D \arrow[r, "i_D"] & C \sqcup D \end{tikzcd}$$

are pullback squares.

(2) For a functor $h\colon E \to C \sqcup D$, the functor

$$\langle \mathrm{pr}_E, \mathrm{pr}_E \rangle \colon (E \times_{C \sqcup D} C) \sqcup (E \times_{C \sqcup D} D) \to E$$

is an equivalence.

Remark 2.3.21. To make sense of these two axioms, we may think of them as the categorical versions of the following two facts about sets:

(1) If a set S admits a map $S \to \emptyset$ to the empty set, then it must itself be empty;

(2) If a set S admits a map $S \to T_0 \sqcup T_1$ to a disjoint union of two sets, then we may write S as the disjoint union of the preimages of T_0 and T_1.

Lemma 2.3.22. *For functors $f\colon C \to \Gamma$, $g\colon D \to \Gamma$ and $\varphi\colon \Gamma' \to \Gamma$, the commutative square*

$$\begin{tikzcd} (C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') \arrow[r] \arrow[d] & C \sqcup D \arrow[d, "\langle f, g \rangle"] \\ \Gamma' \arrow[r, "\varphi"] & \Gamma \end{tikzcd}$$

is a pullback square.

Proof. We need to show that the induced functor

$$(C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') \to (C \sqcup D) \times_\Gamma \Gamma'$$

is an equivalence. But this is a special case of Axiom B.4' applied to the functor $f = \mathrm{pr}_1\colon (C \sqcup D) \times_\Gamma \Gamma' \to C \sqcup D$: by the pasting law of pullback squares there are equivalences

$$C \times_{C \sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq C \times_\Gamma \Gamma' \qquad \text{and} \qquad D \times_{C \sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq D \times_\Gamma \Gamma'.$$

This finishes the proof. ◻

Corollary 2.3.23 (Distributivity)**.** *For ∞-categories C, D and E, the functor*

$$\langle E \times i_C, E \times i_D \rangle \colon E \times C \sqcup E \times D \to E \times (C \sqcup D)$$

is an equivalence.

Proof. This is an instance of the previous lemma by taking $\Gamma = *$, $\Gamma' = E$, $f = p_C\colon C \to *$, $g = p_D\colon D \to *$, and $\varphi = p_E\colon E \to *$. ◻

2.3.4 Functor categories

Given two ∞-categories C and D, we are interested in forming the *functor category* $\mathrm{Fun}(C,D)$ whose objects are precisely the functors from C to D.

Axiom B.6 (Functor category)**.** The *functor category* between two ∞-categories C and D is an ∞-category denoted by $\mathrm{Fun}(C,D)$. Given another ∞-category E, we obtain for every functor $f\colon E\times C\to D$ a functor $f_c\colon E\to \mathrm{Fun}(C,D)$ obtained from f by *currying*. Conversely, there is for every functor $g\colon E\to \mathrm{Fun}(C,D)$ a functor $g^u\colon E\times C\to D$ obtained by *uncurrying*. These operations come equipped with natural isomorphisms $f\cong (f_c)^u$ and $g\cong (g^u)_c$. In other words, currying and uncurrying determine a 'one-to-one correspondence' between functors $E\to \mathrm{Fun}(C,D)$ and functors $E\times C\to D$.

The operations of currying and uncurrying respect natural isomorphisms: given a natural isomorphism $\alpha\colon f\cong f'$ of functors $f\colon E\times C\to D$, we obtain a natural isomorphism $\alpha^c\colon f_c\cong f'_c$ between their curried functors. Conversely, an isomorphism $\beta\colon g\cong g'$ gives $\beta^u\colon g^u\cong (g')^u$. Again, we demand that[5] $(\alpha_c)^u\cong\alpha$ and $(\beta^u)_c\cong\beta$.

By considering the case of the terminal category $E=*$, we observe that objects $*\to \mathrm{Fun}(C,D)$ of the functor category correspond to functors $C\simeq C\times *\to D$ from C to D, justifying the notation.

Axiom B.7 (Functoriality uncurrying in E)**.** The operation of uncurrying is assumed to be functorial in E. More precisely, given functors $g\colon E\to \mathrm{Fun}(C,D)$ and $h\colon E'\to E$, then the uncurrying of the composite $g\circ h\colon E'\to \mathrm{Fun}(C,D)$ comes with a preferred isomorphism to the composite $E'\times C\xrightarrow{h\times \mathrm{id}_C} E\times C\xrightarrow{g^u} D$. Similarly, given an isomorphism $\beta\colon g\cong g'$, the uncurrying of the isomorphism $\beta\circ h\colon g\circ h\cong g'\circ h$ is isomorphic to $\beta^u\circ(h\times\mathrm{id}_C)\colon g^u\circ(h\times\mathrm{id}_C)\cong (g')^u\circ(h\times\mathrm{id}_C)$.

Exercise 2.3.24. Show that also the operation of currying is functorial in E: given functors $f\colon E\times C\to D$ and $h\colon E'\to E$, there is a preferred isomorphism between the currying of the functor $E'\times C\xrightarrow{h\times\mathrm{id}_C} E\times C\xrightarrow{f} D$ and the composite $E'\xrightarrow{h} E\xrightarrow{f_c}\mathrm{Fun}(C,D)$.

Construction 2.3.25 (Evaluation functor)**.** Given ∞-categories C and D, the *evaluation functor* $\mathrm{ev}\colon \mathrm{Fun}(C,D)\times C\to D$ is defined as $\mathrm{ev}:=(\mathrm{id}_{\mathrm{Fun}(C,D)})^u$, i.e. as the functor obtained by uncurrying the identity functor $\mathrm{id}_{\mathrm{Fun}(C,D)}\colon \mathrm{Fun}(C,D)\to\mathrm{Fun}(C,D)$. Given an object $x\in C$, we write $\mathrm{ev}_x\colon \mathrm{Fun}(C,D)\to D$ for the composite

$$\mathrm{ev}_x\colon \mathrm{Fun}(C,D)\simeq \mathrm{Fun}(C,D)\times *\xrightarrow{\mathrm{id}\times x}\mathrm{Fun}(C,D)\times C\xrightarrow{\mathrm{ev}} D,$$

and refer to it as the *evaluation functor at x*.

[5]These isomorphisms should really be interpreted as commutative squares of natural isomorphisms.

Definition 2.3.26 (Arrow category)**.** Let C be an ∞-category. We refer to the functor category $\mathrm{Fun}([1], C)$ as the *arrow category* of C. Observe that objects of $\mathrm{Fun}([1], C)$ are precisely morphisms ('arrows') in C.

Definition 2.3.27 (Hom anima)**.** Let C be any ∞-category. Given objects x and y of C, we define the *hom anima* $\mathrm{Hom}_C(x,y)$ via the following pullback square:

$$\begin{tikzcd} \mathrm{Hom}_C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ * \arrow[r, "(x,y)"] & C \times C. \end{tikzcd}$$

Observe that the objects of $\mathrm{Hom}_C(x,y)$ are triples (f,α,β), where $f\colon x' \to y'$ is a morphism in C and $\alpha\colon x \cong x'$ and $\beta\colon y \cong y'$ are isomorphisms in C. We will often abuse notation and pretend that $x = x'$ and $y = y'$.

The use of the word 'anima' will be explained below, see Exercise 2.6.3.

Definition 2.3.28 (Natural transformations)**.** If C and D are ∞-categories, we define a *natural transformation* of functors $C \to D$ to be a morphism in $\mathrm{Fun}(C,D)$. By (un)currying, this may equivalently be encoded as a functor $[1] \times C \to D$. We denote the hom anima in $\mathrm{Fun}(C,D)$ by $\mathrm{Nat}(f,g)$:

$$\mathrm{Nat}(f,g) := \mathrm{Hom}_{\mathrm{Fun}(C,D)}(f,g).$$

The functoriality of the product naturally provides functoriality for the functor categories:

Construction 2.3.29 (Functoriality of functor categories in C and D)**.** Given a functor $g\colon D \to D'$, we define the functor $g \circ -\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D')$ as the currying of the composite

$$\mathrm{Fun}(C,D) \times C \xrightarrow{\mathrm{ev}} D \xrightarrow{g} D'.$$

Similarly, given a functor $f\colon C' \to C$, we define the functor $- \circ f\colon \mathrm{Fun}(C',D) \to \mathrm{Fun}(C,D)$ as the currying of the composite

$$\mathrm{Fun}(C',D) \times C \xrightarrow{\mathrm{id} \times f} \mathrm{Fun}(C',D) \times C' \xrightarrow{\mathrm{ev}} D.$$

In a completely analogous way, every natural isomorphism $\beta\colon g \cong g'$ of functors $D \to D'$ induces a natural isomorphism $(\alpha \circ -)\colon (g \circ -) \cong (g' \circ -)$ of functors $\mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D')$, and similarly for the construction $- \circ f$.

Alternative notations for these functors that we will frequently use are g_* and f^*:

$$g_* := g \circ -\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D'), \qquad f^* := - \circ f\colon \mathrm{Fun}(C',D) \to \mathrm{Fun}(C,D).$$

Exercise 2.3.30. Formulate and prove that the assignments $g \mapsto (g \circ -)$ and $f \mapsto (- \circ f)$ are functorial, in the sense that they respect identity functors and composition of functors.

Proposition 2.3.31. *Let T, C, D and E be ∞-categories, and let $f\colon C \to E$ and $g\colon D \to E$ be functors.*

(1) The functor $p_{\mathrm{Fun}(T,*)}\colon \mathrm{Fun}(T,*) \to *$ is an equivalence.

(2) The functor

$$(\mathrm{pr}_C \circ -, \mathrm{pr}_D \circ -)\colon \mathrm{Fun}(T, C\times D) \to \mathrm{Fun}(T,C)\times \mathrm{Fun}(T,D)$$

is an equivalence.

(3) The functor

$$(\mathrm{pr}_C \circ -, \mathrm{pr}_D \circ -)\colon \mathrm{Fun}(T, C\times_E D) \to \mathrm{Fun}(T,C)\times_{\mathrm{Fun}(T,E)} \mathrm{Fun}(T,D)$$

induced by the isomorphism $(f\circ -)\circ(\mathrm{pr}_C \circ -) \cong (g\circ -)\circ(\mathrm{pr}_D\circ -)$ is an equivalence.

(4) The functor $\mathrm{Fun}(\emptyset, T) \to *$ is an equivalence.

(5) The functor

$$(-\circ i_C, -\circ i_D)\colon \mathrm{Fun}(C\sqcup D, T) \to \mathrm{Fun}(C,T)\times\mathrm{Fun}(D,T)$$

is an equivalence.

(6) The functor $\mathrm{ev}_*\colon \mathrm{Fun}(*,C) \to C$ is an equivalence.

(7) The functor

$$\mathrm{Fun}(T,\mathrm{Fun}(C,D)) \to \mathrm{Fun}(T\times C, D)$$

obtained by currying the composite

$$\mathrm{Fun}(C,\mathrm{Fun}(D,E))\times C\times D \xrightarrow{\mathrm{ev}\times\mathrm{id}_D} \mathrm{Fun}(D,E)\times D \xrightarrow{\mathrm{ev}} E.$$

is an equivalence.

Proof. Each of these claims is a direct (though somewhat tedious) consequence of the universal properties of all the constructions involved. Because of time reasons, we will not prove most of these claims and instead refer the reader to [Cis+24, Section 1.4]. But to give just a flavor of the proof, let us show how to prove (1).

We will show that an inverse to the functor $\mathrm{Fun}(T,*) \to *$ is given by the functor $(p_{*\times C})_c\colon * \to \mathrm{Fun}(C,*)$, obtained by uncurrying the functor $p_{*\times C}\colon *\times C \to *$. The composite $(p_{*\times C})_c \circ p_{\mathrm{Fun}(C,*)}\colon * \to *$ is automatically isomorphic to $\mathrm{id}_*\colon * \to *$, hence it remains to show that the composite $p_{\mathrm{Fun}(C,*)} \circ (p_{*\times C})_c\colon \mathrm{Fun}(C,*) \to \mathrm{Fun}(C,*)$ is isomorphic to $\mathrm{id}_{\mathrm{Fun}(C,*)}$. It will suffice to prove that the two uncurried functors $\mathrm{Fun}(C,*)\times C \to *$ are isomorphic. But this is again automatic by the universal property of $*$. $\square$

2.3.5 Pushouts of ∞-categories

Using functor categories, we may dualize the definition of a pullback square of ∞-categories:

Definition 2.3.32 (Pushout square)**.** A commutative square of ∞-categories

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "v"] & D \end{tikzcd}$$

is called a *pushout square* (or *cocartesian*) if, for every ∞-category E, the induced square

$$\begin{tikzcd} \mathrm{Fun}(D,E) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Fun}(D',C) \arrow[d, "(f')^*"] \\ \mathrm{Fun}(C,E) \arrow[r, "u^*"] & \mathrm{Fun}(C',E) \end{tikzcd}$$

is a pullback square.

Lemma 2.3.33 (Pasting lemma for pushout squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\ D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3. \end{tikzcd}$$

If the left-hand square is a pushout square, then the right-hand square is a pushout square if and only if the outer rectangle is a pushout square.

Proof. This follows immediately from Lemma 2.3.19. □

2.3.6 Summary

Here is a brief summary of the constructions of ∞-categories we have introduced so far:

Axiom B. We demand the existence of the following ∞-categories:

(B.1) A *terminal ∞-category* $*$: for every C there is a functor $p_C\colon C \to *$, and for any two functors $f,g\colon C \to *$ there is a natural isomorphism $f \cong g$.

(B.2) An *initial ∞-category* $\emptyset$: for every C there is a functor $\emptyset \to C$, and for any two functors $f,g\colon \emptyset \to C$ there is a natural isomorphism $f \cong g$.

(B.2') The ∞-category $\emptyset$ is strictly initial: every functor $C \to \emptyset$ is an equivalence.

(B.3) For every two ∞-categories C and D a *product* $C \times D$ with functors $\mathrm{pr}_C\colon C \times D \to C$ and $\mathrm{pr}_D\colon C \times D \to D$: functors $E \to C \times D$ are specified by giving the two components $E \to C$ and $E \to D$, and natural isomorphisms between functors $E \to C \times D$ are specified by giving natural isomorphisms between the components.

(B.4) Similarly there is a *coproduct* $C \sqcup D$ with functors $i_C\colon C \sqcup D$ and $i_D\colon D \to C \sqcup D$: functors $C \sqcup D \to E$ are specified by giving the two components $C \to E$ and $D \to E$, and natural isomorphisms between functors $C \sqcup D \to E$ are specified by giving natural isomorphisms between the components.

(B.4') Coproducts are universal: a functor $E \to C \sqcup D$ into a coproduct can essentially uniquely be written in the form $E_C \sqcup E_D \to C \sqcup D$ for two functors $E_C \to C$ and $E_D \to D$.

(B.5) For functors $f\colon C \to E$ and $g\colon D \to E$ there exists a *pullback* $C \times_E D$ with functors $\mathrm{pr}_C\colon C \times_E D \to C$ and $\mathrm{pr}_D\colon C \times_E D \to D$ and a natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$. Functors $T \to C \times_E D$ are specified by giving two components $t\colon T \to C$ and $s\colon T \to D$ together with a natural isomorphism $f \circ t \cong g \circ s$. One may similarly specify natural isomorphisms between two functors $T \to C \times_E D$.

(B.6) For ∞-categories C and D there is a *functor category* $\mathrm{Fun}(C,D)$. Every functor $f\colon E \times C \to D$ gives rise to a *curried functor* $f_c\colon E \to \mathrm{Fun}(C,D)$ and conversely every functor $g\colon E \to \mathrm{Fun}(C,D)$ gives rise to an *uncurried functor* $g^u\colon E \times C \to D$, satisfying $f \cong (f_c)^u$ and $g \cong (g^u)_c$. We may similarly curry/uncurry natural isomorphisms between functors. Currying is demanded to be functorial in E.

2.4 The commutative square, Segal and Rezk axioms

In Section 2.2.2 we introduced the notions of morphisms and commutative triangles in an ∞-category C. We will now impose three additional axioms that guarantee that these notions behave as expected, which may informally be stated as follows:

(D) *Commutative square axiom*: the data of a commutative square $[1] \times [1] \to C$ is the same as that of two commutative triangles $[2] \to C$ that agree along their diagonal;

(E) *Segal axiom*: for every pair of composable morphisms $f\colon x \to y$ and $g\colon y \to z$, there is an essentially unique commutative triangle $\sigma\colon [2] \to C$ of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & z. \end{tikzcd}$$

This leads to a *composition map* $-\circ-\colon \operatorname{Hom}_C(y,z)\times\operatorname{Hom}_C(x,y)\to\operatorname{Hom}_C(x,z)$.

(F) *Rezk axiom*: the data of an isomorphism $x\cong y$ between two objects x and y is the same as that of a morphism $f\colon x\to y$ that admits an inverse $g\colon y\to x$ satisfying $gf\cong \mathrm{id}_x$ and $fg\cong\mathrm{id}_y$.

In the next three subsections we will provide precise formulations of these three axioms.

2.4.1 The commutative square axiom

Throughout, C denotes an ∞-category.

Definition 2.4.1 (Commutative square)**.** We define a *commutative square in C* to be a functor $[1]\times[1]\to C$. We will often display commutative squares as follows:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w. \end{tikzcd}$$

Remark 2.4.2. Once we have the ∞-category of ∞-categories available, see Section 2.6.8 below, the meta-theoretical notion of a commutative square of functors from Definition 2.2.2 will be an instance of Definition 2.4.1 applied to $C=\mathrm{Cat}_\infty$.

Definition 2.4.3. We define two functors $j_0,j_1\colon [2]\to[1]\times[1]$ as

$$j_0 := (s_0,s_1) \qquad\text{and}\qquad j_1 := (s_1,s_0).$$

These two commutative triangles in $[1]\times[1]$ may be visualized as follows:

$$j_0 \quad = \quad \begin{tikzcd} (0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \\ & (1,1), \end{tikzcd} \qquad\qquad j_1 \quad = \quad \begin{tikzcd} (0,0) \arrow[d] \arrow[dr] & \\ (1,0) \arrow[r] & (1,1). \end{tikzcd}$$

The simplicial relations $s_0\circ d_1\cong\mathrm{id}_{[1]}\cong s_1\circ d_1$ provide an isomorphism making the following square commute:

$$\begin{tikzcd} {[1]} \arrow[r, "d_1"] \arrow[d, "d_1"'] & {[2]} \arrow[d, "j_0"] \\ {[2]} \arrow[r, "j_1"'] & {[1]\times[1]}. \end{tikzcd}$$

Axiom D (Commutative square axiom)**.** For every ∞-category C, the commutative square

$$\begin{tikzcd}
\mathrm{Fun}([1]\times[1],C) \arrow[r, "j_1^*"] \arrow[d, "j_0^*"'] & \mathrm{Fun}([2],C) \arrow[d, "d_1^*"] \\
\mathrm{Fun}([2],C) \arrow[r, "d_1^*"] & \mathrm{Fun}([1],C)
\end{tikzcd}$$

is a pullback square.

In other words, providing a commutative square in C is the same as providing two commutative triangles that agree on their diagonals:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\
z \arrow[r, "k"'] & w
\end{tikzcd}
\quad \leftrightarrow \quad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "l"] & y \arrow[d, "h"] \\
z \arrow[r, "k"'] & w.
\end{tikzcd}$$

Corollary 2.4.4. *For every ∞-category C the restriction functors*

$$j_0^*\colon \mathrm{Fun}([1]\times[1],C) \to \mathrm{Fun}([2],C) \qquad \textit{and} \qquad j_1^*\colon \mathrm{Fun}([1]\times[1],C) \to \mathrm{Fun}([2],C)$$

admit sections

$$p_0^*\colon \mathrm{Fun}([2],C) \to \mathrm{Fun}([1]\times[1],C) \qquad \textit{and} \qquad p_2^*\colon \mathrm{Fun}([2],C) \to \mathrm{Fun}([1]\times[1],C)$$

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "h"'] & y \arrow[d, "g"] \\
& z
\end{tikzcd}
\mapsto
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "h"] & y \arrow[d, "g"] \\
x \arrow[r, "h"'] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[d, "f"'] \arrow[dr, "h"] & \\
y \arrow[r, "g"'] & z
\end{tikzcd}
\mapsto
\begin{tikzcd}
x \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "h"] & z \arrow[d, "\mathrm{id}_z"] \\
y \arrow[r, "g"'] & z.
\end{tikzcd}$$

Proof. Under the equivalence $\mathrm{Fun}([1]\times[1],C) \xrightarrow{\sim} \mathrm{Fun}([2],C)\times_{\mathrm{Fun}([1],C)}\mathrm{Fun}([2],C)$, we take one of the components to be the identity of $\mathrm{Fun}([2],C)$ and we take the other one to be given by the degenerate triangle, i.e. given by precomposition with

$$[2] \xrightarrow{s_0} [1] \xrightarrow{d_1} [2] \qquad \text{or} \qquad [2] \xrightarrow{s_1} [1] \xrightarrow{d_1} [2]. \qquad \square$$

Remark 2.4.5. Applying the corollary to $C = [2]$, we in particular obtain functors

$$p_0 := p_0^*(\mathrm{id}_{[2]})\colon [1]\times[1] \to [2] \qquad \text{and} \qquad p_2 := p_2^*(\mathrm{id}_{[2]})\colon [1]\times[1] \to [2],$$

and in fact p_0^* and p_2^* are given by precomposition with p_0 and p_2, respectively. We may pictorially represent these two commutative squares in $[2]$ by the following diagrams:

$$p_0 \quad = \quad
\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] \arrow[dr] & 1 \arrow[d] \\
0 \arrow[r] & 2,
\end{tikzcd}
\qquad
p_2 \quad = \quad
\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d, equal] \\
1 \arrow[r] & 2.
\end{tikzcd}$$

2.4.2 The Segal axiom

Given a commutative triangle $\sigma\colon [2] \to C$ we may extract two morphisms $f := \sigma \circ d_2$ and $g := \sigma \circ d_0$:

$$\sigma \qquad = \qquad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr] & & z. \end{tikzcd}$$

By the simplicial identities we have $d_2 \circ d_0 = d_0 \circ d_1\colon [0] \to [2]$, so that the target of f does indeed agree with the source of g.

Axiom E (Segal axiom)**.** For every ∞-category C, restriction functor

$$(d_0^*, d_2^*)\colon \operatorname{Fun}([2], C) \to \operatorname{Fun}([1], C) \times_C \operatorname{Fun}([1], C), \qquad \sigma \mapsto (g, f)$$

is an equivalence of ∞-categories.

Given morphisms $f\colon x \to y$ and $g\colon y \to z$, the axiom provides an essentially unique commutative triangle $\sigma\colon [2] \to C$ satisfying $d_2^*(\sigma) \cong f$ and $d_0^*(\sigma) \cong g$. The resulting morphism $g \circ f := d_1^*(\sigma)\colon [1] \to C$ is called the *composite of f and g*. The assignment $(g, f) \mapsto g \circ f$ is functorial:

Construction 2.4.6 (Composition functor)**.** Consider the zig-zag

$$\operatorname{Fun}([1], C) \times_C \operatorname{Fun}([1], C) \xleftarrow[\sim]{(d_0^*, d_2^*)} \operatorname{Fun}([2], C) \xrightarrow{(d_1^*)^*} \operatorname{Fun}([1], C).$$

The first map is an equivalence by the Segal axiom, and hence it admits an inverse, providing a composite functor

$$- \circ -\colon \operatorname{Fun}([1], C) \times_C \operatorname{Fun}([1], C) \xrightarrow{\sim} \operatorname{Fun}([2], C) \xrightarrow{(d_1^*)^*} \operatorname{Fun}([1], C).$$

Passing to fibers over $(x, y, z) \in C^3$ and $(x, z) \in C^2$, respectively, then produces a functor

$$- \circ -\colon \operatorname{Hom}_C(y, z) \times \operatorname{Hom}_C(x, y) \to \operatorname{Hom}_C(x, z), \qquad (g, f) \mapsto g \circ f,$$

which we call the *composition functor*.

We will now show that composition in an ∞-category is unital and associative.

Lemma 2.4.7 (Unitality)**.** *For every morphism $f\colon x \to y$ in an ∞-category C, there are natural isomorphisms*

$$\mathrm{id}_y \circ f \cong f \cong f \circ \mathrm{id}_x$$

in $\operatorname{Fun}([1], C)$*. Moreover, these isomorphisms are natural in f, in the sense that they form natural isomorphisms of functors* $\operatorname{Fun}([1], C) \to \operatorname{Fun}([1], C)$.

Proof. For a fixed morphism f, the isomorphisms are an immediate consequence of Exercise 2.2.14, using the degenerate commutative triangles $s_1^*(f) := f \circ s_1$ and $s_0^*(f) := f \circ s_0$. For the naturality of the relation $f \circ \mathrm{id}_x \cong f$ we have to show that the composite

$$\mathrm{Fun}([1],C) \xrightarrow{\sim} \mathrm{Fun}([1],C) \times_C C \xrightarrow{1\times p_{[1]}^*} \mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1]) \xrightarrow{-\circ-} \mathrm{Fun}([1],C)$$

is isomorphic to the identity functor. This follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r, "s_1^*"] \arrow[d, "\simeq"'] \arrow[rr, bend left, equal] & \mathrm{Fun}([2],C) \arrow[r, "d_1^*"] \arrow[d, "\simeq"] & \mathrm{Fun}([1],C) \arrow[d, equal] \\
\mathrm{Fun}([1],C) \times_C C \arrow[r, "1\times p_{[1]}^*"'] & \mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1]) \arrow[r, "-\circ-"'] & \mathrm{Fun}([1],C).
\end{tikzcd}$$

A similar discussion applies to the relation $f \cong \mathrm{id}_y \circ f$. $\square$

Proposition 2.4.8 (Associativity)**.** *For composable morphisms $f\colon x \to y$, $g\colon y \to z$ and $h\colon z \to w$ in an ∞-category C, there is a natural isomorphism*

$$h \circ (g \circ f) \cong (h \circ g) \circ f$$

in $\mathrm{Fun}([1],C)$.

Proof. Consider the following two morphisms in $\mathrm{Fun}([1],C)$, regarded as objects of $\mathrm{Fun}([1],\mathrm{Fun}([1],C)) \simeq \mathrm{Fun}([1]\times[1],C)$:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "g\circ f"] & y \arrow[d, "g"] \\
x \arrow[r, "g\circ f"'] & z
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
y \arrow[r, "h\circ g"] \arrow[d, "g"'] \arrow[dr, "h\circ g"] & w \arrow[d, "\mathrm{id}_w"] \\
z \arrow[r, "h"'] & w.
\end{tikzcd}$$

These morphisms are natural in f, g and h: we have functors

$$\begin{array}{ccccc}
\mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1]) & \xrightarrow{\simeq} & \mathrm{Fun}([2],C) & \to & \mathrm{Fun}([1]\times[1],C), \\
\begin{tikzcd} x \arrow[r, "f"] & y \arrow[d, "g"] \\ & z \end{tikzcd} & \mapsto & \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "g\circ f"'] & y \arrow[d, "g"] \\ & z \end{tikzcd} & \mapsto & \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "g\circ f"] & y \arrow[d, "g"] \\ x \arrow[r, "g\circ f"'] & z, \end{tikzcd}
\end{array}$$

and similarly for the other diagram. Forming the composite of these two morphisms in $\mathrm{Fun}([1],C)$, we thus obtain a commutative square in C of the form

$$\begin{tikzcd}
x \arrow[r, "(h\circ g)\circ f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr] & w \arrow[d, "\mathrm{id}_w"] \\
x \arrow[r, "h\circ(g\circ f)"'] & w,
\end{tikzcd}$$

natural in f, g and h. Using Lemma 2.4.7, the diagonal of this square is then naturally isomorphic to both $(h \circ g) \circ f$ and $h \circ (g \circ f)$, finishing the proof. □

2.4.3 The Rezk axiom

The notions of identity morphisms and composition of morphisms in ∞-categories, provided by the Segal axiom, lead to the notion of an *isomorphism* in an ∞-category:

Definition 2.4.9 (Isomorphisms)**.** Consider a morphism $f\colon x \to y$ in an ∞-category C. We say that f is *invertible*, or that it is an *isomorphism*,[6] if there are commutative triangles in C of the form

$$\begin{tikzcd} y \arrow[r, "g"] \arrow[dr, "\mathrm{id}_y"'] & x \arrow[d, "f"] \\ & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "\mathrm{id}_x"'] & y \arrow[d, "h"] \\ & x. \end{tikzcd}$$

A priori, this use of the word 'isomorphism' clashes with the other notion of 'isomorphism' that we have, namely that of natural isomorphisms between functors $x, y\colon * \to C$. For this reason, we will now introduce the *Rezk axiom*, which enforces that these two notions of isomorphisms agree with each other. We start by defining the ∞-category $\mathrm{Iso}(C)$ of isomorphisms in C.

Definition 2.4.10. Given a ∞-category C, we define the ∞-category $\mathrm{Iso}(C)$ via the following pullback diagram:

$$\begin{tikzcd} \mathrm{Iso}(C) \arrow[r] \arrow[dd] \arrow[rr, bend left, dashed, "\pi_{\mathrm{Iso}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([2],C) \times_{d_0,\mathrm{Fun}([1],C),d_2} \mathrm{Fun}([2],C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\Delta"] \\ & \mathrm{Fun}([2],C) \times \mathrm{Fun}([2],C) \arrow[r, "d_0^* \times d_2^*"] \arrow[d, "d_1^* \times d_1^*"] & \mathrm{Fun}([1],C) \times \mathrm{Fun}([1],C) \\ C \times C \arrow[r, "(x,y) \mapsto (\mathrm{id}_y, \mathrm{id}_x)"] & \mathrm{Fun}([1],C) \times \mathrm{Fun}([1],C). & \end{tikzcd}$$

We may think of the objects of $\mathrm{Iso}(C)$ as commutative diagrams in C of the form

$$\begin{tikzcd} & x \arrow[dd, "f"] \arrow[dr, "\mathrm{id}_x"] & \\ y \arrow[ur, "g"] \arrow[dr, "\mathrm{id}_y"'] & & x. \\ & y \arrow[ur, "h"'] & \end{tikzcd}$$

[6] In the literature these are often called 'equivalences'. I might sometimes accidentally call them 'equivalences' as well.

(Note that this is not a commutative square!) More formally, an object of $\mathrm{Iso}(C)$ consists of objects x and y of C, a morphism f in C and two commutative triangles σ_1 and σ_2 in C together with isomorphisms $d_0^*(\sigma_1) \cong f \cong d_2^*(\sigma_2)$, $d_1^*(\sigma_1) \cong \mathrm{id}_x$ and $d_1^*(\sigma_2) \cong \mathrm{id}_y$.
We denote the composite functor at the top of the diagram by

$$\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1], C).$$

As one expects, the identity map $\mathrm{id}_x \colon x \to x$ is invertible in C for every term x of C:

Construction 2.4.11. We construct a functor $i \colon C \to \mathrm{Iso}(C)$ lifting the functor $C \to \mathrm{Fun}([1], C) \colon x \mapsto \mathrm{id}_x$. Informally, i is given by sending an object x to the following commutative diagram in C:

$$\begin{tikzcd} & x \arrow[dd, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_x"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_x"'] & & x. \\ & x \arrow[ur, "\mathrm{id}_x"'] & \end{tikzcd}$$

More formally, restriction along the map $p_{[2]} \colon [2] \to [0]$ induces a functor $p_{[2]}^* \colon C \to \mathrm{Fun}([2], C)$, giving a functor $(p_{[2]}^*, p_{[2]}^*) \colon C \to \mathrm{Fun}([2], C) \times_{\mathrm{Fun}([1], C)} \mathrm{Fun}([2], C)$. Since the two relevant edges of this diagram are both the identity on x, this functor factors through $\mathrm{Iso}(C)$, providing the desired functor $i \colon C \to \mathrm{Iso}(C)$.

Axiom F (Rezk axiom)**.** For every ∞-category C the functor $i \colon C \to \mathrm{Iso}(C)$ is an equivalence.

The Rezk axiom expresses that two objects x and y in C may be identified with each other as soon as we find an invertible morphism $f \colon x \to y$ between them:

Corollary 2.4.12. *Let x and y be objects of C and assume there exists an invertible morphism $f \colon x \to y$. Then there exists an isomorphism $x \cong y$ in C.*

Proof. Since the functor $i \colon C \to \mathrm{Iso}(C)$ is an equivalence, there exists a term z of C together with an isomorphism $i(z) \cong f$ in $\mathrm{Iso}(C)$. Using the source and target functors $s, t \colon \mathrm{Iso}(C) \to C$, this isomorphism in $\mathrm{Iso}(C)$ produces isomorphisms $z \cong x$ and $z \cong y$ in C. By inversion we obtain $x \cong z$ and by composition we obtain $x \cong y$, as desired. □

Remark 2.4.13. To be completely honest with you: the Rezk axiom doesn't *really* identify the two notions of isomorphisms that we have, it really just says that every isomorphism in C is isomorphic to an identity map. In the way I have set things up in these lectures, it is difficult to do much better than that: the notion of "natural isomorphism" is simply a primitive notion and we cannot really say very much about it. A more satisfying approach would have been to not just introduce the notion of a natural isomorphism $\alpha\colon f \cong g$, but to instead introduce a whole ∞-category $f \cong g$ of natural isomorphisms, and then define a single natural isomorphism as a functor $\alpha\colon * \to f \cong g$. If one proceeds this way, one can just ask this ∞-category to fit into a pullback square of the form

$$\begin{tikzcd} f \cong g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(\mathrm{Fun}(C,D)) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ * \arrow[r, "(f,g)"] & \mathrm{Fun}(C,D)\times\mathrm{Fun}(C,D), \end{tikzcd}$$

and then one has *truly* identified the primitive notion of natural isomorphism with that of an invertible natural transformation (which is what I had always intended).

I don't want to change Axiom A.1 anymore so we'll just leave it like it is, but from now on it should be understood that the notion of a natural isomorphism is supposed to be just the same as that of an invertible natural transformation!

Axioms E and F are inspired by Rezk's notion of *complete Segal spaces* [Rez01].

2.5 Digression: Quasicategories

So far, we have been treating ∞-category theory very axiomatically, focusing on the essential features of the theory itself and ignoring how precisely one might model these objects within ordinary mathematics. For completeness, we will now discuss the most common choice of model for ∞-categories: *quasicategories*. I will give a definition of these objects, give the main examples, and sketch why these objects do indeed satisfy the axioms we have introduced so far.

Remark 2.5.1. Some words on history are in order. Quasicategories have played a pivotal role in the acceptance of ∞-category theory as a legitimate part of mathematics. The conceptual idea of 'doing category theory up to homotopy' was around for a while, but a major difficulty was how to make this idea rigorous within ordinary mathematics. While the definition of a quasicategory already appeared in the work of Boardman and Vogt [BV73] under the name 'weak Kan complexes', it was through deep insights and important foundational work by André Joyal [Joy08] and subsequently Jacob Lurie [Lur09; Lur17] that the theory obtained a rigorous, well-developed and practical form that could immediately be used by researchers.

At this point in time, the language of ∞-category theory is so well-established that we no longer need to worry about implementations in set theory, and for reasons I explained at the end of Section 2.1 I think that learning about quasicategories is not relevant anymore for learning about ∞-category theory. I am still including this material mostly for completeness, but except for Definition 2.5.4 it will not be relevant for the remainder of this course and it will not appear in the exam.

Definition 2.5.2. The *simplex category* $\mathbf{\Delta}$ is the category of non-empty finite linearly ordered sets and order-preserving maps between them.

Observe that any object in $\mathbf{\Delta}$ is isomorphic to the linear order $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for some n, and that the isomorphism is unique if it exists. We will therefore frequently denote general objects of $\mathbf{\Delta}$ simply by $[n]$.

Observation 2.5.3. Every morphism in $\mathbf{\Delta}$ can be written as a finite composite of face maps $d_i \colon [n-1] \to [n]$ and degeneracy maps $s_i \colon [n+1] \to [n]$ (see Notation 2.2.9). The identities these maps satisfy are known as the *simplicial identities*.

Definition 2.5.4. Let C be an ∞-category. A *simplicial object in C* is defined to be a functor $\mathbf{\Delta}^{\mathrm{op}} \to C$. We will denote the ∞-category of simplicial objects by

$$\mathrm{s}C := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C).$$

In light of Observation 2.5.3, we may informally think of a simplicial object in C as a collection of objects X_n of C for every natural number n together with face maps $d_i \colon X_n \to X_{n-1}$ and degeneracy maps $s_i \colon X_n \to X_{n_1}$ satisfying the simplicial identities. We frequently draw simplicial objects as follows:

$$\cdots \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} X_2 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} X_1 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} X_0.$$

Since this looks somewhat chaotic, it is common to leave out the degeneracy maps and only draw the face maps:

$$\cdots \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} X_2 \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} X_1 \substack{\longrightarrow \\ \longrightarrow} X_0.$$

We will now specialize to $C = \mathrm{Set}$. For a simplicial set $X \in \mathrm{sSet}$, we refer to the set X_n as its *set of n-simplices*.

Definition 2.5.5 (Simplices)**.** For every natural number n, we define the *n-simplex* Δ^n as the simplicial set represented by $[n]$:

$$\Delta^n := \mathrm{Hom}_{\mathrm{sSet}}(-, [n]) \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}.$$

By the Yoneda lemma this defines a fully faithful functor

$$\Delta^\bullet \colon \mathbf{\Delta} \hookrightarrow \mathrm{sSet}.$$

Definition 2.5.6 (Horns)**.** For $n \geq 1$ and $0 \leq k \leq n$, we define the *k-horn* Λ^n_k as the subsimplicial set

$$\Lambda^n_k \subseteq \Delta^n$$

whose m-simplices are those maps $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ such that there is some element $i \in [n] \setminus \{k\}$ which is not in the image of φ. Equivalently, it is the union inside Δ^n of all of its faces except for the k-th face.

Example 2.5.7. For $n = 1$, the two horns Λ^1_0 and Λ^1_1 may be displayed as follows:

$$\begin{tikzcd} 0 & \hookrightarrow & 0 \arrow[r] & 1 & \hookleftarrow & 1 \\ \Lambda^1_0 & & \Delta^1 & & & \Lambda^1_1 \end{tikzcd}$$

Example 2.5.8. For $n = 2$ the three horns Λ^2_0, Λ^2_1 and Λ^2_2 may be displayed as follows:

$$\begin{tikzcd}[column sep=small] & 1 & & & 1 \arrow[dr] & & & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 & 0 \arrow[ur] & & 2 & 0 \arrow[rr] & & 2 \\ & \Lambda^2_0 & & & \Lambda^2_1 & & & \Lambda^2_2 & \end{tikzcd}$$

Definition 2.5.9. Let X be a simplicial set.

(1) We say that X is a *Kan complex* if for every $n \geq 1$ and $0 \leq k \leq n$, every morphism of simplicial sets $\Lambda^n_k \to X$ extends to a morphism $\Delta^n \to X$:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n. \arrow[ur, dashed] & \end{tikzcd}$$

(2) We say that X is a *quasicategory* if the above condition holds for all $n \geq 1$ but only for $0 < k < n$.

We denote by

$$\mathrm{Kan} \subseteq \mathrm{qCat} \subseteq \mathrm{sSet}$$

the full subcategories on the Kan complexes and the quasicategories, respectively.

We will now give the two main examples of quasicategories:

Example 2.5.10 (Singular complex)**.** Let X be a topological space. We will construct a simplicial set $\mathrm{Sing}(X) \in \mathrm{sSet}$ called the *singular complex of X*.

- For every $n \geq 0$, let $|\Delta^n|$ denote the topological n-simplex:

$$|\Delta^n| := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1,\ x_i \geq 0 \text{ for } i = 0, \dots, n.\}$$

- For every order-preserving map $\varphi\colon [n] \to [m]$ we get a continuous map

$$|\Delta^\varphi|\colon |\Delta^n| \to |\Delta^m|, \quad (x_0, \dots, x_n) \mapsto (y_0, \dots, y_m), \qquad y_i := \sum_{j \in \varphi^{-1}(i)} x_j.$$

  This defines a functor

$$|\Delta^\bullet|\colon \mathbf{\Delta} \to \mathrm{Top}, \qquad [n] \mapsto |\Delta^n|, \qquad \varphi \mapsto |\Delta^\varphi|.$$

- We now define the simplicial set $\mathrm{Sing}(X)$ as

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X),$$

  i.e. it is the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{|\Delta^\bullet|} \mathrm{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Top}}(-,X)} \mathrm{Set}$.

You may know this simplicial set from your algebraic topology courses: it is used in the definition of the *singular (co)homology* of X.

Exercise 2.5.11. Show that the singular complex $\mathrm{Sing}(X)$ of a topological space X is a Kan complex.

Example 2.5.12 (Nerve of a category)**.** Let C be a (small) ordinary category. We define the *nerve of* C as the simplicial set $N(C) \in \mathrm{sSet}$ given by

$$N(C)_n := \mathrm{Hom}_{\mathrm{Cat}}([n], C),$$

i.e. it is the composite $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}}(-,C)} \mathrm{Set}$. Here Cat denotes the ordinary category of small categories.

Exercise 2.5.13. Show that the nerve $N(C)$ of a category C is a quasicategory. Provide an example for which it is not a Kan complex.

We will now discuss the *homotopy-coherent nerve* of a Kan-enriched category.

Definition 2.5.14. A *Kan-enriched category* (resp. a simplicially enriched category) is a category $\mathbf{C}$ such that for all objects x, y of $\mathbf{C}$ the hom set $\mathrm{Hom}_{\mathbf{C}}(x, y)$ is the set of vertices of some Kan complex $\mathrm{Hom}^\Delta(x, y)$ (resp. simplicial set) and the composition in $\mathbf{C}$ is induced by a morphism of Kan complexes (resp. simplicial sets)

$$- \circ -\colon \mathrm{Hom}^\Delta(y, z) \times \mathrm{Hom}^\Delta(x, y) \to \mathrm{Hom}^\Delta(x, z)$$

which satisfy unitality and associativity.

Example 2.5.15. The category sSet has an internal Hom: for a simplicial set X, the functor $X \times - \colon \text{sSet} \to \text{sSet}$ admits a right adjoint that we denote by

$$F(X,-)\colon \text{sSet} \to \text{sSet}.$$

Using this, the category Kan can be turned into a Kan-enriched category **Kan** by taking

$$\text{Hom}^{\Delta}_{\mathbf{Kan}}(X,Y) := F(X,Y),$$

the internal Hom in Kan.

Example 2.5.16. Similarly, the category qCat can be turned into a Kan-enriched category **qCat** by taking

$$\text{Hom}^{\Delta}_{\mathbf{qCat}}(C,D) := F(C,D)^{\simeq},$$

where $C^{\simeq} \subseteq C$ denotes the largest Kan complex contained in C (consisting of those n-simplices all of whose edges are invertible in C).

The vertices of the Kan complex $\text{Hom}^{\Delta}_{\mathbf{C}}(x,y)$ are simply the morphisms $f\colon x \to y$ in $\mathbf{C}$. We refer to 1-simplices in $\text{Hom}^{\Delta}_{\mathbf{C}}(x,y)$ as *homotopies* between morphisms, denoted $f \sim g$. We would now like to construct a variant of the nerve for a Kan-enriched category in which the diagrams 'commute up to homotopy'. For example, the 2-simplices of this homotopy-coherent nerve should consist of three morphisms

$$\begin{tikzcd} & x_1 \arrow[dr, "f_{12}"] & \\ x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & x_2 \end{tikzcd}$$

together with a homotopy $f_{012}\colon f_{12} \circ f_{01} \simeq f_{02}$. Similarly, the 3-simplices should consist of the data of six morphisms

$$\begin{tikzcd} & x_1 \arrow[rr, "f_{13}"] \arrow[dr, "f_{12}"] & & x_3 \\ x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] \arrow[urrr, "f_{03}"] & & x_2 \arrow[ur, "f_{23}"'] & \end{tikzcd}$$

together with four homotopies

$$f_{012}\colon f_{12} \circ f_{01} \simeq f_{02}, \quad f_{013}\colon f_{13} \circ f_{01} \simeq f_{03}, \quad f_{023}\colon f_{23} \circ f_{02} \simeq f_{03}, \quad f_{123}\colon f_{23} \circ f_{12} \simeq f_{13}$$

and a homotopy of homotopies $f_{0123}\colon \Delta^1 \times \Delta^1 \to \text{Hom}^{\Delta}_{\mathbf{C}}(x_0,x_3)$ representing the following diagram

$$\begin{tikzcd} f_{23} \circ f_{12} \circ f_{01} \arrow[r, "f_{123} \circ f_{01}"] \arrow[d, "f_{23} \circ f_{012}"'] & f_{13} \circ f_{01} \arrow[d, "f_{013}"] \\ f_{23} \circ f_{02} \arrow[r, "f_{023}"'] & f_{03}. \end{tikzcd}$$

Definition 2.5.17. For every $n \geq 0$, we denote by $\mathfrak{C}[\Delta^n]$ the simplicially enriched category defined as follows:

- The objects of $\mathfrak{C}[\Delta^n]$ are the elements of $[n]$;
- Given $i, j \in [n]$, we define the simplicial set $\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i, j)$ to be empty if $j < i$, and otherwise as the nerve of the poset $P_{i,j}$ defined by

  $$P_{i,j} := \{I \subseteq \{i, i+1, \ldots, j-1, j\} \mid i, j \in I\} \subseteq \mathcal{P}(\{i, \ldots, j\}),$$

  ordered by *reverse* inclusion.
- For $i, j, k \in [k]$, the composition map

  $$\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(j, k) \times \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i, j) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i, k)$$

  is induced from taking the nerve of the map of partially ordered sets $P_{j,k} \times P_{i,j} \to P_{i,k}$ which sends (I_1, I_2) to the union $I_1 \cup I_2$.

This defines a functor

$$\mathfrak{C}[\Delta^\bullet] \colon \mathbf{\Delta} \to \mathrm{Cat}_\Delta$$

from $\mathbf{\Delta}$ to the category of (small) simplicially enriched categories.

Definition 2.5.18. Let $\mathbf{C}$ be a simplicially enriched category. We define its *homotopy coherent nerve* $N^\Delta(\mathbf{C}) \in \mathrm{sSet}$ by

$$N^\Delta(\mathbf{C})_n := \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathbf{C}),$$

i.e. as the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathfrak{C}[\Delta^\bullet]} (\mathrm{Cat}_\Delta)^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}_\Delta}(-,\mathbf{C})} \mathrm{Set}$.

Proposition 2.5.19 (Cordier and Porter [CP86])**.** *Assume that* $\mathbf{C}$ *is Kan-enriched. Then the simplicial set* $N^\Delta(\mathbf{C})$ *is a quasicategory.*

Exercise 2.5.20. Give a proof of Proposition 2.5.19.

We now relate the theory of quasicategories to our axioms of ∞-category theory:

Proposition 2.5.21. *Quasicategories satisfy axioms A-F.*

Proof sketch.

- A *functor* of quasicategories is a morphism $f \colon C \to D$ of simplicial sets. Composition is strictly unital and associative;
- The ∞-categories $*$ and $\emptyset$ are the initial and terminal objects of sSet;

- Products and coproducts are given by categorical products and coproducts of simplicial sets;
- Pullbacks are given by *homotopy pullbacks* of quasicategories;
- The functor category $\mathrm{Fun}(C, D)$ is the internal hom in qCat: the functor $\mathrm{Fun}(C, -)\colon$ qCat $\to$ qCat is right adjoint to $C \times -\colon$ qCat $\to$ qCat;
- For every quasicategory C we may define an quasicategory $\mathrm{Iso}(C)$ just like in Definition 2.4.10, but where now all the pullback are *strict* pullbacks taken in sSet. It comes with a forgetful functor $(s,t)\colon \mathrm{Iso}(C) \to C \times C$. We define a *natural isomorphism* $\alpha\colon f \cong g$ to be a functor $\alpha\colon C \to \mathrm{Iso}(D)$ such that $s(\alpha) = f$ and $t(\alpha) = g$. Similarly, we define an *n-isomorphism* for $n \geq 3$ to be a functor $C \to \mathrm{Iso}^{n-1}(D) = \mathrm{Iso}(\ldots \mathrm{Iso}(D))$.
- One can check that commutative square axiom, Segal axiom and Rezk axiom are satisfied. Since this requires a lot of simplicial combinatorics not relevant to the actual practice of ∞-category theory, we will not go into the details here. □

Remark 2.5.22. One can prove that a simplicial set X is a quasicategory if and only if the morphism of simplicial sets

$$F(\Delta^2, X) \to F(\Lambda^2_1, X) \cong F(\Delta^1, X) \times_X F(\Delta^1, X)$$

induced by the inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ is a so-called 'trivial fibration' of simplicial sets. As a consequence this morphism is an equivalence of quasicategories for every quasicategory X. So even though a priori a quasicategory only guarantees the *existence* of a 2-simplex given any pair of composable 1-simplices $f\colon x \to y$ and $g\colon y \to z$, the filling conditions for all the higher-dimensional horns guarantee that this 2-simplex is in some sense 'unique up to homotopy', which is precisely what the Segal axiom is expressing.

From this point forward, the reader who chooses to do so may think of ∞-categories as quasicategories. Nevertheless, I will continue writing everything in a way that does not refer to simplicial sets and respects the fundamental principle of homotopy theory.

2.6 More aspects of ∞-category theory

The theory of ∞-category theory is vast, and I could keep telling you about more things for the rest of the course if I wanted. But since this course is ultimately about stable homotopy theory, we do not have the time to go into all possible aspects of the theory. In this section I will quickly go through the remaining constructions of ∞-category that we need, and state various important results as black boxes.

2.6.1 Animae

The notion of a 'groupoid' from ordinary category theory has a natural analogue in the setting of ∞-categories:

Definition 2.6.1. An ∞-category C is called an *anima*, or an ∞-*groupoid*, if every morphism in C is invertible.

Remark 2.6.2. The plural of 'anima' is either 'anima' or 'animae', depending on taste. (I personally prefer 'animae' but since many people use 'anima' I will probably often mix them up.)

The terminology 'anima' was introduced relatively recently by Clausen and Scholze. For historical reasons, the word 'space' is often used for 'anima' in the literature: it turns out that the ∞-category An of animae can be obtained from the ordinary category of topological spaces by inverting the weak homotopy equivalences, see Theorem 2.6.27 below. While it is certainly useful at times to think of animae as spaces (in the sense of Convention 1.2.3) it is such a fundamental concept in homotopy theory that it definitely deserves its own special name as well.

Exercise 2.6.3 (See Exercise 6.6 on page 145 for some hints)**.** Let C be an ∞-category and let x and y be objects in C. Show that the Hom anima $\mathrm{Hom}_C(x,y)$ is indeed an anima.

Recall that every category C has a largest subgroupoid $C^{\simeq}$ contained in it, called the *core* of C, which has the same objects as C but whose morphisms are only the *isomorphisms* in C. It is uniquely characterized by the property that the inclusion $C^{\simeq} \hookrightarrow C$ is universal among functors from a groupoid into C. We will use this universal property to axiomatize the ∞-categorical version of the core:

Axiom G (Groupoid core axiom)**.** For every ∞-category C, there is an anima $C^{\simeq}$ called the *(groupoid) core of* C, which comes equipped with a functor $\gamma_C \colon C^{\simeq} \to C$. Every functor $F \colon X \to C$ from an anima X factors through γ_C, and for functors $G, H \colon X \to C^{\simeq}$ every natural isomorphism $\gamma_C \circ G \cong \gamma_C \circ H$ may be lifted to a natural isomorphism $G \cong H$.

Furthermore, the groupoid core of $[1]$ is equivalent to $* \sqcup *$.

Definition 2.6.4. For ∞-categories C and D, we write $\mathrm{Map}(C,D)$ for the anima $\mathrm{Fun}(C,D)^{\simeq}$.

Exercise 2.6.5. Show that for an anima X and an ∞-category C the inclusion $C^{\simeq} \hookrightarrow C$ induces an equivalence

$$\mathrm{Map}(X, C^{\simeq}) \xrightarrow{\sim} \mathrm{Map}(X, C).$$

2.6.2 Fully faithful and essentially surjective functors are equivalences

Recall that a functor between ordinary categories is an equivalence if and only if it is both fully faithful and essentially surjective. The same statement is true for ∞-categories as well, but we will treat it as a black box since its proof requires some techniques that we haven't introduced.

Definition 2.6.6. A functor $F\colon C \to D$ is called *fully faithful* if for every pair of objects x and y of C, the induced map of animae

$$F\colon \operatorname{Hom}_C(x,y) \to \operatorname{Hom}_D(Fx,Fy)$$

is an equivalence.

Definition 2.6.7. A functor $F\colon C \to D$ is called *essentially surjective* if for every object y of D there exists an object x of C together with an isomorphism $Fx \cong y$.

Theorem 2.6.8 (Joyal, [Lan21, Theorem 2.3.20], [Cis+24, Theorem 6.3.1])**.** *A functor $F\colon C \to D$ is an equivalence if and only if it is fully faithful and essentially surjective.*

2.6.3 Pointwise natural isomorphisms are invertible

Another statement we will treat as a black box is the following:

Theorem 2.6.9 (Joyal, [Lan21, Theorem 2.2.1], [Cis+24, Theorem 6.2.10])**.** *Let $F,G\colon C \to D$ be two functors and let $\alpha\colon F \Rightarrow G$ be a natural transformation. Then α is a natural isomorphism if and only if for every object x of C the morphism $\alpha(x)\colon F(x) \to G(x)$ is an isomorphism in D.*

2.6.4 Opposite categories

Axiom H. For every ∞-category C there is an *opposite category* C^{op}. Similarly every functor $F\colon C \to D$ induces an opposite functor $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$. We have equivalences $(C^{\mathrm{op}})^{\mathrm{op}} \simeq C$ and $(F^{\mathrm{op}})^{\mathrm{op}} \simeq F$.

If C and D are ordinary categories then C^{op} and F^{op} agree with the usual constructions of opposite categories.

If X is an anima then we have an equivalence $X^{\mathrm{op}} \simeq X$.

Note that functors $[1] \to C^{\mathrm{op}}$ correspond to functors $[1] \simeq [1]^{\mathrm{op}} \to (C^{\mathrm{op}})^{\mathrm{op}} \simeq C$, so that morphisms in C^{op} are just morphisms in C but with source and target flipped. Also since $[2] \simeq [2]^{\mathrm{op}}$, this correspondence between morphisms in C and morphisms in C^{op} preserves composition. In particular C is an anima if and only if C^{op} is an anima.

Proposition 2.6.10. *The construction $C \mapsto C^{\mathrm{op}}$ is compatible with all constructions of ∞-categories:*

(1) There are equivalences $*^{\mathrm{op}} \simeq *$ and $\emptyset^{\mathrm{op}} \simeq \emptyset$.

(2) For ∞-categories C and D there are equivalences

$$(C \times D)^{\mathrm{op}} \xrightarrow{\sim} C^{\mathrm{op}} \times D^{\mathrm{op}},$$
$$C^{\mathrm{op}} \sqcup D^{\mathrm{op}} \xrightarrow{\sim} (C \sqcup D)^{\mathrm{op}},$$
$$\mathrm{Fun}(C^{\mathrm{op}}, D^{\mathrm{op}}) \simeq \mathrm{Fun}(C, D)^{\mathrm{op}}.$$

(3) For functors $F\colon C \to E$ and $G\colon D \to E$ there is an equivalence

$$(C \times_E D)^{\mathrm{op}} \simeq C^{\mathrm{op}} \times_{E^{\mathrm{op}}} D^{\mathrm{op}}.$$

(4) For an ∞-category C there is an equivalence $(C^{\mathrm{op}})^{\simeq} \simeq (C^{\simeq})^{\mathrm{op}}$.

Proof. All of these properties follow readily from the universal properties of these constructions. For illustration, let us merely sketch the proof that $(-)^{\mathrm{op}}$ preserves products: functors $T \to (C \times D)^{\mathrm{op}}$ correspond to functors $T^{\mathrm{op}} \to C \times D$, which in turn are pairs of functors $(T^{\mathrm{op}} \to C, T^{\mathrm{op}} \to D)$, which in turn translate back into pairs of functors $(T \to C^{\mathrm{op}}, T \to D^{\mathrm{op}})$, i.e. functors $T \to C^{\mathrm{op}} \times D^{\mathrm{op}}$. □

Corollary 2.6.11. *For objects x and y in an ∞-category C we have an equivalence*

$$\mathrm{Hom}_{C^{\mathrm{op}}}(x, y) \simeq \mathrm{Hom}_C(y, x).$$

Proof. Applying $(-)^{\mathrm{op}}$ to the pullback square defining $\mathrm{Hom}_C(y,x)$ and using the equivalence $\mathrm{Fun}([1], C)^{\mathrm{op}} \simeq \mathrm{Fun}([1]^{\mathrm{op}}, C^{\mathrm{op}}) \simeq \mathrm{Fun}([1], C^{\mathrm{op}})$, we obtain a pullback square of the form

$$\begin{tikzcd}
\mathrm{Hom}_C(y,x)^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C^{\mathrm{op}}) \arrow[d, "(\mathrm{ev}_1, \mathrm{ev}_0)"] \\
* \arrow[r, "(y,x)"] & C^{\mathrm{op}} \times C^{\mathrm{op}},
\end{tikzcd}$$

where the evaluation at 0 and 1 have swapped roles because of the equivalence $[1]^{\mathrm{op}} \simeq [1]$ that was used. This shows that $\mathrm{Hom}_{C^{\mathrm{op}}}(x, y) \simeq \mathrm{Hom}_C(y,x)^{\mathrm{op}}$. But since $\mathrm{Hom}_C(y,x)$ is an anima by Exercise 2.6.3, we have $\mathrm{Hom}_C(y,x) \simeq \mathrm{Hom}_C(y,x)^{\mathrm{op}}$, finishing the proof. □

2.6.5 Limits and colimits

Very important in category theory are the notions of limits and colimits:

Definition 2.6.12 (Limit/colimit)**.** Let I and C be ∞-categories and let $F\colon I \to C$ be a functor.

(1) A *cone on F* is an object X in C together with a natural transformation $\varepsilon\colon \operatorname{const}_X \to F$ of functors $I \to C$.

(2) A cone (X, ε) is called a *limit cone* if for every other object Y of C the composite

$$\operatorname{Hom}_C(Y, X) \xrightarrow{\text{const}} \operatorname{Nat}(\operatorname{const}_Y, \operatorname{const}_X) \xrightarrow{\varepsilon\circ -} \operatorname{Nat}(\operatorname{const}_Y, F)$$

is an equivalence.

In this case, the object X is called the *limit* of F. It is unique up to contractible choice, and will be denoted either by $\lim_I F$ or $\lim_{i\in I} F(i)$.

Dually, one defines a *cocone* to be a natural transformation $\eta\colon F \to \operatorname{const}_X$. It is a *colimit cone* if for every Y the composite

$$\operatorname{Hom}_C(X, Y) \xrightarrow{\text{const}} \operatorname{Nat}(\operatorname{const}_X, \operatorname{const}_Y) \xrightarrow{-\circ\eta} \operatorname{Nat}(F, \operatorname{const}_Y)$$

is equivalence.

Observation 2.6.13. Note that a functor $F\colon I \to C$ admits a limit in C if and only if the opposite functor $F^{\text{op}}\colon I^{\text{op}} \to C^{\text{op}}$ admits a colimit in C^{op}.

Definition 2.6.14. We say an ∞-category C *admits all I-indexed (co)limits* if for every functor $F\colon I \to C$ there exists a (co)limit (co)cone for F. If $G\colon C \to D$ is a functor between ∞-categories that admit I-indexed (co)limits, then we say that G *preserves I-indexed (co)limits* if for every $F\colon I \to D$, the functor G sends (co)limit (co)cones of F in C to (co)limit (co)cones of GF in D.

The notion of limits and colimits are closely related to that of *adjunctions*, introduced below in Section 2.8. We discuss the relations between (co)limits and adjunctions in Section 2.8.3 below.

2.6.6 Subcategories

Next up, we would like to define subcategories.

Definition 2.6.15. A functor $F\colon C \to D$ is called a *monomorphism*, or an *embedding*, if the commutative square

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "F"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

is a pullback square, i.e. the functor $\Delta_F := (\mathrm{id}_C, \mathrm{id}_C)\colon C \to C \times_D C$ is an equivalence of ∞-categories.

Remark 2.6.16. Note that this means that two functors $G, H\colon T \to C$ are naturally isomorphic whenever the functors $F \circ G$ and $F \circ H$ are naturally isomorphic, hence this is a homotopical version of the usual notion of monomorphisms in a category.

We denote embeddings as $F\colon C \hookrightarrow D$. If y is an object of D, we say that *y is in C*, sometimes written as $y \in C$, if there exists some object x of C satisfying $F(x) \cong y$. By the following lemma this object is then unique up to contractible choice and we will often abusively again denote it by y.

Lemma 2.6.17. *Let $F\colon C \hookrightarrow D$ be a monomorphism, and let y be an object of D which is contained in C. Then the fiber C_y of F at y is contractible.*

Proof. By definition C_y sits in a pullback square as follows:

$$\begin{tikzcd} C_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "F"] \\ * \arrow[r, "y"] & D. \end{tikzcd}$$

Since y is contained in C, there exists some object $x\colon * \to C_y$ of C_y. We claim that this functor is an equivalence between C_y and $*$. To this end, consider the following commutative diagram:

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "x"'] & C_y \arrow[r] \arrow[d, "{(x, \mathrm{id}_{C_y})}"] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "x"] \\ C_y \arrow[r, "\Delta_{C_y}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_y \times C_y \arrow[r, "\mathrm{pr}_1"] \arrow[d] & C_y \\ C \arrow[r, "\Delta_F", "\simeq"'] & C \times_D C. & \end{tikzcd}$$

Since the bottom square is a pullback square and Δ_F is an equivalence, also Δ_{C_y} is an equivalence. To show that $x\colon * \to C_y$ is an equivalence, it thus remains to show that the top left square is a pullback square. But this follows from the pasting law of pullback squares, since both the top right square and the top outer rectangle are pullback squares. □

Definition 2.6.18. Let C be an ∞-category. A *collection of morphisms in C* is a monomorphism $M \hookrightarrow \mathrm{Map}([1], C)$. We say it is *closed under composition* if the following two conditions are satisfied:

- For a morphism $f\colon x \to y$ in C, if $f \in M$ then also $\mathrm{id}_x \in M$ and $\mathrm{id}_y \in M$;
- For morphisms $f\colon x \to y$ and $g\colon y \to z$ in C, if $g, f \in M$ then also $g \circ f \in M$.

Axiom I (Subcategory axiom)**.** Consider a ∞-category C equipped with a collection of morphisms $m\colon M \hookrightarrow \mathrm{Map}([1],C)$ of C which is closed under composition. Then there exists a ∞-category $\langle M\rangle_C$ equipped with a functor $i_M\colon \langle M\rangle_C \to C$ satisfying:

- The induced functor $(i_M)_*\colon\ \mathrm{Map}([1],\langle M\rangle_C) \to \mathrm{Map}([1],C)$ factors through M;
- The functor i_M is *universal* with respect to the previous property: every other functor $f\colon D \to C$ whose induced functor $f_*\colon\ \mathrm{Map}([1],D) \to \mathrm{Map}([1],C)$ factors through M admits an essentially unique factorization through $\langle M\rangle_C$.

The last condition means more precisely that there is a pullback square of the form

$$\begin{tikzcd}
\mathrm{Map}(D,\langle M\rangle_C) \arrow[r] \arrow[d] & \mathrm{Map}(D,C) \arrow[d] \\
\mathrm{Map}(\mathrm{Map}([1],D),M) \arrow[r, hook] & \mathrm{Map}(\mathrm{Map}([1],D),\mathrm{Map}([1],C)).
\end{tikzcd}$$

We refer to $\langle M\rangle_C$ as the *(non-full) subcategory spanned by the morphisms in M*.

Exercise 2.6.19. Show that the induced functor $(i_M)_*\colon\ \mathrm{Map}([1],\langle M\rangle_C) \to M$ is an equivalence.

We may also form *full* subcategories in which we only select a collection of objects and then take *all* morphisms between those objects:

Construction 2.6.20. Consider an embedding $\Gamma \hookrightarrow C^{\simeq}$. We define M_Γ as the following pullback:

$$\begin{tikzcd}
M_\Gamma \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1],C) \arrow[d, "{(s,t)}"] \\
\Gamma\times\Gamma \arrow[r, hook] & C^{\simeq}\times C^{\simeq}.
\end{tikzcd}$$

We refer to $\langle M_\Gamma\rangle_C$ as the *full subcategory spanned by the objects in* Γ.

2.6.7 Localizations

Axiom J.1 (Localization axiom)**.** For every ∞-category C and every collection of morphisms $W \hookrightarrow \mathrm{Map}([1],C)$, there exists a functor $l\colon C \to C[W^{-1}]$ exhibiting $C[W^{-1}]$ as a localization of C at the morphisms in W:

- The functor l sends the morphisms in W to isomorpisms $C[W^{-1}]$;

- The functor l is *universal* with respect to the previous property: for every other ∞-category D, precomposition with l induces a fully faithful functor

$$l^*\colon \operatorname{Fun}(C[W^{-1}],D) \hookrightarrow \operatorname{Fun}(C,D)$$

whose essential image is the full subcategory of $\operatorname{Fun}(C,D)$ spanned by those functors $f\colon C \to D$ which sends morphisms of C contained in W to isomorphisms in D.

The last condition in particular means that there is a pullback square of the form

$$\begin{tikzcd} \operatorname{Map}(C[W^{-1}],D) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Map}(C,D) \arrow[d] \\ \operatorname{Map}(W,\operatorname{Iso}(D)^{\simeq}) \arrow[r, hook] & \operatorname{Map}(W,\operatorname{Map}([1],D)). \end{tikzcd}$$

Definition 2.6.21. Given a collection of morphisms W in an ∞-category C, we say that a functor $l\colon C \to D$ *exhibits D as a localization of C at the morphisms in W* if it satisfies properties (1) and (2) of the axiom. In this case, l induces a functor $C[W^{-1}] \to D$ which is an equivalence.

Example 2.6.22 (Exercise 6.4)**.** The functor $p_{[1]}\colon [1] \to *$ exhibits $*$ as a localization of $[1]$ at all its morphisms.

Example 2.6.23 (Exercise 5.2)**.** Recall the functors

$$p_0\colon [1]\times[1] \to [2] \qquad \text{and} \qquad p_2\colon [1]\times[1] \to [2]$$

from Remark 2.4.5, pictorially represented by the following two diagrams:

$$p_0 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad\qquad p_2 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

Then p_0 exhibits $[2]$ as a localization of $[1]\times[1]$ at the morphism $(0,0) \to (0,1)$, while p_2 exhibits $[2]$ as a localization of $[1]\times[1]$ at the morphism $(0,1) \to (1,1)$.

In the case where W consists of *all* morphisms in C, we denote the localization by $|C|$ and call it the *geometric realization of C*.

Axiom J.2 (Geometric realization axiom)**.** For every ∞-category C, the geometric realization $|C|$ is an anima.

2.6.8 The ∞-categories Cat_∞ and An

Finally we will introduce the ∞-categories Cat_∞ and An of (small) ∞-categories and (small) animae.

Axiom K. There is an ∞-category Cat_∞ whose objects are called *small ∞-categories*. Every object $c \in \mathrm{Cat}_\infty$ defines an ∞-category C. For objects $c, d \in \mathrm{Cat}_\infty$ corresponding to ∞-categories C and D, there is an equivalence of animae

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(c,d) \xrightarrow{\sim} \mathrm{Map}(C,D),$$

and in particular morphisms in Cat_∞ correspond to functors of ∞-categories. Composition in Cat_∞ corresponds to composition of functors.

The ∞-category Cat_∞ admits all I-limits and I-colimits for every small ∞-category I.

The ∞-category Cat_∞ is closed under all constructions of ∞-categories introduced so far: initial/terminal ∞-categories, (co)products, pullbacks, functor categories, groupoid cores, subcategories, and localizations. The ∞-categories $[n]$ are small for all $n \geq 0$.

Remark 2.6.24. Axiom K is a somewhat ad hoc and informal approximation of a 'true' axiom introducing Cat_∞ via a concept called 'directed univalence'. Setting up the required theory here would take too much time, so we will have to do with Axiom K. The interested reader may consult [Cis+24, Chapter 7] for more details.

Remark 2.6.25. There is a functor $\mathrm{qCat} \to \mathrm{Cat}_\infty$ which exhibits Cat_∞ as the localization of qCat at the equivalence of quasicategories.

In terms of quasicategories, the ∞-category Cat_∞ may be modeled as the homotopy-coherent nerve $N^\Delta(\mathbf{qCat})$ of the Kan-enriched category $\mathbf{qCat}$ from Example 2.5.16.

Definition 2.6.26. We define the *∞-category of animae* as the full subcategory

$$\mathrm{An} \subseteq \mathrm{Cat}_\infty$$

spanned by the animae.

Theorem 2.6.27. *The subcategory* $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ *is closed under all limits and colimits. Each of the following four vertical functors*

$$\begin{tikzcd}
\mathrm{CW} \arrow[r, hook] \arrow[drr, "\Pi_\infty"'] & \mathrm{Top} \arrow[r, "\mathrm{Sing}"] \arrow[dr, "\Pi_\infty"] & \mathrm{Kan} \arrow[r, hook] \arrow[d, "\Pi_\infty"] & \mathrm{sSet} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}) \arrow[dl, "{|-|:=\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}}"] \\
 & & \mathrm{An} &
\end{tikzcd}$$

is a localization, inducing equivalences of ∞-categories

$$\begin{aligned}\mathrm{An} &\simeq \mathrm{CW}[\text{homotopy equivalences}^{-1}] \\ &\simeq \mathrm{Top}[\text{weak homotopy equivalences}^{-1}] \\ &\simeq \mathrm{Kan}[\text{homotopy equivalences}^{-1}] \\ &\simeq \mathrm{sSet}[\text{weak equivalences}^{-1}].\end{aligned}$$

When X is a topological space/CW-complex/Kan complex, we refer to the anima $\Pi_\infty(X)$ as its *fundamental ∞-groupoid* or *underlying anima*.

Remark 2.6.28. Since the ∞-category An is a localization of the category of topological spaces, it is often called the *∞-category of spaces* in the literature and is denoted either by $\mathcal{S}$ or by Spc. This explains Convention 1.2.3: we use the word 'topological space' if we want to think of X as an object of Top, but if we use the word 'space' then it means we are thinking of it as an object of An. Throughout the rest of the course, we will mostly use the word 'anima' instead of 'space', though at times it is certainly useful for intuition to think of an anima as a space.

Remark 2.6.29. In terms of quasicategories, the ∞-category An can be modeled by the quasicategory $N^\Delta(\mathbf{Kan})$, the homotopy-coherent nerve of the Kan-enriched category **Kan** from Example 2.5.15.

Remark 2.6.30. The Hom animae $\mathrm{Hom}_C(X,Y)$ assemble into a *Hom functor*[7]

$$\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}.$$

In particular every object $X \in C$ defines a functor $Y_X := \mathrm{Hom}_C(-,X)\colon C^{\mathrm{op}} \to \mathrm{An}$.

The following theorem is again taken as a black box:

Theorem 2.6.31 (Yoneda lemma)**.** *For a functor $F\colon C^{\mathrm{op}} \to \mathrm{An}$ and an object $X \in C$ there is an equivalence*

$$\mathrm{Nat}(Y_X, F) \xrightarrow{\sim} F(X), \qquad (\varphi\colon Y_X \to F) \mapsto \varphi(\mathrm{id}_X).$$

In particular the Yoneda embedding

$$Y\colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}), \qquad X \mapsto \mathrm{Hom}_C(-,X)$$

is fully faithful.

Theorem 2.6.32. *The functor* $\mathrm{Hom}_C(-,-)\colon C^{\mathrm{op}} \times C \to \mathrm{An}$ *preserves limits in both variables separately.*

[7] Formally defining this functor takes some work.

Proof sketch. The claim means that for an object X in C the functors

$$\mathrm{Hom}_C(X,-)\colon C \to \mathrm{An} \qquad \text{and} \qquad \mathrm{Hom}_C(-,X)\colon C^{\mathrm{op}} \to \mathrm{An}$$

preserve limits. Note that the claim for $\mathrm{Hom}_C(-,X)$ may be reduced to that for $\mathrm{Hom}_C(X,-)$ by replacing C by C^{op}. For $\mathrm{Hom}_C(X,-)$, we must show that for any functor $F\colon I \to C$ whose limit exists in C the canonical map

$$\mathrm{Hom}_C(X, \lim_{i\in I} F(i)) \to \lim_{i\in I} \mathrm{Hom}_C(X, F(i))$$

is an equivalence. By definition of a limit, we know that the map of animae

$$\mathrm{Hom}_C(X, \lim_{i\in I}) \to \mathrm{Nat}(\mathrm{const}_X, F)$$

is an equivalence, where the right-hand side denotes the Hom anima in $\mathrm{Fun}(I,C)$. The proof thus essentially boils down to computing Hom animae in functor categories. This is non-trivial to do, but let us just say that in this case there is an equivalence

$$\mathrm{Nat}(\mathrm{const}_X, F) \xrightarrow{\sim} \lim_{i\in I} \mathrm{Hom}_C(X, F(i))$$

compatible with the two maps from $\mathrm{Hom}_C(X, \lim_{i\in I}(F(i)))$. □

Appendix: Even more ∞-category theory

Although I said in the previous section that the content of that section contained the 'remaining constructions of ∞-category that we need', this turned out to be too optimistic: at various points during the rest of the lectures it turned out that I needed more basic ∞-category theory that I had not explicitly introduced. To keep the notes well organized, I will collect those additional definitions and results here in this Appendix to Chapter 2. Because of time reasons, the contents of this chapter were not discussed in full detail during the lectures; I have included additional details here for the interested students.

Remark 2.6.33. For those students who are taking the exam for this course: in the exam you are expected to have a working understanding of ∞-category theory, but you do not need to learn the proofs of all the results contained in this chapter. Most important is that you develop an intuition for which statements from ordinary category theory you may import to the setting of ∞-categories and for which statements you need to be more careful.

2.7 (Full) Subcategories

In Section 2.6.6 we introduced a way of associating a subcategory $\langle M\rangle_C$ to a given collection of morphisms M in an ∞-category C which is closed under composition. In this section we will see that a functor $F\colon D \to C$ is of the form $\langle M\rangle_C \hookrightarrow C$ for some M if and only if F is a monomorphism (Corollary 2.7.2). Furthermore, we will see that F exhibits D as a *full* subcategory if and only if F is fully faithful (Corollary 2.7.3).

Proposition 2.7.1. *A functor $F\colon C \to D$ is an equivalence if and only if the induced functor*

$$F_*\colon \mathrm{Map}([1], C) \to \mathrm{Map}([1], D)$$

is an equivalence.

Proof. If F is an equivalence, it is clear that F_* is an equivalence. Conversely, assume that F_* is an equivalence. By Theorem 2.6.8, it will suffice to show that F is essentially surjective and fully faithful. Consider the following commutative diagram:

$$\begin{tikzcd}
C^{\simeq} \arrow[r, "p_{[1]}^*"] \arrow[d, "F^{\simeq}"'] & \mathrm{Map}([1],C) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "F_*"] & C^{\simeq} \arrow[d, "F^{\simeq}"] \\
D^{\simeq} \arrow[r, "p_{[1]}^*"] & \mathrm{Map}([1],D) \arrow[r, "\mathrm{ev}_0"] & D^{\simeq}.
\end{tikzcd}$$

Since the horizontal composites are the identity, it follows that $F^{\simeq}$ is a retract of F_*, and thus in particular it is an equivalence. In particular, f is essentially surjective. Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}([1],C) \arrow[r, "f_*"] \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & \mathrm{Map}([1],D) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] \\
C^{\simeq}\times C^{\simeq} \arrow[r, "f^{\simeq}\times f^{\simeq}"] & D^{\simeq}\times D^{\simeq}.
\end{tikzcd}$$

Since the top and bottom maps are equivalences, it follows that the square is a pullback square. In particular, the induced maps on fibers over all objects $(x,y) \in C^{\simeq}\times C^{\simeq}$ is an equivalence. But this induced map on fibers is precisely the map

$$\mathrm{Hom}_C(x,y) \to \mathrm{Hom}_D(Fx,Fy),$$

showing that F is fully faithful. □

Corollary 2.7.2. *A functor $F\colon C \to D$ is a monomorphism if and only if the induced functor*

$$F_*\colon M := \mathrm{Map}([1],C) \to \mathrm{Map}([1],D)$$

is an embedding and F induces an equivalence $C \xrightarrow{\sim} \langle M\rangle_D$.

Proof. If F_* is an embedding, then the functor

$$\mathrm{Map}([1],C) \to \mathrm{Map}([1],C)\times_{\mathrm{Map}([1],D)} \mathrm{Map}([1],C) \simeq \mathrm{Map}([1],C\times_D C)$$

is an equivalence, hence by Proposition 2.7.1 so is $C \to C\times_D C$, showing that F is an embedding. Conversely, if F is an embedding, so is F_*. To see that $C \to \langle M\rangle_D$ is an equivalence, it suffices by Proposition 2.7.1 to show that $\mathrm{Map}([1],C) \to \mathrm{Map}([1],\langle M\rangle_D)$ is an equivalence, which is true since both sides are M. □

Corollary 2.7.3. *A functor $f\colon C \to D$ is fully faithful if and only if the induced functor*

$$f^{\simeq}\colon \Gamma := C^{\simeq} \to D^{\simeq}$$

is an embedding and f induces an equivalence $C \xrightarrow{\sim} \langle M_\Gamma\rangle_D$.

Proof. See [Cis+24, Proposition 6.4.2]. $\square$

Definition 2.7.4. A *subcategory* of an ∞-category D is an ∞-category C equipped with a monomorphism $C \hookrightarrow D$. It is a *full subcategory* if this monomorphism is fully faithful.

Warning 2.7.5. In the literature this is often called a *replete* subcategory: if an object X is contained in the subcategory, and $X \xrightarrow{\sim} Y$ is an isomorphism in D, then also Y and the isomorphism are contained in the subcategory. When working 'model-independently' as we are doing in this course, it does not really make sense to talk about non-replete subcategories, and it would go against the fundamental principle of homotopy theory. So for this reason I'm dropping the word 'replete'.

2.8 Adjunctions

In this section we will introduce the important notion of adjunctions in the setting of ∞-categories. We start in Section 2.8.1 with the definition and various equivalent characterizations. In Section 2.8.2 we discuss some important examples of adjunctions: the path component functor $\pi_0 \colon \mathrm{An} \to \mathrm{Set}$ as a left adjoint to the inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ and the homotopy category functor $\mathrm{h} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}$ left adjoint to the inclusion $\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$. In Section 2.8.3 we discuss the relation between (co)limits and adjunctions, and deduce various useful consequences about limits and colimits. In Section 2.8.4 we discuss Kan extensions and their pointwise formulas. In Section 2.8.5 we briefly introduce the notion of cofinality in ∞-category theory.

2.8.1 Definition and characterizations

Definition 2.8.1. Let $F \colon C \to D$ and $G \colon D \to C$ be two functors. An *adjunction* between C and D consists of a natural isomorphism

$$\mathrm{Hom}_D(F(-),-) \cong \mathrm{Hom}_C(-,G(-))$$

of functors $C^{\mathrm{op}} \times D \to \mathrm{An}$. We often write $F \dashv G$ to indicate that such an isomorphism as been given.

Proposition 2.8.2. *Consider two functors $F \colon C \to D$ and $G \colon D \to C$. The following data are equivalent:*

(1) The data of an ajunction $F \dashv G$;

(2) The data of a natural transformation $\eta \colon \mathrm{id}_C \to GF$, called the unit, such that for all objects X of C and Y of D the induced map

$$\mathrm{Hom}_D(FX,Y) \xrightarrow{G} \mathrm{Hom}_C(GFX,GY) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_C(X,GY)$$

is an equivalence;

(3) The data of a natural transformation $\varepsilon\colon FG \to \mathrm{id}_D$, called the counit, such that for all objects X of C and Y of D the induced map

$$\mathrm{Hom}_C(X,GY) \xrightarrow{F} \mathrm{Hom}_D(FX,FGY) \xrightarrow{\varepsilon_Y\circ -} \mathrm{Hom}_D(FX,Y)$$

is an equivalence;

(4) The data of both a unit $\eta\colon \mathrm{id} \to GF$ and a counit $\varepsilon\colon FG \to \mathrm{id}$ together with commutative triangles as follows, called the triangle identities:[8]

$$\begin{tikzcd} & FGF \arrow[dr, "\varepsilon F"] & \\ F \arrow[ur, "F\eta"] \arrow[rr, equal] & & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & GFG \arrow[dr, "G\varepsilon"] & \\ G \arrow[ur, "\eta G"] \arrow[rr, equal] & & G. \end{tikzcd}$$

Proof sketch. First assume (1), so that there is a natural equivalence $\mathrm{Hom}_D(FX,Y) \simeq \mathrm{Hom}_C(X,GY)$. If we fix $X \in C$, then by the Yoneda lemma the natural transformation $\mathrm{Hom}_D(FX,-) \xrightarrow{\sim} \mathrm{Hom}_D(X,G(-))$ corresponds to a map $\eta_X\colon X \to GFX$: we obtain this map by taking $Y = FX$ and letting η_X be the map that under this equivalence corresponds to the identity map $\mathrm{id}_{FX}\colon FX \to FX$. Conversely, the map η_X determines the transformation as the composite

$$\mathrm{Hom}_D(FX,-) \xrightarrow{G} \mathrm{Hom}_C(GFX,G(-)) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_C(X,-).$$

The naturality of the equivalence in X corresponds to the naturality of $\eta\colon \mathrm{id}_C \to GF$, showing the equivalence between (1) and (2).

The equivalence between (1) and (3) is proved similarly, where now the counit map $\varepsilon_Y\colon FGY \to Y$ corresponds to the identity map $\mathrm{id}_{GY}\colon GY \to GY$.

To show that these conditions are also equivalent to (4), assume first that (1)-(3) are satisfied. For the first triangle identity, observe that by definition the map $\eta_X\colon X \to GFX$ is a preimage of $\mathrm{id}_{FX}\colon FX \to FX$ under the equivalence

$$\mathrm{Hom}_C(X,GFX) \xrightarrow{F} \mathrm{Hom}_D(FX,FGFX) \xrightarrow{\varepsilon_{FX}\circ -} \mathrm{Hom}_D(FX,FX).$$

This precisely says that the composite $\varepsilon_{FX}\circ F\eta_X$ is equivalent to id_{FX}, naturally in X, which is the first triangle identity. The second triangle identity similarly follows from the fact that the morphism $\varepsilon_Y\colon FGY \to Y$ is a preimage of $\mathrm{id}_{GY}\colon GY \to GY$ under the equivalence

$$\mathrm{Hom}_D(FGY,Y) \xrightarrow{G} \mathrm{Hom}_C(GFGY,GY) \xrightarrow{-\circ\eta_{GY}} \mathrm{Hom}_C(GY,GY).$$

[8]The details aren't super relevant here, but strictly speaking the data of an adjunction should only contain either the first or the second triangle identity, while for the other triangle we only demand the *existence*; see e.g. [Lur24, Tag 02F4].

Finally, assume that (4) is satisfied. An easy check shows that in this case the map defined in (2) is inverse to the map defined in (3), producing the desired natural equivalence $\mathrm{Hom}_D(FX,Y) \simeq \mathrm{Hom}_C(X,GY)$. □

It turns out that we may detect left/right functors 'objectwise':

Definition 2.8.3. Let $F\colon C \to D$ be a functor and let Y be an object of D.

(1) An object Z of C is called a *right adjoint object to Y under F* if it comes equipped with a map $\varepsilon_Y\colon FZ \to Y$ such that for every other object X of C the induced map

$$\mathrm{Hom}_C(X,Z) \xrightarrow{F} \mathrm{Hom}_D(FX,FZ) \xrightarrow{\varepsilon_Y\circ -} \mathrm{Hom}_D(FX,Y)$$

is an equivalence of animae.

(2) Dually, Z is called a *left adjoint object to Y under F* if it comes equipped with a map $\eta_Y\colon Y \to FZ$ such that for every other object X of C the induced map

$$\mathrm{Hom}_C(Z,X) \xrightarrow{F} \mathrm{Hom}_D(FZ,FX) \xrightarrow{-\circ\eta_Y} \mathrm{Hom}_D(Y,FZ)$$

is an equivalence of animae.

Lemma 2.8.4. *Consider a functor $F\colon C \to D$ and let $D_R \subseteq D$ be a full subcategory. Assume that for every $Y \in D_R$ there exists a right adjoint object Z to Y under F. Then these right adjoint objects assemble into a functor $G\colon D \to C$ which comes equipped with a natural isomorphism*

$$\mathrm{Hom}_D(F(-),-) \cong \mathrm{Hom}_C(-,G(-))$$

of functors $C^{\mathrm{op}} \times D_R \to \mathrm{An}$.

Dually, if $D_L \subseteq D$ is a subcategory such that every $Y \in D_L$ admits a left adjoint object under F, then these left adjoint objects assemble into a functor $L\colon D_L \to C$ that comes equipped with a natural isomorphism

$$\mathrm{Hom}_C(L(-),-) \cong \mathrm{Hom}_{D_L}(-,F(-))$$

of functors $D_L^{\mathrm{op}} \times C \to \mathrm{An}$.

Proof. We will prove the claim for right adjoints; the claim for left adjoints is dual. The assumption on F is that for every Y in D_R the functor $\mathrm{Hom}_D(F(-),Y)\colon C^{\mathrm{op}} \to \mathrm{An}$ is representable, in the sense that it lies in the image of the Yoneda embedding $Y\colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}},\mathrm{An})$. Since the Yoneda embedding is fully faithful, this means there exists a functor

$$G\colon D_R \to C$$

making the following diagram commute:

$$\begin{tikzcd} & C \arrow[d, hook, "Y"] \\ D_R \arrow[ur, dashed, "G"] \arrow[r, "Y \mapsto \mathrm{Hom}_D(F(-),Y)"'] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}). \end{tikzcd}$$

Here the bottom functor is the one obtained from currying the functor $\mathrm{Hom}_D(F(-),-)\colon C^{\mathrm{op}} \times D_R \to \mathrm{An}$. Since $Y(X) = \mathrm{Hom}_C(-,X)$, this commutative triangle precisely encodes a natural isomorphism

$$\mathrm{Hom}_C(F(-),-) \cong \mathrm{Hom}_{D_R}(-,G(-))$$

of functors $C^{\mathrm{op}} \times D_R \to \mathrm{An}$, as desired. □

Corollary 2.8.5. *A functor $F\colon C \to D$ admits a right adjoint if and only if every object of D admits a right adjoint object under F. Dually, F admits a left adjoint if and only if every object of D admits a left adjoint object under F.*

Proof. We again only prove the claim about right adjoints. If a right adjoint $G\colon D \to C$ exists, then by Proposition 2.8.2 we see that the object $Z := GY$ together with the counit map $\varepsilon_Y\colon FGY \to Y$ provides a right adjoint object to Y under F. Conversely, if all the right adjoint objects exists, then we may apply Lemma 2.8.4 with $D = D_R$ to obtain a functor $G\colon D \to C$ together with a natural isomorphism

$$\mathrm{Hom}_C(F(-),-) \cong \mathrm{Hom}_D(-,G(-)),$$

which means that G is a right adjoint to F. □

Every adjunction automatically induces new adjunctions at the level of functor categories:

Lemma 2.8.6. *Let $F \dashv G$ be an adjunction of ∞-categories. Then:*

(1) For every ∞-category E the functors

$$G^*\colon \mathrm{Fun}(C,E) \rightleftarrows \mathrm{Fun}(D,E) : F^*$$

form an adjunction, with unit and counit given by

$$G^*F^* = (FG)^* \xrightarrow{\varepsilon^*} \mathrm{id}_D^* = \mathrm{id}_{\mathrm{Fun}(D,E)} \qquad \textit{and} \qquad \mathrm{id}_{\mathrm{Fun}(C,E)} = \mathrm{id}_C^* \xrightarrow{\eta^*} (GF)^* = F^*G^*.$$

(2) For every ∞-category E the functors

$$F_*\colon \mathrm{Fun}(E,C) \rightleftarrows \mathrm{Fun}(E,D) : G_*$$

form an adjunction, with unit and counit given by

$$F_*G_* = (FG)_* \xrightarrow{\varepsilon_*} (\mathrm{id}_D)_* = \mathrm{id}_{\mathrm{Fun}(D,E)} \qquad \textit{and} \qquad \mathrm{id}_{\mathrm{Fun}(C,E)} = (\mathrm{id}_C)_* \xrightarrow{\eta_*} (GF)_* = G_*F_*.$$

Proof. The triangle identities for ε^* and η^*, resp. ε_* and η_*, follow immediately from the triangle identities for ε and η. We leave the details to the reader. □

2.8.2 Path components and homotopy categories

I want to record some relations between sets and animae. Because these lie in the intersection between the axiomatic approach and the classical approach I will refer to them as 'facts'.

Fact 2.8.7. Every set may be regarded as a discrete groupoid and hence as a discrete ∞-groupoid/anima. The inclusion

$$\mathrm{Set} \hookrightarrow \mathrm{An}$$

is fully faithful and admits a left adjoint

$$\pi_0 \colon \mathrm{An} \to \mathrm{Set}.$$

We call $\pi_0(X)$ the *set of path components* of the anima X.

Definition 2.8.8. We say an anima X is *connected* if the set $\pi_0(X)$ has a single element.

We will often abuse terminology and refer to an anima as a *set* if it is in the essential image of the inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$.

Definition 2.8.9 (Path components of an anima)**.** For an anima X, the unit of the adjunction from Fact 2.8.7 gives a map of animae

$$X \to \pi_0(X).$$

Since $\pi_0(X)$ is a disjoint union of points, this decomposes X as a disjoint union of individual path componens: for $[x] \in \pi_0(X)$ we define the *path component of X at* $[x]$ as the pullback

$$\begin{tikzcd}
X_{[x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
* \arrow[r, "{[x]}"] & \pi_0(X).
\end{tikzcd}$$

Fact 2.8.10. An ∞-category C is equivalent to an ordinary category if and only if the Hom animae $\mathrm{Hom}_C(X,Y)$ is a set for all objects X and Y of C.

From now on we will drop the distinction between ordinary categories and ∞-categories whose Hom animae are sets.

Definition 2.8.11. We denote by

$$\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$$

the full subcategory spanned by the ordinary categories. The objects and morphisms of Cat are ordinary categories and ordinary functors. For categories C and D, the Hom anima $\mathrm{Hom}_{\mathrm{Cat}}(C,D)$ is the groupoid $\mathrm{Fun}(C,D)^{\simeq}$ whose objects are the functors $C \to D$ and whose isomorphisms are the natural isomorphisms $F \cong G$.

Fact 2.8.12. The inclusion $\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$ admits a left adjoint

$$\mathrm{h}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}$$

called the *homotopy category functor*. Given an ∞-category C, the homotopy category $\mathrm{h}C$ has the same objects as C, but has Hom sets given by the set of path components of the Hom anima of C:

$$\mathrm{Hom}_{\mathrm{h}C}(X,Y) \cong \pi_0(\mathrm{Hom}_C(X,Y)).$$

Lemma 2.8.13. *Let C be an ordinary category and let W be a collection of morphisms in C. Then the homotopy category $\mathrm{h}(C[W^{-1}])$ of the ∞-categorical localization $C[W^{-1}]$ of C at the morphisms in W has the universal property of the 1-categorical localization.*

Proof. For any ordinary category D we have an equivalence $\mathrm{Fun}(\mathrm{h}(C[W^{-1}]),D) \xrightarrow{\sim} \mathrm{Fun}(C[W^{-1}]),D)$, so $\mathrm{h}(C[W^{-1}])$ simply inherits its universal property from $C[W^{-1}]$. $\square$

Corollary 2.8.14. *The homotopy category* hAn *of the ∞-category of animae is equivalent to the homotopy category* $\mathrm{h}\mathcal{S}$ *of spaces, defined in Convention 1.2.3:*

$$\mathrm{hAn} \simeq \mathrm{h}\mathcal{S}.$$

Similarly we have an equivalence $\mathrm{hAn}_* \simeq \mathrm{h}\mathcal{S}_*$.

Proof. By Theorem 2.6.27 we have $\mathrm{An} \simeq \mathrm{CW}[\text{homotopy equivalences}^{-1}]$, and hence hAn is the 1-categorical localization of the category of CW-complexes at the homotopy equivalences. But this is in essence precisely how we defined $\mathrm{h}\mathcal{S}$. $\square$

2.8.3 (Co)limits and adjunctions

Just like for ordinary categories, we may describe the existence of (co)limits in terms of adjunctions:

Lemma 2.8.15. *Let I and C be ∞-categories and let $F\colon I \to C$ be a functor.*

(1) An object W in C equipped with a cone $\varepsilon_F\colon \mathrm{const}_W \to F$ is a limit of F in C, in the sense of Definition 2.6.12, if and only if ε_F exhibits W as a right adjoint object to F under the functor $\mathrm{const}\colon C \to \mathrm{Fun}(I,C)$, in the sense of Definition 2.8.3.

(2) Dually, an object W equipped with a cocone $\eta_F\colon F \to \mathrm{const}_W$ is a colimit of F in C if and only if it exhibits W as a left adjoint object to F under $\mathrm{const}\colon C \to \mathrm{Fun}(I,C)$.

Proof. This is simply a matter of unwinding definitions: the condition in (1) that W is a right adjoint object to F under const means that for every other object X of C the composite

$$\mathrm{Hom}_C(X,W) \xrightarrow{\text{const}} \mathrm{Nat}(\mathrm{const}_X, \mathrm{const}_W) \xrightarrow{\varepsilon} \mathrm{Nat}(\mathrm{const}_X, F)$$

is an equivalence, where we recall that we write $\mathrm{Nat}(-,-)$ for the Hom anima in $\mathrm{Fun}(I,C)$. But this is precisely the universal property of the limit from Definition 2.6.12! □

Corollary 2.8.16. *Let I and C be ∞-categories. Then C admits all I-indexed limits if and only if the functor*

$$\mathrm{const}\colon C \to \mathrm{Fun}(I,C), \qquad W \mapsto \mathrm{const}_W$$

admits a right adjoint

$$\lim_I\colon \mathrm{Fun}(I,C) \to C.$$

Dually, C admits all I-indexed colimits if and only if $\mathrm{const}\colon C \to \mathrm{Fun}(I,C)$ *admits a left adjoint*

$$\mathrm{colim}_I\colon \mathrm{Fun}(I,C) \to C.$$

Proof. In light of Lemma 2.8.15 this is an immediate consequence of the pointwise criterion for adjoints from Corollary 2.8.5. □

As a particular consequence of the previous corollary, we see that limit cones and colimit cocones are fully functorial in the functor $F\colon I \to C$: the transformations

$$\mathrm{const}_{\lim F} \to F \qquad \text{and} \qquad F \to \mathrm{const}_{\mathrm{colim}\, F}$$

are given by the counit and unit, respectively, of the adjunctions $\mathrm{const} \dashv \lim_I$ and $\mathrm{colim}_I \dashv \mathrm{const}$.

Lemma 2.8.17. *Let $F\colon C \to D$ be a functor and let I be an ∞-category such that both C and D admit I-indexed limits. Then F preserves I-indexed limits if and only if the 'canonical' natural transformation*

$$F(\lim_I(-)) \to \lim_I(F_*(-))$$

of functors $\mathrm{Fun}(I,C) \to D$ *is a natural isomorphism.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\mathrm{const}"'] & D \arrow[d, "\mathrm{const}"] \\ \mathrm{Fun}(I,C) \arrow[r, "F_*"] & \mathrm{Fun}(I,D). \end{tikzcd}$$

The 'canonical' transformation above is the one that under the adjunction $\mathrm{const} \dashv \lim_I$ corresponds to the transformation

$$\mathrm{const}_{F(\lim_I(-))} \cong F_*(\mathrm{const}_{\lim_I(-)}) \to F_*(-)$$

induced by the counit $\mathrm{const}_{\lim_I(X)} \to X$. By Theorem 2.6.9 we may check that this map is an isomorphism for every fixed diagram $X\colon I \to C$. But now we observe that the map $F(\lim_I(X)) \to \lim_I(F_*(-))$ is an isomorphism if and only if the associated cone $\mathrm{const}_{F(\lim_I(X))} \to F_*(X) = F \circ X$ is a limit cone, giving the claim. $\square$

Lemma 2.8.18. *Let $F\colon C \rightleftarrows D :G$ be an adjunction of ∞-categories. Then F preserves all colimits that exist in C and G preserves all limits that exist in D.*

Proof. We prove that F preserves colimits; the proof for G preserving limits is dual. Let $X\colon I \to C$ be a diagram that admits a colimit in C, and let $\eta\colon X \to \mathrm{const}_{\mathrm{colim}_I X}$ denote the associated colimit cone. We need to show that the cocone

$$F(\eta)\colon F \circ X \to F(\mathrm{const}_{\mathrm{colim}_I X}) = \mathrm{const}_{F(\mathrm{colim}_I X)}$$

is a colimit cone in D. By definition, this means that for every other object W in D the map

$$\mathrm{Hom}_D(F(\mathrm{const}_{\mathrm{colim}_I X}), W) \to \mathrm{Nat}(F \circ X, \mathrm{const}_W)$$

is an equivalence of animae. But under the adjunction $F \dashv G$ and the induced adjunction $F \circ - \dashv G \circ -$ from Lemma 2.8.6, this map is equivalent to the map

$$\mathrm{Hom}_C(\mathrm{const}_{\mathrm{colim}_I X}, G(W)) \to \mathrm{Nat}(X, G \circ \mathrm{const}_W) = \mathrm{Nat}(X, \mathrm{const}_{G(W)}).$$

Since the latter map is an equivalence by the universal property of the colimit $\mathrm{colim}_I X$, this finishes the proof. $\square$

Lemma 2.8.19. *Consider ∞-categories C and I, and assume that C admits all I-indexed limits (resp. colimits).*

(1) For every ∞-category E, the functor category $\mathrm{Fun}(E, C)$ admits I-indexed limits (resp. colimits).

(2) For every functor $E' \to E$, the restriction functor $\mathrm{Fun}(E, C) \to \mathrm{Fun}(E', C)$ preserves I-indexed limits (resp. colimits).

(3) In particular, for every object X of E the functor

$$\mathrm{ev}_X\colon \mathrm{Fun}(E, C) \to C$$

preserves I-indexed limits (resp. colimits).

Proof. We treat the case for limits; the case for colimits is dual. By Corollary 2.8.16 there is an adjunction

$$\mathrm{const}\colon C \rightleftarrows \mathrm{Fun}(I, C) : \lim_I .$$

By Lemma 2.8.6 this adjunction induces an adjunction on functor categories of the form

$$\mathrm{Fun}(E, C) \rightleftarrows \mathrm{Fun}(E, \mathrm{Fun}(I, C)) \simeq \mathrm{Fun}(I, \mathrm{Fun}(E, C)).$$

By applying Corollary 2.8.16 another time this says that $\mathrm{Fun}(E, C)$ admits I-indexed limits. Observe that the limit-functor for $\mathrm{Fun}(E, C)$ is given as the composite

$$\mathrm{Fun}(I, \mathrm{Fun}(E, C)) \simeq \mathrm{Fun}(E, \mathrm{Fun}(I, C)) \xrightarrow{(\lim_I)_*} \mathrm{Fun}(E, C).$$

This description makes it clear that precomposition with any functor $E' \to E$ preserves I-limits. □

A fact which will frequently be useful is that colimits along an indexing category I which itself may be written as a colimit may be computed as an iterated colimit:

Theorem 2.8.20. *Let J be a small ∞-category and consider a J-indexed family of small ∞-categories $I_\bullet\colon J \to \mathrm{Cat}_\infty$. Define $I := \mathrm{colim}_j I_j$, and let $\varepsilon\colon I_\bullet \to \mathrm{const}_I$ denote the colimit cone, giving functors $\varepsilon_j\colon I_j \to I$. Let C be an ∞-category and let $F\colon I \to C$ be a functor.*

(1) Assume that the restricted diagram

$$F|_{I_j}\colon I_j \xrightarrow{\varepsilon_j} I \xrightarrow{F} C$$

admits a colimit for every $j \in J$. Then these colimits assemble into a functor

$$F'\colon J \to C, \qquad j \mapsto \mathrm{colim}(F_{I_j}\colon I_j \to C).$$

(2) In this case, the functor F admits a colimit in C if and only if the functor F' admits a colimit in C, and then we have

$$\mathrm{colim}_{i \in I} F(i) \simeq \mathrm{colim}_{j \in J} \mathrm{colim}(F_{I_j}).$$

Proof. We do not have the techniques available to prove this result. Perhaps a proof will be given in Section 2.9 at some point, after having introduced straightening/unstraightening. □

2.8.4 Kan extensions

Definition 2.8.21. Let I, J and C be ∞-categories and let $\varphi\colon I \to J$ be a functor.

(1) Given a functor $F\colon I \to C$, a *left Kan extension of F along φ* is a left adjoint object to F under the restriction functor $\varphi^*\colon \mathrm{Fun}(J,C) \to \mathrm{Fun}(I,C)$. More explicitly, it is a functor $\varphi_!(F)\colon J \to C$ equipped with a natural transformation $\eta_F\colon F \to \varphi^*\varphi_!(F)$ such that for every other functor $G\colon J \to C$ the composite

$$\mathrm{Nat}(\varphi_!(F), G) \xrightarrow{\varphi^*} \mathrm{Nat}(\varphi^*\varphi_!(F), \varphi^*G) \xrightarrow{-\circ\eta_F} \mathrm{Nat}(F, \varphi^*G)$$

is an equivalence.

$$\begin{tikzcd} I \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & C \\ & J \arrow[ur, "G"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} I \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\Downarrow\eta_F"] & C \\ & J \arrow[ur, "\varphi_!(F)" description] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\overset{\overline{\alpha}}{\Rightarrow}"] & \end{tikzcd}$$

(2) Dually, a *right Kan extension of F along φ* is a right adjoint object to F under φ^*, i.e. a functor $\varphi_*(F)\colon J \to C$ equipped with a transformation $\eta_F\colon \varphi^*\varphi_*(F) \to F$ inducing equivalences

$$\mathrm{Nat}(G, \varphi_*(F)) \xrightarrow{\sim} \mathrm{Nat}(\varphi^*, G)$$

for all $G\colon J \to C$.

If every functor $F\colon I \to C$ admits a left Kan extension along φ, then by Corollary 2.8.5 these left Kan extensions assemble into a left adjoint

$$\varphi_!\colon \mathrm{Fun}(I,C) \to \mathrm{Fun}(J,C)$$

to the restriction functor $\varphi^*\colon \mathrm{Fun}(J,C) \to \mathrm{Fun}(I,C)$, called the functor of *left Kan extension along φ*. Dually, if every functor F admits a *right* Kan extension along φ then φ^* admits a right adjoint

$$\varphi_*\colon \mathrm{Fun}(I,C) \to \mathrm{Fun}(J,C)$$

called *right Kan extension along φ*.

Observation 2.8.22. Note that the left Kan extension of a functor $F\colon I \to C$ along the functor $p_C\colon C \to *$ is precisely a colimit of F. Dually, a right Kan extension of F along p_C is a limit of F.

The most useful and common form of Kan extensions are the *pointwise Kan extensions*: those that can be computed using explicit pointwise formulas. In practice, the only Kan extensions one cares about are the pointwise Kan extensions. To discuss these, we first need to introduce slice categories:

Definition 2.8.23 (Slice category)**.** Let C be an ∞-category and let x be an object in C. We define the *slice categories* $C_{/x}$ and $C_{x/}$, also known as the *over category* and the *under category*, respectively, via the following two pullback squares:

$$\begin{tikzcd} C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_1"] \\ * \arrow[r, "x"] & C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_0"] \\ * \arrow[r, "x"] & C. \end{tikzcd}$$

Note that an object of $C_{/x}$ is a pair $(y, f\colon y \to x)$ of an object y equipped with a morphism to x, and dually objects of $C_{x/}$ are pairs $(y, f: x \to y)$. As a result of Exercise 5.2 morphisms $(y, f) \to (y', f')$ in $C_{/x}$ may equivalently be encoed by commutative triangles of the form

$$\begin{tikzcd} y \arrow[rr, "g"] \arrow[dr, "f"'] & & y' \arrow[dl, "f'"] \\ & x, & \end{tikzcd}$$

and dually for morphisms in $C_{x/}$.

We denote by

$$s\colon C_{/x} \to C \qquad \text{and} \qquad t\colon C_{x/} \to C$$

the assignments $s(y, f\colon y \to x) = y$ and $t(y, f: x \to y) = y$, i.e. we first include the slice into the arrow category $\mathrm{Fun}([1],C)$ and then apply the source/target functor.

Definition 2.8.24. For a functor $F\colon C \to D$ and an object d of D, we define the *relative slice categories* $C_{/d}$ and $C_{d/}$ via the following two pullback squares:

$$\begin{tikzcd} C_{/d} \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_{/d} \arrow[d, "s"] \\ C \arrow[r, "F"] & D \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C_{d/} \arrow[r] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_{d/} \arrow[d, "t"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

Objects of $C_{/d}$ consist of pairs $(y, f\colon F(y) \to d)$, and objects of $C_{d/}$ consist of pairs $(y, f\colon d \to F(y))$.

Remark 2.8.25. These relative slice categories are sometimes also denoted by F/d and d/F, which makes their dependencies on F more transparent.

With this in place, we can come to the pointwise formulas for Kan extensions:

Theorem 2.8.26. *Let $F\colon I \to C$ and $\varphi\colon I \to J$ be functors.*

(1) Assume that for every object j of J the composite functor

$$I_{/j} \xrightarrow{s} I \xrightarrow{F} C$$

admits a colimit in C. Then these colimits assemble into a functor

$$\varphi_!(F)\colon J \to C.$$

Furthermore, the maps

$$F(i) \simeq \operatorname{colim}_{i' \in I_{/i}} F(i') \to \operatorname{colim}_{i' \in I_{/\varphi(i)}} F(i') = \varphi_!(F)(i)$$

assemble into a natural transformation $\eta_F\colon F \to \varphi^*\varphi_!(F)$ exhibiting $\varphi_!(F)$ it as a left Kan extension of F along φ.

(2) Dually, if for every j in J the composite

$$I_{j/} \xrightarrow{t} I \xrightarrow{F} C$$

admits a limit in C, then these limits assemble into functor

$$\varphi_*(F)\colon J \to C$$

which is a right Kan extension to F.

Proof. Unfortunately we do not have the techniques yet to be able to prove this result. I may or may not type up a proof of this result in Section 2.9 in the future. □

Definition 2.8.27. If the condition in (1) is satisfied, we say that *F admits a pointwise left Kan extension along φ*, and if the condition in (2) is satisfied we say that *F admits a pointwise right Kan extension along φ.*

In general, the Kan extension of F might not actually be an 'extension', in the sense that the unit map $F \to \varphi^*(\varphi_!F)$ and counit map $\varphi^*(\varphi_*F) \to F$ may not be invertible. However, this *is* the case whenever φ is fully faithful, explaining the terminology. For this we need the following alternative characterization of fully faithful functors:

Lemma 2.8.28. *Let $F\colon C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor F is fully faithful;

(2) For every object y of C the commutative square

$$\begin{tikzcd} C_{/y} \arrow[r, "F"] \arrow[d, "s"] & C_{/Fy} \arrow[d, "s"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

is a pullback square;

(3) *For every object x of C the commutative square*

$$\begin{tikzcd} C_{y/} \arrow[r, "F"] \arrow[d, "t"] & C_{Fx/} \arrow[d, "t"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

is a pullback square.

Proof sketch. We prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. The two vertical functors in the commutative square in (2) are *right fibrations*, in a sense to be explained below in Section 2.9. One can prove that a functor between two right fibrations is an equivalence if and only if it induces equivalences on all fibers. Since the induced map on fibers is precisely the map $\mathrm{Hom}_C(x,y) \to \mathrm{Hom}_C(Fx,Fy)$, it follows that condition (2) is equivalent to fully faithfulness of F. □

Lemma 2.8.29. *Assume that $\varphi\colon I \hookrightarrow J$ is a fully faithful functor, and let $F\colon I \to C$ be a functor.*

(1) *Assume that F admits a pointwise left Kan extension $\varphi_!(F)$ along φ. Then the unit map $\eta_F\colon F \to \varphi^*\varphi_!(F)$ is a natural isomorphism.*

(2) *Dually, if F admits a pointwise right Kan extension $\varphi_*(F)$ along φ then the counit $\varepsilon_F\colon \varphi^*\varphi_*(F) \to F$ is a natural isomorphism.*

Proof. We prove (1); the proof of (2) is dual. By Theorem 2.6.9 we may check that η_F is a pointwise isomorphism. For an object i of I, the map $F(i) \to \varphi^*\varphi_!(F)$ is induced by the map of slices

$$I_{/i} \to I_{/\varphi(i)}, \quad (i', f\colon i' \to i) \mapsto (i', F(f)\colon F(i') \to F(i)).$$

But since φ is fully faithful, this functor is an equivalence by Lemma 2.8.28, and it follows that $\eta_F(i)$ is an isomorphism. □

2.8.5 Cofinal functors

We will briefly discuss the useful notion of *initial/final functors*:

Theorem 2.8.30 (Joyal's version of Quillen's Theorem A)**.** *For a functor $\alpha\colon I \to J$, the following two conditions are equivalent:*

(1) *A functor $F\colon J \to C$ has a colimit if and only if the functor $F \circ \alpha\colon I \to C$ has a colimit, and in this case the canonical map*

$$\mathrm{colim}_I(F \circ \alpha) \to \mathrm{colim}_J F$$

is an isomorphism in C;

(2) *For every object j of J, the relative slice category $I_{j/} = I \times_J J_{j/}$ is weakly contractible, meaning that its geometric realization $|I_{/j}|$ is contractible.*

The dual result is also true, where in (1) we use limits and in (2) we use the under categories $I_{/j}$.

Proof. For a proof, see [Lur09, Theorem 4.1.3.1]. □

Definition 2.8.31. A functor α satisfying the conditions of the theorem is called *final*. We say that α is *initial* if $\alpha^{\mathrm{op}} \colon I^{\mathrm{op}} \to J^{\mathrm{op}}$ is final (i.e. each relative slice $I_{j/}$ is contractible, or equivalently limits of functors $F \colon J \to C$ may be computed as limits of $F \circ \alpha \colon I \to C$).

Warning 2.8.32. The terminology regarding final functors is completely messed up in the literature: because of historic reasons, final functors are also called *co*final functors, where the 'co' roughly means 'jointly final' and has nothing to do with the usual meaning of 'co' in category theory. Nevertheless, some authors use the words 'final' and 'cofinal' for the two notions from Definition 2.8.31, but there is no consensus as to which of the two words should refer to which of the two notions.

We think that out convention is easy to remember in light of the following lemma:

Lemma 2.8.33. *Consider an object j of an ∞-category J. Then $j \colon * \to C$ is an initial functor if and only if j is an initial object, and it is a final functor if and only if j is a final (a.k.a. terminal) object of J.*

Proof. Given an object x of C, unwinding definitions reveals that the relative slice category of the functor $j \colon * \to C$ is equivalent to the Hom anima $\mathrm{Hom}_C(j,x)$. Since this is already an anima, it is equivalent to its geometric realization, and hence it is weakly contractible if and only if it is contractible. So we see that j is an initial functor if and only if the Hom anima $\mathrm{Hom}_C(j,x)$ is contractible for every object x of C, which by is the case by definition precisely if j is an initial object. □

A common way of showing that the relative slice categories $I_{j/}$ or $I_{/j}$ are weakly contractible is by showing they admit an initial or terminal object.

Lemma 2.8.34. *Let C be an ∞-category that admits an initial (resp. terminal) object $x \colon * \to C$. Then C is weakly contractible: $|C| \simeq *$.*

Proof. By Remark 3.1.3 we have an adjunction $x \colon * \rightleftarrows C : p_C$. Passing to geometric realizations, this induces two functos $|x| \colon * \simeq |*| \rightleftarrows |C| : |p_C|$. The counit $\mathrm{const}_x \to \mathrm{id}_C$ induces a natural transformation $|\mathrm{const}_x| = \mathrm{const}_x \to \mathrm{id}_{|C|}$, which is automatically a natural isomorphism since $|C|$ is an anima. This shows that $|p_C| \colon |C| \to *$ is an equivalence, as desired. □

Corollary 2.8.35. *If a functor $\alpha\colon I \to J$ admits a left adjoint, then α is a final functor. Dually, if it admits a right adjoint it is an initial functor.*

Proof. We need to show that for every object j the relative slice category $I_{j/}$ is weakly contractible. Left $\beta^L\colon J \to I$ be a left adjoint to α. Then the relative slice category $I_{j/}$ is equivalent to the slice category $I_{\beta(j)/}$ of I. But this admits an initial object given by $(\beta(j), \mathrm{id}_{\beta(j)})$, hence is weakly contractible by Lemma 2.8.34. □

2.9 Straightening/unstraightening

2.9.1 Cartesian and cocartesian fibrations

TO DO.

2.9.2 Descent

An important property that distinguishes the ∞-category An from other ∞-categories is a property called *descent*.

TO DO: expand this. (For now this is explained in a somewhat ad hoc way in Theorem 4.2.6.)

3 Stable ∞-categories

In this chapter we introduce the notion of a *stable* ∞-category. A generic reference for this material is Lurie [Lur17, Chapter 1].

3.1 Foundations

Initial and terminal objects

Definition 3.1.1 (Initial/terminal objects)**.** An object x of an ∞-category C is called *terminal* if for every other object y the Hom anima $\mathrm{Hom}_C(y,x)$ is contractible. We say that x is *initial* if for every other y the Hom anima $\mathrm{Hom}_C(x,y)$ is contractible.

Remark 3.1.2. Since $\mathrm{Fun}(\emptyset, C) \xrightarrow{\sim} *$, one observes that terminal objects are precisely the limits of the functor $\emptyset \to C$, while the initial objects are precisely the colimits of the functor $\emptyset \to C$.

Remark 3.1.3. For an object x of C, it follows from Corollary 2.8.16 that X is terminal if and only if the functor

$$x\colon * \to C$$

is right adjoint to the functor $p_C\colon C \to *$. Dually, x is initial if and only if it is *left* adjoint to $p_C\colon C \to *$.

Lemma 3.1.4. *Let i be an initial object of an ∞-category I. Then every ∞-category C admits I-indexed limits, and the limit functor is given by evaluation at i:*

$$\lim_I = \mathrm{ev}_i\colon \mathrm{Fun}(I, C) \to C.$$

Dually, if i is terminal in I then C admits I-indexed colimits which are given by evaluating at i.

Proof. By Corollary 2.8.16 it suffices to show that ev_i defines a right adjoint to $\mathrm{const}\colon C \to \mathrm{Fun}(I, C)$. This adjunction may be obtained by considering the adjunction $p_I\colon I \rightleftarrows * :i$ and passing to functor categories using Lemma 2.8.6.

(An alternative proof would be to cite Lemma 2.8.33.) $\square$

Definition 3.1.5 (Zero object)**.** We call an ∞-category C *pointed* if it has a *zero object*, i.e. an object 0 that is both initial and terminal.

Example 3.1.6. The ∞-category An_* of pointed animae is pointed. More generally, if C is an ∞-category with a terminal object $*$, then C_* is pointed. Here C_* is defined as the slice category of C over $*$, i.e. as the following pullback:

$$\begin{tikzcd} C_* \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_0"] \\ * \arrow[r, "*"] & C. \end{tikzcd}$$

Objects of C_* are objects X of C equipped with a map $* \to X$ from the terminal object.

Pullbacks and pushouts

Definition 3.1.7. We define the ∞-category $\lrcorner$ via the following pushout square of ∞-categories:

$$\begin{tikzcd} {[0]} \arrow[r, "1"] \arrow[d, "1"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & \lrcorner . \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

In particular, a functor $F\colon \lrcorner \to C$ by definition corresponds to a pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ in C. The ∞-category $\lrcorner$ turns out to be an ordinary category: it has three objects 0, 1 and 2 and two non-identity morphisms $0 \to 2$ and $1 \to 2$:

$$\lrcorner \qquad = \qquad \begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

Given another object W, a cone on F a priori consists of a commutative diagram

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d] & W \arrow[r, equal] \arrow[d] & W \arrow[d] \\ X \arrow[r, "f"] & Z & Y. \arrow[l, "g"'] \end{tikzcd}$$

The following lemma lets us equivalently encode this data as a commutative square.

Lemma 3.1.8. *The functor $\lrcorner \times [1] \to [1] \times [1]$ described informally by the diagram*

$$\begin{tikzcd} (0,0) \arrow[r, equal] \arrow[d] & (0,0) \arrow[r, equal] \arrow[d] & (0,0) \arrow[d] \\ (1,0) \arrow[r] & (1,1) & (0,1) \arrow[l] \end{tikzcd}$$

exhibits $[1]\times[1]$ *as a localization of* $\lrcorner\times[1]$ *at the morphisms* $(0,0)\to(2,0)$ *and* $(1,0)\to(2,0)$.

Proof. Since $\lrcorner\times[1]$ is a pushout of two copies of $[1]\times[1]$ and $[1]\times[1]$ is a pushout of two copies of $[2]$, we may construct such a functor by gluing two copies of the functor $p_0\colon [1]\times[1]\to[2]$ from Remark 2.4.5. We need to show that for any ∞-category C, precomposition with this functor induces a fully faithful functor

$$\mathrm{Fun}([1]\times[1],C)\hookrightarrow\mathrm{Fun}(\lrcorner\times[1],C)$$

whose essential image consists of those functors $\lrcorner\times[1]\to C$ inverting the two morphisms $(0,0)\to(2,0)$ and $(1,0)\to(2,0)$. Since the ∞-categories $[1]\times[1]$ and $\lrcorner\times[1]$ are pushouts, we may rewrite this map (up to equivalence) as follows:

$$p_0^*\times_{\mathrm{id}}p_0^*\colon\ \mathrm{Fun}([2],C)\times_{\mathrm{Fun}([1],C)}\mathrm{Fun}([2],C)\to\mathrm{Fun}([1]\times[1])\times_{\mathrm{Fun}([1],C)}\mathrm{Fun}([1]\times[1],C).$$

The claim now follows from the fact that $p_0\colon[1]\times[1]\to[2]$ is a localization at the morphism $(0,0)\to(0,1)$, see Example 2.6.23. $\square$

Corollary 3.1.9. *Consider a functor* $F\colon\lrcorner\to C$*, corresponding to morphisms* $f\colon X\to Z$ *and* $g\colon Y\to Z$*, and let* W *be an object of* C*. Then the functor from the lemma induces an equivalence of* ∞*-categories*

$$\left\{\text{Commutative squares}\ \begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}\right\}\xrightarrow{\ \sim\ }\mathrm{Nat}\left(\begin{tikzcd} & W \arrow[d, equal] & & X \arrow[d, "f"] \\ W \arrow[r, equal] & W, & Y \arrow[r, "g"] & Z \end{tikzcd}\right).$$

Proof. By definition, the right-hand side is defined as the Hom anima in $\mathrm{Fun}(\lrcorner,C)$ between the diagrams const_W and F. Because we have an equivalence

$$\mathrm{Fun}([1],\mathrm{Fun}(\lrcorner,C))\xrightarrow{\sim}\mathrm{Fun}(\lrcorner\times[1],C),$$

this means that the right-hand side is equivalent to the fiber over (const_W,F) of the functor

$$\mathrm{Fun}(\lrcorner\times[1],C)\to\mathrm{Fun}(\lrcorner\times\{0\},C)\times\mathrm{Fun}(\lrcorner\times\{1\},C).$$

By the previous lemma, this fiber is equivalent to the fiber over (W,F) of the functor

$$\mathrm{Fun}([1]\times[1],C)\to\mathrm{Fun}(\{(0,0)\},C)\times\mathrm{Fun}(\lrcorner,C),$$

which is by definition what the left-hand side of the corollary means. $\square$

Definition 3.1.10. We say a commutative square in C of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

is a *pullback square* if the corresponding cone $\mathrm{const}_W \to F$ is a limit cone in C. In this case, we also write $W = X \times_Z Y$.

The story for pushouts is endirely dual:

Definition 3.1.11 (Pushout squares)**.** We define the ∞-category $\ulcorner$ as the following pushout in Cat_∞:

$$\begin{tikzcd} {[0]} \arrow[r, "0"] \arrow[d, "0"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \ulcorner . \end{tikzcd}$$

Note that it is the opposite of $\lrcorner$. We may display as follows:

$$\ulcorner \qquad = \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 2. \end{tikzcd}$$

This time a functor $F\colon \ulcorner \to C$ corresponds to a pair of morphisms $f\colon Z \to X$ and $g\colon Z \to Y$. By dualizing the argument for cones on $\lrcorner$ from before (replacing the functor p_0 by $p_2\colon [1] \times [1] \to [2]$) one sees that a cocone $F \to \mathrm{const}_W$ may equivalently be encoded as a commutative square in C of the form

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r] & W. \end{tikzcd}$$

We say that it is a *pushout square* if it corresponds to a colimit cocone. In this case we write $W = X \sqcup_Z Y$.

Stable ∞-categories

We are now ready to define stable ∞-categories:

Definition 3.1.12 (Stable ∞-category)**.** An ∞-category C is called *stable* if:

(1) There exists a zero object $0 \in C$;

(2) Pushouts and pullbacks exist in C;

(3) A commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ W \arrow[r, "k"] & Z \end{tikzcd}$$

in C is a pullback square if and only if it is a pushout square.

The squares in (3) are called *exact squares*.

Important examples of exact squares are the *exact sequences*:

Definition 3.1.13 (Exact sequence)**.** A *null-sequence* in a pointed ∞-category C is a commutative square in C of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z, \end{tikzcd}$$

where 0 is a zero object. We call it a *fiber sequence* if it is a pullback square in C and a *cofiber sequence* if it is a pushout square. In these cases, we write $X = \mathrm{fib}(g)$ and $Z = \mathrm{cofib}(f)$, respectively. We will call it an *exact sequence* if it is both a fiber sequence as well as a cofiber sequence. We will frequently use the simplified notation

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

for null-sequences, leaving the null-homotopy of $gf\colon X \to Z$ implicit in the notation.

Definition 3.1.14 (Loop and suspension)**.** Let C be a pointed ∞-category that admits fibers. For an object $X \in C$ we define ΩX as a fiber of $0 \to X$:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & X. \end{tikzcd}$$

These objects assemble into a functor $\Omega\colon C \to C$. Dually, if C admits cofibers, then we define the suspension ΣX of X as a cofiber of $X \to 0$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, very near end, "\ulcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X. \end{tikzcd}$$

We obtain a functor $\Sigma\colon C \to C$.

Exercise 3.1.15. Let C be a pointed ∞-category that admits fibers and cofibers. Show that the functor $\Sigma\colon C \to C$ is left adjoint to $\Omega\colon C \to C$: for all $X, Y \in C$ there is a natural equivalence

$$\mathrm{Hom}_C(\Sigma X, Y) \simeq \mathrm{Hom}_C(X, \Omega Y).$$

It turns out that stability may be completely characterized in terms of fibers and cofibers:

Theorem 3.1.16. *For a pointed ∞-category C, the following three conditions are equivalent:*

(1) The ∞-category C is stable;

(2) The ∞-category C admits both fibers and cofibers, and a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence if and only if it is a cofiber sequence;

(3) The ∞-category C admits fibers and the functor $\Omega\colon C \to C$ is an equivalence;

(4) The ∞-category C admits cofibers and the functor $\Sigma\colon C \to C$ is an equivalence.

Proof. It is clear that (1) implies (2) by taking $W = 0$ in the defining property of stability. If we assume (2), then the fiber sequence

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

is by assumption also a pushout square and thus the counit map $\Sigma\Omega X \to X$ is an equivalence. Similarly the unit map $X \to \Omega\Sigma X$ is an equivalence. It follows that Σ and Ω are inverse equivalences, so that (3) and (4) hold.

We now show that (3) implies (1); applying the argument to C^{op} will then show that also (4) implies (1). We will first show that C admits pushouts and that pushout squares agree with pullback squares. To this end, consider the full subcategory

$$P \subseteq \mathrm{Fun}([1] \times [1], C)$$

spanned by the pullback squares. We then have the following claim:

Claim: The restriction functor $P \to \mathrm{Fun}(\ulcorner, C)$ is an equivalence.

Proof of claim: To construct an inverse, consider a diagram $z \leftarrow x \to y$ in C, and consider the following pullback diagram (ignoring the dashed morphisms):

$$\begin{tikzcd}
\Omega x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] & \\
\Omega z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
0 \arrow[r] & c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[r] \arrow[d] & y \arrow[d, dashed] \\
 & 0 \arrow[r] & z \arrow[r, dashed] & \Omega^{-1} a.
\end{tikzcd}$$

The desired inverse functor is now given by

$$\left(\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd} \right) \quad \mapsto \quad \Omega^{-1} \left(\begin{tikzcd} \Omega x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega y \arrow[d] \\ \Omega z \arrow[r] & a \end{tikzcd} \right).$$

Note that this functor does indeed land in P since the equivalence $\Omega^{-1}\colon C \xrightarrow{\sim} C$ preserves pullback squares. Furthermore, it provides an extension of the original diagram, hence defines a right inverse. Finally, if we start with any pullback square, we may rewrite it in the form

$$\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d] \\
z \arrow[r] & \Omega^{-1} a,
\end{tikzcd}$$

and we see that the above procedure recovers the original pullback square, showing that it is also a left inverse. This finishes the proof of the claim. $\square$

We will now use the claim to produce pushouts in C. Consider again a diagram $z \leftarrow x \to y$ in C, and extend it uniquely to a pullback diagram:

$$\begin{tikzcd}
x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d] \\
z \arrow[r] & w.
\end{tikzcd}$$

We claim that this square is also a pushout square in C. Indeed, for every other object a in C we have

$$\mathrm{Nat}\left(\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd}, \begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] & a \\ a & \end{tikzcd} \right) \simeq \mathrm{Nat}\left(\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd}, \begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & a \arrow[d, equal] \\ a \arrow[r, equal] & a \end{tikzcd} \right) \simeq \mathrm{Hom}_C(w, a),$$

where the last equivalence holds because $[1]\times[1]$ has a terminal object $(1,1)$. This computation shows that C admits pushouts and that pushout squares agree with pullback squares.

Note that in particular C admits cofibers and that $\Sigma\colon C \to C$ is an equivalence, so applying the above logic to C^{op} we see that C also admits all pullbacks, and hencce is stable. This finishes the proof. $\square$

Remark 3.1.17 (Abelian categories)**.** To get some intuition for stable ∞-categories, observe that condition (2) in the previous proposition is very similar to the definition of an abelian category. Recall that an abelian category $\mathcal{A}$ is an ordinary category satisfying:

(1) $\mathcal{A}$ is semiadditive: it admits a zero object 0 and binary biproducts $X \oplus Y$;

(2) $\mathcal{A}$ admits all kernels and cokernels;

(3) All monomorphisms and epimorphisms in $\mathcal{A}$ are normal:

   a) Every monomorphism $i\colon A \hookrightarrow B$ is the kernel of the quotient map $B \twoheadrightarrow \mathrm{coker}(i)$;
   b) Every epimorphism $p\colon B \twoheadrightarrow C$ is the cokernel of the inclusion $\ker(p) \hookrightarrow B$.

Note that condition (3) is equivalent to the condition that for every pair of morphisms

$$A \overset{i}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C$$

satisfying $pi = 0$, if i is a monomorphism and p is an epimorphism then this sequence is a fiber sequence (i.e. $A \cong \ker(p)$) if and only if it is a cofiber sequence (i.e. $C = \mathrm{coker}(i)$). In a stable ∞-category, the notions of 'monomorphism' and 'epimorphism' are not well-behaved, so we simply demand condition (3) for *all* morphisms i and p. It turns out that in this case we may weaken the assumption in (1): it turns out that stable ∞-categories are automatically additive (see Lemma 3.2.3 below).

Warning 3.1.18. Based on questions during/after the lecture, let me emphasize: when we regard an abelian category $\mathcal{A}$ as an ∞-category, it is *never* stable unless it is the trivial category (since the loop functor $\Omega\colon \mathcal{A} \to \mathcal{A}$ is the zero-functor but by Theorem 3.1.16 it needs to be an equivalence in order for $\mathcal{A}$ to be stable). More generally, an ordinary category can never be a stable ∞-category. The previous remark merely illustrates an analogy which may help to evoke some intuitions.

Something we *can* do is that we may associate to every abelian category $\mathcal{A}$ a stable ∞-category $\mathcal{D}(\mathcal{A})$ called the *derived ∞-category* of $\mathcal{A}$, obtained from the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ by inverting the quasi-isomorphisms:

$$\mathcal{D}(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

It comes with a fully faithful inclusion $\mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$. Usual constructions from homological algebra (like derived tensor products or derived Hom (a.k.a. Ext) functors) are naturally formulated within $\mathcal{D}(\mathcal{A})$.

Exact functors

When working with stable ∞-categories, we usually want to restrict attention to *exact* functors between them:

Definition 3.1.19 (Exact and pointed functors)**.** A functor $F\colon C \to D$ between pointed ∞-categories is *pointed* if it preserves zero objects. If C and D are stable, then a functor $F\colon C \to D$ is called *exact* if it is pointed and sends exact sequences to exact sequences.

Proposition 3.1.20. *Let $F\colon C \to D$ be a functor between stable ∞-categories. Then the following are equivalent:*

(1) The functor F is exact;

(2) The functor F preserves initial objects and pushouts;

(3) The functor F preserves terminal objects and pullbacks.

Proof. Since C and D are pointed, it is clear that F is pointed in each of the three cases. The argument from Theorem 3.1.16 shows that a pointed functor preserves exact sequences if and only if it preserves pushout squares if and only if it preserves pullback squares. □

Since exact functors are clearly closed under composition, the following definition makes sense:

Definition 3.1.21. We define the *∞-category of small stable ∞-categories* as the (non-full) subcategory

$$\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty$$

spanned by the stable ∞-categories and the exact functors between them.

Theorem 3.1.22 ([Lur17, Theorem 1.1.4.4])**.** *The ∞-category $\mathrm{Cat}_\infty^{\mathrm{st}}$ admits small limits, and the inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \hookrightarrow \mathrm{Cat}_\infty$ preserves small limits.*

3.2 The structure of stable ∞-categories

Let us now establish some general properties of stable ∞-categories.

3.2.1 Additivity

We start with additivity:

Definition 3.2.1 ((Semi)additive ∞-category)**.** An ∞-category C is *semiadditive* if

(1) It admits finite products and coproducts;

(2) It has a zero-object 0;

(3) It admits *biproducts*: for any $X, Y \in C$, the map
$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \sqcup Y \to X \times Y$$
is an equivalence.

In this case, we will write $X \oplus Y$ for the biproduct of X and Y. We call C *additive* if it is semiadditive and the following further condition is satisfied:

(4) For any $X \in C$, the "shear map"
$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix} : X \sqcup X \to X \times X$$
is an equivalence.

Remark 3.2.2. As we will see in Section 4.3, the Hom anima $\mathrm{Hom}_C(X,Y)$ of a semiadditive ∞-category automatically becomes a *commutative monoid*. The sum $f+g$ of two morphisms $f, g \in \mathrm{Hom}_C(X,Y)$ is given by the composite
$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y,$$
where Δ and ∇ denote the diagonal and codiagonal. If C is additive, then $\mathrm{Hom}_C(X,Y)$ is even a commutative *group*. See Exercise 7.6 for details.

Lemma 3.2.3. *Every stable ∞-category C is additive.*

Proof. Products and coproducts are special cases of pullbacks and pushouts, and the zero-object exists by pointedness of C. To show C has biproducts, let X and Y be objects in C, and consider the following commutative square:
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "(\mathrm{id},0)"] \\ Y \arrow[r, "(0,\mathrm{id})"] & X \times Y. \end{tikzcd}$$

Since it is a product of the two pullback squares

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ 0 & \longrightarrow & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\mathrm{id}} & Y, \end{array}$$

it is again a pullback square. Since C is stable, it is also a pushout square. This means that the map

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \sqcup Y \to X \times Y$$

is an equivalence, i.e. that C is semiadditive. For additivity, we consider the commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},\mathrm{id})} \\ X & \xrightarrow{(0,\mathrm{id})} & X \times X. \end{array}$$

We claim that this is a pullback square as well. To see this, consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{id},\mathrm{id})} & & \downarrow \\ X & \xrightarrow{(0,\mathrm{id})} & X \times X & \xrightarrow{\mathrm{pr}_2} & X \\ & & \downarrow{\scriptstyle pr_1} & & \downarrow \\ & & X & \longrightarrow & 0. \end{array}$$

The right bottom square, the right rectangle and the upper rectangle are pullback squares. By the pasting-property of pullback diagrams, the upper right square is a pullback square and thus the upper left square is a pullback square. Since C is stable, it means that this is also a pushout square, and so the map

$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix} : X \sqcup X \to X \times X$$

is an equivalence. This finishes the proof that C is additive. □

3.2.2 Functor categories

Proposition 3.2.4. *Let C be a stable ∞-category and I an arbitrary ∞-category. Then* $\mathrm{Fun}(I,C)$ *is a stable ∞-category.*

Idea of proof. For this proof needs the following facts:

(1) If C admits K-indexed (co)limits, then so does $\mathrm{Fun}(I,C)$;

(2) For every $i \in I$ the functor $\mathrm{ev}_i\colon \mathrm{Fun}(I,C) \to C$ preserves all K-limits and K-colimits;

(3) The functors $(\mathrm{ev}_i)_{i\in I}$ are jointly conservative: a natural transformation $\alpha\colon F \to G$ is a natural isomorphism (i.e. an isomorphism in $\mathrm{Fun}(I,C)$) if and only if each morphism $\alpha(i)\colon F(i) \to G(i)$ is an isomorphism in C.

It then follows that the constant functor $\mathrm{const}_0\colon I \to C$ is both terminal and initial in $\mathrm{Fun}(I,C)$, hence a zero-object. Furthermore, the ∞-category $\mathrm{Fun}(I,C)$ admits pushouts and pullbacks. Given a commutative square

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

in $\mathrm{Fun}(I,C)$, it is a pushout/pullback square if and only if it is so after evaluating at each $i \in I$, and hence by stability of C these two conditions are equivalent. □

3.2.3 Finite limits and colimits

Stable ∞-categories admit all finite limits and colimits:

Definition 3.2.5 (Finite ∞-categories)**.** We denote by

$$\mathrm{Cat}_\infty^{\mathrm{fin}} \subseteq \mathrm{Cat}_\infty$$

the smallest full subcategory which contains the ∞-categories $\emptyset$, $*$ and $[1]$, and which is closed under pushouts of ∞-categories. A small ∞-category I is called *finite* if it is contained in $\mathrm{Cat}_\infty^{\mathrm{fin}}$.

Example 3.2.6. The following ∞-categories are finite:

- The ∞-categories $\emptyset$, $*$ and $[1]$;
- The ∞-category $* \sqcup *$;
- The ∞-categories $\lrcorner$ and $\ulcorner$;
- The ∞-categories $[n]$. (It doesn't follow from the axioms I stated here, but it's a refined version of the Segal axiom that $[n]$ is a pushout of $[n-1]$ and $[1]$ for all n.)

Remark 3.2.7. There is a functor $(-)^{\mathrm{op}}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ implementing the assignment $C \mapsto C^{\mathrm{op}}$. This functor is an equivalence, hence in particular preserves pushouts and full subcategories. Since each of the ∞-categories $\emptyset$, $*$ and $[1]$ are dual to their opposites, it follows that an ∞-category C is finite if and only if its opposite C^{op} is finite.

Definition 3.2.8. An ∞-category C is called *left exact*, or said to *admit finite limits*, if it admits all I-indexed limits for all finite ∞-categories I. Similarly, a functor $F\colon C \to D$ between left exact ∞-categories is called *left exact* if it preserves I-indexed limits for all $I \in \mathrm{Cat}_\infty^{\mathrm{fin}}$. We denote by

$$\mathrm{Cat}_\infty^{\mathrm{lex}} \subseteq \mathrm{Cat}_\infty$$

the subcategory consisting of the left exact ∞-categories and left exact functors.

Theorem 3.2.9 (Lurie [Lur09, Corollary 4.4.2.4])**.** *(1) An ∞-category C admits finite limits and only if it has a terminal object and admits pullbacks.*

(2) A functor $F\colon C \to D$ is left exact if and only if it preserves terminal objects and pullbacks.

Dually C has finite colimits if and only if it has an initial object and admits pushouts, and F preserves finite colimits if and only if it preserves initial objects and pushouts.

Idea of proof. One direction is clear. The other direction needs the non-trivial fact that when the ∞-category I can be written as a colimit $I = \mathrm{colim}_j I_j$, we get an alternative description of $\lim_{i\in I} F(i)$ as $\lim_{j\in J^{\mathrm{op}}} \lim_{i\in I_j} F(i)$. $\square$

Corollary 3.2.10. *Every stable ∞-category admits finite limits and colimits. Every exact functor $F\colon C \to D$ between stable ∞-categories preserves finite limits and colimits.* $\square$

We record an alternative characterization of stable ∞-categories, due to Moritz Groth [Gro16]:[1]

Proposition 3.2.11. *An ∞-category C is stable if and only if it has all finite limits and colimits and finite limits commute with finite colimits.*

Proof. Let C be a stable ∞-category. If I is a finite ∞-category, then C has I-indexed colimits by the previous corollary, hence there is a functor $\mathrm{colim}\colon \mathrm{Fun}(S, C) \to C$. Its domain is also stable and colim preserves finite colimits, because colimits commute with colimits (exercise). So the functor is exact and hence preserves finite limits as well.

Conversely, assume that C is finitely complete and cocomplete and that finite limits commute with colimits in C. Considering the empty category, the fact that empty limits commute with empty colimits expresses that C is pointed. To show that C is stable, consider for every

[1] This result was not discussed during the lectures and will not be part of the exam material.

$X \in C$ the following diagram:

$$\begin{tikzcd}
X \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & X \arrow[l] \arrow[r] & 0 \\
0 \arrow[u] & X \arrow[l] \arrow[r] \arrow[u] & X. \arrow[u]
\end{tikzcd}$$

Taking pushouts of the rows we get the diagram $0 \to \Sigma X \leftarrow 0$ whose pullback is $\Omega\Sigma X$. If instead we take pullbacks of the columns, we get $X \leftarrow X \to X$, whose pushout is X. Pullbacks commuting with pushouts tell us then the canonical map $X \to \Omega\Sigma X$ is an equivalence. A similar argument shows that $\Sigma\Omega X \to X$ is an equivalence, so that C is stable by Theorem 3.1.16. □

3.2.4 Exercises on stable ∞-categories

Recall that there are exercise sheets at the end of the notes. Since some of the exercises from sheet 7 are especially important, I will also state them here as part of the lecture notes:

Exercise 3.2.12. Let $f\colon X \to Y$ be a morphism in a stable ∞-category. Show that there is an isomorphism

$$\mathrm{cofib}(f) \cong \mathrm{fib}(f)[1].$$

Exercise 3.2.13. Let $f\colon X \to Y$ be a morphism in a stable ∞-category C. Show that the following are equivalent:

(1) The morphism f is an isomorphism in C;

(2) The cofiber $\mathrm{cofib}(f)$ of f is a zero object in C;

(3) The fiber $\mathrm{fib}(f)$ of f is a zero object in C.

Recall that the *splitting lemma* provides equivalent conditions for a short exact sequence in an abelian category to be a *split short exact sequence*. In the following exercise we will prove a splitting lemma for stable ∞-categories:

Exercise 3.2.14. Let C be a stable ∞-category and let $X \xrightarrow{i} Y \xrightarrow{p} Z$ be an exact sequence in C. Show that the following are equivalent:

(1) The associated morphism $\mathrm{fib}(i) \to X$ is null-homotopic;

(2) The associated morphism $Z \to \mathrm{cofib}(p)$ is null-homotopic;

(3) The map i admits a retract: there is a morphism $r\colon Y \to X$ satisfying $ri \simeq \mathrm{id}_X$;

(4) The map p admits a section: there is a morphism $s\colon Z \to Y$ satisfying $ps \simeq \mathrm{id}_Z$;

(5) There is an isomorphism $Y \cong X \oplus Z$ in C making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "(\mathrm{id}_X,0)"'] & Y \arrow[r, "p"] \arrow[d, "\cong"] & Z \\ & X \oplus Z. \arrow[ur, "\mathrm{pr}_Z"'] & \end{tikzcd}$$

For the following exercise, you may use the fact (to be proved in two different ways in Exercise 7.6 and Exercise 7.7 below) that in a stable ∞-category C the set $[X,Y] := \pi_0 \operatorname{Hom}_C(X,Y)$ of homotopy classes of morphisms from X to Y in C admits a canonical abelian group structure, and that the composition maps

$$- \circ f\colon [X,Y] \to [X',Y] \qquad \text{and} \qquad g \circ -\colon [X,Y] \to [X,Y']$$

are group homomorphisms for all morphisms $f\colon X' \to X$ and $g\colon Y \to Y'$ in C.

Exercise 3.2.15. Let C be a stable ∞-category and consider a commutative square in C of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "g"'] & W. \end{tikzcd}$$

Show that the following conditions are equivalent:

(1) The square is exact;

(2) The sequence

$$X \xrightarrow{(f,u)} Y \oplus Z \xrightarrow{(v,-g)} W$$

is an exact sequence.

3.3 Stabilization and the ∞-category of spectra

Given any ∞-category with finite limits, there is a universal way to turn it into a stable ∞-category:

Definition 3.3.1. Let C be an ∞-category admitting finite limits. We define its *stabilization* $\mathrm{Sp}(C)$ as the $\mathbb{N}^{\mathrm{op}}$-indexed limit

$$\mathrm{Sp}(C) := \lim(\ldots \xrightarrow{\Omega} C_* \xrightarrow{\Omega} C_* \xrightarrow{\Omega} C_*)$$

in the (very large[2]) ∞-category Cat_∞ of ∞-categories. An object of $\mathrm{Sp}(C)$ is called a *spectrum object in* C.

The forgetful functor to the copy of C_* indexed by $n \in \mathbb{N}$ is denoted by

$$\Omega^{\infty-n}\colon \mathrm{Sp}(C) \to C_*.$$

We write $\Omega^\infty = \Omega^{\infty-0}\colon \mathrm{Sp}(C) \to C_*$ for the forgetful functor to the final copy.

The assignment $C \mapsto \mathrm{Sp}(C)$ defines a functor $\mathrm{Sp}\colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$.

Definition 3.3.2. We define the ∞-category Sp of *spectra* as the stabilization of the ∞-category An:

$$\mathrm{Sp} := \mathrm{Sp}(\mathrm{An}) = \lim(\cdots \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_*).$$

Remark 3.3.3. By the universal property of the limit, an object X of $\mathrm{Sp}(C)$ consists of a sequence $(X_n)_{n\in\mathbb{N}}$ of objects in C_* equipped with isomorphisms $\sigma_n\colon X_n \xrightarrow{\cong} \Omega X_{n+1}$ in C_*. For $C = \mathrm{An}$ this is precisely the definition of a spectrum we saw in Definition 1.3.1!

Similarly, given spectrum objects $X = (X_n)_{n\in\mathbb{N}}$ and $Y = (Y_n)_{n\in\mathbb{N}}$ in C, the Hom anima in $\mathrm{Sp}(C)$ may be computed as

$$\mathrm{Hom}_{\mathrm{Sp}(C)}(X,Y) \simeq \lim_n \mathrm{Hom}_{C_*}(X_n, Y_n),$$

so that a morphism $X \to Y$ consists of a collection of morphisms $f_n\colon X_n \to Y_n$ in C_* together with a collection of homotopies filling the following squares:

$$\begin{tikzcd} X_n \arrow[r, "\cong"] \arrow[d, "f_n"'] & \Omega X_{n+1} \arrow[d, "\Omega f_{n+1}"] \\ Y_n \arrow[r, "\cong"] & \Omega Y_{n+1}. \end{tikzcd}$$

Again for $C = \mathrm{An}$ these are precisely the morphisms of spectra from Definition 1.3.1!

Let us prove that $\mathrm{Sp}(C)$ is automatically stable. We start by discussing limits and colimits in $\mathrm{Sp}(C)$:

Lemma 3.3.4. *Let I and C be ∞-categories and assume C has finite limits.*

[2] I'm sweeping set-theoretical issues under the rug here: since I want to allow C to be a large ∞-category, I should work in a 'very large' universe $\widehat{\mathrm{Cat}}_\infty$ whose objects are the large ∞-categories. For those interested, the key word to look up here is 'Grothendieck universes'.

(1) *If C has I-limits, then also* $\mathrm{Sp}(C)$ *has I-limits and the functors* $\Omega^{\infty-n}\colon \mathrm{Sp}(C) \to C$ *preserves them.*

(2) *If C_* has I-colimits that are preserved by* $\Omega\colon C_* \to C_*$, *then* $\mathrm{Sp}(C)$ *has I-colimits and* $\Omega^{\infty-n}\colon \mathrm{Sp}(C) \to C_*$ *preserves them.*

Proof. This is a special case of a the more general fact that if $C_\bullet\colon K \to \mathrm{Cat}_\infty$ is a diagram of small ∞-categories C_k for $k \in K$ such that C_k admits I-(co)limits for all k and such that $C_k \to C_{k'}$ preserves I-(co)limits for all $k \to k'$ in K, then also the limit $\lim_{k\in K} C_k$ in Cat_∞ has I-(co)limits and each functor $\lim_{k\in K} C_k \to C_k$ preserves them. By the description of limits in terms of an adjunction on functor categories from Corollary 2.8.16, the claim then follows by combining the equivalence

$$\mathrm{Fun}(I, \lim_{k\in K} C_k) \simeq \lim_{k\in K} \mathrm{Fun}(I, C_k)$$

with the fact that a limit of adjunctions is an adjunction. We omit the details.

In the case at hand, if C has I-limits then so does C_*, and since $\Omega\colon C_* \to C_*$ preserves all I-limits (as it is a right adjoint) also $\mathrm{Sp}(C)$ has all limits. For the colimits, we need to specifically demand that Ω preserves them. □

We are now ready to prove that $\mathrm{Sp}(C)$ is stable. The proof is a bit more tricky than one might naively except, and hence we will provide a more detailed proof than usual. I would like to thank Marc Hoyois for pointing out to me that my original proof sketch was too naive.

Theorem 3.3.5. *For every ∞-category C with finite limits, the stabilization* $\mathrm{Sp}(C)$ *is a stable ∞-category.*

Proof. By the previous corollary, the ∞-category $\mathrm{Sp}(C)$ has finite limits. The terminal spectrum object is at each leven n by the terminal object of C_*, but since this is also an initial object and $\Omega\colon C_* \to C_*$ preserves initial objects we get from Lemma 3.3.4 that it is also an initial object, so that $\mathrm{Sp}(C)$ is pointed.

By part (3) of Theorem 3.1.16, it thus remains to show that the functor

$$\Omega\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C)$$

is an equivalence. Let us start by giving a more explicit description of the spectrum object $\Omega(X)$ for $X \in \mathrm{Sp}(C)$. Since each of the functors $\Omega^{\infty-n}\colon \mathrm{Sp}(C) \to C_*$ preserve limits, they will commute with taking loops, and hence we get

$$\Omega(X)_n = \Omega(X_n)$$

for all n. We need to be careful with the structure maps though: since $\Omega\colon C_* \to C_*$ preserves limits, we have for all finite categories I a commutative square

$$\begin{tikzcd} \mathrm{Fun}(I, C_*) \arrow[r, "\lim_I"] \arrow[d, "\Omega"'] & C_* \arrow[d, "\Omega"] \\ \mathrm{Fun}(I, C_*) \arrow[r, "\lim_I"] & C_*, \end{tikzcd}$$

and in particular there is a commutative square

$$\begin{tikzcd} C_* \arrow[r, "\Omega"] \arrow[d, "\Omega"'] & C_* \arrow[d, "\Omega"] \\ C_* \arrow[r, "\Omega"] & C_*. \end{tikzcd}$$

We refer to the resulting natural isomorphism $\mathrm{swap}\colon \Omega^2 \cong \Omega^2$ as the *swap of coordinates*, since it swaps the roles of the two instances of Ω. Most importantly: it is *not* the identity isomorphism! All in all, we see that the structure morphisms of the spectrum object $\Omega(X)$ are given by

$$\sigma_n^{\Omega(X)}\colon \Omega(X)_n = \Omega(X_n) \xrightarrow[\cong]{\Omega(\sigma_n^X)} \Omega(\Omega(X_{n+1})) \xrightarrow[\cong]{\mathrm{swap}} \Omega(\Omega(X_{n+1})) = \Omega(\Omega(X)_{n+1}).$$

Now, to show that $\Omega\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C)$ is an equivalence, we claim that an inverse is given by the 'shift functor'

$$[1]\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C), \qquad X \mapsto X[1] = \{X_{n+1}\}_{n\in\mathbb{N}},$$

where the structure maps for $X[1]$ are the isomorphisms

$$\sigma_n^{X[1]}\colon X[1]_n = X_{n+1} \xrightarrow[\cong]{\sigma_{n+1}^X} \Omega(X_{n+2}) = \Omega(X[1]_{n+1})$$

inherited from X. For this, first note that the shift functor is an equivalence of ∞-categories. Indeed, consider the other shift functor

$$[-1]\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C), \qquad X \mapsto X[-1] := \{Y_n\}_{n\in\mathbb{N}}, \qquad Y_n := \begin{cases} X_{n-1} & n \geq 1 \\ \Omega X_0 & n = 0, \end{cases}$$

where the structure maps are the isomorphisms

$$\sigma_n^{X[-1]}\colon X[-1]_n = X_{n-1} \xrightarrow[\cong]{\sigma_{n-1}^X} \Omega(X_n) = \Omega(X[-1]_{n+1})$$

for $n \geq 1$, and the identity map $\sigma_0^{X[-1]} := \mathrm{id}_{\Omega X_0}$ when $n = 0$. Since we are simply shifting up and down it is easy to produce natural isomorphisms

$$X \cong X[1][-1] \qquad \text{and} \qquad X \cong X[-1][1],$$

showing that the two shift functors are inverse to one another.

To show that Ω is also inverse to the shift functor, it hence remains to construct a natural isomorphism

$$\alpha\colon X \xrightarrow{\cong} \Omega(X[1])$$

of functors $\mathrm{Sp}(C) \to \mathrm{Sp}(C)$; by uniqueness of left-inverses of the shift functor this then results in a natural isomorphism $\Omega \cong [-1]$ and hence Ω is also a *right*-inverse of $[1]$. To construct α, note that for any n we have an isomorphism $\sigma_n\colon X_n \xrightarrow{\cong} \Omega(X_{n+1}) = \Sigma(X[1])_n$. However, due to the swap maps that we have in the structure maps of $\Omega(X[1])$ these maps will not immediately define an isomorphism of spectrum objects. Instead we are going to use the following trick to get rid of the swap maps:

Claim: The forgetful functors $\Omega^{\infty-2k}\colon \mathrm{Sp}(C) \to C_*$ induce an equivalence

$$\mathrm{Sp}(C) \simeq \lim(\dots \xrightarrow{\Omega^2} C_* \xrightarrow{\Omega^2} C_* \xrightarrow{\Omega^2} C_*).$$

Indeed, this is an immediate consequence of the fact that the even natural numbers are cofinal in $\mathbb{N}$, so any $\mathbb{N}^{\mathrm{op}}$-indexed limit may be computed as a limit over just the even natural numbers. In particular, a spectrum object in C may equivalently be described as a collection $(X_{2k})_{k\in\mathbb{N}}$ of pointed objects of C together with structure isomorphisms $\sigma'_{2k}\colon X_{2k} \xrightarrow{\cong} \Omega^2 X_{2k+2}$. The key fact is now the following lemma:

Lemma 3.3.6. *For any pointed category D with finite limits, the natural isomorphism*

$$\Omega(\Omega^2 X) \xrightarrow{\cong} \Omega^2(\Omega X)$$

coming from the fact that Ω commutes with Ω^2 is homotopic to the identity map on $\Omega^3(X)$.

Proof sketch. There is a *universal* pointed ∞-category with finite limits: the opposite $(\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}$ of the ∞-category of finite pointed animae. Universality means that evaluation at the object $S^0 \in \mathrm{An}_*^{\mathrm{fin}}$ determines an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, D) \xrightarrow{\sim} D$$

between D and the ∞-category of left-exact functors $(\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}} \to D$. Universality is satisfies essentially by definition of $\mathrm{An}_*^{\mathrm{fin}}$: it is the smallest subcategory of An_* generated under finite colimits by S^0. Given an object X of D, we denote the resulting functor by

$$X^{(-)}\colon (\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}} \to D,$$

and for a pointed anima A we refer to the object X^A as the *tensoring of X by A*. It will then suffice to show that the relation is satisfied in $(\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}$. Passing to the opposite category, this means that we must show that the canonical comparison map

$$\Sigma^2(\Sigma(S^0)) \xrightarrow{\cong} \Sigma(\Sigma^2(S^0))$$

is homotopic to the identity map. Now recall that the ∞-category An_* is a localization of the category Top_* at the weak homotopy equivalences. The above map is in the essential image of this map, if we interpret both sides as the actual suspension at the level of topological spaces, rather than the ∞-categorically defined suspension in An_*. The claim thus follows from the fact that the map

$$S^1 \wedge S^1 \wedge S^1 \to S^1 \wedge S^1 \wedge S^1, \quad (x, y, z) \mapsto (z, x, y)$$

is homotopic to the identity map. This finishes the proof of the lemma. $\square$

Now to finish the proof of the theorem, i.e. to produce the natural isomorphism $\alpha\colon X \xrightarrow{\cong} \Omega(X[1])$, we consider the maps

$$\alpha_{2k} := \sigma_{2k}\colon X_{2k} \xrightarrow{\cong} \Omega(X_{2k+1}) = \Omega(X[1])_{2k}.$$

We must then produce commutative diagrams as follows:

$$\begin{tikzcd}
X_{2k} \arrow[r, "\sigma_{2k}^X"] \arrow[d, "\sigma_{2k}^X"', "\cong"] & \Omega(X_{2k+1}) \arrow[d, "\sigma_{2k}^{\Omega(X[1])}"] \\
\Omega(X_{2k+1}) \arrow[d, "\Omega(\sigma_{2k+1}^X)"', "\cong"] & \Omega^2(X_{2k+2}) \arrow[d, "\Omega(\sigma_{2k+1}^{\Omega(X[1])})"] \\
\Omega^2(X_{2k+2}) \arrow[r, "\Omega^2(\sigma_{2k+2}^X)"] & \Omega^3(X_{2k+3}).
\end{tikzcd}$$

Unwinding definitions, we see that the two maps differ only by the swap map $\Omega(\Omega^2(X_{2k+3})) \cong \Omega^2(\Omega(X_{2k+3}))$, hence the claim follows from Lemma 3.3.6. $\square$

Notation 3.3.7. The 'shift notation' is actually convenient in an arbitrary stable ∞-category C: for $n \geq 0$ we write

$$[n] := \Sigma^n\colon C \xrightarrow{\sim} C.$$

We also write

$$[-n] := \Omega^n\colon C \xrightarrow{\sim} C.$$

Proposition 3.3.8. *The ∞-category* Sp *of spectra admits all small limits and filtered colimits.*[3] *In particular,* Sp *admits all sequential (i.e. $\mathbb{N}$-indexed) colimits.*

Proof. The statement about limits is an instance of Lemma 3.3.4: An admits all small limits and colimits. For the statement about limits, we need the following non-trivial claim, which Lurie proves in [Lur09]:

[3] An ∞-category K is called *filtered* if for every finite diagram $F\colon I \to K$ in K there exists an object $W \in K$ with a cone $F \to \mathrm{const}_W$.

Claim: an ∞-category K is filtered if and only if the functor $\mathrm{colim}_K \colon \mathrm{Fun}(K, \mathrm{An}) \to \mathrm{An}$ preserves finite limits.

In other words: finite limits commute with filtered colimits in An. Since $\Omega \colon \mathrm{An}_* \to \mathrm{An}_*$ is a finite limit, it thus commutes with filtered colimits, and so we may apply the corollary to get that Sp admits filtered colimits. $\square$

We have already seen an example of a spectrum: the Eilenberg MacLane spectrum HA with

$$(HA)_n = K(A, n).$$

Another important class of examples is suspension spectra:

Construction 3.3.9 (Suspension spectrum)**.** Consider the functor $Q \colon \mathrm{An}_* \to \mathrm{An}_*$ defined by the formula

$$Q(X) := \mathrm{colim}_{n \geq 0} \Omega^n \Sigma^n X,$$

where the comparison maps in the diagram are induced by the unit $X \to \Omega\Sigma X$ of the adjunction $\Sigma \dashv \Omega$. Again using the fact that Ω preserves filtered colimits, we see that

$$\Omega Q(\Sigma(X)) = \Omega(\mathrm{colim}_{n\geq 0} \Omega^n \Sigma^{n+1} X) \simeq \mathrm{colim}_{n \geq 0} \Omega^{n+1} \Sigma^{n+1} \simeq Q(X).$$

Hence we obtain a spectrum $\Sigma^\infty(X)$, called the *suspension spectrum of* X:

$$\Sigma^\infty(X)_n := Q(\Sigma^n X).$$

This construction is functorial in X, giving rise to a functor

$$\Sigma^\infty \colon \mathrm{An}_* \to \mathrm{Sp}.$$

Proposition 3.3.10. *The suspension spectrum functor* $\Sigma^\infty \colon \mathrm{An}_* \to \mathrm{Sp}$ *is left adjoint to* $\Omega \colon \mathrm{Sp} \to \mathrm{An}_*$.

Proof. We will show condition (2) from Proposition 2.8.2. For a pointed anima X, we have a map

$$\eta_X \colon X = \Omega^0 \Sigma^0 X \to \mathrm{colim}_n \Omega^n \Sigma^n X = QX = \Omega^\infty \Sigma^\infty (X).$$

We claim that this is the unit map for an adjunction. In other words, we have to show that for every spectrum Y the composite

$$\mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, Y) \xrightarrow{\Omega^\infty} \mathrm{Hom}_{\mathrm{An}_*}(\Omega^\infty \Sigma^\infty X, \Omega^\infty Y) \xrightarrow{-\circ \eta_X} \mathrm{Hom}_{\mathrm{An}_*}(X, \Omega^\infty(Y))$$

is an equivalence. To see this, consider the following commutative diagram of animae:

$$
\begin{tikzcd}
\ddots \arrow[dr, "\simeq"'] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
 & \mathrm{Hom}_{\mathrm{An}_*}(\Sigma^2 X, Y_2) \arrow[r, "\Omega"] \arrow[dr, "\simeq"'] & \mathrm{Hom}_{\mathrm{An}_*}(\Omega\Sigma^2 X, Y_1) \arrow[r, "\Omega"] \arrow[d] & \mathrm{Hom}_{\mathrm{An}_*}(\Omega^2\Sigma^2 X, Y_0) \arrow[d] \\
 & & \mathrm{Hom}_{\mathrm{An}_*}(\Sigma X, Y_1) \arrow[r, "\Omega"] \arrow[dr, "\simeq"'] & \mathrm{Hom}_{\mathrm{An}_*}(\Omega\Sigma X, Y_0) \arrow[d] \\
 & & & \mathrm{Hom}_{\mathrm{An}_*}(X, Y_0).
\end{tikzcd}
$$

The bottom-right corner is $\mathrm{Hom}_{\mathrm{An}_*}(X, \Omega^\infty(Y))$, and since the diagonal maps are all equivalences we see that the limit of the diagonals is also equivalent to $\mathrm{Hom}_{\mathrm{An}_*}(X, \Omega^\infty(Y))$. But an alternative way of computing this limit is by first taking the limit of the vertical columns and then taking a limit along the resulting horizontal diagram. For the limit of the k-th column we have

$$\lim_n \mathrm{Hom}_{\mathrm{An}_*}(\Omega^n \Sigma^{n+k} X, Y_k) \simeq \mathrm{Hom}_{\mathrm{An}_*}(\mathrm{colim}_n \Omega^n \Sigma^n \Sigma^k X, Y_k) \simeq \mathrm{Hom}_{\mathrm{An}_*}(Q\Sigma^k(X), Y_k).$$

The resulting horizontal limit then gives

$$\lim_k \mathrm{Hom}_{\mathrm{An}_*}(Q\Sigma^k(X), Y_k) = \mathrm{Hom}_{\lim_k \mathrm{An}_*}((Q\Sigma^k(X))_{k\in\mathbb{N}}, (Y_k)_{k\in\mathbb{N}}) = \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, Y).$$

This finishes the proof. □

Definition 3.3.11 (Unreduced suspension spectrum)**.** Let X be an anima. We define its *unreduced suspension spectrum* $\mathbb{S}[X] \in \mathrm{Sp}$ as

$$\mathbb{S}[X] := \Sigma^\infty(X_+),$$

where $X_+ := X \sqcup *$. This defines a functor

$$\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}.$$

We define the *sphere spectrum* as

$$\mathbb{S} := \mathbb{S}[\mathrm{pt}] = \Sigma^\infty(S^0).$$

By the following corollary, we may think of the unreduced suspension spectrum as the 'free spectrum on X':

Corollary 3.3.12. *The functor* $\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}$ *is left adjoint to* $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$.

Proof. This is immediate from Proposition 3.3.10 in light of the observation that the functor

$$(-)_+ \colon \mathrm{An} \to \mathrm{An}_*, \qquad X \mapsto X_+$$

is a left adjoint to the forgetful functor $\mathrm{An}_* \to \mathrm{An}$. $\square$

We have the following table with analogies between ordinary algebra and 'higher algebra':

Ordinary algebra	Higher algebra
Abelian group	Spectrum
Integers $\mathbb{Z}$	Sphere spectrum $\mathbb{S}$
Underlying set	Underlying anima $X_0 = \Omega^\infty(X)$
Free abelian group $\mathbb{Z}[S]$	Unreduced suspension spectrum $\mathbb{S}[X]$.

Recall that a *presentation* of an abelian group A is a way to write A as a cokernel of a map of free abelian groups:

$$A = \operatorname{coker}(\mathbb{Z}[T] \xrightarrow{f} \mathbb{Z}[S]).$$

There is always a *standard presentation* where we take S to be the underlying set of A, and take T the underlying set of $\mathbb{Z}[A]$. We will now discuss an analogous story for spectra.

Definition 3.3.13 (Prespectrum)**.** A *prespectrum* is a collection $(X_n)_{n\in\mathbb{N}}$ of pointed animae together with *non-invertible* maps $\sigma_n \colon X_n \to \Omega(X_{n+1})$. A morphism $f \colon X \to Y$ of prespectra is a collection of pointed maps $f_n \colon X_n \to Y_n$ together with homotopies making the following squares commute:

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\sigma_n^X"'] & Y_n \arrow[d, "\sigma_n^Y"] \\ \Omega(X_{n+1}) \arrow[r, "\Omega(f_{n+1})"] & \Omega(Y_{n+1}). \end{tikzcd}$$

We want to think of a prespectrum as a *presentation* of a spectrum. Note that every spectrum is canonically a prespectrum. Conversely we can turn every prespectrum into a spectrum as follows:

Construction 3.3.14. Given a prespectrum X, we define its *associated spectrum* X^{sp} as the following colimit in Sp:

$$X^{\mathrm{sp}} := \operatorname{colim}(\Sigma^\infty X_0 \to (\Sigma^\infty X_1)[-1] \to (\Sigma^\infty X_2)[-2] \to (\Sigma^\infty X_3)[-3] \to \dots),$$

where the comparison maps in the colimit are given levelwise by

$$(\Sigma^\infty X_n)[-n]_k = (\Sigma^\infty X_n)_{k-n} = Q\Sigma^{k-n} X_n \xrightarrow{\sigma_n} Q\Sigma^{k-n}\Omega X_{n+1} \xrightarrow{\varepsilon} Q\Sigma^{k-(n+1)} X_{n+1} = (\Sigma^\infty X_{n+1})[-(n+1)]_k,$$

where ε is the counit of the suspension-loop adjunction on An_*.

Remark 3.3.15. One may define an ∞-category PSp of prespectra, and one can show that the above construction defines a left adjoint

$$(-)^{\mathrm{sp}}\colon \mathrm{PSp} \to \mathrm{Sp}$$

to the inclusion $\mathrm{Sp} \hookrightarrow \mathrm{PSp}$.

Proposition 3.3.16 (Standard presentation)**.** *Every spectrum X is naturally isomorphic to the spectrum associated to its underlying prespectrum:*

$$\mathrm{colim}_n((\Sigma^\infty X_n)[-n]) \xrightarrow{\cong} X.$$

Proof. By the Yoneda lemma, it suffices to show that the two functors $\mathrm{Sp}^{\mathrm{op}} \to \mathrm{An}$ represented by both sides are equivalent. For a spectrum Y, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}}(\mathrm{colim}_n((\Sigma^\infty X_n)[-n]),Y) &\simeq \lim_n \mathrm{Hom}_{\mathrm{Sp}}((\Sigma^\infty X_n)[-n],Y) \simeq \lim_n \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X_n, Y[n]) \\ &\simeq \lim_n \mathrm{Hom}_{\mathrm{An}_*}(X_n, \Omega^\infty Y[n]) = \lim_n \mathrm{Hom}_{\mathrm{An}_*}(X_n, Y_n) \simeq \mathrm{Hom}_{\mathrm{Sp}}(X,Y).\end{aligned}$$

All of this is natural in Y, finishing the proof. □

Corollary 3.3.17. *The ∞-category* Sp *admits all small colimits: given a diagram $X\colon I \to$* Sp*, their colimit is given by*

$$\mathrm{colim}_{i\in I} X(i) = ((\mathrm{colim}_{i\in I} X(i)_n)_{n\in\mathbb{N}})^{\mathrm{sp}},$$

where the structure maps of the prespectrum are the maps

$$\mathrm{colim}_{i\in I} X(i)_n \xrightarrow[\cong]{\mathrm{colim}_i(\sigma_n)} \mathrm{colim}_{i\in I} \Omega X(i)_{n+1} \to \Omega(\mathrm{colim}_{i\in I} X(i)_{n+1}).$$ □

3.4 Tensor products of spectra

Using the standard presentation of spectra, we can now define the *tensor product* of spectra. This is supposed to be a functor

$$-\otimes-\colon \mathrm{Sp}\times\mathrm{Sp} \to \mathrm{Sp}$$

which preserves colimits in both variables. By resolving every spectrum as a colimit of suspension spectra via their standard presentations, it thus suffices to define the tensor product of suspension spectra, where we may simply take

$$\Sigma^\infty(X)\otimes\Sigma^\infty(Y) := \Sigma^\infty(X\wedge Y).$$

Here the smash product $X \wedge Y$ of two pointed animae is defind via the following pushout square:

$$\begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \wedge Y. \end{tikzcd}$$

All in all, we see that the following definition is forced upon us:

Definition 3.4.1. The *tensor product* $X \otimes Y$ of two spectra X and Y is defined as the following colimit in Sp:

$$X \otimes Y := \operatorname{colim}_{n\in\mathbb{N}} \operatorname{colim}_{m\in\mathbb{N}} \left(\Sigma^\infty(X_n \wedge X_m)[-(n+m)]\right).$$

This defines a functor

$$-\otimes -\colon \mathrm{Sp}\times\mathrm{Sp} \to \mathrm{Sp}.$$

We will now prove that $-\otimes Y\colon \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits for every spectrum Y by producing an explit right adjoint:

Construction 3.4.2. Let Y and Z be spectra. We define the *mapping spectrum* $\operatorname{map}(Y,Z) \in \mathrm{Sp}$ as

$$\operatorname{map}(Y,Z)_n := \operatorname{Hom}_{\mathrm{Sp}}(Y, Z[n]) \simeq \lim_k \operatorname{Hom}_{\mathrm{An}_*}(Y_k, Z_{k+n}),$$

where the structure maps are given by

$$\operatorname{Hom}_{\mathrm{Sp}}(Y, Z[n]) \cong \operatorname{Hom}_{\mathrm{Sp}}(Y, \Omega(Z[n+1])) \cong \Omega \operatorname{Hom}_{\mathrm{Sp}}(Y, Z[n+1]).$$

This defines a functor

$$\operatorname{map}\colon \mathrm{Sp}^{\mathrm{op}}\times\mathrm{Sp} \to \mathrm{Sp}.$$

Proposition 3.4.3. *For every spectrum Y, the functor* $\operatorname{map}(Y,-)\colon \mathrm{Sp}\to\mathrm{Sp}$ *is right adjoint to* $-\otimes Y\colon \mathrm{Sp}\to\mathrm{Sp}$*:*

$$\operatorname{Hom}_{\mathrm{Sp}}(X\otimes Y, Z) \cong \operatorname{Hom}_{\mathrm{Sp}}(X, \operatorname{map}(Y,Z)).$$

Proof. Plugging in the definition of $X\otimes Y$ and unwinding everything, we get

$$\begin{aligned} \operatorname{Hom}_{\mathrm{Sp}}(X\otimes Y, Z) &= \operatorname{Hom}_{\mathrm{Sp}}(\operatorname{colim}_{n,m} \Sigma^\infty(X_n \wedge Y_m)[-(n+m)], Z) \\ &\cong \lim_{n,m} \operatorname{Hom}_{\mathrm{Sp}}(\Sigma^\infty(X_n\wedge Y_m), Z[n+m]) \\ &\cong \lim_{n,m} \operatorname{Hom}_{\mathrm{An}_*}(X_n \wedge Y_m, Z_{n+m}) \\ &\cong \lim_n \lim_m \operatorname{Hom}_{\mathrm{An}_*}(X_n, \operatorname{Hom}_{\mathrm{An}_*}(Y_m, Z_{n+m})) \\ &\cong \lim_n \operatorname{Hom}_{\mathrm{An}_*}(X_n, \lim_m \operatorname{Hom}_{\mathrm{An}_*}(Y_m, Z_{n+m})) \\ &\cong \lim_n \operatorname{Hom}_{\mathrm{An}_*}(X_n, \operatorname{map}(Y,Z)_n) \\ &\cong \operatorname{Hom}_{\mathrm{Sp}}(X, \operatorname{map}(Y,Z)), \end{aligned}$$

finishing the proof. $\square$

The tensor product and mapping spectrum constructions give convenient ways to define the *homology* and *cohomology* of an anima:

Definition 3.4.4 (Homotopy groups)**.** For a spectrum X, we define its *k-th homotopy group* $\pi_k(X) \in \mathrm{Ab}$ as

$$\pi_k(X) := [\mathbb{S}[k], X] := \pi_0 \operatorname{Hom}_{\mathrm{Sp}}(\mathbb{S}[k], X).$$

If E is a spectrum and X is an anima, we define the *E-homology of X* as the homotopy groups of $\Sigma^\infty(X) \otimes E$:

$$E_k(X) := \pi_k(\Sigma^\infty(X) \otimes E).$$

Dually we define the *E-cohomology groups of X* as the homotopy groups of the mapping spectrum $\mathrm{map}(\Sigma^\infty(X), E)$:[4]

$$E^k(X) := \pi_{-k}(\mathrm{map}(\Sigma^\infty(X), E)).$$

Exercise 3.4.5. Show that this definition of $E^k(X)$ agrees with the one from Construction 1.3.7.

[4]The minus sign is mostly for historical reasons: this expression shows that it would be more natural to always use homological grading and simply put cohomology in negative degrees.

4 Algebraic structures in homotopy theory

As discussed in the introduction of these notes, the fundamental principle of homotopy theory implies that homotopical analogues of familiar algebraic structures often come with infinite hierarchies of coherence conditions: each time we demand a certain relation to hold, this relation will be exhibited by some homotopy, and then these homotopies themselves need to satisfy even higher coherences, ad infinitum. For this reason, we cannot generally define algebraic structures in homotopy theory by writing down a list of relations that need to be satisfied; we need to find cleverer ways to encode all this data. In this chapter we will discuss some example of such algebraic structures: monoids, commutative monoids and commutative algebras.

4.1 Monoids and groups

We start with the notion of a monoid.

Definition 4.1.1 (Monoid)**.** Let C be an ∞-category that admits finite products. We define a *monoid in C*, also known as an *E_1-monoid*, to be a simplicial object

$$M\colon \mathbf{\Delta}^{\mathrm{op}} \to C, \quad [n] \mapsto M_n := M([n]),$$

satisfying the *Segal condition*: for every $[n] \in \mathbf{\Delta}$ the map

$$(e_i^*)_{i=1}^n \colon M_n \to \prod_{i=1}^n M_1$$

induced by the n inclusion maps $e_i\colon [1] \cong \{i-1 \leq i\} \hookrightarrow [n]$ is an isomorphism. In particular, taking $n = 0$ we see that M_0 is a terminal object of C.
We write

$$\mathrm{Mon}(C) \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$$

for the full subcategory spanned by the monoids in C. We refer to M_1 as the *underlying object* of M.

Note that the unique map $s_0\colon [1] \to [0]$ provides a map

$$e := s_0^*\colon * = M_0 \to M_1$$

which we will call the *unit* of M. Similarly, the inclusion $d_1\colon [1] \cong \{0 \le 2\} \hookrightarrow [2]$ induces a map

$$m := d_1^*\colon M_1 \times M_1 \simeq M_2 \to M_1$$

which we call the *multiplication* of M. The unitality and associativity are encoded by the following commutative diagrams in $\mathbf{\Delta}$:

$$\begin{tikzcd} & {[1]} \arrow[dl, equal] \arrow[d, "(0,2)"] \arrow[dr, equal] & \\ {[1]} & {[2]} \arrow[l, "(0,0,1)"] \arrow[r, "(0,1,1)"'] & {[1]} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} {[1]} \arrow[r, "(0,2)"] \arrow[d, "(0,2)"'] & {[2]} \arrow[d, "(0,2,3)"] \\ {[2]} \arrow[r, "(0,1,3)"] & {[3]}. \end{tikzcd}$$

Similarly, all the expected 'higher coherences' relating the various ways of multiplying four, five, or more elements in M will all follow from suitable higher dimensional diagrams in the category $\mathbf{\Delta}$ by composing with $M\colon \mathbf{\Delta}^{\mathrm{op}} \to C$.

Instead of merely encoding the two-fold multiplication $M_1 \times M_1 \to M_1$, the monoid M encodes all n-fold multiplications $\prod_{i=1}^n x_i$ simultaneously, together with all their compatibilities: for a map $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ we should think of the resulting map $\varphi^*\colon M^n \to M^m$ as follows:

$$\varphi^*(x_1,\ldots,x_n) := (y_1,\ldots,y_m), \qquad y_i := \prod_{\varphi(i-1)<j\le\varphi(i)} m_j.$$

Exercise 4.1.2. Let C be an ordinary category. Show that the category $\mathrm{Mon}(C)$ is equivalent to the usual category of monoids in C.

Definition 4.1.3. We say that a monoid $M \in \mathrm{Mon}(C)$ is *grouplike*, or a *group in C*, if the so-called 'shear map'

$$(\mathrm{pr}_1, m)\colon M \times M \to M \times M, \qquad (x,y) \mapsto (x,xy)$$

is an isomorphism in C. We denote by

$$\mathrm{Grp}(C) \subseteq \mathrm{Mon}(C)$$

the full subcategory spanned by the groups in C.

Recall from Exercise 7.7 that for an ∞-category C with finite limits the loop object ΩX of a pointed object X admits a canonical structure of a group object in the homotopy category $\mathrm{Ho}(C)$. We will now show that this can be enhanced to a group object in C itself:

Construction 4.1.4 (Group structure on loop space)**.** Let C be an ∞-category with finite limits, and consider a pointed object X with base point $x\colon * \to X$. For an object $[n] \in \mathbf{\Delta}$, denote by I_n the following category:

$$I_n \qquad := \qquad \begin{tikzcd} 0 \arrow[ddrr] & & \\ 1 \arrow[drr] & & \\ \vdots & & \infty \\ n-1 \arrow[urr] & & \\ n \arrow[uurr] & & \end{tikzcd}$$

In other words, I_n consists of $n+1$ non-identity morphisms whose targets agree. We may consider for every n the functor $\tilde{X}_n\colon I_n \to C$ which sends each of the $n+1$ non-identity morphisms of I_n to the map $x\colon * \to X$, and we define

$$(\overline{\Omega}X)_n := \lim(\tilde{X}_n\colon I_n \to C) \quad \in \quad C.$$

Observe that the categories I_n are covariantly functorial in $[n] \in \mathbf{\Delta}$: for a morphism $\varphi\colon [n] \to [m]$ in $\mathbf{\Delta}$ we obtain a functor

$$I_\varphi\colon I_n \to I_m, \qquad I_\varphi(i) := \varphi(i), \qquad I_\varphi(\infty) = \infty.$$

It follows that $(\overline{\Omega}X)_n$ is contravariantly functorial in $[n] \in \mathbf{\Delta}$, and hence defines a functor $\overline{\Omega}X\colon \mathbf{\Delta}^{\mathrm{op}} \to C$. All in all, we obtain

$$\overline{\Omega}\colon C_* \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C).$$

Note that we have $(\overline{\Omega}_X)_1 \simeq * \times_X * = \Omega$, so the functor $\overline{\Omega}$ lifts the loop functor Ω.

Lemma 4.1.5. *For every pointed object X in C the simplicial object $\overline{\Omega}X$ is a group object in C, giving rise to a functor*

$$\overline{\Omega}\colon C_* \to \mathrm{Grp}(C)$$

which lifts the loop functor $\Omega\colon C_* \to C_*$.

Proof. To show that $\overline{\Omega}X$ is a monoid, we will show by induction that for every n the map

$$(e_i^*)_{i=1}^n\colon (\overline{\Omega}X)_n \to \prod_{i=1}^n \Omega X$$

is an isomorphism in C. For $n = 0$, $(\overline{\Omega}X)_0$ is the limit of the diagram $* \to X$, but since the category $[1]$ has 0 as an initial object this limit is simply given by $*$, see Lemma 3.1.4. For

$n \geq 1$, note that we may write I_{n+1} as a pushout of I_n and I_1 along I_0:

$$\begin{tikzcd} I_0 \arrow[r] \arrow[d] & I_n \arrow[d] \\ I_1 \arrow[r, "I_{e_{n+1}}"'] & I_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By applying Theorem 2.8.20(2) with $J = \ulcorner$, we thus obtain a pullback square of the form

$$\begin{tikzcd} \lim(\overline{X}_{n+1} \colon I_{n+1} \to C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim(\overline{X}_n \colon I_n \to C) \arrow[d] \\ \lim(\overline{X}_1 \colon I_1 \to C) \arrow[r] & \lim(\overline{X}_0 \colon I_0 \to C). \end{tikzcd}$$

By induction, this pullback square is isomorphic to the commutative square

$$\begin{tikzcd} (\overline{\Omega}X)_{n+1} \arrow[r, "(e_i^*)_{i=1^n}"] \arrow[d, "e_{n+1}^*"'] & \prod_{i=1}^n \Omega X \arrow[d] \\ \Omega X \arrow[r] & *, \end{tikzcd}$$

and so this being a pullback square means that the map $(\overline{\Omega}X)_{n+1} \xrightarrow{(e_i^*)_{i=1^{n+1}}} \prod_{i=1}^{n+1} \Omega X$ is an isomorphism, finishing the induction step. This shows that $\overline{\Omega}X$ is a monoid in C.

To see that it is even a group, we must show that the shear map $(\mathrm{id}, m) \colon \overline{\Omega}X \times \overline{\Omega}X \to \overline{\Omega}X \times \overline{\Omega}X$ is an isomorphism in C, or equivalently that it is an isomorphism in the homotopy category $\mathrm{Ho}(C)$. For this we observe that the monoid in $\mathrm{Ho}(C)$ induced by the monoid $\overline{\Omega}X \in \mathrm{Mon}(C)$ is precisely the monoid structure constructed on ΩX in Exercise 7.7 (see the solution sheet on GRIPS), and it was shown there that ΩX is even a group object. This finishes the proof. □

Remark 4.1.6. In practice, the functor $\overline{\Omega} \colon C_* \to \mathrm{Grp}(C)$ is usually simply denoted again by Ω. We will temporarily use the unusual notation $\overline{\Omega}$ to distinguish the group object $\overline{\Omega}(X)$ from its underlying pointed object ΩX.

The loop functor admits a left adjoint:

Construction 4.1.7 (Classifying space of a monoid)**.** Let C be an ∞-category admitting geometric realizations (i.e. $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits). We define a functor

$$B \colon \mathrm{Mon}(C) \to C_*$$

by setting

$$BM := |M| := \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} M_n,$$

which comes equipped with the canonical basepoint $* = M_0 \to \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} M_n$. We refer to BM as the *classifying space of M*.

Proposition 4.1.8. *Assume that C admits both finite limits and geometric realizations. Then the functors B and $\overline{\Omega}$ define an adjunction*

$$B\colon \operatorname{Mon}(C) \rightleftarrows C_* : \overline{\Omega}.$$

Proof. Let $\mathbf{\Delta}_+$ denote the *augmented simplex category*, which in addition to the objects $[n]$ for $n \geq 0$ also has an object $[-1] = \emptyset$ that has a unique map to each $[n]$. There are fully faithful inclusions

$$i\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_+ \qquad \text{and} \qquad j\colon [1] \hookrightarrow \mathbf{\Delta}_+, \quad j(0) = [-1], \quad j(1) = [0].$$

A functor $(\mathbf{\Delta}_+)^{\mathrm{op}} \to C$ is precisely the same data as functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ equipped with a cocone $X \to \mathrm{const}_W$. We may now consider the following two adjunctions:

$$\begin{tikzcd} \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \arrow[r, shift left=1.5ex, "i_!"] \arrow[r, phantom, "\perp"] & \operatorname{Fun}((\mathbf{\Delta}_+)^{\mathrm{op}}, C) \arrow[l, shift left=1.5ex, "i^*"] \arrow[r, shift left=1.5ex, "j^*"] \arrow[r, phantom, "\perp"] & \operatorname{Fun}([1]^{\mathrm{op}}, C). \arrow[l, shift left=1.5ex, "j_*"] \end{tikzcd}$$

The functors $i_!$ and j_* are called *left Kan extension along i* and *right Kan extension along j*; I have written down some additional explanations on Kan extensions in Section 2.8.4. Unwinding the pointwise formulas for Kan extensions explained in that section, one observes that:

- The composite $j^* i_!$ sends a simplicial object X in C to the map $X_0 \to |X| := \operatorname{colim} X$;
- The composite $i^* j_*$ sends a morphism $f\colon Y \to X$ in C to its so-called *Čech nerve*:

  $$\check{C}(f)\colon \mathbf{\Delta}^{\mathrm{op}} \to C, \qquad \check{C}(f)_n := Y^{\times_X^n}.$$

  (Observe that the categories I_n used above are precisely the relative slice categories for the inclusion $[1]^{\mathrm{op}} \hookrightarrow {\mathbf{\Delta}_+}^{\mathrm{op}}$, and the pointwise limits computing the right Kan extension j_* are given by I_n-indexed limits, that just as in the proof of Lemma 4.1.5 can be computed to be an n-fold fiber product of Y with itself over X.)

So we see that the functors B and $\overline{\Omega}$ are precisely the restrictions of these functors to the subcategories

$$\operatorname{Mon}(C) \subseteq \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \qquad \text{and} \qquad C_* \subseteq \operatorname{Fun}([1], C),$$

and the adjunction $j^* i_! \dashv i^* j_*$ accordingly restricts to an adjunction $B \dashv \overline{\Omega}$. □

4.2 The recognition principle for loop spaces

The goal of this section is to prove the following theorem:

Theorem 4.2.1 (Recognition principle for loop spaces, May [May72], [BV73], Lurie [Lur09, Theorem 7.2.2.11])**.** *Let M be a monoid in* An *and let X be a pointed anima.*

(1) The unit map $M \to \Omega BM$ is an equivalence if and only if M is a group in An*;*

(2) The pointed anima BM is connected;

(3) The counit $B\Omega X \to X$ is an inclusion of the connected component of the basepoint of X.

The original formulation by May was in terms of a notion of 'operads': it says that if a topological space M comes equipped with certain algebraic structure making it a 'grouplike E_1-space', then it is equivalent to ΩBM. Our formulation above is a refined version of this theorem in the setting of ∞-categories, which has the additional advantage that it immediately provides us with an equivalence of ∞-categories between groups in An and connected pointed animae:

Corollary 4.2.2. *The adjunction $B \dashv \Omega$ restricts to an equivalence of ∞-categories*

$$B\colon \mathrm{Grp}(\mathrm{An}) \underset{}{\overset{\simeq}{\rightleftarrows}} \mathrm{An}_*^{\geq 0} :\Omega$$

between groups in An *and connected pointed animae.*

Proof. The adjunction from Proposition 4.1.8 restricts to groups and pointed connected animae since $\Omega(X)$ is always a group and BM is always connected. The unit $M \to \Omega BM$ is always an isomorphism by part (1) of the theorem, while the counit $B\Omega X \to X$ is always an isomorphism by part (3) of the theorem. This shows that the two functors are in fact inverse to one another. □

For the proof of the theorem we will need some preliminaries.

Extra degeneracies

Definition 4.2.3. We define the subcategory $\mathbf{\Delta}^{\mathrm{deg}} \subseteq \mathbf{\Delta}$ as the subcategory consisting of *all* objects $[n]$, but with only those morphisms $\varphi\colon [n] \to [m]$ satisfying $\varphi(n) = m$. Note that there exists a functor $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ given on objects by $[n] \mapsto [n+1]$, and on morphisms by sending $\varphi\colon [n] \to [m]$ to the map

$$[n+1] \to [m+1], \qquad i \mapsto \begin{cases} \varphi(i) & 0 \leq i \leq n \\ m+1 & i = n+1. \end{cases}$$

We wish to think of $\mathbf{\Delta}^{\mathrm{deg}}$ as an enlargement of the augmented simplex category $\mathbf{\Delta}_+$ in which there exist 'extra degeneracies'. Indeed, note that all the face maps that are allowed in $\mathbf{\Delta}^{\mathrm{deg}}$ are actually already in the image of the functor $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$, but the degeneracy map $s_n\colon [n+1] \to [n]$ is an example of a morphism in $\mathbf{\Delta}^{\mathrm{deg}}$ which does not lie in this image. This means that we may think of functors $Y\colon (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \to C$ as *augmented simplicial objects with extra degeneracies*. We may draw them as follows, with the extra degeneracies marked with dashed arrows:

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \dashleftarrow} Y_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \dashleftarrow} Y_1 \;\substack{\longrightarrow\\ \dashleftarrow} Y_0.$$

The non-dashed arrows define its *underlying augmented simplicial object*, obtained by precomposition with $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$. It is useful in practice to have extra degeneracies, since it leads to the following easy computation of the geometric realization:

Lemma 4.2.4 (Extra degeneracy argument)**.** *Consider an augmented simplicial object $X\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to C$, and assume that it admits 'extra degeneracies', in the sense that it is of the form $X_n = Y_{n+1}$ for some functor $Y\colon (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \to C$, i.e. X is given by the following composite:*

$${\mathbf{\Delta}_+}^{\mathrm{op}} \xrightarrow{(+1)} (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \xrightarrow{Y} C.$$

Then X is a colimit diagram, in the sense that the corresponding cocone $X|_{\mathbf{\Delta}^{\mathrm{op}}} \to \mathrm{const}_{X_{-1}}$ is a colimit cocone:

$$\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n \xrightarrow{\sim} X_{-1}.$$

Proof. We need to show that the preferred map from $\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n$ to X_{-1} induced by the cocone is an isomorphism.

To this end, observe that the inclusion functor $(+1)\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ admits a right adjoint given by the inclusion $\mathbf{\Delta}^{\mathrm{deg}} \hookrightarrow \mathbf{\Delta}$: for natural numbers $n, m \geq 0$, providing a map $\varphi\colon [n] \to [m]$ of partially ordered sets is the same as providing a map $\overline{\varphi}\colon [n+1] \to [m]$ of posets satisfying the condition that $\overline{\varphi}(n+1) = m$. It follows from Corollary 2.8.35 that the functor $(+1)\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ is initial, and hence its dual $(+1)^{\mathrm{op}}\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}$ is final. By Theorem 2.8.30, this means that the geometric realization of X may be computed as

$$\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n = \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} Y_{n+1} \simeq \operatorname{colim}_{[m]\in(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}} Y_m.$$

But since $\mathbf{\Delta}^{\mathrm{deg}}$ has an initial object given by $[0]$, $(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}$ has a terminal object, and so it follows from Lemma 3.1.4 that the colimit $\operatorname{colim}_{[m]\in(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}} Y_m$ is equivalent to $Y_0 = X_{-1}$ as desired. □

Descent

The ∞-category An has a special feature that distinguishes it from other ∞-categories: it satisfies *descent*.

Definition 4.2.5 (Cocone ∞-categories)**.** For an ∞-category I, we define the ∞-category $I^{\triangleright}$, called the *cocone of* I, via the following pushout square of ∞-categories:

$$\begin{tikzcd} I\times\{1\} \arrow[r, hook] \arrow[d] & I\times[1] \arrow[d] \\ * \arrow[r] & I^{\triangleright}. \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

We will often denote the object classified by the bottom morphism as ∞. Note that a functor $\overline{F}\colon I^{\triangleright} \to C$ consists of a functor $I\times[1] \to C$ whose restriction to $I\times\{1\}$ is constant. Equivalently, it consists of a functor $F := \overline{F}|_{I\times\{0\}}\colon I \to C$ and an object $W := \overline{F}(\infty)$ in C together with a cocone $F \to \mathrm{const}_W$.

We say that a functor $\overline{F}\colon I^{\triangleright} \to C$ is a *colimit diagram* if the associated cocone $F \to \mathrm{const}_W$ is a colimit cocone.

We dually define $I^{\triangleleft}$, called the *cone of* I, as the following pushout of ∞-categories:

$$\begin{tikzcd} I\times\{0\} \arrow[r, hook] \arrow[d] & I\times[1] \arrow[d] \\ * \arrow[r] & I^{\triangleleft}. \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

A functor $\overline{F}\colon I^{\triangleleft} \to C$ consists of a functor $F\colon I \to C$ together with a cone $\mathrm{const}_W \to F$. We say that $\overline{F}$ is a *limit diagram* if this cone is a limit cone.

Theorem 4.2.6 (Descent for colimits)**.** *Let I be a small ∞-category, let $\overline{F},\overline{G}\colon I^{\triangleright} \to \mathrm{An}$ be functors and let $\overline{\alpha}\colon \overline{F} \Rightarrow \overline{G}$ be a natural transformation such that the restriction $\alpha := \overline{\alpha}|_I\colon F \Rightarrow G$ is a cartesian transformation, in the sense that for every morphism $i \to j$ in I the commutative square*

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d, "\alpha(i)"'] & F(j) \arrow[d, "\alpha(j)"] \\ G(i) \arrow[r] & G(j) \end{tikzcd} \tag{4.1}$$

is a pullback square. Assume that $\overline{G}$ is a colimit diagram. Then $\overline{F}$ is a colimit diagram is and only if $\overline{\alpha}$ is a cartesian transformation, i.e. also all the squares

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d, "\alpha(i)"'] & \overline{F}(\infty) \arrow[d, "\overline{\alpha}(\infty)"] \\ G(i) \arrow[r] & \overline{G}(\infty) = \mathrm{colim}_I\, G. \end{tikzcd}$$

are pullback squares.

Proof. This is [Lur09, Theorem 6.1.3.9]. □

Let us unwind what the above result is saying:

- First assume that both $\overline{F}$ and $\overline{G}$ are colimit diagrams, i.e. $\overline{G}(\infty) = \operatorname{colim}_I G$ and $\overline{F}(\infty) = \operatorname{colim}_I F$. If we assume that each of the squares (4.1) is a pullback square, then we may recover the whole diagram $F(i)$ from just the map $\operatorname{colim}_i F(i) \to \operatorname{colim}_i G(i)$ on colimits induced by α.
- Conversely, if we only know that $\overline{G}$ is a colimit diagram, but we assume that $\overline{\alpha}$ is a cartesian transformation, then we may deduce that the value $\overline{F}$ must be the colimit of $F\colon I \to \mathrm{An}$.

We do not yet have enough theory to do this, but assume for a moment that we had a functor of the form

$$\mathrm{An}_{/-}\colon \mathrm{An}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \qquad X \mapsto \mathrm{An}_{/X}, \qquad (f\colon X \to Y) \mapsto (f^*\colon \mathrm{An}_{/Y} \to \mathrm{An}_{/X}).$$

Then the above result will say that this functor preserves small limits: for every functor $G\colon I \to \mathrm{An}$ we have an equivalence of ∞-categories

$$\mathrm{An}_{/\operatorname{colim}_{i\in I} G(i)} \xrightarrow{\sim} \lim_{i\in I^{\mathrm{op}}} \mathrm{An}_{/G(i)}.$$

In other words: the data of an anima X equipped with a map $X \to \operatorname{colim}_{i\in i} G(i)$ is the same as a compatible family of animae X_i equipped with maps $X_i \to G(i)$.

Example 4.2.7 (Universality of initial anima)**.** Consider $I = \emptyset$, so that $I^{\triangleright} = *$. In this case $\overline{\alpha}$ is just a morphism $X \to Y$ of animae, and the condition that $\overline{G}$ is a colimit diagram means that $Y = \emptyset$. Since $I^{\triangleright}$ has no morphisms, $\overline{\alpha}$ is always cartesian, hence by the theorem also $\overline{F}$ is a colimit diagram, i.e. we also have $X = \emptyset$. So descent for colimits in this case precisely boils down to the universality of the empty category from Axiom B.2'.

Example 4.2.8 (Universality of coproducts)**.** Consider $I = * \sqcup *$, so that $I^{\triangleright} = \lrcorner$. The transformation $\overline{\alpha}$ will thus be a diagram of the form

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "\alpha_0"] & X_2 \arrow[d, "\alpha_2"] & X_1 \arrow[l] \arrow[d, "\alpha_1"] \\
Y_0 \arrow[r] & Y_0 \sqcup Y_1 & Y_1, \arrow[l]
\end{tikzcd}$$

where in the bottom we get a coproduct, reflecting the condition that $\overline{G}$ is a colimit diagram. The theorem then expresses that the following two conditions are equivalent:

- The two squares in the diagram are both pullback squares;
- The functor $X_0 \sqcup X_1 \to X_2$ is an isomorphism.

But the fact that these two conditions are equivalent is precisely an instance of the universality of coproducts from Axiom B.4', with the two parts of the axiom corresponding to the two implications.

Whitehead's theorem

One final ingredient we will need is a version of Whitehead's Theorem for An.

Definition 4.2.9. For a pointed anima (X,x) and a natural number $n \geq 1$, we define the *n-th homotopy group of* X as

$$\pi_n(X,x) := [S^n, X]_* := \pi_0 \operatorname{Hom}_{\mathrm{An}_*}(S^n, X) = \operatorname{Hom}_{\mathrm{Ho}(\mathrm{An}_*)}(S^n, X).$$

Since $S^n = \Sigma^n(S^0)$, this may also be written as the set of path components of the n-fold loop space of X:

$$\pi_n(X,x) \cong \pi_0(\Omega^n X).$$

Since $\Omega^n X$ is a group object in C, $\pi_n(X,x)$ is a group, and is abelian for $n \geq 2$.

To prove Whitehead's theorem for animae, we will need to lift any morphism of animae to a morphism of CW-complexes. This will be achieved via the following proposition:

Proposition 4.2.10 (Cisinski [Cis19]). *For CW-complexes* $\overline{X}$ *and* $\overline{Y}$ *with underlying animae* $X = \Pi_\infty(\overline{X})$ *and* $Y = \Pi_\infty(\overline{Y})$*, there is an equivalence*

$$\operatorname{Hom}_{\mathrm{An}}(X,Y) \simeq \operatorname{colim}_{\overline{Y} \overset{\simeq}{\hookrightarrow} \overline{Z}} \operatorname{Hom}_{\mathrm{CW}}(\overline{X}, \overline{Z}),$$

where the colimit runs over the subcategory of $\mathrm{CW}_{\overline{Y}/}$ *spanned by the trivial cofibrations* $\overline{Y} \overset{\simeq}{\hookrightarrow} \overline{Z}$*, i.e. those cofibrations that are homotopy equivalences.*

Proof. This is the dual of [Cis19, Theorem 7.2.8], applied to left calculus of fractions coming from the class of trivial cofibrations via [Cis19, Corollary 7.2.18]. □

Corollary 4.2.11. *The localization functor* $\Pi_\infty \colon \mathrm{CW} \to \mathrm{An}$ *is essentially surjective on objects and on morphisms:*

- *Every object in* An *is isomorphic to* $\Pi_\infty(X)$ *for some* $X \in \mathrm{CW}$*;*
- *Every morphism in* An *is isomorphic to* $\Pi_\infty(f)$ *for some morphism* $f \colon X \to Y$ *of CW-complexes.*

Proof. For essential surjectivity, note that the essential image Π_∞ again satisfies the universal property of the localization, hence must be equivalent to An. For essential surjectivity on morphisms, consider a morphism $f \colon X \to Y$ of animae. We may write X and Y as the underlying animae of two CW-complexes $\overline{X}$ and $\overline{Y}$. By Proposition 4.2.10 there exists a trivial cofibration $\overline{Y} \hookrightarrow \overline{Z}$ and a continuous map $g \colon \overline{X} \to \overline{Z}$ that induces the map f after applying $\Pi_\infty(-)$. This finishes the proof. □

Proposition 4.2.12 (Cisinski [Cis19])**.** *The localization functor* $\Pi_\infty\colon \mathrm{CW} \to \mathrm{An}$ *sends homotopy pushouts of CW-complexes to pushouts of animae.*

Proof. This is (dual of) [Cis19, Theorem 7.2.25] applied to the class of trivial cofibrations of CW-complexes. □

Proposition 4.2.13 (Whitehead's theorem for An)**.** *A morphism of animae* $f\colon X \to Y$ *is an isomorphism if and only if it induces a bijection* $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ *on sets of path components, and if for every point* x *of* X *the map* $\pi_n(X,x) \to \pi_n(Y, f(x))$ *is an isomorphism of groups.*

Proof. By Corollary 4.2.11, we may write f as the map on animae induced by a continuous map $\overline{f}\colon \overline{X} \to \overline{Y}$ of CW-complexes. Note that the homotopy groups of $\overline{X}$ are the same as the homotopy groups of its underlying anima X: we have

$$\pi_n(\overline{X}) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{CW}_*)}(S^n, \overline{X}) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{An}_*)}(S^n, X) = \pi_n(X),$$

due to the equivalence $\mathrm{Ho}(\mathrm{CW}_*) \simeq \mathrm{Ho}(\mathrm{An}_*)$ from Corollary 2.8.14. From the assumptions on f, it thus follows that the map $\overline{f}\colon \overline{X} \to \overline{Y}$ induces isomorphisms on all homotopy groups (and a bijection between sets of path components), and hence by the ordinary Whitehead theorem, Theorem 1.2.2, is a homotopy equivalence of CW-complexes. But since the localization functor $\Pi_\infty(-)\colon \mathrm{CW} \to \mathrm{An}$ inverts homotopy equivalences, this means that $f = \Pi_\infty(\overline{f})$ is an isomorphism of animae, as desired. □

Corollary 4.2.14. *Let* $f\colon X \to Y$ *be a morphism of connected pointed animae. Then* f *is an isomorphism if and only if the induced map* $\Omega f\colon \Omega X \to \Omega Y$ *is an isomorphism of animae.*

Proof. The 'only if'-direction is clear. For the converse, assume that Ωf is an isomorphism. To show that f is an isomorphism, it suffices by Proposition 4.2.13 to show that the map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection and that each $\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism of groups for $n \geq 0$. The former is clear, since $\pi_0(X)$ and $\pi_0(Y)$ are both assumed to be a point. For the letter we use that $\pi_n(X) \cong \pi_{n-1}(\Omega X)$, and similarly the map $\pi_n(f)$ is isomorphic to the map $\pi_{n-1}(\Omega f)\colon \pi_{n-1}(\Omega X) \to \pi_{n-1}(\Omega Y)$. But this is an isomorphism since Ωf is an isomorphism. □

Proof of the Recognition Principle

We are now ready to prove the recognition principle.

Proof of Theorem 4.2.1. We start with (2): the fact that BM is connected for any monoid object M in An. By definition, BM is connected if and only if its set of path components

$\pi_0(BM)$ is a single point. Since $\pi\colon \mathrm{An} \to \mathrm{Set}$ is a left adjoint, it preserves colimits by Lemma 2.8.18, and thus $\pi_0(BM)$ is a colimit in sets of the following simplicial diagram:

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; \pi_0(M_2) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \pi_0(M_1) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \pi_0(M_0).$$

But $M_0 \cong *$ hence $\pi_0(M_0) = *$ and since the resulting map $* = \pi_0(M_0) \to \operatorname{colim}_{[n]} \pi_0(M_n) \cong \pi_0(BM)$ is surjective it follows that $\pi_0(BM)$ has a single element.

Let us assume (1) for a moment and use it to deduce (3), the fact that the counit $\varepsilon_X\colon B\Omega X \to X$ is the inclusion of the path component X_0 of the basepoint of X. Since $B\Omega X$ is connected, the counit map certainly lands in X_0 and we need to show that the resulting map $B\Omega X \to X_0$ is an isomorphism of animae. Since both sides are connected pointed animae, it suffices by Corollary 4.2.14 to show that the map $\Omega(\varepsilon_X)\colon \Omega(B\Omega X) \to \Omega(X_0) \simeq \Omega(X)$ is an isomorphism. By the triangle identities for the adjunction, there is a commutative triangle of the form

$$\begin{tikzcd} & \Omega(B\Omega X) \arrow[dr, "\Omega(\varepsilon_X)"] & \\ \Omega X \arrow[ur, "\eta_{\Omega X}"] \arrow[rr, equal] & & \Omega X, \end{tikzcd}$$

and so $\Omega(\varepsilon_X)$ is an isomorphism if and only if $\eta_\Omega\colon \Omega X \to \Omega(B\Omega X)$ is an isomorphism. But since ΩX is a group object, this is an instance of part (1).

It remains to prove part (1), i.e. that M is a group if and only if the map $\eta_M\colon M \to \Omega BM$ is an isomorphism of monoids. We will deduce this as an application of Theorem 4.2.6. For this, we need to rephrase both conditions in terms of suitable pullback squares.

First, note that $\eta_M\colon M \to \Omega BM$ is an isomorphism if and only if the map $(\eta_M)_1\colon M_1 \to (\Omega BM)_1$ on underlying animae is an isomorphism: since $M_n \simeq M_1^{\times n}$ and $(\Omega BM)_n \simeq ((\Omega BM)_1)^{\times n}$ we then get that it is an isomorphism at every level. The map $M_1 \to \Omega BM$ is induced by the commutative square

$$\begin{tikzcd} M_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & M_0 \simeq * \arrow[d] \\ M_0 \simeq * \arrow[r] & BM \end{tikzcd}$$

and so is an isomorphism if and only if this is a pullback square of animae.

Second, note that M is a group if and only if the commutative square

$$\begin{tikzcd} M_1^2 \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"'] & M_1 \arrow[d] \\ M_1 \arrow[r] & * \end{tikzcd} \tag{4.2}$$

is a pullback square. Let us rephrase this condition a bit. Consider the functor

$$PM\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}, \qquad [n] \mapsto M_{n+1}$$

and consider the natural transformation $\alpha\colon PM \to M$ whose components are the maps $d_{n+1}\colon M_{n+1} \to M_n$.

Claim: M is a group if and only if α is a cartesian natural transformation, in the sense that for every morphism $\varphi\colon [n] \to [m]$ the square

$$\begin{tikzcd} PM_m \arrow[r, "\varphi^*"] \arrow[d, "\alpha_m"'] & PM_n \arrow[d, "\alpha_n"] \\ M_m \arrow[r, "\varphi^*"] & M_n \end{tikzcd} \tag{4.3}$$

is a pullback square.

Proof. One direction is clear: by taking φ to be the map $d_1\colon [0] \to [1]$, the square (4.3) reduces to (4.2). Conversely, assume (4.2) is a pullback square. We start by showing that (4.3) is a pullback square for $\varphi = d_i\colon [n-1] \to [n]$, where $n \geq 1$ and $0 \leq i \leq n$. In the case where $i = 0$, then under the isomorphisms $M_n \cong M_1^n$ coming from the Segal condition this square takes the form

$$\begin{tikzcd} M_1^{n+2} \arrow[r, "(m_1,\ldots,m_{n+2})\mapsto(m_2,\ldots,m_{n+2})"] \arrow[d, "(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_{n+1})"'] & M_1^{n+1} \arrow[d, "(m_2,\ldots,m_{n+2})\mapsto(m_2,\ldots,m_{n+1})"] \\ M_1^{n+1} \arrow[r, "(m_1,\ldots,m_{n+1})\mapsto(m_2,\ldots,m_{n+1})"'] & M^n, \end{tikzcd}$$

which is always a pullback square. In the case $0 < i < n$, the square (4.3) takes the form

$$\begin{tikzcd} M_1^{n+2} \arrow[r, "(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_i m_{i+1},\ldots,m_{n+2})"] \arrow[d, "(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_{n+1})"'] & M_1^{n+1} \arrow[d, "(x_1,\ldots,x_{n+1})\mapsto(x_1,\ldots,x_n)"] \\ M_1^{n+1} \arrow[r, "(m_1,\ldots,m_{n+1})\mapsto(m_1,\ldots,m_i m_{i+1},\ldots,m_{n+1})"'] & M^n, \end{tikzcd}$$

which is also always a pullback square. Finally, for $i = n$ the square takes the form

$$\begin{tikzcd} M_1^{n+2} \arrow[r, "(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_n,m_{n+1}m_{n+2})"] \arrow[d, "(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_{n+1})"'] & M_1^{n+1} \arrow[d, "(x_1,\ldots,x_{n+1})\mapsto(x_1,\ldots,x_n)"] \\ M_1^{n+1} \arrow[r, "(m_1,\ldots,m_{n+1})\mapsto(m_1,\ldots,m_n)"'] & M^n. \end{tikzcd}$$

Note that none of the maps afffect the first n coordinates, and that for the last two coordinates m_{n+1} and m_{n+2} this is precisely the square (4.2). So this square is a product of a degenerate pullback square and the pullback square from (4.2) and hence is also a pullback square. This shows the claim for all maps φ of the form d_i.

Now, when φ is a degeneracy map of the form $s_i\colon [n+1] \to [n]$, then it will have a section of the form $d_i\colon [n] \to [n+1]$, and it follows from the pasting law of pullback squares that the square (4.3) is a pullback square:

$$\begin{tikzcd}
PM_n \arrow[r, "s_i^*"] \arrow[d, "\alpha_n"'] & PM_{n+1} \arrow[r, "d_i^*"] \arrow[d, "\alpha_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & PM_n \arrow[d, "\alpha_n"] \\
M_n \arrow[r, "s_i^*"] & M_{n+1} \arrow[r, "d_i^*"] & M_n.
\end{tikzcd}$$

This shows the claim when φ is either a face map d_i or a degeneracy map s_i. Since every map $\varphi\colon [n] \to [m]$ is an iterated composite of face maps and degeneracy maps, we then get the claim for general φ by using the pasting law of pullback squares again. □

We may now finish the proof of the theorem. By taking the colimit of $M\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$, we may extend this simplicial object to an augmented simplicial object $\overline{M}\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to \mathrm{An}$ with $\overline{M}_{-1} = BM = \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} M_n$. The simplicial object PM then inherits an extension to an augmented simplicial object $\overline{PM}\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to \mathrm{An}$ via $\overline{PM}_n := M_{n+1}$. Since $\overline{PM}$ has extra degeneracies, it is a colimit diagram by Lemma 4.2.4. The transformation $\alpha\colon PM \to M$ extends to a transformation $\overline{\alpha}\colon \overline{PM} \to \overline{M}$. We are now finally in a position to use descent for colimits from Theorem 4.2.6:

- First assume that M is a group. By the above claim, this means that $\alpha\colon PM \to M$ is a cartesian transformation. Since $\overline{PM}$ and $\overline{M}$ are colimit diagrams, Theorem 4.2.6 then implies that $\overline{\alpha}$ is also a cartesian transformation, hence in particular the square

$$\begin{tikzcd}
PM_0 = M_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & \overline{PM}_{-1} \simeq M_0 \simeq * \arrow[d, "d_0"] \\
M_0 \simeq * \arrow[r, "d_0"] & \overline{M}_{-1} \simeq BM
\end{tikzcd}$$

  is a pullback square, expressing the fact that the map $M_1 \to \Omega BM$ is an isomorphism.

- Conversely, if the previous square is a pullback square, then by the pasting law of pullback squares we get that each outer square

$$\begin{tikzcd}
PM_n = M_{n+1} \arrow[r, "d_0^*"] \arrow[d, "d_{n+1}^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & PM_0 = M_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & \overline{PM}_{-1} \simeq M_0 \simeq * \arrow[d, "d_0"] \\
M_n \arrow[r, "d_0^*"] & M_0 \simeq * \arrow[r, "d_0"] & \overline{M}_{-1} \simeq BM
\end{tikzcd}$$

is a pullback square, since the left-hand square is always a pullback square. But then another application of the pasting law of pullback squares applied to

$$\begin{tikzcd}
PM_m \arrow[r, "\varphi^*"] \arrow[d, "\alpha_m"'] & PM_n \arrow[r, "d_0"] \arrow[d, "\alpha_n"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{PM}_{-1} \arrow[d, "\overline{\alpha}_{-1}"] \\
M_m \arrow[r, "\varphi^*"] & M_n \arrow[r, "d_0"] & \overline{M}_{-1}
\end{tikzcd}$$

shows that each of the squares (4.3) is a pullback square, hence M is a group by the Claim.

This finishes the proof of the Theorem. □

4.3 Commutative monoids and commutative groups

We now move to *commutative* monoids in an ∞-category.

Construction 4.3.1 (Span category)**.** There is an ∞-category Span(Fin), called the *span category of finite sets*. Its objects are finite sets, and for finite sets S and T its Hom animae are given by the groupoids

$$\mathrm{Hom}_{\mathrm{Span(Fin)}}(S,T) \quad := \quad \left\{ \begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\cong"] & \\ S & & T \\ & U' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \right\}$$

of *spans from S to T*; so the objects of this groupoid are spans $S \leftarrow U \rightarrow T$ and the morphisms are isomorphisms of spans. The identity span is given by $S \xleftarrow{\mathrm{id}_S} S \xrightarrow{\mathrm{id}_S} S$, and composition of spans is defined by taking pullbacks:

$$\begin{tikzcd}
 & & U \times_T V \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] & & \\
 & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\
S & & T & & R.
\end{tikzcd}$$

Remark 4.3.2. For every ∞-category with pullbacks C there exists a span ∞-category $\mathrm{Span}(C)$, and the above construction is a special case where $C = \mathrm{Fin}$ is the ordinary category of finite sets. (Note that Span(Fin) is no longer an ordinary category!) It is best to define $\mathrm{Span}(C)$ as a complete Segal animae, but this requires some more theory we haven't discussed so we will not go into this.

Lemma 4.3.3. *The ∞-category* Span(Fin) *is semiadditive, with biproducts given by disjoint unions of finite sets.*

Proof. Consider finite sets S_i for $i = 1, \ldots, n$. We have spans of the form

$$S_i \xleftarrow{\mathrm{id}_{S_i}} S_i \hookrightarrow \bigsqcup_{i=1}^{n} S_i.$$

We claim that these morphisms exhibit $\bigsqcup_{i=1}^n S_i$ as a coproduct of the S_i in Span(Fin). Indeed, note that for every map $U \to \bigsqcup_{i=1}^n S_i$ of finite sets we may write $U = \bigsqcup_{i=1}^n U_i$, where U_i is the preimage of S_i. This gives the desired equivalence of groupoids

$$\mathrm{Hom}_{\mathrm{Span(Fin)}}(T, \bigsqcup_{i=1}^{n} S_i) = \left\{ \begin{tikzcd} & \bigsqcup_{i=1}^n U_i \arrow[dl, "f"'] \arrow[dd, "\cong"] \arrow[dr, "\bigsqcup_{i=1}^n g_i"] & \\ T & & \bigsqcup_{i=1}^n S_i \\ & \bigsqcup_{i=1}^n U_i' \arrow[ul, "f'"] \arrow[ur, "\bigsqcup_{i=1}^n g_i'"'] & \end{tikzcd} \right\} \simeq \prod_{i=1}^{n} \mathrm{Hom}_{\mathrm{Span}}(T, S_i).$$

By dualizing this argument (using that spans from S to T are the same as spans from T to S) we see that similarly the spans

$$\bigsqcup_{i=1}^{n} S_i \hookleftarrow S_i \xrightarrow{\mathrm{id}_{S_i}} S_i$$

exhibits $\bigsqcup_{i=1}^n S_i$ also as the *product* of the S_i in Span(Fin). This shows that Span(Fin) is semiadditive. $\square$

Definition 4.3.4 (Commutative monoid)**.** Let C be an ∞-category with finite products. We define a *commutative monoid* in C to be a product-preserving functor

$$M \colon \mathrm{Span(Fin)} \to C.$$

We denote by

$$\mathrm{CMon}(C) \subseteq \mathrm{Fun}(\mathrm{Span(Fin)}, C)$$

the full subcategory spanned by the commutative monoids. We refer to the evaluation $M(*)$ at the one-point set as the *underlying object* of M. We will sometimes abuse notation and simply refer to $M(*)$ as M.

Remark 4.3.5 (Restriction and addition maps)**.** Let $M \colon \mathrm{Span(Fin)} \to C$ be a commutative monoid. Every morphism $f \colon S \to T$ of finite sets determines two 'half-degenerate' spans

$$f^* \quad = \quad (T \xleftarrow{f} S \xrightarrow{\mathrm{id}_S} S) \qquad \text{and} \qquad f_\oplus \quad = \quad (S \xleftarrow{\mathrm{id}_S} S \xrightarrow{f} T),$$

and hence the functoriality of M provides two maps

$$f^*\colon M(T) \to M(S) \qquad \text{and} \qquad f_\oplus\colon M(S) \to M(T).$$

Since products in Span(Fin) are computed as disjoint unions of finite sets, the condition that M preserves finite products means that for every finite set S the map

$$(e_s^*)_{s\in S}\colon M(S) \to M^S := \prod_{s\in S} M(*)$$

is an isomorphism, where $e_s\colon \{s\} \hookrightarrow S$ is the inclusion of the element s into S. We will use this identification from now on to identify $M(S)$ with the S-fold product of M in C. Under these identifications, we should think of the maps $f^*\colon M^T \to M^S$ and $f_\oplus\colon M^S \to M^T$ as the *restriction* and the *addition* maps, respectively:

$$f^*(x_t)_{t\in T} = (x_{f(s)})_{s\in S} \qquad \text{and} \qquad f_\oplus(x_s)_{s\in S} = (y_t)_{t\in T}, \quad y_t := \sum_{s\in f^{-1}(t)} x_s.$$

Specializing to the case $T = *$ and $S = * \sqcup *$, we obtain an addition map

$$+\colon M \times M = M^{*\sqcup *} \to M,$$

and similarly the map $\emptyset \to *$ in Fin gives rise to the zero map $0\colon * \to M$. Just as with non-commutative monoids, we are not just encoding the binary addition map $+\colon M\times M \to M$, but all n-fold addition maps $M^n \to M$ together will all possible compatibilities between them.

Remark 4.3.6 (Comparison to usual definition)**.** To state the condition that $M(S)$ is isomorphic to $\prod_{s\in S} M(*)$, one only needs the contravariant functoriality of M along the inclusion maps $e_s\colon \{s\} \hookrightarrow S$. Instead of working with the whole span category Span(Fin), we could then also have worked with its subcategory

$$\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \subseteq \mathrm{Span}(\mathrm{Fin})$$

which contains all the objects but only those spans of the form $S \hookleftarrow U \to T$ where the left-pointing morphism is an injection. More precisely, if we say that a functor $M\colon \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to C$ satisfies the *Segal condition* if the induced maps $(e_s^*)_{s\in S}\colon M(S) \to \prod_{s\in S} M(*)$ are isomorphisms, then one can show that restriction along the inclusion induces an equivalence

$$\mathrm{CMon}(C) = \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}), C) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}), C),$$

with inverse given by right Kan extension along the inclusion; see [BH21, Proposition C.1]. The category Span(Fin, inj, all) may be thought of as the category of finite sets and *partially* *defined maps*: a morphism from S to T in this category is by definition a choice of subset U of S together with a map $U \to T$.

Note that the category $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ is equivalent to the category Fin_* of finite *pointed* sets: there is a functor

$$\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to \mathrm{Fin}_*, \qquad S \mapsto S_+ := S \sqcup *$$

which on objects equips every finite set S with an additional basepoint, and which on morphisms sends the partially defined map from S to T to the pointed map $S_+ \to T_+$ which on U is just given by the given map $U \to T$ and on the complement of U sends everything to the basepoint of T_+. The inverse to this equivalence is given on objects by removing the basepoint, and on morphisms by taking the partial map defined on those elements of S_+ that are not sent to the basepoint in T_+.

Warning 4.3.7. The ∞-category $\mathrm{CMon}(C)$ is usually defined as $\mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Fin}_*, C)$ in the literature; this was also the original approach taken by Segal [Seg68] after which this condition is named. I find the approach with $\mathrm{Span}(\mathrm{Fin})$ a bit more conceptual, hence have chosen to present it this way.

Definition 4.3.8. A commutative monoid M is called a *commutative group* if the 'shear map' $(\mathrm{pr}_1, +)\colon M \times M \to M \times M$ is an isomorphism. We denote by

$$\mathrm{CGrp}(C) \subseteq \mathrm{CMon}(C)$$

the full subcategory spanned by the commutative groups in C.

Recall that the ordinary category of abelian monoids is semiadditive: the product $A \otimes B$ of two abelian monoids is also the *co*product, and the resulting biproduct is usually denoted $A \oplus B$. We will now show that $\mathrm{CMon}(C)$ is semiadditive for arbitrary C. We start with two preliminary lemmas:

Lemma 4.3.9. *Let C be an ∞-category with finite products. Then the ∞-category $\mathrm{CMon}(C)$ has finite products and the evaluation map $\mathrm{ev}_T\colon \mathrm{CMon}(C) \to C$ preserves finite products for each $T \in \mathrm{Fin}$.*

Proof. The functor category $\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$ admits finite products (Lemma 2.8.19) which are computed pointwise, and the subcategory $\mathrm{CMon}(C)$ is closed under finite products. $\square$

Lemma 4.3.10. *Let C be an ∞-category with finite products. Then there exists a cotensoring functor*

$$(-)^{(-)}\colon \mathrm{Span}(\mathrm{Fin}) \times \mathrm{CMon}(C) \to \mathrm{CMon}(C), \qquad (S, X) \mapsto X^S,$$

defined by $X^S(T) := X(S \times T)$, which preserves finite products in both variables and whose restriction to $\{*\} \times \mathrm{CMon}(C)$ is the identity functor.

Proof. For a finite set S and a commutative monoid X we define the cotensoring by $X^S(T) := X(S\times T)$ for $T \in \mathrm{Fin}$. The functoriality in S and T comes from the observation that the product functor $-\times-\colon \mathrm{Fin}\times\mathrm{Fin} \to \mathrm{Fin}$ induces a functor on span categories by taking products of spans:

$$-\times-\colon\ \mathrm{Span}(\mathrm{Fin})\times\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Span}(\mathrm{Fin}), \qquad (S,T)\mapsto S\times T.$$

(This is no longer the categorical product in $\mathrm{Span}(\mathrm{Fin})$!)

- Since we have a natural bijection $*\times T \cong T$, it is clear that the cotensoring is the identity on $\{*\}\times\mathrm{CMon}(C)$.
- Fixing $S\in\mathrm{Fin}$, the functor $(-)^S\colon\ \mathrm{CMon}(C)\to\mathrm{CMon}(C)$ preserves finite products: we may check this pointwise for every $T\in\mathrm{Fin}$, where it is by definition given by the evaluation $X\mapsto X(S\times T)$, which preserves finite products.
- Fixing $X\in\mathrm{CMon}(C)$, the functor $X^{(-)}\colon\ \mathrm{Span}(C)\to\mathrm{CMon}(C)$ preserves finite products: again we may check this pointwise for every $T\in\mathrm{Fin}$, where we may write it as the composite
$$\mathrm{Span}(\mathrm{Fin}) \xrightarrow{-\times T} \mathrm{Span}(\mathrm{Fin}) \xrightarrow{X} C.$$
But X preserves finite products by assumption and $-\times T$ preserves finite products since the functor $-\times T\colon \mathrm{Fin}\to\mathrm{Fin}$ preserves finite *co*products: $\bigsqcup_{i=1}^n S_i\times T \xrightarrow{\sim} (\bigsqcup_{i=1}^n S_i)\times T$.

In particular we see that X^S is the S-fold product of X in $\mathrm{CMon}(C)$, explaining the notation.

□

Proposition 4.3.11. *Let C be an ∞-category with finite products. Then the ∞-category* $\mathrm{CMon}(C)$ *is semiadditive. The subcategory* $\mathrm{CGrp}(C)$ *is even additive.*

Proof. The main point is showing that $\mathrm{CMon}(C)$ is semiadditive; it is then essentially by definition true that the subcategory $\mathrm{CGrp}(C)$ is even additive.

We first show that $\mathrm{CMon}(C)$ is pointed by showing that the constant functor $\mathrm{const}_*\colon\ \mathrm{Span}(\mathrm{Fin})\to C$ is a zero object. Note that const_* is a terminal object in the functor category (Lemma 2.8.19) and hence also in $\mathrm{CMon}(C)$. To see it is also an initial object, consider any commutative monoid M. Since $\mathrm{Span}(\mathrm{Fin})$ has an initial object given by the empty set, limits over $\mathrm{Span}(\mathrm{Fin})$ are given by evaluation at the empty set (see Lemma 3.1.4) and henc we get

$$\mathrm{Hom}_{\mathrm{CMon}(C)}(\mathrm{const}_*, M) \simeq \mathrm{Hom}_C(*, \lim_{S\in\mathrm{Span}(\mathrm{Fin})} M(S)) \simeq \mathrm{Hom}_C(*, M(\emptyset)) \simeq \mathrm{Hom}_C(*,*) \simeq *,$$

where we also use that $M(\emptyset)\simeq *$ because M preserves finite products.

To see that $\mathrm{CMon}(C)$ is semiadditive, consider two commutative monoids X and Y. We need to show that the maps $(\mathrm{id}_X, 0)\colon X \to X\times Y$ and $(0,\mathrm{id}_Y)\colon Y \to X\times Y$ exhibit the product $X\times Y$ also as a *co*product. In other words, we need to show that the corresponding natural transformation

$$((\mathrm{id}_X,0),(0,\mathrm{id}_Y))\colon (X,Y) \to (X\times Y, X\times Y) = \Delta(X\times Y)$$

is the unit of an adjunction between the product functor and the diagonal functor $\Delta\colon \mathrm{CMon}(C) \to \mathrm{CMon}(C)\times\mathrm{CMon}(C)$ (with the product being the *left* adjoint!). To this end, we have to produce for every third commutative monoid Z a compatible counit map

$$m_Z\colon Z\times Z \to Z$$

satisfying the triangle identities. Using the cotensoring map from Lemma 4.3.10, we may construct the map m_Z as the following composite:

$$m_Z\colon Z\times Z \cong Z^{*\sqcup *} \to Z^* \cong Z,$$

where the middle map uses functoriality of the cotensoring in the forward-ponting span associated to $*\sqcup * \to *$. The triangle identities take the following form:

$$\begin{tikzcd} & (Z\times Z, Z\times Z) \arrow[dr, "(m_Z,m_Z)"] & \\ (Z,Z) \arrow[ur, "(\mathrm{id}_Z,0)\times(0,\mathrm{id}_Z)"] \arrow[rr, equal] & & (Z,Z) \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} & (X\times Y)\times(X\times Y) \arrow[dr, "m_{X\times Y}"] & \\ X\times Y \arrow[ur, "(\mathrm{id}_X,0)\times(0,\mathrm{id}_Y)"] \arrow[rr, equal] & & X\times Y. \end{tikzcd}$$

For the right-hand triangle, we use that the cotensoring preserves products, so that the map $m_{X\times Y}$ may be rewritten as the product map

$$m_X\times m_Y\colon (X\times X)\times(Y\times Y) \to X\times Y.$$

We then see that to produce both triangles, it suffices to produce for every $X\in\mathrm{CMon}(C)$ two natural homotopies between the composites

$$X \xrightarrow{(\mathrm{id}_X,0)} X\times X \xrightarrow{m_X} X, \qquad X \xrightarrow{(0,\mathrm{id}_X)} X\times X \xrightarrow{m_X} X$$

and the identity on X. For this, observe that we may write the zero map $0\colon \mathrm{const}_* \to X$ as the map $X^{\emptyset} \to X^*$ induced on cotensorings by the inclusion $\emptyset \hookrightarrow *$ in Fin, so the claim boils down to functoriality of the cotensoring in the $\mathrm{Span}(\mathrm{Fin})$-variable, together with the fact that the composites

$$* = *\sqcup\emptyset \hookrightarrow *\sqcup * \to * \qquad\text{and}\qquad * = \emptyset\sqcup * \hookrightarrow *\sqcup * \to *$$

are the identity maps in Fin. This finishes the proof. $\square$

4.4 Symmetric monoidal ∞-categories and ∞-operads

4.5 The recognition principle

4.6 The group completion theorem

5 Vector bundles and complex K-theory

5.1 Vector bundles and classifying spaces

5.2 The spectrum KU and complex K-theory

Bott periodicity

Real topological K-theory

6 Thom spectra

7 Localization and completion

7.1 Bousfield localizations

We will introduce the notion of an *E-local spectrum* for a spectrum E.

Example 7.1.1. The *p-localization* $X_{(p)}$ of a spectrum X.

Example 7.1.2. The *$K(n)$-localization* $L_{K(n)}X$ of a spectrum X.

Example 7.1.3. Inverting a prime p: $X[1/p] = L_{\mathbb{S}[1/p]}X$.

7.2 Completion of spectra

We will discuss the notion of *p-completion* of a spectrum and its relation to p-localization. We will also discuss the *arithmetic fracture square*.

7.3 Chromatic homotopy theory

Depending on how much time there is, we might give a brief introduction to chromatic homotopy theory.

8 Manifolds and duality

8.1 Categorical duality

8.2 Atiyah duality

Exercise sheets

Exercise sheet 1

Exercise 1.1. Consider continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$, and let T be a topological space. Make precise the bijection between the set of continuous maps $T \to X \times_Z^h Y$ and the set of triples (t_X, t_Y, H) with $t_X\colon T \to X$, $t_Y\colon T \to Y$ and $H\colon T \times [0,1] \to Z$ satisfying $H_0 = f \circ t_X$ and $H_1 = g \circ t_Y$, i.e. the set of homotopy commutative diagrams

$$\begin{tikzcd} T \arrow[r, "t_X"] \arrow[d, "t_Y"'] & X \arrow[d, "f"] \arrow[dl, phantom, "\swarrow H"] \\ Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Formulate and prove the dual analogue for homotopy pushouts.

Exercise 1.2. Consider a diagram of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Z \arrow[d, "\gamma"] & Y \arrow[l, "g"'] \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Z' & Y' \arrow[l, "g'"'] \end{tikzcd}$$

which commutes up to homotopy, in the sense that we are given homotopies $H\colon \gamma \circ f \sim f' \circ \alpha$ and $K\colon \gamma \circ g \sim g' \circ \beta$.

- Construct a map $\alpha \times_\gamma \beta\colon X \times_Z^h Y \to X' \times_{Z'}^h Y'$. (This map will depend on the choices of homotopies H and K, though this is not reflected in our notation.)

- Given a second diagram of topological spaces

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\alpha'"'] & Z' \arrow[d, "\gamma'"] & Y' \arrow[l, "g'"'] \arrow[d, "\beta'"] \\ X'' \arrow[r, "f''"] & Z'' & Y'' \arrow[l, "g''"'] \end{tikzcd}$$

which commutes up to homotopy, with homotopies $H' \colon \gamma' \circ f' \sim f'' \circ \alpha'$ and $K' \colon \gamma' \circ g' \sim g'' \circ \beta'$, show that the pasted diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha' \circ \alpha"'] & Z \arrow[d, "\gamma' \circ \gamma"] & Y \arrow[l, "g"'] \arrow[d, "\beta' \circ \beta"] \\ X'' \arrow[r, "f''"] & Z'' & Y'' \arrow[l, "g''"'] \end{tikzcd}$$

again commutes up to homotopy.

- Show that there is a homotopy between $(\alpha' \times_{\gamma'} \beta') \circ (\alpha \times_\gamma \beta)$ and $(\alpha' \circ \alpha) \times_{\gamma' \circ \gamma} (\beta' \circ \beta)$.
- Show that $\alpha \times_\gamma \beta$ is a homotopy equivalence whenever α, β and γ are homotopy equivalences.

Exercise 1.3. Formulate and prove the analogous statement for homotopy pushouts.

Recall that we call a topological space a *space* if it is homotopy equivalent to a CW-complex.

Exercise 1.4. Show that if $f \colon Z \to X$ and $g \colon Z \to Y$ are maps of spaces, then the homotopy pushout $X \sqcup^h_Z Y$ is again a space.
Hint: use Exercise 1.2.5 and the cellular approximation theorem.

Exercise 1.5 (Bonus exercise)**.** Using the theorem by Milnor stated below, show that for maps of spaces $f \colon X \to Z$ and $g \colon Y \to Z$ the homotopy pullback $X \times^h_Z Y$ is again a space.

Theorem (Milnor)**.** *Let X and Y be CW-complexes, and let $A_1, A_2 \subseteq X$ and $B_1, B_2 \subseteq Y$ be subcomplexes. Let* $\mathrm{Map}((X, A_1, A_2), (Y, B_1, B_2))$ *be the subspace of* $\mathrm{Map}(X,Y)$ *(equipped with the open-compact topology) consisting of those continuous maps $f \colon X \to Y$ satisfying $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$. If X is a finite CW-complex, then* $\mathrm{Map}((X, A_1, A_2), (Y, B_1, B_2))$ *is homotopy-equivalent to a CW-complex.*

Exercise sheet 2

Exercise 2.1 (Mapping telescope as homotopy pushout)**.** Recall that the *mapping telescope* of a sequence

$$Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} \ldots$$

of pointed spaces is defined as

$$\mathrm{hocolim}_n Z_n := \left(\bigvee_{n \in \mathbb{N}} Z_n \wedge [n, n+1]_+ \right) / \sim,$$

where the equivalence relation $\sim$ is given by $(x_n, n+1) \sim (f(x_n), n+1)$.

(1) For a space T, show that a map $t\colon \operatorname{hocolim}_n Z_n \to T$ corresponds to a family of pointed maps $t_n\colon Z_n \to T$ together with pointed homotopies $H_n\colon t_n \sim t_{n+1}\circ f_n$.

(2) Let $t'\colon \operatorname{hocolim}_n Z_n \to T$ be another such map, corresponding to the family (t'_n, H'_n). Show that a homotopy $t \sim t'$ corresponds to a family of homotopies $K_n\colon t_n \sim t'_n$ together with a homotopy of homotopies $Z_n \times [0,1]\times[0,1] \to T$ which restricts to H_n and H'_n on $Z_n \times [0,1]\times\{0,1\}$ and to K_n and $K_{n+1}\circ(f_n \times \mathrm{id}_{[0,1]})$ on $Z_n \times \{0,1\}\times[0,1]$.

(3) Define Z as the following homotopy pushout:

$$\begin{tikzcd} \bigvee_{n\geq 0} Z_n \arrow[r] \arrow[d] & \bigvee_{k\geq 0} Z_{2k} \arrow[d] \\ \bigvee_{k\geq 0} Z_{2k+1} \arrow[r] & Z. \end{tikzcd}$$

Here the top horizontal map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k}$ for $n = 2k$ even, and the map $f_{2k+1}\colon Z_{2k+1} Z_{2k+2}$ for $n = 2k+1$ odd. The left vertical map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k+1}$ for $n = 2k+1$ odd, an the map $f_{2k}\colon Z_{2k} \to Z_{2k+1}$ for $n = 2k$ even.

Show that one can describe maps $t\colon Z \to T$ with the same data as in (1). Show that one describe homotopies between maps $t, t'\colon Z \to T$ with the same data as in (2).

(4) Conclude that there is a homotopy equivalence $Z \simeq \operatorname{hocolim}_n Z_n$.

Exercise 2.2 (Pasting law for homotopy pushout squares)**.** Consider a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_X"'] & Y \arrow[r, "g"] \arrow[d, dashed, "\varphi_Y"] \arrow[dl, phantom, "\swarrow\!\!\!\swarrow \sim"] & Z \arrow[d, "\varphi_Z"] \arrow[dl, phantom, "\swarrow\!\!\!\swarrow \sim"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'. \end{tikzcd}$$

(1) Use the universal properties of the homotopy pushouts to construct maps

$$Z \sqcup^h_Y (Y \sqcup^h_X X') \to Z \sqcup^h_X X' \qquad \text{and} \qquad Z \sqcup^h_X X' \to Z \sqcup^h_Y (Y \sqcup^h_X X')$$

and show that they are (homotopy) inverse to each other.

(2) Assume that the left square is a homotopy pushout square (i.e. the map $Y \sqcup^h_X X' \to Y'$ induced by the universal property of $Y \sqcup^h_X X'$ is a homotopy equivalence.) Show that the right square is a homotopy pushout square if and only if the outer composite rectangle is a homotopy pushout square.

Hint: Use part (1), and use Exercise 1.3 from last sheet to obtain a homotopy equivalence $Z \sqcup^h_Y (Y \sqcup^h_X X') \xrightarrow{\sim} Z \sqcup^h_Y Y'$.

(3) Formulate the pasting law for homotopy *pullback* squares by dualizing the statement in (2). Give a brief proof sketch by indicating how to dualize parts (1) and (2).

Exercise 2.3 (Puppe sequence)**.** Consider a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of pointed spaces.

(1) Using Exercise 2.2[1], show that the (reduced[2]) cofiber of g is homotopy equivalent to ΣX.

(2) Show similarly that the cofiber of the resulting map $Z \to \Sigma X$ is homotopy equivalent to ΣY.

(3) Explain how to obtain from this the *Puppe sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \to \Sigma^2 X \to \ldots,$$

in which every pair of composable maps that appear form a cofiber sequence.

(4) Deduce that every cohomology theory E^*, as defined in the lecture, gives rise to a long exact sequence of the form

$$\ldots \xrightarrow{g^*} E^{n-1}(Y) \xrightarrow{f^*} E^{n-1}(X) \to E^n(Z) \xrightarrow{g^*} E^n(Y) \xrightarrow{f^*} E^n(X) \to E^{n+1}(Z) \xrightarrow{g^*} E^{n+1}(Y) \xrightarrow{f^*} \ldots$$

for every cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Exercise 2.4 (Eilenberg-MacLane spectra)**.** By Brown representability, reduced singular cohomology $\widetilde{H}^*(-;A)$ with coefficients in an abelian group A is represented by some spectrum E. Show that E is the Eilenberg-MacLane spectrum HA (up to homotopy equivalence).

Exercise 2.5 (Mayer-Vietoris sequence)**.** Consider a homotopy pushout square of pointed spaces

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "l"'] & A \arrow[d, "i"] \\ B \arrow[r, "j"'] & X. \end{tikzcd}$$

[1]Strictly speaking it is the version of Exercise 2.2 for *reduced* homotopy pushouts, i.e. for homotopy pushouts of *pointed* spaces. The proof is the same as that for Exercise 2.2 and you may simply use it in Exercise 2.3.

[2]From now on, all cofibers of pointed maps are understood to be *reduced* cofibers.

(1) Show that the cofiber of the map $(i,j)\colon A\vee B\to X$ is homotopy equivalent to $\Sigma(C)$. *Hint: you may use the Fact below.*

(2) Show similarly that the cofiber of the resulting map $X\to\Sigma(C)$ is homotopy equivalent to $\Sigma A\vee\Sigma B$.

(3) Conclude that for a functor $F\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}}\to\mathrm{Ab}$ satisfying both the Wedge axiom and the Mayer-Vietoris property, we obtain a long exact sequence

$$\cdots\to F(\Sigma X)\xrightarrow{(\Sigma i^*,\Sigma j^*)} F(\Sigma A)\times F(\Sigma B)\xrightarrow{(\Sigma k)^*-(\Sigma l)^*} F(\Sigma C)\xrightarrow{\partial} F(X)\xrightarrow{(i^*,j^*)} F(A)\times F(B)\xrightarrow{k^*-l^*} F(C).$$

Fact. Consider a homotopy commutative diagram

$$\begin{tikzcd}
X_2 \arrow[rr, "f_2"] \arrow[dr, "k"] \arrow[dd, "l"'] & & Y_2 \arrow[dr, "k'"] \arrow[dd, "l'"'] & \\
& X_0 \arrow[rr, "f_0"] \arrow[dd, "i"] & & Y_0 \arrow[dd, "i'"] \\
X_1 \arrow[rr, "f_1"] \arrow[dr, "j"'] & & Y_1 \arrow[dr, "j'"] & \\
& X \arrow[rr, "f"'] & & Y,
\end{tikzcd}$$

meaning that all six faces commute up to given homotopy and that there is a homotopy of homotopies between the two induced homotopies $i'\circ f_0\circ k\sim j'\circ f_1\circ l$ of maps $X_2\to Y$. Then passing to horizontal cofibers induces a homotopy commutative square

$$\begin{tikzcd}
C(f_2) \arrow[r] \arrow[d] & C(f_0) \arrow[d] \\
C(f_1) \arrow[r] & C(f).
\end{tikzcd}$$

If the left and right vertical face of the original cube are both homotopy pushout squares, then the induced square on homotopy cofibers is also a homotopy pushout square.

While one can prove this fact by hand, it will be an easy consequence of the general theory of colimits in ∞-categories, hence we will postpone its proof.

Exercise 2.6 (Bonus exercise)**.** Convince yourself that the Fact is true.

Exercise sheet 3

Exercise 3.1 (Yoneda lemma)**.** Let C be an ordinary category and consider an object X of C. Consider the functor

$$y_X\colon C^{\mathrm{op}}\to\mathrm{Set},\qquad Y\mapsto \mathrm{Hom}_C(Y,X).$$

For any other functor $F\colon C^{\mathrm{op}} \to \mathrm{Set}$, show that evaluation at the identity $\mathrm{id}_X \in \mathrm{Hom}_C(X,X) = y_X(X)$ induces a bijection

$$\mathrm{Nat}(y_X, F) \xrightarrow{\cong} F(X),$$

where the left-hand side denotes the set of natural transformations $y_X \to F$.

Conclude from this that for two pointed spaces X and Y, any natural transformation $[-,X]_* \to [-,Y]_*$ of functors $\mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Set}$ is given by postcomposition with a pointed map $f\colon X \to Y$, and that this condition uniquely determines f up to homotopy.

Exercise 3.2. Let C and D be ∞-categories. Show that a functor $f\colon C \to D$ is an equivalence if and only if it admits both a section (i.e. a functor $g\colon D \to C$ such that $\mathrm{id}_D \cong fg$) and a retraction (i.e. a functor $h\colon D \to C$ such that $\mathrm{id}_C \cong hf$), and that in this case there is a natural isomorphism $g \cong h$.

Conclude that an inverse to f is unique up to isomorphism if it exists.

Exercise 3.3. Show that if $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.

Exercise 3.4. Show that equivalences of ∞-categories satisfy the 2-out-of-6 property: given functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.

Exercise 3.5. Construct for every morphism $f\colon x \to y$ commutative triangles in C of the form

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y. \end{tikzcd}$$

Exercise 3.6 (Bonus exercise)**.** Show that products of ∞-categories are associative: given ∞-categories C, D and E, produce an equivalence of ∞-categories

$$C \times (D \times E) \xrightarrow{\sim} (C \times D) \times E.$$

Exercise 3.7 (Bonus exercise)**.** Also the coproduct of ∞-categories is unital, associative and commutative. Given ∞-categories C, D and E, construct the necessary functors:

$$\begin{array}{llll} C \to C \sqcup \emptyset, & C \to \emptyset \sqcup C, & C \sqcup D \to D \sqcup C, & (C \sqcup D) \sqcup E \to C \sqcup (D \sqcup E), \\ C \sqcup \emptyset \to C, & C \to \emptyset \sqcup C, & D \sqcup C \to C \sqcup D, & C \sqcup (D \sqcup E) \to (C \sqcup D) \sqcup E. \end{array}$$

Verify for yourself that these functors are indeed inverse to one another. (You don't need to write out this verification.)

Exercise sheet 4

Exercise 4.1. Show that coproducts of ∞-categories are *disjoint*: given ∞-categories C and D, the commutative square

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & C \arrow[d, "i_C"] \\
D \arrow[r, "i_D"] & C \sqcup D
\end{tikzcd}$$

is a pullback square. In similar spirit, show that for every functor $f\colon C \to D$ the commutative square

$$\begin{tikzcd}
\emptyset \arrow[r, "\mathrm{id}_\emptyset"] \arrow[d] & \emptyset \arrow[d] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

is a pullback square. (In both cases also explain which square I mean when I say *the* commutative square.)

Exercise 4.2. Consider ∞-categories C and D, and let $p_C\colon C \to *$ and $p_D\colon D \to *$ be their functors to the terminal ∞-category. Construct a commutative square of the form

$$\begin{tikzcd}
C \times D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_C"'] & C \arrow[d, "p_C"] \\
D \arrow[r, "p_D"] & *
\end{tikzcd}$$

and show it is a pullback square.

Exercise 4.3. Consider a commutative square

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\
E \arrow[r, "k"] & F
\end{tikzcd}$$

and assume that h is an equivalence. Show that g is an equivalence if and only if the square is a pullback square.

The goal of the following two exercises is to give an alternative description of the pullback $C \times_E D$ of two functors $f\colon C \to E$ and $g\colon D \to E$.

Exercise 4.4. Consider a functor $g\colon D \to E$ between ∞-categories.

(1) Construct for every ∞-category C a commutative diagram of the form

$$\begin{tikzcd} C\times D \arrow[r, "\mathrm{id}_C\times g"] \arrow[d, "\mathrm{pr}_D"'] & C\times E \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_E"] & C \arrow[d, "p_C"] \\ D \arrow[r, "g"] & E \arrow[r, "p_E"] & *. \end{tikzcd}$$

Combining the pasting lemma with Exercise 4.2, deduce that the left-hand square is a pullback square.

(2) Construct a commutative diagram of the following form:

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "{(g,\mathrm{id}_D)}"'] & E \arrow[d, "{(\mathrm{id}_E,\mathrm{id}_E)}"] \\ E\times D \arrow[r, "\mathrm{id}_E\times g"] \arrow[d, "\mathrm{pr}_D"'] & E\times E \arrow[d, "\mathrm{pr}_2"] \\ D \arrow[r, "g"] & E, \end{tikzcd}$$

where $\mathrm{pr}_2\colon E\times E\to E$ means the projection to the second factor of E. Show that the vertical composite of these squares

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "{\mathrm{pr}_D\circ(g,\mathrm{id}_D)}"'] & E \arrow[d, "{\mathrm{pr}_1\circ(\mathrm{id}_E,\mathrm{id}_E)}"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

is a pullback square. *Hint:* use Exercise 4.3.

(3) Combine parts (1) and (2) with the pasting law of pullback squares to conclude that the commutative square

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "{(g,\mathrm{id}_D)}"'] & E \arrow[d, "{(\mathrm{id}_E,\mathrm{id}_E)}"] \\ E\times D \arrow[r, "\mathrm{id}_E\times g"] & E\times E \end{tikzcd}$$

is a pullback square.

Exercise 4.5. Consider two functors $f\colon C\to E$ and $g\colon D\to E$ between ∞-categories.

(1) Construct a commutative diagram of the following form:

$$\begin{tikzcd} C\times_E D \arrow[r, "\mathrm{pr}_D"] \arrow[d, "{(\mathrm{pr}_C,\mathrm{pr}_D)}"'] & D \arrow[r, "g"] \arrow[d, "{(g,\mathrm{id}_D)}"] & E \arrow[d, "{(\mathrm{id}_E,\mathrm{id}_E)}"] \\ C\times D \arrow[r, "f\times\mathrm{id}_D"] \arrow[d, "\mathrm{pr}_C"'] & E\times D \arrow[r, "\mathrm{id}_E\times g"] \arrow[d, "\mathrm{pr}_E"] & E\times E \\ C \arrow[r, "f"] & E. & \end{tikzcd}$$

(In other words, produce for each of the three squares a natural isomorphism which makes the square commute.)

(2) Composing the two squares on the left, we obtain a commutative square of the form

$$\begin{tikzcd} C \times_E D \arrow[r, "\mathrm{pr}_D"] \arrow[d, "\mathrm{pr}_C \circ (\mathrm{pr}_C, \mathrm{pr}_D)"'] & D \arrow[d, "\mathrm{pr}_E \circ (g, \mathrm{id}_D)"] \\ C \arrow[r, "f"] & E. \end{tikzcd}$$

Show that this square is a pullback square (for example by proving this square is actually isomorphic to the canonical one defining $C \times_E D$.)

(3) Recall from Exercise 4.4 that the bottom-left and top-right squares in this diagram are also pullback squares. Deduce from the pasting law of pullback squares that the commutative square

$$\begin{tikzcd} C \times_E D \arrow[r] \arrow[d, "(\mathrm{pr}_C, \mathrm{pr}_D)"'] & E \arrow[d, "(\mathrm{id}_E, \mathrm{id}_E)"] \\ C \times D \arrow[r, "f \times g"] & E \times E \end{tikzcd}$$

is a pullback square.

Exercise 4.6 (Bonus exercise)**.** Explain heuristically why the pullback $(C \times D) \times_{E \times E} E$ appearing in part (3) of the previous exercise should be a fiber product of C and D over E: explain how we may think of a functor into it as specifying functors into C and D which agree after mapping further to E.

(Warning: this heuristic argument would not constitute a proof and is just meant to aid intuition.)

Exercise sheet 5

Exercise 5.1. Use the commutative square axiom to define functors

$$\min\colon [1] \times [1] \to [1], \qquad \text{and} \qquad \max\colon [1] \times [1] \to [1]$$

that behave like the 'maximum' and 'minimum' functors on the partial order $\{0 \leq 1\}$, in the sense that they satisfy the equations

$$\begin{aligned} \max(x,0) &= x = \max(0,x), & \max(x,1) &= 1 = \max(1,x), \\ \min(x,0) &= 0 = \min(0,x), & \min(x,1) &= x = \min(1,x). \end{aligned}$$

(Also show that the functors you define actually satisfy these equations.)

In the lecture we used the commutative square axiom to construct functors $p_0, p_2\colon [1]\times[1] \to [2]$, pictorially represented by the following diagrams:

$$p_0 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] \arrow[dr] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad\qquad p_2 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

As mentioned in the lecture, these functors provide a way of regarding commutative triangles in C as certain degenerate forms of commutative squares: a commutative triangle $\sigma\colon [2] \to C$ of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"] & & z \end{tikzcd}$$

may alternatively be encoded as a commutative square $[1]\times[1] \to C$ whose restriction $\{0\}\times[1]$ is the identity

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] & y \arrow[d, "g"] \\ x \arrow[r, "gf"] & z. \end{tikzcd}$$

Alternatively we may encode it as a commutative square $[1]\times[1] \to C$ whose restriction to $[1]\times\{1\}$ is the identity

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "gf"'] & y \arrow[d, "g"] \\ z \arrow[r, "\mathrm{id}_z"] & z. \end{tikzcd}$$

The following exercise makes this precise:

Exercise 5.2. Show that for every ∞-category C the following two commutative squares are pullback squares:

$$\begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_0^*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1]\times[1],C) \arrow[d, "\mathrm{res}"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}(\{0\}\times[1],C) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_2^*"] \arrow[d, "\mathrm{ev}_2"'] & \mathrm{Fun}([1]\times[1],C) \arrow[d, "\mathrm{res}"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}([1]\times\{1\},C). \end{tikzcd}$$

(It suffices to prove it for one of them; the other case is symmetric.) The functors labeled 'res' are given by precomposition with the respective inclusions of $\{0\}\times[1]$ and $[1]\times\{1\}$ into $[1]\times[1]$.

Exercise 5.3. Let C_0 and C_1 be ∞-categories and consider morphisms $f_0\colon x_0 \to y_0$ and $g_0\colon y_0 \to z_0$ in C_0 and morphisms $f_1\colon x_1 \to y_1$ and $g_1\colon y_1 \to z_1$ in C_1. Show that the composite $(g_0, g_1)\circ(f_0, f_1)\colon (x_0, x_1) \to (z_0, z_1)$ in $C_0 \times C_1$ is given by $(g_0 \circ f_0, g_1 \circ f_1)$.

Exercise 5.4. Let X be a topological space. Show that its singular complex $\mathrm{Sing}(X) \in \mathrm{sSet}$ is a Kan complex.

Exercise 5.5. Let C be a small category.

- Show that its nerve $N(C) \in \mathrm{sSet}$ is a quasicategory;
- Provide an example for which $N(C)$ is not a Kan complex;
- Give a complete characterization of those categories C for which $N(C)$ is a Kan complex.

Exercise 5.6 (Bonus exercise)**.** Let C be a Kan-enriched category. Show that its homotopy-coherent nerve $N^{\Delta}(C) \in \mathrm{sSet}$ is a quasicategory.

Exercise sheet 6

Exercise 6.1. Show that for every ∞-category C and every anima X the inclusion $\gamma_C \colon C^{\simeq} \to C$ induces an equivalence

$$(\gamma_C)_* \colon \mathrm{Map}(X, C^{\simeq}) \xrightarrow{\sim} \mathrm{Map}(X, C).$$

For the next exercise, we need the following fact:[3]

Fact. For every ∞-category C, the functor $p_{[1]}^* \colon C \to \mathrm{Fun}([1], C)$ is fully faithful.

Exercise 6.2. Let C be an ∞-category. Show that the following statements are equivalent:

(1) The ∞-category C is an anima;

(2) The functor $p_{[1]}^* \colon C \to \mathrm{Fun}([1], C)$ is an equivalence;

(3) The functor $\pi_{\mathrm{Iso}} \colon \mathrm{Iso}(C) \to \mathrm{Fun}([1], C)$ is an equivalence.

Exercise 6.3. Show that a morphism $f \colon X \to Y$ of animae is an equivalence iff the induced map $f^* \colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is an equivalence for every anima Z.

Exercise 6.4. Show that the functor $p_{[1]} \colon [1] \to *$ exhibits the terminal ∞-category as a localization of $[1]$ at all its morphisms, i.e. show that $|[1]| \simeq *$.
Hint: use Exercise 6.2 and Exercise 6.3.

[3] This fact follows from Exercise 6.4. However, this reasoning is circular since the fact is indirectly used to prove Exercise 6.4. If we had taken characterization (2) in Exercise 6.2 as our definition of animae then the circularity can be avoided.

Exercise 6.5. Let C be a pointed ∞-category that admits fibers and cofibers. Show that the functor $\Sigma\colon C \to C$ is left adjoint to $\Omega\colon C \to C$: for all $X, Y \in C$ there is a natural equivalence

$$\mathrm{Hom}_C(\Sigma X, Y) \simeq \mathrm{Hom}_C(X, \Omega Y).$$

Exercise 6.6 (Bonus exercise)**.** Let C be an ∞-category and let x and y be objects of C. The goal of this exercise is to show that the Hom anima $\mathrm{Hom}_C(x,y)$ is indeed an anima.

(1) For an object z in an ∞-category D, we define the *slice category* $D_{/z}$ via the following pullback square:

$$\begin{tikzcd} D_{/z} \arrow[r] \arrow[d] & \mathrm{Fun}([1], D) \arrow[d, "\mathrm{ev}_1"] \\ * \arrow[r, "z"] & D. \end{tikzcd}$$

In other words, objects of $D_{/z}$ are morphisms $f\colon w \to z$ in D and morphisms are commutative squares in D of the form

$$\begin{tikzcd} w \arrow[r] \arrow[d] & w' \arrow[d] \\ z \arrow[r, equal] & z, \end{tikzcd}$$

or equivalently (by Exercise 5.2) commutative triangles in D of the form

$$\begin{tikzcd} w \arrow[rr] \arrow[dr] & & w' \arrow[dl] \\ & z. & \end{tikzcd}$$

Show that for every other ∞-category E there is an equivalence

$$\mathrm{Fun}(E, D_{/z}) \xrightarrow{\sim} \mathrm{Fun}(E, D)_{/\mathrm{const}_z}.$$

(2) Let $\pi\colon C_{/y} \to C$ denote the composite functor

$$C_{/y} \hookrightarrow \mathrm{Fun}([1], C) \xrightarrow{\mathrm{ev}_0} C.$$

Show that the following commutative square is a pullback square:

$$\begin{tikzcd} \mathrm{Fun}([1], C_{/y}) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_1"] \\ C_{/y} \arrow[r, "\pi"] & C. \end{tikzcd}$$

Hint: Use the Segal axiom and Exercise 5.2.

(3) Show that there is a pullback square

$$\begin{tikzcd}
\mathrm{Hom}_C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{/y} \arrow[d, "\pi"] \\
* \arrow[r, "x"] & C.
\end{tikzcd}$$

(4) Show that $\mathrm{Hom}_C(x, y)$ is an anima.

Hint: Use Exercise 6.2.

Exercise sheet 7

Exercise 7.1. Let $f\colon X \to Y$ be a morphism in a stable ∞-category. Show that there is an isomorphism[4]

$$\mathrm{cofib}(f) \cong \mathrm{fib}(f)[1].$$

Exercise 7.2. Let $f\colon X \to Y$ be a morphism in a stable ∞-category C. Show that the following are equivalent:

(1) The morphism f is an isomorphism in C;

(2) The cofiber $\mathrm{cofib}(f)$ of f is a zero object in C;

(3) The fiber $\mathrm{fib}(f)$ of f is a zero object in C.

Recall that the *splitting lemma* provides equivalent conditions for a short exact sequence in an abelian category to be a *split short exact sequence*. In the following exercise we will prove a splitting lemma for stable ∞-categories:

Exercise 7.3. Let C be a stable ∞-category and let $X \xrightarrow{i} Y \xrightarrow{p} Z$ be an exact sequence in C. Show that the following are equivalent:

(1) The associated morphism $\mathrm{fib}(i) \to X$ is null-homotopic;

(2) The associated morphism $Z \to \mathrm{cofib}(p)$ is null-homotopic;

(3) The map i admits a retract: there is a morphism $r\colon Y \to X$ satisfying $ri \simeq \mathrm{id}_X$;

(4) The map p admits a section: there is a morphism $s\colon Z \to Y$ satisfying $ps \simeq \mathrm{id}_Z$;

[4] Recall that the notation $X[1]$ is an alternative notation for $\Sigma(X)$ in a stable ∞-category.

(5) There is an isomorphism $Y \cong X \oplus Z$ in C making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "(\mathrm{id}_X,0)"'] & Y \arrow[r, "p"] \arrow[d, "\cong"] & Z \\ & X \oplus Z. \arrow[ur, "\mathrm{pr}_Z"'] & \end{tikzcd}$$

For the following exercise, you may use the fact (to be proved in two different ways in Exercise 7.6 and Exercise 7.7 below) that in a stable ∞-category C the set $[X,Y] := \pi_0 \mathrm{Hom}_C(X,Y)$ of homotopy classes of morphisms from X to Y in C admits a canonical abelian group structure, and that the composition maps

$$- \circ f \colon [X,Y] \to [X',Y] \qquad \text{and} \qquad g \circ - \colon [X,Y] \to [X,Y']$$

are group homomorphisms for all morphisms $f \colon X' \to X$ and $g \colon Y \to Y'$ in C.

Exercise 7.4. Let C be a stable ∞-category and consider a commutative square in C of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "g"'] & W. \end{tikzcd}$$

Show that the following conditions are equivalent:

(1) The square is exact;

(2) The sequence

$$X \xrightarrow{(f,u)} Y \oplus Z \xrightarrow{(v,-g)} W$$

is an exact sequence.

Exercise 7.5. Let C and D be ∞-categories with finite limits. Show that there is an equivalence

$$\mathrm{Sp}(C \times D) \xrightarrow{\sim} \mathrm{Sp}(C) \times \mathrm{Sp}(D).$$

Exercise 7.6 (Bonus exercise)**.** Let C be a semiadditive ∞-category.

(1) Show that for all objects X and Y in C the set $[X,Y] := \pi_0 \mathrm{Hom}_C(X,Y)$ of homotopy classes of morphisms from X to Y admits a canonical structure of an abelian monoid, where addition of $f,g \in [X,Y]$ is given by the composite

$$f+g\colon \quad X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

(2) Show that for all morphisms $f\colon X' \to X$ and $g\colon Y \to Y'$ in C the composition maps

$$- \circ f\colon [X,Y] \to [X',Y] \qquad \text{and} \qquad g \circ -\colon [X,Y] \to [X,Y']$$

are homomorphisms of abelian monoids.

(3) Let M be an abelian monoid. Show that M is an abelian group if and only if the map

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\colon M \oplus M \to M \oplus M, \qquad (x,y) \mapsto (x, x+y)$$

is a bijection of sets.

(4) Conclude that the semiadditive ∞-category C is in fact *additive* if and only if the abelian monoid $[X,Y]$ is an abelian group for all X and Y.

Exercise 7.7 (Bonus exercise)**.** Let C be a pointed ∞-category with finite limits.

(1) Show that for every object X, the loop space ΩX admits the structure of a group object in the homotopy category $\mathrm{Ho}(C)$: there are maps

$$e\colon * \to \Omega X, \qquad m\colon \Omega X \times \Omega X \to \Omega X, \qquad \text{and} \qquad i\colon \Omega X \to \Omega X$$

that satisfy the usual group relations up to homotopy.

(2) Show that for objects X and Y in C, the set $[X, \Omega Y]$ is canonically a group. Show that $[X, \Omega^2 Y]$ is an abelian group.

(3) Deduce that when C is stable, the Hom-set $[X,Y]$ in $\mathrm{Ho}(C)$ is an abelian group for all X and Y, and that composition in C defines group homomorphisms.

(4) Show that the resulting abelian group structure on $[X,Y]$ agrees with the one defined in Exercise 7.6.

Remark 7.8. Applying the previous exercise to C^{op}, we deduce that for every X the suspension ΣX is a *cogroup object* in $\mathrm{Ho}(C)$, and hence that $[\Sigma X, Y]$ admits an abelian group structure for all Y. Applying this to $C = \mathrm{An}_*$, this provides a proof of the claim in Remark 1.3.10.

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