

# INTRODUCTION TO QUASICATEGORIES

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Note: this is a major revision of the previous version of these notes. I'll leave a copy of the previous version (last modified in June 2021) online for a while, but this version is much better.

Note: this is a draft, which can change daily, though it also sometimes goes months without any change. I hope to add additional material on (co)Cartesian fibrations.

## 1. INTRODUCTION TO $\infty$ -CATEGORIES

I'll give a brief discussion to motivate the notion of  $\infty$ -categories.

**1.1. Groupoids.** Modern mathematics is based on sets. The most familiar way of constructing new sets is as sets solutions to equations. For instance, given a commutative ring  $R$ , we can consider the set  $X(R)$  of tuples  $(x, y, z) \in R^3$  which satisfy the equation  $x^5 + y^5 = z^5$ . We can express such sets as limits; for instance,  $X(R)$  is the pullback of the diagram of sets

$$R \times R \xrightarrow{(x,y) \mapsto x^5 + y^5} R \xleftarrow{z^5 \mapsto z} R.$$

Another way to construct new sets is by taking “quotients”; e.g., as sets of equivalence classes of an equivalence relation. This is in some sense much more subtle than sets of solutions to equations: mathematicians did not routinely construct sets this way until they were comfortable with the set theoretic formalism introduced by the end of the 19th century.

Some sets of equivalence classes are nothing more than that; but some have “higher” structure standing behind them, which is often encoded in the form of a *groupoid*<sup>1</sup>. Here are some examples.

- Given a topological space  $X$ , we can define an equivalence relation on the set of points, so  $x \sim x'$  if and only if there is a continuous path connecting them. The set of equivalence classes is the set  $\pi_0 X$  of path components. Standing behind this equivalence relation is the *fundamental groupoid*  $\Pi_1 X$ , whose objects are points of  $X$ , and whose morphisms are path-homotopy classes of paths between two points.
- Given any category  $C$ , there is an equivalence relation on the collection of objects, so that  $X \sim Y$  if there exists an isomorphism between them. Equivalence classes are the isomorphism classes of objects. Standing behind this equivalence relation is the *core* of  $C$  (also called the *maximal subgroupoid*), which is a groupoid having the same objects as  $C$ , but having as morphisms only the isomorphisms in  $C$ .
- As a special case of the above, let  $C = \text{Vect}_F$  be the category of finite dimensional vector spaces and linear maps over some field  $F$ . Then isomorphism classes of objects correspond to non-negative integers, via the notion of dimension. The core  $\text{Vect}_F^{\text{core}}$  is a groupoid whose objects are finite dimensional vector spaces, and whose morphisms are *invertible* linear maps.

Note that many interesting problems are about describing isomorphism classes; e.g., classifying finite groups of a given order, or principal  $G$ -bundles on a space. In practice, one learns that when you try to classify some type of objects up to isomorphism, you will need to have a good handle on the isomorphisms between such objects, including the groups of automorphisms of such objects. So you will likely need to know about the groupoid, even if it is not the primary object of interest.

For instance, a problem such as: “classify principal  $G$ -bundles on a space  $M$  up to isomorphism” naturally leads you to consider the problem: “describe the groupoid  $\text{Bun}_G(M)$  of principal  $G$ -bundles on a space  $M$ ”. This kind of problem can be thought of as a more sophisticated analogue of one like: “find the set  $X(R)$  of solutions to  $x^5 + y^5 = z^5$  in the ring  $R$ ”. (In fact, the theory of “moduli stacks” exactly develops this analogy between the two problems.) To do this, you can imagine having a “groupoid-based mathematics”, generalizing the usual set-based one. Here are some observations about this.

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<sup>1</sup>I assume familiarity with basic categorical concepts, such as in Chapter 1 of [Rie16].

- We regard two sets as “essentially the same” if they are *isomorphic*, i.e., if there is a bijection  $f: X \rightarrow X'$  between them. Any such bijection has a unique inverse bijection  $f^{-1}: X' \rightarrow X$ .

On the other hand, we regard two categories as “essentially the same” if they are merely *equivalent*, i.e., if there is a functor  $f: C \rightarrow C'$  which admits an inverse *up to natural isomorphism*. It is not the case that such an inverse up to natural isomorphism is itself unique. These same remarks apply in particular to equivalences of groupoids.

Although any equivalence of categories admits some kind of inverse, the failure to be unique leads to complications. For example, one goal of every course in abstract linear algebra is to demonstrate and exploit an equivalence of categories

$$f: \text{Mat}_F \rightarrow \text{Vect}_F.$$

Here  $\text{Vect}_F$  is the category of finite dimensional vector spaces over  $F$ , while  $\text{Mat}_F$  is the *matrix* category, whose objects are non-negative integers, and whose morphisms  $n \rightarrow m$  are  $m \times n$ -matrices with entries in  $F$ . The functor  $f$  is defined by an explicit construction; e.g., it sends the object  $n$  to the vector space  $F^n$ . However, there is no completely “natural” way to construct an inverse functor  $f^{-1}: \text{Vect}_F \rightarrow \text{Mat}_F$ : producing such an inverse functor requires making an arbitrary choice, for each abstract vector space  $V$ , of a basis for  $V$ .

- We can consider “solutions to equations” in groupoids (e.g., limits). However, the naive construction of limits of groupoids may not preserve equivalences of groupoids; thus, we need to consider “weak” or “homotopy” limits.

For example, suppose  $M$  is a space which is a union of two open subsets  $U$  and  $V$ . The *homotopy pullback* of

$$\text{Bun}_G(U) \rightarrow \text{Bun}_G(U \cap V) \leftarrow \text{Bun}_G(V)$$

is a groupoid, whose objects are triples  $(P, Q, \alpha)$ , where  $P \rightarrow U$  and  $Q \rightarrow U$  are  $G$ -bundles, and  $\alpha: P|_{U \cap V} \xrightarrow{\sim} Q|_{U \cap V}$  is an isomorphism of  $G$ -bundles over  $U \cap V$ ; the morphisms  $(P, Q, \alpha) \rightarrow (P', Q', \alpha')$  are pairs  $(f: P \rightarrow P', g: Q \rightarrow Q')$  are pairs of bundle maps which are compatible over  $U \cap V$  with the isomorphisms  $\alpha, \alpha'$ . Compare this with the *strict pullback*, which consists of  $(P, Q)$  such that  $P|_{U \cap V} = Q|_{U \cap V}$  as bundles; in particular,  $P|_{U \cap V}$  and  $Q|_{U \cap V}$  must be *identical sets*.

A basic result about bundles is that  $\text{Bun}_G(M)$  is equivalent to this homotopy pullback. The strict limit may fail to be equivalent to this. Note that it is impossible to describe the strict pullback without knowing precisely what definition of  $G$ -bundle we are using: in this case we need to be able to say when two bundles are *equal*, rather than *isomorphic*. The homotopy pullback is however relatively insensitive to the precise definition of  $G$ -bundle. (The point being, there can exist many non-identical “precise definitions of  $G$ -bundle”, because what we really care about in the end is understanding  $\text{Bun}_G(M)$  up to *equivalence*, rather than up to isomorphism.)

These kinds of issues persist when dealing with higher groupoids and categories.

**1.2. Higher groupoids.** There is a category  $\text{Gpd}$  of groupoids, whose objects are groupoids and whose morphisms are functors. However, there is even more structure here; there are *natural transformations* between functors  $f, f': G \rightarrow G'$  of groupoids. That is,  $\text{Fun}(G, G')$  forms not merely a set, but a category. We can consider the collection consisting of (0) groupoids, (1) *equivalences* between groupoids, and (2) natural isomorphisms between equivalences; this is an example of a *2-groupoid*<sup>2</sup>. There is no reason to stop at 2-groupoids: there are  $n$ -groupoids, the totality of which are an example of an  $(n+1)$ -groupoid. (In this hierarchy, 0-groupoids are sets, and 1-groupoids are groupoids.) We might as well take the limit, and consider  $\infty$ -groupoids.

<sup>2</sup>More precisely, a “quasistrict 2-groupoid”.

It turns out to be difficult (though not impossible) to construct an “algebraic” definition of  $n$ -groupoid. The approach which in seems to work best in practice is to use homotopy theory. We start with the observation that every groupoid  $G$  has a *classifying space*  $BG$ . This is defined explicitly as a quotient space

$$G \mapsto BG := \left( \coprod_{x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n} \Delta_{\text{top}}^n \right) / \sim,$$

where we glue in a topological  $n$ -simplex  $\Delta_{\text{top}}^n$  for each  $n$ -fold sequence of composable arrows in  $G$ , modulo certain identifications. It turns out (i) the fundamental groupoid of  $BG$  is equivalent to  $G$ , and (ii) the higher homotopy groups  $\pi_k$  of  $BG$  are trivial, for  $k \geq 2$ . A space with property (ii) is said to be *1-truncated*. Furthermore, (iii) there is a bijection between *equivalence classes of groupoids up to equivalence* and *CW-complexes which are 1-types, up to homotopy equivalence*. (More is true, but I’ll stop there for now.)

The conclusion is that groupoids and equivalences between them are modelled by 1-types and homotopy equivalences between them. This suggests that we should *define*  $n$ -groupoids as  $n$ -types (CW complexes with trivial homotopy groups in dimensions  $> n$ ), with equivalences being homotopy equivalences. Removing the restriction on homotopy groups leads to modelling  $\infty$ -groupoids by CW-complexes up to homotopy equivalence.

There is a different approach, which we will follow. It uses the fact that the classifying space construction factors through a “combinatorial” construction, called the “nerve”. That is, we have

$$(G \in \text{Gpd}) \mapsto (NG \in \text{sSet}) \mapsto (\|NG\| = BG \in \text{Top}),$$

where  $NG$  is the *nerve* of the groupoid, and is an example of a *simplicial set*;  $\|X\|$  denotes the *geometric realization* of a *simplicial set*  $X$ . In fact, the nerve of a groupoid is a particular kind of simplicial set called a *Kan complex*. It is a classical fact of homotopy theory that Kan complexes model all homotopy types. Thus, we will choose our definitions so that  $\infty$ -groupoids are precisely the Kan complexes.

**1.3.  $\infty$ -categories.** An  $\infty$ -category is a generalization of  $\infty$ -groupoid in which morphisms are no longer required to be invertible in any sense.

There are a number of approaches to defining  $\infty$ -categories. Here are two which build on top of the identification of  $\infty$ -groupoids with Kan complexes.

- A category  $C$  consists of a set  $\text{ob } C$  of objects, and for each pair of objects a set  $\text{hom}_C(x, y)$  of maps from  $x$  to  $y$ . If we replace the set  $\text{hom}_C(x, y)$  with a Kan complex (or more generally a simplicial set)  $\text{map}_C(x, y)$ , we obtain a category *enriched* over Kan complexes (or simplicial sets). This leads to one model for  $\infty$ -categories: categories enriched over simplicial sets.
- The nerve construction makes sense for categories: given a category  $C$ , we have a simplicial set  $NC$ . In general,  $NC$  is not a Kan complex; however, it does land in a special class of simplicial sets, which are called *quasicategories*. This leads to another model for  $\infty$ -categories: *quasicategories*.

In this paper we focus on the second case: the quasicategory model for  $\infty$ -categories.

**1.4. Historical remarks.** Quasicategories were invented by Boardman and Vogt [BV73, §IV.2], under the name *restricted Kan complex*. They did not use them to develop a theory of  $\infty$ -categories. This development began with the work of Joyal, first published in [Joy02]. Much of the material in this course was developed first by Joyal, in published papers and unpublished manuscripts [Joy08a], [Joy08b], [JT08]. Lurie [Lur09] gives a thorough treatment of quasicategories (which he simply calls “ $\infty$ -categories”), recasting and extending Joyal’s work significantly. There is been much work since then which has refined our understanding even more.

There are significant differences between the ways that Joyal and Lurie develop the theory. In particular, they give different definitions of the notion of a “categorical equivalence” between simplicial sets, though they do in fact turn out to be equivalent [Lur09, §2.2.5]. The approach I follow here is essentially that of Joyal. It is also basically the same as the approach Lurie takes in his reworking of the foundations at [kerodon.net](http://kerodon.net).

I have tried to generally adopt the terminology and notation of [Lur09] in most places.

**1.5. Goal of this book.** The goal of this book is to give a reasonably approachable introduction to the subject of higher category theory. In particular, I am writing with the following ideas in mind.

- The prerequisites are merely some basic notions of category theory, as seen in a first year algebraic topology or algebraic geometry course. No advanced training in homotopy theory is assumed: in particular, no knowledge of simplicial sets or model categories is assumed. You will learn what you need to know about these by reading this book.
- The book is written in “lecture notes” style rather than “textbook” style. That is, I will try to avoid introducing a lot of theory in section 3 which is only to be used in section 42, even if that is the “natural” place for it. The goal is to introduce new ideas near where they are first used, so that motivations are clear.
- The structure of the exposition is organized around the following type of question: Here is a [definition we can make/theorem we can prove] for ordinary categories; how do we generalize it to quasicategories? In some cases the answer is easy. In others, it can require a significant detour.
- The exposition is largely from the bottom up, rather than from the top down. Thus, I attempt to give complete details about everything I prove, so that nothing is relegated to references. (The current document does not achieve this yet, but that is the plan; in some cases, such details will be put into appendices.)
- The idea is that, after you have read this book, you will be well-prepared to dip into the main references on quasicategories (e.g., Lurie’s books) without too much difficulty. Note that this book is not meant to (and does not) supplant any such reference.

**1.6. Prerequisites.** I assume only familiarity with basic concepts of category theory, such as from [Lei14], or as discussed in the first few chapters of [Rie16]. Some categorical prerequisites: you should be at least aware of the following notions (or know where to turn to in order to learn them):

- categories, functors, and natural transformations;
- full subcategories;
- groupoids;
- products and coproducts;
- pushouts and pullbacks;
- general colimits and limits.
- adjoint functors.

It is also helpful, but not essential, to know a little algebraic topology (such as fundamental groups and groupoids, and the definition of singular homology, as described in Chs. 1–3 of Hatcher’s textbook)

**1.7. References and other sources.** As noted, the material depends mainly on the work of Joyal and Lurie.

- Joyal’s first paper [Joy02] on the subject explicitly introduces quasicategories as a model for  $\infty$ -categories. It is worth looking at.
- There are several versions of unpublished lecture notes by Joyal [Joy08a], [Joy08b], which develop the theory of quasicategories from scratch. Also note the paper by Joyal and Tierney [JT08], which gives a summary of some of this unpublished work.

- Lurie’s “Higher topos theory” [Lur09] gives a complete development of  $\infty$ -categories, including many topics not even touched in this book. The main general material on  $\infty$ -categories is in Chapters 1–4, together with quite a bit of material from the appendices. It is also worth looking at Chapter 5, which develops the very important notions of accessible and presentable  $\infty$ -categories. The final two chapters apply these ideas to the theory of  $\infty$ -topoi.
- Lurie’s “Higher algebra” [Lur12] treats a number of “advanced topics”, including *stable  $\infty$ -categories* (the  $\infty$ -categorical foundations underlying derived categories in homological algebra and stable homotopy), various notions of monoidal structures on  $\infty$ -categories (via the theory of  $\infty$ -operads), and other topics.
- After I came up with the first version of these notes, Cisinski published the book “Higher Categories and Homotopical Algebra”. It covers much of the material in these notes (and much more), on roughly similar lines: in his book model categories play a more prominent role from the start than they do here.
- Bergner’s “The homotopy theory of  $(\infty, 1)$ -categories” is a survey of various approaches to higher categories and their interrelationships.
- Groth’s note “A short course on  $\infty$ -categories” provides a brief survey to some of the basic ideas about quasicategories and their applications. It is not a complete treatment, but it does get very quickly to some of the more advanced topics.
- Riehl and Verity ...
- Kerodon ...

1.8. **Things to add.** This is a place for me to remind myself of things I might add.

- A discussion of  $n$ -truncation and  $n$ -groupoids, including the equivalence of ordinary groupoids to 1-groupoids (so connecting with the introduction).
- Pointwise criterion for limits/colimits: Show that  $S^{\triangleright} \rightarrow \text{Fun}(D, C)$  is a colimit cone if each projection to  $S^{\triangleright} \rightarrow \text{Fun}(\{d\}, C) \approx C$  is one.

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## Part 1. Simplicial sets and nerves of categories

### 2. SIMPLICIAL SETS

In the subsequent sections, we will define quasicategories as a generalization of the notion of a category. To accomplish this, we will recharacterize categories as a particular kind of *simplicial set*; relaxing this characterization will lead us to the definition of quasicategories.

Simplicial sets were introduced as a combinatorial framework for the homotopy theory of spaces. There are a number of treatments of simplicial sets from this point of view. I recommend Greg Friedman’s survey [Fri12] as a starting place for learning about this viewpoint. Here I’ll focus on what we need in order to develop quasicategories.

2.1. **The simplicial indexing category  $\Delta$ .** We write  $\Delta$  for the category whose

- **objects** are the finite and non-empty totally ordered sets  $[n] := \{0 < 1 < \cdots < n\}$  for  $n \geq 0$ , and
- **morphisms**  $f: [n] \rightarrow [m]$  are weakly monotone functions, i.e., such that  $x \leq y$  implies  $f(x) \leq f(y)$ .



Note that we exclude the empty set from  $\Delta$ . Morphisms in  $\Delta$  are often called **simplicial operators**. simplicial operators

Because  $[n]$  is an ordered set, you can also think of it as a category: the objects are the elements of  $[n]$ , and there is a morphism (necessarily unique)  $i \rightarrow j$  if and only if  $i \leq j$ . Thus, morphisms in the category  $\Delta$  are precisely the functors between the categories  $[n]$ . We can, and will, also think of  $[n]$  as the category “freely generated” by the picture

$$\boxed{0} \rightarrow \boxed{1} \rightarrow \cdots \rightarrow \boxed{n-1} \rightarrow \boxed{n}.$$

Arbitrary non-identity morphisms  $i \rightarrow j$  in  $[n]$  can be expressed uniquely as iterated composites of the arrows  $i \rightarrow i+1$  which are displayed in the picture.

I will often use the following notation for morphisms in  $\Delta$ :

$$f = \langle f_0 \cdots f_n \rangle: [n] \rightarrow [m] \quad \text{with } f_0 \leq \cdots \leq f_n \quad \text{represents the function } k \mapsto f_k.$$

2.2. *Remark.* There are distinguished simplicial operators called **face** and **degeneracy** operators: face operator  
degeneracy operator

$$d^i := \langle 0, \dots, \widehat{i}, \dots, n \rangle: [n-1] \rightarrow [n], \quad 0 \leq i \leq n,$$

$$s^i := \langle 0 \dots i, i, \dots, n \rangle: [n+1] \rightarrow [n], \quad 0 \leq i \leq n.$$

All maps in  $\Delta$  can be obtained as a composition of face and degeneracy operators, and in fact  $\Delta$  can be described as the category *generated* by the above symbols, subject to a set of *relations* called the “simplicial identities”, which can be found in various places, e.g., [Fri12, Def. 3.2].

2.3. **Simplicial sets.** A **simplicial set** is a functor  $X: \Delta^{\text{op}} \rightarrow \text{Set}$ , i.e., a contravariant functor (or “presheaf”) from  $\Delta$  to sets. simplicial set

It is typical to write  $X_n$  for  $X([n])$ , and call it the set of  $n$ -*simplices* in  $X$ . I generally prefer to call it the set of  $n$ -**dimensional cells** (or just  $n$ -**cells**) of  $X$  instead (because the word “simplices” also applies to the so called “standard  $n$ -simplices” defined below (3.1), and I would like to avoid confusion between them). I will also sometimes speak of the set of all cells of  $X$ , i.e., of the disjoint union  $\coprod_{n \geq 0} X_n$  the sets  $X_n$ . n-dimensional cells  
n-cells

The 0-dimensional cells of a simplicial set are also called **vertices**, while the 1-dimensional cells are also called **edges**. vertices  
edges

Given a cell  $a \in X_n$  and a simplicial operator  $f: [m] \rightarrow [n]$ , I will write  $af \in X_m$  as shorthand for  $X(f)(a)$ . That is, I’ll think of simplicial operators as acting on cells from the right; this is a convenient choice given that  $X$  is a *contravariant* functor. In this language, a simplicial set consists of

- a sequence of sets  $X_0, X_1, X_2, \dots$ ,
- functions  $a \mapsto af: X_n \rightarrow X_m$  for each simplicial operator  $f: [m] \rightarrow [n]$ , such that
- $a \text{ id} = a$ , and  $(af)g = a(fg)$  for any cell  $a$  and simplicial operators  $f$  and  $g$  whenever this makes sense.

If I need to have the simplicial operator act from the left, I’ll write  $f^*(a) := af$ . Thus, a simplicial operator  $f: [m] \rightarrow [n]$  induces a function  $f^*: X_n \rightarrow X_m$  for any simplicial set  $X$ .

Sometimes I’ll use a subscript notation when speaking of the action of particular simplicial operators. So, given a simplicial operator of the form  $f = \langle f_0 \cdots f_m \rangle: [m] \rightarrow [n]$ , we can indicate the action of  $f$  on cells using subscripts:

$$a_{f_0 \cdots f_m} := af = a\langle f_0 \cdots f_m \rangle.$$

In particular, applying simplicial operators of the form  $\langle i \rangle: [0] \rightarrow [n]$  to an  $n$ -dimensional cell  $a \in X_n$  gives vertices  $a_0, \dots, a_n \in X_0$ , which we call the “*vertices of  $a$* ”, while applying simplicial operators of the form  $\langle ij \rangle: [1] \rightarrow [n]$  for  $0 \leq i \leq j \leq n$  gives edges  $a_{ij} \in X_1$ , which we call the “*edges of  $a$* ”.

**2.4. The category of simplicial sets.** A simplicial set is a functor. A **map of simplicial sets** (or **simplicial map**) is a natural transformation of functors. Explicitly, a map  $\phi: X \rightarrow Y$  between simplicial sets is a collection of functions  $\phi: X_n \rightarrow Y_n$ ,  $n \geq 0$ , which commute with simplicial operators:

map of simplicial sets  
simplicial map

$$(\phi a)f = \phi(af) \quad \text{for all simplicial operators } f \text{ and cells } a \text{ in } X, \text{ when this makes sense.}$$

I'll write  $s\text{Set}$  for the category of simplicial sets and maps between them<sup>3</sup>.

**2.5. Remark.** A simplicial set is not the same thing as a *abstract simplicial complex*, though there are some relationships between the two notions; see (6.14).

**2.6. Discrete simplicial sets.** A simplicial set  $X$  is **discrete** if every simplicial operator  $f$  induces a bijection  $f^*: X_n \rightarrow X_m$ .

discrete simplicial set

Every set  $S$  gives us a discrete simplicial set  $S^{\text{disc}}$ , defined so that  $(S^{\text{disc}})_n = S$ , and so that each simplicial operator acts according to the identity map of  $S$ . This construction defines a functor  $S \mapsto S^{\text{disc}}: \text{Set} \rightarrow s\text{Set}$ .

**2.7. Exercise.** Show that for any set  $S$  and simplicial set  $X$  there is a bijection  $\text{Hom}_{s\text{Set}}(S^{\text{disc}}, X) \rightarrow \text{Hom}_{\text{Set}}(S, X_0)$ .

**2.8. Exercise** (Discrete simplicial sets come from sets). Show that (i) every discrete simplicial set  $X$  is isomorphic to  $S^{\text{disc}}$  for the set  $S = X_0$ , and (ii) for every pair of sets  $S$  and  $T$ , the evident function  $\text{Hom}_{\text{Set}}(S, T) \rightarrow \text{Hom}_{s\text{Set}}(S^{\text{disc}}, T^{\text{disc}})$  is a bijection.

Let  $s\text{Set}^{\text{disc}}$  denote the full subcategory of  $s\text{Set}$  spanned by discrete simplicial sets. That is, objects of  $s\text{Set}^{\text{disc}}$  are discrete simplicial sets, and morphisms of  $s\text{Set}^{\text{disc}}$  are all simplicial maps between them. Then (2.8) amounts to saying that *the full subcategory of discrete simplicial sets is equivalent to the category of sets*.

For this reason, it is often convenient to (at least informally) “identify” sets with their corresponding discrete simplicial sets. Thus, given a set  $S$  we will abuse notation and also write  $S$  for the discrete simplicial set  $S^{\text{disc}}$  defined above.

### 3. STANDARD SIMPLICIES

**3.1. Standard  $n$ -simplex.** The **standard  $n$ -simplex**  $\Delta^n$  is the simplicial set defined by

standard  $n$ -simplex

$$\Delta^n := \text{Hom}_{\Delta}(-, [n]).$$

That is, the standard  $n$ -simplex is exactly the functor *represented* by the object  $[n]$ . Explicitly, this means that

$$(\Delta^n)_m = \text{Hom}_{\Delta}([m], [n]) = \{\text{simplicial operators } a: [m] \rightarrow [n]\},$$

while the action of simplicial operators on cells of  $\Delta^n$  is given by composition:  $f: [m'] \rightarrow [m]$  sends  $(a: [m] \rightarrow [n]) \in (\Delta^n)_m$  to  $(af: [m'] \rightarrow [n]) \in (\Delta^n)_{m'}$ .

The **generator** of  $\Delta^n$  is the cell

generator of a cell

$$\iota_n := \langle 01 \dots n \rangle = \text{id}_{[n]} \in (\Delta^n)_n$$

corresponding to the identity map of  $[n]$ .

The **Yoneda lemma** (applied to the category  $\Delta$ ) asserts that the function

Yoneda lemma

$$\begin{aligned} \text{Hom}_{s\text{Set}}(\Delta^n, X) &\rightarrow X_n, \\ g &\mapsto g(\iota_n), \end{aligned}$$

is a bijection for every simplicial set  $X$ . (*Exercise:* if this fact is not familiar to you, prove it.)

<sup>3</sup>Lurie [Lur09] uses  $\text{Set}_{\Delta}$  to denote the category of simplicial sets.

The Yoneda lemma can be stated this way: for each  $n$ -dimensional cell  $a \in X_n$  there exists a *unique* map  $f_a: \Delta^n \rightarrow X$  of simplicial sets which sends the generator to it, i.e., such that  $f_a(\iota_n) = a$ . We call the map  $f_a$  the **representing map** of the cell  $a$ .

representing map of a cell

We will often use the bijection provided by the Yoneda lemma implicitly. In particular, instead of using notation such as  $f_a$ , we will typically abuse notation and write  $a: \Delta^n \rightarrow X$  for the representing map of the cell  $a \in X_n$ . We reiterate that the map  $a: \Delta^n \rightarrow X$  is characterized as the *unique* map sending the generator  $\iota_n$  of  $\Delta^n$  to  $a$ . Thus with our notation we have  $a = a(\iota_n)$ , where the two appearances of “ $a$ ” denote respectively the cell of  $X_n$  and the representing morphism  $\Delta^n \rightarrow X$ .

**3.2. Exercise.** Show that the representing map  $f: \Delta^n \rightarrow X$  of  $a \in X_n$  sends  $\langle f_0 \cdots f_k \rangle \in (\Delta^n)_k$  to  $a\langle f_0 \cdots f_k \rangle \in X_k$ .

Note that if  $X = \Delta^m$  is also a standard simplex, then the Yoneda lemma gives a bijection

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \Delta^m) \xrightarrow{\sim} (\Delta^m)_n = \mathrm{Hom}_{\Delta}([n], [m]).$$

The inverse of this bijection sends a simplicial operator  $f: [n] \rightarrow [m]$  to the map  $\Delta^f: \Delta^n \rightarrow \Delta^m$  of simplicial sets defined on cells  $g \in (\Delta^n)_k = \mathrm{Hom}_{\Delta}([k], [n])$  by  $g \mapsto fg$ . (*Exercise:* prove this.)

I will commonly abuse notation, and write  $f: \Delta^n \rightarrow \Delta^m$  instead of  $\Delta^f$  for the map induced by the simplicial operator  $f$ , as it is also the representing map of the corresponding  $n$ -dimensional cell  $f \in (\Delta^m)_n$ .

**3.3. Terminal and initial simplicial sets.** The standard 0-simplex  $\Delta^0$  is the terminal object in  $\mathrm{sSet}$ ; i.e., for every simplicial set  $X$  there is a unique map  $X \rightarrow \Delta^0$ . Sometimes I write  $*$  instead of  $\Delta^0$  for this object. Note that  $\Delta^0$  is the only standard  $n$ -simplex which is discrete.

The **empty simplicial set**  $\emptyset$  is the functor  $\Delta^{\mathrm{op}} \rightarrow \mathrm{Set}$  sending each  $[n]$  to the empty set. It is the initial object in  $\mathrm{sSet}$ ; i.e., for every simplicial set  $X$  there is a unique map  $\emptyset \rightarrow X$ .

empty simplicial set

**3.4. Exercise.** Show that a simplicial set  $X$  is isomorphic to the empty simplicial set if and only if  $X_0$  is isomorphic to the empty set.

**3.5. Standard simplices on totally ordered sets.** The definition of the standard simplices  $\Delta^n$  can be extended to simplicial sets “generated” by arbitrary totally ordered sets. Thus, from any totally ordered set  $S$  we get a simplicial set  $\Delta^S$  with  $(\Delta^S)_n = \{\text{order preserving } [n] \rightarrow S\}$ .

Note that for any *non-empty and finite* totally ordered set  $S = \{s_0 < s_1 < \cdots < s_n\}$ , there is a *unique* order preserving bijection  $[n] \xrightarrow{\sim} S$  for a unique  $n \geq 0$ , so that there is a *unique* isomorphism  $\Delta^S \approx \Delta^n$  of simplicial sets (*Exercise:* prove this). When  $S = \emptyset$  we have  $(\Delta^\emptyset)_k = \emptyset$  for all  $k$ , so  $\Delta^\emptyset$  is the empty simplicial set.

This notation is especially convenient for subsets  $S \subseteq [n]$  with induced ordering, as the simplicial set  $\Delta^S$  is in a natural way a *subcomplex* of  $\Delta^n$  (i.e., a collection of cells of  $\Delta^n$  closed under action of simplicial operators; we will return to the notion of subcomplex below §6.5).

Furthermore, any simplicial operator  $f: [m] \rightarrow [n]$  factors through its image  $S = f([m]) \subseteq [n]$ , giving a factorization

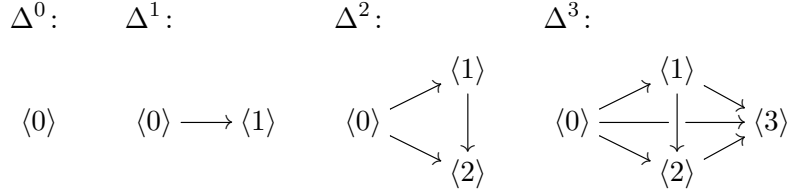
$$[m] \xrightarrow{f_{\mathrm{surj}}} S \xrightarrow{f_{\mathrm{inj}}} [n]$$

of maps between ordered sets, and thus a factorization  $\Delta^m \xrightarrow{\Delta^{f_{\mathrm{surj}}}} \Delta^S \xrightarrow{\Delta^{f_{\mathrm{inj}}}} \Delta^n$  of the induced map  $\Delta^f$  of simplicial sets, where  $\Delta^S$  can be regarded as a subcomplex of  $\Delta^n$ .

**3.6. Exercise.** Show that  $\Delta^{f_{\mathrm{inj}}}$  and  $\Delta^{f_{\mathrm{surj}}}$  respectively induce maps between simplicial sets which are (respectively) injective and surjective on sets of  $k$ -cells for all  $k$ . (The case of  $\Delta^{f_{\mathrm{inj}}}$  is formal, but the case of  $\Delta^{f_{\mathrm{surj}}}$  is not completely formal.)

**3.7. Pictures of standard simplices.** When we draw a “picture” of  $\Delta^n$ , we draw a geometric  $n$ -simplex: the convex hull of  $n + 1$  points in general position, with vertices labelled by  $0, \dots, n$ . The faces of the geometric simplex correspond exactly to *injective* simplicial operators into  $[n]$ : these cells are called *non-degenerate*. For each non-degenerate cell  $f$  in  $\Delta^n$ , there is an infinite collection of *degenerate* cells with the same “image” as  $f$  (when viewed as a simplicial operator with target  $[n]$ ).

Here are some “pictures” of standard simplices, which show their non-degenerate cells. Note that we draw the 1-cells of  $\Delta^n$  as arrows; this lets us easily see the total ordering on the vertices of  $\Delta^n$ .



We’ll extend the terminology of “degenerate” and “non-degenerate” cells to arbitrary simplicial sets in (17.5).

#### 4. THE NERVE OF A CATEGORY

The nerve of a category is a simplicial set which retains all the information of the original category. In fact, the nerve construction provides a full embedding of  $\mathbf{Cat}$ , the category of (small) categories, into  $\mathbf{sSet}$ , which means that we are able to think of categories as just a special kind of simplicial set.

**4.1. Construction of the nerve.** Given a category  $C$ , the **nerve** of  $C$  is the simplicial set  $NC$  defined so that

$$(NC)_n := \mathrm{Hom}_{\mathbf{Cat}}([n], C),$$

the set of functors from  $[n]$  to  $C$ , and so that simplicial operators  $f: [m] \rightarrow [n]$  act by precomposition:  $a \mapsto af$  for an element  $a: [n] \rightarrow C$  in  $(NC)_n$ .

**4.2. Example.** There is an evident isomorphism  $N[n] \approx \Delta^n$ , which is in fact the unique isomorphism between these two simplicial sets.

Given a functor  $F: C \rightarrow D$  between categories, we obtain a map  $NF: NC \rightarrow ND$  of simplicial sets, sending  $(a: [n] \rightarrow C) \in (NC)_n$  to  $(Fa: [n] \rightarrow D) \in (ND)_n$ . Thus the nerve construction defines a functor  $N: \mathbf{Cat} \rightarrow \mathbf{sSet}$ .

**4.3. Structure of the nerve.** We observe the following, whose verification we leave to the reader.

- $(NC)_0$  is canonically identified with the set of objects of  $C$ .
- $(NC)_1$  is canonically identified with the set of morphisms of  $C$ .
- The operators  $\langle 0 \rangle^*, \langle 1 \rangle^*: (NC)_1 \rightarrow (NC)_0$  assign to a morphism its source and target respectively.
- The operator  $\langle 00 \rangle^*: (NC)_0 \rightarrow (NC)_1$  assigns to an object its identity morphism.
- $(NC)_2$  is in bijective correspondence with the set of pairs  $(f, g)$  of morphisms such that  $gf$  is defined, i.e., such that the target of  $f$  is the source of  $g$ . This bijection is given by sending  $a \in (NC)_2$  to  $(a_{01}, a_{12}) \in (NC)_1 \times (NC)_1$ .
- The operator  $\langle 02 \rangle^*: (NC)_2 \rightarrow (NC)_1$  assigns, to a 2-cell corresponding to a pair  $(f, g)$  of morphisms, the composite morphism  $gf$ .

In particular, you can recover the category  $C$  from its nerve  $NC$ , up to isomorphism, since the nerve contains all information about objects, morphisms, identity morphisms, and composition of morphisms in  $C$ .

We have the following general description of  $n$ -dimensional cells in the nerve.

**4.4. Proposition.** *Let  $C$  be a category.*

(1) *There is a bijective correspondence*

$$(NC)_n \xrightarrow{\sim} \{ (g_1, \dots, g_n) \in (\text{mor } C)^{\times n} \mid \text{target}(g_{i-1}) = \text{source}(g_i) \},$$

*which sends  $(a: [n] \rightarrow C) \in (NC)_n$  to the sequence  $(a\langle 0, 1 \rangle, \dots, a\langle n-1, n \rangle)$*

(2) *With respect to the correspondence of (1), the map  $f^*: (NC)_n \rightarrow (NC)_m$  induced by a simplicial operator  $f: [m] \rightarrow [n]$  coincides with the function*

$$(g_1, \dots, g_n) \mapsto (h_1, \dots, h_m), \quad h_k = \begin{cases} \text{id} & \text{if } f(k-1) = f(k) \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k). \end{cases}$$

*Proof.* For (1), one verifies that an inverse is given by the function which sends a sequence  $(g_1, \dots, g_n)$  to  $(a: [n] \rightarrow C) \in (NC)_n$  defined on objects by  $a(k) = \text{target}(g_{k-1}) = \text{source}(g_k)$ , and on morphisms by  $a(\langle ij \rangle) = g_j g_{j-1} \cdots g_{i+1}$  for  $i < j$ . For (2), note that for  $a \in (NC)_n$  corresponding to the tuple  $(g_1, \dots, g_n)$  we can compute

$$(af)\langle k-1, k \rangle = a\langle f(k-1), f(k) \rangle = \begin{cases} \text{id} & \text{if } f(k-1) = f(k), \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k). \end{cases}$$

□

**4.5. Remark.** It is clear from the above remarks that most of the information in the nerve of  $C$  is redundant: we only needed  $(NC)_k$  for  $k = 0, 1, 2$  and certain simplicial operators between them to recover complete information about the category  $C$ .

**4.6. Exercise.** Show that for any discrete simplicial set  $X$  there exists a category  $C$  and an isomorphism  $NC \approx X$ .

**4.7. Characterization of nerves.** This leads to the question: given a simplicial set  $X$ , how can we detect that it is isomorphic to the nerve of some category?

**4.8. Proposition.** *A simplicial set  $X$  is isomorphic to the nerve of some category if and only if for all  $n \geq 2$  the function*

$$\phi_n: X_n \rightarrow \{ (g_1, \dots, g_n) \in (X_1)^{\times n} \mid g_{i-1}\langle 1 \rangle = g_i\langle 0 \rangle, 1 \leq i \leq n \}$$

*which sends  $a \in X_n$  to  $(a_{0,1}, \dots, a_{n-1,n})$  is a bijection.*

*Proof.* First, suppose  $X = NC$  for some category  $C$ . Then the function  $\phi_n$  is precisely the bijection of (4.4)(1). Thus, if  $X$  is isomorphic to the nerve of some category then its  $\phi_n$  are bijections.

Now suppose  $X$  is a simplicial set such that the  $\phi_n$  are bijections. We *define* a category  $C$ , with

$$(\text{objects of } C) = X_0, \quad (\text{morphisms of } C) = X_1,$$

following the discussion in (4.3). Thus, the source and target of  $g \in X_1$  are  $g_0$  and  $g_1$  in  $X_0$  respectively, the identity map of  $x \in X_0$  is  $x_{00} \in X_1$ , while the composite of  $(g, h)$  such that  $g_1 = h_0$  is  $a_{02}$ , where  $a \in X_2$  is the unique 2-cell with  $a_{01} = g$  and  $a_{12} = h$ . We leave the remaining details (e.g., unit and associativity properties) to the reader, though we note that proving associativity requires consideration of cells of  $X_3$ . (Or look ahead to (7.10), where we carry out the argument explicitly in a slightly different context.)

Next, we claim that for  $a \in X_n$ , and for  $0 \leq i \leq j \leq k \leq n$ , we have that

$$a_{i,k} = a_{j,k} a_{i,j},$$

where  $a_{i,k}, a_{i,j}, a_{j,k} \in X_1$  are images of  $a$  under face operators  $[1] \rightarrow [n]$ , and right-hand side represents composition of two morphisms in  $C$ . To see this, note first that for  $b \in X_2$ , we have  $b_{0,2} = b_{1,2} b_{0,1}$  by construction of  $C$ . The general case follows from this by setting  $b = a_{i,j,k}$ .

Now we can define maps  $\psi_n: X_n \rightarrow (NC)_n$  by sending  $a \in X_n$  to  $\psi(a): [n] \rightarrow C$  defined by  $\psi(a)(i \rightarrow j) = a_{i,j}$ , which is a functor by the above remarks. These maps  $\psi_n$  are seen to be bijections using the bijections  $\phi_n$  and (4.4), since  $\psi_n(a)(i-1 \rightarrow i) = a_{i-1,i}$ . If  $f: [m] \rightarrow [n]$  is a simplicial operator, then we compute

$$\psi_m(af)(i \rightarrow j) = (af)_{i,j} = a_{f(i),f(j)} = (\psi_n(a))(f(i) \rightarrow f(j)) = (\psi_n(a)f)(i \rightarrow j),$$

whence  $\psi$  is a map of simplicial sets. We have thus constructed an isomorphism  $\phi: X \rightarrow NC$  of simplicial sets, as desired.  $\square$

**4.9. A characterization of maps between nerves.** Maps between nerves are the same as functors between categories.

**4.10. Proposition.** *The nerve functor  $N: \text{Cat} \rightarrow s\text{Set}$  is fully faithful. That is, every simplicial set map  $g: NC \rightarrow ND$  between nerves is of the form  $g = N(f)$  for a unique functor  $f: C \rightarrow D$ .*

*Proof.* We need to show that  $\text{Hom}_{\text{Cat}}(C, D) \rightarrow \text{Hom}_{s\text{Set}}(NC, ND)$  induced by the functor  $N$  is a bijection for all categories  $C$  and  $D$ . *Injectivity* is clear, as a functor  $f$  is determined by its effect on objects and morphisms, which is exactly the effect of  $N(f)$  on 0- and 1-cells of the nerves.

For *surjectivity*, observe that for any map  $g: NC \rightarrow ND$  of simplicial sets, we can define a candidate functor  $f: C \rightarrow D$ , defined on objects and morphisms by the action of  $g$  on 0-dimensional and 1-dimensional cells. That  $F$  has the correct action on identity maps follows from the fact that  $g$  commutes with the simplicial operator  $\langle 00 \rangle: [1] \rightarrow [0]$ . That  $f$  preserves composition uses (4.4) and the fact that  $g$  commutes with the simplicial operator  $\langle 02 \rangle: [1] \rightarrow [2]$ .

Note that given  $g: NC \rightarrow ND$  and  $f: C \rightarrow D$  as constructed above, the maps  $g, N(f): NC \rightarrow ND$  coincide on 0-dimensional and 1-dimensional cells by construction. It follows that  $g = N(f)$  by (4.11) below. Thus, we have shown that  $N: \text{Hom}_{\text{Cat}}(C, D) \rightarrow \text{Hom}_{s\text{Set}}(NC, ND)$  is surjective as desired.  $\square$

**4.11. Exercise** (Maps to a nerve are determined by edges). Show that if  $C$  is a category and  $X$  is any simplicial set (not necessarily a nerve), then two maps  $g, g': X \rightarrow NC$  are equal if and only if  $g_1 = g'_1: X_1 \rightarrow (NC)_1$ , i.e.,  $g$  and  $g'$  are equal if and only if they coincide on edges. (Hint: use (4.4).)

## 5. SPINES

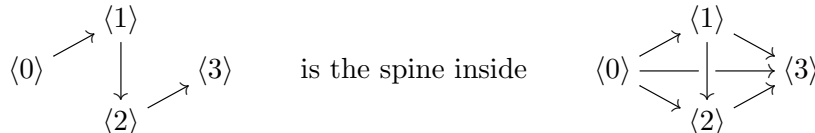
In this section we will restate our characterization of simplicial sets which are isomorphic to nerves, in terms of a certain “extension” condition. To state this condition we need the notion of a “spine” of a standard  $n$ -simplex.

**5.1. The spine of an  $n$ -simplex.** The **spine** of the  $n$ -simplex  $\Delta^n$  is the simplicial set  $I^n$  defined by

$$(I^n)_k = \{ \langle a_0 \cdots a_k \rangle \in (\Delta^n)_k \mid a_k \leq a_0 + 1 \}.$$

That is, a  $k$ -dimensional cell of  $I^n$  is a simplicial operator  $a: [k] \rightarrow [n]$  whose image is of the form either  $\{j\}$  or  $\{j, j+1\}$ . The action of simplicial operators on cells of  $I^n$  is induced by their action on  $\Delta^n$ . (To see that this action is well defined, observe that for  $a: [k] \rightarrow [n]$  in  $(I^n)_k$  and  $f: [p] \rightarrow [k]$ , the image of the simplicial operator  $af$  is contained in the image of  $a$ .)

There is an evident injective map  $I^n \rightarrow \Delta^n$  of simplicial sets. (In fact,  $I^n$  is another example of a *subcomplex* of  $\Delta^n$ , see below (6.5).) Here is a picture of  $I^3$  in  $\Delta^3$ :



Note that  $I^0 = \Delta^0$  and  $I^1 = \Delta^1$ .

The key property of the spine is the following.

**5.2. Proposition.** *Given a simplicial set  $X$ , for every  $n \geq 0$  there is a bijection*

$$\mathrm{Hom}(I^n, X) \xrightarrow{\sim} \{ (a_1, \dots, a_n) \in (X_1)^{\times n} \mid a_i \langle 1 \rangle = a_{i+1} \langle 0 \rangle \},$$

*defined by sending  $f: I^n \rightarrow X$  to  $(f(\langle 01 \rangle), f(\langle 12 \rangle), \dots, f(\langle n-1, n \rangle))$ . (In the case  $n = 0$ , the target of the bijection is taken to be the set  $X_0$  of vertices of  $X$ , and the bijection in this case sends  $f \mapsto f \langle 0 \rangle$ .)*

We will give the proof at the end of the next section (6.17), after we describe  $I^n$  as a colimit of a diagram of standard simplices; specifically, as a collection of 1-simplices “glued” together at their ends.

**5.3. Nerves are characterized by unique spine extensions.** We can now state our new characterization of nerves: they are simplicial sets such that every map  $I^n \rightarrow X$  from a spine extends *uniquely* along  $I^n \subseteq \Delta^n$  to a map from the standard  $n$ -simplex. That is, nerves are precisely the simplicial sets with “unique spine extensions”.

**5.4. Proposition.** *A simplicial set  $X$  is isomorphic to the nerve of some category if and only if the restriction map  $\mathrm{Hom}(\Delta^n, X) \rightarrow \mathrm{Hom}(I^n, X)$  along  $I^n \subseteq \Delta^n$  is a bijection for all  $n \geq 2$ .*

*Proof.* Immediate from (5.2) and (4.8). □

## 6. COLIMITS OF SIMPLICIAL SETS AND SUBCOMPLEXES

**6.1. Colimits of sets and simplicial sets.** Given any functor  $F: C \rightarrow \mathbf{Set}$  from a small category to sets, there is a “simple formula” for its colimit. First consider the coproduct (i.e., disjoint union)  $\coprod_{c \in \mathrm{ob} C} F(c)$  of the values of the functor; I’ll write  $(c, x)$  for a typical element of this coproduct, with  $c \in \mathrm{ob} C$  and  $x \in F(c)$ . Consider the relation  $\sim$  on this defined by

$$(c, x) \sim (c', x') \quad \text{if } \exists \alpha: c \rightarrow c' \text{ in } C \text{ such that } F(\alpha)(x) = x'.$$

Define

$$X := \left( \coprod_{c \in \mathrm{ob} C} F(c) \right) / \sim,$$

the set obtained as the quotient by the equivalence relation “ $\sim$ ” which is *generated* by the relation “ $\sim$ ”. For each object  $c$  of  $C$  we have a function  $i_c: F(c) \rightarrow X$  defined by  $i_c(x) := [(c, x)]$ , sending  $x$  to the equivalence class of  $(c, x)$ . Then the data  $(X, \{i_c\})$  is a colimit of the functor  $F$ : i.e., for any set  $S$  and collection of functions

$$f_c: F(c) \rightarrow S \quad \text{such that} \quad f_{c'} \circ F(\alpha) = f_c \quad \text{for all } \alpha: c \rightarrow c'$$

there exists a *unique* function  $f: X \rightarrow S$  such that  $f \circ i_c = f_c$ .

**6.2. Example.** Verify that  $(X, \{i_c\})$  is in fact a colimit of  $F$ .

We write  $\mathrm{colim}_C F$  for a chosen colimit of  $F$ .

Note that if the relation “ $\sim$ ” is not itself an equivalence relation, it can be difficult to figure out what “ $\approx$ ” actually is: the simple formula may not be so simple in practice.

**6.3. Exercise.** If  $C$  is a groupoid, then the relation  $\sim$  is already an equivalence relation.

There are cases when things are more tractable.

**6.4. Proposition.** *Let  $\mathcal{A}$  be a collection of subsets of a set  $S$ . Regard  $\mathcal{A}$  as a partially ordered set under “ $\subseteq$ ”, and hence as a category. Suppose  $\mathcal{A}$  has the following property: for all  $s \in S$ , and  $T, U \in \mathcal{A}$  such that  $s \in T \cap U$ , there exists  $V \in \mathcal{A}$  such that  $s \in V \subseteq T \cap U$ . Then the tautological map*

$$\operatorname{colim}_{T \in \mathcal{A}} T \rightarrow \bigcup_{T \in \mathcal{A}} T$$

*(sending  $[(T, t)] \mapsto t$ ) is a bijection.*

*Proof.* Show that  $(T, t) \approx (T', t')$  if and only if  $t = t'$ . The remaining details are left for the reader.  $\square$

Note: an easy way to satisfy the hypothesis of (6.4) is to show that  $\mathcal{A}$  is closed under pairwise intersection, i.e., that  $T, U \in \mathcal{A}$  implies  $T \cap U \in \mathcal{A}$ .

**6.5. Subcomplexes.** Given a simplicial set  $X$ , a **subcomplex** is just a subfunctor of  $X$ ; i.e., a collection of subsets  $A_n \subseteq X_n$  which are closed under the action of simplicial operators, and thus form a simplicial set so that the inclusion  $A \rightarrow X$  is a morphism of simplicial sets. We typically write  $A \subseteq X$  when  $A$  is a subcomplex of  $X$ . subcomplex

**6.6. Example.** Examples we have already seen include the spines  $I^n \subseteq \Delta^n$  and the  $\Delta^S \subseteq \Delta^n$  associated to subsets  $S \subseteq [n]$ .

**6.7. Exercise.** For any map  $f: X \rightarrow Y$  of simplicial sets, the image  $f(X) \subseteq Y$  of  $f$  is a subcomplex of  $Y$ .

For every set  $S$  of cells of a simplicial set, there is a smallest subcomplex which contains the set, namely the intersection of all subcomplexes containing  $S$ .

**6.8. Example.** For a vertex  $x \in X_0$ , we write  $\{x\} \subseteq X$  for the smallest subcomplex which contains  $x$ . This subcomplex has exactly one  $n$ -dimensional cell for each  $n \geq 0$ , namely  $x\langle 0 \cdots 0 \rangle$ , and thus is isomorphic to  $\Delta^0$ .

More generally, for a collection of vertices  $a, b, c, \dots \in X_0$ , we write  $\{a, b, c, \dots\} \subseteq X$  for the smallest subcomplex which contains  $a, b, c, \dots$ . This subcomplex is a discrete simplicial set. This choice of notation is supported by our informal identification of discrete simplicial sets with sets (2.6).

The result (6.4) carries over to simplicial sets, where the role of subsets is replaced by subcomplexes.

**6.9. Proposition.** *Let  $\mathcal{A}$  be a collection of subcomplexes of a simplicial set  $X$ . Regard  $\mathcal{A}$  partially ordered set under “ $\subseteq$ ”, and hence as a category. Suppose  $\mathcal{A}$  has the following property: for all  $n \geq 0$ , all  $x \in X_n$ , and all  $K, L \in \mathcal{A}$  such that  $x \in K_n \cap L_n$ , there exists  $M \in \mathcal{A}$  such that  $x \in M_n$  and  $M \subseteq K \cap L$ . Then the tautological map*

$$\operatorname{colim}_{K \in \mathcal{A}} K \rightarrow \bigcup_{K \in \mathcal{A}} K$$

*is a bijection.*

*Proof.* Because simplicial sets are actually functors  $\Delta^{\text{op}} \rightarrow \text{Set}$ , colimits in simplicial sets are “computed degreewise”. That is, if  $F: C \rightarrow s\text{Set}$  is a functor with colimit  $Y = \operatorname{colim}_{c \in C} F(c) \in s\text{Set}$ , then for each  $n \geq 0$  there is a canonical bijection

$$Y_n \approx \operatorname{colim}_{c \in C} F(c)_n.$$

The proposition follows using this observation and (6.4).  $\square$



6.10. *Remark* (Pushouts of subcomplexes). A special case of (6.9) applied to simplicial sets which we will use often is the following. If  $K$  and  $L$  are subcomplexes of a simplicial set  $X$ , then so are both  $K \cap L$  and  $K \cup L$ , and furthermore the evident commutative square

$$\begin{array}{ccc} K \cap L & \hookrightarrow & L \\ \downarrow & & \downarrow \\ K & \hookrightarrow & K \cup L \end{array}$$

is a pushout square in simplicial sets, i.e.,  $K \cup L \approx \operatorname{colim}[K \leftarrow K \cap L \rightarrow L]$ . (Proof: (6.9) with  $\mathcal{A} = \{K, L, K \cap L\}$ .)

6.11. **Subcomplexes of  $\Delta^n$ .** For each  $S \subseteq [n]$  we have a subcomplex  $\Delta^S \subseteq \Delta^n$ . The following says that every subcomplex of  $\Delta^n$  is a union of  $\Delta^S$ s.

6.12. **Lemma.** *Let  $K \subseteq \Delta^n$  be a subcomplex. If  $(f: [m] \rightarrow [n]) \in K_m$  with  $f([m]) = S$ , then  $f \in (\Delta^S)_m$  and  $\Delta^S \subseteq K$ .*

This proof uses the following elementary fact.

6.13. **Lemma.** *Any order preserving surjection  $f: S \rightarrow T$  between finite totally ordered sets admits an order preserving section, i.e.,  $s: T \rightarrow S$  such that  $fs = \operatorname{id}_T$ .*

*Proof.* Let  $s(t) = \min \{s \in S \mid f(s) = t\}$ . □

*Proof of (6.12).* Choose a section  $t: S \rightarrow [m]$  of  $f_{\operatorname{surj}}: [m] \rightarrow S$  (6.13). Consider a cell  $\bar{g} \in (\Delta^S)_k \subseteq (\Delta^n)_k$ , represented by a map  $\bar{g}: [k] \rightarrow [n]$  whose image is contained in  $S$ . We get a commutative diagram

$$\begin{array}{ccccc} & & [k] & & \\ & s:=tg \swarrow & \downarrow g & \searrow \bar{g} & \\ [m] & \xrightarrow{f_{\operatorname{surj}}} & S & \xrightarrow{f_{\operatorname{inj}}} & [n] \end{array}$$

so  $\bar{g} = fs$  and hence is a cell of the subcomplex  $K$  since  $f$  is. Thus  $\Delta^S \subseteq K$ , and it is immediate that  $f \in (\Delta^S)_m$ . □

6.14. *Remark.* Thus, a subcomplex  $K \subseteq \Delta^n$  determines and is determined by a collection  $\mathcal{K}$  of non-empty subsets of  $[n]$  with the property that  $T \subseteq S$  and  $S \in \mathcal{K}$  implies  $T \in \mathcal{K}$ : namely,

$$\mathcal{K} = \{S \subseteq [n] \mid \Delta^S \subseteq K\} \quad \text{and} \quad K = \bigcup_{S \in \mathcal{K}} \Delta^S.$$

In other words, a subcomplex of  $\Delta^n$  is the “same thing” as an *abstract simplicial complex* whose vertex set is a subset of  $[n]$ .

We can sharpen (6.12): every subcomplex of  $\Delta^n$  is a *colimit* of subcomplexes of the form  $\Delta^S$ .

6.15. **Proposition.** *Let  $K \subseteq \Delta^n$  be a subcomplex. Let  $\mathcal{A}$  be the poset of all non-empty subsets  $S \subseteq [n]$  such that  $\Delta^S \subseteq K$ . Then the tautological map*

$$\operatorname{colim}_{S \in \mathcal{A}} \Delta^S \rightarrow K$$

*is an isomorphism.*

*Proof.* We must show that for each  $m \geq 0$ , the map  $\operatorname{colim}_{S \in \mathcal{A}} (\Delta^S)_m \rightarrow K_m$  is a bijection. Each  $(\Delta^S)_m = \{f: [m] \rightarrow [n] \mid f([m]) \subseteq S\}$  is a distinct subset of  $K_m \subseteq (\Delta^n)_m$ ; i.e.,  $S \neq S'$  implies  $(\Delta^S)_m \neq (\Delta^{S'})_m$ . In view of (6.9), it suffices to show that for each  $f \in K_m$  there is a minimal  $S$  in  $\mathcal{A}$  such that  $f \in (\Delta^S)_m$ . This is immediate from (6.12), which says that  $f \in (\Delta^S)_m$  and  $\Delta^S \subseteq K$  where  $S = f([m])$ , and it is obvious that this  $S$  is minimal with this property. □

The **product**  $X$  and  $Y$  is defined to be their product as functors, so the set of  $n$ -cells of the product  $X \times Y$  is the set  $X_n \times Y_n$  of pairs  $(a, b)$  of  $n$ -cells in  $X$  and  $Y$ , with simplicial operators  $f$  acting by  $(a, b) \mapsto (af, bf)$ . product of simplicial sets

6.16. *Exercise* (Very important: product of 1-simplices). Let  $X = \Delta^1 \times \Delta^1$ . Let  $A, B \subseteq X$  be subcollections of cells in  $X$  defined so that

$$(f, g) \in \begin{cases} A & \text{if } f(i) \leq g(i) \text{ for all } i = 0, \dots, n, \\ B & \text{if } f(i) \geq g(i) \text{ for all } i = 0, \dots, n. \end{cases}$$

where  $f, g: [n] \rightarrow [1]$  are simplicial operators. Show that  $A$  and  $B$  are subcomplexes of  $X$  which are each isomorphic to  $\Delta^2$ , and that  $A \cap B$  is isomorphic to  $\Delta^1$ . Use this to show that the following is a pushout square of simplicial sets.

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{\langle 02 \rangle} & \Delta^2 \\ \langle 02 \rangle \downarrow & & \downarrow \langle \langle 011 \rangle, \langle 001 \rangle \rangle \\ \Delta^2 & \xrightarrow{\langle \langle 001 \rangle, \langle 011 \rangle \rangle} & \Delta^1 \times \Delta^1 \end{array}$$

That is, in simplicial sets a “square” can be obtained by gluing two “triangles” along a common edge, as suggested by the following picture.

$$\begin{array}{ccc} (0, 1) & \longrightarrow & (1, 1) \\ \uparrow & \nearrow & \uparrow \\ (0, 0) & \longrightarrow & (1, 0) \end{array}$$

6.17. **Proof of (5.2).** Now we can prove our claim about maps out of a spine, using an explicit description of a spine as a colimit.

*Proof of (5.2).* Let  $\mathcal{A}$  be the poset of all non-empty  $S \subseteq [n]$  such that  $\Delta^S \subseteq I^n$ ; i.e., subsets of  $[n]$  of the form  $\{j\}$  or  $\{j, j+1\}$ . Explicitly the poset  $\mathcal{A}$  has the form

$$\{0\} \rightarrow \{0, 1\} \leftarrow \{1\} \rightarrow \{1, 2\} \leftarrow \{2\} \rightarrow \dots \leftarrow \{n-1\} \rightarrow \{n-1, n\} \leftarrow \{n\}.$$

By (6.15),  $\operatorname{colim}_{S \in \mathcal{A}} \Delta^S \rightarrow I^n$  is an isomorphism. Thus  $\operatorname{Hom}(I^n, X) \approx \operatorname{Hom}(\operatorname{colim}_{S \in \mathcal{A}} \Delta^S, X) \approx \lim_{S \in \mathcal{A}} \operatorname{Hom}(\Delta^S, X)$ , and an elementary argument gives the result.  $\square$

## 7. HORNS AND INNER HORNS

We now are going to give another (less obvious!) characterization of nerves, in terms of “extending inner horns”, rather than “extending spines”. It will be this characterization that we “weaken” to obtain the definition of a quasicategory.

7.1. **Definition of horns.** We define a collection of subobjects of the standard simplices, called “horns”. For each  $n \geq 1$ , these are subcomplexes  $\Lambda_j^n \subset \Delta^n$  for each  $0 \leq j \leq n$ . The **horn**  $\Lambda_j^n$  is the subcomplex of  $\Delta^n$  defined by horn

$$(\Lambda_j^n)_k = \{ f: [k] \rightarrow [n] \mid ([n] \setminus \{j\}) \not\subseteq f([k]) \}.$$

Using the fact (6.15) that subcomplexes of  $\Delta^n$  are always unions of  $\Delta^S$ s, we see that  $\Lambda_j^n$  is the union of “faces”  $\Delta^{[n] \setminus i}$  of  $\Delta^n$  other than the  $j$ th face:

$$\Lambda_j^n = \bigcup_{i \neq j} \Delta^{[n] \setminus i} \subset \Delta^n.$$

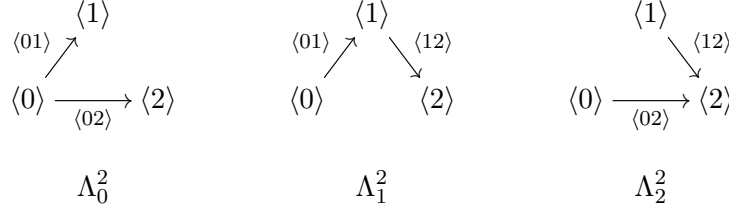
When  $0 < j < n$  we say that  $\Lambda_j^n \subset \Delta^n$  is an **inner horn**. We also say it is a **left horn** if  $j < n$  and inner horn  
left horn

a **right horn** if  $0 < j$ . Sometimes I'll speak of an **outer horn**, meaning a horn  $\Lambda_j^n$  with  $j \in \{0, n\}$ , i.e., a non-inner horn.

right horn  
outer horn

7.2. *Example.* The horns inside  $\Delta^1$  are just the vertices viewed as subobjects:  $\Lambda_0^1 = \Delta^{\{0\}} = \{0\} \subset \Delta^1$  and  $\Lambda_1^1 = \Delta^{\{1\}} = \{1\} \subset \Delta^1$ . Neither is an inner horn, the first is a left horn, and the second is a right horn.

7.3. *Example.* These are the three horns inside the 2-simplex.



Only  $\Lambda_1^2$  is an inner horn, while  $\Lambda_0^2$  and  $\Lambda_1^2$  are left horns, and  $\Lambda_1^2$  and  $\Lambda_2^2$  are right horns. Note that  $\Lambda_1^2$  is the same as the spine  $I^2$ .

7.4. *Exercise.* Visualize the four horns inside the 3-simplex. The simplicial set  $\Lambda_j^3$  actually kind of looks like a horn: you blow into the vertex  $\langle j \rangle$ , and sound comes out of the opposite missing face  $\Delta^{[3] \setminus j}$ .

7.5. *Exercise.* Show that  $\Lambda_j^n$  is the largest subcomplex of  $\Delta^n$  which does not contain the cell  $\langle 0 \cdots \widehat{j} \cdots n \rangle \in (\Delta^n)_{n-1}$ , the “face opposite the vertex  $j$ ”.

We note that inner horns always contain spines:  $I^n \subseteq \Lambda_j^n$  if  $0 < j < n$ . This is also true for outer horns if  $n \geq 3$ , but not for outer horns with  $n = 1$  or  $n = 2$ .

7.6. **The inner horn extension criterion for nerves.** We can now characterize nerves as those simplicial sets which admit “unique inner horn extensions”; this is different than, but analogous to, the characterization in terms of unique spine extensions (5.4).

7.7. **Proposition.** *A simplicial set  $X$  is isomorphic to the nerve of a category, if and only if  $\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_j^n, X)$  is a bijection for all  $n \geq 2$ ,  $0 < j < n$ .*

The proof will take up the rest of the section.

7.8. **Nerves have unique inner horn extensions.** First we show that nerves have unique inner horn extensions.

7.9. **Proposition.** *If  $C$  is a category, then for every inner horn  $\Lambda_j^n \subset \Delta^n$  the evident restriction map*

$$\text{Hom}(\Delta^n, NC) \rightarrow \text{Hom}(\Lambda_j^n, NC)$$

*is a bijection.*

*Proof.* Since inner horns contain spines, we can consider restriction along  $I^n \subseteq \Lambda_j^n \subseteq \Delta^n$ . The composite

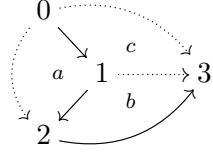
$$\text{Hom}(\Delta^n, NC) \rightarrow \text{Hom}(\Lambda_j^n, NC) \xrightarrow{r} \text{Hom}(I^n, NC)$$

of restriction maps is a bijection (5.4), so  $r$  is a surjection. Thus, it suffices to show that  $r$  is injective.

Recall (4.11) that a map of simplicial sets  $X \rightarrow NC$  to a nerve is uniquely determined by its values on edges. Thus we can reduce to the following: given a map  $f: \Lambda_j^n \rightarrow NC$  from an inner horn to a nerve, the values  $f$  takes on arbitrary edges  $\langle a, b \rangle \in (\Lambda_j^n)_1$  are uniquely determined by the values it takes on edges in the spine  $(I^n)_1$ , i.e., edges of the form  $\langle a, a \rangle$  and  $\langle a, a+1 \rangle$ . There are three cases.

$n = 2$ : The claim is immediate since  $I^2 = \Lambda_1^2$ .

$n = 3$ : Here is a picture of  $I^3 \subset \Lambda_1^3 \subset \Delta^3$ , showing all nondegenerate 0,1, and 2 cells.



Note that  $\Lambda_1^3$  is the smallest subcomplex of  $\Delta^3$  containing the 2-cells  $a, b, c$ . Since the target of  $f: \Lambda_1^3 \rightarrow NC$  is the nerve of a category, we can check by hand that the values of  $f$  on edges is determined by its value on  $I^3$ . In fact, only three edges of  $\Lambda_1^3$  are not in its spine, and we have

$$f\langle 02 \rangle = f\langle 12 \rangle \circ f\langle 01 \rangle, \quad f\langle 13 \rangle = f\langle 23 \rangle \circ f\langle 12 \rangle, \quad f\langle 03 \rangle = f\langle 13 \rangle \circ f\langle 01 \rangle,$$

using the presence of the 2-cells  $\langle 012 \rangle, \langle 123 \rangle, \langle 013 \rangle$  in  $\Lambda_1^3$ . A similar argument applies for  $\Lambda_2^3 \subseteq \Delta^3$ .

$n \geq 4$ : In this case we have that the subcomplex  $\Lambda_j^n \subset \Delta^n$  contains all 0, 1, and 2 dimensional cells of  $\Delta^n$ . The argument that the value of a map  $f: \Lambda_j^n \rightarrow NC$  on edges is determined by its value on the spine proceeds much as the case  $n = 3$ : compute  $f\langle x, y \rangle$  by induction on the value of  $y - x \geq 1$ .

□

**7.10. Nerves are characterized by unique inner horn extension.** Let  $X$  be an arbitrary simplicial set, and suppose it has unique inner horn extensions, i.e., each restriction map  $\text{Hom}(\Delta^n, X) \rightarrow \text{Hom}(\Lambda_j^n, X)$  is a bijection for all  $0 < j < n$  with  $n \geq 2$ .

Observe that unique extension along  $\Lambda_1^2 \subset \Delta^2$ , defines a “composition law” on the set  $X_1$ . That is, given  $f, g \in X_1$  such that  $f_1 = g_0$  in  $X_0$ ,<sup>4</sup> there is a unique map

$$u: \Lambda_1^2 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \xrightarrow{(f,g)} X \quad \text{such that} \quad \langle 01 \rangle \mapsto f \in X_1, \quad \langle 12 \rangle \mapsto g \in X_1.$$

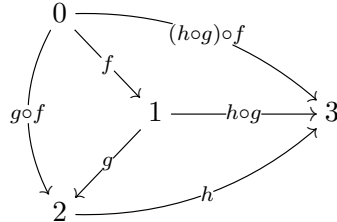
Let  $\tilde{u}: \Delta^2 \rightarrow X$  be the *unique* extension of  $u$  along  $\Lambda_1^2 \subset \Delta^2$ , and define the “composite”

$$g \circ f := \tilde{u}_{02}.$$

The 2-cell  $\tilde{u}$  is uniquely characterized by:  $\tilde{u}_{01} = f$ ,  $\tilde{u}_{12} = g$ , and  $g \circ f$  is defined by  $\tilde{u}_{02} = g \circ f$ .

This composition law is automatically unital. Given  $x \in X_0$ , write  $1_x := x\langle 00 \rangle \in X_1$ , so that  $(1_x)_0 = x = (1_x)_1$ . Then applying the composition law gives  $1_x \circ f = f$  and  $g \circ 1_x = g$ . (Proof: consider the 2-cells  $f\langle 011 \rangle, g\langle 001 \rangle \in X_2$ , and use the fact that their representing maps  $\Delta^2 \rightarrow X$  are the unique extensions of their restrictions to  $\Lambda_1^2 \subset \Delta^2$ .)

Now consider  $\Lambda_1^3 \subset \Delta^3$ . Recall (6.15) that  $\Lambda_1^3$  is a union (and colimit) of  $\Delta^S \subseteq \Delta^3$  such that  $S \not\supseteq \{0, 2, 3\}$ . A map  $v: \Lambda_1^3 \rightarrow X$  can be pictured as



so that the planar 2-cells in the picture correspond to non-degenerate 2-cells of  $\Delta^3$  which are contained in  $\Lambda_1^3$ , while the edges are labelled according to their images in  $X$ , using the composition

<sup>4</sup>Recall that  $f_1 = f\langle 1 \rangle$  and  $g_0 = g\langle 0 \rangle$ , regarded as maps  $\Delta^0 \rightarrow X$  and thus as elements of  $X_0$ , using the notation discussed in (2.3).

law defined above. Let  $\tilde{v}: \Delta^3 \rightarrow X$  be any extension of  $v$  along  $\Lambda_1^3 \subset \Delta^3$ , and consider the restriction  $w := \tilde{v}\langle 023 \rangle: \Delta^2 \rightarrow X$  to the face  $\Delta^2 \approx \Delta^{\{0,2,3\}} \subset \Delta^3$ . Then  $w_{01} = g \circ f$ ,  $w_{12} = h$ , and  $w_{02} = (h \circ g) \circ f$ , and thus the existence of  $w$  demonstrates that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

In other words, the *existence* of extensions along  $\Lambda_1^3 \subset \Delta^3$  implies that the composition law we defined above is associative. (We could carry out this argument using  $\Lambda_2^3 \subset \Delta^3$  instead.)

Thus, given an  $X$  with unique inner horn extensions, we can construct a category  $C$ , so that objects of  $C$  are elements of  $X_0$ , morphisms of  $C$  are elements of  $X_1$ , and composition is given as above.

Next we construct a map  $X \rightarrow NC$  of simplicial sets. There are obvious maps  $\alpha_n: X_n \rightarrow (NC)_n$ , corresponding to restriction along spines  $I^n \subseteq \Delta^n$ ; i.e.,  $\alpha(x) = (x_{01}, \dots, x_{n-1,n})$ . These maps are compatible with simplicial operators, so that they define a map  $\alpha: X \rightarrow NC$  of simplicial sets. *Proof:* For any  $n$ -cell  $x \in X_n$ , all of its edges are determined by edges on its spine via the (associative) composition law:  $x_{ij} = x_{j-1,j} \circ x_{j-2,j-1} \circ \dots \circ x_{i,i+1}$ , for all  $0 \leq i \leq j \leq n$ . Thus for  $f: [m] \rightarrow [n]$  we have  $\alpha(xf) = ((xf)_{01}, \dots, (xf)_{m-1,m}) = (x_{f_0f_1}, \dots, x_{f_{m-1}f_m}) = (x_{01}, \dots, x_{n-1,n})_{f_0 \dots f_m} = (\alpha x)f$ .

Now we can prove that nerves are characterized by unique extension along inner horns.

*Proof of (7.7).* We have already shown (7.9) that nerves have unique extensions for inner horns. Consider a simplicial set  $X$  which has unique inner horn extension. By the discussion above, we obtain a category  $C$  and a map  $\alpha: X \rightarrow NC$  of simplicial sets, which is clearly a bijection in degrees  $\leq 1$ . We will show  $\alpha_n: X_n \rightarrow (NC)_n$  is bijective by induction on  $n$ .

Fix  $n \geq 2$ , and consider the commutative square

$$\begin{array}{ccc} \mathrm{Hom}(\Delta^n, X) & \xrightarrow{\sim} & \mathrm{Hom}(\Lambda_1^n, X) \\ \alpha_{\Delta^n} \downarrow & & \downarrow \alpha_{\Lambda_1^n} \\ \mathrm{Hom}(\Delta^n, NC) & \xrightarrow{\sim} & \mathrm{Hom}(\Lambda_1^n, NC) \end{array}$$

The vertical maps are induced by post-composition with  $\alpha: X \rightarrow NC$ . The horizontal maps are induced by restriction along  $\Lambda_1^n \subset \Delta^n$ , and are bijections (top by hypothesis, bottom by (7.9)). Because  $\Lambda_1^n$  is a colimit of standard simplices of dimension  $< n$  (6.15), the map  $\alpha_{\Lambda_1^n}$  is a bijection by the induction hypothesis. Therefore so is  $\alpha_{\Delta^n}$ .  $\square$

## Part 2. Quasicategories

### 8. DEFINITION OF QUASICATEGORIES

We can now define the notion of a quasicategory, by removing the uniqueness part of the inner horn extension criterion for nerves.

**8.1. Identifying categories with their nerves.** From this point on, I will (at least informally) often not distinguish a category  $C$  from its nerve. In particular, I may assert something like “let  $C$  be a simplicial set which is a category”, which should be read as “ $C$  is a simplicial set which is isomorphic to the nerve of some category”. This should not lead to much confusion, due to the fact that the nerve functor is a fully faithful embedding of  $\mathbf{Cat}$  into  $\mathbf{sSet}$  (4.10). Note that under this informal identification, the linearly ordered category  $[n]$  is identified with the standard  $n$ -simplex  $\Delta^n$ .

**8.2. Definition of quasicategory.** A **quasicategory**<sup>5</sup> is a simplicial set  $C$  such that for every map  $f: \Lambda_j^n \rightarrow C$  from an inner horn, there *exists* an extension of it to  $g: \Delta^n \rightarrow C$ . That is,  $C$  is a quasicategory if the function  $\mathrm{Hom}(\Delta^n, C) \rightarrow \mathrm{Hom}(\Lambda_j^n, C)$  induced by restriction along  $\Lambda_j^n \subset \Delta^n$

quasicategory

<sup>5</sup>This is the same as what Lurie [Lur09] and many others call an  $\infty$ -category.

is *surjective* for all  $0 < j < n$ ,  $n \geq 2$ , so there always exists a dotted arrow in any commutative diagram of the form

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & C \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

Any category (more precisely, the nerve of any category) is a quasicategory. In fact, by what we have shown (7.7) a category is precisely a quasicategory for which there exist *unique* extensions of inner horns.

Let  $C$  be a quasicategory. We refer to elements of  $C_0$  as the **objects** of  $C$ , and elements of  $C_1$  as the **morphisms** of  $C$ . Every morphism  $f \in C_1$  has a **source** and **target**, namely its vertices  $f_0 = f\langle 0 \rangle, f_1 = f\langle 1 \rangle \in C_0$ . For  $f \in C_1$  we write  $f: f_0 \rightarrow f_1$ , just as we would for morphisms in a category. Likewise, for every object  $x \in C_0$ , there is a distinguished morphism  $1_x: x \rightarrow x$ , called the **identity morphism**, defined by  $1_x = x_{00} = x\langle 00 \rangle$ . When  $C$  is (the nerve of) a category, all the above notions coincide with the usual ones. Note, however, that we cannot generally define *composition* of morphisms in a quasicategory in the same way we do for a category.

objects  
morphisms  
source  
target  
  
identity morphism

We now describe some basic categorical notions which admit immediate generalizations to quasicategories. Many of these generalizations apply to arbitrary simplicial sets.

**8.3. Products of quasicategories.** Simplicial sets are functors, so the product of simplicial sets  $X$  and  $Y$  is just the product of the functors. Thus,  $(X \times Y)_n = X_n \times Y_n$ , with the evident action of simplicial operators:  $(x, y)f = (xf, yf)$ .

**8.4. Proposition.** *The product of two quasicategories (as simplicial sets) is a quasicategory.*

*Proof.* Exercise, using the bijective correspondence between the sets of (i) maps  $K \rightarrow X \times Y$  and (ii) pairs of maps  $(K \rightarrow X, K \rightarrow Y)$ .  $\square$

**8.5. Exercise.** If  $C$  and  $D$  are categories, then  $N(C \times D) \approx NC \times ND$ . Thus, the notion of product of quasicategories generalizes that of categories.

**8.6. Coproducts of quasicategories.** The coproduct of simplicial sets  $X$  and  $Y$  is just the coproduct of functors, whence  $(X \amalg Y)_n = X_n \amalg Y_n$ , i.e., the set  $n$ -cells of the coproduct is the disjoint union of the sets of  $n$ -cells of  $X$  and  $Y$ . More generally,  $(\coprod_s X_s)_n = \coprod_s (X_s)_n$  for an indexed collection  $\{X_s\}$  of simplicial sets.

**8.7. Proposition.** *The coproduct of any indexed collection of quasicategories is a quasicategory.*

To prove this, we introduce the set of **connected components** of a simplicial set. Given a simplicial set  $X$ , define an equivalence relation  $\approx$  on the set  $\coprod_{n \geq 0} X_n$  of cells of  $X$ , generated by the relation

$$a \sim af \quad \text{for all } n \geq 0, a \in X_n, f: [m] \rightarrow [n].$$

An equivalence class for  $\approx$  is called a **connected component** of  $X$ , and we write  $\pi_0 X$  for the set of connected components. This construction defines a functor  $\pi_0: s\text{Set} \rightarrow \text{Set}$ .

connected components

connected component

**8.8. Exercise** (Connected components are subcomplexes). Show that each connected component  $C$  (i.e., equivalence class of cells) of a simplicial set  $X$  is a subcomplex of  $X$ .

**8.9. Exercise.** Show that any map  $f: X \rightarrow Y$  of simplicial sets carries each connected component  $C \subseteq X$  into some connected component of  $Y$ , and thus induces a map  $\pi_0 X \rightarrow \pi_0 Y$ .

**8.10. Exercise** (A simplicial set is a coproduct of its connected components). Show that for any simplicial set  $X$  the map  $\coprod_{C \in \pi_0 X} C \rightarrow X$  of simplicial sets induced by inclusions of connected components is an isomorphism.

8.11. *Exercise* (Connected components are path components). Show that there is a canonical bijection

$$(X_0 / \approx_1) \xrightarrow{\sim} \pi_0 X,$$

where the left-hand side denotes the set of equivalence classes in the vertex set  $X_0$  with respect to the equivalence relation  $\approx_1$  which is *generated* by the relation  $\sim_1$  on  $X_0$ , defined by

$$a \sim_1 b \quad \text{iff there exists } e \in X_1 \text{ such that } a = e_0, b = e_1.$$

8.12. *Exercise* ( $\pi_0$  is the colimit). Show that there is a bijection  $\text{colim}_{\Delta^{\text{op}}} X \xrightarrow{\sim} \pi_0 X$ , between the set of connected components of  $X$  and the colimit of the functor  $X: \Delta^{\text{op}} \rightarrow \text{Set}$ .

We say that a simplicial set  $X$  is **connected** if  $\pi_0 X$  is a singleton.

connected

8.13. *Exercise*. Show that every standard simplex  $\Delta^n$  is connected, and that every horn  $\Lambda_j^n$  is connected.

*Proof of (8.7).* Consider a disjoint union  $X = \coprod_s X_s$  of simplicial sets. If  $x$  is a cell of  $X$  contained in some  $X_s$ , then any simplicial operator must carry  $x$  to another cell contained in  $X_s$ . Therefore any connected component of  $X$  is contained inside one of the  $X_s$  summands. The proof is now straightforward, using (8.9) and the fact that horns and standard simplices are connected (8.13).  $\square$

8.14. *Exercise* (Important). Show that the evident map  $\pi_0(X \times Y) \rightarrow \pi_0 X \times \pi_0 Y$  induced by projections is a bijection.

8.15. *Exercise* (Colimits of functors to sets via  $\pi_0$ ). Given a functor  $F: C \rightarrow \text{Set}$  from an ordinary category to sets, we can describe its colimit as the set of connected components of the nerve of its *point category*.

(1) Show that there is a 1-category  $C_F$  (the **point category** of  $F$ ) so that:

point category

- *objects* are pairs  $(c, x)$  where  $c \in C_0$  and  $x \in F(c)$ , and
- *morphisms*  $(c, x) \rightarrow (c', x')$  are maps  $\alpha: c \rightarrow c' \in C_1$  such that  $F(\alpha)(x) = x'$ .

(2) Show that there is an isomorphism  $\pi_0 N(C_F) \xrightarrow{\sim} \text{colim}_C F$ .

8.16. **Subcategories of quasicategories.** We say that a subcomplex  $C' \subseteq C$  of a quasicategory  $C$  is a **subcategory** if for all  $n \geq 2$  and  $0 < k < n$ , every  $f: \Delta^n \rightarrow C$  such that  $f(\Lambda_k^n) \subseteq C'$  satisfies  $f(\Delta^n) \subseteq C'$ . That is, *all* inner horn extensions in  $C$  along “horns in  $C'$ ” are themselves contained in  $C'$ . It is clear that a subcategory is in fact a quasicategory.

subcategory

8.17. *Exercise*. Let  $C$  be a quasicategory, and consider  $S \subseteq C_1$  a collection of morphisms in  $C$ . Define  $C'_n := \{a \in C_n \mid a_{ij} \in S \text{ for all } 0 \leq i < j \leq n\}$ . Show that the  $C'_n$ s describe a subcomplex  $C'$  of  $C$  if and only if  $f_{00}, f_{11} \in S$  for all  $f \in S$ . Show that furthermore  $C'$  is a subcategory if and only if, in addition, for all  $u \in C_2$  we have that  $u_{01}, u_{12} \in S$  implies  $u_{02} \in S$ .

When  $C$  is an ordinary category, a subcategory of  $C$  in the above sense is the same as a subcategory in the usual sense, which correspond exactly to subsets  $S \subseteq C_1$  of morphisms for which (i) if  $(x \xrightarrow{a} y) \in S$  then  $\text{id}_x, \text{id}_y \in S$ , and (ii)  $S$  is closed under composition.

8.18. *Remark*. In general, if  $C' \subseteq C$  is a subcomplex and  $C$  and  $C'$  are quasicategories, it need not be the case that  $C'$  is a subcategory of  $C$ . See (10.5) below.

8.19. **Full subcategories of quasicategories.** We say that a subcomplex  $C' \subseteq C$  of a quasicategory  $C$  is a **full subcategory** if for all  $n$  and all  $a \in C_n$ , we have that  $a \in C'_n$  if and only if  $a_i \in C'_0$  for all  $i = 0, \dots, n$ .

full subcategory

8.20. *Exercise*. Show that a full subcategory  $C' \subseteq C$  is in fact a subcategory as defined in (8.16), and thus in particular a full subcategory  $C'$  is itself a quasicategory.

Given a quasicategory  $C$  and a set  $S \subseteq C_0$  of vertices, let

$$C'_n = \{a \in C_n \mid a_j \in S \text{ for all } j = 0, \dots, n\},$$

the set of  $n$ -dimensional cells whose vertices are in  $S$ . This is evidently a full subcategory of  $C$ , called the **full subcategory spanned by  $S$** .

full  
subcategory  
spanned by  $S$

When  $C$  is an ordinary category, a full subcategory of  $C$  in the above sense is the same as a full subcategory in the usual sense.

**8.21. Opposite of a quasicategory.** Given a category  $C$ , the *opposite* category  $C^{\text{op}}$  has  $\text{ob } C^{\text{op}} = \text{ob } C$ , and  $\text{Hom}_{C^{\text{op}}}(x, y) = \text{Hom}_C(y, x)$ , and the sense of composition is reversed:  $g \circ_{C^{\text{op}}} f = f \circ_C g$ .

This concept also admits a generalization to quasicategories, which we define using a non-trivial involution  $\text{op}: \Delta \rightarrow \Delta$  of the category  $\Delta$ . This is the functor which on objects sends  $[n] \mapsto [n]$ , and on morphisms sends  $\langle f_0, \dots, f_n \rangle: [n] \rightarrow [m]$  to  $\langle m - f_n, \dots, m - f_0 \rangle$ , i.e.,  $\text{op}(f)(x) = m - f(n - x)$ .

**8.22. Remark.** You can visualize this involution as the functor which “reverses the ordering” of the totally-ordered sets  $[n]$ . Note that the totally ordered set “[ $n$ ] with the order of its elements reversed” isn’t actually an object of  $\Delta$ , but rather is uniquely isomorphic to  $[n]$ , via the function  $x \mapsto n - x$ .

The **opposite** of a simplicial set  $X: \Delta^{\text{op}} \rightarrow \text{Set}$  is the composite functor  $X^{\text{op}} := X \circ \text{op}$ . We have a unique isomorphism  $(\Delta^n)^{\text{op}} \approx \Delta^n$ , and this isomorphism restricts to  $(\Lambda_j^n)^{\text{op}} \approx \Lambda_{n-j}^n$ , so that the opposite of an inner horn is another inner horn. As a consequence, the opposite of a quasicategory is a quasicategory. It is straightforward to verify that  $(NC)^{\text{op}} = N(C^{\text{op}})$ , so the notion of opposite quasicategory generalizes the notion of opposite category. The functor  $\text{op}: \Delta \rightarrow \Delta$  satisfies  $\text{op} \circ \text{op} = \text{id}_\Delta$ , so  $(X^{\text{op}})^{\text{op}} = X$ .

opposite

## 9. FUNCTORS AND NATURAL TRANSFORMATIONS

**9.1. Functors.** A **functor** between quasicategories is merely a map  $f: C \rightarrow D$  between the simplicial sets.

functor

We write  $\text{qCat}$  for the category of quasicategories and functors between them. Clearly  $\text{qCat} \subset \text{sSet}$  is a full subcategory. Because the nerve functor is a full embedding of  $\text{Cat}$  into  $\text{qCat}$ , any functor between ordinary categories is also a functor between quasicategories.

**9.2. Exercise** (Mapping property of a full subcategory). Let  $C$  be a quasicategory, and  $C' \subseteq C$  the full subcategory spanned by some subset  $S \subseteq C_0$ . Show that a functor  $f: D \rightarrow C$  factors through a functor  $f': D \rightarrow C' \subseteq C$  if and only if  $f(D_0) \subseteq S$ .

**9.3. Natural transformations.** Given functors  $F, G: C \rightarrow D$  between categories, a *natural transformation*  $\phi: F \Rightarrow G$  is a choice, for each object  $c$  of  $C$ , of a map  $\phi(c): F(c) \rightarrow G(c)$  in  $D$ , such that for every morphism  $\alpha: c \rightarrow c'$  in  $C$  the square

$$\begin{array}{ccc} F(c) & \xrightarrow{\phi(c)} & G(c) \\ f(\alpha) \downarrow & & \downarrow g(\alpha) \\ F(c') & \xrightarrow{\phi(c')} & G(c') \end{array}$$

commutes in  $D$ .

There is a standard convenient reformulation of this: a natural transformation  $\phi: F \Rightarrow G$  is the same thing as a functor

$$H: C \times [1] \rightarrow D,$$

so that  $H|_{C \times \{0\}} = F$ ,  $H|_{C \times \{1\}} = G$ , and  $H|_{\{c\} \times [1]} = \alpha(c)$  for each  $c \in \text{ob } C$ . (Here we make implicit use of the evident isomorphisms  $C \times \{0\} \approx C \approx C \times \{1\}$ .)



This reformulation admits a straightforward generalization to quasicategories. A **natural transformation**  $h: f_0 \Rightarrow f_1$  of functors  $f_0, f_1: C \rightarrow D$  between quasicategories is defined to be a **natural transformation** map

$$h: C \times N[1] = C \times \Delta^1 \rightarrow D$$

of simplicial sets such that  $h|_{C \times \{i\}} = f_i$  for  $i = 0, 1$ . For ordinary categories this coincides with the classical notion.

## 10. EXAMPLES OF QUASICATEGORIES

There are many ways to produce quasicategories, as we will see. Unfortunately, “hands-on” constructions of quasicategories which are not ordinary categories are relatively rare. Here I give a few reasonably explicit examples to play with. For the most part these are not used in the subsequent text, except as an occasional source of examples.

**10.1. Large vs. small.** I have been implicitly assuming that certain categories are **small**; i.e., they have *sets* of objects and morphisms. For instance, for the nerve of a category  $C$  to be a simplicial set, we need  $C_0 = \text{ob } C$  to be a set. small

However, in practice many categories of interest are only **locally small**; i.e., the collection of objects is not a set but is a “proper class”, although for any pair of objects  $\text{Hom}_C(X, Y)$  is a set. For instance, the category *Set* of sets is of this type: there is no set of all sets. Other examples include the categories of abelian groups, topological spaces, (small) categories, simplicial sets, etc. It is also possible to have categories which are not even locally small, e.g., the category of locally small categories. These are called **large** categories. locally small  
large

We would like to be able to talk about large categories in exactly the same way we talk about small categories. This is often done by positing a hierarchy of (Grothendieck) “universes”. A universe  $U$  is (informally) a collection of sets which is closed under the operations of set theory. We additionally assume that for any universe  $U$ , there is a larger universe  $U'$  such that  $U \in U'$ . Thus, if by “set” we mean “ $U$ -set”, then the category *Set* is a “ $U'$ -category”. This idea can be implemented in the usual set theoretic foundations by postulating the existence of suitable *strongly inaccessible cardinals*.

The same distinctions occur for simplicial sets. For instance, the nerve of a small category is a small simplicial set (i.e., the elements form a set), while the nerve of a large category is a large simplicial set.

I’m not going to be pedantic about this. I’ll usually assume categories like *Set*, *Cat*, *sSet*, etc., are categories whose objects are “small” sets/categories/simplicial sets/whatever, i.e., are built from sets in a fixed universe  $U$  of “small sets”. However, I sometimes need to consider examples of sets/categories/simplicial sets/whatever which are not small. I leave it to the reader to determine when this is the case.

In practice, a main point of concern involves constructions such as limits and colimits. Many typical examples of categories  $C = \text{Set}, \text{Cat}, \text{sSet}$ , etc., in which objects are built out of small sets are **small complete** and **small cocomplete**: any functor  $F: D \rightarrow C$  from a *small* category  $D$  has a limit and a colimit in  $C$ . This is *not* true if  $D$  is not assumed to be small. In this case care about the small/large distinction is necessary. small complete  
small cocomplete

**10.2. The quasicategory of categories.** This is an example of a quasicategory in which *objects* are (small) categories, *morphisms* are functors between categories, and *2-dimensional cells* are certain kinds of natural isomorphisms of functors.

Define a simplicial set  $\text{Cat}_1$  so that  $(\text{Cat}_1)_n$  is a set whose elements are data  $x := (C_i, F_{ij}, \zeta_{ijk})$  where

- (0) for each  $i \in [n]$ ,  $C_i$  is a (small) category,
- (1) for each  $i \leq j$  in  $[n]$ ,  $F_{ij}: C_i \rightarrow C_j$  is a functor, and

(2) for each  $i \leq j \leq k$  in  $[n]$ ,  $\zeta_{ijk}: F_{ik} \Rightarrow F_{jk}F_{ij}$  is a natural *isomorphism* of functors  $C_i \rightarrow C_k$ , such that

- for each  $i$  in  $[n]$ ,  $F_{ii}: C_i \rightarrow C_i$  is the identity functor  $\text{Id}_{C_i}$  of  $C_i$ ,
- for each  $i \leq j$  in  $[n]$ ,  $\zeta_{iij}: F_{ij} \Rightarrow F_{ij} \text{Id}_{C_i}$  and  $\zeta_{ijj}: F_{ij} \Rightarrow \text{Id}_{C_j} F_{ij}$  are the identity natural isomorphism of  $F_{ij}$ , and
- for each  $i \leq j \leq k \leq \ell$ , the diagram

$$(10.3) \quad \begin{array}{ccc} F_{i\ell} & \xRightarrow{\zeta_{ij\ell}} & F_{j\ell}F_{ij} \\ \zeta_{ik\ell} \downarrow & & \downarrow \zeta_{j k \ell} F_{ij} \\ F_{k\ell}F_{ik} & \xRightarrow{F_{k\ell}\zeta_{ijk}} & F_{k\ell}F_{jk}F_{ij} \end{array}$$

of natural isomorphisms commutes.

For a simplicial operator  $\delta: [m] \rightarrow [n]$  define

$$(C_i, F_{ij}, \zeta_{ijk})\delta = (C_{\delta(i)}, F_{\delta(i)\delta(j)}, \zeta_{\delta(i)\delta(j)\delta(k)}).$$

10.4. *Example.* An 0-cell in  $\text{Cat}_1$  is a category. A 1-cell in  $\text{Cat}_1$  is a functor. A 2-cell in  $\text{Cat}_1$  is a natural isomorphism  $F_{02} \Rightarrow F_{12}F_{01}$  from a functor to a composite of two functors.

I claim that  $\text{Cat}_1$  is a quasicategory. Fillers for  $\Lambda_1^2 \subset \Delta^2$  always exist: a map  $\Lambda_1^2 \rightarrow \text{Cat}_1$  is a choice of data  $(C_0 \xrightarrow{F_{01}} C_1 \xrightarrow{F_{12}} C_2)$ , and an extension to  $\Delta^2$  can be given by setting  $F_{02} = F_{12}F_{01}$  and  $\zeta_{012} = \text{id}_{F_{02}}$ . Note that this is not the only possible extension: we can take  $F_{02}$  to be any functor which is naturally isomorphic to  $F_{12}F_{01}$ , and  $\zeta_{012}$  can be any such natural isomorphism.

Fillers for  $\Lambda_1^3 \subset \Delta^3$  and  $\Lambda_2^3 \subset \Delta^3$  always exist, and are unique: finding a filler amounts to choosing isomorphisms  $\zeta_{023} = \zeta_{ik\ell}$  (for  $\Lambda_1^3$ ) or  $\zeta_{013} = \zeta_{ij\ell}$  (for  $\Lambda_2^3$ ) making (10.3) commute. All fillers for inner horns  $\Lambda_j^n \subset \Delta^n$  in higher dimensions  $n \geq 4$  exist and are unique: there is no additional data to supply in these cases, and all properties of the data are automatically satisfied.

10.5. *Exercise.* Note that the (nerve of) the category  $\text{Cat}$  of small categories and functors between them is isomorphic to the subcomplex of  $\text{Cat}_1$  whose cells are  $(C_i, F_{ij}, \zeta_{ijk})$  such that  $F_{ik} = F_{jk}F_{ij}$  and  $\zeta_{ijk} = \text{id}_{F_{ik}}$ . Show that this subcomplex is a quasicategory which not a subcategory in the sense of (8.16). (“The category of categories is not a subcategory of the quasicategory of categories.”)

10.6. **Singular complex of a space.** The **topological  $n$ -simplex** is

topological  $n$ -simplex

$$\Delta_{\text{top}}^n := \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0 \right\},$$

the convex hull of the standard basis vectors in  $(n+1)$ -dimensional Euclidean space. These fit together to give a functor  $\Delta_{\text{top}}: \Delta \rightarrow \text{Top}$  to the category of topological spaces and continuous maps, with  $\Delta_{\text{top}}([n]) = \Delta_{\text{top}}^n$ . A simplicial operator  $f: [m] \rightarrow [n]$  sends  $(x_0, \dots, x_m) \in \Delta_{\text{top}}^m$  to  $(y_0, \dots, y_n) \in \Delta_{\text{top}}^n$  with  $y_j = \sum_{f(i)=j} x_i$ .

For a topological space  $T$ , we define its **singular complex**  $\text{Sing } T$  to be the simplicial set with cells  $[n] \mapsto \text{Hom}_{\text{Top}}(\Delta_{\text{top}}^n, T)$ , with the evident action of simplicial operators.

singular complex

Define **topological horns**

topological horns

$$(\Lambda_j^n)_{\text{top}} := \{ x \in \Delta_{\text{top}}^n \mid \exists i \in [n] \setminus \{j\} \text{ such that } x_i = 0 \} \subset \Delta_{\text{top}}^n,$$

and observe that continuous maps  $(\Lambda_j^n)_{\text{top}} \rightarrow T$  correspond in a natural way with maps  $\Lambda_j^n \rightarrow \text{Sing } T$ .

10.7. *Exercise.* Prove the previous statement: show that  $(\Lambda_j^n)_{\text{top}}$  is a colimit in topological spaces of a functor  $S \mapsto \Delta_{\text{top}}^S$ , where  $S$  ranges over non-empty subsets of  $[n]$  such that  $\Delta^S \subseteq \Lambda_j^n$ , and use (6.15).

There exists a continuous retraction  $\Delta_{\text{top}}^n \rightarrow (\Lambda_j^n)_{\text{top}}$ , and thus we see that

$$\text{Hom}(\Delta^n, \text{Sing } T) \rightarrow \text{Hom}(\Lambda_j^n, \text{Sing } T)$$

is surjective for every horn (not just inner ones).

10.8. *Exercise.* Describe a continuous retraction  $\Delta_{\text{top}}^n \rightarrow (\Lambda_j^n)_{\text{top}}$ .

10.9. *Remark* (Kan complexes). A simplicial set  $X$  which has extensions for all horns is called a **Kan complex**. Thus,  $\text{Sing } T$  is a Kan complex, and so in particular is a quasicategory (and as we will see below, a “quasigroupoid” (12.13)). Kan complex

10.10. **Eilenberg-MacLane object.** Fix an abelian group  $A$  and an integer  $d \geq 0$ . We define a simplicial set  $K = K(A, d)$ , so that  $K_n$  is a set whose elements are data  $a = (a_{i_0 \dots i_d})$  consisting of

- for each  $0 \leq i_0 \leq \dots \leq i_d \leq n$ , an element  $a_{i_0 \dots i_d} \in A$ , such that
- $a_{i_0 \dots i_d} = 0$  if  $i_{u-1} = i_u$  for any  $u$ , and
- for each  $0 \leq j_0 \leq \dots \leq j_{d+1} \leq n$  we have  $\sum_u (-1)^u a_{j_0 \dots \widehat{j_u} \dots j_{d+1}} = 0$ .

(Here “ $j_0 \dots \widehat{j_u} \dots j_{d+1}$ ” is shorthand for the subsequence  $j_0, j_1, \dots, j_{u-1}, j_{u+1}, \dots, j_d, j_{d+1}$  with  $j_u$  omitted.)

For a map  $\delta: [m] \rightarrow [n]$  we define

$$(a\delta)_{i_0 \dots i_d} = a_{\delta(i_0) \dots \delta(i_d)}.$$

The object  $K(A, d)$  is a Kan complex, and hence a quasicategory (and in fact a quasigroupoid as we will see). When  $d = 0$ , this is just a discrete simplicial set, equal to  $A$  in each dimension.

10.11. *Exercise.* Show that  $K(A, 1)$  is isomorphic to the nerve of a groupoid, namely the nerve of the group  $A$  regarded as a category with one object.

10.12. *Exercise.* Show that  $K(A, d)$  is a Kan complex, i.e., that  $\text{Hom}(\Delta^n, K(A, d)) \rightarrow \text{Hom}(\Lambda_j^n, K(A, d))$  is surjective for all horns  $\Lambda_j^n \subset \Delta^n$ . In fact, this map is *bijective* unless  $n = d$ . (Hint: there are four distinct cases to check, namely  $n < d$ ,  $n = d$ ,  $n = d + 1$ , and  $n > d + 1$ .)

10.13. *Exercise.* Given a simplicial set  $X$ , a **normalized  $d$ -cocycle** with values in an abelian group  $A$  is a function  $f: X_d \rightarrow A$  such that normalized  $d$ -cocycle

- (1)  $f(x_{0, \dots, i, \dots, d-1}) = 0$  for all  $x \in X_{d-1}$  and  $0 \leq i \leq d-1$ , and
- (2)  $\sum (-1)^i f(x_{0, \dots, \widehat{i}, \dots, d+1}) = 0$  for all  $x \in X_{d+1}$  and  $0 \leq i \leq d+1$ .

Show that the set  $Z_{\text{norm}}^d(X; A)$  of normalized  $d$ -cocycles on  $X$  is in bijective correspondence with  $\text{Hom}_{\text{Set}}(X, K(A, d))$ . (Hint: an element  $a \in K_n$  is uniquely determined by the collection of elements  $a\delta \in K_d = A$ , as  $\delta$  ranges over injective maps  $[d] \rightarrow [n]$ .)

10.14. *Remark.* Eilenberg-MacLane objects are an example of a *simplicial abelian group*: the map  $+: K \times K \rightarrow K$  defined in each dimension by  $(a + b)_{i_0 \dots i_d} = a_{i_0 \dots i_d} + b_{i_0 \dots i_d}$  is a map of simplicial sets which satisfies the axioms of an abelian group, reflecting the fact that  $Z_{\text{norm}}^d(X; A)$  is an abelian group.

10.15. **Topological nerve of a topological category.** A **topological category** (or **category enriched over topological spaces**) is a category such that each hom-set is equipped with a topology, and so that composition is continuous. That is, a topological category  $\mathcal{C}$  consists of a set of objects, and for each pair of objects  $X, Y$  a topological space  $\mathcal{C}(X, Y)$ , with distinguished points  $1_X \in \mathcal{C}(X, X)$  and continuous maps  $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$  satisfying the axioms for a category. topological category  
category enriched over  
topological spaces

10.16. *Example.* A topological category with one object is the same thing as a topological monoid. This class of examples includes topological groups.

10.17. *Example.* The category of locally compact Hausdorff topological spaces has the structure of a topological category, where the space of maps from  $X$  to  $Y$  is the set of continuous maps equipped with the “compact-open topology”.

There are many variations on this example. The most commonly seen one is the category of *compactly generated spaces*, which is also made into a topological category by endowing sets of continuous maps with a suitable topology.

A continuous map  $[0, 1] \rightarrow \mathcal{C}(X, Y)$  from the unit interval is called a **homotopy** between its endpoints. homotopy

Given a topological category  $\mathcal{C}$ , we define a simplicial set  $N(\mathcal{C})$ , so that  $N(\mathcal{C})_n$  is a set whose elements are data  $f := (X_i, f_{ij})$  where

- (0) for each  $i \in [n]$ ,  $X_i$  is an object of  $\mathcal{C}$ ,
  - (1) for each  $i < j$  in  $[n]$ ,  $f_{ij}: I^{j-i-1} \rightarrow \mathcal{C}(X_i, X_j)$  is a continuous map (where  $I$  = unit interval),
- such that

- for each  $i < j < k$  in  $[n]$ , we have an identity

$$f_{jk}(t_{j+1}, \dots, t_{k-1}) \circ f_{ij}(t_{i+1}, \dots, t_{j-1}) = f_{ik}(t_{i+1}, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_{k-1})$$

for all  $t_{i+1}, \dots, t_{j-1}, t_{j+1}, \dots, t_{k-1} \in I$ .

For a simplicial operator  $\delta: [m] \rightarrow [n]$ , define

$$(X_i, f_{ij})\delta = (X_{\delta(i)}, f'_{ij})$$

where  $f'_{ij}: I^{j-i-1} \rightarrow \mathcal{C}(X_{\delta(i)}, X_{\delta(j)})$  is defined so that

- if  $\delta(i) = \delta(j)$ , then  $f'_{ij} = \text{id}_{X_{\delta(i)}}$ , and
- if  $\delta(i) < \delta(j)$ , then

$$f'_{ij}(t_{i+1}, \dots, t_{j-1}) = f_{\delta(i)\delta(j)}(u_{\delta(i)+1}, \dots, u_{\delta(j)-1}), \quad \text{where } u_k = \min \{ t_i \mid \delta(i) = k \},$$

or  $u_k = 1$  if  $\delta^{-1}(k)$  is empty.

10.18. *Example.* A 0-cell is an object of  $\mathcal{C}$ . A 1-cell is a morphism of  $\mathcal{C}$ . A 2-cell is a homotopy between a morphism  $h$  and a composite  $gf$  of morphisms in  $\mathcal{C}$ .

A 3-cell is a collection of morphisms

$$g_{01}: X_0 \rightarrow X_1, \quad g_{12}: X_1 \rightarrow X_2, \quad g_{23}: X_2 \rightarrow X_3, \quad g_{02}: X_0 \rightarrow X_2, \quad g_{13}: X_1 \rightarrow X_3, \quad g_{03}: X_0 \rightarrow X_3,$$

together with prescribed homotopies

$$h_{02} \text{ from } g_{12}g_{01} \text{ to } g_{02}, \quad h_{13} \text{ from } g_{23}g_{12} \text{ to } g_{13},$$

and a prescribed map  $k: [0, 1] \times [0, 1] \rightarrow \text{Map}(X_0, X_3)$  from a square which restrict to the homotopies  $g_{23}h_{02}$  and  $h_{13}g_{01}$  on its bottom and left edges respectively.

$$\begin{array}{ccc} g_{13}g_{01} & \xrightarrow{\quad\quad\quad} & g_{03} \\ \downarrow h_{13}g_{01} & & \downarrow \\ g_{23}g_{12}g_{01} & \xrightarrow[g_{23}h_{02}]{} & g_{23}g_{02} \end{array}$$

I claim that  $N(\mathcal{C})$  is a quasicategory. For  $0 < i < n$ , let  $J_i^{n-1} \subset I^{n-1}$  is the subspace consisting of  $(t_1, \dots, t_{n-1})$  such that (i)  $t_k \in \{0, 1\}$  for at least one  $k \in \{1, \dots, n-1\}$ , and (ii)  $t_i \neq 1$ . That is,  $J_i^{n-1}$  is an “open box”.

**10.19. Lemma.** *For any  $0 < i < n$ , the data of a map  $\Lambda_i^n \rightarrow N(\mathcal{C})$  is almost identical to the data of an element  $N(\mathcal{C})_n$ , the only difference being that  $f_{0n}$  is defined only on the subspace  $J_i^{n-1} \subset I^{n-1}$ . Furthermore, for each  $\delta: \Delta^m \rightarrow \Lambda_i^n \subset \Delta^n$  the induced map  $\text{Hom}(\Lambda_i^n, N(\mathcal{C})) \rightarrow \text{Hom}(\Delta^m, N(\mathcal{C})) = N(\mathcal{C})_m$  is described by the same formulas as the map  $\delta^*: N(\mathcal{C})_n \rightarrow N(\mathcal{C})_m$ .*

Using this, we see that fillers for inner horns in  $N(\mathcal{C})$  always exist, as producing such a filler amounts to extending a continuous map  $J_i^{n-1} \rightarrow \mathcal{C}(X_0, X_n)$  along the inclusion  $J_i^{n-1} \subset I^{n-1}$ , which is clear since there exists a retraction of  $I^{n-1}$  to the subspace  $J^{n-1}$ .

## 11. HOMOTOPY CATEGORY OF A QUASICATEGORY

Our next goal is to define the notion of an *isomorphism* in a quasicategory. This notion behaves much like that of *homotopy equivalence* in topology. We will define isomorphism by means of the *homotopy category* of a quasicategory. If we think of a quasicategory as “an ordinary category with higher structure”, then its homotopy category is the ordinary category obtained by “flattening out the higher structure”.

**11.1. The fundamental category of a simplicial set.** The homotopy category of a quasicategory is itself a special case of the notion of the *fundamental category* of a simplicial set, which we turn to first.

A **fundamental category** for a simplicial set  $X$  consists of (i) a category  $hX$ , and (ii) a map  $\alpha: X \rightarrow N(hX)$  of simplicial sets, such that for every category  $C$ , the map

$$\alpha^*: \text{Hom}(N(hX), NC) \rightarrow \text{Hom}(X, NC)$$

induced by restriction along  $\alpha$  is a bijection. This is a universal property which characterizes the fundamental category up to unique isomorphism, if it exists.

**11.2. Proposition.** *Every simplicial set has a fundamental category.*

*Proof sketch.* Given  $X$ , we construct  $hX$  by generators and relations. First, consider the **free category**  $F$ , whose objects are the set  $X_0$ , and whose morphisms are finite “composable” sequences  $[a_n, \dots, a_1]$  of edges of  $X_1$ . Thus, morphisms in  $F$  are “words”, whose “letters” are edges  $a_i$  with  $(a_{i+1})_0 = (a_i)_1$ , and composition is concatenation of words; the element  $[a_n, \dots, a_1]$  is then a morphism  $(a_1)_0 \rightarrow (a_n)_1$ . We must also suppose that there is an empty sequence  $[]_x$  in  $F$  for each vertex  $x \in X_0$ ; these correspond to identity maps in  $F$ .

Then  $hX$  is defined to be the largest quotient category of  $F$  subject to the following relations on the set of morphisms:

- $[a] \sim []_x$  for each  $x \in X_0$  where  $a = x_{00} \in X_1$ , and
- $[g, f] \sim [h]$  whenever there exists  $a \in X_2$  such that  $a_{01} = f$ ,  $a_{12} = g$ , and  $a_{02} = h$ .

The map  $\alpha: X \rightarrow N(hX)$  sends  $x \in X_n$  to the equivalence class of  $[x_{n-1,n}, \dots, x_{0,1}]$ . Given this, verifying the desired universal property of  $\alpha$  is formal.

(We will give another construction of the fundamental category in (15.27).) □

**11.3. Exercise.** Complete the proof of (11.2) by showing that  $\alpha^*: \text{Hom}(N(hX), NC) \rightarrow \text{Hom}(X, NC)$  is a bijection for any category  $C$ .

As a consequence, the fundamental category construction describes a functor  $h: s\text{Set} \rightarrow \text{Cat}$ , which is left adjoint to the nerve functor  $N: \text{Cat} \rightarrow s\text{Set}$ , so that there is an isomorphism

$$\text{Hom}_{\text{Cat}}(hX, C) \xrightarrow{\sim} \text{Hom}_{s\text{Set}}(X, NC)$$

natural in  $X$  and  $C$ .

In general, the fundamental category of a simplicial set is not an easy thing to get a hold of explicitly, because it is difficult to give an explicit description of a “quotient category” induced by a relation on its morphisms. We will not be making much use of it. When  $C$  is a quasicategory, there

is a more concrete construction of  $hC$ , which in this context is called the *homotopy category* of  $C$ . *Warning:* Sometimes people will not distinguish “fundamental category” from “homotopy category” as I have here, and just call either the homotopy category.

**11.4. The homotopy relation on morphisms.** Fix a quasicategory  $C$ . For  $x, y \in C_0$ , let  $\text{hom}_C(x, y) := \{ f \in C_1 \mid f_0 = x, f_1 = y \}$  denote the set of “morphisms” in  $C$  from  $x$  to  $y$ . We write  $1_x$  for the element  $x_{00} \in \text{hom}_C(x, x)$ .

Define relations  $\sim_\ell, \sim_r$  on  $\text{hom}_C(x, y)$  (called **left homotopy** and **right homotopy**) by

- $f \sim_\ell g$  iff there exists  $a \in C_2$  with  $a_{01} = 1_x, a_{02} = f, a_{12} = g$ ,
- $f \sim_r g$  iff there exists  $b \in C_2$  with  $b_{12} = 1_y, b_{01} = f, b_{02} = g$ .

left homotopy  
right homotopy

Pictorially:

$$f \sim_\ell g: \quad \begin{array}{ccc} x & & y \\ & \searrow f & \\ 1_x \downarrow & a & \nearrow \\ & \nearrow g & \\ x & & y \end{array} \quad f \sim_r g: \quad \begin{array}{ccc} & & y \\ & \nearrow f & \\ x & b & \downarrow 1_y \\ & \searrow g & \\ & & y \end{array}$$

Note that  $f \sim_\ell g$  in  $\text{hom}_C(x, y)$  if and only if  $g \sim_r f$  in  $\text{hom}_{C^{\text{op}}}(y, x)$ .

**11.5. Remark.** If  $C$  is an ordinary category, then the left homotopy and right homotopy relations reduce to the equality relation on the set  $\text{Hom}_C(x, y)$ .

**11.6. Proposition.** *The relations  $\sim_\ell$  and  $\sim_r$  are equal to each other, and are an equivalence relation on  $\text{hom}_C(x, y)$ .*

*Proof.* Given  $f, g, h: x \rightarrow y$  in a quasicategory  $C$ , we will prove

- (1)  $f \sim_\ell f$ ,
- (2)  $f \sim_\ell g$  and  $g \sim_\ell h$  imply  $f \sim_\ell h$ ,
- (3)  $f \sim_\ell g$  implies  $f \sim_r g$ ,
- (4)  $f \sim_r g$  implies  $g \sim_\ell f$ .

Statements (3) and (4) combine to show that  $\sim_\ell$  is symmetric, and thus with (1) and (2) that  $\sim_\ell$  is an equivalence relation. Statements (3) and (4) and symmetry imply that  $\sim_r$  and  $\sim_\ell$  coincide. The idea is to use the inner-horn extension condition for  $C$  to produce the appropriate relations.

Statement (1) is exhibited by  $f_{001} \in C_2$ .

$$\begin{array}{ccc} & x & \\ & \searrow f & \\ 1_x & f_{001} & y \\ & \nearrow f & \\ & x & \end{array}$$

Statements (2), (3), and (4) are demonstrated by the following diagrams, which present a map from an inner horn of  $\Delta^3$  (respectively  $\Lambda_1^3$ ,  $\Lambda_1^3$ , and  $\Lambda_2^3$ ) to  $C$  constructed from the given data. The restriction of any extension to  $\Delta^3$  along the remaining face (respectively  $\Delta^{\{023\}}$ ,  $\Delta^{\{023\}}$ , and  $\Delta^{\{013\}}$ ) gives the conclusion.

$$\begin{array}{ccc} \begin{array}{ccc} x & & y \\ & \searrow f & \\ 1_x \downarrow & x_{000} & \nearrow \\ & \nearrow g & \\ x & & y \end{array} & \begin{array}{ccc} x & & y \\ & \searrow g & \\ f \downarrow & a & \nearrow \\ & \nearrow g & \\ y & & y \end{array} & \begin{array}{ccc} x & & y \\ & \searrow g & \\ 1_x \downarrow & f_{001} & \nearrow \\ & \nearrow f & \\ x & & y \end{array} \end{array}$$

□

**11.7. Composition of homotopy classes of morphisms.** We now define  $f \approx g$  to mean  $f \sim_\ell g$  (equivalently  $f \sim_r g$ ). We speak of **homotopy classes**  $[f]$  of morphisms  $f \in \text{hom}_C(x, y)$ , meaning equivalence classes under  $\approx$ . Next we observe that we can compose homotopy classes.

homotopy classes

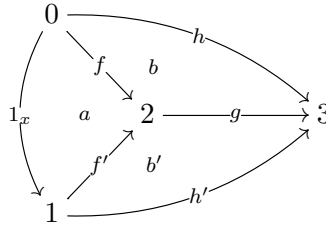
Given  $f \in \text{hom}_C(x, y)$ ,  $g \in \text{hom}_C(y, z)$ ,  $h \in \text{hom}_C(x, z)$ , we say that  $h$  is a **composite** of  $(g, f)$  if there exists a 2-cell  $a \in C_2$  with  $a\langle 01 \rangle = f$ ,  $a\langle 12 \rangle = g$ ,  $a\langle 02 \rangle = h$ ; thus composition is a three-fold relation on  $\text{hom}(x, y) \times \text{hom}(y, z) \times \text{hom}(x, z)$ . The composition relation is compatible with the homotopy relation in the following sense.

composite

**11.8. Lemma.** *If  $f \approx f'$ ,  $g \approx g'$ ,  $h$  a composite of  $(g, f)$ , and  $h'$  a composite of  $(g', f')$ , then  $h \approx h'$ .*

*Proof.* Since  $\approx$  is an equivalence relation, it suffices prove the special cases (a)  $f = f'$ , and (b)  $g = g'$ . We prove case (b), as case (a) is analogous.

Let  $a \in C_2$  exhibit  $f \sim_\ell f'$ , and let  $b, b' \in C_2$  exhibit  $h$  as a composite of  $(g, f)$  and  $h'$  as a composite of  $(g, f')$  respectively. The inner horn  $\Delta^3_2 \rightarrow C$  defined by



extends to  $u: \Delta^3 \rightarrow C$ , and  $u|_{\Delta^{\{0,1,3\}}}$  exhibits  $h \sim_\ell h'$ .  $\square$

Thus, composites of  $(g, f)$  live in a unique homotopy class of morphisms in  $C$ , which only depends on the homotopy classes of  $g$  and  $f$ . I will write  $[g] \circ [f]$  for the homotopy class containing composites of  $(g, f)$ .

I'll leave the following as exercises; the proofs are much like what we have already seen.

**11.9. Lemma.** *Given  $f: x \rightarrow y$ , we have  $[f] \circ [1_x] = [f] = [1_y] \circ [f]$ .*

**11.10. Lemma.** *If  $[g] \circ [f] = [u]$ ,  $[h] \circ [g] = [v]$ , then  $[h] \circ [u] = [v] \circ [f]$ .*

**11.11. The homotopy category of a quasicategory.** For any quasicategory, we define its **homotopy category**  $hC$ , with object set  $\text{ob}(hC) := C_0$ , and with morphism sets  $\text{hom}_{hC}(x, y) := \text{hom}_C(x, y) / \approx$ , with composition defined by  $[g] \circ [f]$ . The above lemmas (11.9) and (11.10) exactly imply that  $hC$  is a category.

homotopy category

We define a map  $\pi: C \rightarrow N(hC)$  of simplicial sets as follows. On vertices,  $\pi$  is the identity map  $C_0 = N(hC)_0 = \text{ob } hC$ . On edges, the map is defined by the tautological quotient maps  $\text{hom}_C(x, y) \rightarrow \text{hom}_C(x, y) / \approx$  sending  $f \mapsto [f]$ . The map  $\pi$  sends an  $n$ -cell  $a \in C_n$  to the unique  $\pi(a) \in N(hC)_n$  such that  $\pi(a)_{i-1, i} = \pi(a_{i-1, i})$ . These functions are seen to be compatible with simplicial operators using the following exercise.

**11.12. Exercise.** Let  $C$  be a quasicategory and  $a \in C_n$  an  $n$ -cell, and define  $f_i := a_{i-1, i} \in C_1$  for  $i = 1, \dots, n$  and  $g := a_{0, n} \in C_1$ . Show that  $[f_n] \circ \dots \circ [f_1] = [g]$  in the homotopy category  $hC$ .

Note that if  $C$  is an ordinary category, then  $f \approx g$  if and only if  $f = g$ . Thus,  $\pi: C \rightarrow N(hC)$  is an isomorphism of simplicial sets if and only if  $C$  is isomorphic to the nerve of a category.

The following says that the homotopy category of a quasicategory is its fundamental category, justifying the notation “ $hC$ ”.

**11.13. Proposition.** *Let  $C$  be a quasicategory and  $D$  a small category, and let  $\phi: C \rightarrow N(D)$  be a map of simplicial sets. Then there exists a unique map  $\psi: N(hC) \rightarrow N(D)$  such that  $\psi\pi = \phi$ .*

*Proof.* We first show existence, by constructing a suitable map  $\psi$ , which being a map between nerves can be described as a functor  $hC \rightarrow D$ . On objects, let  $\psi$  send  $x \in \text{ob}(hC) = C_0$  to  $\phi(x) \in \text{ob}(D) = (ND)_0$ . On morphisms, let  $\psi$  send  $[f] \in \text{hom}_{hC}(x, y)$  to  $\phi(f) \in \text{hom}_D(\phi(x), \phi(y)) \subseteq (ND)_1$ . Observe that the function on morphisms is well-defined since if  $f \sim_\ell f'$ , exhibited by some  $a \in C_2$ , then  $\phi(a) \in (ND)_2$  exhibits the identity  $\phi(f) = \phi(f')\phi(1_x) = \phi(f')$  in  $D$ . It is straightforward to show that  $\psi$  so defined is actually a functor, and that  $\psi\pi = \phi$  as maps  $C \rightarrow N(D)$ .

The functor  $\psi$  defined above is the unique solution: the value of  $\psi$  on objects and morphisms is uniquely determined, and  $\pi: C_k \rightarrow (hC)_k$  is bijective for  $k = 0$  and surjective for  $k = 1$ .  $\square$

In particular, the homotopy category construction gives a pair of adjoint functors

$$h: \mathbf{qCat} \rightleftarrows \mathbf{Cat} : N.$$

11.14. *Exercise.* Understand the homotopy categories of the various examples of quasicategories described in (10).

11.15. *Exercise* (Easy but important: homotopy category of a product). Show that for quasicategories  $C$  and  $D$  there is an isomorphism of categories  $hC \times hD \approx h(C \times D)$ .

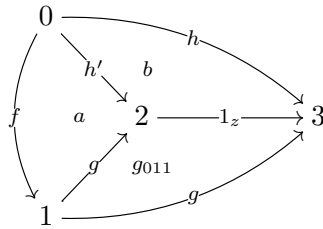
11.16. **A criterion for composition.** We have observed that for morphisms  $f: x \rightarrow y$  and  $g: y \rightarrow z$  in a quasicategory that we can define a composite “ $g \circ f$ ” using extension along  $\Lambda_1^2 \subset \Delta^2$ , and that though such compositions are not unique, they are unique up to homotopy, so we get a well-defined homotopy class  $[g] \circ [f]$ . The following proposition says that every element in this homotopy class is obtained from this construction.

11.17. **Proposition.** *If  $f: x \rightarrow y$ ,  $g: y \rightarrow z$ , and  $h: x \rightarrow z$  are morphisms in a quasicategory  $C$ , then  $[h] = [g] \circ [f]$  if and only if there exists  $u: \Delta^2 \rightarrow C$  such that*

$$u|_{\Delta^{\{0,1\}}} = f, \quad u|_{\Delta^{\{1,2\}}} = g, \quad u|_{\Delta^{\{0,2\}}} = h.$$

*Thus, every morphism in the homotopy class of  $h$  can be interpreted as a composite of  $g$  with  $f$ .*

*Proof.* Clearly if  $u$  exists then  $[h] = [g] \circ [f]$ . Conversely, suppose given  $f, g, h$  with  $h \in [g] \circ [f]$ , and choose some  $a: \Delta^2 \rightarrow C$  with  $a_{01} = f$  and  $a_{12} = g$ , whence  $[g] \circ [f] = [h']$  for  $h' = a_{02}$ . Since  $h \in [h']$  there is a  $b \in C_2$  witnessing the relation  $h' \sim_r h$ , and using this we can construct a map  $\Lambda_2^3 \rightarrow C$  according to the diagram



Extend to a map  $v: \Delta^3 \rightarrow C$ ; then  $u = v|_{\Delta^{\{0,1,3\}}}$  exhibits  $h$  as a composite of  $(g, f)$  as desired.  $\square$

11.18. *Exercise.* Let  $C' \subseteq C$  be a subcategory (8.16) of a quasicategory  $C$ . Show that if  $f, g: x \rightarrow y$  are morphisms of  $C$  which are homotopic in  $C$ , then  $f \in C'_1$  if and only if  $g \in C'_1$ . Use this to show that there is a bijective correspondence

$$(\text{subcategories of } C) \leftrightarrow (\text{subcategories of } hC),$$

and also a bijective correspondence

$$(\text{full subcategories of } C) \leftrightarrow (\text{full subcategories of } hC).$$



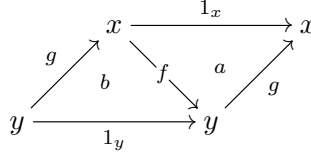
## 12. ISOMORPHISMS IN A QUASICATEGORY

Let  $C$  be a quasicategory. We say that an edge  $f \in C_1$  is an **isomorphism**<sup>6</sup> if its image in the homotopy category  $hC$  is an isomorphism in the usual sense of category theory. isomorphism

Explicitly,  $f: x \rightarrow y$  is an isomorphism if and only if there exists an edge  $g: y \rightarrow x$  such that  $[g] \circ [f] = [1_x]$  and  $[f] \circ [g] = [1_y]$ , where equality is in the homotopy category  $hC$ .

12.1. *Example.* Consider  $f \in C_1$ . If we can produce  $g \in C_1$  and  $a, b \in C_2$  such that

$$a_{01} = f = b_{12}, \quad a_{12} = g = b_{01}, \quad a_{02} = x_{00}, \quad b_{02} = y_{00} :$$



then  $[g] \circ [f] = [1_x]$  and  $[f] \circ [g] = [1_y]$ , so  $f$  isomorphism. The converse also holds: if  $f$  is an isomorphism, then there exist  $g \in C_1$  and  $a, b \in C_2$  as above, which can be proved using (11.17).

12.2. *Example* (Identity maps are isomorphisms). For every  $x \in C_0$  the identity map  $1_x: x \rightarrow x$  is an isomorphism: for instance, use  $a = b = x_{00}$  in the above diagram.

12.3. *Exercise.* Show that any functor  $f: C \rightarrow D$  between quasicategories sends isomorphisms to isomorphisms.

12.4. **Preinverses and postinverses.** Let  $C$  be a quasicategory. Given  $f: x \rightarrow y \in C_1$ , a **postinverse**<sup>7</sup> of  $f$  is a  $g: y \rightarrow x \in C_1$  such that  $[g] \circ [f] = [1_x]$ , and a **preinverse**<sup>8</sup> of  $f$  is an  $e: y \rightarrow x \in C_1$  such that  $[f] \circ [e] = [1_y]$ . An **inverse** is an  $f' \in C_1$  which is both a postinverse and a preinverse. The following is trivial, but very handy. postinverse  
preinverse  
inverse

12.5. **Proposition.** *In a quasicategory  $C$  consider  $f \in C_1$ . The following are equivalent.*

- $f$  is an isomorphism.
- $f$  admits an inverse  $f'$ .
- $f$  admits a postinverse  $g$  and a preinverse  $e$ .
- $f$  admits a postinverse  $g$  and  $g$  admits a postinverse  $h$ .
- $f$  admits a preinverse  $e$  and  $e$  admits a preinverse  $d$ .

*If these equivalent conditions apply, then  $f \approx d \approx h$  and  $f' \approx e \approx g$ , and all of them are isomorphisms.*

*Proof.* All of these are equivalent to the corresponding statements about morphisms in the homotopy category  $hC$ , where they are seen to be equivalent to each other by elementary arguments.  $\square$

Note that inverses to a morphism in a quasicategory are generally *not unique*, though necessarily they are *unique up to homotopy*.

12.6. **Quasigroupoids.** A **quasigroupoid**<sup>9</sup> is a quasicategory  $C$  such that  $hC$  is a groupoid, i.e., a quasicategory in which every morphism is an isomorphism. quasigroupoid

12.7. *Exercise.* If every morphism in a quasicategory admits a preinverse, then it is a quasigroupoid. Likewise if every morphism admits a postinverse.

**12.8. The core of a quasicategory.** For an ordinary category  $A$ , the **core** (or **interior**, or **maximal subgroupoid**) of  $A$  is the subcategory  $A^{\text{core}} \subseteq A$  consisting of all the objects, and all the *isomorphisms* between the objects. core  
interior  
maximal subgroupoid

For a quasicategory  $C$ , we define the <sup>10</sup> $C^{\text{core}} \subseteq C$  to be the subcomplex consisting of cells all of whose edges are all isomorphisms. That is,  $C^{\text{core}}$  is defined so that the diagram

$$\begin{array}{ccc} C^{\text{core}} & \xrightarrow{\quad} & C \\ \downarrow & & \downarrow \pi \\ (hC)^{\text{core}} & \xrightarrow{\quad} & hC \end{array}$$

is a pullback of simplicial sets. Observe that  $N(A^{\text{core}}) = (NA)^{\text{core}}$  for a category  $A$ .

**12.9. Proposition.** *Given a quasicategory  $C$ , its core  $C^{\text{core}}$  is a subcategory and a quasigroupoid, and every subcomplex of  $C$  which is a quasigroupoid is contained in  $C^{\text{core}}$ .*

*Proof.* First we show that  $C^{\text{core}}$  is a subcategory (8.16). Suppose  $f: \Delta^n \rightarrow C$  such that  $f(\Lambda_k^n) \subseteq C^{\text{core}}$  for  $n \geq 2$  and  $0 < k < n$ . When  $n \geq 3$  then  $(\Lambda_k^n)_1 = (\Delta^n)_1$ , so clearly  $f(\Delta^n) \subseteq C^{\text{core}}$ . When  $n = 2$  we have that  $f(\Delta^n) \subseteq C^{\text{core}}$  because the composite of two isomorphisms is an isomorphism.

It follows that  $C^{\text{core}}$  is a quasicategory, and clearly it is a quasigroupoid. The final statement is clear: if  $G \subseteq C$  is a subcomplex which is a quasigroupoid, then every edge in  $G$  has an inverse in  $G$ , and hence an inverse in  $C$ .  $\square$

**12.10. Exercise.** Show that if  $C$  is a quasicategory, there is an isomorphism  $(hC)^{\text{core}} \approx h(C^{\text{core}})$ .

**12.11. Kan complexes.** Recall that a Kan complex (10.9) is a simplicial set which has the extension property with respect to all horns, not just inner horns. That is,  $K$  is a Kan complex iff

$$\text{Hom}(\Delta^n, K) \rightarrow \text{Hom}(\Lambda_j^n, K)$$

is surjective for all  $0 \leq j \leq n$ ,  $n \geq 1$ .

**12.12. Exercise.** Show that every simplicial set  $X$  has extensions for 1-dimensional horns; i.e., every  $\Lambda_j^1 \rightarrow X$  extends over  $\Lambda_j^1 \subset \Delta^1$ , where  $j \in \{0, 1\}$ . Thus,  $X$  is a Kan complex if and only if it has extensions just for the horns inside simplices of dimension  $\geq 2$ .

**12.13. Proposition.** *Every Kan complex is a quasigroupoid.*

*Proof.* It is immediate that a Kan complex  $K$  is a quasicategory. To show  $K$  is a quasigroupoid, note that the extension condition for  $\Lambda_0^2 \subset \Delta^2$  implies that every morphism in  $hK$  admits a postinverse. Explicitly, if  $f: x \rightarrow y$  is an edge in  $K$ , let  $u: \Lambda_0^2 \rightarrow K$  with  $u_{01} = f$  and  $u_{02} = f_{00} = 1_x$ , so there is an extension  $v: \Delta^2 \rightarrow K$  and  $g := v_{12}$  satisfies  $gf \approx 1_x$ . Use (12.7).  $\square$

This proposition has a converse: *quasigroupoids are precisely the Kan complexes*. This is a very important technical result, and it is not trivial; it is the main result of [Joy02]. We will prove this as (33.2).

Recall (10.6) that we observed that the singular complex  $\text{Sing } T$  of a topological space is a Kan complex, and therefore a quasigroupoid. It is reasonable to think of  $\text{Sing } T$  as the **fundamental quasigroupoid** of the space  $T$ . fundamental  
groupoid      quasi-

<sup>6</sup>Lurie [Lur09, §1.2.4] uses the term “equivalence” for this. I prefer to go with “isomorphism” here, because it is in fact a generalization of the classical notion of isomorphism in a category, and also because so many other things get to be called some kind of equivalence. Other authors also use “isomorphism” in this context.

<sup>7</sup>or *left inverse*, or *retraction*,

<sup>8</sup>or *right inverse*, or *section*,

<sup>9</sup>This is the same as what Lurie [Lur09] and many others call an  $\infty$ -*groupoid*.

<sup>10</sup>Lurie (along with many others) uses the notation  $C^\simeq$  for what we are calling  $C^{\text{core}}$ .

12.14. *Exercise* (for topologists). Show that if  $T$  is a topological space, then  $h\text{Sing} T$ , the homotopy category of the singular complex of  $T$ , is precisely the usual fundamental groupoid of  $T$ .

12.15. **Quasigroupoids, components, and isomorphism classes.** We say that two objects in a quasicategory are **isomorphic** if there exists an isomorphism between them. This is an equivalence relation on  $C_0$ , and thus we speak of **isomorphism classes** of objects.

isomorphic  
isomorphism classes

Recall (8.11) that the set of connected components of a simplicial set is given by

$$\pi_0 X \approx \left( \left( \coprod_{n \geq 0} X_n \right) / \sim \right) \approx (X_0 / \sim_1),$$

the equivalence classes of cells of  $X$  under the equivalence relation generated by “related by a simplicial operator”, or equivalently the equivalence classes of vertices of  $X$  under the equivalence relation generated by “connected by an edge”. Note that if  $T$  is a topological space, then elements of  $\pi_0 \text{Sing} T$  correspond exactly to path components of  $T$ .

For *quasigroupoids*,  $\pi_0$  recovers the set of isomorphism classes of objects.

12.16. **Proposition.** *If  $C$  is a quasicategory, then*

$$\pi_0(C^{\text{core}}) \approx \{\text{isomorphism classes of objects of } C\}.$$

*Proof.* Straightforward: edges in  $C^{\text{core}}$  are precisely the isomorphisms in  $C$ .  $\square$

12.17. *Exercise.* Show that for a quasicategory  $C$ ,  $\pi_0(C^{\text{core}}) \approx \pi_0(h(C^{\text{core}})) \approx \pi_0((hC)^{\text{core}})$ .

### 13. FUNCTION COMPLEXES AND THE FUNCTOR QUASICATEGORY

Given ordinary categories  $C$  and  $D$ , the *functor category*  $\text{Fun}(C, D)$  has

- as objects, the functors  $C \rightarrow D$ , and
- as morphisms  $f \rightarrow f'$ , natural transformations of functors.

Furthermore, for any category  $A$  there is a bijective correspondence between sets of functors

$$\{A \times C \rightarrow D\} \iff \{A \rightarrow \text{Fun}(C, D)\}.$$

Explicitly, a functor  $\phi: A \rightarrow \text{Fun}(C, D)$  corresponds to  $\tilde{\phi}: A \times C \rightarrow D$ , given on objects by  $\tilde{\phi}(a, c) = \phi(a)(c)$  for  $a \in \text{ob } A$  and  $c \in \text{ob } C$ , and on morphisms by  $\tilde{\phi}(\alpha, \gamma) = \phi(a')(\gamma) \circ \phi(a)(c) = \phi(\alpha)(c') \circ \phi(a)(\gamma): \phi(a)(c) \rightarrow \phi(a')(\gamma) \circ \phi(a)(c) = \phi(a')(\gamma)$  for  $\alpha: a \rightarrow a' \in \text{mor } A$  and  $\gamma: c \rightarrow c' \in \text{mor } C$ .

The generalization of the functor category to quasicategories admits a similar adjunction, and in fact can be defined for arbitrary simplicial sets.

13.1. **Function complexes.** Given simplicial sets  $X$  and  $Y$ , we may form the **function complex**  $\text{Fun}(X, Y)$ . This is a simplicial set with

function complex

$$\text{Fun}(X, Y)_n = \text{Hom}(\Delta^n \times X, Y),$$

so that the action of a simplicial operator  $\delta: [m] \rightarrow [n]$  on  $\text{Fun}(X, Y)$  is induced by

$$\text{Hom}(\delta \times \text{id}_X, Y): \text{Hom}(\Delta^n \times X, Y) \rightarrow \text{Hom}(\Delta^m \times X, Y).$$

In particular, the set  $\text{Fun}(X, Y)_0$  of vertices of the function complex is precisely the set of maps  $X \rightarrow Y$  of simplicial sets.

13.2. *Remark.* There are many alternate notations for the function complex. A common one is  $\text{Map}(X, Y)$ , because it is the “object of maps” from  $X$  to  $Y$ , and is sometimes called the *mapping space*.

13.3. **Proposition.** *The function complex construction defines a functor*

$$\text{Fun}: s\text{Set}^{\text{op}} \times s\text{Set} \rightarrow s\text{Set}.$$

*Proof.* Left as an exercise. □

By construction, for each  $n$ , there is a bijective correspondence

$$\{\Delta^n \times X \rightarrow Y\} \longleftrightarrow \{\Delta^n \rightarrow \text{Fun}(X, Y)\}.$$

In fact, we can replace  $\Delta^n$  with an arbitrary simplicial set.

**13.4. Proposition.** *For simplicial sets  $X, Y, Z$ , there is a bijection*

$$\text{Hom}(X \times Y, Z) \xrightarrow{\sim} \text{Hom}(X, \text{Fun}(Y, Z))$$

*natural in all three variables.*

*Proof.* The bijection sends  $f: X \times Y \rightarrow Z$  to  $\tilde{f}: X \rightarrow \text{Fun}(Y, Z)$  defined so that for  $x \in X_n$ , the element  $\tilde{f}(x) \in \text{Fun}(Y, Z)_n$  is represented by the composite

$$\Delta^n \times Y \xrightarrow{x \times \text{id}} X \times Y \xrightarrow{f} Z.$$

The inverse of this bijection sends  $g: X \rightarrow \text{Fun}(Y, Z)$  to  $\tilde{g}: X \times Y \rightarrow Z$ , defined so that for  $(x, y) \in X_n \times Y_n$ , the element  $\tilde{g}(x, y) \in Z_n$  is represented by

$$\Delta^n \xrightarrow{(\text{id}, y)} \Delta^n \times Y \xrightarrow{g(x)} Z.$$

The proof amounts to showing that both  $\tilde{f}$  and  $\tilde{g}$  are in fact maps of simplicial sets, and that the above constructions are in fact inverse to each other. This is left as an exercise, as is the proof of naturality. □

**13.5. Exercise.** Show, using the previous proposition, that there are natural isomorphisms

$$\text{Fun}(X \times Y, Z) \approx \text{Fun}(X, \text{Fun}(Y, Z)).$$

of simplicial sets. This implies that the function complex construction makes  $s\text{Set}$  into a *cartesian closed category*. (Hint: show that both objects represent *isomorphic* functors  $s\text{Set}^{\text{op}} \rightarrow \text{Set}$ , and apply the Yoneda lemma.)

**13.6. Remark.** The construction of the function complex is not special to simplicial sets. The construction of  $\text{Fun}(X, Y)$  (and its properties as described above) works the same way in any category of functors  $C^{\text{op}} \rightarrow \text{Set}$ , where  $C$  is a small category (e.g.,  $C = \Delta$ ). In this general setting, the role of the standard  $n$ -simplices is played by the representable functors  $\text{Hom}_C(-, c): C^{\text{op}} \rightarrow \text{Set}$ .

**13.7. Functor quasicategories.** Thus, we may expect the generalization of functor category to quasicategories to be defined by the function complex. In fact, if  $C$  and  $D$  are quasicategories, then the vertices of  $\text{Fun}(C, D)$  are precisely the functors  $C \rightarrow D$ , and the edges of  $\text{Fun}(C, D)$  are precisely the natural transformations. Furthermore, for ordinary categories, the function complex recovers the functor category.

**13.8. Exercise.** Show that for ordinary categories  $C$  and  $D$  that  $N\text{Fun}(C, D) \approx \text{Fun}(NC, ND)$ . (Hint: use that  $N([n]) = \Delta^n$ , and the fact that the nerve preserves finite products (8.5).)

It turns out that a *function complex between quasicategories is again a quasicategory*. To prove this, we will need to take a detour to develop some technology about “weakly saturated” classes of maps and “lifting properties”. After this, we will complete the proof as (20.4).

### Part 3. Lifting properties

#### 14. WEAKLY SATURATED CLASSES AND INNER-ANODYNE MAPS

Quasicategories are defined by an “extension property”: they are the simplicial sets  $C$  such that any map  $K \rightarrow C$  extends over  $L$ , whenever  $K \subset L$  is an inner horn inclusion  $\Lambda_j^n \subset \Delta^n$ . The set of inner horns “generates” a larger class of maps (which will be called the class of *inner anodyne* maps), which “automatically” shares the extension property of the inner horns. This class of inner anodyne maps is called the *weak saturation* of the set of inner horns.

For instance, we will later (14.12) observe that the spine inclusions  $I^n \subset \Delta^n$  are inner anodyne, so that quasicategories admit “spine extensions”, i.e., any  $I^n \rightarrow C$  extends over  $I^n \subset \Delta^n$  to a map  $\Delta^n \rightarrow C$ .

**14.1. Weakly saturated classes.** Consider a category (such as  $s\text{Set}$ ) which has all small colimits. A **weakly saturated class** is a class  $\mathcal{A}$  of morphisms in the category, which

weakly saturated class

- (1) contains all isomorphisms,
- (2) is closed under cobase change,
- (3) is closed under composition,
- (4) is closed under transfinite composition,
- (5) is closed under coproducts, and
- (6) is closed under retracts.

Given a class of maps  $S$ , its **weak saturation**  $\bar{S}$  is the smallest weakly saturated class containing  $S$ .

weak saturation

We need to explain some of the elements of this definition.

- **Closed under cobase change** is also called **closed under pushout**: it means that if  $f'$  is the pushout of  $f: X \rightarrow Y$  along some map  $g: X \rightarrow Z$ , then  $f \in \mathcal{A}$  implies  $f' \in \mathcal{A}$ .
- **Closed under composition** means that if  $g, f \in \mathcal{A}$  and  $gf$  is defined, then  $gf \in \mathcal{A}$ .
- We say that  $\mathcal{A}$  is **closed under countable composition** if given a countable sequence of composable morphisms, i.e., maps

Closed under cobase change  
closed under pushout  
Closed under composition  
closed under countable composition

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \dots$$

such that each  $f_k \in \mathcal{A}$  for all  $k \in \mathbb{Z}_{>0}$ , the induced map  $X_0 \rightarrow \text{colim}_k X_k$  to the colimit is in  $\mathcal{A}$ .

The notion **closed under transfinite composition** is a generalization of this, in which  $\mathbb{N}$  is replaced by an arbitrary ordinal  $\lambda$  (i.e., a well-ordered set). This means that for any ordinal  $\lambda$  and any functor  $X: \lambda \rightarrow s\text{Set}$ , if for every  $i \in \lambda$  with  $i \neq 0$  the evident map

closed under transfinite composition

$$(\text{colim}_{j < i} X(j)) \rightarrow X(i)$$

is in  $\mathcal{A}$ , then the induced map  $X(0) \rightarrow \text{colim}_{j \in \lambda} X(j)$  is in  $\mathcal{A}$ .

- **Closed under coproducts** means that if  $\{f_i: X_i \rightarrow Y_i\}$  is a set of maps in  $\mathcal{A}$ , then  $\coprod_i f_i: \coprod_i X_i \rightarrow \coprod_i Y_i$  is in  $\mathcal{A}$ .
- We say that  $f$  is a **retract** of  $g$  if there exists a commutative diagram in  $C$  of the form

Closed under coproducts

retract

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & X' & \xrightarrow{\quad} & X \\ f \downarrow & & g \downarrow & & f \downarrow \\ Y & \xrightarrow{\quad} & Y' & \xrightarrow{\quad} & Y \end{array} \quad \begin{array}{c} \text{id} \\ \text{id} \end{array}$$

This is really a special case of the notion of a retract of an object in the functor category  $\text{Fun}([1], s\text{Set})$ . We say that  $\mathcal{A}$  is **closed under retracts** if for every diagram as above,

closed under retracts

$g \in \mathcal{A}$  implies  $f \in \mathcal{A}$ .

14.2. *Remark.* This list of properties is not minimal: (3) is the special case of (4) when  $\lambda = [2]$ , and (5) can be deduced from (2) and (4). *Exercise:* Show this.

14.3. *Example.* Consider the category of sets. The class of all surjective maps is weakly saturated, and in fact is the weak saturation of  $\{\{0, 1\} \rightarrow \{1\}\}$ . Likewise, the class of injective maps is weakly saturated, and in fact is the weak saturation of  $\{\emptyset \rightarrow \{1\}\}$ .

14.4. *Example.* The classes of injections and surjections of simplicial sets are weakly saturated classes. Later we will identify the class of monomorphisms of simplicial sets as the weak saturation of the set of “cell inclusions” (18.5).

14.5. **Proposition.** *Fix a collection  $\mathcal{C}$  of simplicial sets (e.g., the class of quasicategories). Let  $\mathcal{A}$  be the class of maps of simplicial sets  $i: A \rightarrow B$  such that every map  $f: A \rightarrow C$  to an element  $C \in \mathcal{C}$  admits an extension to  $g: B \rightarrow C$  such that  $gi = f$ . Then  $\mathcal{A}$  is a weakly saturated class.*

*Proof.* This is a straightforward and instructive exercise, which we leave to the reader. It is highly recommended that you work through the argument if you haven’t seen it before.  $\square$

There is a dual notion of a **weakly cosaturated class**: a weakly cosaturated class is the same thing as a weakly saturated class in the opposite category, and is characterized by being closed under properties formally dual to (1)–(6).

**weakly  
class**      **cosaturated**

14.6. **Classes of “anodyne” morphisms.** We use the following notation for sets of types of horns:

$$\begin{aligned} \text{InnHorn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 < k < n, n \geq 2 \}, & (\text{inner horns}), \\ \text{LHorn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 \leq k < n, n \geq 1 \}, & (\text{left horns}), \\ \text{RHorn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 < k \leq n, n \geq 1 \}, & (\text{right horns}), \\ \text{Horn} &:= \{ \Lambda_k^n \subset \Delta^n \mid 0 \leq k \leq n, n \geq 1 \}, & (\text{horns}). \end{aligned}$$

The weak saturation of each of these sets will play an important role in what follows. Right now, we focus on the smallest of these classes, namely the weak saturation  $\overline{\text{InnHorn}}$  of the set of inner horns, which is called the class of **inner anodyne**<sup>11</sup> morphisms. The weak saturations of the other sets are the classes of “left anodyne”, “right anodyne”, and plain old “anodyne” morphisms, about which we have more to say later. Note that anodyne morphisms are always monomorphisms, since monomorphisms of simplicial sets themselves form a weakly saturated class.

**inner anodyne**

14.7. **Proposition.** *If  $C$  is a quasicategory and  $A \subseteq B$  is an inner anodyne inclusion, then any  $f: A \rightarrow C$  admits an extension to  $g: B \rightarrow C$  so that  $g|_A = f$ .*

*Proof.* We know that the class  $\mathcal{A}$  of maps  $i: A \rightarrow B$  such that every map from  $A$  to a quasicategory extends along  $i$  is weakly saturated (14.5). The claim follows immediately since  $\text{InnHorn} \subseteq \mathcal{A}$  by definition.  $\square$

14.8. *Exercise* (Easy but important). Show that every inner anodyne map induces a bijection on vertices. (Hint: show that the class of maps of simplicial sets which are a bijection on vertices is weakly saturated.)

14.9. **Examples of inner anodyne morphisms.** It is crucial to be able to prove that certain explicit maps are inner anodyne.

Let  $S \subseteq [n]$ . The associated **generalized horn** is the subcomplex  $\Lambda_S^n \subset \Delta^n$  defined by

**generalized horn**

<sup>11</sup>The “anodyne” terminology for the weak saturation of a set of horns was introduced by Gabriel and Zisman [GZ67]. “Anodyne” derives from ancient Greek, meaning “without pain”; we leave it to the reader to decide whether this choice of terminology is appropriate.

$$\Lambda_S^n := \bigcup_{i \in S} \Delta^{[n] \setminus i},$$

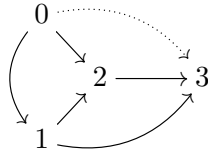
i.e., the union of codimension one faces of the  $n$ -simplex indexed by elements of  $S$ . In particular,  $\Lambda_{[n] \setminus \{j\}}^n$  is the usual horn  $\Lambda_j^n$ . I'll generalize this notation to arbitrary totally ordered sets, so  $\Lambda_S^T = \bigcup_{i \in S} \Delta^{T \setminus i}$  when  $S \subseteq T$ .

We call  $\Lambda_S^n \subset \Delta^n$  a **generalized inner horn** if  $S$  is not an “interval” in  $[n]$ , i.e., if there exist  $s < t < s'$  with  $s, s' \in S$  and  $t \notin S$ . generalized inner horn

**14.10. Lemma.** *All generalized inner horn inclusions  $\Lambda_S^n \subset \Delta^n$  are inner anodyne.*

There is a slick proof of this given by Joyal [Joy08a, Prop. 2.12], which we present in the appendix (61.1).

**14.11. Example.** Consider  $\Lambda_{\{0,3\}}^3$ , which can be pictured as the solid diagram in



We can get from this to  $\Delta^3$  in two steps:

$$\begin{array}{ccc} \Lambda_{\{0,3\}}^{\{0,2,3\}} & \hookrightarrow & \Delta^{\{0,2,3\}} \\ \downarrow & & \downarrow \\ \Lambda_{\{0,3\}}^3 & \hookrightarrow & \Lambda_2^3 \hookrightarrow \Delta^3 \end{array}$$

The square is a pushout of subcomplexes since  $\Lambda_{\{0,3\}}^3 \cap \Delta^{\{0,2,3\}} = \Lambda_{\{0,3\}}^{\{0,2,3\}}$ , and the map along the top is isomorphic to  $\Lambda_1^2 \subset \Delta^2$ , an inner horn inclusion. This proves that  $\Lambda_{\{0,3\}}^3 \subset \Delta^3$  is inner anodyne.

Recall that every standard  $n$ -simplex contains a spine  $I^n \subseteq \Delta^n$ .

**14.12. Lemma.** *The spine inclusions  $I^n \subset \Delta^n$  are inner anodyne for all  $n$ . Thus, for a quasicategory  $C$ , any  $I^n \rightarrow C$  extends to  $\Delta^n \rightarrow C$ .*

This is proved in [Joy08a, Prop. 2.13]; we give the proof in the appendix (61.2).

**14.13. Example.** To show that  $I^3 \subset \Delta^3$  is inner anodyne, observe that we can get from  $I^3$  to a generalized inner horn in two steps by attaching 2-cells along inner horn inclusions:

$$\begin{array}{ccccc} \Lambda_{\{0,2\}}^{\{0,1,2\}} & \hookrightarrow & \Delta^{\{0,1,2\}} & & \Lambda_{\{1,3\}}^{\{1,2,3\}} & \hookrightarrow & \Delta^{\{1,2,3\}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ I^3 & \hookrightarrow & I^3 \cup \Delta^{\{0,1,2\}} & = & I^3 \cup \Delta^{\{0,1,2\}} & \hookrightarrow & \Lambda_{\{0,3\}}^3 \end{array}$$

since  $I^3 \cap \Delta^{\{0,1,2\}} = \Lambda_{\{0,2\}}^{\{0,1,2\}}$  and  $(I^3 \cup \Delta^{\{0,1,2\}}) \cap \Delta^{\{1,2,3\}} = \Lambda_{\{1,3\}}^{\{1,2,3\}}$ .

**14.14. Exercise.** Use (14.12) to show that the tautological map  $\pi: C \rightarrow N(hC)$  from a quasicategory to (the nerve of) its homotopy category is surjective in every degree.

## 15. LIFTING CALCULUS AND INNER FIBRATIONS

We have defined quasicategories by an “extension property”: in general, we say that  $X$  satisfies the extension property for  $f: A \rightarrow B$  if for any diagram

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ f \downarrow & \nearrow s & \\ B & & \end{array}$$

there exists a morphism  $s$  making the diagram commute. In this section, we discuss a “relative” version of this, called a “lifting property”.

**15.1. The lifting relation.** Given morphisms  $f: A \rightarrow B$  and  $g: X \rightarrow Y$  in a category, a **lifting problem** for  $(f, g)$  is a pair of morphisms  $(u, v)$  such that  $vf = gu$ . That is, a lifting problem is any commutative square of solid arrows of the form

lifting problem

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ f \downarrow & \nearrow s & \downarrow g \\ B & \xrightarrow{v} & Y \end{array}$$

A **lift** for the lifting problem is a morphism  $s$  such that  $sf = u$  and  $gs = v$ , i.e., a dotted arrow making the diagram commute.

lift

We may thus define the **lifting relation** on morphisms in our category: we write “ $f \sqsubset g$ ” if *every* lifting problem for  $(f, g)$  admits a lift<sup>12</sup>. Equivalently,  $f \sqsubset g$  exactly if

lifting relation

$$\mathrm{Hom}(B, X) \xrightarrow{sf \mapsto (sf, gs)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A, Y)} \mathrm{Hom}(B, Y)$$

is a surjection, where the target is the set of pairs  $(u: A \rightarrow X, v: B \rightarrow Y)$  such that  $gu = vf$  (i.e., the target is exactly the set of lifting problems for  $(f, g)$ ).

When  $f \sqsubset g$  holds, one sometimes says  $f$  has the **left lifting property** relative to  $g$ , or that  $g$  has the **right lifting property** relative to  $f$ . Or we just say that  $f$  **lifts against**  $g$ .

left lifting property  
right lifting property  
 $f$  lifts against  $g$ .

We extend the notation to classes of maps, so “ $\mathcal{A} \sqsubset \mathcal{B}$ ” means:  $a \sqsubset b$  for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ .

Note: I will also sometimes speak of a **lifting problem of type**  $f \sqsubset g$ , by which I mean a pair  $(u, v)$  which is a lifting problem for  $(f, g)$ .

lifting problem of type  
 $f \sqsubset g$ 

**15.2. Exercise.** Show that  $f \sqsubset f$  if and only if  $f$  is an isomorphism.

Given a class of morphisms  $\mathcal{A}$ , define the **right complement**  $\mathcal{A}^\square$  and **left complement**  ${}^\square\mathcal{A}$  by

right complement  
left complement

$$\mathcal{A}^\square = \{ g \mid a \sqsubset g \text{ for all } a \in \mathcal{A} \}, \quad {}^\square\mathcal{A} = \{ f \mid f \sqsubset a \text{ for all } a \in \mathcal{A} \}.$$

**15.3. Proposition.** For any class  $\mathcal{A}$ , the left complement  ${}^\square\mathcal{A}$  is a weakly saturated class, and the right complement  $\mathcal{A}^\square$  is a weakly cosaturated class.

**15.4. Exercise (Important).** Prove (15.3). (This is a “relative” version of the proof of (14.5).)

**15.5. Exercise (Easy).** Prove that if  $\mathcal{A} \subseteq \mathcal{B}$ , then  $\mathcal{A}^\square \supseteq \mathcal{B}^\square$  and  ${}^\square\mathcal{A} \supseteq {}^\square\mathcal{B}$ . Use this to show  $\mathcal{A}^\square = ({}^\square(\mathcal{A}^\square))^\square$  and  ${}^\square\mathcal{A} = {}^\square({}^\square({}^\square\mathcal{A}))$ .

**15.6. Exercise** (for those who know a little homological algebra). Fix an abelian category  $\mathcal{C}$  (e.g., the category of modules over some ring  $R$ ). Let  $\mathcal{P}$  be the class of morphisms in  $\mathcal{C}$  of the form  $0 \rightarrow P$  where  $P$  is projective, and let  $\mathcal{B}$  be the class of epimorphisms in  $\mathcal{C}$ . Show that  $\mathcal{P} \sqsubset \mathcal{B}$ ; also, show that  $\mathcal{B} = \mathcal{P}^\square$  if  $\mathcal{C}$  has enough projectives.

**15.7. Exercise.** In the setting of the previous exercise, identify the class  ${}^\square\mathcal{B}$ .

<sup>12</sup>Sometimes one sees the notation “ $f \perp g$ ” or “ $f \pitchfork g$ ” used instead. Our notation is taken from [Rie14, §11].



**15.8. Inner fibrations.** A map  $p$  of simplicial sets is an **inner fibration** if  $\text{InnHorn} \sqsupset p$ . The class of inner fibrations  $\text{InnFib} = \text{InnHorn}^\sqsupset$  is thus the right complement of the set of inner horns. Note that  $C$  is a quasicategory if and only if  $C \rightarrow *$  is an inner fibration. inner fibration

Because  $\text{InnFib}$  is a right complement, it is weakly cosaturated. In particular, it is closed under composition. This implies that if  $p: C \rightarrow D$  is an inner fibration and  $D$  is a quasicategory, then  $C$  is also a quasicategory. Also note that since the left complement of  $\text{InnFib}$  is weakly saturated, we have  $\overline{\text{InnHorn}} \sqsupset \text{InnFib}$ .

**15.9. Exercise.** Show that if  $f: C \rightarrow D$  is any functor from a quasicategory  $C$  to a category  $D$ , then  $f$  is an inner fibration. In particular, all functors between categories are automatically inner fibrations. (Hint: use the fact that all inner horns mapping to a category have *unique* extensions to simplices.)

**15.10. Exercise.** Show that any inclusion  $C' \subseteq C$  of a subcomplex of a quasicategory is an inner fibration if and only if  $C'$  is a subcategory (8.16) of  $C$ .

**15.11. Exercise.** Let  $p: C \rightarrow D$  be a functor between quasicategories, and let  $p^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$  be the restriction of  $p$  to cores (12.8). Show that if  $p$  is an inner fibration then  $p^{\text{core}}$  is also an inner fibration. (Hint. There are two distinct cases of lifting problems  $(\Lambda_k^n \subset \Delta^n) \sqsupset p^{\text{core}}$ , namely  $n = 2$  and  $n \geq 3$ .)

**15.12. Exercise.** Consider a pullback square of simplicial sets

$$\begin{array}{ccc} X' & \longrightarrow & X \\ p' \downarrow & & \downarrow p \\ Y' & \xrightarrow{\pi} & Y \end{array}$$

such that  $\pi$  is a *surjective* map. Show that if  $p'$  is an inner fibration then so is  $p$ .

**15.13. Example** (Campbell's example). Here is an example of an inner fibration whose target is not a quasicategory. Let  $H^\ell := \Delta^2 / \Delta^{\{0,1\}}$ , i.e., the pushout of the diagram  $\Delta^2 \leftarrow \Delta^{\{0,1\}} \rightarrow \Delta^0$  of simplicial sets, and let  $f: \Delta^1 \rightarrow H^\ell$  be the composite  $\Delta^1 \xrightarrow{\langle 02 \rangle} \Delta^2 \xrightarrow{\pi} H^\ell$ , where  $\pi$  is the evident projection. Then  $f$  is an inner fibration. To see this, note that the base-change of  $f$  along the projection map  $\pi$  is the inclusion  $\Lambda_0^2 \subseteq \Delta^2$ , which is an inner fibration since both source and target are categories (15.10). Thus  $f$  is an inner fibration by (15.12).

However,  $H^\ell$  is not a quasicategory: there is a map  $\Delta_1^2 \rightarrow H^\ell$  which does not extend over  $\Lambda_1^2 \subset \Delta^2$  (*Exercise*: find it).

The map  $f$  has been observed by Alexander Campbell [Cam19] to be a counterexample to a number of plausible-sounding statements, some of which we will discuss later (38.16).

**15.14. Exercise.** Show that the map  $f: \Delta^1 \rightarrow H^\ell$  of (15.13) is not inner anodyne. (Hint: (15.2).)

**15.15. Factorizations.** It turns out that we can always factor any map of simplicial sets into an inner anodyne map followed by an inner fibration. This is a consequence of the following general observation.

**15.16. Proposition** (“Small object argument”). *Let  $S$  be a set of morphisms in  $s\text{Set}$ . Every map  $f$  between simplicial sets admits a factorization  $f = pj$  with  $j \in \overline{S}$  and  $p \in S^\sqsupset$ .*

The proof of this proposition is by means of what is known as the “small object argument”. I’ll give the proof in the next section. For now we record a consequence.

**15.17. Corollary.** *For any set  $S$  of morphisms in  $s\text{Set}$ , we have that  $\overline{S} = \sqsupset(S^\sqsupset)$ .*

*Proof.* That  $\overline{S} \subseteq {}^\square(S^\square)$  is immediate from (15.3). Given  $f$  such that  $f \sqsupseteq S^\square$ , use the small object argument (15.16) to choose  $f = pj$  with  $j \in \overline{S}$  and  $p \in S^\square$ . We have a commutative diagram of solid arrows

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & \nearrow & & \searrow & \\
 \bullet & \xrightarrow{\text{id}} & \bullet & \xrightarrow{\text{id}} & \bullet \\
 \downarrow f & & \downarrow j & & \downarrow f \\
 \bullet & \xrightarrow{s} & \bullet & \xrightarrow{p} & \bullet \\
 & \searrow & \text{id} & \nearrow & \\
 & & & & 
 \end{array}$$

A map  $s$  exists making the diagram commute, because  $f \sqsupseteq p$ , so there is a lift in

$$\begin{array}{ccc}
 \bullet & \xrightarrow{j} & \bullet \\
 \downarrow f & \nearrow s & \downarrow p \\
 \bullet & \xrightarrow{\text{id}} & \bullet
 \end{array}$$

The diagram exhibits  $f$  as a retract of  $j$ , whence  $f \in \overline{S}$  since weak saturations are closed under retracts.  $\square$

15.18. *Remark* (Retract trick). The proof of the corollary is called the “retract trick”: given  $f = pj$ ,  $f \sqsupseteq p$  implies that  $f$  is a retract of  $j$ , while  $j \sqsupseteq f$  implies that  $f$  is a retract of  $p$ .

In the case we are currently interested in, we have that  $\overline{\text{InnHorn}} = {}^\square\text{InnFib}$  and  $\overline{\text{InnHorn}}^\square = \text{InnFib}$ , and thus any map can be factored into an inner anodyne map followed by an inner fibration.

15.19. **Weak factorization systems.** A **weak factorization system** in a category is a pair  $(\mathcal{L}, \mathcal{R})$  of classes of maps such that

weak factorization system

- every map  $f$  admits a factorization  $f = r\ell$  with  $r \in \mathcal{R}$  and  $\ell \in \mathcal{L}$ , and
- $\mathcal{L} = {}^\square\mathcal{R}$  and  $\mathcal{R} = \mathcal{L}^\square$ .

Thus, in any weak factorization the “left” class  $\mathcal{L}$  is weakly saturated and the “right” class  $\mathcal{R}$  is weakly cosaturated. The small object argument implies that  $(\overline{S}, S^\square)$  is a weak factorization in  $s\text{Set}$  for every set of maps  $S$ . In particular,  $(\overline{\text{InnHorn}}, \text{InnFib})$  is a weak factorization system.

15.20. *Exercise* (for those who know some homological algebra). In an abelian category, let  $\mathcal{A}$  be the class of monomorphisms with projective cokernel, and let  $\mathcal{B}$  be the class of epimorphisms. Show that the pair  $(\mathcal{A}, \mathcal{B})$  is a weak factorization system if and only if the category has enough projectives. (This exercise is related to (15.6).)

15.21. *Exercise* (Goodwillie). Classify all weak factorization systems on the category of sets. (There are exactly six.)

15.22. **Uniqueness of liftings.** The relation  $f \sqsupseteq g$  says that lifting problems admit solutions, but not that the solutions are unique. However, we can incorporate uniqueness into the lifting calculus if our category has pushouts.

Given a map  $f: A \rightarrow B$ , let  $f^\vee := (\text{id}_B, \text{id}_B): B \amalg_A B \rightarrow B$  be the “fold” map, i.e., the unique map such that the composition with either of the canonical maps  $B \rightarrow B \amalg_A B$  is  $f$ . It is straightforward to show that for a map  $g: X \rightarrow Y$  we have that  $\{f, f^\vee\} \sqsupseteq g$  if and only if in every commutative square

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 \downarrow f & \nearrow s & \downarrow g \\
 B & \longrightarrow & Y
 \end{array}$$

there exists a *unique* lift  $s$ .

15.23. *Example.* Consider the category of topological spaces. Let  $\mathcal{A}$  be the class of morphisms of the form  $A \times \{0\} \rightarrow A \times [0, 1]$ , where  $A$  is an arbitrary space. Then  $(\mathcal{A} \cup \mathcal{A}^\vee)^\square$  contains all covering maps (by the “Covering Homotopy Theorem”).

A weak factorization system  $(\mathcal{L}, \mathcal{R})$  in which liftings are always unique is called an **orthogonal factorization system**.

orthogonal  
factoriza-  
tion system

15.24. *Exercise.* Show that in an orthogonal factorization system, the factorizations  $f = r\ell$  with  $\ell \in \mathcal{L}$  and  $r \in \mathcal{R}$  are unique up to *unique isomorphism*.

15.25. *Exercise.* Show that  $(\{\text{surjections}\}, \{\text{injections}\})$  is an orthogonal factorization system for  $\text{Set}$ .

15.26. *Exercise.* Let  $S$  be a set of maps of simplicial sets, and consider the weak factorization system  $(\bar{S} \cup \bar{S}^\vee, (S \cup S^\vee)^\square)$ . Show that this is in fact an orthogonal factorization system. (Hint: show that that for any map  $g$ , the class of maps  $f$  such that the lifting problem for  $(f, g)$  has a *unique* solution is a weakly saturated class.)

15.27. *Example* (The fundamental category via an orthogonal factorization system). In simplicial sets, the projection map  $C \rightarrow *$  is in the right complement to  $S := \text{InnHorn} \cup \text{InnHorn}^\vee$  if and only if  $C$  is *isomorphic* to a nerve of a category (7.7). The small object argument using  $S$ , applied to a projection  $X \rightarrow *$ , thus produces a morphism  $\pi: X \rightarrow Y$  in  $\bar{S}$  with  $Y$  the nerve of a category.

Uniqueness of liftings in this case implies that  $\pi: X \rightarrow Y$  has precisely the universal property of the fundamental category of  $X$  defined in (11.1): given  $f: X \rightarrow C$  with  $C$  a category, a unique extension of  $f$  over  $X \rightarrow Y$  exists. Thus, the small object argument applied to  $S$  gives another construction of the fundamental category (11.1) of an arbitrary simplicial set  $S$ .

15.28. *Exercise.* Prove that if  $f: X \rightarrow Y$  is any inner anodyne map, then the induced functor  $h(f): hX \rightarrow hY$  between fundamental categories is an isomorphism. (Hint: use the universal property of fundamental categories to construct an inverse to  $h(f)$ .)

## 16. THE SMALL OBJECT ARGUMENT

In this section we give the proof of (15.16), i.e., that given a fixed set  $S = \{s_i: A_i \rightarrow B_i\}$  of maps of simplicial sets, we can factor any map  $f: X \rightarrow Y$  as  $f = pj$  with  $j \in \bar{S}$  and  $p \in S^\square$ . For the reader: it may be helpful to first work through the special case where  $Y = \Delta^0$  (the terminal object in simplicial sets). Also, it is worth noting that we will only rarely need to know any details of this proof in the subsequent text.

16.1. **A factorization construction.** Given any map  $f: X \rightarrow Y$ , we first produce a factorization

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y, \quad (Rf)(Lf) = f$$

as follows. Consider the set

$$[S, f] := \{ (s_i, u, v) \mid s_i \in S, fu = vs_i \} = \left\{ \begin{array}{ccc} A_i & \xrightarrow{u} & X \\ s_i \downarrow & & \downarrow f \\ B_i & \xrightarrow{v} & Y \end{array} \right\}$$

of all commutative squares which have an arrow from  $S$  on the left-hand side, and  $f$  on the right-hand side, i.e., the set of all lifting problems of type  $s_i \sqsubset f$  for some  $s_i \in S$ . We define  $Ef$ ,  $Lf$ , and  $Rf$

using the diagram

$$\begin{array}{ccc}
 \coprod_{(s_i, u, v)} A_i & \xrightarrow{(u)} & X \\
 \downarrow \coprod s_i & & \downarrow Lf \\
 \coprod_{(s_i, u, v)} B_i & \xrightarrow{\quad} & Ef \\
 & \searrow (v) & \nearrow Rf \\
 & & Y
 \end{array}
 \quad \begin{array}{c}
 \nearrow f \\
 \\
 \\
 \end{array}$$

where the coproducts are indexed by the set  $[S, f]$ , and the square is a pushout. Note that  $Lf \in \bar{S}$  by construction; however, we do not expect that  $Rf$  in  $S^\square$ .

We can iterate the construction:

$$\begin{array}{ccccccc}
 X & & & & & & \\
 \parallel & \searrow Lf & \searrow L^2 f & \searrow L^3 f & \searrow L^\omega f & & \\
 X & \xrightarrow{Lf} & Ef & \xrightarrow{LRf} & E^2 f & \xrightarrow{LR^2 f} & E^3 f \longrightarrow \dots \longrightarrow E^\omega f \\
 \downarrow f & \nearrow Rf & \nearrow R^2 f & \nearrow R^3 f & \nearrow R^\omega f & & \\
 Y & & & & & & 
 \end{array}$$

Here each triple  $(E^\alpha f, L^\alpha f, R^\alpha f)$  is obtained by factoring the “ $R$ ” map of the previous one, so that

$$(16.2) \quad E^{\alpha+1} f := E(R^\alpha f), \quad L^{\alpha+1} f := L(R^\alpha f) \circ (L^\alpha f), \quad R^{\alpha+1} f := R(R^\alpha f).$$

Taking direct limits gives a factorization  $X \xrightarrow{L^\omega f} E^\omega f \xrightarrow{R^\omega f} Y$  of  $f$ , with  $E^\omega f = \text{colim}_{n \rightarrow \infty} E^n f$ .

We can go even further, using the magic of transfinite induction, and define compatible factorizations  $(E^\lambda f, L^\lambda f, R^\lambda f)$  for each ordinal<sup>13</sup>  $\lambda$ . For successor ordinals  $\alpha + 1$  use the prescription of (16.2), while for limit ordinals  $\beta$  take a direct limit  $E^\beta f := \text{colim}_{\alpha < \beta} E^\alpha f$  as in the construction of  $E^\omega f$  above.

It is immediate that every  $L^\alpha f \in \bar{S}$ , because weak saturations are closed under transfinite composition. The maps  $R^\alpha f$  are not generally contained in  $S^\square$ , though they do satisfy a “partial lifting property”: whenever  $\alpha < \beta$  there exists *by construction* a dotted arrow making

$$\begin{array}{ccccc}
 A_i & \xrightarrow{u'} & E^\alpha f & \longrightarrow & E^{\alpha+1} f & \longrightarrow & E^\beta f \\
 s_i \downarrow & & \nearrow \text{dotted} & & \downarrow R^\beta f & & \\
 B_i & \xrightarrow{\quad} & & \xrightarrow{v} & & \longrightarrow & Y
 \end{array}$$

commute, for any  $u'$  and  $v$  making the square commute. Thus, we get a solution to a lifting problem  $(u, v)$  of  $s_i$  against  $R^\beta f$  whenever the map  $u: A_i \rightarrow E^\beta f$  on the top of a commutative square that we want a lift for can be factored through one of the maps  $E^\alpha f \rightarrow E^\beta f$  with  $\alpha < \beta$ . This is so exactly because  $E^{\alpha+1} f$  was obtained from  $E^\alpha f$  by “formally adjoining” a solution to *every* such lifting problem.

The “small object argument” amounts to the following.

**Claim.** There exists an ordinal  $\kappa$  such that for every domain  $A_i$  of a map in  $S$ , every map  $A_i \rightarrow E^\kappa f$  factors through some  $E^\alpha f \rightarrow E^\kappa f$  with  $\alpha < \kappa$ .

Given this, it follows from the “partial lifting property” that  $S \square R^\kappa f$ , and so we obtain the desired factorization:  $f = (R^\kappa f) \circ (L^\kappa f)$  with  $L^\kappa f \in \bar{S}$  and  $R^\kappa f \in S^\square$ .

It remains to prove the claim, which we will do by choosing  $\kappa$  to be a *regular cardinal* which is “bigger” than all the simplicial sets  $A_i$ .

<sup>13</sup>For a treatment of ordinals, see for instance the chapter on sets in [TS14].

**16.3. Regular cardinals.** The **cardinality** of a set  $X$  is the smallest ordinal  $\lambda$  such that there exists a bijection between  $X$  and  $\lambda$ ; we write  $|X|$  for this. Ordinals which can appear this way are called **cardinals**. For instance, the first infinite ordinal  $\omega$  is the countable cardinal.

cardinality

cardinals

Note: the class of infinite cardinals is an unbounded subclass of the ordinals, so is well-ordered and can be put into bijective correspondence with ordinals. The symbol  $\aleph_\alpha$  denotes the  $\alpha$ th infinite cardinal, e.g.,  $\aleph_0 = \omega$ .

Say that  $\lambda$  is a **regular cardinal**<sup>14</sup> if it is an infinite cardinal, and if for every set  $A$  of ordinals such that (i)  $\alpha < \lambda$  for all  $\alpha \in A$ , and (ii)  $|A| < \lambda$ , we have that  $\sup A < \lambda$ . For instance,  $\omega$  is a regular cardinal, since any finite collection of finite ordinals has a finite upper bound. Not every infinite cardinal is regular<sup>15</sup>; however, there exist arbitrarily large regular cardinals<sup>16</sup>.

regular cardinal

Every ordinal  $\alpha$  defines a category, which is the poset of ordinals strictly less than  $\alpha$ . Colimits of functors  $Y: \kappa \rightarrow \mathbf{Set}$  with  $\kappa$  a regular cardinal have the following property: the map

$$(16.4) \quad \operatorname{colim}_{\alpha < \kappa} \operatorname{Hom}(X, Y_\alpha) \rightarrow \operatorname{Hom}(X, \operatorname{colim}_{\alpha < \kappa} Y_\alpha)$$

is a bijection whenever  $|X| < \kappa$ . This generalizes the familiar case of  $\kappa = \omega$ : any map of a finite set into the colimit of a countable sequence factors through a finite stage.

**16.5. Exercise.** Prove that (16.4) is a bijection when  $|X| < \kappa$ .

**16.6. Small simplicial sets.** Given a regular cardinal  $\kappa$ , we say that a simplicial set is  **$\kappa$ -small** if it is isomorphic to the colimit of some functor  $F: C \rightarrow s\mathbf{Set}$ , such that (i)  $|\operatorname{ob} C|, |\operatorname{mor} C| < \kappa$ , and (ii) each  $F(c)$  is isomorphic to a standard simplex  $\Delta^n$ . Morally, we are saying that a simplicial set is  $\kappa$ -small if it can be “presented” with fewer than  $\kappa$  generators and fewer than  $\kappa$  relations.

 $\kappa$ -small

Given a functor  $Y: \kappa \rightarrow s\mathbf{Set}$  and a  $\kappa$ -small simplicial set  $X$ , we have a bijection as in (16.4). (This is sometimes phrased as:  *$\kappa$ -small simplicial sets are  $\kappa$ -compact*.) Thus, to prove the claim about the small object argument, we simply choose a regular cardinal  $\kappa$  greater than  $\sup\{|(A_i)_n|\}$ , where  $(A_i)_n$  is the set of  $n$ -cells of the simplicial set  $A_i$ , which ranges over the set of domains of morphisms in  $S$ .

**16.7. Example.** The standard simplices  $\Delta^n$ , as well as any subcomplex such as the horns  $\Lambda_j^n$ , are  $\omega$ -small: this is a consequence of (6.15). Thus, when we carry out the small object argument for  $S = \operatorname{InnHorn}$ , we can take  $(E^\omega f, L^\omega f, R^\omega f)$  to be the desired factorization.

**16.8. Remark.** The small object argument can be carried out in a very large class of categories, including the *locally presentable categories*. This class includes familiar algebraic examples, such as categories of groups, rings, lie algebras, modules, etc., along with many others. With a little more care the argument can sometimes be carried out even more generally (possibly under additional hypotheses on  $S$ ), for instance in the category of topological spaces

**16.9. Functoriality.** The construction  $f \mapsto (X \xrightarrow{L^f} Ef \xrightarrow{R^f} Y)$  is a functor  $\operatorname{Fun}([1], s\mathbf{Set}) \rightarrow \operatorname{Fun}([2], s\mathbf{Set})$ , and it follows that so is  $f \mapsto (X \xrightarrow{L^\alpha f} E^\alpha f \xrightarrow{R^\alpha f} Y)$  for any  $\alpha$ . Because the choice of regular cardinal  $\kappa$  depends only on  $S$ , not on the map  $f$ , we see that the small object argument actually produces a *functorial factorization* of a map into a composite of an element of  $\overline{S}$  with an element  $S^\square$ . We will have use of this later in (64).

## 17. DEGENERATE AND NON-DEGENERATE CELLS

We have noted that monomorphisms of simplicial sets form a weakly saturated class. Here we identify an important set of maps called  $\operatorname{Cell}$ , so that the weak saturation of  $\operatorname{Cell}$  is precisely the

<sup>14</sup>In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.

<sup>15</sup>For instance,  $\aleph_\omega = \sup\{\aleph_k \mid k < \omega\}$  is not regular.

<sup>16</sup>For instance, every successor cardinal  $\aleph_{\alpha+1}$  is regular.

class of monomorphisms. We do so by getting a very explicit handle on monomorphisms of simplicial sets. This will involve the notion of *degenerate* and *non-degenerate* cells of a simplicial set.

**17.1. Boundary of a standard simplex.** For each  $n \geq 0$ , we define

$$\partial\Delta^n := \bigcup_{k \in [n]} \Delta^{[n] \setminus \{k\}} \subset \Delta^n,$$

the union of all codimension-one faces of the  $n$ -simplex. Equivalently,

$$(\partial\Delta^n)_k = \{f: [k] \rightarrow [n] \mid f([k]) \neq [n]\}.$$

We call  $\partial\Delta^n$  the **boundary** of  $\Delta^n$ . Note that  $\partial\Delta^0 = \emptyset$  and  $\partial\Delta^1 = \Delta^{\{0\}} \amalg \Delta^{\{1\}}$ .

boundary

**17.2. Exercise.** Show that  $\partial\Delta^n$  is the largest subcomplex of  $\Delta^n$  which does not contain the “generator”  $\iota_n = \langle 0 \dots n \rangle \in (\Delta^n)_n$ . In other words,  $\partial\Delta^n$  is the maximal *proper* subcomplex of  $\Delta^n$ .

**17.3. Exercise.** Show that if  $C$  is a category, then the evident maps  $\text{Hom}(\Delta^n, C) \rightarrow \text{Hom}(\partial\Delta^n, C)$  defined by restriction are isomorphisms when  $n \geq 3$ , but not necessarily when  $n \leq 2$ .

**17.4. Trivial fibrations and monomorphisms.** Let  $\text{Cell}$  be the set consisting of the inclusions  $\partial\Delta^n \subset \Delta^n$  for  $n \geq 0$ . The resulting right complement is  $\text{TrivFib} := \text{Cell}^\square$ , the class of **trivial fibrations** (also sometimes called **acyclic fibrations**). By the small object argument (15.16), we obtain a weak factorization system  $(\overline{\text{Cell}}, \text{TrivFib})$ .

trivial fibrations  
acyclic fibrations

Since the elements of  $\text{Cell}$  are monomorphisms, and the class of all monomorphisms is weakly saturated, we see that all elements of  $\overline{\text{Cell}}$  are monomorphisms. We are going to prove the converse, i.e., we will show that  $\overline{\text{Cell}}$  is precisely equal to the class of monomorphisms.

**17.5. Degenerate and non-degenerate cells.** Recall  $\Delta^{\text{surj}}, \Delta^{\text{inj}} \subset \Delta$ , the subcategories of the category  $\Delta$  of simplicial operators, consisting of all the objects and the *surjective* and *injective* order-preserving maps respectively, and that every operator factors uniquely as  $f = f^{\text{inj}} f^{\text{surj}}$ , a surjection followed by an injection.

A cell  $a \in X_n$  is said to be **degenerate** if there exists a *non-injective* simplicial operator  $f \in \Delta$  and an element  $b$  in  $X$  such that  $a = bf$ . In view of the factorization  $f = f^{\text{inj}} f^{\text{surj}}$ , we see that  $a$  is degenerate if and only if there exists a *non-identity surjective* simplicial operator  $f \in \Delta^{\text{surj}}$  and a cell  $b$  in  $X$  such that  $a = bf$ .

degenerate

Likewise, a cell  $a \in X_n$  is said to be **non-degenerate** if it is not degenerate, i.e., if  $a = bf$  for some  $f$  in  $\Delta$  and  $b$  in  $X$  we must have  $f \in \Delta^{\text{inj}}$ . Equivalently,  $a$  is non-degenerate if  $a = bf$  for some  $f$  in  $\Delta^{\text{surj}}$  and  $b$  in  $X$  implies  $f = \text{id}$ .

non-degenerate

We write  $X_n = X_n^{\text{deg}} \amalg X_n^{\text{nd}}$  for the decomposition of  $X_n$  into complementary subsets of degenerate and non-degenerate cell. Note that if  $f: A \rightarrow X$  is a map of simplicial sets, then  $f(A_n^{\text{deg}}) \subseteq X_n^{\text{deg}}$ , while  $f^{-1}(X_n^{\text{nd}}) \subseteq A_n^{\text{nd}}$ . Note that neither  $X_n^{\text{deg}}$  nor  $X_n^{\text{nd}}$  assemble to give a subcomplex of  $X$  (unless  $X$  is empty).

**17.6. Proposition.** *If  $X$  is a simplicial set and  $A \subseteq X$  is a subcomplex, then  $A_n^{\text{nd}} = X_n^{\text{nd}} \cap A_n$  and  $A_n^{\text{deg}} = X_n^{\text{deg}} \cap A_n$ .*

*Proof.* The first statement is a consequence of the second, since subsets of degenerate and non-degenerate cells are complementary. It is clear that  $A_n^{\text{deg}} \subseteq X_n^{\text{deg}} \cap A_n$ . Conversely, suppose  $a \in X_n^{\text{deg}} \cap A_n$ , so  $a \in A_n$  and  $a = bg$  for some non-identity  $g: [n] \rightarrow [k] \in \Delta^{\text{surj}}$  and  $b \in X_k$ . Any surjection in  $\Delta$  has a section (6.13), so there exists  $s: [k] \rightarrow [n]$  such that  $gs = 1_{[k]}$ . Then  $b = bgs = as \in A_k$ , whence  $a \in A_n^{\text{deg}}$  as desired.  $\square$

**17.7. Exercise (easy).** For any simplicial set  $X$ , we have  $X_0^{\text{deg}} = \emptyset$  and  $X_0^{\text{nd}} = X_0$ , while  $X_1^{\text{deg}}$  is the image of  $\langle 00 \rangle^*: X_0 \rightarrow X_1$  (which is an injective function) and  $X_1^{\text{nd}}$  is its complement.

17.8. *Example.* Here are all cells in the standard 2-simplex up to dimension 3, with the non-degenerate ones indicated by a box.

$(\Delta^2)_0$	$(\Delta^2)_1$	$(\Delta^2)_2$	$(\Delta^2)_3$
$\langle 0 \rangle$	$\langle 00 \rangle$	$\langle 000 \rangle$	$\langle 0000 \rangle$
$\langle 1 \rangle$	$\langle 11 \rangle$	$\langle 111 \rangle$	$\langle 1111 \rangle$
$\langle 2 \rangle$	$\langle 22 \rangle$	$\langle 222 \rangle$	$\langle 2222 \rangle$
	$\langle 01 \rangle$	$\langle 001 \rangle \langle 011 \rangle$	$\langle 0001 \rangle \langle 0011 \rangle \langle 0111 \rangle$
	$\langle 02 \rangle$	$\langle 002 \rangle \langle 022 \rangle$	$\langle 0002 \rangle \langle 0022 \rangle \langle 0222 \rangle$
	$\langle 12 \rangle$	$\langle 112 \rangle \langle 122 \rangle$	$\langle 1112 \rangle \langle 1122 \rangle \langle 1222 \rangle$
		$\langle 012 \rangle$	$\langle 0012 \rangle \langle 0112 \rangle \langle 0122 \rangle$

17.9. *Exercise.* Describe the degenerate and non-degenerate cells of all the standard  $n$ -simplices  $\Delta^n$ .

17.10. *Exercise.* For every  $n \geq 0$ , let  $\Delta^n / \partial \Delta^n$  be the pushout of the diagram  $\Delta^n \leftarrow \partial \Delta^n \rightarrow \Delta^0$ , where  $\partial \Delta^n \hookrightarrow \Delta^n$  is the usual inclusion and  $\partial \Delta^n \rightarrow \Delta^0$  is the unique map to the terminal object. Describe all degenerate and non-degenerate cells of  $\Delta^n / \partial \Delta^n$ .

17.11. *Exercise.* Show that if  $C$  is an ordinary category, then a cell  $a \in N(C)_k$  of the nerve with  $k > 0$  is non-degenerate if and only if it is represented by a composable sequence of *non-identity* maps  $c_0 \rightarrow \cdots \rightarrow c_k$  in the category  $C$ .

17.12. *Exercise.* Let  $X$  be a simplicial set. Show that

$$X_n^{\text{deg}} = \{ af \mid a \in X_k, f: [n] \rightarrow [k], k < n \}.$$

17.13. **Simplicial sets are canonically free with respect to surjective operators.** The key observation is that degenerate cells in a simplicial set are precisely determined by knowledge of the non-degenerate cells.

17.14. **Proposition** (Eilenberg-Zilber lemma). *Let  $a$  be a cell of  $X$ . Then there exists a unique pair  $(b, \sigma)$  consisting of a non-degenerate cell  $b$  and a map  $\sigma$  in  $\Delta^{\text{surj}}$  such that  $a = b\sigma$ .*

*Proof.* [GZ67, §II.3]. First note that for degenerate  $a$  such a pair  $(b, \sigma)$  exists by definition, while for nondegenerate  $a$  we can take the pair  $(a, \text{id})$ .

Given  $\sigma: [n] \rightarrow [m]$ , let  $\Gamma(\sigma) = \{ \delta: [m] \rightarrow [n] \mid \sigma\delta = \text{id}_{[m]} \}$  denote the set of sections of  $\sigma$ . The sets  $\Gamma(\sigma)$  is non-empty when  $\sigma \in \Delta^{\text{surj}}$  (6.13). We note the following elementary observation, whose proof is left for the reader:

$$\text{If } \sigma, \sigma' \in \Delta^{\text{surj}} \text{ are such that } \Gamma(\sigma) = \Gamma(\sigma'), \text{ then } \sigma = \sigma'.$$

Let  $a \in X_n$  be such that  $a = b_i \sigma_i$  for  $b_i \in X_{m_i}^{\text{nd}}$ ,  $\sigma_i \in \Delta^{\text{surj}}([n], [m_i])$ , for  $i = 1, 2$ . We want to show that  $m_1 = m_2$ ,  $b_1 = b_2$ , and  $\sigma_1 = \sigma_2$ .

Pick any  $\delta_1 \in \Gamma(\sigma_1)$  and  $\delta_2 \in \Gamma(\sigma_2)$ . Then we have

$$b_1 = b_1 \sigma_1 \delta_1 = a \delta_1 = b_2 \sigma_2 \delta_1, \quad b_2 = b_2 \sigma_2 \delta_2 = a \delta_2 = b_1 \sigma_1 \delta_2,$$

so  $b_1$  and  $b_2$  are related by the simplicial operators  $\sigma_2 \delta_1$  and  $\sigma_1 \delta_2$ . Since  $b_1$  and  $b_2$  are both non-degenerate,  $\sigma_2 \delta_1: [m_1] \rightarrow [m_2]$  and  $\sigma_1 \delta_2: [m_2] \rightarrow [m_1]$  must be injective. This implies  $m_1 = m_2$ , and since the only order-preserving injective map  $[m] \rightarrow [m]$  is the identity map, we must have  $\sigma_2 \delta_1 = \text{id} = \sigma_1 \delta_2$ , from which it follows that  $b_1 = b_2$ . This also shows that  $\delta_1 \in \Gamma(\sigma_2)$  and  $\delta_2 \in \Gamma(\sigma_1)$ . Since  $\delta_1$  and  $\delta_2$  were arbitrarily chosen sections, we have shown  $\Gamma(\sigma_1) = \Gamma(\sigma_2)$ , and therefore  $\sigma_1 = \sigma_2$ .  $\square$

We can reinterpret the Eilenberg-Zilber lemma as follows.

17.15. **Corollary.** *For any simplicial set  $X$ , the evident maps*

$$\coprod_{j \geq 0} X_j^{\text{nd}} \times \text{Hom}_{\Delta^{\text{surj}}}([n], [j]) \rightarrow X_n$$

*defined by  $(j, x, \sigma) \mapsto x\sigma$  are bijections. Furthermore, these bijections are natural with respect to surjective simplicial operators  $[n'] \rightarrow [n]$ , and with respect to maps  $X \rightarrow Y$  of simplicial sets.*

*Proof.* The bijection is a restatement of (17.14). For the second statement, note that if  $\tau: [n'] \rightarrow [n]$  is a surjective simplicial operator, then  $(k, x, \sigma\tau) \mapsto (x\sigma)\tau$ . The third statement is straightforward.  $\square$

Another way to say this: the restricted functor  $X|(\Delta^{\text{surj}})^{\text{op}}: (\Delta^{\text{surj}})^{\text{op}} \rightarrow \text{Set}$  is *canonically isomorphic* to a coproduct of representable functors  $\text{Hom}_{\Delta^{\text{surj}}}(-, [k])$  indexed by the nondegenerate cells of  $X$ . Or more simply: simplicial sets are *canonically free* with respect to surjective simplicial operators.

The following exercises show that the subcomplexes of a simplicial set  $X$  can be completely characterized by the sets of non-degenerate cells of  $X$  that they contain.

17.16. *Exercise.* Let  $X^{\text{nd}} = \coprod_{n \geq 0} X_n^{\text{nd}}$  be the set of non-degenerate cells of  $X$ . For  $x, y \in X^{\text{nd}}$  write  $y \leq x$  if there exists  $f \in \Delta$  such that  $y = xf$ . Show that “ $\leq$ ” is a partial order on the set  $X^{\text{nd}}$ ; it is called the *face relation*.

17.17. *Exercise.* Show that if  $xf = yg$  for some  $x, y \in X^{\text{nd}}$ ,  $f \in \Delta$  and  $g \in \Delta^{\text{surj}}$ , then  $y \leq x$ .

17.18. *Exercise.* Let  $S \subseteq X^{\text{nd}}$  be a subset of non-degenerate cells which is closed downward under “ $\leq$ ”, i.e.,  $y \leq x$  and  $x \in S$  implies  $y \in S$ . Show that there exists a unique subcomplex  $A \subseteq X$  such that  $A^{\text{nd}} = S$ . (Hint: the cells of  $A$  are of the form  $xg$  where  $x \in S$  and  $g \in \Delta$ .)

17.19. *Remark.* A simplicial set can be recovered up to isomorphism if you only know (i) its sets of non-degenerate cells, and (ii) the faces of the non-degenerate cells. The proposition we proved above tells how to reconstruct the degenerate cells; simplicial operators on degenerate cells are computed using the fact that any simplicial operator factors into a surjection followed by an injection.

*Warning.* The faces of a non-degenerate cell can be degenerate; this happens for instance for  $\Delta^n / \partial\Delta^n$  in (17.10) when  $n \geq 2$ . If  $X$  is such that all faces of non-degenerate cells are also non-degenerate, then we get a functor  $X^{\text{nd}}: (\Delta^{\text{inj}})^{\text{op}} \rightarrow \text{Set}$ , and the full simplicial set  $X$  can be recovered from  $X^{\text{nd}}$ . For instance, this is so for the standard simplices  $\Delta^n$ , as well as any subcomplexes of such. Functors  $(\Delta^{\text{inj}})^{\text{op}} \rightarrow \text{Set}$  are the combinatorial data behind the notion of a  $\Delta$ -*complex*, as seen in Hatcher’s textbook on algebraic topology [Hat02, Ch. 2.1].

The following exercises give a different point of view of this principle.

17.20. *Exercise.* Fix an object  $[n]$  in  $\Delta$ , and consider the category  $\Delta_{[n]/}^{\text{surj}}$ , which has

- **objects** the surjective morphisms  $\sigma: [n] \rightarrow [k]$  in  $\Delta$ , and
- **morphisms** commutative triangles in  $\Delta$  of the form

$$\begin{array}{ccc} & & [k] \\ & \nearrow \sigma & \\ [n] & & \downarrow \tau \\ & \searrow \sigma' & \\ & & [k'] \end{array}$$

Show that the category  $\Delta_{[n]/}^{\text{surj}}$  is *isomorphic* to the poset  $\mathcal{P}(\underline{n})$  of subsets of the set  $\underline{n} = \{1, \dots, n\}$ . In particular,  $\Delta_{[n]/}^{\text{surj}}$  is a lattice (i.e., has finite products and coproducts, called *meets* and *joins* in this context).



17.21. *Exercise.* Let  $X$  be a simplicial set. Given  $n \geq 0$  and  $\sigma: [n] \rightarrow [k]$  in  $\Delta^{\text{surj}}$ , let  $X_n^\sigma := \sigma^*(X_k)$ , the image of the operator  $\sigma^*$  in  $X_n$ . Show that  $X_n^{\sigma \vee \sigma'} = X_n^\sigma \cap X_n^{\sigma'}$ , where  $\sigma \vee \sigma'$  is join in the lattice  $\Delta_{[n]}^{\text{surj}}$ . Conclude that for each  $x \in X_n$  there exists a maximal  $\sigma$  such that  $x \in X_n^\sigma$ .

## 18. THE SKELETAL FILTRATION

18.1. **Skeleta.** Given a simplicial set  $X$ , the  $k$ -**skeleton**  $\text{Sk}_k X \subseteq X$  is the subcomplex with  $n$ -cells  $k$ -skeleton

$$(\text{Sk}_k X)_n = \bigcup_{0 \leq j \leq k} \{yf \mid y \in X_j, f: [n] \rightarrow [j] \in \Delta\}.$$

It is immediate that this defines a subcomplex of  $X$ , which is in fact the smallest subcomplex containing all cells of dimensions  $\leq k$ . Note that  $\text{Sk}_{k-1} X \subseteq \text{Sk}_k X$  and  $X = \bigcup_k \text{Sk}_k X$ , and that a map  $X \rightarrow Y$  of simplicial sets restricts to a map  $\text{Sk}_k X \rightarrow \text{Sk}_k Y$ . The skeleta constructions define functors  $\text{Sk}_k: s\text{Set} \rightarrow s\text{Set}$ .

In view of (17.14) and (17.15), we see that

$$(\text{Sk}_k X)_n \approx \coprod_{0 \leq j \leq k} X_j^{\text{nd}} \times \text{Hom}_{\Delta^{\text{surj}}}([n], [j]).$$

The complement of the set of cells of  $\text{Sk}_{k-1} X$  in  $\text{Sk}_k X$  consists precisely of the *nondegenerate*  $k$ -cells of  $X$  together with their associated degenerate cells (in dimensions  $> k$ ).

18.2. *Example.* The  $(n-1)$ -skeleton of the standard  $n$ -simplex is precisely what we have called its boundary:  $\text{Sk}_{n-1} \Delta^n = \partial \Delta^n$ . The only cells of  $\Delta^n$  not contained in its boundary are the generator  $\iota = \langle 0 \dots n \rangle \in (\Delta_n)_n$  together with the degenerate cells associated to it.

18.3. **Proposition.** *The evident square*

$$\begin{array}{ccc} \coprod_{a \in X_k^{\text{nd}}} \partial \Delta^k & \longrightarrow & \text{Sk}_{k-1} X \\ \downarrow & & \downarrow \\ \coprod_{a \in X_k^{\text{nd}}} \Delta^n & \longrightarrow & \text{Sk}_k X \end{array}$$

is a pushout of simplicial sets. More generally, for any subcomplex  $A \subseteq X$ , the evident square

$$\begin{array}{ccc} \coprod_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \partial \Delta^k & \longrightarrow & A \cup \text{Sk}_{k-1} X \\ \downarrow & & \downarrow \\ \coprod_{a \in X_k^{\text{nd}} \setminus A_k^{\text{nd}}} \Delta^k & \longrightarrow & A \cup \text{Sk}_k X \end{array}$$

is a pushout.

*Proof.* In each of the above squares, the complements of the vertical inclusions coincide precisely. In particular, the complement of the inclusion  $(A \cup \text{Sk}_{k-1} X)_n \subseteq (A \cup \text{Sk}_k X)_n$  is in bijective correspondence with  $(X_k^{\text{nd}} \setminus A_k^{\text{nd}}) \times \text{Hom}_{\Delta^{\text{surj}}}([n], [k])$ , and thus the square is a pushout (18.4).  $\square$

In the proof, we used the following fact which generalizes (6.10), which is worth recording.

18.4. **Lemma.** *If*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ i \downarrow & & \downarrow j \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a pullback of simplicial sets such that (i)  $j$  is a monomorphism, and (ii)  $f$  induces in each degree  $n$  a bijection  $Y'_n \setminus i(X'_n) \xrightarrow{\sim} Y_n \setminus j(X_n)$ , then the square is a pushout square.

*Proof.* Verify the analogous statement for a pullback square of sets.  $\square$

18.5. **Corollary.**  $\overline{\text{Cell}}$  is precisely the class of monomorphisms.

*Proof.* We know all elements of  $\overline{\text{Cell}}$  are monomorphisms. Any monomorphism is isomorphic to an inclusion  $A \subseteq X$  of a subcomplex, so we only need show that such inclusions are contained in  $\overline{\text{Cell}}$ . Since  $X \approx \text{colim}_k A \cup \text{Sk}_k X$ , (18.3) exhibits the inclusion as a countable composite of pushouts along coproducts of elements of  $\text{Cell}$ .  $\square$

18.6. **Geometric realization.** Recall the singular complex functor  $\text{Sing}: \text{Top} \rightarrow s\text{Set}$  (10.6). This functor has a left adjoint  $\|-\| : s\text{Set} \rightarrow \text{Top}$ , called **geometric realization**, constructed explicitly by

$$(18.7) \quad \|X\| := \text{Cok} \left[ \coprod_{f: [m] \rightarrow [n]} X_n \times \Delta_{\text{top}}^m \rightrightarrows \coprod_{[p]} X_p \times \Delta_{\text{top}}^p \right];$$

that is, take a collection of topological simplices indexed by cells of  $X$ , and make identifications according to the simplicial operators in  $X$ . (Here the symbol “Cok” represents taking a “coequalizer”, i.e., the colimit of a diagram of shape  $\bullet \rightrightarrows \bullet$ .)

18.8. *Exercise.* Describe the two unlabelled maps in (18.7). Then show that  $\|-\|$  is in fact left adjoint to  $\text{Sing}$ .

Because geometric realization is a left adjoint, it commutes with colimits. It is straightforward to check that  $\|\Delta^n\| \approx \Delta_{\text{top}}^n$ , and that  $\|\partial\Delta^n\| \approx \partial\Delta_{\text{top}}^n$ , where the latter is the subspace

$$\partial\Delta_{\text{top}}^n := \{ (x_0, \dots, x_n) \in \Delta_{\text{top}}^n \mid \exists k \text{ such that } x_k = 0 \} \subset \Delta_{\text{top}}^n.$$

Applying this to the skeletal filtration, we discover that there are pushouts

$$\begin{array}{ccc} \coprod_{a \in X_k^{\text{nd}}} \partial\Delta_{\text{top}}^k & \longrightarrow & \|\text{Sk}_{k-1} X\| \\ \downarrow & & \downarrow \\ \coprod_{a \in X_k^{\text{nd}}} \Delta_{\text{top}}^k & \longrightarrow & \|\text{Sk}_k X\| \end{array}$$

of spaces, and that  $\|X\| = \bigcup \|\text{Sk}_k X\|$  with the direct limit topology. Thus,  $\|X\|$  is presented to us as a *CW-complex*, whose cells are in an evident bijective correspondence with the set of non-degenerate cells of  $X$ .

## 19. PUSHOUT-PRODUCT AND PULLBACK-HOM

We are going to prove several “enriched” versions of lifting properties associated to inner anodyne maps and inner fibrations. As a consequence we’ll be able to prove that function complexes of quasicategories are themselves quasicategories.

19.1. **Definition of pushout-product and pullback-hom.** Given maps  $f: A \rightarrow B$ ,  $g: K \rightarrow L$  and  $h: X \rightarrow Y$  of simplicial sets, we define new maps  $f \square g$  and  $g^{\square h}$  called the **pushout-product**<sup>17</sup> and the **pullback-hom**<sup>18</sup>. The pushout-product  $f \square g: (A \times L) \amalg_{A \times K} (B \times K) \rightarrow B \times L$  is the

pushout-product  
pullback-hom

<sup>17</sup>This is sometimes called the *box-product*. Some also call it the *Leibniz-product*, as its form is that of the Leibniz rule for boundary of a product space:  $\partial(X \times Y) = (\partial X \times Y) \cup_{\partial X \times \partial Y} (X \times \partial Y)$  (which is itself reminiscent of the original Leibniz rule  $D(fg) = (Df)g + f(Dg)$  of calculus).

unique map fitting in the diagram

$$\begin{array}{ccc}
 A \times K & \xrightarrow{\text{id} \times g} & A \times L \\
 f \times \text{id} \downarrow & & \downarrow \\
 B \times K & \longrightarrow & (A \times L) \amalg_{A \times K} (B \times K) \\
 & \searrow f \sqcap g & \downarrow f \times \text{id} \\
 & & B \times L
 \end{array}$$

$\text{id} \times g$  (curved arrow from  $B \times K$  to  $B \times L$ )

while the pullback-hom  $h \sqcap g: \text{Fun}(L, X) \rightarrow \text{Fun}(K, X) \times_{\text{Fun}(K, Y)} \text{Fun}(L, Y)$  is the unique map fitting in the diagram

$$\begin{array}{ccccc}
 \text{Fun}(L, X) & \xrightarrow{\text{Fun}(g, X)} & & & \text{Fun}(K, X) \\
 \downarrow h \sqcap g & & \text{Fun}(K, X) \times_{\text{Fun}(K, Y)} \text{Fun}(L, Y) & \longrightarrow & \downarrow \text{Fun}(\text{id}, h) \\
 & & \downarrow & & \text{Fun}(K, Y) \\
 \text{Fun}(L, X) & \xrightarrow{\text{Fun}(L, h)} & \text{Fun}(L, Y) & \xrightarrow{\text{Fun}(g, \text{id})} & \text{Fun}(K, Y)
 \end{array}$$

19.2. *Remark.* Typically we form  $f \sqcap g$  when  $f$  and  $g$  are monomorphisms, in which case  $f \sqcap g$  is also a monomorphism. In this case, the cells  $(b, \ell) \in B \times L$  which are *not* in the image of  $f \sqcap g$  are exactly those such that  $b \in B \setminus A$  and  $\ell \in L \setminus K$ .

19.3. *Remark* (Important!). On vertices, the pullback-hom  $h \sqcap g$  is just the “usual” map  $\text{Hom}(L, X) \rightarrow \text{Hom}(K, X) \times_{\text{Hom}(K, Y)} \text{Hom}(L, Y)$  sending  $s \mapsto (sg, hs)$ . Thus,  $h \sqcap g$  is surjective on vertices if and only if  $g \sqcap h$ .

We think of the pullback-hom as encoding an “enriched” version of the lifting problem for  $(g, h)$ . Thus, the target of  $h \sqcap g$  is an object which “parameterizes families” of commutative squares involving  $g$  and  $h$ . Similarly, the source of  $h \sqcap g$  “parameterizes families” of such commutative squares together with lifts.

19.4. *Remark.* The pushout-product construction is symmetric:  $f \sqcap g$  is isomorphic to  $g \sqcap f$  in the arrow category  $\text{Fun}([1], \text{sSet})$ . Ultimately, this is because product is symmetric. The pullback-hom construction however is not symmetric.

The product/function complex adjunction gives rise to the following relationship between lifting problems, which we may refer to as **adjunction of lifting problems**.

adjunction of lifting problems

19.5. **Proposition.** *We have that  $(f \sqcap g) \sqcap h$  if and only if  $f \sqcap (h \sqcap g)$ .*

*Proof.* Compare the two lifting problems using the product/map adjunction.

$$\begin{array}{ccc}
 (A \times L) \amalg_{A \times K} (B \times K) & \xrightarrow{(u, v)} & X \\
 f \sqcap g \downarrow & \searrow s & \downarrow h \\
 B \times L & \xrightarrow{w} & Y
 \end{array}
 \iff
 \begin{array}{ccc}
 A & \xrightarrow{\tilde{u}} & \text{Fun}(L, X) \\
 f \downarrow & \searrow \tilde{s} & \downarrow h \sqcap g \\
 B & \xrightarrow{(\tilde{v}, \tilde{w})} & \text{Fun}(K, X) \times_{\text{Fun}(K, Y)} \text{Fun}(L, Y)
 \end{array}$$

On the left-hand side are maps

$$u: A \times L \rightarrow X, \quad v: B \times K \rightarrow X, \quad w: B \times L \rightarrow Y, \quad s: B \times L \rightarrow X,$$

<sup>18</sup>Sometimes called the *box-power* or *pullback-power*. A common alternate notation is  $g \pitchfork h$ . This may also be called the *Leibniz-hom*, though I don’t know what rule of calculus it is related to.

while on the right-hand side are maps

$$\tilde{u}: A \rightarrow \text{Fun}(L, X), \quad \tilde{v}: B \rightarrow \text{Fun}(K, X), \quad \tilde{w}: B \rightarrow \text{Fun}(L, Y), \quad \tilde{s}: B \rightarrow \text{Fun}(L, X).$$

The data of  $(u, v, w)$  giving a commutative square as on the left corresponds bijectively to data  $(\tilde{u}, \tilde{v}, \tilde{w})$  giving a commutative square as on the right. Similarly, lifts  $s$  correspond bijectively to lifts  $\tilde{s}$ .  $\square$

It is important to note the special cases where one or more of  $A = \emptyset$ ,  $K = \emptyset$ , or  $Y = *$  hold. For instance, if  $K = \emptyset$  and  $Y = *$ , the proposition implies

$$(A \times L \xrightarrow{f \times L} B \times L) \sqsubset (X \rightarrow *) \quad \text{iff} \quad (A \xrightarrow{f} B) \sqsubset (\text{Fun}(L, X) \rightarrow *).$$

This is the kind of case we are interested in for proving that  $\text{Fun}(K, C)$  is a quasicategory whenever  $C$  is. The more general statement of the proposition is a kind of “relative” version of the thing we want; it is especially handy for carrying out inductive arguments.

**19.6. Exercise** (if you like monoidal categories). Let  $\mathcal{C} := \text{Fun}([1], \text{sSet})$ , the “arrow category” of simplicial sets. Show that  $\square: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  defines a symmetric monoidal structure on  $\mathcal{C}$ , with unit object  $(\emptyset \subset \Delta^0)$ . Furthermore, show that this is a *closed* monoidal structure, with  $-\square g$  left adjoint to  $(-)^{\square g}: \mathcal{C} \rightarrow \mathcal{C}$ .

**19.7. Inner anodyne maps and pushout-products.** The key fact we want to prove is the following.

**19.8. Proposition.** *We have that  $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ , i.e., that  $i \square j$  is inner anodyne whenever  $i$  is inner anodyne and  $j$  is a monomorphism.*

To set up the proof we need the following.

**19.9. Proposition.** *For any sets of maps  $S$  and  $T$ , we have  $\overline{S \square T} \subseteq \overline{S} \square \overline{T}$ .*

*Proof.* Let  $\mathcal{F} = (S \square T)^{\square}$ . From the small object argument we have that  $\overline{S \square T} = {}^{\square} \mathcal{F}$  (15.17), so we will show  $(\overline{S \square T}) \sqsubset \mathcal{F}$ . First we show that  $(\overline{S \square T}) \sqsubset \mathcal{F}$ . Consider

$$\begin{aligned} \mathcal{A} &:= \{ a \mid (a \square T) \sqsubset \mathcal{F} \} \\ &\approx \{ a \mid a \sqsubset (\mathcal{F}^{\square T}) \} \end{aligned}$$

by correspondence between lifting problems for pushout-products and pullback-homs (19.5). Thus  $\mathcal{A}$  is a left complement, and so is weakly saturated. Since  $S \subseteq \mathcal{A}$  then  $\overline{S} \subseteq \mathcal{A}$ , i.e.,  $(\overline{S \square T}) \sqsubset \mathcal{F}$ . The same idea applied to

$$\mathcal{B} := \{ b \mid (\overline{S \square T}) \sqsubset b \} \approx \left\{ b \mid b \sqsubset (\mathcal{F}^{\square \overline{S}}) \right\},$$

gives  $\overline{T} \subseteq \mathcal{B}$ , whence  $(\overline{S \square T}) \sqsubset \mathcal{F}$ .  $\square$

**19.10. Lemma.** *We have  $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ .*

*Proof.* This is a calculation, given in [Joy08a, App. H], and presented in the appendix (62.3).  $\square$

*Proof of (19.8).* We have that

$$\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}.$$

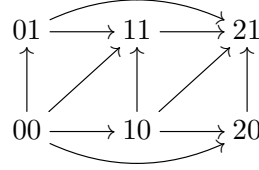
The first inclusion is (19.9), while the second is an immediate consequence of  $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$  (19.10).  $\square$

19.11. *Example.* Let's carry out a proof of (19.10) explicitly in one case, by showing that  $(\Lambda_1^2 \subset \Delta^2) \square (\partial \Delta^1 \subset \Delta^1)$  is inner anodyne. This map is the inclusion

$$(\Lambda_1^2 \times \Delta^1) \cup_{\Lambda_1^2 \times \partial \Delta^1} (\Delta^2 \times \partial \Delta^1) \subset \Delta^2 \times \Delta^1,$$

whose target is a “prism”, and whose source is a “trough”. To show this is in  $\overline{\text{InnHorn}}$ , we'll give an explicit procedure for constructing the prism from the trough by succesively attaching simplices along inner horns.

Note that  $\Delta^2 \times \Delta^1 = N([2] \times [1])$ , so we are working inside the nerve of a poset, whose elements (objects) are “ $ij$ ” with  $i \in \{0, 1, 2\}$  and  $j \in \{0, 1\}$ . Here is a picture of the trough, showing all the non-degenerate cells as the planar 2-cells of the graph.



The complement of this in the prism consists of three non-degenerate 3-cells, five non-degenerate 2-cells (two of which form the “lid” of the trough, while the other three are in the interior of the prism), and one non-degenerate edge cell (separating the two 2-cells which form the lid).

The following chart lists all non-degenerate cells in the complement of the trough, along with their codimension one faces (in order). The “ $\sqrt{\phantom{x}}$ ” marks cells which are contained in the trough.

$\langle 00, 21 \rangle$	$\langle 00, 20, 21 \rangle$	$\langle 00, 01, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 10, 20, 21 \rangle$	$\langle 00, 10, 11, 21 \rangle$	$\langle 00, 01, 11, 21 \rangle$
$\sqrt{\langle 21 \rangle}$	$\sqrt{\langle 20, 21 \rangle}$	$\sqrt{\langle 01, 21 \rangle}$	$\sqrt{\langle 10, 21 \rangle}$	$\sqrt{\langle 11, 21 \rangle}$	$\sqrt{\langle 10, 20, 21 \rangle}$	$\sqrt{\langle 10, 11, 21 \rangle}$	$\sqrt{\langle 01, 11, 21 \rangle}$
$\sqrt{\langle 10 \rangle}$	$\langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\langle 00, 21 \rangle$	$\langle 00, 20, 21 \rangle$	$\langle 00, 11, 21 \rangle$	$\langle 00, 11, 21 \rangle$
	$\sqrt{\langle 00, 20 \rangle}$	$\sqrt{\langle 00, 01 \rangle}$	$\sqrt{\langle 00, 10 \rangle}$	$\sqrt{\langle 00, 11 \rangle}$	$\langle 00, 10, 21 \rangle$	$\langle 00, 10, 21 \rangle$	$\langle 00, 01, 21 \rangle$
					$\sqrt{\langle 00, 10, 20 \rangle}$	$\sqrt{\langle 00, 10, 11 \rangle}$	$\sqrt{\langle 00, 01, 11 \rangle}$

Note that the cells  $\langle 00, 21 \rangle$ ,  $\langle 00, 10, 21 \rangle$ , and  $\langle 00, 11, 21 \rangle$  of the complement appear multiple times as faces. We can attach simplices to the domain in the following order:

$$\textcircled{1} \langle 00, 10, 21 \rangle, \quad \textcircled{2} \langle 00, 10, 20, 21 \rangle, \quad \textcircled{3} \langle 00, 10, 11, 21 \rangle, \quad \textcircled{4} \langle 00, 01, 11, 21 \rangle.$$

In each case, the intersection of the simplex with (domain+previously attached simplices) is an inner horn. This directly exhibits  $(\Lambda_1^2 \subset \Delta^2) \square (\partial \Delta^1 \subset \Delta^1)$  as an inner anodyne map.

## 20. FUNCTION COMPLEXES OF QUASICATEGORIES ARE QUASICATEGORIES

**20.1. Enriched lifting properties.** We record the immediate consequences of  $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$  (19.8).

### 20.2. Proposition.

(1) If  $i: A \rightarrow B$  is inner anodyne and  $j: K \rightarrow L$  a monomorphism, then

$$i \square j: (A \times L) \cup_{A \times K} (B \times K) \rightarrow B \times L$$

is inner anodyne.

(2) If  $j: K \rightarrow L$  is a monomorphism and  $p: X \rightarrow Y$  is an inner fibration, then

$$p^{\square j}: \text{Fun}(L, X) \rightarrow \text{Fun}(K, X) \times_{\text{Fun}(K, Y)} \text{Fun}(L, Y)$$

is an inner fibration.

(3) If  $i: A \rightarrow B$  is inner anodyne and  $p: X \rightarrow Y$  is an inner fibration, then

$$p^{\square i}: \text{Fun}(B, X) \rightarrow \text{Fun}(A, X) \times_{\text{Fun}(A, Y)} \text{Fun}(B, Y)$$

is a trivial fibration.

These can be summarized as

$$\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}, \quad \text{InnFib}^{\square \overline{\text{Cell}}} \subseteq \text{InnFib}, \quad \text{InnFib}^{\square \overline{\text{InnHorn}}} \subseteq \text{TrivFib}.$$

Statement (1) is just restating (19.8). The other two statements follow from (1) using the correspondence between lifting problems for pushout-products and pullback-homs (19.5), together with the facts that  $\text{InnFib} = \text{InnHorn}^{\square}$  and  $\text{TrivFib} = \text{Cell}^{\square}$ . For instance, (2) follows from the observation that  $i \not\square p^{\square j}$  iff  $(i \square j) \not\square p$ , and that  $i \in \overline{\text{InnHorn}}$  and  $j \in \overline{\text{Cell}}$  imply  $i \square j \in \overline{\text{InnHorn}}$ . Likewise (3) follows a similar argument using that  $j \not\square p^{\square i}$  iff  $(i \square j) \not\square p$ .

We are going to use these consequences all the time. To announce that I am using any of these, I will simply assert “ $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ ” without other explanation; sometimes, to indicate an application of statements (2) and (3), I will call it “enriched lifting”. The following gives the most general statement, of which (19.8) amounts to the special case of  $S = U = \text{InnHorn}$  and  $T = \text{Cell}$ .

**20.3. Proposition.** *Let  $S, T$ , and  $U$  be sets of morphisms in  $s\text{Set}$ . Write  $\overline{S}, \overline{T}$ , and  $\overline{U}$  for the weak saturations of these sets, and let  $S^{\square} := S^{\square}$ ,  $T^{\square} := T^{\square}$ , and  $U^{\square} := U^{\square}$  denote the respective right complements. If  $S \square T \subseteq \overline{U}$ , then*

$$\overline{S} \square \overline{T} \subseteq \overline{U}, \quad U\text{Fib}^{\square \overline{T}} \subseteq S\text{Fib}, \quad U\text{Fib}^{\square \overline{S}} \subseteq T\text{Fib}.$$

*Proof.* Exercise using (19.5). □

There are many useful special cases of (20.2), obtained by taking the domain of a monomorphism to be empty, or the target of an inner fibration to be terminal.

- If  $i: A \rightarrow B$  is inner anodyne, so is  $i \times \text{id}_L: A \times L \rightarrow B \times L$ .
- If  $p: X \rightarrow Y$  is an inner fibration, then so is  $\text{Fun}(L, p): \text{Fun}(L, X) \rightarrow \text{Fun}(L, Y)$ .
- If  $j: K \rightarrow L$  is a monomorphism and  $C$  a quasicategory, then  $\text{Fun}(j, C): \text{Fun}(L, C) \rightarrow \text{Fun}(K, C)$  is an inner fibration.
- If  $i: A \rightarrow B$  is inner anodyne and  $C$  a quasicategory, then  $\text{Fun}(i, C): \text{Fun}(B, C) \rightarrow \text{Fun}(A, C)$  is a trivial fibration.

In particular, we can now prove that function complexes between quasicategories are quasicategories.

**20.4. Theorem.** *For  $C$  a quasicategory and  $L$  a simplicial set,  $\text{Fun}(L, C)$  is a quasicategory.*

*Proof.* Let’s spell this out with a little detail. Because  $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ , we have (19.8) that

$$(\Lambda_j^n \subset \Delta^n) \square (\emptyset \subseteq K) = (\Lambda_j^n \times K \rightarrow \Delta^n \times K)$$

is inner anodyne for any  $K$  and  $0 < j < n$ . Thus, for any diagram

$$\begin{array}{ccc} \Lambda_j^n \times K & \longrightarrow & C \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n \times K & & \end{array}$$

with  $C$  a quasicategory, a dotted arrow exists. By adjunction, this is the same as saying we can extend  $\Lambda_j^n \rightarrow \text{Fun}(K, C)$  along  $\Lambda_j^n \subset \Delta^n$ . That is, we have proved that  $\text{Fun}(K, C)$  is a quasicategory. □

**20.5. Remark.** Most weakly saturated classes  $\overline{S}$  that we will explicitly discuss in these notes will have the property that  $S \square \text{Cell} \subseteq \overline{S}$ , and thus analogues of the above remarks will hold for such classes.

**20.6. Exercise (Important).** Show that  $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ . (Hint: (18.5).) State the analogue of (20.2) associated to this inclusion.

**20.7. Exercise.** As a consequence of the previous exercise, show that every trivial fibration admits a section.

**20.8. Composition functors.** We can use the above theory to construct “composition functors”. If  $C$  is an ordinary category, the operation of composing a sequence of  $n$  maps can be upgraded to a functor

$$\mathrm{Fun}([1], C) \times_C \mathrm{Fun}([1], C) \rightarrow \mathrm{Fun}([1], C)$$

which on *objects* describes composition of a sequence of maps. The source of this functor is the evident inverse limit in  $\mathrm{Cat}$  of

$$\mathrm{Fun}([1], C) \xrightarrow{\langle 1 \rangle^*} \mathrm{Fun}([0], C) \xleftarrow{\langle 0 \rangle^*} \mathrm{Fun}([1], C),$$

which is isomorphic to  $\mathrm{Fun}(I^2, C)$ .

We can generalize this to quasicategories, with the proviso that the composition functor we produce is not uniquely determined. We use the following observation: any trivial fibration admits a section (20.7).

Let  $C$  be a quasicategory. Then map  $r: \mathrm{Fun}(\Delta^2, C) \rightarrow \mathrm{Fun}(I^2, C)$  induced by restriction along  $I^2 \subseteq \Delta^2$  is a trivial fibration by (20.2), since  $I^2 \subset \Delta^2$  is an inner-horn inclusion. Therefore  $r$  admits a section  $s$ , so we get a diagram

$$\mathrm{Fun}(I^2, C) \xleftarrow[r]{} \mathrm{Fun}(\Delta^2, C) \xrightarrow{s} \mathrm{Fun}(\Delta^{\{0,2\}}, C)$$

where  $r'$  is restriction along  $\Delta^{\{0,2\}} \subset \Delta^2$ . The composite  $r's$  can be thought of as a kind of “composition” functor. It is not unique, since  $s$  isn't, but we'll see (23.13) that this is ok: all functors constructed this way are “naturally isomorphic” to each other.

The same argument gives rise to a (non-unique) “ $n$ -fold composition functor”

$$\mathrm{Fun}([1], C) \times_C \cdots \times_C \mathrm{Fun}([1], C) \rightarrow \mathrm{Fun}([1], C),$$

whose source is isomorphic to  $\mathrm{Fun}(I^n, C)$ , using that spine inclusions are inner anodyne (14.12).

**20.9. A useful variant.** The proof of (19.8) actually proves something a little stronger.

**20.10. Proposition** ([Joy08a, §2.3.1], [Lur09, §2.3.2]). *We have that  $\overline{\{\Lambda_1^2 \subset \Delta^2\} \square \mathrm{Cell}} = \overline{\mathrm{InnHorn}}$ .*

*Proof.* We give a proof in the appendix (62.3).  $\square$

A consequence of this is another characterization of quasicategories.

**20.11. Corollary.** *A simplicial set  $C$  is a quasicategory if and only if  $f: \mathrm{Fun}(\Delta^2, C) \rightarrow \mathrm{Fun}(\Lambda_1^2, C)$  is a trivial fibration.*

*Proof.* First notice that  $(\partial\Delta^k \subset \Delta^k) \sqsupseteq f$  for all  $k \geq 0$  iff  $(\partial\Delta^k \subset \Delta^k) \square (\Lambda_1^2 \subset \Delta^2) \sqsupseteq (C \rightarrow *)$  for all  $k \geq 0$ , since  $f = (C \rightarrow *) \square \{\Lambda_1^2 \subset \Delta^2\}$ . Therefore  $f \in \mathrm{TrivFib} = \mathrm{Cell}^\square$  if and only if  $(C \rightarrow *) \in (\mathrm{Cell} \square \{\Lambda_1^2 \subset \Delta^2\})^\square$ . The conclusion immediately follows using (20.10).  $\square$

## Part 4. Categorical equivalence of quasicategories

### 21. NATURAL ISOMORPHISMS

**21.1. Natural isomorphisms of functors.** Let  $C$  and  $D$  be quasicategories. Recall that a natural transformation between functors  $f_0, f_1: C \rightarrow D$  is defined to be a morphism  $\alpha: f_0 \rightarrow f_1$  in the functor quasicategory  $\mathrm{Fun}(C, D)$ , or equivalently a map  $\tilde{\alpha}: C \times \Delta^1 \rightarrow D$  such that  $\tilde{\alpha}|_{C \times \Delta^{\{i\}}} = f_i$ ,  $i = 0, 1$ .

Say that  $\alpha: f_0 \rightarrow f_1$  is a **natural isomorphism** if  $\alpha$  is an isomorphism in the quasicategory of functors  $\mathrm{Fun}(C, D)$ . Thus,  $\alpha$  is a natural isomorphism iff there exists a natural transformation  $\beta: f_1 \rightarrow f_0$  such that  $\beta\alpha \approx 1_{f_0}$  and  $\alpha\beta \approx 1_{f_1}$ , where “ $\approx$ ” is homotopy between morphisms in the quasicategory  $\mathrm{Fun}(C, D)$ .

natural isomorphism

This notion of natural isomorphism corresponds with the usual one for ordinary categories, since in that case homotopy of morphisms is the same as equality of morphisms.

Observe that “there exists a natural isomorphism  $f_0 \rightarrow f_1$ ” is an equivalence relation on the set of all functors  $C \rightarrow D$ , as this relation precisely coincides with “there exists an isomorphism  $f_0 \rightarrow f_1$ ” in the category  $h\text{Fun}(C, D)$ . We say that  $f_0$  and  $f_1$  are **naturally isomorphic** functors.

naturally isomorphic

Furthermore, the “naturally isomorphic” relation is compatible with composition: if  $f, f'$  are naturally isomorphic and  $g, g'$  are naturally isomorphic, then so are  $gf$  and  $g'f'$ . You can read this off from the fact the operation of composition of functors extends to a functor  $\text{Fun}(D, E) \times \text{Fun}(C, D) \rightarrow \text{Fun}(C, E)$  between quasicategories, and so induces a functor

$$h\text{Fun}(D, E) \times h\text{Fun}(C, D) \approx h(\text{Fun}(D, E) \times \text{Fun}(C, D)) \rightarrow h\text{Fun}(C, E).$$

(This uses (11.15) to identify the homotopy category of the product with the product of homotopy categories.)

**21.2. Pointwise criterion for natural isomorphisms.** Recall that if  $C$  and  $D$  are ordinary categories, a natural transformation  $\alpha: f_0 \rightarrow f_1$  between functors  $f_0, f_1: C \rightarrow D$  is a natural isomorphism iff and only if  $\alpha$  is a “pointwise isomorphism” (or “objectwise isomorphism”); i.e., if for each object  $c$  of  $C$  the evident map  $\alpha(c): f_0(c) \rightarrow f_1(c)$  is an isomorphism in  $D$ . That natural isomorphisms are “pointwise isomorphisms” is immediate. The opposite implication follows from the fact that a natural transformation between functors of ordinary values can be completely recovered from its “values on objects”. Thus, given  $\alpha: f_0 \rightarrow f_1$  such that each  $\alpha(c): f_0(c) \rightarrow f_1(c)$  is an isomorphism, we may explicitly construct an inverse transformation  $\beta: f_1 \rightarrow f_0$  by setting  $\beta(c) := \alpha(c)^{-1}: f_1(c) \rightarrow f_0(c)$ . Note that this  $\beta$  is in fact the *unique* inverse to  $\alpha$  (since inverses to morphisms are unique when they exist).

One of these directions is straightforward for quasicategories.

**21.3. Proposition.** *Let  $C$  and  $D$  be quasicategories. If  $\alpha: C \times \Delta^1 \rightarrow D$  is a natural isomorphism between functors  $f_0, f_1: C \rightarrow D$ , then for each object  $c$  of  $C$  the induced map  $\alpha(c): f_0(c) \rightarrow f_1(c)$  is an isomorphism in  $D$ .*

*Proof.* The map  $\text{Fun}(C, D) \rightarrow \text{Fun}(\{c\}, D) = D$  induced by restriction along  $\{c\} \subseteq C$  is a functor between quasicategories, so it takes isomorphisms to isomorphisms (12.3). It sends  $\alpha$  to  $\alpha(c)$ .  $\square$

The converse to this proposition is also true: *A natural transformation  $\alpha: C \times \Delta^1 \rightarrow D$  of functors between quasicategories is a natural isomorphism if and only if each of the maps  $\alpha(c)$  are isomorphisms in  $D$ .* Unfortunately, this is much more subtle to prove, as it requires using the existence of inverses to the  $\alpha(c)$ s to produce an inverse to  $\alpha$ , which though it exists is not at all unique. We will prove this converse later as (35.2).

**21.4. Remark.** An immediate consequence of the pointwise criterion is that if  $D$  is a *quasigroupoid*, then so is  $\text{Fun}(C, D)$ .

**21.5. Remark.** The pointwise criterion can be reformulated in terms of homotopy categories. The homotopy category construction takes quasicategories to categories, and takes functors to functors. Furthermore, given a natural transformation  $\alpha: f_0 \rightarrow f_1$  of functors  $f_0, f_1: C \rightarrow D$  between quasicategories (i.e., a functor  $\alpha: C \times \Delta^1 \rightarrow D$  such that  $(\alpha|_{C \times \{j\}}) = f_j$ ), we obtain an induced transformation  $h\alpha: hf_0 \rightarrow hf_1$  of functors  $hf_0, hf_1: hC \rightarrow hD$  between their homotopy categories (so that the value of  $h\alpha$  at an object  $c \in \text{ob } hC = C_0$  is the homotopy class of the edge  $\alpha(\{c\} \times \Delta^1) \subseteq D$ ). Then the pointwise criterion asserts that  $\alpha$  is a natural isomorphism of functors between quasicategories if and only if  $h\alpha$  is a natural isomorphism of functors between ordinary categories.



## 22. CATEGORICAL EQUIVALENCE

We are now in position to define the correct generalization of the notion of “equivalence” of categories. This will be called *categorical equivalence* of quasicategories, and will be a direct generalization of the classical notion.

Given this, we use it to define a notion of categorical equivalence which applies to arbitrary maps of simplicial sets which may not be categories. Finally, we will show that the two definitions agree for maps between quasicategories.

**22.1. Categorical equivalences between quasicategories.** A **categorical inverse** to a functor  $f: C \rightarrow D$  between quasicategories is a functor  $g: D \rightarrow C$  such that  $gf$  is naturally isomorphic to  $1_C$  and  $fg$  is naturally isomorphic to  $1_D$ . We provisionally say that a functor  $f$  between quasicategories is a **categorical equivalence** if it admits a categorical inverse.

categorical inverse

categorical equivalence

**22.2. Remark.** Categorical equivalence between quasicategories is a kind of “homotopy equivalence”, where homotopies are natural isomorphisms between functors.

If  $C$  and  $D$  are nerves of ordinary categories, then natural isomorphisms between functors in our sense are precisely natural isomorphisms between functors in the classical sense, and that categorical equivalence between nerves of categories coincides precisely with the usual notion of equivalence of categories.

If quasicategories are equivalent, then their homotopy categories are equivalent.

**22.3. Proposition.** *If  $f: C \rightarrow D$  is a categorical equivalence between quasicategories, then  $h(f): hC \rightarrow hD$  is an equivalence of categories.*

*Proof.* Immediate, given that natural isomorphisms  $f \Rightarrow g: C \rightarrow D$  induce natural isomorphisms  $h(f) \Rightarrow h(g): hC \rightarrow hD$ .  $\square$

Note: the converse is not at all true. For instance, there are many examples of quasicategories which are not equivalent to  $\Delta^0$ , but whose homotopy categories are: e.g.,  $\text{Sing } T$  for any non-contractible simply connected space  $T$  (10.6), or  $K(A, d)$  for any non-trivial abelian group  $A$  and  $d \geq 2$  (10.10).

**22.4. Exercise** (Categorical inverses are unique up to natural isomorphism). Let  $f: C \rightarrow D$  be a functor between quasicategories, and suppose  $g, g': D \rightarrow C$  are both categorical inverses to  $f$ . Show that  $g$  and  $g'$  are naturally isomorphic.

**22.5. General categorical equivalence.** We can extend the notion of categorical equivalence to maps between arbitrary simplicial sets. Say that a map  $f: X \rightarrow Y$  between arbitrary simplicial sets is a **categorical equivalence** if for every quasicategory  $C$ , the induced functor  $\text{Fun}(f, C): \text{Fun}(Y, C) \rightarrow \text{Fun}(X, C)$  of quasicategories admits a categorical inverse.

categorical equivalence

We claim that on maps between quasicategories this general definition of categorical equivalence coincides with the provisional notion described earlier.

**22.6. Lemma.** *For a map  $f: C \rightarrow D$  between quasicategories, the two notions of categorical equivalence described above coincide. That is, the following are equivalent:*

- (1)  $f$  admits a categorical inverse.
- (2) For every quasicategory  $E$ , the functor  $\text{Fun}(f, E): \text{Fun}(D, E) \rightarrow \text{Fun}(C, E)$  admits a categorical inverse.

To prove this, we will need the following observation. The construction  $X \mapsto \text{Fun}(X, E)$  is a functor  $\text{sSet}^{\text{op}} \rightarrow \text{sSet}$ , and so in particular induces a natural map

$$\gamma_0: \text{Hom}(X, Y) \rightarrow \text{Hom}(\text{Fun}(Y, E), \text{Fun}(X, E))$$

of sets, which sends  $f: X \rightarrow Y$  to  $\text{Fun}(f, E): \text{Fun}(Y, E) \rightarrow \text{Fun}(X, E)$ . The observation we need is that this construction admits an “enrichment”, to a map

$$\gamma: \text{Fun}(X, Y) \rightarrow \text{Fun}(\text{Fun}(Y, E), \text{Fun}(X, E)),$$

which coincides with  $\gamma_0$  on vertices. The map  $\gamma$  in question is adjoint to the “composition” map  $\text{Fun}(X, Y) \times \text{Fun}(Y, E) \rightarrow \text{Fun}(X, E)$ . (*Exercise:* Describe explicitly what  $\gamma$  does to  $n$ -dimensional cells.) We say that the functor  $\text{Fun}(-, E)$  is an *enriched* functor, as it gives not merely a map between hom-sets (i.e., acts on vertices in function complexes), but a map between function complexes.

*Proof.* (1)  $\implies$  (2). When  $C$ ,  $D$ , and  $E$  are quasicategories so are the function complexes between them (20.4). In this case, the above map  $\gamma$  takes functors  $C \rightarrow D$  to functors  $\text{Fun}(D, E) \rightarrow \text{Fun}(C, E)$  between quasicategories, natural transformations of such functors to natural transformations, and natural isomorphisms of such functors to natural isomorphisms. Using this observation, it is straightforward to show that a categorical inverse  $g: D \rightarrow C$  to  $f: C \rightarrow D$  gives rise to a categorical inverse  $\text{Fun}(g, E): \text{Fun}(C, E) \rightarrow \text{Fun}(D, E)$  to the induced functor  $\text{Fun}(f, E): \text{Fun}(D, E) \rightarrow \text{Fun}(C, E)$ .

(2)  $\implies$  (1). Conversely, suppose  $f: C \rightarrow D$  is a categorical equivalence in the general sense, so that  $f^* = \text{Fun}(f, E)$  admits a categorical inverse for every quasicategory  $E$ , which implies that each functor

$$h(f^*): h\text{Fun}(D, E) \rightarrow h\text{Fun}(C, E)$$

is an equivalence of ordinary categories (22.3). In particular, it follows that  $f^*$  induces a bijection of sets

$$\widehat{f}^*: \pi_0(\text{Fun}(D, E)^{\text{core}}) \xrightarrow{\sim} \pi_0(\text{Fun}(C, E)^{\text{core}});$$

recall that  $\pi_0(\text{Fun}(D, E)^{\text{core}}) \approx \pi_0((h\text{Fun}(D, E))^{\text{core}})$  is precisely the set of *natural isomorphism classes* of functors  $D \rightarrow E$ .

Taking  $E = C$ , surjectivity of  $\widehat{f}^*$  implies that there is a functor  $g \in \text{Fun}(D, C)_0$  together with a natural isomorphism  $gf \rightarrow \text{id}_C$  in  $\text{Fun}(C, C)_1$ . Taking  $E = D$ , we note that since

$$f^*(\text{id}_D) = \text{id}_D f = f \text{id}_C \approx f g f = f^*(f g),$$

injectivity of  $\widehat{f}^*$  implies that  $\text{id}_D \approx f g$ , i.e., there exists a natural isomorphism  $\text{id}_D \rightarrow f g$  in  $\text{Fun}(D, D)_1$ . Thus, we have shown that  $g$  is a categorical inverse of  $f$ , as desired.  $\square$

**22.7. Remark.** The definition of categorical equivalence we are using here is *very* different to the definition adopted by Lurie in [Lur09, §2.2.5]. It is also slightly different from the definition of “weak categorical equivalence” used by Joyal [Joy08a, 1.20]. Lurie adopts a definition closely related to Joyal’s in *kerodon.net*. As we will show soon (25.13), weak categorical equivalence and the definition used in *kerodon* are equivalent to our definition of categorical equivalence. The discussion around [Lur09, 2.2.5.8] shows that Joyal’s definitions is equivalent to the one used in [Lur09], and so they are both equivalent to the one we have used.

## 23. TRIVIAL FIBRATIONS AND INNER ANODYNE MAPS

Inner anodyne maps and trivial fibrations are particular kinds of categorical equivalences.

**23.1. Trivial fibrations to the terminal object.** Recall that a trivial fibration  $p: X \rightarrow Y$  of simplicial sets is a map such that  $(\partial\Delta^k \subset \Delta^k) \sqsubset p$  for all  $k \geq 0$ . That is,  $\text{TrivFib} = \text{Cell}^{\sqsubset}$ , so  $p$  is a trivial fibration if and only if  $\text{Cell} \sqsubset p$ .

**23.2. Exercise.** Consider an indexed collection of trivial fibrations  $p_i: X_i \rightarrow Y_i$ . Show that  $p := \coprod p_i: \coprod X_i \rightarrow \coprod Y_i$  is a trivial fibration. (Hint: see proof of (8.7).)

**23.3. Proposition.** *Let  $X$  be a simplicial set and  $p: X \rightarrow *$  be a trivial fibration whose target is the terminal simplicial set. Then  $X$  is a Kan complex (and thus a quasigroupoid) and  $p$  is a categorical equivalence.*

*Proof.* Since  $\text{InnHorn} \subset \overline{\text{Cell}}$ , it is immediate that  $X$  is a Kan complex. Next note that  $X$  only has one isomorphism class of objects. To show this, note that since  $X$  is a quasigroupoid, it suffices to produce for any pair of objects  $a, b \in X_0$  a morphism  $a \rightarrow b$ . This amounts to producing a lift in

$$\begin{array}{ccc} \partial\Delta^1 = \{0, 1\} & \xrightarrow{(a,b)} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^1 & \xrightarrow{\quad} & * \end{array}$$

which exists because  $(\partial\Delta^1 \subset \Delta^1) \in \text{Cell}$ .

To show that  $p$  is a categorical equivalence, first note that  $X$  is non-empty, since  $\text{Hom}(\Delta^0, X) \rightarrow \text{Hom}(\partial\Delta^0, X) = *$  is surjective. Choose any  $s \in \text{Hom}(\Delta^0, X)$ , so  $ps = \text{id}_{\Delta^0}$ . The composite  $sp: X \rightarrow X$  is an object of  $\text{Fun}(X, X)$ , and we want to show it is naturally isomorphic to  $\text{id}_X$ . Enriched lifting (20.3) applied to  $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$  (20.6) implies that if  $i: \emptyset \rightarrow X$ , the pullback-hom map

$$p^{\square i} = \text{Fun}(i, X): \text{Fun}(X, X) \rightarrow \text{Fun}(\emptyset, X) \times_{\text{Fun}(\emptyset, *)} \text{Fun}(X, *) = *$$

is a trivial fibration. Thus by what we have proved, any two objects of  $\text{Fun}(X, X)$  are isomorphic, so in particular  $sp \approx \text{id}_X$  as desired.  $\square$

We will prove a partial converse to this later (38.11): *a quasicategory  $C$  is categorically equivalent to  $*$  if and only if  $C \rightarrow *$  is a trivial fibration.*

**23.4. Preisomorphisms.** We need a way to produce categorical equivalences between simplicial sets which are not necessarily quasicategories.

Let  $X$  be a simplicial set. Say that an edge  $a \in X_1$  is a **preisomorphism**<sup>19</sup> if it projects to an isomorphism under  $\alpha: X \rightarrow hX$ , the tautological map to the (nerve of the) fundamental category (11.1). If  $X$  is actually a quasicategory, the preisomorphisms are just the isomorphisms (since in that case the fundamental category is the same as the homotopy category). Note that degenerate edges are always preisomorphisms, since they go to identity maps in the fundamental category.

preisomorphism

**23.5. Proposition.** *An edge  $a \in X_1$  is a preisomorphism if and only if for every map  $g: X \rightarrow C$  to a quasicategory  $C$ , the image  $g(a)$  is an isomorphism in  $C$ .*

*Proof.* Isomorphisms in  $C$  are exactly the edges which are sent to isomorphisms under  $\gamma: C \rightarrow hC$ . Given this the proof is straightforward, using the fact that the formation of fundamental categories is functorial, and that  $hX$  is itself a category and hence a quasicategory.  $\square$

As a consequence, any map  $X \rightarrow Y$  of simplicial sets takes preisomorphisms to preisomorphisms. In particular, any map *from* a quasicategory takes isomorphisms to preisomorphisms. We will use this observation below.

**23.6. Example.** Consider the subcomplex  $\Lambda_{\{0,3\}}^3 = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$  of  $\Delta^3$ . Define  $Z$  to be the pushout of the diagram

$$\Lambda_{\{0,3\}}^3 \xleftarrow{j} \Delta^{\{0,2\}} \amalg \Delta^{\{1,3\}} \xrightarrow{p} \Delta^{\{x\}} \amalg \Delta^{\{y\}}$$

where  $j$  is the evident inclusion,  $\Delta^{\{x\}}$  and  $\Delta^{\{y\}}$  are simplicial sets isomorphic to  $\Delta^0$ , but with vertices labelled “ $x$ ” and “ $y$ ” respectively, and  $p$  is induced by the evident projections  $\Delta^{\{0,2\}} \rightarrow \Delta^{\{x\}}$  and  $\Delta^{\{1,3\}} \rightarrow \Delta^{\{y\}}$ . The resulting complex  $Z$  looks like

$$\begin{array}{ccc} y & \xrightarrow{y_{00}} & y \\ g \downarrow & \nearrow b & \downarrow h \\ x & \xrightarrow{x_{00}} & x \end{array}$$

<sup>19</sup>How about *protoisomorphism* instead?

with seven non-degenerate cells:  $x, y \in Z_0$ ,  $f, g, h \in Z_1$ ,  $a, b \in Z_2$ . The simplicial set  $Z$  is not a quasicategory (*Exercise*: why not?).

However, a map  $\phi: Z \rightarrow C$  to a quasicategory corresponds exactly a choice of:

- objects  $\phi(x), \phi(y) \in C_0$ ,
- morphisms  $\phi(f): \phi(x) \rightarrow \phi(y)$  and  $\phi(g), \phi(h): \phi(y) \rightarrow \phi(x)$  in  $C_1$ , and
- 2-cells  $\phi(a), \phi(b) \in C_2$  exhibiting  $f \sim_\ell h$  and  $g \sim_\ell f$  respectively.

In particular,  $\phi(g)$  is a preinverse of  $\phi(f)$  and  $\phi(h)$  is a postinverse of  $\phi(f)$ , so all of these edges are isomorphisms in  $C$ . Therefore all edges of  $Z$  are preisomorphisms.

**23.7. Example.** Here is a variant of the previous example. Consider the subcomplex  $\Lambda_{\{0,3\}}^3 = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$  of  $\Delta^3$ . Define  $Z'$  to be the pushout of the diagram

$$\Lambda_{\{0,3\}}^3 \xleftarrow{j} \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,3\}} \cup \Delta^{\{2,3\}} \xrightarrow{p} \Delta^{\{y<x\}},$$

where  $j$  is the evident inclusion,  $\Delta^{\{y<x\}}$  is a simplicial set isomorphic to  $\Delta^1$  but with vertices labelled “ $y$ ” and “ $x$ ” instead of “0” and “1”, and  $p$  is the unique map which on vertices sends  $0, 2 \mapsto y$ ,  $1, 3 \mapsto x$ . The resulting complex  $Z'$  looks like

$$\begin{array}{ccc} y & \xrightarrow{y_{00}} & y \\ g \downarrow & \nearrow f & \downarrow g \\ x & \xrightarrow{x_{00}} & x \end{array}$$

with six non-degenerate cells:  $x, y \in Z'_0$ ,  $f, g \in Z'_1$ ,  $a, b \in Z'_2$ . Again,  $Z'$  is not a quasicategory.

A map  $\phi: Z' \rightarrow C$  to a quasicategory corresponds exactly to a choice of:

- objects  $\phi(x), \phi(y) \in C_0$ ,
- morphisms  $\phi(f): \phi(x) \rightarrow \phi(y)$  and  $\phi(g): \phi(y) \rightarrow \phi(x)$  in  $C_1$ , and
- 2-cells  $\phi(a), \phi(b) \in C_2$  exhibiting  $f \sim_\ell g$  and  $g \sim_\ell f$  respectively.

Thus as in the previous example, every edge of  $Z'$  is a preisomorphism.

**23.8. Exercise.** Consider the inclusions  $i: \Delta^{\{x,y\}} \rightarrow Z$  and  $i': \Delta^{\{x,y\}} \rightarrow Z'$  in (23.6) and (23.7). Show that an edge in a quasicategory  $C$  is an isomorphism if and only if its representing map  $f: \Delta^{\{x,y\}} \rightarrow C$  extends along  $i$ , and if and only if its representing map extends along  $i'$ .

Say that vertices in a simplicial set  $X$  are **preisomorphic** if they can be connected by a chain of preisomorphisms (which can point in either direction). Clearly, any map  $g: X \rightarrow C$  to a quasicategory takes preisomorphic vertices of  $X$  to isomorphic objects of  $C$ .

We can apply this to function complexes. If two maps  $f_0, f_1: X \rightarrow Y$  are preisomorphic (viewed as vertices in  $\text{Fun}(X, Y)$ ), then for any quasicategory  $C$ , the induced functors  $\text{Fun}(f_0, C), \text{Fun}(f_1, C): \text{Fun}(Y, C) \rightarrow \text{Fun}(X, C)$  are naturally isomorphic. To see this, consider

$$\Delta^1 \xrightarrow{a} \text{Fun}(X, Y) \xrightarrow{b} \text{Fun}(\text{Fun}(Y, C), \text{Fun}(X, C))$$

where  $b$  is adjoint to the composition map  $\text{Fun}(Y, C) \times \text{Fun}(X, Y) \rightarrow \text{Fun}(X, C)$ . If  $a$  represents a preisomorphism  $f_0 \rightarrow f_1$  in  $\text{Fun}(X, Y)$ , then  $ba$  represents an isomorphism  $\text{Fun}(f_0, C) \rightarrow \text{Fun}(f_1, C)$ , since the target of  $b$  is a quasicategory. As a consequence we get the following.

**23.9. Lemma.** *If  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$  are maps of simplicial sets such that  $gf$  is preisomorphic to  $\text{id}_X$  in  $\text{Fun}(X, X)$  and  $fg$  is preisomorphic to  $\text{id}_Y$  in  $\text{Fun}(Y, Y)$ , then  $f$  and  $g$  are categorical equivalences.*

It is important to note that this is a sufficient condition for a map to be a categorical equivalence, but not a necessary one: there are many categorical equivalences of simplicial sets to which the lemma cannot be applied (see (24.3) below).

### 23.10. Trivial fibrations are always categorical equivalences.

**23.11. Proposition.** *Every trivial fibration between simplicial sets is a categorical equivalence.*

Here is some notation. Given maps  $f: A \rightarrow Y$  and  $g: B \rightarrow Y$ , we write  $\text{Fun}_Y(f, g)$  or  $\text{Fun}_Y(A, B)$  for the simplicial set defined by the pullback square

$$\begin{array}{ccc} \text{Fun}_Y(A, B) & \longrightarrow & \text{Fun}(A, B) \\ \downarrow & & \downarrow g_* = \text{Fun}(A, g) \\ \{f\} & \longrightarrow & \text{Fun}(A, Y) \end{array}$$

Note that vertices of  $\text{Fun}_Y(A, B)$  correspond exactly to “sections of  $g$  over  $f$ ”, i.e., to  $s: A \rightarrow B$  such that  $gs = f$ . You can think of  $\text{Fun}_Y(A, B)$  as a simplicial set which “parameterizes” sections of  $g$  over  $f$ . I’ll call this the **relative function complex over  $Y$** .

relative function complex over  $Y$

**23.12. Exercise.** Show that  $n$ -dimensional cells of  $\text{Fun}_Y(A, B)$  correspond to maps  $a: \Delta^n \times A \rightarrow B$  such that  $ga = \pi(\text{id} \times f)$ , where  $\pi: \Delta^n \times Y \rightarrow Y$  is the projection.

*Proof of (23.11).* Fix a trivial fibration  $p: X \rightarrow S$ . We regard both  $X$  and  $S$  as objects *over*  $S$ , via  $p$  and  $\text{id}_S$ , and consider various relative function complexes over  $S$ .

Note that since  $p$  is a trivial fibration, so are  $\text{Fun}(X, p) = p^{\square(\emptyset \subset X)}$  and  $\text{Fun}(S, p) = p^{\square(\emptyset \subset X)}$  by enriched lifting  $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ . The maps

$$\text{Fun}_S(S, X) \rightarrow \text{Fun}_S(S, S) = * \quad \text{and} \quad \text{Fun}_S(X, X) \rightarrow \text{Fun}_S(X, S) = *$$

are (by construction) base changes of  $\text{Fun}(S, p)$  and  $\text{Fun}(X, p)$  respectively, and so are also trivial fibrations since  $\text{TrivFib}$  is closed under base change. It follows from (23.3) that both  $\text{Fun}_S(S, X)$  and  $\text{Fun}_S(X, X)$  are quasigroupoids which are categorically equivalent to the terminal object (and so are non-empty and such that all objects are isomorphic). Note that these are isomorphic to subcomplexes of simplicial sets  $\text{Fun}(S, X)$  and  $\text{Fun}(X, X)$  respectively, which however need not be quasicategories. However all edges of  $\text{Fun}_S(S, X)$  and  $\text{Fun}_S(X, X)$  are necessarily preisomorphisms in  $\text{Fun}(S, X)$  and  $\text{Fun}(X, X)$ .

Pick any vertex  $s$  of  $\text{Fun}_S(S, X)$ , so that  $s$  can be regarded as a map  $s: S \rightarrow X$  such that  $ps = \text{id}_S$ . Pick any isomorphism  $a: \text{id}_X \rightarrow sp$  in  $\text{Fun}_S(X, X)$ , which is hence a preisomorphism in  $\text{Fun}(X, X)$ . Thus, we have exhibited maps  $p$  and  $s$  whose composites are preisomorphic to identity functors, and therefore they are categorical equivalences by (23.9).  $\square$

**23.13. Remark** (“Uniqueness” of sections of trivial fibrations). Suppose that  $p: C \rightarrow D$  is a trivial fibration between quasicategories. As we have noted, the relative function complex  $\text{Fun}_D(D, C)$  “parameterizes sections of  $p$ ”. Since this is a quasigroupoid equivalent to the terminal quasicategory (23.11), not only is  $p$  a categorical equivalence, but also

- $p$  admits a section, which is a categorical inverse to  $p$ , and
- any two sections of  $p$  are naturally isomorphic.

We will often make use of this observation.

**23.14. Exercise.** Let  $p: C \rightarrow D$  be a functor between ordinary categories. Show that  $p$  is a trivial fibration if and only if (i) it is surjective on objects, and (ii)  $\text{hom}_C(x, y) \rightarrow \text{hom}_D(px, py)$  is a bijection for all objects  $x, y \in C_0$ .

**23.15. Exercise.** Let  $p: C \rightarrow D$  be a trivial fibration between ordinary categories. Show that  $S := \text{Fun}_D(D, C)$  is an ordinary category which is equivalent to the terminal category, and that the set of objects of  $S$  is in bijective correspondence with the set of sections of the map  $p: C_0 \rightarrow D_0$ .

**23.16. Inner anodyne maps are always categorical equivalences.**

**23.17. Proposition.** *Every inner anodyne map between simplicial sets is a categorical equivalence.*

*Proof.* Let  $j: X \rightarrow Y$  be a map in  $\overline{\text{InnHorn}}$ , and let  $C$  be any quasicategory. The induced map  $\text{Fun}(j, C): \text{Fun}(Y, C) \rightarrow \text{Fun}(X, C)$  is a trivial fibration by enriched lifting and  $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$  (20.2), and therefore is a categorical equivalence.  $\square$

**23.18. Every simplicial set is categorically equivalent to a quasicategory.**

**23.19. Proposition.** *Fix a simplicial set  $X$ .*

- (1) *There exists a quasicategory  $C$  and an inner anodyne map  $f: X \rightarrow C$ , which is therefore a categorical equivalence.*
- (2) *For any two  $f_i: X \rightarrow C_i$  as in (1), there exists a categorical equivalence  $g: C_1 \rightarrow C_2$  such that  $gf_1 = f_2$ .*
- (3) *Any two categorical equivalences  $g_1, g_2: C_1 \rightarrow C_2$  such that  $g_i f_1 = f_2$  are naturally isomorphic.*

Here is some more notation. Given maps  $f: X \rightarrow A$  and  $g: X \rightarrow B$ , we write  $\text{Fun}_{X/}(f, g)$  or  $\text{Fun}_{X/}(A, B)$  for the simplicial set defined by the pullback square

$$\begin{array}{ccc} \text{Fun}_{X/}(A, B) & \longrightarrow & \text{Fun}(A, B) \\ \downarrow & & \downarrow f^* = \text{Fun}(f, B) \\ \{g\} & \longrightarrow & \text{Fun}(X, B) \end{array}$$

This is the **relative function complex under  $X$** .

relative function complex under  $X$

**23.20. Exercise.** Show that  $n$ -cells of  $\text{Fun}_{X/}(A, B)$  correspond to maps  $a: \Delta^n \times A \rightarrow B$  such that  $a(\text{id} \times f) = g\pi$ , where  $\pi: \Delta^n \times X \rightarrow X$  is the projection

*Proof of (23.19).* (1) By the small object argument (15.16), we can factor  $X \rightarrow *$  into  $X \xrightarrow{j} C \xrightarrow{p} *$  where  $j \in \overline{\text{InnHorn}}$  and  $p \in \text{InnFib}$ . The inner anodyne map  $j$  is the desired categorical equivalence to a quasicategory.

(2) For  $i, j \in \{1, 2\}$ , we have a restriction map  $f_{i,j}^*: \text{Fun}(C_i, C_j) \rightarrow \text{Fun}(X, C_j)$ , which is necessarily a trivial fibration by enriched lifting since  $\overline{\text{Horn}} \square \overline{\text{Cell}} \subseteq \overline{\text{Horn}}$ . Therefore the maps  $\text{Fun}_{X/}(C_i, C_j) \rightarrow *$  (obtained by base-change from the  $f_{i,j}^*$ ) are all trivial fibrations, i.e., each  $\text{Fun}_{X/}(C_i, C_j)$  is a quasigroupoid with only one isomorphism class of objects (23.3). As in the proof of (23.11) we construct  $g: C_1 \rightarrow C_2$  and  $g': C_2 \rightarrow C_1$  which are categorically inverse to each other; details are left to the reader.

(3) The maps  $g_1, g_2$  correspond to vertices in  $\text{Fun}_{X/}(C_1, C_2)$ , which as we have observed is a quasigroupoid with only one isomorphism class of objects.  $\square$

Thus, we can always “replace” a simplicial set  $X$  by a categorically equivalent quasicategory  $C$ . Although such  $C$  is not unique, it is unique up to categorical equivalence.

You can think of such a replacement  $X \rightarrow C$  of  $X$  as a quasicategory “freely generated” by the simplicial set  $X$ , an idea which is validated by the fact that  $\text{Fun}(j, D): \text{Fun}(C, D) \rightarrow \text{Fun}(X, D)$  is a trivial fibration for every quasicategory  $D$ .

## 24. SOME EXAMPLES OF CATEGORICAL EQUIVALENCES

**24.1. Free monoid on one generator.** Let  $F$  denote the free monoid on one generator  $g$ . This is a category with one object  $x$ , and morphism set  $\{g^n \mid n \geq 0\}$ .

Associated to the generator  $g$  is a map

$$\gamma: S^1 := \Delta^1 / \partial \Delta^1 \rightarrow N(F)$$

sending the image of the generator  $\iota \in (\Delta^1)_1$  in  $S^1$  to  $g$ . (We use “ $L/K$ ” as a shorthand for “ $L \amalg_K *$ ” whenever  $K \subseteq L$ . The object  $S^1$  is called the “simplicial circle”, which has exactly two nondegenerate simplicies, one in dimension 0 and one in dimension 1.)

It is not hard to see that  $F$  is “freely generated” as a category by  $S^1$ , in the sense that  $h(S^1) = F$  (the fundamental category of  $S^1$  is  $F$ ). It turns out that  $N(F)$  is actually “freely generated as a *quasicategory*” by  $S^1$ .

**24.2. Proposition.** *The map  $\gamma: S^1 \rightarrow N(F)$  is a categorical equivalence, and in fact is inner anodyne.*

*Proof.* This is an explicit calculation. Note that a general cell in  $N(F)_d$  corresponds to a sequence  $(g^{m_1}, \dots, g^{m_d})$  of elements of the monoid  $F$ , where  $m_1, \dots, m_d \geq 0$ . Let  $a_k \in N(F)_k$  denote the  $k$ -cell corresponding to the sequence  $(g, g, \dots, g)$ , and let  $Y_k \subseteq N(F)$  denote the subcomplex which is the image of the representing map  $a_k: \Delta^k \rightarrow N(F)$ . For  $f: [d] \rightarrow [k]$  we compute that  $a_k f = (g^{m_1}, \dots, g^{m_d})$  where  $m_i = f(i) - f(i-1)$ , so that

$$(Y_k)_d = \{ a_k f \mid f: [d] \rightarrow [k] \} = \{ (g^{m_1}, \dots, g^{m_d}) \mid m_1 + \dots + m_d \leq k \},$$

Clearly  $N(F) = \bigcup_{k \geq 1} Y_k$ , with  $Y_1 \approx S^1$  and  $Y_2 \approx Y_1 \cup_{\Lambda^2} \Delta^2$ . Furthermore we have the following:

- A simplicial operator  $f: [d] \rightarrow [k]$  (i.e., element of  $(\Delta^k)_d$ ) is such that  $a_k f$  is in the subcomplex  $Y_{k-1}$  of  $Y_k$  if and only if  $f(d) - f(0) < k$ , if and only if either  $f(d) < k$  or  $f(0) > 0$ , i.e., if and only if  $f$  is in the subcomplex  $\Lambda_{\{0,k\}}^k = \Delta^{\{0, \dots, k-1\}} \cup \Delta^{\{1, \dots, k\}}$  of  $\Delta^k$ .
- Every cell  $y$  of  $Y_k$  not in  $Y_{k-1}$  is the image under  $a_k$  of a *unique* cell in  $\Delta^k$ . (I.e., if  $f: [d] \rightarrow [k]$ , then  $m_1 + \dots + m_d = f(d) - f(0)$ , which is equal to  $k$  if and only if  $f(0) = 0$  and  $f(d) = k$ , and if this is the case then  $f(i) = m_1 + \dots + m_i$ .)

In other words, the square

$$\begin{array}{ccc} \Lambda_{\{0,k\}}^k & \longrightarrow & Y_{k-1} \\ \downarrow & & \downarrow \\ \Delta^k & \xrightarrow{a_k} & Y_k \end{array}$$

is a pullback, and  $a_k$  induces in each dimension  $d$  a bijection  $(\Delta^k)_d \setminus (\Lambda_{\{0,k\}}^k)_d \xrightarrow{\sim} (Y_k)_d \setminus (Y_{k-1})_d$ . It follows (18.4) that the square is a pushout.

The inclusion  $\Lambda_{\{0,k\}}^k \subset \Delta^k$  is a *generalized inner horn*, and we have noted this is inner anodyne when  $k \geq 2$  (14.10). It follows that each  $Y_{k-1} \rightarrow Y_k$  is inner anodyne for  $k \geq 2$ , whence  $S^1 \rightarrow N(F)$  is inner anodyne.  $\square$

**24.3. Remark.** This gives an explicit example of a categorical equivalence to which (23.9) does not apply:  $\gamma$  does not admit an “inverse up to preisomorphisms”. There is only one map  $\delta: N(F) \rightarrow S^1$ , namely the composite  $N(F) \rightarrow * \rightarrow S^1$ , and it is clear that neither  $\gamma\delta: N(F) \rightarrow N(F)$  nor  $\delta\gamma: S^1 \rightarrow S^1$  are preisomorphic to identity functors.

**24.4. Free categories.** We can generalize the above to free monoids with arbitrary sets of generators, and in fact to free categories. Let  $S$  be a **1-dimensional** simplicial set, i.e., one such that  $S = \text{Sk}_1 S$ . These are effectively the same thing as *directed graphs* (allowed to have multiple parallel edges and loops):  $S_0$  corresponds to the set of vertices of the directed graph, and  $S_1^{\text{nd}}$  corresponds to the set of edges of the directed graph.

Let  $F := hS$ . We call  $F$  the **free category** on the 1-dimensional simplicial set  $S$ . In this case, the morphisms of the fundamental category are precisely the words in the edges  $S_1^{\text{nd}}$  of the directed graph (including empty words for each vertex, corresponding to identity maps). That is, it is precisely the free category described in the proof of (11.2).

**1-dimensional**

**free category**

**24.5. Proposition.** *The evident map  $\gamma: S \rightarrow N(F)$  is a categorical equivalence, and in fact is inner anodyne.*

*Proof.* This is virtually the same as the proof of (24.2). In this case,  $Y_k \subseteq N(F)$  is the subcomplex generated by all  $a: \Delta^k \rightarrow N(F)$  such that each spine-edge  $a_{i-1,i}$  is in  $S_1^{\text{nd}}$ , and  $Y_k$  is obtained by attaching a generalized horn to  $Y_{k-1}$  for each such  $a$ .  $\square$

As a consequence, it is “easy” to construct functors  $F \rightarrow C$  from a free category to a quasicategory: start with a map  $S \rightarrow C$ , which amounts to specifying vertices and edges in  $C$  corresponding to elements  $S_0$  and  $S_1^{\text{nd}}$ , and extend over  $S \subseteq F$ . The evident restriction map  $\text{Fun}(F, C) \rightarrow \text{Fun}(S, C)$  is a categorical equivalence, and in fact a trivial fibration. In other words, free categories are also “free quasicategories”.

**24.6. Exercise.** Describe the ordinary category  $A := h\Lambda_0^3$  “freely generated” by  $\Lambda_0^3$ . Show that the tautological map  $\Lambda_0^3 \rightarrow N(A)$  is inner anodyne.

**24.7. Free commutative monoids.** Let  $F$  be the free monoid on one generator again, with generator corresponding to simplicial circle  $S^1 = \Delta^1/\partial\Delta^1 \subset N(F)$ . Thus  $F^{\times n}$  is the free commutative monoid on  $n$  generators. Recall that the nerve functor preserves products, so  $N(F^{\times n}) \approx N(F)^{\times n}$ . We obtain a map

$$\delta = \gamma^{\times n}: (S^1)^{\times n} \rightarrow N(F^{\times n})$$

from the “simplicial  $n$ -torus”.

**24.8. Proposition.** *The map  $\delta: (S^1)^{\times n} \rightarrow N(F^{\times n})$  is a categorical equivalence, and in fact is inner anodyne.*

*Proof.* This is a consequence of the fact that if  $j: A \rightarrow B$  is inner anodyne and  $K$  an arbitrary simplicial set, then  $j \times K \times: A \times K \rightarrow B \times K$  is inner anodyne (because  $\overline{\text{InnHorn}} \square \text{Cell} \subseteq \overline{\text{InnHorn}}$ ). It follows that  $A^{\times n} \rightarrow B^{\times n}$  is a composite of inner anodyne maps, and so is inner anodyne and thus a categorical equivalence (23.17). Also use the fact that the nerve construction preserves products (8.5), so  $N(F^{\times n}) = N(F)^{\times n}$ .  $\square$

**24.9. Exercise.** Let  $S^1 \vee S^1 \subset (S^1)^{\times 2}$  be the subcomplex obtained as the evident “one-point union” of the two “coordinate circles”; i.e.,  $S^1 \vee S^1 = (S^1 \times \{*\}) \cup (\{*\} \times S^1)$ . Suppose given a map  $\phi: S^1 \vee S^1 \rightarrow C$  to a quasicategory  $C$ , corresponding to a choice of object  $x \in C_0$  together with two morphisms  $f, g: x \rightarrow x$  in  $C_1$ . Show that there exists an extension of  $\phi$  along  $S^1 \vee S^1 \subset N(F^{\times 2})$  if and only if  $[f][g] = [g][f]$  in  $hC$ .

**24.10. Remark.** The analogue of the above exercise for  $n = 3$  isn’t true. That is, consider the subcomplex  $S^1 \vee S^1 \vee S^1 \subset (S^1)^{\times 3}$  which is a one-point union of three circles, suppose we have  $S^1 \vee S^1 \vee S^1 \rightarrow C$  corresponding to three morphisms  $f, g, h: x \rightarrow x$  in  $C$ , and suppose we also know that  $[f][g] = [g][f]$ ,  $[g][h] = [h][g]$ , and  $[f][h] = [h][f]$  in  $hC$ . Then you can show that there exists an extension to a map  $K \rightarrow C$  as in (24.9), where  $K \subseteq (S^1)^{\times 3}$  is the subcomplex  $(S^1 \times S^1 \times \{*\}) \cup (S^1 \times \{*\} \times S^1) \cup (\{*\} \times S^1 \times S^1)$ . However, there need not exist an extension to a map  $(S^1)^{\times 3} \rightarrow C$ , and thus there may not exist an extension to a map  $N(F^{\times 3}) \rightarrow C$ . (For an explicit example where this fails, take  $C = \text{Sing } T$ , where  $T \subseteq (S_{\text{top}}^1)^{\times 3}$  is the subspace of the topological 3-torus consisting of tuples  $(x_1, x_2, x_3)$  such that at least one  $x_i$  is the basepoint of  $S_{\text{top}}^1$ .)

Thus, this is a situation where the “higher structure” of a quasicategory plays a role. When  $C$  is an ordinary category, it is easy to show that the desired extension does always exist. However, for a general quasicategory  $C$ , three pairwise-commuting endomorphisms of an object do not generally give rise to a functor  $N(F^{\times 3}) \rightarrow C$  from the free commutative monoid on 3 generators.



**24.11. Finite groups are not finite.** If  $A$  is any ordinary category, then  $\mathrm{Sk}_2 N(A)$  “freely generates  $N(A)$  as a category”, in the sense that  $h(\mathrm{Sk}_2 N(A)) \approx A$ , or equivalently that  $\mathrm{Fun}(N(A), N(B)) \rightarrow \mathrm{Fun}(\mathrm{Sk}_2 N(A), N(B))$  is an isomorphism for any category  $B$ . However, it is often the case that *no finite dimensional simplicial set* “freely generates  $N(A)$  as a quasicategory”. In fact, this is the case for every non-trivial finite group.

**24.12. Example.** Let  $G$  be the finite group of order 2, with generator  $g$ , and consider  $C = NG$ , the nerve of  $G$  viewed as a category with one object. Then  $C$  has exactly one non-degenerate cell in each dimension:

$$C_k^{\mathrm{nd}} = \{a_k := (g, \dots, g)\}.$$

Thus, to give a map  $\phi: C \rightarrow D$  to some other quasicategory requires describing the infinite list of data  $\phi(a_k) \in D_k$ ,  $k \geq 0$ , which must necessarily satisfy a number of compatibilities in order that  $\phi$ . It turns out that there is no way to reduce this to a finite list of data and conditions, in the following sense: there is *no finite dimensional simplicial set* which is categorically equivalent to  $C$ .

To prove this, relies on some topology, together with a fact to be proved later<sup>20</sup>: if  $f: X \rightarrow Y$  is any categorical equivalence of simplicial sets, then the induced map  $\|f\|: \mathrm{realiz} X \rightarrow \mathrm{realiz} Y$  of geometric realizations must be a homotopy equivalence of spaces. The realization  $\|NG\|$  is the *classifying space* of  $G$ . The topology we need is that the cohomology of this space is non-zero in arbitrarily large dimensions:  $H^{2k}(\|NG\|, \mathbb{Z}) \approx \mathbb{Z}/2 \not\approx 0$  for all  $k \geq 0$ . On the other hand, if  $X$  is a simplicial set with  $X = \mathrm{Sk}_d X$ , then  $H^k(X, \mathbb{Z}) = 0$  for  $k > d$ .

A similar observation applies for any non-trivial *finite* group  $G$ . Thus, non-trivial finite groups are near “freely generated as a quasicategory” by finite dimensional complexes. In the world of  $\infty$ -categories, finite groups are fundamentally infinite dimensional objects.

## 25. THE HOMOTOPY CATEGORY OF QUASICATEGORIES

**25.1. The homotopy category of  $\mathbf{qCat}$ .** The homotopy category  $h\mathbf{qCat}$  of quasicategories is defined as follows. The objects of  $h\mathbf{qCat}$  are the quasicategories. Morphisms  $C \rightarrow D$  in  $h\mathbf{qCat}$  are natural isomorphism classes of functors. That is,

$$\mathrm{Hom}_{h\mathbf{qCat}}(C, D) := \text{isomorphism classes of objects in } h\mathrm{Fun}(C, D) = \pi_0(\mathrm{Fun}(C, D)^{\mathrm{core}}).$$

That this defines a category results from the fact that composition of functors passes to a functor  $h\mathrm{Fun}(D, E) \times h\mathrm{Fun}(C, D) \rightarrow h\mathrm{Fun}(C, E)$ , and thus is compatible with natural isomorphism.

It comes with an obvious functor  $\mathbf{qCat} \rightarrow h\mathbf{qCat}$ . Note that a map  $f: C \rightarrow D$  of quasicategories is a categorical equivalence if and only if its image in  $h\mathbf{qCat}$  is an isomorphism.

**25.2. Remark.** We can similarly define a category  $h\mathbf{Cat}$ , whose objects are ordinary categories and whose morphisms are *isomorphism classes* of functors. The nerve functor evidently induces a full embedding  $h\mathbf{Cat} \rightarrow h\mathbf{qCat}$ .

**25.3. Warning.** Although we use the word “homotopy”, the definition of  $h\mathbf{qCat}$  given above is not an example of the notion of the homotopy category of a quasicategory defined in (11):  $\mathbf{qCat}$  is a (large) ordinary category, so is isomorphic to its own homotopy category in that sense. Here we are using the equivalence relation on morphisms(=functors) defined by natural isomorphism.

We define  $h\mathbf{Kan} \subset h\mathbf{qCat}$  to be the full subcategory of the homotopy category spanned by quasicategories which are Kan complexes.

For future reference, we note that  $h\mathbf{qCat}$  and  $h\mathbf{Kan}$  have finite products, which just amount to the usual products of simplicial sets.

**25.4. Proposition.** *The terminal simplicial set  $\Delta^0$  is a terminal object in  $h\mathbf{qCat}$ . If  $C_1, C_2$  are quasicategories, then the projection maps exhibit  $C_1 \times C_2$  as a product in  $h\mathbf{qCat}$ .*

<sup>20</sup>I don’t know if this will actually get proved later. It is proved in [GJ09].

*Proof.* This is straightforward. The key observation for the second statement is the fact that isomorphism classes of objects in a product of quasicategories correspond to pairs of isomorphism classes in each (8.14), and the fact that  $\text{Fun}(X, C_1 \times C_2) \xrightarrow{\sim} \text{Fun}(X, C_1) \times \text{Fun}(X, C_2)$ .  $\square$

25.5. *Exercise* (Products of categorical equivalences). Let  $f: X \rightarrow Y$  and  $f': X' \rightarrow Y'$  be categorical equivalences of simplicial sets. Show that  $f \times f': X \times X' \rightarrow Y \times Y'$  is a categorical equivalence. (Hint: reduce to the case where one of the maps is identity.)

25.6. **The 2-out-of-6 and 2-out-of-3 properties.** A class of morphisms  $\mathcal{W}$  in a category is said to satisfy the **2-out-of-6 property** if (i)  $\mathcal{W}$  contains all identity maps, and (ii) given sequence  $(h, g, f)$  of maps such that the composites  $gf$  and  $hg$  are defined, if  $gf, hg \in \mathcal{W}$  then also  $f, g, h, hgf \in \mathcal{W}$ . 2-out-of-6 property

A class of morphisms  $\mathcal{W}$  in a category is said to satisfy the **2-out-of-3 property** if (i)  $\mathcal{W}$  contains all identity maps, and (ii) given a sequence  $(g, f)$  of maps such that the composite  $gf$  is defined, if any two of  $(f, g, gf)$  are in  $\mathcal{W}$ , so is the third. 2-out-of-3 property

25.7. *Example.* In any category, the class of isomorphisms satisfies 2-out-of-6 property and the 2-out-of-3 property. The class of identity maps satisfies 2-out-of-3, but does not generally satisfy 2-out-of-6.

25.8. **Proposition.** *If  $\mathcal{W}$  satisfies 2-out-of-6, then it satisfies 2-out-of-3.*

*Proof.* Given  $f, g$  such that  $gf$  is defined, apply 2-out-of-6 to the composable sequences  $(\text{id}, g, f)$ ,  $(g, \text{id}, f)$ ,  $(g, f, \text{id})$ .  $\square$

25.9. *Exercise.* Given a functor  $f: C \rightarrow D$  between categories, let  $\mathcal{W}$  be the class of maps in  $C$  that  $f$  takes to isomorphisms in  $D$ . Show that  $\mathcal{W}$  satisfies 2-out-of-6, and thus 2-out-of-3.

25.10. *Example* (2-out-of-6 for equivalences of categories). In  $\text{Cat}$ , the category of small categories and functors, the class of equivalences satisfies 2-out-of-6, and thus 2-out-of-3.

To see this, first suppose  $(h, g, f)$  is a triple of functors such that there are natural isomorphisms  $gf \approx \text{id}$  and  $hg \approx \text{id}$ . Then, since (i) natural isomorphism is an equivalence relation on functors and (ii) is compatible with composition, we see that

$$h = h \text{id} \approx h(gf) = (hg)f \approx \text{id} f = f,$$

and thus that  $g$  is an equivalence since  $hg \approx \text{id}$  and  $gh \approx gf \approx \text{id}$ .

Next, note that composites of equivalences are equivalences, by a straightforward argument: if  $g$  and  $f$  are equivalences and composable, and  $g'$  and  $f'$  are categorical inverses to them, then  $f'g'$  is easily seen to be a categorical inverse to  $gf$ .

Now suppose that  $(h, g, f)$  are such that  $gf$  and  $hg$  are categorical equivalences. Choose categorical inverses  $u$  and  $v$  for these, so that

$$gfu \approx \text{id}, \quad ugf \approx \text{id}, \quad hgv \approx \text{id}, \quad vhg \approx \text{id}.$$

Apply the above remarks to the triples  $(ug, f, ug)$ ,  $(vh, g, vu)$ ,  $(gv, h, gv)$ , and  $(ugv, hgf, vgu)$  to show that  $f, g, h$  are equivalences, where we use that

$$fug \approx (vhg)fug = vh(gfu)g \approx vhg \approx \text{id}, \quad gvh \approx gvh(gfu) = g(vhg)fu \approx gfu \approx \text{id}.$$

It follows that the composite  $hgf$  is also an equivalence.

Alternately, we can apply (25.9) to the tautological functor  $\text{Cat} \rightarrow h\text{Cat}$ , which sends a functor to an isomorphism in  $h\text{Cat}$  if and only if it is an equivalence.

25.11. **Proposition.** *The class  $\text{CatEq}$  of categorical equivalences in  $s\text{Set}$  satisfies 2-out-of-6, and thus 2-out-of-3.*

*Proof.* It is immediate that the identity map of a simplicial set is a categorical equivalence.

Next consider functors  $f, g, h$  between *quasicategories* such that  $gf$  and  $hg$  are defined and are categorical equivalences. Then  $f, g, h$  and  $hgf$  are categorical equivalences by an argument which is word-for-word the same as in (25.10).

For the general case, we reduce to the quasicategory case by applying  $\text{Fun}(-, C)$ , where  $C$  is an arbitrary quasicategory.  $\square$

**25.12. Other characterizations of categorical equivalence.** It turns out that we can replace the condition in the definition of categorical equivalence with some seemingly weaker conditions.

**25.13. Proposition.** *Let  $f: X \rightarrow Y$  be a map of simplicial sets. The following are equivalent.*

- (1) *The map  $f$  is a categorical equivalence: i.e., for every quasicategory  $C$ , the functor  $\text{Fun}(Y, C) \rightarrow \text{Fun}(X, C)$  induced by restriction along  $f$  admits a categorical inverse.*
- (2) *For every quasicategory  $C$ , the map  $h\text{Fun}(Y, C) \rightarrow h\text{Fun}(X, C)$  induced by restriction along  $f$  is an equivalence of ordinary categories.*
- (3) *For every quasicategory  $C$ , the map  $\pi_0(\text{Fun}(Y, C)^{\text{core}}) \rightarrow \pi_0(\text{Fun}(X, C)^{\text{core}})$  induced by restriction along  $f$  is a bijection of sets.*

*Proof.* (1)  $\Rightarrow$  (2) is immediate from (22.3), while (2)  $\Rightarrow$  (3) is immediate, since an equivalence of ordinary categories induces a bijection on isomorphism classes of objects. We prove that (3) implies (1).

In the case that  $f$  is a map between *quasicategories*, this is really what the second half of the proof of (22.6) actually shows. That is, we let  $C$  be either  $X$  or  $Y$ , and use the bijections

$$\widehat{f}^*: \pi_0(\text{Fun}(Y, X)^{\text{core}}) \xrightarrow{\sim} \pi_0(\text{Fun}(X, X)^{\text{core}}), \quad \widehat{f}^*: \pi_0(\text{Fun}(Y, Y)^{\text{core}}) \xrightarrow{\sim} \pi_0(\text{Fun}(X, Y)^{\text{core}}),$$

to (a) produce a  $g: Y \rightarrow X$  such that  $gf \approx \text{id}_X$ , and (b) show that  $fgf \approx f \text{id}_X = \text{id}_Y f$  implies  $fg \approx \text{id}_Y$ .

We reduce the case of a general map  $f$  to that of a map  $f'$  between quasicategories as follows. Use factorization to construct a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ u \downarrow & & \downarrow v \\ X' & \xrightarrow{f'} & Y' \end{array}$$

so that  $u$  and  $v$  are inner anodyne (and so categorical equivalences), and  $X'$  and  $Y'$  are quasicategories. If we apply  $\text{Fun}(-, C)$  to the square with  $C$  a quasicategory, the vertical maps become trivial fibrations, and hence induce bijections on isomorphism classes of objects. Therefore  $\text{Fun}(f, C)$  induces a bijection on isomorphism classes of objects if and only if  $\text{Fun}(f', C)$  does.  $\square$

Joyal [Joy08a, 1.20] singles out statement (2) of (25.13) as his basic notion of equivalence, which he calls **weak categorical equivalence**<sup>21</sup>. In *kerodon.net* Lurie singles out statement (3) as the basic notion of equivalence. We see that either of these are equivalent to the definition of categorical equivalence we are using.

weak categorical equivalence

Finally, we note one more criterion for categorical equivalence between quasicategories, which is in some sense “dual” to the one given above.

**25.14. Proposition.** *Let  $f: C \rightarrow D$  be a functor between quasicategories. The following are equivalent*

- (1) *The map  $f$  is a categorical equivalence.*

<sup>21</sup>This is not to be confused with “weak homotopy equivalence”, which we will talk about later (49).

- (2) For every simplicial set  $X$ , the map  $f_*: \text{Fun}(X, C) \rightarrow \text{Fun}(Y, D)$  induced by composition with  $f$  is a categorical equivalence.
- (3) For every simplicial set  $X$ , the map  $(hf)_*: h\text{Fun}(X, C) \rightarrow h\text{Fun}(X, D)$  induced by composition with  $f$  is an equivalence of categories.
- (4) For every simplicial set  $X$ , the map  $\pi_0(\text{Fun}(X, C)^{\text{core}}) \xrightarrow{ra} \pi_0(\text{Fun}(X, D)^{\text{core}})$  induced by composition with  $f$  is a bijection of sets.

*Proof.* (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) are straightforward. So we need to prove (4) implies (1). The proof is analogous to what we have already done: let  $X$  be either  $C$  or  $D$ , and use the bijections

$$\widehat{f}_*: \pi_0(\text{Fun}(D, C)^{\text{core}}) \xrightarrow{\sim} \pi_0(\text{Fun}(D, D)^{\text{core}}), \quad \widehat{f}_*: \pi_0(\text{Fun}(C, C)^{\text{core}}) \xrightarrow{\sim} \pi_0(\text{Fun}(C, D)^{\text{core}})$$

to (a) produce a  $g: D \rightarrow C$  such that  $fg \approx \text{id}_D$ , and (b) show that  $fgf \approx \text{id}_D f = f \text{id}_C$  implies  $gf \approx \text{id}_C$ .  $\square$

**25.15. The homotopy 2-category of  $\text{qCat}$ .** A **2-category**  $E$  is a category which is itself “enriched” over  $\text{Cat}$ . That is,

- for each pair of objects  $x, y \in \text{ob } E$ , there is a category  $\underline{\text{Hom}}_E(x, y)$ , so that
- the objects of  $\underline{\text{Hom}}_E(x, y)$  are precisely the set  $\text{Hom}_E(x, y)$  of morphisms of  $E$ , and
- there are “composition functors”  $\underline{\text{Hom}}_E(y, z) \times \underline{\text{Hom}}_E(x, y) \rightarrow \underline{\text{Hom}}_E(x, z)$  for all  $x, y, z \in \text{ob } E$  which on objects is just ordinary composition of morphisms in  $E$ , which
- is unital and associative in the evident sense.

One refers to the objects of  $\underline{\text{Hom}}_E(x, y)$  as **1-morphisms**  $f: x \rightarrow y$  of  $E$ , and the morphisms of  $\underline{\text{Hom}}_E(x, y)$  as **2-morphisms**  $\alpha: f \Rightarrow g$  of  $E$ . The **underlying category** of  $E$  consists of the objects and 1-morphisms only.

**1-morphisms**  
**2-morphisms**  
**underlying category**

The standard example of a 2-category is  $\text{Cat}$ , the category of categories, with objects=categories, 1-morphisms=functors, 2-morphisms=natural transformations.

We can enlarge the category  $\text{qCat}$  of quasicategories to a **homotopy 2-category**  $h_2\text{qCat}$ , so that

$$\text{Hom}_{h_2\text{qCat}}(C, D) := h\text{Fun}(C, D).$$

That is,

- **objects** of  $h_2\text{qCat}$  are quasicategories,
- **1-morphisms** of  $h_2\text{qCat}$  are functors between quasicategories,
- **2-morphisms** of  $h_2\text{qCat}$  are *isomorphism classes* of natural transformations of functors.

Note that  $\text{qCat}$  sits inside  $h_2\text{qCat}$  as its underlying category; thus,  $h_2\text{qCat}$  contains all the information of  $\text{qCat}$ . On the other hand  $h\text{qCat}$  is obtained from  $h_2\text{qCat}$  by first identifying 1-morphisms (functors) which are 2-isomorphic (i.e., naturally isomorphic), and then throwing away the 2-morphisms; thus,  $h_2\text{qCat}$  contains all the information of  $h\text{qCat}$ .

## Part 5. Joins, slices, and Joyal’s extension and lifting theorems

In this part we describe and apply two methods to construct new quasicategories from old, called “joins” and “slices”. They are both generalizations of constructions which can be carried out on categories: the most familiar of these classical constructions is *slice category*  $C_{/x}$  associated to an object  $x$  of a category  $C$ , in which objects of the slice  $C_{/x}$  are morphisms  $c \rightarrow x$  in  $C$ , and morphisms of  $C_{/x}$  are commutative triangles in  $C$ .

With these constructions in hand, we will be able to define notions of *limits* and *colimits* in quasicategories. We will also be able to prove some of the results we have deferred up until now, including the equivalence of quasigroupoids and Kan complexes (33.2) and the pointwise criterion for natural isomorphisms (35.2). Much of the material in this part comes from Joyal’s seminal paper [Joy02].

## 26. JOINS

**26.1. Join of categories.** If  $A$  and  $B$  are ordinary categories, we can define a category  $A \star B$  called the join. This has

$$\text{ob}(A \star B) = \text{ob } A \amalg \text{ob } B, \quad \text{mor}(A \star B) = \text{mor } A \amalg (\text{ob } A \times \text{ob } B) \amalg \text{mor } B,$$

so that we put in a *unique* map from each object of  $A$  to each object of  $B$ . Explicitly,

$$\text{Hom}_{A \star B}(x, y) := \begin{cases} \text{Hom}_A(x, y) & \text{if } x, y \in \text{ob } A, \\ \text{Hom}_B(x, y) & \text{if } x, y \in \text{ob } B, \\ \{*\} & \text{if } x \in \text{ob } A, y \in \text{ob } B, \\ \emptyset & \text{if } x \in \text{ob } B, y \in \text{ob } A, \end{cases}$$

with composition defined so that the evident inclusions  $A \rightarrow A \star B \leftarrow B$  are functors, and in fact are isomorphisms to full subcategories of  $A \star B$ . (Check that this really defines a category, and that  $A$  and  $B$  are identified with full subcategories of  $A \star B$ .)

**26.2. Example.** We have that  $[p] \star [q] \approx [p + 1 + q]$ .

**26.3. Exercise** (Functors from a join of categories). Show that functors  $f: A \star B \rightarrow C$  are in bijective correspondence with triples  $(f_A: A \rightarrow C, f_B: B \rightarrow C, \gamma: f_A \circ \pi_A \Rightarrow f_B \circ \pi_B)$ , where  $f_A$  and  $f_B$  are functors, and  $\gamma$  is a natural transformation of functors  $A \times B \rightarrow C$ , where  $\pi_A: A \times B \rightarrow A$  and  $\pi_B: A \times B \rightarrow B$  denote the evident projection functors.

**26.4. Exercise** (Functors to a join of categories). Show that functors  $f: C \rightarrow A \star B$  are in bijective correspondence with triples of functors  $(\pi: C \rightarrow [1], f_{\{0\}}: C^{\{0\}} \rightarrow A, f_{\{1\}}: C^{\{1\}} \rightarrow B)$ , where  $C^{\{j\}} := \pi^{-1}(\{j\}) \subseteq C$  is the fiber of  $\pi$  over  $j \in \text{ob}[1]$ , i.e., the subcategory of  $C$  consisting of objects which  $\pi$  sends to  $j$  and morphisms which  $\pi$  sends to  $\text{id}_j$ .

**26.5. Exercise.** Describe an isomorphism  $(A \star B)^{\text{op}} \approx B^{\text{op}} \star A^{\text{op}}$ .

**26.6. Cones on categories.** An important special case are the **left cone** and **right cone** of a category, defined by  $A^\triangleleft := [0] \star A$  and  $A^\triangleright := A \star [0]$ . For instance, the right cone  $A^\triangleright$  is the category obtained by adjoining one additional object  $v$  to  $A$ , as well as a unique map  $x \rightarrow v$  for each object  $x$  of  $A^\triangleright$ . In this case,  $v$  becomes a terminal object for  $A^\triangleright$ , and we can say that  $A \mapsto A^\triangleright$  *freely adjoins a terminal object* to  $A$ . (Note that a terminal object of  $A$  will not be terminal in  $A^\triangleright$  anymore.) Likewise,  $A \mapsto A^\triangleleft$  *freely adjoins an initial object* to  $A$ .

left cone  
right cone

Limits and colimits of functors can be characterized using cones: if  $p: A \rightarrow C$  is a functor, a *colimit* of  $p$  is a functor  $\hat{p}: A^\triangleright \rightarrow C$  which is initial among functors which extend  $p$ , and likewise, a *limit* of  $p$  is a functor  $\hat{p}: A^\triangleleft \rightarrow C$  which is terminal among functors which extend  $p$ .

**26.7. Remark.** It is worthwhile to spell this out in detail. Given a functor  $p: A \rightarrow C$ , to describe a functor  $q: A^\triangleright \rightarrow C$  which *extends*  $p$ , it suffices to give

- (1) an object  $q(v)$  in  $C$ ,
- (2) for each object  $a \in \text{ob } A$  a morphism  $q(a \rightarrow v): p(a) \rightarrow q(v)$  in  $C$ , such that
- (3) for each morphism  $\alpha: a \rightarrow a'$  in  $A$  we have an equality  $q(a' \rightarrow v) \circ p(\alpha) = q(a \rightarrow v)$  of morphisms  $p(a) \rightarrow q(v)$  in  $C$ .

$$\begin{array}{ccc} a & & p(a) \\ \alpha \downarrow & \Rightarrow & \downarrow p(\alpha) \\ a' & & p(a') \end{array} \quad \begin{array}{ccc} & & q(a \rightarrow v) \\ & \searrow & \\ & & q(v) \\ & \nearrow & \\ & & q(a' \rightarrow v) \end{array}$$

Given functors  $q, q': A^\triangleright \rightarrow C$ , we may consider natural transformations  $\phi: q \rightarrow q'$  which *extend the identity transformation* of  $p$ . Explicitly, such a transformation  $\phi$  is exactly determined by

- (1) a morphism  $\phi(v): q(v) \rightarrow q'(v)$  in  $C$  such that
- (2) for each object  $a \in \text{ob } A$  we have an equality  $q'(a \rightarrow v) = \phi(v) \circ q(a \rightarrow v)$  of morphisms  $p(a) \rightarrow q'(v)$  in  $C$ .

$$a \quad \Longrightarrow \quad \begin{array}{ccc} & q(a \rightarrow v) & q(v) \\ & \nearrow & \downarrow \phi(v) \\ p(a) & & \\ & \searrow & \downarrow \\ & q'(a \rightarrow v) & q'(v) \end{array}$$

An extension  $\widehat{p}: A^\triangleright \rightarrow C$  of  $p$  is a *colimit* of  $p$  if for every  $q$  extending  $p$  there exists a unique map  $\phi(v): \widehat{p}(v) \rightarrow q(v)$  in  $C$  such that  $q(a \rightarrow v) = \phi(v) \circ \widehat{p}(a \rightarrow v)$  for all  $a \in \text{ob } A$ . The object  $\widehat{p}(v)$  is what is colloquially known as “the colimit of  $p$ ”, although the full data of a colimit of  $p$  is actually the functor  $\widehat{p}$ . We will call the functor  $\widehat{p}$  a *colimit cone* in what follows.

**26.8. Ordered disjoint union.** As noted above (26.2), the join operation on categories effectively descends to  $\Delta$ . We will call this the **ordered disjoint union**. It is a functor  $\sqcup: \Delta \times \Delta \rightarrow \Delta$ , defined so that  $[p] \sqcup [q] := [p + 1 + q]$ , to be thought of as the disjoint union of underlying sets, ordered so that the subsets  $[p]$  and  $[q]$  retain their ordering, and elements of  $[p]$  come before elements of  $[q]$ .

ordered disjoint union

It is handy to extend this to the category  $\Delta_+$ , the full subcategory of ordered sets obtained by adding the empty set  $[-1] := \emptyset$  to  $\Delta$ . The functor  $\sqcup$  extends in an evident way to  $\sqcup: \Delta_+ \times \Delta_+ \rightarrow \Delta_+$ .

This extended functor makes  $\Delta_+$  into a (strict, but nonsymmetric) monoidal category, with unit object  $[-1]$ .

Note that for any map  $f: [p] \rightarrow [q_1] \sqcup [q_2]$  in  $\Delta_+$ , there is a *unique* decomposition  $[p] = [p_1] \sqcup [p_2]$  such that  $f = f_1 \sqcup f_2$  for some (necessarily unique)  $f_i: [p_i] \rightarrow [q_i]$  in  $\Delta_+$ . (We need an object  $[-1]$  to be able to say this, even if  $p, q_1, q_2 \geq 0$ ; if  $f([p]) \subseteq [q_1]$  then  $p_2 = -1$ .)

**26.9. Join of simplicial sets.** Let  $X$  and  $Y$  be simplicial sets. The **join** of  $X$  and  $Y$  is a simplicial set  $X \star Y$  defined as follows. It has  $n$ -dimensional cells

join

$$(X \star Y)_n := \coprod_{[n]=[n_1] \sqcup [n_2]} X_{n_1} \times Y_{n_2},$$

where  $[n_1], [n_2] \in \text{ob } \Delta_+$ , and we declare  $X_{-1} = * = Y_{-1}$  to be a one-point set. The action of simplicial operators is defined in the evident way, using the observation of the previous paragraph: for  $(x, y) \in X_{n_1} \times Y_{n_2} \subseteq (X \star Y)_n$  and  $f: [m] \rightarrow [n]$ , we have  $(x, y)f = (xf_1, yf_2) \in X_{m_1} \times Y_{m_2} \subseteq (X \star Y)_m$ , where  $f = f_1 \sqcup f_2$ ,  $f_j: [m_j] \rightarrow [n_j]$  is the unique decomposition of  $f$  over  $[n] = [n_1] \sqcup [n_2]$ .

**26.10. Exercise.** Check that the above defines a simplicial set.

In particular,

$$\begin{aligned} (X \star Y)_0 &= X_0 \amalg Y_0, \\ (X \star Y)_1 &= X_1 \amalg X_0 \times Y_0 \amalg Y_1, \\ (X \star Y)_2 &= X_2 \amalg X_1 \times Y_0 \amalg X_0 \times Y_1 \amalg Y_2, \end{aligned}$$

and so on.

Note that there are evident maps  $X \rightarrow X \star Y \leftarrow Y$ , which give isomorphisms from  $X$  and  $Y$  to subcomplexes of  $X \star Y$ , and these subcomplexes are disjoint from each other.

There are isomorphisms

$$(X \star Y) \star Z \xrightarrow{\sim} X \star (Y \star Z),$$

natural in  $X, Y, Z$ : on either side, the set of  $n$ -cells can be described as  $\coprod_{[n]=[n_1] \sqcup [n_2] \sqcup [n_3]} X_{n_1} \times Y_{n_2} \times Z_{n_3}$ . Together with the evident isomorphisms  $\emptyset \star X \approx X \approx X \star \emptyset$ , the join gives a monoidal structure on  $s\text{Set}$  with unit object  $\Delta^{-1} := \emptyset$ . Note that  $\star$  is *not* symmetric monoidal, though it is true that  $(Y \star X)^{\text{op}} \approx X^{\text{op}} \star Y^{\text{op}}$ . (*Exercise:* verify this.)

**26.11. Joins of simplices.** We have the (unique) isomorphism

$$\Delta^p \star \Delta^q \approx \Delta^{p+1+q}.$$

Furthermore, if  $f: [p'] \rightarrow [p]$  and  $g: [q'] \rightarrow [q]$  are simplicial operators, then the induced map  $f \star g: \Delta^{p'} \star \Delta^{q'} \rightarrow \Delta^p \star \Delta^q$  between joins of simplices is uniquely isomorphic to  $(f \sqcup g): \Delta^{p'+1+q'} \rightarrow \Delta^{p+1+q}$ .

In particular, if  $S \subseteq [p]$  and  $T \subseteq [q]$  are subsets, giving rise to subcomplexes  $\Delta^S \subseteq \Delta^p$  and  $\Delta^T \subseteq \Delta^q$ , then the evident map  $\Delta^S \star \Delta^T \rightarrow \Delta^p \star \Delta^q \approx \Delta^{p+1+q}$  realizes the inclusion of the subcomplex  $\Delta^{S \sqcup T} \subseteq \Delta^{p+1+q}$  associated to the subset  $S \sqcup T \subseteq [p] \sqcup [q] = [p+1+q]$ . This makes it relatively straightforward to describe the join of subcomplexes of standard simplices.

**26.12. Left and right cones of simplicial sets.** An important example of joins of simplicial sets are the cones. The **left cone** and **right cone** of a simplicial set  $X$  are

$$X^{\triangleleft} := \Delta^0 \star X, \quad X^{\triangleright} := X \star \Delta^0.$$

left cone  
right cone

Note that outer horns are examples of cones:

$$(\partial \Delta^n)^{\triangleleft} = \Delta^0 \star \partial \Delta^n \approx \Lambda_0^{n+1}, \quad (\partial \Delta^n)^{\triangleright} = \partial \Delta^n \star \Delta^0 \approx \Lambda_{n+1}^{n+1}.$$

It is straightforward to show that the nerve takes joins of categories to joins of simplicial sets:  $N(A \star B) \approx N(A) \star N(B)$ , and thus  $N(A^{\triangleleft}) \approx (NA)^{\triangleleft}$  and  $N(A^{\triangleright}) \approx (NA)^{\triangleright}$ . (*Exercise:* prove this.)

**26.13. The join of quasicategories is a quasicategory.** Here is a handy rule for constructing maps into a join (compare (26.4)). Note that every join admits a canonical map  $\pi: X \star Y \rightarrow \Delta^0 \star \Delta^0 \approx \Delta^1$ , namely the join applied to the projections  $X \rightarrow \Delta^0$  and  $Y \rightarrow \Delta^0$ .

**26.14. Lemma** ([Joy08a, Prop. 3.5], compare (26.4)). *Maps  $f: K \rightarrow X \star Y$  are in bijective correspondence with the set of triples*

$$(\pi: K \rightarrow \Delta^1, \quad f_{\{0\}}: K^{\{0\}} \rightarrow X, \quad f_{\{1\}}: K^{\{1\}} \rightarrow Y),$$

where  $K^{\{j\}} := \pi^{-1}(\{j\}) \subseteq K$ , the pullback of  $\{j\} \rightarrow \Delta^1$  along  $\pi$ .

*Proof.* This is a straightforward exercise. In one direction, the correspondence sends  $f$  to  $(\bar{\pi}f, f|_{K^{\{0\}}}, f|_{K^{\{1\}}})$ , where  $\bar{\pi}: X \star Y \rightarrow \Delta^0 \star \Delta^0 = \Delta^1$ .  $\square$

**26.15. Proposition.** *If  $C$  and  $D$  are quasicategories, so is  $C \star D$ .*

*Proof.* Use the previous lemma (26.14), together with the observations (which we leave as an exercise) that for any map  $\pi: \Lambda_j^n \rightarrow \Delta^1$  from an *inner* horn, the preimages  $\pi^{-1}(\{0\})$  and  $\pi^{-1}(\{1\})$  are either inner horns, standard simplices, or are empty, and for any map  $\pi: \Delta^n \rightarrow \Delta^1$  from a standard simplex, the preimages are either a standard simplex or empty.  $\square$

**26.16. Exercise.** Let  $f: [m] \rightarrow [n]$  be any simplicial operator. Show that the induced map  $f: \Delta^m \rightarrow \Delta^n$  on standard simplices is uniquely isomorphic to a join of maps  $f_0 \star f_1 \star \cdots \star f_n$ , with  $f_j: \Delta^{m_j} \rightarrow \Delta^0$ , where each  $m_j \geq -1$ .

**26.17. Exercise.** Show that (26.14) implies the following: there is an adjoint pair of functors

$$i^*: \mathbf{sSet}_{/\Delta^1} \rightleftarrows \mathbf{sSet}_{/\partial \Delta^1} : i_*$$

where the left adjoint  $i^*$  is the functor defined by pullback along the inclusion  $\partial \Delta^1 \hookrightarrow \Delta^1$ , and the right adjoint  $i_*$  sends  $p: K \rightarrow \Delta^1$  to  $(K^{\{0\}} \star K^{\{1\}} \rightarrow \Delta^1)$ , where  $K^{\{j\}} := p^{-1}(j) \subseteq K$ . This gives another characterization of join, as “pushforward along  $i$ ”.

## 27. SLICES

**27.1. Slices of categories.** Given an ordinary category  $C$ , and an object  $x \in \text{ob } C$ , we may form the **slice categories**  $C_{x/}$  and  $C_{/x}$ , (also called *undercategory* and *overcategory*, or *slice-over category* and *slice-under category*).

For instance, the slice-over category  $C_{/x}$  is the category whose *objects* are maps  $f: c \rightarrow x$  with target  $x$ , and whose *morphisms*  $(f: c \rightarrow x) \rightarrow (f': c' \rightarrow x)$  are maps  $g: c \rightarrow c'$  such that  $f'g = f$ .

This can be reformulated in terms of joins. Let “ $T$ ” denote the terminal category (isomorphic to  $[0]$ ). Note that  $\text{ob } C_{/x}$  corresponds to the set of functors  $f: [0] \star T \rightarrow C$  such that  $f|T = x$ , and  $\text{mor } C_{/x}$  corresponds to the set of functors  $g: [1] \star T \rightarrow C$  such that  $g|T = x$ .

More generally, given a functor  $p: A \rightarrow C$  of categories, we obtain slice categories  $C_{p/}$  and  $C_{/p}$  defined as follows. The category  $C_{/p}$  has

- **objects:** functors  $f: [0] \star A \rightarrow C$  such that  $f|A = p$ ,
- **morphisms**  $f \rightarrow f'$ : functors  $g: [1] \star A \rightarrow C$  such that  $g|A = p$ .

Likewise, the category  $C_{p/}$  has

- **objects:** functors  $f: A \star [0] \rightarrow C$  such that  $f|A = p$ ,
- **morphisms**  $f \rightarrow f'$ : functors  $g: A \star [1] \rightarrow C$  such that  $g|A = p$ .

**27.2. Exercise.** Describe composition of morphisms in  $C_{/p}$  and  $C_{p/}$ .

**27.3. Exercise.** Show that  $(C_{p/})^{\text{op}} \approx (C^{\text{op}})_{/p^{\text{op}}}$  (isomorphism of categories).

**27.4. Exercise.** Fix a functor  $p: A \rightarrow C$ , and let  $B$  be a category. Construct bijections

$$\{\text{functors } f: B \rightarrow C_{p/}\} \leftrightarrow \{\text{functors } g: B \star A \rightarrow C \text{ s.t. } g|A = p\}$$

and

$$\{\text{functors } f: B \rightarrow C_{p/}\} \leftrightarrow \{\text{functors } g: A \star B \rightarrow C \text{ s.t. } g|A = p\}.$$

**27.5. Remark.** The notions of limits and colimits can be formulated very compactly in terms of the general notion of slices. Thus, given a functor  $p: A \rightarrow C$ , a *colimit* of  $p$  is the same data as an initial object of  $C_{p/}$ , while a *limit* of  $p$  is the same data as a terminal object of  $C_{/p}$ . (*Exercise:* prove this; we will directly generalize this formulation to define limits and colimits for quasicategories. Compare (26.7).)

**27.6. Joins and colimits of simplicial sets.** The join functor  $\star: s\text{Set} \times s\text{Set} \rightarrow s\text{Set}$  is in some ways analogous to the product functor  $\times$ , e.g., it is a monoidal functor.

The product operation  $(-) \times (-)$  on simplicial sets commutes with colimits in each input, and the functors  $X \times -$  and  $- \times X$  admit right adjoints (in both cases, the right adjoint is  $\text{Fun}(X, -)$ ). The join functor does not commute with colimits in each variable, but *almost* does so; the only obstruction is the value on the initial object

More precisely, the functors  $X \star -$  and  $- \star X: s\text{Set} \rightarrow s\text{Set}$  do not preserve the initial object, since  $X \star \emptyset \approx X \approx \emptyset \star X$ . However, (the identity map of)  $X$  is tautologically the initial object of  $s\text{Set}_{X/}$ , the slice category of simplicial sets under  $X$ .

**27.7. Proposition.** *For every simplicial set  $X$ , the induced functors*

$$X \star -, \quad - \star X: s\text{Set} \rightarrow s\text{Set}_{X/}$$

*preserve colimits.*

*Proof.* This follows from the degreewise formula for the join, which has the form:

$$(X \star Y)_n = X_n \amalg (X_{n-1} \times Y_0) \amalg \cdots \amalg (X_0 \times Y_{n-1}) \amalg Y_n = X_n \amalg (\text{terms which are “linear” in } Y).$$

That is, for each  $n \geq 0$  the functor  $Y \mapsto (X \star Y)_n: s\text{Set} \rightarrow \text{Set}_{X_n/}$  is seen to be colimit preserving, since each functor  $X_k \times (-): \text{Set} \rightarrow \text{Set}$  is colimit preserving.  $\square$



27.8. *Exercise* (Trivial, but important). Show that the functors  $X \star -$  and  $- \star X: s\text{Set} \rightarrow s\text{Set}$  preserve *pushouts*.

27.9. **Slices of simplicial sets.** We have seen that the functors

$$S \star -: s\text{Set} \rightarrow s\text{Set}_{S/} \quad \text{and} \quad - \star T: s\text{Set} \rightarrow s\text{Set}_{T/}$$

preserve colimits, and therefore we predict that they admit right adjoints. These exist, and are called **slice** functors, denoted

$$(p: S \rightarrow X) \mapsto X_{p/}: s\text{Set}_{S/} \rightarrow s\text{Set}$$

and

$$(q: T \rightarrow X) \mapsto X_{/q}: s\text{Set}_{T/} \rightarrow s\text{Set}.$$

I will sometimes distinguish these as **slice-under** and **slice-over**, respectively. Explicitly, there are bijective correspondences

$$(27.10) \quad \left\{ \begin{array}{ccc} S & \xrightarrow{p} & X \\ \downarrow & \searrow & \uparrow \\ S \star K & & \end{array} \right\} \Longleftrightarrow \{K \dashrightarrow X_{p/}\}, \quad \left\{ \begin{array}{ccc} T & \xrightarrow{q} & X \\ \downarrow & \searrow & \uparrow \\ K \star T & & \end{array} \right\} \Longleftrightarrow \{K \dashrightarrow X_{/q}\}.$$

Here we write “ $S \rightarrow S \star K$ ” and “ $T \rightarrow K \star T$ ” for the inclusions  $S \star \emptyset \subseteq S \star K$  and  $\emptyset \star T \subseteq K \star T$ , using the canonical isomorphisms  $S \star \emptyset = S$  and  $\emptyset \star T = T$ .

Taking  $K = \Delta^n$  we obtain the formulas

$$(X_{p/})_n = \text{Hom}_{s\text{Set}_{S/}}(S \star \Delta^n, X), \quad (X_{/q})_n = \text{Hom}_{s\text{Set}_{T/}}(\Delta^n \star T, X),$$

which we regard as the definition of slices. (I.e., these formulas specify the  $n$ -cells of the slices, and naturality in “ $\Delta^n$ ” specifies the action of simplicial operators.)

27.11. *Exercise.* Given this explicit definition of slices in terms of their cells and the action of simplicial operators, verify the bijective correspondences (27.10).

In particular, we note the special cases associated to  $x: \Delta^0 \rightarrow X$ :

$$\text{Hom}_{s\text{Set}}(K, X_{x/}) = \text{Hom}_{s\text{Set}_{\Delta^0/}}(\Delta^0 \star K, X) \approx \text{Hom}_{s\text{Set}_*}((K^\triangleleft, v), (X, x)),$$

$$\text{Hom}_{s\text{Set}}(K, X_{/x}) = \text{Hom}_{s\text{Set}_{\Delta^0/}}(K \star \Delta^0, X) \approx \text{Hom}_{s\text{Set}_*}((K^\triangleright, v), (X, x)).$$

The notation  $(X, x)$  with  $x \in X_0$  represents a **pointed simplicial set**, the category of which is  $s\text{Set}_* := s\text{Set}_{\Delta^0/}$ . We write  $v$  for the cone point of  $K^\triangleleft$  and  $K^\triangleright$ .

The slice construction for simplicial sets agrees with that for categories.

27.12. **Proposition.** *The nerve preserves slices; i.e., if  $p: A \rightarrow C$  is a functor between 1-categories, then  $N(C_{p/}) \approx (NC)_{Np/}$  and  $N(C_{/p}) \approx (NC)_{/Np}$ .*

*Proof.* Left as an exercise. □

27.13. **Slice as a functor.** The function complex construction  $\text{Fun}(-, -)$  is a functor in two variables, contravariant in the first and covariant in the second. The slice constructions also behave something like a functor of two variables, though it is a little more complicated, because the slice constructions also depend on a map between the two objects. A precise statement is that every diagram on the left gives rise to commutative diagrams as on the right.

$$\begin{array}{ccc} S & \xrightarrow{p} & X \\ j \uparrow & & \downarrow f \\ T & \xrightarrow{f p j} & Y \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} X_{/p} & \longrightarrow & Y_{/f p} \\ \downarrow & & \downarrow \\ X_{/p j} & \longrightarrow & Y_{/f p j} \end{array} \quad \begin{array}{ccc} X_{p/} & \longrightarrow & Y_{f p/} \\ \downarrow & & \downarrow \\ X_{p j/} & \longrightarrow & Y_{f p j/} \end{array}$$

There seems to be no decent notation for the maps in the right-hand squares. The whole business of joins and slices can get pretty confusing because of this.

27.14. *Remark.* A very precise formulation is that each kind of slice defines a functor  $\text{Tw}(s\text{Set}) \rightarrow s\text{Set}$  from the *twisted arrow category* of simplicial sets, whose objects are maps  $p$  of simplicial sets, and whose morphisms are pairs  $(j, f): p \rightarrow fpj$ , where  $j$  and  $f$  are themselves maps of simplicial sets.

Let's spell this out in terms of the correspondence between “maps into slices” and “maps from joins”. Given  $T \xrightarrow{j} S \xrightarrow{f} X \xrightarrow{p} Y$ , consider “restriction map”  $X_{f/} \rightarrow Y_{pfj/}$ . The composite of a map  $u: K \rightarrow X_{f/}$  with this restriction map is described in terms of the bijection of (27.10) as follows. The map  $u$  corresponds to a dotted arrow in

$$\begin{array}{ccccc} T & \xrightarrow{j} & S & \xrightarrow{f} & X \xrightarrow{p} Y \\ \downarrow & & \downarrow & \nearrow \tilde{u} & \\ T \star K & \xrightarrow{j \star K} & S \star K & & \end{array}$$

The composite  $K \xrightarrow{u} X_{f/} \rightarrow Y_{pfj/}$  corresponds to  $p\tilde{u}(j \star K)$ .

A particular special case which we will see a lot of are the “restriction” maps

$$X_{/f} \rightarrow X \quad \text{and} \quad X_{f/} \rightarrow X$$

induced by sequence  $\emptyset \rightarrow S \xrightarrow{f} X \rightarrow \Delta^0$ , using that  $X_{/\emptyset} = X = X_{\emptyset/}$ . For instance,  $X_{/f} \rightarrow X$  sends an  $n$ -cell  $x \in (X_{/f})_n$  corresponding to  $\tilde{x}: \Delta^n \star S \rightarrow X$  extending  $f$  to the  $n$ -cell of  $X$  represented by the map  $\tilde{x}|(\Delta^n \star \emptyset)$  defined as the composite

$$\Delta^n = \Delta^n \star \emptyset \rightarrow \Delta^n \star S \xrightarrow{\tilde{x}} X.$$

Another special case of interest are the “projection” functors

$$X_{/f} \rightarrow Y_{/pf} \quad \text{and} \quad X_{f/} \rightarrow Y_{pf/}$$

induced by the sequence  $\emptyset \rightarrow S \xrightarrow{f} X \xrightarrow{p} Y$ . For instance,  $X_{/f} \rightarrow Y_{/fp}$  sends an  $n$ -cell  $x \in (X_{/f})_n$  corresponding to  $\tilde{x}: \Delta^n \star S \rightarrow X$  extending  $f$  to the  $n$ -cell of  $Y_{pf/}$  represented by  $p\tilde{x}: \Delta^n \star S \rightarrow Y$ .

27.15. *Exercise.* Let  $f: S \rightarrow X$  and  $g: T \rightarrow X$  be maps of simplicial sets. Describe and prove bijections between the following sets of solutions to lifting problems:

$$\left\{ \begin{array}{ccc} & X_{f/} & \\ \nearrow & \downarrow & \\ T & \xrightarrow{g} & X \end{array} \right\} \iff \left\{ \begin{array}{ccc} S \amalg T & \xrightarrow{(f,g)} & X \\ \downarrow & \nearrow & \\ S \star T & & \end{array} \right\} \iff \left\{ \begin{array}{ccc} & X_{/g} & \\ \nearrow & \downarrow & \\ S & \xrightarrow{f} & X \end{array} \right\}$$

Here  $X_{f/} \rightarrow X$  and  $X_{/g} \rightarrow X$  are the evident restriction maps, and  $S \amalg T \rightarrow S \star T$  the tautological inclusion.

## 28. SLICES OF QUASICATEGORIES

In this section we show that, given a quasicategory  $C$  and an object  $x \in C_0$ , both  $C_{/x}$  and  $C_{x/}$  are also quasicategories.

We recall the sets **left horns**

$$\text{LHorn} := \{ \Lambda_k^n \subset \Delta^n \mid 0 \leq k < n, n \geq 1 \} = \text{InnHorn} \cup \{ \Lambda_0^n \subset \Delta^n \mid n \geq 1 \}$$

and the **right horns**

$$\text{RHorn} := \{ \Lambda_k^n \subset \Delta^n \mid 0 < k \leq n, n \geq 1 \} = \text{InnHorn} \cup \{ \Lambda_n^n \subset \Delta^n \mid n \geq 1 \}.$$

The associated weak saturations  $\overline{\text{LHorn}}$  and  $\overline{\text{RHorn}}$  are the **left anodyne** and **right anodyne**

**left horns**

**right horns**

**left anodyne**  
**right anodyne**

maps. The associated right complements

$$\text{LFib} := \text{LHorn}^{\square}, \quad \text{RFib} := \text{RHorn}^{\square}$$

are the **left fibrations** and **right fibrations**. Note that

$$\overline{\text{InnHorn}} \subseteq \overline{\text{LHorn}} \cap \overline{\text{RHorn}} \quad \text{and} \quad \text{LFib} \cup \text{RFib} \subseteq \text{InnFib}.$$

left fibrations  
right fibrations

These classes correspond to each other under the opposite involution  $(-)^{\text{op}}: s\text{Set} \rightarrow s\text{Set}$ ; i.e.,  $\text{LHorn}^{\text{op}} = \text{RHorn}$ ,  $\text{LFib}^{\text{op}} = \text{RFib}$ .

**28.1. Proposition.** *Let  $C$  be a quasicategory and  $x \in C_0$ . The evident maps  $C_{x/} \rightarrow C$  and  $C_{/x} \rightarrow C$  which “forget  $x$ ” (i.e., induced by the sequence  $\emptyset \rightarrow \{x\} \rightarrow C$ ) are left fibration and right fibration respectively. In particular,  $C_{x/}$  and  $C_{/x}$  are also quasicategories.*

*Proof.* I claim that  $\pi: C_{/x} \rightarrow C$  is a right fibration. Explicitly, this map sends the  $n$ -cell  $a: \Delta^n \rightarrow C_{/x}$ , which corresponds to  $\tilde{a}: \Delta^n \star \Delta^0 \rightarrow C$  such that  $\tilde{a}|(\emptyset \star \Delta^0) = x$ , to the  $n$ -cell represented by  $\tilde{a}|(\Delta^n \star \emptyset) \rightarrow C$ . Using the join/slice adjunction, there is a bijective correspondence between lifting problems

$$\begin{array}{ccc} \Lambda_j^n & \xrightarrow{f} & C_{/x} \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \xrightarrow{g} & C \end{array} \iff \begin{array}{ccc} \emptyset \star \Delta^0 & \xrightarrow{\quad} & (\Lambda_j^n \star \Delta^0) \cup_{\Lambda_j^n \star \emptyset} (\Delta^n \star \emptyset) \xrightarrow{(\tilde{f}, g)} C \\ & \downarrow & \nearrow \\ & \Delta^n \star \Delta^0 & \end{array}$$

Note that there is a unique isomorphism  $\Delta^n \star \Delta^0 \approx \Delta^{n+1}$ . For any subset  $S \subset [n]$ , the above isomorphism identifies the subcomplex  $\Delta^S \star \Delta^0 \subset \Delta^n \star \Delta^0$  with  $\Delta^{S \cup \{n+1\}} \subset \Delta^{n+1}$ , while  $\Delta^S \star \emptyset \subset \Delta^n \star \Delta^0$  is identified with  $\Delta^S \subseteq \Delta^{n+1}$ . Since  $\Lambda_j^n = \bigcup_{k \in [n] \setminus j} \Delta^{[n] \setminus k}$ , we see that

- (1) the subcomplex  $(\Lambda_j^n \star \Delta^0) \cup_{\Lambda_j^n \star \emptyset} (\Delta^n \star \emptyset)$  of  $\Delta^n \star \Delta^0$  is the horn  $\Lambda_j^{n+1} \subset \Delta^{n+1}$ , and
- (2) the subcomplex  $\emptyset \star \Delta^0$  of  $\Delta^n \star \Delta^0$  is the vertex  $\{n+1\}$ .

Thus, the right hand diagram above is isomorphic to

$$\begin{array}{ccc} \{n+1\} & \xrightarrow{\quad} & \Lambda_j^{n+1} \xrightarrow{\quad} C \\ & \downarrow & \nearrow \\ & \Delta^{n+1} & \end{array}$$

If  $C$  is a quasicategory, then an extension exists for  $0 < j \leq n$ .

Since right fibrations are inner fibrations, the composite  $C_{/x} \rightarrow C \rightarrow *$  is an inner fibration, and thus  $C_{/x}$  is a quasicategory.

The case of  $C_{x/} \rightarrow C$  is similar, using the correspondence

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & C_{x/} \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \longrightarrow & C \end{array} \iff \begin{array}{ccc} \{0\} & \xrightarrow{\quad} & \Lambda_{j+1}^{n+1} \xrightarrow{\quad} C \\ & \downarrow & \nearrow \\ & \Delta^{n+1} & \end{array}$$

□

**28.2. Exercise.** Let  $p: C \rightarrow D$  be an inner fibration between simplicial sets. Show that  $p$  is a left fibration if and only if the induced maps  $C_{x/} \rightarrow D_{px/}$  on slices are trivial fibrations for all  $x \in C_0$ , and that  $p$  is a right fibration if and only if the induced maps  $C_{/x} \rightarrow D_{/px}$  on slices are trivial fibrations for all  $x \in C_0$ .

## 29. INITIAL AND TERMINAL OBJECTS

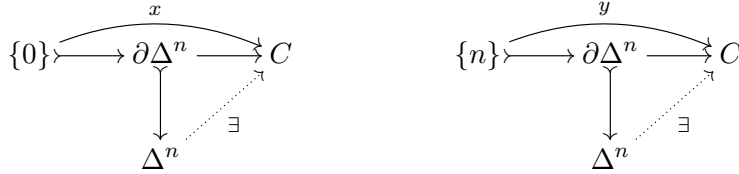
We give the definition of initial and terminal object in a quasicategory, and we reformulate it in terms of slices.

**29.1. Initial and terminal objects.** An **initial object**<sup>22</sup> of a quasicategory  $C$  is an  $x \in C_0$  such that every  $f: \partial\Delta^n \rightarrow C$  (for all  $n \geq 1$ ) such that  $f|_{\{0\}} = x$ , there exists an extension  $f': \Delta^n \rightarrow C$ .

initial object

A **terminal object** of  $C$  is an initial object of  $C^{\text{op}}$ . That is, a  $y \in C_0$  such that every  $f: \partial\Delta^n \rightarrow C$  with  $f|_{\{n\}} = y$  extends to  $\Delta^n$ .

terminal object



Let's spell out the first parts of the definition of initial object applied to  $x \in C_0$ :

- The condition for  $n = 1$  says that for every object  $c$  in  $C$  there exists  $f: x \rightarrow c$ ,
- The condition for  $n = 2$  says that for every triple of maps  $f: x \rightarrow c$ ,  $g: c \rightarrow c'$ , and  $h: x \rightarrow c'$ , we must have  $[h] = [g][f]$ . In particular (taking  $f = 1_x$ ), we see there is at most one homotopy class of maps from  $x$  to any object.

If  $C$  is the nerve of an ordinary category, then  $\text{Hom}(\Delta^n, C) \xrightarrow{\sim} \text{Hom}(\partial\Delta^n, C)$  for all  $n \geq 3$ . Thus, for ordinary categories, this definition coincides with the usual notion of initial object.

For general quasicategories, we see that an initial object  $x \in C_0$  necessarily satisfies  $\text{Hom}_{hC}(x, y) \approx *$  for all  $y \in C_0$ , so that  $x$  represents an initial object in the homotopy category  $hC$ , but this is not sufficient to be initial in  $C$ : there are also an infinite sequence of “higher” conditions that an initial object of a quasicategory must satisfy.

We will now reformulate these notions using slice categories.

**29.2. Reformulation of initial/terminal via slices.** We can restate the definition of initial/terminal object using the “forgetful” functor of the relevant slice.

**29.3. Proposition.** *If  $C$  is a quasicategory, then  $x \in C_0$  is initial if and only if  $C_{x/} \rightarrow C$  is a trivial fibration, and terminal if and only if  $C_{/x} \rightarrow C$  is a trivial fibration.*

*Proof.* This is an application of the join/slice adjunction. Applied to  $\partial\Delta^n \subset \Delta^n$  with  $n \geq 0$  and  $C_{x/} \rightarrow C$ , this has the form

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & C_{x/} \\ \downarrow & \nearrow & \downarrow \pi \\ \Delta^n & \xrightarrow{g} & C \end{array} \iff \begin{array}{ccc} \Delta^0 \star \emptyset & \xrightarrow{\quad} & (\Delta^0 \star \partial\Delta^n) \cup_{\emptyset \star \partial\Delta^n} (\emptyset \star \Delta^n) \xrightarrow{(\tilde{f}, g)} C \\ & & \downarrow \\ & & \Delta^0 \star \Delta^n \end{array}$$

The right-hand diagram is isomorphic to

$$\begin{array}{ccc} \{0\} & \xrightarrow{\quad} & \partial\Delta^{n+1} \xrightarrow{\quad} C \\ & & \downarrow \\ & & \Delta^{n+1} \end{array}$$

<sup>22</sup>We use Joyal's definition of initial and terminal object [Joy02, §4] here. Lurie's definition [Lur09, 1.2.12.1] is different, but is equivalent to what we use, by [Lur09, 1.2.12.5] and (29.3).

Thus  $C_{x/} \rightarrow C$  is in  $\text{TrivFib} = \text{Cell}^\square$  if and only if  $x$  is an initial object of  $C$ , as desired.  $\square$

**29.4. Remark.** This implies that if  $x$  is initial, then  $C_{x/} \rightarrow C$  is a categorical equivalence. Later (38.8) we'll be able to show the converse: if  $C_{x/} \rightarrow C$  is a categorical equivalence, then  $x$  is initial.

**29.5. Uniqueness of initial and terminal objects.** A crucial fact about initial and terminal objects in an ordinary category is that they are *unique up to unique isomorphism*. One way to formulate this is as follows: given a category  $C$ , let  $C^{\text{init}} \subseteq C$  be the full subcategory spanned by the initial objects. Then one of two cases applies: either there are no initial objects, so  $C^{\text{init}}$  is empty, or there is at least one initial object, and  $C^{\text{init}}$  is equivalent to the terminal category  $[0]$ .

This leads to an analogous formulation for quasicategories.

**29.6. Proposition.** *Let  $C$  be a quasicategory. Let  $C^{\text{init}}$  and  $C^{\text{term}}$  denote respectively the full subcategories spanned by initial objects and terminal objects. Then (i) either  $C^{\text{init}}$  is empty or is categorically equivalent to the terminal quasicategory  $\Delta^0$ , and (ii) either  $C^{\text{term}}$  is empty or is categorically equivalent to the terminal quasicategory  $\Delta^0$ .*

*Proof.* Since  $C^{\text{term}} = ((C^{\text{op}})^{\text{init}})^{\text{op}}$ , we just need to consider the case of initial objects. By definition of initial object, any  $f: \partial\Delta^n \rightarrow C^{\text{init}}$  with  $n \geq 1$  can be extended to  $g: \Delta^n \rightarrow C$ , and the image of  $g$  must lie in the full subcategory  $C^{\text{init}}$  since all of its vertices do. If  $C^{\text{init}} \neq \emptyset$ , then this extension condition also holds for  $n = 0$ , whence  $C^{\text{init}} \rightarrow \Delta^0$  is a trivial fibration, and thus  $C^{\text{init}}$  is categorically equivalent to  $\Delta^0$  by (23.1).  $\square$

There are some seemingly obvious facts about initial and terminal objects that we can't prove just yet. For instance:

- (1) Given a quasicategory  $C$  with object  $x$ , an object  $\tilde{f} \in (C_{x/})_0$  of the slice under  $x$  is initial if and only if the corresponding morphism  $f: x \rightarrow c$  in  $C$  is an isomorphism. Likewise, an object  $\tilde{g} \in (C_{/x})_0$  of the slice over  $x$  is terminal if and only if the corresponding morphism  $g: c \rightarrow x$  in  $C$  is an isomorphism.
- (2) In a quasicategory, every object which is isomorphic to an initial object is initial, and any object isomorphic to a terminal object is terminal.

Proofs will be given as (33.8) and (33.10).

**29.7. Initial and terminal objects in functor categories.** Here is a sample of a property of initial/terminal objects that we can now prove. A functor between ordinary categories whose values are all initial (or terminal) objects is itself initial (or terminal) as an object of the functor category. The same holds with categories replaced by quasicategories.

**29.8. Proposition.** *Consider a map  $f: X \rightarrow C$  from a simplicial set to a quasicategory. Suppose that for every vertex  $x \in X_0$  the object  $f(x) \in C_0$  is initial (resp. terminal) in  $C$ . Then the functor  $f$  is initial (resp. terminal) viewed as an object of  $\text{Fun}(X, C)$ .*

As a consequence, if  $C$  has an initial (or terminal) object  $c_0$ , then the “constant” map (defined as the composite  $X \rightarrow \{c_0\} \rightarrow C$ ) is an initial (or terminal) object of  $\text{Fun}(X, C)$ .

**29.9. Remark.** In other words, there is an inclusion  $\text{Fun}(X, C^{\text{init}}) \subseteq \text{Fun}(X, C)^{\text{init}}$  of the full subcategories of “objectwise initial functors” and “initial functors” in  $\text{Fun}(X, C)$ . Once we know that any object isomorphic to an initial object is initial (33.10), it will follow that when  $C^{\text{init}}$  is non-empty then  $\text{Fun}(X, C^{\text{init}}) = \text{Fun}(X, C)^{\text{init}}$ . To see this, pick an initial object  $c_0 \in C^{\text{init}}$  and let  $f_0: X \rightarrow C$  be the constant map with image  $\{c_0\} \subseteq C$ . Since any two initial objects are isomorphic, every  $f \in \text{Fun}(X, C)^{\text{init}}$  is naturally isomorphic to  $f_0$ , and therefore  $f(x)$  is isomorphic to  $f_0(x) = c_0$  for every  $x \in X_0$ . By (33.10),  $f(x)$  must be initial in  $C$ , so  $f \in \text{Fun}(X, C^{\text{init}})_0$ .

On the other hand, it is possible for  $\text{Fun}(X, C)^{\text{init}}$  to be non-empty when  $C^{\text{init}}$  is empty. (*Exercise:* give an example. *Hint:* think small.)

*Proof.* (29.8) Assume  $f(x) \in C_0$  is initial in  $C$  for all  $x \in X_0$ . Suppose given  $g: \partial\Delta^n \rightarrow \text{Fun}(X, C)$  with  $n \geq 1$  and  $g|_{\{0\}} = f$ . We want to show that there exists an extension  $g': \Delta^n \rightarrow \text{Fun}(X, C)$  of  $g$ . We convert this to the adjoint lifting problem:

$$\begin{array}{ccc} & & f \\ & \nearrow & \\ \{0\} \times X & \longrightarrow & \partial\Delta^n \times X \xrightarrow{\tilde{g}} C \\ & \downarrow & \nearrow \tilde{g}' \\ & \Delta^n \times X & \end{array}$$

The strategy is to construct the extension by inductively constructing extensions  $\tilde{g}_k: F_k \rightarrow C$  where  $F_k = (\partial\Delta^n \times X) \cup \text{Sk}_k(\Delta^n \times X)$ ,  $k \geq 0$  is the skeletal filtration (18.3) of the inclusion  $\partial\Delta^n \times X \rightarrow \Delta^n \times X$ . That is, we need to inductively construct lifts  $\tilde{g}_k$  in

$$\begin{array}{ccccc} \coprod_{h \in F_k^{\text{nd}} \setminus F_{k-1}^{\text{nd}}} \partial\Delta^k & \xrightarrow{(h|\partial\Delta^k)} & F_{k-1} & \xrightarrow{\tilde{g}_{k-1}} & C \\ \downarrow & & \downarrow & \nearrow \tilde{g}_k & \\ \coprod_{h \in F_k^{\text{nd}} \setminus F_{k-1}^{\text{nd}}} \Delta^k & \xrightarrow{(h)} & F_k & & \end{array}$$

for all  $k \geq 0$ . For  $k = 0$  we have  $F_{-1} = F_0$ , since  $n \geq 1$  so  $(\partial\Delta^n \times X)_0 = (\Delta^n \times X)_0$ .

For  $k \geq 1$ , note that a  $k$ -dimensional cell  $h = (a, b): \Delta^k \rightarrow \Delta^n \times X$  is not contained in the subcomplex  $\partial\Delta^n \times X$  if and only if  $a \in (\Delta^n)_k \setminus (\partial\Delta^n)_k$ , i.e., if the corresponding simplicial operator  $a: [k] \rightarrow [n]$  is surjective. Therefore such  $a: \Delta^k \rightarrow \Delta^n$  sends the vertex  $0 \in (\Delta^k)_0$  to  $0 \in (\Delta^n)_0$ . Therefore, each composite

$$\partial\Delta^k \xrightarrow{h|\partial\Delta^k} F_{k-1} = (\partial\Delta^n \times X) \cup \text{Sk}_{k-1}(\Delta^n \times X) \xrightarrow{\tilde{g}_{k-1}} C$$

sends the vertex 0 to  $\tilde{g}_{k-1}(0, b(0)) = \tilde{g}(0, b(0)) = f(0, b(0))$ , which by hypothesis is an initial object of  $C$ . Therefore an extension of  $(\tilde{g}_{k-1}h)|_{\partial\Delta^k}$  along  $\partial\Delta^k \subset \Delta^k$  exists as desired.  $\square$

### 30. JOINS AND SLICES IN LIFTING PROBLEMS

Recall that for an object  $x$  in a quasicategory  $C$ , the slice objects  $C_{x/}$  and  $C_{/x}$  are also quasicategories. It turns out that the conclusion remains true for more general kinds of slices of quasicategories.

**30.1. Proposition.** *Let  $f: S \rightarrow C$  be a map of simplicial sets, and suppose  $C$  is a quasicategory. Then both  $C_{f/}$  and  $C_{/f}$  are quasicategories.*

The proof is just like that of (28.1): we will show below (30.15) that  $C_{f/} \rightarrow C$  is a left fibration and  $C_{/f} \rightarrow C$  is a right fibration.

To set this up, we need a little technology about how joins interact with lifting problems.

**30.2. Pushout-joins.** We define an analogue of the pushout-product for the the join. Given maps  $i: A \rightarrow B$  and  $j: K \rightarrow L$  of simplicial sets, the **pushout-join** (or **box-join**)  $i \boxtimes j$  is the map

$$i \boxtimes j: (A \star L) \amalg_{A \star K} (B \star K) \xrightarrow{(i \star L, B \star j)} B \star L.$$

pushout-join  
box-join

**30.3. Warning.** Unlike the pushout-product, the pushout-join is not symmetric, since the join is not symmetric:  $i \boxtimes j \not\approx j \boxtimes i$ .

30.4. *Example.* We have already observed examples of pushout-joins in the proof of (28.1), namely

$$(\Lambda_j^n \subset \Delta^n) \boxtimes (\emptyset \subset \Delta^0) \approx (\Lambda_j^{n+1} \subset \Delta^{n+1}), \quad (\emptyset \subset \Delta^0) \boxtimes (\Lambda_j^n \subset \Delta^n) \approx (\Lambda_{1+j}^{1+n} \subset \Delta^{1+n}),$$

and also

$$(\emptyset \subset \Delta^0) \boxtimes (\partial \Delta^n \subset \Delta^n) \approx (\partial \Delta^{1+n} \subset \Delta^{1+n}), \quad (\partial \Delta^n \subset \Delta^n) \boxtimes (\emptyset \subset \Delta^0) \approx (\partial \Delta^{n+1} \subset \Delta^{n+1})$$

in the proof of (29.3). These generalize to arbitrary horns and cells. The pushout-join of a horn with a cell is always a horn:

$$\begin{aligned} (\Lambda_j^n \subset \Delta^n) \boxtimes (\partial \Delta^k \subset \Delta^k) &\approx (\Lambda_j^{n+1+k} \subset \Delta^{n+1+k}), \\ (\partial \Delta^k \subset \Delta^k) \boxtimes (\Lambda_j^n \subset \Delta^n) &\approx (\Lambda_{k+1+j}^{k+1+n} \subset \Delta^{k+1+n}). \end{aligned}$$

Also, the pushout-join of a cell with a cell is always a cell:

$$(\partial \Delta^n \subset \Delta^n) \boxtimes (\partial \Delta^k \subset \Delta^k) \approx (\partial \Delta^{n+1+k} \subset \Delta^{n+1+k}).$$

We leave proofs as an exercise for the reader.

30.5. *Exercise.* Prove the isomorphisms asserted in (30.4). (*Hint:* use (26.11).)

30.6. *Remark.* Both pushout-product and pushout-join are special cases of a general construction: given any functor  $F: s\text{Set} \times s\text{Set} \rightarrow s\text{Set}$  of two variables, you get a corresponding “pushout- $F$ ” functor:  $F_{\square}: \text{Fun}([1], s\text{Set}) \times \text{Fun}([1], s\text{Set}) \rightarrow \text{Fun}([1], s\text{Set})$ .

30.7. **Pullback-slices.** Just as the pushout-product is associated to the pullback-hom, so the pushout-join is associated to two kinds of **pullback-slices**. Given a sequence of maps  $T \xrightarrow{j} S \xrightarrow{p}$  pullback-slices  
 $X \xrightarrow{f} Y$ , we define the map

$$f^{\boxtimes pj}: X_{/p} \rightarrow X_{/pj} \times_{Y_{/fpj}} Y_{/fp},$$

where the maps defining the pullback and the components of  $f^{\boxtimes pj}$  are the evident maps induced from the sequence, as described in (27.13). In a similar way, we define the map

$$f^{j\boxtimes p}: X_{p/} \rightarrow X_{pj/} \times_{Y_{fpj/}} Y_{fp/}.$$

30.8. *Remark.* When  $Y = *$ , these pullback-slice maps are just the restriction maps  $X_{/p} \rightarrow X_{/pj}$  and  $X_{p/} \rightarrow X_{pj/}$ . When  $T = \emptyset$ , these pullback-slice maps have the form  $X_{/p} \rightarrow X \times_Y Y_{/fp}$  and  $X_{p/} \rightarrow X \times_Y Y_{fp/}$ . When both  $Y = *$  and  $T = \emptyset$ , we get  $X_{/p} \rightarrow X$  and  $X_{p/} \rightarrow X$ .

30.9. *Remark.* Both pullback-hom and pullback-slices are special cases of a general construction: given any functor  $F: \text{Tw}(s\text{Set}) \rightarrow s\text{Set}$  from the twisted arrow category (27.14), you get a corresponding “pullback- $F$ ” functor  $F^{\square}: \text{Tw}(s\text{Set}) \rightarrow s\text{Set}$ . In the case of pullback-hom, the  $F$  in question is a composite functor  $\text{Tw}(s\text{Set}) \rightarrow s\text{Set}^{\text{op}} \times s\text{Set} \xrightarrow{\text{Fun}} s\text{Set}$ .

30.10. **Joins, slices, and lifting problems.** The pushout-join and pullback-slice interact with lifting problems in much the same way that pushout-product and pullback-hom do.

30.11. **Proposition.** *Given  $i: A \rightarrow B$ ,  $j: K \rightarrow L$ , and  $h: X \rightarrow Y$ , the following are equivalent.*

- (1)  $(i \boxtimes j) \square h$ .
- (2)  $i \square (h^{\boxtimes qj})$  for all  $q: L \rightarrow X$ .
- (3)  $j \square (h^{i\boxtimes p})$  for all  $p: B \rightarrow X$ .

*Proof.* A straightforward exercise. The equivalence of (1) and (2) looks like:

$$\begin{array}{ccc} \begin{array}{ccc} A & \xrightarrow{\quad} & X/q \\ i \downarrow & \nearrow & \downarrow h^{\boxtimes qj} \\ B & \xrightarrow{\quad} & X_{/qj} \times_{Y_{/hqj}} Y_{/hq} \end{array} & \Longleftrightarrow & \begin{array}{ccc} \emptyset \star L & \xrightarrow{\quad q \quad} & X \\ \downarrow i \boxtimes j & \nearrow & \downarrow h \\ B \star L & \xrightarrow{\quad} & Y \end{array} \end{array}$$

□

Now we can set up “join/slice analogues” of the “enriched lifting theory” we have seen for products and function complexes.

**30.12. Proposition.** *Let  $S$  and  $T$  be sets of maps in  $s\text{Set}$ . Then  $\overline{S} \boxtimes \overline{T} \subseteq \overline{S \boxtimes T}$ .*

*Proof.* This is formal and nearly identical to the proof of the weak saturation result for box-products (19.9). □

**30.13. Proposition.** *We have*

$$\overline{\text{Cell}} \boxtimes \overline{\text{Cell}} \subseteq \overline{\text{Cell}}, \quad \overline{\text{RHorn}} \boxtimes \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}, \quad \text{and} \quad \overline{\text{Cell}} \boxtimes \overline{\text{LHorn}} \subseteq \overline{\text{InnHorn}}.$$

*Proof.* Immediate from (30.4) and (30.12). □

**30.14. Proposition.** *Given  $T \xrightarrow{i} S \xrightarrow{f} X \xrightarrow{p} Y$ , consider the pullback-slice maps*

$$\ell: X_{f/} \rightarrow X_{fi/} \times_{Y_{pfi/}} Y_{pf/}, \quad r: X_{/f} \rightarrow X_{/fi} \times_{Y_{/pfi}} Y_{/pf}.$$

*We have the following.*

- (1)  $i \in \overline{\text{Cell}}, p \in \text{TrivFib}$  implies  $\ell, r \in \text{TrivFib}$ .
- (2)  $i \in \overline{\text{Cell}}, p \in \text{InnFib}$  implies  $\ell \in \text{LFib}, r \in \text{RFib}$ .
- (3)  $i \in \overline{\text{RHorn}}, p \in \text{InnFib}$  implies  $\ell \in \text{TrivFib}$ .
- (4)  $i \in \overline{\text{LHorn}}, p \in \text{InnFib}$  implies  $r \in \text{TrivFib}$ .

*Proof.* Exercise, using (30.13). □

We are mostly interested in special cases when  $X = C$  is a quasicategory, and  $Y = *$ .

**30.15. Corollary.** *Given  $T \xrightarrow{j} S \xrightarrow{f} C$  with  $C$  a quasicategory, consider the pullback-slice maps*

$$\ell: C_{f/} \rightarrow C_{fj/}, \quad r: C_{/f} \rightarrow C_{/fj}.$$

*We have the following.*

- (1)  $j \in \overline{\text{Cell}}$  implies  $\ell \in \text{LFib}, r \in \text{RFib}$ .
- (2)  $j \in \overline{\text{RHorn}}$  implies  $\ell \in \text{TrivFib}$ .
- (3)  $j \in \overline{\text{LHorn}}$  implies  $r \in \text{TrivFib}$ .

*In particular, (1) when  $T = \emptyset$  gives*

- (1')  $\ell: C_{p/} \rightarrow C$  is a left fibration and  $r: C_{/p} \rightarrow C$  is a right fibration.

Here is another useful special case when  $T = \emptyset$ : slices preserve trivial fibrations.

**30.16. Corollary.** *Given  $S \xrightarrow{f} X \xrightarrow{p} Y$  where  $p$  is a trivial fibration, all of the maps in*

$$X_{f/} \rightarrow X \times_Y Y_{pf/} \rightarrow Y_{pf/} \quad \text{and} \quad X_{/f} \rightarrow X \times_Y Y_{/pf} \rightarrow Y_{/pf}$$

*are trivial fibrations.*

*Proof.* The two pullback-slice maps are trivial fibrations by (30.14). The projections are each base changes of the trivial fibration  $f$ , and so are trivial fibrations. □

We'll also meet the following consequence now and again: joins preserve monomorphisms.

**30.17. Proposition.** *If  $i: A \rightarrow B$  is a monomorphism of simplicial sets, then so are  $S \star i: S \star A \rightarrow S \star B$  and  $i \star S: A \star S \rightarrow B \star S$  for any  $S$ .*



*Proof.* The map  $S \star i$  is the composite

$$S \star A \rightarrow (S \star A) \cup_{\varnothing \star A} (\varnothing \star B) \xrightarrow{(\varnothing \subseteq S) \boxtimes i} S \star B.$$

the second map is a monomorphism by  $\overline{\text{Cell}} \boxtimes \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$  (30.13), while the first map is a cobase change of the monomorphism  $i$ .

Note: that join preserves monomorphisms can also be proved directly from the definition of join (26.9), or the identification of join as a right adjoint (26.17).  $\square$

**30.18. Composition functors for slices.** Here is a nice consequence of the above. Let  $C$  be a quasicategory and  $f: x \rightarrow y$  a morphism in it; we represent  $f$  by a map  $\Delta^1 \rightarrow C$  of simplicial sets, which we also call  $f$ . We obtain two restriction functors

$$C_{/x} \xleftarrow{r_0} C_{/f} \xrightarrow{r_1} C_{/y}$$

associated to the inclusions  $\{0\} \subset \Delta^1 \supset \{1\}$ . The first inclusion  $\{0\} \subset \Delta^1$  is a left-horn inclusion, and thus by (30.15) the restriction map  $r_0$  is a trivial fibration, and hence we can choose a section  $s: C_{/x} \rightarrow C_{/f}$  of  $r_0$ .

The resulting composite  $r_1 s: C_{/x} \rightarrow C_{/y}$  can be thought of as a functor realizing the operation which sends an object  $(c \xrightarrow{g} x)$  of  $C_{/x}$  to an object  $(c \xrightarrow{f \circ g} y)$  of  $C_{/y}$  defined by “composing  $f$  and  $g$ ” (but remember that such composition is not uniquely defined in a quasicategory  $C$ ; the choice of section  $s$  gives a collection of such choices for all  $g$ .)

**30.19. Exercise.** Show that if  $C$  is a category, then  $r_0$  is an isomorphism, and that  $r_1 s$  is precisely the functor  $C_{/x} \rightarrow C_{/y}$  described above.

### 31. LIMITS AND COLIMITS IN QUASICATEGORIES

**31.1. Definition of limits and colimits.** Now we can define the notion of a limit and colimit of a functor between quasicategories (and in fact of a map from a simplicial set to a quasicategory). Given a map  $f: K \rightarrow C$  where  $C$  is a quasicategory, a **colimit** of  $f$  is defined to be an initial object of the slice quasicategory  $C_{f/}$ . Explicitly, a colimit of  $f: K \rightarrow C$  is a map  $\hat{f}: K \star \Delta^0 = K^\triangleright \rightarrow C$  extending  $f$ , such that for  $n \geq 1$  a lift exists in every diagram of the form

$$\begin{array}{ccc} K \star \{0\} & \xrightarrow{\quad} & K \star \partial \Delta^n \\ & \searrow \hat{f} & \downarrow \\ & & K \star \Delta^n \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \\ \nearrow \end{array} \quad \begin{array}{c} C \\ \nearrow \end{array}$$

Sometimes it is better to call  $\hat{f}$  a **colimit cone** for  $f$ , in which case the restriction  $\hat{f}|_{\varnothing \star \Delta^0}$  to the cone point is an object in  $C$  which can be called a “colimit of  $f$ ”.

Similarly, a **limit** of  $f$  is a terminal object of  $C_{f/}$ ; explicitly, this is a map  $\hat{f}: \Delta^0 \star K = K^\triangleleft \rightarrow C$  extending  $f$  such that for  $n \geq 1$  a lift exists in every diagram of the form

$$\begin{array}{ccc} \{n\} \star K & \xrightarrow{\quad} & \partial \Delta^n \star K \\ & \searrow \hat{f} & \downarrow \\ & & \Delta^n \star K \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \\ \nearrow \end{array} \quad \begin{array}{c} C \\ \nearrow \end{array}$$

Again, we will also sometimes refer to  $\hat{f}: \Delta^0 \star K = K^\triangleleft \rightarrow C$  as a **limit cone** for  $f$ , while the restriction  $\hat{f}|_{\Delta^0 \star \varnothing}$  can be called the “limit of  $f$ ”.

31.2. *Example.* Consider the empty simplicial set  $K = \emptyset$  and the unique map  $f: \emptyset \rightarrow C$ . Then  $C_{f/} = C$ , so a colimit of  $f$  is precisely the same as an initial object of  $C$ . Likewise, a limit of  $p$  is precisely the same as a terminal object of  $C$ .

31.3. *Example.* Consider  $K = \Lambda_0^2$ , which is the nerve of a category which we can draw as the picture  $(1 \leftarrow 0 \rightarrow 2)$ . Then  $(\Lambda_0^2)^\triangleright \approx \Delta^1 \times \Delta^1$  is also an ordinary category; explicitly it has the form of a commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & \searrow & \downarrow \\ 2 & \longrightarrow & v \end{array}$$

where  $v$  is the “cone vertex”. A colimit cone  $(\Lambda_0^2)^\triangleright \rightarrow C$  is called a **pushout diagram** in  $C$ .

pushout diagram

Similar considerations give  $(\Lambda_2^2)^\triangleleft \approx \Delta^1 \times \Delta^1$ ; a limit cone  $(\Lambda_2^2)^\triangleleft \rightarrow C$  is called a **pullback diagram** in  $C$ .

pullback diagram

31.4. *Exercise* (Colimits in full subcategories). Let  $C' \subseteq C$  be an inclusion of a full subcategory. Show that if  $f: K \rightarrow C'$  has a colimit  $\hat{f}$  in  $C$ , and if the image of  $\hat{f}$  is contained in  $C'$ , then  $\hat{f}$  is in fact a colimit of  $f$  in  $C'$ .

31.5. **Uniqueness of limits and colimits.** Limits and colimits are unique if they exist.

31.6. **Proposition.** Let  $f: K \rightarrow C$  be a map to a quasicategory, and let  $(C_{f/})^{\text{colim}} \subseteq C_{f/}$  and  $(C_{/f})^{\text{lim}} \subseteq C_{/f}$  denote the full subcategories spanned by colimit cones and limit cones respectively. Then (i) either  $(C_{f/})^{\text{colim}}$  is empty or is categorically equivalent to  $\Delta^0$ , and (ii) either  $(C_{/f})^{\text{lim}}$  is empty or is categorically equivalent to  $\Delta^0$ .

*Proof.* This is just the uniqueness of initial and terminal objects (29.6), since  $(C_{f/})^{\text{colim}} = (C_{f/})^{\text{init}}$  and  $(C_{/f})^{\text{lim}} = (C_{/f})^{\text{term}}$ .  $\square$

We have noted above (29.3) that an object  $x$  in a quasicategory  $C$  is initial iff  $C_{x/} \rightarrow C$  is a trivial fibration, and terminal iff  $C_{/x} \rightarrow C$  is a trivial fibration. There is a similar characterization of limit and colimit cones.

31.7. **Proposition.** Let  $C$  be a quasicategory. Let  $\tilde{f}: K^\triangleright \rightarrow C$  be a map, and write  $f := \tilde{f}|_K$ . Then  $\tilde{f}$  is a colimit diagram if and only if  $C_{\tilde{f}/} \rightarrow C_{f/}$  is a trivial fibration.

Likewise, let  $\tilde{g}: K^\triangleleft \rightarrow C$  be a map, and write  $g := \tilde{g}|_K$ . Then  $\tilde{g}$  is a limit diagram if and only if  $C_{/\tilde{g}} \rightarrow C_{/g}$  is a trivial fibration.

*Proof.* I'll just do the case of colimits.

We make an elementary observation about iterated slices (see (31.8) below). There is an isomorphism  $(C_{f/})_{\tilde{f}/} \approx C_{\tilde{f}/}$ , where the symbol “ $\tilde{f}$ ” refers to both a morphism  $\tilde{f}: K^\triangleright \rightarrow C$  (on the right-hand side of the isomorphism) and the corresponding object  $\tilde{f} \in (C_{f/})_0$  (on the left-hand side of the isomorphism). The point is that in either simplicial set, a  $k$ -dimensional cell corresponds to a map  $K \star \Delta^0 \star \Delta^k \rightarrow C$  which restricts to  $\tilde{f}$  on  $K \star \Delta^0 \star \emptyset$ .

Using this, the statement amounts to the special case for initial and terminal objects (29.3).  $\square$

31.8. *Exercise* (Iterated slices). Let  $f: A \star B \rightarrow C$  be a map of simplicial sets. Describe isomorphisms

$$C_{f/} \approx (C_{f_A/})_{\tilde{f}_B/}, \quad C_{/f} \approx (C_{/f_B})_{/\tilde{f}_A},$$

where  $f_A: A \rightarrow C$  and  $f_B: B \rightarrow C$  are the evident restrictions of  $f$  to subcomplexes, and  $\tilde{f}_A: A \rightarrow C_{/f_B}$  and  $\tilde{f}_B: B \rightarrow C_{f_A/}$  are the adjoints to  $f$ .

**31.9. Limits and colimits in slices.** Given a map  $f: S \rightarrow C$  to a quasicategory, we have “forgetful functors”  $\pi: C_{/f} \rightarrow C$  and  $\pi: C_{f/} \rightarrow C$  from the slices to  $C$ .

The following proposition says that an initial object of  $C$  implies a compatible initial object of  $C_{/f}$ , and a terminal object of  $C$  implies a compatible terminal object of  $C_{f/}$ . Note that when  $C$  is an ordinary category this is entirely straightforward: e.g., given an initial object  $c_0$  of  $C$ , there is a *unique* cone  $\tilde{f}: S^\triangleleft \rightarrow C$  extending a given  $f: S \rightarrow C$  which sends the cone vertex to  $c_0$ , and it’s an easy exercise to show that  $\tilde{f}$  represents an initial object of the slice  $C_{/f}$ .

**31.10. Proposition.** *Let  $f: S \rightarrow C$  be a map from a simplicial set to a quasicategory.*

- (1a) *If  $x \in (C_{/f})_0$  is an object such that  $\pi(x) \in C_0$  is initial in  $C$ , then  $x$  is initial in  $C_{/f}$ .*
- (1b) *If  $C$  has an initial object then so does  $C_{/f}$ .*
- (2a) *If  $x \in (C_{f/})_0$  is an object such that  $\pi(x) \in C_0$  is terminal in  $C$ , then  $x$  is terminal in  $C_{f/}$ .*
- (2b) *If  $C$  has a terminal object then so does  $C_{f/}$ .*

*Proof.* (See [Lur09, 1.2.13.8].) I’ll only prove (1a) and (1b), as the other parts are analogous.

To prove (1a), let  $x \in (C_{/f})_0$  and  $y = \pi(x) \in C_0$ ; we need to show that if  $y$  is initial then so is  $x$ . To show that  $x$  is initial we must produce a lift in any diagram of the form

$$\begin{array}{ccc} & x & \\ \Delta^0 \star \emptyset \twoheadrightarrow & (\Delta^0 \star \partial \Delta^n) \cup_{\emptyset \star \partial \Delta^n} (\emptyset \star \Delta^n) \twoheadrightarrow & C_{/f} \\ & \downarrow & \nearrow \\ & \Delta^0 \star \Delta^n & \end{array}$$

for  $n \geq 0$ , using the identification  $(\emptyset \subset \Delta^0) \boxtimes (\partial \Delta^n \subset \Delta^n) \approx (\partial \Delta^{n+1} \subset \Delta^{n+1})$ . This lifting problem is equivalent to one of the form

$$\begin{array}{ccc} & x' & \\ \Delta^0 \star \emptyset \star S \twoheadrightarrow & (\Delta^0 \star \partial \Delta^n \star S) \cup_{\emptyset \star \partial \Delta^n \star S} (\emptyset \star \Delta^n \star S) \twoheadrightarrow & C \\ & \downarrow & \nearrow \\ & \Delta^0 \star \Delta^n \star S & \end{array}$$

(because  $(-)\star S$  preserves *pushouts* (27.8)), which in turn is equivalent to one of the form

$$\begin{array}{ccc} & x'' & \\ S \twoheadrightarrow & \partial \Delta^n \star S \twoheadrightarrow & C_{y/} \\ & \downarrow & \downarrow q \\ & \Delta^n \star S \twoheadrightarrow & C \end{array}$$

(In these diagrams the maps marked  $x, x', x''$  are all adjoints of each other.) Since  $y$  is initial,  $q$  is a trivial fibration (29.3), and therefore a lift exists since  $\partial \Delta^n \star S \rightarrow \Delta^n \star S$  is a monomorphism, because joins preserve monomorphisms (30.17). We conclude that  $x$  is initial when  $y$  is.

Next we prove (1b). Suppose  $y \in C_0$  is an initial object. This implies  $q: C_{y/} \rightarrow C$  is a trivial fibration (29.3). In particular, a lift exists in

$$\begin{array}{ccc} & C_{y/} & \\ x'' \nearrow & \downarrow q & \\ S & \xrightarrow{p} & C \end{array}$$

By an adjunction argument (27.15),  $x''$  corresponds to a map  $x: \Delta^0 \rightarrow C_{/f}$  such that  $\pi(x) = y$ . By what we have already proved,  $x$  must be initial since  $\pi(x) = y$  is initial.  $\square$

31.11. *Remark.* In fact, the converses of (1a) and (2a) in (31.10) are also true, as long as we assume that  $C$  has an initial/terminal object. The proof of these converses requires (33.10), which we have not established yet.

We can now generalize the above to arbitrary limits and colimits.

The following proposition says that colimits in  $C_{/f}$  or limits in  $C_{f/}$  can be “computed in the underlying quasicategory”  $C$  (if the corresponding colimit or limit in  $C$  exists).

31.12. **Proposition.** *Let  $p: S \rightarrow C$  be a map from a simplicial set to a quasicategory.*

- (1) *Let  $f: K \rightarrow C_{/p}$  be a map such that the composite map  $f_0 = \pi f: K \xrightarrow{f} C_{/p} \xrightarrow{\pi} C$  has a colimit cone in  $C$ . Then*
  - (a)  *$f$  admits a colimit cone, and*
  - (b) *if  $\tilde{f}: K^{\triangleright} \rightarrow C_{/p}$  is such that the composite map  $K^{\triangleright} \xrightarrow{\tilde{f}} C_{/p} \rightarrow C$  is a colimit cone, then  $\tilde{f}$  is a colimit cone.*
- (2) *Let  $f: K \rightarrow C_{p/}$  be a map such that the composite map  $f_0 = \pi f: K \xrightarrow{f} C_{p/} \xrightarrow{\pi} C$  has a limit cone in  $C$ . Then*
  - (a)  *$f$  admits a limit cone, and*
  - (b) *if  $\tilde{f}: K^{\triangleleft} \rightarrow C_{p/}$  is such that the composite map  $K^{\triangleleft} \xrightarrow{\tilde{f}} C_{p/} \rightarrow C$  is a limit cone, then  $\tilde{f}$  is a limit cone.*

The proof will make use an observation sketched in the following exercise: any composite of a slice-over followed by a slice-under can be reinterpreted as a slice-under followed by a slice-over.

31.13. *Exercise (Two-sided slice).* Fix a map  $p: A \star B \rightarrow X$  of simplicial sets. Describe a simplicial set  $X_{/p/}$  which admits bijective correspondences

$$\left\{ \begin{array}{ccc} A \star B & \xrightarrow{p} & X \\ \downarrow & \searrow & \uparrow \\ A \star K \star B & & \end{array} \right\} \Longleftrightarrow \{K \dashrightarrow X_{/p/}\},$$

natural in  $K$ . Then construct natural isomorphisms

$$(X_{p_A/})_{/\tilde{p}_B} \approx X_{/p/} \approx (X_{/p_B})_{\tilde{p}_A/},$$

where  $p_A: A \rightarrow X$  and  $p_B: B \rightarrow X$  are the evident restrictions of  $p$  to subcomplexes, and  $\tilde{p}_A: A \rightarrow X_{/p_B}$  and  $\tilde{p}_B: B \rightarrow X_{p_A/}$  are adjoints to  $p$ .

*Proof of (31.12).* I prove (1), as (2) is analogous. Note that  $f: K \rightarrow C_{/p}$  is adjoint to a map  $g: K \star S \rightarrow C$  extending  $p$ , which in turn is adjoint to a map  $q: S \rightarrow C_{f_0/}$ . Colimit cones of  $f_0$  correspond precisely to initial objects of  $C_{f_0/}$ ; in particular, the hypothesis of (1) asserts that  $C_{f_0/}$  has an initial object. Likewise, colimit cones of  $f$  correspond exactly to initial objects of  $(C_{/p})_{f/}$ . As in (31.13) we have isomorphisms

$$(C_{/p})_{f/} \approx C_{/g/} \approx (C_{f_0/})_{/q}.$$

To prove (1a) here it suffices to show that  $(C_{f_0/})_{/q}$  has an initial object, which follows by an application of (31.10)(1b) to the restriction functor  $(C_{f_0/})_{/q} \rightarrow C_{f_0/}$  to “lift” an initial object of  $C_{f_0/}$  to  $(C_{f_0/})_{/q}$ .

To prove (1b) here it suffices to show that the restriction functor  $(C_{f_0/})_{/q} \rightarrow C_{f_0/}$  has the property that objects sent to initial objects of  $C_{f_0/}$  are initial in  $(C_{f_0/})_{/q}$ , which is immediate from (31.10)(1a).  $\square$

**31.14. Invariance of limits and colimits.** There are some seemingly obvious facts about invariance of limits and colimits which we cannot prove yet.

- (1) Limits and colimits are invariant under categorical equivalence. For instance, if  $f: C \rightarrow D$  is a categorical equivalence of quasicategories, and  $u: K \rightarrow C$  is some map, then  $p$  admits a colimit in  $C$  if and only if  $fu$  admits a colimit in  $D$ , and the induced functor  $C_{u/} \rightarrow D_{fu/}$  preserves colimit cones. We will prove these as (41.5) and (41.6).
- (2) Limits and colimits are invariant under natural isomorphism. For instance, if  $\alpha: f_0 \rightarrow f_1$  is a natural isomorphism of maps  $f_0, f_1: K \rightarrow C$ , then  $f_0$  admits a colimit if and only if  $f_1$  does, and if  $\hat{f}_0$  and  $\hat{f}_1$  are colimit cones for  $f_0$  and  $f_1$  respectively, there exists an isomorphism  $\hat{\alpha}: \hat{f}_0 \rightarrow \hat{f}_1$  extending  $\alpha$ . We will prove this as (??).

## 32. THE JOYAL EXTENSION AND LIFTING THEOREMS

We are now at the point where we can state and prove Joyal's theorems about extending or lifting maps along outer horns. This will allow us to prove several of the results whose proofs we have deferred up to now.

**32.1. Joyal extension theorem.** Joyal's theorem gives precise criteria for extending maps from *outer horns* into a quasicategory.

**32.2. Theorem** (Joyal extension). [Joy02, Thm. 1.3] *Let  $C$  be a quasicategory, and fix a map  $f: \Delta^1 \rightarrow C$ . The following are equivalent.*

- (1) *The edge represented by  $f$  is an isomorphism in  $C$ .*
- (2) *Every  $a: \Lambda_0^n \rightarrow C$  with  $n \geq 2$  such that  $f = a|_{\Delta^{\{0,1\}}}: \Delta^1 \rightarrow C$  admits an extension to a map  $\Delta^n \rightarrow C$ .*
- (3) *Every  $b: \Lambda_n^n \rightarrow C$  with  $n \geq 2$  such that  $f = b|_{\Delta^{\{n-1,n\}}}: \Delta^1 \rightarrow C$  admits an extension to a map  $\Delta^n \rightarrow C$ .*

I'll call  $\langle 01 \rangle \in \Delta^n$  the **leading edge**, and  $\langle n-1, n \rangle \in \Delta^n$  the **trailing edge**. Thus, the implications (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (3) say that we can always extend  $\Lambda_0^n \rightarrow C$  to an  $n$ -simplex if the *leading edge* goes to an isomorphism in  $C$ , and extend  $\Lambda_n^n \rightarrow C$  to an  $n$ -simplex if the *trailing edge* goes to an isomorphism in  $C$ .

leading edge  
trailing edge

The implications (2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (1) are easy, and are left as an exercise.

**32.3. Exercise** (Easy part of Joyal extension). Suppose  $C$  is a quasicategory with edge  $f \in C_1$ , and suppose that every map  $a: \Lambda_0^n \rightarrow C$  with  $n \in \{2, 3\}$  and  $f = a|_{\Delta^{\{0,1\}}}$  admits an extension along  $\Lambda_0^n \subset \Delta^n$ . Prove that  $f$  is an isomorphism.

The proof of the Joyal extension theorem will be an application of the fact that left fibrations and right fibrations are *conservative isofibrations*.

**32.4. Conservative functors.** A functor  $p: C \rightarrow D$  between categories is **conservative** if whenever  $f$  is a morphism in  $C$  such that  $p(f)$  is an isomorphism in  $D$ , then  $f$  is an isomorphism in  $C$ . The definition of a conservative functor between *quasicategories* is precisely the same.

conservative

**32.5. Proposition.** *All left fibrations and right fibrations between quasicategories are conservative.*

*Proof.* Consider a right fibration  $p: C \rightarrow D$ , and a morphism  $f: x \rightarrow y$  in  $C$  such that  $p(f)$  is an isomorphism. We first show that  $f$  admits a preinverse.

Let  $a: \Lambda_2^2 \rightarrow C$  such that  $a_{12} = f$  and  $a_{02} = 1_y$ . Let  $b: \Delta^2 \rightarrow C$  be any 2-dimensional cell exhibiting a preinverse of  $p(f)$ , i.e., such that  $b_{12} = p(f)$  and  $b_{02} = 1_{p(y)}$ , so that  $b_{01}$  is a preinverse.

Now we have a commutative diagram

$$\begin{array}{ccc} \Lambda_2^2 & \xrightarrow{a} & C \\ \downarrow & \nearrow s & \downarrow p \\ \Delta^2 & \xrightarrow{b} & D \end{array}$$

which admits a lift since  $p$  is a right fibration. The lift  $s$  exhibits a preinverse  $g := s|_{\Delta^{\{0,1\}}}$  for  $f$ .

Because  $p(f)$  was assumed to be an isomorphism in  $D$ , its preinverse  $b_{01} = p(g)$  is also an isomorphism, and therefore by the above argument  $g$  admits a preinverse as well. We conclude that  $f$  is an isomorphism by (12.5).  $\square$

**32.6. Isofibrations.** We say that a functor  $p: C \rightarrow D$  of quasicategories is an **isofibration**<sup>23</sup> if isofibration

- (1)  $p$  is an inner fibration, and
- (2) we have “isomorphism lifting” along  $p$ . That is, for any  $c \in C_0$  and *isomorphism*  $g: p(c) \rightarrow d'$ , there exists a  $c' \in C_0$  and *isomorphism*  $f: c \rightarrow c'$  such that  $p(f) = g$ .

Condition (2) is illustrated by the diagram

$$\begin{array}{ccccc} \{0\} & \xrightarrow{c} & C^{\text{core}} & \longrightarrow & C \\ \downarrow & \nearrow f & \downarrow p^{\text{core}} & & \downarrow p \\ \Delta^1 & \xrightarrow{g} & D^{\text{core}} & \longrightarrow & D \end{array}$$

Recall that if  $C$  and  $D$  are nerves of ordinary categories, then any functor  $C \rightarrow D$  is an inner fibration. Thus in the case of ordinary categories, being an isofibration amounts to condition (2) only. Also, it is clear that in the case of ordinary categories condition (2) is equivalent to

- (2') for any  $c \in C_0$  and *isomorphism*  $g': d' \rightarrow p(c)$ , there exists a  $c' \in C_0$  and *isomorphism*  $f': c' \rightarrow c$  such that  $p(f) = g'$ .

To derive (2) from (2') for ordinary categories, just apply (2') to the (unique) inverse of  $g$ .

The symmetry between (2) and (2') also holds for functors between quasicategories, by the following.

**32.7. Proposition.** *An inner fibration  $p: C \rightarrow D$  between quasicategories is an isofibration if and only if  $h(p): h(C) \rightarrow h(D)$  is an isofibration of ordinary categories.*

*Proof.* ( $\implies$ ) Straightforward. ( $\impliedby$ ) Suppose given an isomorphism  $g: p(c) \rightarrow d'$  in  $D$ . If  $h(p): hC \rightarrow hD$  is an isofibration, there exists an isomorphism  $f': c \rightarrow c'$  in  $C$  such that  $p(f') \sim_r g$ . Now choose a lift in

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{a} & C \\ \downarrow & \nearrow s & \downarrow p \\ \Delta^2 & \xrightarrow{b} & D \end{array}$$

where  $b$  exhibits  $p(f') \sim_r g$  and  $a(\langle 01 \rangle) = f'$  and  $a(\langle 12 \rangle) = 1_{c'}$ . The edge  $f = s_{02}$  is a lift of  $g$ , and is an isomorphism since  $f' \sim_r f$ .  $\square$

**32.8. Example.** For any quasicategory  $C$ , the tautological map  $C \rightarrow \Delta^0$  to the terminal category is an isofibration.

<sup>23</sup>Joyal uses the term “quasifibration” in [Joy02]. Later in [Joy08a] this is called a “pseudofibration”. Lurie uses this notion in [Lur09], but never names it. The term “isofibration” is used by Riehl and Verity [RV15].

32.9. *Exercise.* (i) Let  $\mathbf{Group}$  denote the category of groups, whose objects are pairs  $G = (S, \mu)$  consisting of a set  $S$  and a function  $\mu: S \times S \rightarrow S$  satisfying a well-known list of axioms. Show that the functor  $U: \mathbf{Group} \rightarrow \mathbf{Set}$  which on objects sends  $(S, \mu) \mapsto S$  is an isofibration between ordinary categories.

(ii) Consider the functor  $U': \mathbf{Group} \rightarrow \mathbf{Set}$  defined on objects by  $G \mapsto \mathrm{Hom}(\mathbb{Z}, G)$ . Explain why, although  $U'$  is naturally isomorphic to  $U$ , you don't know how to show whether  $U'$  is an isofibration without explicit reference to the axioms of your set theory. (Note that  $S$  and  $\mathrm{Hom}(\mathbb{Z}, S)$  may be distinct sets, even though they are in bijective correspondence.) The moral is that the property of being an isofibration is not “natural isomorphism invariant”.

### 32.10. Left and right fibrations are isofibrations.

32.11. **Proposition.** *All left fibrations and right fibrations between quasicategories are isofibrations.*

*Proof.* Suppose  $p: C \rightarrow D$  is a right fibration (and hence an inner fibration) between quasicategories, and consider

$$\begin{array}{ccc} \{1\} & \xrightarrow{\quad} & C \\ \downarrow & \nearrow f & \downarrow p \\ \Delta^1 & \xrightarrow{g} & D \end{array}$$

where  $g$  represents an isomorphism in  $D$ . Because  $p$  is a right fibration and  $(\{1\} \subset \Delta^1) \in \mathrm{RHorn}$ , there exists a lift  $f$ . Because right fibrations are conservative,  $f$  represents an isomorphism.  $\square$

Note that the above proof explicitly checked isofibration condition (2') for right fibrations; thus, by symmetry we conclude that isofibration condition (2) holds for right fibrations. It seems difficult to give an elementary *direct* proof that right-fibrations satisfy (2).

### 32.12. Proof of the Joyal extension theorem.

*Proof of (32.2).* We prove  $(1) \Rightarrow (2)$ . Suppose given  $a: \Lambda_0^n \rightarrow C$  such that  $f = a|_{\Delta^{\{0,1\}}}$  represents an isomorphism. Observe (30.4) that  $(\Lambda_0^n \subset \Delta^n)$  is the pushout-join of a 1-horn with an  $(n-2)$ -cell:

$$(\Lambda_0^n \subset \Delta^n) \approx (\Delta^{\{0\}} \subset \Delta^{\{0,1\}}) \boxtimes (\partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}),$$

since  $\Lambda_0^n \approx (\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}}) \cup (\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})$  inside  $\Delta^n \approx \Delta^{\{0,1\}} \star \Delta^{\{2,\dots,n\}}$ . Using this, we get a correspondence of lifting problems

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & \Lambda_0^n \xrightarrow{a} C \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & & \Delta^n \end{array} \iff \begin{array}{ccc} \Delta^{\{0\}} & \xrightarrow{h} & C/(a|\Delta^{\{2,\dots,n\}}) \\ \downarrow & \nearrow & \downarrow q \\ \Delta^{\{0,1\}} & \xrightarrow{g} & C/(a|\partial\Delta^{\{2,\dots,n\}}) \\ & \searrow f & \downarrow p \\ & & C \end{array}$$

where  $g$  is adjoint to  $a|_{(\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})}$ , and  $h$  is adjoint to  $a|_{(\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}})}$ . Because  $C$  is a quasicategory, and because  $p$  and  $q$  are restrictions along monomorphisms  $\emptyset \subset \partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}$ , both  $p$  and  $q$  are right fibrations (30.15), and therefore are conservative isofibrations (32.5), (32.11). Thus since  $f$  represents an isomorphism, so does  $g$  since  $p$  is conservative, and therefore a lift exists since  $q$  is an isofibration.

The proof of  $(2) \Rightarrow (1)$  is left as an exercise (32.3). The proof of  $(1) \iff (3)$  is similar.  $\square$

**32.13. The Joyal lifting theorem.** There is a relative generalization, which we will have use of in the future.

**32.14. Theorem** (Joyal lifting). *Let  $p: C \rightarrow D$  be an inner fibration between quasicategories, and let  $f \in C_1$  be an edge such that  $p(f)$  is an isomorphism in  $D$ . The following are equivalent.*

- (1) *The edge  $f$  is an isomorphism in  $C$ .*
- (2) *For all  $n \geq 2$ , every diagram of the form*

$$\begin{array}{ccccc} \Delta^{\{0,1\}} & \xrightarrow{\quad} & \Lambda_0^n & \xrightarrow{\quad} & C \\ & & \downarrow & \nearrow & \downarrow p \\ & & \Delta^n & \xrightarrow{\quad} & D \end{array}$$

$f$  (curved arrow from  $\Delta^{\{0,1\}}$  to  $C$ )

*admits a lift.*

- (3) *For all  $n \geq 2$ , every diagram of the form*

$$\begin{array}{ccccc} \Delta^{\{n-1,n\}} & \xrightarrow{\quad} & \Lambda_n^n & \xrightarrow{\quad} & C \\ & & \downarrow & \nearrow & \downarrow p \\ & & \Delta^n & \xrightarrow{\quad} & D \end{array}$$

$f$  (curved arrow from  $\Delta^{\{n-1,n\}}$  to  $C$ )

*admits a lift.*

*Proof.* The implications (2)  $\Rightarrow$  (1) and (3)  $\Rightarrow$  (1) are elementary, as in (32.3).

For (1)  $\Rightarrow$  (2), the first step is to prove that

$$C_{/(a|\Delta^{\{2,\dots,n\}})} \xrightarrow{q} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p} C$$

are both right fibrations. For instance, the map  $q$  is the pullback-slice of the inner fibration  $p$  by a monomorphism, so is a right fibration by (30.14). The map  $p$  is the composite

$$C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p'} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \xrightarrow{p''} C,$$

where  $p'$  is the base change of the right fibration  $D_{/(pa|\Delta^{\{2,\dots,n\}})} \rightarrow D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}$ , and  $p''$  is a right fibration (in both cases by (30.15)) Then the proof of (1)  $\Rightarrow$  (2) proceeds exactly as in (32.2).  $\square$

### 33. APPLICATIONS OF THE JOYAL EXTENSION THEOREM

We can now prove a number of statements whose proofs we have deferred until now, as well as some others.

**33.1. Quasigroupoids are Kan complexes.** First we prove the identification of quasigroupoids with Kan complexes.

**33.2. Theorem** (Joyal [Joy08a]). *Every quasigroupoid is a Kan complex.*

*Proof.* In a quasigroupoid, the Joyal extension property (32.2) applies to all maps from  $\Lambda_0^n$  and  $\Lambda_n^n$  with  $n \geq 2$ , since every edge is an isomorphism. (Recall that all simplicial sets automatically have extensions for 1-horns (12.12).)  $\square$

From now on we will use terms “quasigroupoid” and “Kan complex” interchangeably.



**33.3. Invariance of slice categories.** Here is an equivalent reformulation of the Joyal extension theorem in terms of maps between slices.

**33.4. Proposition** (Reformulation of Joyal extension). *If  $f: x \rightarrow y$  is an edge in a quasicategory  $C$ , then the following are equivalent: (1)  $f$  is an isomorphism; (2)  $C_{f/} \rightarrow C_{x/}$  is a trivial fibration; (3)  $C_{f/} \rightarrow C_{y/}$  is a trivial fibration.*

*Proof.* For all  $n \geq 0$  we have a correspondence of lifting problems

$$\begin{array}{ccc} \partial\Delta^n \longrightarrow C_{f/} & & \\ \downarrow \dashrightarrow & \iff & \Delta^1 \star \emptyset \longrightarrow (\Delta^1 \star \partial\Delta^n) \cup (\{0\} \star \Delta^n) \longrightarrow C \\ \Delta^n \longrightarrow C_{x/} & & \downarrow \\ & & \Delta^1 \star \Delta^n \end{array}$$

and  $((\Delta^1 \star \partial\Delta^n) \cup (\{0\} \star \Delta^n)) \subset \Delta^1 \star \Delta^n \approx (\Lambda_0^{1+1+n} \subseteq \Delta^{1+1+n})$ . The lifting problems on the right-hand side are precisely those of statement (2) of the Joyal extension theorem (32.2).  $\square$

**33.5. Exercise** (Reformulation of Joyal lifting). Let  $p: C \rightarrow D$  be an inner fibration, and  $f: x \rightarrow y$  an edge in  $C$  such that  $p(f) \in D_1$  is an isomorphism. Show that the following are equivalent: (1)  $f$  is an isomorphism in  $C$ ; (2)  $C_{f/} \rightarrow C_{x/} \times_{D_{p(x)/}} D_{p(f)/}$  is a trivial fibration; (3)  $C_{f/} \rightarrow C_{y/} \times_{D_{p(y)/}} D_{p(f)/}$  is a trivial fibration.

**33.6. Corollary.** *If  $f: x \rightarrow y$  is an isomorphism in a quasicategory  $C$ , then  $C_{x/}$  and  $C_{y/}$  are categorically equivalent, and  $C_{/x}$  and  $C_{/y}$  are categorically equivalent.*

*Proof.* Consider  $C_{/x} \xleftarrow{r_0} C_{f/} \xrightarrow{r_1} C_{/y}$ . We have already observed (30.15) that  $r_0 \in \text{TrivFib}$ , since  $\{0\} \subset \Delta^0$  is left anodyne. The reformulation of Joyal extension (33.4) implies that  $r_1 \in \text{TrivFib}$  when  $f$  is an isomorphism. Therefore  $C_{/x}$  and  $C_{/y}$  are connected by a chain of categorical equivalences. The proof for slice-under categories is analogous.  $\square$

**33.7. Invariance of initial objects.** Now we prove some additional facts about initial and terminal objects. We will explicitly prove the statements about initial objects, as the case of terminal objects is similar.

**33.8. Proposition.** *Let  $f: x \rightarrow y$  be a morphism in a quasicategory  $C$ , and let  $\tilde{f} \in (C_{x/})_0$  be the object of the slice which corresponds to  $f \in C_1$ . Then  $\tilde{f}$  is initial in  $C_{x/}$  if and only if  $f$  is an isomorphism.*

*Proof.* For all  $n \geq 1$  we have a correspondence of lifting problems

$$\begin{array}{ccc} \{0\} \longrightarrow \partial\Delta^n \longrightarrow C_{x/} & & \\ \downarrow \dashrightarrow & \iff & \Delta^0 \star \{0\} \longrightarrow \Delta^0 \star \partial\Delta^n \longrightarrow C \\ \Delta^n & & \downarrow \\ & & \Delta^0 \star \Delta^n \end{array}$$

and  $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) \approx (\Lambda_0^{1+n} \subseteq \Delta^{1+n})$ , so a lift exists in either if and only if  $f$  is an isomorphism, by the Joyal extension theorem applied to the right-hand lifting problem.

(Alternately, we can note that  $\tilde{f}$  is initial if and only if  $\pi: (C_{x/})_{\tilde{f}/} \rightarrow C_{x/}$  is a trivial fibration (29.3), and that  $\pi$  is isomorphic to  $C_{f/} \rightarrow C_{x/}$  (31.8), so the claim follows from (33.4).)  $\square$

**33.9. Remark.** Note that (33.8) implies that the slice  $C_{x/}$  for any  $x \in C_0$  in a quasicategory necessarily has an initial object, namely the vertex corresponding to the edge  $1_x \in C_1$ . Likewise, the slice  $C_{/x}$  has a terminal object corresponding to  $1_x \in C_1$ .

**33.10. Proposition.** *Any object in a quasicategory isomorphic to an initial object is also initial.*

*Proof.* Let  $x$  be an initial object in  $C$ , and let  $c$  be an object isomorphic to  $x$ . It is easy to see that  $x$  is initial in the homotopy category  $hC$ , and therefore  $c$  is initial in  $hC$  also. This has a useful consequence: *any* map between  $x$  and  $c$  (in either direction) must be an isomorphism in  $C$ .

We next note another fact: if  $x$  is initial, any map  $f: S \rightarrow C$  extends along  $S \subset \Delta^0 \star S$  to a map  $f': \Delta^0 \star S \rightarrow C$  such that  $f'|_{\Delta^0}$  represents  $x$ . This is a consequence of the fact (29.3) that  $p: C_{x/} \rightarrow C$  is a trivial fibration, whence (23.13) there exists a map  $s: C \rightarrow C_{x/}$  such that  $ps = \text{id}_C$ ; set  $f'$  be the adjoint to  $sf: S \rightarrow C_{x/}$ .

To show  $c$  is initial in  $C$ , we need to extend any  $a: \partial\Delta^n \rightarrow C$  with  $a_0 = c$  to a map  $\tilde{a}: \Delta^n \rightarrow C$ . This follows from a succession of two extension problems:

$$\begin{array}{ccc}
 \emptyset \star \partial\Delta^n & \xrightarrow{\quad} & (\Delta^0 \star \emptyset) \amalg (\emptyset \star \partial\Delta^n) \xrightarrow{(x,a)} C \\
 \downarrow & & \downarrow \quad \quad \quad \searrow g \\
 & & \Delta^0 \star \partial\Delta^n \quad \quad \quad \searrow h \\
 \emptyset \star \Delta^n & \xrightarrow{\quad} & \Delta^0 \star \Delta^n
 \end{array}$$

The extension  $g: \Lambda_0^{n+1} \rightarrow C$  exists by the remarks of the previous paragraph since  $x$  is initial. The extension  $h$  exists because the leading edge of  $g$  is a map  $x \rightarrow c$  in  $C$ , which is an isomorphism by the remarks of the first paragraph. The desired extension  $\tilde{a}$  is  $h|(\emptyset \star \Delta^n)$ .  $\square$

**33.11. Remark** (Slices of quasigroupoids are quasigroupoids). If  $C$  is a quasigroupoid, and  $x \in C_0$  an object, then the slices  $C_{/x}$  and  $C_{x/}$  are quasigroupoids. This is immediate from the fact that the restriction maps  $C_{/x} \rightarrow C$  and  $C_{x/} \rightarrow C$  are conservative, being respectively right and left fibrations (30.15) (32.5).

**33.12. Remark** (Initial and terminal objects in quasigroupoids). If  $C$  is a quasigroupoid with object  $x \in C_0$ , then (33.8) and its analogue for final objects, together with the fact that slices of quasigroupoids are quasigroupoids (33.11), implies that *every* object of  $C_{x/}$  is initial, and *every* object of  $C_{/x}$  is terminal. That is,  $C_{x/} = (C_{x/})^{\text{init}}$  and  $C_{/x} = (C_{/x})^{\text{term}}$ , and they are non-empty (33.9), so both  $C_{x/}$  and  $C_{/x}$  are categorically equivalent to the terminal quasicategory.

#### 34. POINTWISE NATURAL ISOMORPHISMS

Recall (20.4) that if  $C$  is a quasicategory then so is any function complex  $\text{Fun}(X, C)$  for an arbitrary simplicial set  $X$ . In this setting, say that an edge in  $\text{Fun}(X, C)_1$  is a **pointwise isomorphism** of maps  $X \rightarrow C$  if for each vertex  $x \in X_0$ , the composite  $\Delta^1 \xrightarrow{f} \text{Fun}(X, C) \xrightarrow{\text{res}} \text{Fun}(\{x\}, C) \approx C$  represents an isomorphism in  $C$ , where  $f$  is the representing map of the edge.

Note that any isomorphism in  $\text{Fun}(X, C)$  is automatically a pointwise isomorphism, as isomorphisms are preserved by the restriction functor (12.3). Our next goal is to prove the converse holds, so that pointwise isomorphisms in a functor category are the same as isomorphisms. We will prove this as (35.2), after some preliminary work.

**34.1. A lifting property for pointwise isomorphisms.** We establish a “lifting property” for pointwise isomorphisms.

**34.2. Proposition.** *Let  $p: C \rightarrow D$  be an inner fibration between quasicategories, and let  $i: S \rightarrow T$  be a monomorphism of simplicial sets such that  $i_0: S_0 \rightarrow T_0$  is a bijection. Then a lift exists in any*

pointwise isomorphism

diagram of the form

$$\begin{array}{ccc} \{0\} & \xrightarrow{\quad} & \text{Fun}(T, C) \\ \downarrow & \nearrow t & \downarrow p^{\square i} \\ \Delta^1 & \xrightarrow{v} & \text{Fun}(S, C) \times_{\text{Fun}(S, D)} \text{Fun}(T, D) \end{array}$$

where the composite  $\Delta^1 \rightarrow \text{Fun}(S, C) \times_{\text{Fun}(S, D)} \text{Fun}(T, D) \rightarrow \text{Fun}(S, C)$  represents a pointwise isomorphism. Necessarily any such lift  $t$  itself represents a pointwise isomorphism since  $S_0 \xrightarrow{\sim} T_0$ .

In particular, when  $D = \Delta^0$ , this says if  $C$  is a quasicategory,  $f: T \rightarrow C$  a map of simplicial sets, and  $S \subseteq T$  a subcomplex containing all vertices of  $T$ , then any pointwise isomorphism  $\bar{\alpha}: f|_S \rightarrow \bar{g}$  of maps  $S \rightarrow C$  extends to a pointwise isomorphism  $\alpha: f \rightarrow g$  of maps  $T \rightarrow C$ .

We need to introduce a number of ideas before we prove this. Recall (18.5) that the class of monomorphisms of simplicial sets is precisely the weak saturation  $\overline{\text{Cell}}$  of  $\text{Cell}$ . The same idea shows that the class of monomorphisms which are bijections on vertices is precisely the weak saturation  $\overline{\text{Cell}}_{\geq 1}$  of  $\text{Cell}_{\geq 1} := \{(\partial\Delta^n \subset \Delta^n) \mid n \geq 1\}$ .

**34.3. Path category.** Next we define the **path category** of a quasicategory  $C$ . This is the full subcategory

$$\text{Fun}^{\text{iso}}(\Delta^1, C) \subseteq \text{Fun}(\Delta^1, C)$$

spanned by objects corresponding to functors  $\Delta^1 \rightarrow C$  which represent an isomorphism in  $C$ . Note that any functor  $p: C \rightarrow D$  between quasicategories induces a functor  $p_*: \text{Fun}^{\text{iso}}(\Delta^1, C) \rightarrow \text{Fun}^{\text{iso}}(\Delta^1, D)$  on these subcategories, and also that restriction along  $\{x\} \rightarrow \Delta^1$  for  $x = 0, 1$  induces restriction functors  $r_x: \text{Fun}^{\text{iso}}(\Delta^1, C) \rightarrow \text{Fun}(\{x\}, C) = C$ .

**34.4. Lemma.** *Let  $C$  be a quasicategory and  $X$  a simplicial set. Then the standard bijection  $\text{Hom}(X, \text{Fun}(\Delta^1, C)) \approx \text{Hom}(\Delta^1, \text{Fun}(X, C))$  restricts to a bijection between (i) the set of maps  $X \rightarrow \text{Fun}^{\text{iso}}(\Delta^1, C)$  and (ii) the set of maps  $\Delta^1 \rightarrow \text{Fun}(X, C)$  which represent pointwise isomorphisms in  $\text{Fun}(X, C)$ .*

*Proof.* Consider  $f: X \rightarrow \text{Fun}(\Delta^1, C)$ , and write  $f': \Delta^1 \rightarrow \text{Fun}(X, C)$  and  $f'': X \times \Delta^1 \rightarrow C$  for its adjoints. Then it is straightforward to check that  $f$  is in the set (i) iff (iii): for each  $x \in X_0$  the composite  $\{x\} \times \Delta^1 \hookrightarrow X \times \Delta^1 \xrightarrow{f''} C$  represents an isomorphism in  $C$ . Likewise it is similarly straightforward to check that  $f$  is in the set (ii) iff (iii) holds.  $\square$

Using this, we can reformulate the statement of (34.2) as follows: given an inner fibration  $p: C \rightarrow D$  and a monomorphism  $i: S \rightarrow T$  of simplicial sets which induces a bijection of vertices, we need to show there exists a lift in every commutative square of the form

$$\begin{array}{ccc} S & \xrightarrow{\quad} & \text{Fun}^{\text{iso}}(\Delta^1, C) \\ i \downarrow & \nearrow & \downarrow q \\ T & \xrightarrow{\quad} & \text{Fun}(\{0\}, C) \times_{\text{Fun}(\{0\}, D)} \text{Fun}^{\text{iso}}(\Delta^1, D) \end{array}$$

where  $q$  is the evident restriction of  $p^{\square(\{0\} \subset \Delta^1)}$ .

*Proof of (34.2).* Let  $\mathcal{C}$  be the class of monomorphisms of simplicial sets  $i: S \rightarrow T$  such that  $i \boxtimes q$  for every map  $q: \text{Fun}^{\text{iso}}(\Delta^1, C) \rightarrow \text{Fun}(\{0\}, C) \times_{\text{Fun}(\{0\}, D)} \text{Fun}^{\text{iso}}(\Delta^1, D)$  obtained by restriction from  $p^{\square(\{0\} \subset \Delta^1)}$ , for every inner fibration  $p: C \rightarrow D$  between quasicategories. It is clear from its definition that  $\mathcal{C}$  is a weakly saturated class, as it is defined as a left complement to a class of maps.

To prove the claim, it suffices to show that  $\text{Cell}_{\leq 1} \subseteq \mathcal{C}$ , whence  $\overline{\text{Cell}}_{\geq 1} \subseteq \mathcal{C}$ . That is, it suffices to show that for any  $n \geq 1$ , a lift exists in any commutative square of the form

$$\begin{array}{ccc} (\{0\} \times \Delta^n) \cup_{\{0\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) & \xrightarrow{\tilde{u}} & C \\ \downarrow & \nearrow & \downarrow p \\ \Delta^1 \times \Delta^n & \longrightarrow & D \end{array}$$

where  $p$  is an inner fibration of quasicategories, and  $\tilde{u}$  is such that  $\tilde{u}|_{\Delta^1 \times \{y\}}$  represents an isomorphism for all  $y \in (\Delta^n)_0$ . This follows from the following proposition (34.5) in the case of  $(x, y) = (0, 0)$ .  $\square$

Thus we have reduced to the following proposition, which is a kind of “pushout-product” version of Joyal lifting, where we replace the horn inclusion  $\Lambda_0^n \subset \Delta^n$  with the inclusion  $(\{0\} \subset \Delta^1) \square (\partial \Delta^n \subset \Delta^n)$ , with the role of the “leading edge” played by  $\Delta^1 \times \{0\} \subset \Delta^1 \times \Delta^n$ ; or alternately, replace the horn inclusion  $\Lambda_n^n \subset \Delta^n$  with the inclusion  $(\{1\} \subset \Delta^1) \square (\partial \Delta^n \subset \Delta^n)$ , with the role of the “trailing edge” played by  $\Delta^1 \times \{n\} \subset \Delta^1 \times \Delta^n$ .

**34.5. Proposition** (Pushout-product Joyal lifting). *Suppose  $p: C \rightarrow D$  is an inner fibration of quasicategories, and suppose  $n \geq 1$ , and either  $(x, y) = (0, 0)$  or  $(x, y) = (1, n)$ . For any diagram*

$$\begin{array}{ccc} \Delta^1 \times \{y\} & \xrightarrow{f} & (\{x\} \times \Delta^n) \cup_{\{x\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) \\ & & \downarrow \\ & & \Delta^1 \times \Delta^n \longrightarrow D \end{array}$$

$\downarrow p$   
 $C$

such that  $f$  represents an isomorphism in  $C$ , a lift exists.

*Proof.* This is a calculation, given in the appendix (62.5), which itself relies on Joyal lifting.  $\square$

**34.6. Example.** To give an idea of the proof (34.5), consider the case of  $n = 1$  and  $(x, y) = 0$ , in which case  $K = (\{0\} \times \Delta^1) \cup_{\{0\} \times \partial \Delta^1} (\Delta^1 \times \partial \Delta^1)$  can be pictured the solid-arrow part of the diagram

$$\begin{array}{ccc} (0, 1) & \longrightarrow & (1, 1) \\ \uparrow & \nearrow a & \uparrow \\ (0, 0) & \xrightarrow[e]{} & (1, 0) \end{array}$$

To lift to a map  $\Delta^1 \times \Delta^1 \rightarrow C$ , we first choose a lift on the 2-cell  $a$ , which is attached along an inner horn  $\Lambda_1^2 \subset \Delta^2$ ; then we choose a lift on the 2-cell  $b$ , which is a non-inner horn  $\Lambda_0^2 \subset \Delta^2$  such that  $K \rightarrow C$  sends its leading edge (marked  $e$ ) to an isomorphism in  $C$ , so Joyal-lifting applies.

### 35. PROOF OF THE POINTWISE CRITERION FOR NATURAL ISOMORPHISMS

In this section we will prove the following.

**35.1. Proposition.** *Let  $j: K \rightarrow L$  be a monomorphism of simplicial sets such that  $j: K_0 \xrightarrow{\sim} L_0$  is a bijection. Then for every quasicategory  $C$  the restriction map  $\text{Fun}(j, C): \text{Fun}(L, C) \rightarrow \text{Fun}(K, C)$  is conservative.*

Given this, the pointwise criterion follows easily.

**35.2. Theorem** (Pointwise criterion for isomorphisms in functor categories). *Let  $C$  be a quasicategory and  $X$  a simplicial set. Then an edge of  $\text{Fun}(X, C)$  is an isomorphism if and only if it is a pointwise isomorphism.*

*Proof using (35.1).* Consider the inclusion  $j: \text{Sk}_0 X \hookrightarrow X$  of the 0-skeleton (18.1), so that  $\text{Sk}_0 X = \coprod_{x \in X_0} \Delta^0$ . Then  $j^*: \text{Fun}(X, C) \rightarrow \text{Fun}(\text{Sk}_0 X, C)$  is conservative by (35.1). So it suffices to show pointwise isomorphisms in  $\text{Fun}(\text{Sk}_0 X, C)$  are isomorphisms. This is clear from the evident isomorphism

$$\text{Fun}(\text{Sk}_0 X, C) \approx \prod_{x \in X_0} C$$

which implies an isomorphism  $h \text{Fun}(\text{Sk}_0 X, C) \approx \prod_{x \in X_0} hC$ , so the claim follows from the “pointwise criterion” for ordinary categories.  $\square$

We now prove (35.1), using ideas from [Lur09, §3.1.1].

*Proof of (35.1).* Suppose  $C$  is a quasicategory, and  $j: K \rightarrow L$  a monomorphism of simplicial sets which is a bijection on vertices. We want to show that  $p = \text{Fun}(j, C)$  is conservative. By the Joyal lifting theorem (32.14) it suffices to show that for every  $n \geq 2$  and every commutative diagram of the form

$$\begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & \Lambda_0^n \xrightarrow{u} \text{Fun}(L, C) \\ & & \downarrow \text{ } \nearrow a \\ & & \Delta^n \xrightarrow{v} \text{Fun}(K, C) \end{array} \quad \begin{array}{c} \downarrow p = \text{Fun}(j, C) \end{array}$$

a lift  $a$  exists whenever  $pf = v|_{\Delta^{\{0,1\}}}: \Delta^1 \rightarrow \text{Fun}(K, C)$  represents an isomorphism in  $\text{Fun}(K, C)$ .

To do this we will “factor” this lifting problem through a commutative diagram of the form

$$\begin{array}{ccccc} \Delta^{\{0,1\}} & \xrightarrow{f} & \Lambda_0^n & \xrightarrow{u'} & \text{Fun}(V, C) \longrightarrow \text{Fun}(L, C) \\ & & \downarrow & \nearrow b & \downarrow q = \text{Fun}(h, C) \downarrow p = \text{Fun}(j, C) \\ & & \Delta^n & \xrightarrow{v'} & \text{Fun}(U, C) \longrightarrow \text{Fun}(K, C) \end{array}$$

and produce a lift  $b$  in the left-hand square. Here  $h: U \rightarrow V$  is the evident inclusion

$$h: (\{0\} \times L) \cup (\Delta^1 \times K) \rightarrow \Delta^1 \times L,$$

and the right-hand square is obtained by the evident restriction to the inclusion of subcomplexes  $\{1\} \times K \rightarrow \{1\} \times L$ , which is canonically isomorphic to  $j$ .

Let  $r: \Delta^1 \times \Delta^n \rightarrow \Delta^n$  be the unique functor given on objects by  $r(x, y) = y$  if  $(x, y) \neq (0, 1)$ , and  $r(0, 1) = 0$ , i.e., the unique natural transformation  $\langle 0023 \dots n \rangle \rightarrow \langle 0123 \dots n \rangle = \text{id}_{\Delta^n}$  of functors  $\Delta^n \rightarrow \Delta^n$ . Note that  $r(\{0\} \times \Delta^n) \subseteq \Lambda_0^n$  and  $r(\Delta^1 \times \Lambda_0^n) \subseteq \Lambda_0^n$ . We define the left-hand square in the diagram above to be adjoint to the large rectangle in

$$\begin{array}{ccccc} (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n) & \xrightarrow{r} & \Lambda_0^n & \xrightarrow{u} & \text{Fun}(L, C) \\ \downarrow & & \downarrow & \nearrow b' & \downarrow p = \text{Fun}(j, C) \\ \Delta^1 \times \Delta^n & \xrightarrow{r} & \Delta^n & \xrightarrow{v} & \text{Fun}(K, C) \end{array}$$

so that the desired lift  $b$  will correspond to a choice of lift  $b'$ . Next note that in turn this new lifting problem is adjoint to one of the form

$$\begin{array}{ccc} \{0\} & \longrightarrow & \text{Fun}(T, C) \\ \downarrow & \nearrow \bar{b} & \downarrow \text{Fun}(i, C) \\ \Delta^1 & \xrightarrow{\bar{v}} & \text{Fun}(S, C) \end{array}$$

where  $i: S \rightarrow T$  is the evident inclusion  $(\Delta^n \times K) \cup (\Lambda_0^n \times L) \rightarrow \Delta^n \times L$ . Note that  $i$  is a bijection on vertices since  $j$  is, and therefore the proposition (34.2) on lifting pointwise isomorphisms applies (with  $D = *$ ). Thus, to finish the proof, it suffices to show that  $\bar{v}$  represents a pointwise isomorphism

To prove this, it suffices to show that for every vertex  $y \in (\Delta^n)_0$ , the composition  $\Delta^1 \xrightarrow{\bar{v}} \text{Fun}(S, C) \rightarrow \text{Fun}(\{y\} \times K, C)$  represents an isomorphism (since all vertices of  $S$  are in  $\Delta^n \times K$ ). By adjunction this composition is isomorphic to the composite  $g_y$  of

$$\Delta^1 \times \{y\} \rightarrow \Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n \xrightarrow{v} \text{Fun}(K, C).$$

When  $y \neq 1$  we have that  $r|\Delta^1 \times \{y\} = \langle yy \rangle$ , and so  $g_y$  represents an identity morphism in  $\text{Fun}(K, C)$ , which is automatically an isomorphism. On the other hand, when  $y = 1$ , we have that  $r|\Delta^1 \times \{1\} = \langle 01 \rangle$ , and we see that  $g_1$  is exactly the map  $pf$ , which represents an isomorphism by hypothesis.  $\square$

**35.3. Exercise.** Show that if  $f, g: C \rightarrow D$  are naturally isomorphic functors between quasicategories, then their restrictions  $f^{\text{core}}, g^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$  to cores are also naturally isomorphic. Conclude that if  $f: C \rightarrow D$  is a categorical equivalence between quasicategories, then the restriction  $f^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$  of  $f$  to cores is a categorical equivalence of quasigroupoids.

## Part 6. Isofibrations

### 36. MORE RESULTS ON ISOMORPHISMS AND ISOFIBRATIONS

**36.1. Pullbacks of cores.** Recall that for a quasicategory  $C$ , the core  $C^{\text{core}} \subseteq C$  is the maximal quasigroupoid in  $C$  (12.8). The following says that maximal quasigroupoids are preserved by certain kinds of pullbacks.

**36.2. Proposition.** *Let*

$$\begin{array}{ccc} C' & \xrightarrow{u} & C \\ q \downarrow & & \downarrow p \\ D' & \xrightarrow{v} & D \end{array}$$

*be a pullback square of simplicial sets such that the objects are quasicategories and  $p$  is an inner fibration. An edge  $f \in C'_1$  is an isomorphism in  $C'$  if and only if  $u(f) \in C_1$  and  $q(f) \in D'_1$  are isomorphisms in  $C$  and  $D'$  respectively. Thus the induced map  $(C')^{\text{core}} \rightarrow C^{\text{core}} \times_{D^{\text{core}}} (D')^{\text{core}}$  on cores is an isomorphism.*

*Proof.* This is a straightforward application of Joyal lifting (32.14): as  $q$  is an inner fibration and  $q(f)$  is an isomorphism, to show  $f$  is an isomorphism we must produce a lift in every lifting problem

described by the left-hand square in

$$\begin{array}{ccccc}
 \Delta^{\{0,1\}} & \xrightarrow{\quad} & \Lambda_0^n & \xrightarrow{\quad} & C' & \xrightarrow{u} & C \\
 & \searrow f & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow p \\
 & & \Delta^n & \longrightarrow & D' & \longrightarrow & D
 \end{array}$$

Because  $u(f)$  is an isomorphism, we know a lift exists in the large rectangle by Joyal lifting, and the desired lift exists because the right-hand square is a pullback.

Recall that an  $n$ -dimensional cell  $a$  of a quasicategory is in the core if and only if all of its edges  $a_{ij}$  are isomorphisms. Given this, the assertion about pullbacks of cores is immediate.  $\square$

36.3. *Exercise.* Show that the inclusion  $C^{\text{core}} \rightarrow C$  of the core of a quasicategory is an isofibration.

36.4. **The walking isomorphism.** Let

$$\text{Horn} = \{ \Lambda_j^n \subset \Delta^n \mid n \geq 1, 0 \leq j \leq n \} = \text{RHorn} \cup \text{LHorn}$$

denote the set of all horn inclusions. A map is **anodyne** if it is in  $\overline{\text{Horn}}$ , the weak saturation of the set of horn inclusions, and is a **Kan fibration** if it is in  $\text{KanFib} := \text{Horn}^\square$ .

**anodyne**  
**Kan fibration**

36.5. *Example.* A simplicial set  $X$  is a Kan complex iff  $X \rightarrow \Delta^0$  is a Kan fibration.

Let  $\text{Iso}$  be the **walking isomorphism**, i.e., the category with two objects 0 and 1, and a unique isomorphism between them. Its nerve  $N\text{Iso}$  is a simplicial set, which by abuse of notation I will also denote  $\text{Iso}$ . Let  $u: \Delta^1 \rightarrow \text{Iso}$  be the inclusion representing the unique map  $0 \rightarrow 1$  in  $\text{Iso}$ .

**walking isomorphism**

36.6. **Proposition.** *The map  $u: \Delta^1 \rightarrow \text{Iso}$  is anodyne.*

*Proof.* The  $k$ -dimensional cells of  $\text{Iso}$  are in one-to-one correspondence with sequences  $(x_0 x_1 \cdots x_k)$  with  $x_i \in \{0, 1\}$ . For each  $k \geq 0$  there are exactly two *non-degenerate*  $k$ -dimensional cells  $u_k$  and  $v_k$ , corresponding respectively the alternating sequences  $(0101 \dots)$  and  $(1010 \dots)$  of length  $k+1$ . We also write  $u_k, v_k: \Delta^k \rightarrow \text{Iso}$  for the maps representing these non-degenerate cells.

Let  $F_k \subset \text{Iso}$  be the smallest subcomplex containing  $u_k$ . Observe that for a simplicial operator  $f: [d] \rightarrow [k]$  we have  $u_k f = (x_0 x_1 \cdots x_d)$  with  $x_i \equiv f(i) \pmod{2}$ . In particular,

- $u_k \langle 1 \dots k \rangle = v_{k-1}$ ,
- $u_k \langle 0 \dots k-1 \rangle = u_{k-1}$ ,
- $u_k \langle 0, 1, \dots, \widehat{i}, \dots, k-1, k \rangle$  is a degenerate cell associated to  $u_{k-2}$  if  $i = 1, \dots, k-1$ .

From this we can see that the only non-degenerate cells of  $F_k \setminus F_{k-1}$  are  $u_k$  and  $v_{k-1} = u_k \langle 1 \dots k \rangle$ . Therefore  $\text{Iso} = \bigcup_k F_k$ ,  $F_1 = u(\Delta^1)$ , and the commutative square

$$\begin{array}{ccc}
 \Lambda_0^k & \longrightarrow & F_{k-1} \\
 \downarrow & & \downarrow \\
 \Delta^k & \xrightarrow{u_k} & F_k
 \end{array}$$

is a pushout square for all  $k \geq 1$  by (18.4), since it is a pullback, and any cell in the complement of  $F_{k-1} \subset F_k$  is the image of a unique cell under the map  $u_k$ . It follows that  $u$  is anodyne.  $\square$

As an immediate consequence, any map  $f: \Delta^1 \rightarrow C$  can be extended over  $\text{Iso}$  when  $C$  is a quasigroupoid, since these are Kan complexes (33.2). We can easily refine this to give a criterion for  $f$  to represent an isomorphism in a general quasicategory.

36.7. **Proposition.** *Let  $C$  be a quasicategory, and  $f: \Delta^1 \rightarrow C$  a map. Then there exists  $f': \text{Iso} \rightarrow C$  with  $f'u = f$  if and only if  $f$  represents an isomorphism in  $C$ .*

*Proof.* ( $\implies$ ) Clear: consider induced maps  $[1] \xrightarrow{u} \text{Iso} \rightarrow hC$  on homotopy categories. ( $\impliedby$ ) If  $f$  represents an isomorphism then it factors  $\Delta^1 \rightarrow C^{\text{core}} \subseteq C$  through the core, which is a Kan complex, so an extension along the anodyne map  $u$  to a map  $\text{Iso} \rightarrow C^{\text{core}} \subseteq C$  exists.  $\square$

36.8. *Exercise.* Let  $Z$  be the complex of (23.6), and let  $F: \Delta^1 \rightarrow Z$  be the map representing the edge  $f \in Z_1$ . Show that  $F$  is anodyne, and state and prove an analogue of (36.7) with  $Z$  in place of  $\text{Iso}$ .

36.9. *Remark.* Let  $Z'$  be the complex of (23.7), and let  $F': \Delta^1 \rightarrow Z'$  be the map representing the edge  $f \in Z'_1$ . Note that  $Z'$  is isomorphic to the subcomplex  $X \subset \text{Iso}$  which is the union of the images of 2-dimensional cells (010) and (101), and that under this isomorphism  $F'$  corresponds to  $u$ . It is straightforward to show that  $F'$  also satisfies an analogue of (36.7), by the same idea as the proof of (36.8).

However, it turns out that  $F': \Delta^1 \rightarrow Z'$  is *not* anodyne, and that  $Z' \approx X \rightarrow \text{Iso}$  is not a categorical equivalence. In particular, a map  $X \rightarrow C$  to a quasicategory can fail to extend along  $X \subset \text{Iso}$ .

36.10. **Isofibrations and Kan fibrations.** As we have seen, a functor  $f: C \rightarrow D$  between quasicategories is an isofibration if (1) it is an inner fibration, and (2) every diagram

$$\begin{array}{ccccc} \{j\} & \longrightarrow & C^{\text{core}} & \longrightarrow & C \\ \downarrow & \nearrow g & \downarrow p^{\text{core}} & & \downarrow p \\ \Delta^1 & \xrightarrow{f} & D^{\text{core}} & \longrightarrow & D \end{array}$$

with  $j = 0$  admits a lift  $g$ . Furthermore, it is equivalent to require (2') instead of (2), where (2') is the same statement with  $j = 1$ .

In particular,  $C \rightarrow *$  is an isofibration for any quasicategory  $C$  (because identity maps are isomorphisms). Given a functor  $p: C \rightarrow D$  between quasicategories, we write  $p^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$  for its restriction to cores.

36.11. **Proposition.** *Let  $p: C \rightarrow D$  be an inner fibration between quasicategories. Then the following are equivalent.*

- (1)  $p$  is an isofibration.
- (2)  $p^{\text{core}}$  is an isofibration.
- (3)  $p^{\text{core}}$  is a Kan fibration.

*In particular, an inner fibration between Kan complexes is an isofibration if and only if it is a Kan fibration.*

*Proof.* (1)  $\iff$  (2). That  $p^{\text{core}}$  is an inner fibration is an elementary argument (15.11). It is also immediate that condition (2) holds for  $p$  iff it holds for  $p^{\text{core}}$ .

(2)  $\iff$  (3). Suppose  $q$  is an inner fibration between Kan complexes, e.g.,  $q = p^{\text{core}}$ . Then Joyal lifting (32.14) implies that  $(\Lambda_j^n \subset \Delta^n) \sqcap q$  for all  $n \geq 2$ , and all  $0 \leq j \leq n$ . The claim follows from the observation that  $q$  is an isofibration iff  $(\Lambda_j^1 \subset \Delta^1) \sqcap q$  for either of (or both of)  $j = 0$  or  $j = 1$ .  $\square$

36.12. *Exercise.* Give an example of an inner fibration between Kan complexes which is not a Kan fibration.

We have another “lifting criterion” for isofibrations involving the walking isomorphism.

36.13. **Proposition.** *A map  $p$  between quasicategories is an isofibration iff (1) it is an inner fibration, and (2'')  $(\{0\} \subset \text{Iso}) \sqcap p$ .*



*Proof.* ( $\Leftarrow$ ) Straightforward, using the fact (36.7) that every  $f: \Delta^1 \rightarrow D$  representing an isomorphism factors through a map  $N(\text{Iso}) \rightarrow D$ .

( $\Rightarrow$ ) Solve a lifting problem ( $a: \{0\} \rightarrow C, b: \text{Iso} \rightarrow D$ ) of type  $(\{0\} \subset \text{Iso}) \boxtimes p$  by solving two lifting problems in sequence

$$\begin{array}{ccc} \{0\} & \xrightarrow{a} & C^{\text{core}} \rightarrow C \\ \downarrow & \nearrow t & \downarrow p^{\text{core}} \downarrow p \\ \Delta^1 & \xrightarrow{b'u} & D^{\text{core}} \rightarrow D \end{array} \quad \begin{array}{ccc} \Delta^1 & \xrightarrow{t} & C^{\text{core}} \rightarrow C \\ \downarrow u & \nearrow s & \downarrow p^{\text{core}} \downarrow p \\ \text{Iso} & \xrightarrow{b'} & D^{\text{core}} \rightarrow D \end{array}$$

$b': \text{Iso} \rightarrow D^{\text{core}}$  is the factorization of  $b$  through the core, and  $u: \Delta^1 \rightarrow \text{Iso}$  represents the morphism  $0 \rightarrow 1$  in  $\text{Iso}$ . Since  $p$  is an isofibration, there exists a lift  $t$  in the left-hand square which represents an isomorphism in  $C$ . Both  $t$  and  $b$  land in the relevant cores, and so it suffices to produce a lift in the right-hand diagram, which exists because  $u$  is anodyne (36.6) and  $p^{\text{core}}$  is a Kan fibration by (36.11).  $\square$

In other words, the isofibrations are precisely the maps *between quasicategories* which are contained in  $(\text{InnHorn} \cup \{\{0\} \subset \text{Iso}\})^{\boxtimes}$ . As an immediate consequence we get the following.

**36.14. Proposition.** *If  $p: C \rightarrow D$  is a trivial fibration between quasicategories, then it is an isofibration.*

*Proof.* All inner horn inclusions as well as  $\{0\} \rightarrow \text{Iso}$  are monomorphisms, so this follows from (36.13).  $\square$

**36.15. Remark.** We have deliberately excluded maps between non-quasicategories from the definition of isofibration. The correct generalization of isofibration to arbitrary simplicial sets is called “categorical fibration”, and will be discussed later (39).

### 37. LIFTING PROPERTIES FOR ISOFIBRATIONS

**37.1. A useful lifting result.** In view of the pointwise criterion for natural isomorphisms (35.2), the lifting property for pointwise isomorphisms (34.2) can be reformulated as follows.

**37.2. Proposition.** *Let  $p: C \rightarrow D$  be an inner fibration between quasicategories, and let  $i: S \rightarrow T$  be a monomorphism of simplicial sets such that  $i_0: S_0 \rightarrow T_0$  is a bijection. Then the induced pullback-hom map  $p^{\square i}: \text{Fun}(T, C) \rightarrow \text{Fun}(S, C) \times_{\text{Fun}(S, D)} \text{Fun}(T, D)$  is a conservative isofibration.*

*Proof.* We know that  $p^{\square i}$  is an inner fibration between quasicategories. Consider a commutative square of the form

$$\begin{array}{ccc} \{0\} & \xrightarrow{\quad} & \text{Fun}(T, C) \\ \downarrow & \nearrow t & \downarrow p^{\square i} \\ \Delta^1 & \xrightarrow{v} & \text{Fun}(S, C) \times_{\text{Fun}(S, D)} \text{Fun}(T, D) \end{array}$$

where  $v$  represents an isomorphism. Then the composite of  $v$  with the projection to  $\text{Fun}(S, C)$  certainly represents a pointwise isomorphism, and thus (34.2) applies to give a lift  $t$  which is necessarily a pointwise isomorphism, and hence an isomorphism by (35.2). That the functor  $p^{\square i}$  is also clear, e.g., using the pointwise criterion.  $\square$

As a consequence we get the following.

**37.3. Corollary.** *Let  $C$  be a quasicategory, and  $i: S \rightarrow T$  a monomorphism of simplicial sets which induces a bijection on vertices. Then the fibers of  $\text{Fun}(T, C) \rightarrow \text{Fun}(S, C)$  over any vertex of the target are quasigroupoids.*

*Proof.* An immediate consequence of (37.2).  $\square$

**37.4. Enriched lifting for isofibrations.** Using the same ideas that prove (37.2), we can show the following.

**37.5. Proposition.** *Let  $p: C \rightarrow D$  be an isofibration between quasicategories, and  $i: K \rightarrow L$  any monomorphism of simplicial sets. Then the induced pullback-hom map*

$$p^{\square i}: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$$

*is an isofibration.*

*Proof.* Let  $\mathcal{C}$  be the class of monomorphisms  $i$  of simplicial sets such that  $p^{\square i}$  is an isofibration whenever  $p: C \rightarrow D$  is an isofibration between quasicategories. I claim that the class  $\mathcal{C}$  is weakly saturated. First note that since  $p$  is an inner fibration between quasicategories, so is  $p^{\square i}$  for any monomorphism  $i$  since  $\overline{\text{InnHorn}} \square \text{Cell} \subseteq \overline{\text{InnHorn}}$  (20.2). Given this,  $p^{\square i}$  is an isofibration iff  $(\{0\} \subset \text{Iso}) \square p^{\square i}$  (36.13). Finally, recall that for any map  $j$  we have  $j \square p^{\square i}$  iff  $(i \square j) \square p$  iff  $i \square p^{\square j}$  (19.5), and so  $i \in \mathcal{C}$  iff (1)  $i \in \overline{\text{Cell}}$  and (2)  $i \square p^{\square(\{0\} \subset \text{Iso})}$  for every isofibration  $p$ .

Thus, to prove the proposition it suffices to show  $\text{Cell} \subseteq \mathcal{C}$ . We have that  $(\partial \Delta^n \subset \Delta^n)$  is in  $\mathcal{C}$  when  $n \geq 1$  by (37.2) and the fact that isofibrations are inner fibrations by definition, while  $(\partial \Delta^0 \subset \Delta^0)$  is in  $\mathcal{C}$  tautologically since  $p^{\square(\partial \Delta^0 \subset \Delta^0)} = p$  is an isofibration by hypothesis.  $\square$

**37.6. Example.** If  $C$  is a quasicategory and  $i: K \rightarrow L$  a monomorphism, then  $i^*: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C)$  is an isofibration by (37.5) and the fact that  $C \rightarrow *$  is an isofibration.

**37.7. Covering homotopy extension property.** Here is a very handy consequence of enriched lifting for isofibrations (37.5). Consider maps  $i: K \rightarrow L$  and  $p: C \rightarrow D$  of simplicial sets, with pullback-hom map

$$p^{\square i}: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D).$$

A vertex  $(u, v)$  in the target of  $p^{\square i}$  corresponds to a lifting problem of type  $i \square p$ , and this lifting problem has a solution if and only if the vertex  $(u, v)$  is in the image of a vertex  $s$  in  $\text{Fun}(L, C)$ .

An edge  $e$  in the target of  $p^{\square i}$  from vertex  $(u_0, v_0)$  to vertex  $(u_1, v_1)$  corresponds to a commutative square

$$\begin{array}{ccc} K \times \Delta^1 & \xrightarrow{\tilde{u}} & C \\ i \times \text{id} \downarrow & & \downarrow p \\ L \times \Delta^1 & \xrightarrow{\tilde{v}} & D \end{array}$$

such that  $\tilde{u}|_{K \times \{x\}} = u_x$  and  $\tilde{v}|_{L \times \{x\}} = v_x$  for  $x = 0, 1$ . We think of such an edge  $e$  as a “deformation” relating the two lifting problems  $(u_0, v_0)$  and  $(u_1, v_1)$ . In certain circumstances, a lifting problem admits a solution if it admits a suitable deformation to a solvable lifting problem. In our setting this happens when a lifting problem is *isomorphic* to a solvable one, a principle called “covering homotopy extension”.

**37.8. Proposition** (Covering homotopy extension for isofibrations). *Let  $i: K \rightarrow L$  be a monomorphism of simplicial sets, and  $p: C \rightarrow D$  an isofibration of quasicategories. If two lifting problems  $(u_0, v_0)$  and  $(u_1, v_1)$  of type  $i \square p$  are represented by isomorphic objects of  $\text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$ , then  $(u_0, v_0)$  admits a lift if and only if  $(u_1, v_1)$  admits a lift.*

*Proof.* Let  $e$  be such an isomorphism. I’ll show that if  $(u_0, v_0)$  admits a lift  $s: L \rightarrow C$ , then  $(u_1, v_1)$  also admits a lift. The hypotheses on  $i$  and  $p$ , together with the fact that  $p^{\square i}$  is an isofibration (37.5),

imply that a lift  $t$  exists in the commutative square

$$\begin{array}{ccc} \{0\} & \xrightarrow{s} & \mathrm{Fun}(L, C) \\ \downarrow & \nearrow t & \downarrow p \square i \\ \Delta^1 & \xrightarrow{e} & \mathrm{Fun}(K, C) \times_{\mathrm{Fun}(K, D)} \mathrm{Fun}(L, D) \end{array}$$

Then the vertex  $t(1) \in \mathrm{Fun}(L, C)_0$  gives the desired lift for  $(u_1, v_1)$ . The proof of the reverse direction is similar, using  $(\{1\} \subset \Delta^1)$  instead of  $(\{0\} \subset \Delta^1)$ .  $\square$

**37.9. Remark.** In the proof of (37.8), the isofibration condition is used to lift the edge  $e$  to a suitable edge  $t$ , but the lift  $t$  need not itself be an isomorphism. In fact, an analogous covering homotopy extension property can be proved in other contexts where the edge  $e$  can be shown to lift.

### 38. ISOFIBRATIONS AND CATEGORICAL EQUIVALENCES

The goal of this section is to show that the isofibrations which are categorical equivalences are exactly the trivial fibrations between quasicategories.

**38.1. Fiberwise categorical equivalence.** Say that a functor  $p: C \rightarrow D$  between quasicategories is a **fiberwise categorical equivalence**<sup>24</sup> if there exists

fiberwise categorical  
equivalence

- $s: D \rightarrow C$  such that  $ps = \mathrm{id}_D$ , and
- $h: C \times \Delta^1 \rightarrow D$  representing a natural isomorphism  $\mathrm{id}_C \rightarrow sp$ , such that  $ph = p\pi_C$ , so that the diagram

$$\begin{array}{ccc} C \times \partial\Delta^1 & \xrightarrow{(\mathrm{id}_C, sp)} & C \\ \downarrow & \nearrow h & \downarrow p \\ C \times \Delta^1 & \xrightarrow{\pi_C} C \xrightarrow{p} & D \end{array}$$

commutes.

Any fiberwise categorical equivalence is a categorical equivalence, since  $s$  is a categorical inverse to  $p$ .

**38.2. Remark.** Here is one way to think about the identity  $ph = p\pi_C$ : it says that the map  $p_*: \mathrm{Fun}(C, p): \mathrm{Fun}(C, C) \rightarrow \mathrm{Fun}(C, D)$  sends the isomorphism represented by  $h$  to the identity map of the object  $p$ .

**38.3. Exercise.** Show that fiberwise categorical equivalence can be reformulated in terms of the relative function complex of (23.10): a functor  $p: C \rightarrow D$  is a fiberwise categorical equivalence iff there exists (i) an object  $s \in \mathrm{Fun}_{/D}(D, C)_0$  and (ii) an isomorphism  $h: \mathrm{id}_C \rightarrow sp$  in  $\mathrm{Fun}_{/C}(C, C)_1$ .

**38.4. Exercise.** Show that the term “fiberwise” is justified: for each object  $d \in D_0$  the projection  $p^{-1}(d) \rightarrow \{d\}$  of a fiber to its image is a categorical equivalence.

**38.5. Exercise.** Give an example of a functor  $p: C \rightarrow D$  between quasicategories such that  $p^{-1}(d) \rightarrow \{d\}$  is a categorical equivalence for each  $d \in D_0$ , but  $p$  is not itself a fiberwise categorical equivalence. (Hint: there are examples where  $C$  and  $D$  are ordinary categories.)

**38.6. Exercise.** Show that any base change of a fiberwise categorical equivalence  $p$  along a functor from a quasicategory is also a fiberwise categorical equivalence.

<sup>24</sup>I should change this terminology. It could be confused with the “fiberwise criterion” for homotopy equivalence”. How about “vertical categorical equivalence”?

### 38.7. Isofibrations which are categorical equivalences are trivial fibrations.

**38.8. Proposition.** *A functor  $p: C \rightarrow D$  of quasicategories is a trivial fibration if and only if it is both a categorical equivalence and an isofibration.*

*Proof.* We have already seen that trivial fibrations of quasicategories are categorical equivalences (23.11) and isofibrations (36.14). The other direction is a consequence of the two following propositions (38.9) and (38.10), which show that for an isofibration  $p$ , categorical equivalence  $\Rightarrow$  fiberwise categorical equivalence  $\Rightarrow$  trivial fibration.  $\square$

**38.9. Proposition.** *If  $p: C \rightarrow D$  is an isofibration and a categorical equivalence, then it is a fiberwise categorical equivalence.*

*Proof.* Choose a categorical inverse  $g: D \rightarrow C$  for  $p$ , for which there are natural isomorphisms  $gp \approx \text{id}_C$  and  $pg \approx \text{id}_D$ . We will “deform”  $g$  to a functor  $s: D \rightarrow C$  equipped with natural isomorphisms  $1_{\text{id}_D}: ps \rightarrow \text{id}_D$  and  $h: sp \rightarrow \text{id}_C$  which exhibit  $p$  as a fiberwise categorical equivalence.

*Step 1.* Choose  $v: D \times \Delta^1 \rightarrow D$  representing a natural isomorphism  $pg \rightarrow \text{id}_D$ . Since  $p$  is an isofibration so is  $\text{Fun}(D, p): \text{Fun}(D, C) \rightarrow \text{Fun}(D, D)$  (37.5), and so a lift  $\alpha$  representing an isomorphism exists in

$$\begin{array}{ccc} \{0\} & \xrightarrow{g} & \text{Fun}(D, C) \\ \downarrow & \nearrow \alpha & \downarrow p^* \\ \Delta^1 & \xrightarrow{\tilde{v}} & \text{Fun}(D, D) \end{array}$$

Let  $s := \alpha(1) \in \text{Fun}(D, C)_0$ , so that  $s: D \rightarrow C$  is a functor such that  $ps = \text{id}_D$ , and  $\alpha: g \rightarrow s$  is a natural isomorphism. Thus we have natural isomorphisms  $\text{id}_C \approx gp \approx sp$  of functors  $C \rightarrow C$ , i.e., there exists a natural isomorphism  $w: sp \rightarrow \text{id}_C$ , represented by an edge  $w \in \text{Fun}(C, C)_1$ .

*Step 2.* We have functors  $\text{Fun}(C, C) \xrightarrow{p_*} \text{Fun}(C, D) \xrightarrow{s_*} \text{Fun}(C, C)$  induced by postcomposition with  $p$  and  $s$ . We can apply various iterations of these functors to the natural isomorphism  $w: sp \rightarrow \text{id}_C$ , some of which are pictured in the following solid arrow diagram of objects and isomorphisms in  $\text{Fun}(C, C)$  and  $\text{Fun}(C, D)$ :

$$\begin{array}{ccc} \begin{array}{ccc} & \text{id}_C & \\ w \nearrow & & \searrow h \\ sp & & \\ (sp)_*(w) \searrow & & \nearrow sp \end{array} & \in \text{Fun}(C, C) & \xRightarrow{p_*} & \begin{array}{ccc} & p & \\ p_*(w) \nearrow & & \searrow 1_p \\ p = psp & & p \\ p_*(w) = (psp)_*(w) \searrow & & \nearrow p \end{array} \\ & & & \in \text{Fun}(C, D) \end{array}$$

Note that  $ps = \text{id}_D$  implies that  $p_*(sp) = p$  and  $(sp)_*(sp) = sp$ , and that  $(psp)_*(w) = p_*(w)$ . The right hand diagram “commutes” in  $\text{Fun}(C, D)$ , i.e., it represents the boundary of an element  $b \in \text{Fun}(C, D)_2$ , namely the degenerate cell  $b := (p_*(w))_{011}$  associated to the edge  $p_*(w): psp = p \rightarrow p$ .

The above picture is represented by a commutative square

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{a} & \text{Fun}(C, C) \\ \downarrow & \nearrow t & \downarrow \text{Fun}(X, p) = p_* \\ \Delta^2 & \xrightarrow{b} & \text{Fun}(C, D) \end{array}$$

in simplicial sets. Since  $p$  is an inner fibration and  $a|_{\Delta^{\{0,1\}}} = w$  represents an isomorphism, a lift  $t$  exists by Joyal lifting (32.14). Thus  $h := t|_{\Delta^{\{1,2\}}}: \Delta^1 \rightarrow \text{Fun}(C, C)$  is a natural isomorphism  $h: \text{id}_C \rightarrow sp$  such that  $p_*(h) = 1_p$ , i.e.,  $ph = p\pi_C$ . We have thus produced  $s: D \rightarrow C$  and  $h: C \times \Delta^1 \rightarrow C$  exhibiting  $p$  as a fiberwise categorical equivalence, as desired.  $\square$

**38.10. Proposition.** *If  $p: C \rightarrow D$  is an isofibration and a fiberwise categorical equivalence, then  $p$  is a trivial fibration.*

*Proof.* Given such a functor  $p$ , consider a lifting problem

$$\begin{array}{ccc} K & \xrightarrow{u} & C \\ i \downarrow & & \downarrow p \\ L & \xrightarrow{v} & D \end{array}$$

with  $i$  a monomorphism. Since  $p$  is an isofibration, the covering homotopy extension property (37.8) applies, so it suffices to show that this lifting problem is isomorphic to one we can solve. In fact, the data  $(s: D \rightarrow C, h: C \times \Delta^1 \rightarrow C)$  of a fiberwise categorical equivalence provides us with such an isomorphism, via the commutative rectangle

$$\begin{array}{ccccc} K \times \Delta^1 & \xrightarrow{u \times \text{id}} & C \times \Delta^1 & \xrightarrow{h} & C \\ \downarrow i \times \text{id} & & \downarrow p \times \text{id} & & \downarrow p \\ L \times \Delta^1 & \xrightarrow{v \times \text{id}} & D \times \Delta^1 & \xrightarrow{\pi_D} & D \end{array}$$

(Note that  $\pi_D(p \times \text{id}) = p\pi_C = ph$ .) Over  $\{0\} \subset \Delta^1$  this is the original lifting problem  $(u, v)$ , while over  $\{1\} \subset \Delta^1$  we get a lifting problem  $(spu, v)$  since  $sp = h|_{C \times \{1\}}$  and  $pspu = pu = vi$ . The diagram provides a morphism  $e: (u, v) \rightarrow (spu, v)$  in  $\text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$ , whose projection to  $\text{Fun}(L, D)$  is the identity map of  $v$ , and whose projection to  $\text{Fun}(K, C)$  is represented by  $h(u \times \text{id})$ , which is an isomorphism since  $h$  is. Thus  $e$  is itself an isomorphism by (36.2).

Finally, we know a lift for  $(spu, v)$ , namely  $sv: L \rightarrow C$  (since  $svi = spu$  and  $psv = v$ ).  $\square$

**38.11. Corollary.** *A quasicategory  $C$  is categorically equivalent to the terminal category  $\Delta^0$  if and only if  $C \rightarrow \Delta^0$  is a trivial fibration.*

*Proof.* Immediate from (38.8) and the fact that  $C \rightarrow \Delta^0$  is an isofibration (32.8).  $\square$

**38.12. Exercise.** Let  $C$  be a quasicategory and let  $\pi: C \rightarrow hC$  be the tautological map to its homotopy category. Show that

- (1)  $\pi$  is an isofibration, and
- (2)  $(\partial\Delta^n \subset \Delta^n) \boxtimes \pi$  for  $n = 0, 1, 2$ .

Conclude that  $\pi$  is a categorical equivalence if and only if  $(\partial\Delta^n \subset \Delta^n) \boxtimes \pi$  for all  $n \geq 3$ .

**38.13. Monomorphisms which are categorical equivalences.** We can now give the following “lifting characterization” of monomorphisms which are categorical equivalences.

**38.14. Proposition.** *Let  $j: K \rightarrow L$  be a monomorphism of simplicial sets. Then  $j$  is a categorical equivalence if and only if  $\text{Fun}(j, C): \text{Fun}(L, C) \rightarrow \text{Fun}(K, C)$  is a trivial fibration for all quasicategories  $C$ .*

*Proof.* Straightforward, using the fact that  $\text{Fun}(j, C)$  is an isofibration since  $j$  is mono (37.5), and that isofibrations which are categorical equivalences are trivial fibrations (38.8).  $\square$

**38.15. Remark.** The class  $\text{CatEq} \cap \overline{\text{Cell}}$  of monomorphisms which are categorical equivalences is a weakly saturated class: (38.14) says it is the intersection of  $\overline{\text{Cell}}$  with the left complement of  $\{p^{\square \text{Cell}} \mid p: C \rightarrow *, C \in \text{qCat}\}$ . Clearly  $\overline{\text{InnHorn}} \subseteq \text{CatEq} \cap \overline{\text{Cell}}$  by (23.17).

However,  $\overline{\text{InnHorn}} \neq \text{CatEq} \cap \overline{\text{Cell}}$ . For instance, every inner anodyne map is a bijection on vertices, but the monomorphism  $\{0\} \rightarrow \text{Iso}$  is not bijective on vertices but is a categorical equivalence. Even if we restrict to morphisms in  $\text{CatEq} \cap \overline{\text{Cell}}$  which are bijections on vertices, we need not have an inner anodyne map, as the following example shows (38.16).

38.16. *Example* (Campbell's counterexample [Cam19]). Recall the simplicial set  $H^\ell = \Delta^2 / \Delta^{\{0,1\}}$  of (15.13), and write  $\pi: \Delta^2 \rightarrow \Delta^{\{0,1\}}$  for the evident quotient map. We have a commutative diagram

$$\begin{array}{ccccc} \Delta^1 & \xrightarrow{f} & H^\ell & \xleftarrow{g} & \Delta^1 \\ & \searrow \text{id} & \downarrow h & \swarrow \text{id} & \\ & & \Delta^1 & & \end{array}$$

where  $f$  and  $g$  represent the edges  $\pi(\langle 02 \rangle)$  and  $\pi(\langle 12 \rangle)$  in  $H^\ell$ , and  $h$  is the unique map such that  $h\pi = \langle 001 \rangle$ . Note that the maps  $f, g, h$  induce bijections on vertices.

The map  $g$  is actually inner anodyne: it is isomorphic to the cobase-change of the horn inclusion  $\Lambda_1^2 \rightarrow \Delta^2$  along the unique map  $\Lambda_1^2 \rightarrow \Delta^1$  which on vertices sends  $0, 1 \mapsto 0$  and  $2 \mapsto 1$ .

Thus  $g$  is a categorical equivalence, whence so are  $h$  and hence  $f$  by the 2-out-of-3 property (25.11). On the other hand, we have observed that  $f$  is not an inner anodyne map (15.14). Therefore  $f$  is a categorical equivalence which is a monomorphism and a bijection on vertices, but is not an inner anodyne map.

### 39. CATEGORICAL FIBRATIONS

We can now give the following “lifting characterization” of isofibrations.

**39.1. Proposition.** *A map  $p: C \rightarrow D$  with  $D$  a quasicategory is an isofibration if and only if  $j \sqsupset p$  for every  $j: K \rightarrow L$  which is both a monomorphism and a categorical equivalence.*

*Proof.* ( $\Leftarrow$ ) Immediate from the characterization of isofibrations as maps between quasicategories in the right complement of  $\text{InnHorn} \cup \{\{0\} \subset \text{Iso}\}$  (36.13). (Note that  $p$  must in particular be an inner fibration, so  $C$  must be a quasicategory, since the inner horn inclusions are monomorphisms and categorical equivalences.)

( $\Rightarrow$ ) Suppose  $p$  is an isofibration. We have a commutative diagram

$$\begin{array}{ccccc} & & \text{Fun}(j, C) & & \\ & \text{Fun}(L, C) & \xrightarrow{\quad} & \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D) & \xrightarrow{q} \text{Fun}(K, C) \\ & \searrow p^{\square j} & & \downarrow & \downarrow \\ & & \text{Fun}(L, D) & \xrightarrow{\text{Fun}(j, D)} & \text{Fun}(K, D) \end{array}$$

in which  $p^{\square j}$ ,  $\text{Fun}(j, C)$ , and  $\text{Fun}(j, D)$  are isofibrations by (37.5), and  $\text{Fun}(j, C)$  and  $\text{Fun}(j, D)$  are categorical equivalences since  $j$  is. Therefore  $\text{Fun}(j, C)$  and  $\text{Fun}(j, D)$ , and hence the base-change  $q$ , are trivial fibrations by (38.8). Therefore  $q$  is a categorical equivalence (23.11), whence  $p^{\square j}$  is a categorical equivalence by 2-out-of-3 (25.11) and so a trivial fibration by (38.8). It follows that  $p^{\square j}$  is surjective on vertices, i.e.,  $j \sqsupset p$  as desired.  $\square$

Say that a map  $p: X \rightarrow Y$  of simplicial sets is a **categorical fibration** (or **Joyal fibration**) if and only if  $j \sqsupset p$  for all  $j$  which are monomorphisms and categorical equivalences. Then (39.1) says that isofibrations are precisely the categorical fibrations between quasicategories. I'll write  $\text{CatFib} = (\text{InnHorn} \cup \{\{0\} \rightarrow \text{Iso}\})^\square$  for the class of categorical fibrations.

**categorical fibration**  
**Joyal fibration**

**39.2. Proposition.** *A map  $p: X \rightarrow Y$  of simplicial sets is a trivial fibration if and only if it is a categorical fibration and a categorical equivalence.*

*Proof.* ( $\Rightarrow$ ) We know trivial fibrations are categorical equivalences (23.11), and it is clear that they are categorical fibrations by definition.

( $\Leftarrow$ ) If  $p$  is a categorical fibration and a categorical equivalence, factor  $p$  as  $X \xrightarrow{j} Z \xrightarrow{q} Y$  with  $j$  a monomorphism and  $q$  a trivial fibration, by the small object argument applied to  $\text{Cell}$ . In particular

$q$  is a categorical equivalence (23.11), and thus so is  $j$  by 2-out-of-3 (25.11), and therefore  $j \boxdot p$  by hypothesis. Thus the “retract trick” (15.18) exhibits  $p$  as a retract of  $q$ , whence  $p$  is also a trivial fibration.  $\square$

We have an enriched lifting property relating  $\overline{\text{Cell}}$  and  $\text{CatFib}$ .

**39.3. Proposition.** *If  $p: X \rightarrow Y$  is a categorical fibration and  $j: K \rightarrow L$  is a monomorphism, then*

$$q = p^{\square j}: \text{Fun}(L, X) \rightarrow \text{Fun}(K, X) \times_{\text{Fun}(K, Y)} \text{Fun}(L, Y)$$

*is a categorical fibration. Furthermore, if either  $j$  or  $p$  is also a categorical equivalence, then  $q$  is a trivial fibration and hence a categorical equivalence.*

*Proof.* To show that  $q$  is a categorical fibration, consider  $i: A \rightarrow B$  a monomorphism which is a categorical equivalence. We have  $i \boxdot q$  iff  $(i \square j) \boxdot p$ , so since  $p$  is a categorical fibration it suffices to show that the monomorphism  $i \square j$  is a categorical equivalence, and by (38.14) it suffices to show  $\text{Fun}(i \square j, C)$  is a trivial fibration for every quasicategory  $C$ . This map is isomorphic to  $r^{\square j}$  where  $r = \text{Fun}(i, C)$ . Note that  $r := \text{Fun}(i, C)$  is an isofibration (37.5) and a categorical equivalence since  $i$  is, and therefore a trivial fibration (38.8). Thus  $r^{\square j}$  is also a trivial fibration using  $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ , and hence  $\text{Fun}(i \square j, C)$  is a trivial fibration as desired.

If  $p$  is also a categorical equivalence, then it is a trivial fibration by (39.2), so  $q$  is a trivial fibration by  $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ .

If  $j$  is also a categorical equivalence, we want to show  $i \boxdot p^{\square j}$  for any monomorphism  $i$ . But we have  $i \boxdot p^{\square j}$  iff  $(i \square j) \boxdot p$  iff  $j \boxdot (p^{\square i})$ . By what we have just proved  $p^{\square i}$  is a categorical fibration, and therefore  $j \boxdot p^{\square i}$  by definition.  $\square$

#### 40. PATH FACTORIZATION

Recall the path category of a quasicategory  $D$ , defined to be the full subcategory

$$\text{Fun}^{\text{iso}}(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D)$$

spanned by the objects corresponding to functors  $\Delta^1 \rightarrow D$  which represent isomorphisms in  $D$ . I will sometimes write  $\widehat{D} := \text{Fun}^{\text{iso}}(\Delta^1, D)$  as a shorthand for this. The restriction maps along  $\{0\} \subset \Delta^1 \supset \{1\}$  induce functors  $D \xleftarrow{r_0} \widehat{D} \xrightarrow{r_1} D$ . Recall also (34.4) that functors  $\widehat{H}: C \rightarrow \text{Fun}^{\text{iso}}(\Delta^1, D)$  correspond exactly to maps  $H: C \times \Delta^1 \rightarrow D$  representing a natural isomorphism  $f_0 \rightarrow f_1$  of functors  $C \rightarrow D$  where  $f_i = r_i \widehat{H}_i$ .

**40.1. Remark.** If  $D$  is a Kan complex (i.e., a quasigroupoid), then  $\text{Fun}^{\text{iso}}(\Delta^1, D) = \text{Fun}(\Delta^1, D)$ .

**40.2. Warning.** The path category  $\text{Fun}^{\text{iso}}(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D)$  is not the same as the core  $\text{Fun}(\Delta^1, D)^{\text{core}} \subseteq \text{Fun}(\Delta^1, D)$ , and neither of these are the same as  $\text{Fun}(\Delta^1, D^{\text{core}})$ , unless  $D$  is a quasigroupoid: the path category is always a full subcategory, whereas the core is typically not a full subcategory.

**40.3. Lemma.** *Let  $D$  be a quasicategory. Then both restriction functors*

$$D \xleftarrow{r_0} \text{Fun}^{\text{iso}}(\Delta^1, D) \xrightarrow{r_1} D$$

*from the path category are trivial fibrations.*

*Proof.* We need to solve the lifting problem

$$\begin{array}{ccc} \partial \Delta^n & \longrightarrow & \text{Fun}^{\text{iso}}(\Delta^1, D) \twoheadrightarrow \text{Fun}(\Delta^1, D) \\ \downarrow & \nearrow & \downarrow r_i \\ \Delta^n & \longrightarrow & \text{Fun}(\{i\}, D) \end{array}$$

for all  $n \geq 0$  and  $i = 0, 1$ . When  $n = 0$  this is easy: any object of  $D$  is the source and target of an isomorphism in  $D$ , namely its identity map. For  $n \geq 1$  it suffices to find a lifting in the adjoint lifting problem

$$\begin{array}{ccc} \Delta^1 \times \{y\} & \xrightarrow{\quad f \quad} & D \\ & \searrow \scriptstyle (\{x\} \times \Delta^n) \cup_{\{x\} \times \partial \Delta^n} (\Delta^1 \times \partial \Delta^n) & \nearrow \\ & \Delta^1 \times \Delta^n & \end{array}$$

where  $y = 0$  if  $x = 0$  and  $y = n$  if  $x = 1$ . In either case we know by hypothesis that  $f$  represents an isomorphism in  $D$ , so a lift exists by the “pushout-product version” of Joyal lifting (34.5).  $\square$

**40.4. Lemma.** *If  $D$  is a quasicategory, then the map  $r = (r_0, r_1): \text{Fun}^{\text{iso}}(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D) \rightarrow D \times D$  from the path category induced by restriction along  $\partial \Delta^1 \subset \Delta^1$  is an isofibration.*

*Proof.* First note that both maps in the sequence  $\text{Fun}^{\text{iso}}(\Delta^1, D) \rightarrow \text{Fun}(\Delta^1, D) \rightarrow D \times D$  are inner fibrations, whence the composite  $r$  is an inner fibration. The first map is an inner fibration because it is an inclusion of a subcategory (15.10), while the second is so by enriched lifting and  $\overline{\text{InnHorn}} \square \text{Cell} \subseteq \overline{\text{InnHorn}}$  (20.2).

To prove that  $r$  is an isofibration, note that it is the composite of two maps  $\text{Fun}^{\text{iso}}(\Delta^1, D) \rightarrow \text{Fun}(\Delta^1, D) \rightarrow \text{Fun}(\partial \Delta^1, D)$  which are isofibrations: the first by an elementary argument (36.3), the second by (37.5).  $\square$

**40.5. The path factorization construction.** For a functor  $f: C \rightarrow D$  between quasicategories, we define a factorization  $C \xrightarrow{j} P(f) \xrightarrow{p} D$  by means of the commutative diagram

$$\begin{array}{ccccc} C & \xrightarrow{j} & P(f) & \xrightarrow{\quad p \quad} & D \\ & \searrow & \downarrow s_0 & \downarrow r_0 & \\ & & C & \xrightarrow{f} & D \end{array}$$

in which the square is a pullback square. The map  $j$  is the unique one so that  $s_0 j = \text{id}_C$  and  $t j = \tilde{\pi} f$ , where  $\tilde{\pi}: D \rightarrow \text{Fun}^{\text{iso}}(\Delta^1, D) \subseteq \text{Fun}(\Delta^1, D)$  is adjoint to the projection  $D \times \Delta^1 \rightarrow D$ .

**40.6. Example.** The path factorization of  $\text{id}_D$  is just  $D \xrightarrow{\tilde{\pi}} \text{Fun}^{\text{iso}}(\Delta^1, D) \xrightarrow{r_1} D$ .

**40.7. Remark.** Note that the objects of  $P(f)$  are pairs  $(c, \alpha)$  consisting of an object  $c \in C_0$  and an isomorphism  $\alpha: f(c) \rightarrow d$  in  $D$ . The map  $j$  sends an object  $c$  to  $(c, 1_{f(c)})$ , while  $p$  sends  $(c, \alpha)$  to  $d$ .

**40.8. Exercise.** Show that if  $f: C \rightarrow D$  is a functor between ordinary categories, then  $P(f)$  is also an ordinary category.

The properties of this construction are summarized by the following.

**40.9. Proposition.** *In the path factorization of  $f$ , the simplicial set  $P(f)$  is a quasicategory, the map  $j$  is a categorical equivalence, and  $p$  is an isofibration. Furthermore  $s_0$  is a trivial fibration.*

*Proof.* From (40.3) we know that both  $r_0$  and  $r_1$  are trivial fibrations. Therefore the base change  $s_0$  of  $r_0$  is a trivial fibration, and hence an inner fibration, which implies that  $P(f)$  is a quasicategory.

Since  $s_0$  is a trivial fibration it is a categorical equivalence (23.11), and thus  $j$  is a categorical equivalence by 2-out-of-3 (25.11).



To show that  $p$  is an isofibration, observe that there is actually a pullback square of the form

$$\begin{array}{ccc} P(f) & \xrightarrow{t} & \mathrm{Fun}^{\mathrm{iso}}(\Delta^1, D) \\ s=(s_0, p) \downarrow & & \downarrow r=(r_0, r_1) \\ C \times D & \xrightarrow{f \times \mathrm{id}_D} & D \times D \end{array}$$

(To see this, use patching of pullback squares where we regard  $C \times D$  as a pullback of  $C \xrightarrow{f} D \leftarrow D \times D$ .) Since  $r$  is an isofibration (40.4), its base-change  $s$  is also an isofibration, and since the projection  $\pi: C \times D \rightarrow D$  is an isofibration the composite  $p = \pi s$  is an isofibration as desired. (I have here used several facts about isofibrations which are left as an exercise (40.10).)  $\square$

40.10. *Exercise* (Some properties of isofibrations). Prove the following facts about isofibrations.

- (1) The composite of two isofibrations is an isofibration.
- (2) Any base-change of an isofibration  $p: C \rightarrow D$  along a map  $D' \rightarrow D$  from a quasicategory is also an isofibration. (Hint: use (36.2).)
- (3) For any quasicategories  $C$  and  $D$  the projection  $C \times D \rightarrow D$  is an isofibration.

#### 41. INVARIANCE PROPERTIES OF SLICES AND LIMITS

That isofibrations which are categorical equivalences are trivial fibrations (38.8) has a number of useful consequences. For instance, we can reformulate the notion of limit or colimit of a functor using only the notions of slices and of categorical equivalence.

**41.1. Proposition.** *Let  $C$  be a quasicategory. Then a map  $\tilde{p}: K^\triangleright \rightarrow C$  is a colimit diagram iff the forgetful functor  $C_{\tilde{p}/} \rightarrow C_{p/}$  is a categorical equivalence, where  $p := \tilde{p}|_K$ . Likewise, a map  $\tilde{q}: K^\triangleleft \rightarrow C$  is a limit diagram iff the forgetful functor  $C_{/\tilde{q}} \rightarrow C_{/q}$  is a categorical equivalence, where  $q := \tilde{q}|_K$ .*

*Proof.* Immediate using the characterization of limits and colimits in terms of trivial fibrations (31.7), and the fact that the indicated forgetful functors are either left or right fibrations (30.15), and therefore are isofibrations (32.11).  $\square$

**41.2. Invariance of slice categories under categorical equivalence.** We can now show that a categorical equivalence between quasicategories induces equivalences of its slice categories.

**41.3. Proposition.** *Let  $f: C \rightarrow D$  be a categorical equivalence of quasicategories. For any map  $q: K \rightarrow C$  of simplicial sets, the induced maps  $C_{q/} \rightarrow D_{f q/}$  and  $C_{/q} \rightarrow D_{/f q}$  on slice categories are also categorical equivalences.*

*Proof.* I'll prove the slice-under case; the slice-over case is exactly the same. Consider the path factorization (40.9) of  $f$ , which gives a commutative diagram

$$\begin{array}{ccccc} & & f & & \\ & \nearrow & & \searrow & \\ C & \xrightarrow{j} & P(f) & \xrightarrow{p} & D \\ & \searrow & \downarrow s_0 & & \\ & & C & & \end{array}$$

where  $j$  is a categorical equivalence,  $p$  is an isofibration, and  $s_0$  a trivial fibration. The hypothesis that  $f$  is a categorical equivalence implies that  $p$  is a categorical equivalence by 2-out-of-3 (25.11), and therefore that  $p$  is a trivial fibration by (38.8).

Recall that if  $f$  is a trivial fibration, then so is the induced map  $C_{q/} \rightarrow D_{fq/}$  by  $\overline{\text{Cell}} \boxtimes \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$  on slices (30.13). Taking slices in the above diagram gives

$$\begin{array}{ccccc}
 & & \tilde{f} & & \\
 & \nearrow & & \searrow & \\
 C_{q/} & \xrightarrow{\tilde{j}} & P(f)_{j_{q/}} & \xrightarrow{\tilde{p}} & D_{fq/} \\
 & \searrow & \downarrow \tilde{s}_0 & & \\
 & & C_{q/} & & 
 \end{array}$$

in which both  $\tilde{p}$  and  $\tilde{s}_0$  are trivial fibrations and thus categorical equivalences (23.11). Applying the 2-out-of-3 property shows that  $\tilde{f}$  is a categorical equivalence as desired.  $\square$

#### 41.4. Invariance of limits and colimits under categorical equivalence.

**41.5. Proposition.** *Let  $f: C \rightarrow D$  be a categorical equivalence between quasicategories. A map  $\hat{p}: K^\triangleright \rightarrow C$  is a colimit cone in  $C$  if and only if  $f\hat{p}$  is a colimit cone in  $D$ , and a map  $\hat{q}: K^\triangleleft \rightarrow C$  is a limit cone in  $C$  if and only if  $f\hat{q}$  is a colimit cone in  $D$ ,*

*Proof.* We prove the case of colimits. Consider the commutative diagram

$$\begin{array}{ccc}
 C_{\hat{p}/} & \xrightarrow{f''} & D_{f\hat{p}/} \\
 \pi \downarrow & & \downarrow \pi' \\
 C_{p/} & \xrightarrow{f'} & D_{fp/}
 \end{array}$$

where the horizontal maps are induced by  $f$ , and the vertical ones by restriction along  $K \subset K^\triangleleft$ . Since  $f$  is a categorical equivalence,  $f'$  and  $f''$  are also categorical equivalences by (41.3). Therefore by 2-out-of-3 for categorical equivalences (25.11)  $\pi$  is a categorical equivalence if and only if  $\pi'$  is, and the claim follows from (41.1).  $\square$

We can also show that the existence of colimit and limit cones is reflected by categorical equivalences.

**41.6. Proposition.** *Let  $f: C \rightarrow D$  be a categorical equivalence between quasicategories. A map  $u: K \rightarrow C$  admits a colimit cone in  $C$  if and only if  $fu$  admits a colimit cone in  $D$ , and  $f$  admits a limit cone in  $C$  if and only if  $fu$  admits a colimit cone in  $D$ .*

*Proof.* We prove the case of colimits. The forward direction is immediate from (41.5), so we just prove the reverse direction. Suppose given  $u: K \rightarrow C$  such that  $w = fu$  has a colimit cone  $\hat{w}: K^\triangleright \rightarrow D$ . Use a path factorization (40.9) to construct a commutative diagram of solid arrows

$$\begin{array}{ccccc}
 K & & & & K^\triangleright \\
 \downarrow u & \searrow v & & \swarrow \hat{v} & \downarrow \hat{w} \\
 C & \xrightarrow{i} & P(f) & \xrightarrow{p} & D \\
 & \searrow & \downarrow s_0 & & \\
 & & C & & 
 \end{array}$$

with  $pi = f$ , so that  $p$  and  $s_0$  are trivial fibrations and  $i$  a categorical equivalence. Since  $p$  is a trivial fibration a lift  $\hat{v}$  exists in its square, which by (41.5) applied to the categorical equivalence  $p$  must

be a colimit cone of  $v := iu$ . Therefore again by (41.5) applied to the categorical equivalence  $s_0$ , we see that  $\hat{u} := s_0\hat{v}$  must be a colimit cone of  $s_0v = s_0iu = u$ , so  $u$  admits a colimit as desired.  $\square$

## Part 7. The fundamental theorem

### 42. THE FUNDAMENTAL THEOREM OF CATEGORY THEORY

Recall that a functor  $f: C \rightarrow D$  between quasicategories is said to be an *equivalence* there exists a  $g: D \rightarrow C$  such that  $gf$  and  $fg$  are naturally isomorphic to the respective identity functors. When  $C$  and  $D$  are ordinary categories, there is a well-known criterion for the existence of such a  $g$ , namely:  *$f$  is an equivalence if and only if  $f$  is fully faithful and essentially surjective*. Here

- *fully faithful* means that  $\text{Hom}_C(x, y) \rightarrow \text{Hom}_D(f(x), f(y))$  is a bijection of sets for every pair of objects  $x, y \in \text{ob } C$ , and
- *essentially surjective* means that for every object  $d \in \text{ob } D$  there exists an object  $c \in \text{ob } C$  such that  $f(c)$  is isomorphic to  $d$ .

I like to call this fact the *Fundamental Theorem of Category Theory*. This is non-standard and frankly pretentious terminology<sup>25</sup>, though I am unaware of any standard abbreviated name for this result<sup>26</sup>. I want to give this fact a fancy name in order to signpost it, as it is quite *nonconstructive*: to prove it requires making a *choice* for each object  $d$  in  $D$  of an object  $c$  of  $C$  and an isomorphism  $f(c) \approx d$  (so it in fact relies on an appropriate form of the axiom of choice).

42.1. *Exercise*. Prove the “Fundamental Theorem” for ordinary categories as follows: given  $f: C \rightarrow D$  which is fully faithful and essentially surjective, make a choice of object  $g(d) \in \text{ob } C$  and isomorphism  $\alpha(d): f(g(d)) \rightarrow d$  for each object of  $d$ , and extend this to the data of a categorical inverse of  $f$ .

42.2. *Example*. Fix a field  $k$ . Let  $\text{Mat}$  be the category whose objects are non-negative integers  $n \geq 0$ , and whose morphisms  $A: n \rightarrow m$  are  $(m \times n)$ -matrices with entries in  $k$ , so that composition is matrix multiplication. Let  $\text{Vect}$  be the category of finite dimensional  $k$ -vector spaces and linear maps. Every basic class in linear algebra proves that the evident functor  $F: \text{Mat} \rightarrow \text{Vect}$  is fully faithful and essentially surjective. Therefore  $F$  is an equivalence of categories. However, there is no *canonical* choice of an inverse functor, whose construction amounts to making an arbitrary choice of basis for each vector space.

We are going to state and then prove an analogue of this result for functors between quasicategories. This will first require an analogue of hom-sets, namely the *quasigroupoid* of maps between two objects, also called the *mapping space*.

### 43. MAPPING SPACES OF A QUASICATEGORY

Given a quasicategory  $C$  and objects  $x, y \in C_0$ , the **mapping space** (or **mapping quasigroupoid**) from  $x$  to  $y$  is the simplicial set defined by the pullback square

$$\begin{array}{ccc} \text{map}_C(x, y) & \longrightarrow & \text{Fun}(\Delta^1, C) \\ \downarrow & & \downarrow \\ \{(x, y)\} & \longrightarrow & C \times C \end{array}$$

mapping space  
mapping  
groupoid      quasi-

That is,  $\text{map}_C(x, y)$  is the fiber of the restriction map  $\text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\partial\Delta^1, C)$  over the point  $(x, y) \in (C \times C)_0$ , where we use the isomorphism  $\text{Fun}(\partial\Delta^1, C) \approx C \times C$  induced by the isomorphism  $\partial\Delta^1 \approx \Delta^0 \amalg \Delta^0$ .

<sup>25</sup>E.g., the Fundamental Theorems of Arithmetic, Algebra, Calculus, etc. But if they can have a Fundamental Theorem, why can't we? (It should be mentioned that there is another candidate for a fundamental theorem, namely the Yoneda lemma.)

<sup>26</sup>I also don't know when it was first formulated, or who first stated it.

If  $C = N(A)$  is the nerve of a category, then  $\text{map}_C(x, y)$  is a discrete simplicial set (2.6) corresponding to the set  $\text{Hom}_C(x, y)$ .

**43.1. Mapping spaces are Kan complexes.** The terminology “space” is justified by the following

**43.2. Proposition.** *The simplicial sets  $\text{map}_C(x, y)$  are quasigroupoids (and hence Kan complexes by (33.2)).*

*Proof.* This is a consequence of (37.3), since the inclusion  $i: \partial\Delta^1 \rightarrow \Delta^1$  induces a bijection on vertices, and so restriction along  $i$  is conservative.  $\square$

**43.3. Mapping spaces and homotopy classes.** The set of morphisms  $x \rightarrow y$  in a quasicategory  $C$  is precisely the set of objects of  $\text{map}_C(x, y)$ . Two such are isomorphic as objects in  $\text{map}_C(x, y)$  if and only if they are homotopic in  $C$ .

**43.4. Proposition.** *Let  $C$  be a quasicategory. For any two maps  $f, g: x \rightarrow y$  in  $C$ , we have that  $f \approx g$  (equivalence under the relation used to define the homotopy category  $hC$ ) if and only if  $f$  and  $g$  are isomorphic as objects of the quasigroupoid  $\text{map}_C(x, y)$ . That is,*

$$\text{Hom}_{hC}(x, y) \approx \pi_0 \text{map}_C(x, y)$$

for every pair  $x, y$  of objects of  $C$ .

*Proof.* Suppose  $f, g \in \text{map}_C(x, y)_0$  are isomorphic, so that in particular there is a morphism  $f \rightarrow g$  in the quasigroupoid  $\text{map}_C(x, y)$ . This amounts to a map  $\Delta^1 \times \Delta^1 \rightarrow C$  which can be represented by a diagram of cells of  $C$  of the form:

$$\begin{array}{ccc} x & \xrightarrow{g} & y \\ 1_x \uparrow & \nearrow a & \uparrow 1_y \\ & h & \\ x & \xrightarrow{b} & y \\ & \searrow f & \end{array}$$

This explicitly exhibits a chain  $f \sim_r h \sim_\ell g$  of homotopies, so  $f \approx g$  as desired.

Conversely, if  $f \approx g$ , we can explicitly construct a map  $H: f \rightarrow g$  in  $\text{map}_C(x, y)$ : in terms of the above picture, let  $h = g$ , let  $b$  be an explicit choice of right-homotopy  $f \sim_r g$ , and let  $a = g_{001}$ .  $\square$

**43.5. Extended mapping spaces and composition.** Given a finite list  $x_0, \dots, x_n \in C_0$  of objects in a quasicategory, we have an **extended mapping space**. These are the simplicial sets defined by the pullback squares

$$\begin{array}{ccc} \text{map}_C(x_0, \dots, x_n) & \longrightarrow & \text{Fun}(\Delta^n, C) \\ \downarrow & & \downarrow \\ \{(x_0, \dots, x_n)\} & \longrightarrow & C^{\times(n+1)} \end{array}$$

where the right-hand vertical arrow is induced by restriction along  $\text{Sk}_0 \Delta^n \rightarrow \Delta^n$ , using the isomorphism  $\text{Sk}_0 \Delta^n \approx \coprod_{n+1} \Delta^0$ , whence  $\text{Fun}(\text{Sk}_0 \Delta^n, C) \approx C^{\times(n+1)}$ . By (37.3) the extended mapping spaces are quasigroupoids.

On the other hand, we may consider the fibers of  $\text{Fun}(I^n, C) \rightarrow C^{\times(n+1)}$  defined by restriction along  $\text{Sk}_0 \Delta^n = \text{Sk}_0 I^n \rightarrow I^n$ , where  $I^n \subset \Delta^n$  is the spine. The fibers of this map are isomorphic to  $n$ -fold products of mapping spaces  $\text{map}_C(x_{n-1}, x_n) \times \dots \times \text{map}_C(x_0, x_1)$ .

**43.6. Lemma.** *The map*

$$g_n: \text{map}_C(x_0, \dots, x_n) \rightarrow \text{map}_C(x_{n-1}, x_n) \times \dots \times \text{map}_C(x_0, x_1)$$

*induced by restriction along the spine inclusion  $I^n \subseteq \Delta^n$  is a trivial fibration. In particular, this map is a categorical equivalence between quasigroupoids.*

*Proof.* The map  $g_n$  is a base change of  $p: \text{Fun}(\Delta^n, C) \rightarrow \text{Fun}(I^n, C)$ . Since  $I^n \subset \Delta^n$  is inner anodyne (14.12), and  $C$  is a quasicategory, the map  $p$  is a trivial fibration by enriched lifting using  $\overline{\text{InnHorn}} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$  (19.8).  $\square$

The inclusions  $I^2 \subset \Delta^2 \supset \Delta^{\{0,2\}}$  induce restriction maps

$$\text{Fun}(I^2, C) \xleftarrow{\sim} \text{Fun}(\Delta^2, C) \rightarrow \text{Fun}(\Delta^{\{0,2\}}, C)$$

in which the first map is a trivial fibration. As noted earlier (20.8) by choosing a categorical inverse to the first map (e.g., a section, since it is a trivial fibration) we obtain a “composition functor”  $\text{Fun}(I^2, C) \rightarrow \text{Fun}(\Delta^1, C)$ .

For any triple  $(x_0, x_1, x_2)$  of objects of  $C$ , the above maps restrict to maps between subcomplexes:

$$\text{map}_C(x_1, x_2) \times \text{map}_C(x_0, x_1) \xleftarrow{g_2} \text{map}_C(x_0, x_1, x_2) \rightarrow \text{map}_C(x_0, x_2).$$

As  $g_2$  is a trivial fibration (43.6), we can choose a section for it, and so by the same construction as (20.8) we obtain a “composition” functor

$$(43.7) \quad \text{comp}: \text{map}_C(x_1, x_2) \times \text{map}_C(x_0, x_1) \rightarrow \text{map}_C(x_0, x_2).$$

Again, this depends on a choice of categorical inverse to  $g_2$ . However, any two categorical inverses to  $g_2$  are naturally isomorphic (22.4), and therefore  $\text{comp}$  is defined up to natural isomorphism. That is, it is a well-defined map in  $h\text{Kan}$ , the homotopy category of Kan complexes (25.1). These noncanonical composition maps are “associative up to homotopy”.

**43.8. Proposition.** *The two maps obtained by composing the sides of the square*

$$\begin{array}{ccc} \text{map}_C(x_2, x_3) \times \text{map}_C(x_1, x_2) \times \text{map}_C(x_0, x_1) & \xrightarrow{\text{id} \times \text{comp}} & \text{map}_C(x_2, x_3) \times \text{map}_C(x_0, x_2) \\ \downarrow \text{comp} \times \text{id} & & \downarrow \text{comp} \\ \text{map}_C(x_1, x_3) \times \text{map}_C(x_0, x_1) & \xrightarrow{\text{comp}} & \text{map}_C(x_0, x_3) \end{array}$$

are naturally isomorphic. That is, the diagram commutes in  $h\text{Kan} \subset h\text{qCat}$ .

*Proof.* Here is a diagram of Kan complexes which actually commutes “on the nose”, i.e., not merely in the homotopy category, but in  $s\text{Set}$ . I use “ $\langle x, y, z \rangle$ ” as shorthand for “ $\text{map}_C(x, y, z)$ ”, etc.

$$\begin{array}{ccccc} \langle x_2, x_3 \rangle \times \langle x_1, x_2 \rangle \times \langle x_0, x_1 \rangle & \xleftarrow{\sim} & \langle x_2, x_3 \rangle \times \langle x_0, x_1, x_2 \rangle & \rightarrow & \langle x_2, x_3 \rangle \times \langle x_0, x_2 \rangle \\ \uparrow \sim & & \uparrow \sim & & \uparrow \sim \\ \langle x_1, x_2, x_3 \rangle \times \langle x_0, x_1 \rangle & \xleftarrow{\sim} & \langle x_0, x_1, x_2, x_3 \rangle & \longrightarrow & \langle x_0, x_2, x_3 \rangle \\ \downarrow & & \downarrow & & \downarrow \\ \langle x_1, x_3 \rangle \times \langle x_0, x_1 \rangle & \xleftarrow{\sim} & \langle x_0, x_1, x_3 \rangle & \longrightarrow & \langle x_0, x_3 \rangle \end{array}$$

The maps labelled “ $\xrightarrow{\sim}$ ” are trivial fibrations, and so are categorical equivalences. All the maps in the above diagram are obtained via restriction along inclusions in

$$\begin{array}{ccccc} \Delta^{\{2,3\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\sim} & \Delta^{\{2,3\}} \cup \Delta^{\{0,1,2\}} & \longleftarrow & \Delta^{\{2,3\}} \cup \Delta^{\{0,2\}} \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \Delta^{\{1,2,3\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\sim} & \Delta^3 & \longleftarrow & \Delta^{\{0,2,3\}} \\ \uparrow & & \uparrow & & \uparrow \\ \Delta^{\{1,3\}} \cup \Delta^{\{0,1\}} & \xrightarrow{\sim} & \Delta^{\{0,1,3\}} & \longleftarrow & \Delta^{\{0,3\}} \end{array}$$

where the maps labelled “ $\xrightarrow{\sim}$ ” are inner anodyne (being generalized inner horn inclusions (14.10)), and which therefore give rise to trivial fibrations in the previous diagram by the same argument we used to define  $\text{comp}$ . After passing from  $\text{Kan} \subset s\text{Set}$  to  $h\text{Kan}$  the categorical equivalences become isomorphisms, and the result follows.  $\square$

**43.9. Segal categories.** Thus, a quasicategory does not quite give rise in this way to a category “enriched over Kan complexes”. Although we can define a composition law, it is not uniquely determined, and is only associative “up to homotopy”.

What we do get is a Segal category. A **Segal category** is a functor

Segal category

$$M: \Delta^{\text{op}} \rightarrow s\text{Set}$$

such that

- (1) the simplicial set  $M([0])$  is discrete, i.e.,  $M([0]) = \text{Sk}_0 M([0])$ ,
- (2) for each  $n \geq 1$ , the simplicial set  $M([n])$  is a Kan complex, and
- (3) for each  $n \geq 1$  the “Segal map”

$$M([n]) \xrightarrow{\langle (n-1,n)^*, \dots, (0,1)^* \rangle} M([1]) \times_{M([0])} \cdots \times_{M([0])} M([1])$$

is a categorical equivalence.

Given a quasicategory  $C$ , we obtain a functor  $M_C: \Delta^{\text{op}} \rightarrow s\text{Set}$  by

$$\begin{aligned} M_C([0]) &:= \text{Sk}_0 C, \\ M_C([n]) &:= \text{Fun}(\Delta^n, C) \times_{\text{Fun}(\text{Sk}_0 \Delta^n, C)} \text{Fun}(\text{Sk}_0 \Delta^n, \text{Sk}_0 C) \\ &\approx \coprod_{x_0, \dots, x_n \in C_0} \text{map}_C(x_0, \dots, x_n). \end{aligned}$$

This object encodes all the structure we used above. For instance, the zig-zag

$$M_C([1]) \times_{M_C([0])} M_C([1]) \xleftarrow{\langle (12)^*, (01)^* \rangle} M_C([2]) \xrightarrow{(02)^*} M_C([1])$$

is a coproduct over all triples  $x_0, x_1, x_2 \in C_0$  of the zig-zag (43.7) used to define “composition”.

You also get a Segal category from a “simplicially enriched” category. For instance, suppose  $\mathcal{C}$  is a (small) category which is enriched over the category of Kan complexes, with object set  $\text{ob } \mathcal{C}$ , and function objects  $\mathcal{C}(x, x') \in \text{Kan}$  for each  $x, x'$ . Then we can define  $M_{\mathcal{C}}: \Delta^{\text{op}} \rightarrow s\text{Set}$  by

$$\begin{aligned} M_{\mathcal{C}}([0]) &:= \text{ob } \mathcal{C}, \\ M_{\mathcal{C}}([n]) &:= \coprod_{x_0, \dots, x_n \in \text{ob } \mathcal{C}} \mathcal{C}(x_{n-1}, x_n) \times \cdots \times \mathcal{C}(x_0, x_1). \end{aligned}$$

We thus obtain functors

$$\text{qCat} \rightarrow \text{SeCat} \leftarrow s\text{Cat}^{\text{Kan}}$$

relating quasicategories, Segal categories, and Kan-complex-enriched categories. Simplicially enriched categories were proposed as a model for  $\infty$ -categories by Dwyer and Kan<sup>27</sup>, while Segal categories were proposed as a model for  $\infty$ -categories by Hirschowitz and Simpson [HS01]<sup>28</sup>. These models are all known to be equivalent to quasicategories in a suitable sense; see [Ber10] for more about these models and their comparison.

**43.10. The enriched homotopy category of a quasicategory.** Given a quasicategory  $C$  we can produce a vestigial version of a category enriched over quasigroupoids, called the **enriched homotopy category** of  $C$  and denoted  $\mathcal{HC}$ .<sup>29</sup> This object will be a category *enriched over*  $h\text{Kan}$ , where  $h\text{Kan}$  is the full subcategory of  $hq\text{Cat}$  spanned by Kan complexes. The underlying category of the enriched category  $\mathcal{HC}$  will just be the homotopy category  $hC$  of  $C$ .

enriched  
category      homotopy

<sup>27</sup>They called them “homotopy theories” instead of “ $\infty$ -categories”; see [DS95, §11.6].

<sup>28</sup>In fact, they generalize this to “Segal  $n$ -categories”, which were the first effective model for  $(\infty, n)$ -categories.

<sup>29</sup>Lurie usually calls this “ $hC$ ” in [Lur09], though he also uses that notation for the ordinary homotopy category of  $C$  that we have already discussed. I prefer to have two separate notations.

We now define  $\mathcal{H}C$ . The objects of  $\mathcal{H}C$  are just the objects of  $C$ . For any two objects  $x, y \in C_0$ , we have the quasigroupoid

$$\mathcal{H}C(x, y) := \text{map}_C(x, y)$$

which we will regard as an object of the homotopy category  $h\text{Kan}$  of Kan complexes. Composition  $\mathcal{H}C(x_1, x_2) \times \mathcal{H}C(x_0, x_1) \rightarrow \mathcal{H}C(x_0, x_2)$  is the composition map defined above (43.7), which is well-defined as a morphism in  $h\text{Kan}$ . Composition is associative as shown above (43.8). (Remember that  $h\text{qCat}$ , and thus also the full subcategory  $h\text{Kan}$ , has finite products, which coincide with products of simplicial sets (25.4).)

The *underlying* ordinary category of  $\mathcal{H}C$  is just the ordinary homotopy category  $hC$ , since

$$\text{Hom}_{h\text{Kan}}(\Delta^0, \text{map}_C(x, y)) \approx \pi_0 \text{map}_C(x, y) \approx \text{Hom}_{hC}(x, y).$$

43.11. *Warning.* A quasicategory  $C$  *cannot* be recovered from its enriched homotopy category  $\mathcal{H}C$ , not even up to equivalence. Furthermore, there exist  $h\text{Kan}$ -enriched categories which do not arise as  $\mathcal{H}C$  for any quasicategory  $C$ . A proof is outside the scope of these notes: counterexamples may be produced (for instance) from examples of associative  $H$ -spaces which are not loop spaces, and examples of spaces which admit several inequivalent loop space structures.

43.12. *Exercise.* Let  $C$  and  $D$  be quasicategories. Show that there is an isomorphism  $\mathcal{H}(C \times D) \approx \mathcal{H}C \times \mathcal{H}D$  of  $h\text{Kan}$ -enriched categories.

#### 44. THE FUNDAMENTAL THEOREM OF QUASICATEGORY THEORY

44.1. **Fully faithful and essentially surjective functors between quasicategories.** Note that any functor  $f: C \rightarrow D$  of quasicategories induces functors  $\text{map}_C(x, y) \rightarrow \text{map}_D(f(x), f(y))$  for every pair of objects  $x, y$  in  $C$ . We say that a functor  $f: C \rightarrow D$  between quasicategories is

- **fully faithful** if for every pair  $c, c' \in C_0$ , the resulting map  $\text{map}_C(c, c') \rightarrow \text{map}_D(fc, fc')$  is **fully faithful** a categorical equivalence, and
- **essentially surjective** if for every  $d \in D_0$  there exists a  $c \in C_0$  together with an isomorphism  $fc \rightarrow d$  in  $D$ ; that is, if the induced functor  $hf: hC \rightarrow hD$  of ordinary categories is **essentially surjective**.

44.2. *Remark.* Another way to say this is as follows:  $f: C \rightarrow D$  is fully faithful and essentially surjective iff the induced  $h\text{Kan}$ -enriched functor  $\mathcal{H}f: \mathcal{H}C \rightarrow \mathcal{H}D$  is an equivalence of *enriched* categories.

44.3. **Proposition.** *If  $f: C \rightarrow D$  is a categorical equivalence between quasicategories, then  $f$  is fully faithful and essentially surjective.*

*Proof.* To prove essential surjectivity, choose any categorical inverse  $g$  to  $f$  and natural isomorphism  $\alpha: fg \rightarrow \text{id}_D$ . Then for any  $d \in D_0$  we get an object  $c := g(d) \in C_0$  and an isomorphism  $\alpha(d): f(c) \rightarrow d$  in  $D$ .

To show that  $f$  is fully faithful, choose a categorical inverse  $g$  of  $f$ . Given  $x, y \in C_0$ , consider the induced diagram of quasigroupoids

$$\begin{array}{ccccccc} & & gf & & & & \\ & \nearrow & & \searrow & & & \\ \text{map}_C(x, y) & \xrightarrow{f} & \text{map}_D(fx, fy) & \xrightarrow{g} & \text{map}_C(gfx, gfy) & \xrightarrow{f} & \text{map}_D(fgfx, fgfy) \\ & & & \searrow & & \nearrow & \\ & & & fg & & & \end{array}$$

By the 2-out-of-6 property for categorical equivalences (25.11), it will suffice to show that the maps marked  $gf$  and  $fg$  are categorical equivalences between the respective mapping spaces. Since  $gf: C \rightarrow C$  and  $fg: D \rightarrow D$  are naturally isomorphic to the identity maps of  $C$  and  $D$  respectively, the claim follows from (44.4) which we prove below.  $\square$

**44.4. Proposition.** *If  $f_0, f_1: C \rightarrow D$  are functors which are naturally isomorphic, then  $f_0$  is fully faithful if and only if  $f_1$  is.*

To prove this we need to apply the following to the path category.

**44.5. Lemma.** *Any trivial fibration  $p: C \rightarrow D$  between quasicategories is fully faithful.*

*Proof.* For  $x, y \in C_0$  we have a diagram of pullback squares

$$\begin{array}{ccccc}
 \text{map}_C(x, y) & \longrightarrow & \text{Fun}(\Delta^1, C) & & \\
 \downarrow q & & \downarrow p^{\square(\partial\Delta^1 \subset \Delta^1)} & & \\
 \text{map}_D(px, py) & \longrightarrow & \text{Fun}(\partial\Delta^1, C) \times_{\text{Fun}(\partial\Delta^1, D)} \text{Fun}(\Delta^1, D) & \longrightarrow & \text{Fun}(\Delta^1, D) \\
 \downarrow & & \downarrow & & \downarrow \\
 \{(x, y)\} & \longrightarrow & \text{Fun}(\partial\Delta^1, C) & \longrightarrow & \text{Fun}(\partial\Delta^1, D)
 \end{array}$$

The pullback-hom  $p^{\square(\partial\Delta^1 \subset \Delta^1)}$  is a trivial fibration using  $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ , so  $q$  is a trivial fibration and thus a categorical equivalence (23.11).  $\square$

*Proof of (44.4).* Consider a natural isomorphism  $H: C \times \Delta^1 \rightarrow D$  between  $f_0$  and  $f_1$ , and write  $\tilde{H}: C \rightarrow \hat{D} \subseteq \text{Fun}(\Delta^1, D)$  for its adjoint, where  $\hat{D} = \text{Fun}^{\text{iso}}(\Delta^1, D)$ . Then lemma (40.3) implies that in the commutative diagram

$$\begin{array}{ccc}
 & & f_0 \rightarrow \text{Fun}(\{0\}, D) = D \\
 & \nearrow f_0 & \\
 C & \xrightarrow{\tilde{H}} \hat{D} & \\
 & \searrow f_1 & \\
 & & f_1 \rightarrow \text{Fun}(\{1\}, D) = D
 \end{array}$$

both  $r_0$  and  $r_1$  are trivial fibrations. Because  $r_0$  and  $r_1$  are trivial fibrations, for any  $x, y \in C_0$  we get a commutative diagram

$$\begin{array}{ccccc}
 & & f_0 & \longrightarrow & \text{map}_D(f_0(x), f_0(y)) \\
 & \nearrow f_0 & & & \uparrow \sim \\
 \text{map}_C(x, y) & \longrightarrow & \text{map}_{\hat{D}}(\tilde{H}(x), \tilde{H}(y)) & & \\
 & \searrow f_1 & & & \downarrow \sim \\
 & & f_1 & \longrightarrow & \text{map}_D(f_1(x), f_1(y))
 \end{array}$$

in which the maps indicated by “ $\sim$ ” are categorical equivalences by (44.5). Using the 2-out-of-3 property of categorical equivalences (25.11), we see that the map marked  $f_0$  is a categorical equivalence if and only if the map marked  $f_1$  is. Thus we have shown that  $f_0: C \rightarrow D$  is fully faithful if and only if  $f_1: C \rightarrow D$  is fully faithful.  $\square$

We’ve finished proving the lemma we needed for the proof that categorical equivalences are fully faithful (44.3).

We note a useful fact: to check that a functor is fully faithful, it suffices to check the defining property on *representatives of isomorphism classes* of objects.

**44.6. Proposition.** *Let  $f: C \rightarrow D$  be a functor between quasicategories, and let  $S \subset C_0$  be a subset of objects which includes a representative of every isomorphism class in  $C$ . Then  $f$  is fully faithful if and only if  $\text{map}_C(c, c') \rightarrow \text{map}_D(fc, fc')$  is a categorical equivalence for all  $c, c' \in S$ .*



*Proof.* The only-if direction is immediate from the definition of fully faithful. To prove the if direction, let  $x, x' \in C_0$  and choose isomorphisms  $\alpha: x \rightarrow c$  and  $\alpha': x' \rightarrow c'$  where  $c, c' \in S$ . We may interpret  $\alpha$  and  $\alpha'$  as objects of  $\widehat{C} = \text{Fun}^{\text{iso}}(\Delta^1, C) \subseteq \text{Fun}(\Delta^1, C)$ . We obtain a commutative diagram

$$\begin{array}{ccccc} \text{map}_C(x, x') & \xleftarrow{r_0} & \text{map}_{\widehat{C}}(\alpha, \alpha') & \xrightarrow{r_1} & \text{map}_C(c, c') \\ f \downarrow & & \widehat{f} \downarrow & & f \downarrow \\ \text{map}_D(fx, fx') & \xleftarrow{r_0} & \text{map}_{\widehat{D}}(\widehat{f}\alpha, \widehat{f}\alpha') & \xrightarrow{r_1} & \text{map}_D(fc, fc') \end{array}$$

where the vertical arrows are induced by  $f: C \rightarrow D$  and  $\widehat{f}: \widehat{C} \rightarrow \widehat{D}$ , where  $\widehat{f}$  is the restriction of  $\text{Fun}(\Delta^1, f): \text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\Delta^1, D)$  to the path categories  $\widehat{C} = \text{Fun}^{\text{iso}}(\Delta^1, C)$  and  $\widehat{D} = \text{Fun}^{\text{iso}}(\Delta^1, D)$ . The maps marked  $r_0$  and  $r_1$  are categorical equivalences by (40.3) and (44.5). Therefore the left-hand vertical arrow is a categorical equivalence using the hypothesis on  $f$  and 2-out-of-3 for categorical equivalences (25.11).  $\square$

**44.7. The fundamental theorem for quasicategories.** The converse to (44.3) is also true, whence: *A map  $f: C \rightarrow D$  between quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

This is a non-trivial result. It gives a necessary and sufficient condition for  $f: C \rightarrow D$  to admit a categorical inverse, but it does not spell out how to construct such an inverse. After some preliminaries, we will prove this as (46.2).

**44.8. 2-out-of-6 for fully faithful essentially surjective functors.** The following result will be useful in the proof of the fundamental theorem. Recall the 2-out-of-6 and 2-out-of-3 properties of a class of morphisms (25.6), and that the class of categorical equivalences has these properties (25.11).

**44.9. Proposition.** *The class  $\mathcal{C}$  of fully faithful and essentially surjective functors between quasicategories satisfies the 2-out-of-6 property, and thus the 2-out-of-3 property.*

*Proof.* Any identity functor  $\text{id}: C \rightarrow C$  is manifestly fully faithful and essentially surjective.

Next note that if a functor  $f: C \rightarrow D$  between quasicategories is fully faithful and essentially surjective, then the induced  $hf: hC \rightarrow hD$  is an equivalence of ordinary categories. Conversely, if  $hf$  is an equivalence, then  $f$  is essentially surjective.

Suppose  $C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} F$  is a sequence of functors between quasicategories such that  $gf$  and  $hg$  are fully faithful and essentially surjective. The induced sequence  $hC \rightarrow hD \rightarrow hE \rightarrow hF$  of functors on homotopy categories has the same property, and thus all the functors between homotopy categories are equivalences. From this we conclude immediately that  $f, g, h, hgf$  are essentially surjective.

Given objects  $x, y \in C_0$ , we have induced maps

$$\begin{array}{ccccccc} & & gf & & & & \\ & \curvearrowright & & \curvearrowleft & & & \\ \text{map}_C(x, y) & \xrightarrow{f} & \text{map}_D(fx, fy) & \xrightarrow{g} & \text{map}_E(gfx, gfy) & \xrightarrow{h} & \text{map}_F(hgfx, hgy) \\ & & & & hg & & \end{array}$$

The hypothesis that  $gf$  and  $hg$  are fully faithful implies that the indicated arrows are categorical equivalences, and hence all arrows are by (25.11). Because  $f$  and  $gf$  are essentially surjective, the collections of objects  $\{fx \mid x \in C_0\} \subseteq D_0$  and  $\{gfx \mid x \in C_0\} \subseteq E_0$  include representatives of every isomorphism class of  $D$  and  $E$  respectively, and thus (44.6) implies that  $f, g, h$ , and therefore  $hgf$ , are fully faithful.  $\square$

**44.10. Reduction steps.** To prove the fundamental theorem of quasicategories for a general map between quasicategories, we can reduce to the special case of isofibrations.

**44.11. Lemma.** *To prove that every fully faithful and essentially surjective functor of quasicategories is a categorical equivalence, it suffices to prove it for the special case of isofibrations.*

*Proof.* Let  $f: C \rightarrow D$  be a functor which is fully faithful and essentially surjective. Consider the path factorization

$$C \xrightarrow{j} P(f) \xrightarrow{p} D$$

of  $f$ , with  $j$  a categorical equivalence and  $p$  an isofibration (40.9). Recall that the class categorical equivalences satisfies 2-out-of-3 (25.11), as does the class of functors which are fully faithful and essentially surjective (44.9). Since every categorical equivalence (such as  $j$ ) is fully faithful and essentially surjective (44.3), the claim follows.  $\square$

We will prove the special case of isofibrations by showing that if an isofibration  $p$  is fully faithful and essentially surjective, then it is a trivial fibration, i.e. so that  $\text{Cell} \sqsupset p$ . First note the following.

**44.12. Proposition.** *An isofibration  $p$  is essentially surjective if and only if it is surjective on vertices, i.e., iff  $(\partial\Delta^0 \subset \Delta^0) \sqsupset p$ .*

*Proof.* The  $\Leftarrow$  implication is obvious, while  $\Rightarrow$  is a straightforward exercise.  $\square$

Thus, to complete the proof of the fundamental theorem, it suffices to show that if an isofibration  $p$  is fully faithful, then  $\text{Cell}_{\geq 1} \sqsupset p$ , which will prove as (46.1).

#### 45. A FIBERWISE CRITERION FOR TRIVIAL FIBRATIONS TO QUASIGROUPOIDS

We give a criterion for an isofibration to be a trivial fibration when the target is a *quasigroupoid*. This criterion is in terms of its fibers. The **fiber** of a map  $p: X \rightarrow Y$  over a vertex  $y \in Y_0$  is defined to be the pullback of  $p$  along  $\{y\} \rightarrow Y$ . We will write  $p^{-1}(y) = \{y\} \times_Y X$  for the fiber of  $p$  over  $y$ .

Recall that a quasicategory  $C$  is categorically equivalent to the terminal category  $\Delta^0$  if and only if  $C \rightarrow \Delta^0$  is a trivial fibration (38.11). We call such an object a **contractible Kan complex**. If  $p: X \rightarrow Y$  is a trivial fibration, then since  $\text{TrivFib} = \text{Horn}^\square$  we see immediately that every projection  $p^{-1}(y) \rightarrow *$  from a fiber is a trivial fibration; i.e., the fibers of a trivial fibration are necessarily contractible Kan complexes. The “fiberwise criterion” asserts the converse for isofibrations to Kan complexes.

**45.1. Proposition.** *Let  $p: C \rightarrow D$  be an isofibration in which  $D$  is a quasigroupoid. Then  $p$  is a trivial fibration if and only if every fiber of  $p$  is a contractible Kan complex.*

*Proof.* We have just observed  $(\Rightarrow)$ , so we prove  $(\Leftarrow)$ . So suppose  $p$  is an isofibration to a quasigroupoid whose fibers are contractible Kan complexes, and consider a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{a} & C \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{b} & D \end{array}$$

We will “deform” the lifting problem  $(a, b)$  to one of the same type which lives inside a single fiber of  $p$ . As such lifting problems have solutions by the hypothesis that the fibers of  $p$  are contractible Kan complexes, the covering homotopy extension property (37.8) implies that the original lifting problem has a solution.

Let  $\gamma: \Delta^n \times \Delta^1 \rightarrow \Delta^n$  be the unique map which on vertices is given by  $\gamma(k, 0) = k$  and  $\gamma(k, 1) = n$ , i.e., the unique natural transformation  $\gamma: \text{id}_{\Delta^n} \rightarrow \langle n \dots n \rangle$  of functors  $\Delta^n \rightarrow \Delta^n$ . Since  $p$  is an

isofibration, so is  $\text{Fun}(\partial\Delta^n, p)$  (37.5), and since  $\text{Fun}(\partial\Delta^n, D)$  is a quasigroupoid, we can construct a lift  $u$  in

$$\begin{array}{ccc} \partial\Delta^n \times \{0\} & \xrightarrow{a} & C \\ \downarrow & \nearrow u & \downarrow p \\ \partial\Delta^n \times \Delta^1 & \xrightarrow{\gamma} \Delta^n \times \Delta^1 \xrightarrow{\gamma} \Delta^n & \xrightarrow{b} D \end{array}$$

which represents a natural isomorphism of functors  $\partial\Delta^n \rightarrow C$ . The resulting commutative square

$$\begin{array}{ccc} \partial\Delta^n \times \Delta^1 & \xrightarrow{u} & C \\ \downarrow & & \downarrow p \\ \Delta^n \times \Delta^1 & \xrightarrow{b\gamma} & D \end{array}$$

represents an morphism  $e$  in  $\text{Fun}(\partial\Delta^n, C) \times_{\text{Fun}(\partial\Delta^n, D)} \text{Fun}(\Delta^n, D)$  with vertex  $e_0 = (a, b)$  the original lifting problem, and vertex  $e_1 = (a', b')$  where  $b' = b\gamma|_{\Delta^n \times \{1\}}$  factors as  $\Delta^n \rightarrow \{b(n)\} \rightarrow D$ . Furthermore  $e$  is an isomorphism by (36.2) since its images  $u$  and  $b\gamma$  are isomorphisms. By the covering homotopy extension property (37.8) it suffices to produce a lift in the rectangle

$$\begin{array}{ccccc} \partial\Delta^n & \xrightarrow{a'} & p^{-1}(b(n)) & \xrightarrow{\gamma} & C \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{\gamma} & \{b(n)\} & \xrightarrow{\gamma} & D \end{array}$$

which amounts to producing a lift in the left-hand square, which exists because  $p^{-1}(b(n))$  is a contractible Kan complex.  $\square$

We often apply the fiberwise criterion in the following way.

**45.2. Corollary.** *Suppose we have a pullback square of the form*

$$\begin{array}{ccc} \coprod_{\alpha \in I} C'_\alpha & \longrightarrow & C \\ \coprod p'_\alpha \downarrow & & \downarrow p \\ \coprod_{\alpha \in I} D'_\alpha & \xrightarrow{g} & D \end{array}$$

such that (1)  $D$  is a quasigroupoid, (2)  $p$  is an isofibration, and (3) the map  $g$  is surjective on vertices. Then  $p$  is a trivial fibration if and only if every  $p'_\alpha: C'_\alpha \rightarrow D'_\alpha$  is a trivial fibration.

*Proof.* The fibers of  $p$  all appear as fibers of the  $p'_\alpha$  by (3), so this is immediate from the fiberwise criterion (45.1).  $\square$

**45.3. Remark.** The proof of (45.1) actually shows something a little stronger: If  $p: C \rightarrow D$  is an isofibration to a quasigroupoid, then for any fixed  $n \geq 0$  we have that  $(\partial\Delta^n \subset \Delta^n) \boxtimes p$  if and only if  $(\partial\Delta^n \subset \Delta^n) \boxtimes (p^{-1}(y) \rightarrow \{y\})$  for all  $y \in D_0$ .

The hypothesis that the target is a quasigroupoid is necessary: there is no “fiberwise criterion” for an arbitrary isofibration between quasicategories to be a trivial fibration.

**45.4. Exercise.** Give an example of an isofibration between quasicategories whose fibers are all categorically equivalent to  $\Delta^0$ , but is not a categorical equivalence, and hence not a trivial fibration. (Hint: think small.)

**45.5. Pullback-hom criterion for fully faithful isofibrations.** Using the fiberwise criterion, we obtain a new criterion for an isofibration to be fully faithful.

**45.6. Proposition.** *Let  $p: C \rightarrow D$  be an isofibration between quasicategories. Then  $p$  is fully faithful if and only if*

$$(p^{\square(\partial\Delta^1 \subset \Delta^1)})^{\text{core}}: \text{Fun}(\Delta^1, C)^{\text{core}} \rightarrow ((C \times C) \times_{(D \times D)} \text{Fun}(\Delta^1, D))^{\text{core}}$$

*is a trivial fibration.*

*Proof.* We can form a commutative diagram

$$\begin{array}{ccccc} \coprod_{(c,c') \in C_0 \times C_0} \text{map}_C(c, c') & \xrightarrow{j'} & \text{Fun}(\Delta^1, C) & & \\ \downarrow q_{c,c'} & & \downarrow q & & \\ \coprod_{(c,c') \in C_0 \times C_0} \text{map}_D(pc, pc') & \xrightarrow{j} & (C \times C) \times_{(D \times D)} \text{Fun}(\Delta^1, D) & \longrightarrow & \text{Fun}(\Delta^1, D) \\ \downarrow & & \downarrow & & \downarrow \\ (\text{Sk}_0 C) \times (\text{Sk}_0 C) & \xrightarrow{i} & C \times C & \xrightarrow{p \times p} & D \times D \end{array}$$

in which: each square is a pullback and the map,  $q = p^{\square(\partial\Delta^1 \subset \Delta^1)}$  is the pullback-hom map which is an isofibration (37.5), and the horizontal maps  $i, j, j'$  are surjective on vertices. Since mapping spaces are Kan complexes, the maps  $i, j$ , and  $j'$  factor through cores, and the resulting square

$$\begin{array}{ccc} \coprod_{(c,c') \in C_0 \times C_0} \text{map}_C(c, c') & \longrightarrow & \text{Fun}(\Delta^1, C)^{\text{core}} \\ \downarrow q_{c,c'} & & \downarrow q^{\text{core}} \\ \coprod_{(c,c') \in C_0 \times C_0} \text{map}_D(pc, pc') & \xrightarrow{\tilde{j}} & ((C \times C) \times_{(D \times D)} \text{Fun}(\Delta^1, D))^{\text{core}} \end{array}$$

is a pullback square. The map  $q^{\text{core}}$  is an isofibration between quasigroupoids (36.11), so the fiberwise criterion (45.2) applies to show that  $q^{\text{core}}$  is a trivial fibration if and only if each  $q_{c,c'}: \text{map}_C(c, c') \rightarrow \text{map}_D(pc, pc')$  is a trivial fibration, and therefore if and only if each  $q_{c,c'}$  is a categorical equivalence by (38.8). The map  $q_{c,c'}$  is precisely the one induced by the functor  $p$ , so the proposition is proved.  $\square$

## 46. PROOF OF THE FUNDAMENTAL THEOREM

In this section, we will prove the following.

**46.1. Proposition.** *If  $p: C \rightarrow D$  is an isofibration which is fully faithful, then  $\text{Cell}_{\geq 1} \boxtimes p$ .*

As discussed in (44.10), this proves the following.

**46.2. Theorem** (Fundamental theorem of quasicategories). *A functor  $f: C \rightarrow D$  of quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

**46.3. The class  $\mathcal{C}_p$ .** Let  $p: C \rightarrow D$  be an isofibration. We define the class

$$\mathcal{C}_p := \{ i \in \overline{\text{Cell}} \mid (p^{\square i})^{\text{core}} \in \text{TrivFib} \}$$

of monomorphisms such that the restriction of the pullback-hom map  $p^{\square i}$  to cores is a trivial fibration. Note that if  $i \in \mathcal{C}_p$ , then in particular  $(p^{\square i})^{\text{core}}$  is surjective on vertices, whence  $p^{\square i}$  is also surjective on vertices and thus  $i \boxtimes \mathcal{C}_p$ . Thus to prove (46.1), it suffices to show  $\text{Cell}_{\geq 1} \subseteq \mathcal{C}_p$ .

46.4. **The class  $\mathcal{C}_p$  is weakly saturated.** First we need to show that  $\mathcal{C}_p$  is weakly saturated.

46.5. **Lemma.** *Let  $i: K \rightarrow L$  be a monomorphism of simplicial sets. Then there exists a monomorphism  $i'$  such that, for any isofibration  $p: C \rightarrow D$  between quasicategories, we have  $i \sqsubset p^{\text{core}}$  if and only if  $i' \sqsubset p$ .*

*Proof.* Given  $i$  we construct  $i'$  as in the following diagram

$$\begin{array}{ccccc} K & \xrightarrow{u} & K' & & \\ \downarrow i & & \downarrow j & \searrow i' & \\ L & \xrightarrow{v} & P & \xrightarrow{k} & L' \end{array}$$

where we

- (1) choose an anodyne map  $u: K \rightarrow K'$  to a Kan complex  $K'$ ,
- (2) we form the pushout  $P$  of  $u$  along  $i$ , and
- (3) choose an anodyne map  $k: P \rightarrow L'$  to a Kan complex  $L'$ ,

whence  $i' := kj$  is a monomorphism. The choices of  $u$  and  $k$  can be made using the small object argument (15.16) applied to the set Horn of all horn inclusions. We need to show  $i \sqsubset p^{\text{core}}$  iff  $i' \sqsubset p$  for any isofibration  $p$ .

( $\implies$ ) Suppose  $i \sqsubset p^{\text{core}}$ , and consider a lifting problem of type  $i' \sqsubset p$ . Since  $K'$  and  $L'$  are Kan complexes, any lifting problem of type  $i' \sqsubset p$  factors through cores:

$$\begin{array}{ccccc} K' & \longrightarrow & C^{\text{core}} & \longrightarrow & C \\ \downarrow i' & \nearrow & \downarrow p^{\text{core}} & & \downarrow p \\ L' & \longrightarrow & D^{\text{core}} & \longrightarrow & D \end{array}$$

so it suffices to show  $i' \sqsubset p^{\text{core}}$ . Since  $j$  is a cobase change of  $i$  we have  $j \sqsubset p^{\text{core}}$ , while  $k \sqsubset p^{\text{core}}$  since  $k$  is anodyne and  $p^{\text{core}}$  is a Kan fibration (39.1). Therefore  $i' = kj \sqsubset p^{\text{core}}$  as desired.

( $\impliedby$ ) Suppose  $i' \sqsubset p$ , and consider a lifting problem  $(a: K \rightarrow C^{\text{core}}, b: L \rightarrow D^{\text{core}})$  of type  $i \sqsubset p^{\text{core}}$ . We factor this lifting problem through a diagram of the following form

$$\begin{array}{ccccccc} & & a & & & & \\ & \searrow & & \nearrow & & & \\ K & \xrightarrow{u} & K' & \xrightarrow{a'} & C^{\text{core}} & \longrightarrow & C \\ \downarrow i & & \downarrow j & & \downarrow p^{\text{core}} & & \downarrow p \\ L & \xrightarrow{v} & P & \xrightarrow{k} & L' & \xrightarrow{b''} & D^{\text{core}} & \longrightarrow & D \\ & \nearrow & & \searrow & & & & & \\ & & b & & & & & & \end{array}$$

as follows.

- (1) Since  $u$  is anodyne and  $C^{\text{core}}$  is a Kan complex, we can factor  $a = a'u$  for some  $a': K' \rightarrow C^{\text{core}}$ .
- (2) There is a unique map  $b': P \rightarrow D^{\text{core}}$  such that  $b'v = b$  and  $b'j = a'$  since  $P$  is a pushout.
- (3) Since  $k$  is anodyne and  $D^{\text{core}}$  is a Kan complex, we can factor  $b' = b''k$  for some  $b'': L' \rightarrow D^{\text{core}}$ .

By hypothesis a lift  $t$  exists. Since  $L'$  is a Kan complex the lift  $t$  factors through  $C^{\text{core}} \subseteq C$ , so we have a map  $t': L' \rightarrow C^{\text{core}}$  such that  $t'i' = a'$  and  $p^{\text{core}}t' = b''$ . Then the composite  $s := t'kv: L \rightarrow C^{\text{core}}$  is the desired solution to the lifting problem  $(a, b)$ .  $\square$

46.6. **Lemma.** *For an isofibration  $p$  the class  $\mathcal{C}_p$  is weakly saturated.*

*Proof.* For each  $j_n: \partial\Delta^n \rightarrow \Delta^n$ , choose a map  $j'_n$  as in (46.5) so that  $j_n \sqsubset p^{\text{core}}$  if and only if  $j'_n \sqsubset p$ , and therefore  $j_n \sqsubset (p^{\square i})^{\text{core}}$  if and only if  $j'_n \sqsubset p^{\square i}$  if and only if  $i \sqsubset p^{\square j'_n}$ . Then  $\mathcal{C}_p = \overline{\text{Cell}} \cap \{ p^{\square j'_n} \mid n \geq 0 \}$ , and so is weakly saturated.  $\square$

**46.7. The class  $\mathcal{C}_p$  has precancellation.** We will need the following result which relates pullback-homs and composition of maps. You can think of it as an “enriched” version of the fact that  $i \sqsubset p$  and  $j \sqsubset p$  imply  $ji \sqsubset p$ .

**46.8. Proposition** (Transitivity triangle for pullback-homs). *Let  $A \xrightarrow{i} B \xrightarrow{j} C$  and  $p: X \rightarrow Y$  be maps of simplicial sets. Then there is a factorization*

$$p^{\square(j \circ i)} = q \circ p^{\square j}$$

where  $q$  is a base-change of  $p^{\square i}$ .

*Proof.* I use “[ $A, X$ ]” as a shorthand for “ $\text{Fun}(A, X)$ ”. Form the commutative diagram

$$\begin{array}{ccccc}
 & & [j, X] & & \\
 & \searrow & \xrightarrow{\quad} & \searrow & \\
 [C, X] & \xrightarrow{p^{\square j}} [B, X] \times_{[B, Y]} [C, Y] & \xrightarrow{\quad} & [B, X] & \xrightarrow{[i, X]} [A, X] \\
 & \searrow p^{\square ji} & \downarrow q & \downarrow p^{\square i} & \downarrow [A, p] \\
 & [A, X] \times_{[A, Y]} [C, Y] & \xrightarrow{\quad} & [A, X] \times_{[A, Y]} [B, Y] & \xrightarrow{\quad} [A, X] \\
 & \downarrow [C, p] & & \downarrow [B, p] & \\
 & [C, Y] & \xrightarrow{[j, Y]} [B, Y] & \xrightarrow{[i, Y]} [A, Y] & 
 \end{array}$$

in which all three squares are pullbacks, whence in particular  $q$  is a base-change of  $p^{\square i}$ . The claim follows.  $\square$

**46.9. Exercise.** Prove the following transitivity-triangles:

- (1)  $(i \circ j) \square f = k \circ (i \square f)$  where  $k$  is a cobase-change of  $j \square f$ .
- (2)  $(q \circ p)^{\square i} = r \circ p^{\square i}$  where  $r$  is a base-change of  $q^{\square i}$ .

Next, we show that  $\mathcal{C}_p$  has the following “precancellation” property.

**46.10. Proposition.** *Let  $p: C \rightarrow D$  be an isofibration between quasicategories. If  $i: K \rightarrow K'$  and  $j: K' \rightarrow K''$  are monomorphisms, then  $i, ji \in \mathcal{C}_p$  implies  $j \in \mathcal{C}_p$ .*

*Proof.* By (46.8) we have  $p^{\square ji} = q \circ p^{\square j}$  where  $q$  is a base-change of  $p^{\square i}$ . Restricting to cores gives a factorization  $(p^{\square ji})^{\text{core}} = q^{\text{core}} \circ (p^{\square i})^{\text{core}}$ . Furthermore  $q^{\text{core}}$  is a base-change of  $(p^{\square i})^{\text{core}}$  as (36.2) applies since  $p^{\square i}$  is an inner fibration between quasicategories (19.8).

We have that  $(p^{\square ji})^{\text{core}}$ ,  $(p^{\square j})^{\text{core}}$ ,  $(p^{\square i})^{\text{core}}$ , and hence  $q^{\text{core}}$  are isofibrations (36.11). Since  $ji, i \in \mathcal{C}_p$ , we have that  $(p^{\square ji})^{\text{core}}$ ,  $(p^{\square i})^{\text{core}}$  and hence  $q^{\text{core}}$  are trivial fibrations, and therefore are categorical equivalences, whence  $p^{\square j}$  is also a weak equivalence by 2-out-of-3 (25.11), and therefore  $p^{\square j}$  is a trivial fibration since it is an isofibration between quasicategories (38.8).  $\square$

**46.11. The end of the proof.** We can now prove (46.1), using the following lemma to show that if  $p$  is a fully faithful isofibration, then  $\mathcal{C}_p$  contains  $\overline{\text{Cell}}_{\geq 1}$ , whence  $\overline{\text{Cell}}_{\geq 1} \sqsubset p$  as desired. We write  $j_n: \partial\Delta^n \rightarrow \Delta^n$  for the  $n$ th cell inclusion.

**46.12. Lemma.** *Let  $\mathcal{C}$  be a weakly saturated class of monomorphism which has precancellation with respect to the class of monomorphisms, and which contains all inner horn inclusions. If  $\mathcal{C}$  also contains some cell inclusion  $j_n: \partial\Delta^n \rightarrow \Delta^n$ , then  $\text{Cell}_{\geq n} \subseteq \mathcal{C}$ .*

*Proof.* We show that  $j_m \in \mathcal{C}$  for  $m > n$  by induction on  $n$ . For any  $m \geq 1$  we have a commutative diagram

$$\begin{array}{ccccc} \partial\Delta^{m-1} & \longrightarrow & \Lambda_1^m & & \\ j_{m-1} \downarrow & & i \downarrow & \searrow j_m i & \\ \Delta^{m-1} & \xrightarrow{\langle 023\dots m \rangle} & \partial\Delta^m & \xrightarrow{j_m} & \Delta^m \end{array}$$

in which the left-hand square is a pushout. By induction we have that  $j_{m-1} \in \mathcal{C}$ , whence  $i \in \mathcal{C}$  since it is weakly saturated. We have that  $j_m i \in \mathcal{C}$  since it is an inner horn inclusion. Therefore  $j_m \in \mathcal{C}$  as desired by precancellation, since  $i$  and  $j_m$  are monomorphisms.  $\square$

*Proof of (46.1).* Let  $p$  be a fully faithful isofibration. To show  $\text{Cell}_{\geq 1} \sqsubset p$  it suffices to show  $\text{Cell}_{\geq 1} \subseteq \mathcal{C}_p$ . We have  $(\partial\Delta^1 \subset \Delta^1) \in \mathcal{C}_p$  by (45.6), and we know that  $\mathcal{C}_p$  is weakly saturated (46.5) and has precancellation with respect to monomorphisms (46.10). Finally, note that  $\mathcal{C}_p$  has inner horn inclusions since  $p$  is an inner fibration, so  $p^{\square i} \in \text{TrivFib}$  for  $i \in \text{InnHorn}$  since  $\text{InnHorn} \square \text{Cell} \subseteq \text{InnHorn}$ , and therefore  $(p^{\square i})^{\text{core}} \in \text{TrivFib}$  (46.13). Thus we can apply the lemma (46.12).  $\square$

46.13. *Exercise.* Let  $p: C \rightarrow D$  be a trivial fibration between quasicategories. Show that  $p^{\text{core}}: C^{\text{core}} \rightarrow D^{\text{core}}$  is also a trivial fibration.

## Part 8. Model categories

### 47. THE JOYAL MODEL STRUCTURE ON SIMPLICIAL SETS

47.1. **Model categories.** A **model category** (in the sense of Quillen) is a category  $\mathcal{M}$  with three classes of maps:  $W$ ,  $\text{Cof}$ ,  $\text{Fib}$ , which I will call **weak equivalences**, **cofibrations**, and **fibrations** respectively, satisfying the following axioms.

model category  
weak equivalences  
cofibrations  
fibrations

- $\mathcal{M}$  has all small limits and colimits.
- $W$  satisfies the 2-out-of-3 property.
- $(\text{Cof} \cap W, \text{Fib})$  and  $(\text{Cof}, \text{Fib} \cap W)$  are weak factorization systems (15.19).

An object  $X$  is **cofibrant** if the map from the initial object is a cofibration, and **fibrant** if the map to the terminal object is a fibration. A map in  $\text{Cof} \cap W$  is called a **trivial cofibration**, and a map in  $\text{Fib} \cap W$  is called a **trivial fibration**.

cofibrant  
fibrant  
trivial cofibration  
trivial fibration

47.2. *Warning.* Do not confuse the general notion of “weak equivalence” in an arbitrary model category with the specific notion of “weak homotopy equivalence of simplicial sets” defined in (49).

47.3. *Remark.* The third axiom implies that  $\text{Cof}$ ,  $\text{Cof} \cap W$ ,  $\text{Fib}$ , and  $\text{Fib} \cap W$  are closed under retracts.

47.4. *Exercise.* Show that in a model category (as defined above), the class of weak equivalences is closed under retracts. *Hint:* if  $f$  is a retract of a weak equivalence  $g$ , construct a factorization of  $f$  which is itself a retract of a factorization of  $g$ <sup>30</sup>.

<sup>30</sup>In many formulations of model categories, closure of weak equivalences under retracts is taken as one of the axioms. The formulation we use is described in Riehl, “A concise definition of a model category” [Rie09], which gives a solution to this exercise.

The formulation we are using is equivalent to ones used most often today, but it’s worthwhile to note that there is some variation: for instance, some may weaken the definition by merely requiring existence of *finite* limits and colimits (as in Quillen’s original definition), or strengthen it by requiring the existence of *functorial* factorization in the weak factorizations (as in Hovey’s book [Hov99].)

47.5. *Exercise* (Slice model categories). Let  $\mathcal{M}$  be a model category, and let  $X$  be an object of  $\mathcal{M}$ . Show that the slice categories  $\mathcal{M}_{X/}$  and  $\mathcal{M}_{/X}$  admit model category structures, in which the weak equivalences, cofibrations, and fibrations are precisely the maps whose images under  $\mathcal{M}_{/X} \rightarrow \mathcal{M}$  or  $\mathcal{M}_{X/} \rightarrow \mathcal{M}$  are weak equivalences, cofibrations, and fibrations in  $\mathcal{M}$ .

47.6. *Exercise* (Goodwillie). Classify all model category structures on the category of sets. (There are exactly nine. *Hint*: use (15.21).)

47.7. **Categorical fibrations and the small object argument.** As we have seen, the class  $\text{CatEq} \cap \overline{\text{Cell}}$  of monomorphisms which are categorical equivalences is weakly saturated (38.15), with right complement  $\text{CatFib}$ , the class of categorical fibrations (39). In fact, the pair  $(\text{CatEq} \cap \overline{\text{Cell}}, \text{CatFib})$  is a weak factorization system, as a consequence of the small object argument (15.16) and the following.

47.8. **Proposition.** *There exists a set  $S$  of maps of simplicial sets such that  $\overline{S} = \overline{\text{Cell}} \cap \text{CatEq}$ , whence  $S^\square = \text{CatFib}$ .*

Unfortunately, it's apparently not known how to write down an explicit set of maps  $S$  so that  $S^\square = \text{CatFib}$ . What is known is that such a set *exists*. We give a proof of this fact in the appendix (64).

47.9. **The Joyal model structure.**

47.10. **Theorem** (Joyal). *The category of simplicial sets admits a model structure, in which*

- $\text{W} = \text{CatEq}$ , the class of categorical equivalences,
- $\text{Cof} = \overline{\text{Cell}}$ , the class of monomorphisms,
- $\text{Fib} = \text{CatFib}$ , the class of categorical fibrations.

*Furthermore, the fibrant objects are precisely the quasicategories, and the fibrations with target a fibrant object are precisely the isofibrations.*

*Proof.* Categorical equivalences satisfy 2-out-of-3 by (25.11). We have that

- $\overline{\text{Cell}} = \text{monomorphisms}$  by (18.5),
- $\text{Fib} \cap \text{W} = \text{CatFib} \cap \text{CatEq} = \text{TFib} = \text{Cell}^\square$  by (39.2),
- $\text{Cof} \cap \text{W} = \overline{\text{Cell}} \cap \text{CatEq} = \overline{S}$  for some set  $S$  (47.8),
- $\text{Fib} = \text{CatFib} = (\text{Cof} \cap \text{W})^\square = S^\square$  by definition,

so both  $(\text{Cof} \cap \text{W}, \text{Fib})$  and  $(\text{Cof}, \text{Fib} \cap \text{W})$  are weak factorization systems via the small object argument (15.16). Thus, we get a model category.

We have shown (39.1) that the categorical fibrations  $p: C \rightarrow D$  with  $D$  a quasicategory are precisely the isofibrations. Applied when  $D = *$ , this implies that quasicategories are exactly the fibrant objects, and thus that fibrations with fibrant target are precisely the isofibrations.  $\square$

47.11. *Remark.* It is a fact that a model category structure is uniquely determined by its cofibrations and fibrant objects [Joy08a, Prop. E.1.10]. Thus, the Joyal model structure is the unique model structure on simplicial sets with  $\text{Cof} = \text{monomorphisms}$  and with fibrant objects the quasicategories.

47.12. **Cartesian model categories.** Recall that the category of simplicial sets is *cartesian closed*. A **cartesian model category** is a model category which is cartesian closed, such that the terminal object is cofibrant, and with the following properties. Suppose  $i: A \rightarrow B$  and  $j: K \rightarrow L$  are cofibrations and  $p: X \rightarrow Y$  is a fibration.

cartesian model category

•

$$i \square j: (A \times L) \cup_{A \times K} (B \times K) \rightarrow B \times L$$

is a cofibration, and is in addition a weak equivalence if either  $i$  or  $j$  is also a weak equivalence, and



$$p^{\square j}: \text{Fun}(L, X) \rightarrow \text{Fun}(K, X) \times_{\text{Fun}(K, Y)} \text{Fun}(L, Y)$$

is a fibration, and is in addition a weak equivalence if either  $j$  or  $p$  is also a weak equivalence.

In fact, we only need to specify *one* of the above two properties, as they imply each other.

**47.13. Proposition.** *The Joyal model structure is cartesian.*

*Proof.* This is just (39.3). □

#### 48. MODEL CATEGORIES AND HOMOTOPY COLIMITS

We are going to exploit these model category structures now. The main purpose of model categories is to give tools for showing that a given construction preserves certain kinds of equivalence.

**48.1. Creating new model categories.** Given a model category  $\mathcal{M}$ , many other categories related to it can also be equipped with model category structures, such as functor categories  $\text{Fun}(C, \mathcal{M})$  where  $C$  is a small category. We won't consider general formulations of this here, but rather will set up some special cases.

As an example, we consider the case of  $C = [1] = \{0 \xrightarrow{01} 1\}$ .

**48.2. Proposition.** *There exists a model structure on  $\mathcal{N} := \text{Fun}([1], \mathcal{M})$  in which a map  $\alpha: X \rightarrow X'$  is*

- *a weak equivalence if  $\alpha(i): X(i) \rightarrow X'(i)$  is a weak equivalence in  $\mathcal{M}$  for  $i = 0, 1$*
- *a cofibration if both  $\alpha(0)$  and the map  $(\alpha(1), X(01)): X(1) \cup_{X(0)} X'(0) \rightarrow X'(1)$  are cofibrations in  $\mathcal{M}$ , and*
- *a fibration if  $\alpha(i)$  is a fibration in  $\mathcal{M}$  for  $i = 0, 1$ .*

*Proof.* It is clear that  $\mathcal{N}$  has small limits and colimits, and that weak equivalences in it have the 2-out-of-3 property. It remains to show that  $(\text{Cof} \cap W, \text{Fib})$  and  $(\text{Cof}, \text{Fib} \cap W)$  are weak factorization systems, where  $W$ ,  $\text{Cof}$ ,  $\text{Fib}$  are the of maps in  $\mathcal{N}$  defined in the statement of the proposition.

We start with the following observation about lifting in  $\mathcal{N} = \text{Fun}([1], \mathcal{M})$ : given maps  $j: A \rightarrow B$  and  $p: X \rightarrow Y$  in  $\mathcal{N}$ , we can solve a lifting problem  $(u, v)$  of type  $j \square p$  in  $\mathcal{N}$  by solving a sequence of two lifting problems in  $\mathcal{M}$ , namely

$$\begin{array}{ccc} A(0) & \xrightarrow{u(0)} & X(0) \\ j(0) \downarrow & \nearrow s(0) & \downarrow p(0) \\ B(0) & \xrightarrow{v(0)} & Y(0) \end{array} \quad \text{and} \quad \begin{array}{ccc} A(1) \cup_{A(0)} B(0) & \xrightarrow{(u(1), X(01) \circ s(0))} & X(1) \\ (j(1), B(01)) \downarrow & \nearrow s(1) & \downarrow p(1) \\ B(1) & \xrightarrow{v(1)} & Y(1) \end{array}$$

where the second problem depends on the solution  $s(0)$  to the first problem. Then the maps  $s(0)$  and  $s(1)$  fit together to give a map  $s: B \rightarrow X$  in  $\mathcal{N}$  which solve the original lifting problem.

Given this, it is not hard to prove that  $\text{Cof} \cap W \square \text{Fib}$  and  $\text{Cof} \square \text{Fib} \cap W$ , using the definitions and the fact that  $\mathcal{M}$  is a model category. The trickiest point is to observe that if  $j: A \rightarrow B$  is both a cofibration and a weak equivalence in  $\mathcal{N}$ , then  $(j(1), B(01))$  is a trivial cofibration in  $\mathcal{M}$ : this uses 2-out-of-3 for weak equivalences in  $\mathcal{M}$  and the fact that  $A(1) \rightarrow A(1) \cup_{A(0)} B(0)$  must be a trivial cofibration in  $\mathcal{M}$ , being a cobase-change of  $j(0)$ .

Next, observe that to describe a factorization of a map  $f: X \rightarrow Y$  in  $\mathcal{N}$  into  $f = pj$  with  $j: X \rightarrow U$  and  $p: U \rightarrow Y$ , it suffices to describe a sequence of two factorizations in  $\mathcal{M}$ , namely

$f(0) = p(0) \circ j(0)$  and  $h = p(1) \circ g$ , as in

$$\begin{array}{ccc}
 X(0) & \xrightarrow{j(0)} & U(0) & \xrightarrow{p(0)} & Y(0) \\
 \downarrow X(01) & & \downarrow \eta' & & \downarrow Y(01) \\
 X(1) & \xrightarrow{\eta} & X(1) \cup_{X(0)} U(0) & \xrightarrow{g} & U(1) & \xrightarrow{p(1)} & Y(1) \\
 & & & \searrow h & & & \\
 & & & & & & 
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 X(0) & \xrightarrow{j(0)} & U(0) & \xrightarrow{p(0)} & Y(0) \\
 & & & & \\
 X(0) & \xrightarrow{j(0)} & U(0) & \xrightarrow{p(0)} & Y(0)
 \end{array}$$

where  $h = (f(1), Y(01) \circ p(0))$ , so that  $j(1) = g \circ \eta$  and  $U(01) = g \circ \eta'$ .

To factor  $f = pj$  in  $\mathcal{N}$  with  $j \in \text{Cof} \cap W$  and  $p \in \text{Fib}$ , it suffices to successively choose factorizations of  $f(0)$  and  $h$  of this type. Likewise, to factor  $f = pj$  in  $\mathcal{N}$  with  $j \in \text{Cof}$  and  $p \in \text{Fib} \cap W$ , it suffices to successively choose factorizations of  $f(0)$  and  $h$  of this type.

It remains to show that  $\text{Cof} \cap W = {}^{\square}\text{Fib}$ ,  $\text{Fib} = \text{Cof} \cap W^{\square}$ ,  $\text{Cof} = {}^{\square}\text{Fib} \cap W$ , and  $\text{Cof}^{\square} = \text{Fib} \cap W$ . This is an immediate consequence of the “retract trick” (15.17), together with the easily checked fact that  $\text{Cof}$ ,  $\text{Cof} \cap W$ ,  $\text{Fib}$ , and  $\text{Fib} \cap W$  are closed under retracts, which can be proved directly using the definition and the fact that the analogous classes in  $\mathcal{M}$  are closed under retracts (47.3).  $\square$

The opposite of a model category is also a model category, by switching the roles of fibrations and cofibrations. Therefore, there is another model structure on  $\text{Fun}([1], \mathcal{M}) = (\text{Fun}([1], \mathcal{M}^{\text{op}}))^{\text{op}}$ .

**48.3. Ken Brown lemma.** The “Ken Brown lemma” gives an explicit criterion for a functor to preserve weak equivalences between large classes of objects.

**48.4. Proposition** (Ken Brown lemma). *Let  $F: \mathcal{M} \rightarrow \mathcal{N}$  be a functor between model categories.*

- (1) *If  $F$  takes trivial cofibrations to weak equivalences, then  $F$  takes weak equivalences between cofibrant objects to weak equivalences.*
- (2) *If  $F$  takes trivial fibrations to weak equivalences, then  $F$  takes weak equivalences between fibrant objects to weak equivalences.*

*Proof.* I prove (1); the proof of (2) is formally dual.

Let  $f: X \rightarrow Y$  be a weak equivalence between cofibrant objects in  $\mathcal{M}$ . Form the commutative diagram

$$\begin{array}{ccccccc}
 \emptyset & \longrightarrow & Y & & & & \\
 \downarrow & & \downarrow & \searrow b & & \searrow \text{id}_Y & \\
 X & \longrightarrow & X \amalg Y & \xrightarrow{i} & C & \xrightarrow{p} & Y \\
 & \searrow a & & & \nearrow f & & \\
 & & & & & & 
 \end{array}$$

where the square is a pushout, and we have chosen a factorization of  $(f, \text{id}_Y): X \amalg Y \rightarrow Y$  as  $pi$ , a cofibration  $i$  followed by a weak equivalence  $p$  (e.g., a trivial fibration). Because  $X$  and  $Y$  are cofibrant, the maps  $X \rightarrow X \amalg Y \leftarrow Y$  are cofibrations. Using this and the 2-out-of-3 property for weak equivalences, we see that  $a$  and  $b$  are trivial cofibrations. Applying  $F$  gives

$$\begin{array}{ccccc}
 & & F(Y) & & \\
 & & \downarrow F(b) & \searrow \text{id} & \\
 F(X) & \xrightarrow{F(a)} & F(C) & \xrightarrow{F(p)} & F(Y)
 \end{array}$$

in which  $F(b)$  and  $F(a)$  are weak equivalences by hypothesis, whence  $F(p)$  is a weak equivalence by 2-out-of-3, and therefore  $F(f) = F(p)F(a)$  is a weak equivalence, as desired.  $\square$

**48.5. Quillen pairs.** Given an adjoint pair of functors  $F: \mathcal{M} \rightleftarrows \mathcal{N} : G$  between model categories, we see from the properties of weak factorization systems that

- $F$  preserves cofibrations if and only if  $G$  preserves trivial fibrations, and
- $F$  preserves trivial cofibrations if and only if  $G$  preserves fibrations.

If both of these are true, we say that  $(F, G)$  is a **Quillen pair**.

Quillen pair

Note that if  $(F, G)$  is a Quillen pair, then the Ken Brown lemma (48.4)(1) applies to  $F$ , while (48.4)(2) applies to  $G$ .

**48.6. Good colimits.** We can apply the above to certain examples of colimit functors, which we will refer to generically as “good colimits”. There are three types of these: arbitrary coproducts of cofibrant objects, countable sequential colimits of cofibrant objects along cofibrations, and pushouts of cofibrant objects along a cofibration. We will show that “good colimits are weak equivalence invariant”.

**48.7. Exercise.** Let  $S$  be a small discrete category (i.e., all maps are identities). Show that if  $\mathcal{M}$  is a model category, then  $\text{Fun}(S, \mathcal{M})$  is a model category in which  $\alpha: X \rightarrow X'$  is

- a weak equivalence, cofibration, or fibration iff each  $\alpha_s: X_s \rightarrow X'_s$  is one in  $\mathcal{M}$ .

Then show that  $\text{colim}: \text{Fun}(S, \mathcal{M}) \rightleftarrows \mathcal{M} : \text{const}$  is a Quillen pair, and use this to prove the next proposition.

**48.8. Proposition** (Good coproducts). *Given a collection  $f_s: X_s \rightarrow X'_s$  of weak equivalences between cofibrant objects in  $\mathcal{M}$ , the induced map  $\coprod f_s: \coprod X_s \rightarrow \coprod X'_s$  is a weak equivalence.*

*Proof.* Apply the Ken Brown lemma (48.4) to the coproduct functor  $\text{Fun}(S, \mathcal{M}) \rightarrow \mathcal{M}$ , using the model structure of (48.7).  $\square$

**48.9. Exercise.** Let  $\omega$  be the category

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$$

with objects indexed by natural numbers. Show that if  $\mathcal{M}$  is a model category, then  $\text{Fun}(\omega, \mathcal{M})$  is a model category in which  $\alpha: X \rightarrow X'$  is

- a weak equivalence if each  $\alpha(i)$  is a weak equivalence in  $\mathcal{M}$ ,
- a cofibration if (i)  $\alpha(0)$  is a cofibration in  $\mathcal{M}$ , and  $X'(i) \cup_{X(i)} X(i+1) \rightarrow X'(i+1)$  is a cofibration in  $\mathcal{M}$  for all  $i \geq 0$ , and
- a fibration if each  $\alpha(i)$  is a fibration in  $\mathcal{M}$ .

Then show that  $\text{colim}: \text{Fun}(\omega, \mathcal{M}) \rightleftarrows \mathcal{M} : \text{const}$  is a Quillen pair, and use this to prove the next proposition.

**48.10. Proposition** (Good sequential colimits). *Give a natural transformation  $\alpha: X \rightarrow X'$  of functors  $\omega \rightarrow \mathcal{M}$  such that all maps  $\alpha(i): X(i) \rightarrow X'(i)$  are weak equivalences, all objects  $X(i)$  and  $X'(i)$  are cofibrant, and the maps  $X(i) \rightarrow X(i+1)$  and  $X'(i) \rightarrow X'(i+1)$  are cofibrations, the induced map  $\text{colim}_\omega X \rightarrow \text{colim}_\omega X'$  is a weak equivalence.*

*Proof.* Apply the Ken Brown lemma (48.4) to the colimit functor  $\text{Fun}(\omega, \mathcal{M}) \rightarrow \mathcal{M}$ , using the model structure of (48.9).  $\square$

**48.11. Exercise.** Recall that  $\Lambda_0^2$  is a category:

$$1 \xleftarrow{01} 0 \xrightarrow{12} 2.$$

Show that if  $\mathcal{M}$  is a model category, then  $\text{Fun}(\Lambda_0^2, \mathcal{M})$  is a model category in which  $\alpha: X \rightarrow X'$  is

- a weak equivalence if  $\alpha(i): X(i) \rightarrow X'(i)$  is a weak equivalence in  $\mathcal{M}$  for  $i = 0, 1, 2$  (i.e., an **objectwise weak equivalence**),

objectwise weak equivalence

- a cofibration if  $\alpha(0)$ ,  $\alpha(1)$ , and the evident map  $X(2) \cup_{X(0)} X'(0) \rightarrow X'(2)$  are cofibrations in  $\mathcal{M}$ , and
- a fibration if  $\alpha(1)$ ,  $\alpha(2)$ , and the evident map  $X(0) \rightarrow X'(0) \times_{X'(1)} X(1)$  are fibrations in  $\mathcal{M}$ .

Then show that  $\text{colim}: \text{Fun}(\Lambda_0^2, \mathcal{M}) \rightleftarrows \mathcal{M} : \text{const}$  is a Quillen pair, and use this to prove the next proposition.

**48.12. Proposition** (Good pushouts). *Given a natural transformation  $\alpha: X \rightarrow X'$  of functors  $\Lambda_0^2 \rightarrow \mathcal{M}$ , i.e., a diagram*

$$\begin{array}{ccccc} X(1) & \longleftarrow & X(0) & \xrightarrow{X(02)} & X(2) \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ X'(1) & \longleftarrow & X'(0) & \xrightarrow{X'(02)} & X'(2) \end{array}$$

*in which the vertical maps are weak equivalences, all objects  $X(i)$  and  $X'(i)$  are cofibrant, and the maps  $X(02)$  and  $X'(02)$  are cofibrations, the induced map  $\text{colim}_{\Lambda_0^2} X \rightarrow \text{colim}_{\Lambda_0^2} X'$  is a weak equivalence.*

*Proof.* Apply the Ken Brown lemma (48.4) to the colimit functor  $\text{Fun}(\Lambda_0^2, \mathcal{M}) \rightarrow \mathcal{M}$ , using the model structure of (48.11).  $\square$

In the Joyal model structure on  $s\text{Set}$ , all objects are automatically cofibrant, which makes the above propositions especially handy.

We will call any colimit diagram in a model category, satisfying the hypotheses of one of (48.8), (48.12), (48.10) a **good colimit**. Thus, we see that good colimits are weak equivalence invariant. These “good colimits” are examples of what are called *homotopy colimits*.

good colimit

Since the opposite of a model category is also a model category, all of the results of this section admit dual formulations, leading to the observation that **good limits** are homotopy invariant.

good limits

**48.13. Exercise.** State and prove the dual versions of all the results in this section.

**48.14. Exercise.** Recall the relative function complex (23.18), which for objects  $p: S \rightarrow K$  and  $q: S \rightarrow C$  in  $s\text{Set}_S$  is the simplicial set

$$\text{Fun}_{S/}(K, C) = \text{Fun}(K, C) \times_{\text{Fun}(S, C)} \{q\}.$$

Show that if  $f: K \rightarrow L$  is a categorical equivalence,  $C$  is a quasicategory, and both  $p$  and  $fp$  are monomorphisms, then the induced map  $f^*: \text{Fun}_{S/}(L, C) \rightarrow \text{Fun}_{S/}(K, C)$  on relative function complexes is a categorical equivalence. (Hint: Both source and target of  $f^*$  can be described via good pullbacks with respect to the Joyal model structure.)

## Part 9. Quasigroupoids and weak homotopy equivalence

### 49. WEAK HOMOTOPY EQUIVALENCE

Say that a map  $f: X \rightarrow Y$  of simplicial sets is a **weak homotopy equivalence** if and only for every  $\infty$ -groupoid (i.e., Kan complex)  $C$  the induced functor  $\text{Fun}(f, C): \text{Fun}(Y, C) \rightarrow \text{Fun}(X, C)$  is a categorical equivalence. It is immediate that every categorical equivalence is a weak homotopy equivalence, but the converse is not so.

weak homotopy equivalence

**49.1. Exercise.** Show that either inclusion  $\Delta^0 \rightarrow \Delta^1$  is a weak homotopy equivalence but not a categorical equivalence.

**49.2. Remark.** A more logical name for weak homotopy equivalence might be “groupoidal equivalence”, by analogy with categorical equivalence.

49.3. *Exercise.* Show that the class of weak homotopy equivalences satisfies 2-out-of-6, and hence 2-out-of-3.

49.4. **Proposition.** *Let  $i: X \rightarrow Y$  be a monomorphism of simplicial sets. Then the following are equivalent.*

- (1) *The map  $i$  is a weak homotopy equivalence.*
- (2) *For every isofibration  $p: C \rightarrow D$  between quasigroupoids, the pullback-hom map  $p^{\square i}$  is a trivial fibration.*
- (3) *For every isofibration  $p: C \rightarrow D$  between quasigroupoids we have  $i \sqsupseteq p$ .*

*Then  $i$  is a weak homotopy equivalence if and only if  $i \sqsupseteq p$  for every isofibration  $p: C \rightarrow D$  between quasigroupoids.*

*Proof.* (1  $\implies$  2): Consider the commutative diagram

$$\begin{array}{ccccc}
 & & p_* & & \\
 \text{Fun}(Y, C) & \xrightarrow{\quad} & \text{Fun}(X, C) \times_{\text{Fun}(X, D)} \text{Fun}(Y, D) & \xrightarrow{\quad} & \text{Fun}(Y, D) \\
 & \searrow p^{\square i} & \downarrow q & & \downarrow i^* \\
 & & \text{Fun}(X, C) & \xrightarrow{\quad} & \text{Fun}(X, D) \\
 & \swarrow i^* & & & 
 \end{array}$$

in which the square is a pullback. All the maps in this diagram are isofibrations by (37.5) (in fact, they are Kan fibrations), while the maps marked  $i^*$  are categorical equivalences by hypothesis. In particular, the maps marked  $i^*$  are trivial fibrations by (38.8), and thus so is the map  $q$  obtained by basechange. Thus  $p^{\square i}$  is a categorical equivalence by 2-out-of-3 and hence is a trivial fibration by (38.8).

(2  $\implies$  3): The trivial fibration  $p^{\square i}$  is surjective on vertices, giving  $i \sqsupseteq p$ .

(3  $\implies$  2): If  $p: C \rightarrow D$  is an isofibration between quasigroupoids, then so is  $p^{\square(\partial\Delta^n \subset \Delta^n)}$  for any  $n \geq 0$ , using (38.8). Thus (2) implies that  $i \sqsupseteq p^{\square(\partial\Delta^n \subset \Delta^n)}$ , which is equivalent to  $(\partial\Delta^n \subset \Delta^n) \sqsupseteq p^{\square i}$ , whence  $p^{\square i}$  is a trivial fibration as desired.

(2  $\implies$  1): The hypothesis implies in particular that  $\text{Fun}(Y, C) \rightarrow \text{Fun}(X, C)$  is a trivial fibration for every quasigroupoid  $C$ , and hence a categorical equivalence (23.11).  $\square$

49.5. **Corollary.** *Every anodyne map (i.e., element of  $\overline{\text{Horn}}$ ) is a weak homotopy equivalence.*

*Proof.* Anodyne maps  $i$  are monomorphisms such that  $i \sqsupseteq p$  for every Kan fibration. Thus statement (3) of (49.4) applies since isofibrations between quasigroupoids are Kan fibrations (36.11).  $\square$

## 50. GROUPOID COMPLETION

50.1. **Functors into the core of a quasicategory.** Given a quasicategory  $C$  and a simplicial set  $X$ , let

$$\text{Fun}^{\text{iso}}(X, C) \subseteq \text{Fun}(X, C)$$

denote the full subcategory spanned by objects which are functors  $f: X \rightarrow C$  with the property that  $f(X) \subseteq C^{\text{core}}$ .

50.2. *Example.* When  $X = \Delta^1$ , then this is precisely the path category  $\text{Fun}^{\text{iso}}(\Delta^1, C)$  introduced in (34.1).

Note that  $\text{Fun}^{\text{iso}}(X, C)$  is not necessarily a quasigroupoid, unless  $C$  itself is a quasigroupoid.

We have a convenient characterization of maps into  $\text{Fun}^{\text{iso}}(X, C)$ .

50.3. **Proposition.** *For any quasicategory  $C$  and simplicial sets  $X$  and  $S$ , the evident bijection  $\text{Hom}(S, \text{Fun}(X, C)) \approx \text{Hom}(X, \text{Fun}(S, C))$  restricts to a bijection*

$$\left\{ S \longrightarrow \text{Fun}^{\text{iso}}(X, C) \right\} \longleftrightarrow \left\{ X \longrightarrow \text{Fun}(S, C)^{\text{core}} \right\}.$$

*Proof.* (This is a generalization of (34.4).) Consider  $f: S \rightarrow \text{Fun}(X, C)$ , and write  $f': X \rightarrow \text{Fun}(S, C)$  and  $f'': S \times X \rightarrow C$  for its adjoints. We have the following observations (which make use of the pointwise criterion for natural isomorphisms (35.2)).

- (1) The map  $f$  factors through  $\text{Fun}^{\text{iso}}(X, C) \subseteq \text{Fun}(X, C)$  if and only if for each vertex  $s \in S_0$  the induced map  $f(s): X \rightarrow C$  factors through  $C^{\text{core}} \subseteq C$ . This amounts to saying that for each edge  $g \in X_1$ , each map  $f(s)$  sends  $g$  to an isomorphism in  $C$ .
- (2) The map  $f'$  factors through  $\text{Fun}(S, C)^{\text{core}} \subseteq \text{Fun}(S, C)$  if and only if for each edge  $g \in X_1$  the image  $f'(g) \in \text{Fun}(S, C)_1$  represents an isomorphism in  $\text{Fun}(S, C)$ . By the objectwise criterion (35.2), this amounts to saying that  $f'(g)$  sends each vertex  $s \in S_0$  to an isomorphism in  $C$ .

It is thus apparent that conditions (1) and (2) are equivalent: both are amount to the requirement that  $\Delta^0 \times \Delta^1 \xrightarrow{s \times g} S \times X \xrightarrow{f''} C$  represent an isomorphism in  $C$  for every  $s \in S_0$  and  $g \in X_1$ .  $\square$

For any map  $f: X \rightarrow Y$  of simplicial sets and any quasicategory  $C$ , the induced functor  $\text{Fun}(f, C)$  restricts to a functor  $\text{Fun}^{\text{iso}}(Y, C) \rightarrow \text{Fun}^{\text{iso}}(X, C)$  between full subcategories.

**50.4. Proposition.** *Let  $i: X \rightarrow Y$  be any map of simplicial sets which is a monomorphism and a weak homotopy equivalence. Then for any quasicategory  $C$ , the restriction map*

$$i^*: \text{Fun}^{\text{iso}}(Y, C) \rightarrow \text{Fun}^{\text{iso}}(X, C)$$

*is a trivial fibration, and thus in particular a categorical equivalence between quasicategories.*

*Proof.* We need to solve lifting problems

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{u} & \text{Fun}^{\text{iso}}(Y, C) \\ \downarrow & \nearrow \tilde{s} & \downarrow i^* \\ \Delta^n & \xrightarrow{v} & \text{Fun}^{\text{iso}}(X, C) \end{array}$$

for all  $n \geq 0$ . Using (50.3) we can replace this with the adjoint lifting problem

$$\begin{array}{ccc} X & \xrightarrow{\tilde{v}} & \text{Fun}(\Delta^n, C)^{\text{core}} \\ \downarrow i & \nearrow \tilde{s} & \downarrow p^{\text{core}} \\ Y & \xrightarrow{\tilde{u}} & \text{Fun}(\partial\Delta^n, C)^{\text{core}} \end{array}$$

where  $p^{\text{core}}$  is induced by the restriction map  $p: \text{Fun}(\Delta^n, C) \rightarrow \text{Fun}(\partial\Delta^n, C)$ . By (37.5) the map  $p$  is an isofibration, and thus  $p^{\text{core}}$  is an isofibration between quasigroupoids (36.11). Therefore a lift exists by (49.4).  $\square$

**50.5. Groupoid completion.** For any simplicial set  $X$ , we can always construct a monomorphism  $i: X \rightarrow X'$  to a Kan complex which is a weak homotopy equivalence. For instance, factor  $X \rightarrow *$  into an anodyne map followed by a Kan fibration. Any such map provides an example of a **groupoid completion** of  $X$ .

groupoid completion

**50.6. Proposition.** *Suppose  $i: X \rightarrow X'$  is a monomorphism of simplicial sets which is a weak homotopy equivalence, with  $X'$  a quasigroupoid. Then for any quasicategory  $C$ , restriction along  $i$  induces a trivial fibration*

$$p: \text{Fun}(X', C) \rightarrow \text{Fun}^{\text{iso}}(X, C).$$

*In particular, any map  $f: X \rightarrow X^{\text{core}} \subseteq C$ , extends over  $i$  to a map  $g: X' \rightarrow C^{\text{core}}$ , and any two such extensions are naturally isomorphic in  $\text{Fun}(X, C)$ .*

*Proof.* That  $p$  is a trivial fibration is immediate from (50.4) and the fact that  $\mathrm{Fun}^{\mathrm{iso}}(X', C) = \mathrm{Fun}(X', C)$  since  $X'$  is a quasigroupoid. The fiber of  $p$  over a vertex representing  $f$  is thus a contractible Kan complex, so any two objects in this fiber are isomorphic, and hence correspond to isomorphic objects of  $\mathrm{Fun}(X', C)$ .  $\square$

Although the groupoid completion isn't unique, it is unique up to categorical equivalence.

50.7. *Exercise.* Let  $f_i: X \rightarrow X_i$  be groupoid completions of  $X$ , for  $i = 1, 2$ . Show that there exists a categorical equivalence  $g: X_1 \rightarrow X_2$  such that  $gf_1 = f_2$ , and that any two such are naturally isomorphic. (Hint: proof of (23.19) and (38.14).)

We can apply this construction when  $X$  is a quasicategory, or even when  $X$  is the nerve of an ordinary category, and obtain interesting new quasigroupoids.

50.8. *Example.* It turns out that every simplicial set is weakly equivalent to the nerve of some ordinary category, and in fact to the nerve of some poset [Tho80]. Thus, for every Kan complex  $K$ , there exists an ordinary category  $A$  and a weak equivalence  $NA \rightarrow K$ , which therefore induces categorical equivalences  $\mathrm{Fun}(K, C) \approx \mathrm{Fun}^{\mathrm{iso}}(NA, C)$  for every quasicategory  $C$ .

We note that there is also a classical groupoid completion construction, which given an ordinary category  $A$  produces an ordinary groupoid  $A_{\mathrm{Gpd}}$  by “formally inverting all maps”. We have that  $h((NA)_{\mathrm{Kan}}) \approx N(A_{\mathrm{Gpd}})$ , but in general  $(NA)_{\mathrm{Kan}}$  is not weakly equivalent to  $N(A_{\mathrm{Gpd}})$ .

50.9. *Exercise.* Let  $A$  be the poset of *proper and non-empty* subsets of  $\{0, 1, 2, 3\}$ . Show that  $A_{\mathrm{Gpd}}$  is equivalent to the one-object category, but that  $(NA)_{\mathrm{Kan}}$  is not equivalent to the one-object category. (In the second case, you can prove non-equivalence by showing  $\pi_0 \mathrm{Fun}(NA, K(\mathbb{Z}, 2)) \approx \mathbb{Z}$ , using the Eilenberg-MacLane object of (10.10).)

## 51. LOCALIZATION OF QUASICATEGORIES

There is a generalization of groupoid completion, which applies to a simplicial set  $X$  equipped with a subcomplex  $W \subseteq X$ . For a quasicategory  $C$ , let

$$\mathrm{Fun}^{W\mathrm{iso}}(X, C) \subseteq \mathrm{Fun}(X, C)$$

denote the full subcategory spanned by objects  $f: X \rightarrow C$  such that  $f(W) \subseteq C^{\mathrm{core}}$ . (Note that this condition is satisfied if and only if  $f$  maps the edges of  $W$  to isomorphisms.) Clearly  $\mathrm{Fun}^{W\mathrm{iso}}(X, C)$  is the preimage of  $\mathrm{Fun}^{\mathrm{iso}}(W, C)$  along the restriction map  $\mathrm{Fun}(X, C) \rightarrow \mathrm{Fun}(W, C)$ .

Given a subcomplex  $W \subseteq X$ , we may define a **localization** of  $X$  with respect to  $W$ . This is any map  $X \rightarrow X_{(W)}$  constructed as follows.

(1) Choose a groupoid completion  $i: W \rightarrow W'$  of  $W$ , i.e., a monomorphism to a Kan complex.

(2) Choose an inner anodyne map  $j: X \cup_W W' \rightarrow X_{(W)}$  to a quasicategory  $X_{(W)}$ .

If  $W = X$  then  $X \rightarrow X_{(X)}$  is an example of a groupoid completion of  $X$  as discussed above.

**51.1. Proposition.** *For any localization  $X \rightarrow X_{(W)}$  as defined above, and any quasicategory  $C$ , the restriction map  $\mathrm{Fun}(X_{(W)}, C) \rightarrow \mathrm{Fun}(X, C)$  induces a trivial fibration*

$$\mathrm{Fun}(X_{(W)}, C) \rightarrow \mathrm{Fun}^{W\mathrm{iso}}(X, C).$$

*In particular, any map  $f: X \rightarrow C$  such that  $f(W) \subseteq C^{\mathrm{core}}$  extends to a functor  $g: X_{(W)} \rightarrow C$ , and any two such extensions are naturally isomorphic.*

*Proof.* Consider

$$\begin{array}{ccccccc}
 \mathrm{Fun}(X_{(W)}, C) & \xrightarrow{j^*} & \mathrm{Fun}(X \cup_W W', C) & \xrightarrow{p} & \mathrm{Fun}^{W_{\mathrm{iso}}}(X, C) & \twoheadrightarrow & \mathrm{Fun}(X, C) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Fun}(W', C) & \xrightarrow{i^*} & \mathrm{Fun}^{W_{\mathrm{iso}}}(W, C) & \twoheadrightarrow & \mathrm{Fun}(W, C)
 \end{array}$$

in which both squares are pullbacks. The map  $j^*$  is a trivial fibration since  $g$  is inner anodyne, using  $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ , while  $i^*$  is a trivial fibration (50.6) since  $\mathrm{Fun}^{W_{\mathrm{iso}}}(W, C) = \mathrm{Fun}^{\mathrm{iso}}(W, C)$ , whence  $p$  is a trivial fibration.  $\square$

**51.2. Quasicategories from relative categories.** A **relative category** is a pair  $W \subseteq C$  consisting of an *ordinary category*  $C$  and a subcategory  $W$  containing all the objects of  $C$ . The above construction gives, for any relative category, a map

$$C \rightarrow C_{(W)},$$

unique up to categorical equivalence. We may call  $C_{(W)}$  the **localization** of  $C$  with respect to  $W$ . localization

It turns out that all quasicategories, up to categorical equivalence, arise as localizations of relative categories in this way [BK11].

## 52. THE KAN-QUILLEN MODEL STRUCTURE ON SIMPLICIAL SETS

Say that map  $p: X \rightarrow Y$  is a **groupoidal fibration** if  $j \sqsubset p$  for all  $j$  which are monomorphisms and weak equivalences. I write  $\mathrm{GpdFib}$  for the class of groupoidal fibrations. As with the class of categorical fibrations, there is a set of maps  $T$  such that  $\mathrm{GpdFib} = T^{\square}$ ; see (64). groupoidal fibration

### 52.1. The Kan-Quillen model structure.

**52.2. Theorem.** *The category of simplicial sets admits a model structure, in which*

- $W = \text{weak homotopy equivalences (WHEq)},$
- $\mathrm{Cof} = \text{monomorphisms } (\overline{\mathrm{Cell}}),$
- $\mathrm{Fib} = \text{groupoidal fibrations (GpdFib)}.$

*Furthermore, the fibrant objects are precisely the Kan complexes, and the fibrations with target a fibrant object are precisely the Kan fibrations.*

*Proof.* Weak equivalences satisfy 2-out-of-3 by (49.3). We have that

- $\mathrm{Cof} = \overline{\mathrm{Cell}}$  by definition,
- $\mathrm{Fib} \cap W = \mathrm{GpdFib} \cap \mathrm{WHEq} = \mathrm{TFib} = \mathrm{Cell}^{\square}$  by (39.2),
- $\mathrm{Cof} \cap W = \overline{\mathrm{Cell}} \cap \mathrm{WHEq} = \overline{T}$  for some set  $T$ , as noted above.
- $\mathrm{Fib} = \mathrm{GpdFib} = (\mathrm{Cof} \cap W)^{\square} = T^{\square}$  by definition,

so both  $(\mathrm{Cof} \cap W, \mathrm{Fib})$  and  $(\mathrm{Cof}, \mathrm{Fib} \cap W)$  are weak factorization systems via the small object argument (15.16). Thus, we get a model category.

We have seen that Kan fibrations between Kan complexes (which are exactly the isofibrations between Kan complexes) have the lifting property of groupoidal fibrations (49.4), so the statements about fibrant objects and fibrations to fibrant objects follow just as in the categorical case.  $\square$

### 52.3. Proposition. *The Quillen model structure is cartesian.*

*Proof.* We must show that  $p^{\square j}$  is a groupoidal fibration if  $j$  is a monomorphism and  $p$  a groupoidal fibration, and also that it is a weak equivalence if either  $j$  or  $p$  is. This is proved by an argument nearly identical to the proof of (39.3).  $\square$



**52.4. Kan fibrations are groupoidal fibrations.** The proof of the Quillen model structure we gave above relied on (64) to produce a set  $T$  such that  $\overline{T} = \text{Cof} \cap \text{WHEq}$ . In fact, more is true. It turns out that we can take  $T = \text{Horn}$ , so that  $\text{GpdFib} = \text{KanFib}$ . It was in this form that the model structure was first constructed by Quillen.

We will not give a proof of this here. The non-trivial part is to show that  $\text{KanFib} \subseteq \text{GpdFib}$ . This proposition is usually proved via an argument (due to Quillen) based on the theory of *minimal fibrations*. See for instance Quillen's original argument [Qui67, §II.3] or [GJ09, Ch. 1].

These arguments work by showing that  $\text{KanFib}$  is the weak cosaturation of the class of *Kan fibrations between Kan complexes*, which we know are groupoidal fibrations. In fact one can even show that every Kan fibration is a *base change* of a Kan fibration between Kan complexes, see [KLV12].

The observation that the Kan-Quillen model structure can be constructed without first showing  $\text{GpdFib} = \text{KanFib}$ , and thus (52.2) in the form I have stated it, is due to Cisinski [Cis06].

### 53. WEAK HOMOTOPY EQUIVALENCE AND HOMOTOPY GROUPS

**53.1. Pointed simplicial sets and pointed function complexes.** Given a simplicial set  $X$  and a vertex  $x \in X_0$ , I'll write  $(X, x)$  for the corresponding **pointed** simplicial set, i.e., object of  $s\text{Set}_* := s\text{Set}_{\Delta^0/}$ . Given pointed simplicial sets  $(X, x)$ ,  $(Y, y)$ , I'll write

$$\text{Fun}_*((X, x), (Y, y)) := \text{Fun}_{\Delta^0/}((X, x), (Y, y))$$

for the relative function complex, and call it the pointed function complex. I'll often omit mention of the basepoints and write  $\text{Fun}_*(X, Y)$ . This defines a functor  $\text{Fun}_*: (s\text{Set}_*)^{\text{op}} \times s\text{Set}_* \rightarrow s\text{Set}_*$ , where the basepoint of  $\text{Fun}_*(X, Y)$  is represented by the constant map  $X \rightarrow \{y\} \rightarrow Y$  to the basepoint of  $Y$ .

Say that a map of pointed simplicial sets is a weak homotopy equivalence if the underlying map of simplicial sets is a weak homotopy equivalence.

**53.2. Proposition.** *The pointed function complex is weak homotopy equivalence invariant when the target is a Kan complex. That is,*

- (1) *if  $f: A \rightarrow B$  is a weak homotopy equivalence of pointed simplicial sets, and  $X$  is a pointed Kan complex, then  $f^*: \text{Fun}_*(B, X) \rightarrow \text{Fun}_*(A, X)$  is a weak homotopy equivalence, and*
- (2) *if  $A$  is any pointed simplicial set and  $g: X \rightarrow Y$  is a weak homotopy equivalence of pointed Kan complexes, then  $g_*: \text{Fun}_*(A, X) \rightarrow \text{Fun}_*(A, Y)$  is a weak homotopy equivalence.*

*Proof.* The pointed function complex  $\text{Fun}_*(A, X)$  is defined by a pullback  $\text{Fun}(A, X) \times_{\text{Fun}(\{a\}, X)} \{\bar{x}\}$ , which is good pullback when  $X$  is a Kan complex since  $\{a\} \rightarrow A$  is always a monomorphism. The claim follows from the fact that  $\text{Fun}$  is weak homotopy invariant whenever the target  $X$  is a Kan complex.  $\square$

**53.3. Homotopy sets.** Given a Kan complex  $X$ , for each  $n \geq 0$  and each vertex  $x \in X_0$  we define the  *$n$ th homotopy set* to be

$$\pi_n(X, x) := \pi_0 \text{Fun}_*(\Delta^n / \partial \Delta^n, X).$$

These define functors  $\pi_n: s\text{Set}_* \rightarrow \text{Set}_*$  from pointed simplicial sets to pointed sets, where the basepoint of  $\pi_n(X, x)$  is the path component containing the constant map.

**53.4. Remark.** In general,  $\pi_n(X, x)$  is a pointed set. In fact, when  $n \geq 1$  it has a natural structure of a group, which is abelian when  $n \geq 2$ .

**53.5. Exercise.** Let  $X$  be a Kan complex and  $x \in X_0$ . Show that  $\pi_1(X, x) \approx \text{Hom}_{hX}(x, x)$ , so that  $\pi_1(X, x)$  has a group structure defined by composition in  $hX$ , and that this group structure is natural with respect to maps between pointed Kan complexes.

We present a proof that  $\pi_n(X, x)$  are groups for all  $n \geq 1$  in the appendix (63).

53.6. *Remark.* If  $T$  is a topological space with basepoint  $t \in T$ , then the homotopy sets  $\pi_n(\text{Sing}(T), t)$  are in natural bijective correspondence with the “usual” homotopy sets (groups)  $\pi_n(T, t)$  of the space  $T$ . This is a straightforward consequence of the observation that  $\|\Delta^n / \partial\Delta^n\|$  is homeomorphic to an  $n$ -dimensional sphere.

53.7.  **$\pi_*$ -equivalences.** Say that a map  $f: X \rightarrow Y$  between Kan complexes is a  **$\pi_*$ -equivalence** if for all  $k \geq 0$  and all  $x \in X_0$ , the induced map  $\pi_k(X, x) \rightarrow \pi_k(Y, f(x))$  is a bijection. It is clear from (53.2) that every weak equivalence of Kan complexes is a  $\pi_*$ -equivalence. In fact the converse is also true.

53.8. **Theorem.** *A map  $f: X \rightarrow Y$  between Kan complexes is a weak homotopy equivalence if and only if it is a  $\pi_*$ -equivalence.*

We give a proof in an appendix (63).

#### 54. EVERY QUASIGROUPOID IS EQUIVALENT TO ITS OPPOSITE

Every ordinary groupoid  $C$  is equivalent, and in fact isomorphic, to its opposite: there is a functor  $C \rightarrow C^{\text{op}}$  which is the identity on objects, and which sends each morphism to its inverse. We cannot define such a functor for quasigroupoids, since inverses of morphisms in a quasigroupoid are not unique. However, it is the case that any quasigroupoid is equivalent to its opposite.

We will produce for each quasicategory  $C$  a quasigroupoid  $\mathcal{S}(C)$ , together with a trivial fibration  $\mathcal{S}(C) \rightarrow C^{\text{core}}$ , with the property that  $\mathcal{S}(C)$  and  $\mathcal{S}(C^{\text{op}})$  are isomorphic as simplicial sets.

54.1. **The functor  $\mathcal{S}$ .** Given a set  $S$ , let  $\text{Iso}^S$  denote the simplicial set with

$$(\text{Iso}^S)_n := \text{Hom}_{\text{Set}}([n] = \{0, 1, \dots, n\}, S),$$

with simplicial operators induced in the evident way. Observe that

$$\text{Hom}_{s\text{Set}}(X, \text{Iso}^S) \approx \text{Hom}_{\text{Set}}(X_0, S),$$

so that we have a functor  $\text{Iso}^\bullet: \text{Set} \rightarrow s\text{Set}$  which is right adjoint to  $X \mapsto X_0: s\text{Set} \rightarrow \text{Set}$ . In particular, we have

$$\text{Hom}_{s\text{Set}}(\text{Iso}^S, \text{Iso}^T) \approx \text{Hom}_{\text{Set}}(S, T).$$

Recall that  $X \mapsto X_0$  also admits a left adjoint  $S \mapsto S_{\text{disc}}$ , sending any set to the corresponding discrete simplicial set (2.7).

Note that the simplicial set  $\text{Iso}^S$  is the nerve of a category, with object set  $S$  and a unique morphism  $x \rightarrow y$  for every pair  $x, y \in S$ . This is in fact a groupoid, which when  $S$  is non-empty is equivalent to the trivial groupoid. For instance,  $\text{Iso}^{\{0,1\}}$  is precisely the walking isomorphism  $\text{Iso}$  discussed in (36.4).

We may compose  $\text{Iso}^\bullet$  with the evident functor  $\Delta \rightarrow \text{Set}$  sending the ordered set  $[n]$  to its underlying set  $\{0, 1, \dots, n\}$ , and in this way obtain a functor  $\text{Iso}: \Delta \rightarrow s\text{Set}$ , so that  $(\text{Iso}^{[n]})_m = \text{Hom}_{\text{Set}}([m], [n])$ . This in turn induces by restriction a functor  $\mathcal{S}: s\text{Set} \rightarrow s\text{Set}$ , with

$$\mathcal{S}(C)_n = \text{Hom}(\text{Iso}^{[n]}, C).$$

54.2. **Proposition.** *There is a natural isomorphism  $\mathcal{S}(C) \rightarrow \mathcal{S}(C^{\text{op}})$  of functors  $s\text{Set} \rightarrow s\text{Set}$ .*

*Proof.* It suffices to describe a natural isomorphism  $\text{Iso} \rightarrow \text{Iso} \circ \text{op}$  of functors  $\Delta \rightarrow s\text{Set}$ . On each object  $[n]$  in  $\Delta$  this is the map  $\text{Iso}^r: \text{Iso}^{[n]} \rightarrow \text{Iso}^{[n]}$  induced by the order reversing bijection  $r: [n] \rightarrow [n]$ ,  $r(x) := n - x$ .  $\square$

We write  $\eta_n: \Delta^n \rightarrow \text{Iso}^{[n]}$  for the map representing the  $n$ -cell corresponding to the identity function  $[n] \rightarrow [n]$ . Then we get a natural transformation  $\epsilon: \mathcal{S} \rightarrow \text{id}_{s\text{Set}}$  of functors  $s\text{Set} \rightarrow s\text{Set}$ , which for a simplicial set  $C$  and  $n \geq 0$  is given by the function  $\text{Hom}(\text{Iso}^n, C) \rightarrow C_n$  which sends  $f: \text{Iso}^n \rightarrow C$  to the  $n$ -cell represented by  $f\eta_n: \Delta^n \rightarrow C$ . Note that  $\epsilon$  induces a bijection on vertices, since  $\eta_0: \Delta^0 \rightarrow \text{Iso}^{[0]}$  is an isomorphism.

We are going to show that  $\epsilon: \mathcal{S}(C) \rightarrow C$  is a trivial fibration whenever  $C$  is a quasigroupoid. Together with (54.2) this gives a sequence of categorical equivalences

$$C \xleftarrow{\epsilon_C} \mathcal{S}(C) \approx \mathcal{S}(C^{\text{op}}) \xrightarrow{\epsilon_{C^{\text{op}}}} C^{\text{op}}$$

between quasigroupoids. In particular, by choosing any section  $s$  of the trivial fibration  $\epsilon_C$  we get a categorical equivalence  $C \rightarrow C^{\text{op}}$  which is identity on objects.

**54.3. The functor  $\mathcal{R}$ .** The functor  $\mathcal{S}$  admits a left adjoint  $\mathcal{R}$ , which we can describe explicitly. We will make use of the evident identification

$$\gamma \mapsto \tilde{\gamma}: \text{Hom}_{s\text{Set}}([n], \Delta^m) \approx \text{Hom}_{s\text{Set}}(\text{Iso}^{[n]}, \text{Iso}^{[m]}).$$

Note that  $\tilde{\iota}_n = \text{id}_{\text{Iso}^{[n]}}$  where  $\iota_n: [n] \rightarrow \Delta^n$  is the evident function sending  $k \mapsto \langle k \rangle \in (\Delta^n)_0$ .

Define  $\mathcal{R}(X)$  to be the simplicial set with  $n$ -dimensional cells

$$\mathcal{R}(X)_n := \{[n] \xrightarrow{\gamma} \Delta^m \xrightarrow{x} X\} / \sim,$$

where we quotient by the equivalence relation generated by  $(xf, \gamma) \sim (x, f\gamma)$  for every simplicial operator  $f$ . The simplicial operators act on  $\mathcal{R}(X)$  in the evident way:  $g: [n'] \rightarrow [n]$  sends an equivalence class  $[x, \gamma] \in \mathcal{R}(X)_n$  to  $[x, \gamma g] \in \mathcal{R}(X)_{n'}$ .

**54.4. Proposition.** *The functor  $\mathcal{R}$  is left adjoint to  $\mathcal{S}$ , so that a map  $f: X \rightarrow \mathcal{S}(C)$  corresponds to a map  $g: \mathcal{R}(X) \rightarrow C$  which sends  $[x, \gamma] \in \mathcal{R}(X)_n$  to  $f(x)\tilde{\gamma} \in C_n$ .*

*Proof.* We just need to verify that the correspondence is well-defined, a bijection, and natural in  $X$  and  $C$ . That it is well-defined is clear, since if  $[x\delta, \gamma] = [x, \delta\gamma]$  for some simplicial operator  $\gamma$ , we have  $f(x\delta)\tilde{\gamma} = f(x)\delta\tilde{\gamma} = f(x)(\delta\tilde{\gamma})$ . To see that it is a bijection, note that an inverse is given by sending  $g: \mathcal{R}(X) \rightarrow C$  to the map  $f: X \rightarrow \mathcal{S}(C)$  sending  $x \in X_n$  to  $g([x, \iota_n])$ . Naturality is a straightforward verification.  $\square$

**54.5. Example.** Suppose  $X = \Delta^n$ . Then

$$\text{Hom}(\mathcal{R}(\Delta^n), C) \approx \text{Hom}(\Delta^n, \mathcal{S}(C)) \approx \text{Hom}(\text{Iso}^{[n]}, C)$$

so  $\mathcal{R}(\Delta^n) \approx \text{Iso}^{[n]}$  by Yoneda, and in fact  $\mathcal{R}: s\text{Set} \rightarrow s\text{Set}$  extends  $\text{Iso}: \Delta \rightarrow s\text{Set}$ .

We can also see this from the explicit description of  $\mathcal{R}$ . Each equivalence class  $[x, \gamma] \in \mathcal{R}(\Delta^n)_k$  contains a unique element of the form  $([k] \xrightarrow{\iota_k} \Delta^k \rightarrow \Delta^n)$ , so  $\mathcal{R}(\Delta^n)_k$  is in natural bijective correspondence with the set of all functions  $\gamma: [k] \rightarrow [n]$  (not necessarily order preserving), i.e., with the set of  $k$ -cells of  $\text{Iso}^{[n]}$ .

In the following, it will be convenient to represent a  $k$ -cell in  $\mathcal{R}(\Delta^n) \approx \text{Iso}^{[n]}$  by a sequence  $(a_0 \dots a_k)$  of elements  $a_i \in [k]$ . Note that such a  $k$ -cell is non-degenerate if and only if the sequence has no consecutive repetition, i.e.,  $a_{i-1} \neq a_i$  for all  $i = 1, \dots, k$ .

**54.6. Lemma.** *Let  $K \subseteq \Delta^n$  be a subcomplex. Then the induced map  $\mathcal{R}(K) \rightarrow \mathcal{R}(\Delta^n)$  is injective, whose image consists exactly of the cells represented by sequences  $(a_0 \dots a_n)$  such that  $\Delta^{\{a_0, \dots, a_n\}} \subseteq K$ .*

*Proof.* If  $K = \Delta^S$  for some  $S \subseteq [n]$  this is immediate from the explicit description of cells of  $\mathcal{R}(\Delta^n)$ , in which case it is convenient to identify  $\mathcal{R}(\Delta^S)$  with the corresponding subcomplex of  $\mathcal{R}(\Delta^n)$ . It is then clear that for  $S, T \subseteq [n]$  we have  $\mathcal{R}(\Delta^S) \cap \mathcal{R}(\Delta^T) = \mathcal{R}(\Delta^{S \cap T})$ .

In general we can write  $K \approx \text{colim}_{\Delta^S \subseteq K} \Delta^S$ , a colimit over a poset of some subsets  $S \subseteq [n]$  (6.15). Since  $\mathcal{R}$  is a left adjoint it preserves colimits, so  $\mathcal{R}(K) \approx \text{colim}_{\Delta^S \subseteq K} \mathcal{R}(\Delta^S)$ . Then the claim follows from (6.9).  $\square$

**54.7. The proof.** The natural map  $\epsilon: \mathcal{S}(C) \rightarrow C$  is adjoint to a natural map  $\eta: X \rightarrow \mathcal{R}(X)$ , which sends  $x \in X_n$  to the element  $[x, \iota_n] \in \mathcal{R}(X)_n$ . When  $X = \Delta^n$  this is just the tautological map  $\eta_n: \Delta^n \rightarrow \mathcal{R}(\Delta^n) = \text{Iso}^{[n]}$  described earlier.

For an arbitrary map  $f: X \rightarrow Y$  of simplicial sets, we define maps

$$f_{\mathcal{R}} := (\mathcal{R}(f), \eta_Y): \mathcal{R}(X) \cup_X Y \rightarrow \mathcal{R}(Y), \quad g_{\mathcal{S}} := (\mathcal{S}(f), \epsilon_X): \mathcal{S}(X) \rightarrow \mathcal{S}(Y) \times_Y X.$$

**54.8. Lemma.** *For any maps  $i: K \rightarrow L$  and  $p: C \rightarrow D$  of simplicial sets, we have that  $i_{\mathcal{R}} \boxtimes p$  if and only if  $i \boxtimes p_{\mathcal{S}}$ .*

*Proof.* This is a straightforward verification of the equivalence of lifting problems.

$$\begin{array}{ccc} \mathcal{R}(X) \cup_X Y & \xrightarrow{(u,v)} & C \\ i_{\mathcal{R}} \downarrow & \nearrow s & \downarrow p \\ \mathcal{R}(Y) & \xrightarrow{w} & D \end{array} \iff \begin{array}{ccc} X & \xrightarrow{\tilde{u}} & \mathcal{S}(C) \\ i \downarrow & \nearrow \tilde{s} & \downarrow p_{\mathcal{S}} \\ Y & \xrightarrow{(\tilde{w}, v)} & \mathcal{S}(D) \times_D C \end{array}$$

□

**54.9. Proposition.** *If  $f: K \rightarrow L$  is a monomorphism, so is  $f_{\mathcal{R}}$ .*

*Proof.* Let  $i_n: \partial\Delta^n \rightarrow \Delta^n$  be the cell inclusion. We already know that  $\mathcal{R}(i_n)$  is a monomorphism by (54.6), which also explicitly describes the image of this map. Using this it is straightforward to show that  $\eta_n: \Delta^n \rightarrow \mathcal{R}(\Delta^n)$  is also injective, and that the pullback of  $\mathcal{R}(i_n)$  along  $\eta_n$  is precisely  $\partial\Delta^n \subset \Delta^n$ , from which it follows that  $(i_n)_{\mathcal{R}}$  is a monomorphism.

Let  $\mathcal{C}$  be the class of maps  $f: K \rightarrow L$  such that  $f_{\mathcal{R}} \in \overline{\text{Cell}}$ . By (54.8) and  $\overline{\text{Cell}} = \boxtimes \text{TrivFib}$  it follows that

$$\mathcal{C} = \boxtimes (\text{TrivFib}_{\mathcal{S}}) = \{ f \mid f \boxtimes p_{\mathcal{S}} \text{ for all } p \in \text{TrivFib} \},$$

so  $\mathcal{C}$  is weakly saturated. Since all  $i_n \in \mathcal{C}$ , we have that  $\overline{\text{Cell}} \subseteq \mathcal{C}$  and the claim follows. □

**54.10. Lemma.** *For each horn inclusion  $j_{n,k}: \Lambda_k^n \subset \Delta^n$ ,  $0 \leq k \leq n$ , the induced map  $(j_{n,k})_{\mathcal{R}}: \mathcal{R}(\Lambda_k^n) \cup_{\Lambda_k^n} \Delta^n \rightarrow \mathcal{R}(\Delta^n)$  is anodyne.*

*Proof.* This is an explicit calculation, which generalizes (36.6) which is the case of  $(n, k) = (1, 0)$ .

Let  $T_m := \mathcal{R}(\Delta^n)_m^{\text{nd}} \setminus \mathcal{R} := \mathcal{R}(\Lambda_k^n)_m^{\text{nd}}$  be the set of nondegenerate  $m$ -cells of  $\mathcal{R}(\Delta^n)$  not contained in the subcomplex. In terms of representing sequences,  $(a_0 \dots a_m) \in T_m$  if and only if it has no consecutive repetitions, and if  $[n] \setminus \{k\} \subseteq \{a_0, \dots, a_m\} \subseteq [n]$ . Note in particular that  $T_m = \emptyset$  if  $m < n - 1$ .

Partition this set as  $T_m = T_m^1 \amalg T_m^2$ , where  $(a_0 \dots a_m) \in T_m^1$  iff  $a_0 = k$ , and  $(a_0 \dots a_m) \in T_m^2$  iff  $a_0 \neq k$ . Recall the notation  $d^i := \langle 0 \dots \hat{i} \dots m \rangle$  for the simplicial face operator  $[m-1] \rightarrow [m]$  (2.2). The verification of the following two statements is immediate.

- (1)  $d^0$  restricts to a bijection  $T_m^1 \rightarrow T_{m-1}^2$ .
- (2) For each  $0 < i \leq m$ ,  $d^i(T_m^1) \cap T_{m-1}^2 = \emptyset$ .

Given this, define  $F_m \subseteq \mathcal{R}(\Delta^n)$  to be the smallest subcomplex containing  $\mathcal{R}(\Lambda_k^n)$  and the sets  $T_i^1$  for  $0 \leq i \leq m$  (and hence the sets  $T_i^2$  for  $0 \leq i \leq m-1$ ). Then it is apparent that each inclusion  $F_{m-1} \subseteq F_m$  is obtained by cobase change along a coproduct of horn inclusions  $\Lambda_0^m \subset \Delta^m$ , with one copy for each element of  $T_m^1$ , and since  $F_{n-1} = \mathcal{R}(\Lambda_k^n)$  and  $\bigcup_m F_m = \mathcal{R}(\Delta^n)$  it follows that  $\mathcal{R}(\Lambda_k^n) \rightarrow \mathcal{R}(\Delta^n)$  is anodyne.

To show that  $\mathcal{R}(\Lambda_k^n) \cup_{\Lambda_k^n} \Delta^n \rightarrow \mathcal{R}(\Delta^n)$  is anodyne just note that the domain is contained in  $F_n$ , and is the smallest subcomplex containing  $F_{n-1}$  and the  $n$ -cell  $(01 \dots m)$ , which is an element of  $T_n^1$ . □

Let  $\mathcal{C}$  denote the class of monomorphisms  $i: K \rightarrow L$  of simplicial sets such that  $i_{\mathcal{R}}$  is a weak homotopy equivalence.

**54.11. Proposition.** *The class  $\mathcal{C}$  contains all monomorphisms.*

*Proof.* We are going to apply (46.12), so we must show that  $\mathcal{C}$  contains inner horn inclusions, is weakly saturated, has precancellation with respect to monomorphisms, and contains  $(\partial\Delta^0 \subset \Delta^0)$ .

First note that since anodyne maps are weak homotopy equivalences,  $\mathcal{C}$  contains all horn inclusions by (54.10).

Next we show that  $\mathcal{C}$  is weakly saturated. Let  $i$  be a monomorphism of simplicial sets, and recall that so is  $i_{\mathcal{R}}$  (54.9). Then  $i_{\mathcal{R}}$  is a weak homotopy equivalence if and only if  $i_{\mathcal{R}} \boxdot p$  for all Kan fibrations  $p: C \rightarrow D$  between Kan complexes (49.4). Thus  $i \in \mathcal{C}$  if and only if  $i \boxdot p_S$ , so  $\mathcal{C}$  is weakly saturated.

Next we show that  $\mathcal{C}$  has precancellation, i.e., that  $i, ji \in \mathcal{C}$  imply  $j \in \mathcal{C}$ . For monomorphisms  $A \xrightarrow{i} B \xrightarrow{j} C$  we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{R}(A) \cup_A B & \xrightarrow{i_{\mathcal{R}}} & \mathcal{R}(B) \\ \downarrow & & \downarrow \\ \mathcal{R}(A) \cup_A C & \xrightarrow{k} \mathcal{R}(B) \cup_B C & \xrightarrow{j_{\mathcal{R}}} \mathcal{R}(C) \\ & \searrow (ji)_{\mathcal{R}} & \nearrow \end{array}$$

in which the square is a pushout. If  $i \in \mathcal{C}$ , then  $i_{\mathcal{R}} \in \overline{\text{Cell}} \cap \text{GpdEq}$ , whence  $k \in \overline{\text{Cell}} \cap \text{GpdEq}$  since this class is saturated. Since also  $(ji)_{\mathcal{R}} \in \text{GpdEq}$ , we have  $j_{\mathcal{R}}$  in  $\text{GpdEq}$  by 2-out-of-3.

Finally note that  $j := (\partial\Delta^0 \subset \Delta^0) \in \mathcal{C}$ , since in fact  $j_{\mathcal{R}}$  is an isomorphism. □

**54.12. Proposition.** *For every Kan fibration  $p: C \rightarrow D$  between Kan complexes, we have that  $p_S: \mathcal{S}(C) \rightarrow \mathcal{S}(D) \times_D C$  is a trivial fibration. In particular,  $\mathcal{S}(C) \rightarrow C$  is a trivial fibration for every quasigroupoid  $C$ .*

*Proof.* It suffices to show that  $i \boxdot p_S$  for every monomorphism  $i$ , or equivalently that  $i_{\mathcal{R}} \boxdot p$ . That this is so is because  $i_{\mathcal{R}}$  is a weak homotopy equivalence by (54.11), so  $i_{\mathcal{R}} \boxdot p$  by (49.4). □

Note that if  $i: \emptyset \rightarrow X$ , then  $i_{\mathcal{R}} = \eta_X: X \rightarrow \mathcal{R}(X)$ , and this map is a weak homotopy equivalence by (54.11). Thus for any simplicial set  $X$ , we get a diagram

$$X \xleftarrow{\eta_X} \mathcal{R}(X) \approx \mathcal{R}(X^{\text{op}}) \xrightarrow{\eta_{X^{\text{op}}}} X^{\text{op}}$$

in which the maps are weak homotopy equivalences. Thus we learn that every simplicial set is weakly homotopy equivalent to its opposite.

## Part 10. Understanding join and slice

### 55. THE ALTERNATE SLICE

Given a quasicategory  $C$  and an object  $x \in C_0$ , we have constructed the slice quasicategories  $C_{x/}$  and  $C_{/x}$ , which come equipped with forgetful functors  $C_{x/} \rightarrow C$  and  $C_{/x} \rightarrow C$ . When  $C$  is an ordinary category, the slices can also be described as pullbacks. For instance, for an ordinary category  $C$ , the slice  $C_{/x}$  is isomorphic to the fiber of the restriction functor  $\text{Fun}(\Delta^1, C) \rightarrow \text{Fun}(\{1\}, C)$  over the vertex representing  $x \in C_0$ .

**55.1. Exercise.** Prove that if  $C$  is an ordinary category, this fiber is isomorphic to  $C_{/x}$ .

**55.2. Alternate slices over and under an object.** For a general simplicial set, we take this as the definition of the **alternate slice**. Thus, given a simplicial set  $C$  and  $x \in C_0$ , we define simplicial sets  $C^{x/}$  and  $C^{/x}$  together with maps  $p: C^{x/} \rightarrow C$  and  $q: C^{/x} \rightarrow C$  via the pullback squares

alternate slice

$$\begin{array}{ccccc} C^{x/} & \longrightarrow & \text{Fun}(\Delta^1, C) & \longleftarrow & C^{/x} \\ p \downarrow & & \downarrow & & \downarrow q \\ \{x\} \times C & \xrightarrow{j_0} & \text{Fun}(\partial\Delta^1, C) & \xleftarrow{j_1} & \{x\} \times C \end{array}$$

where the maps  $j_k$ ,  $k = 0, 1$ , are induced by the inclusions  $\{x\} \rightarrow \text{Fun}(\{k\}, C) = C$ .

These alternate slices are not generally isomorphic to the slice we have already defined, but note that there are evident bijections  $(C_{x/})_0 \approx (C^{x/})_0$  and  $(C_{/x})_0 \approx (C^{/x})_0$  on sets of vertices. We will eventually show that if  $C$  is a quasicategory, then there are categorical equivalences  $C_{x/} \rightarrow C^{x/}$  and  $C_{/x} \rightarrow C^{/x}$ .

A key feature of alternate slices is that, unlike ordinary slices, it is straightforward to relate them to the mapping spaces of a quasicategory.

**55.3. Proposition.** *Given a quasicategory  $C$  and objects  $x, y \in C_0$ , there are pullback squares*

$$\begin{array}{ccc} \text{map}_C(x, y) & \longrightarrow & C^{x/} \\ \downarrow & & \downarrow p \\ \{y\} & \longrightarrow & C \end{array} \quad \begin{array}{ccc} \text{map}_C(x, y) & \longrightarrow & C^{/y} \\ \downarrow & & \downarrow q \\ \{x\} & \longrightarrow & C \end{array}$$

That is, the fibers of the forgetful functors of alternate slices of  $C$  are mapping spaces of  $C$ .

*Proof.* Immediate from the definitions of mapping spaces and alternate slices.  $\square$

**55.4. Remark.** The fibers of the forgetful functors  $C_{x/} \rightarrow C$  over  $y$  and  $C_{/y} \rightarrow C$  over  $x$  for the usual slices are called the **right and left mapping spaces** respectively, and are denoted  $\text{map}_C^R(x, y)$  and  $\text{map}_C^L(x, y)$ . For a quasicategory  $C$  they are categorically equivalent to the usual mapping space  $\text{map}_C(x, y)$ , as a consequence of the equivalence slices and alternate slices which we will prove.

right and left mapping spaces

Given a map  $p: C \rightarrow D$  and  $x \in C_0$  we have evident restriction maps  $p': C^{x/} \rightarrow C \times_D D^{px/}$  and  $p'': C^{/x} \rightarrow C \times_D D^{/px}$ . This “pullback-alternate-slice” map is closely related to the pullback-hom map.

**55.5. Lemma.** *For any map  $p: C \rightarrow D$  and  $x \in C_0$ , there are pullback squares of the form*

$$\begin{array}{ccccc} C^{x/} & \longrightarrow & \text{Fun}(\Delta^1, C) & \longleftarrow & C^{/x} \\ p' \downarrow & & \downarrow p^{\square(\partial\Delta^1 \subset \Delta^1)} & & \downarrow p'' \\ C \times_D D^{px/} & \longrightarrow & \text{Fun}(\partial\Delta^1, C) \times_{\text{Fun}(\partial\Delta^1, D)} \text{Fun}(\Delta^1, D) & \longleftarrow & C \times_D D^{/px} \end{array}$$

*Proof.* This is a straightforward exercise. For instance,  $C \times_D D^{px/}$  is seen to be the pullback of  $\text{Fun}(\partial\Delta^1, C) \times_{\text{Fun}(\partial\Delta^1, D)} \text{Fun}(\Delta^1, D) \rightarrow (C \times C) \times_{D \times D} (D \times D) \leftarrow (\{x\} \times C) \times_{\{px\} \times D} (\{px\} \times D)$ .

Precomposing the left arrow with  $p^{\square(\partial\Delta^1 \subset \Delta^1)}$  and using the obvious isomorphisms gives the pullback square defining  $C^{x/}$ .  $\square$

**55.6. Alternate slice for arbitrary maps.** We can generalize the alternate slice construction to arbitrary maps of simplicial sets. Thus, suppose given a map  $f: K \rightarrow C$ , which corresponds to a vertex  $\tilde{f} \in \text{Fun}(K, C)_0$ . We define the general **alternate slices**  $C^{f/}$  and  $C/^f$  via the pullback squares

$$\begin{array}{ccccccc} C^{f/} & \longrightarrow & \text{Fun}(K, C)^{\tilde{f}/} & \longrightarrow & \text{Fun}(K \times \Delta^1, C) & \longleftarrow & \text{Fun}(K, C)^{/ \tilde{f}} \longleftarrow C/^f \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{\tilde{\pi}_K} & \{\tilde{f}\} \times \text{Fun}(K, C) & \xrightarrow{j_0} & \text{Fun}(K \times \partial\Delta^1, C) & \xleftarrow{j_1} & \text{Fun}(K, C) \times \{\tilde{f}\} \xleftarrow{\tilde{\pi}_K} C \end{array}$$

where the maps  $j_k$ ,  $k = 0, 1$  are induced by the inclusions  $\{\tilde{f}\} \rightarrow \text{Fun}(K \times \{k\}, C)$ , and  $\tilde{\pi}_K$  is adjoint to the projection  $\pi: C \times K \rightarrow C$ . Thus, the general alternate slices are obtained by base-change from the special case for functors from  $\Delta^0$ .

The fibers of general alternate slices can also be described as mapping spaces, namely as spaces of natural transformations to or from a constant functor.

**55.7. Proposition.** *Given a quasicategory  $C$ , a map  $f: K \rightarrow C$  of simplicial sets, and  $x, y \in C_0$ , there are pullback squares*

$$\begin{array}{ccc} \text{map}_{\text{Fun}(K, C)}(f, \tilde{\pi}_K y) & \longrightarrow & C^{f/} \\ \downarrow & & \downarrow \\ \{y\} & \longrightarrow & C \end{array} \quad \begin{array}{ccc} \text{map}_{\text{Fun}(K, C)}(\tilde{\pi}_K x, f) & \longrightarrow & C/^f \\ \downarrow & & \downarrow \\ \{x\} & \longrightarrow & C \end{array}$$

where  $\tilde{\pi}_K x: K \rightarrow C$  represents the constant map with value  $x$ , i.e., the composite  $K \rightarrow \{x\} \rightarrow C$ .

Given a sequence of maps  $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$  of simplicial sets, we have evident restriction maps  $C^{f/} \rightarrow C^{fi/} \times_{D^{pfi/}} D^{pf/}$  and  $C/^f \rightarrow C/^fi \times_{D/^pfi} D/^pf$ . These are alternate analogs of the pullback-slice maps of (30.14). Applied to a sequence of the form  $\emptyset \rightarrow \{x\} \rightarrow C \xrightarrow{p} D$  this gives the special case already discussed. It turns out that the general case can be obtained via base-change from the special case.

**55.8. Proposition.** *There are pullback squares of the form*

$$\begin{array}{ccc} C^{f/} & \longrightarrow & U^{\tilde{f}/} \\ \downarrow & & \downarrow \\ C^{fi/} \times_{D^{pfi/}} D^{pf/} & \longrightarrow & U \times_V U^{(\tilde{f}i, \tilde{p}f)/} \end{array} \quad \begin{array}{ccc} C/^f & \longrightarrow & U/^{\tilde{f}} \\ \downarrow & & \downarrow \\ C/^fi \times_{D/^pfi} D/^pf & \longrightarrow & U \times_V V^{(\tilde{f}i, \tilde{p}f)} \end{array}$$

where  $U \rightarrow V$  is the pullback-hom map  $q = p^{\square i}: \text{Fun}(L, C) \rightarrow \text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D)$ , and the vertical maps in each square are alternate pullback-slice maps associated to the sequences  $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$  and  $\emptyset \rightarrow \{\tilde{f}\} \rightarrow U \xrightarrow{q} V$  respectively.

*Proof.* This is a difficult to visualize but ultimately straightforward argument. A key observation is that the four alternate-slice objects associated to  $f$ ,  $pf$ ,  $fi$ , and  $pfi$  are each described by a pullback square  $[1]^{\times 2} \rightarrow \text{sSet}$ , which therefore fit together to give a functor  $[1]^{\times 4} \rightarrow \text{sSet}$ , which is a limit cone. Decomposing this 4-dimensional cartesian cube in a different way gives the desired pullback description. It may be helpful to note that the lower horizontal map in the left square is really a

map of the form

$$C \times_{(C \times_D D)} (C^{fi/} \times_{D^{pf i/}} D^{pf /}) \rightarrow \text{Fun}(L, C) \times_{(\text{Fun}(K, C) \times_{\text{Fun}(K, D)} \text{Fun}(L, D))} (\text{Fun}(K, C)^{\tilde{f}i/} \times_{\text{Fun}(K, D)^{\tilde{p}f i/}} \text{Fun}(L, D)^{\tilde{p}f /}),$$

involving 14 of the 16 vertices of the 4-dimensional cube.  $\square$

## 56. THE ALTERNATE JOIN

Just as the usual slices are adjoint to a join construction, the alternate slices are adjoint to an alternate join construction.

**56.1. Definition of alternate join.** Given simplicial sets  $X$  and  $Y$ , we define the **alternate join** alternate join by the pushout diagram

$$\begin{array}{ccc} (X \times \{0\} \times Y) \amalg (X \times \{1\} \times Y) & \longrightarrow & (X \times \{0\} \times \Delta^0) \amalg (\Delta^0 \times \{1\} \times Y) \\ \downarrow & & \downarrow \\ X \times \Delta^1 \times Y & \xrightarrow{\quad} & X \diamond Y \end{array}$$

where the map on the left is induced by inclusion  $\partial\Delta^1 \subset \Delta^1$ , and the map on the top by the projections  $X \rightarrow \Delta^0$  and  $Y \rightarrow \Delta^0$ .

**56.2. Example.** We have isomorphisms  $X \diamond \emptyset \approx \emptyset \approx \emptyset \diamond X$ .

**56.3. Example.** We have isomorphisms

$$X \diamond \Delta^0 \approx (X \times \Delta^1)/(X \times \{1\}), \quad \Delta^0 \diamond Y \approx (\Delta^1 \times Y)/(\{0\} \times Y).$$

**56.4. Remark.** Unlike the join, the alternate join is not monoidal:  $(X \diamond Y) \diamond Z \not\approx X \diamond (Y \diamond Z)$  in general. Also, the alternate join of two quasicategories is not usually a quasicategory.

Viewed as a functor of either variable, the alternate join gives functors

$$S \diamond -: s\text{Set} \rightarrow s\text{Set}_{S/}, \quad - \diamond T: s\text{Set} \rightarrow s\text{Set}_{T/}$$

from simplicial sets to the evident slice categories.

**56.5. Proposition.** *The alternate join is left adjoint to the alternate slices, in the sense that  $S \diamond -: s\text{Set} \rightarrow s\text{Set}_{S/}$  and  $- \diamond T: s\text{Set} \rightarrow s\text{Set}_{T/}$  are left adjoint to*

$$(f: S \rightarrow C) \mapsto C^{f/}: s\text{Set}_{S/} \quad \text{and} \quad (g: T \rightarrow C) \mapsto C^{/g}: s\text{Set}_{T/} \rightarrow s\text{Set}$$

*respectively. Thus, we have natural bijections*

$$\text{Hom}(K, C^{f/}) \approx \text{Hom}_{S/}(S \diamond K, C), \quad \text{Hom}(K, C^{/g}) \approx \text{Hom}_{T/}(K \diamond T, C).$$

*Proof.* Straightforward.  $\square$

**56.6. Enriched adjunction for alternate join/slice.** In fact, we can do a little better: the alternate join and slices participate in **enriched adjunctions** enriched adjunctions involving the relative function complex (23.18), which for objects  $p: S \rightarrow K$  and  $f: S \rightarrow C$  in  $s\text{Set}_{S/}$  is a simplicial set

$$\text{Fun}_{S/}(K, C) = \text{Fun}(K, C) \times_{\text{Fun}(S, C)} \{f\}.$$

First, note that for any simplicial sets  $S$ ,  $K$ , and  $X$ , we have pushout squares of the form

$$\begin{array}{ccc} X \times S & \longrightarrow & S \\ \downarrow & & \downarrow \\ X \times (S \diamond K) & \xrightarrow{\ell} & S \diamond (X \times K) \end{array} \quad \begin{array}{ccc} X \times S & \longrightarrow & S \\ \downarrow & & \downarrow \\ X \times (K \diamond S) & \xrightarrow{r} & (X \times K) \diamond S \end{array}$$



where the top horizontal maps are projections, the vertical maps are the evident inclusions, and the bottom horizontal maps are the evident ones induced by the fact that the objects are quotients of the same product  $X \times S \times K \times \Delta^1$  (up to permuting the factors). Using this, for any  $f: K \rightarrow C$  we obtain isomorphisms

$$\mathrm{Fun}(K, C^{f/}) \xrightarrow{\sim} \mathrm{Fun}_{S/}(S \diamond K, C), \quad \mathrm{Fun}(K, C^{f/}) \xrightarrow{\sim} \mathrm{Fun}_{S/}(K \diamond S, C).$$

In the left-hand case, we define the map by describing a natural transformation

$$\left\{ X \xrightarrow{\quad} \mathrm{Fun}(K, C^{f/}) \right\} \Rightarrow \left\{ X \xrightarrow{\quad} \mathrm{Fun}_{S/}(S \diamond K, C) \right\}$$

sending

$$\left\{ \begin{array}{ccc} S & & \\ \downarrow & \searrow f & \\ S \diamond (X \times K) & \xrightarrow{\quad} & C \end{array} \right\} \Rightarrow \left\{ \begin{array}{ccc} X \times S & \xrightarrow{\mathrm{proj}} & S \\ \downarrow & & \downarrow f \\ X \times (S \diamond K) & \xrightarrow{\quad} & C \end{array} \right\}$$

by precomposing with  $\ell$  defined above. The transformation is an isomorphism exactly because the square defining  $\ell$  is a pushout square.

We note that this provides us with another description of the alternate slice.

**56.7. Proposition.** *For any map  $f: S \rightarrow C$  of simplicial sets, we have isomorphisms*

$$C^{f/} \approx \mathrm{Fun}_{S/}(S \diamond \Delta^0, C), \quad C^{f/} \approx \mathrm{Fun}_{S/}(\Delta^0 \diamond S, C).$$

*Proof.* These are the enriched adjunction maps for alternate join/slice for  $K = \Delta^0$ .  $\square$

**56.8. Enriched adjunction for join/slice.** In an analogous way we can define enriched adjunction maps for the usual join and slices, using commutative squares

$$\begin{array}{ccc} X \times S & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \\ X \times (S \star K) & \xrightarrow{\ell'} & S \star (X \times K) \end{array} \quad \begin{array}{ccc} X \times S & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \\ X \times (K \star S) & \xrightarrow{r'} & (X \times K) \star S \end{array}$$

which however are not pushouts in general (though they are trivially pushouts when  $X = \Delta^0$ ). For instance the map  $\ell'$  is the one corresponding by (26.14) to the triple of maps

$$\pi: X \times (S \star K) \rightarrow X \times \Delta^1 \rightarrow \Delta^1, \quad \pi^{-1}(\{0\}) = X \times S \xrightarrow{\mathrm{proj}} S, \quad \pi^{-1}(\{1\}) = X \times K \xrightarrow{\mathrm{id}} X \times K.$$

Following the same recipe as for alternate join and slices, given  $f: K \rightarrow C$  we have natural maps

$$\mathrm{Fun}(K, C_{f/}) \rightarrow \mathrm{Fun}_{S/}(S \star K, C), \quad \mathrm{Fun}(K, C_{f/}) \rightarrow \mathrm{Fun}_{S/}(K \star S, C),$$

which we call **enriched adjunction maps** for join and slices. These are not in general isomorphisms, but they are bijections on vertices. Although they are not isomorphisms, we will later show that these enriched adjunction maps are categorical equivalences when  $C$  is a quasicategory. For now, we show that when  $C$  is a quasicategory they induce bijections on isomorphism classes of objects.

**56.9. Proposition.** *For  $K$  a simplicial set and  $p: S \rightarrow C$  a map to a quasicategory  $C$ , the enriched adjunction map for join/slice induces bijections*

$$\pi_0(\mathrm{Fun}(K, C_{p/})^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}_{S/}(S \star K, C)^{\mathrm{core}}), \quad \pi_0(\mathrm{Fun}(X, C_{p/})^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}_{S/}(K \star S, C)^{\mathrm{core}}),$$

*Proof.* We give the proof in the slice-under case. Since the enriched adjunction map gives a bijection on objects, it suffices to prove injectivity on sets of isomorphism classes. Thus, we suppose given  $f_0, f_1: X \rightarrow C_{p/}$  representing objects of  $\mathrm{Fun}(X, C_{p/})$  such that the corresponding objects  $\tilde{f}_0, \tilde{f}_1: S \star X \rightarrow C$  represent isomorphic objects of  $\mathrm{Fun}_{S/}(S \star X, C)$ , and show that there is a natural isomorphism  $f_0 \rightarrow f_1$ .

enriched  
maps      adjunction

Let  $C \xrightarrow{i} \widehat{C} \xrightarrow{r=(r_0, r_1)} C \times C$  be the standard path category for  $C$  (34.3), with  $\widehat{C} = \text{Fun}^{\text{iso}}(\Delta^1, C)$ . An isomorphism  $\tilde{f}_0 \rightarrow \tilde{f}_1$  in  $\text{Fun}_S(S \star K, C)$  amounts to a choice of lift  $\tilde{f}$  in

$$\begin{array}{ccccc} S & \xrightarrow{p} & C & \xrightarrow{i} & \widehat{C} \\ \downarrow & & & \nearrow \tilde{f} & \downarrow r \\ S \star X & \xrightarrow{(\tilde{f}_0, \tilde{f}_1)} & C \times C & & \end{array}$$

Applying the join/slice adjunction to the lower triangle gives a diagram

$$\begin{array}{ccc} & \widehat{C}_{pi/} & \\ f \nearrow & \downarrow r'=(r'_0, r'_1) & \\ X & \xrightarrow{(\tilde{f}_0, \tilde{f}_1)} & C_{p/} \times C_{/p} \end{array}$$

Since  $r_0, r_1: \widehat{C} \rightarrow C$  are trivial fibrations (40.3), so are the induced maps  $r'_0, r'_1: \widehat{C}_{pi/} \rightarrow C_{/p}$  on slices (30.16). Thus every functor in

$$\begin{array}{ccccc} & & \text{id} & \xrightarrow{\quad} & \text{Fun}(X, C_{p/}) \\ & \nearrow & & \nearrow & \\ \text{Fun}(X, C_{p/}) & \xrightarrow{i'_*} & \text{Fun}(X, \widehat{C}_{pi/}) & \xrightarrow{(r'_0)_*} & \text{Fun}(X, C_{p/}) \\ & \searrow & & \searrow & \\ & & \text{id} & \xrightarrow{\quad} & \text{Fun}(X, C_{p/}) \end{array}$$

is a categorical equivalence, whence  $r'_0$  and  $r'_1$  induces the same bijection on isomorphism classes of objects. we see that every arrow in this diagram is a categorical equivalence, and therefore both  $\pi_0$  and  $\pi_1$  induce *the same bijection* on isomorphism classes on objects, and thus  $f_0 = (r'_0)_*(F)$  and  $f_1 = (r'_1)_*(F)$  are isomorphic as desired.  $\square$

56.10. *Exercise.* Construct a natural “distributivity” map  $X \times (S \star T) \rightarrow (X \times S) \star (X \times T)$ .

## 57. EQUIVALENCE OF JOIN AND ALTERNATE JOIN

The proof that slice and alternate slice are equivalent will rely on an equivalence between join and alternate join.

There is a canonical comparison map  $X \diamond Y \rightarrow X \star Y$ , natural in both variables, which by (26.14) corresponds to the triple of maps

$$\pi: X \diamond Y \rightarrow \Delta^1, \quad \pi^{-1}(\{0\}) = X \xrightarrow{\text{id}} X, \quad \pi^{-1}(\{1\}) = Y \xrightarrow{\text{id}} Y.$$

**57.1. Proposition.** *The canonical comparison map  $X \diamond Y \rightarrow X \star Y$  is a categorical equivalence for all simplicial sets  $X$  and  $Y$ .*

We will give the proof at the end of this section.

**57.2. Categorical invariance of joins.** First we note that the alternate join is a “categorically invariant” construction.

**57.3. Proposition.** *The alternate join  $\diamond$  preserves categorical equivalences in either variable. That is, if  $Y \rightarrow Y'$  is a categorical equivalence, then so are  $X \diamond Y \rightarrow X \diamond Y'$  and  $Y \diamond Z \rightarrow Y' \diamond Z$ .*

*Proof.* The  $\diamond$  functor is constructed (56.1) using finite products and a “good” pushout, i.e., a pushout along a cofibration (=monomorphism). The result follows because both finite products (25.5) and good pushouts (48.12) preserve categorical equivalences.  $\square$

Once we prove equivalence of join and alternate join, this will imply the categorical invariance of the usual join.

**57.4. Corollary.** *The join  $\star$  preserves categorical equivalences in either variable. That is, if  $Y \rightarrow Y'$  is a categorical equivalence, then so are  $X \star Y \rightarrow X \star Y'$  and  $Y \star Z \rightarrow Y' \star Z$ .*

*Proof.* Immediate using (57.1), the invariance of the alternate join under categorical equivalence (57.3), and the 2-out-of-3 property of categorical equivalences (25.11).  $\square$

**57.5. Skeletal induction.** To prove (57.1) we will use the following strategy.

**57.6. Proposition** (Skeletal induction). *Let  $\mathcal{C}$  be a class of simplicial sets with the following properties.*

- (1) *If  $X \in \mathcal{C}$ , then every object isomorphic to  $X$  is in  $\mathcal{C}$ .*
- (2) *Every  $\Delta^n \in \mathcal{C}$ .*
- (3) *The class  $\mathcal{C}$  is closed under good colimits. That is:*
  - (a) *any coproduct of objects of  $\mathcal{C}$  is in  $\mathcal{C}$ ;*
  - (b) *any pushout of a diagram  $X_0 \leftarrow X_1 \rightarrow X_2$  of objects in  $\mathcal{C}$  along a monomorphism  $X_1 \rightarrow X_2$  is in  $\mathcal{C}$ ;*
  - (c) *any colimit of a countable sequence  $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$  of objects in  $\mathcal{C}$ , such that each  $X_k \rightarrow X_{k+1}$  is a monomorphism, is in  $\mathcal{C}$ .*

*Then  $\mathcal{C}$  is the class of all simplicial sets.*

*Proof.* This is a straightforward consequence of the skeletal filtration (18.3). To show  $X \in \mathcal{C}$ , it suffices to show each  $\text{Sk}_n X \in \mathcal{C}$  by (3c). So we show that all  $n$ -skeleta are in  $\mathcal{C}$  by induction on  $n$ , with base case  $n = -1$  (the empty simplicial set), which is really a special case of (3a). Since  $\text{Sk}_{n-1} X \subseteq \text{Sk}_n X$  is a pushout along a coproduct of maps  $\partial\Delta^n = \text{Sk}_{n-1} \Delta^n \rightarrow \Delta^n$ , this follows using (2), (3a), (3b), and the inductive hypothesis, which tells us that  $\partial\Delta^n \in \mathcal{C}$ .  $\square$

We will use skeletal induction to show that certain natural transformations from simplicial sets to a model category take values in weak equivalences.

**57.7. Proposition.** *Let  $\alpha: F \rightarrow F'$  be a natural transformation between functors  $s\text{Set} \rightarrow \mathcal{M}$ , where  $\mathcal{M}$  is some model category. If*

- (1)  *$F$  and  $F'$  preserve colimits,*
- (2)  *$F$  and  $F'$  take monomorphisms to cofibrations,*
- (3)  *$F$  and  $F'$  take inner anodyne maps to weak equivalences in  $\mathcal{M}$ , and*
- (4)  *$\alpha(\Delta^1): F(\Delta^1) \rightarrow F'(\Delta^1)$  is a weak equivalence in  $\mathcal{M}$ ,*

*then  $\alpha(X): F(X) \rightarrow F'(X)$  is a weak equivalence in  $\mathcal{M}$  for all simplicial sets  $X$ .*

*Proof.* [Lur09, 4.2.1.2] Consider the class of simplicial sets  $\mathcal{C} := \{X \mid \alpha(X) \text{ is a weak equivalence}\}$ . We use skeletal induction (57.6) to show that  $\mathcal{C}$  contains all simplicial sets.

It is clear that  $\mathcal{C}$  is closed under isomorphic objects. Because  $F$  and  $F'$  preserve colimits (1) and cofibrations (2), they take good colimit diagrams in  $s\text{Set}$  to good colimit diagrams in  $\mathcal{M}$ . Since good colimits are weak equivalence invariant (48.8), (48.12), (48.10), we see that  $\mathcal{C}$  is closed under forming good colimits. It remains to show that  $\Delta^n \in \mathcal{C}$  for all  $n$ .

We have  $\Delta^1 \in \mathcal{C}$  by (4). Since  $\Delta^0$  is a retract of  $\Delta^1$ , we get that  $\Delta^0 \in \mathcal{C}$  since weak equivalences in  $\mathcal{M}$  are closed under retracts (47.4).

The spines  $I^n$  can be built from  $\Delta^0$  and  $\Delta^1$  by a sequence of good pushouts (glue on one 1-simplex at a time), so the  $I^n \in \mathcal{C}$ . The inclusions  $I^n \subset \Delta^n$  are inner anodyne (14.12), so by (3) and the 2-out-of-3 property of weak equivalences in  $\mathcal{M}$  it follows that  $\Delta^n \in \mathcal{C}$ .  $\square$

**57.8. Proof of the equivalence.** We will apply this idea to functors  $s\text{Set} \rightarrow s\text{Set}_{X/}$ , where the slice category  $s\text{Set}_{X/}$  inherits its model structure from the Joyal model structure on  $s\text{Set}$  (47.5).

*Proof of (57.1).* The functors  $X \diamond (-), X \star (-), (-) \diamond X, (-) \star X: s\text{Set} \rightarrow s\text{Set}_{X/}$  satisfy the first three properties required of the functors in the previous proposition (57.7). That is, they (1) preserve colimits, (2) take monomorphisms to cofibrations, and (3) take inner anodyne maps to categorical equivalences. Condition (3) for  $\diamond$  follows from (57.3), while condition (3) for  $\star$  follows from (30.13) since  $\overline{\text{InnHorn}} \subseteq \overline{\text{LHorn}} \cap \overline{\text{RHorn}}$ .

Thus, to show  $X \diamond Y \rightarrow X \star Y$  is a categorical equivalence for a fixed  $X$  and arbitrary  $Y$ , it suffices by the previous proposition to show that  $X \diamond \Delta^1 \rightarrow X \star \Delta^1$  is a categorical equivalence. The same argument in the other variable lets us reduce to the case when  $X = \Delta^1$ , i.e., to showing that a single map  $\bar{f}: \Delta^1 \diamond \Delta^1 \rightarrow \Delta^1 \star \Delta^1$  is a categorical equivalence, which is the following lemma (57.9).  $\square$

**57.9. Lemma.** *The canonical comparison map  $\bar{f}: \Delta^1 \diamond \Delta^1 \rightarrow \Delta^1 \star \Delta^1$  is a categorical equivalence.*

*Proof.* We will show  $\bar{f}$  is a categorical equivalence by producing a map  $\bar{g}: \Delta^1 \star \Delta^1 \rightarrow \Delta^1 \diamond \Delta^1$  such that  $\bar{f}\bar{g} = \text{id}_{\Delta^1 \star \Delta^1}$  and  $\bar{g}\bar{f}$  is preisomorphic to the identity map of  $\Delta^1 \diamond \Delta^1$ , via (23.9).

Since  $\Delta^1 \diamond \Delta^1$  is a quotient of a cube, we start with maps involving the cube. Write vertices in  $(\Delta^1)^{\times 3}$  as sequences  $(x, t, y)$  where  $x, t, y \in \{0, 1\}$ . Let

$$f: (\Delta^1)^{\times 3} \rightarrow \Delta^1 \star \Delta^1 = \Delta^3$$

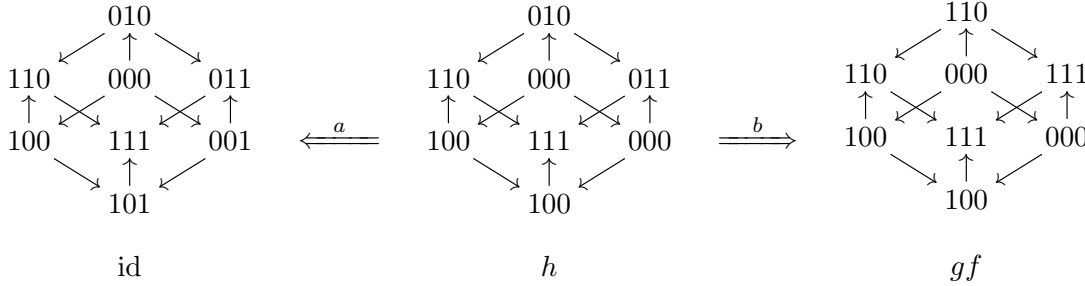
be the map which on vertices sends

$$(x, t, y) \mapsto (1-t)x + t(2+y) = \begin{cases} x & \text{if } t = 0, \\ 2+y & \text{if } t = 1. \end{cases}$$

On passage to quotients this gives the comparison map  $\bar{f}: \Delta^1 \diamond \Delta^1 \rightarrow \Delta^1 \star \Delta^1$  we want.

Let  $g: \Delta^3 \rightarrow (\Delta^1)^{\times 3}$  be the map classifying the cell  $\langle (000), (100), (110), (111) \rangle$ , and let  $\bar{g}: \Delta^3 \rightarrow \Delta^1 \diamond \Delta^1$  be the composite with the quotient map. We have  $f g = \text{id}_{\Delta^3} = \bar{f} \bar{g}$ .

Let  $h \in \text{Fun}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_0$  and  $a, b \in \text{Fun}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_1$  be as indicated in the following picture.



These pass to elements  $\bar{h}, \bar{a}, \bar{b}$  in  $\text{Fun}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$ . The edges  $\bar{a}$  and  $\bar{b}$  are preisomorphisms, as one sees that for each vertex  $v \in (\Delta^1 \diamond \Delta^1)$ , the induced maps  $\Delta^1 \times \{v\} \subset \Delta^1 \times (\Delta^1 \diamond \Delta^1) \xrightarrow{\bar{a} \text{ or } \bar{b}} \Delta^1 \diamond \Delta^1$  represent degenerate edges of  $\Delta^1 \diamond \Delta^1$ . Thus  $\bar{f}\bar{g}$  and  $\bar{g}\bar{f}$  are preisomorphic to identity maps, and hence  $\bar{f}$  is a categorical equivalence as desired.  $\square$

## 58. EQUIVALENCE OF SLICE AND ALTERNATE SLICE

To compare slices with alternate slices, we will use the following commutative diagram of function complexes.

**58.1. Proposition.** *For all simplicial sets  $K$  and all maps  $f: S \rightarrow C$  of simplicial sets, we have commutative squares*

$$\begin{array}{ccc} \mathrm{Fun}(K, C_{f/}) & \longrightarrow & \mathrm{Fun}_{S/}(S \star K, C) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(K, C^{f/}) & \xrightarrow{\approx} & \mathrm{Fun}_{S/}(S \diamond K, C) \end{array} \quad \begin{array}{ccc} \mathrm{Fun}(K, C_{/f}) & \longrightarrow & \mathrm{Fun}_{S/}(K \star S, C) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(K, C^{/f}) & \xrightarrow{\approx} & \mathrm{Fun}_{S/}(K \diamond S, C) \end{array}$$

in which the horizontal maps are the respective enriched adjunction maps, the left vertical map in each square is induced by the slice comparison maps  $C_{f/} \rightarrow C^{f/}$  or  $C_{/f} \rightarrow C^{/f}$ , and the right vertical map in each square is induced by the join comparison maps  $S \diamond K \rightarrow S \star K$  or  $K \diamond S \rightarrow K \star S$ .

*Proof.* This is straightforward from the definitions of the enriched adjunction maps and the comparison maps for joins and slices.  $\square$

## 58.2. Equivalence of slice and alternate slice.

**58.3. Proposition.** *For any quasicategory  $C$  and map  $f: S \rightarrow C$ , the comparison maps  $C_{f/} \rightarrow C^{f/}$  and  $C_{/f} \rightarrow C^{/f}$  are categorical equivalences.*

*Proof.* [Lur09, 4.2.1.5] We do the first case. First recall that if  $f: A \rightarrow B$  is a functor between quasicategories, then  $f$  is a categorical equivalence if and only if the induced maps  $\pi_0(\mathrm{Fun}(X, A)^{\mathrm{core}}) \rightarrow \pi_0(\mathrm{Fun}(X, B)^{\mathrm{core}})$  are bijections for all simplicial sets  $X$  (25.13).

We refer to the left-hand commutative square of (58.1). We know that the bottom horizontal map is an isomorphism (56.6). By (56.9) the top map is a bijection on isomorphism classes of objects. By (57.1)  $S \diamond X \rightarrow S \star X$  is a categorical equivalence, and therefore the right-hand vertical map induces a categorical equivalence using (48.14).

Therefore, both horizontal maps and the right-hand vertical map induce bijections on isomorphism classes of objects, and hence so does the left-hand vertical map as desired. Since this holds for every simplicial set  $K$ , it follows that  $C_{f/} \rightarrow C^{f/}$  is a categorical equivalence by the criterion of (25.14). The proposition is proved.  $\square$

**58.4. Corollary.** *For any quasicategory  $C$  map  $f: S \rightarrow C$ , and simplicial set  $K$ , the enriched adjunction maps  $\mathrm{Fun}(K, C_{f/}) \rightarrow \mathrm{Fun}_{S/}(S \star K, C)$  and  $\mathrm{Fun}(K, C_{/f}) \rightarrow \mathrm{Fun}_{S/}(K \star S, C)$  are categorical equivalences.*

*Proof.* Immediate from the proof of (58.3)  $\square$

As a consequence, we obtain another variant of the slice construction. Let  $C$  be a quasicategory and  $S$  a simplicial set, and consider the forgetful functors

$$p: \mathrm{Fun}(S^{\triangleright}, C) \rightarrow \mathrm{Fun}(S, C), \quad q: \mathrm{Fun}(S^{\triangleleft}, C) \rightarrow \mathrm{Fun}(S, C).$$

For a given map  $f: S \rightarrow C$ , the fibers of  $p$  and  $q$  over the vertex of  $\mathrm{Fun}(S, C)$  corresponding to  $f$  are precisely the relative function complexes  $\mathrm{Fun}_{S/}(S^{\triangleright}, (C, f))$  and  $\mathrm{Fun}_{S/}(S^{\triangleleft}, (C, f))$ . These fibers are in fact equivalent to the evident slice categories.

**58.5. Corollary.** *For any map  $f: S \rightarrow C$  to a quasicategory, we have commutative squares*

$$\begin{array}{ccc} C_{f/} & \longrightarrow & \mathrm{Fun}_{S/}(S^{\triangleright}, (C, f)) \\ \downarrow & & \downarrow \\ C^{f/} & \xrightarrow{\approx} & \mathrm{Fun}_{S/}(S \diamond \Delta^0, (C, f)) \end{array} \quad \begin{array}{ccc} C_{/f} & \longrightarrow & \mathrm{Fun}_{S/}(S^{\triangleleft}, (C, f)) \\ \downarrow & & \downarrow \\ C^{/f} & \xrightarrow{\approx} & \mathrm{Fun}_{S/}(\Delta^0 \diamond S, (C, f)) \end{array}$$

in which the lower horizontal maps are isomorphisms, the top horizontal maps are bijections on sets of objects, and every map is a categorical equivalence.

*Proof.* These are just the commutative squares of (58.1) with  $K = \Delta^0$ , together with (58.3) and (58.4).  $\square$

### 59. PROPERTIES OF THE ALTERNATE SLICE

Recall that given a sequence of maps  $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$  where  $i$  is a monomorphism and  $p$  an inner fibration, the induced pullback-slice maps  $C_{f/} \rightarrow C_{fi/} \times_{D_{pfi/}} D_{pf/}$  and  $C_{/f} \rightarrow C_{/fi} \times_{D_{/pfi}} D_{/pf}$  are left fibration and right fibration respectively (30.14), and therefore in particular are conservative isofibrations (32.5) (32.11). Furthermore, they are trivial fibrations if either  $p$  is a trivial fibration or if  $i$  is right or left anodyne respectively (30.14). We will show that the alternate pullback-slice maps share these properties, at least when  $C$  and  $D$  are quasicategories.

**59.1. Proposition.** *Let  $p: C \rightarrow D$  be an inner fibration between quasicategories. Then for any object  $x \in C_0$ , the evident induced map  $p': C^{x/} \rightarrow C \times_D D^{px/}$  is a left fibration, and  $p'': C^{/x} \rightarrow C \times_D D^{/px}$  is a right fibration.*

*Proof.* We deal with the case of  $p'$ , as the case of  $p''$  is similar. Since  $p'$  is a base-change of  $p^{\square(\partial\Delta^1 \subset \Delta^1)}$ , it is an inner fibration. Thus we need to produce a lift in

$$\begin{array}{ccccc} \Lambda_0^n & \xrightarrow{\quad} & C^{x/} & \xrightarrow{\quad} & \text{Fun}(\Delta^1, C) \\ \downarrow & \nearrow \text{dotted} & \downarrow p' & & \downarrow p^{\square(\partial\Delta^1 \subset \Delta^1)} \\ \Delta^n & \xrightarrow{\quad} & C \times_D D^{px/} & \xrightarrow{\quad} & \text{Fun}(\partial\Delta^1, C) \times_{\text{Fun}(\partial\Delta^1, D)} \text{Fun}(\Delta^1, D) \end{array}$$

for  $n \geq 1$ . This lifting problem is equivalent to one of the form

$$\begin{array}{ccc} (\partial\Delta^1 \times \Delta^n) \times_{\partial\Delta^1 \times \Lambda_0^n} (\Delta^1 \times \Lambda_0^n) & \xrightarrow{t} & C \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta^1 \times \Delta^n & \xrightarrow{\quad} & D \end{array}$$

in which the map  $t$  sends  $\{0\} \times \Delta^n$  to  $\{x\} \subseteq C$ . The claim follows from the following (59.2).  $\square$

**59.2. Lemma** (Another pushout-product version of Joyal lifting). *Let  $p: C \rightarrow D$  be an inner fibration between quasicategories. Then if  $n \geq 0$ , and for either: (i)  $(x, y) = (0, 0)$ ,  $\{a, b\} = \{0, 1\}$ , or (ii)  $(x, y) = (1, n)$ ,  $\{a, b\} = \{n-1, n\}$ , a lift exists in any diagram of the form*

$$\begin{array}{ccc} \{x\} \times \Delta^{\{a, b\}} & \xrightarrow{\quad} & (\partial\Delta^1 \times \Delta^n) \times_{\partial\Delta^1 \times \Lambda_y^n} (\Delta^1 \times \Lambda_y^n) \xrightarrow{\quad} C \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta^1 \times \Delta^n & \xrightarrow{\quad} & D \end{array}$$

$f$

*Proof.* We give a proof in the appendix as another application of Joyal lifting (62.6).  $\square$

As a special case (i.e., if  $D = \Delta^0$ ), we learn that for a quasicategory  $C$  we get a left fibration  $C^{x/} \rightarrow C$  and a right fibration  $C^{/x} \rightarrow C$ , which are therefore both conservative isofibrations.

In fact, we have the same property for the general alternate pullback slice map.

**59.3. Corollary.** *Consider a sequence  $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$  where  $i$  is a monomorphism and  $p$  is an inner fibration between quasicategories. Then the alternate pullback slice map  $p': C^{f/} \rightarrow C^{fi/} \times_{D_{pfi/}} D^{pf/}$  is a left fibration and the alternate pullback slice map  $p'': C^{/f} \rightarrow C^{/fi} \times_{D_{/pfi}} D^{/pf}$  is a right fibration. In particular, both  $p'$  and  $p''$  are conservative isofibrations.*

*Proof.* We consider the slice-under case. Consider the pullback-hom map  $q := p^{\square i}: U \rightarrow V$ . This is an inner fibration by (20.2). Thus for any object  $u \in U_0$  the induced map  $q': U^{u/} \rightarrow U \times_V V^{qu/}$  is a left fibration (59.1). The claim follows because  $p'$  is a basechange of  $q'$  (55.8).  $\square$

We note the following special case.

**59.4. Corollary.** *For a sequence  $K \xrightarrow{i} L \xrightarrow{f} C$  where  $i$  is a monomorphism and  $C$  is a quasicategory, the restriction functor  $C^{f/} \rightarrow C^{fi/}$  is a left fibration and  $C^{/f} \rightarrow C^{/fi}$  is a right fibrations, hence both are conservative isofibrations.*

*Proof.* Immediate from (59.3) (where  $D = \Delta^0$ ), and (32.5) and (32.11).  $\square$

The equivalence between slice and alternate slice extends to the target of the pullback-slice maps.

**59.5. Proposition.** *Consider a sequence  $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$  where  $i$  is a monomorphism and  $p$  is an inner fibration between quasicategories. Then the horizontal maps in the commutative squares*

$$\begin{array}{ccc} C_{f/} & \xrightarrow{\quad} & C^{f/} \\ \downarrow & & \downarrow \\ C_{fi/} \times_{D_{pfi/}} D_{pf/} & \xrightarrow{\quad} & C^{fi/} \times_{D^{pfi/}} D^{pf/} \end{array} \quad \begin{array}{ccc} C_{/f} & \xrightarrow{\quad} & C^{/f} \\ \downarrow & & \downarrow \\ C_{/fi} \times_{D_{/pfi}} D_{/pf} & \xrightarrow{\quad} & C^{/fi} \times_{D^{/pfi}} D^{/pf} \end{array}$$

*induced by the comparison between slice and alternate slice are all categorical equivalences. In particular, under these hypotheses the slice-pullback map is a categorical equivalence (and hence a trivial fibration) if and only if the corresponding alternate slice-pullback map is a categorical equivalence (and hence a trivial fibration).*

*Proof.* The first statement is immediate from the categorical equivalence of slice and alternate slice (58.3) once we see that the pullbacks along the bottom row are all good pullbacks with respect to the Joyal model structure. This is the case because all the slices and alternate slices are quasicategories and thus fibrant objects in the Joyal model structure, and because the restriction maps  $D_{pf/} \rightarrow D_{pfi/}$  and  $D_{/pf} \rightarrow D_{/pfi}$  (by (30.15))  $D^{pf/} \rightarrow D^{pfi/}$  and  $D^{/pf} \rightarrow D^{/pfi}$  (by (59.4)) are left fibrations, and hence isofibrations, and hence fibrations in the Joyal model structure. The final statement is a consequence of 2-out-of-3 (25.11), and the fact that the pullback-slice and alternate pullback-slice maps are isofibrations and so are categorical equivalence if and only if they are trivial fibrations (38.8).  $\square$

**59.6. Corollary.** *Consider a sequence  $K \xrightarrow{i} L \xrightarrow{f} C \xrightarrow{p} D$  where  $i$  is a monomorphism and  $p: C \rightarrow D$  is an inner fibration between quasicategories. If  $i$  is right anodyne then the alternate pullback-slice map  $C^{f/} \rightarrow C^{fi/} \times_{D^{pfi/}} D^{pf/}$  is a trivial fibration, and if  $i$  is left anodyne then the alternate pullback-slice map  $C^{/f} \rightarrow C^{/fi} \times_{D^{/pfi}} D^{/pf}$  is a trivial fibration.*

*Proof.* Immediate from (59.5) and the corresponding facts for the corresponding pullback-slice maps (30.14).  $\square$

We will often combine the above with the following.

**59.7. Proposition.** *For any monomorphism  $j: K \rightarrow L$  of simplicial sets, the map  $\Delta^0 \star j: K^{\triangleleft} \rightarrow L^{\triangleleft}$  is left anodyne, and the map  $j \star \Delta^0: K^{\triangleright} \rightarrow L^{\triangleright}$  is right anodyne.*

*Proof.* We prove the first case. Note that for any map  $p: C \rightarrow D$  of simplicial sets, we have that  $(\Delta^0 \star j) \square p$  if and only if  $j \square (C_{x/} \rightarrow D_{px/})$  for all vertices  $x \in C_0$ . Since  $(\overline{\text{LHorn}}, \text{LFib})$  is a factorization system, we see that the class of maps  $j$  such that  $j \star \Delta^0$  is weakly saturated, so it suffices to show that the cell inclusions have this property. But we know that  $\Delta^0 \star (\partial \Delta^n \subset \Delta^n) = (\Lambda_0^{n+1} \subset \Delta^{n+1})$ .  $\square$

As a consequence we get the following, which says that slices *under a left cone* or *over a right cone* are equivalent to the corresponding slices under or over the “cone point”.

**59.8. Corollary.** *For any map  $\hat{f}: S^\triangleright \rightarrow C$  to a quasicategory, the restriction maps  $C_{\hat{f}/} \rightarrow C_{\hat{f}(v)/}$  and  $C^{\hat{f}/} \rightarrow C^{\hat{f}(v)/}$  induced by restriction along the inclusion of the cone point of  $S^\triangleright$  are trivial fibrations. Likewise, for any map  $\hat{g}: S^\triangleleft \rightarrow C$  to a quasicategory, the restriction maps  $C_{/\hat{g}} \rightarrow C_{/\hat{g}(v)}$  and  $C^{/\hat{g}} \rightarrow C^{/\hat{g}(v)}$  induced by restriction along the inclusion of the cone point of  $S^\triangleright$  are trivial fibrations.*

*As a consequence, for each  $c \in C_0$  the induced maps*

$$\mathrm{map}_{\mathrm{Fun}(S^\triangleright, C)}(\hat{f}, \tilde{\pi}_{S^\triangleright} c) \rightarrow \mathrm{map}_C(\hat{f}(v), c) \quad \text{and} \quad \mathrm{map}_{\mathrm{Fun}(S^\triangleleft, C)}(\tilde{\pi}_{S^\triangleleft} c, \hat{g}) \rightarrow \mathrm{map}_C(c, \hat{g}(v))$$

*are equivalences.*

*Proof.* We do the case of  $\hat{f}$ . By (59.7) the inclusion  $\{v\} \rightarrow S^\triangleright$  is right anodyne. Hence the restriction map  $C_{\hat{f}/} \rightarrow C_{\hat{f}(v)/}$  is a trivial fibration by (30.14), and  $C^{\hat{f}/} \rightarrow C^{\hat{f}(v)/}$  is a trivial fibration by (59.6). The equivalence of mapping spaces is immediate from (55.7) and the fact that all restriction maps to  $C$  are isofibrations.  $\square$

## 60. LIMITS, COLIMITS, AND MAPPING SPACES

**60.1. Pushout products and right and left anodyne maps.** Recall that  $\overline{\mathrm{InnHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$  (19.8) and  $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ . We have an analogous fact for left or right anodyne maps.

**60.2. Proposition.** *We have that  $\overline{\mathrm{LHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{LHorn}}$  and  $\overline{\mathrm{RHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{RHorn}}$ .*

*Proof.* This is a calculation. See the appendix (62).  $\square$

Note that the above proposition implies  $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ .

**60.3. Fiberwise criterion for trivial fibrations, revisited.** We note the following “fiberwise” criterion for a left or right fibration to be a trivial fibration (and hence a categorical equivalence), which is analogous to the one we proved for isofibrations to quasigroupoids (45.1). Recall that the fibers of any left or right fibration between simplicial sets are Kan complexes.

**60.4. Proposition.** *Let  $p: X \rightarrow Y$  be either a left or right fibration of simplicial sets. Then  $p$  is a trivial fibration if and only if it has contractible fibers.*

*Proof.* [Lur09, 2.1.3.4]. We consider the case of a left fibration, and note that the direction  $(\implies)$  is immediate.

Suppose given a left fibration  $p$  with contractible fibers. We will show that  $(\partial\Delta^n \subset \Delta^n) \boxtimes p$  for all  $n \geq 0$ , by a variant of the covering homotopy extension technique we used to prove (45.1), to “deform” a given lifting problem to one which lives in a single fiber.

As in the proof of (45.1) we consider a lifting problem of type  $(\partial\Delta^n \subset \Delta^n) \boxtimes p$ , i.e., a vertex  $(a, b) \in \mathrm{Fun}(\partial\Delta^n, X) \times_{\mathrm{Fun}(\partial\Delta^n, Y)} \mathrm{Fun}(\Delta^n, Y)$ . Let  $\gamma: \Delta^n \times \Delta^1 \rightarrow \Delta^n$  be the unique map given on vertices by  $\gamma(k, 0) = k$  and  $\gamma(k, 1) = n$ . Then there exists a lift  $u$  in

$$\begin{array}{ccc} \partial\Delta^n \times \{0\} & \xrightarrow{a} & X \\ \downarrow & \nearrow u & \downarrow p \\ \partial\Delta^n \times \Delta^1 & \xrightarrow{\gamma} & \Delta^n \times \Delta^1 \xrightarrow{\gamma} \Delta^n \xrightarrow{b} Y \end{array}$$

since  $\overline{\mathrm{LHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{LHorn}}$  (60.2) so  $\partial\Delta^n \{0\} \subset \partial\Delta^n \times \Delta^n$  is left anodyne. The lower right triangle represents an edge  $e$  in  $\mathrm{Fun}(\partial\Delta^n, X) \times_{\mathrm{Fun}(\partial\Delta^n, Y)} \mathrm{Fun}(\Delta^n, Y)$  connecting the vertex  $e_0 = (a, b)$  to a



vertex  $e_1 = (a', b')$ , where  $b' = b\gamma|_{\Delta^n \times \{1\}}$  factor as  $\Delta^n \rightarrow \{b(n)\} \rightarrow Y$ . As the lifting problem  $(a', b')$  lives in a single fiber, by hypothesis it admits a solution  $t: \Delta^n \rightarrow X$ .

Thus we have a solid arrow commutative diagram

$$\begin{array}{ccc} (\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{(u,t)} & X \\ \downarrow & \nearrow s & \downarrow p \\ \Delta^n \times \Delta^1 & \xrightarrow{b\gamma} & Y \end{array}$$

If we can produce a lift  $s$ , then the restriction  $s|_{\Delta^n \times \{0\}}$  is the desired solution to the original lifting problem  $(a, b)$ .

Form the diagram

$$\begin{array}{ccccc} & & \xrightarrow{(u,t)} & & \\ & & \text{---} & & \\ (\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) & \xrightarrow{g} & C & \xrightarrow{h} & X \\ \downarrow & \nearrow s' & \downarrow p' & & \downarrow p \\ \Delta^n \times \Delta^1 & \xrightarrow{\gamma} & \Delta^n & \xrightarrow{b} & Y \end{array}$$

where the right-hand square is a pullback. Observe that (i)  $p'$  is a left fibration, and hence an inner fibration, between quasicategories, and that (ii)  $\gamma$  sends the edge  $\{n\} \times \Delta^1$  to the degenerate edge  $\langle nn \rangle$  in  $\Delta^n$ . Therefore  $g$  sends the edge  $\{n\} \times \Delta^1$  into the fiber of  $p'$  over  $n \in (\Delta^n)_0$ , which is isomorphic to the fiber of  $p$  over  $b(n)$ , which is by hypothesis a contractible Kan complex. Thus,  $g|_{\{n\} \times \Delta^1}$  represents an isomorphism in the quasicategory  $C$ . Therefore the pushout-product version of Joyal lifting (34.5) gives a lift  $s'$ , and so  $s := hs'$  is the desired lift.  $\square$

Note that (60.4) includes (45.1) as a special case, since a Kan fibration is a left (and right) fibration.

**60.5. Initial and terminal objects via mapping spaces.** We can apply this fiberwise criterion to pullback-slice maps or their alternate analogs, since these are often either left or right fibrations.

**60.6. Proposition.** *Let  $C$  be a quasicategory and  $x \in C_0$  an object of  $C$ .*

- (1) *The object  $x$  is initial in  $C$  if and only if for every object  $c \in C_0$  the mapping space  $\text{map}_C(x, c)$  is contractible.*
- (2) *The object  $x$  is terminal in  $C$  if and only if for every object  $c \in C_0$  the mapping space  $\text{map}_C(c, x)$  is contractible.*

*Proof.* I'll prove case (1). Consider the commutative diagram

$$\begin{array}{ccc} C_{x/} & \xrightarrow{\gamma} & C^{x/} \\ p \searrow & & \swarrow q \\ & C & \end{array}$$

where  $p$  and  $q$  are the evident forgetful functors and  $\gamma$  is the comparison map. The object  $x$  is initial if and only if  $p$  is a categorical equivalence (41.1), hence by 2-out-of-3 (25.11) if and only if  $q$  is a categorical equivalence. Since  $q$  is a left fibration and thus an isofibration (32.11), this is so if and only if  $q$  is a trivial fibration (38.8), and this is the case if and only if the fibers of  $q$  are contractible (60.4). The claim follows because the fibers of  $q$  are precisely the mapping spaces  $\text{map}_C(x, c)$  (55.3).  $\square$

**60.7. Limits and colimits via mapping spaces.** We have a similar result for general limits and colimits.

**60.8. Proposition.**

- (1) For any map  $\widehat{f}: S^{\triangleright} \rightarrow C$  to a quasicategory with  $\widehat{f}|_S = f$ , the following are equivalent.
- (a) The slice restriction functor  $C_{\widehat{f}/} \rightarrow C_{f/}$  is a trivial fibration, i.e.,  $\widehat{f}$  is a colimit cone.
  - (b) The slice restriction functor  $C_{\widehat{f}/} \rightarrow C_{f/}$  is a categorical equivalence.
  - (c) The alternate slice restriction functor  $C^{\widehat{f}/} \rightarrow C^{f/}$  is a trivial fibration.
  - (d) The alternate slice restriction functor  $C^{\widehat{f}/} \rightarrow C^{f/}$  is a categorical equivalence.
  - (e) For each object  $c \in C_0$ , the restriction map

$$\mathrm{map}_{\mathrm{Fun}(S^{\triangleright}, C)}(\widehat{f}, \widetilde{\pi}_{S^{\triangleright}} c) \rightarrow \mathrm{map}_{\mathrm{Fun}(S, C)}(f, \widetilde{\pi}_S c)$$

is an equivalence, where  $\widetilde{\pi}_{S^{\triangleright}}: C \rightarrow \mathrm{Fun}(S^{\triangleright}, C)$  and  $\widetilde{\pi}_S: C \rightarrow \mathrm{Fun}(S, C)$  are adjoints to projection.

Furthermore, if any of these hold, then there are equivalences  $\mathrm{map}_{\mathrm{Fun}(S, C)}(f, \widetilde{\pi}_S c) \approx \mathrm{map}_C(\widehat{f}(v), c)$ .

- (2) For any map  $\widehat{g}: S^{\triangleleft} \rightarrow C$  to a quasicategory with  $\widehat{g}|_S = f$ , the following are equivalent.
- (a) The slice restriction functor  $C_{/\widehat{g}} \rightarrow C_{/g}$  is a trivial fibration, i.e.,  $\widehat{g}$  is a limit cone.
  - (b) The slice restriction functor  $C_{/\widehat{g}} \rightarrow C_{/g}$  is a categorical equivalence.
  - (c) The alternate slice restriction functor  $C^{/\widehat{g}} \rightarrow C^{/g}$  is a trivial fibration.
  - (d) The alternate slice restriction functor  $C^{/\widehat{g}} \rightarrow C^{/g}$  is a categorical equivalence.
  - (e) For each object  $c \in C_0$ , the restriction map

$$\mathrm{map}_{\mathrm{Fun}(S^{\triangleright}, C)}(\widetilde{\pi}_{S^{\triangleright}} c, \widehat{g}) \rightarrow \mathrm{map}_{\mathrm{Fun}(S, C)}(\widetilde{\pi}_S c, g)$$

is an equivalence, where  $\widetilde{\pi}_{S^{\triangleright}}: C \rightarrow \mathrm{Fun}(S^{\triangleright}, C)$  and  $\widetilde{\pi}_S: C \rightarrow \mathrm{Fun}(S, C)$  are adjoints to projection.

Furthermore, if any of these hold, then there are equivalences  $\mathrm{map}_{\mathrm{Fun}(S, C)}(\widetilde{\pi}_S c, g) \approx \mathrm{map}_C(c, \widehat{g}(v))$ .

*Proof.* We prove case (1), following the same strategy as the proof of (60.6). The equivalence of (a)–(d) is straightforward using (59.5) to compare pullback-slices with alternate pullback-slices, and the fact that each of the maps is an isofibration. For the equivalence with (e) we refer to the diagram of pullback squares (55.7)

$$\begin{array}{ccccc} \mathrm{map}_{\mathrm{Fun}(S^{\triangleright}, C)}(\widehat{f}, \widetilde{\pi}_{S^{\triangleright}} c) & \xrightarrow{p_c} & \mathrm{map}_{\mathrm{Fun}(S, C)}(f, \widetilde{\pi}_S c) & \longrightarrow & \{c\} \\ \downarrow & & \downarrow & & \downarrow \\ C^{\widehat{f}/} & \xrightarrow{p} & C^{f/} & \longrightarrow & C \end{array}$$

Since all vertices of  $C^{f/}$  are contained in  $\mathrm{map}_{\mathrm{Fun}(S, C)}(f, \widetilde{\pi}_S c)$  for some  $c \in C_0$ , we deduce from the fiberwise criterion (60.4) that  $p$  is a trivial fibration if and only if each  $p_c$  is a trivial fibration. This is the case if and only if each  $p_c$  is a categorical equivalence, since each  $p_c$  is a pullback of the left fibration  $p$  and so is an isofibration. The claim that  $\mathrm{map}_{\mathrm{Fun}(S, C)}(f, \widetilde{\pi}_S c) \approx \mathrm{map}_C(\widehat{f}(v), c)$  if  $\widehat{f}$  is a colimit cone is immediate from (59.8).  $\square$

## Part 11. Appendices

### 61. APPENDIX: GENERALIZED HORNS

A **generalized horn**<sup>31</sup> is a subcomplex  $\Lambda_S^n \subset \Delta^n$  of the standard  $n$ -simplex, where  $S \subseteq [n]$  and **generalized horn**

$$(\Lambda_S^n)_k := \{ f: [k] \rightarrow [n] \mid S \not\subseteq f([k]) \}.$$

In other words, a generalized horn is a union of some codimension 1 faces of the  $n$ -simplex:

$$\Lambda_S^n = \bigcup_{s \in S} \Delta^{[n] \setminus s}.$$

In particular,

$$\Lambda_{[n]}^n = \partial \Delta^n, \quad \Lambda_{[n] \setminus j}^n = \Lambda_j^n, \quad \Lambda_{\{j\}}^n = \Delta^{[n] \setminus j}, \quad \Lambda_\emptyset^n = \emptyset.$$

In general  $S \subseteq T$  implies  $\Lambda_S^n \subseteq \Lambda_T^n$ .

**61.1. Proposition** (Joyal [Joy08a, Prop. 2.12]). *Let  $S \subsetneq [n]$  be a proper subset.*

- (1)  $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{Horn}}$  if  $S \neq \emptyset$ .
- (2)  $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{LHorn}}$  if  $n \in S$ .
- (3)  $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{RHorn}}$  if  $0 \in S$ .
- (4)  $(\Lambda_S^n \subset \Delta^n) \in \overline{\text{InnHorn}}$  if  $S$  is not an “interval”; i.e., if there exist  $a < b < c$  with  $a, c \in S$  and  $b \notin S$ .

*Proof.* We start with an observation. Consider  $S \subsetneq [n]$  and  $t \in [n] \setminus S$ . Observe the diagram

$$\begin{array}{ccccc} \Delta^{[n] \setminus t} \cap \Lambda_S^n & \xrightarrow{\sim} & \Delta^{[n] \setminus t} & & \\ \downarrow & & \downarrow & & \\ \Lambda_S^n & \xrightarrow{\quad} & \Lambda_{S \cup t}^n & \xrightarrow{\quad} & \Delta^n \end{array}$$

in which the square is a pushout, and the top arrow is isomorphic to the generalized horn  $\Lambda_S^{[n] \setminus t} \subset \Delta^{[n] \setminus t}$ . Thus,  $(\Lambda_S^n \subset \Delta^n)$  is contained in the weak saturation of any set containing the two inclusions

$$\Lambda_S^{[n] \setminus t} \subset \Delta^{[n] \setminus t} \quad \text{and} \quad \Lambda_{S \cup t}^n \subset \Delta^n.$$

Each of the statements of the proposition is proved by an evident induction on the size of  $[n] \setminus S$ , using the above observation. I'll do case (4), as the other cases are similar. If  $S \subset [n]$  is not an interval, there exists some  $s < u < s'$  with  $s, s' \in S$  and  $u \notin S$ . If  $[n] \setminus S = \{u\}$  then we already have an inner horn. If not, then choose  $t \in [n] \setminus (S \cup \{u\})$ , in which case  $S \cup t$  is not an interval in  $[n]$ , and  $S$  is not an interval in  $[n] \setminus t$ . Therefore both  $\Lambda_S^{[n] \setminus t} \subset \Delta^{[n] \setminus t}$  and  $\Lambda_{S \cup t}^n \subset \Delta^n$  are inner anodyne by the inductive hypothesis. The proofs of the other cases are similar.  $\square$

**61.2. Proposition** (Joyal [Joy08a, Prop. 2.13]). *For all  $n \geq 2$ , we have that  $(I^n \subset \Delta^n) \in \overline{\text{InnHorn}}$ .*

*Proof.* We can factor the spine inclusion as  $h_n = g_n f_n$ :

$$I^n \xrightarrow{f_n} \Delta^{\{1, \dots, n\}} \cup I^n \xrightarrow{g_n} \Delta^n.$$

We show by induction on  $n$  that  $f_n, g_n, h_n \in \overline{\text{InnHorn}}$ , noting that the case  $n = 2$  is immediate.

To show that  $f_n \in \overline{\text{InnHorn}}$ , consider the pushout square

$$\begin{array}{ccc} I^{\{1, \dots, n\}} & \xrightarrow{\quad} & \Delta^{\{1, \dots, n\}} \\ \downarrow & & \downarrow \\ I^n & \xrightarrow{f_n} & \Delta^{\{1, \dots, n\}} \cup I^n \end{array}$$

in which the top arrow is isomorphic to  $h_{n-1}$ , which is in  $\overline{\text{InnHorn}}$  by induction.

<sup>31</sup>This notion is from [Joy08a, §2.2.1]. However, I have changed the sense of the notation: our  $\Lambda_S^n$  is Joyal's  $\Lambda^{[n] \setminus S}$ . I find my notation easier to follow, but note that it does conflict with the standard notation for horns. Maybe I should use something like  $\Lambda^{n, S}$ ?

To show that  $g_n \in \overline{\text{InnHorn}}$ , consider the diagram

$$\begin{array}{ccc} \Delta^{\{1,\dots,n-1\}} \cup I^{\{0,\dots,n-1\}} & \xrightarrow{g_{n-1}} & \Delta^{\{0,\dots,n-1\}} \\ \downarrow & & \downarrow \\ \Delta^{\{1,\dots,n\}} \cup I^n & \xrightarrow{\quad} & \Delta^{\{1,\dots,n\}} \cup \Delta^{\{0,\dots,n-1\}} \xrightarrow{\quad} \Delta^n \end{array}$$

in which the square is a pushout, the top horizontal arrow is isomorphic to  $g_{n-1}$ , an element of  $\overline{\text{InnHorn}}$  by induction, and the bottom right horizontal arrow is equal to  $\Lambda_{\{0,n\}}^n \subset \Delta^n$ , which is in  $\overline{\text{InnHorn}}$  by (61.1)(4). □

## 62. APPENDIX: BOX PRODUCT LEMMAS

Here is where I'll prove various statements mentioned in the text.

- $\text{LHorn} \square \text{Cell} \subseteq \overline{\text{LHorn}}$  (60.2), proved in (62.1) below.
- $\text{RHorn} \square \text{Cell} \subseteq \overline{\text{RHorn}}$  (60.2), proved in (62.1) below.
- $\text{Horn} \square \text{Cell} \subseteq \overline{\text{Horn}}$ , is a consequence of the above, since  $\text{Horn} = \text{LHorn} \cup \text{RHorn}$  and  $\overline{\text{LHorn}} \cup \overline{\text{RHorn}} \subseteq \overline{\text{Horn}}$ .
- $\text{InnHorn} \square \text{Cell} \subseteq \overline{\text{InnHorn}}$  (19.10), proved in (62.3) below.

**62.1. Left and right horns.** We prove the case of  $\text{LHorn} \square \text{Cell} \subseteq \overline{\text{LHorn}}$  here. Given this  $\text{RHorn} \square \text{Cell} \subseteq \overline{\text{RHorn}}$  follows since  $\text{op}: s\text{Set} \rightarrow s\text{Set}$  carries LHorn to RHorn and preserves Cell.

Joyal [Joy08a, 2.25]<sup>32</sup> observes that  $(\Lambda_k^n \subset \Delta^n)$  is a retract of  $(\Lambda_k^n \subset \Delta^n) \square (\{0\} \subset \Delta^1)$  when  $0 \leq k < n$ . The retraction is

$$\Delta^n \xrightarrow{s} \Delta^n \times \Delta^1 \xrightarrow{r} \Delta^n$$

defined by  $s(x) = (x, 1)$  and

$$r(x, 0) = \begin{cases} x & \text{if } x \leq k, \\ k & \text{if } x \geq k, \end{cases} \quad r(x, 1) = x.$$

Note that  $r(\Delta^{[n] \setminus j} \times \Delta^1) = \Delta^{[n] \setminus j}$  if  $j \neq k$ , and  $r(\Delta^n \times \{0\}) = \Delta^{\{0,\dots,k\}} \subseteq \Delta^{[n] \setminus (k+1)}$ , so this gives the desired retraction.

The existence of the retraction reduces showing  $\text{LHorn} \square \text{Cell} \subseteq \overline{\text{LHorn}}$  to proving

$$(\{0\} \subset \Delta^1) \square \text{Cell} \subseteq \overline{\text{LHorn}},$$

since  $(\Lambda_k^n \subset \Delta^n) \in \overline{\text{Cell}}$  and thus  $(\Lambda_k^n \subset \Delta^n) \square \text{Cell} \subseteq \overline{\text{Cell}}$ .

**62.2. Lemma.** *We have that  $(\{0\} \subset \Delta^1) \square \text{Cell} \subseteq \overline{\text{LHorn}}$ .*

*Proof.* ... Let  $K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial \Delta^n)$ , so that  $(\{0\} \subset \Delta^1) \square (\partial \Delta^n \subset \Delta^n)$  is the inclusion  $K \rightarrow \Delta^1 \times \Delta^n$ . We will show that we can build  $\Delta^1 \times \Delta^n$  from  $K$  by an explicit sequence of steps, where in each case we attach an  $(n+1)$ -sequence along a left horn.

For each  $0 \leq a \leq n$  let  $\tau_a$  be the  $(n+1)$ -dimensional cell of  $\Delta^1 \times \Delta^n$  defined by

$$\tau_a = \langle (0, 0), \dots, (0, a), (1, a), \dots, (1, n) \rangle.$$

We obtain an ascending filtration of  $\Delta^1 \times \Delta^n$  by starting with  $K$  and attaching simplices in the following order:

$$\tau_n, \tau_{n-1}, \dots, \tau_1, \tau_0.$$

The  $\tau$ s range through all non-degenerate  $(n+1)$ -dimensional cells of  $\Delta^1 \times \Delta^n$ , so  $K \cup \bigcup \tau_a = \Delta^1 \times \Delta^n$ . (Here I am using the same notation for elements  $\tau_a \in (\Delta^1 \times \Delta^n)_{n+1}$  and for the corresponding subcomplex of  $\Delta^1 \times \Delta^n$  which is isomorphic to  $\Delta^{n+1}$ .)

<sup>32</sup>Lurie [Lur09, 2.1.2.6] states this incorrectly in my edition.

The claim is that each attachment is along a specified horn inclusion. More precisely, for  $a \in [n]$  the simplex  $\tau_a$  is attached to  $K \cup \bigcup_{k>a} \tau_k$  along the horn at the vertex  $(0, a)$  in  $\tau_a$ , i.e., via a  $\Lambda_a^{n+1} \subset \Delta^n$  horn inclusion. Note that if when  $a > 0$  this is an inner horn, while when  $a = 0$  this is the inclusion  $\Lambda_0^{n+1} \subset \Delta^n$ ; in either case, it is a left horn. Given the claim, it follows that  $(\{0\} \subset \Delta^1) \square (\partial \Delta^n \subset \Delta^n) \in \overline{\text{LHorn}}$  as desired.

The proof of the claim amounts to the following list of elementary observations about  $\tau_a$ :

- Every codimension-one face is contained in  $\Delta^1 \times \partial \Delta^n$  except: the face opposite vertex  $(0, a)$ , and the face opposite vertex  $(1, a)$ .
- The face opposite vertex  $(1, a)$  is contained in  $\{0\} \times \Delta^n$  if  $a = n$ , or is a face of  $\tau_{a+1}$  if  $a < n$ .
- The face opposite vertex  $(0, a)$  is not contained in  $\Delta^1 \times \partial \Delta^n$ , nor in  $\{0\} \times \Delta^n$ . Nor is it contained in any  $\tau_i$  with  $i > a$  (because the vertex  $(1, a)$  is in this face but not in  $\tau_i$  with  $i > a$ ).

Taken together these show that  $\tau_a \cap (K \cup \bigcup_{k>a} \tau_k)$  is the  $a$ th horn in the  $(n+1)$ -simplex  $\tau_a$ .  $\square$

**62.3. Inner horns.** Here is an argument for the key case for inner horns.

Consider  $\Delta^n \xrightarrow{s} \Delta^2 \times \Delta^n \xrightarrow{r} \Delta^n$ , the unique maps which are given on vertices by

$$s(y) = \begin{cases} (0, y) & \text{if } y < j, \\ (1, y) & \text{if } y = j, \\ (2, y) & \text{if } y > j, \end{cases} \quad r(x, y) = \begin{cases} y & \text{if } x = 0 \text{ and } y < j, \\ y & \text{if } x = 2 \text{ and } y > j, \\ j & \text{otherwise.} \end{cases}$$

These explicitly exhibit  $(\Lambda_j^n \subset \Delta^n)$  as a retract of  $(\Lambda_1^2 \subset \Delta^2) \square (\Lambda_j^n \subset \Delta^n)$ , so

$$\text{InnHorn} \subseteq \{\Lambda_1^2 \subset \Delta^2\} \square \overline{\text{Cell}}.$$

We have (20.6) that  $\overline{\text{Cell}} \square \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ , so the above implies that  $\text{InnHorn} \square \overline{\text{Cell}} \subseteq \{\Lambda_1^2 \subset \Delta^2\} \square \overline{\text{Cell}}$ . Thus the assertions “ $\text{InnHorn} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ ” and “ $\{\Lambda_0^2 \subset \Delta^2\} \square \overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ ” are equivalent. Thus both assertions follow from the following.

**62.4. Lemma.** *For all  $n \geq 0$  we have that  $(\Lambda_1^2 \subset \Delta^2) \square (\partial \Delta^n \subset \Delta^n) \in \overline{\text{InnHorn}}$ .*

*Proof.* [Lur09, 2.3.2.1].

For each  $0 \leq a \leq b < n$ , let  $\sigma_{ab}$  be the  $(n+1)$ -simplex of  $\Delta^2 \times \Delta^n$  defined by

$$\sigma_{ab} = \langle (0, 0), \dots, (0, a), (1, a), \dots, (1, b), (2, b+1), \dots, (2, n) \rangle.$$

For each  $0 \leq a \leq b \leq n$ , let  $\tau_{ab}$  be the  $(n+2)$ -cell of  $\Delta^2 \times \Delta^n$  defined by

$$\tau_{ab} = \langle (0, 0), \dots, (0, a), (1, a), \dots, (1, b), (2, b), \dots, (2, n) \rangle.$$

The set  $\{\tau_{ab}\}$  consists of all the non-degenerate  $(n+2)$ -dimensional cells. Note that  $\sigma_{ab}$  is a face of  $\tau_{ab}$  and of  $\tau_{a,b+1}$ , but not a face of any other  $\tau$ .

We attach simplices to  $K := (\Lambda_1^2 \times \Delta^n) \cup (\Delta^2 \times \partial \Delta^n)$  in the following order:

$$\sigma_{00}, \sigma_{01}, \sigma_{11}, \sigma_{02}, \sigma_{12}, \sigma_{22}, \dots, \sigma_{0,n-1}, \dots, \sigma_{n-1,n-1},$$

followed by

$$\tau_{00}, \tau_{01}, \tau_{11}, \tau_{02}, \tau_{12}, \tau_{22}, \dots, \tau_{0,n}, \dots, \tau_{n,n}.$$

The  $\tau$ s range through all the non-degenerate  $(n+2)$ -dimensional cells of  $\Delta^2 \times \Delta^n$ , so that  $K \cup \bigcup \sigma_{a,b} \cup \bigcup \tau_{a,b} = \Delta^2 \times \Delta^n$ .

The claim is that each attachment is along an inner horn inclusion. More precisely, each  $\sigma_{ab}$  gets attached along the horn at the vertex  $(1, a)$  in  $\sigma_{ab}$ , i.e., via a  $\Lambda_{a+1}^{n+1} \subset \Delta^{n+1}$  horn inclusion, which is always inner since  $a \leq b < n$ . Likewise, each  $\tau_{ab}$  gets attached along the horn at vertex  $(1, a)$  in  $\tau_{ab}$ , i.e., via a  $\Lambda_{a+1}^{n+2} \subset \Delta^{n+2}$  horn inclusion, which is always inner since  $a \leq b \leq n$ .

The proof of the claim amounts to the following lists of elementary observations.

For  $\sigma_{a,b}$ :

- Every codimension-one face is contained in  $\Delta^2 \times \partial\Delta^n$ , except the following: the face opposite vertex  $(0, a)$ , and the face opposite vertex  $(1, a)$ .
- The face opposite vertex  $(0, a)$  is either contained in  $\Lambda_0^2 \times \Delta^n$  if  $a = 0$ , or a face of  $\sigma_{a-1,b}$  if  $a > 0$ .
- The face of  $\sigma_{a,b}$  opposite vertex  $(1, a)$  is not contained in  $\Delta^2 \times \partial\Delta^n$ , nor in  $\Lambda_0^2 \times \Delta^n$ , nor in any  $\sigma_{i,b}$  with  $i < a$  (because of the vertex  $(0, a)$ ), nor in any  $\sigma_{i,j}$  with  $i \leq j < b$  (because of the vertex  $(1, b)$  if  $a < b$ , or the vertex  $(0, a)$  if  $a = b$ ).

For  $\tau_{a,b}$  when  $a < b$ :

- Every codimension-one face is contained in  $\Delta^2 \times \partial\Delta^n$  except the following: the face opposite vertex  $(0, a)$ , the face opposite vertex  $(1, a)$ , the face opposite vertex  $(1, b)$ , and the face opposite vertex  $(2, b)$ .
- The face opposite vertex  $(2, b)$  is  $\sigma_{a,b}$ , while the face opposite vertex  $(1, b)$  is  $\sigma_{a,b-1}$ .
- The face opposite vertex  $(0, a)$  is either contained in  $\Lambda_1^2 \times \Delta^n$  if  $a = 0$ , or is a face of  $\tau_{a-1,b}$  if  $a > 0$ .
- The face opposite vertex  $(1, a)$  is not contained in  $\Delta^2 \times \partial\Delta^n$ , nor in  $\Lambda_1^2 \times \Delta^n$ , nor in any  $\sigma_{i,j}$  (because of the vertices  $(1, b)$  and  $(2, b)$ ), nor in any  $\tau_{i,b}$  with  $i < b$  (because of the vertex  $(0, a)$ ), nor in any  $\tau_{i,j}$  with  $i \leq j < b$  (because of the vertex  $(1, b)$ ).

For  $\tau_{a,b}$  when  $a = b$ :

- Every codimension-one face is contained in  $\Delta^2 \times \partial\Delta^n$  except the following: the face opposite vertex  $(0, a)$ , the face opposite vertex  $(1, a) = (1, b)$ , and the face opposite vertex  $(2, b)$ .
- The face opposite vertex  $(2, b)$  is  $\sigma_{a,b}$ .
- The face opposite vertex  $(0, a)$  is contained in  $\Lambda_1^2 \times \Delta^n$  if  $a = 0$ , or is a face of  $\tau_{a-1,b}$  if  $a > 0$ .
- The face opposite vertex  $(1, a) = (1, b)$  is not contained in  $\Delta^2 \times \partial\Delta^n$ , nor in  $\Lambda_1^2 \times \Delta^n$ , nor in any  $\sigma_{i,j}$  (because of the vertices  $(0, a)$  and  $(2, b)$ ), nor in any  $\tau_{i,b}$  with  $i < b$  (because of the vertex  $(0, a)$ ), nor in any  $\tau_{i,j}$  with  $i \leq j < b$  (because of the vertex  $(0, a)$ ).

□

**62.5. A pushout-product version of Joyal lifting.** We now give a proof of (34.5): we will prove the case of  $(x, y) = (0, 0)$ , i.e., given  $p: C \rightarrow D$  an inner fibration of quasicategories,  $n \geq 1$ , and

$$\begin{array}{ccc}
 \Delta^1 \times \{0\} & \xrightarrow{\quad f \quad} & C \\
 \downarrow & \searrow & \downarrow p \\
 \Delta^1 \times \Delta^n & \xrightarrow{\quad \quad \quad} & D
 \end{array}$$

$(\{0\} \times \Delta^n) \cup_{\{0\} \times \partial\Delta^n} (\Delta^1 \times \partial\Delta^n)$

such that  $f$  represents an isomorphism in  $C$ , we will construct a lift. (Note that if  $n = 0$  such a lift does not generally exist.)

We refer to the proof of (62.2), where we observed that we can build  $\Delta^1 \times \Delta^n$  from  $K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n)$  by successively attaching a sequence  $\tau_n, \dots, \tau_0$  of  $(n+1)$ -simplices along horns; in particular,  $\tau_a$  is attached to  $K \cup \bigcup_{k>a} \tau_k$  along a horn inclusion isomorphic to  $\Lambda_a^{n+1} \subset \Delta^{n+1}$ .

Given this, we thus construct the desired lift by inductively choosing a lift defined on each  $\tau_a$  relative to the given lift on its  $\Lambda_a^{n+1}$ -horn. When  $a > 0$  such a lift exists because  $p$  is an inner fibration and  $\tau_a$  is attached along an inner horn, while when  $a = 0$  a lift exists by Joyal lifting (32.14), as  $\Delta^1 \times \{0\}$  is the leading edge of  $\tau_0$ .

**62.6. Another pushout-product version of Joyal lifting.** We show that given  $p: C \rightarrow D$  an inner fibration of categories,  $n \geq 1$ , and

$$\begin{array}{ccc}
 \{0\} \times \Delta^{\{0,1\}} & \xrightarrow{f} & (\partial\Delta^1 \times \Delta^n) \cup_{\partial\Delta^1 \times \Lambda_0^n} (\Delta^1 \times \Lambda_0^n) \xrightarrow{\quad} C \\
 & \downarrow & \searrow \text{dotted} \\
 & \Delta^1 \times \Delta^n & \xrightarrow{\quad} D
 \end{array}$$

$p$

such that  $f$  represents an isomorphisms in  $C$ , we can construct a lift.

We refer to the notation of the proof of (62.2), so that  $\tau_0, \dots, \tau_n$  are the nondegenerate  $(n+1)$ -cells of  $\Delta^1 \times \Delta^n$ . I claim that  $\Delta^1 \times \Delta^n$  can be built from  $K = (\partial\Delta^1 \times \Delta^n) \cup (\Delta^1 \times \Lambda_0^n)$  by successively attaching the sequence  $\tau_0, \dots, \tau_n$  along *generalized horns*, so that

- $\tau_0$  is attached along a horn inclusion isomorphic to  $\Lambda_1^{n+1}$ ,
- $\tau_a$ ,  $0 < a < n$ , is attached along a generalized horn inclusion isomorphic to  $\Lambda_{[n+1] \setminus \{0, k+1\}}^{n+1}$ , and
- $\tau_n$  is attached along a horn inclusion isomorphic to  $\Lambda_0^{n+1}$ .

In each except the last case the inclusion is a generalized *inner* horn, while the leading edge of  $\tau_n$  is precisely  $\Delta^1 \times \{0\}$ .

### 63. APPENDIX: WEAK EQUIVALENCES AND HOMOTOPY GROUPS

In this appendix we give a proof of (53.8), that the weak homotopy equivalences between Kan complexes are precisely the  $\pi_*$ -equivalences.

**63.1. Models for the simplicial sphere.** We first note that we can replace  $\Delta^n/\partial\Delta^n$  in the definition of  $\pi_n(X, x) := \pi_0 \text{Fun}_*(\Delta^n/\partial\Delta^n, X)$  with any other “model” of a simplicial  $n$ -sphere.

**63.2. Proposition.** *Let  $(S, s)$  be a pointed simplicial set which is weakly homotopy equivalent to  $(\Delta^n/\partial\Delta^n, *)$  in  $s\text{Set}_*$ , i.e., such that there exists a zig-zag of basepoint preserving weakly homotopy equivalences*

$$(S, s) \leftarrow (S_1, s_1) \rightarrow (S_2, s_2) \leftarrow \cdots \leftarrow (\Delta^n/\partial\Delta^n, *).$$

*Then there exists a bijection  $\pi_n(X, x) \approx \pi_0 \text{Fun}_*(S, X)$ , functorial in  $X$ .*

*Proof.* Immediate using (53.2)(1). □

For instance, the boundary of an  $(n+1)$ -simplex is a simplicial  $n$ -sphere.

**63.3. Proposition.** *There is a weak homotopy equivalence  $(\partial\Delta^{n+1}, \{0\}) \rightarrow (\Delta^n/\partial\Delta^n, *)$  of pointed simplicial sets, and thus natural isomorphisms  $\pi_n(X, x) \approx \pi_0 \text{Fun}_*((\partial\Delta^{n+1}, \{0\}), (X, x))$  for Kan complexes  $X$ .*

*Proof.* The inclusion  $\{0\} \subseteq \Lambda_0^{n+1}$  is anodyne (63.4) and thus a weak homotopy equivalence (49.5). Therefore by application of good pushouts (48.12) the induced map  $\partial\Delta^{n+1} = \partial\Delta^{n+1}/\{0\} \rightarrow \partial\Delta^{n+1}/\Lambda_0^{n+1}$  is a weak homotopy equivalence. The claim follows because we have an isomorphism  $\partial\Delta^{n+1}/\Lambda_0^{n+1} \approx \Delta^n/\partial\Delta^n$ , induced by  $\langle 1, \dots, n+1 \rangle: \Delta^n \rightarrow \partial\Delta^{n+1}$ . The description of  $\pi_n$  is immediate from (53.2)(1). □

**63.4. Exercise.** Define subcomplexes  $F_k \subseteq \Delta^n$  for  $0 \leq k \leq n$ , so that  $F_k$  is the union of all  $\Delta^S \subseteq \Delta^n$  such that (i)  $0 \in S \subseteq [n]$  and (ii)  $|S| \leq k+1$ . Show that each inclusion  $\Delta^{\{0\}} = F_0 \subset F_1 \subset \cdots \subset F_n = \Lambda_0^n$  is anodyne, whence  $\{0\} \subseteq \Lambda_0^n$  is anodyne.

### 63.5. $\pi_*$ -equivalences.

**63.6. Proposition.** *The class of  $\pi_*$ -equivalences between Kan complexes satisfies 2-out-of-6, and thus satisfies 2-out-of-3.*

*Proof.* This is much like the proof that functors which are essentially surjective and fully faithful share this property. One ingredient is to prove that if  $f_0, f_1: X \rightarrow Y$  are functors which are naturally isomorphic, then  $f_0$  is a  $\pi_*$ -equivalence if and only if  $f_1$  is. Another ingredient is the observation that to check that  $f$  is a  $\pi_*$ -equivalence, it suffices to check  $\pi_k(X, x) \rightarrow \pi_k(Y, fx)$  for  $x \in S$  where  $S \subseteq X_0$  is a set of representatives of  $\pi_0 X$ .  $\square$

Since every weak homotopy equivalence  $f: X \rightarrow Y$  between Kan complexes is a  $\pi_*$ -equivalence, to show (53.8) we can reduce to the case when  $f$  is a Kan fibration. In fact, we will show that  $f$  is a trivial fibration, using the following.

**63.7. Proposition.** *Let  $p: X \rightarrow Y$  be a Kan fibration between Kan complexes, and consider  $n \geq 0$ . Then  $\text{Cell}_{\leq n} \sqsupset p$  if and only if, for all  $0 \leq k < n$  and all  $x \in X_0$ , the induced map*

$$\pi_k(X, x) \rightarrow \pi_k(Y, p(x))$$

*is a bijection, and is a surjection for  $k = n$ .*

*Proof.* This will be immediate from (??) and (63.9) below.  $\square$

In the following, given a vertex  $x \in X_0$ , we write  $\bar{x}: K \rightarrow X$  for any constant map with value  $x$ , i.e., any map of the form  $K \rightarrow \{x\} \rightarrow X$ .

**63.8. Lemma.** *Let  $p: X \rightarrow Y$  be a Kan fibration between Kan complexes. Let  $n \geq 0$  be such that  $p_*: \pi_n(X, x) \rightarrow \pi_n(Y, px)$  is surjective for all  $x \in X_0$ . Then  $(* \subset \Delta^n / \partial \Delta^n) \sqsupset p$ .*

*Proof.* Suppose given a lifting problem  $(u, v)$  of type  $(* \subset \Delta^n / \partial \Delta^n) \sqsupset p$ . Note that  $u = \bar{x}$  for some  $x \in X$ . We will show that we can deform this lifting problem to one of the form  $(\bar{x}, \bar{y})$  where  $y = f(x) \in Y_0$ , which tautologically admits a lift (i.e., the constant map  $\bar{x}: \Delta^n / \partial \Delta^n \rightarrow X$ ), so that the claim follows by covering homotopy extension (37.8).

The hypothesis that  $p_*$  is surjective says exactly that there exists a map  $\tilde{e}: (\Delta^n / \partial \Delta^n) \times \Delta^1 \rightarrow Y$  such that (i)  $\tilde{e}|(\Delta^n / \partial \Delta^n) \times \{0\} = v$ , (ii)  $\tilde{e}|(\Delta^n / \partial \Delta^n) \times \{1\} = \bar{y}$ , where  $\bar{y}$  is the unique map factoring through  $\{y = px\} \rightarrow Y$ , and (iii)  $\tilde{e}|* \times \Delta^1 = \bar{y}$  is also a constant map. By adjunction, we see that this exactly gives the data of an edge  $e$  in  $\text{Fun}(*, X) \times_{\text{Fun}(*, Y)} \text{Fun}(\Delta^n / \partial \Delta^n)$  with  $e_0 = (\bar{x}, v)$ , and  $e_1 = (\bar{x}, \bar{y})$ , as desired.  $\square$

**63.9. Lemma.** *Let  $p: X \rightarrow Y$  be a Kan fibration between Kan complexes. Let  $n \geq 0$  be such that  $\pi_n(X, x) \rightarrow \pi_n(Y, px)$  is injective for all  $x \in X_0$ . Then  $(* \subset \Delta^{n+1} / \partial \Delta^{n+1}) \sqsupset p$  implies  $(\partial \Delta^{n+1} \subset \Delta^{n+1}) \sqsupset p$ .*

*Proof.* Suppose given a lifting problem  $(u, v)$  of type  $(\partial \Delta^{n+1} \subset \Delta^{n+1}) \sqsupset p$ . We show that this can be deformed to a lifting problem  $(u', v')$  such that  $u' = \bar{x}$  for some  $x \in X$ , i.e., so that the lifting problem  $(u', v')$  factors as

$$\begin{array}{ccccc} \partial \Delta^n & \longrightarrow & * & \xrightarrow{\bar{x}} & X \\ \downarrow & & \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \Delta^n / \partial \Delta^n & \xrightarrow{v''} & Y \end{array}$$

Then by hypothesis the lifting problem  $(\bar{x}, v'')$  admits a solution, so the claim follows using covering homotopy extension (37.8).



Let  $x := u(\langle 0 \rangle) \in X_0$  and  $y = p(x)$ . Recall that  $\partial\Delta^{n+1}$  is weakly homotopy equivalent to  $\Delta^n/\partial\Delta^n$  (63.3), so that we have natural isomorphism  $\pi_0 \text{Fun}_*((\partial\Delta^{n+1}, \{0\}), (T, t)) = \pi_n(T, t)$  (63.2). In particular,  $u: \partial\Delta^{n+1} \rightarrow X$  represents an element of  $\pi_n(X, x)$ . Consider the composite

$$\partial\Delta^n \times \Delta^1 \rightarrow \Delta^n \times \Delta^1 \xrightarrow{\gamma} \Delta^n \xrightarrow{v} Y$$

where  $\gamma$  represents the natural transformation  $\text{id}_{\Delta^n} \rightarrow \langle n \dots n \rangle$ . This gives an edge in  $\text{Fun}_*((\partial\Delta^n, \{0\}), (Y, y))$  connecting  $pu$  with  $\bar{y}$ , i.e.,  $pu$  represents the trivial element of  $\pi_n(Y, y)$ . By hypothesis we conclude that  $u$  represents the trivial element of  $\pi_n(X, x)$ , so there exists  $h: \partial\Delta^{n+1} \times \Delta^1 \rightarrow X$  so that  $h|_{\partial\Delta^{n+1} \times \{0\}} = u$ ,  $h|_{\partial\Delta^{n+1} \times \{1\}} = \bar{x}$ , and  $h|_{\{0\} \times \Delta^1} = \bar{x}$ . Now consider

$$\begin{array}{ccc} (\partial\Delta^{n+1} \times \Delta^1) \cup_{\partial\Delta^{n+1} \times \{0\}} \Delta^{n+1} \times \{0\} & \xrightarrow{(ph, v)} & Y \\ \downarrow i & \nearrow k & \\ \Delta^{n+1} \times \Delta^1 & & \end{array}$$

which is well-defined since  $ph|_{\partial\Delta^{n+1} \times \{0\}} = pu = v|_{\partial\Delta^{n+1} \times \{0\}}$ . Note that since  $i$  is the pushout product  $(\partial\Delta^{n+1} \subset \Delta^{n+1}) \square (\{0\} \subset \Delta^1)$  which is in  $\text{Cell} \square \text{Horn} \subseteq \overline{\text{Horn}}$  (??), and  $Y$  is a Kan complex, a lift  $k$  exists. Therefore the pair  $(h, k)$  represents an edge in  $\text{Fun}(\partial\Delta^{n+1}, X) \times_{\text{Fun}(\partial\Delta^{n+1}, Y)} \text{Fun}(\Delta^{n+1}, Y)$  connecting  $(u, v)$  with  $(\bar{x}, v')$  where  $v' = k|_{\Delta^{n+1} \times \{1\}}$ . This is what we needed.  $\square$

**63.10. Group structures.** We give some exercises which lead to a proof that  $\pi_n(X, x)$  is a group when  $n \geq 1$ , and abelian when  $n \geq 2$ . We have already noted that  $\pi_1(X, x) = \text{Hom}_{hX}(x, x)$ , so has a natural group structure (53.5).

Write  $\partial(\Delta^p \times \Delta^q) := (\Delta^p \times \partial\Delta^q) \cup (\partial\Delta^p \times \Delta^q)$ .

**63.11. Exercise.** Show that  $\partial(\Delta^n \times \Delta^1)$ , with any choice of basepoint, is weakly homotopy equivalent to  $\Delta^n/\partial\Delta^n$  in  $s\text{Set}_*$ . (Hint: let  $S := (\Delta^n \times \{0\}) \cup (\partial\Delta^n \times \Delta^1)$ , and note that  $S$  is weak homotopy equivalent to  $\Delta^0$ , and that  $\partial(\Delta^n \times \Delta^1)/S \approx \Delta^n/\partial\Delta^n$ .)

**63.12. Exercise.** Show that  $(\Delta^n \times \Delta^1)/\partial(\Delta^n \times \Delta^1)$  is weakly homotopy equivalent to  $\Delta^{n+1}/\partial\Delta^{n+1}$  in  $s\text{Set}_*$ . (Hint: use (63.3).)

**63.13. Exercise.** Show that  $\pi_n(X, x) \approx \pi_{n-1}(\text{map}_X(x, x), 1_X)$  for all  $n \geq 1$ . Conclude that  $\pi_n(X, x)$  is a group if  $n \geq 1$ .

#### 64. APPENDIX: SETS GENERATING WEAKLY SATURATED CLASSES

We show that the weakly saturated classes  $\text{CatEq} \cap \overline{\text{Cell}}$  and  $\text{GpdEq} \cap \overline{\text{Cell}}$  are each generated by some set  $S$ , and so in particular are parts of weak factorization systems

$$(\text{CatEq} \cap \overline{\text{Cell}}, \text{CatFib}) \quad \text{and} \quad (\text{GpdEq} \cap \overline{\text{Cell}}, \text{GpdFib}).$$

In either case, we will show that the weakly saturated class is generated by the class of injective maps  $K \hookrightarrow L$  in the class for which the number of cells in  $K$  and  $L$  is bounded by some explicit regular cardinal. We obtain  $S$  by choosing one representative for each isomorphism class in this class; then  $S$  is a set because of the cardinality bound.

**64.1. Lemma.** *Let  $U$  be any weakly saturated class of maps of simplicial sets. Suppose  $Y$  is a simplicial set with subcomplex  $X \subseteq Y$ . Then there exists a subcomplex  $X' \subseteq Y$  which is maximal with respect to the properties that (i)  $X \subseteq X' \subseteq Y$  and (ii)  $(X \rightarrow X') \in U$ .*

*Proof.* Let  $\mathcal{P}$  be the set of all subcomplexes  $Z$  of  $Y$  such that  $X \subseteq Z$  and  $(X \rightarrow Z) \in U$ . Say that  $Z \leq Z'$  for  $Z, Z' \in \mathcal{P}$  if  $Z \subseteq Z'$  and  $(Z \rightarrow Z') \in U$ . Then  $\mathcal{P}$  is a partially ordered set since  $U$  is closed under composition (but note that  $Z \subseteq Z'$  need not imply  $Z \leq Z'$ ). Furthermore,  $\mathcal{P}$  is non-empty since  $X \in \mathcal{P}$ .

I claim that  $\mathcal{P}$  satisfies the hypothesis of Zorn's lemma. In fact, suppose  $\mathcal{C} \subseteq \mathcal{P}$  is a non-empty chain. Using the axiom of choice we can choose an ordinal  $\lambda$  and a cofinal map  $f: \lambda \rightarrow \mathcal{C}$  (i.e., one such that for all  $Z \in \mathcal{C}$  there exists  $\alpha \in \lambda$  with  $Z \leq f(\alpha)$ ). Then  $B := \operatorname{colim}_{\alpha < \lambda} f(\alpha) = \bigcup_{\alpha < \lambda} f(\alpha)$  is such that  $(f(0) \rightarrow B) \in U$  since  $U$  is closed under transfinite composition, and thus  $B \in \mathcal{P}$ . Since  $B$  is clearly an upper bound for  $\mathcal{C}$ , Zorn's lemma applies, and  $\mathcal{P}$  has a maximal element  $X'$ .  $\square$

**64.2. Lemma.** *Let  $T$  be a weakly saturated class of monomorphisms. Suppose that there exists a regular cardinal  $\kappa$  with the following property: for any inclusion  $X \subseteq Y$  of simplicial sets with  $(X \rightarrow Y) \in T$ ,  $Y$  is equal to the union of the collection of all subcomplexes  $B \subseteq Y$  such that (i)  $B$  is  $\kappa$ -small, and (ii)  $(B \cap X \rightarrow B) \in T$ . Then  $T = \overline{S}$  where  $S \subseteq T$  is the subclass of maps whose codomains are  $\kappa$ -small simplicial sets.*

*Proof.* Let  $S$  be the class described in the statement. We want to show that any element of  $T$  is in  $\overline{S}$ . Since the maps in  $T$  are monomorphisms, it suffices to consider inclusions of subcomplexes  $K \subseteq Y$  which are in  $T$ , and to show that these are in  $\overline{S}$ .

Apply (64.1) with the class  $\overline{S}$  to obtain  $K \subseteq K' \subseteq Y$  maximal with respect to the property that  $(K \rightarrow K') \in \overline{S}$ . If  $K' = Y$  we are done. If not, then by hypothesis applied to  $K' \subseteq Y$  there exists a  $\kappa$ -small subcomplex  $B \subseteq Y$  with  $B \not\subseteq K'$  and  $(B \cap K' \rightarrow B) \in S \subseteq T$ . This implies  $(K' \rightarrow B \cup K') \in \overline{S}$  since this is a cobase change of  $B \cap K' \rightarrow B$ , and therefore  $(K \rightarrow B \cup K') \in \overline{S}$ . This contradicts the maximality of  $K'$ .  $\square$

**64.3. Detection functors.** Let  $\mathcal{C}$  be a class of morphisms in  $s\mathbf{Set}$ . A **detection functor** for  $\mathcal{C}$  is a functor  $F: \operatorname{Fun}([1], \mathbf{Set}) \rightarrow \operatorname{Fun}([1], s\mathbf{Set})$  on arrow categories such that there exists a regular cardinal  $\kappa$ , with the following properties:

detection functor

- (1) For any map  $f$  of simplicial sets (i.e., object of  $\operatorname{Fun}([1], s\mathbf{Set})$ ) we have  $f \in \mathcal{C}$  if and only if  $F(f): F_0(f) \rightarrow F_1(f)$  is a bijection of sets.
- (2) The functor  $F$  commutes with  $\kappa$ -filtered colimits.
- (3) The functor  $F$  takes maps between  $\kappa$ -small simplicial sets to maps between  $\kappa$ -small sets.

**64.4. Proposition.** *Let  $F$  be a detection functor for  $\mathcal{C}$ , with associated infinite regular cardinal  $\kappa$ . Suppose  $f: X \subseteq Y$  is an inclusion of a subcomplex which is an element of  $\mathcal{C}$ . Then  $Y$  is a union of all subcomplexes  $B \subseteq Y$  such that (i)  $B$  is  $\kappa^+$ -small and (ii)  $(B \cap X \rightarrow B) \in \mathcal{C}$ . (Here  $\kappa^+$  is the successor cardinal to  $\kappa$ , which is also regular.)*

*Proof.* (Adapted from [Joy08a, D.2.16].) Let  $\mathcal{P}$  be the poset of subcomplexes of  $Y$ , so that the detection functor gives a composite functor

$$\mathcal{P} \rightarrow \operatorname{Fun}([1], \mathbf{Set}), \quad A \mapsto F(f_A), \quad f_A: A \cap X \rightarrow A,$$

which commutes with  $\kappa$ -filtered colimits since  $F$  does. For any cardinal  $\alpha$  let  $\mathcal{P}_\alpha \subseteq \mathcal{P}$  be the subset consisting of subcomplexes  $A \subseteq Y$  with  $|A| < \alpha$ . We will show that every  $A \in \mathcal{P}_\kappa$  is contained in some  $B \in \mathcal{P}_{\kappa^+}$  such that  $F(f_B)$  is a bijection. Since  $Y = \bigcup_{A \in \mathcal{P}_\kappa} A = \operatorname{colim}_{A \in \mathcal{P}_\kappa} A$  this proves the claim.

Suppose given  $A \in \mathcal{P}_\kappa$ . Note that  $\mathcal{P}_\kappa$  is  $\kappa$ -filtered and thus

$$\operatorname{colim}_{A \in \mathcal{P}} F(f_A) \approx F(f)$$

since  $F$  preserves  $\kappa$ -filtered colimits. Furthermore since  $f \in \mathcal{C}$  we have that  $F(f): F_0(f) \rightarrow F_1(f)$  is a bijection of sets. Therefore, for any  $A \in \mathcal{P}_\kappa$  we can choose  $A^+ \in \mathcal{P}_\kappa$  with  $A \subseteq A^+$  such that a lift exists in

$$\begin{array}{ccc} F_0(f_A) & \longrightarrow & F_0(f_{A^+}) \\ \downarrow & \nearrow s & \downarrow \\ F_1(f_A) & \longrightarrow & F_1(f_{A^+}) \end{array}$$

Now define a functor  $A_\bullet: \kappa \rightarrow \mathcal{P}_\kappa$  by transfinite induction, so that

- $A_0 := A$ ,
- $A_{\lambda+1} := A^+$ ,
- $A_\lambda := \operatorname{colim}_{i < \lambda} A_i$  if  $\lambda \leq \kappa$  is a limit ordinal.

Let  $B := \operatorname{colim}_{i < \kappa} A_i = \bigcup_{i < \kappa} A_i$ , which will be an element of  $\mathcal{P}_{\kappa^+}$ . Since  $F$  preserves  $\kappa$ -filtered colimits, we have  $F(f_B) = \operatorname{colim}_{i < \kappa} F(f_{A_i})$ , which is seen to be a bijection. Thus we have proved that  $A \subseteq B$  with  $|B| < \kappa^+$  and  $(B \cap X \rightarrow X) \in \mathcal{C}$  as desired.  $\square$

**64.5. Corollary.** *Suppose  $\mathcal{C}$  is a class of maps in  $s\mathbf{Set}$  for which there exists a detection functor, and suppose  $T := \mathcal{C} \cap \overline{\mathbf{Cell}}$  is weakly saturated. Then  $T = \overline{S}$  for some set  $S$ .*

**64.6. Construction of detection functors.** It remains to construct detection functors for the classes  $\mathbf{CatEq}$  and  $\mathbf{GpdEq}$ . We obtain the detection functor as a composite of several intermediate steps, so  $F := F^{(4)}F^{(3)}F^{(2)}F^{(1)}$ .

*Step 1:* Recall (16.9) that the small object argument gives a *functorial* way to factor a map  $f$  as  $f = pi$ , with  $i \in \overline{S}$  and  $p \in S^\square$  for some set  $S$ .

We can apply this with  $S = \mathbf{InnHorn}$ , so that we obtain a functor  $s\mathbf{Set} \rightarrow \mathbf{Fun}([1], s\mathbf{Set})$  sending  $X$  to  $i_X: X \rightarrow X'$ , where  $i_X$  is a categorical equivalence and  $X'$  is a quasicategory. Alternately we can apply this with  $S = \mathbf{Horn}$ , so that  $i_X$  is a weak homotopy equivalence and  $X'$  a Kan complex.

In either case, for any map of simplicial sets  $f: X \rightarrow Y$  we obtain a commutative square

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X' \\ f \downarrow & & \downarrow f' \\ Y & \xrightarrow{i_Y} & Y' \end{array}$$

so that we get a functor  $F^{(1)}: \mathbf{Fun}([1], s\mathbf{Set}) \rightarrow \mathbf{Fun}([1], \mathbf{qCat}) \subseteq \mathbf{Fun}([1], s\mathbf{Set})$ , which on objects sends  $f \mapsto f'$ . If  $S = \mathbf{InnHorn}$  we have that  $f'$  is a categorical equivalence if and only if  $f$  is, while if  $S = \mathbf{Horn}$  we have that  $f'$  is a categorical equivalence if and only if  $f$  is a weak homotopy equivalence.

*Step 2:* Define a functor  $F^{(2)}: \mathbf{Fun}([1], \mathbf{qCat}) \rightarrow \mathbf{Fun}([1], \mathbf{qCat})$  which on objects sends  $f: C \rightarrow D$  to the path factorization  $p = F^{(2)}(f): \mathbf{Fun}^{\mathbf{iso}}(\Delta^1, C) \times_C D \rightarrow D$ . We have that  $p$  is an isofibration, and is a categorical equivalence if and only if  $f$  is. Therefore,  $f$  is a categorical equivalence if and only if  $p$  is a trivial fibration (38.8).

*Step 3:* Define a functor  $F^{(3)}: \mathbf{Fun}([1], \mathbf{qCat}) \rightarrow \mathbf{Fun}([1], \mathbf{Set})$  sending  $f: X \rightarrow Y$  to the map of sets

$$F^{(3)}(f): \prod_{n \geq 0} \mathbf{Hom}(\Delta^n, X) \rightarrow \prod_{n \geq 0} \mathbf{Hom}(\partial \Delta^n, X) \times_{\mathbf{Hom}(\partial \Delta^n, Y)} \mathbf{Hom}(\Delta^n, Y).$$

Thus,  $f$  is a trivial fibration if and only if  $F^{(3)}(f)$  is surjective.

*Step 4:* Define a functor  $F^{(4)}: \mathbf{Fun}([1], \mathbf{Set}) \rightarrow \mathbf{Fun}([1], \mathbf{Set})$  sending  $f: X \rightarrow Y$  to

$$F^{(4)}(f): \operatorname{colim}[X \times_Y X \rightrightarrows X] \rightarrow Y.$$

Thus,  $f$  is a surjection if and only if  $F^{(4)}(f)$  is a bijection.

It is clear that the composite functor  $F$  is such that  $F(f)$  is a bijection if and only if  $f$  is a categorical equivalence (or weak homotopy equivalence). We can choose an infinite regular cardinal such that each of  $F^{(i)}$  preserves  $\kappa$ -filtered colimits and takes  $\kappa$ -small simplicial sets to  $\kappa$ -small sets or simplicial sets as the case may be. In fact, any infinite regular cardinal  $> \omega$  satisfies when  $i = 2, 3, 4$ , while for  $i = 1$  we choose  $\kappa$  greater than the size of the domains and codomains of objects in  $S$ .

## 65. STUFF

Just dumping some stuff here that may make it into a future new section.

**65.1. Proposition.** *Let  $C$  be a quasicategory, and let  $i$  be the inclusion*

$$i: A = \partial\Delta^m \times \Delta^n \times \Delta^1 \cup \Delta^m \times \partial\Delta^n \times \Delta^1 \cup \Delta^m \times \Delta^n \times \{1\} \rightarrow \Delta^m \times \Delta^n \times \Delta^1,$$

*with  $m, n \geq 1$ . Then any  $f: A \rightarrow C$  which sends the edge  $\{(m, n)\} \times \Delta^1$  to an isomorphism in  $C$  extends over  $i$ .*

*Proof.* We have already proved the case of  $n = 0$ . Prove the general case by building  $\Delta^m \times \Delta^n$  from its boundary by attaching cells.  $\square$

Here's a better statement of the previous.

**65.2. Proposition.** *Let  $C$  be a quasicategory and  $i: K \rightarrow L$  a monomorphism which is a bijection on vertices. Let  $S \subseteq L_0$  be the subset of vertices of the form  $a_k$ , for all  $a \in L_k^{\text{nd}} \setminus K_k^{\text{nd}}$ . Then in any diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & \text{Fun}(\Delta^1, C) \\ \downarrow & \nearrow & \downarrow \\ L & \longrightarrow & \text{Fun}(\{1\}, C) \end{array}$$

*if  $f$  sends each vertex in  $S$  to a vertex in  $\text{Fun}(\Delta^1, C)$  corresponding to an isomorphism in  $C$ , then a lift exists.*

**65.3. Corollary.** *Let  $C$  be a category, and let  $p: \text{Fun}(\Delta^n, C) \rightarrow \text{Fun}(\partial\Delta^n, C)$  be the restriction map, with  $n \geq 1$ . Then an edge  $\alpha$  of  $\text{Fun}(\Delta^n, C)$  is  $p$ -Cartesian iff its corresponding to natural transformation  $\alpha: f \Rightarrow f'$  of functors  $\Delta^n \rightarrow C$  is such that  $\alpha(n): f(n) \rightarrow f'(n)$  is an isomorphism in  $C$ .*

**65.4. Proposition.** *Let  $C$  be a quasicategory and  $K$  a simplicial set, and consider the restriction functor*

$$p: \text{Fun}(K^\triangleright, C) \rightarrow \text{Fun}(K \cup \{v\}, C),$$

*where  $v$  is the cone point of  $K^\triangleright$ . Then for any edge  $\alpha \in \text{Fun}(K^\triangleright, C)_1$ , if  $\alpha(v)$  is an isomorphism in  $C$ , then  $\alpha$  is  $p$ -Cartesian.*

**65.5. Corollary.** *If  $C$  is a quasicategory and  $K$  a simplicial set, the restriction functor*

$$q: \text{Fun}(K^\triangleright, C) \rightarrow \text{Fun}(K, C)$$

*is a Cartesian fibration.*

**65.6. Corollary.** *If  $C$  is a quasicategory and  $K$  a simplicial set, and  $q: \text{Fun}(K^\triangleright, C) \rightarrow \text{Fun}(K, C)$  is restriction, then an object in  $\text{Fun}(K^\triangleright, C)$  relatively  $q$ -initial iff it is an initial object in its fiber.*

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