SPECTRA ARE YOUR FRIENDS

A LEISURELY STROLL THROUGH THE LAND OF SPHERES

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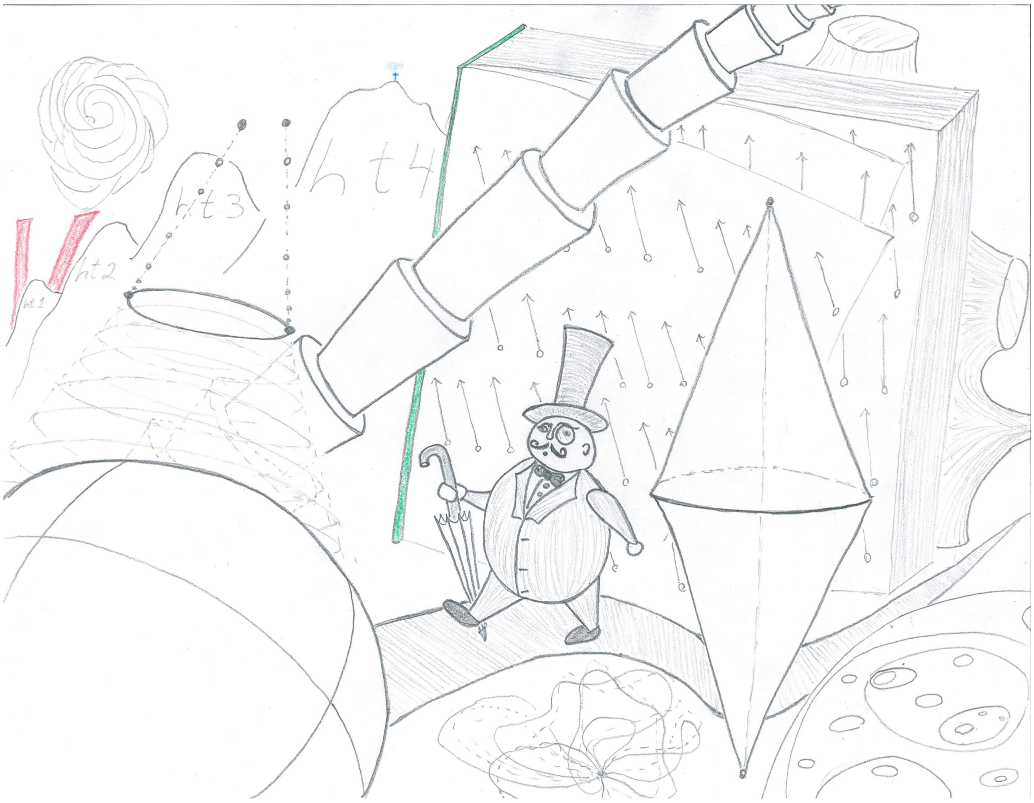

This is a collection of notes on stable homotopy theory. They came about after Tom Gannon, a fellow grad student at UT, asked me to write him an email to tell him a little about the sphere spectrum. As one email grew into many, the recipient list expended as well, and finally I was convinced to make these notes available publically.

The key thing these notes strive for is a friendly, informal, and conversational style. We do not strive to be exhaustive, nor do we strive to be concise. If more words allow us to shed more light on something, we will rarely pass down the opportunity to do so.

What follows contains hardly any proofs, but hopefully ample motivation behind every idea. The hope is that, given the birds-eye-view for orientation and layout of the land, the proofs and details will be easy(er) to pick up if and when needed. Essentially everything we mention, especially in Part 1, is elaborated on in full rigorous glory in a measure 0 subset of Jacob Lurie's treatise *Higher Algebra*.

On the use of ∞*-categories.* The perspective we take is unapologetically ∞-categorical. That partially betrays the author's personal preferences and beliefs, but is also a consequence of how these notes came to be. This is because Tom Gannon, the original email's

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recepient and target audience, works in the Gaitsgorian denomination of the Geometric Langlands Program, where ∞-categorical technology is an all-pervasive state religion.

However, we believe this should not be an obstacle for other interested readers either. Our use of ∞-category theory is exclusively to enable abstract nonsense arguments for homotopical objects. In particular, we will never need to "look under the hood" into the finer points of the simplicial nitty-gritty that oils the machine of ∞-categories from the quasi-categorical approach. We merely assume it is there and runs as smoothly as it should, and direct any suspicious reader to drown their doubts in the depths of Lurie's *Higher Topos Theory.*

In conclusion we hope that our decision to use ∞-categories will not throw other interested readers off too much. But as stated, we refuse to apologize for it.

Warning. These notes are little more than a transcription of some emails between friends on fun math, and as such should not be taken too seriously. In particular, we can not vouch that everything contained in them is correct, nor even that the mistakes are restricted to the few typos and small fixable gaffs that permiate every mathematical text. Use with caution and at your own risk!

Contents

Part 1. Spectra among stable ∞-categories

One aspect of stable homotopy theory that can be perceived either as annoying or as amazing is that there exist a number of different perspectives on spectra, and all are worth juggling simultaneously for the different insights they bring.

1.1. Stable ∞-categories

Recall the notion of a stable ∞-category - a good place to do homological algebra. There are several different versions on which things to give as definitions and which to derive as consequences, but here is one:

Definition 1. An ∞-category $\mathcal{C}$ is stable if it satisfies the following conditions:

(1) It contains finite limits and colimits.
(2) It has a zero object $0 \in \mathcal{C}$.

(3) Fiber sequences and cofiber sequences in $\mathcal{C}$ coincide.

In particular, the suspension and loops functors $\Sigma X = 0 \coprod_X 0$ and $\Omega X = 0 \times_X 0$, which always form an adjunction $\Sigma \dashv \Omega$, is an adjoint equivalence of $\mathcal{C}$. In the homological grading (which is the common-sense one in homotopy theory when disusing homotopy groups) there correspond to shifts $\Sigma = [1]$ and $\Omega = [-1]$.

The condition 3. above could be replaced with requiring that either Σ or Ω is an equivalence of ∞-categories $\mathcal{C} \to \mathcal{C}$. Unlike the above definition, which emphasizes the analogy with abelian categories, this way of defining a sthetable ∞-category would put front and center that everything is *stable* under suspension (or loops) - hence the name!

This alternative definition of stability will be useful in the next subsection, as it is somewhat less to check (is certain functor an equivalence) than the above definition (are all fiber and cofiber sequences the same).

1.1.1. **Triangulated categories.** Recall that any ∞-category $\mathcal{C}$ gives rise to an ordinary category, denoted either $\mathrm{h}\mathcal{C}$ or $\mathrm{Ho}(\mathcal{C})$, and called the *homotopy category* of $\mathcal{C}$. It is obtained by keeping the same set of objects, but by quotienting out all the homotopy equivalences, which is to say that we set

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y).$$

When you were first studying the derived category, e.g. in a course or textbook on homological algebra, there were probably attempts to indoctrinate you into the language of triangulated categories. The notion of a triangulated category is as old as that of the derived category. Both are due to Verdier, from his study of what we now call Verdier duality. He encountered the derived category, noticed in dismay that it was not an abelian category, and as such scrambled to find a good notion which would include the derived category as its example. He came up with triangulated categories, and since that notion pleased Grothendieck, everyone was pleased.

But triangulated categories aren't all that they're made out to be: there are some issues with the functoriality of cones and cocones, and also the definition itself includes the infamous octahedral axiom, while just being a version of the Second Isomorphism Theorem, is still far from the most obvious thing ever. Contrast this with the definition of a stable ∞-category, which is genuinely one of the most natural things ever.

Well, the point is that whenever $\mathcal{C}$ is a stable ∞-category, its homotopy category $\mathrm{h}\mathcal{C}$ carries a natural triangulated category structure. The shifts $[1]$ are defined to be the suspension functors Σ, the distinguished triangles are defined to be cofiber (or fiber; they agree) sequences, and everything else comes for free. Furthermore, just about every triangulated category that we have ever encountered actually extends in a canonical way to a stable ∞-category, the homotopy category of which it is. Thus perhaps a philosophy on stable ∞-categories might be that they, and not triangulated categories as Verdier thought, are the actual correct generalization of the notion of an abelian category.

1.2. Stabilization of the ∞-category $\mathcal{S}$

Most ∞-categories are not stable. Perhaps they might be missing some finite limits of colimits. But even if they do have them, such as the ∞-category $\mathcal{S}$ of spaces (or, if you want, ∞-groupoids), they might not have a zero object.

1.2.1. **Instability due to the Hopf fibration.** Passing to the ∞-category $\mathcal{S}_*$ of pointed spaces solves this issue, but it still isn't stable. Indeed, the functors Σ and Ω are not equivalences of pointed spaces - for instance, while there exist no non-trivial maps $S^1 \to S^0$, suspending twice leads us to consider maps $S^3 \to S^2$. Here we have a famous example of a homotopically non-trivial map, the Hopf fibration, which may be constructed for instance as

$$S^3 \subset \mathbf{C}^2 - \{0\} \to (\mathbf{C}^2 - \{0\})/\mathbf{C}^\times = \mathbf{CP}^1 \simeq S^2,$$

where the inclusion identifies the 3-sphere as the unit sphere of $\mathbf{R}^4 = \mathbf{C}^2$.

1.2.2. **Brute force inverting the functor Ω.** But now that we know what the issue is, it isn't hard to fix it. We want $\Omega : \mathcal{S}_* \to \mathcal{S}_*$ to be an equivalence? Fine, we can make it be, by passing to

$$\mathrm{Sp} = \varprojlim(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \cdots)$$

and this is the famous ∞-category of spectra. This is precisely analogous to how one forms localizations of rings in commutative algebra, i.e. how we invert some elements in a ring.

1.2.3. **Classical interpretation I: Ω-spectra.** Unwinding the definition, a spectrum X may thus (up to an appropriate notion of homotopy equivalence of spectra) be presented as a sequence $\{X_i\}_i$ of pointed spaces together with homotopy equivalences $X_i \simeq \Omega X_{i+1}$ for all i. This is one of the classical definitions of spectra, called Ω-spectra in the older literature.

Note also that the structure maps give rise by adjunction to pointed maps $\Sigma X_i \to X_{i+1}$, conjuring to mind an even more classical definition of spectra and likely the first definition you've ever seen. Let us call those sort of objects *sequential spectra*. The issue with sequential spectra is that it is quite hard to make sense of what the correct notion of homotopy equivalence is in that case to present the same objects (or even to obtain a notion of a map of spectra). Of course, given a sequential spectrum, which is to say a sequence of spaces $\{X_i\}$ with structure maps $\Sigma X_i \to X_{i+1}$, we may obtain a homotopy equivalent Ω-spectrum $\{X_i'\}$ by setting

$$X_i' = \varinjlim_k \Omega^k X_{i+k},$$

with the sequential colimit on the right coming from the structure maps of the sequential spectrum by the adjunction $\Sigma \dashv \Omega$.

1.2.4. **Classical interpretation II: infinite loop spaces.** Another perspective on spectra that the above definition might suggest to us is that of infinite loop spaces. Indeed, what should that be? Well, likely a (pointed) space X_0, such that there exists another space X_1 for which $X_0 \simeq \Omega X_1$, and for which there exists another space X_2 for which $X_1 \simeq \Omega X_2$, etc. Surely we're collecting precisely the data of an Ω-spectrum.

But there is a slight difference: in an Ω-spectrum there is no natural space X_0 to begin with. In effect, there exists also further deloopings $X_{-1}, X_{-2}, \dots of X_0$. Indeed, there are more spectra than there are infinite loop spaces, and the latter account only (but precisely) for the connective ones.

1.2.5. **The functors Ω^∞ and Σ^∞.** In light of the previous subsection, for any spectrum X, the component space X_0 is an infinite loop space. This might explain why we denote the functor $X \mapsto X_0$ by $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ and call it the *underlying infinite loop space functor*. Often times, one skips the "infinite loop" part.

Another justification for the notation Ω^∞ is that this functor admits a left adjoint $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$. The latter functor is most easily described as a sequential spectrum, associating to a pointed space X the sequence $\{\Sigma^i X\}_i$ with structure maps $\Sigma(\Sigma^i X) \simeq \Sigma^{i+1} X$ (and then passing via the procedure desribed in 1.2.3 to the associated Ω-spectrum). The spectrum $\Sigma^\infty X$ so-obtained is called the *suspension spectrum* of the pointed space X, and its key example is $S := \Sigma^\infty S^0$, the *sphere spectrum*. That's a good name, since its constituent spaces are $\Sigma^i S^0 \simeq S^i$, the spheres.

Some (a non-overwhelming majority of) people use $\mathbb{S}$ to denote the sphere spectrum, in analogy to using $\mathbb{Z}$ to denote the integers. But since I prefer $\mathbf{Z}$ for the latter, and also because I prefer to agree with the majority in writing the sphere spaces as S^i instead of $\mathbb{S}^i$, I will follow Lurie and the current trends in just using S (but this is one of those things where I don't really have any issues with either side).

Another, slightly less exciting example of a suspension spectrum, is that of a point. Since Σ^∞ preserves colimits, it preserves the zero objects, and so $\Sigma^\infty(*) \simeq 0$, the zero spectrum.

1.2.6. **A small variation on the theme of Σ^∞.** Instead of considering the adjunction between $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$ and $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$, we can pass to unpointed spaces along the forgetful functor $\mathcal{S}_* \to \mathcal{S}$. This forgetful functor obviously admits a left adjoint $X \mapsto X_+ = X \coprod *$ of adjoining a disjoint base-point. Combining the two adjunctions into one, we get that the composite functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ (forgetting the base-point of $\Omega^\infty X$) has a left adjoint $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$. It sends a (non-pointed) space X to its suspension spectrum $\Sigma^\infty_+ X = \Sigma^\infty(X_+)$.

In various contexts, a more evocative notation for $\Sigma^\infty_+ X$ is $S[X]$ (we will freely switch between both). This is supposed to evoke the idea of group algebras. Indeed, jumping ahead a little, if X possesses a homotopy group structure (more precisely, is a $\mathbb{E}_1$-group), this equips $S[X]$ with the structure of a $\mathbb{E}_1$-ring, literally being the group algebra over S.

The functor $S[-] : \mathcal{S} \to \mathrm{Sp}$ also admits a more explicit description. This comes about since $\mathcal{S}$ is the free ∞-category generated by a single object under colimits (this may be seen an incarnation of all spaces admiting CW complex representatives, and the latter begin gluings of spheres along discs, since $S^0 = * \coprod *$ and $S^n \simeq \Sigma^n S^0$ are all just colimits of points, and gluing is also a colimit). Since the suspension spectrum $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ commutes with colimits, it is therefore essentially uniquely determined by where it sends the point. As $\Sigma^\infty_+(*) \simeq \Sigma^\infty S^0 \simeq S$, the suspension spectrum of a general space X may be identified with the colimit

$$S[X] \simeq \varinjlim_X S$$

of the trivial diagram $X \to \mathrm{Sp}$ (where we view X as an ∞-groupoid) with constant value S.

1.2.7. **How about inverting Σ?** The question in the title of this subsection seems very sensible. We know that stability can be characterized by either of Σ or Ω being an equivalence (as the other, being its adjoint, will become one also automatically). In 1.2.3 we constructed the ∞-category of spectra from spaces by inverting Ω. Could we have analogously inverted suspension?

The problem is that imitating 1.2.3 with Σ in place of Ω (and revering arrows) will result in an ∞-category missing some limits. Nonetheless, there is a way around this, but it requires us to be a little more clever. As insane as it seems, this was actually the historically first way of constructing spectra (modulo the ∞-business).

Let $\mathcal{S}^\omega_*$ denote the ∞-category of compact objects in spaces. Then $\mathcal{S} \simeq \mathrm{Ind}(\mathcal{S}^\omega)$ (i.e. spaces are compactly generated - this is roughly what presentability technically boils down to), and we plan to imitate this in the stable world. We define the *Spanier-Whitehead* ∞-category of finite spectra as

$$\mathcal{SW} = \varinjlim(\cdots \xrightarrow{\Sigma} \mathcal{S}^\omega_* \xrightarrow{\Sigma} \mathcal{S}^\omega_* \xrightarrow{\Sigma} \cdots).$$

This evidently achieves the dream of "inverting suspension", though only at the level of compact spaces. Then the ∞-category of spectra may be recovered as $\mathrm{Sp} \simeq \mathrm{Ind}(\mathcal{SW})$.

1.3. Stabilization of a general ∞-category

The stabilization procedure applied in section 1.2.2 to $\mathcal{S}$ to obtain Sp applies to other ∞-categories as well.

1.3.1. **Stabilization by inverting Ω.** Indeed, let $\mathcal{C}$ be an ∞-category with all finite limits. In particular, it has a terminal object $* \in \mathcal{C}$, so we may pass to its pointification $\mathcal{C}_* := \mathcal{C}_{*/}$, i.e. consider pointed objects. The ∞-category $\mathcal{C}_*$ has $*$ as a zero object, and since $\mathcal{C}$ has finite limits, so does $\mathcal{C}_*$. It in particular has based loops $\Omega X = * \times_X *$, allowing us to form the *stabilization* of $\mathcal{C}$ as

$$\mathrm{Sp}(\mathcal{C}) = \varprojlim(\cdots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \cdots).$$

As in the case of spaces, which obviously recovers spectra as $\mathrm{Sp}(\mathcal{S}) \simeq \mathrm{Sp}$, the ∞-category $\mathrm{Sp}(\mathcal{C})$ is stable. Furthermore it supports a limit-preserving functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ by projecting on a fixed factor $\mathcal{C}_*$ in the sequential colimit, entirely analogously to section 1.2.5 above. This functor expresses the universal property of stabilization.

Definition 2. Let $\mathcal{C}$ be an ∞-category with all finite limits. The stabilization of $\mathcal{C}$ is an initial object among limit-preserving functors $\mathcal{D} \to \mathcal{C}$ into stable ∞-categories $\mathcal{D}$.

More precisely, if $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is the stabilization and $F : \mathcal{D} \to \mathcal{C}$ is any limit-preserving functor from a stable ∞-category, then there exists an essentially unique exact functor $\hat{F} : \mathcal{D} \to \mathrm{Sp}(\mathcal{C})$ with a coherent homotopy $\hat{F} \circ \Omega^\infty \simeq F$.

This should make sense: the stabilization is the closest stable ∞-category to the one you are starting with. The above construction of $\mathrm{Sp}(\mathcal{C})$ may be viewed as constructing an explicit model for constructing a stabilization $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ satisfying the universal property in the definition. We will mention two other approaches to construct stabilizations in the following sections.

An obvious observation based on the definition is that the process of stabilization leaves those ∞-categories which are already stable unchanged. More precisely, if $\mathcal{C}$ is stable, then the identity functor exhibits it as its own stabilization.

1.3.2. **So that's Ω^∞, but where is Σ^∞_+?** In analogy with the case of spaces in subsection 1.2.5 above, you might wonder whether the functor Ω^∞ admits an adjoint, which we would be tempted to denote Σ^∞.

Well, the functor Ω^∞ commutes with all limits by construction, so the answer will surely be affirmative by the Adjoint Functor Theorem, provided we add the assumption that the ∞-category $\mathcal{C}$ is presentable. In that case, its stabilization $\mathrm{Sp}(\mathcal{C})$ will also be presentable.

In that case, the functor $\Sigma^\infty_+ : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ (defined as the left adjoint to Ω^∞) may be used to characterize stabilization via a universal property, just like Ω^∞ was in 1.3.2. The only difference is that now we are mapping stable ∞-categories into $\mathcal{C}$, and the functors preserve colimits instead of limits.

1.3.3. **The presentable case.** As suggested in the previous paragraph, having all limits and colimits at one's disposal is useful. Thus it's convenient to consider the ∞-categories $\mathcal{P}\mathrm{r}^L$ and $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$ of presentable and presentable stable ∞-categories respectively, with colimit-preserving morphisms between them (i.e. with left adjoints as morphisms - hence the superscript L). If we analogously took limit-preserving morphisms (i.e. right adjoints), we would get $\mathcal{P}\mathrm{r}^R$ and $\mathcal{P}\mathrm{r}^R_{\mathrm{st}}$.

Now stabilization of presentable ∞-categories may be expressed neatly as a right adjoint to the inclusion $\mathcal{P}\mathrm{r}^L_{\mathrm{st}} \to \mathrm{Pr}^L$ or equivalently, the left adjoint to the inclusion $\mathrm{Pr}^R_{\mathrm{st}} \to \mathcal{P}\mathrm{r}^R$. The unit and counit of these two indicated adjunctions are garnered by the functors Σ^∞_+ and Ω^∞. This should come as no shock - it is a general truth that working with presentable ∞-categories makes "large scale" category-theoretic nonsense easier.

Also, since any presentable ∞-category is compactly generated (and that is essentially the definition of presentability), we can play a game analogous to subsection 1.2.7 to obtain an alternative description of stabilization as

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Ind}\big(\varinjlim(\cdots \xrightarrow{\Sigma} \mathcal{C}^\omega \xrightarrow{\Sigma} \mathcal{C}^\omega \xrightarrow{\Sigma} \cdots\big),$$

analogous to the Spanier-Whitehead approach to defining spectra.

1.4. Weren't spectra supposed to be cohomology theories?

So far we've been doing little more in this section 1.3 than observing what in the previous section 1.2 works for general ∞-categories. But though we have offered a few different perspectives on spectra (and stabilization more generally) already, we are missing an important one. Possibly the first one that one gets to hear: "spectra are cohomology theories". Let's see how this works.

1.4.1. **What even is a (co)homology theory.** It is more convenient to work with homology than cohomology theories, so let us do that. Traditionally (see below the axiomatization of Eilenberg-Steenrod, for the purposes of enabling the statement of which the whole field of category theory was created!!!) these consist of a functor from certain kinds of topological spaces into chain complexes, satisfying certain properties such as homotopy invariance and excision (a fancy, albeit more descriptive name for what amounts essentially to Mayer-Vietoris). The precise choice of details varies a little depending on whether we are asking for a homology theory on all topological spaces or only say finite CW complexes, whether we are looking at reduced, non-reduced theories, theories on pairs, etc.

Let us commit ourselves to the reduced setting and where we are trying to evaluate homology theories on finite CW complexes.

Definition 3 (Eilenberg-Steenrod axioms)**.** An (reduced extraordinary) homology theory is a sequence of functors $E_i : \mathcal{CW}_*^{\mathrm{fin}} \to \mathcal{A}\mathrm{b}$ that satisfies the following properties:

(1) (Homotopy invariance) A homotopy equivalence of pointed finite CW complexes $f : X \simeq Y$ induces an isomorphism $E_i(f) : E_i(X) \cong E_i(Y)$ for all i.
(2) (Additivity) For any two finite CW complexes X and Y, the canonical map exhibits the isomorphism $E_i(X \vee Y) \cong E_i(X) \oplus E_i(Y)$ for all i.
(3) (Suspension) For any pointed finite CW complex X there is a natural isomorphism $E_{i+1}(\Sigma X) \cong E_i(X)$ for all i.
(4) (Exactness) For any map of pointed finite CW complexes $f : X \to Y$, let $\mathrm{cofib}(f)$ denote its homotopy cofiber (classically called the mapping cone). Then the sequence
$$E_i(X) \to E_i(Y) \to E_i(\mathrm{cofib}(f))$$
is exact for all i.

Here in the additivity axiom the "wedge" of pointed spaces $X \vee Y$ is obtained by pinching together the spaces X and Y at the basepoints, and is the coproduct in $\mathcal{CW}_*^{\mathrm{fin}}$. You may be used to a "long exact sequence of a pair" axiom there, but it is not hard to derive it from the combination of Axioms 3. and 4.

In the next few subsections, we embark on an journey to find an ∞-categorical refinement of the above definition, which will ultimately yield another approach to stabilization. Firstly, since a homology theory is supposed to be homotopy invariant (Axiom 1. above), we should take the domain to be an appropriate ∞-category of finite spaces.

1.4.2. **Finite spaces vs compact spaces.** You might half-expect that we should take finite spaces here to mean $\mathcal{S}^\omega$, the compact spaces. Alas, this is incorrect. Unlike what you might expect, finite spaces, which is to say spaces homotopy equivalent to a finite CW-complex, do not coincide with the categorically compact spaces. There are, as it turns out, more of the latter. The failure of compact space to be finite is measured by Wall's finiteness obstruction, a topic of deep and surprising connection to manifold theory, that we will say nothing more about.

So let $\mathcal{S}_*^{\mathrm{fin}}$ denote the ∞-category of finite spaces. Technically speaking, just as $\mathcal{S}$ is the free ∞-category spanned by colimits from one object $*$, so we get $\mathcal{S}^{\mathrm{fin}}$ by using only finite colimits (surely you can formulate this as a universal property for yourself, if you want).

Further let $\mathcal{S}^{\mathrm{fin}}_*$ denote the pointed objects in $\mathcal{S}^{\mathrm{fin}}$, an instance of pointification as discussed in subsection 1.3.1. This is going to be the domain ∞-category for our homology theories.

1.4.3. **Upgrading to taking values in chain complexes.** Traditionally homology theories take values in abelian groups, but we can demand instead that there exists a chain complex $C_*(X;E)$ from which the homology groups will be obtained as $E_i(X) = \mathrm{H}_i(C_*(X;E))$, where H_i on the right denotes ordinary chain complex homology. Of course the chain complex $C_*(X;E)$ satisfying this will not be unique, but it will be unique up to quasi-isomorphism (almost by definition: we're specifying its homology groups!). Trying to work ∞-categorically, that's all we can either expect or want anyway.

In light of "chains for the homology theory" perspective, the suspension axiom (Axiom 3. above) amounts to requiring that $C_*(\Sigma X;E)[-1] \simeq C_*(X;E)$ (homological indexing on shifts), and the additivity axiom (Axiom 2. above) to $C_*(X \vee Y;E) \simeq C_*(X;E) \oplus C_*(Y;E)$.

1.4.4. **Dold-Kan says spaces of chains are fine too.** By the Dold-Kan philosophy, equating chain complexes and simplicial abelian groups, we could just as well work with chains valued in spaces instead.

So we will want associate to every finite space X a "space of chains" $E(X)$, corresponding by Dold-Kan (this is more of an analogy than an actual theorem) to $C_*(X;E)$. In particular, since H_i corresponds to π_i through Dold-Kan, the homology groups should be expressible as $E_i(X) \simeq \pi_i E(X)$.

This will technically only work for $i \geq 0$, but the next subsection shows that by passing to a high enough suspension, we will be able to make sense of $E_{-N}(X)$ too for arbitrarily big N.

1.4.5. **Suspension axiom rephrased.** The key thing is that, since using the $\Sigma \dashv \Omega$ adjunction and the fact that $\Sigma S^i \simeq S^{i+1}$ gives the isomorphism

$$\pi_i(\Omega Y) = \pi_0 \mathrm{Map}_{\mathcal{S}_*}(S^i, \Omega Y) \cong \pi_0 \mathrm{Map}_{\mathcal{S}_*}(\Sigma S^i, Y) = \pi_{i+1}(Y),$$

the suspension axiom (Axiom 3. above) will from the perspective of "spaces of chains" amount to requiring that

Definition 4 (Suspension Axiom)**.** For any pointed finite space X the canonical map exihbits an equivalence

$$\Omega E(\Sigma X) \simeq E(X).$$

Equivalently, Dold-Kan exchanges the shift $[-1]$ of chain complexes with loops of based spaces, and then this becomes this becomes the chain complex formulation of the axiom that we saw in subsection 1.4.3.

BTW, this is why we could get away with only taking spaces of chains, instead of having to take simplicial abelian groups. Because for any $X \in \mathcal{S}^{\mathrm{fin}}_*$, the equivalence $E(X) \simeq \Omega^i E(\Sigma^i X)$ equips the space of chains with a $\mathbb{E}_i$-structure for all i. These structures are compatible with each other, making $E(X)$ ultimately into a $\mathbb{E}_\infty$-space. Thus its homotopy groups are all abelian groups, and we are good.

1.4.6. **Additivity axioms rephrased.** So far we have decided that ∞-categorical homology theories should be functors $E : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{S}$ (this already subsumes the homotopy invariance axiom) that satisfy certain properties, and we have identified the suspension axiom. It remains to identify the additivity and exactness axioms (Axioms 2. and 4. above), and we do so in this section and the next.

The additivity axiom, as remarked in subsection 1.4.3, translates on the level of chain complexes to the claim that $X \mapsto C_*(X;E)$ takes $\vee$ to $\oplus$. But the direct product $\oplus$ is both a product and a coproduct in abelian groups, so it might be unclear in which role it is appearing here. The Dold-Kan shift of gears is once again useful: now we are talking about spaces, and we need to decide whether $E(X \vee Y)$ should be $E(X) \times E(Y)$ or $E(X) \coprod E(Y)$.

Note that homotopy groups of spaces satisfy $\pi_i(Y \times Y') = \pi_i(Y) \oplus \pi_i(Y')$, so the first choice will surely do.

On the other hand, coproducts would backfire something nasty. Indeed, a homotopy group always implicitly involves a base-point, so it only knows (excluding $i = 0$, of course) only about the connected component of the base-point. Not only that, even supposing you chose to work in the based setting (which we are not in the codomain of E!) and could take the wedge $\vee$ to fix the issue, it would still not be good. To see that, remember from your basic alg top class that the most trivial corrolary of Van Kampen's theorem computes the fundamental group of $S^1 \vee S^1$ as the free group on two generators - quite far from the friendly $\mathbf{Z} \oplus \mathbf{Z}$!

This all leads us to state the additivity axiom as:

Definition 5 (Additivity axiom)**.** For any two pointed finite spaces X and Y, the canonical map exhibits a homotopy equivalence $E(X \vee Y) \simeq E(X) \times E(Y)$.

There is technically a wee bit more encoded in the version of Additivity encoded in Axiom 2. above. That is, choosing one of the spaces to be a point, it allows us to conclude that $E_i(X) \simeq E_i(X) \oplus E_i(*)$ for all i and all X, and as such that $E_i(*) = 0$. This is indeed the defining property of a reduced homology theory. The space-of-chains level Additivity statement fails to take this into account, so we will need to impose it additionally at the end in 1.4.8.

1.4.7. **Exactness axiom rephrased.** To phrase exactness, we need to decide if we are viewing the exactness of the sequence appearing in Axiom 4. above as a statement about kernels or as a statement about cokernels. In the former case, the spaces-of-chains-level statement will involve the (homotopy) fiber, in the latter a the (homotopy) cofiber.

For this, note that $Y \mapsto \pi_0 \mathrm{Map}_{\mathcal{S}_*}(S^i, Y)$ exibits homotopy groups as a glorified Hom functor. Since the second factor of Hom takes limits to limits (in the ∞-categorical sense here, but then π_0 returns their un-derived ordinary analogues), we find that $\pi_i(\mathrm{fib}(f)) \cong \mathrm{Ker}(\pi_i(f))$. So choosing the fiber will surely work.

Conversely, picking the cofiber would not work, as the covariant Hom does not preserve colimits (it does preserve filtered ones because S^i is compact, but not cofibers). Combining what we have figured out, the exactness axiom may be phrased as:

Definition 6 (Exactness axiom)**.** For any map of pointed finite spaces $f : X \to Y$ the canonical map

$$E(X) \to \mathrm{fib}(E(Y) \to E(\mathrm{cofib}(f)))$$

induced by f is a homotopy equivalence.

1.4.8. **Combining the Axioms 2. − 4. into one.** One nice thing is that all three of the suspension, additivity, and exactness axioms can be elegantly stated simultaneously in this "space of chains" context. Note that, since $\vee$ is the coproduct in $\mathcal{S}_*^{\mathrm{fin}}$, all three statements are about evaluating E on colimits and obtaining limits. In fact, here is a common way to generalize all three of them:

Definition 7. A functor $F : \mathcal{C} \to \mathcal{D}$ is excisive if for every diagram of the form $Y \leftarrow X \to Z$ in $\mathcal{C}$, the canonical map $F(X) \to F(Y) \times_{F(Y \coprod_X Z)} F(Z)$ is an equivalence in $\mathcal{D}$. That is to say, F sends pushout squares in $\mathcal{C}$ to pullback squares in $\mathcal{D}$.

An excisive functor $E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ is now almost the same as the above properties, only with, one difference: we have no control over the space $E(*)$, and if it is not contractible, we will get different things. So let us suppose that indeed (as follows from the Eilenberg-MacLane axioms anyway) that the functor E is *pointed*, in the sense that it sends to zero object $* \in \mathcal{S}_*^{\mathrm{fin}}$ to the terminal object $* \in \mathcal{S}$.

To show that a pointed excisive functor $E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ satisfies the above-stated Additivity axiom, consider the diagram $X \leftarrow * \to Y$ in $\mathcal{S}_*^{\mathrm{fin}}$, to show the Exactness axioms, consider

$* \leftarrow X \xrightarrow{f} Y$, and finally for Suspension, consider the diagram $* \leftarrow X \to *$. It is also quite clear that, if the conclusion of excision holds for these three types of diagrams, they will hold for all - this is roughly due to pointedness giving us access to $*$, the third diagram giving us access to suspension, and so now we can build any S^i, and finally the first two diagrams give us access to coproducts and pushouts, which together generate all finite colimits. Since any space in $\mathcal{S}_*^{\mathrm{fin}}$ is built out of finite colimits of spheres, we are golden.

1.4.9. **Stabilization as excisive functors.** In summary, a good ∞-categorical notion of a homology theory is a pointed excisive functor $E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$. If we denote the full subcategory that the latter span in $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$ by $\mathrm{Exc}_*(\mathcal{S})$, then there is a canonical equivalence of ∞-categories $\mathrm{Exc}_*(\mathcal{S}) \simeq \mathrm{Sp}$. Its inverse is given by sending a spectrum E into the functor $X \mapsto E[X] := E \otimes S[X]$, where $\otimes$ stands for the as-of-yet-unmentioned smash product of spectra. Alternatively, in line with subsection 1.2.6, we could have also directly defined $E[X] \simeq \varinjlim_X E$, the colimit of the constant diagram from the ∞-groupoid X into spectra with constant value E.

Of course there was nothing special about the ∞-category $\mathcal{S}$ here. If $\mathcal{C}$ is an ∞-category with all finite colimits, letting $\mathrm{Exc}_*(\mathcal{C})$ denote the ∞-category of pointed excisive functors $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$, there is an identification $\mathrm{Exc}_*(\mathcal{C}) \simeq \mathrm{Sp}(\mathcal{C})$. Thus the stabilization of an ∞-category always amounts to considering homology theories on pointed finite spaces with values in said ∞-category.

In fact, the equivalence $\mathrm{Exc}_*(\mathcal{C}) \simeq \mathrm{Sp}(\mathcal{C})$ may be proved without insane difficulty by checking that the functor $\mathrm{Exc}_*(\mathcal{C}) \to \mathcal{C}$, sending $E \mapsto E(S^0)$, satisfies the universal property for the stabilization functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$. From the perspective of homology theories and spaces of chains, the underlying infinite loop space of a spectrum E is expressed as $\Omega^\infty E \simeq E(S^0)$.

While the excisive functor description of stabilization may seem like the most arcane among the several approaches we have so far seen, it can in fact be the most useful one for proving various fun abstract things about stabilization - see Higher Algebra for a spectacular demonstration.

1.4.10. **Recovering an Ω-spetrum from an excisive functor.** Let us say a few words about how to recover an Ω-spectrum from an excisive functor E. Since an excisive functor satisfies the rephrased Suspension Axiom that we gave in subsection 1.4.6, we have $\Omega E(\Sigma X) \simeq E(X)$ for any X. Choosing $X \simeq S^0$ and iterating, we find that $E(S^i) \simeq \Omega^i E(S^0)$. Thus we obtain an Ω-spectrum by setting $E_i := E(S^i)$, and this is the Ω-spectrum that represents the same spectrum as the excisive functor E.

It might at first sight seem highly implausible that the collection of spaces $E(S^i)$ together with the equivalences $\Omega^i E(S^j) \simeq E(S^{j-i})$ for all $j \geq i$ should determine the values $E(X)$ for any finite pointed space X, let alone the functoriality of the whole excisive functor E. Alas, excisiveness is a strong condition.

The point is that any space may be obtained by gluing together dijsoint unions of spheres and filling them in by discs (that is to say, any space may be presented as a CW complex), and since $S^i \simeq \Sigma^i S^0$ is a colimit, and taking disjoint unions and gluing are also colimits, this means that any pointed space may be built from S^0 by (homotopy) colimits. Indeed, we defined finite spaces as those whih may be obtained by finite colimits! Now, any finite colimit may be obtained as a sequence of pushouts, and the excisiveness coniditon tells us how to evaluate E on pushouts. Hopefully this makes the claim that the Ω-spectrum $E_i \simeq E(S^i)$ determines the whole excisive functor E less surprising.

1.4.11. **Brown's Representability Theorem.** We saw that spectra (or more general stabilization of an ∞-catgory) may be expressed in terms of excisive functors. We encountered the latter by discussing a way to rephrase the notion of a homology theory in an ∞-categorically-friendly way. By taking homotopy groups of an excisive functor, it is not too hard to make a homology theory out of one. In fact, there is an essentially unique way of doing that - this is the content of the following celebrated *Brown Representability Theorem.*

Theorem 8 (Brown). *Let $E_i : \mathcal{CW}_*^{\text{fin}} \to \mathcal{A}\text{b}$ be a homology theory, i.e. let it satisfy the Eilenberg-Steenrod axioms. Then there exists an Ω-spectrum $\{E_i\}_{i\in\mathbf{Z}}$ such that*

$$E_i(X) \simeq \varinjlim_k \pi_{i+k}(E_k \wedge X)$$

for any finite pointed CW complex X.

Here $E_k \wedge X$ denotes the smash product of the pointed spaces E_k and X - see subsection 1.5.1 for a review of that basic operation on pointed spaces. The colimit, which ranges as $k \to \infty$ and makes sense at least for $k \geq -i$, is taken along homomorphisms

$$\pi_{i+k}(E_k \wedge X) \simeq \pi_{i+k}(\Omega E_{k+1} \wedge X) \to \pi_{i+k}(\Omega(E_{k+1} \wedge X)) \simeq \pi_{i+k+1}(E_{k+1} \wedge X),$$

where the first equivalence comes from the structure maps of the Ω-spectrum, the second map is induced by the smash product as

$$\Omega Y \wedge X \simeq \mathrm{Map}_{\mathcal{S}_*}(S^1, Y) \wedge \mathrm{Map}_{\mathcal{S}_*}(S^0, X) \to \mathrm{Map}_{\mathcal{S}_*}(S^1 \wedge S^0, X \wedge Y) \simeq \Omega(Y \wedge X),$$

and the final map takes into account that $\pi_n(\Omega Y) \simeq \pi_{n+1}(Y)$.

The Ω-spectrum $\{E_i\}$ in the statement of the Theorem turns out to be unique up to homotopy equivalence. Conversely given any Ω-spectrum $\{E_i\}_{i\in\mathbf{Z}}$, we may define $E_i(X) := \pi_0(E_i \wedge X)$ just as in the Theorem statement to obtain a homology theory.

Thus spectra and homology theoreories are in bijection up to homotopy equivalence, perhaps leading one to wonder why we consider the more complicated spectra in the first place. The answer is that while they have the same objects, the category of homology theories is allegedly[1] a mess, while the category of spectra hSp (and even more so the associated ∞-category Sp) is terrifically well-behaved - the whole field of stable homotopy theory is a justification of this claim.

In fact, Brown's Representability Theorem is a little stronger than the statement we gave above. It shows that some light conditions on a functor $F : \mathcal{CW}_*^{\text{fin}} \to \mathcal{S}\text{et}$ guarantee that there exists a space Y such that $F(X) \simeq \pi_0(Y \wedge X)$ for all finite pointed CW-complexes X. When F is a component functor of a homology theory, these spaces Y together assemble into an Ω-spectrum. The dual statement is where the theorem gets its name: under some light conditions on a functor $F : (\mathcal{CW}_*^{\text{fin}})^{\text{op}} \to \mathcal{S}\text{et}$, there exists a space Y such that $F(X) \simeq \pi_0\mathrm{Map}_{\mathcal{S}_*}(X, Y)$. Since we have $\pi_0\mathrm{Map}_{\mathcal{S}_*}(X, Y) = \mathrm{Hom}_{\mathcal{CW}_*}(X, Y)$, this is a representability result.

This contravariant version of Brown's Representability specializes to give representability of any cohomology theory by spectra as well. We will not define cohomology theories fully, instead remarking that they are contravariant functors required to satisfy a similar set of Eilenberg-Steenrod axioms as we listed in 1.4.1 for their covariant homology theory cousins.

Theorem 9 (Brown). *Let $E^i : (\mathcal{CW}_*^{\text{fin}})^{\text{op}} \to \mathcal{A}\text{b}$ be a cohomology theory. Then there exists an Ω-spectrum $\{E_i\}_{i\in\mathbf{Z}}$ such that $E^i(X) \simeq \pi_0\mathrm{Map}_{\mathcal{S}_*}(X, E_i)$ for any finite pointed CW complex X.*

We mentioned this cohomological version of Brown's Representability because we will use it extensively in our discussion of topological K-theory in section 2.2.

In the language of the ∞-category of spectra, the conclusion of Brown's Representability Theorem may be rephrased as saying that there exists a spectrum $E \in \mathrm{Sp}$ such that in the

[1]This is an oft-repeated claim that is rarely substantiated. In an interesting MathOverflow post, Peter May claims that this was already a classical folklore fact the time he was in grad school.

cohomological case

$$E^i(X) \simeq \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty X, \Sigma^i E) \simeq \pi_{-i}\underline{\operatorname{Map}}_S(\Sigma^\infty X, E)$$

where $\underline{\operatorname{Map}}_S$ denotes the internal mapping spectrum, and in the homological case that

$$E_i(X) \simeq \pi_i(\Sigma^\infty X \otimes E),$$

where $\otimes$ denotes the smash product of spectra, to be discussed next.

1.5. Smash product of spectra

One of the key structures that spectra should come equipped with is the smash product.

1.5.1. **Smash product of spaces.** The smash product of spectra should be compatible (and extend) the smash product of pointed spaces, where recall that it is defined as $X \wedge Y = (X \times Y)/(X \vee Y)$ - take the product, and then pinch everything that has either X or Y's base-point on either coordinate together into a single base-point.

An important thing to note is that, while equipping the ∞-category $\mathcal{S}_*$ with a symmetric monoidal structure, the smash product is not actually the categorical product in it. It nonetheless is a categorically meaningful construction: for any $X, Y, Z \in \mathcal{S}_*$, there is a natural equivalence

$$\operatorname{Map}_{\mathcal{S}_*}(X \wedge Y, Z) \simeq \operatorname{Map}_{\mathcal{S}_*}(X, \operatorname{Map}_{\mathcal{S}_*}(Y, Z)),$$

where the mapping space $\operatorname{Map}_{\mathcal{S}_*}(Y, Z)$ is pointed with the constant map to the base-point of Z as the base-point. This is the sense in which $\wedge$ is the"correct" sort of product to consider in the based setting.

1.5.2. **Yet another approach to spectra.** One reason to care about the smash product is that suspension may be expressed through it as $\Sigma X = S^1 \wedge X$. In that sense, passing from spaces to spectra is all about inverting the object S^1 with respect to the symmetric monoidal structure $\wedge$ on $\mathcal{S}_*$.

Suppose for a moment that we already have a well-developed theory of the smash product of spectra, making Sp into a symmetric monoidal ı-category $\mathrm{Sp}^\otimes$. Then the idea that spectra are all about inverting S^1 with respect to $\wedge$ is in fact a theorem:

Theorem 10 (Hovey). *The left adjoint functor $\Sigma^\infty_+ : \mathcal{S}_* \to \mathrm{Sp}$ exhibits the ∞-category of spectra as a localization $\mathrm{Sp} \simeq (\mathcal{S}_*)^\wedge[(S^1)^{-1}]$ of the presentably symmetric monoidal category $\mathcal{S}_*$ with respect to the smash product at the object S^1.*

More formally, that is to say that for every presentably symmetric monoidal ∞-category $\mathcal{C}^\otimes$ and symmetric monoidal left adjoint functor $F : \mathcal{S}^\wedge_* \to \mathcal{C}^\otimes$ for which the object $F(S^1)$ is invertible (admits an inverse with respect to $\otimes$), there exists an essentially unique symmetric monoidal left adjoint functor $\hat{F} : \mathrm{Sp}^\otimes \to \mathcal{C}^\otimes$ for which $F \simeq \hat{F} \circ \Sigma^\infty_+$.

Note that working in the setting of presentably symmetric monoidal ∞-categories is essential for this to work, i.e. the theorem is false in bigger categories. Also allow me to explain that "*presentably symmetric monoidal*" means that we are dealing with a presentable symmetric monoidal ∞-category equipped with a symmetric monoidal operation $(X, Y) \mapsto X \otimes Y$, which preserves colimits separately in each variable. This is a very common assumption to make on a symmetric monoidal ∞-category - presentably symmetric monoidal ones are to symmetric monoidal ones as presentable ∞-categories are to all ∞-categories.

I credit Hovey with the theorem, because he was the first one to make a similar statement work in a model categorical setting, and proposed an analogous procedure as a form of stabilization (though it often disagrees with "real" stabilization as we know it!). That said, there were additional difficulties in making this work in the ∞-categorical setting, where the result is due to Barthel and friends. They were also the ones to formalize how a similar procedure yields various analogues of spectra which are "richer" than just stabilization, such as motivic spectra and genuine equivariant spectra.

1.5.3. **A quick peak at the genuine world.** Since we're already here, let's just sketch roughly how this works in the genuine equivariant world, i.e. how the technique of the previous subsection gives rise to a good ∞-category of G-spectra. *Nothing in this subsection will have any bearing on the subsequent ones, and it can (and maybe should) safely be skipped.*

1.5.3.1. *Genuine equivariant spaces.* You start off with a finite group G (some of it goes through for a compact Lie group too, but let's not go there). The game we're playing is that we wish to keep track not just of G-equivariance, but of H-equivariance with respect to all subgroups $H \subseteq G$ at once. Slightly more formally, G-spaces can be made by gluing together G-equivariant cells of the form $\Sigma^i(G/H)$. Contrast this with just the $\Sigma^i(*) \simeq S^i$-shaped cells that we use when setting up usual homotopy theory. A theorem due to Elmendorf gives a slightly neater description of the ∞-category $\mathcal{S}_G$ of G-spaces as

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S})$$

where $\mathcal{O}_G$ is the orbit category of G, i.e. the full subcategory of $\mathcal{S}\mathrm{et}_G^{\mathrm{fin}}$ (the ordinary category of usual finite G-sets, i.e. finite sets with a G-action) spanned by "orbits" G/H for all subgroups $H \subseteq G$. The point to take away is mostly just that G-spaces make perfectly good sense as a nice presentable ∞-category.

1.5.3.2. *Pointed equivariant spaces and representation spheres.* Pointed G-spaces are easy: the ∞-category $\mathcal{S}_{G*}$ is obtained simply as the pointification of G-spaces, i.e. $\mathcal{S}_{G*} \simeq (\mathcal{S}_G)_{*/}$. They even carry a symmetric monoidal smash product $\wedge$ defined in an analogous way to the one in ordinary based spaces.

A key family of examples of based G-spaces is: take any finite-dimensional (real, or if you insist, orthogonal) representation V of G, and let S^V denote the one-point compactification of V. This is the *representation sphere* associated to the rep V, with the added compactifying point-at-infinty as the base-point. When $V = \mathbf{R}^i$ is the trivial G-rep, this recovers the ordinary sphere S^i, and just as usual spheres satisfy $S^i \wedge S^j \simeq S^{i+j}$, we have $S^V \wedge S^W \simeq S^{V \oplus W}$ in the G-world.

Is the fact that we have two different sorts of spheres running around, ones built from orbits and the others built from representations, bother you? Welcome to equivariant homotopty theory.

1.5.3.3. *Genuine equivariant spectra.* As part of the welcome package, please enjoy your complimentary definition of G-equivariant spectra:

Definition 11. The ∞-category of (genuine) G-spectra as a localization

$$\mathrm{Sp}_G \simeq (\mathcal{S}_{G*})^\wedge[\{(S^V)\}^{-1}_{V \in \mathrm{Rep}(G)}]$$

of the presentably symmetric monoidal category $\mathcal{S}_{G*}$ with respect to the smash product at the "multiplicative subset" of representation spheres.

Of course it would suffice to just invert all irreps (since rep spheres are as observed additive in the rep). And all irreps can be found as subreps of the one rep to rule them all, one rep to find them, one rep to bring them all, and in the darkness bind them: the regular representation $\mathbf{R}[G]$. In that sense, the regular representation sphere $S^{\mathbf{R}[G]}$ contains all the "possible shapes" (distinct irreps) that representation spheres can take. So if it tickles your pickle to only invert one representation sphere, then perhaps you will enjoy the description $\mathrm{Sp}_G \simeq (\mathcal{S}_{G*})^\wedge[(S^{\mathbf{R}[G]})^{-1}]$.

1.5.3.4. *Spectral Mackey functors.* Btw, just before we depart from these unwelcoming G-lands, let's mention another way of defining the ∞-category of G-spectra. For motivation, recall Elmendorf's theorem for the unstable G-spaces from above. We can rephrase it as $\mathcal{S}_G \simeq \mathrm{Fun}_\Sigma((\mathcal{S}\mathrm{et}_G^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{S})$, identifying G-spaces with finite coproduct preserving functors from finite G-sets to spaces. To get G-spectra, we make two changes: values should be taken in spectra instead of in spaces, and secondly we wish to keep track of more equivariance. The latter is encoded by replacing our domain ∞-category with the correspondence ∞-category $\mathrm{Corr}(\mathcal{S}\mathrm{et}_G^{\mathrm{fin}})$, which is in the local parlance known as the Burnside category of G. Then G-spectra amount to

$$\mathrm{Sp}_G \simeq \mathrm{Fun}_\Sigma(\mathrm{Corr}(\mathcal{S}\mathrm{et}_G^{\mathrm{fin}}), \mathrm{Sp}),$$

where coproduct preservation works the same before, with $\coprod$ giving a symmetric monoidal structure to $\mathrm{Corr}(\mathcal{S}\mathrm{et}_G^{\mathrm{fin}})$ (this sort of a game should be well familiar to you if you've ever looked into the "meat" of Gaitsgory-Rozenblyum, vol 1). This description of G-spectra is known in the field as "spectral Mackey functors", for what that's worth.

Honsetly, I don't really know why you would care about anything in this section. Maybe you like to see how some rep theoretic notions get (ab)used in random other fields? Maybe for mathematical culture? In any case, let us linger in this equivariant realm no longer - we have already stayed past our welcome!

1.5.4. **Desiderata for the smash product.** Back to sanity! While pointed spaces have a smash product, constructing a good smash product on spectra turned out to be quite a hurdle in the development of stable homotopy. In retrospect, this is tied to the classical workers in the field emphasizing the importance of "strict models", where all the homotopy coherence was (in various different highly intelligent ways) eliminated, and set-theoretic models could be employed. That is surely an approach to facilitate computations, but for abstract things such as have to do with homotopy coherence (homotopy limits and colimits, etc) it can be quite inconvenient.

Anyway, what should we demand of the smash product of spectra? It should be a symmetric monoidal structure $\otimes$ on the ∞-category Sp, for which the suspension spectrum functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$ will be symmetric monoidal. That is to say, we want that $\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty X \otimes \Sigma^\infty Y$, and that the sphere spectrum S is the unit for $\otimes$. In fact, we imagine the smash product as an analogue of the tensor product of modules, but where modules are over the sphere spectrum.

Furthermore we want the smash product to commute with colimits - the smash product should more accurately be an analogue of the derived tensor products, just as the ∞-category of spectra, being an ∞-category, will be the derived (DG-)category of S-modules.

1.5.5. **Out of thin air.** One of the most amazing things about the ∞-categorical approaches to stable homotopy theory (and its subsection that used to be known as "brave new algebra") as developed in Higher Algebra, is that the smash product comes almost entirely for free. This is in great contrast to previous approaches to spectra, where obtaining it was a major technical achievement. To get the smash product however, we need to consider some rather abstract nonsense, to which we dedicate the next few subsections.

1.5.6. **The Lurie tensor product.** The ∞-category of stable presentable ∞-categories $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$, just as well as the bigger $\mathcal{P}\mathrm{r}^L$ of not-necessarily stable ones, carries a particularly nice symmetric monoidal structure: the Lurie tensor product, given by

$$\mathcal{C} \otimes \mathcal{D} := \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D}).$$

Of course the Lurie tensor product is very natural. It fulfills in the context of $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$ or $\mathcal{P}\mathrm{r}^L$ respectively, the analogous universal property that the usual tensor product does in modules. More precisely, for any triple of (stable) presentable ∞-categories $\mathcal{C}, \mathcal{D}$, and $\mathcal{E}$ there is a canonical equivalence of ∞-categories

$$\mathrm{Fun}^L(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}^L(\mathcal{C}, \mathrm{Fun}^L(\mathcal{D}, \mathcal{E})).$$

This is nothing but a version of the venerated tensor-Hom adjunction. If you wish, you can further identify the functor ∞-category of functors with the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by all the functors $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits in each variable separately. In this sense, the Lurie tensor product, like the tensor product of modules, encodes "bilinear" maps in term of "linear" ones (where linearity in this context stands for colimit preservation). This makes among other things rather obvious the fact that $\otimes$ is symmetric.

1.5.7. **Stabilization as tensoring with spectra.** The magic of the Lurie tensor product is this: the ∞-category Sp is its unit in $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$. This, or actually something a bit more general, actually isn't hard to show either. In fact, let's do it!

Let $\mathcal{C}$ be an arbitrary presentable ∞-category. Now let us split things into "steps":

- *Step 1:* Let's actually spell out some details of the already-mentioned observation that the ∞-category $\mathcal{S}$ is freely generated under colimits from a single object. Indeed, we have $\mathcal{S} \simeq \mathcal{P}(*)$ where $\mathcal{P}$ denotes the presheaf functor $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, and the presheaf functor has the universal property that $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D})$ for any ∞-category $\mathcal{C}$ and any ∞-category with all colimits $\mathcal{D}$. That is to say, $\mathcal{P}(\mathcal{C})$ is the freely generated by colimits from $\mathcal{C}$, the universal arrow of this universal property of course being given by the Yoneda embedding.
- *Step 2:* Note that, via passing to opposite categories which switches left and right adjoints, we have

  $$\mathcal{C} \otimes \mathcal{S} \simeq \mathrm{Fun}^R(\mathcal{S}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}^L(\mathcal{S}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathcal{C},$$

  where the second-to-last equivalence follows from Step 1. We conclude that the ∞-category $\mathcal{S}$ is the unit for $\otimes$ in $\mathcal{P}\mathrm{r}^L$.
- *Step 3:* For any ∞-category $\mathcal{D}$ the pointification procedure as described in subsection 1.3.1 easily satisfies the property that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})_* \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}_*)$. The same holds if we adorn the functor ∞-categories with R (all these things are simple exercises in category theory). Applying this to $\mathcal{D} \simeq \mathcal{S}$, we find that

  $$\mathcal{C} \otimes \mathcal{S}_* \simeq \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*) \simeq \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_* \simeq (\mathcal{S} \otimes \mathcal{C})_* \simeq \mathcal{C}_*,$$

  where the last equivalence is where we used Step 2. Thus tensoring with $\mathcal{S}_*$ amounts to pointification of a presentable ∞-category.
- *Step 4:* Remember that $\mathrm{Sp} \simeq \varprojlim \mathcal{S}_*$, the limit of sequential diagram with all the maps Ω. Since the Hom preserves limits in its second factor (and the subscript R means we are looking at limit preserving functors too), this implies that for every presentable ∞-category we have

  $$\mathcal{C} \otimes \mathrm{Sp} \simeq \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \varprojlim \mathcal{S}_*) \simeq \varprojlim \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*) \simeq \varprojlim \mathcal{C}_* \simeq \mathrm{Sp}(\mathcal{C})$$

  where in the second-to-last term the limit is taken over suspension functors in the ∞-category $\mathcal{C}^{\mathrm{op}}$. The last equivalence is of course just the construction of stabilization from subsection 1.3.1.

  Since we know that $\mathrm{Sp}(\mathcal{C}) \simeq \mathcal{C}$ for any stable ∞-category $\mathcal{C}$, it follows that $\mathcal{C} \otimes \mathrm{Sp} \simeq \mathcal{C}$ for all $\mathcal{C} \in \mathcal{P}\mathrm{r}^L_{\mathrm{st}}$. We conclude, as promised that the ∞-category of spectra is the unit for the Lurie tensor product on $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$.

Btw, just note that Step 4. offers us yet another different contruction of stabilization for a presentable ∞-category. We have so many of those now, so many different perspectives, yay! :)

1.5.8. **Unveiling the smash product.** OK, now we know that Sp is the $\otimes$-unit in $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$. "Then what?!" I hear you say. "You promised me a smash product of spectra!". Alas, we have already obtained it, we just don't know yet!

Indeed, here is an obvious fact: if $\mathcal{C}$ is a symmetric monoidal ∞-category and $\mathbf{1} \in \mathcal{C}$ is a unit object, then there exists an essentially unique commutative algebra structure on $\mathbf{1}$. That is to say, we may view the symmetric monoidal unit as $\mathbf{1} \in \mathrm{CAlg}(\mathcal{C})$. Duh: a commutative algebra structure on $\mathbf{1}$ is about multiplication maps $\mathbf{1} \otimes \cdots \otimes \mathbf{1} \to \mathbf{1}$, but these objects are canonically equivalent, and this canonical equivalence may be chosen as the multiplication.

This is the most trivial and un-interesting category-theoretic observation, but here is shines: apply it to the unit Sp in the ∞-category $\mathcal{P}\mathrm{r}^L_{\mathrm{st}}$ with respect to the Lurie tensor product. Thus there is a canonical way in which $\mathrm{Sp} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^L_{\mathrm{st}})$, and the objects of the latter ∞-category may be canonically identified with the stable presentably symmetric monoidal ∞-categories (recall the latter notion from subsection 1.5.2). That is to say, Sp carries a canonical symmetric monoidal structure, which preserves colimits in each variable. We denote the symmetric monoidal operation by $\otimes$ and call it the smash product of spectra.

1.5.9. **Checking the desiderata.** The smash product constructed in this abstract way automatically preserves colimits in each variable. That is one of the desired properties for it, that we listed in subsection 1.5.4. The other one is that the supsnesion spectrum functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$ be symmetric monoidal with respect to the smash product on both sides. To verify this, we must delve yet a little more deeper into the abstract nonsense.

Note that, as we applied the argument in subsection 1.5.8 to the unit object $\mathrm{Sp} \in \mathcal{P}\mathrm{r}^L_{\mathrm{st}}$ with respect to the Lurie tensor product, so could we apply it entirely analogously to the $\mathcal{S}$ and $\mathcal{S}_*$, the Lurie-tensor-product-unit objects (according to Steps 2. and 3. in subsection 1.5.7) of $\mathcal{P}\mathrm{r}^L$ and $\mathcal{P}\mathrm{r}^L_*$ respectively. This equips $\mathcal{S}$ with its usual Cartesian product structure (that's easy enough to check). Now the symmetric monoidal structures on $\mathcal{S}_*$ and Sp are all of the same form, arising from the Lurie tensor product, which implies that the canonical functors between them in $\mathcal{P}\mathrm{r}^L$ will preserve it, i.e. be symmetric monoidal. By "canonical functors" here we mean the left adjoints (since we're working in $\mathcal{P}\mathrm{r}^L$ and not in $\mathcal{P}\mathrm{r}^R$) of the reverse "forgetful functors", i.e. the the pointification functor $\mathcal{S} \to \mathcal{S}_*$ given by $X \mapsto X_+$ and the suspension spectrum functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$.

To recognize the symmetric monoidal structure on pointed spaces, note that the smash product of spaces indeed satisfies the condition that $(X \times Y)_+ \simeq X_+ \wedge Y_+$ for any two un-based spaces X and Y (go ahead, draw a picture to convince yourself! Feel glorious for being able to draw a picture for a proof, as if you were some kind of a real topologist and not actually neck-deep down this ∞-categorical mess.). Since all spaces in $\mathcal{S}_*$ are built out of colimits from $S^0 \simeq *_+$, e.g. the spheres are obtainable as $S^n \simeq \Sigma^n S^0$, a colimit if ever there was one, the condition that the symmetric monoidal structure on $\mathcal{S}_*$ preserves colimits in each variable (automatic due to working in the presentable setting, as explained in subsection 1.5.8) implies that what happens to X_+ for all spaces $X \in \mathcal{S}$ (and furthermore sufficiently just for $X \simeq *$) entirely deterimes the symmetric monoidal structure. Thus it follows that the symmetric monoidal structure obtained on $\mathcal{S}_*$ is just the usual smash product of pointed spaces.

This means that the suspension spectrum construction is $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$ is symmetric monoidal with respect to smash product on both sides. That amounts to the claim that $\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty X \otimes \Sigma^\infty Y$ for all pointed spaces X and Y, as well as the claim that the sphere spectrum $S \simeq \Sigma^\infty S^0$ is the unit for the smash product of spectra. Thus we have fulfilled all the desired conditions of subsection 1.5.4.

1.6. Why did people ever come up with spectra?

These days, the study of spectra and stable ∞-categories more generally might be seen (and is so seen by many of the practitioners) as its own end. Spectra are rich and interesting objects with deep and fundamental connections to homological algebra, the study of manifolds, number theory, and algebraic geometry all in one - what's not to love!

1.6.1. **Stable homotopy groups.** The first and most decisive development that led to spectra was Freudenthal's Suspension Theorem. It goes something like this: given a pointed space X, the counit of the adjunction $\Sigma^k \dashv \Omega^k$ between k-fold iterated suspensions and based loops is a canonical map of the form $X \to \Omega^k\Sigma^k X$. Furthermore observe that for $k \leq l$, these maps factor through $\Omega^k\Sigma^k X \to \Omega^l\Sigma^l X$ and give rise to a tower of based maps

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \Omega^3\Sigma^3 X \to \cdots.$$

Recalling the canonical isomorphism $\pi_i(\Omega^k Y) \cong \pi_{i+k}(Y)$ (which we have already encountered at the start of subsection 1.4.5), we obtain by applying π_i to the above tower a tower of abelian groups

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \pi_{i+3}(\Sigma^3 X) \to \cdots.$$

Freudenthal's Theorem now guarantees that (at least for X a finite space, or if you want, finite CW complex) this tower stabilizes, i.e. all the maps from a certain one onward are isomorphisms.

Therefore it makes sense to consider the *i-th stable homotopy group of* X defined as $\pi_i^s(X) = \varinjlim_k \pi_{i+k}(\Sigma^k X)$. Freudenthal's theorem just says that this is the same as dropping the limit if you take a big enough value of k. The reason for people's interest in stable homotopy groups was simple: often times, whey were much simpler to compute than the famously computationally-inaccessible higher homotopy groups, and certain information about the latter could be derived from knowledge of the former.

1.6.2. **Homotopy groups of spectra.** Of course these days, we rarely speak about stable homotopy groups. Instead we usually talk about homotopy groups of spectra, which encapsulate the latter because $\pi_i(\Sigma^\infty X) \simeq \pi_i^s(X)$. In analogy with the above, if a spectrum X is given by a sequence of spaces $\{X_k\}$ equipped with structure maps $\Sigma X_k \to X_{k+1}$, its homotopy groups may be defined as $\pi_i(X) \coloneqq \varinjlim_k \pi_{i+k} X_k$.

If we wish to be a little less anachronistic, we may simply define in complete analogy with spaces $\pi_i(X) = \pi_0 \mathrm{Map}_{\mathrm{Sp}}(\Sigma^i S, X)$.

Note thus that for all $i \geq 0$, this gives $\pi_i(X) \simeq \pi_i(\Omega^\infty X)$ with π_i on the left standing for homotopy groups of spectra and on the right for ordinary homotopy groups of spaces. This shows that the "underlying infinite loop space" $\Omega^\infty X$ of a spectrum X is usually a rather complicated space even for very simple spectra X. For example, $\Omega^\infty S$ is the free loop space on one generator, a space whose homotopy groups are the stable homotopy groups of spheres. The latter not being known, it is clear that it has to be one wild space.

1.6.3. **Spectra also have to do with other things.** Freudenthal suspension was the seminal phenomenon which made homotopy theorists consider studying phenomena stable under suspension, inching in light of subsection 1.5.2 towards spectra. A number of other things homotopy theoriests were interested turned out to also be related to spectra:

- Extraordinary cohomology theories were proved by Brown to all be representable by spectra. Viewing them as spectra also solved a peering problem that the category of cohomology theories themselves was rather well-behaved, and so a poor place to try to use universal properties.
- Infinite loop spaces are best studied as spectra. We touched upon this in 1.2.4, but it is worthwhile pointing out, because the intuition of infinite loop spaces is and was quite different from that of cohomology theories.

Really I would say that the first of these two point is the key: people cared (and still do - ask Arun for example) about cohomology theories, and we've now known for a while that having good categories of things we want to study makes life easier. So spectra were just the convenient place to do it.

But none the less, let us remark a little about the second of the two points above, namely the connection to loop space theory, in the following 3 subsections.

1.6.4. **Recognition theorem for iterated loop spaces.** These days, Peter May is most famous for his textbook on algebraic topology. But there was a time when he was a mighty force of nature in the field. He was one of the staunchest proponents of spectra, and spent a great deal of his career looking for ever-better models for them. In particular, the great EKMM paper (book, really), whose great success it was to give the first good category of spectra with a smash product (a really big deal at the time, the early 90s, and culmination of more than two decades of difficulties) has May as the last and decisive M.

But what I want to mention here isn't the EKMM construction (which is very artistic and esoteric, indexing spectra on vector spaces with inner products, and relying crucially on the properties of the linear isometries operad for the construction of the smash product), but instead another landmark result of May, which greatly clarified and cemented the role of spectra (though we will get to that only in 1.6.6).

Theorem 12 (Boardman-Vogt; May)**.** *For any $k \geq 0$, the k-fold iterated loop spaces coincide with groups-up-to-homotopy which are commutative up to k-th order homotopy coherences.*

More formally: the k-fold iterated based loops functor $\Omega^k : \mathcal{S}_* \to \mathcal{S}_*$ extends to an equivalence of ∞-categories

$$\mathcal{S}_*^{\geq k} \simeq \mathrm{Mon}_{\mathbb{E}_k}^{\mathrm{gp}}$$

between k-connective based spaces on the left (first non-trivial homotopy group is in degree k) and grouplike $\mathbb{E}_k$-spaces on the right. Here an $\mathbb{E}_k$-space X is said to be grouplike if $\pi_0(X)$, with the monoid structure that the $\mathbb{E}_k$-space structure on X induces, is actually a group.

The slightly technical notion of a grouplike $\mathbb{E}_k$-space in the formal statement is the rigorous meaning behind the heuristic "groups-up-to-homotopy which are commutative up to k-th order homotopy coherences" appearing in the informal statement.

The take-away is that any group operation arises up to homotopy from concatenation of loops, and that homotopy analogues of commutativity ($\mathbb{E}_k$-ness) correspond to how many-fold the loop space in question is.

1.6.5. **What are $\mathbb{E}_k$-spaces anyway.** The notion of $\mathbb{E}_k$-structure is the standard homotopy business: commutativity from classical algebra can be required on several levels. Not at all: that is $\mathbb{E}_1$ - that is to say, $\mathbb{E}_1$ only means that we have a homotopy-associative operation, and we can say $\mathbb{E}_1 \simeq \mathbb{A}_\infty$ where $\mathbb{A}_n$ are homotopy-notions of associativity (we won't be encountering these associativity fellows from here on). Commutativity can be demanded on just the level of π_0 - that is $\mathbb{E}_2$. Next $\mathbb{E}_3$ amounts to commutativity up to first-level homotopy coherence, e.g. the choice of how to pass between the permutations of the different three-factor products should be contractible. But that doesn't say anything about four-factor product, as the contractibility of the space of those would amount to $\mathbb{E}_4$. In general an $\mathbb{E}_k$ structure means that we have a monoid operation and the products of up to k elements can be put in whatever order up to a contractible choice (e.g. switching between them can not create non-trivial isomorphisms).

In classical algebra we of course only have two possibilities form monoids, groups, rings and the like: either $\mathbb{E}_1$, which means associative, or $\mathbb{E}_2 = \mathbb{E}_3 = \cdots = \mathbb{E}_\infty$, which means commutative. In retrospect, that is a consequence of working in $\mathcal{S}\mathrm{et}, \mathcal{A}\mathrm{b}$ or the likes, which are all 1-categories.

When one works in a 2-category setting, there are three possibilities: either $\mathbb{E}_1$ meaning associativity, or $\mathbb{E}_2$ which is partial commutativity, or $\mathbb{E}_3 = \mathbb{E}_4 = \cdots = \mathbb{E}_\infty$ which is full commutativity. This too is familiar: applied to the case of the 2-category $\mathcal{C}\mathrm{at}$ of categories, these three notions of a monoid structure correspond to monoidal categories, braided monoidal categories, and symmetric monoidal categories respectively.

Since we are working in a genuine ∞-categorical setting in homotopy theory, there are infinitely many distinct possibilities for homotopy monoids

$$\mathbb{E}_1 \subsetneq \mathbb{E}_2 \subsetneq \mathbb{E}_3 \subsetneq \cdots \subsetneq \mathbb{E}_\infty := \bigcup_{k\geq 1} \mathbb{E}_k.$$

In particular, $\mathbb{E}_\infty$, which means homotopy commutativity up to all orders of homotopy coherence, is the "real", or better most complete, analogue of commutativity.

BTW, just to have some intuition on what $\mathbb{E}_n$ means, let me mention the Dunn Additivity Theorem. It says that $\mathbb{E}_k \simeq \mathbb{E}_1^{\otimes k}$, or informally that a $\mathbb{E}_k$-structure is equivalent to a set of k-many compatible $\mathbb{E}_1$-structures (of course to make this rigorous, the language of ∞-operads is needed). Applying this for $k = 2$ in the classical context of sets say, we recover the classical Eckmann-Hilton argument; indeed, Dunn Additivity is merely a far-reaching refinement thereof.

1.6.6. **Recognition theorem for infinite loop spaces.** Of course the Recognition theorem of subsection 1.6.4 didn't really say anything about spectra, only about iterated loop spaces. But recall that spectra, at least the connective ones, may be identified with infinite loop spaces. This amounts to passing to the limit $k \to \infty$ in the above statement, and gives May's theorem:

Theorem 13 (May)**.** *Infinite loop spaces coincide with groups-up-to-homotopy which are commutative up to all orders of homotopy coherences. More formally: the underlying infinite loop space functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ extends to an equivalence of ∞-categories*

$$\mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty} = \mathrm{CMon}^{\mathrm{gp}}$$

between connective spectra on the left (no negative homotopy groups) and grouplike $\mathbb{E}_\infty$-spaces on the right.

This version of the recognition theorem can be derived from the previous one with some ∞-categorical dexterity, using the fact that $\mathrm{Sp} \simeq \varprojlim(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \cdots)$. In fact, this limit definition of spectra may well be motivated from the perspective of this theorem as the analogue of pointed spaces which makes the theorem, analogue of the Recognition Theorem of subsection 1.6.4, work for $\mathbb{E}_\infty$-groups.

If the take-away of the iterated loop space Recognition Theorem in 1.6.4 was that the operation in any $\mathbb{E}_k$-group comes from k-fold composition of loops, then the take-away here is that the operation in any $\mathbb{E}_\infty$-group comes from addition in a (connective) spectrum.

May's Theorem may be seen as giving justification to the claim that spectra play the analogous role with respect to spaces, or in homotopical mathematics, as abelian groups do with respect to sets, or in ordinary mathematics. So indeed at this point, homotopy theorists could do little else but acknowledge that spectra were a key notion and here to stay.

This theorem is also the first strong indication of the claim that stable homotopy theory is in fact all about algebra (albeit algebra in a certain homotopy-invariant setting). We will discuss this in much more detail in the next section.

1.7. Brave New Algebra

The term "Brave New Algebra" was given to stable homotopy theory (or at least to a certain subfield there-of) half-mockingly by Waldhousen, one of its leading practitioners. The point was something like this: stable homotopy theory developed out of the study of spaces (see the previous section), but by that point in time (the early 90s I think) it had "degenerated" into what was just-about pure algebra. Whatever algebraists could do with modules, stable homotopy theorists were able to do with spectra, albeit usually with considerably more effort. The name "brave new algebra" was supposed to acknowledge this aspect of the field, encapsulated in the motto: Stable homotopy theory is about algebra over the spehere spectrum.

These days this is not merely a motto, but a perfectly rigorous fact, though it is only when "thinking with ∞-categories" that it really becomes useful.

1.7.1. **Literary allusion.** Before we get to explaining this in more detail, let us acknowledge that the name "brave new algebra" in intentionally evocative of Huxley's landmark novel "Brave New World". This novel is considered a dystopia, and as such the implied suggestion was that perhaps we shouldn't be too quick to leave behind the algebraic topology that birthed the subject in favor of pure abstract algebra.

Fair, as the complaint may be, it is my humble hope that the warning proved to not have much substance. Connections to algebraic toplogy have since failed to yield many significant new insights, while analogies with algebra, algebraic geometry, etc. have. This goes so far that some albeit rare people these days (Sam being an example) take it to the extreme and claim that thinking about spaces through topology is useless and misplaced, and that instead thinking of them only as ∞-groupoids is the way to go.

I don't personally endorse such a perspective, partially because I find it silly to try to forget the myriad of intuitions and ideas that went into where the subject is today, but also because the interplay between category theory, algebra, and toplogical ideas, is one of the aspects of the subject that I most appreciate and find most surprising and fascinating! Why discard something so rich?

1.7.2. **Abelian groups are discrete spectra.** Here is a fact: discrete spectra, i.e. spectra which satisfy $\pi_i(X) \simeq 0$ for all $i \neq 0$, are uniquely determined by the abelian group $\pi_0(X)$. In fact, the functor $X \mapsto \pi_0(X)$ from the subcategory $\mathrm{Sp}^\heartsuit \subset \mathrm{Sp}$ spanned by all such spectra to $\mathcal{A}\mathrm{b}$ is an equivalence of categories. Switching the perspective, the inverse functor of this equivalence allow us to identify abelian groups with a full subcategory of the ∞-category of spectra.

In the future we will make use of this fully and *not distinguish the abelian group A from the discrete spectrum whose π_0 is A.* In this and the next paragraph alone, to ease you into it, we abide by the more classical tradition of denoting the corresponding spectrum by $\mathrm{H}A$ and calling it the *Eilenberg-MacLane spectrum of A.*

There are several different perspectives on what this identification of abelian groups is about. Here is a non-exhaustive list:

- Viewing spectra as homology theories (section 1.1.4), the Eilenberg-MacLane spectrum $\mathrm{H}A$ corresponds to ordinary homology with values in A, sending a space X to $\mathrm{H}_*(X; A)$.
- Recall that stable ∞-categories are an analogue of abelian categories. The equivalences $\mathrm{Sp} \simeq \mathrm{Sp}(\mathcal{S})$ (section 1.1.1) and $\mathcal{S}^\heartsuit \simeq \mathcal{S}\mathrm{et}$, the latter being the identification of discrete spaces with sets, then leads us to expect the promised identification $\mathrm{Sp}^\heartsuit \simeq \mathrm{Ab}(\mathcal{S}\mathrm{et}) \simeq \mathcal{A}\mathrm{b}$, where the middle term denotes the abelianization.
- Under the equivalence $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{CMon}^\infty$ discussed in subsection 1.6.6, passing to discrete objects (i.e. such that all non-zero homotopy groups vanish) on both sides gives precisely the desired equivalence $\mathrm{Sp}^\heartsuit \simeq \mathcal{A}\mathrm{b}$ again.

As remarked, the notation $\mathrm{H}A$ for the Eilenberg-MacLane spectrum is supposed to indicate that this is the spectrum representing ordinary homology. But we prefer to think about it in parallel with the third of the above perspectives, in which case the name A seems more appropriate for both the spectrum and the abelian group.

1.7.3. **Smash product vs tensor product of abelian groups.** The embedding $\mathcal{A}\mathrm{b} \to \mathrm{Sp}$ is fully faithful, but not monoidal. Indeed, if $\otimes$ denotes as before the smash product of spectra, and $\otimes^\heartsuit$ denotes the ordinary tensor product of abelian groups, then we have $\pi_0(A \otimes B) \simeq A \otimes^\heartsuit B$ for all $A, B \in \mathrm{Sp}^\heartsuit \simeq \mathcal{A}\mathrm{b}$, but $A \otimes B$ might not be discrete anymore.

Example. for $A = B = \mathbf{F}_2$, the graded commutative algebra $\pi_*(\mathbf{F}_2 \otimes \mathbf{F}_2)$ is isomorphic to $\mathbf{F}_2[\xi_1, \xi_2, \ldots]$ for generators ξ_i of degree $2^i - 1$. This is the famous dual Steenrod algebra, often denoted $\mathcal{A}^\vee$ or $\mathcal{A}_*$. It is of great computational importance in homotopy theory - if it were all concentrated in degree 0, homotopy theory would a significantly less rich and more boring subject!

Nonetheless, the canonical map $A \otimes B \to \pi_0(A \otimes B) \simeq A \otimes^\heartsuit B$ exhibits the embedding $\mathcal{A}\mathrm{b} \to \mathrm{Sp}$ as lax symmetric monoidal (lax vs strict: there exists such a map vs it must also be an equivalence). Lax symmetric monoidal structure is enough to preserve commutative algebra objects (we will pay a debt and discuss those in some detail in the next subsection), so this induces a map

$$\mathrm{CAlg}^\heartsuit := \mathrm{CAlg}(\mathcal{A}\mathrm{b}) \to \mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp}).$$

Here $\mathrm{CAlg}^\heartsuit$ is the category of commutative rings, while CAlg is the ∞-category of $\mathbb{E}_\infty$-rings (originally known as "highly commutative ring spectra"). The latter is the notion of a commutative ring native to Sp, when spectra are viewed as the correct ∞-categorical analogue of abelian groups. In particular, commutative rings may be viewed as special cases of $\mathbb{E}_\infty$-rings.

How to see for a fact that the embedding $\mathcal{A}\mathrm{b} \simeq \mathrm{Sp}^\heartsuit \hookrightarrow \mathrm{Sp}$ is lax symmetric monoidal? Well, here's a general fact, easy to prove by abstract nonsense: the right adjoint of a symmetric monoidal functor is always lax symmetric monoidal. Now the left adjoint to the inclusion functor in question is given by $\pi_0 : \mathrm{Sp} \to \mathcal{A}\mathrm{b}$, and since $\pi_0(X \otimes Y) \simeq \pi_0(X) \otimes^\heartsuit \pi_0(Y)$ for all spectra X and Y, this functor is indeed symmetric monoidal.

1.7.4. **Digression: commutative algebra objects.** We've used the notation $\mathrm{CAlg}(\mathcal{C})$ in the previous section, and at certain times much earlier (e.g. subsection 1.5.8), waiving it around like any sane person should instinctively know precisely what that is. Perhaps it's time to settle the debt and spell out what this is about.

Given a symmetric monoidal ∞-category $\mathcal{C}^\otimes$, which is to say an ∞-category $\mathcal{C}$ with a symmetric monoidal operation $\otimes$ on $\mathcal{C}$, a commutative algebra in $\mathcal{C}$ informally, this consists of an object $A \in \mathcal{C}$ together with a "multiplication" map $A \otimes A \to A$, which is unital, associative, and commutative, all up to arbitrarily high homotopy coherence.

A bit more formally, there is a monad Sym^* on $\mathcal{C}$ given by $X \mapsto \mathrm{Sym}^*(X) = \coprod_n (X^{\otimes n})_{\Sigma_n}$, and $\mathrm{CAlg}(\mathcal{C})$ is the ∞-categories of modules (or in the more traditional categorical parlance: algebras) for this monad Sym^*.

In any case, the commutativity here is understood in the ∞-categorical sense, or equivalentnly homotopical $\mathbb{E}_\infty$-sense, so another name could be an $\mathbb{E}_\infty$-algebra object in $\mathcal{C}$. But since this just is the organic notion of commutativity that we get by working ∞-categorically, we prefer to stick to the simpler language.

Some examples of commutative algebra objects:

(1) Let R be a commutative ring and $\mathrm{Mod}_R^\heartsuit$ the category of (ordinary) R-modules, made symmetric monoidal through the (non-derived) tensor product $\otimes_R^\heartsuit$. Then

$$\mathrm{CAlg}(\mathrm{Mod}_R^\heartsuit) \simeq \mathrm{CAlg}_R^\heartsuit$$

is the category of (ordinary) commutative R-algebras.

(2) A noteworthy special case of the above with $R = \mathbf{Z}$, we have $\mathrm{CAlg}(\mathcal{A}\mathrm{b}) \simeq \mathrm{CAlg}^\heartsuit$, the category of commutative rings.

(3) Viewing the ∞-category of spectra Sp as symmetric monoidal with the smash product $\otimes$, the commutative algebras are

$$\mathrm{CAlg}(\mathrm{Sp}) \simeq \mathrm{CAlg},$$

the $\mathbb{E}_\infty$-rings (of more traditionally: $\mathbb{E}_\infty$-ring spectra). This is of course more of a definition than a theorem though. Make the ∞-category of spaces $\mathcal{S}$ symmetric monoidal by equipping it with the Cartesian symmetric monoidal structure, which is to say $X \otimes Y := X \times Y$. Then we have

$$\mathrm{CAlg}(\mathcal{S}) \simeq \mathrm{CMon},$$

the ∞-category of $\mathbb{E}_\infty$-spaces. Commutative algebras for a Cartesian structure are often called commutative monoids, which explains the notation used in subsection 1.6.6.

(4) In the 1-categorical analogue of the previous example, considering $\mathcal{S}\mathrm{et}$ with its Cartesian symmetric monoidal structure, we get

$$\mathrm{CAlg}(\mathcal{S}\mathrm{et}) \simeq \mathrm{CMon}^\heartsuit,$$

by which we have denoted the category of (ordinary) commutative monoids.

(5) Equipping the category of categories $\mathcal{C}\mathrm{at}$ or ∞-category of ∞-categories $\mathcal{C}\mathrm{at}_\infty$ respectively with the Cartesian symmetric monoidal structure, the commutative algebra objects $\mathrm{CAlg}(\mathcal{C}\mathrm{at})$ and $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ are symmetric monoidal categories and symmetric monoidal ∞-categories respectively.

(6) As mentioned in section 1.1.5, the commutative algebras $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^L)$ in the Cartesian symmetric monoidal ∞-category of presentable symmetric monoidal ∞-categories $\mathcal{P}\mathrm{r}^L$ amounts to a stably symmetric monoidal presentable ∞-category. That is to say, a presentable ∞-category together with a symmetric monoidal structure which factor-wise preserves colimits.

The third of the examples on this list, $\mathbb{E}_\infty$-rings, will feature prominently, playing the role of homotopical commutative rings, from here on.

Surely the utility of the notion of commutative algebra objects is now clear beyond any doubt, but hopefully it also seems like a rather natural concept.

1.7.5. **In higher algebra, like in the alphabet, S comes before Z.** Enough digression, back to our regularly scheduled business!

At the end of subsection 1.7.4, we saw that the functor $\pi_0 : \mathrm{Sp} \to \mathcal{A}\mathrm{b}$ is symmetric monoidal. In particular, it sends the unit S for the smash product of spectra to the unit $\mathbf{Z}$ for the tensor product of abelian groups. That is to say, we have a canonical isomorphism $\pi_0(S) \simeq \mathbf{Z}$.

This might look wrong at first glance, since we have $\pi_0(S^0) = \mathbf{Z}/2$, but note that it's actually OK, since $\pi_1(S^1) = \mathbf{Z}$ and similarly $\pi_2(S^2) = \pi_3(S^3) = \cdots = \pi_0^s(S^0) = \pi_0(S)$, where we have recalled stable homotopy groups from 1.6.1 and their replationship to homotopy groups of spectra from 1.6.2.

In particular, there is a canonical map of $\mathbb{E}_\infty$-rings $S \to \pi_0(S) \simeq \mathbf{Z}$, witnessing that the sphere spectrum "quotients down" to the integers. To interpret this, recal that the ring $\mathbf{Z}$ is the initial object in the category of commutative rings. Well, enlarging from ordinary rings to $\mathbb{E}_\infty$-rings, their homotopy analogues, the initial object becomes S. From the POV of algebraic geometry, this means that the point $\operatorname{Spec}\mathbf{Z}$ is no longer the "smallest possible" point (i.e. terminal for everything), as there is a point "beneath" it: $\operatorname{Spec} S$. This is a perspective on SAG (spectral algebraic geometry) championed in particular in a paper by Toen titled "*Under* $\operatorname{Spec}\mathbf{Z}$" (though, of course, in French).

But note that it is a little different from another slightly-more-conjectural version of AG "under" $\operatorname{Spec}\mathbf{Z}$, the geometry over $\mathbf{F}_1$, the fictional field with 1 element. In particular, we are not claiming that S is a model for $\mathbf{F}_1$, just that both are certain analogues of commutative rings which map homomorphically onto $\mathbf{Z}$.

1.7.6. **Modules and tensor products over an $\mathbb{E}_\infty$-ring.** We saw in subsection 1.7.3 that the embedding $\mathcal{A}\mathrm{b} \hookrightarrow \mathrm{Sp}$ yields an equally fully faithful embedding $\mathrm{CAlg}^\heartsuit \hookrightarrow \mathrm{CAlg}$, identifying ordinary commutative ring with discrete $\mathbb{E}_\infty$-rings. Thus things we can do with ordinary rings, we might as well try do with an arbitrary $\mathbb{E}_\infty$-rings R.

For isntance, we can speak of R-modules (since we are in the commutative setting, differentiating left and right modules is unnecessary): that consists of an underlying spectrum M together with a multiplication map $R \otimes M \to M$, which satisfies the module axioms with respect to the $\mathbb{E}_\infty$-structure on M, of course up to coherent homotopy. The R-modules (more classically called R-module spectra) form an ∞-category Mod_R, where morphisms are spectrum maps $M \to N$ which make the appropriate diagrams including module multiplications on M and N commute up to coherent homotopy.

The smash product of spectra also gives rise to a relative tensor product on Mod_R. For any two R-modules M and N, we define

$$M \otimes_R N := \varinjlim \big(\cdots M \otimes R \otimes R \otimes N \Rrightarrow M \otimes R \otimes N \rightrightarrows M \otimes N \big)$$

where $\rightrightarrows$ denotes two parallel arrows, $\Rrightarrow$ denotes three parallel arrows, and we are signifying a simplicial diagram. The morphisms in this simplicial diagram all come from the R-module structure on M and N, and the multiplication on R itself (the above-undenoted opposite-direction-going degenericies come from the "unit element inclusion" map $S \xrightarrow{1} R$).

This definition might look a little hardcore or even batshit insane, but it really just generalizes the fact that for an ordinary commutative ring R and two ordinary R-modules M and N, their relative tensor product $M \otimes_R^\heartsuit M$ (denoted in analogy with our convention that $\otimes^\heartsuit$ denotes the tensor product of abelian groups) may be constructed as the coequalizer

$$M \otimes_R^\heartsuit N = \mathrm{Coeq}(M \otimes^\heartsuit R \otimes^\heartsuit N \rightrightarrows M \otimes^\heartsuit N)$$

of the maps $x \otimes a \otimes y \mapsto ax \otimes y$ and $x \otimes a \otimes y \mapsto x \otimes ay$ for all $x \in M, y \in N$ and $a \in R$. In this 1-categorical case, it sufficed to only consider the first level, but in the ∞-categorical setting of the previous paragraph, we needed to consider higher levels too. That's yet another incarnation of the already-encountered fact that in ordinary algebra, if you have $a(bc) = (ab)c$, then associativity will hold for any number of factors, while homotopically, where such equivalence need to be specified and are data instead of structure, this is no longer so (compare to analogous situation for commutativity we discussed in subsection 1.6.5).

The relative tensor product makes Mod_R into a stable presentably symmetric monoidal ∞-category. So it's a nice category to do commutative algebra in, and we obtained it essentially by saying the word "R-modules" and interpreted it in the ∞-categorical setting. Abelian groups became replaced with spectra, commutative rings with $\mathbb{E}_\infty$-rings, and that was it.

1.7.7. **Derived category of modules inside spectra.** Let us restrict the ∞-categorical notion discussed for an arbitrary $\mathbb{E}_\infty$-ring R to the case where R belongs to the full subcategory $\mathrm{CAlg}^\heartsuit \subset \mathrm{CAlg}$ of ordinary commutative rings.

You might (based on notation) perhaps first guess that Mod_R will reproduce the category $\mathrm{Mod}_R^\heartsuit$ of ordinary R-modules. It indeed shares some of its properties, such as containing $0, R, R^{\oplus n}$, kernels and cokernels (which in homotopy-land we might prefer to call fibers and cofibers), etc. Alas, being a stable ∞-category, it also possesses certain things that $\mathrm{Mod}_R^\heartsuit$ doesn't: for instance, shifts $\Sigma^n R$. The relative tensor product $\otimes_R$ is also a little different than its usual cousin $\otimes_R^\heartsuit$, because while the latter is only right-exact, the former is fully exact (in fact, it commutes with all colimits in each variable, almost by definition).

The resolution of this apparent mystery is to recall that we also know an upgraded analogue of $\mathrm{Mod}_R^\heartsuit$ in "classical algebra" (by which I mean, non-∞-categorical nonsense): the (unbounded) derived category of R-modules $\mathcal{D}(R)$. Objects therein are chain complexes of ordinary R-modules, up to quasi-isomorphisms, shifting complexes to the right gives a functor $[1]$, and the ordinary tensor product $\otimes_R^\heartsuit$ on $\mathrm{Mod}_R^\heartsuit$ lifts to a derived tensor product $\otimes_R^L$ on $\mathcal{D}(R)$ which is exact.

One slightly subtle point is that $\mathcal{D}(R)$ is often viewed as an ordinary category in basic treatments of homological algebra, i.e. the equivalence relation of quasi-isomorphisms is quotiented out set-theoretically. If on the other hand we view it as the same sort of defined-up-to-equivalence as we have in homotopy theory all the time (a "space" is really only defined up to homotopy equivalence, etc.), then we get $\mathcal{D}(R)$ as an ∞-category. One approach to this is to go through dg-categories (which are themselves models for linear presentable ∞-categories), another by working directly in ∞-cat-land as done in Higher Algebra, but in whichever way, when we say $\mathcal{D}(R)$, we mean the derived ∞-category of R-modules. This changes little-to-nothing: it's still the same construction as always of the derived category, just the POV is slightly shifted.

Theorem 14 (Shipley)**.** *For an ordinary commutative ring R, there is a canonical equivalence of symmetric monoidal ∞-categories $\mathrm{Mod}_R^{\otimes_R} \simeq \mathcal{D}(R)^{\otimes_R^L}$.*

Thus instead of imagining an element of $\mathcal{D}(R)$ as a chain complex of ordinary R-modules, defined only up to quasi-isomorphism, we may via the forgetful functor $\mathrm{Mod}_R \to \mathrm{Sp}$ think of it as a spectrum (an object inherently defined ∞-categorically and thus only up to an appropriate notion of homotopy equivalence) together with an additional structure, namely that of an R-module, with respect to the smash product $\otimes$.

In general, the above result (which was first proved by Shipley, but becomes essentially tautological with modern ∞-categorical tools) may be interpreted as saying that stable homotopy theory contains and subsumes all ordinary homological algebra.

1.7.8. **Modules over the sphere.** Now that we've seen what modules over a discrete $\mathbb{E}_\infty$-ring are, and recognized it as the derived category, let us turn our attention to the mother of all non-discrete $\mathbb{E}_\infty$-rings: the sphere spectrum S.

The result is in some way super boring: given a spectrum M, an S-module structure on it would consist of a multiplication map $S \otimes M \to M$ satisfying various properties. But of course, since S is the monoidal unit for the smash product, there is a canonical equivalence of spectra $S \otimes M \simeq M$ for any $M \in \mathrm{Sp}$, and taking these to be the module structure maps will surely satisfy all the requirements. Furthermore, all S-module structures are of this form, which is to say that the forgetful functor $\mathrm{Mod}_S \to \mathrm{Sp}$ is an equivalence of ∞-categories.

It is furthermore easy to see that $\otimes_S \simeq \otimes$, i.e. that the relative tensor product over the sphere is just the smash product. Indeed, in the colimit definition of $\otimes_S$ in subsection 1.7.6, we see that all the terms in the simplicial diagram are equivalent and all the maps between them these canonical equivalences. As such, $\mathrm{Mod}_S^{\otimes_S} \simeq \mathrm{Sp}^\otimes$ is an equivalence of symmetric monoidal ∞-categories.

1.7.9. **List of analogies.** The content of the previous section gives us the last new (and in my mind, the most useful) perspective on spectra: they are modules over the sphere S. Or we can be slightly more precise and think of Sp as analogous to $\mathcal{D}(R)$ for any ordinary commutative ring R. Thus we imagine in decreasing order of correctness the heuristic

$$\mathrm{Sp} \approx \mathcal{D}(R) \approx \mathrm{Mod}_R^\heartsuit$$

and as already mentioned, this leads to a long list of analogies, some making sense only for $\mathcal{D}(R)$ or just $\mathrm{Mod}_R^\heartsuit$, and some for both. Here are a few:

- the sphere spectrum S is like the base ring R

- the smahs product $\otimes$ is like the derived tensor product $\otimes_R^L$ is like the ordinary tensor product $\otimes_R^\heartsuit$
- fib and cofib are like hKer and hCoker (in certain UT faculty member's notation, more classically cone and cocone) are like Ker and Coker respectively
- $\Sigma^n M$ and $\Omega^n M$ are like shifts $M[n]$ and $M[-n]$
- $\pi_n : \mathrm{Sp} \to \mathcal{A}\mathrm{b}$ is like $H^{-n} : \mathcal{D}(R) \to \mathrm{Mod}_R^\heartsuit$ (the minus is there due to homological vs cohomological grading)
- (co)fiber sequences are like distinguished triangles are like short exact sequences
- the functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ is like the forgetful functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{et}$
- the functor $\Sigma_+^\infty : \mathcal{S} \to \mathrm{Sp}$ is like the free R-module functor $X \mapsto R^{\oplus X}$

Of these only the last two probably require some additional justification. Recall from section 1.2.5 the adjunction $\Sigma_+^\infty \dashv \Omega^\infty$, which together with the fact that $S \simeq \Sigma_+^\infty(*)$ implies that

$$\mathrm{Map}_{\mathrm{Sp}}(S, X) \simeq \mathrm{Map}_{\mathcal{S}}(*, \Omega^\infty X) \simeq \Omega^\infty X$$

for any spectrum X (since obviously $\mathrm{Map}_{\mathcal{S}}(*, Y) \simeq Y$ for any space Y). That gives a very explicit understanding of the functor Ω^∞ as the functor corepresented by the sphere spectrum. But in terms of the analogy $\mathrm{Sp} \simeq \mathrm{Mod}_R^\heartsuit$ it exhibits $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ as analogous to the functor $\mathrm{Mod}_R^\heartsuit \to \mathcal{S}\mathrm{et}$ given by

$$M \mapsto \mathrm{Hom}_{\mathrm{Mod}_R^\heartsuit}(R, M) = M,$$

sending the ordinary R-module M to its underlying set M. That settles the pentultimate point on the above list. The ultimate one is just the observation that the free functor $\mathcal{S}\mathrm{et} \to \mathrm{Mod}_R^\heartsuit$ is the left adjoint to the forgetful functor $\mathrm{Mod}_R^\heartsuit \to \mathcal{S}\mathrm{et}$, just like Σ_+^∞ is the left adjoint to Ω^∞.

Part 2. **Some examples of spectra**

So far the impression of stable homotopy theory you may have acquired from Part 1 may very well be that it is all about categorical nonsense, abstract universal properties, etc. And while that is certainly true to some extent, the field is also highly computational. In fact, much of our knowledge and understanding of spectra comes from trying to make sense of tons of ingenious but puzzling computations.

In order to do computations, one of course needs some objects to work with. This is why in this section we will collect some examples of spectra that people often care most about. That said, I make no promise of this being an exhaustive list! It is just some of the coolest and most traditional examples.

2.1. Examples stemming from what we know so far

We have seen a rich number of perspectives on what the ∞-category of spectra is, but have seen little actual different examples of spectra. So far we have two families of examples:

- Suspension spectra $\Sigma^\infty X$ for pointed spaces X, most prominently $S \simeq \Sigma^\infty S^0$, the sphere spectrum. Their homotopy groups are stable homotopy groups of spaces, i.e.

$$\pi_i(\Sigma^\infty X) = \pi_i^s(X),$$

  recall subsection 1.6.1 for that. In particular, note that $\pi_i(\Sigma^\infty(X)) = 0$ for all $i \le 0$ - suspension spectra are always *connective.*
- Abelian groups as discrete spectra (or Eilenberg-MacLane spectra, if you prefer). Their homotopy groups are $\pi_0(A) = A$ and $\pi_i(A) = 0$ for all $i \neq 0$.

- For any commutative ring R and any chain complex up to quasiisomorphism $M \in \mathcal{D}(R)$, we know that M can also be viewed as an R-module spectrum. Its homotopy groups give (opposite-graded) chain-complex cohomology, which is to say that
$$\pi_i(M) \simeq H^{-i}(M).$$
- The spectrum M will thus be connective if and only if the chain complex M is concentrated in negative degrees.

In many ways, this is a very rich set of examples, but our goal here is to mention a few more archetypal examples.

2.1.1. **Algebraic constructions.** The thesis of the previous email was that $\mathrm{Sp} \simeq \mathrm{Mod}_S$, or in words, that spectra behave much like nice algebraic modules. As such, there are a bunch of algebraic constructions that we can perform with spectra already at our disposal to produce new ones. We explore a few of them in the next few subsections.

2.1.2. **The symmetric algebra.** One of the most basic operations in algebra is the formation of polynomial algebras. This can be done in the land of spectra too. Indeed, there is a symmetric algebra functor $\mathrm{Sym}^* : \mathrm{Sp} \to \mathrm{CAlg}$, given explicitly by $\mathrm{Sym}^*(M) = \bigoplus_{n\geq 0} M^{\otimes n}_{\Sigma_n}$. Here the action of the symmetric group Σ_n on the n-fold smash power $M^{\otimes n}$ is through permuting the factors. This is precisely analogous to the corresponding construction of the symmetric algebra in usual algebra, only that everything in sight carries its natural ∞-categorical meaning.

Imitating usual algebra further, we may define an analogue of polynomial ring over the sphere spectrum as the free $\mathbb{E}_\infty$-ring $S\{t\} := \mathrm{Sym}^*(S)$. On the level of homotopy we get $\pi_0(S\{t\}) \simeq \pi_0(S)[t] = \mathbf{Z}[t]$ with the generator t in degree 0. To put the generator t into any degree $d \in \mathbf{Z}$, we may form $S\{t^d\} = \mathrm{Sym}^*(\Sigma^d S)$, and to consider multi-variable analogues, we coud take $S\{t_1, \ldots, t_n\} = \mathrm{Sym}^*(S^{\oplus n})$.

The symmetric algebra functor $\mathrm{Sym}^* : \mathrm{Sp} \to \mathrm{CAlg}$ satisfies the expected universal property, which is to say that it is left adjoint to the forgetful functor $\mathrm{CAlg} \to \mathrm{Sp}$. In fact, since all the ∞-categories in sight are presentable and the Adjoint Functor Theorem is therefore available, a relatively easy way of showing the existence and functoriality of the symmetric algebra functor is to show that the forgetful functor preserves limits.

A useful upshot is that limits of $\mathbb{E}_\infty$-rings can therefore be computed in spectra. The analogous claim for colimits fails epically. For instance, just as the ordinary relative tensor product $A \otimes^\heartsuit_R B$ is computes pushouts of span $A \leftarrow R \to B$ in the ordinary category of commutative rings $\mathrm{CAlg}^\heartsuit$, so does the relative smash product $A \otimes_R B$ compute the pushout of an eponymous span in the ∞-category of $\mathbb{E}_\infty$-rings CAlg.

Of course everything that we said in this subsection for $\mathrm{Sp} \simeq \mathrm{Mod}_S$ works just as well in the context of Mod_R for any $\mathbb{E}_\infty$-ring R. But that is enough said about symmetric algebras; let us move on to analogues of other algebraic constructions.

2.1.3. **Fibers and cofibers.** Taking kernels and cokernels is replaced by taking fibers and cofibers of morphisms of spectra. In particular, if you have a map $f : M \to N$ of spectra, which you may wish to think of as an inclusion, then you can form the "quotient" as $M/N = \mathrm{cofib}(f)$.

That said, people will rarely denote cofibers by quotient notation, as the notion of taking quotients is always a little dicey in homotopy theory. Our usual intuition on quotients usually requires us to quotient by a submodule or something like that, and that becomes problematic in spectra-land.

Indeed, if you want to actualize the idea that f might be an inclusion by requiring that $\mathrm{fib}(f) \simeq 0$, then you will not be in a very exciting place. Indeed, $0 \to M \to N$ will be a cofiber sequence, so it will induce a long exact sequence with
$$0 = \pi_i(0) \to \pi_i(M) \to \pi_i(M) \to \pi_{i+1}(0) = 0,$$

showing that f induces an equivalence on all homotopy groups and is as such an equivalence of spectra. Thus the condition $\mathrm{fib}(f) \simeq 0$ is equivalent to f being an equivalence, and much stronger than we might hope the correct analogue of monomorphisms should be.

This has some interesting consequences: maps you knew in algebra to be monomorphisms suddenly have kernels, albeit concentrated in higher degrees (and as the preceding paragraph shows, they have to have such "higher kernels" else they be isomorphisms). For instance, the fiber of the map $\mathbf{Z} \to \mathbf{Q}$, viewed as a map of (discrete, i.e. Eilenberg-MacLane) spectra, is $\Sigma^{-1}(\mathbf{Q}/\mathbf{Z})$, where $\mathbf{Q}/\mathbf{Z} \subseteq \mathbf{R}/\mathbf{Z} = S^1$ is the torsion subgroup of the circle.

Even slightly more shockingly, for any prime p the fiber of the map $\mathbf{Z}_p \to \mathbf{Q}_p$ is equivalent to $\Sigma^{-1}\mathbf{Q}_p/\mathbf{Z}_p \simeq \Sigma^{-1}\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$. (In the literature the notation $\mathbf{Z}/p^\infty$ is not uncommon for the Pruffer group $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$, since the latter is the colimit $\varinjlim \mathbf{Z}/p^n$, but a certain P. Scholze is especially vehement about that being bad notation, so we try our best not to use it.) This example is behind an often unintuitive equivalence between p-localization and p-completion in homotopy theory, that we may or may not talk more about at some point.

2.1.4. **Mod p Moore spectrum.** One popular example where a quotient notation similar to the one discussed in the previous subsection is actually used in practice is the *mod p Moore spectrum S/p*. This spectrum is defined as the cofiber in Sp of the multiplication-by-p-map $p : S \to S$. To get hold of this map, note that we may identify by all the things we know so far

$$p \in \mathbf{Z} \simeq \pi_0(S) \simeq \pi_0(\Omega^\infty S) \simeq \pi_0\mathrm{Map}_{\mathrm{Sp}}(S, S),$$

sending up with a homotopy class of a spectrum map $p : S \to S$ as promised. Of course we only get a homotopy class of such maps, but more than that is unfeasible to expect - the "points" of the space $\mathrm{Map}_{\mathrm{Sp}}(S, S)$ do not have any good meaning in the ∞-category land, as all is defined and considered only up to homotopy.

2.1.5. **Universal property of the Moore spectrum.** The mod p Moore spectrum S/p satisfies a universal property. Indeed, let $M \in \mathrm{Sp}$ be arbitrary. Since Map takes colimits in its first factor out to be limits, and $S/p \simeq \mathrm{cofib}(S \xrightarrow{p} S)$ is a particular kind of colimit, we find canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{Sp}}(S/p, M) \simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Sp}}(S, M) \xrightarrow{p^*} \mathrm{Map}_{\mathrm{Sp}}(S, M)) \simeq \mathrm{fib}(\Omega^\infty M \xrightarrow{p} \Omega^\infty M).$$

Thus, if we view the infinite loop space $\Omega^\infty M$ as a grouplike $\mathbb{E}_\infty$-space (recall this as the May Recognition Principle, that we talked about way back in subsection 1.6.6), thus a homotopy-coherently analogue of a commutative monoid, spectrum maps $S/p \to M$ correspond to the p-torsion in $\Omega^\infty M$.

But that doesn't just mean "points which vanish upon multiplication by p" (ignoring for the moment that "points" are not really what we should be discussing anyway), but instead a specified homotopy $p \simeq 0$ between the multiplication by p map and the multiplication by 0 map. This is an instance of a general feature of life in ∞-category land: properties often become extra structure.

Before you go thinking that the mod p Moore spectrum S/p is very much like $\mathbf{Z}/p \simeq \mathbf{F}_p$ though, allow me to dash your dreams: I believe that S/p does not admit an $\mathbb{E}_\infty$-ring structure[2]. This is a prime example of the general fact that quotienting in stable homotopy land is dicey business!

[2]There is actually a bit of a mindmelting story here. We may form the *versal quotient* $S/\!/_{\mathbb{E}_n} p$ to be the universal $\mathbb{E}_n$-ring with an $\mathbb{E}_n$-ring map from S and a nillhomotopy $p \simeq 0$. In fact, versal quotients may be constructed using Thom spectrum techniques that we will discuss in a subsequent section. The crazy thing now is that while the $\mathbb{E}_1$-versal quotient produces the Moore spectrum $S/\!/_{\mathbb{E}_1} p \simeq S/p$, the $\mathbb{E}_2$-versal quotient is $S/\!/_{\mathbb{E}_2} p \simeq \mathbf{F}_p$, the usual Eilenberg-MacLane spectrum we would expect. But $\mathbb{E}_n$-versal quotients $S/\!/_{\mathbb{E}_n} p$ for higher $n \geq 3$ *are different*! This is very exciting: it is saying that our algebraic intuition of $\mathbf{F}_2$ as the universal commutative ring of characteristic 2 is deceptive, stemming from the fact that $\mathbb{E}_2 = \mathbb{E}_\infty$ in

2.1.6. **Moore spectra, more fun!** An analogous construction, replacing multiplication by p with multiplication by n, produces Moore spectra S/n for any $n \geq 0$. Here we must for $n = 0$ interpret this as $S/0 = S$.

A slightly elaboration on this construction can make sense of a Moore spectrum SA for any abelian group A. Indeed, an arbitrary abelian group A is a colimit in the category of abelian groups of various copies of $\mathbf{Z}$, indexed on the diagram of homomorphisms $\mathbf{Z} \to A$ which pick out elements in A. (When the corresponding element of A is n-torsion, this map factors through $\mathbf{Z}/n \to A$. But that's OK, since $\mathbf{Z}/n$ itself is the cofiber of the multiplication by n map $\mathbf{Z} \xrightarrow{n} \mathbf{Z}$.) When working with spectra, the sphere spectrum S should play the role that the abelian group $\mathbf{Z}$ does in algebra. This suggests taking the colimit over the same indexing category as produces A out of copies of $\mathbf{Z}$, but with copies of S inserted instead. This produces the Moore spectrum SA.

As noted, the standard presentation of the finite group $\mathbf{Z}/n \simeq \mathrm{cofib}(\mathbf{Z} \xrightarrow{n} \mathbf{Z})$ shows that our previous definition of mod n Moore spectra is the special case $S/n \simeq S\mathbf{Z}/n$. In particular, $S\mathbf{Z} \simeq S$.

Moore spectra are distinguished by the property that $SA \otimes \mathbf{Z} \simeq A$. Indeed, we constructed them by replacing $\mathbf{Z}$ in a colimit computing A by S. Since smashing commutes with colimits in the first variable (and the second too, but that doesn't matter here), we find that applying $\otimes \mathbf{Z}$ just replaces the copies of S again with $\mathbf{Z}$. And so the colimit returns A once again.

2.1.7. **Localization of $\mathbb{E}_\infty$-rings.** Let R be an $\mathbb{E}_\infty$-ring. Given an element $x \in \pi_i(R)$, the localization of R at x is defined to be an initial object $R \to R[x^{-1}]$ among $\mathbb{E}_\infty$-ring maps $R \to A$ under which x is sent to an invertible element in $\pi_*(A)$. In ∞-categorical parlance, the rigorous statement would be that we wish the map $R \to R[x^{-1}]$ to induce for any $\mathbb{E}_\infty$-algebra map $R \to A$ an equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_R}(R[x^{-1}], A) \simeq \mathrm{Map}_{\mathrm{CAlg}}(R, A) \times_{\pi_2(\mathrm{A})} \pi_*(A)^\times,$$

where the pullback is along the map sending $f : R \to A$ to $\pi_2(f)(x)$, and the map $\pi_*(A)^\times \hookrightarrow \pi_*(A) \to \pi_2(A)$ composing the obvious inclusion with the obvious projection.

Evidently this is just the usual universal property of localization, and if $i = 0$, then $\pi_0(R[x^{-1}]) \simeq \pi_0(R)[x^{-1}]$. The underlying classical ring of the $\mathbb{E}_\infty$-ring localization is thus the ordinary localization of rings.

2.1.8. **Localization of modules.** Whenever M is an R-module (recall: always means "R-module *spectrum*"), and as before $x \in \pi_i(R)$, the R-module localization $M \to M[x^{-1}]$ may be defined through a similar universal property as for rings above, with seeking the module action of x to invertible. Alternatively, we can simply set $M[x^{-1}] \simeq M \otimes_R R[x^{-1}]$ and recover the same object.

For an explicit construction of localization, let us note that the element $x \in \pi_i(R)$ may be identified with an R-linear map $R \xrightarrow{x} \Sigma^{-i} R$. Aplying the relative smash product $- \otimes_R M$, we obtain a morphism $M \xrightarrow{x} \Sigma^{-i} M \in \mathrm{Mod}_R$. Then we may obtain the localization explicitly as the colimit in the ∞-category Mod_R of the form

$$M[x^{-1}] \simeq \varinjlim(M \xrightarrow{x} \Sigma^i M \xrightarrow{\Sigma^i x} \Sigma^2 M \to \cdots).$$

Note that this is analogous to one way to comute localization in usual algebra too (where the are of course no suspensions, as there is only π_0).

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the usual 1-categorical world where such intuition stemms from. On the other hand, to actually achieve $p \simeq 0$ in $\mathbb{E}_\infty$-ring land, more homotopies need to be specified!

2.1.9. p-**localization.** We can play the localization game with repsect to an arbitrary prime $p \in \mathbf{Z} \simeq \pi_0(S)$ to localize any spectrum M to $M[p^{-1}]$. When plugging in an $\mathbb{E}_\infty$-ring, of course the localization $R[p^{-1}]$ will remain such.

Now note this funny thing: a spectrum M being p-local, which is to say that $M \simeq M[p^{-1}]$, is equivalent to asking that multiplication by p act invertibly on M (in fact, it's even enough to ask that it acts invertibly on all the homotopy groups $\pi_i(M)$). That is equivalent by the discussion in subsection 2.1.2 to asking that $\mathrm{cofib}(M \xrightarrow{p} M) \simeq 0$. But of course the smash product of spectra commutes with colimits in each factor by definition, and so $\mathrm{cofib}(M \xrightarrow{p} M) \simeq \mathrm{cofib}(S \xrightarrow{p} S) \otimes M \simeq S/p \otimes M$.

In conclusion: a spectrum M is p-local if and only if $S/p \otimes M \simeq 0$. This sort of phrasing of locality in terms of smash-vanishing is the starting point of Bousfield localization, a way to localize at any spectrum. But perhaps let us not get into that now.

2.1.10. **Rationalization.** The localization procedure outlined in the previous few sections could of course be carried out for several elements at the same time (or iteratively, if you prefer). Doing it all the primes $p \in \mathbf{Z}$ produces what is called *rationalization*, and for a spectrum M we denote its rationalization as $M_{\mathbf{Q}}$.

Recall however the well-known fact that stable homotopy groups of spheres $\pi_i(S)$ are all torsion for $i \geq 1$. This implies that $S_{\mathbf{Q}}$ possesses no homotopy groups but the 0-th one. As such it is a discrete spectrum, and more precisely $S_{\mathbf{Q}} \simeq \mathbf{Q}$.

This has the consequence that the rationalization $M_{\mathbf{Q}}$ of any spectrum M comes naturally equipped with an $S\mathbf{Q} \simeq \mathbf{Q}$-module structure, showing that rationality immediately reduces spectra to chain complexes of ordinary $\mathbf{Q}$-modules. The interesting parts of stable homotopy theory thus lie over primes in the land of torsion.

For any M we also have $M_{\mathbf{Q}} \simeq M \otimes \mathbf{Q}$, which we say in fancy words as it being a "smashing localization". A similar thing holds for the p-localization of the previous subsection, where the formula is $M[p^{-1}] \simeq M \otimes S[p^{-1}]$, which we have seen in subsection 2.1.8.

2.2. Topological K-theory

Since most of the initial interest in spectra was from the perspective of cohomology theories, it is not surprising that that is where some of the first interesting examples of spectra arise from. The first extraordinary cohomology theory was complex K-theory, stemming essentially from Grothendieck's work on the Riemann-Roch theorem (though that was the algebraic analogue, and the topological is due to Atiyah and Hirzebruch a year or two later).

2.2.1. **The 0-th complex K-theory.** For a pointed finite CW complex X, we set $\mathrm{KU}^0(X)$ to be the set of complex vector bundles E over X (of finite rank) modulo the equivalence relation under which two complex bundles E and E' on X are equivalent if and only if there exist two trivial complex bundles ε_1 and ε_2 on X and a vector bundle isomorphism $E \oplus \varepsilon_1 \cong E' \oplus \varepsilon_2$. This equivalence is called stable isomorphism, so $\mathrm{KU}^0(X)$ consists of stable isomorphism classes of vector bundles on X.

Direct sum of vector bundles makes $\mathrm{KU}^0(X)$ into a commutative monoid, but as is a little less obvious, it is in fact a group. Indeed, since X is compact by assumption, we can collect local trivializations of a vector bundle E together into an embedding $E \hookrightarrow \mathbf{C}^N_X$ into a trivial bundle on X of some sufficiently high rank N. This trivial complex vector bundle carries a natural inner product structure (take the usual Hermitian inner product on $\mathbf{C}^N_X$), allowing us to form the orthogonal complement $E^\perp$ fiber-wise. This is evidently also a vector bundle over X, and since it satisfies the isomorphism $E \oplus E^\perp \cong \mathbf{C}^N_X$, we see that $E^\perp$ is the stable-isomorphism inverse to E.

2.2.2. **Bott periodicity.** We have only defined the 0-th group of complex K-theory so far. Instead of defining $\mathrm{KU}^i(X)$ explicitly for all $i \in \mathbf{Z}$, we instead have a theorem take us the rest of the way. The theorem in question is the following landmark result in the history of algebraic topology and homotopy theory alike:

Theorem 15 (Bott Periodicity)**.** *For any pointed finite CW complex X there is a canonical isomorphism* $\mathrm{KU}^0(\Sigma^2 X) \cong \mathrm{KU}^0(X)$.

As complex K-theory KU should be a cohomology theory, it should satisfy the Eilenberg-Steenrod axioms. The relevant one here is the suspension axiom, requiring that

$$\mathrm{KU}^{i+1}(\Sigma X) \simeq \mathrm{KU}^i(X)$$

for all $i \in \mathbf{Z}$.

Since we already know KU^0, we can use Bott periodicity to build the suspension axiom in "by force" and define even-graded cohomology groups as $\mathrm{KU}^{2n}(X) := \mathrm{KU}^0(X)$ and odd-graded ones as $\mathrm{KU}^{2n+1}(X) := \mathrm{KU}^0(\Sigma X)$. Checking the Eilenberg-Steenrod axioms is now a breeze. Recall from subsection 1.4.11 the Brown Representability Theorem for cohomolology theories. Through it, the cohomology theory KU^i defines a perfectly good spectrum, which we shall denote KU, and call the *complex topological K-theory spectrum.*

2.2.3. **Unreduced 0-th complex K-theory & Grothendieck group.** You might have been a little surprised by the previous subsection. Indeed, you might have heard before that complex K-theory sends a space to the Grothendieck group of vector bundles on it. Let's briefly recall how that works.

Let $\mathrm{Vect}_{\mathbf{C}}(X)$ denote the set of isomorphism classes of complex vector bundles on a (non-pointed) finite CW-complex X. Direct sum of vector bundles makes it into a monoid, from which we can "group complete", i.e. adjoin formal inverses -E for every (iso class of a) complex vector bundle E on X satisfying by definition $E \oplus (-E) \cong 0$. This has the effect of allowing us to multiply vector bundles by integers, with $nE = E^{\oplus n}$ for $n \geq 1$. In particular, we obtain an abelian group, and this is $\mathrm{KU}^0_{\mathrm{unred}}(X)$, the *unreduced 0-th complex K-theory group of X.*

This relates to the reduced version of complex K-theory that we discussed in the previous section through the canonical isomorphism

$$\mathrm{KU}^0_{\mathrm{unred}}(X) \cong \mathrm{KU}^0(X_+).$$

Indeed, that is how reduced and unreduced cohomology theory coincide with each other in general. From the point of view of the complex K-theory spectrum KU, we have for all $i \in \mathbf{Z}$

$$\mathrm{KU}^i(X) \simeq \pi_{-i}\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, \mathrm{KU})\mathrm{KU}^i_{\mathrm{unred}}(X) \simeq \pi_{-i}\mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, \mathrm{KU}).$$

Thus we see that KU encodes complex K-theory reduced and unreduced alike, and the difference is only that once we are mapping Σ^∞ a space and once Σ^∞_+ of a pointed space into it.

Though it is really the spectum KU that we are after interested in here, the idea of obtaining K-theory by taking a monoid of bundles under direct sum and group completing it to obtain an abelian group, will have future significance. Namely, we will encounter it again, when we discuss another class of examples of spectra: algebraic K-theory.

2.2.4. **A few words on classifying spaces.** If you are fond of classifying spaces, there is a more concise and more elegant way of phrasing complex K-theory. So let's say a few words about topological classifying spaces, that will be pertinent in what follows.

Let BU be the classifying space for the infinite unitary group $\mathrm{U} = \varinjlim_n \mathrm{U}(n)$, i.e. the homotopy quotient $*/\mathrm{U}$ (or more classically: choose a contractible space EU with a free U-action, and form the usual quotient $\mathrm{BU} := \mathrm{EU}/\mathrm{U}$. But this is really just using replacement to compute homotopy colimits - the map $\mathrm{EU} \to *$ is a cofibrant replacement in the model category of CW-complexes with U-action).

Converely, you can construct $\mathrm{BU} = \varinjlim_n \mathrm{BU}(n)$ directly as a colimit (without passing through the infinite unirary group U). This has the advantage that everything is sight is about finite-rank vector bundles: the maps $X \to \mathrm{BU}(n)$ are in natural equivalence with rank n complex vector bundles on X (this is the universal property of a classifying space, afterall), and the maps $\mathrm{BU}(n) \to \mathrm{BU}(n+1)$ correspond on the level of bundles to the "stabilization" map $E \to E \oplus \mathbf{C}_X$, in the sense of bundle stabilization as discussed in subsection 2.2.1. In this sense, BU may be viewed as the classifying space of stable-isomorphism-classes of complex vector bundles (technically we only get the rank 0 stable vector bundles, but let us ignote that for the moment).

On the other hand (and we will make no use of this here, but it's cool) the colimit description of BU gives an interpretation of this classifying space in terms of perhaps more familiar objects. Indeed, we may identify $\mathrm{BU}(n) \simeq \mathrm{Gr}(n, \mathbf{C}^\infty)$ the classifying space of rank n-vector bundles with the Grassmannian of n-dimensional complex linear subspaces in the infinite dimensional space $\mathbf{C}^\infty$. That should make sense; the Grassmanian $\mathrm{Gr}(n, \mathbf{C}^\infty)$ has as points n-dimensional complex vector spaces, so what should a map $X \to \mathrm{Gr}(n, \mathbf{C}^\infty)$ be but a way to associate to every point a vector space in a continuous fashion - lo and behold the universal property of the classifying space $\mathrm{BU}(n)$. So BU is in some sense $\mathrm{Gr}(\infty, \mathbf{C}^\infty)$, but unlike $\mathrm{Gr}(n, \mathbf{C}^n)$ which is boring, $\mathbf{C}^\infty$ admits a lot of different copies of itself inside it, so this Grassmanian is interesting.

Applying the formula from the previous paragraph for $n = 1$, we get the fundamental equivalence $\mathrm{BU}(1) \simeq \mathbf{CP}^\infty$, which shows up often in homotopy theory. This space has many other names too, btw: since $\mathrm{U}(1) \simeq S^1 \simeq \mathrm{B}\mathbf{Z}$, this is also $\mathrm{B}S^1 \simeq \mathrm{B}^2\mathbf{Z} \simeq K(\mathbf{Z}, 2)$. It also has more esoteric names such as $\mathrm{PU}(\mathcal{H})$ for a separable infintie-dimensional Hilbert space $\mathcal{H}$, but let's leave it at that.

2.2.4.1. *Complex K-theory in terms of classifying spaces.* In light of the discussion in the previous subsection, the definition of the 0-th K-theory group of a finite pointed space X that we gave in subsection 2.2.1 above amounts to saying that $\mathrm{KU}^0(X) = \pi_0\mathrm{Map}_{\mathcal{S}_*}(X, \mathrm{BU} \times \mathbf{Z})$, the homotopy classes of pointed maps[3] $X \to \mathrm{BU} \times \mathbf{Z}$, where we choose the trivial bundle as the basepoint in BU (the copy of $\mathbf{Z}$ keeps track of the rank of a "virtual" stable vector bundle).

Bott periodicity then follows from and is equivalent to the classifying space result that $\Omega^2\mathrm{BU} \simeq \mathrm{BU} \times \mathbf{Z}$ or equivalently $\Omega^2\mathrm{U} \simeq \mathrm{U}$. Indeed, it is in this form that Bott originally stated his periodicity result.

2.2.5. **Homotopy groups of** KU. We will use the classifying space approach to complex K-theory, as given in the previous section, to obtain a straightforward computation of the homotopy groups of KU. Note from plugging $X = *$ into the correct formulas in subsection 2.2.3, that we may express the homotopy groups of the complex toplogical K-theory spectrum as

$$\pi_i(\mathrm{KU}) = \mathrm{KU}^{-i}(S^0) = \mathrm{KU}^{-i}_{\mathrm{unred}}(*),$$

allowing us to think of the homotopy groups as the value of the associated unreduced cohomology theory on the point. This works just as well for any spectrum, viewed as a cohomology theory.

Since the cohomology theory in question is periodic, we find that the even homotopy groups of KU are

$$\pi_{\mathrm{ev}}(\mathrm{KU}) \simeq \mathrm{KU}^{\mathrm{ev}}(S^0) \simeq \mathrm{KU}^0(S^0) \simeq \pi_0\mathrm{Map}_{\mathcal{S}_*}(S^0, \mathrm{BU} \times \mathbf{Z}) \simeq \pi_0(\mathrm{BU} \times \mathbf{Z}) \simeq \mathbf{Z}.$$

[3] A popular and traditional notation for the set of homotopy classes $\pi_0\mathrm{Map}_{\mathcal{S}_*}(X, Y)$ or for $\pi_0\mathrm{Map}_{\mathcal{S}}(X, Y)$ is $[X, Y]$. These are the Hom sets in the homotopy categories $\mathrm{Ho}(\mathcal{S}_*)$ and $\mathrm{Ho}(\mathcal{S})$ respectively, but since we are viewing the underlying ∞-categories as more fundamental in these emails, we will prefer the more explicit notation.

To figure out the odd ones, we proceed similarly by identifying $\pi_{\text{odd}}(\mathrm{KU}) \simeq \mathrm{KU}^{-1}(S^0) \simeq \mathrm{KU}^0(S^1)$ through either the suspension axiom, or the definition of KU^i we gave in subsection 2.2.2. Using once again the classifying space approach from subsection 2.2.4, we get

$$\mathrm{KU}^0(S^1) \simeq \pi_0\mathrm{Map}_{\mathcal{S}_*}(S^1, \mathrm{BU}\times\mathbf{Z}) \simeq \pi_0(\Omega(\mathrm{BU}\times\mathbf{Z})) \simeq \pi_0\mathrm{U} \simeq \varinjlim_n \pi_0\mathrm{U}(n) \simeq 0,$$

since[4] all the unitary groups $\mathrm{U}(n)$ are connected. Together we find that

$$\pi_{\text{odd}}(\mathrm{KU}) \simeq 0.$$

2.2.6. **The underlying infinite loop space of** KU**.** Playing a similar game to the previous subsection, we will identify the underlying infinite loop space $\Omega^\infty\mathrm{KU}$ (note that up until now, we were only talking about its homotopy groups, which is to say about the cohomology theory, not about the spectrum itself). By the adjunction between Σ^∞ and Ω^∞, we find for any pointed space X a canonical equivalence

$$\mathrm{Map}_{\mathcal{S}_*}(X, \Omega^\infty\mathrm{KU}) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty X, \mathrm{KU})$$

to which we apply the functor π_i (as these are homotopy groups of spaces, we must have $i \geq 0$) to find

$$\pi_i\mathrm{Map}_{\mathcal{S}_*}(X, \Omega^\infty\mathrm{KU}) \simeq \mathrm{KU}^{-i}(X) \simeq \mathrm{KU}^0(\Sigma^i X) \simeq \pi_0\mathrm{Map}_{\mathcal{S}_*}(\Sigma^i X, \mathrm{BU}\times\mathbf{Z}).$$

Suspension is a limit and as such goes out the first factor of Hom to become a colimit, so remembering the definition of higher homotopy groups, we obtain further natural equivalences

$$\pi_0\mathrm{Map}_{\mathcal{S}_*}(\Sigma^i X, \mathrm{BU}\times\mathbf{Z}) \simeq \pi_0\Omega^i\mathrm{Map}_{\mathcal{S}_*}(X, \mathrm{BU}\times\mathbf{Z}) \simeq \pi_i\mathrm{Map}_{\mathcal{S}_*}(X, \mathrm{BU}\times\mathbf{Z}).$$

Connecting all these isomorphisms, we can recognize them as stemming from a map $\mathrm{BU}\times\mathbf{Z} \to \Omega^\infty\mathrm{KU}$, and since this map then induces isomorphisms on all homotopy groups, it must be an equivalence. Thus in summary we have $\Omega^\infty\mathrm{KU} \simeq \mathrm{BU}\times\mathbf{Z}$, which is indeed an infinite loop space by Bott periodicity.

2.2.7. **Other versions of topological K-theory.** Throughout everything above, we insistently considered only complex vector bundles. Alas, there is nothing special about $\mathbf{C}$, and we could have played the same game with $\mathbf{R}$. In that case the classifying space BU above becomes replaced with BO, the classifying space of the infinite orthogonal group $\mathrm{O} = \varinjlim_n \mathrm{O}(n)$. This space too satisfies a Bott periodicity, but with a period of 8 instead of 2. That is to say, we have an equivalence $\Omega^8\mathrm{O} \simeq \mathrm{O}$ or equivlanetnly

$$\Omega^8\mathrm{BO} \simeq \mathrm{BO}\times\mathbf{Z}.$$

This allows us to use the same trick as before and define $\mathrm{KO}^i(X) = \mathrm{KO}^{i+8}(X)$, and in conjunction with the suspension axiom for a cohomology theory, we get a spectrum KO. It is sometimes called real topological K-theory, but the name "real K-theory" is a little disputed. Where KU has to do with complex vector bundles, KO has to do with real ones. Its underlying loop space is $\Omega^\infty\mathrm{KO} \simeq \mathrm{BO}\times\mathbf{Z}$ and its homotopy groups, being 8-periodic, are

$$\begin{array}{llll} \pi_0(\mathrm{KO}) = \mathbf{Z}, & \pi_1(\mathrm{KO}) = \mathbf{Z}/2, & \pi_2(\mathrm{KO}) = \mathbf{Z}/2, & \pi_3(\mathrm{KO}) = 0, \\ \pi_4(\mathrm{KO}) = \mathbf{Z}, & \pi_5(\mathrm{KO}) = 0, & \pi_6(\mathrm{KO}) = 0, & \pi_7(\mathrm{KO}) = 0. \end{array}$$

Of course a similar game could be played with certain other groups G, leading to topological K-theory spectra KG, e.g. KSp for the symplectic group (it is a wonderful, though rarely genuinely problematic, accident that Sp is both the standard notation for the ∞-category of spectra and for the symplectic group). Unlike KU and KO, which

[4]The observant reader might also inquire about why we were to commute the colimit past π_0. Well, the functor $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$ is a left adjoint to the inclusion $\mathcal{S}\mathrm{et} \hookrightarrow \mathcal{S}$ identifying sets with discrete spaces. And left adjoints of course always commute with colimits. :)

are both landmark examples, these other topological K-theories are more of a curiosity though.

2.2.8. **Ring structure on topological K-theory.** Topological K-theory, real and complex alike, is built out of vector bundles. The spectrum addition is represented by the direct sum of vector bundles, but what does the tensor product represent?

The answer, of course, is a ring structure. More precisely, both KU and KO are $\mathbb{E}_\infty$-rings. On the level of cohomology theories, this implies that $\mathrm{KU}^*(X)$ and $\mathrm{KO}^*(X)$ (here $*$ means implicit summation over all possible values $* = i$) are graded rings for any finite space X, and on the level of vector bundle representatives for elements of these rings, the ring multiplication is indeed given by the tensor product of vector bundles.

2.2.9. **The conjugation action.** The $\mathbb{E}_\infty$-ring structures on KU and KO are very similar, the first one arising from $\otimes_{\mathbf{C}}$ and the second one from $\otimes_{\mathbf{R}}$. In fact, the complexification map $V \mapsto V \otimes_{\mathbf{R}} \mathbf{C}$ (of vector spaces, or if you want, vector bundles; really a map $\mathrm{BO}(n) \to \mathrm{BU}(n)$ of classifying spaces) induces a $\mathbb{E}_\infty$-ring map $c : \mathrm{KO} \to \mathrm{KU}$.

In fact, the conjugation action of the cyclic group $C_2 \cong \mathbf{Z}/2$ acting on $\mathbf{C}$, and through it on any complex vector space and bundle alike, induces a C_2-action on KU in the ∞-category of spectra. That means no more and no less than a functor $\mathrm{B}C_2 \to \mathrm{Sp}$, where $\mathrm{B}C_2$ is viewed as an ∞-groupoid and in particular as an ∞-category, and where the restriction $* \to \mathrm{B}C_2 \to \mathrm{Sp}$ gives rise to the "underlying object" on which the group C_2 acts, in our case the spectrum KU. Passing to the limit of the functor $\mathrm{B}C_2 \to \mathrm{Sp}$ produces the (homotopy) fixed-points KU^{C_2} (in more traditional literature denoted KU^{hC_2}. I will switch back and forth depending on the mood. But as usual, the "homotopy" fixed-points are the natural ones that we get by trying to say "fixed points" in our ∞-categorical setting. From this perspective, the h-less notation seems to be more sensible.)

Now the $\mathbb{E}_\infty$-ring map $c : \mathrm{KO} \to \mathrm{KU}$ given by complexification is C_2-equivariant with respect to the just-described conjugation C_2-action on KU and the trivial C_2-action on KO (given by the constant functor $\mathrm{B}C_2 \to * \xrightarrow{\mathrm{KO}} \mathrm{Sp}$). In fact, more is true: it is the universal such map from a trivial C_2-action. That is to say, the map c exhibits an equivalence

$$\mathrm{KO} \simeq \mathrm{KU}^{hC_2}.$$

This is closely analogous to how algebraic geometry over $\mathbf{R}$ is nothing but algebraic geometry over $\mathbf{C}$, conscious of a C_2-action. It is also analogous of Galois theory, where a field extension L/K being Galois implies among other things that $K \simeq L^{\mathrm{Gal}(L/K)}$. This analogy has been made precise by Rognes, who developed a theory of Galois extensions of $\mathbb{E}_\infty$-rings which also encapsulates a number of other exciting examples, and of which $c : \mathrm{KO} \to \mathrm{KU}$ is a prime example (other than, boringly, ordinary Galois extensions viewed as discrete spectra).

2.2.10. **The Chern character.** Grothendieck initially invented K-theory (in the algebraic setting, and only the 0-th one) in the course of stating and proving what is today known as the Grothendieck-Riemann-Roch Theorem. This theorem is all about a certain construction called the Chern character, and we will discuss its analogue in algebraic topology (i.e. for manifolds, not for varieties) here.

On the most basic level, the Chern character is a ring homomorphism $\mathrm{ch} : \mathrm{KU}^0_{\mathrm{unred}}(X) \to \mathrm{H}^*(X; \mathbf{Q})$, sending (the isomorphism class of) a complex line bundle L on X to

$$\mathrm{ch}(L) := e^{c_1(L)} = \sum_{0 \le n \le \dim X} \tfrac{1}{n!} c_1(L)^n.$$

Note the $\frac{1}{n!}$-factors - they necessitate the Chern character to take values in cohomology with $\mathbf{Q}$-coefficients, i.e. ch it doesn't factor through $\mathrm{H}^*(X; \mathbf{Z})$. The Chern character is furthermore required to be compatible with pullbacks along maps $f : X \to Y$ on both sides, qualifying it as a characteristic class. Though we've only explicitly specified it on line bundles, compatibility with pullback and it being a ring homomorphism determine ch completely for all vector bundles, due to a certain result called the Splitting Principle, of which we shall say little more than that it allows for reduction to sums of line bundles.

The cool thing for us here is that this extends to a spectrum-level $\mathbb{E}_\infty$-ring map $\mathrm{ch} : \mathrm{KU} \to \bigoplus_{i\in\mathbf{Z}} \Sigma^{2i}\mathbf{Q}$, which on π_0 (and evaluated on a space) recovers the Chern character discussed in the previous paragraph. The reason for the weird-looking direct sum and suspensions on the RHS is because KU is a 2-periodic spectrum, so we must also appropriately 2-periodize the Eilenberg-MacLane spectrum $\mathbf{Q}$ in order to make it capable of receiving a map from KU. In particular, the RHS spectrum may be identified with $\mathbf{Q}[\beta^{\pm 1}]$, the $\mathbb{E}_\infty$-ring of Laurent polynomials with coefficients in $\mathbf{Q}$ in a degree 2 variable β. Thus we obtain the Chern character, incarnated as an $\mathbb{E}_\infty$-ring map $\mathrm{ch} : \mathrm{KU} \to \mathbf{Q}[\beta^{\pm 1}]$.

The Riemann-Roch Theorem became in Grothendieck's hands an upon-rationalizing isomorphism $K^0(X)_\mathbf{Q} \simeq A^*(X)_\mathbf{Q}$ of a prescribed form (Chern character + Todd class ...). The analogous statement in algebraic topology is that the above-discussed Chern character map induces an equivalence

$$\mathrm{KU}_\mathbf{Q} \simeq \mathbf{Q}[\beta^{\pm 1}],$$

where the left hand side is $\mathrm{KU}_\mathbf{Q} \simeq \mathrm{KU} \otimes \mathbf{Q}$, the smash product of KU with the rationals, which is to say, the rationalization of the complex topological K-theory spectrum.

2.2.11. **Snaith's Theorem.** Before we leave the wonderful world of topological K-theory, let us reflect upon what makes it so interesting. The answer is surely Bott periodicity, or from the perspective of the spectrum KU, its 2-periodicity. Let us discuss Snaith's Theorem, which is essentially a claim about Bott periodicity determining KU.

For any $\mathbb{E}_\infty$-ring R, the 0-th homotopy group $\pi_0(R)$ inherits a commutative ring structure, while the other homotopy groups $\pi_i(R)$ carry a canonical $\pi_0(R)$-module structure. As such, the Bott periodicity isomorphism $\pi_0(\mathrm{KU}) \simeq \pi_2(\mathrm{KU})$ may be viewed as a $\pi_0(\mathrm{KU}) = \mathbf{Z}$-linear map. That is to say, it specifies, as the image of $1 \in \mathbf{Z}$ under it, an element $\beta \in \pi_2(\mathrm{KU})$. By construction of complex topological K-theory, the element β is invertible in the graded ring $\pi_*(\mathrm{KU})$.

But what is this Bott element β, geometrically speaking? Well, consider the classifying space BU. It contains inside the classifying space of complex line bundles $\mathrm{BU}(1) \simeq \mathbf{CP}^\infty$. In its guise as the infinite complex projective space, we can find an non-trivial element $\beta \in \pi_2(\mathbf{CP}^\infty)$: consider the inclusion

$$S^2 \simeq \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty.$$

The homotopy class of this map is the promised element $\beta \in \pi_0\mathrm{Map}_{\mathcal{S}_*}(S^2, \mathbf{CP}^\infty) = \pi_2(\mathbf{CP}^\infty)$. The assertion that this homotopy group element is non-trivial follows by recognizing the map in question $S^2 \to \mathbf{CP}^\infty \simeq \mathrm{B}S^1$ as the classifying map for the Hopf fibration $S^3 \to S^2$. Non-triviality of the Hopf fibration is now equivalent to the fact that $\beta \neq 0$.

Now consider the composite map of pointed spaces

$$\mathbf{CP}^\infty \simeq \mathrm{BU}(1) \hookrightarrow \mathrm{BU} \simeq \mathrm{BU} \times \{0\} \hookrightarrow \mathrm{BU} \times \mathbf{Z} \simeq \Omega^\infty\mathrm{KU}.$$

On π_2 this sends the Hopf fibration $\beta \in \pi_2(\mathbf{CP}^\infty)$ to the invertible element $\beta \in \pi_0(\mathrm{KU})$, and furthermore the map in question is a nice map of $\mathbb{E}_\infty$-spaces (recall: these are spaces with a homotopy coherently commutative monoid structure). By adjunction this map corresponds to a $\mathbb{E}_\infty$-ring map $S[\mathbf{CP}^\infty] \to \mathrm{KU}$. On the level of π_2, this map becomes $\mathbf{Z}[\pi_2(\mathbf{CP}^\infty)] \to \pi_2(\mathrm{KU})$, once more sending β to β. But since β is invertible in $\pi_*(\mathrm{KU})$, the universal property of localization gives rise to a $\mathbb{E}_\infty$-ring map $(S[\mathbf{CP}^\infty])[\beta^{-1}] \to \mathrm{KU}$.

Theorem 16 (Snaith). *The described map is an equivalence of $\mathbb{E}_\infty$-rings*

$$\mathrm{KU} \simeq \big(S[\mathbf{CP}^\infty]\big)[\beta^{-1}].$$

If you wish, you can view this as an alternative characterization of complex topological K-theory. As expected out of such a non-trivial theorem, it makes several other hard theorem easy to prove. It also shows that topological K-theory, though initially constructed and viewed differently, still essentially belongs to the setting discussed in section 1.2.1. That is to say, it can be obtained via algebraic operations from suspension spectra of familiar spaces.

2.3. Thom spectra

In the previous section we saw how vector bundles may be used to give rise to topological K-theory. But there is another way to create spectra out of vector bundles, and it goes by the name of this section.

The idea of Thom spectra, or at least of the preceding Thom spaces, was first extensively studied and used to great avail in Rene Thom's thesis, a document that many have called the true birthplace of modern homotopy theory. So, you know, no pressure with your thesis!

2.3.1. **The easy post-modren approach.** Though Thom spectra are quite old, the most elegant approach to constructing them that I am aware of is due to Ando-Blumberg-Gepner-Hopkins-Rezk, using a heavy dose of ∞-categorical machinery. We discuss this first (as it's quite easy) and only later indicate the slightly more intricate classical construction.

2.3.2. **Local systems of spectra.** Fix for a moment a space X. We wish to consider local systems of spectra on X. Naively these should be families of spectra $\{E_x\}_{x\in X}$ such that

- every point $x \in X$ gives rise to a spectrum E_x
- every path $x \to y$ in X gives rise to an equivalence of spectra $E_x \simeq E_y$
- every 2-simplex (or if you want, 2-cell) in X with vertices x, y, z gives rise to a homotopy exhibiting commutativity of the relevant triangle of maps between E_x, E_y, and E_z in Sp
- etc.

With ∞-categories at our disposal, this is almost trivial to formalize: we view X as an ∞-groupoid, and define a local system of spectra on X to be a functor $E : X \to \mathrm{Sp}$. They clearly form an ∞-category, which is nothing but $\mathrm{Fun}(X, \mathrm{Sp})$.

These also go by the name parametrized spectra, and have originally been studied in an explicit point-set model (without ∞-categorical machinery) in a tour-de-force book of May-Sigurdsson. But then the ABGHR boys came together and rephrased everything in extremely elegant terms, and we are following them here.

Of course the choice of taking values in the ∞-category Sp is arbitrary. Nothing would change if we considered local systems of R-modules for any $\mathbb{E}_\infty$-ring R. Because it's all the same, we stick to the case $R \simeq S$ here.

2.3.3. **Functoriality of local systems.** We don't to know much about the techonology of local systems of spectra, so we will be brief. A map of spaces $f : X \to Y$ induces a number of maps between local systems of spectra, just as you would expect:

- A pullback $f^* : \mathrm{Fun}(Y, \mathrm{Sp}) \to \mathrm{Fun}(X, \mathrm{Sp})$, given by composing a functor $E : Y \to \mathrm{Sp}$ with f.
- A "left" pushforward $f_! : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Fun}(Y, \mathrm{Sp})$, given by left Kan extension along f.
- A "right" pushforward $f_* : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Fun}(Y, \mathrm{Sp})$, given by right Kan extension along f.

By the definition of Kan extensions, we find that these functors form adjunctions $f_! \dashv f^* \dashv f_*$. They also satisfy the base-change formula you would expect, etc.

The case of most interest to us is when we consider the terminal map $p : X \to *$. Then the functorialities become

- The pullback $p^* : \mathrm{Sp} \to \mathrm{Fun}(X, \mathrm{Sp})$ sends a spectrum E to the constant local system with value E.
- The "left" pushforward $p_! : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$ sends a local system of spectra $E : X \to \mathrm{Sp}$ to the colimit $\varinjlim E \in \mathrm{Sp}$. More poetically we can write $p_! E \simeq \varinjlim_{x \in X} E_x$.
- The "right" pushforward $p_* : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$ sends a local system of spectra $E : X \to \mathrm{Sp}$ to the limit $\varprojlim E \in \mathrm{Sp}$. More poetically we can write $p_* E \simeq \varprojlim_{x \in X} E_x$.

Analogy with usual local systems dictates that we think of p_* and $p_!$ as two versions of global sections, perhaps one viewed as "with compact support" and the other one without. But let us not take all of this this too seriously.

Cohomology with compact support appears in the version of Poincare duality for non-compact manifolds. As such, another reasonably popular set of terminology and notations is to call $C_*(X; E) := p_! E$ and $C^*(X; E) := p_* E$ the *chains* and *cochains on X with coefficients in E* respectively. When $E \simeq p^* A$ is the constant local system with the value $A \in \mathcal{A}\mathrm{b} \simeq \mathrm{Sp}^{\heartsuit} \subseteq \mathrm{Sp}$, this recovers the usual meaning of chains and cochains, hence being a sensible terminology. Furthermore if $E \in \mathrm{Sp}$ is any spectrum, identified with a local system of spectra via the pullback p^*, we have $E_i(X) \simeq \pi_i C_*(X; E)$ and $E^i(X) \simeq \pi_{-i} C^*(X; E)$ for all $i \in \mathbf{Z}$, where E_i and E^i denote the (non-reduced) homology and cohomology theory corresponding to the spectrum E.

2.3.4. **Example: spectra with a G-action.** When we consider local systems of spectra on the classifying space $X \simeq BG$ of a group (or, if you prefer, grouplike $\mathbb{E}_1$-space) G, we recover the theory of G-actions of spectra. Indeed, both ∞-categories were defined to be $\mathrm{Fun}(BG, \mathrm{Sp})$. That is to say, a local system of spectra on BG is the same thing as a spectrum with a G-action, just as it surely should be.

In that case the functoriality with respect to the terminal map $p : BG \to *$ recovers

- The spectrum M with a trivial G-action as $p^* M$.
- The homotopy coinvariants (or quotient) $E_{hG} \simeq p_! E$ for any $E \in \mathrm{Fun}(BG, \mathrm{Sp})$.
- The homotopy invariants (or fixed-points) $E^{hG} \simeq p_* E$ for any $E \in \mathrm{Fun}(BG, \mathrm{Sp})$.

2.3.5. **The J-homomorphism.** Now we are almost ready to discuss the construction of Thom spectra, but for one thing: we must familiarize ourselves with the J-homomorphism. That is a wonderful and classical map in homotopy theory, which arises as follows.

Consider the n-sphere S^n as the one-point compactification of $\mathbf{R}^n$. The isometry group $\mathrm{O}(n)$ of the latter naturally extends to act on S^n by fixing the point at infinity. Thus choosing the point at infinity as the basepoint for S^n, any isometry $f \in \mathrm{O}(n)$ gives rise to a map $f : S^n \to S^n$. That is to say, we obtain a map $\mathrm{O}(n) \to \Omega^n S^n$ (where we recall that based loops, as their name suggests, may be given as $\Omega X \simeq \mathrm{Map}_{\mathcal{S}_*}(S^1, X)$, and likewise for Ω^n with S^n). The action of $\mathrm{O}(n)$ on S^n is compatible in passage $n \mapsto n+1$ through the isometric isomorphism $\mathbf{R} \oplus \mathbf{R}^n = \mathbf{R}^{n+1}$. We may therefore pass to the colimit as $n \to \infty$ of the maps $\mathrm{O}(n) \to \Omega^n S^n$ to obtain a map $J : \mathrm{O} \to \Omega^\infty S$. This is the most basic form of the J-homomorphism.

Passing to homotopy, we obtain an explicit family of maps $J : \pi_k(\mathrm{O}(n)) \to \pi_{k+n}(S^n)$, compatible with varying n. By taking n to be big enough, the codomain will stabilize to the stable homotopy group $\pi_k^s(S^0) = \pi_k(S)$. On the other hand, the left-hand-side will still be homotopy groups of orthogonal groups, well understood by Bott periodicity phenomena (if nothing else). In this way, the J homomorphism traces out stable homotopy classes, possibly in high-degree homotopy groups of spheres, and indeed the major application of it in stable homotopy theory has been to try to bootstrap computations of homotopy groups of spheres off it.

2.3.6. **The J-homomorphism not a homomorphism.** But back to $J : \mathrm{O} \to \Omega^\infty S$ in the abstract. Since $\mathrm{O}(n)$ is a group, and that the construction of the J-homomorphism was stated purely in terms of actions, it seems plausible to expect that the procedure explained in the last subsection would leave us with something like a group homomorphism at the end. The name "the J-*homomorphism*" sure suggests so too. And while we may recall that $\Omega^\infty S$, by the virtue of being an infinite loop space and May's Recognition Theorem, carries the structure of an $\mathbb{E}_\infty$-space, the J-homomorphism in the form $J : \mathrm{O} \to \Omega^\infty S$ fails to be a homomorphism of $\mathbb{E}_1$-spaces (the "greatest common denominator" between a group and an $\mathbb{E}_\infty$-space).

The issue is that the $\mathbb{E}_\infty$-structure on $\Omega^\infty S$ that we are discussing comes from the spectrum structure of S, i.e. is in some sense additive. Instead, the J-homomorphism sends the group operation in O into a "multiplicative" $\mathbb{E}_\infty$-structure on Ω^∞. Of course this exists, and is inherited from the $\mathbb{E}_\infty$-ring structure on S, but it is very far from being grouplike; indeed, $\pi_0 S \simeq \mathbf{Z}$ fails to be a group under multiplication in an epic way.

The solution is to modify the target, replacing Ω^∞ with $\mathrm{GL}_1(S) \simeq \mathrm{Aut}_{\mathrm{Sp}}(S)$, the "automorphism group" of the sphere spectrum. We will talk more about it in the next few sections, but the take-away is that it produces a map $J : \mathrm{O} \to \mathrm{GL}_1(S)$ of grouplike $\mathbb{E}_1$-spaces, and this is how we understand the J-homomorphism from here on.

2.3.7. **The ∞-group $\mathrm{GL}_1(S)$.** Let us discuss the grouplike $\mathbb{E}_1$-space $\mathrm{GL}_1(S)$ with a little more rigor. In fact, since it is absolutely no harder, let us discuss $\mathrm{GL}_1(R)$ for any $\mathbb{E}_\infty$-ring R.

We may proceed like this: let $\mathrm{Mod}_R^{\simeq} \subset \mathrm{Mod}_R$ denote the subcategory of the ∞-category of R-modules where we discard all morphisms that are not equivalences. In this way we obtain an ∞-groupoid, or equivalently a space. Its objects are the same as of Mod_R, so we may consider the full subcategory of $\mathrm{Mod}_R^{\simeq}$ spanned by the unit R-module R. Since this ∞-groupoid has only a single object R, it corresponds to a pointed (with base-point R) connected space. Now recall the Boardman-Vogt-May Recognition Theorem for Loop Spaces from 1.6.4. It shows that the connected space in question is in fact of the form $\mathrm{B}G$ for some uniquely-determined grouplike $\mathbb{E}_1$-space G. This we finally set to be the sought-after $\mathrm{GL}_1(R) := G$.

That was of course just a fancy way to say that $\mathrm{GL}_1(R) \simeq \mathrm{Map}_{\mathrm{Mod}_R^{\simeq}}(R, R)$, the space of R-linear equivalences $R \simeq R$, in full analogy with how $\mathrm{GL}_1(R)$ is defined for an ordinary ring R. The key is merely that the above description also specifies the $\mathbb{E}_1$-structure, and since we are working ∞-categorically, that is a rather formidable accomplishment.

2.3.8. **Alternative construction of $\mathrm{GL}_1(S)$.** Another approach is to recall that the set $\pi_0 R \simeq \pi_0(\Omega^\infty R)$ inherits a commutative ring structure from the $\mathbb{E}_\infty$-ring structure on R. Thus we can define $\mathrm{GL}_1(R)$ as the pullback of the cospan $\Omega^\infty R \to \pi_0(R) \leftarrow \pi_0(R)^\times$ in the ∞-category CMon of $\mathbb{E}_\infty$-spaces. (A basic property of the latter is that the limits in it are preserved under the forgetful functor $\mathrm{CMon} \to \mathcal{S}$, thus the underlying space of this $\mathbb{E}_\infty$-space is obtained by merely taking the same pullback in the ∞-category as spaces.)

To see that this is the same as the previous constrction relies on observing that $\Omega^\infty R \simeq \mathrm{Map}_{\mathrm{Mod}_R}(R, R)$. The advantage of the approach outlined in this paragraph though is that it automatically equips $\mathrm{GL}_1(R)$ with an $\mathbb{E}_\infty$-structure, not merely an $\mathbb{E}_1$-structure. Also, note that this construction explicitly addresses the issue of non-grouplikeness of $\Omega^\infty S$, raised in subsection 2.3.3, making it seem like a sensible target for the J-homomorphism.

2.3.9. **Digression: the spectrum $\mathrm{gl}_1(R)$.** As such we may use the May Recognition Principle for infinite loop spaces from 1.6.6 to obtain an essentially unique connective spectrum $\mathrm{gl}_1(R)$ for which there is an equivalence of grouplike $\mathbb{E}_\infty$-spaces $\Omega^\infty \mathrm{gl}_1(R) \simeq \mathrm{GL}_1(R)$. So we got another family of examples of spectra, which this section is supposed to be all about! Sweet!

From the pullback description in the previous paragraph (and since ordinary homotopy groups of $\mathrm{GL}_1(R)$ are the same as the homotopy groups of the spectrum $\mathrm{gl}_1(R)$, it is easy to determine the homotopy groups of this spectrum as $\pi_i(\mathrm{gl}_1(R)) = \pi_i(R)$ for all $i \geq 1$, then $\pi_0(\mathrm{gl}_1(R)) \simeq (\pi_0 R)^\times$, and finally $\pi_i(\mathrm{gl}_1(R)) = 0$ for all $i < 0$.

But let's get back to business:

2.3.10. **The definition of Thom spectra.** At long last, we can explain how to form Thom spectra out of vector bundles. This will bring together what we've discussed about local systems of spectra and the J-homomorphism, and then we'll be done.

Start with a vector bundle $E \to X$ of rank r. It is classified by a map $X \to \mathrm{BO}(r)$, and composing with the canonical map $\mathrm{BO}(r) \to \mathrm{BO}$ corresponds in light of the discussion of BO in the last section to passage from E to the associated stable (in the sense of arbitrary addition of summands $\mathbf{R}$) vector bundle. Now we can apply the J-homomorphism $\mathrm{BO} \to \mathrm{BGL}_1(S)$, which geometrically corresponds to passing to the associated spherical bundle (indeed, remember that the J homomorphism was about one-point compacitying copies of $\mathbf{R}^r$ into S^r).

Altogether, we obtain a map $X \to \mathrm{BGL}_1(S)$, but recall from subsection 2.3.4 that the ∞-groupoid $\mathrm{BGL}_1(S)$ is by definition equivalent to the full subcategory of $\mathrm{Sp}^{\simeq}$ spanned by S. As such, we can compose with the inclusions $\mathrm{BGL}_1(S) \subset \mathrm{Sp}^{\simeq} \subset \mathrm{Sp}$ to end up with a functor $X \to \mathrm{Sp}$, which is to say, a local system of spectra on X. Intuitively, this local system has at the point $x \in X$ value $S[E_x]$, where E_x is the fiber of the vector bundle we started with, and the fact that we are applying the functor $S[-] \simeq \Sigma^\infty_+$ has to do with respect to + with the one-point compacitification, and then stabilizing.

Definition 17. The Thom spectrum of the vector bundle $E \to X$, denoted variously by X^E or $\mathrm{Th}(E)$, is obtained by applying the functor $p_! : \mathrm{Fun}(X, \mathrm{Sp}) \to \mathrm{Sp}$ of left pushforward along the terminal map $p : X \to *$ to the local system of spectra associated to E. Explicitly, that means that the Thom spectrum is given by

$$\mathrm{Th}(E) \simeq X^E := \varinjlim(X \xrightarrow{E} \mathrm{BO}(r) \hookrightarrow \mathrm{BO} \xrightarrow{J} \mathrm{BGL}_1(S) \hookrightarrow \mathrm{Sp}).$$

This definition may strike you as somewhat hardcore: so many functors, so many things - but it's really super simple. You start of with a vector bundle E on a space X, view it as a map to classifying space $X \to \mathrm{BO}$, compose with the J-homomorphism to land in the ∞-category of spectra, and take the colimit. Easy-peasy!

2.3.11. **Thom spectra are similar to suspension spectra.** To convince yourself that performing this construction might be sensible, recall that the Thom spectrum X^E is roughly $\varinjlim_{x \in X} S[E_x]$. Well, if we didn't have the suspension spectrum in there, this would be the colimit $\varinjlim_{x\in X} E_x$. But since the fiber E_x is is equivalent to $\mathbf{R}^r \simeq *$, this is the same as $\varinjlim_{x\in X} * \simeq X$. Thus, since the functor $S[-] : \mathcal{S} \to \mathrm{Sp}$ is a left adjoint and as such preserves colimits, the Thom spectrum X^E is roughly like the suspension spectrum $S[X]$.

But in fact, X^E *isn't* just $\varinjlim_{x\in X} S[E_x]$, and that's the whole point - it can twist the fibers a bit before combining them! And that's why it's interesting. :)

The question for which bundles $E \to X$ we do have $X^E \simeq S[X]$ is a very profound one, leading to the theory of orientations. Indeed, for any $\mathbb{E}_\infty$-ring R a good notion of *R-orientation* for a bundle $E \to X$ is the requirement that $X^E \otimes R \simeq R[X]$. That is to say, the answer to the question is affirmative upon smashing with R. This has been thoroughly studied in stable homotopy theory, from the perspective we are pursuing most notably by Ando-Blumberg-Gepner.

In particular, any trivial vector bundle is S-oriented, so that all suspension spectra are examples of Thom spectra.

2.3.12. **Mahowald's Theorem.** Indeed, many spectra can be viewed as examples of Thom spectra. Andrew has a motto about that, which goes something like: "All spectra are Thom spectra, except the ones that aren't." In fact, a certain portion of his career has been devoted to proving that certain spectra *can not* be viewed as Thom spectra.

The historically first example (and by far the simplest, so the one that I shall restrict to telling here) of this principle was Mahowald's Theorem, exhibiting the Eilenberg-MacLane spectrum $\mathbf{F}_2$ as a Thom spectrum.

How does this work? Well, note first that

$$\pi_1(\mathrm{BO}) \simeq \pi_0(\Omega\mathrm{BO}) = \pi_0(\mathrm{O}) = \mathbf{Z}/2,$$

the last isomorphism following easily from the fact that $\mathrm{O}(n)$ have two components for all $n \geq 1$: the orientation-preserving and the orientation-reversing isometries. Thus there is only a single non-trivial homotopy class of pointed maps $S^1 \to \mathrm{BO}$, of course corresponding to $1 \in \mathbf{Z}/2$. But note that by Bott periodicity BO is an infinite loop space. In particular, it is a 2-fold loop space.

Now we need a rather easy fact about iterated loop spaces: the forgetful functor from the ∞-category of n-fold loop spaces (and n-fold loop space maps between them, i.e. maps which respect the deloopings) to $\mathcal{S}_*$ admits a left adjoint. This functor, which we can call the free n-fold loop space, sends a pointed space X to the n-fold loop space $\Omega^n\Sigma^n X$, and the universal arrow $X \to \Omega^n\Sigma^n X$ is just the unit of the adjunction $\Sigma^n \dashv \Omega^n$. Thus if Y is an n-fold loop space, any pointed map $X \to Y$ induces an essentially unique n-fold loop space map $\Omega^n\Sigma^n X \to Y$.

Applying this to the map $S^1 \to \mathrm{BO}$, we obtain a 2-fold loop space map $\Omega^2 S^3 \simeq \Omega^2\Sigma^2 S^1 \to \mathrm{BO}$, which we may view as a stable vector bundle on $\Omega^2 S^3$. We can take its Thom spectrum like before: compose with the J-homomorphism and then take the colimit in the ∞-category of spectra. Well, Mahowald's Theorem says that the Thom spectrum produced this way is the Eilenberg-MacLane spectrum $\mathbf{F}_2$.

Variants of this Theorem, found later by Hopkins and others, tell how to construct Eilenberg-Maclane spectra $\mathbf{F}_p$, $\mathbf{Z}_p$, and $\mathbf{Z}$ as Thom spectra as well, but the constructions are much more involved (one needs to work in p-complete spectra, etc.) so we do not go into them here.

2.3.13. **Traditional examples.** Mahowald's Theorem is interesting and unexpected, but it ends up producing a spectrum we already knew. Instead the more traditional examples of Thom spectra are ones we haven't encountered before.

Let G be a group (compact Lie, say, or maybe finite) and $\rho : G \to \mathrm{O}(n)$ be an orthogonal representation thereof. This is equivalent to a rank n vector bundle on $\mathrm{B}G$, given by its classifying map $\mathrm{B}G \to \mathrm{BO}(n)$. This gives rise to the Thom spectrum that is usually denoted just $\mathrm{M}G$, despite technically depending on the choice of the underlying representation ρ.

Of course if the group G admits a particularly canonical (in that case usually also faithful) representation ρ of this form, the symbol $\mathrm{M}G$ should be reserved for the Thom spectrum with respect to that ρ. Examples are $\mathrm{MO}(n), \mathrm{MU}(n), \mathrm{MSO}(n), \mathrm{MSU}(n), \mathrm{MSp}(n)$, etc.

Playing the same game with stable vector bundles instead of actual ones allows us to form the particularly important MO and MU. Just to unravel what's going on, note that the former of the two is given by

$$\mathrm{MO} \simeq \varinjlim(\mathrm{BO} \xrightarrow{J} \mathrm{BGL}_1(S) \hookrightarrow \mathrm{Sp}),$$

literally the colimit of the J-homomorphism in spectra. The spectrum MU is obtained by merely pre-composing with the map $\mathrm{BU} \to \mathrm{BO}$, coming from the inclusions $\mathrm{U}(n) \hookrightarrow \mathrm{O}(2n)$, before applying the colimit. Since the construction of the Thom spectrum commutes with colimits in the group (easy check with our definition of Thom spectra), these are also equivalent to $\mathrm{MO} \simeq \varinjlim \mathrm{MO}(n)$ and $\mathrm{MU} \simeq \varinjlim \mathrm{MU}(n)$.

2.3.14. **Thom spectra and cobordisms.** You might be surprised to learn the names that MO and MU carry. They are the real and the complex cobordism spectrum respectively. This is due to the highly non-obvious fact that the cohomology theories that they correspond to are the theory of cobordisms of real and complex manifolds repsectively. In particular, elements in $\pi_n(\mathrm{MO})$ and $\pi_n(\mathrm{MU})$ correspond with cobordism classes of closed n-dimensional manifolds, real or complex[5] respectively.

We have little to say about this, other to mention the theorem of Galatius-Madsen-Tillmann-Weiss, which among other things shows that this identification also happens on the level of underlying infinite loop spaces. More precisely, if $\mathrm{Bord}_{\mathbf{R}}$ and $\mathrm{Bord}_{\mathbf{C}}$ are the ∞-categories of bordisms of manifolds (here the n-morphisms are given by n-dimensional manifolds, viewed as bordisms), then the underlying ∞-groupoids $\mathrm{Bord}_{\mathbf{R}}^{\simeq}$ and $\mathrm{Bord}_{\mathbf{C}}^{\simeq}$ inherit an infinite loop space structure from the symmetric monoidal structure given by disjoint union on bordisms. With this structure, we have

$$\Omega^\infty \mathrm{MO} \simeq \mathrm{Bord}_{\mathbf{R}}^{\simeq}, \qquad \Omega^\infty \mathrm{MU} \simeq \mathrm{Bord}_{\mathbf{C}}^{\simeq}.$$

Since both spectra are connective, this characterizes them essentially uniquely. For what it's worth, let us also point out that their homotopy groups (isomorphic by the above to cobordism groups, whose determination can be pawned off as a problem for geometric topologists) are given by the polynomial rings

$$\pi_*(\mathrm{MO}) \simeq \mathbf{F}_2[x_n | n \geq 2, n \neq 2^n - 1], \qquad \pi_*(\mathrm{MU}) \simeq \mathbf{Z}[u_1, u_2, \ldots]$$

on generators x_n in degree n and generators u_n in degree 2^n respectively. From this, we may observes that homtopy groups form graded rings. Is there any reason for that, we might ask.

2.3.15. **Ring structure on Thom spectra.** Indeed, there is a $\mathbb{E}_\infty$-ring structure on MO and MU. This makes sense from the Galatius-Madsen-Tillmann-Weiss perspective: the disjoint union of manifolds gives rise to the "additive" spectrum structure, so the product of manifolds should equip it with an appropriate commutative ring structure (since $\times$ distributes over $\coprod$ in the usual way).

As pointed out in light of the ABGHR perspective, the ring structure may be seen as coming in a more general way from the construction of Thom spectra. This goes roughly as follows: let $E : X \to \mathrm{BO}$ be an n-fold loop space map (recall: BO is an infinite loop space by Bott), so in particular X has to be an n-fold loop space itself. To obtain an $\mathbb{E}_n$-structure on the associated Thom spectrum, we procede in steps.

- By the Recognition Principle that should be familiar by now, n-fold loop spaces are paritcular cases of $\mathbb{E}_n$-spaces, so we are asking for the classifying map E to be an $\mathbb{E}_n$-map.
- We compose with the J-homomorphism $J : \mathrm{BO} \to \mathrm{BGL}_1(S)$, itself an $\mathbb{E}_\infty$-map and so an $\mathbb{E}_n$-map for every n. Thus we have a $\mathbb{E}_n$-map structure on the composite $J \circ E : X \to \mathrm{BGL}_1(S)$.
- Recall that the $\mathbb{E}_\infty$-structure on $\mathrm{GL}_1(S)$ comes from the "mutiplicative" structure on the sphere spectrum. More precisely, if we view $\mathrm{BGL}_1(S)$ as a symmetric monoidal ∞-groupoid (∞-groupoid, which is also a symmetric monoidal ∞-category), then the inclusion functor $\mathrm{BGL}_1(S) \hookrightarrow \mathrm{Sp}^\otimes$ is symmetric monoidal with respect to the smash product.
- Altogether, we find that the associated local system of spectra $J \circ E : X \to \mathrm{Sp}$ is an $\mathbb{E}_n$-monoidal functor, equipping its colimit $X^E = \varinjlim J \circ E$ with a natural structure of a $\mathbb{E}_n$-ring.

Since both the identity map $\mathrm{BO} \to \mathrm{BO}$, as well as the map $\mathrm{BU} \to \mathrm{BO}$, are infinite loop space maps, this procedure applies to exhibit a $\mathbb{E}_\infty$-ring structure on cobordism spectra MO and MU as promised.

[5] Technically the relevant structure is not quite a complex one, but instead a stably almost complex one. That is to say, a complex structure on some sum of the tangent bundle with a trivial bundle.

2.3.16. **Other species of cobordisms.** Let us return to the setting of subsection 2.2.3.12. Essentially through the Galatius-Madsen-Tillmann-Weiss identification of $\Omega^\infty\mathrm{MO}$ with $\mathrm{Bord}^{\simeq}_{\mathbf{R}}$ (though this was known much before and requires much less profound technology), we can obtain a cobordism interpretation of various other variants $\mathrm{M}G$ of Thom spectra.

Here G is a group, and to have any hope of forming a Thom spectrum, it must come equipped with a homomorphism $G \to \mathrm{O}$. We can interpret this as a type of tangential structure: a condition that one might consider requiring on a tangent bundle[6] of a manifold M through its classifying map $M \xrightarrow{TM} \mathrm{BO}$, by asking it to factor through $\mathrm{B}G \to \mathrm{BO}$. For example:

- If $G = \mathrm{O}$, then the requirement is void.
- If $G = \mathrm{Spin} := \varinjlim_{n\geq 0} \mathrm{Spin}(n)$, it is asking for a spin structure on the manifold M.
- If $G = \mathrm{U}$, this is the requirement that TM carry the structure of a complex vector bundle. Equivalently, this is asking for an (almost) complex structure on the manifold M.
- If $G = *$ is the trivial group, the requirement is that the tangent bundle TM is trivial, i.e. asking that the manifold M is framed.

This defined a class of manifolds, equipped with the prescribed extra structure, and called G-manifolds. The underlying loop space $\Omega^\infty\mathrm{M}G$ of the relevant Thom spectrum is then equivalent to $\mathrm{Bord}^{\simeq}_G$, the space of cobordisms of G-manifolds (*Correction:* I am told this is not known, only conjectured, and known for several groups G that one cares about). This principle goes by the name of Thom's Theorem.

Its perhaps most surprising application comes when applied to $G = *$. The relevant tangential structure is framing, so the relevant Thom spectrum is denoted MFr. This Thom spectrum is by definition the colimit of the composite functor

$$* \to \mathrm{BO} \xrightarrow{J} \mathrm{BGL}_1(S) \hookrightarrow \mathrm{Sp},$$

which is just a very fancy way of picking out the sphere spectrum $S \in \mathrm{Sp}$. It follows that $\mathrm{MFr} \simeq S$, and consequently

$$\Omega^\infty S \simeq \mathrm{Bord}^{\simeq}_{\mathrm{fr}},$$

identifying (the underlying loop space of) the sphere spectrum with the space of framed bordisms. In this way, perhaps somewhat unexpectedly, framed bordisms know about the sphere spectrum.

2.3.17. **The original approach to Thom spectra.** What we discussed so far in this section was from the ABGHR ∞-categorical perspective. But Thom spectra much predate this. Though I think we gained as ample an understanding as possible, the little Arun voice inside my head would kill me in my sleep if I didn't at least mention the classical construction.

Fix a vector bundle $E \to X$. We can form its *Thom space* $T(E)$, which is just a fancy name for the one-point compactification of the total space E. Alternatively, if you are in the setting of smooth manifolds, and pick a fiber-wise inner product on E, you can form $T(E) = D(E)/S(E)$, that is by quotienting the inclusion $S(E) \subset D(E)$ of the unit sphere bundle into the closed unit disc bundle. In any case, there is a canonical equivalence $T(E \oplus \mathbf{R}) \simeq \Sigma T(E)$.

Now let $E_n \to \mathrm{BO}(n)$ be the universal n-dimensional vector bundle. Explicitly it's the associated bundle $E_n = \mathrm{EO}(n) \times_{\mathrm{O}(n)} \mathbf{R}^n$ to the universal principal $\mathrm{O}(n)$-bundle $\mathrm{EO}(n)$ on $\mathrm{BO}(n)$, if we try to be precise for once. Anyhow, we define the n-th space of the Thom spectrum MO by $\mathrm{MO}_n = T(E_n)$, and its structure maps are

$$\Sigma\mathrm{MO}_n = \Sigma T(E_n) = T(E_n \oplus \mathbf{R}) \to T(E_{n+1}) = \mathrm{MO}_{n+1}.$$

[6]Really it is all about the stable tangent bundle, i.e. there can be hidden trivial bundle summands, a difficulty that we choose to ignore in this discussion.

The middle map heuristically comes from the fact that the vector bundle $E_n \oplus \mathbf{R}$ has rank $n+1$, and as such admits a map into the universal rank $n+1$ bundle E_{n+1}. More precisely, it is the map $\mathrm{EO}(n) \times_{\mathrm{O}(n)} \mathbf{R}^n \times \mathbf{R} \to \mathrm{EO}(n+1) \times_{\mathrm{O}(n+1)} \mathbf{R}^{n+1}$ coming from the block inclusion $\mathrm{O}(n) \to \mathrm{O}(n+1)$, compatible with the inclusion $\mathbf{R}^n \cong \mathbf{R}^n \oplus 0 \hookrightarrow \mathbf{R}^{n+1}$.

In any even, this is the classical constrution of the Thom spectrum MO (and MU would be entirely analogous with $\mathbf{C}^n$ in place of $\mathbf{R}^n$). You can't say I didn't tell it to you. :)

2.4. Truncation of spectra

Unlike all the somewhat fancier and involved things we've seen so far, such as topological K-theory and Thom spectra, let us spend this section discussing a very simple way of getting new examples of spectra from old ones - by cutting away a bunch of their homotopy groups!

2.4.1. **Truncating a chain complex.** Under the analogy between Sp and the derived category $\mathcal{D}(R)$, truncation of spectra should be like truncating a (co)chain complex of (ordinary) R-modules. Let us take a few subsections to discuss how this works in detail.

2.4.2. **Attempt 1: stupid truncation.** Given such a complex $M^\bullet$, we could try to cut it off by merely defining the i-th truncation $\tau_s^{\geq i} M^\bullet$ to be the chain complex

$$\cdots \to 0 \to 0 \to M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} M^{i+2} \to \cdots$$

with $d^j : M^j \to M^{j+1}$ the differentials of the complex. As the index s indicates, the complex $\tau_s^{\geq i} M^\bullet$ thus produced is called the *stupid truncation* (actual name, as used in the papers of Bhargav Bhatt and others). It's really not a very smart construction, as it's not invariant under quasi-isomorphisms - a quasi-iso may very well change the i-th component M^i of $M^\bullet$. As such, $\tau_s^{\geq i}$ doesn't really exist on the level of the derived category $\mathcal{D}(R)$.

2.4.3. **Attempt 2: actual truncation.** So let's try again. Since we should only take the weak-homotopy class of $M^\bullet$ into account, it seems sensible to demand that $\tau^{\geq i} M^\bullet$ has the same cohomology groups as $M^\bullet$ in degrees $\geq i$, and that its cohomology vanish in all smaller degrees.

This is not very hard to accomplish: set $\tau^{\geq i} M^\bullet$ to be the complex

$$\cdots \to 0 \to 0 \to \operatorname{Coker} d^{i-1} \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} M^{i+2} \to \cdots.$$

This clearly is but a chain complex model, but it now gives a well-defined element in $\mathcal{D}(R)$. It comes equipped with a map $M^\bullet \to \tau^{\geq i} M^\bullet$, which exhibits its universal property.

2.4.4. **Universal property of truncation.** Indeed, let $\mathcal{D}(R)^{\geq i} \subset \mathcal{D}(R)$ denote the full subcategory of complexes (with cohomology) concentrated in degree $\geq i$. The truncation $\tau^{\geq i} M^\bullet$ is initial among objects in $\mathcal{D}(R)^{\geq i}$ with a map from $M^\bullet$. That is to say, the construction $\tau^{\geq i} : \mathcal{D}(R) \to \mathcal{D}(R)^{\geq i}$ provides a right adjoint to the inclusion $\mathcal{D}(R)^{\geq i} \hookrightarrow \mathcal{D}(R)$.

Analogously defining $\mathcal{D}(R)^{\leq i} \subset \mathcal{D}(R)$ to consist of complexes with chomology purely in degree $\leq i$, we get truncation in the other direction $\tau^{\leq i} : \mathcal{D}(R) \to \mathcal{D}(R)^{\leq i}$ as the left adjoint to the inclusion. We could also explicitly construct a chain model for $\tau^{\leq i} M^\bullet$ as

$$\cdots \to M^{i-2} \xrightarrow{d^{i-2}} M^{i-1} \xrightarrow{d^{i-1}} \operatorname{Ker} d^i \to 0 \to 0 \to \cdots.$$

Furthermore truncations in one direction may be expressed in terms of truncations in the other one: clearly $\tau^{\leq i} M^\bullet \to M^\bullet \to \tau^{\geq (i+1)} M^\bullet$ is a (co)fiber sequence.

2.4.5. **Truncation of spectra.** Though the analogy between the derived category of modules and the ∞-category of spectra is imperfect in the sense that we can not make sense of the chain-complex-level constructions such as in 2.4.3 in Sp, the universal property from 2.4.4 works flawlessly. We just make one slight cosmetic change - since spectra are graded homologically, all the indices will lower and all the inequlities will reverse.

We define the subcategory of *i-connective spectra* $\mathrm{Sp}_{\geq i} \subset \mathrm{Sp}$ to be the full subcategory spanned by spectra X which have $\pi_j(X) = 0$ for all $j < i$. The *i-truncated spectra* $\mathrm{Sp}_{\leq i} \subset \mathrm{Sp}$ are defined analogously to consist of X with $\pi_j(X) = 0$ for all $j > i$. Both of these subcategory inclusions admit adjoints (Adjoint Functor Theorem wonders wants me to ask who called), with the left adjoint $\tau_{\geq i} : \mathrm{Sp} \to \mathrm{Sp}_{\geq i}$ usually called the *i-connective cover*, and the right adjoint $\tau_{\leq i} : \mathrm{Sp} \to \mathrm{Sp}_{\leq i}$ called the *i-truncation*.

For any spectrum $X \in \mathrm{Sp}$ we have $\pi_j(X) = \pi_j(\tau_{\leq i}X)$ for all $j \leq i$ and $\pi_j(X) = \pi_j(\tau_{\geq i}X)$ for all $j \geq i$ as expected. Just as before, we get a fiber sequence $\tau_{\geq i}X \to X \to \tau_{\leq(i+1)}X$ for every i.

Though we won't need to know any of the technicalities, allow me to point out that the structure we are observing here on the ∞-category Sp falls under the heading of a *t-structure*, a structure already well-studied in the land of triangulated categories.

2.4.6. **Space-level constructions.** In analogy with the chain complex picture we have been propagating so far, it might seem strange to give the two opposite-directed truncations different names. It makes sense in terms of the analogous space-level construction though.

We may define subcategories $\mathcal{S}_{\geq i}$ and $\mathcal{S}_{\leq i}$ as above in terms of the ordinary homotopy groups of spaces. Unlike in the case of spectra though, there is in this setting some asymmetry between the two directions, since we have $\mathcal{S} = \mathcal{S}_{\geq 0}$ while the subcategories $\mathcal{S}_{\leq -1} = \mathcal{S}_{\leq -2} = \cdots$ are all empty. The adjoints $\tau_{\geq i} : \mathcal{S} \to \mathcal{S}_{\geq i}$ and $\tau_{\leq i} : \mathcal{S} \to \mathcal{S}_{\leq i}$ exist as above due to abstract nonsense.

Given a space X, let us suppose it comes presented as a CW complex. Then the truncation $\tau_{\leq i}X$ may be obtained by taking the i-skeleton, then for each non-trivial element of $\pi_{i+1}X$ gluing onto it an $(i+2)$-cell contracting it. This may introduce some non-trivial elements in π_{i+2}, which we kill by gluing in $(i+3)$-cells. Continuing inductively, we obtain $\tau_{\geq i}X$. This procedure is known classically as "killing homotopy groups".

In low degrees, we get

- $\tau_{\leq 0}X$ is the connected components $\pi_0(X)$.
- $\tau_{\leq 1}X$ is, under an equivalence of categories between 1-truncated CW complexes and groupoids, the fundamental groupoid $\pi_{\leq 1}X$, also sometimes denoted $\Pi(X)$.
- In particular, if X is connected, then its 1-truncation is $\tau_{\leq 1}X \simeq \mathrm{B}\pi_1(X)$, the classifying space of the fundamental group. $\tau_{\geq 2}X$ is the universal cover of X.

The last of these cases is especially telling as to why the functor $\tau_{\geq i}$ is called the i-connected *cover.*

2.4.7. **Connective spectra.** Since all spaces are connective, a distinguished role is played among spectra which are 0-connective. In that case we simply say that they are connective, and employ special notation $\mathrm{Sp}^{\mathrm{cn}} = \mathrm{Sp}_{\geq 0}$. Another reason for this preferential treatment is, as we already discussed, that the functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ restricts to an equivalence to infinite loop spaces only on the connective part of Sp. As such, connective spectra are more easily understood as ∞-categorical abelian groups, while the non-connective part is a bit more mysterious.

Most spectra we encounter in our day-to-day life (e.g. the sphere) will probably be connective, unless we desuspend them too much. One big exception is topological K-theory. Indeed, we saw that as a consequence of Bott Periodicity, KU and KO are 2-periodic and 8-periodic (in homotopy groups) respectively. Of course if we knew in advance that they are periodic, we could recover them from their connective covers. The latter are called *connective complex* and real topological K-theory respectively, and are denoted $\mathrm{ku} := \tau_{\geq 0}\mathrm{KU}$ and $\mathrm{ko} := \tau_{\geq 0}\mathrm{KO}$. Indeed, this is a case of a common paradigm, where capital letters denote non-connective (usually periodic) spectra, while their small-letter analogues refer to their connective covers (compare with $\mathrm{gl}_1(R)$ from 2.3.7).

In fact, the connective cover functor $\tau_{\geq 0} : \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{cn}}$ determines the whole t-structure on Sp. Indeed, We have $\mathrm{Sp}_{\geq n} \simeq \mathrm{Sp}^{\mathrm{cn}}[n]$, the n-connective cover is given in terms of the connective cover as $\tau_{\geq n}X \simeq \tau_{\geq 0}(X[-n])[n]$, truncated spectra may be obtained as $\mathrm{Sp}_{\leq n} \simeq \mathrm{fib}(\mathrm{Sp} \xrightarrow{\tau_{\geq(n+1)}} \mathrm{Sp}_{\geq(n+1)})$, and finally n-truncation my be obtained as the cofiber $\tau_{\leq n}X \simeq \mathrm{cofib}(\tau_{\geq(n-1)}X \to X)$. In this way, connective spectra $\mathrm{Sp}^{\mathrm{cn}} \subset \mathrm{Sp}$ know everything about the t-structure.

2.4.8. **Non-connective spectra are weird.** The distinguished role of connective spectra is also seen in spectral algebraic geometry, where although most of the definitions make sense for non-connective $\mathbb{E}_\infty$-rings just as well, they only have nice behavior, which is to say, exhibit properties familiar from usual algebraic geometry, under the additional assumption of connectivity.

Perhaps the most poignant demonstration is this: for a connective spectrum X, the truncation map $X \to \tau_{\leq 0}X \simeq \pi_0(X)$ exhibits the map to the "underlying ordinary abelian group" $\pi_0(X)$ of X. When X is not connective, the interpretation of $\pi_0(X)$ as an underlying abelian group is a lot less tangible, since the natural maps only go

$$X \to \tau_{\leq 0}X \leftarrow \tau_{\geq 0}\tau_{\leq 0}X \simeq \pi_0(X).$$

This may not seem so bad, but if you want to interpret a spectral scheme as some sort of "higher nilpotent thickening" of an underlying ordinary one, it is quite unfortunate if there is no canonical map from the underlying ordinary scheme into its supposed thickening.

2.4.9. **The Postnikov tower.** Truncations are often used to inductively study a spectrum (or space) X through its Postnikov tower

$$\cdots \to \tau_{\leq 2}X \to \tau_{\leq 1}X \to \tau_{\leq 0}X \to \tau_{\leq(-1)}X \to \tau_{\leq(-2)}X \to \cdots$$

whose "associated graded" is (i.e. the fibers are) $\Sigma^i\pi_i(X)$ (or the iterated classifying space $\mathrm{B}^i\pi_i(X)$ in the case of spaces). Spectra with various desired properties can be built successively by constructing their i-truncation, and then checking in terms of $\pi_i(X)$ that the extension problem to ascend the tower is verified. Furthermore we have a convergence result $X \simeq \varprojlim_{k\to\infty} \tau_{\leq k}X$, allowing us to reduce the study of any construction that preserves filtered limits entirely to what it does to truncated spectra. This is a technique that is very often immensely useful.

2.5. Algebraic K-theory

In section 1.2.2 we discussed topological K-theory, constructed out of topological vector bundles. We mentioned however that the origins of K-theory link it to Grothendieck's work on the Riemann-Roch theorem. In this section we will briefly review that story, and then explain how it extends to give rise to algebraic K-theory spectra.

2.5.1. **The Grothendieck group of a variety.** Let X be a smooth variety over a field k. Let $\mathrm{Coh}(X)$ denote the category of (non-derived) coherent sheaves on X. Let $\mathcal{A} = \mathbf{Z}\langle \mathrm{Coh}(X)^{\simeq}\rangle$ denote the free abelian group generated by isomorphism classes $[\mathscr{F}]$ of coherent sheaves $\mathscr{F}$ on X, and let $\mathcal{R} \subseteq \mathcal{A}$ denote the subgroup generated by elements $[\mathscr{F}'] - [\mathscr{F}'] + [\mathscr{F}'']$ for all short exact sequences

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

of coherent sheaves on X. The Grothendieck group of X is defined to be the quotient group $\mathrm{K}_0(X) := \mathcal{A}/\mathcal{R}$.

2.5.2. **Universal property of the Grothendieck group.** In light of split short exact sequences, we see that the addition on the Grothendieck group is given by $[\mathscr{F}]+[\mathscr{F}'] = [\mathscr{F}\oplus\mathscr{F}']$. In this way, we may interpret the construction of the Grothendieck group as the universal way of making all short exact sequences of quasi-coherent sheaves behave as if they were split. This can be easily made precise as a universal property: $\mathrm{K}_0(X)$ is initial among abelian groups A with a (set-theoretic) map $f : \mathrm{Coh}(X)^{\simeq} \to A$ satisfying $f(\mathscr{F}) = f(\mathscr{F}') + f(\mathscr{F}'')$ for every short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

of coherent sheaves on X.

This is why Grothendieck initially introduced his group (and with this, K-theory). In the Riemann-Roch story, as by that time re-interpreted by Serre, Weil, and Hirzebruch, the goal was to compare various "characteristic classes" of coherent sheaves, satisfying the above-described additivity property wrt short exact sequences. Grothendieck's innocuous idea was to take this seriously and consider these functors as group homomorphisms from $\mathrm{K}_0(X)$. The advantage is that $\mathrm{K}_0(X)$ itself behaves a lot like a cohomology theory for schemes, which could be exploited. And so, K-theory was born.

2.5.3. **Grothendieck group of vector bundles.** Suppose that X is a smooth variety. In that case, any coherent sheaf admits a resolution

$$0 \to \mathscr{E}_r \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$$

by locally free sheaves (always of finite rank) $\mathscr{E}_i$. Exactness of this sequence implies the equality $[\mathscr{F}] = \sum_{0\le i\le r}(-1)^i[\mathscr{E}_i]$ in $\mathrm{K}_0(X)$, thus showing that the Grothendieck group is generated by (the image of) the subgroup $\mathrm{Vect}(X)^{\simeq} \subseteq \mathrm{Coh}(X)^{\simeq}$ of locally free sheaves, i.e. vector bundles, inside coherent sheaves.

This was behind the definition of topological K-theory in section 1.2.2: since every short exact sequence of topological vector bundles on a manifold splits, additivity was reduced to $[\mathscr{E}\oplus\mathscr{E}'] = [\mathscr{E}]+[\mathscr{E}']$. Alas, a short exact sequence of algebraic vector bundles need not split algebraically, so the more complicated definition is necessary. That is, it does not split unless ..

2.5.4. **The affine case.** If $X = \operatorname{Spec} A$ is an affine scheme, then every short exact sequence of vector bundles does indeed split. (The reason that this always happens in the algebro-topological case is that, from many points of view, all topological manifolds "topologically affine" - that's one perspective on the Whitney Embedding Theorem, anyway.) Indeed, in terms of the equivalence of categories $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_A^{\heartsuit}$ between (ordinary, non-derived) quasi-coherent sheaves on the affine and modules, vector bundles correspond to projective A-modules. The latter are defined by the fact that they split short exact sequences.

Consequently the Grothendieck group $\mathrm{K}_0(A) := \mathrm{K}_0(X)$ is the free abelian group generated by classes $[M]$ of projective modules $M \in \mathrm{Mod}_A^{\mathrm{proj}}$ under the relation that $[M\oplus M'] = [M]\oplus[M']$. All that taking the free abelian group accomplishes is thus to add in formal inverses $[M]$ for each projective (discrete) A-module M.

That is to say, consider the set $(\mathrm{Mod}_A^{\mathrm{proj}})^{\simeq}$ of isomorphism classes of projective A-modules. The operation $\oplus$ of direct sum makes it into a commutative monoid. Then $\mathrm{K}_0(A)$ is the group completion of this monoid.

2.5.5. **Group completion.** Group completion is the left adjoint to the inclusion $\mathcal{A}\mathrm{b} \subseteq \mathrm{CMon}^{\heartsuit}$ of the category of abelian groups into the category of commutative monoids. That is to say, given a commutative monoid M, its group completion M^{gp} is an abelian group together with a homomorphism $M \to M^{\mathrm{gp}}$ of commutative monoids, and initial among abelian groups with such a homomorphism.

The Grothendieck group of a commutative ring is then nothing but

$$\mathrm{K}_0(A) \simeq ((\mathrm{Mod}_A^{\mathrm{proj}})^{\simeq})^{\mathrm{gp}}.$$

Of course we are describing this because it will generalize in a simple way to the ∞-categorical setting.

2.5.6. **Toward the K-theory spectrum.** Let us turn our attention now to constructing algebraic K-theory, as an ∞-categorical analogue of the preceding discussion. In accordance with the philosophy that we have encountered several times now, we replace in the above discussion commutative rings with $\mathbb{E}_\infty$-rings, commutative monoids with $\mathbb{E}_\infty$-spaces, and abelian groups with grouplike $\mathbb{E}_\infty$-spaces (which, we know, amounts to the same thing as connective spectra). Let us carry out this program. In the next few subsections.

2.5.7. **Group completion for $\mathbb{E}_\infty$-spaces.** Group completion of $\mathbb{E}_\infty$-spaces may be defined analogously to the discrete case in the previous section, as the left adjoint to the inclusion of ∞-categories $\mathrm{CMon}^{\mathrm{gp}} \to \mathrm{CMon}$ of grouplike $\mathbb{E}_\infty$-spaces into not-necessarily-grouplike ones.

Restricted to discrete objects $\mathrm{CMon}^{\heartsuit} \subseteq \mathrm{CMon}$, the group completion in this sense agrees with the one from subsection 2.6.5. In particular, it lands inside the subcategory of discrete objects $\mathcal{A}\mathrm{b} \subseteq \mathrm{CMon}$.

An explicit construction of group completion may be given as $M^{\mathrm{gp}} \simeq \Omega \mathrm{B} M$, which passes through the Boardman-Vogt Recognition Principle identifying loop spaces and grouplike $\mathbb{E}_1$-spaces. Since we will not need this, let us not go in more detail.

2.5.8. **Perfect modules.** What will we use in place of projective modules over the discrete commutative ring A, that were used to construct $\mathrm{K}_0(A)$? The answer is that for an $\mathbb{E}_\infty$-ring A, we should consider the full subcategory $\mathrm{Mod}_A^{\mathrm{perf}} \subseteq \mathrm{Mod}_A$ of *perfect A-modules*. This is a ubiquitous condition to put on a module in this setting, and as such there is a myriad of perspectives on it.

- On the one hand, $\mathrm{Mod}_A^{\mathrm{perf}}$ is the smallest stable subcategory of Mod_A containing A itself and retracts. That is to say, any perfect A-module M may be built out of A by a finite process involving only $\oplus$, Σ, fibers, and cofibers.
- Saying essentially the same thing a bit differently, an A-module is perfect iff it can be written as a retract of some module of the form $\Sigma^{i_1} A \oplus \cdots \oplus \Sigma^{i_k} A$ for $i_j \in \mathbf{Z}$. Compare this to projective modules (over a classical commutative ring, if you insist), which are only retracts of $A \oplus \cdots \oplus A$, so no shifts allowed.
- Yet differently, $\mathrm{Mod}_A^{\mathrm{perf}}$ is the category of compact objects in Mod_A. This is under the categorical meaning of compactness: an object K in an ∞-category $\mathcal{C}$ is said to be compact if the Yoneda functor $C \mapsto \mathrm{Map}_C(K, C)$ commutes with filtered colimits. The idea is that the filtered limit might be something like an ascending chain of open inclusions in some ambient topological space $U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$, and if K is a compact subset of the same space, then $K \subseteq \bigcup_{i \geq 0} U_i$ implies that there is some index k such that $K \subseteq U_k$.
- Or one can ask for the tensor product $N \mapsto N \otimes_A M$ to preserve limits (as it already preserves colimits). That is equivalent to dualizability, in the sense of there existing a dual module $M^\vee$ for which there is an equivalence
$$\mathrm{Map}_{\mathrm{Mod}_A}(N \otimes_A M, L) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(N, M^\vee \otimes_A L)$$
natural in arbitrary $N, L \in \mathrm{Mod}_A$.
- Under a Noetherian hypothesis on connective $\mathbb{E}_\infty$-ring A (precisely: $\pi_0(A)$ is a Noetherian ring and all the modules $\pi_i(A)$ are finitely generated), we can characterize an A-module M as perfect iff it is flat, the $\pi_0(R)$-modules $\pi_i(M)$ are finitely generated for all i, and vanish for i sufficiently small.
- When A is an ordinary commutative ring, viewed as a discrete $\mathbb{E}_\infty$-ring, the subcategory $\mathrm{Mod}_A^{\mathrm{perf}} \subseteq \mathrm{Mod}_A \simeq \mathcal{D}(A)$ consists of perfect complexes, i.e. chain complexes of A-modules whose cohomology modules are finitely generated projective,

and vanish outside a finite range of degrees. In short: it is the derived category analogue of projective modules, as it should be.

With so many nice characterizations and properties of perfect modules, surely we are happy to feed them into the machine to produce algebraic K-theory.

2.5.9. **Algebraic K-theory space.** The time has come to unveil algebraic K-theory of an $\mathbb{E}_\infty$-ring A. We proceed in tight analogy with subsection 2.6.6. Instead of taking isomorphism classes, we should discard all the non-equivalence morphisms in $\mathrm{Mod}_A^{\mathrm{perf}}$. This leaves us with the maximal contained ∞-groupoid $(\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq}$, and via the usual identification between ∞-groupoids and spaces, we may consider it as a space.

Furthermore the construction $\mathcal{C} \to \mathcal{C}^{\simeq}$ is symmetric monoidal as a functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$, if both ∞-categories are equipped with the Cartesian symmetric monoidal structure (products are just the categorical products). Thus it induces a functor $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty) \to \mathrm{CMon}$ from symmetric monoidal ∞-categories to $\mathbb{E}_\infty$-spaces, since both are the commutative algebra objects in the repsective ∞-categories. That is to say that a symmetric monoidal structure on $\mathcal{C}$ descends to give an $\mathbb{E}_\infty$-structure on the space $\mathcal{C}^{\simeq}$.

We apply this to the $\mathrm{Mod}_A^{\mathrm{perf}}$, equipped with the symmetric monoidal structure given by $\oplus$. This makes $(\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq}$ into an $\mathbb{E}_\infty$-space. Finally we group complete to obtain the algebraic K-theory space of A as $\Omega^\infty \mathrm{K}(A) \coloneqq ((\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq})^{\mathrm{gp}}$.

2.5.10. **Algebraic K-theory spectrum.** By design the algebraic K-theory space $\Omega^\infty\mathrm{K}(A)$ is a grouplike $\mathbb{E}_\infty$-space. Recall that the functor Ω^∞ induces an equivalence of ∞-categories $\mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{CMon}^{\mathrm{gp}}$ by the May Recognition Principle. Thus there exists an essentially unique connective spectrum $\mathrm{K}(A)$ with the K-theory space of A as its underlying infinite loop space, and this spectrum we call the algebraic K-theory spectrum of A.

We obviously have $\pi_0(\mathrm{K}(A)) = \pi_0(\Omega^\infty\mathrm{K}(A)) \simeq \mathrm{K}_0(A)$, recovering the Grothendieck group, while the higher homology groups $\mathrm{K}_i(A) \coloneqq \pi_i(\mathrm{K}(A))$ are known as higher algebraic K-theory.

You might ask what some examples of algebraic K-theory spectra are, but of course if you have heard anything about algebraic K-theory, you have probably heard that it's hard to compute. Instead, it contains much interesting information about the commutative algebra of the rings in question.real

Do note however that algebraic K-theory fails to share the most distinguishing feature of its topological cousin: there is no analogue of Bott periodicity. In many ways, this is why algebraic K-theory is hard, and also why it took much longer for people to figure out how to even correctly define higher algebraic K-theory - there was no Bott periodicity cheating available!

2.5.11. **Ring structure on algebraic K-theory.** Note that for a commutative ring A, the Grothendieck group $\mathrm{K}_0(A)$ actually carries a ring structure. Indeed, the multiplication comes from the tensor product of projective A-modules.

The situation is fully analogous in the ∞-categorical situation: the relative smash product $\otimes_A$ equips $\mathrm{K}(A)$ with an $\mathbb{E}_\infty$-ring structure.

In fact, up until now we have in this chapter never made use of the $\mathbb{E}_\infty$-structure on A. Thus everything would work just as fine for an $\mathbb{E}_n$-ring A for any $n \geq 1$. The only thing that would change is that the relative smash product $\otimes_A$ would only make $\mathrm{Mod}_A^{\mathrm{perf}}$ into an $\mathbb{E}_{n-1}$-monoidal ∞-category, and as such $\mathrm{K}(A)$ would be itself an $\mathbb{E}_{n-1}$-ring. We conclude that K-theory reduces commutativity by one.

2.5.12. **An analogous construction of topological K-theory.** A highly analogous aproach to how we defined algebraic K-theory can be taken to obtain topological K-theory as well.

Indeed, let $\mathrm{Vect}_{\mathbf{C}}^{\mathrm{fd}}$ denote the ∞-category whose objects are finite dimensional complex vector spaces, and whose mapping spaces are the spaces of linear maps, equipped with their usual topology, inherited from that on $\mathbf{C}$. If we did not demand finite dimensionality we would instead obtain a bigger ∞-category $\mathrm{Vect}_{\mathbf{C}}$, of which $\mathrm{Vect}_{\mathbf{C}}^{\mathrm{fd}} \subseteq \mathrm{Vect}_{\mathbf{C}}$ is the full subcategory of compact objects (alternatively: of dualizable objects). In this way the inclusion $\mathrm{Vect}_{\mathbf{C}}^{\mathrm{fd}} \subseteq \mathrm{Vect}_{\mathbf{C}}$ is analogous to $\mathrm{Mod}_A^{\mathrm{perf}} \subseteq \mathrm{Mod}_A$ discussed in 2.6.8.

Pushing this analogy further, direct sum of complex vector spaces makes $\mathrm{Vect}_{\mathbf{C}}^{\mathrm{fd}}$ into a symmetric monoidal ∞-category, and makes its maximal ∞-subgroupoid $(\mathrm{Vect}_{\mathbf{C}}^{\mathrm{fd}})^{\simeq}$ into an $\mathbb{E}_\infty$-space. Group completing produces the space $((\mathrm{Vect}_{\mathbf{C}}^{\mathrm{fd}})^{\simeq})^{\mathrm{gp}}$, which we easily recognize as the underlying infinite loop space $\Omega^\infty\mathrm{KU}$ of complex topological K-theory.

Using May Recognition Principle, this recovers the connective complex K-theory spectrum ku. Applying the same construction with $\mathrm{Vect}_{\mathbf{R}}^{\mathrm{fd}}$ finite dimensional real vector spaces would produce ko, the connective real topological K-theory spectrum. In this way, algebraic K-theory is more an analogue of ku and ko than of KU and KO.

2.5.13. **Algebraic K-theory of a category.** Note that nothing in the construction of algebraic K-theory, as outlined in subsections 2.6.9. and 2.6.9, used any special properties of the ∞-category $\mathrm{Mod}_A^{\mathrm{perf}}$. We may generalize it to construct K-theory $\mathrm{K}(\mathcal{C})$ of any presentably symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ as the composite functor

$$\mathrm{K} : \mathrm{CAlg}(\mathrm{Pr}^L) \xrightarrow{(-)^\omega} \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty) \xrightarrow{(-)^{\simeq}} \mathrm{CMon} \xrightarrow{(-)^{\mathrm{gp}}} \mathrm{CMon}^{\mathrm{gp}} \xleftarrow{\Omega^\infty} \mathrm{Sp}^{\mathrm{cn}},$$

where we use that the last functor is an equivalence of ∞-categories. The first functor in the composition is one induced on commutative algebras by the symmetric monoidal functor $\mathrm{Pr}^L \to \mathcal{C}\mathrm{at}_\infty$ of passage to subcategory of compact objects $\mathcal{C} \mapsto \mathcal{C}^\omega$. The rest of the functors we already discussed.

This puts all the versions of (connective) K-theory that we encountered so far on the same footing: algebraic K-theory is $\mathrm{K}(A) \simeq \mathrm{K}(\mathrm{Mod}_A)$ and topological K-theory is $\mathrm{ku} \simeq \mathrm{K}(\mathrm{Vect}_{\mathbf{C}})$ and $\mathrm{ko} \simeq \mathrm{K}(\mathrm{Vect}_{\mathbf{R}})$.

2.5.14. **The Barrat-Quillen-Priddy Theorem.** The incarnation of K-theory for a symmetric monoidal ∞-category from the previous section, also appears in the following celebrated Theorem:

Theorem 18 (Barrat-Quillen-Priddy)**.** *There is a canonical equivalence* $\mathrm{K}(\mathcal{S}\mathrm{et}) \simeq S$.

Chasing through the definitions, the theorem identifies the $\mathbb{E}_\infty$-space $\Omega^\infty S$ with the group completion of $(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\simeq}$, the (nerve of the category of) finite sets with bijections between them. A finite set is determined up to bijection by its cardinality, and the bijections of an n-element set form the symmetric group Σ_n, so we have $(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\simeq} \simeq \coprod_{n\geq 0} \mathrm{B}\Sigma_n$.

The Barrat-Quillen-Priddy Theorem is super easy to prove in our context. Here's the idea: recall that the free $\mathbb{E}_\infty$-space functor $\mathcal{S} \to \mathrm{CMon}$ is given by $X \mapsto \coprod_{n\geq 0} X^n_{h\Sigma_n}$, where $X^n_{h\Sigma_n}$ is the homotopy quotient of the permutation-of-factors action of $\Sigma_n on X^n$. Thus $(\mathcal{S}\mathrm{et})^{\simeq}$ is the free $\mathbb{E}_\infty$-space on a single generator. By thinking about adjoints, it is clear that group completion takes free $\mathbb{E}_\infty$-spaces into free grouplike $\mathbb{E}_\infty$-spaces on the same generators. Thus it remains to prove that $\Omega^\infty S$ is the free grouplike $\mathbb{E}_\infty$-space on a single generator. We can use May's Recognition Principle to reduce this to saying that S is the free connective spectrum on a single generator. Here "free spectrum functor" is the left adjoint of the "forgetful functor" $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{S}$, i.e. it is given by $X \mapsto S[X]$. Finally indeed $S \simeq S[*]$, and the theorem is proved. See, too easy not to prove!

2.5.15. **Counting with the sphere spectrum.** As consequence of Barrat-Quillen-Priddy, the sphere spectrum S is the group completion of $(\mathcal{S}\mathrm{et}^{\mathrm{fin}})^{\simeq} \simeq \coprod_{n\geq 0} \mathrm{B}\Sigma_n$. Note that on π_0 this reproduces the counting numbers $\mathbf{Z}_{\geq 0}$, just as $\pi_0(S) \simeq \mathbf{Z}$.

This is the content of the following allegory, allegedly due to Lars Hasselholt, but that I learned from Yuri Sulyma: “When the prehistoric shepherds were on the right track when they chose to count sheep with numbers and permutations, but went astray when they added the negative numbers only on π_0, forgetting about the permutations. It took humanity millennia afterwards to realize that we shouldn't be counting with the integers, but with the sphere spectrum.”

Let us point out that, since the rationalization of the sphere spectrum is $S_{\mathbf{Q}} \simeq \mathbf{Q}$, the “difference” goes away the moment we allow ourselves to also divide by non-zero numbers. The difference between $\mathbf{Z}$ and S, between ordinary algebra and homotopy theory, is in that sense only about the way in which group completion is applied. Better: if we don't want to forget permutations when we start counting with negative numbers, we arrive at the sphere spectrum.

2.5.16. **Other variants of algebraic K theory.** We have spent a fair while discussing an analogue of the Grothendieck group of an affine scheme. But as we saw in subsections 2.5.1 - 2.5.3, the Grothendieck group of a non-affine scheme is much more complicated.

There exists an analogous construction of algebraic K-theory, via the so-called Waldhausen $S_\bullet$-construction. We will not go into any detail, other than to remark that the construction is a careful elaboration on the idea from subsection 2.5.1 of splitting certain pre-specified sequences.

Indeed, it is this Waldhausen version of K-theory that is usually meant as algebraic K-theory, and is the better-behaved notion for non-affine schemes, spectral or otherwise, and other spectrally-enriched categories in general. (When the two disagree, i.e. outside the affine situation, algebraic K-theory as we have discussed is usually called “direct sum K-theory”.)

A still slightly further elaboration exists in the form of non-connective K-theory. As the name suggests, this sometimes produces negative K-theory groups, agreeing with ones that algebraists had predicted long ago, before it was even clear how to correctly define higher K-theory groups. For this version, notations $\mathbf{K}(A)$ and $\mathbb{K}(A)$ are common. A result of Blumberg-Gepner-Tabuada is that both Waldhausen K-theory and non-connective K-theory admit characterizations by universal properties in terms of non-commutative motives.

Non-connective K-theory was first introduced in a paper by Thomason-Trobaugh, which is notable among much else for this simultaneously haunting and charming dedication:

> The first author must state that his coauthor and close friend, Tom Trobaugh, quite intelligent, singularly original, and inordinately generous, killed himself consequent to endogenous depression. Ninety-four days later, in my dream, Tom's simulacrum remarked, “The direct limit characterization of perfect complexes shows that they extend, just as one extends a coherent sheaf.” Awaking with a start, I knew this idea had to be wrong, since some perfect complexes have a non-vanishing K_0 obstruction to extension. I had worked on this problem for 3 years, and saw this approach to be hopeless. But Tom's simulacrum had been so insistent, I knew he wouldn't let me sleep undisturbed until I had worked out the argument and could point to the gap. This work quickly led to the key results of this paper. To Tom, I could have explained why he must be listed as a coauthor. During his lifetime, Tom also pointed out the interesting comparison of the careers of Grothendieck and Newton.

What a quaint note to end this section on!

2.6. Topological Hochschild homology

After the ever-profound and mysterious algebraic K-theory, let us tackle a most fashionable example: topological Hochschild homology. Though this spectrum has been around for a long time, essentially as long as the subject, it has attracted a lot of attention in recent years when the influential series of papers by Bhatt-Morrow-Scholze used THH first as inspiration and later made an explicit connection to various arithmetic cohomology theories. For those who know more than me, I should start saying things like AΩ, prisms, and I don't know what else; but I really don't know, so let us stop there.

In this section however, we will see none of the flashy connections to arithmetic geometry. Instead, we merely recount the beautiful classical tale of introducing Hochschild homology, topological or otherwise, and leave discussion of some of its finer structure to the next section.

2.6.1. **Classical Hochscild homology.** The classical definition of the *i-th Hochshild homology group* of a commutative R-algebra A is as

$$\mathrm{HH}_i(A) = \mathrm{Tor}_i^R(A, A).$$

If we wish to emphasize the dependence on the underlying ring R, the notations $\mathrm{HH}_i(A/R)$ and $\mathrm{HH}_i^R(A)$ are also not uncommon.

This can be expressed more elegantly using the technology of the derived category. Indeed, let us denote the derived tensor product on $\mathcal{D}(R)$ by $\otimes_R^{\mathrm{L}}$. Then we may identify Hochshild homology as the homology groups of the derived tensor product $A \otimes^{\mathrm{L}}_{A\otimes_R A} A$. Let us denote this element of $\mathcal{D}(A)$ as $\mathrm{HH}(A)$ (or $\mathrm{HH}(A/R)$, if we wish to emphasize R) and abusively refer to it as the *Hochschild homology of* A.

2.6.2. **Derived Hochschild homology.** The classical treatments of Hochshild homology one finds in the literature usually insist that A be a smooth, or at the very least flat, R-algebra. Without that assumption Hochschild homology $\mathrm{HH}(A) = A \otimes^{\mathrm{L}}_{\mathrm{A}\otimes_R\mathrm{A}} A$ fails to exhibit much nice behavior. Of course, the reason for this is quite transparent from the derived perspective: the tensor product below is not derived.

To fix this, one may define derived Hochschild homology to be $A \otimes^{\mathrm{L}}_{A\otimes^{\mathrm{L}}_R A} A$, which may look a little intimidating but is actually a very friendly object.

For the majority of practitioners of Hochschild homology these days, this is the correct definition of Hochschild homology for a non-flat R-algebra A anyway, so the adjective derived is usually dropped (and the non-derived version never considered). We follow this and boldly recycle the notation $\mathrm{HH}(A)$ (or $\mathrm{HH}(A/R)$).

2.6.3. **Topological Hochschild homology.** In accordance with our usual perspective of treating $\mathrm{Sp}^{\otimes} \simeq \mathrm{Mod}_S^{\otimes_S}$ as a close analogue of the derived category $\mathcal{D}(R)^{\otimes^{\mathrm{L}}_R} \simeq \mathrm{Mod}_R^{\otimes_R}$ for any discrete commutative ring R, we define the *topological Hochschild homology* of any $\mathbb{E}_\infty$-ring A as

$$\mathrm{THH}(A) \coloneqq A \otimes_{A\otimes A} A.$$

That is to say, topological Hochshild homology is nothing more and nothing less than $\mathrm{HH}(A/S)$, (derived) Hochschild homology over the sphere spectrum.

Note that, since $A \otimes_R B$ is the pushout of the diagram $A \leftarrow R \rightarrow B$ in the ∞-category CAlg, topological Hoschshild homology of A naturally comes equipped with an $\mathbb{E}_\infty$-structure. It also carries two A-module structures (one from the first and one from the second copy of A), but we fix one of them and just view it as an A-module (an $\mathbb{E}_\infty$-algebra over A, even).

2.6.4. THH **for $\mathbb{E}_1$-rings, I.** At this point we should admit that we did not actually need A to be a $\mathbb{E}_\infty$-ring. Recall that $\mathbb{E}_1$-rings (also known as $\mathbb{A}_\infty$-rings in older literature) are the homotopy coherent versions (in spectra) of associative rings. The construction of THH works in the setting $\mathbb{E}_1$-rings also, albeit we need to be a little careful. This is because an $\mathbb{E}_n$-ring structure on A only implies that the relative smash product $\otimes_A$ induces an $\mathbb{E}_{n-1}$-monoidal structure on the ∞-category of (left, say) A-modules Mod_A. Thus an $\mathbb{E}_1$-structurcture does not induce even a monoidal (which is to say, $\mathbb{E}_1$-monoidal) structure on A-modules, making it a little more difficult to form the tensor products that we need to define THH.

2.6.5. **Digression: Relative smash product and bimodules.** The way to go is to observe what the natural domain and codomain of the relative smash product actually are. Fix three $\mathbb{E}_1$-rings A, B, and C. Let ${}_A\mathrm{BMod}_B$ denote the ∞-category of (A, B)-bimodules, that is to say, informally, spectra M together with a left action maps $A \otimes M \otimes B \to M$. Equivalently: M has a compatible left A-module and right B-module structure. The relative smash product is then most organically viewed as a functor

$$\otimes_B : {}_A\mathrm{BMod}_B \times {}_B\mathrm{BMod}_C \to {}_A\mathrm{BMod}_C.$$

When we plug in $A \simeq C \simeq S$, we recover the usual smash product of a right B-module with a left B-module, but without the presence of an $\mathbb{E}_2$-structure on B, the B-module structure is not preserved.

The only remaining fact to note about bimodules, relevant for constructing THH, is that ${}_A\mathrm{BMod}_B$ is canonically equivalent to the left module ∞-category $\mathrm{Mod}_{A \otimes B^{\mathrm{op}}}$, where B^{op} is an $\mathbb{E}_1$-ring with the same underlying spectrum as B, only with the order of multiplication reversed.

2.6.6. THH **for $\mathbb{E}_1$-rings, II.** Thus to form topological Hochschild homology of an $\mathbb{E}_1$-ring A, we should consider A as an (A, A)-bimodule. Informally we may define an action of $A \otimes A$ on A by $(a_L \otimes a_R)a := a_L a a_R$. Then we define THH as

$$\mathrm{THH}(A) := A \otimes_{A \otimes A^{\mathrm{op}}} A.$$

Note that, unlike when A is an $\mathbb{E}_\infty$-ring, for an $\mathbb{E}_1$-ring $\mathrm{THH}(A)$ is merely a spectrum. This lack of a ring structure is compatible with remarks made about $\mathbb{E}_1$-rings in subsection 2.7.4.

Unwinding the definition of the relative smash product as a colimit of a simplicial diagram, we may express

$$\mathrm{THH}(A) \simeq \varinjlim(\cdots \Rrightarrow A \otimes A \rightrightarrows A)$$

showing THH itself to be a colimit (or as we would say in this case, geometric realization) of a particularly natural simplicial diagram. This formula of course works for $\mathbb{E}_\infty$-rings just as well as for $\mathbb{E}_1$-ones.

That said, while topological Hochschild homology exists for any $\mathbb{E}_1$-ring, and even has a lot of nice properties, it behaves best in the $\mathbb{E}_\infty$-case, so we will mostly (possibly fully) restrict ourselves to that for the remainder of this section.

2.6.7. **Geometric interpretation: self-intersection of the diagonal.** THH admits a number of beautiful and sometimes useful (some dispute this last bit) interpretations in terms of spectral algebraic geometry.

We need not go into details of SAG for this, all we need to assume that such a thing exists, that to every $\mathbb{E}_\infty$-ring we associate an affine spectral scheme $\mathrm{Spec}\, A$, and that the functors $\mathrm{Spec} : (\mathrm{CAlg})^{\mathrm{op}} \leftrightarrow \mathrm{Aff}^{\mathrm{nc}} : \mathscr{O}$ is an equivalence of ∞-categories. The subscript "nc" indicates that we are doing a non-connective version of this story; were we trying to do real algebraic geometry with this, it would probably be better to add a connectivity assumption (remember: nonconnective spectra are weird).

Thus let $X = \operatorname{Spec} A$ be an affine spectral scheme (in usual terminology: affine nonconnective spectral scheme) for an $\mathbb{E}_\infty$-ring A. The ring multiplication map $A \otimes A \to A$ corresponds geometrically to the diagonal map $X \to X \times X$. Then we get that $X \times_{X\times X} X \simeq \operatorname{Spec}(A \otimes_{A\otimes A} A) \simeq \operatorname{Spec} \mathrm{THH}(A)$, since pullback in affine spectral schemes correspond to pushouts of $\mathbb{E}_\infty$-rings, and the latter are formed by relative smash products. That is to say that THH is given by (the functions on) the self-intersection of the diagonal of X inside $X \times X$. Such an intersection would not be very interesting in classical algebraic geometry, but in SAG (as in DAG), it is highly interesting. This is due to it being very far from transverse, and one perspective on derived pullbacks is that they are derived functors of ordinary pullbacks, agreeing with them when the intersection is transversal, but computing the "correct" intersection otherwise.

2.6.8. **Geometric interpretation: the free loop space.** There is another algebro-geometric interpretation of THH, or perhaps the same one, but evoking different intuition in light of classical analogies. We need three preliminaries:

(1) Recall that a circle may be glued together from two intervals, which intersect each other in a disjoint pair of intervals. Since an interval is contractible as a space, this exhibits a presentation
$$S^1 \simeq * \coprod_{* \amalg *} *$$
of the circle as a pushout. Of course this pushout has to be considered in its homotopical, which is to say ∞-categorical, incarnation.

(2) Note that spectral stacks should from the "functor of points" perspective be construed as certain sorts of functors $\mathrm{CAlg} \to \mathcal{S}$. In particular, we can define for any space $K \in \mathcal{S}$ the constant functor like that with value K (or possibly sheafification thereof, if you insist) and view it as some sort of a spectral stack.

(3) Given any pair of spectral stacks X and Y (irrelevant of whatever that should mean), we may consider the mapping stack $\underline{\mathrm{Map}}(X, Y)$ defined by the requirement that for any spectral stack Z there is a natural homotopy equivalence
$$\mathrm{Map}_{\mathrm{SpSt}}(Z, \underline{\mathrm{Map}}(X, Y)) \simeq \mathrm{Map}_{\mathrm{SpSt}}(Z \times X, Y).$$
We make no promises that $\underline{\mathrm{Map}}(X, Y)$ is itself a spectral stack (in any of the requirements that should imply), but a variant of a result by Toen in the DAG setting should get you far. All that is relevant for us is that this thing is a functor $\mathrm{CAlg} \to \mathcal{S}$, which we choose to think of as a spectral stack.

Combining the 2. and 3. together, we may for any spectral stack X define its free loop space to be $\mathcal{L}X := \underline{\mathrm{Map}}(S^1, X)$, where S^1 is the circle viewed as a constant spectral stack. By 3. and some basic properties of the mapping stack construction (it takes pushouts in the first variable to pullbacks, and $\underline{\mathrm{Map}}(*, X) \simeq X$ for any X), we find that
$$\mathcal{L}X \simeq \underline{\mathrm{Map}}(* \coprod_{* \amalg *} *, X) \simeq X \times_{X\times X} X.$$
That means that, from the derived perspective, the free loop space coincides with the self-intersection of the diagonal. Since the latter is an incarnation of THH, so is the former. More precisely, there is an equivalence of $\mathbb{E}_\infty$-rings
$$\mathrm{THH}(A) \simeq \mathcal{O}(\mathcal{L}X)$$
for an affine spectral scheme $X = \operatorname{Spec} A$.

If you were to run this same reasoning in derived algebraic geometry over a ring R, you would arrive at a geometric interpretation of $\mathrm{HH}(A/R)$ as functions on the (derived) free loop space on $\operatorname{Spec} A$ in the context of derived R-stacks, i.e. derived stacks over $\operatorname{Spec} R$.

2.6.9. **Digression: tensoring with a space.** There is a way to express the contents of the previous paragraph entirely without the language of algebraic geometry, at the cost of perhaps even a little more categorical nonsense.

Recall that the ∞-category of spaces $\mathcal{S}$ is generated by a single generator, the contractible space $*$, under colimits. Let $\mathcal{C}$ be any ∞-category which has all colimits. Then the previous generation statement translates into that a colimit preserving functor $F : \mathcal{S} \to \mathcal{C}$ is specified essentially uniquely and entirely by specifying the (equivalence class of, as is always implicit,) the object $F(*) \in \mathcal{C}$. The *tensoring of an object* $C \in \mathcal{C}$ *with spaces* is defined as the colimit-preserving functor $- \otimes C : \mathcal{S} \to \mathcal{C}$ specified by $* \otimes C \simeq C$.

This admits a more explicit description. Recall that any space $X \in \mathcal{S}$ may be written as

$$X \simeq \varinjlim_{x \in X} \{x\} \simeq \varinjlim_{X} *;$$

this is a slight extension of the claim that any space admits a CW complex model, since gluing is a form of a colimit, and the spheres are $S^0 \simeq * \coprod *$ and $S^i \simeq \Sigma^i S^0$, all created by colimits from a point. The tensoring of an object $C \in \mathcal{C}$ by a space $X \in \mathcal{S}$ is then the object $X \otimes C \in \mathcal{C}$ given by $X \otimes C \simeq \varinjlim_X C$.

You might begin to notice that the tensor product symbol $\otimes$ is much overloaded in higher algebra. Thus we will sometimes denote tensoring by spaces in the ∞-category $\mathcal{C}$ by $\otimes_{\mathcal{C}}$ when wishing to emphasize the context.

2.6.10. THH **as tensoring with the circle.** Now that we know what tensoring with a space is, we claim that for any $\mathbb{E}_\infty$-ring A, the topological Hochschild homology of A is equivalent to $S^1 \otimes_{\mathrm{CAlg}} A$.

Indeed, this is easy: recall that $S^1 \simeq * \coprod_{* \coprod *} *$. Since pushouts in the ∞-category of $\mathbb{E}_\infty$-rings CAlg are given by relative smash product, we find that

$$S^1 \otimes_{\mathrm{CAlg}} A \simeq A \otimes_{A \otimes A} A \simeq \mathrm{THH}(A).$$

2.6.11. **The circle action.** Viewing $S^1 \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$ as a group, the tensoring construction of the previous subsection can be reinterpreted as saying that $\mathrm{THH}(A)$ is initial among $\mathbb{E}_\infty$-algebras over A with an S^1-action through $\mathbb{E}_\infty$-maps.

Here the S^1-action comes through the equivalence $\mathrm{THH}(A) \simeq S^1 \otimes A$ from S^1 acting on itself by left multiplication. Under the geometric interpretation $\mathrm{THH}(A) \simeq \mathscr{O}(\mathcal{L}X)$ for $X = \operatorname{Spec} A$, it comes from rotation of loops.

Though the contents of the previous few paragraphs hold exclusively for $\mathbb{E}_\infty$-rings, the spectrum $\mathrm{THH}(A)$ still carries a canonical S^1-action (now only through spectrum maps, as there is no guaranteed ring structure in sight!) for any $\mathbb{E}_1$-ring A. The origin of the S^1-action can in that case be traced to geometric realization presentation of THH we encountered in subsection 2.7.6.

The circle action is a rather crucial aspect of the structure on topological Hochschild homology, and will be especially crucial in a future section where we outline a construction of topological cyclic homology.

2.6.12. **THH is computable.** One great thing about THH is that given A, by an large it is possible to compute $\mathrm{THH}(A)$. As mentioned in the last section, that is in start contrast with algebraic K-theory.

For instance, let X be an arbitrary connected space. The based loop space ΩX carries a natural $\mathbb{E}_1$-space structure, coming from concatenation of loops. This makes its suspension spectrum $S[\Omega X]$ into an $\mathbb{E}_1$-ring. Its topological Hochschild homology is then

$$\mathrm{THH}(S[\Omega X]) \simeq S[\mathcal{L}X],$$

where $\mathcal{L}X := \mathrm{Map}_{\mathcal{S}}(S^1, X)$ is the free loop space. Note the all-but-accidental analogy with the algebro-geometric interpretation of THH from 2.7.8.

Another example, which is at the heart of why THH is interesting to aritmetically-minded people, is this: a simple computation of ordinary Hochschild homology shows that

$$\mathrm{HH}_*(\mathbf{F}_p) \simeq \Gamma_{\mathbf{F}_p}(u)$$

is a divided power algebra on a single generator u of degree 2. On the other hand, we a much more sophisticated landmark computation due to Bokstedt identified the homotopy ring of topological Hochschild homology of the Eilenberg-MacLane spectrum of $\mathbf{F}_p$ as

$$\pi_*(\mathrm{THH}(\mathbf{F}_p)) \simeq \mathbf{F}_p[u],$$

a polynomial algebra on a same degree 2 generator u. Since polynomial algebras are in very many ways much better behaved than divided power algebras, this is very useful.

In light of the this subsection, it is quite amazing that THH carries a distinguished map from K-theory, and that this map often knows quite a lot about K-theory itself. Alas, it exists, and is called the *Dennis trace map*. We will discuss it in the next section.

2.7. Traces and topological cyclic homology

The division of this section and the previous is rather artificial. Indeed, we will mention results and notions from the previous section constantly. The reason for the split is primarily to punctuate a perspective shift, but also so as hopefully not ruin the impression of accessibility of THH that the previous section hoped to instill.

Thus now that we know what topological Hoschschild homology is, this section is dedicated to discussing its relationship with algebraic K-theory. As alluded to at the end of the last section, the relationship stems from a *trace map* from K-theory to Hochschild homology. Let us explain where this comes from.

2.7.1. **Chern character and loop spaces.** A construction of a slightly weaker trace map that I am quite fond of is this: start with our affine spectral scheme $X = \operatorname{Spec} A$ and a rank r vector bundle $\mathscr{E}$ on X. It is classified by a map of spectral stacks $X \xrightarrow{\mathscr{E}} \mathrm{BGL}_r$, where the RHS denotes the classifying stack of the spectral algebraic group GL_r. Passing to free loop spaces gives rise to a map of spectral stacks $\mathcal{L}X \xrightarrow{\mathcal{L}\mathscr{E}} \mathcal{L}\mathrm{BGL}_r$.

Now for any group scheme G, spectral or otherwise, the derived free loop space may be identified as $\mathcal{L}\mathrm{B}G \simeq G/_{\mathrm{conj}}G$, the quotient of G by the action of itself under conjugation. Functions $\mathcal{L}\mathrm{B}G$ are thus equivalent to conjugation-invariant functions on G itself. When G is a matrix group, the trace map $\mathrm{tr} : G \to \mathbf{A}^1$ (note that functions on X are the same things as maps $X \to \mathbf{A}^1$, if you wish by the universal property of the affine line) is a prime example of such a map.

Putting this together, starting with a rank r vector bundle $\mathscr{E}$ on X, we obtain a function on $\mathcal{L}X$ given by

$$\mathcal{L}X \xrightarrow{\mathcal{L}\mathscr{E}} \mathcal{L}\mathrm{BGL}_r \simeq \mathrm{GL}_r/_{\mathrm{conj}}\mathrm{GL}_r \xrightarrow{\mathrm{tr}} \mathbf{A}^1.$$

This is a trace construction $\mathrm{ch} : \mathrm{Vect}_r(X) \to \mathscr{O}(\mathcal{L}X)$, which we will call the *Chern character.*

Examples:

- Under the Hochschild-Kostant-Rosenberg isomorphism (which is a fascinating story upon itself that I don't wish to talk too much about here - a story for another day!), which for a smooth algebra A over a characteristic 0 field k identifies Hochschild homology with (Kahler) differential forms as

  $$\mathrm{HH}(A/k) \simeq \mathrm{Sym}^*_k(\Omega_{A/k}[1]) \simeq \bigoplus_{i\geq 0} \Omega^i_{A/k}[i],$$

  this trace construction is identified with the classical (algebro-geometric) Chern character that we briefly touched on in 2.2.10.

- Conversely, let $X \simeq \mathrm{B}G$ be the classifying stack of an algebraic group (or group scheme alike) G. Then the Chern character maps

  $$\mathrm{Rep}_r(G) \simeq \mathrm{Vect}_r(\mathrm{B}G) \xrightarrow{\mathrm{ch}} \mathscr{O}(\mathcal{L}\mathrm{B}G) \simeq \mathscr{O}(G/_{\mathrm{conj}}G) \simeq \mathscr{O}(G)^G =: \mathrm{Cl}(G)$$

  from rank r-representations of G to the class functions on G. Tracing through the construction, we may recognize that it sends a representation to its character, thus justifying the name Chern *character*.

BTW: the HKR Theorem can be extended to non-smooth algebras (or even $\mathbb{E}_\infty$-algebras over k) at the cost of replacing Kahler differentials $\Omega_{A/k}$ with the cotangent complex $L_{A/k}$. Since the HKR Theorem will not be used for anything other than motivation here, we do not go into more detail, but it is a really neat story.

2.7.2. **Trace in the affine case.** In the affine case when $X \simeq \mathrm{Spec}\, A$ for an $\mathbb{E}_\infty$-ring A, the Chern character map may be viewed in light of subsection 2.7.8 as map from the full subspace $\mathrm{BGL}_r(A)$ (where unlike in the last subsection, with some potential for confusion, this stands not for the stacky quotient, but a classifying space of a grouplike $\mathbb{E}_\infty$-space) of $\mathrm{Mod}_A^{\simeq}$, spanned by the object $A^{\oplus r}$, to $\mathrm{THH}(A)$.

The subspace of perfect A-modules $(\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq} \subseteq \mathrm{Mod}_A^{\simeq}$ is generally bigger than just $\coprod_{r\geq 0} \mathrm{BGL}_r(A)$, as there are more perfect A-modules then merely $A^{\oplus r}$ (unless say A is a field), but it does behave much like it. In particular, the trace maps on $\mathrm{BGL}_r(A)$ all come from a trace map on $(\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq}$. Given a perfect A-module E, we may construct an A-linear map

$$A \to \mathrm{End}_A(E) \simeq E^\vee \otimes_A E \to A$$

in which the first map is the inclusion of the identity morphism, the second map is an equivalence that follows from E being perfect (more precisely: dualizable), and the last map is the evaluation map, viewing the dual $E^\vee \simeq \underline{\mathrm{Map}}_A(E, A)$. as A-linear functionals on E. This assembles into a map

$$(\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq} \to \mathrm{Map}_{\mathrm{Mod}_A}(A, A) \simeq \Omega^\infty A$$

and it is not hard to convince oneself that this is a map of $\mathbb{E}_\infty$-spaces.

Viewing the left-hand side as a spectral stack $\mathrm{Perf}^{\simeq} : \mathrm{CAlg} \to \mathcal{S}$, sending $A \mapsto (\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq}$, and right-hand side as $\mathbf{A}^1(A) \simeq \Omega^\infty A$, the naturality in A of this construction shows that this trace is a map of spectral stacks $\mathrm{tr} : \mathrm{Perf}^{\simeq} \to \mathbf{A}^1$.

2.7.3. **Cyclic symmetry of the trace.** Just as in the $\mathrm{BGL}_r(A)$ situation, the trace map $\mathrm{tr} : \mathrm{Perf}^{\simeq} \to \mathbf{A}^1$ possesses a cyclic symmetry. Very informally and naively: that means that $\mathrm{tr}(fgh) = \mathrm{tr}(hfg)$, a property surely familiar from linear algebra. Less informally but also likely less insightfully: it is an S^1-equivariance structure supplied, in light of dualizability of perfect complexes, by the famous Cobordism Hypothesis. Formally this means that the trace map in fact lifts to a map of stacks $\mathrm{tr} : \mathcal{L}\mathrm{Perf}^{\simeq} \to \mathbf{A}^1$.

2.7.4. **The Dennis trace map.** We may now repeat the arguments from 2.8.1 with $\mathrm{Perf}^{\simeq}$ in place of BGL_r to obtain a "character" map

$$(\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq} \to \Omega^\infty \mathscr{O}(\mathcal{L}\mathrm{Spec}\, A) \simeq \Omega^\infty \mathrm{THH}(A).$$

This is furthermore a map of $\mathbb{E}_\infty$-spaces, where the structure on the left-hand side is given by direct sum $\oplus$. Group completing leads to a map

$$\Omega^\infty \mathrm{K}(A) \simeq ((\mathrm{Mod}_A^{\mathrm{perf}})^{\simeq})^{\mathrm{gp}} \to \Omega^\infty \mathrm{THH}(A)$$

of grouplike $\mathbb{E}_\infty$-spaces. Under the equivalence $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{CMon}^{\mathrm{gp}}$ of May's Recognition Theorem, we obtain a map of spectra

$$\mathrm{tr} : \mathrm{K}(A) \to \mathrm{THH}(A).$$

This, at last, is the Dennis trace. As is clear from the preceding discussion, the Dennis trace map is yet another analogue of the Chern character.

2.7.5. **Topological negative cyclic homology.** The S^1-equivariance that went in subsection 2.8.3 into the construction of the Dennis trace map makes it quite clear that the $\mathrm{tr} : \mathrm{K}(A) \to \mathrm{THH}(A)$ factors through the circle action invariants $\mathrm{TC}^-(A) := \mathrm{THH}(A)^{hS^1}$. This spectrum is called the *topological negative cyclic homology* spectrum.

Under the geometric interpretation $\mathrm{THH}(A) \simeq \mathscr{O}(\mathcal{L}X)$ with $X \simeq \operatorname{Spec} A$, topological negative cyclic homology is given as

$$\mathrm{TC}^-(A) \simeq \mathscr{O}((\mathcal{L}X)/S^1),$$

the (stacky) quotient of the free loop space $\mathcal{L}X$ by its S^1-action given by rotation of loops. At first glance one might expect that $(\mathcal{L}X)/S^1$ might be very close to X itself, but in fact it is "a bit more fuzzy". It is none the less closer to it that $\mathcal{L}X$ is, which is to say that $\mathrm{TC}^-(A)$ is a finer invariant of A than $\mathrm{THH}(A)$.

2.7.6. **(Non-topological) cyclic homology.** The analogue of topological cyclic homology over an ordinary ring k instead of over the sphere spectrum S is $\mathrm{HC}(A/k) := \mathrm{HH}(A/k)^{hS^1}$, known simply as *cyclic homology of A*. It is a confusing but entrenched state of terminology that the direct topological analogue of cyclic homology is called "topological negative cyclic homology", while the simpler name "topological cyclic homology" is reserved for a more sophisticated construction.

A landmark result of Goodwillie asserts that the descended Dennis trace map $\mathrm{tr} : \mathrm{K}(A) \to \mathrm{HC}(A/k)$ (sometimes called the Goodwillie-Jones trace) is quite close to being a rational equivalence:

Theorem 19 (Goodwillie)**.** *The rationalized trace map* $\mathrm{tr} : \mathrm{K}(A) \otimes \mathbf{Q} \to \mathrm{HC}(A/k) \otimes \mathbf{Q}$ *is locally constant.*

That is to say, let $A \to A'$ be nilpotent extension (i.e. surjection with a nilpotent kernel) of commutative k-algebras (or connective $\mathbb{E}_\infty$-algebras, or just connective $\mathbb{E}_1$-algebras). The map that the trace map induces between the cofibers of $\mathrm{K}(A) \to \mathrm{K}(A')$ *and* $\mathrm{HC}(A/k) \to \mathrm{HC}(A'/k)$ *is an equivalence after smashing with* $\mathbf{Q}$.

This may be viewed as saying that (rationally) K and HC^- are uniformly apart. This is very computationally powerful, as it allows for computation of the rational part of algebraic K-theory by ascending towers of nilpotent extensions.

When A is a smooth k-algebra and k a field of characteristic zero, cyclic homology has an HKR description. Recall from subsection 2.8.1 that $\mathrm{HH}(A/k) \simeq \bigoplus \Omega^i_{A/k}[i]$, which may by Dold-Kan be viewed as a chain complex with differential i-forms in the i-th degree and the zero differential between them. The circle action, through the identification $C^*(S^1;k) \simeq \mathrm{H}^*(S^1 : k) \simeq k[x]$ (the first equivalence is due to what is called *the rational formality of the circle*, and it is what makes this story work in char 0 but not outside it) with x in degree 1, corresponds to a degree 1 map on the chain complex. That is nothing but the de Rham differential. Passing to homotopy invariants is related to building this differential in and viewing the result as a new chain complex. Thus $\mathrm{HC}(A/k)$ is related to the de Rham chain complex

$$\Omega^0_{A/k} \xrightarrow{d} \Omega^1_{A/k} \xrightarrow{d} \Omega^2_{A/k} \to \cdots$$

and is as such a good analogue of the de Rham cohomology of A.

2.7.7. **Periodic cyclic homology.** Really there is still some refinement available: the "relationship" between the de Rham complex and $\mathrm{HC}(A/k)$ is slightly more complicated than might have come across from the remarks in the previous subsection.

In particular, what is needed to actually compare them is to invert the action of the generator inducing the S^1-action, which under the equivalence $\mathrm{H}^*(\mathrm{B}S^1;k) \simeq k[u]$ corresponds to degree 2 element u. This comes at the expense of introducing a lot of redundancy in the cohomology, effectively making it periodic. For this reason, the result is called the *periodic cyclic homology* $\mathrm{HP}(A/k)$. The extension of the HKR Theorem, alluded to in the previous subsecton, is an equivalence

$$\mathrm{HP}_*(A/k) \simeq \mathrm{H}^*_{\mathrm{dR}}(A/k)[u^{-1}]$$

between periodic cyclic homology groups and periodicized (algebraic) de Rham cohomology groups.

The analogous construction can be done over the sphere spectrum too. We start off with $\mathrm{THH}(A)$ with its S^1-action as before. But to explain what we do next, i.e. in what way we should periodicize the S^1-action, we need to dip our toes in a slight digression.

2.7.8. **Digression: the Tate construction.** Whenever G is a compact Lie group acting on a spectrum M, there exists a distinguished map of spectra

$$\mathrm{Nm} : \Sigma^{\mathfrak{g}}(M_{hG}) \to M^{hG},$$

called the *norm map*, where $\mathfrak{g}$ is the Lie algebra of G. Its cofiber is called the *Tate construction* and denoted M^{tG}.

This map is probably the most familiar in the case of a finite group G, where $\mathfrak{g} \simeq 0$ and so the suspension disappears. Then the norm map $\mathrm{Nm} : M_{hG} \to M^{hG}$ is given informally by $[x] \mapsto \sum_{g \in G} gx$, i.e. sending an orbit to the sum of its elements. Of course the actual formal ∞-categorical construction of the norm map is quite a fair bit more involved. Lurie does it in HA in an inductive way, but there is a more traditional way of doing it through genuine equivariant homotopy theory - pick your poison!

When G is a non-discrete Lie group, the sum should be replaced by integration over G, which at least heuristically explains the shift to get things in the top degree, that being $\dim \mathfrak{g} = \dim G$, so as to make things fit to be integrated over G.

Let G be a finite (or profinite) group, and M an abelian group with a G-action (i.e. a $\mathbf{Z}[G]$-module). The just as the homotopy groups of the homotopy invariants M^{hG} are group cohomology $\mathrm{H}^*(G; M)$, and homotopy groups of homotopy coinvariants M_{hG} are group homology $\mathrm{H}_*(G; M)$, the homotopy groups of the Tate construction M^{tG} give rise to Tate cohomology $\hat{\mathrm{H}}^*(G; M)$. The latter might perhaps be familiar from class field theory, for the purposes of which Tate introduced it. It intertwines group homology and cohomology (as seen in the definition of the Tate construction above), agreeing with $\mathrm{H}^*(G; M)$ in positive degrees, and with $\mathrm{H}_{-*-1}(G; M)$ in negative degrees. This sort of intertwining of degrees and smearing homotopy groups accross all degrees is the periodization proceedure that we need.

2.7.9. **Topological periodic cyclic homology.** Thus *topological periodic cyclic homology* of an $\mathbb{E}_\infty$-ring (or $\mathbb{E}_1$-ring) A is defined as $\mathrm{TP}(A) := \mathrm{THH}(A)^{tS^1}$. Indeed, the most succinct definition of periodic cyclic homology over a commutative ring k is also $\mathrm{HP}(A/k) \simeq \mathrm{HH}(A/k)^{tS^1}$.

And though this is a wonderfully complicated spectrum, knowing much about A and being quite close to algebraic K-theory, we must work a little harder still to define the coveted topological cyclic homology. The construction of the latter, as we shall see, essentially uses the Tate construction of subsection 2.8.8, as well as $\mathrm{TP}(A)$ itself, so the discussion of the periodic version was no detour, but rather a necessary pit-stop on the rout toward $\mathrm{TC}(A)$.

2.7.10. **The Dundas-Goodwillie-McCarthy Theorem.** Before we actually go through the motions of creating this Frankenstein-like horror, let us first say what it is good for. So assume that we already have $\mathrm{TC}(A)$, whatever it is, with a factorization of the Dennis trace map into the cyclotomic trace $\mathrm{trc} : \mathrm{K}(A) \to \mathrm{TC}(A)$. With this, the Goodwille's Theorem, mentioned in 2.8.6, admits an integral (as opposed to rational) refinement:

Theorem 20 (Dundas-Goodwillie-McCarthy). *The cyclotomic trace map* $\mathrm{trc} : \mathrm{K}(A) \to \mathrm{TC}(A)$ *is locally constant.*

That is to say, let $A \to A'$ be a map of connective $\mathbb{E}_\infty$-rings (or $\mathbb{E}_1$-rings), such that $\pi_0(A) \to \pi_0(A')$ is a nilpotent extension (i.e. surjection with a nilpotent kernel) The map that the trace map induces between the cofibers of $\mathrm{K}(A) \to \mathrm{K}(A')$ *and* $\mathrm{TC}(A) \to \mathrm{TC}(A')$ *is an equivalence.*

This theorem is quite amazing. As with Goodwillie's Theorem, it allows to extend computation of K-theory from simpler rings to more complicated ones via ascending along towers of nilpotent extension. But now there is no rationality assumptions - we are obtaining full torsion information as well! This is great: though the definition of $\mathrm{TC}(A)$ is, as we shall see in the next few subsections, a fair bit more involved than that of $\mathrm{THH}(A)$, it is still essentially a very computable spectrum. That it remains "a constant distance away from" algebraic K-theory, a highly non-computable spectrum, is quite an amazing miracle, and most exploitable.

For a pleasantly readable proof of the Dundas-Goodwillie-McCarthy Theorem, see the exposition by Sam Raskin (though beware of some non-conventional choices, such as grading spectra cohomologically).

2.7.11. **A roadmap to** TC**.** To construct topological cyclic homology, we follow an approach of Blumberg-Mandell, which we outline here. Then we will sketch two ways in historical order of supplying the details: first (and with hardly any details) via genuine equivariant homotopy theory, and then (with slightly more details) a naive approach due to Nikolaus-Scholze.

We start off by defining the ∞-category of *cyclotomic spectra* CycSp. This should be something slightly stronger than spectra with an S^1-action. In particular, the sphere spectrum with its trivial action should give rise to an object $S \in \mathrm{CycSp}$. Next we upgrade the circle action on $\mathrm{THH}(A)$ to a cyclotomic structure. Finally we define topological cyclic homology as the mapping spectrum (as CycSp, being a stable ∞-category, will possess a natural enrichment in Sp)

$$\mathrm{TC}(A) \coloneqq \underline{\mathrm{Map}}_{\mathrm{CycSp}}(S, \mathrm{THH}(A)).$$

This definition may seem quite indirect, and justly so. Following the Nikolaus-Scholze approach, will enable us to provide a somewhat more explicit formula later on.

The yoke of the job is thus to define cyclotomic spectra.

2.7.12. **Cyclotomic spectra via genuine S^1-equivariant spectra, I.** If you have skipped subsection 1.5.3, where we briefly dipped our toes into genuine equivariant homotopy theory, then perhaps you may wish to skip this subsection as well. Note however that we will not be using much genuine technology, so you may as well stick around.

Well, other than the following piece of equivariant technology, that we haven't encountered before:

2.7.13. **Intermezzo: Geometric fixed points.** Let G be a compact Lie group (the one we have in mind is $S^1 \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$). Then recall that a genuine G-space X (say pointed, though this makes no difference) is really a certain sort of functor, and in particular for any closed normal subgroup $H \subseteq G$, it produces a (pointed) genuine G/H-space X^H, its *H-fixed points* (in fact, this exists for non-normal subgroups too, but we will only need it for normal ones). A similar construction works for G-spectra, giving rise for a genuine G-spectrum M to a genuine G/H-spectrum M^H, which is called the *categorical H-fixed-points* of M.

Just as ordinary pointed spaces admit suspension spectra, giving rise to the functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$, so does this happen in the G-world, and there is an analogous G-suspension functor $\Sigma^\infty_G : \mathcal{S}_{G*} \to \mathrm{Sp}_G$. Alas, this functor *is not compatible* with the fixed points discussed in the previous paragraph.

Thus we *define a new* fixed-point functor for genuine G-spectra to bridge this gap. The *geometric fixed-points* functor $\Phi^H : \mathrm{Sp}_G \to \mathrm{Sp}_{G/H}$ is defined by the requirements that

- For any pointed genuine G-space X we have
$$\Phi^H(\Sigma^\infty_G X) \simeq \Sigma^\infty_{G/H}(X^H).$$
- The functor Φ^H is symmetric monoidal (with respect to the genuine smash products) and preserves colimits.

This specifies geometric fixed-points essentially uniquely. The discrepancy between $\Phi^H(M)$ and M^H is behind many of the more unpleasant (or charming, depending on ones perspective no doubt) aspects of genuine equivariant homotopy theory.

2.7.14. **Cyclotomic spectra via genuine S^1-equivariant spectra, II.** Defining cyclotomic spectra is easy now. The data of a cyclotomic spectrum consists of a genuine S^1-equivariant spectrum M together with a system of compatible S^1-equivariant equivalences $\Phi^{C_n}(M) \simeq M$ for all $n \geq 0$. Here the geometric fixed points are taken along the inclusion $C_n \subseteq S^1$ of the cyclic group of order n, embedded as n-th roots of unity into $\mathrm{U}(1) \simeq S^1$. Since the equivalence $S^1/C_n \simeq S^1$ is exhibited by the n-th power map $z \mapsto z^n$, we may indeed view $\Phi^{C_n}(M)$ as an S^1-spectrum.

This is a neat enough definition, claiming invariance under taking (geometric) fixed points along arbitrary-order roots of unity inside the circle, hence the number theoretic term "cyclotomic". The annoying part of this is the homotopy-coherence mess that specifies the appropriate "compatibility" between the equivalences $\Phi^{C_n}(M) \simeq M$ and $\Phi^{C_m}(M) \simeq M$ for various n and m. Not intractible, just a little impractical.

It remains to exhibit a cyclotomic structure on $\mathrm{THH}(A)$ and for S with the constant S^1-action, which Bokstedt, Goodwillie, Waldhausen, Hesselholt, and other friends did.

2.7.15. **The Nikolaus-Scholze naive approach.** When studying all this, Peter Scholze observed that much of the above could be rephrased without explicit mention of geometric fixed points, and furthermore without using any genuine S^1-equivariant structure. This was carried out in the rather influential joint paper with Nikolaus.

The idea is roughly to employ the Tate construction to rephrase things without explicit mention of geometric fixed-points. This is because the Tate construction, though the approach to it that we indicted in subsection 2.8.8 used only naive actions, also admits an genuine equivariant approach. We will not explain anything more about how to pass between the Nikolaus-Sholze construction and the genuine equivariant one though, and will instead refer any interested reader to Nikolaus and Scholze's wonderful paper.

2.7.16. **p-typical cyclotomic spectra.** Let p be any fixed prime. In the genuine approach, outlined in subsection 2.8.14, we could have defined p-typical cyclotomic spectra as genuine S^1-spectra M together with an equivalence $\Phi^{C_p}(M) \simeq M$. This "one prime at a time" approach is ill-suited to that approach, however, as it lacks the compatibility data between the different cyclotomic structure maps required.

The naive definition of *p-typical cyclotomic spectra* is that such a spectrum consists of a spectrum with an S^1-action M (i.e. a naive equivariant spectrum) and an S^1-equivariant map $\varphi_p : M \to M^{tC_p}$. Here the Tate construction is taken with respect to the inherited C_p-action, coming from the standard copy $C_p \simeq \mu_p \subset \mathrm{U}(1) \simeq S^1$ of the p-th roots of unity inside the unit circle.

This time we do not require the cyclotomic structure maps to be equivalences. The key thing is though that unlike the genuine cyclotomic structure maps $M \simeq \Phi^{C_p}(M)$, the naive ones $\varphi_p : M \to M^{tC_p}$ are entirely independent of each other!

2.7.17. **Naive approach to cyclotomic spectra.** This allows us to define CycSp to have for objects spectra M with an S^1-action, equipped with a family of S^1-equivariant maps $\varphi_p : M \to M^{tC_p}$ for all primes p. That is it - easy peasy!

We would be remiss not to point out that this definition of CycSp only agrees with the genuine one from 2.8.14 on essentially connective (if you want: bounded below) objects. But since those are the only ones that come into question for the construction of $\mathrm{TC}(A)$ (at least for A connective), this more than suffices.

2.7.18. **Explicit formula for topological cyclic homology.** The rather concrete naive definition of cyclotomic spectra also allows us, following Nikolaus-Scholze, to give a rather concrete description of the mapping spectrum $\underline{\mathrm{Map}}_{\mathrm{CycSp}}(S, M)$, for any cyclotomic spectrum M (the reason this is so interesting is of course that TC is a special case). Passing to (ordinary, i.e. homotopy - no genuine equivariant business here!) S^1-fixed-points from the cyclotomic structure map φ_p gives rise to maps $\varphi_p^{hS^1} : M^{hS^1} \to (M^{tC_p})^{hS^1}$. But spectrum maps of that form can also be obtained just from the S^1-action as

$$\mathrm{can}_p : M^{hS^1} \simeq (M^{hC_p})^{h(S^1/C_p)} \to (M^{tC_p})^{hS^1},$$

in which the last map is obtained by simultaneously passing through the map $M^{hC_p} \to M^{tC_p}$, from the definition of the Tate construction, and using the equivalence $S^1/C_p \simeq S^1$ in the external homotopy fixed-points. The mapping spectrum is then given as the ∞-categorical equalizer

$$\underline{\mathrm{Map}}_{\mathrm{CycSp}}(S, M) \simeq \mathrm{Eq}\Big(M^{hS^1} \rightrightarrows \prod_p (M^{tC_p})^{hS^1}\Big)$$

of the structure maps $\prod_p \varphi_p^{hS^1}$ and the cannical maps $\prod_p \mathrm{can}_p$. We may identify the codomain of the equalizer with a profinite completion $(M^{tS^1})^\wedge$ of the S^1-Tate construction.

When specializing to $M = \mathrm{THH}(A)$, with its yet-to-be-discussed cyclotomic structure, we obtain the formula for topological cyclic homology as the equalizer

$$\mathrm{TC}(A) \simeq \mathrm{Eq}(\mathrm{TC}^-(A) \rightrightarrows \mathrm{TP}(A)^\wedge)$$

of the cyclotomic structure maps and the canonical maps, both viewed as mapping into the profinite completion of the topological periodic cyclic homology.

2.7.19. **The cyclotomic structure on THH, I.** One piece of the puzzle remains, and that is to exhibit a cyclotomic structure on topological Hochschild homology. This is one more of those things that is perfectly doable for $\mathbb{E}_1$-rings, but simplifies substantially for $\mathbb{E}_\infty$-rings. Thus we only discuss the latter situation.

Let A be an arbitrary fixed $\mathbb{E}_\infty$-ring, and p a fixed prime. To exhibit a cyclotomic structure on $\mathrm{THH}(A)$, we must specify an S^1-equivariant map $\mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$. Suppose further that this map of spectra will in fact be a map of $\mathbb{E}_\infty$-rings. Then we can use the fact we learned in subsection 2.7.11 that $\mathrm{THH}(A)$ is initial among $\mathbb{E}_\infty$-algebras over A with an S^1-action, to reduce ourselves to constructing an $\mathbb{E}_\infty$-ring map $A \to \mathrm{THH}(A)^{tC_p}$. To find such a map, we use a key construction available in the ∞-category of spectra Sp that is not available in a derived category $\mathcal{D}(R) \simeq \mathrm{Mod}_R$ for any ordinary commutative ring R:

2.7.20. **The Tate Diagonal.** Let M be a spectrum, and consider its p-th smash power $M^{\oplus p}$. Cyclic permutation of smash factors induces an action of C_p on $M^{\oplus p}$. Thus we can form the Tate construction $T_p(M) := (M^{\otimes p})^{tC_p}$, which has a rich history in homotopy theory, having been studied by Lunoe-Nielsen and Rognes under the name *topological Singer construction*. The *Tate diagonal* is a map of spectra $\Delta_p : M \to T_p(M)$, natural in M.

The simple desiderata of such a non-trivial map is impossible to satisfy in $\mathcal{D}(R)$ for any commutative ring R; indeed, any natural transformation $M \to T_p(M)$ is trivial in $\mathcal{D}(R) \simeq \mathrm{Mod}_R$. The point is that the Tate diagonal Δ_p can not be made to be R-linear for any ordinary commutative ring R. This is the real thing that $\mathrm{Sp} \simeq \mathrm{Mod}_S$ has going for it that ordinary derived categories of modules do not, and why certain things, such as an "integral" version of the Goodwillie Theorem, only work over the sphere (in said case, the Dundas-Goodwillie-McCarthy Theorem), but not over $\mathbf{Z}$.

The existence of the Tate diagonal is one of those landmark super-easy-to-prove things that is easier to prove than to not prove. Assume that the functor $T_p : \mathrm{Sp} \to \mathrm{Sp}$ is exact - there is something to check here, but it boils down to simple combinatorics and the observation that the Tate construction vanishes on induced representations. The Tate diagonal natural transformation Δ_p that we seek should live in the space $\mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp},\mathrm{Sp})}(\mathrm{id}_{\mathrm{Sp}}, T_p)$. Recall from the universal property of stabilization that composing with the functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ induces an equivalence between $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ for any stable ∞-category $\mathcal{C}$. Thus we have homotopy equivalences

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp},\mathrm{Sp})}(\mathrm{id}_{\mathrm{Sp}}, T_p) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathcal{S})}(\Omega^\infty, \Omega^\infty T_p) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathcal{S})}(\mathrm{Map}_{\mathrm{Sp}}(S,-), \Omega^\infty T_p).$$

Now we may invoke the Yoneda lemma, which identifies for any ∞-category $\mathcal{C}$, any functor $F : \mathcal{C} \to \mathcal{S}$ and any object $C \in \mathcal{C}$ a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathcal{S})}(\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathrm{Map}_{\mathcal{C}}(C,-), F) \simeq F(C),$$

to conclude that

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Sp},\mathcal{S})}(\mathrm{Map}_{\mathrm{Sp}}(S,-), \Omega^\infty T_p) \simeq \Omega^\infty T_p(S) \simeq \mathrm{Map}_{\mathrm{Sp}}(S, T_p(S)) \simeq \mathrm{Map}_{\mathrm{Sp}}(S, S^{tC_p}).$$

The last equivalence comes from the fact that, due to the sphere spectrum being a unit for the smash product, having an identification $S^{\otimes p} \simeq S$ with the sphere spectrum with the trivial C_p-action. It follows that we are reduced to finding a map of spectra $S \to S^{tC_p}$. For this, note that the homotopy invariants funtor $M \mapsto M^{hC_p}$ is symmetric monoidal, and as such preserves commutative algebra objects. This means that S^{hC_p} carries a canonical $\mathbb{E}_\infty$-ring structure, and as such receives an essentially unique $\mathbb{E}_\infty$-ring map $S \to S^{hC_p}$ of "inclusion of the multiplicative unit". We compose this map with the canonical quotient map $S^{hC_p} \to S^{tC_p}$, coming from the definition of the Tate constriction, to obtain the desired map $S \to S^{tC_p}$. Following the chain of equivalences we have woven, this produces the Tate diagonal transformation $\Delta_p : M \to T_p(M)$, natural in $M \in \mathrm{Sp}$.

Though we will not show it (Nikolaus-Scholze provide highly recommendable clean and meticulous exposition), both the Tate construction itself, as well as the Tate diagonal transformation, are in fact lax symmetric monoidal. In particular, though this does not mean that it preserves the smash product, it is enough to show that it preserves commutative algebras. Hence $T_p(A)$ is an $\mathbb{E}_\infty$-ring whenever A is an $\mathbb{E}_\infty$-ring, and the Tate diagonal map $\Delta_p : A \to T_p(A)$ is a map of $\mathbb{E}_\infty$-rings.

2.7.21. **The cyclotomic structure on** THH, **II.** We promised to use the Tate diagonal to construct the cyclotomic structure on topological Hochschild homology of an $\mathbb{E}_\infty$-ring A. In subsection 2.8.20 we already reduced this task to choosing an $\mathbb{E}_\infty$-ring map $A \to \mathrm{THH}(A)^{tC_p}$. Using the Tate diagonal we may obtain

$$A \xrightarrow{\Delta_p} T_p(A) \simeq (A^{\otimes p})^{tC_p} \simeq (C_p \otimes_{\mathrm{CAlg}} A)^{tC_p} \to (S^1 \otimes_{\mathrm{CAlg}} A)^{tC_p} \simeq \mathrm{THH}(A)^{tC_p},$$

where the second map (first equivalence) is merely the definition of T_p, the third map (second equivalence) is the observation that, based on the definition of tensoring with spaces from subsection 2.7.9, the smash power $A^{\oplus p}$ coincides with the tensor $C_p \otimes A$ in the ∞-category CAlg (since $C_p \simeq \coprod_{1 \le i \le p} *$ and the coproduct in CAlg is given by the smash product), the thirst map comes from the inclusion $C_p \subseteq S^1$, and the final map (equivalence) comes from the identification between THH and tensoring with S^1 in $\mathbb{E}_\infty$-rings from subsection 2.7.10.

Thus (modulo assuming the lax symmetric monoidality of the Tate diagonal) we have shown how to construct the cyclotomic structure on the topological Hochschild homology of an $\mathbb{E}_\infty$-ring. With that concludes our tour of THH and its many variants. But before we end the section, since we are right here at the gates, les us shoot but a sneak peak at another application of the Tate diagonal.

2.7.22. **The Tate-valued Frobenius of $\mathbb{E}_\infty$-rings.** As mentioned above, the Tate diagonal is behind much of what makes the theory of spectra richer than that of chain complexes of modules (this is partially why we chose to go down the rabbit-hole of topological cyclic homology - to naturally encounter this structure). In particular, it gives something very exciting when applied to $\mathbb{E}_\infty$-rings.

Let A be an $\mathbb{E}_\infty$-ring. Then composing the Tate diagonal with the multiplication of p factors map $\mu : A^{\otimes p} \to A$ (which is C_p-equivariant, and even more, Σ_p-equivariant essentially by definition), we obtain an $\mathbb{E}_\infty$-ring map

$$\varphi : A \xrightarrow{\Delta_p} T_p(A) \simeq (A^{\otimes p})^{tC_p} \xrightarrow{\mu^{tC_p}} A^{tC_p}$$

for every prime p. This is the *Tate-valued Frobenius*, also sometimes called the Nikolaus-Scholze Frobenius. It is the correct notion of the Frobenius map for $\mathbb{E}_\infty$-rings.

2.7.23. **Ordinary Frobenius also takes values in the Tate construction.** The Tate-valued Frobenius might look strange at first sight, namely the codomain might seem all wrong. To convince ourselves that it it all right, let us recall in a bit more detail how the usual Frobenius of commutative rings works. For a commutative ring R, it is a map $R \to R$ given by $x \mapsto x^p$. It is not a ring map, as while perfectly multiplicative, it fails to be additive. Indeed, we have by the Binomial Theorem for any $x, y \in R$

$$(x+y)^p = x^p + \underbrace{p\,x^{p-1}y + \cdots + p\,xy^{p-1}}_{\text{divisible by } p} + y^p$$

and so the Frobenius does descend to a ring map $R \to R/p$. The reason we aren't used to seeing this quotient R/p is that we usually consider the p-Frobenius for a commutative ring R of characteristic p, for which the quotient map is an isomorphism $R \simeq R/p$.

But we saw in subsection 2.1.5 that quotienting by p is not a valid construction to perform with $\mathbb{E}_\infty$-rings, so that does not look promising. The solution is to look at what the ring R/p that appeared really is more closely. Indeed, setting $p = 3$ for simplicity, the above calculation is in more detail

$$(x+y)^3 = \overbrace{x^3 \;+\; \underbrace{xxy + yxx + xyx}_{\text{sum of a } C_3\text{-orbit}} \;+\; \underbrace{yxx + xyx + xxy}_{\text{sum of a } C_3\text{-orbit}} \;+\; y^3}^{\text{invariant under } C_3},$$

showing that the quotiented copy of the ideal in R generated by p is in fact the sum of C_3-orbits (and thinking about the combinatorics behind the Binomial Theorem, we see that the same happens for any prime p), i.e. the image of the norm map $\mathrm{Nm} : R_{C_p} \to R^{C_p}$. The quotient thereof is precisely the Tate construction R^{tC_p}, albeit done in the ordinary category of abelian groups instead of spectra, analogous to where we claim the $\mathbb{E}_\infty$-ring Frobenius takes values. (The key difference is that in the ∞-categorical setting, we must handle the permutations of the facts more carefully, hence why we *must* view the codomain as the Tate construction.) Taking the Tate construction for R with a trivial C_p-action in abelian groups, we have $R_{C_p} \simeq R^{C_p} \simeq R$, and so the norm map may be identified with the map $R \to R$ sending $x \mapsto \sum_{C_p} x = p\,x$. Hence the Tate construction is in this context indeed $R^{tC_p} \simeq R/p$, the codomain of the usual Frobenius map.

2.7.24. **Tate-valued Frobenius and power operations.** This changes when doing the Tate construction in spectra. Even if R is a discrete $\mathbb{E}_\infty$-ring, i.e. an ordinary commutative ring, the spectra R^{hC_p} and R_{hC_p} will generally have a lot of homotopy groups. Those of the former are $\mathrm{H}^*(C_p; R)$, group cohomology of the trivial C_p-module R, and of the latter are $\mathrm{H}_*(C_p; R)$, its group homology. When p is not invertable in R, these groups will generally refuse to vanish - we enter the domain of modular representation theory. This is the reason that the hypothesis that the size of the group not divided the characteristic of the ring of coefficients is so pervasive in basic representation theory of finite groups.

Thus the Tate construction, whose homotopy groups will be the Tate cohomology groups $\hat{\mathrm{H}}^*(C_p; R)$, will usually be quite far from being concentrated in degree 0. That suggests that the Tate-valued Frobenius might be encoding some interesting information.

For instance, when $p = 2$ and $R = \mathbf{F}_2$, we have $\mathbf{F}_2^{tC_2} \simeq \bigoplus_{i\in\mathbf{Z}} \mathbf{F}_2[i]$. The components of the Tate-valued Frobenius

$$\mathbf{F}_2 \xrightarrow{\varphi} \mathbf{F}_2^{tC_2} \simeq \bigoplus_{i\in\mathbf{Z}} \mathbf{F}_2[i] \xrightarrow{\mathrm{pr}_i} \mathbf{F}_2[i]$$

then encodes the data of $\mathrm{Sq}^i : \mathbf{F}_2 \to \mathbf{F}_2[i]$, the *i-th Steenrod square*. In particular, applying this for a fixed space X to the functor of cochains $C^*(X; -)$, we obtain a map of $\mathbf{F}_2$-modules of cochains $C^*(X; \mathbf{F}_2) \to C^*(X; \mathbf{F}_2)[i]$, and passing to homotopy groups π_{-n} we obtain for $i \geq 0$ the Steenrod squares

$$\mathrm{Sq}^i : \mathrm{H}^n(X; \mathbf{F}_2) \to \mathrm{H}^{n+i}(X; \mathbf{F}_2)$$

in their usual form that you likely know and possibly love.

Playing a similar game with a higher prime $p \geq 3$ and $R = \mathbf{F}_p$ gives rise to the *Steenrod extended p-th power operations* P^i and their Bockstein multiplets $\beta\mathrm{P}^i$, the generators of the mod p Steenrod algebra.

When we plug in $R = \mathrm{KU}$, the p-Frobenius will gives rise to the *stable Adams operations* $\psi_p : \mathrm{KU}^*(X) \to \mathrm{KU}^*(X)$, the multiplicative operations determined by the requirement of functoriality and that $\psi_p : \mathrm{KU}^0(X) \to \mathrm{KU}^0(X)$ sends the class of a line bundle $[L]$ to the tensor power $[L^{\otimes p}]$, with respect to tensor product of line bundles on X.

Thus in general, the additional data encoded in the Tate-valued Frobenius of $\mathbb{E}_\infty$-ring spectra has to do with power operations. This is quite exciting, showing where these highly useful computational tools arise from a purely algebraic perspective.