Category,Functors,and Natural Transformations

§1 *Axioms fo Categories*

, , , ” ”. , , , ( )

(metagraph) $a, b, c, \cdots$, $f, g, h, \cdots$, :

Domain( ), f object $a = \mathrm{dom}\ f$;

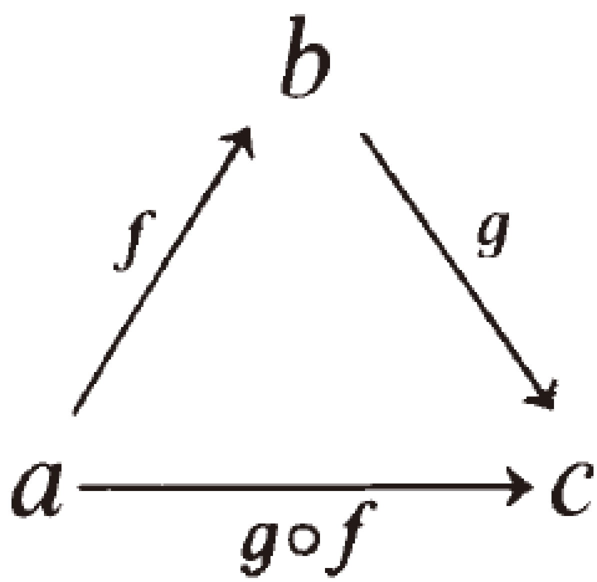
Codomian( ), f object $b = \mathrm{dom}\ f$;

f f (” ”source) (” ”target) .

$$f : a \to b \ a \xrightarrow{f} b$$

: $\cdot \to \cdot \to \cdot \cdot \rightrightarrows \cdot$.

- (Identity), a $\mathrm{id}_a = 1_a : a \to a$;
- (Composition), $<$ g, f $>$ $\mathrm{dom}\ g = \mathrm{cod}\ f$, $g \circ f : \mathrm{dom}\ f \to \mathrm{cod}\ g$.



- (Associativity): (configuration)

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d$$

$$k \circ (g \circ f) = (k \circ g) \circ f \tag{1}$$

((1)), .

$$\begin{tikzcd}
a \arrow[rr, "k \circ (g \circ f) = (k \circ g) \circ f"] \arrow[dd, "f"'] \arrow[ddrr, "g \circ f"'] & & d \\
& & \\
b \arrow[rr, "g"'] \arrow[uurr, "k \circ g"'] & & c \arrow[uu, "k"']
\end{tikzcd}$$

- (Unit law): $f : a \to b$ $g : b \to c$ 1_b

$$1_b \circ f = f \quad g \circ 1_b = g \tag{2}$$

, , b 1_b .

(2) .

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f"'] & b \arrow[d, "1_b"] \arrow[dr, "g"] & \\ & b \arrow[r, "g"'] & c. \end{tikzcd}$$

( ) ( ) . c c', c c' ( ) c c' , .

: .

b C , (2) . , 1_b b , $b : b \to b$. , ,$1_b = b = \mathrm{id}_b$.

. , , , . " " . , $f : X \to Y$ X( ), Y( ) $x \mapsto fx$($< x, fx >$) $x \in X$ $fx \in Y$. . fx, f_x $f(x)$. , S, $s \in S$ $s \mapsto s$ $1_S : S \to S$. S Y , $s \mapsto s$ $S \to Y, S \neq Y$. $f : X \to Y$ $g : Y \to Z$. $g \circ f : X \to Z$ $(g \circ f)x = g(fx)$ $x \in X$. $g \circ f$ f , g $-$, f . , .

, , . : G, H, K; G H $f : G \to H$. : ; Hausdorff ; . (morphism).

. , . , $< g, f >$( ) $< g, f >$ $g \circ f$. "g, f " "$< g, f >$ ".

, C u, $f \circ u$,$f \circ u = f$ $u \circ g$,$u \circ g = g$. , .

1. $(k \circ g) \circ f$ $k \circ (g \circ f)$ ( ). , (kgf)

2. kgf kg gf .

3. C g C u u' $u' \circ g$ $g \circ u$.

, ; u' u 3. , g u' u. . , , ( ) " " ; ,

§2 *Categories*

( ) . , (diagram scheme) O, A

$$a \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} b \tag{1}$$

,

$$A \times_O A = \{< g, f >: g, f \in A \ \mathrm{dom}\ g = \mathrm{cod}\ f\}$$

"O ".

, (gf),

$$\begin{array}{ccc} & O & \\ & \xrightarrow{\mathrm{id}} & \\ A, & A \times_O A & \\ & \xrightarrow{\circ} & \\ c & A, & \quad (2) \\ & c & \\ & \longmapsto & \\ \mathrm{id}_c, & < g, f > & \\ & \longmapsto & \\ & g \circ f & \end{array}$$

$a \in O$ $< g, f > \in A \times_O A$

$$\mathrm{dom}(\mathrm{id}\ a) = a = \mathrm{cod}(\mathrm{id}\ a), \mathrm{dom}(g \circ f) = \mathrm{dom}\ f, \mathrm{cod}(g \circ f) = \mathrm{cod}\ g \quad (3)$$

.

C , A O,

$$c \in C, f \text{ in } C \quad (4)$$

"c C " "f C ".

$$\mathrm{Hom}(b, c) = \{f : f \text{ in } C, \mathrm{dom}\ f = b, \mathrm{cod}\ f = c\} \quad (5)$$

b c .

"Hom " (§8); , ( ) , , . , ( ) $\times_O$. , .

0 (,);

1 $\dot{\circlearrowright}$ · $\circlearrowright$;

2 $\dot{\circlearrowright} \overset{\rightarrow}{} \dot{\circlearrowright}$ a, b $a \to b$;

3 ;

$\downdownarrows$ a, b $a \rightrightarrows b$, .

, .

Discrete Categories( ) , . X ($x \in X$ $x \to x$), ,
Monoid( ) , ( ) . , M, $M \times M -$
Groups( ) , ( )
Matrices( ) K, K ( ) $\mathbf{Matr_K}$; $m, n, \cdots$ $m \times n$ A $n \to m$,
Sets( ) V , $\mathbf{Ens_V}$ V $X \in V$, $f : X \to Y$, . **Ens**
Preorders( ) P, p p' $p \to p'$. P, P $\leq$,$p \leq p'$ P $p \to p'$. (p
( ) (, p, p' $p \leq p'$ $p' \leq p$)
Ordinal Numbers( ) n , $n = \{0, 1, \cdots, n-1\}$; ,0 , $w = \{0, 1, 2, \cdots\}$. n , (
Δ Δ $f : m \to n$ (, m $i \leq j$ n $f_i \leq f_j$) Δ, (simplicial catego
$\mathbf{Finord} = \mathbf{Set}_w$, $f : m \to n$ m n . , n n

Large Categories. ( ) , **Set**, (small) : U, " (universe)", x , " ".

Set( )	: , :
$\mathbf{Set}_*$( )	: ; :
Ens	( ) V
Cat( )	: ; : §3
Mon( )	: ; :
Grp( )	: ; :
Ab(Abel)	: ( )Abel ; :
Rng( )	: ; : ( )
CRng( )	: ; :
$R-$**Mod**(R)	: R R ; :
Mod$-R$(R)	
$K-$**Mod**( )	: K R
Top( )	: ; :
hTop( )	: ; :
$\mathbf{Top}_*$( )	: ; :

( ) , .

§3 *Functors*

(Functors) . , C B, $T : C \to B$ C B (, ,) : (object function)T, C
$c \to c'$ B $Tf : Tc \to Tc'$,

$$T(1_c) = 1_{Tc}, T(g \circ f) = Tg \circ Tf \tag{1}$$

$g \circ f$ C . , , " " : T C f B Tf , C B C $<g, f>$ B $<Tg, Tf>$, $Tg \circ Tf = T(g \circ f)$.

$\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$. X $\mathcal{P}X($ $S \subset X)$. $f : X \to Y$ $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y$ $S \subset X$ $fX \in Y$. $\mathcal{P}(1_X) = 1_{\mathcal{P}X}$ $\mathcal{P}(g \circ f) = \mathcal{P}g \circ \mathcal{P}f$. $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$.

, , . , $n(n$ $)$ Abel $H_n(X)$, X n , f : $X \to Y$ $H_n(f) : H_n(X) \to H_n(Y)$, H_n $\mathbf{Top} \to \mathbf{Ab}$. $: X = Y = S^1$,$H_1(S^1) =$, $H_1(f) :\to$ $d(1$ $)$; $f : S^1 \to S^1$ " (degree)". , $f, g : X \to Y$ $H_n(X) \to H_n(Y)$, H_n $\mathbf{Toph} \to \mathbf{Grp}$. Elienberg-Steenrod , n,H_n $\mathbf{Toph}$; $\mathbf{Grp}$. H_n π_n , .

. K, K $n \times n$ ( ) $\mathrm{GL}_n(K)$. , $f : K \to K'$ $\mathrm{GL}_n f : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K')$. $\mathrm{GL}_n : \mathbf{CRng} \to \mathbf{Grp}$. G, $xyx^{-1}y^{-1}(x, y \in G)$ $[G, G]$, G , . $G \to H$ $(H$ $)$. , $G \mapsto [G, G]$, $\mathbf{Grp} \to \mathbf{Grp}$, $G \mapsto G/[G, G]$ $\mathbf{Grp} \to \mathbf{Ab}$, . , G $Z(G)($ $a \in G$ $ax = xa$ $x \in G$ $)$ $\mathbf{Grp} \to \mathbf{Grp}$, $G \to H$ G H .($G \to H' \leq H$, $G \to H'$).

" (forgets)" (forgetful functor underlying functor). , U : $\mathbf{Grp} \to \mathbf{Set}$ G $UG($, $)$, f $f : G \to G'$, . $U : \mathbf{Rng} \to \mathbf{Ab}$ R . .

. , A, B, C

$$C \xrightarrow{T} B \xrightarrow{S} A$$

c f

$$c \mapsto S(Tc) \quad f \mapsto S(Tf)$$

$S \circ T : C \to A$, ( )S T . . B $\mathrm{id}_B : B \to B$, . , : , . ,
.

(isomorphism)$T : C \to B$ C B ( ) . , $T : C \to B$ $S : B \to C$ $S \circ T$ $T \circ S$; S T (two-sided inverse)$S = T^{-1}$.

, .

C c, c' B $g : Tc \to Tc'$ C $f : c \to c'$ $g = Tf$, $T : C \to B$ (full) , , .

C c, c' $f, f' : c \to c'$ $Tf_1 = Tf_2 : Tc \to Tc'$ $f_1 = f_2$, $T : C \to B$ (faithful) , , . , $\mathbf{Grp} \to \mathbf{Set}$, .

Hom ((2.5)) C $c, c' \in C$, $T : C \to B$ $f : c \to c'$ $Tf : Tc \to Tc'$

$$T_{c,c'} : \mathrm{Hom}(c, c') \to \mathrm{Hom}(Tc, Tc'), f \mapsto Tf$$

,T , ,T . (,) , , , B T .

C S , f dom f cod f(1_s) s $\to$ s' $\to$ s'' . S. , ( ) $S \to C$, , S C . . $S \to C$, S C (full subcategory). C , C s, s' s $\to$ s'. , $\mathbf{Set}_f$ $\mathbf{Set}$

Exercise

1. : ,Lie Lie .

   :

   R, R S, $s \in S$, s^{-1}, $s \cdot s^{-1} = 1_R$. s^{-1} S', $R \cup S'$ $< R \cup S' >$ R , Frac(R) $[r, s]$.

   Frac : $R \to$ Frac(R).

   R, R' $f : R \to R'$. Frac(R) R.

   f , $f(ss') = f(s)f(s')$ $s, s' \in R$.

   $s' \in S'$ $f(ss') = f(1_R) = 1_{R'}$ $f(s') \in$ Frac(R').

   Frac f : Frac$(R) \to$ Frac(R').

   φ : Frac $R \to$ Frac R', $[r, s] \in$ Frac R $[f(r), f(s)] \in$ Frac R', Frac f Frac$(R) \to$ Frac(R') ,

   Frac .

   Lie Lie ,

2. $\mathbf{1} \to C$,$\mathbf{2} \to C$ $\mathbf{3} \to C$ C , .

   $\mathbf{1} \to C$ 1 C c , c , .

   $\mathbf{2} \to C$ 1, 2 C c, c' $c \to c'$, $\mathbf{2} \to C$ C .

   $\mathbf{3}$

3. ” ”:

   a. $T(p \le p'$ $Tp \le Tp')$

   , p, p' f T Tp Tp' .

b. ( )

$T(1) = 1_T$

c. G , $G \to$ **Set** G ,$G \to \mathbf{Matr_K}$ G .

$T : G \to$ **Set** G **Set** S .

G S f, , f , f G .

.

4. **Grp** $\to$ **Ab** G

T : **Grp** $\to$ **Ab**

$G \to Z(G)$.

$S_2 = \{(1), (12)\}$ $S_3 = \{(1), (12), (23), (13), (123), (132)\}$.

$Z(S_3) = \{(1)\}$ $Z(S_2) = S_2$ $S_2 \to S_3 \to S_2 \simeq S_3/\langle(123)\rangle$ $TS_2 \to TS_3 \to TS_2$ $TS_3 \to TS_2$ TS_2, $S_3 \to S_2$ S_2 .

5. T : **Grp** $\to$ **Grp** $T(G) = G$ G .

, , $f : G \to H$ $f : G \to e \lhd H$, $f : G \to H$.

§4 *Natural Transformations*

$S, T : C \to B$, (Natural Transformations)$\tau : S \stackrel{\bullet}{\to} T$ C c B (" ")$\tau_c = \tau c : Sc \to Tc$ C $f : c \to c'$

$$\begin{tikzcd} c \arrow[d, "f"] \\ c' \end{tikzcd} , \qquad \begin{tikzcd} Sc \arrow[r, "\tau c"] \arrow[d, "Sf"'] & Tc \arrow[d, "Tf"] \\ Sc' \arrow[r, "\tau c'"] & Tc' \end{tikzcd} \qquad (1)$$

, $\tau_c : Sc \to Tc$ c . S B C ( ) , τ S ( ) T , ( )

$$\begin{tikzcd} a \arrow[dr, "f"] \arrow[dd, "h"'] & \\ & b \arrow[dl, "g"] \\ c & \end{tikzcd} \qquad \begin{tikzcd} Sa \arrow[rrr, "\tau a"] \arrow[dr, "Sf"] \arrow[dd, "Sh"'] & & & Ta \arrow[dr, "Tf"] \arrow[dd] & \\ & Sb \arrow[rrr, "\tau b"] \arrow[dl, "Sg"] & & & Tb \arrow[dl, "Tg"] \\ Sc \arrow[rrr, "\tau c"'] & & & Tc & \end{tikzcd}$$

$\tau a, \tau b, \tau c, \cdots$ τ (components).

(a morphism of functors): B τc τ (natural equivalence) (natural isomorphism); $\tau : S \cong T$. ,τc $(\tau c)^{-1}$ B $\tau^{-1} : T \stackrel{\bullet}{\to} S$.

. , $\det_K M$ $n\times n$ M , $K.K^*$ K ( ). , M ,$\det_K$ $\mathrm{GL}_n K \to K^*$ ($\mathbf{Grp}$). K, ( ), $f : K \to K'$

$$\begin{tikzcd} \mathrm{GL}_n K \arrow[r, "\det_K"] \arrow[d, "\mathrm{GL}_n f"'] & K^* \arrow[d, "f^*"] \\ \mathrm{GL}_n K' \arrow[r, "\det_{K'}"] & K'^*. \end{tikzcd} \tag{2}$$

.

$\det$ $: \mathrm{GL}_n \cdot \to \cdot^*$ $\mathbf{CRng} \to \mathbf{Grp}$.

$p_G : G \to G/[G,G]$ $\mathbf{Grp} \to \mathbf{Ab} \to \mathbf{Grp}$ p. p , $f : G \to H$ f'

$$\begin{tikzcd} G \arrow[r, "p_G"] \arrow[d, "f"'] & G/[G,G] \arrow[d, "f'"] \\ H \arrow[r, "p_H"] & H/[H,H]\,. \end{tikzcd} \tag{3}$$

(double character group) Abel $\mathbf{Ab}$. $D(G)$ G (character group),$DG = \mathrm{Hom}(G, /)$ $t : G \to /$ (t), $/$ 1 ,

$f : G \to \backslash\{0\}$, $\backslash\{0\} \to S^1 = /$ DG $G \to /$. $f : G \to G'$,$t : G' \to /$ $t \circ f : G \to /$ $t \circ f \in \mathrm{Hom}(G, /) = DG$ $t \in \mathrm{Hom}(G', /) = DG'$ $Df : DG' \to DG($).

$\mathbf{Ab}$ $f : G' \to G$ $\mathbf{Ab}$ $Df : DG \to DG'$ $(Df)t = tf : G' \to /$ t . ,$D(g \circ f) = Df \circ Dg$. ,$D$ (" " , §II.2); DG $G \mapsto D(DG)$ $I(G) = G$ $\mathbf{Ab} \to \mathbf{Ab}$. G

$$\tau_G : G \to D(DG)$$

: $g \in G$, $\tau_G g : DG \to /$ $t \in DG$ $t \mapsto tg$; $(\tau_G g)t = t(g)$. τ $\tau : I \xrightarrow{\bullet} DD$. , G τ , . Abel $\mathbf{Ab}_f$, τ .

, Abel G DG $\sigma_G : G \cong DG$, G , . , D $D' : \mathbf{Ab}_{f,i} \to \mathbf{Ab}_{f,i}$ $\mathbf{Ab}_{f,i}$ $\mathbf{Abel}$, , $D'G = DG$ $D'f = Df^{-1}$. $\sigma_G : G \to D'G$ $\sigma : I \to D'$ $\mathbf{Ab}_{f,i} \to \mathbf{Ab}_{f,i}$, .

.

$\Bbbk$ $\mathbf{Vect}(\Bbbk)$. $\Bbbk$- V,

$$V^\vee := \mathrm{Hom}_\Bbbk(V, \Bbbk) = \{\Bbbk\text{-} \quad V \to \Bbbk\}$$

$f : V_1 \to V_2$

$f^\vee : V_2^\vee \to V_1^\vee$

:V$_2 \to \Bbbk$!

f

$D : V \mapsto V^\vee$,$f \mapsto f^\vee$ $D : \mathbf{Vect}(\Bbbk)^{op} \to \mathbf{Vect}(\Bbbk)$.

$DD^{op} : \mathbf{Vect}_f(\Bbbk) \to \mathbf{Vect}_f(\Bbbk)$. .

n **Finord** (U) $\mathbf{Set}_f$, . $n = \{0, 1, \cdots, n-1\}$, S $S : \mathbf{Finord} \to \mathbf{Set}_f$. , X $n = \#X$, X ( ). X $\theta_X : X \to \#X$. $f : X \to Y$ $\#f = \theta_Y f \theta_X^{-1}$ $\#f : \#X \to \#Y$,

$$\begin{tikzcd} X \arrow[r, "\theta_X"] \arrow[d, "f"] & \# X \arrow[d, "\# f"] \\ Y \arrow[r, "\theta_Y"] & \# Y \end{tikzcd}$$

$\#$ $\# : \mathbf{Set}_f \to \mathbf{Finord}$. X , θ_X . $\# \circ f$ **Finord** I'. , $S \circ \#$ $I : \mathbf{Set}_f \to \mathbf{Set}_f$, X —— X n. , $\theta : I \xrightarrow{\bullet} S\#$. $I \cong S \circ \#, I' = \# \circ S$.

, C D (equivalence) $S : C \to D, T : D \to C$ $I_C \cong T \circ S, I_D \cong S \circ T$. (§IV1.4) " " " " .

" " , Eilenberg-Mac Lane , " " " ", " " " ".

Exercise

1. S ,X^S $h : S \to X$. $X \mapsto X^S$ **Set** $\to$ **Set** , $e(h, s) = h(s)$ $e_X : X^S \times X \xrightarrow{\bullet} X$ h $s \in S$, .

   X^S , $X \mapsto X^S$ **Set** $\to$ **Set**. $f : X \to X'$, X^S $S \to X$, $g \in X^S$ $f \circ g : S \to X' \in X'^S$.

   Set $f : X \to X'$ **Set** $X^S \to X'^S$.

   e_X $e_{X'}$. $e_X : X^S \times S \to X$, id, $X \mapsto X^S \times S$.

   $T : X \mapsto X^S \times S$, .

$$
\begin{tikzcd}
X \arrow[d, "f"] & X^S \times S \arrow[r, "e_X"] \arrow[d, "Tf"] & X \arrow[d, "f"] \\
X' & X'^S \times S \arrow[r, "e_{X'}"] & X'
\end{tikzcd}
$$

, e_X .

2. H , $G \mapsto H \times G$ $H \times \cdot : \mathbf{Grp} \to \mathbf{Grp}$, $f : H \to K$ $H \times \cdot \xrightarrow{\bullet} K \times \cdot$.

   $f : G \to G'$, $H \times f : H \times G \to H \times G'$.

   $H \times f(h, g_1 g_2) = (h, g_1)(h, g_2)$.

   $H \times f$ $H \times G \to H \times G'$.

   $H \times \cdot$.

   $K \times \cdot$.

   $f : H \to K$ $e : H \times \cdot \xrightarrow{\bullet} K \times \cdot$.

$$
\begin{tikzcd}
G \arrow[d, "f"] & H \times G \arrow[r, "e_G"] \arrow[d, "H \times f"] & K \times G \arrow[d, "K \times f"] \\
G' & H \times G' \arrow[r, "e_{G'}"] & K \times G'
\end{tikzcd}
$$

   $e : H \times \cdot \xrightarrow{\bullet} K \times \cdot$.

3. B C ( ) $S, T : B \to C$ ( ), $S \xrightarrow{\bullet} T$ S T , $h \in C$ $Tg = h(Sg)h^{-1}$ $g \in B$.

$$\begin{tikzcd}
Tg \arrow[r] \arrow[d, "\theta_g"'] & Tg \circ h \arrow[d] \\
Sg & h \circ Sg \arrow[l]
\end{tikzcd}$$

$Tg \circ h = h \circ Sg.$

$Tg = h(Sg)h^{-1}.$

4. $S, T : C \to P$ C P , $c \in C$ $Sc \leq Tc$ $S \xrightarrow{\bullet} T$( )

$S \xrightarrow{\bullet} T$ $c, c' \in C$ $f : c \to c'$

$$\begin{tikzcd}
c \arrow[d, "f"'] & & Sc \arrow[r, "\theta_c"] \arrow[d, "Sf"] & Tc \arrow[d, "Tf"] \\
c' & & Sc' \arrow[r, "\theta_{c'}"] & Tc'
\end{tikzcd}$$

$p, p' \in P$ $p \leq p'$ $p \to p'$. P , $\theta_c : Sc \to Tc$ $Sc \leq Tc$. .

5. $\tau : S \xrightarrow{\bullet} T$ (τ) $< g, f >$ $Tg \circ \tau f = \tau(gf) = \tau g \circ Sf$ C $f : c \to c'$ B $\tau f : Sc \to Tc'$. , τ $\tau_c = \tau(1_c)$ ( )

$< g, f >$ $f : c \to c'$ $g : c' \to c''$. $\tau : S \xrightarrow{\bullet} T$.

$$
\begin{tikzcd}
c \arrow[d, "f"] & & Sc \arrow[r, "\tau_c"] \arrow[d, "Sf"'] \arrow[dr, "\tau f"] \arrow[ddr, "Tg \circ \tau f"'] & Tc \arrow[d, "Tf"] \\
c' \arrow[d, "g"] & & Sc' \arrow[r, "\tau_{c'}"] \arrow[d, "Sg"'] \arrow[dr, "\tau g"'] & Tc' \arrow[d, "Tg"] \\
c'' & & Sc'' \arrow[r, "\tau_{c''}"'] & Tc''
\end{tikzcd}
$$

$\tau f := \tau_{c'} \circ Sf$,$\tau g = \tau_{c''} \circ Sg$.

τf τg . , $\tau f : Sc \to Tc'$ $f : c \to c'$.

, τ $\tau_c = \tau(1_c)$. $f = 1_{c'} f$ $f = f 1_c$.

$\tau(1_{c'}) \circ Sf = \tau f$ $Tf \circ \tau(1_c) = \tau f$. $Tf \circ \tau(1_c') = \tau(1_c) \circ Sf$. τ

6. F , F ( ) §2 $\mathbf{Matr}_F$.

(. p40 2.2.15)

§5 *Monic, Epis , and Zeros*

( ) , . , X , Y $Y \to X$. " " .

$e : a \to b$, $e' : b \to a$ $e'e = 1_a$ $ee' = 1_b$ e C . , e' , (, e'' $e''e = 1_a$ $ee'' = 1_b$ $e' = e'1_b = e'(ee'') = e'ee'' = (e'e)e'' = 1_a e'' = e''$), $e' = e^{-1}$. , $e_1 e_2$ e_1, e_2 . $(e_1 e_2)^{-1} = e_2^{-1} e_1^{-1}$. a b C ( )$e : a \to b$; $a \cong b$. , .

$m : a \to b$, $m \circ f_1 = m \circ f_2$ $f_1, f_2 : d \to a$ $f_1 = f_2$. m monic ( ). ,m . **Set** **Grp**

$h : a \to b$, $g_1 \circ h = g_2 \circ h$ $g_1, g_2 : b \to c$ $g_1 = g_2$, h epi ( ). ,h . **Set** (epimor

$h : a \to b$, $hr = 1_b$ $r : b \to a$.h ( ) h (section). h , h . **Set** , **Grp** . ,h h (1_a, g (split) ,h . $f = hg$. , $f : b \to b$ $f^2 = f$. g, h $gh = 1$ $f = hg$, f .

C t, C a $a \to t$, t (terminal). t , $t \to t$, C C . C s, C a $s \to a$, s (initial). , **Set** , $\emptyset$, . **Grp** , .

C z, , z (null) . z , , C a, b $a \to z \to b$(z), a b (zero arrow). . , **Ab** R-**Mod** (0),$\mathbf{Set}_*$ ( ).

, . X (fundamental groupoid)$\pi(X)$.$\pi(X)$ X x, $\pi(X)$ $x \to x'$ f x x' (f $I \to X$,I $I = [0, 1]$, $f(0) = x$, $f(1) = x'$, x x' f, g, $F : I \times I \to X$ $F(t, 0) = f(t)$,$F(t, 1) = g(t)$, $F(1, s) = x$, $F(1, s) = x'$ $s, t \in I$). $g : x' \to x''$ $f : x \to x'$ (composite) h, "f g",

$h(t)$
$=$
$f(2t),$
$0 \leq t \leq 1/2$

$=$
$g(2t-1),$
$1/2 \leq t \leq 1$

, $\pi(X)$ ( )

G ,G x $g : x \to x$ $\mathrm{Hom}_G(x,x)$. $f : x \to x'$, $\mathrm{Hom}_G(x,x)$ $\mathrm{Hom}_G(x',x')$ $g \mapsto fgf^{-1}$( ) . , , (connected). ($\mathrm{Hom}(x,x)$) ( ) . , X π

Exercise

1. ( )

.

.

$\iota :\to$.

$X \to$, $\iota \circ f(x) = f(x)$. $f_1, f_2 : X \to$ $\iota f_1 = \iota f_2$ $f_1 = f_2$. ι .

$f :\to Y$, $f \circ \iota(x) = g \circ \iota(x)$, , $\{x_n\}$ $x_n \to x$.

$f \circ \iota(x_n) = f(x_n)$,$g \circ \iota(x_n) = g(x_n)$.

$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f \circ \iota(x_n) = \lim_{n\to\infty} g \circ \iota(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$

$f = g$.

ι .

2. ,

m, m' $m \circ h_1 = m \circ h_1' \Rightarrow h_1 = h_1'$.

$m \circ m'$, $(m \circ m') \circ h_1 = (m \circ m') \circ h_2$ $(m \circ m') \circ h_1 = m \circ (m' \circ h_1) = (m \circ m')h_2$ $m' \circ h_1 = m' \circ h_2$ $h_1 = h_2$, $m \circ m'$.

.

3. $g \circ f$, f , g

$g \circ f$, h_1, h_2 $(g \circ f) \circ h_1 = (g \circ f) \circ h_2 \Rightarrow h_1 = h_2$. $g \circ (f \circ h_1) = g \circ (f \circ h_2) \Rightarrow h_1 = h_2$.

f , $h_1 = h_2$, h_1, h_2 $h_1 = h_2$. f .

g , $g \circ f \circ h_1 = g \circ f \circ h_2 \Rightarrow f \circ h_1 = f \circ h_2$ h_1, h_2 . k $g \circ f \circ k = g \circ f \circ h_1 \Rightarrow f \circ k \neq f \circ h_1$. f .

$f \circ k \neq f \circ h$ $k \neq h$, $(g \circ f) \circ h_1 = (g \circ f) \circ h_2$ f $h_1 = h_2$.

4. $\rightarrow$ **Rng** .

$\iota :\rightarrow$ $h_1, h_2 :\rightarrow R$ $h_1 \circ \iota = h_2 \circ \iota$ $h_1(\frac{a}{1}) = h_2(\frac{a}{1})$ $a \in$.

$$h_1(\frac{1}{a}\frac{a}{1}) = 1 = h_2(\frac{1}{a}\frac{a}{1})$$

$h_1(\frac{1}{a}) = h_2(\frac{1}{a})$ $a \in$. $\frac{p}{q} \in$ $h_1(\frac{p}{q}) = h_2(\frac{p}{q})$ $h_1 = h_2$.

ι **Rng**

5. **Grp**   ()