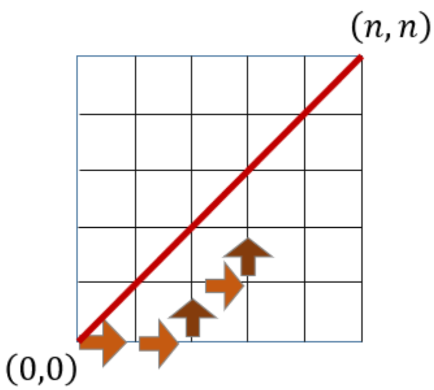
Problem 1 (20%)**.** Let X, Y be discrete random variables. The variance of a random variable X is defined as $\mathrm{Var}[X] := \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$. Prove that

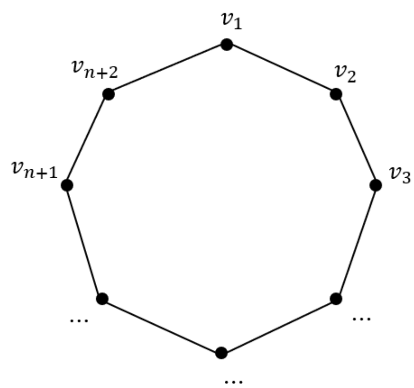
1. $\mathrm{E}[aX + bY] = a \cdot \mathrm{E}[X] + b \cdot \mathrm{E}[Y]$ for any constant a, b.
2. If X and Y are independent, then $\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y]$ and
   $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3. $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. *Hint:* Use the fact that $\mathrm{E}[X \cdot \mathrm{E}[X]] = \mathrm{E}[X]^2$.

Problem 2 (20%)**.** Consider the slides #2. Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Problem 3 (20%)**.** For any integer $n \geq 1$, consider the grid points (r, c) with $1 \leq r, c \leq n$. Let C_n be the number of possible paths from $(0, 0)$ to (n, n) that use only $\rightarrow$ and $\uparrow$ and that never cross the diagonal $r = c$. See also the Figure (a) below. For convenience, define $C_0 := 1$.



(a)



(b)

For any integer $n \geq 2$, consider the convex $(n+2)$-gon with vertices labeled with $v_1, v_2, \ldots, v_{n+2}$. Let P_n denote the number of possible ways to triangulate the polygon. It follows that $P_2 = 2$, $P_3 = 5$, etc. For convenience, also define $P_0 := 1$ and $P_1 := 1$.

1. Prove that for any $n \geq 2$, P_n satisfies the recurrence
$$P_n \;=\; \sum_{0 \leq k < n} P_k \cdot P_{n-k-1}.$$
2. Prove that for any $n \geq 2$, C_n satisfies the same recurrence
$$C_n \;=\; \sum_{0 \leq k < n} C_k \cdot C_{n-k-1}.$$

Note that this proves that P_n also equals the n^{th}-Catalan number.

Problem 4 (20%)**.** Let $\mathcal{F}$ be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \qquad \text{and} \qquad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F},\ A \neq B.$$

Suppose that $\mathcal{F}$ is not 2-colorable. Let x, y be any elements that appear in $\mathcal{F}$, i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:

1. x belongs to at least two members of $\mathcal{F}$.
2. There exists some $C \in \mathcal{F}$ such that $\{x, y\} \subseteq C$.

Hint: Construct proper coloring to prove the properties. For (1), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (1). For (2), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (2).

Problem 5 (20%)**.** Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B. Assume that $|A|d \geq |B|D$.

Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,
2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0, and
3. at least half of the edges leaving A_0 go to B_0.

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0. First prove (3) and then (1).