Problem 1 (20%). Consider the following two problems regarding Markov's and Chebyshev's inequalities.

- For any positive integer k, describe a non-negative random variable X such that

$$\Pr[\, X \geq k \cdot \mathrm{E}[X] \,] = \frac{1}{k}.$$

Note that, this shows that Markov's inequality is as tight as it could possibly be.

- Can you provide an example that shows that Chebyshev's inequality is tight? If not, explain why not.

Problem 2 (20%). Prove that for any two sets I, J with $I \subseteq J$, we have

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

Hint: Rewrite the summation and apply the binomial theorem (in slides # 1a).

Problem 3 (20%). Let $\mathcal{F}$ be a k-uniform k-regular family, i.e., each set has k elements and each element belongs to k sets. Let $k \geq 10$. Show that there exists at least one valid 2-coloring of the elements.

Hint: Define proper events for the sets and apply the symmetric version of the local lemma.

Problem 4 (20%). We proved the asymmetric version of the local lemma in lecture #4. Assume that the statement of this lemma holds. Furthermore, assume that

1. $\Pr[A_i] \leq p$ for all i, and
2. $ep(d+1) \leq 1$.

Prove that $\Pr\left[\bigcap_i \overline{A_i}\right] > 0$, i.e., use Theorem 19.2 to prove the statement of Theorem 19.1.

Hint: Let $x(A_i) = \frac{1}{d+1}$ for all $1 \leq i \leq n$. Use the inequality $\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d$ obtained by the limit formula of $1/e$ and the fact that it converges from the above.

Problem 5 (20%). Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these random bits in any order. Let Y_i be the exclusive-or (XOR) of the i^{th} pair of bits, and let $Y := \sum_{1 \leq i \leq m} Y_i$.

- Show that $Y_i = 0$ and $Y_i = 1$ with probability $1/2$ each.
- Show that $\mathrm{E}[\, Y_i \cdot Y_j \,] = \mathrm{E}[Y_i] \cdot \mathrm{E}[Y_j]$ for any $1 \leq i, j \leq m$ and derive $\mathrm{Var}[Y]$.
- Use Chebyshev's inequality to derive a bound on $\Pr[\, |Y - \mathrm{E}[Y]| \geq n \,]$.