Problem 1 (20%)**.** Show that, for any positive integer n, there is a multiple of n that contains only the digits 7 or 0.

Hint: Consider all the numbers a_i of the form $77\ldots7$, with i sevens, for $i = 1, 2, \ldots n+1$, and the value a_i modulo n.

Problem 2 (20%)**.** Prove that every set of $n+1$ distinct integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of consecutive numbers and a pair whose sum is $2n+1$.

For each n, exhibit two sets of size n to show that the above results are the best possible, i.e., sets of size $n+1$ are necessary.

Hint: Use pigeonholes $(2i, 2i-1)$ and $(i, 2n-i+1)$ for $1 \le i \le n$.

Problem 3 (20%)**.** Let $G = (V, E)$ be a graph. Denote by $\chi(G)$ the minimum number of colors needed to color the vertices in V such that, no adjacent vertices are colored the same.

Prove that, $\chi(G) \le \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices.

Hint: Order the vertices $v_1, v_2, \ldots, v_n$ and use greedy coloring. Show that it is possible to color the graph using $\Delta(G) + 1$ colors.

Problem 4 (20%)**.** Let $\alpha(G)$ be the *independence number* of a graph G, i.e., the maximum size of any independent set of G. Prove the following dual version of Turán's theorem:

> If G is a graph with n vertices and $nk/2$ edges, where $k \ge 1$, then we have
> $$\alpha(G) \ge n/(k+1).$$

Problem 5 (20%)**.** Let X be a finite set and $A_1, A_2, \ldots, A_m$ be a partition of X into mutually disjoint blocks. Given a subset $Y \subseteq X$, consider the partition $Y = B_1 \cup B_2 \cup \cdots \cup B_m$ with the blocks B_i defined as $B_i := A_i \cap Y$. For any $1 \le i \le m$, we say that the block B_i is λ-*large* if

$$\frac{|B_i|}{|A_i|} \;\ge\; \lambda \cdot \frac{|Y|}{|X|}.$$

Show that, for every $\lambda > 0$, at least $(1-\lambda) \cdot |Y|$ elements of Y belong to λ-large blocks.