Problem 1 (20%)**.** How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 21$ with

1. $x_i \geq 0$.
2. $x_i > 0$.
3. $0 \leq x_i \leq 12$.

Problem 2 (20%)**.** Prove the following identities **using path-walking argument**.

1. For any $n, r \in \mathbb{Z}^{\geq 0}$,
$$\sum_{0 \leq k \leq r} \binom{n+k}{k} \; = \; \binom{n+r+1}{r}.$$

2. For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,
$$\sum_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} \; = \; \binom{m+n}{r}.$$

Problem 3 (20%)**.** Let $\mathcal{F}$ be a set family on the ground set X and $d(x)$ be the degree of any $x \in X$, i.e., the number of sets in $\mathcal{F}$ that contains x. Use the double counting principle to prove the following two identities.

$$\sum_{x \in Y} d(x) \; = \; \sum_{A \in \mathcal{F}} |Y \cap A| \; \text{ for any } Y \subseteq X.$$

$$\sum_{x \in X} d(x)^2 \; = \; \sum_{A \in \mathcal{F}} \sum_{x \in A} d(x) \; = \; \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$$

Problem 4 (20%)**.** Let H be a 2α-dense 0-1 matrix. Prove that at least an $\alpha/(1-\alpha)$ fraction of its rows must be α-dense.

Problem 5 (20%)**.** Let $\mathcal{F}$ be a family of subsets defined on an n-element ground set X. Suppose that $\mathcal{F}$ satisfies the following two properties:

1. $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.
2. For any $A \subsetneq X$, $A \notin \mathcal{F}$, there always exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

Prove that
$$2^{n-1} - 1 \; \leq \; |\mathcal{F}| \; \leq \; 2^{n-1}.$$

Hint: Consider any set $A \subseteq X$ and its complement $\overline{A}$. Apply the conditions given above and prove the two inequalities "$\leq$" and "$\geq$" separately.