The permutability of σ_i-sylowizers of some σ_i-subgroups in finite groups

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$, G a finite group and $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(|G|) \neq \emptyset\}$. A subgroup S of a group G is called a σ_i-sylowizer of a σ_i-subgroup R in G if S is maximal in G with respect to having R as its Hall σ_i-subgroup. The main aim of this paper is to investigate the influence of σ_i-sylowizers on the structure of finite groups. We obtained some new characterizations of supersoluble groups by the permutability of the σ_i-sylowizers of some σ_i-subgroups.

1 Introduction

Let π denotes a set of primes. The concept of π-Sylowizers has been introduced by W. Gaschutz [1]. If R is a π-subgroup of the group G, then a π-Sylowizer of R in G is a subgroup S of G maximal with respect to containing R as a Hall π-subgroup.

$\mathbb{P}$ is the set of all primes and n is a natural number. Let $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We write $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [5], two subgroups H and T of a group G are conditionally permutable (or in brevity, c-permutable) in G if there exists an element $x \in G$ such that $HT^x = T^x H$.

2 Preliminaries

Lemma 2.1. Let H be a σ_i-subgroup of G for some $\sigma_i \in \sigma(G)$. Assume that K is a subgroup satisfying $H \leq K \leq G$ and T is a σ_i-sylowizer of H in K. Then there is a σ_i-sylowizer S of H in G such that $T = S \cap K$.

Proof Since H is a Hall σ_i-subgroup of T, there is a σ_i-sylowizer S of H in G such that $S \geq T$. Then H is a Hall σ_i-subgroup of $S \cap K$. Since $T \leq S \cap K$ and T is a σ_i-sylowizer of H in K, we get $T = S \cap K$ by the maximality of T. □

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Lemma 2.2. Let R be a σ_i-subgroup of G for some $\sigma_i \in \sigma(G)$. Assume that N is a normal subgroup of G and R is a Hall σ_i-subgroup of RN. Then S is a σ_i-sylowizer of R in G if and only if S/N is a σ_i-sylowizer of RN/N in G/N.

Proof Let S be a σ_i-sylowizer of R in G. Since R is a Hall σ_i-subgroup of RN, R is a Hall σ_i-subgroup of SN. Thus $N \leq S$ by the maximality of S and so RN/N is a Hall σ_i-subgroup of S/N. If S/N is not a σ_i-sylowizer of RN/N in G/N, then there is a σ_i-sylowizer S_0/N of RN/N in G/N such that $S_0/N > S/N$. Now, $S_0 > S$ and R is a Hall σ_i-subgroup of S_0, which contradicts the fact that S is a σ_i-sylowizer of R in G. Thus S/N is a σ_i-sylowizer of RN/N in G/N.

Conversely, if S/N is a σ_i-sylowizer of RN/N in G/N, then R is a Hall σ_i-subgroup of S. If S is not a σ_i-sylowizer of R in G, then there is a σ_i-sylowizer S_0 of R in G such that $S_0 > S$. Therefore RN/N is a Hall σ_i-subgroup of S_0/N, which contradicts the fact that S/N is a σ_i-sylowizer of RN/N in G/N. Thus S is a σ_i-sylowizer of R in G. □

Lemma 2.3. Let R be a σ_i-subgroup of a σ-full group G for some $\sigma_i \in \sigma(G)$ and S a σ_i-sylowizer of R in G. If S is σ-permutable in G, then $O^{\sigma_i}(G) \leq S$. In particular, $S = RO^{\sigma_i}(G)$ is the unique σ_i-sylowizer of R in G.

Proof Let Q be a Hall σ_j-subgroup of G with $\sigma_j \in \sigma(G)$ and $\sigma_i \cap \sigma_j = \emptyset$. Since S is σ-permutable, we have $SQ \leq G$. Note that since R is a Hall σ_i-subgroup of SQ, we have $QS = S$ by the maximality of S. Hence $Q \leq S$. It shows that $O^{\sigma_i}(G) \leq S$. □

Lemma 2.4. Let R be a σ_i-subgroup of a σ-full group of Sylow type G for some $\sigma_i \in \sigma(G)$ and S a σ_i-sylowizer of R in G. Then S is c-permutable with every Hall σ_j-subgroup of G for all $\sigma_j \in \sigma(G)$ if and only if $|G : S|$ is a σ_i-number.

Proof The sufficiency is evident, we only need to prove the necessity.

Let Q be a Hall σ_j-subgroup of G with $\sigma_j \in \sigma(G)$ and $\sigma_i \cap \sigma_j = \emptyset$. Since S is c-permutable with Q, we have $SQ^x = Q^xS$ for some element $x \in G$. Note that since R is a Hall σ_i-subgroup of SQ^x, we have $Q^xS = S$ by the maximality of S. Hence $Q^x \leq S$. It implies that $|G : S|$ is a σ_i-number. □

3 Results

Theorem 3.1. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, \cdots, H_t\}$ be a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i = 1, \cdots, t$. Suppose that for any $\sigma_i \in \sigma(G)$, every maximal subgroup of any non-cyclic H_i has a σ_i-sylowizer that is c-permutable with every member of $\mathcal{H}$, then G is supersoluble.*

Proof Assume that this is false and let G be a counterexample of minimal order. Then:

(1) *Let N be a minimal normal subgroup of G. Then G is supersoluble.*

We consider the quotient group G/N. It is clear that G/N is a σ-full group of Sylow type and $\mathcal{H}N/N$ is a complete Hall σ-set of G/N such that H_iN/N is nilpotent. Let H/N be a maximal subgroup of H_iN/N and H_{σ_i} be a Hall σ_i-subgroup of H contained in H_i. Then $H = H_{\sigma_i}N$. Since $H_{\sigma_i} \cap N = N_{\sigma_i} = H_i \cap N$, where N_{σ_i} denotes a Hall σ_i-subgroup of N, we have that

$$|H_i : H_{\sigma_i}| = \frac{|H_i||N|}{|H_i \cap N|} \cdot \frac{|H_{\sigma_i} \cap N|}{|H_{\sigma_i}||N|} = |H_iN : H| = q$$

for some $q \in \sigma_i$. This shows that H_{σ_i} is a maximal subgroup of H_i. If H_iN/N is non-cyclic, then so is H_i. Thus if S/N is a σ_i-sylowizer of H/N in G/N, then S is a σ_i-sylowizer of H_{σ_i} in G by Lemma 2.2. Moreover, if S is c-permutable with every member of $\mathcal{H}$, then S/N is c-permutable with every member of $\mathcal{H}N/N$ by Lemma 2.4. It shows that G/N satisfies the hypotheses. Thus G/N is supersoluble by the choice of G.

(2) *N is the unique proper minimal normal subgroup of G and $\Phi(G) = 1$.*

Let p be the smallest prime divisor of G and $p \in \sigma_i$. If H_i is cyclic, then G is p-nilpotent. This shows that G has a proper minimal normal subgroup. Thus we may assume that H_i is non-cyclic. Let M be a maximal subgroup of H_i of index p and S a σ_i-sylowizer of M in G that is c-permutable with every member of $\mathcal{H}$. Then $|G : S| = p$ by Lemma 2.4 and so $S \trianglelefteq G$. Therefore we may choose a proper minimal normal subgroup of G contained in S, say N. By Claim (1), G/N is supersoluble. Moreover, N is the unique minimal normal subgroup of G. Since the class of all supersoluble groups is a saturated formation, we may assume further that $|\Phi(G)| = 1$.

(3) *N is soluble.*

Assume that N is not soluble. Then $p = 2$ and $2||N|$. Let P be a Sylow 2-subgroup of H_i. Then $N_2 = P \cap N$ is a Sylow 2-subgroup of N. If $N_2 \leq \Phi(H_i)$, then $N_2 \leq \Phi(P)$, and so N is 2-nilpotent by Tate's theorem, a contradiction. Hence $N_2 \nleq \Phi(H_i)$. Thus there is a maximal subgroup K of H_i such that $H_i = KN_2$. Let S_0 be a σ_i-sylowizer of K in G that is c-permutable with every member of $\mathcal{H}$. Then $|G : S_0| = 2$ by Lemma 2.4. Thus $G = S_0H_i = S_0N_2 = S_0N$. Now, $|N : N \cap S_0| = |G : S_0| = 2$, which implies that $N \cap S_0 \trianglelefteq N$. Since $N \cap S_0 \trianglelefteq S_0$, we have $N \cap S_0 \trianglelefteq G$. Note that N is a minimal normal subgroup of G, we have $N \cap S_0 = 1$. Thus $|N| = |G : S_0| = 2$, a contradiction.

(4) *Final contradiction.*

By Claim (3), we may assume that N is a q-subgroup for some prime $q \in \sigma_j$. Since $\Phi(G) = 1$, there is a maximal subgroup T of G such that $G = TN$. Let T_{σ_j} be a Hall σ_j-subgroup of T contained in H_j. Then $H_j = T_{\sigma_j}N$ is a Hall σ_j-subgroup of G. If H_j is cyclic, then G is supersoluble by the supersolublity of G/N. Thus we may assume that H_j is non-cyclic. Let $Q \geq T_{\sigma_j}$ be a maximal subgroup of H_j and Y a σ_j-sylowizer of Q in G that is c-permutable with every member of $\mathcal{H}$. Then $|G : Y| = q$ by Lemma 2.4 and $N \nleq Y$. Otherwise $H_j = QN \leq Y$, which contradicts the fact that Q is a Hall σ_j-subgroup of Y. Thus $G = YN$ and so $|N| = |G : Y| = q$. It implies that G is supersoluble, a contradiction. This contradiction completes the proof. $\square$

Theorem 3.2. *Let $\mathfrak{F}$ be a soluble saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H} = \{H_1, \cdots, H_t\}$ is a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i = 1, \cdots, t$. If for any $\sigma_i \in \sigma(E)$, every maximal subgroup of any non-cyclic $H_i \cap E$ has a σ_i-sylowizer that is c-permutable with every member of $\mathcal{H}$, then $G \in \mathfrak{F}$.*

Proof The conclusion holds when $E = G$ by Theorem 3.1, thus we may assume that $E < G$. Let N be a minimal normal subgroup of G contained in E.

(1) *E is supersoluble.*

Let Q be a maximal subgroup of a non-cyclic Hall σ_i-subgroup $H_i \cap E$ of E and S a σ_i-sylowizer of Q in G that is c-permutable with member of $\mathcal{H}$. By Lemma 2.4, $|G : S|$ is a σ_i-number. Let $Y = S \cap E$. Since $|E : Y| = |E : S \cap E| = |SE : S|$ divides $|G : S|$, $|E : Y|$ is a σ_i-number. Hence Y is a σ_i-sylowizer of Q in E and Y is c-permutable with every member of $\mathcal{H} \cap E$ by Lemma 2.4. Thus E is supersoluble by Theorem 3.1.

(2) *N is the unique minimal normal subgroup of G contained in E and $N \cap \Phi(G) = 1$.*

Consider the quotient group G/N, evidently $(G/N)/(E/N) \in \mathfrak{F}$. Since E is supersoluble by Claim (1), we have that N is a p-group for some prime p. Without loss of generality, we may write $E_i = H_i \cap E$ for all $i \in \{1, \cdots, t\}$ and assume that $p \in \sigma_i$ for some i. Let J/N be a maximal subgroup of E_i/N, then J is a maximal subgroup of E_i. If S/N is a σ_i-sylowizer of J/N in G/N, then S is a σ_i-sylowizer of J in G by Lemma 2.2. Moreover, if S is c-permutable with every member of $\mathcal{H}$, then S/N is c-permutable with every member of $\mathcal{H}N/N$ by Lemma 2.4. Let J/N be a maximal subgroup of E_jN/N and J_{σ_j} a Hall σ_j-subgroup of J contained in E_j, where $i \neq j$. Then J_{σ_j} is a maximal subgroup of E_j. If S/N is a σ_j-sylowizer of $J_{\sigma_j}N/N$ in G/N, then S is a σ_j-sylowizer of J_{σ_j} in G by Lemma 2.2. Moreover, if S is c-permutable with every member of $\mathcal{H}$, then S/N is c-permutable with every member of $\mathcal{H}N/N$ by Lemma 2.4. This shows that $(G/N, E/N)$ satisfies the hypotheses. Thus we may have that $G/N \in \mathfrak{F}$ by induction. Moreover, N is the unique minimal normal subgroup of G contained in E and $N \cap \Phi(G) = 1$.

(3) *N is an elementary abelian p-subgroup, where p is the largest prime divisor of $|E|$.*

Since E is supersoluble by Claim (1), the Sylow p-subgroup E_P of E is normal in G. Note that N is the unique minimal normal subgroup of G contained in E, $N \leq E_P$ is an elementary abelian p-subgroup.

(4) *$G \in \mathfrak{F}$.*

Without loss of generality, we may assume that $p \in \sigma_i$. If E_i is cyclic, then $|N| = p$ and so $G \in \mathfrak{F}$. Assume that E_i is non-cyclic. Since $N \not\leq \Phi(G)$, there is a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Thus $E_i = N(M \cap E_i)$ and $H_i = NM \cap H_i = N(M \cap H_i) = NM_i$. Since

$M_i < H_i$, we may choose $P \lessdot H_i$ such that $M_i \leq P$. Since $M \cap E_i \leq P$, $P \cap E_i = P \cap N(M \cap E_i) = (P \cap N)(M \cap E_i)$. Note that $M \cap N = 1$, we have

$$|E_i : E_i \cap P| = |N(M \cap E_i) : (P \cap N)(M \cap E_i)| = |N : P \cap N| = p.$$

Hence $R = E_i \cap P$ is a maximal subgroup of E_i. Let S be a σ_i-sylowizer of R in G that is c-permutable with every member of $\mathcal{H}$. Then $|G : S|$ is a σ_i-number by Lemma 2.4. Since G is soluble, we may write $S = RS_{\sigma_i'}$ and $M = M_i M_{\sigma_i'}$. Note also that $|G : S|$ and $|G : M|$ are σ_i-number in G, $S_{\sigma_i'}$ and $M_{\sigma_i'}$ are also Hall σ_i'-subgroups of G. Thus there is an element g of G such that $S^g_{\sigma_i'} = M_{\sigma_i'}$. Since $G = H_i S^g$, we may write $g = xy$, where $x \in H_i$ and $y \in S^g$. Note that since $R = E_i \cap P \trianglelefteq H_i$, we have $R^y = R^{xy} \leq S^g$ and so $R \leq S^g$. Thus $S^g = RM_{\sigma_i'}$. Since $RM_i = (P \cap E_i)M_i = P \cap E_i M_i = P \cap NM_i = P \leq G$, we have $RM \leq G$. Since M is a maximal subgroup, either $RM = M$ or $RM = G$.

If $RM = G$, then $RM_i = P$ is a Hall σ_i-subgroup of G, which is impossible. Thus $RM = M$ and so $R \leq M \cap E_i$. Since $G = MN = ME_i$, we have $E_i \nleq M$. Note that since $R \lessdot E_i$, we have $R = M \cap E_i$. Thus $|N| = |G : M| = |E_i : E_i \cap M| = |E_i : R| = p$. By [7, Theorem 2], $G \in \mathfrak{F}$, as required. □

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