20241010 MATH3541 NOTE 7[1]

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Contents

1 Introduction

Why shall we investigate topological groups?
(1) Distinguish $(X, \mathcal{O}_X)$ from $\sigma(\mathbb{R})$, where σ is a bijection on X.
(2) Construct continuous functions on X via basic operations.
How to investigate topological groups?
(1) Review basic concepts, especially group action, in group theory.
(2) Force group composition and group inverse to be continuous.
(3) Test the topological properties of these groups.

2 Group Action

2.1 Motivation from Permutation Group

In S_n, notice that:
(1) There exists a well-defined function:

$$\text{Left} : S_n \times \{k\}_{k=1}^n \to \{k\}_{k=1}^n, \text{Left}(\sigma, l) = \sigma(l)$$

(2) For all $\sigma, \sigma' \in S_n$ and $l \in \{k\}_{k=1}^n$:

$$\text{Left}(\sigma, \text{Left}(\sigma', l)) = \text{Left}(\sigma, \sigma'(l)) = \sigma(\sigma'(l)) = \sigma\sigma'(l) = \text{Left}(\sigma\sigma', l)$$

(3) For all $l \in \{k\}_{k=1}^m$:

$$\text{Left}(e, l) = e(l) = l$$

This is exactly why "a group G describes the symmetries on a set X", or "group theory is the language of symmetry". We shall give a name to such function Left.

Definition 2.1. (**Left Action**)
Let G be a group, X be a set, and $\text{Left} : G \times X \to X$ be a function. If:
(1) $\forall g, g' \in G$ and $x \in X, \text{Left}(g, \text{Left}(g', x)) = \text{Left}(gg', x)$;
(2) $\forall x \in X, \text{Left}(e, x) = x$,
then Left is a left action of G on X.

There are some natural group actions.

Proposition 2.2. Let G be a group, and H be a subgroup of G.
Define $\text{Left} : H \times G \to G, (h, g) \mapsto hg$. Left is a left action of H on G.

Proof. We may divide our proof into two parts.
Part 1: For all $h, h' \in H$ and $g \in G$:

$$\text{Left}(h, \text{Left}(h', g)) = \text{Left}(h, h'g) = hh'g = \text{Left}(hh', g)$$

Part 2: For all $g \in G$:

$$\text{Left}(e, g) = eg = g$$

Hence, Left is a left action of H on G. Quod. Erat. Demonstrandum. $\square$

Proposition 2.3. Let G be a group, and N be a normal subgroup of G. Define $\text{Left} : G \times N \to N, (g, n) \mapsto gng^{-1}$. Left is a left action of G on N.

Proof. We may divide our proof into two parts.
Part 1: For all $g, g' \in G$ and $n \in N$:

$$\text{Left}(g, \text{Left}(g', n)) = \text{Left}(g, g'ng'^{-1}) = gg'ng'^{-1}g^{-1} = gg'n(gg')^{-1} = \text{Left}(gg', n)$$

Part 2: For all $n \in N$:

$$\text{Left}(e, n) = ene^{-1} = n$$

Hence, Left is a left action of G on N. Quod. Erat. Demonstrandum. $\square$

2.2 Orbit of an Element

Identify every nontrivial $\sigma \in S_n$ with the set it permutes.
Every $\sigma \in S_n$ is a unique product of disjoint cycles.
Assume that $x \in \sigma$ is in the nontrivial component cycle σ_x of σ.
For all $m \in \mathbb{Z}$, $\sigma^m(x) \in \sigma_x$, so σ_x is fixed under $\{\sigma^m\}_{m \in \mathbb{Z}}$.
We shall describe this phenomenon in general.

Definition 2.4. (**Orbit**)
Let G be a group, X be a set, x be an element of X,
and $\text{Left} : G \times X \to X$ be a left action of G on X.
Define the orbit of G on X through x as:

$$Gx = \text{Left}(G, x) = \{\text{Left}(g, x) \in X : g \in G\}$$

Proposition 2.5. Let G be a group, X be a set,
and $\text{Left} : G \times X \to X$ be a left action of G on X. Define $\sim$ on X by:

$$x \sim x' \text{ if } x \in \text{Left}(G, x')$$

$\sim$ is an equivalence relation on X.

Proof. We may divide our proof into three parts.

Part 1: For all $x \in X$:

$$\exists e \in G, x = \text{Left}(e, x) \implies x \in \text{Left}(G, x) \implies x \sim x$$

Part 2: For all $x, x' \in X$:

$$\begin{aligned} x \sim x' &\implies x \in \text{Left}(G, x') \\ &\implies \exists g' \in G, x = \text{Left}(g', x') \\ &\implies \exists g'^{-1} \in G, x' = \text{Left}(g'^{-1}, \text{Left}(g', x')) = \text{Left}(g'^{-1}, x) \\ &\implies x' \in \text{Left}(G, x) \implies x' \sim x \end{aligned}$$

Part 3: For all $x, x', x'' \in X$:

$$\begin{aligned} x \sim x' \text{ and } x' \sim x'' &\implies x \in \text{Left}(G, x') \text{ and } x' \in \text{Left}(G, x'') \\ &\implies \exists g', g'' \in G, x = \text{Left}(g', x') \text{ and } x' = \text{Left}(g'', x'') \\ &\implies \exists g'g'' \in G, x = \text{Left}(g', \text{Left}(g'', x'')) = \text{Left}(g'g'', x'') \\ &\implies x \in \text{Left}(G, x'') \implies x \sim x'' \end{aligned}$$

Hence, $\sim$ is an equivalence relation on X. Quod. Erat. Demonstrandum. $\square$

Remark: *Notice that:*
(1) ***Lagrange's Theorem*** *can be deduced if we replace (G, X) by (H, G).*
(2) ***Jordan Normal Form*** *can be found if we replace (G, X) by $(\mathbf{GL}_n(\mathbb{C}), \mathbf{M}_n(\mathbb{C}))$.*
(3) ***Cycle Pattern*** *can be found if we replace (G, X) by (S_n, S_n).*

2.3 Stabilizer Subgroup of an Element

In analysis, when solving an equation in $\mathbf{x}$:

$$\mathbf{f}(\mathbf{x}) = \mathbf{y}$$

We usually find a fixed point of another function $\mathbf{T}$ instead:

$$\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n, \mathbf{T}(\mathbf{x}) = \mathbf{x} - A^{-1}[\mathbf{f}(\mathbf{x}) - \mathbf{y}]$$

This is because we have theorems for the existence and uniqueness of fixed points. Hence, we shall study fixed points from different perspectives because they are useful.

Definition 2.6. (**Stabilizer Subgroup**)
Let G be a group, X be a set, x be an element of X,
and $\text{Left} : G \times X \to X$ be a left action of G on X.
Define the stabilizer subgroup of x in G as:

$$G_x = \{g \in G : \text{Left}(g, x) = x\}$$

Proposition 2.7. Let G be a group, X be a set, x be an element of X, and $\mathrm{Left} : G \times X \to X$ be a left action of G on X.

$$G_x \leq G$$

Proof. We may divide our proof into three parts.
Part 1: $\mathrm{Left}(e, x) = x$, so $e \in G_x$.
Part 2: For all $g, g' \in G$:

$$\begin{aligned} g, g' \in G_x &\implies \mathrm{Left}(g, x) = \mathrm{Left}(g', x) = x \\ &\implies \mathrm{Left}(gg', x) = \mathrm{Left}(g, \mathrm{Left}(g', x)) = x \implies gg' \in G_x \end{aligned}$$

Part 3: For all $g \in G$:

$$\begin{aligned} g \in G_x &\implies \mathrm{Left}(g, x) = x \\ &\implies \mathrm{Left}(g^{-1}, x) = \mathrm{Left}(g^{-1}, \mathrm{Left}(g, x)) = x \implies g^{-1} \in G_x \end{aligned}$$

Hence, $G_x \leq G$. Quod. Erat. Demonstrandum. □

Proposition 2.8. Let G be a group, X be a set, x be an element of X, and $\mathrm{Left} : G \times X \to X$ be a left action of G on X.
If we define $G/G_x = \{gG_x\}_{g \in G}$,
then $\sigma_x : G/G_x \to \mathrm{Left}(G, x), gG_x \mapsto \mathrm{Left}(g, x)$ is a bijection.

Proof. We may divide our proof into two parts.
Part 1: For all $gG_x, g'G_x \in G/G_x$, assume that $gG_x = g'G_x$, so $\exists h' \in G_x, g = g'h'$.
This implies:

$$\begin{aligned} \mathrm{Left}(g, x) &= \mathrm{Left}(g'h', x) \\ &= \mathrm{Left}(g', \mathrm{Left}(h', x)) = \mathrm{Left}(g', x) \end{aligned}$$

so σ_x is well-defined.
Part 2: For all $gG_x, g'G_x \in G/G_x$, assume that $\mathrm{Left}(g, x) = \mathrm{Left}(g', x)$.
This implies:

$$\begin{aligned} \mathrm{Left}(g'^{-1}g, x) &= \mathrm{Left}(g'^{-1}, \mathrm{Left}(g, x)) \\ &= \mathrm{Left}(g'^{-1}, \mathrm{Left}(g', x)) = x \end{aligned}$$

As a consequence, $g'^{-1}g \in G_x$, $\exists h' \in G_x, g = g'h'$, $gG_x = g'G_x$, so σ_x is injective.
Hence, the surjective function σ_x is bijective. Quod. Erat. Demonstrandum. □

Remark: *Notice that σ_x is not an isomorphism as $G/G_x, \mathrm{Left}(G, x)$ are not groups.*

3 General Topological Group

In this section, we first study topological properties of topological groups, then study topological properties on corresponding orbit spaces.

3.1 Topological Group and Left Topological Action

Definition 3.1. (**Topological Group**)
Let G be a group and a topological space.
If the following two functions are continuous:
(1) $\text{Comp} : G \times G \to G, (g, g') \mapsto gg'$;
(2) $\text{Inv} : G \to G, g \mapsto g^{-1}$,
then G is a topological group.

Remark: *This concept emphasizes that group and topological space are compatible.*

Proposition 3.2. Let G be a topological group.
(1) For all $g \in G$, $\ell_g : G \to G, g' \mapsto gg'$ is a homeomorphism.
(2) For all $g \in G$, $c_g : G \to G, g' \mapsto gg'g^{-1}$ is a homeomorphism.

Proof. We may divide our proof into two parts.
Part 1: For all $g \in G$, ℓ_g is the restriction of:

$$\text{Comp} : G \times G \to G, (g, g') \mapsto gg'$$

Hence, ℓ_g is continuous.
Part 2: As ℓ_g also has a continuous inverse $\ell_{g^{-1}}$, σ_g is a homeomorphism.
As $c_g = (\text{Inv} \circ \ell_g)^2$ is a composition of homeomorphisms, c_g is a homeomorphism.
Quod. Erat. Demonstrandum. □

Definition 3.3. (**Left Topological Action**)
Let G be a topological group, X be a topological space, and $\text{Left} : G \times X \to X$ be a left action. If Left is continuous, then Left is a left topological action.

3.2 Construction of Topological Group

Definition 3.4. (**Topological Subgroup**)
Let G be a topological group, and H be a subgroup of G.
If we consider the subspace topology of G on H,
then H forms a topological group.
Define H as a topological subgroup of G.

Definition 3.5. (**Topological Product Group**)
Let $(G_\lambda)_{\lambda \in I}$ be an indexed family of topological groups,
and G be the product group of $(G_\lambda)_{\lambda \in I}$.
If we consider the product space topology of $(G_\lambda)_{\lambda \in I}$ on G,
then G forms a topological group.
Define G as the topological product group of $(G_\lambda)_{\lambda \in I}$.

Definition 3.6. (**Topological Quotient Group**)
Let G be a topological group, N be a normal subgroup of G,
and G/N be the quotient group of G modulo N.
If we consider the quotient space topology of N on G,
then G/N forms a topological group.
Define G/N as the topological quotient group of G modulo N.

3.3 Separating a Topological Group

Definition 3.7. (**Symmetric Subset**)
Let G be a group, and U be a nonempty subset of G.
If $U^{-1} = \{u^{-1}\}_{u \in U} = U$, then U is symmetric in U.

Remark: *U is not necessarily a subgroup of G.*

Lemma 3.8. Let G be a topological group, and U be an open neighbour of e.
There exists a symmetric open neighbour N of e, such that $N^{-1}N \subseteq U$.

Proof. Consider the following continuous function:

$$\sigma : G \times G \to G, \sigma(g, g') = g^{-1}g'$$

For given open neighbour U of $e = \sigma(e, e) \in G$,
there exists an open neighbour V of $(e, e) \in G \times G$, such that:

$$\sigma(V) \subseteq U$$

According to the definition of product topology,
there exist open neighbours W, W' of $e \in G$, such that:

$$\begin{aligned} U &\supseteq \sigma(V) \\ &\supseteq \sigma(\pi^{-1}(W) \cap \pi'^{-1}(W')) \\ &= \sigma(W, W') \\ &= W^{-1}W' \end{aligned}$$

Take $N = W \cap W^{-1} \cap W' \cap W'^{-1}$, and we are done.
Quod. Erat. Demonstrandum. $\square$

Definition 3.9. (**Regular Space**)
Let X be a topological space. If $\forall V \in \mathcal{C}_X$ and $v \in V^c, \exists U, u \in \mathcal{O}_X, V \subseteq U$ and $v \in u$ and $U \cap u = \emptyset$, then X is regular.

Remark: *Regular space is stronger than Hausdorff space.*

Proposition 3.10. Let G be a topological group. G is regular.

Proof. For all $V \in \mathcal{C}_X$ and $v \in V^c$, WLOG, assume that $v = e$.
Notice that $U = V^c$ is an open neighbour of e, so there exists a symmetric neighbour N of e, such that $N^{-1}N \subseteq V^c$. This implies:

$$\begin{aligned} V \cap (N^{-1}N) = \emptyset &\implies (NV) \cap N = \emptyset \\ &\implies \exists U = NV, u = N \in \mathcal{O}_G, V \subseteq U \text{ and } v \in u \text{ and } U \cap u = \emptyset \end{aligned}$$

Hence, G is regular. Quod. Erat. Demonstrandum. $\square$

Remark: *It is the group structure of G that separates G.*

Lemma 3.11. Let G be a topological group, X be a topological space, $\mathrm{Left} : G \times X \to X$ be a left topological action, and X/G be the orbit space. $\pi : X \to X/G, x \mapsto \mathrm{Left}(G, x)$ is open.

Proof. For all $U \in \mathcal{O}_X$, we would like to prove that $\pi(U) = \mathrm{Left}(G, U) \in \mathcal{O}_{X/G}$.
It suffices to see $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} \mathrm{Left}(g, U) \in \mathcal{O}_X$. Quod. Erat. Demonstrandum. $\square$

Proposition 3.12. Let G be a topological group, X be a topological space, $\mathrm{Left} : G \times X \to X$ be a left topological action, and X/G be the orbit space. X/G is Hausdorff if and only if $\Delta = \{(x, \mathrm{Left}(g, x))\}_{(x,g) \in X \times G}$ is closed.

Proof. We may divide our proof into two parts.
"if" direction: Assume that Δ is closed.
For all distinct $\pi(x), \pi(x') \in X/G$, (x, x') is an interior point of Δ^c.
There exists $V \in \mathcal{O}_{X \times X}$ with $V \subseteq \Delta^c$, such that $(x, x') \in V$.
WLOG, assume that $V = U \times U'$, where $U, U' \in \mathcal{O}_X$.
Hence, $\pi(x), \pi(x')$ are separated by $\pi(U), \pi(U') \in \mathcal{O}_{X/G}$.
"only if" direction: Assume that X/G is Hausdorff.
For all $(x, x') \in \Delta^c$, $\pi(x), \pi(x')$ are two distinct elements of X/G.
Since X/G is Hausdorff, $\pi(x), \pi(x')$ are separated by $W, W' \in \mathcal{O}_{X/G}$.
This implies $U = \pi^{-1}(W), U' = \pi^{-1}(W') \in \mathcal{O}_X$.

Hence, there exists $V = U \times U' \in \mathcal{O}_{X \times X}$ with $V \subseteq \Delta^c$, such that $(x, x') \in V$.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. □

Remark: *The following proposition specifies the above general criterion.*

Proposition 3.13. Let G be a topological group.
If G has a closed subgroup H, then G/H is Hausdorff.

Proof. It suffices to prove that $\{(g, hg)\}_{(g,h) \in G \times H}$ is closed in G.
Notice that it is the inverse image of H under the following continuous function:

$$\sigma : G \times G \to G, \sigma(g, g') = g'g^{-1}$$

So H is closed implies $\{(g, hg)\}_{(g,h) \in G \times H}$ is closed. Quod. Erat. Demonstrandum. □

3.4 Connected Topological Group

Proposition 3.14. Let G be a topological group.
If G has a proper open subgroup H, then $H \in \mathcal{C}_G$.

Proof. According to **Lagrange's Theorem**, $H^c = \bigcup_{gH \neq eH} gH \in \mathcal{O}_G$, so $H \in \mathcal{C}_G$.
Quod. Erat. Demonstrandum. □

Remark: *Note that H is closed doesn't imply H is open. For example, $\{0\}$ is a closed subgroup in the topological group $\mathbb{R}$, but $\{0\}$ is not open.*

Proposition 3.15. Let G be a topological group.
If G has a proper open subgroup H, then G is not connected.

Proof. As H is clopen in G and $H \neq \emptyset$ and $H \neq G$, G is not connected.
Quod. Erat. Demonstrandum. □

Remark: *Note that G is not connected doesn't imply G has a proper open subgroup.*
Consider the additive group $\mathbb{Z}$ of integers. Define $\sigma : \mathbb{Z} \to \mathbb{R}$ by:

$$\sigma(n) = \begin{cases} 0 & \text{if} \quad n = 0; \\ \frac{1}{n} & \text{if} \quad n \neq 0; \end{cases}$$

Consider the initial metric d_σ of $\mathbb{R}$ on $\mathbb{Z}$ via σ.
For all $n_0, m_0 \in \{0\}^c$, $n_0, m_0, (n_0, m_0)$ are isolated, so it is easy to check:
(1) $\mathrm{Comp} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, (n, m) \mapsto n + m$ *is continuous.*
(2) $\mathrm{Inv} : \mathbb{Z} \to \mathbb{Z}, n \mapsto -n$ *is continuous.*
This implies $\mathbb{Z}$ is a topological group.
As $\mathbb{Z}$ has a nontrivial open partition $\{\{\pm 1\}, \{\pm 1\}^c\}$, so $\mathbb{Z}$ is not connected.
For all proper subgroup H of $\mathbb{Z}$, $0 \in H$ and $0 \in (H^c)'$, so H is not open.

4 Special Topological Group

4.1 Matrix Row Transformation

Definition 4.1. (**Matrix Row Transformation**)
The function $\text{Left} : \mathbf{GL}_n(\mathbb{F}) \times \mathbf{M}_{n,m}(\mathbb{F}) \mapsto \mathbf{M}_{n,m}(\mathbb{F}), (P, A) \mapsto PA$ is a left action of $\mathbf{GL}_n(\mathbb{F})$ on $\mathbf{M}_{n,m}(\mathbb{F})$. Define Left as the matrix row transformation.

Remark: *From now on, we assume that there is a well-defined rank function* $\text{Rank} : \mathbf{M}_{n,m}(\mathbb{F}) \to \mathbb{Z}$*, and we assume all its elementary properties.*

Proposition 4.2. For all $0 < k \leq n$, $\text{Rank}^{-1}([k, +\infty)) \in \mathcal{O}_{\mathbf{M}_{n,m}(\mathbb{R})}$.

Proof. For all $A \in \text{Rank}^{-1}([k, +\infty))$, A has a k by k submatrix A' with $\text{Det}(A') \neq 0$.
As $\sigma' : \mathbf{M}_{n,m}(\mathbb{R}) \to \mathbb{R}, B \mapsto \text{Det}(B')$ is continuous,
A has an open neighbour V, such that each $B \in V$ satisfies $\text{Det}(B') \neq 0$.
Hence, there exists $V \in \mathcal{O}_{\mathbf{M}_{n,m}(\mathbb{R})}$ with $V \subseteq \text{Rank}^{-1}([k, +\infty))$, such that $A \in V$.
Quod. Erat. Demonstrandum. $\square$

Proposition 4.3. $\mathbf{M}_{n,m}(\mathbb{R})/\mathbf{GL}_n(\mathbb{R})$ is not Hausdorff.

Proof. For some $O \in \mathbf{M}_{n,m}(\mathbb{R})$, consider its orbit $\pi(O) \in \mathbf{M}_{n,m}(\mathbb{R})/\mathbf{GL}_n(\mathbb{R})$.
For all $\pi(A) \in \{\pi(O)\}^c$, for open neighbor V of $\pi(O)$, $\pi^{-1}(V)$ is an open neighbour of O.
Take a large enough $N \in \mathbb{N}$, such that $\frac{1}{N}A \in \pi^{-1}(V)$. This implies $\pi(A) = \pi(\frac{1}{N}A) \in V$, so $V = \mathbf{M}_{n,m}(\mathbb{R})/\mathbf{GL}_n(\mathbb{R})$ is the unique open neighbour of $\pi(O)$, $\mathbf{M}_{n,m}(\mathbb{R})/\mathbf{GL}_n(\mathbb{R})$ is not Hausdorff. Quod. Erat. Demonstrandum. $\square$

Remark*: To solve the following problem, recall* ***Least Square Principle****:*
Suppose X is a full column rank real matrix, A is an unknown coefficient column real vector, and Y is a given output column real vector. We have the following results:
(1) $X^T X$ is invertible.
(2) Minimize $\|Y - XA\|_{\text{Euclid}} \iff A = (X^T X)^{-1} X^T Y$.
(3) The minimum $\|Y - X(X^T X)^{-1} X^T Y\|_{\text{Euclid}}$ is continuous with respect to X and Y.

Proposition 4.4. $\text{Rank}^{-1}(\{n\})/\mathbf{GL}_n(\mathbb{R})$ is Hausdorff.

Proof. For all distinct orbits $\pi(A), \pi(B)$, we wish to find disjoint open neighbours.
Step 1: Define the following function f_A.

$$f_A : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}, G \mapsto \|A - GB\|_{\text{Frobenius}}$$

$\pi(A) \neq \pi(B)$ implies f_A is positive definite.
According to **Least Square Principle**, f_A has a positive minimum ϵ_A.

Step 2: Construct a closed ball $\overline{B}(A, \epsilon_A/2)$ in $\mathbf{GL}_n(\mathbb{R})$ under Frobenius norm.
For all $A' \in \overline{B}(A, \epsilon_A/2)$, define the following function $f_{A'}$:

$$f_{A'} : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}, G \mapsto \|A' - GB\|_{\text{Frobenius}}$$

$\forall G \in \mathbf{M}_n(\mathbb{R}), f_{A'}(G) \geq f_A(G) - \|A' - A\|_{\text{Frobenius}} > \epsilon_A/2$ implies $f_{A'}$ is positive definite.
According to **Least Square Principle**, $f_{A'}$ has a positive minimum $\epsilon_{A'}$.
Step 3: Define the following function E.

$$E : \overline{B}\,(A, \epsilon_A/2) \to \mathbb{R}, A' \mapsto \epsilon_{A'}$$

According to **Least Square Principle**, $\text{Rank}(A') = n$ implies E is continuous.
As $\overline{B}(A, \epsilon_A/2)$ is compact, E has a positive minimum $\mu > 0$, which implies:

$$\forall A' \in \overline{B}\,(A, \epsilon_A/2) \text{ and } G \in \mathbf{M}_n(\mathbb{R}), \|A' - GB\|_{\text{Frobenius}} \geq \mu$$

Step 4: We want to find another ball $\overline{B}(B, r)$ centred at B in $\mathbf{GL}_n(\mathbb{R})$ under Frobenius norm, such that for all $A' \in \overline{B}(A, \epsilon_A/2)$ and $B' \in \overline{B}(B, r)$, $\pi(A') \neq \pi(B')$.
Assume to the contrary that such ball doesn't exist, so there exist three sequences:

$$A_k \in \overline{B}(B, \epsilon_A/2), \quad B_k \in \overline{B}(B, 1/k)\backslash\{B\}, \quad G_k \in \mathbf{GL}_n(\mathbb{R})$$

such that each $A_k = G_k B_k$. Notice that:

$$\|G_k\|_{\text{Spectral}} \geq \frac{\|G_k(B_k - B)\|_{\text{Spectral}}}{\|B_k - B\|_{\text{Spectral}}} = \frac{\|A_k - G_k B\|_{\text{Spectral}}}{\|B_k - B\|_{\text{Spectral}}} \geq \frac{\mu}{\sqrt{n}\|B_k - B\|_{\text{Spectral}}}$$

so $\lim\limits_{k \to +\infty} \|B_k - B\|_{\text{Spectral}} = 0$ implies $\lim\limits_{k \to +\infty} \|G_k\|_{\text{Spectral}} = +\infty$.
On one hand, $(A_k)_{k \in \mathbb{N}}$ is bounded, so:

$$\lim_{k \to +\infty} \frac{\|A_k\|_{\text{Spectral}}}{\|G_k\|_{\text{Spectral}}} = 0$$

On the other hand, $\lim\limits_{k \to +\infty} B_k = B$ and $\text{Rank}(B) = n$, so when k is large enough:

$$\frac{\|G_k B_k\|_{\text{Spectral}}}{\|G_k\|_{\text{Spectral}}} \geq \inf \frac{\|B_k \mathbf{x}\|_{\text{Euclid}}}{\|\mathbf{x}\|_{\text{Euclid}}} \geq \frac{1}{2} \inf \frac{\|B\mathbf{x}\|_{\text{Euclid}}}{\|\mathbf{x}\|_{\text{Euclid}}} > 0$$

This contradicts to the following identity:

$$\frac{\|A_k\|_{\text{Spectral}}}{\|G_k\|_{\text{Spectral}}} = \frac{\|G_k B_k\|_{\text{Spectral}}}{\|G_k\|_{\text{Spectral}}}$$

Hence, our assumption is false. Take the following disjoint open neighbours of A, B:

$$U = B(A, \epsilon_A/2), \quad V = B(B, r)$$

By construction, $\pi(U), \pi(V)$ are disjoint open neighbours of $\pi(A), \pi(B)$, so this orbit space is Hausdorff. Quod. Erat. Demonstrandum. $\square$

Remark: *Notice that there exists a bijection* $\tau : \mathbf{M}_{n,m}(\mathbb{R})/\mathbf{GL}_n(\mathbb{R}) \to \mathrm{Spec}(\mathbb{R}^m), \pi(A) \mapsto \mathrm{Row}(A)$*, so we just identify the orbit space* $\mathbf{M}_{n,m}(\mathbb{R})/\mathbf{GL}_n(\mathbb{R})$ *with the spectrum* $\mathrm{Spec}(\mathbb{R}^m)$.

4.2 Matrix Similarity Transformation

Definition 4.5. (**Matrix Similarity Transformation**)
The function $\mathrm{Left} : \mathbf{GL}_n(\mathbb{F}) \times \mathbf{M}_n(\mathbb{F}) \mapsto \mathbf{M}_n(\mathbb{F}), (P, A) \mapsto PAP^{-1}$ is a left action of $\mathbf{GL}_n(\mathbb{F})$ on $\mathbf{M}_n(\mathbb{F})$. Define Left as the matrix similarity transformation.

Remark: *From now on, we assume that there is a well-defined eigenpolynomial function* $\chi : \mathbf{M}_n(\mathbb{F}) \to \mathbb{F}[\lambda]$*, and we assume all its elementary properties.*

Proposition 4.6. The set $\mathbf{D}_n(\mathbb{C})$ of all diagonal matrices is closed in $\mathbf{M}_n(\mathbb{C})$.

Proof. For all convergent sequence $(D_k)_{k\in\mathbb{N}}$ in $\mathbf{D}_n(\mathbb{C})$, its limit D_* is in $\mathbf{D}_n(\mathbb{C})$.
Hence, $\mathbf{D}_n(\mathbb{C})$ is closed in $\mathbf{M}_n(\mathbb{C})$. Quod. Erat. Demonstrandum. $\square$

Proposition 4.7. The set $\mathbf{N}_n(\mathbb{C})$ of all nilpotent matrices is closed in $\mathbf{M}_n(\mathbb{C})$.

Proof. Notice that χ is continuous, so $\mathbf{N} = \chi^{-1}(\{\lambda^n\})$ is closed in $\mathbf{M}_n(\mathbb{C})$.
Quod. Erat. Demonstrandum. $\square$

Remark: *As* $\pi(\mathbf{D}_n(\mathbb{C}))$ *is a union of orbits, the restricted action* $\mathrm{Left} : \mathbf{GL}_n(\mathbb{C}) \times \pi(\mathbf{D}_n(\mathbb{C})) \to \pi(\mathbf{D}_n(\mathbb{C}))$ *is well-defined, so the notation* $\pi(\mathbf{D}_n(\mathbb{C}))/\mathbf{GL}_n(\mathbb{C})$ *makes sense.*

Proposition 4.8. $\pi(\mathbf{D}_n(\mathbb{C}))/\mathbf{GL}_n(\mathbb{C}) \cong \mathbb{C}^n/S_n$.

Proof. Notice that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}^n \arrow[rr, "\chi"] \arrow[d, "\tau"'] & & \mathbb{C}[\lambda]_n & & \pi(\mathbf{D}_n(\mathbb{C})) \arrow[ll, "v"'] \arrow[d, "\pi"] \\
\mathbb{C}^n/S_n \arrow[urr, "\tilde{\chi}"'] & & & & \pi(\mathbf{D}_n(\mathbb{C}))/\mathbf{GL}_n(\mathbb{C}) \arrow[ull, "\tilde{v}"]
\end{tikzcd}$$

(1) According to **Open Mapping Theorem**, eigenpolynomial function χ is surjective, continuous and open, so $\tilde{\chi}$ is a homeomorphism.
(2) According to **Open Mapping Theorem**, Vieta function v is surjective, continuous and open, so $\tilde{v}$ is a homeomorphism.
Combine the two observations above, $\mathbb{C}^n/S_n \cong \mathbb{C}[\lambda]_n \cong \pi(\mathbf{D}_n(\mathbb{C}))/\mathbf{GL}_n(\mathbb{C})$.
Quod. Erat. Demonstrandum. $\square$

Remark: *As* Left *preserves eigenpolynomial, the restricted action* Left : $\mathbf{GL}_n(\mathbb{C}) \times \mathbf{N}_n(\mathbb{C}) \to \mathbf{N}_n(\mathbb{C})$ *is well-defined, so the notation* $\mathbf{N}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C})$ *makes sense.*

Proposition 4.9. $\mathbf{N}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C})$ is closed in $\mathbf{M}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C})$.

Proof. Consider the following matrix $N \in \mathbf{N}_n(\mathbb{C})$:

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

For every possible Jordan form in $\mathbf{N}_n(\mathbb{C})$, such as:

$$\Delta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

there exists a matrix function $P(t)$ on $\mathbb{C}^\times$, such that $\lim\limits_{t\to 0} P(t)NP(t)^{-1} = \Delta$. Here:

$$P(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & t^{-1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & t^{-1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & t^{-2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & t^{-2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^{-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$P(t)NP(t)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

This implies $\overline{\{\pi(N)\}}$ contains $\mathbf{N}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C})$.
Any orbits with different eigenpolynomials can be separated by open sets,
This implies $\mathbf{N}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C})$ contains $\overline{\{\pi(N)\}}$.
To conclude, $\mathbf{N}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C}) = \overline{\{\pi(N)\}}$ is closed in $\mathbf{M}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C})$.
Quod. Erat. Demonstrandum. □

Remark: *By similar construction, one may show that $\pi(N) \subseteq \overline{\pi(M)}$ if and only if:*
(1) $\chi(N) = \chi(M)$.
(2) Every 1 in the Jordan form of N is in the corresponding position of that of M.
so $\pi(N) \leq \pi(M)$ if $\pi(N) \subseteq \overline{\pi(M)}$ is a partial order on $\mathbf{M}_n(\mathbb{C})/\mathbf{GL}_n(\mathbb{C})$.

Proposition 4.10. $\pi(\mathbf{D}_n(\mathbb{C}))$ is closed in $\mathbf{M}_n(\mathbb{C})$.

Proof. For every not diagonalizable matrix J, we wish to separate J from $\pi(\mathbf{D}_n(\mathbb{C}))$.
Step 1: For all $D \in \pi(\mathbf{D}_n(\mathbb{C}))$, define the following continuous function:

$$f_D : \mathbf{M}_n(\mathbb{C}) \to \mathbf{M}_n(\mathbb{C}), H \mapsto \|HD - JH\|_{\text{Frobenius}}$$

J is not diagonalizable implies f_A is positive definite on the following compact set:

$$\mathbf{Q}_n(\mathbb{C}) = \{Q \in \mathbf{M}_n(\mathbb{C}) : QQ^* = Q^*Q = I\}$$

Hence, f_D has a positive minimum $\epsilon_D > 0$ on $\mathbf{Q}_n(\mathbb{C})$.
Step 2: Define the following function E.

$$E : \pi(\mathbf{D}_n(\mathbb{C})) \to \mathbb{R}, D \mapsto \inf_{H \in \mathbf{Q}_n(\mathbb{C})} \|HD - JH\|_{\text{Frobenius}}$$

The following inequality shows that E is $1-$Lipschitz continuous.

$$\begin{aligned}
E(D_1) &= \inf_{H\in\mathbf{Q}_n(\mathbb{C})} \|HD_1 - JH\|_{\text{Frobenius}} \\
&\le \inf_{H\in\mathbf{Q}_n(\mathbb{C})} \|HD_2 - JH\|_{\text{Frobenius}} + \|H(D_1 - D_2)\|_{\text{Frobenius}} \\
&= E(D_2) + \|D_1 - D_2\|_{\text{Frobenius}} \\
E(D_2) &= \inf_{H\in\mathbf{Q}_n(\mathbb{C})} \|HD_2 - JH\|_{\text{Frobenius}} \\
&\le \inf_{H\in\mathbf{Q}_n(\mathbb{C})} \|HD_1 - JH\|_{\text{Frobenius}} + \|H(D_2 - D_1)\|_{\text{Frobenius}} \\
&= E(D_1) + \|D_2 - D_1\|_{\text{Frobenius}} \\
|E(D_2) - E(D_1)| &\le \|D_2 - D_1\|_{\text{Frobenius}}
\end{aligned}$$

The following inequality shows that $\lim\limits_{D\to\infty} E(D) = +\infty$.

$$\begin{aligned}
E(D) &= \inf_{H\in\mathbf{Q}_n(\mathbb{C})} \|HD - JH\|_{\text{Frobenius}} \\
&\ge \inf_{H\in\mathbf{Q}_n(\mathbb{C})} \|HD\|_{\text{Frobenius}} - \|JH\|_{\text{Frobenius}} \\
&= \|D\|_{\text{Frobenius}} - \|J\|_{\text{Frobenius}}
\end{aligned}$$

The two assumptions above shows that E has a positive minimum μ.
Step 3: Construct an open ball $B(J, \mu/2)$.
The following inequality shows $\forall J' \in B(J, \mu/2)$ and $H \in \mathbf{Q}_n(\mathbb{C}), HDH^{-1} \neq J$.

$$\begin{aligned}
\|HDH^{-1} - J'\|_{\text{Frobenius}} &= \|HD - J'H\|_{\text{Frobenius}} \\
&\ge \|HD - JH\|_{\text{Frobenius}} - \|(J' - J)H\|_{\text{Frobenius}} \\
&> \mu - \mu/2 = \mu/2 > 0
\end{aligned}$$

Step 4: Consider the open set $\pi(B(J, \mu/2))$.
As every matrix has a QR-factoization, π "upgrades" the set $B(J, \mu/2)$ of nondiagonalizable matrices via unitary matrix to the set $\pi(B(J, \mu/2))$ of nondiagonalizable matrices via arbitrary matrix, which is a desired open neighbour.
Quod. Erat. Demonstrandum. □

Remark: *This hard proof is given by Prof. Hua.*

4.3 LPU Decomposition

Definition 4.11. (**Lower Triangular Transformation**)
The function $\text{Left} : \mathbf{L}_n(\mathbb{C}) \times \mathbf{GL}_n(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C}), (L, A) \mapsto LA$ is a left action of $\mathbf{L}_n(\mathbb{C})$ on $\mathbf{GL}_n(\mathbb{C})$. Define Left as the lower triangular transformation.

Definition 4.12. (**Upper Triangular Transformation**)
The function Right : $\mathbf{GL}_n(\mathbb{C}) \times \mathbf{U}_n(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C}), (A, U) \mapsto AU$ is a right action of $\mathbf{U}_n(\mathbb{C})$ on $\mathbf{GL}_n(\mathbb{C})$. Define Right as the upper triangular transformation.

Left *reduces every column of* $A \in \mathbf{GL}_n(\mathbb{C})$ *as follows:*

$$\mathbf{Column}_4(A) = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \\ a_{44} \\ a_{54} \\ a_{64} \\ a_{74} \\ a_{84} \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Right *reduces every row of* $A \in \mathbf{GL}_n(\mathbb{C})$ *as follows:*

$$\begin{gathered} \mathbf{Row}_6(A) \\ \parallel \\ \begin{pmatrix} a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & \cdots \end{pmatrix} \\ \downarrow \\ \begin{pmatrix} a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 1 & 0 & 0 & \cdots \end{pmatrix} \end{gathered}$$

Combine the two actions, we can reduce A *into a permutation matrix, so:*

$$\mathbf{L}_n(\mathbb{C}) \backslash \mathbf{GL}_n(\mathbb{C}) / \mathbf{U}_n(\mathbb{C}) \cong S_n$$

Proposition 4.13. If we define $\pi : \mathbf{GL}_n(\mathbb{C}) \to \mathbf{L}_n(\mathbb{C}) \backslash \mathbf{GL}_n(\mathbb{C}) / \mathbf{U}_n(\mathbb{C}), A \mapsto \mathbf{L}_n(\mathbb{C}) A \mathbf{U}_n(\mathbb{C})$ as the double projection map, then $\pi(I)$ is open in $\mathbf{GL}_n(\mathbb{C})$.

Proof. For all permutation matrix $P \in S_n \backslash \{I\}$, take the smallest integer k such that the entry $p_{k,k} = 0$. Notice that the corresponding entry of any LPU is 0, so $\|I - LPU\|_{\text{Frobenius}} \geq 1$. Hence, $B(I, 1) \subseteq \pi(I)$, and $\pi(I)$ is open follows from homogeneity. Quod. Erat. Demonstrandum. $\square$

4.4 Discrete Identification Action

Definition 4.14. (**Klein Bottle**)
Define the following operation $\circ : (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z}) \to (\mathbb{Z} \times \mathbb{Z})$:

$$(m,n)(m',n') = (m + (-1)^n m', n + n')$$

Define the following function $\text{Left} : (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{R} \times \mathbb{R}) \to (\mathbb{R} \times \mathbb{R})$:

$$(m,n)(x,y) = (m + (-1)^n x, n + y)$$

We have the following results:
(1) $\mathbb{Z} \times \mathbb{Z}$ forms a group under $\circ$.
(2) Left is a left action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R} \times \mathbb{R}$.
(3) The edge $\{0\} \times [0,1]$ is glued to the edge $\{1\} \times [0,1]$ without twisting.
(4) The edge $[0,1] \times \{0\}$ is glued to the edge $[0,1] \times \{1\}$ with a half twist.
Define the orbit space $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$ as the Klein bottle $\mathbb{K}^2$.

Proof. We check the claims one by one.
Part 1: For all $(m,n), (m',n') \in \mathbb{Z} \times \mathbb{Z}$ and $(x,y) \in \mathbb{R} \times \mathbb{R}$:

$$\begin{aligned}
[(m,n)(m',n')](x,y) &= (m + (-1)^n m', n + n')(x,y) \\
&= (m + (-1)^n m' + (-1)^{n+n'} x, n + n' + y) \\
&= (m,n)(m' + (-1)^{n'} x, n' + y) \\
&= (m,n)[(m',n')(x,y)]
\end{aligned}$$

Hence, $\circ$ and Left are associative.
Part 2: There exists $(0,0) \in \mathbb{Z} \times \mathbb{Z}$, such that for all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and $(x,y) \in \mathbb{R} \times \mathbb{R}$:

$$\begin{aligned}
(m,n)(0,0) = (m + (-1)^0 0, n + 0) = (m,n) \\
(0,0)(x,y) = (0 + (-1)^0 x, 0 + y) = (x,y)
\end{aligned}$$

Hence, $\circ$ has an identity element $(0,0)$, which is compatible with Left.
Part 3: For all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$, there exists $(-(-1)^n m, -n) \in \mathbb{Z} \times \mathbb{Z}$, such that:

$$\begin{aligned}
(m,n)(-(-1)^n m, -n) &= (m - (-1)^n (-1)^n m, n - n) = (0,0) \\
(-(-1)^n m, -n)(m,n) &= (-(-1)^n m + (-1)^n m, -n + n) = (0,0)
\end{aligned}$$

Hence, every $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ is invertible under $\circ$.
Part 4: For all $(0,y) \in \{0\} \times [0,1]$:

$$(0,y) \sim (1,0)(0,y) = (1 + (-1)^0 0, 0 + y) = (1,y)$$

Hence, the edge $\{0\} \times [0,1]$ is glued to the edge $\{1\} \times [0,1]$ without twisting.

Part 5: For all $(x,0) \in [0,1] \times \{0\}$:

$$(x,0) \sim (1,1)(x,0) = (1+(-1)^1 x, 1+0) = (1-x, 1)$$

Hence, the edge $[0,1] \times \{0\}$ is glued to the edge $[0,1] \times \{1\}$ with a half twist.
To conclude, the claims are valid. Quod. Erat. Demonstrandum. $\square$

Remark: *By adding components, one may generalize Klein bottle:*

$$(m,n,k)(m',n',k') = (m+(-1)^k m', n+(-1)^k n', k+k')$$
$$(m,n,k)(x,y,z) = (m+(-1)^k x, n+(-1)^k y, k+z)$$

By adding a factor, one may construct a different orbit space:

$$(m,n)(m',n') = (m+(-1)^n m', n+(-1)^m n')$$
$$(m,n)(x,y) = (m+(-1)^n x, n+(-1)^m y)$$

However, be careful that this is acturally a disk instead of a projective plane. It is the extra interior identification $(x,y) \sim (1-x, 1-y)$ that makes the difference.

Definition 4.15. (**Lens Space**)
Let p, q be coprime integers, $\zeta = \mathrm{e}^{2\pi\mathrm{i}/p}$ be a unit root,
and $\mathbb{S}^3 = \{(z_1, z_q) \in \mathbb{C} \times \mathbb{C} : |z_1|^2 + |z_q|^2 = 1\}$ be a subspace of $\mathbb{C} \times \mathbb{C}$.
Define the following function Left : $\langle\zeta\rangle \times \mathbb{S}^3 \to \mathbb{S}^3$:

$$\zeta^k(z_1, z_q) = (\zeta^k z_1, \zeta^{qk} z_q)$$

Notice that Left is a left action of $\langle\zeta\rangle$ on $\mathbb{S}^3$.
Define the lens space as the orbit space $L(p,q) = \mathbb{S}^3/\langle\zeta\rangle$.

Proposition 4.16. Every orbit $\pi(z_1, z_q) \in L(p,q)$ contains exactly p elements.

Proof. We may divide our proof into two cases.
Case 1: Assume that $z_1 \neq 0$.
For all $k, k' \in \mathbb{Z}$:

$$\zeta^k = \zeta^{k'} \iff k \equiv k' (\mathrm{mod}\ p)$$

According to the **Division Algorithm**, $\{\zeta^k z_1\}_{k=0}^{p-1}$ contains exactly p elements.
Hence, $\pi(z_1, z_q) = \{(\zeta^k z_1, \zeta^{qk} z_q)\}_{k=0}^{p-1}$ contains exactly p elements.
Case 2: Assume that $z_q \neq 0$. Notice that $q \in \mathbb{Z}_p^\times$ is invertible.

For all $k, k' \in \mathbb{Z}$:

$$\begin{aligned}\zeta^{qk} \equiv \zeta^{qk'} &\iff qk \equiv qk' (\mathrm{mod}\ p)\\ &\iff k \equiv k' (\mathrm{mod}\ p)\end{aligned}$$

According to the **Division Algorithm**, $\{\zeta^{qk} z_q\}_{k=0}^{p-1}$ contains exactly p elements.
Hence, $\pi(z_1, z_q) = \{(\zeta^k z_1, \zeta^{qk} z_q)\}_{k=0}^{p-1}$ contains exactly p elements.
Quod. Erat. Demonstrandum. □

Proposition 4.17. If $p = p'$ and $q \equiv q' (\mathrm{mod}\ p)$, then $L(p, q) \cong L(p', q')$.

Proof. Define a surjective function $\sigma : L(p, q) \to L(p', q'), \pi(z_1, z_q) \mapsto \pi'(z_1, z_q)$.
Part 1: For all $(z_1, z_q), (w_1, w_q) \in \mathbb{S}^3$, notice that $q = q' \in \mathbb{Z}_p^\times$:

$$\begin{aligned}\pi(z_1, z_q) = \pi(w_1, w_q) &\implies \exists k \in \mathbb{Z}, (z_1, z_q) = (\zeta^k w_1, \zeta^{qk} w_q)\\ &\implies \exists k' = k \in \mathbb{Z}, (z_1, z_q) = (\zeta^{k'} w_1, \zeta^{q'k'} w_q)\\ &\implies \pi'(z_1, z_q) = \pi'(w_1, w_q)\end{aligned}$$

Hence, σ is well-defined.
Part 2: For all $(z_1, z_q), (w_1, w_q) \in \mathbb{S}^3$, notice that $q = q' \in \mathbb{Z}_p^\times$:

$$\begin{aligned}\pi'(z_1, z_q) = \pi'(w_1, w_q) &\implies \exists k' \in \mathbb{Z}, (z_1, z_q) = (\zeta^{k'} w_1, \zeta^{q'k'} w_q)\\ &\implies \exists k = k' \in \mathbb{Z}, (z_1, z_q) = (\zeta^k w_1, \zeta^{qk} w_q)\\ &\implies \pi(z_1, z_q) = \pi(w_1, w_q)\end{aligned}$$

Hence, σ is injective.
Part 3: Notice that following diagram commutes:

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r, "\text{The Identity Function } e"] \arrow[d, "\text{The Projection Function } \pi"'] & \mathbb{S}^3 \arrow[d, "\text{The Projection Function } \pi'"] \\
L(p, q) \arrow[r, "\text{The Bijective Function } \sigma"] & L(p', q')
\end{tikzcd}$$

For all open subset U' of $L(p', q')$:

$$\sigma^{-1}(U') = \pi((\sigma \circ \pi)^{-1}(U')) = \pi((\pi' \circ e)^{-1}(U')) \text{ is open in } L(p, q)$$

For all open subset U of $L(p, q)$:

$$\sigma(U) = \pi'((\sigma^{-1} \circ \pi')^{-1}(U)) = \pi'((\pi' \circ e^{-1})^{-1}(U)) \text{ is open in } L(p', q')$$

Hence, σ is a homeomorphism. Quod. Erat. Demonstrandum. □

Proposition 4.18. If $p = p'$ and $qq' \equiv 1 (\text{mod } p)$, then $L(p, q) \cong L(p', q')$.

Proof. Define a surjective function $\sigma : L(p, q) \to L(p', q'), \pi(z_1, z_q) \mapsto \pi'(z_q, z_1)$.
Part 1: For all $(z_1, z_q), (w_1, w_q) \in \mathbb{S}^3$, notice that $qq' = 1 \in \mathbb{Z}_p^\times$:

$$\begin{aligned}\pi(z_1, z_q) = \pi(w_1, w_q) &\implies \exists k \in \mathbb{Z}, (z_1, z_q) = (\zeta^k w_1, \zeta^{qk} w_q) \\ &\implies \exists k' = qk \in \mathbb{Z}, (z_q, z_1) = (\zeta^{k'} w_q, \zeta^{q'k'} w_1) \\ &\implies \pi'(z_q, z_1) = \pi'(w_q, w_1)\end{aligned}$$

Hence, σ is well-defined.
Part 2: For all $(z_1, z_q), (w_1, w_1) \in \mathbb{S}^3$, notice that $qq' = 1 \in \mathbb{Z}_p^\times$:

$$\begin{aligned}\pi'(z_q, z_1) = \pi'(w_q, w_1) &\implies \exists k' \in \mathbb{Z}, (z_q, z_1) = (\zeta^{k'} w_q, \zeta^{q'k'} w_1) \\ &\implies \exists k = q'k' \in \mathbb{Z}, (z_1, z_q) = (\zeta^k w_1, \zeta^{qk} w_q) \\ &\implies \pi(z_1, z_q) = \pi(w_1, w_q)\end{aligned}$$

Hence, σ is injective.
Part 3: Notice that following diagram commutes:

$$\begin{tikzcd} \mathbb{S}^3 \arrow[r, "\text{The Transposition Function } \tau"] \arrow[d, "\text{The Projection Function } \pi"'] & \mathbb{S}^3 \arrow[d, "\text{The Projection Function } \pi'"] \\ L(p, q) \arrow[r, "\text{The Bijective Function } \sigma"] & L(p', q') \end{tikzcd}$$

For all open subset U' of $L(p', q')$:

$$\sigma^{-1}(U') = \pi((\sigma \circ \pi)^{-1}(U')) = \pi((\pi' \circ \tau)^{-1}(U')) \text{ is open in } L(p, q)$$

For all open subset U of $L(p, q)$:

$$\sigma(U) = \pi'((\sigma^{-1} \circ \pi')^{-1}(U)) = \pi'((\pi' \circ \tau^{-1})^{-1}(U)) \text{ is open in } L(p', q')$$

Hence, σ is a homeomorphism. Quod. Erat. Demonstrandum. □

Remark: *The complete classification of lens spaces includes two more results:*
(1) If $p \neq p'$, then $L(p, q) \ncong L(p', q')$.
(2) If $p = p'$ and $q \not\equiv q' (\text{mod } p)$ and $qq' \not\equiv 1 (\text{mod } p)$, then $L(p, q) \ncong L(p', q')$.
However, we need advanced topological invariants in order to show them.

References

[1] H. Ren, “Template for math notes,” 2021.