20241219 MATH3541 Sample Test 2023

1. (1) True. Every open cover of a singleton has a singleton subcover.

(2) True. $x_1, x_2$ are separated by opens $U_1, U_2$ in $X$

$\Rightarrow x_1, x_2$ are separated by opens $U_1' = U_1 \cap X'$, $U_2' = U_2 \cap X'$ in $X' \subseteq X$

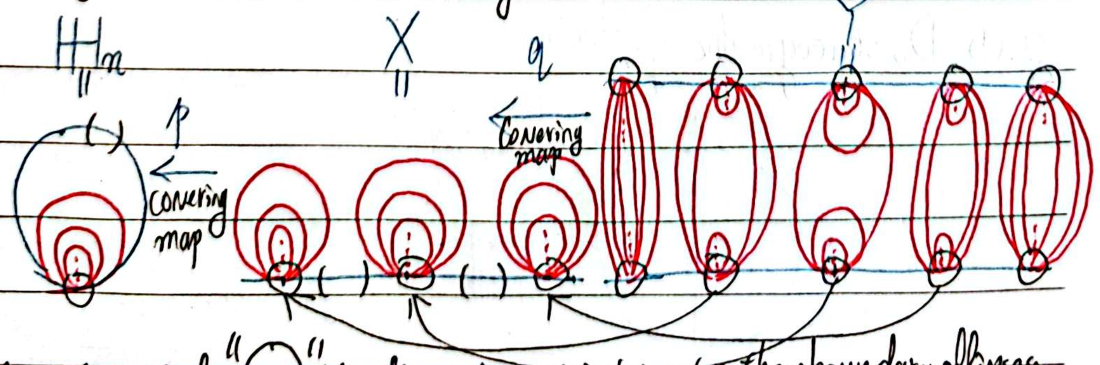
(3) False. Consider the set $X = \{1,2\}$ with non-Hausdorff topology $\mathcal{C}_X = \{\phi, \{1\}, \{1,2\}\}$. The set $\{2\}$ is compact, simply because $|\mathcal{U}| \le |\mathcal{O}_X| = |\mathcal{C}_X| = 3$, but $\{2\} \notin \mathcal{C}_X$, $\{2\}$ is not closed.

(4) False. Consider the covering map $\pi(\vec{x}) = \{+\vec{x}, -\vec{x}\}$ from a simply connected and locally path connected space $X = \mathbb{S}^2$ to $X/G = \mathbb{S}^2/\{\text{identity map, antipodal map}\} = \mathbb{R}P^2$.

As every $\vec{x} \in \mathbb{S}^2$ has an open neighbour $U = \{\vec{z} \in \mathbb{S}^2 : \vec{x} \cdot \vec{z} > 0\}$, such that $U \cap (-U) = \phi$, the action is properly discontinuous, so $\pi_1(X/G) \cong G \cong \mathbb{Z}_2$, which is nontrivial.

(5) True. For all open partition $U, V$ of a path connected space $X$, we prove that $U = \phi$ or $V = \phi$. Assume to the contrary that $U \ni u$ and $V \ni v$. Construct a path $\gamma$ from $u$ to $v$ in $X$, then we get a nontrivial open partition $\gamma^{-1}(U), \gamma^{-1}(V)$ of $[0,1]$, which contradicts with the connexity of $[0,1]$

(6) False. Consider the Hawaii earing:




The preimage of "O" under $p \circ q$ intersects the boundary ellipses countably infinitely many times, which cannot happen on $\mathbb{H}_m$, so $p \circ q$ is not a covering map.

(7) False. $\mathbb{R}^3\setminus\{pt\}$ is homotopic to $\mathbb{S}^2$ via dilation $H(\vec{x},t)=(1-t)\vec{x}+t\frac{\vec{x}}{\|\vec{x}\|}$

Hence, $\pi_1(\mathbb{R}^3\setminus\{pt\})\cong\pi_1(\mathbb{S}^2)\cong\{e\}$.

However, $\pi_1(\mathbb{S})\cong\mathbb{Z}$, so $\mathbb{Z}\not\cong\{e\}$ implies $\mathbb{R}^3\setminus\{pt\}\not\simeq\mathbb{S}$

(8) False. Consider the subspace $\{0,\frac{1}{n}\in\mathbb{R}:n\in\mathbb{N}\}$ of $\mathbb{R}$.

As $\lim_{n\to+\infty}\frac{1}{n}=0$, $\overline{\{\frac{1}{n}\in\mathbb{R}:n\in\mathbb{N}\}}=\{0,\frac{1}{n}\in\mathbb{R}:n\in\mathbb{N}\}$ is closed in $\mathbb{R}$.

According to Heine-Borel theorem, $\{0,\frac{1}{n}\in\mathbb{R}:n\in\mathbb{N}\}$ is compact.

However, every singleton $\{\frac{1}{n}\}$ is a connected component, because $\{\frac{1}{n}\}$ is separated from the remaining points by an open interval $(\frac{1}{n-\frac{1}{2}},\frac{1}{n+\frac{1}{2}})$, so $\{0,\frac{1}{n}\in\mathbb{R};n\in\mathbb{N}\}$ has infinitely many connected components.

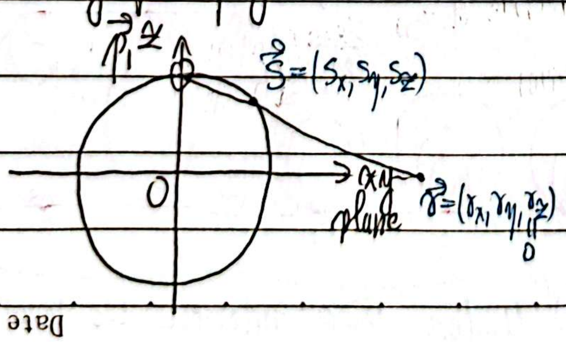
(9) False. For all open disk neighbour $B(0,r)$ of $z^2=0$ downstairs, the restricted map $\sigma|_{B(0,\sqrt{r})}: B(0,\sqrt{r})\to B(0,r)$, $z\mapsto z^2$ fails to be injective, because $\forall z\in B(0,\sqrt{r})$, $\sigma(z)=\sigma(-z)=z^2$.

Hence, $\sigma$ is not a covering map.

(10) False. Consider $X=\mathbb{S}$, $X_1=\mathbb{S}\setminus\{+1\}\cong\mathbb{R}$, $X_2=\mathbb{S}\setminus\{-1\}\cong\mathbb{R}$.

Although $X_1,X_2$ are simply connected and $X_1\cap X_2\ni\pm i$ is nonempty, $\pi_1(X)\cong\mathbb{Z}$ is nontrivial, $X=X_1\cup X_2$ is not simply connected.

2.(1) Do stereographic projection.




$\vec{r}=(0,0,1)+\lambda(s_x,s_y,s_z-1)$

$1+\lambda(s_z-1)=0$, $\lambda=\frac{1}{1-s_z}$

$\vec{r}=\left(\frac{s_x}{1-s_z},\frac{s_y}{1-s_z},0\right)$

$s_x=r_x(1-s_z)$, $s_y=r_y(1-s_z)$

$s_x^2+s_y^2=(r_x^2+r_y^2)(1-s_z)^2=1-s_z^2$

$(r_x^2+r_y^2)(1-s_z)=1+s_z$

$s_z=\frac{-1+r_x^2+r_y^2}{1+r_x^2+r_y^2}$, $s_x=\frac{2r_x}{1+r_x^2+r_y^2}$, $s_y=\frac{2r_y}{1+r_x^2+r_y^2}$

For all $\vec{r}_1=(r_{x,1}, r_{y,1}, 0), \vec{r}_2=(r_{x,2}, r_{y,2}, 0) \in \mathbb{R}^2$:

$$\vec{r}_1=\vec{r}_2 \Rightarrow \vec{s}_1=\left(\frac{2r_{x,1}}{1+r_{x,1}^2+r_{y,1}^2}, \frac{2r_{y,1}}{1+r_{x,1}^2+r_{y,1}^2}, \frac{-1+r_{x,1}^2+r_{y,1}^2}{1+r_{x,1}^2+r_{y,1}^2}\right)$$

$$=\left(\frac{2r_{x,2}}{1+r_{x,2}^2+r_{y,2}^2}, \frac{2r_{y,2}}{1+r_{x,2}^2+r_{y,2}^2}, \frac{-1+r_{x,2}^2+r_{y,2}^2}{1+r_{x,2}^2+r_{y,2}^2}\right)=\vec{s}_2$$

For all $\vec{s}_1=(s_{x,1}, s_{y,1}, s_{z,1}), \vec{s}_2=(s_{x,2}, s_{y,2}, s_{z,2}) \in \mathbb{S}^2 \setminus \{(0,0,1)\}$:

$$\vec{s}_1=\vec{s}_2 \Rightarrow \vec{r}_1=\left(\frac{s_{x,1}}{1-s_{z,1}}, \frac{s_{y,1}}{1-s_{z,1}}, 0\right)=\left(\frac{s_{x,2}}{1-s_{z,2}}, \frac{s_{y,2}}{1-s_{z,2}}, 0\right)=\vec{r}_2$$

Hence, the surjective map $\vec{\sigma}: \vec{r} \mapsto \vec{s}$ is well-defined and bijective.

As the polynomials $2r_x$, $2r_y$, $-1+r_x^2+r_y^2$, $1+r_x^2+r_y^2$ are continuous, and $1+r_x^2+r_y^2 \neq 0$, $\vec{s}=\left(\frac{2r_x}{1+r_x^2+r_y^2}, \frac{2r_y}{1+r_x^2+r_y^2}, \frac{-1+r_x^2+r_y^2}{1+r_x^2+r_y^2}\right)$ is continuous.

As the polynomials $s_x$, $s_y$, $1-s_z$ are continuous, and $1-s_z \neq 0$, $\vec{r}=\left(\frac{s_x}{1-s_z}, \frac{s_y}{1-s_z}, 0\right)$ is continuous, so $\vec{\sigma}: \vec{r} \mapsto \vec{s}$ is a homeomorphism.

(2) Step 1: We do stereographic projection. Note that $\sigma(1,0,0) = \sigma(1,0,0)$

$$\vec{\sigma}: \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{\vec{p}_1\}, \sigma(r_x, r_y, 0)=\left(\frac{2r_x}{1+r_x^2+r_y^2}, \frac{2r_y}{1+r_x^2+r_y^2}, \frac{-1+r_x^2+r_y^2}{1+r_x^2+r_y^2}\right).$$

Step 2: We construct a deformation retraction from $\mathbb{R}^2 \setminus \{\vec{p}_2\}$ to $\mathbb{S}$:

$$\vec{H}: (\mathbb{R}^2 \setminus \{\vec{p}_2\}) \times [0,1] \to \mathbb{R}^2 \setminus \{\vec{p}_2\}, \vec{H}((r_x, r_y, 0), t)=\left((1-t)(r_x-1)+t\frac{r_x-1}{\sqrt{(r_x-1)^2+r_y^2}}+1, (1-t)r_y+t\frac{r_y}{\sqrt{(r_x-1)^2+r_y^2}}, 0\right).$$

Step 3: Do composition, and we get a deformation

$$\vec{\sigma} \circ \vec{H} \circ \vec{\sigma}^{-1} \text{ from } \mathbb{S}^2 \setminus \{\vec{p}_1, \vec{p}_2\} \text{ to } \vec{\sigma}(\mathbb{S}) \cong \mathbb{S}.$$

3.(1) Let $p: \tilde{X} \to X$ be a covering map.
If $\tilde{X}$ is simply connected and locally path connected,
then $p$ is a universal cover.

(2) If for all $m \in M$, there exists an open neighbour $U$ of $m$ in $M$,
such that for all nonidentity element $g \in G$, $U \cap (g * U) = \phi$,
then the action $*$ is properly discontinuous.

(3) As $G$ acts on $\tilde{X}$ free and properly discontinuously,
$p: \tilde{X} \to X$, $\tilde{x}$ (pt upstairs) $\mapsto x$ (pt downstairs = class upstairs)

$$G \circlearrowright (\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$$

quotient map, saturated open to open, $p$ is surjective and continuous.

is a covering map. Fix a base point $x_0$ downstairs,
and fix an arbitrary base point $\tilde{x}_0$ upstairs with $p(\tilde{x}_0) = x_0$.
As $\tilde{X}$ is path connected and locally path connected,
and $p(\pi_1(\tilde{X}, \tilde{x}_0)) = p(\pi_1(\tilde{X}, \tilde{x}_0))$, the universal property
of covering space suggests that $p$ has a unique lift $\tilde{p}_1$ upstairs
with initial point $\tilde{x}_1 \in p^{-1}(\{x_0\})$ under $p$.
For all $[[\gamma]] \in \pi_1(X, x_0)$ downstairs, $[[\gamma]]$ has a unique lift
$[[\tilde{\gamma}]]$ upstairs with initial point $\tilde{x}_0$. Define $\phi([[\gamma]]) =$
[The unique deck transformation $\tilde{p}_1$ that sends $\tilde{x}_0$ to $\tilde{\gamma}(1)$],
it follows that $\phi$ is a group homomorphism from $\pi_1(X, x_0)$ to $G$.

(4) As $\tilde{X}$ is path connected,

for all $\tilde{x}_1$ in the fibre $p^{-1}(\{x_0\})$,

there exists a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$ upstairs.

Project $[[\tilde{\gamma}]]$ downstairs to $[[\gamma]] = p'([[\tilde{\gamma}]]) \in \pi_1(X, x_0)$,

and we've proven that $\phi([[\gamma]]) =$ [The unique deck transformation $\tilde{p}_1$ that sends $\tilde{x}_0$ to $\tilde{\gamma}(1) = \tilde{x}_1$], so $\phi$ is surjective.

As $\tilde{X}$ is simply connected,

for all $[[\gamma]] \in \pi_1(X, x_0)$ downstairs,

$\phi([[\gamma]]) =$ [The unique deck transformation $\tilde{p}$ that sends $\tilde{x}_0$ to $\tilde{\gamma}(1)$] $=$ identity

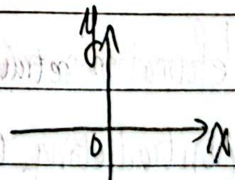
$\Rightarrow \tilde{\gamma}(1) = \tilde{\gamma}(0) = \tilde{x}_0 \Rightarrow [[\tilde{\gamma}]]$ is a loop in $\pi_1(\tilde{X}, \tilde{x}_0)$ upstairs

$\Rightarrow [[\tilde{\gamma}]] \in \pi_1(\tilde{X}, \tilde{x}_0) \cong \{e\}$ is trivial $\Rightarrow [[\gamma]] = p'([[\tilde{\gamma}]])$ is trivial

Hence, $\phi$ is injective, and it follows that $\phi$ is a group isomorphism.

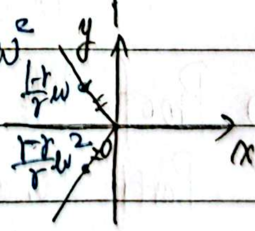
4. (1) $z^2+z+1=0$ iff $z=\omega$ or $z=\omega^2$, where $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$\left(\frac{rz}{1-r}\right)^2+\left(\frac{rz}{1-r}\right)+1=0$ iff $\frac{rz}{1-r}=\omega$ or $\frac{rz}{1-r}=\omega^2$




Case 1: If $r=0$, then no root.

Case 2: If $0<r<1$, then $z=\frac{1-r}{r}\omega$ or $z=\frac{1-r}{r}\omega^2$




(2) For all $(1, z_0) \in \{1\} \times \mathbb{C}^\times$, we wish to check:

$$\lim_{(r,z)\to(1,z_0)} \frac{f\left(\frac{rz}{1-r}\right)}{\left|f\left(\frac{rz}{1-r}\right)\right|} = \frac{z_0^n}{|z_0|^n}$$

It suffices to check:

$$\lim_{\substack{t\to+\infty\\ z\to z_0}} \frac{f(tz)}{|f(tz)|} = \frac{z_0^n}{|z_0|^n}$$

As
$$\lim_{\substack{t\to+\infty\\ z\to z_0}} \frac{|z_0|^n f(tz) - z_0^n |f(tz)|}{|z_0|^n |f(tz)|} = \lim_{\substack{t\to+\infty\\ z\to z_0}} \frac{\operatorname{Det}\begin{pmatrix} |z_0|^n & |t^n z^n + a_{n-1}t^{n-1}z^{n-1}+\cdots| \\ z_0^n & t^n z^n + a_{n-1}t^{n-1}z^{n-1}+\cdots \end{pmatrix}}{|z_0|^n |t^n z^n + a_{n-1}t^{n-1}z^{n-1}+\cdots|}$$

$$= \lim_{\substack{t\to+\infty\\ z\to z_0}} \frac{\operatorname{Det}\begin{pmatrix} |z_0|^n & |z^n| \, \underline{\left|1+\frac{a_{n-1}}{tz}+\cdots\right|}_{\to 1} \\ z_0^n & z^n \, \underline{\left(1+\frac{a_{n-1}}{tz}+\cdots\right)}_{\to 1} \end{pmatrix}_{\to 0}}{\underline{|z_0|^n |z^n| \left|1+\frac{a_{n-1}}{tz}+\cdots\right|}_{\to |z_0|^{2n}}} = 0.$$

Our result follows as a corollary.

Now $f\left(\frac{rz}{1-r}\right) \neq 0$ and $z \neq 0$, the function $F(r,z)$ is well-defined, and the continuity follows from $\lim_{(r,z)\to(1,z_0)} F(r,z) = F(1,z_0)$.

(3). Assume to the contrary that $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $n \geq 1$ has no root.

Now $F(r,z) = \begin{cases} f\left(\frac{rz}{1-r}\right) \Big/ \left|f\left(\frac{rz}{1-r}\right)\right| & \text{if } 0 \leq r < 1 \\ z^n / |z|^n & \text{if } r = 1 \end{cases}$ gives an inverted deformation retraction from a single point $\{0\}$ to the unit circle $S$, contradicting to $\pi_1(\{0\}) \cong \{e\} \ncong \pi_1(S) \cong \mathbb{Z}$.

5.(1) Proof: We may divide our proof into three parts

Part 1: For all $g_1, g_2 \in G$ and $A \in M$, $\sigma(g_1 g_2, A) = (g_1 g_2) A (g_1 g_2)^{-1}$
$= g_1 (g_2 A g_2^{-1}) g_1^{-1} = \sigma(g_1, \sigma(g_2, A))$

Part 2: For all $A \in M$, $\sigma(e, A) = eAe^{-1} = A$.

Part 3: As matrix multiplication and matrix inverse are continuous over $G = GL_n(\mathbb{C})$,
$\sigma(g, A) = gAg^{-1}$ is continuous.

Hence, $*: (g, A) \mapsto \sigma(g, A) = gAg^{-1}$ is a left topological action.

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(2) Proof: Consider the points $[\begin{pmatrix}0&0\\0&0\end{pmatrix}]$, $[\begin{pmatrix}0&1\\0&0\end{pmatrix}]$ downstairs in $M/G$.

For any open neighbour $U$ of $[\begin{pmatrix}0&0\\0&0\end{pmatrix}]$ in $M/G$, we wish to show that $U \ni [\begin{pmatrix}0&1\\0&0\end{pmatrix}]$, thus $M/G$ is not Hausdorff.

Pull back $U$ to $V=\pi^{-1}(U)$ in $M$, where $\pi: M\to M/G$, $A\mapsto[A]$ is the natural projection map. Note that $\lim_{n\to+\infty}\begin{pmatrix}1&0\\0&n\end{pmatrix} * \begin{pmatrix}0&1\\0&0\end{pmatrix} = \lim_{n\to+\infty}\begin{pmatrix}1&0\\0&n\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&1/n\end{pmatrix} = \lim_{n\to+\infty}\begin{pmatrix}0&1/n\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}\in V$,

so for some $n\in\mathbb{N}$, $\begin{pmatrix}0&1/n\\0&0\end{pmatrix}\in V$, $[\begin{pmatrix}0&1/n\\0&0\end{pmatrix}]=[\begin{pmatrix}0&1\\0&0\end{pmatrix}]\in\pi(V)=U$, and we are done.

(3) Proof: As $A$ is diagonalizable $\Rightarrow A=P\Lambda P^{-1}$ for some invertible $P$ and diagonal $\Lambda$ $\Rightarrow [A]=[\Lambda]$ for some diagonal $\Lambda$

$\Rightarrow$ Each $B=QAQ^{-1}$ satisfies $[B]=[A]=[\Lambda]$

$\Rightarrow$ Each $Q*A=B$ is diagonalizable,

the set $M^0$ of all diagonalizable matrices is closed under $*$

For all distinct $[A],[B]\in M^0/G$, $ch(A)\neq ch(B)$,

just choose $U=\{(x,y)\in\mathbb{C}^2: |x-tr(A)|^2+|y-det(A)|^2 < |x-tr(B)|^2+|y-det(B)|^2\}$

$V=\{(x,y)\in\mathbb{C}^2: |x-tr(A)|^2+|y-det(A)|^2 > |x-tr(B)|^2+|y-det(B)|^2\}$

$\in\mathcal{O}_{\mathbb{C}^2}$ to separate $ch(A), ch(B)$ in $\mathbb{C}^2$,

and $\underbrace{\pi(ch^{-1}(U))}_{\text{saturated open in } M^0}$, $\underbrace{\pi(ch^{-1}(V))}_{\text{saturated open in } M^0}$ separate $[A],[B]$ in $M^0/G$ as $ch$ is continuous.

(4).

$$\begin{tikzcd} M^\circ \arrow[d, "\pi"'] \arrow[dr, "ch"] & \\ M^\circ/G \arrow[r, "\phi"] & \mathbb{C}^2 \end{tikzcd}$$

For all $[A] \in M^\circ/G$,

as $ch$ is invariant under $*$,

there exists a unique image $\phi([A]) = ch(A)$,

so $\phi$ is well-defined.

If two classes $[A], [B] \in M^\circ/G$ have the same $ch$,

then $A, B$ are conjugate, $[A]=[B]$, so $\phi$ is injective.

If $(a,b) \in \mathbb{C}^2$, then $\lambda^2 + a\lambda + b = 0$ can be factorized into $(\lambda-\lambda_1)(\lambda-\lambda_2)=0$,

so $(a,b) = ch\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \phi\left(\left[\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\right]\right)$, $\phi$ is surjective.

For all open subset $U$ of $\mathbb{C}^2$, $\phi^{-1}(U) = \pi(\underbrace{ch^{-1}(U)}_{\text{saturated open}}) \in \mathcal{O}_{M^\circ/G}$

Hence, $\phi$ is bijective and continuous.