Prop X. Hausdorff any convergent seq has a unique limit.

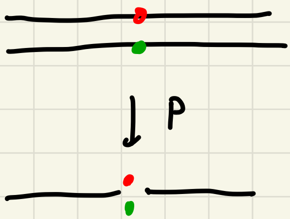
p.f. $x_i \to x_\infty, y_\infty$

$\mathcal{U} \ni x_\infty \quad V \ni y_\infty \qquad U \cap V \neq \emptyset$

Example line with double origin

$$X = \mathbb{R}^1 \sqcup \mathbb{R}^2 \big/ \sim \qquad x \sim y \text{ iff } x = y \neq 0$$

with quotient top



$X \supset \mathcal{U} \ni \bullet \qquad V \ni \bullet$

$p^{-1}(\mathcal{U})$ is open in $\mathbb{R}^1 \sqcup \mathbb{R}^2$

it must contain $(-\varepsilon, \varepsilon) \setminus 0 \sqcup (-\varepsilon, \varepsilon) \setminus 0$ for some ε

since V satisfies a similar property.

$$p^{-1}(U) \cap p^{-1}(V) \neq \emptyset \Rightarrow U \cap V \neq \emptyset.$$

Quotient space of Hausdorff space may not be Hausdorff in general

Example $\mathbb{R}^2/\sim\ =: X \quad v \sim \lambda v \quad \lambda \in \mathbb{R}^\times$

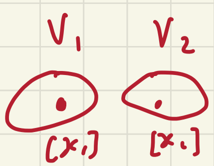
show that X is not Hausdorff.

Prop X Hausdorff $\quad X \xrightarrow{q} [X] = X/\sim$

$[X]$ is Hausdorff iff any two distinct eq. classes in X can be separated by saturated open sets in X

$A \subset X$ is called saturated if $A = q^{-1}(q(A))$.

p.f. $V \subset [X]$ open $\quad q^{-1}(V) = q^{-1}(q(q^{-1}(V)))$



$V_1 \cap V_2 = \emptyset \Rightarrow q^{-1}(V_1) \cap q^{-1}(V_2)$

$$\underset{\substack{\cup \\ x_1}}{A_1} = q^{-1}(q(A_1)) \cap q^{-1}(q(A_2)) = \underset{\substack{\cup \\ x_2}}{A_2} \quad = \emptyset$$

In the example 0 doesn't have any saturated (open) nbhd. except $\mathbb{R}^2$

Collapase subspaces

$A \subset (X, \mathcal{O}_X)$

Define equivalence relation

$x \sim y$ if
1) $x = y \notin A$
2) $x, y \in A$

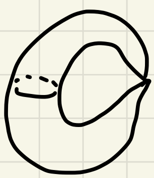
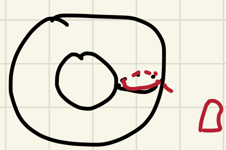
$$X/A := X/\sim$$

Example 1) $X = \overline{D_2} = \{ v \in \mathbb{R}^2 / |v| \leq 1 \}$

homeomorphic. $A = \partial X = \{ v \mid |v| = 1 \}$

$X/A \cong S^2$ as sets (indeed as top spaces)

2) $X = S^2 \quad A = \{p_1, p_2\}$ distinct points

$X/A =$  $= Y/\square \xleftarrow{Y}$  $\square$

3) $Y = \underset{\substack{\cup \\ p}}{S^1} \times S^1 \quad \square = p \times S^1$

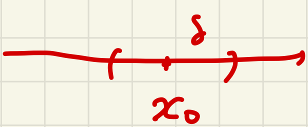
Thm X Hausdorff. X/A is Hausdorff if

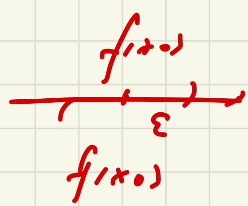
1) A is closed
2) X is T_3, i.e. pts & closed sets can be separated by open sets.

Def'n $f: X \to Y$ $\quad x \in X$

f is continuous at x if $\forall \overset{f(x)}{\overset{\cap}{V}} \in \mathcal{O}_Y \quad f^{-1}(V) \in \mathcal{O}_X.$

Thm $f: (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ is continuous if and only if it's a continuous function in the sense of Calculus.





Same def'n holds for any metric space

$\forall \varepsilon > 0 \ \exists \delta > 0$

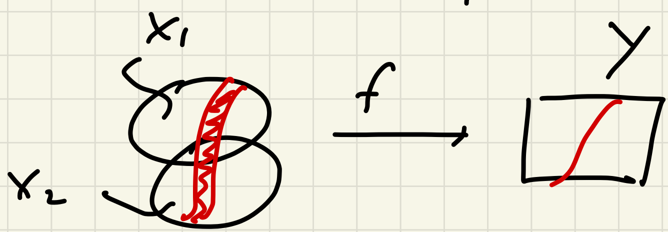
$d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$

Def'n/prop $f: X \to Y$ is continuous $\Longleftrightarrow$ $f^{-1}(V)$ is closed for any closed subset $V \subset Y$

Gluing thm X, Y top space

$X = X_1 \cup X_2$. Suppose X_1, X_2 are open (resp. closed) and $f_i : X_i \to Y$ are cont. s.t. $f_1|_{X_1 \cap X_2} = f_2|_{X_1 \cap X_2}$. Then $f(x) = \begin{cases} f_1 & x \in X_1 \\ f_2 & x \in X_2 \end{cases}$ is a cont. map $: X \to Y$.

p.f.



$V \subset Y$ closed

$$f^{-1}(V) = (f^{-1}(V) \cap X_1) \cup (f^{-1}(V) \cap X_2)$$

closed in X

$$= f_1^{-1}(V) \cup f_2^{-1}(V) = (C_1 \cap X_1) \cup (C_2 \cap X_2)$$

Since f_1 is cont. $f_1^{-1}(V)$ is closed in X_1

$f_1^{-1}(V) = C_1 \cap X_1$ for some closed $C_1 \subset X$

Properties of cont. map

1) [co. Restriction] $f: X \to Y$ cont.

$X \supset A$ subspace, $Y \supset B \supset f(X)$

$f|_A : A \to Y$ is cont.

$f: X \to B$ is cont.

2) [Composition] $X \xrightarrow{f} Y \xrightarrow{g} Z$ cont.

$\Downarrow$

$g \circ f$ cont

3) [Product, quotient]

$f_\alpha \; X \to Y_\alpha$ cont. $\alpha \in \Lambda$

Not true for $\square$

$X \xrightarrow{(f_\alpha)} \prod Y_\alpha$ product top.

is cont

$X \xrightarrow{q} [x]$ quotient top. quotient map is cont.

Def'n $f: X \to Y$ cont. is called a homeomorphism

if f is bijective & f^{-1} is also continuous.

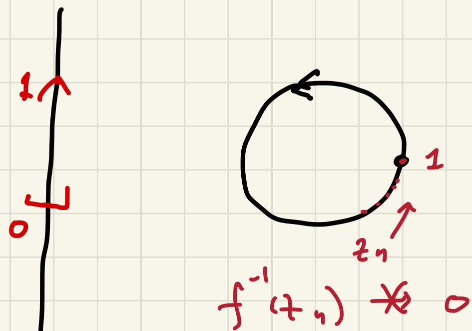
properties that are preserved under homeomorphism

are called topological properties.

Example

1) Cont. bijection does NOT have to be a homeomorphism.

$$f: [0,1) \longrightarrow S^1$$
$$t \longmapsto e^{2\pi i t}$$



$f^{-1}(z) = \frac{1}{2\pi i} \log z$ is not continuous

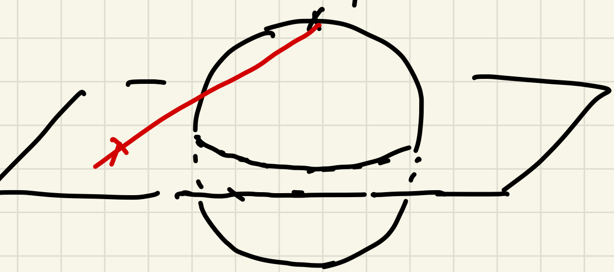
2) The following properties are obviously topological: Hausdorff, compact, connected.

Examples of homeomorphisms

notation for homeo. $\cong$

1) Any open interval in $\mathbb{R}$ $\cong (0,1)$

2) $S^2 \setminus P \cong \mathbb{R}^2$

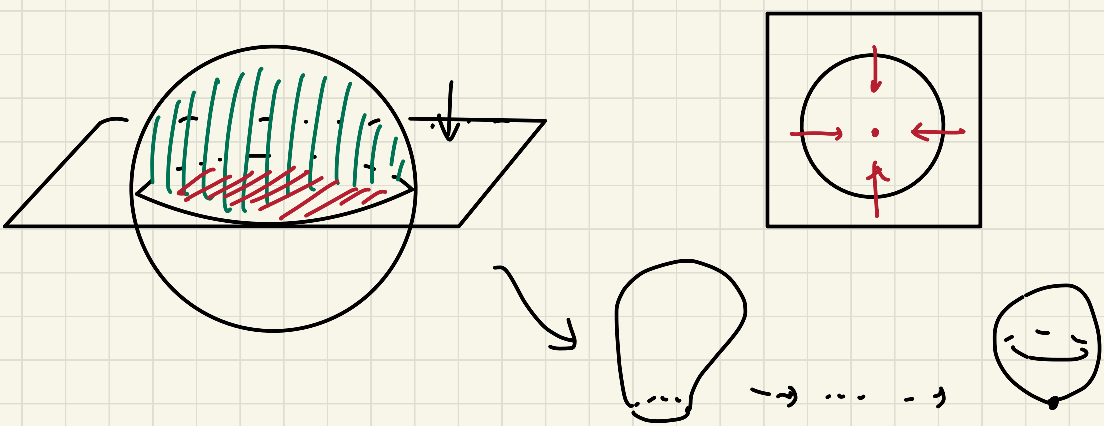


stereographic projection.

Find formula!

3) $X = \overline{D^2} = \{ |z| \leq 1 \}$ $\quad A = \partial D^2$ $\quad S^2 = \{ v \in \mathbb{R}^3 \mid |v| = 1 \}$

$X/A \cong S^2$ homeomorphic



4) $\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{im} z > 0 \}$

$$D^2 = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

Consider the Cayley transform

$$z \longmapsto \frac{z+i}{iz+1} = w$$

$$z + i = (iz+1)w$$

$$z(1-iw) = w - i$$

$$\frac{w-i}{1-iw} \longleftarrow w$$

$$z = x+iy \longmapsto \frac{x+i(y+1)}{(-y+1)+ix} = \frac{\big(x+i(1+y)\big)(1-y-ix)}{(1-y)^2+x^2}$$

$$= \frac{\cancelto{2x}{x(1-y)+(1+y)x} + i(1-y^2-x^2)}{(1-y)^2+x^2}$$

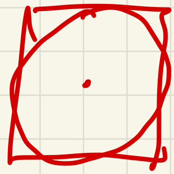
$$|z|^2 = x^2+y^2 < 1$$

$$\Rightarrow \operatorname{im} w = \frac{1-x^2-y^2}{(1-y)^2+x^2} > 0$$

5) L_2-norm and L_∞-norm defines the same top's on $\mathbb{R}^2$

$$\|v\|_2 = \left(\sum v_i^2\right)^{1/2} \qquad \|v\|_\infty = \max_i \{|v_i|\}$$

$$D_2^2 := \{ v \mid \|v\|_2 \leq 1 \}$$



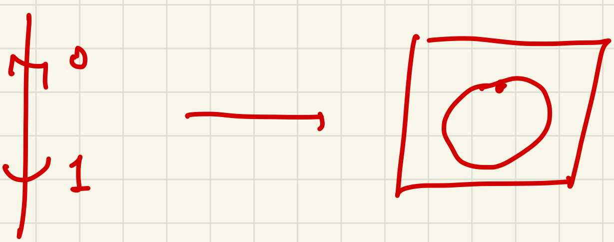
$$D_\infty^2 := \{ v \mid \|v\|_\infty \leq 1 \}$$

Construct a homeomorphism $D_2^2 \cong D_\infty^2$

Def'n $f : X \xrightarrow{\text{cont.}} Y$ is called a top. embed.

if f is inj. $f : X \xrightarrow{\cong} f(X)$

Ex. only inj. is not enough



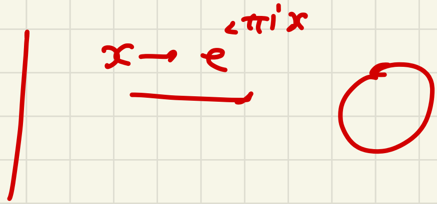
NOT embedding

Def'n $f: X \to Y$ is called **a local homeom.**

if $\forall\, x \in X$ $\exists$ nbhd U of x s.t $f(U) \subset Y$ is open

and. $f|_U : U \xrightarrow{\cong} f(U)$

Ex.



is a local homeomorphism

Thm [Inverse function thm]

$U \overset{\text{open}}{\subset} \mathbb{R}^n$ $\quad F: U \to \mathbb{R}^n$ differentiable (cont.)

$DF: \left[\dfrac{\partial F_j}{\partial x_i} \right]_{\substack{i=1,\dots n \\ j=1,\dots n}}$ has full rk

$\Rightarrow$ F is a local homeomorphism.

Rmk. Local homeom. + bijective

= homeomorphism

Open & Closed Mapping

Def'n A map $f: X \to Y$ is called an not necessarily cont.

open (resp. closed) map if $\forall\, U \overset{\text{open}}{\subset} X$

$f(U)$ is open in Y (resp. $\forall\, V \overset{\text{closed}}{\subset} X$ $f(V) \overset{\text{closed}}{\subset} X$).

Rmk 1) local homeo. is open

$f: X \to Y$ local homeo.

$U \overset{\text{open}}{\subset} X \quad U = \bigcup_\alpha U_\alpha \quad f|_{U_\alpha}: U_\alpha \cong f(U_\alpha)$

$f(\bigcup U_\alpha) = \bigcup f(U_\alpha)$ ✓ but not necessarily closed

$\pi \quad \mathbb{R}^2 \longrightarrow S^1 \times S^1$

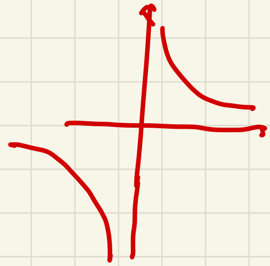
$(x, y) \longmapsto (e^{2\pi i x}, e^{2\pi i y})$

$\lambda \in \mathbb{R} \setminus \mathbb{Q}$

$V = \{ y = \lambda x \} \longrightarrow \pi(V)$ is not closed

2) $\mathbb{R}^2 \to \mathbb{R}$ is open but not closed.

$(x, y) \mapsto x$



3) $U \overset{\text{open}}{\subset} \mathbb{C}$ $f: U \to \mathbb{C}$

non constant holo. function.

$f(U)$ is open in $\mathbb{C}$ (use the fact that non const. holo. function has discrete zeros.)

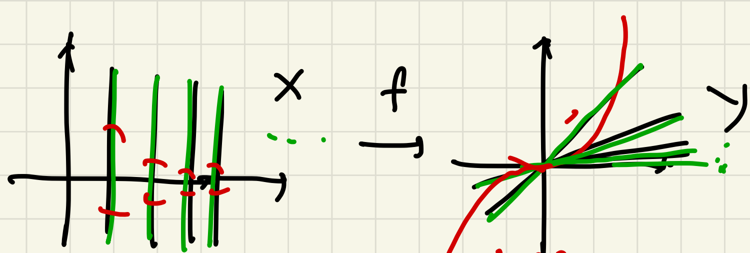
Suppose $f: X \twoheadrightarrow Y$ is a cont. surjection

$x_1 \sim x_2$ if $f(x_1) = f(x_2)$ in Y

then $X/\sim \; \overset{\text{bij.}}{\longleftrightarrow} \; Y$ if is homeo. we call it quotient map!

this is NOT necessarily a homeo.

$y = x^3$



$f^{-1}(A)$ is open A is NOT! $A = \{ y > x^3, x \geq 0 ;\ y < x^3, x \leq 0 \} \cap Y$