20240927 MATH3541 NOTE 6[1]

Author: Be $\sqrt{-1}$maginative, and nothing will be $\frac{\mathrm{d}}{\mathrm{d}x}$ifficult!
Email: u3612704@connect.hku.hk;
Phone: +852 5693 2134; +86 19921823546;

Contents

1 Introduction

Connectedness is harder than compactness in sense that a globally connected set may not be locally connected. In light of this, this note states some important properties of connected and path connected spaces. In addition, as coproduct space is always disconnected, adjunction space is introduced to connect different pieces.

2 Connected Topological Space

2.1 Definition and Criterion

Definition 2.1. (**Connected Topological Space**)
Let X be a topological space.
If all open partition of X is trivial, then X is connected.

Proposition 2.2. Let X be a topological space.
The following three statements are logically equivalent:
(1) X is connected;
(2) $\forall U \subseteq X, U$ is clopen $\implies U = \emptyset$ or $U = X$;
(3) $\forall$ partition $\{U_1, U_2\}$ of $X, \overline{U}_1 \cap U_2 \neq \emptyset$ or $U_1 \cap \overline{U}_2 \neq \emptyset$.

Proof. We may divide our proof into three parts.
(1) $\implies$ (2) : Assume to the contrary that for some clopen $U \subseteq X$, $U \neq \emptyset$ and $U \neq X$. This implies X has a nontrivial open partition $\{U, U^c\}$, so X is not connected.
(2) $\implies$ (3) : Assume to the contrary that for some nonempty $U_1, U_2 \subseteq X$,
$U_1 \cup U_2 = X$ and $\overline{U}_1 \cap U_2 = \emptyset$ and $U_1 \cap \overline{U}_2 = \emptyset$.

$$U_1 \cup U_2 = X \text{ and } \overline{U}_1 \cap U_2 = \emptyset \implies U_1^c \subseteq U_2 \subseteq \overline{U}_1^c \implies U_1^c = U_2 = \overline{U}_1^c \in \mathcal{O}_X$$
$$U_1 \cup U_2 = X \text{ and } U_1 \cap \overline{U}_2 = \emptyset \implies U_2^c \subseteq U_1 \subseteq \overline{U}_2^c \implies U_2^c = U_1 = \overline{U}_2^c \in \mathcal{O}_X$$

Hence, some partition $\{U_1, U_2\}$ of X satisfies $\overline{U}_1 \cap U_2 = \emptyset$ and $U_1 \cap \overline{U}_2 = \emptyset$.
(3) $\implies$ (1) : Assume to the contrary that X is not connected, then X has a nontrivial open partition $\{U_1, U_2\}$. As U_1, U_2 are clopen, $\overline{U}_1 \cap U_2 = \emptyset$ and $U_1 \cap \overline{U}_2 = \emptyset$.
Combine the three parts above, we've proven the logical equivalency.
Quod. Erat. Demonstrandum. □

2.2 Constructions

Remark: *Quotient space inherits connectedness.*

Proposition 2.3. Let X, Y be two topological spaces,
and $\sigma : X \to Y$ be a continuous surjection.
If X is connected, then Y is connected.

Proof. For all open partition $\mathcal{V}$ of Y, $\sigma^{-1}(\mathcal{V})$ is an open partition of X.
Since the saturated open partition $\sigma^{-1}(\mathcal{V})$ is trivial, $\mathcal{V} = \sigma(\sigma^{-1}(\mathcal{V}))$ is also trivial.
Hence, Y is connected. Quod. Erat. Demonstrandum. □

Proposition 2.4. Let X be a topological space, and $\widetilde{X}$ be a quotient space of X. If X is connected, then $\widetilde{X}$ is connected.

Proof. Notice that $\pi : X \to \widetilde{X}, \pi(x) = \widetilde{x}$ is a continuous surjection.
Hence, $\widetilde{X}$ is connected. Quod. Erat. Demonstrandum. □

Definition 2.5. (**Dense Set**)
Let X be a topological space, and X' be a subset of X.
If $\forall U \in \mathcal{O}_X, U \cap X' = \emptyset \implies U = \emptyset$, then X' is dense in X.

Proposition 2.6. Let X be a topological space.
X is connected iff X has a dense connected subspace.

Proof. It suffices to prove the **"if" direction**.
Assume that X has a dense connected subspace X'.
For all $U \subseteq X$:

$$\begin{aligned} U \text{ is clopen in } X &\implies U \cap X' \text{ is clopen in } X' \\ &\implies U \cap X' = \emptyset \text{ or } U \cap X' = X' \\ &\implies U = \emptyset \text{ or } X' \subseteq U \\ &\implies U = \emptyset \text{ or } U = X \end{aligned}$$

Hence, X is connected. Quod. Erat. Demonstrandum. □

Remark: *Coproduct space is not connected.*

Proposition 2.7. Let X_1, X_2 be two topological spaces.
The coproduct space $X = X_1 \sqcup X_2$ of X_1, X_2 is not connected.

Proof. As X has a nontrivial open partition $\{X_1 \times \{1\}, X_2 \times \{2\}\}$,
X is not connected. Quod. Erat. Demonstrandum. □

Remark: *If connected subspaces intersect, then their union is connected.*

Proposition 2.8. Let X be a topological space,
and $(X_\lambda)_{\lambda\in I}$ be an indexed family of subspaces of X. If:
(1) Each X_λ is connected.
(2) $\bigcap_{\lambda\in I} X_\lambda \neq \emptyset$.
Then $\bigcup_{\lambda\in I} X_\lambda$ is connected.

Proof. For all open partition $\mathcal{U}$ of $\bigcup_{\lambda\in I} X_\lambda$, each X_λ is contained in a unique $U_\lambda \in \mathcal{U}$.
Fix $\xi \in \bigcap_{\lambda\in I} U_\lambda \neq \emptyset$. As ξ belongs to a unique $U \in \mathcal{U}$, $\mathcal{U} = \{U\}$ is trivial.
Quod. Erat. Demonstrandum. □

Proposition 2.9. Let X be a topological space,
and $(X_\lambda)_{\lambda\in I}$ be an indexed family of subspaces of X. If:
(1) Each X_λ is connected.
(2) For all $\mu, \nu \in I$, there exists $(\lambda_k)_{k=0}^m$ in I, such that:

$$\lambda_0 = \mu \text{ and } \lambda_m = \nu \text{ and each } X_{\lambda_k} \cap X_{\lambda_{k+1}} \neq \emptyset$$

Then $\bigcup_{\lambda\in I} X'_\lambda$ is connected.

Proof. Fix a kernel X_μ. For all $\nu \in I$, define $Y_\nu = \bigcup_{k=0}^m X_{\lambda_k}$.
Now we get an indexed family of subspaces $(Y_\nu)_{\nu\in I}$ of X, such that:
(1) Each Y_ν is connected.
(2) $\bigcap_{\nu\in I} Y_\nu \neq \emptyset$.
As $\bigcup_{\lambda\in I} X_\lambda = \bigcup_{\nu\in I} Y_\nu$, $\bigcup_{\lambda\in I} X'_\lambda$ is connected.
Quod. Erat. Demonstrandum. □

Remark: *Product space inherits connectedness.*

Lemma 2.10. Let X_1, X_2 be two topological spaces,
and X be the product space of X_1, X_2.
If X_1, X_2 are connected, then X is connected.[2]

Proof. It suffices to notice the following identity:

$$X_1 \times X_2 = \bigcup_{(x_1,x_2)\in X_1\times X_2} (X_1 \times \{x_2\} \cup \{x_1\} \times X_2)$$

Quod. Erat. Demonstrandum. □

Lemma 2.11. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces, X be the product space of $(X_\lambda)_{\lambda\in I}$, and x be an element of X.
For all $J \subseteq I$, define $X_J = \{x' \in X : \forall\lambda \in I\backslash J, x'(\lambda) = x(\lambda)\}$.
$X' = \bigcup_{|J|<+\infty} X_J$ is dense in X.[2]

Proof. Assume to the contrary that $\exists U \in \mathcal{O}_X, U \cap X' = \emptyset$ and $U \neq \emptyset$.
WLOG, assume that $U = \bigcap_{k=1}^m \pi_{\lambda_k}^{-1}(U_{\lambda_k})$, where each $U_{\lambda_k} \in \mathcal{O}_{X_{\lambda_k}}$ is nonempty.
Step 1: Fix $u \in U$, and construct $x' \in X$ by:

$$x'(\lambda) = \begin{cases} u(\lambda_k) & \text{if} \quad \lambda \text{ is equal to some } \lambda_k; \\ x(\lambda) & \text{if} \quad \lambda \text{ is equal to no } \lambda_k; \end{cases}$$

Step 2: State some key properties of x'.
Property 2.1: Each $x'(\lambda_k) = u(\lambda_k) \in U_{\lambda_k}$.
Property 2.2: $J = \{\lambda_k\}_{k=1}^m$ is finite and $\forall\lambda \in I\backslash J, x'(\lambda) = x(\lambda)$.
$x' \in \bigcap_{k=1}^m \pi_{\lambda_k}^{-1}(U_{\lambda_k})$ and $x' \in X_J \subseteq X'$, which contradicts with $U \cap X' = \emptyset$.
Hence, our assumption is false, and we've proven that X' is dense in X.
Quod. Erat. Demonstrandum. □

Proposition 2.12. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces, and X be the product space of $(X_\lambda)_{\lambda\in I}$.
If each X_λ is connected, then X is connected.[2]

Proof. The $X' = \bigcup_{|J|<+\infty} X_J$ constructed in **Lemma 2.11.** satisfies:

$$\bigcap_{|J|<+\infty} X_J = \{x\} \neq \emptyset$$

According to **Proposition 2.8.**, X' is connected.
According to **Proposition 2.6.**, the existence of a dense connected subspace X' implies X is connected. Quod. Erat. Demonstrandum. □

2.3 Connected Component

Definition 2.13. (**Connected Component**)
Let X be a topological space, and x be an element of X. Define the union of all connected subset containing x as the connected component of x in X.

Proposition 2.14. Let X be a topological space.
The set of all connected components $\{X_x\}_{x\in X}$ in X partitions X.

Proof. We may divide our proof into three parts.
Part 1: For all X_x, $\{x\}$ is connected implies $X_x \neq \emptyset$.

Part 2: For all X_{x_1}, X_{x_2}:

$$\begin{aligned} X_{x_1} \cap X_{x_2} \neq \emptyset &\implies X_{x_1} \cup X_{x_2} \text{ is connected} \\ &\implies X_{x_1} \cup X_{x_2} \subseteq X_{x_1} \text{ and } X_{x_1} \cup X_{x_2} \subseteq X_{x_2} \implies X_{x_1} = X_{x_2} \end{aligned}$$

Part 3: For all $x \in X$, there exists X_x, such that $x \in X_x$.
Hence, $\{X_x\}_{x \in X}$ partitions X. Quod. Erat. Demonstrandum. □

2.4 Connectedness as a Topological Invariant

Proposition 2.15. Connectedness is a topological invariant.

Proof. For all X, Y, assume that there exists a homeomorphism $\sigma : X \to Y$.
As σ is surjective and continuous, X is connected implies Y is connected.
As σ^{-1} is surjective and continuous, Y is connected implies X is connected.
Hence, we've proven that connectedness is a topological invariant.
Quod. Erat. Demonstrandum. □

Proposition 2.16. Let X be a topological space.
Each connected component X_x in X is closed in X.

Proof. According to **Proposition 2.6.**, $\overline{X}_x$ is also a connected set containing x, so:

$$\overline{X}_x \subseteq X_x \implies X_x = \overline{X}_x \in \mathcal{C}_x$$

Quod. Erat. Demonstrandum. □

Proposition 2.17. Let X be a topological space.
If X has finitely many connected components,
then each connected component X_x in X is open in X.

Proof. According to **Proposition 2.16.**, X_x^c is a finite union of closed sets, so $X_x^c \in \mathcal{C}_X$ and $X_x \in \mathcal{O}_X$. Quod. Erat. Demonstrandum. □

Proposition 2.18. Let X be a topological space.
If X has a connected basis $\mathcal{B}_X$,
then each connected component X_x in X is open in X.

Proof. We may divide our proof into four steps.
Step 1: $X \in \mathcal{O}_X \implies \exists (U_\lambda)_{\lambda \in I}$ in $\mathcal{B}_X, X = \bigcup_{\lambda \in I} U_\lambda$.
Step 2: X_x is a connected component $\implies \forall \lambda \in I, U_\lambda \subseteq X_x$ or $U_\lambda \cap X_x = \emptyset$.
Step 3: For each $x \in X$, define $I_x = \{\lambda \in I : U_\lambda \subseteq X_x\}$, $\{I_x\}_{x \in X}$ partitions I.
Step 4: Each $X_x = \bigcup_{\lambda \in I_x} U_\lambda \in \mathcal{O}_X$.
Quod. Erat. Demonstrandum. □

Proposition 2.19. Let X be a topological space.
If X has a connected basis $\mathcal{B}_X$, then X is homeomorphic to the coproduct space of its connected component subspaces $(X_\lambda)_{\lambda\in I}$.

Proof. We may divide our proof into two parts.
Part 1: Assume that an arbitrary set U is open in X.
Each $U_\lambda = U \cap X_\lambda$ is open in X_λ.
Hence, $V = \bigcup_\lambda (U_\lambda \times \{\lambda\})$ is open in the coproduct space.
Part 2: Assume that an arbitrary set V is open in the coproduct space.
Each $U_\lambda = \{x \in X_\lambda : (x, \lambda) \in V\}$ is open in X_λ and X.
Hence, $U = \bigcup_{\lambda\in I} U_\lambda$ is open in X.
Quod. Erat. Demonstrandum. □

Remark: *We can always construct a coproduct space out of a family of spaces. However, it is not so trivial to partition a space into the disjoint union of other spaces. For example, the metric space $\mathbb{R}$ is not homeomorphic to $(-\infty, 0) \sqcup [0, +\infty)$, because $[0, 1)$ is not open in $\mathbb{R}$ and $[0, 1) \times \{2\}$ is open in $(-\infty, 0) \sqcup [0, +\infty)$.* ***Proposition 2.18.*** *and* ***Proposition 2.19.*** *suggest that doing such partition is always valid in $\mathbb{R}^n$, so for a differentiable function $f : \Omega \to \mathbb{R}$, if its differential $Df : \Omega \to \mathbb{R}^n$ is given, then we partition its open domain Ω into the disjoint union of connected components $\bigsqcup_{k=1}^{+\infty} \Omega_k$, and add a constant C_k for each Ω_k.*

Definition 2.20. (**Locally Connected Topological Space**)
Let X be a topological space, and x be an element of X.
If every open neighbour U of x contains a connected open neighbour $\mathfrak{U}$ of x, then X is locally connected at x.

Remark: *This is the "correct" definition of local connectedness. We cannot remove the quantifier $\forall$ before U, because the intersection of connected sets may not be connected. It is this key difference that makes connectedness harder to understand.*

Proposition 2.21. Surjective local homeomorphism $\sigma : X \to Y$ preserves local connectedness.

Proof. For all $y \in Y$, for all open neighbour V of y,
we wish to find a connected open neighbour $\mathfrak{V} \subseteq V$ of y.
(1) As σ is surjective, there exists $x \in X$, such that $y = \sigma(x)$.
(2) As σ is a local homeomorphism, there exists an open neighbour U of x, such that $\sigma(U)$ is open in Y, and the restricted map $\sigma|_U : U \to \sigma(U)$ is a homeomorphism.
(3) As X is locally connected, $U \cap \sigma^{-1}(V)$ contains a connected open neighbour $\mathfrak{U}$ of x.
(4) As $\sigma|_U$ is a homeomorphism, V contains a connected open neighbour $\sigma(\mathfrak{U})$ of y.
To conclude, Y is locally connected. Quod. Erat. Demonstrandum. □

Remark: *However, a continuous function doesn't necessarily preserve local connectedness at every point. Construct a continuous function* $\boldsymbol{\gamma} : (0, 6] \to \mathbb{R}^2$ *by:*

$$\boldsymbol{\gamma}(x) = \begin{cases} (x, \sin\frac{\pi}{x}) & \text{if} \quad x \in (0, 2]; \\ (4 - x, 1) & \text{if} \quad x \in [2, 4]; \\ (0, 5 - x) & \text{if} \quad x \in [4, 6]; \end{cases}$$

(1) The domain $(0, 6]$ *of* $\boldsymbol{\gamma}$ *is locally connected at every point.*
(2) The image $\boldsymbol{\gamma}((0, 6])$ *of* $\boldsymbol{\gamma}$ *is not locally connected at* $(0, 0)$*.*

2.5 Examples

Definition 2.22. (**Convex Set**)
Let V be a normed vector space over field $\mathbb{R}$, and Ω be a nonempty subset of V.
If $\forall$ distinct $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$,
$\mathbf{x}_1 \in \Omega$ and $\mathbf{x}_2 \in \Omega \implies \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 \in \Omega$, then Ω is convex.

Proposition 2.23. In a normed vector space V over field $\mathbb{R}$,
every convex subset Ω of V is connected.

Proof. Assume to the contrary that Ω has a nontrivial open partition $\{L, R\}$.
Step 1: Construct $(\mathbf{l}_k)_{k\in\mathbb{N}}$ in L and $(\mathbf{r}_k)_{k\in\mathbb{N}}$ in R.
Fix $\mathbf{l}_1 \in L\backslash R$ and $\mathbf{r}_1 \in R\backslash L$.
$\ell : (1 - t)\mathbf{l}_1 + t\mathbf{r}_1 (t \in \mathbb{R})$ is homeomorphic to $\mathbb{R}$,
so ℓ inherits the completeness and ordering of $\mathbb{R}$.
WLOG, assume that $\mathbf{l}_1 < \mathbf{r}_1$.
For each $k \in \mathbb{N}$, assume that $\mathbf{l}_k < \mathbf{r}_k$ are well-defined.
Define $\mathbf{m}_k = \frac{\mathbf{l}_k + \mathbf{r}_k}{2}$.
Case 1.1: If $\mathbf{m}_k \in L$, then define $\mathbf{l}_{k+1} = \mathbf{m}_k$ and $\mathbf{r}_{k+1} = \mathbf{r}_k$;
Case 1.2: If $\mathbf{m}_k \in R$, then define $\mathbf{l}_{k+1} = \mathbf{l}_k$ and $\mathbf{r}_{k+1} = \mathbf{m}_k$.
Step 2: State some key properties of $(\mathbf{l}_k)_{k\in\mathbb{N}}, (\mathbf{r}_k)_{k\in\mathbb{N}}$.
Property 2.1: $(\mathbf{l}_k)_{k\in\mathbb{N}}$ is increasing with upper bound $\mathbf{r}_1$;
Property 2.2: $(\mathbf{r}_k)_{k\in\mathbb{N}}$ is decreasing with lower bound $\mathbf{l}_1$;
Property 2.3: $\lim\limits_{k\to+\infty} \|\mathbf{r}_k - \mathbf{l}_k\| = 0$.
Hence, $(\mathbf{l}_k)_{k\in\mathbb{N}}, (\mathbf{r}_k)_{k\in\mathbb{N}}$ have the same limit $\boldsymbol{\xi} \in \ell$.
For all open neighbour U of $\boldsymbol{\xi}$, $L \cap U \neq \emptyset$ and $R \cap U \neq \emptyset$,
so $\boldsymbol{\xi} \notin L$ and $\boldsymbol{\xi} \notin R$, which contradicts to $\{L, R\}$ is a partition of Ω.
To conclude, our assumption is false, and we've proven that Ω is connected.
Quod. Erat. Demonstrandum. □

Proposition 2.24. In $\mathbb{R}$, every nonempty connected subset Ω is convex.

Proof. Assume to the contrary that for some $l < m < r$, $l \in \Omega$ and $r \in \Omega$ and $m \notin \Omega$. Now Ω has a nontrivial open partition $\{(-\infty, m) \cap \Omega, (m, +\infty) \cap \Omega\}$, so Ω is not connected. Quod. Erat. Demonstrandum. $\square$

Proposition 2.25. In a normed vector space V over field $\mathbb{R}$,
an arbitrary intersection $\bigcap_{\lambda \in I} \Omega_\lambda$ of convex subsets $(\Omega_\lambda)_{\lambda \in I}$ is convex.

Proof. For all distinct $\mathbf{x}_1, \mathbf{x}_2 \in V$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$:

$$\begin{aligned}\mathbf{x}_1 \in \bigcap_{\lambda \in I} \Omega_\lambda \text{ and } \mathbf{x}_2 \in \bigcap_{\lambda \in I} \Omega_\lambda &\implies \mathbf{x}_1 \text{ is in each } \Omega_\lambda \text{ and } \mathbf{x}_2 \text{ is in each } \Omega_\lambda \\ &\implies \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 \text{ is in each } \Omega_\lambda \implies \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 \in \bigcap_{\lambda \in I} \Omega_\lambda\end{aligned}$$

Hence, $\bigcap_{\lambda \in I} \Omega_\lambda$ is convex. Quod. Erat. Demonstrandum. $\square$

Remark: *However, it is not true that the intersection of connected sets is connected.*

Proposition 2.26. In a normed vector space V over field $\mathbb{R}$,
every convex subset Ω of V is locally connected.

Proof. For all $\mathbf{x} \in \Omega$, for all open neighbour U of $\mathbf{x}$ in Ω,
for some open neighbour U' of $\mathbf{x}$ in V, $U = \Omega \cap U'$.
As U' is open in V , there exists $r > 0$, such that $B(\mathbf{x}, r) \subseteq U'$.
As $B(\mathbf{x}, r) \cap U'$ is convex, U contains a connected open neighbour $B(\mathbf{x}, r) \cap U'$ of $\mathbf{x}$,
so Ω is locally connected. Quod. Erat. Demonstrandum. $\square$

Definition 2.27. (**Convex Function**)
Let V be a normed vector space over field $\mathbb{R}$,
Ω be a convex subset of V, and $f : \Omega \to \mathbb{R}$ be a function.
If $\forall$ distinct $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$,
$\lambda_1 f(\mathbf{x}_1) + \lambda_2 f(\mathbf{x}_2) \geq f(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2)$, then f is convex.

Proposition 2.28. In a normed vector space V over field $\mathbb{R}$, for all convex function $f : \Omega \to \mathbb{R}$ and $\beta \in \mathbb{R}$, the solution set of $f(\mathbf{x}) < \beta$ is convex.

Proof. For all distinct $\mathbf{x}_1, \mathbf{x}_2 \in V$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$:

$$f(\mathbf{x}_1) < \beta \text{ and } f(\mathbf{x}_2) < \beta \implies f(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) \leq \lambda_1 f(\mathbf{x}_1) + \lambda_2 f(\mathbf{x}_2) < \beta$$

Hence, the solution set of $f(\mathbf{x}) < \beta$ is convex. Quod. Erat. Demonstrandum. $\square$

Proposition 2.29. In $\mathbb{R}$, $\mathbb{S} \cong \mathbb{R}/\mathbb{Z}$ is connected.

Proposition 2.30. In $\mathbb{R}$, every connected component of $\mathbb{Q}$ is a singleton.

Proof. Assume to the contrary that $\mathbb{Q}$ has a connected component U with at least two elements r_1, r_2. WLOG, assume that $r_1 < r_2$. Choose an irrational number $s \in (r_1, r_2)$, then U has a nontrivial open partition $\{U \cap (-\infty, s), U \cap (s, +\infty)\}$, a contradiction. Hence, our assumption is false, and we've proven that every connected component is a singleton. Quod. Erat. Demonstrandum. $\square$

Remark: *Connected components are not necessarily open when the space is infinite.*

Proposition 2.31. In a metric space X, $|X| > 1 \implies |X| > \aleph_0$.

Proof. Fix $x_0 \in X$, and project X onto $\mathbb{R}$ by the following map:

$$\sigma : X \to \mathbb{R}, \sigma(x) = d_X(x_0, x)$$

The positive definiteness of d_X implies $|\sigma(X)| > 1$.
The continuity of d_X implies σ is connected.
Proposition 2.24. suggests that $\sigma(X)$ is an interval, so $|X| \geq |\sigma(X)| > \aleph_0$.
Quod. Erat. Demonstrandum. $\square$

Proposition 2.32.

$$\int \frac{1}{1-x^2} \mathrm{d}x = \begin{cases} \text{arcoth } x + C_1 & \text{if} \quad x \in (-\infty, -1); \\ \text{artanh } x + C_2 & \text{if} \quad x \in (-1, +1); \\ \text{arcoth } x + C_3 & \text{if} \quad x \in (+1, +\infty); \end{cases}$$

Proof. We may divide our proof into three steps.
Step 1: Define $f : \Omega \to \mathbb{R}$ and find the natural domain Ω of f.

$$1 - x^2 \neq 0 \iff x \neq \pm 1 \iff x \in \Omega = \mathbb{R} \backslash \{\pm 1\}$$

Step 2: As Ω is open in $\mathbb{R}$, it is homeomorphic to the disjoint union of its connected component subspaces, that is:

$$\Omega \cong \overbrace{(-\infty, -1)}^{\Omega_1} \sqcup \overbrace{(-1, +1)}^{\Omega_2} \sqcup \overbrace{(+1, +\infty)}^{\Omega_3}$$

Step 3: Find an antiderivative for each $f|_{\Omega_k} : \Omega_k \to \mathbb{R}$, $f|_{\Omega_k}(x) = f(x)$.

$$\begin{array}{rclclcl} (\text{arcoth } x)' & = & \frac{1}{\coth'(\text{arcoth } x)} & = & \frac{1}{-\text{csch}^2(\text{arcoth } x)} & = & f|_{\Omega_1}(x) \text{ if } x \in \Omega_1 \\ (\text{artanh } x)' & = & \frac{1}{\tanh'(\text{artanh } x)} & = & \frac{1}{+\text{sech}^2(\text{artanh } x)} & = & f|_{\Omega_2}(x) \text{ if } x \in \Omega_2 \\ (\text{arcoth } x)' & = & \frac{1}{\coth'(\text{arcoth } x)} & = & \frac{1}{-\text{csch}^2(\text{arcoth } x)} & = & f|_{\Omega_3}(x) \text{ if } x \in \Omega_3 \end{array}$$

Step 4: Add a constant for each Ω_k.

$$\int \frac{1}{1-x^2}\mathrm{d}x = \begin{cases} \operatorname{arcoth} x + C_1 & \text{if} \quad x \in (-\infty,-1); \\ \operatorname{artanh} x + C_2 & \text{if} \quad x \in (-1,+1); \\ \operatorname{arcoth} x + C_3 & \text{if} \quad x \in (+1,+\infty); \end{cases}$$

Quod. Erat. Demonstrandum. □

3 Path Connected Topological Space

3.1 Definition and Criterion

Definition 3.1. (**Path**)
Let X be a topological space. If $\gamma : [0,1] \to X$ is a continuous, then γ is a path.

Definition 3.2. (**Path Connected Topological Space**)
Let X be a topological space. If for all $x_1, x_2 \in X$, there exists a path γ in X, such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$, then X is path connected.

Remark: *We shall not assume that γ is injective, otherwise the set of all path connected components will not partition the whole space. For example, in the adjunction space $[\mathbb{R} \sqcup \mathbb{R}] = \{\{(0,1)\},\{(0,2)\},\{(x,1),(x,2)\}\}_{x\in\mathbb{R}\setminus\{0\}}$, the subspace $\{\{(0,1)\},\{(0,2)\}\}$ is not connected, so any path from $\{(0,1)\}$ to $\{(0,2)\}$ in $[\mathbb{R} \sqcup \mathbb{R}]$ fails to be injective.*

Proposition 3.3. Let X be a topological space.
If X is path connected, then X is connected.

Proof. Assume to the contrary that X is not connected.
There exists a nontrivial open partition $\{U_1, U_2\}$ of X.
There exist $x_1 \in U_1$ and $x_2 \in U_2$, such that there is no path from x_1 to x_2 in X.
Hence, X is not path connected. Quod. Erat. Demonstrandum. □

Remark: *In order to "upgrade" connectedness to path connectedness, the concept of local path connectedness should be introduced.*

3.2 Constructions

Remark: *Quotient space inherits path connectedness.*

Proposition 3.4. Let X be a topological space,
and $\sigma : X \to Y$ be a continuous surjection.
If X is path connected, then Y is path connected.

Proof. For all $y_1, y_2 \in Y$, as σ is surjective, corresponding preimages x_1, x_2 exist. There exists a path γ from x_1 to x_2, so there exists a path $\sigma \circ \gamma$ from $y_1 = \sigma(x_1)$ to $y_2 = \sigma(x_2)$. Hence, Y is path connected. Quod. Erat. Demonstrandum. $\square$

Proposition 3.5. Let X be a topological space,
and $\widetilde{X}$ be a quotient space of X.
If X is path connected, then $\widetilde{X}$ is path connected.

Proof. Notice that $\pi : X \to \widetilde{X}, \pi(x) = [\widetilde{x}$ is a continuous surjection.
Hence, $\widetilde{X}$ is path connected. Quod. Erat. Demonstrandum. $\square$

Remark: *The path connectedness of the whole space is unrelated with the existence of a dense path connected subspace. We will illustrate this later by topologist's sine curve.*

Remark: *Coproduct space is not path connected.*

Proposition 3.6. Let X_1, X_2 be two topological spaces. The coproduct space $X = X_1 \sqcup X_2$ of X_1, X_2 is not connected, thus not path connected.

Remark: *If path connected subspaces intersect, then their union is path connected.*

Definition 3.7. (**Concatenation**)
Let X be a topological space, $x_0, x_1, \cdots, x_n$ be a sequence of points,
$0 = c_0 < c_1 < \cdots < c_n = 1$ be a partition of $[0,1]$,
and $\gamma_0, \gamma_1, \cdots, \gamma_{n-1} : [0,1] \to X$ be a sequence of paths satisfying:

$$x_0 = \gamma_0(0), \gamma_0(1) = x_1 = \gamma_1(0), \cdots, \gamma_{n-1}(1) = x_n$$

Define the following path $\gamma = \gamma_0 \star_{c_1} \gamma_1 \star_{c_2} \cdots \star_{c_{n-1}} \gamma_{n-1} : [0,1] \to X$
as the concatenation of $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ at $c_0, c_1, \cdots, c_n$:

$$\gamma(t) = \begin{cases} \gamma_0(\frac{t-c_0}{c_1-c_0}) & \text{if} \quad c_0 \le t \le c_1; \\ \gamma_1(\frac{t-c_1}{c_2-c_1}) & \text{if} \quad c_1 \le t \le c_2; \\ \vdots & \qquad \vdots \\ \gamma_{n-1}(\frac{t-c_{n-1}}{c_n-c_{n-1}}) & \text{if} \quad c_{n-1} \le t \le c_n; \end{cases}$$

Definition 3.8. (**Identity Path**)
Let X be a topological space.
For all $x_0 \in X$, define $e_{x_0} : [0,1] \to X, e_{x_0}(t) = x_0$ as the identity path at x_0.

Definition 3.9. (**Inverse Path**)
Let X be a topologcial space, and γ be a path in X.
Define $\gamma^{-1} : [0, 1] \to X, \gamma^{-1}(t) = \gamma(1 - t)$ as the inverse of γ.

Remark:

Proposition 3.10. Let X be a topological space,
and $(X_\lambda)_{\lambda \in I}$ be an indexed family of subspaces of X. If:
(1) Each X_λ is path connected.
(2) $\bigcap_{\lambda \in I} X_\lambda \neq \emptyset$.
Then $\bigcup_{\lambda \in I} X_\lambda$ is path connected.

Proof. Fix $x \in \bigcap_{\lambda \in I} X_\lambda$. For all $x_0, x_1 \in \bigcup_{\lambda \in I} X_\lambda$, x_0 is in some X_{λ_0} and x_1 is in some X_{λ_1}. There exist a path γ_0 from x_0 to x in X_{λ_0} and a path γ_1 from x to x_1 in X_{λ_1}. Therefore, there exists a path $\gamma_1\gamma_0$ from x_0 to x_1 in $\bigcup_{\lambda \in I} X_\lambda$. Hence, $\bigcup_{\lambda \in I} X_\lambda$ is path connected. Quod. Erat. Demonstrandum. □

Proposition 3.11. Let X be a topological space,
and $(X_\lambda)_{\lambda \in I}$ be an indexed family of subspaces of X. If:
(1) Each X_λ is path connected;
(2) For all $\mu, \nu \in I$, there exists $(\lambda_k)_{k=0}^m$ in I, such that:

$$\lambda_0 = \mu \text{ and } \lambda_m = \nu \text{ and each } X_{\lambda_k} \cap X_{\lambda_{k+1}} \neq \emptyset$$

then $\bigcup_{\lambda \in I} X_\lambda$ is path connected.

Proof. Fix a kernel X_μ. For all $\nu \in I$, define $Y_\nu = \bigcup_{k=0}^m Y_{\lambda_k}$.
Now we get an indexed family of subspaces $(Y_\nu)_{\nu \in I}$ of X, such that:
(1) Each Y_ν is path connected;
(2) $\bigcap_{\nu \in I} Y_\nu \neq \emptyset$.
As $\bigcup_{\lambda \in I} X_\lambda = \bigcup_{\nu \in I} Y_\nu$, we may conclude that $\bigcup_{\lambda \in I} X_\lambda$ is path connected.
Quod. Erat. Demonstrandum. □

Remark: *Product space inherits path connectedness.*

Proposition 3.12. Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces,
and X be the product space of $(X_\lambda)_{\lambda \in I}$.
If each X_λ is path connected, then X is path connected.

Proof. For all $x_0, x_1 \in X$, each $x_0(\lambda), x_1(\lambda)$ are connected by a path γ_λ.
Now the product path (γ_λ) connected x_0, x_1 in X, so X is path connected.
Quod. Erat. Demonstrandum. □

3.3 Path Connected Component

Definition 3.13. (**Path Connected Component**)
Let X be a topological space, and x be an element of X.
Define the union of all path connected subset containing x
as the path connected component of x in X

Proposition 3.14. Let X be a topological space.
The set of all path connected components $\{X_x\}_{x\in X}$ in X partitions X.

Proof. We may divide our proof into three parts.
Part 1: For all X_x, $\{x\}$ is path connected implies $X_x \neq \emptyset$;
Part 2: For all X_{x_1}, X_{x_2}:

$$\begin{aligned} X_{x_1} \cap X_{x_2} \neq \emptyset &\implies X_{x_1} \cup X_{x_2} \text{ is path connected} \\ &\implies X_{x_1} \cup X_{x_2} \subseteq X_{x_1} \text{ and } X_{x_1} \cup X_{x_2} \subseteq X_{x_2} \implies X_{x_1} = X_{x_2} \end{aligned}$$

Part 3: For all $x \in X$, there exists X_x, such that $x \in X_x$.
Hence, $\{X_x\}_{x\in X}$ partitions X. Quod. Erat. Demonstrandum. □

3.4 Path Connectedness as a Topological Invariant

Proposition 3.15. Path connectedness is a topological invariant.

Proof. For all X, Y, assume that there exists a homeomorphism $\sigma : X \to Y$.
As σ is surjective and continuous,
X is path connected implies Y is path connected.
As σ^{-1} is surjective and continuous,
Y is path connected implies X is path connected.
Hence, we've proven that path connectedness is a topological invariant.
Quod. Erat. Demonstrandum. □

Proposition 3.16. Let X be a topological space. Each path connected component is contained in the corresponding connected component.

Proof. Each path connected component is connected,
so it is contained in the corresponding connected component.
Quod. Erat. Demonstrandum. □

Remark: *Path connected component can be a proper subset of the corresponding connected component, so it can be neither open nor closed.*

Definition 3.17. (**Locally Path Connected Topological Space**)
Let X be a topological space, and x be an element of X.
If every open neighbour U of x contains a path connected open neighbour U' of x, then X is locally path connected at x.

Remark: *Again, we cannot remove the quantifier $\forall$ before U, because the intersection of path connected sets may not be path connected.*

Proposition 3.18. Surjective local homeomorphism $\sigma : X \to Y$ preserves local path connectedness.

Proof. For all $y \in Y$, for all open neighbour V of y,
we wish to find a path connected open neighbour V of y.
(1) As σ is surjective, there exists $x \in X$, such that $y = \sigma(x)$.
(2) As σ is a local homeomorphsm, there exists an open neighbour U of x, such that $\sigma(U)$ is open in Y, and the restricted map $\sigma|_U : U \to \sigma(U)$ is a homeomorphism.
(3) As X is locally path connected, $U \cap \sigma^{-1}(V)$ contains a path connected open neighbour $\mathfrak{U}$ of x.
(4) As $\sigma|_U$ is a homemorphism, V contains a path connected open neighbour $\sigma(\mathfrak{U})$ of y. To conclude, Y is locally path connected. Quod. Erat. Demonstrandum. □

Remark: *Again, a continuous function doesn't necessarily preserve local path connectedness at every point. The counterexample is identical, thus omitted.*

Proposition 3.19. Let X be a topological space.
X is connected and locally path connected implies X is path connected.

Proof. Assume to the contrary that X is not path connected.
Fix $x \in X$, its path connected component $X_x \subsetneq X$ generates a partition $\{X_x, X_x^c\}$.
Step 1: As X is connected, $\overline{X_x} \cap X_x^c \neq \emptyset$ or $X_x \cap \overline{X_x^c} \neq \emptyset$.
Step 2: As X is locally path connected,
there exists a path connected open set U that intersects both X_x and X_x^c.
Case 2.1: Assume that $\overline{X_x} \cap X_x^c$ contains some x_1.
As X is locally path connected, choose a path connected open neighbour U_1 of x_1.
As $x_1 \in \overline{X_x}$, U_1 intersects X_x at some x_2.
As X_x, U_1 are path connected, x is connected to x_2,
and x_2 is connected to x_1, contradicting to x is not connected to x_1.
Case 2.2: Assume that $X_x \cap \overline{X_x^c}$ contains some x_1.
As X is locally path connected, choose a path connected open neighbour U_1 of x_1.
As $x_1 \in \overline{X_x^c}$, U_1 intersects X_x^c at some x_2.
As X_x, U_1 are path connected, x is connected to x_1,
and x_1 is connected to x_2, contradicting to x is not connected to x_2.

Hence, our assumption is wrong, and we've proven that X is path connected.
Quod. Erat. Demonstrandum. ☐

3.5 Examples

Proposition 3.20. In a normed vector space V over field $\mathbb{R}$, every convex subset Ω of V is path connected.

Proof. For all $\mathbf{x}_0, \mathbf{x}_1 \in \Omega$, there exists a path γ from $\mathbf{x}_0$ to $\mathbf{x}_1$ in Ω defined by:

$$\gamma(t) = (1-t)\mathbf{x}_0 + t\mathbf{x}_1$$

Hence, Ω is path connected. Quod. Erat. Demonstrandum. ☐

Proposition 3.21. In $\mathbb{R}$, every path connected subset Ω is convex.

Proof. In $\mathbb{R}$, Ω is path connected implies Ω is connected, which further implies Ω is convex. Quod. Erat. Demonstrandum. ☐

Proposition 3.22. In a normed vector space V over field $\mathbb{R}$,
every convex subset Ω is locally path connected.

Proof. For all $\mathbf{x} \in \Omega$, for all open neighbour U of $\mathbf{x}$ in Ω,
for some open neighbour U' of $\mathbf{x}$ in V, $U = \Omega \cap U'$.
As U' is open in V, there exists $r > 0$, such that $B(\mathbf{x}, r) \subseteq U'$.
As $B(\mathbf{x}, r) \cap U'$ is convex, U contains a path connected open neighbour $B(\mathbf{x}, r) \cap U'$ of $\mathbf{x}$, so Ω is locally path connected. Quod. Erat. Demonstrandum. ☐

Proposition 3.23. In the metric space $\mathbb{R}$, the closure of the topologist's sine curve $S = \{(x, \sin \frac{\pi}{x}) \in \mathbb{R}^2 : x > 0\}$ is not path connected.

Proof. Assume to the contrary that $\overline{S}$ is path connected.
There exists a path γ from $(0,0)$ to $(1,0)$ in $\overline{S}$.
Define γ_1, γ_2 as the component functions of γ.
The preimage set $\gamma_1^{-1}(\{0\})$ satisfies:

$$\{0\} \subseteq \gamma_1^{-1}(\{0\}) \subseteq [0, 1)$$

Hence, its supremum β exists in $[0, 1]$.
As $\gamma_1^{-1}(\{0\})$ is closed in $\mathbb{R}$, $\beta \in \gamma_1^{-1}(\{0\})$.
On one hand, γ_2 is continuous, so:

$$\lim_{t \to \beta^+} \gamma_2(t) = \gamma_2(\beta) \text{ exists}$$

On the other hand, γ_1 is continuous, so for all $\epsilon > 0$:

$$\begin{array}{rcccl} \gamma_1([\beta,\beta+\epsilon]) & \supseteq & [\gamma_1(\beta),\gamma_1(\beta+\epsilon)] & = & [0,\gamma_1(\beta+\epsilon)] \\ \gamma_1((\beta,\beta+\epsilon]) & \supseteq & \gamma_1([\beta,\beta+\epsilon])\backslash\{\gamma(\beta)\} & = & (0,\gamma_1(\beta+\epsilon)] \end{array}$$

there exist $t_1, t_2 \in (\beta, \beta+\epsilon]$, such that $\gamma_2(t_1) = +1$ and $\gamma_2(t_2) = -1$, a contradiction.
Hence, our assumption is false, and we've proven that $\overline{S}$ is not path connected.
Quod. Erat. Demonstrandum. $\square$

4 Adjunction Space

4.1 Definition of Adjunction Space

Lemma 4.1. Let X_1, X_2 be two sets, U_1 be a subset of X_1,
and $\sigma : U_1 \to X_2$ be a function. Define $[X_1 \sqcup X_2]_\sigma$ by:
(1) If $x_2 \notin \sigma(U_1)$, then $[(x_2,2)]_\sigma = \{(x_2,2)\}$.
(2) If $x_2 \in \sigma(U_1)$, then $[(x_2,2)]_\sigma = \{(x_1,1),(x_2,2) : \sigma(x_1) = x_2\}$.
(3) If $x_1 \notin U_1$, then $[(x_1,1)]_\sigma = \{(x_1,1)\}$.
(4) If $x_1 \in U_1$, then $[(x_1,1)]_\sigma = [(\sigma(x_1),2)]_\sigma$.
$[X_1 \sqcup X_2]_\sigma$ is a partition of $X_1 \sqcup X_2$.

Definition 4.2. (**Adjunction Space**)
Let X_1, X_2 be two topological spaces,
U_1 be a subset of X_1, and $\sigma : U_1 \to X_2$ be a function.
Define the adjunction space from X_1 to X_2 via σ as $X_1 \cup_\sigma X_2 = [X_1 \sqcup X_2]_\sigma$.

Remark: *When studying $X_1 \cup_\sigma X_2$:*
(1) π_1 means the projection map from X_1 to $X_1 \sqcup X_2$.
(2) π_2 means the projection map from X_2 to $X_1 \sqcup X_2$.
(3) π means the projection map from $X_1 \sqcup X_2$ to $X_1 \cup_\sigma X_2$.

Lemma 4.3. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces,
and ξ be an element of $\prod_{\lambda\in I} X_\lambda$. Define $\sim_\xi$ on $\coprod_{\lambda\in I} X_\lambda$ by:
(1) When $\mu = \mu', (x,\mu) \sim_\xi (x',\mu')$ if $x = x'$.
(2) When $\mu \neq \mu', (x,\mu) \sim_\xi (x',\mu')$ if $x = \xi(\mu)$ and $x' = \xi(\mu')$.
$\sim_\xi$ is an equivalence relation.

Definition 4.4. (**Wedge Sum**)
Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces,
and ξ be an element of $\prod_{\lambda\in I} X_\lambda$.
Define the wedge sum of $(X_\lambda)_{\lambda\in I}$ at ξ as $\bigvee^\xi_{\lambda\in I} X_\lambda = [\coprod_{\lambda\in I} X_\lambda]_\xi$.

Remark: *When studying* $\bigvee_{\lambda\in I}^{\xi} X_\lambda$*:*
(1) π_μ *means the projection map from* X_μ *to* $\coprod_{\lambda\in I} X_\lambda$*.*
(2) π *means the projection map from* $\coprod_{\lambda\in I} X_\lambda$ *to* $\bigvee_{\lambda\in I}^{\xi} X_\lambda$*.*

4.2 Examples of Adjunction Space

Definition 4.5. (**One Line with Two Origins**)
Choose $X_1 = \mathbb{R}, X_2 = \mathbb{R}, U_1 = \mathbb{R}\backslash\{0\}, \sigma : x \mapsto x$.
Define $\mathbb{R} \cup_\sigma \mathbb{R}$ as one line with two origins.

Proposition 4.6. $\mathbb{R} \cup_\sigma \mathbb{R}$ is not Hausdorff.

Proof. We may divide our proof into three steps.
Step 1: For any open neighbour U_1 of $\{(0,1)\}$, construct $(-\epsilon_1, +\epsilon_1)$.

$$\begin{aligned}\{(0,1)\} \in U_1 \in \mathcal{O}_{\mathbb{R}\cup_\sigma\mathbb{R}} &\implies (0,1) \in \pi^{-1}(U_1) \in \mathcal{O}_{\mathbb{R}\sqcup\mathbb{R}} \\ &\implies 0 \in \pi_1^{-1}(\pi^{-1}(U_1)) \in \mathcal{O}_{\mathbb{R}} \\ &\implies \text{Some } (-\epsilon_1, +\epsilon_1) \subseteq \pi_1^{-1}(\pi^{-1}(U_1))\end{aligned}$$

Step 2: For any open neighbour U_2 of $\{(0,2)\}$, construct $(-\epsilon_2, +\epsilon_2)$.

$$\begin{aligned}\{(0,2)\} \in U_2 \in \mathcal{O}_{\mathbb{R}\cup_\sigma\mathbb{R}} &\implies (0,2) \in \pi^{-1}(U_2) \in \mathcal{O}_{\mathbb{R}\sqcup\mathbb{R}} \\ &\implies 0 \in \pi_2^{-1}(\pi^{-1}(U_2)) \in \mathcal{O}_{\mathbb{R}} \\ &\implies \text{Some } (-\epsilon_2, +\epsilon_2) \subseteq \pi_2^{-1}(\pi^{-1}(U_2))\end{aligned}$$

Step 3: Choose $x = \frac{1}{2}\min\{\epsilon_1, \epsilon_2\}$, and prove that $\{(x,1),(x,2)\} \in U_1 \cap U_2$.

$$\begin{aligned}x \in (-\epsilon_1, +\epsilon_1) \text{ and } x \in (-\epsilon_2, +\epsilon_2) &\implies x \in \pi_1^{-1}(\pi^{-1}(U_1)) \text{ and } x \in \pi_2^{-1}(\pi^{-1}(U_2)) \\ &\implies (x,1) \in \pi^{-1}(U_1) \text{ and } (x,2) \in \pi^{-1}(U_2) \\ &\implies \{(x,1),(x,2)\} \in U_1 \cap U_2\end{aligned}$$

As some distinct $\{(0,1)\}, \{(0,2)\} \in \mathbb{R} \cup_\sigma \mathbb{R}$ cannot be separated by disjoint open sets, $\mathbb{R} \cup_\sigma \mathbb{R}$ is not Hausdorff. Quod. Erat. Demonstrandum. ☐

Remark: *This implies* $\mathbb{R} \cup_\sigma \mathbb{R}$ *is not metrizable.*

Proposition 4.7. $\mathbb{R} \cup_\sigma \mathbb{R}$ is not compact.

Proof. Construct an open cover $\mathcal{V} = \pi \circ \pi_1(\mathcal{U}) \cup \pi \circ \pi_2(\mathcal{U})$ of $\mathbb{R} \cup_\sigma \mathbb{R}$,
where $\mathcal{U} = \{(n-1, n+1)\}_{n\in\mathbb{Z}}$. As $\mathcal{V}$ has no finite subcover, $\mathbb{R} \cup_\sigma \mathbb{R}$ is not compact.
Quod. Erat. Demonstrandum. ☐

Remark: $\pi \circ \pi_1, \pi \circ \pi_2$ *are open in this setting, but not open in general.*

Proposition 4.8. $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally compact at every point.

Proof. For all $p \in \mathbb{R} \cup_\sigma \mathbb{R}$, WLOG, assume that $\exists x \in \mathbb{R}, p = \pi(x, 1)$.

$$\begin{aligned}
[x-1, x+1] \text{ is compact } &\implies [x-1, x+1] \sqcup [x-1, x+1] \text{ is compact} \\
&\implies [x-1, x+1] \cup_\sigma [x-1, x+1] \text{ is compact} \\
x \in (x-1, x+1) \in \mathcal{O}_{\mathbb{R}} &\implies (x, 1) \in (x-1, x+1) \sqcup (x-1, x+1) \in \mathcal{O}_{\mathbb{R} \sqcup \mathbb{R}} \\
&\implies p \in (x-1, x+1) \cup_\sigma (x-1, x+1) \in \mathcal{O}_{\mathbb{R} \cup_\sigma \mathbb{R}}
\end{aligned}$$

As each $p \in \mathbb{R} \cup_\sigma \mathbb{R}$ has a compact neighbour $[x-1, x+1] \cup_\sigma [x-1, x+1]$, $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally compact at every point. Quod. Erat. Demonstrandum. $\square$

Proposition 4.9. $\mathbb{R} \cup_\sigma \mathbb{R}$ is path connected.

Proof. π_1, π_2, π are continuous, so $\pi \circ \pi_1, \pi \circ \pi_2$ are continuous.
$\mathbb{R}$ is path connected, so $\pi \circ \pi_1(\mathbb{R}), \pi \circ \pi_2(\mathbb{R})$ are path connected.
$\{(1, 1), (1, 2)\} \in \pi \circ \pi_1(\mathbb{R}) \cap \pi \circ \pi_2(\mathbb{R})$, so $\mathbb{R} \cup_\sigma \mathbb{R} = \pi \circ \pi_1(\mathbb{R}) \cup \pi \circ \pi_2(\mathbb{R})$ is path connected.
Quod. Erat. Demonstrandum. $\square$

Remark: *This implies $\mathbb{R} \cup_\sigma \mathbb{R}$ is connected.*

Proposition 4.10. $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally path connected at every point.

Proof. For all $p \in \mathbb{R} \cup_\sigma \mathbb{R}$, WLOG, assume that $\exists x \in \mathbb{R}, p = \pi(x, 1)$.
Case 1: In this case, $x = 0$.
As every open neighbour U of $\{(0, 1)\}$ contains a path connected open neighbour $V = (-\epsilon, +\epsilon) \cup_\sigma \emptyset$ of $\{(0, 1)\}$, $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally path connected at p.
Case 2: In this case, $x \neq 0$.
As $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally homeomorphic to $\mathbb{R}$ at p, $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally path connected at p.
Hence, $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally path connected at every point. Quod. Erat. Demonstrandum. $\square$

Remark: *This implies $\mathbb{R} \cup_\sigma \mathbb{R}$ is locally connected at every point.*

Definition 4.11. (**A Family of Lines with One Origin**)
Choose $(X_\lambda)_{\lambda \in I} = (\mathbb{R})_{\lambda \in I}, \xi = (0)_{\lambda \in I}$.
Define $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ as a family of lines with one origin.

Lemma 4.12. Define d on $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ by:
(1) When $\mu = \mu'$, $d(\pi(x, \mu), \pi(x', \mu')) = |x - x'|$.
(2) When $\mu \neq \mu'$, $d(\pi(x, \mu), \pi(x', \mu')) = |x| + |x'|$.
d is a metric on $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$.

Proposition 4.13. When I is finite, the following two topologies are equal:
(1) The metric topology of $(\bigvee_{\lambda\in I}^{\xi}\mathbb{R}, d)$.
(2) The wedge sum topology of $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$.

Proof. We may divide our proof into two parts.
Part 1: We prove that the metric topology is coarser than the wedge sum topology.
For each open ball $B(p,r)$ in $(\bigvee_{\lambda\in I}^{\xi}\mathbb{R}, d)$, for each $\mu\in I$:
Situation 1.1: If $\exists x\in\mathbb{R}, p=\pi\circ\pi_\mu(x)$, then $(\pi\circ\pi_\mu)^{-1}(B(p,r))=B(x,r)$.
Situation 1.2: If $\forall x\in\mathbb{R}, p\neq\pi\circ\pi_\mu(x)$, then $(\pi\circ\pi_\mu)^{-1}(B(p,r))=B(0,r-l)$ or $\emptyset$.
Here, l is the distance from p to the origin.
This implies $B(p,r)$ is open in the wedge sum $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$.
Part 2: We prove that the wedge sum topology is coarser than the metric topology.
For each open subset U of $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$, for each $p\in U$:
Situation 2.1: If p is the origin, then each $(\pi\circ\pi_\mu)^{-1}(U)$ contains some $B(0,r_\mu)$.
As I is finite, $r=\min\{r_\lambda\}_{\lambda\in I}>0$, so U contains some open ball at the origin.
Situation 2.2: If p is not the origin, then $\exists!\mu\in I$ and $x\in\mathbb{R}\backslash\{0\}, p=\pi\circ\pi_\mu(x)$.
Choose $r=|x|$, then U contains some open ball at p.
Hence, the two topologies are equal. Quod. Erat. Demonstrandum. □

Remark: *Notice that the same argument won't work if I is infinite.*

Proposition 4.14. $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$ is Hausdorff.

Proof. For all distinct $p=\pi\circ\pi_\mu(x), p'=\pi\circ\pi_{\mu'}(x')\in\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$:
Case 1: In this case, exactly one of p,p' is the origin.
WLOG, assume that p is the origin.
There exist open subsets U,U' of $\mathbb{R}$,
such that $0\in U$ and $x'\in U'$ and $U\cap U'=\emptyset$.
There exist open subsets $V=\bigvee_{\lambda\in I}^{\xi}U, V'=\pi\circ\pi_{\mu'}(U'\backslash\{0\})$ of $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$,
such that $p\in V$ and $p'\in V'$ and $V\cap V'=\emptyset$.
Case 2: In this case, neither of p,p' is the origin.
Situation 2.1: If $\mu=\mu'$, then $x\neq x'$.
There exist open subsets U,U' of $\mathbb{R}$,
such that $x\in U$ and $x'\in U'$ and $U\cap U'=\emptyset$.
There exist open subsets $V=\pi\circ\pi_\mu(U\backslash\{0\}), V'=\pi\circ\pi_{\mu'}(U'\backslash\{0\})$ of $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$,
such that $p\in V$ and $p'\in V'$ and $V\cap V'=\emptyset$
Situation 2.2: If $\mu\neq\mu'$, then:
There exist open subsets $V=\pi\circ\pi_\mu(\mathbb{R}\backslash\{0\}), V'=\pi\circ\pi_{\mu'}(\mathbb{R}\backslash\{0\})$ of $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$,
such that $p\in V$ and $p'\in V'$ and $V\cap V'=\emptyset$.
Hence, $\bigvee_{\lambda\in I}^{\xi}\mathbb{R}$ is Hausdorff. Quod. Erat. Demonstrandum. □

Remark: *As each $\pi\circ\pi_\mu$ is not open, we need to prove by cases.*

Proposition 4.15. $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ is not compact.

Proof. Construct an open cover $\mathcal{V} = \{\bigvee^{\xi}_{\lambda \in I} (-1, +1)\} \cup \bigcup_{\lambda \in I} \pi \circ \pi_{\lambda}(\mathcal{U})$ of $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$, where $\mathcal{U} = \{(n-1, n+1)\}_{n \in \mathbb{Z} \setminus \{0\}}$. As $\mathcal{V}$ has no finite subcover, $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ is not compact. Quod. Erat. Demonstrandum. □

Proposition 4.16. For all $p \in \bigvee^{\xi}_{\lambda \in I} \mathbb{R}$:
(1) If p is the origin and I is finite,
then $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ is locally compact at p.
(2) If p is the origin and I is infinite,
then $\bigvee_{\lambda \in I} \mathbb{R}$ is not locally compact at p.
(3) If p is not the origin,
then $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ is locally compact at p.

Proof. We may prove these statements one by one.
(1) When p is the origin and I is finite:

$$\begin{aligned}
[-1, +1] \text{ is compact } &\implies \coprod_{\lambda \in I} [-1, +1] \text{ is compact} \\
&\implies \bigvee^{\xi}_{\lambda \in I} [-1, +1] \text{ is compact} \\
\bigvee^{\xi}_{\lambda \in I} (-1, +1) = B(p, 1) &\implies \bigvee^{\xi}_{\lambda \in I} (-1, +1) \text{ is an open neighbour of } p
\end{aligned}$$

As $p \in \bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ has a compact neighbour $\bigvee^{\xi}_{\lambda \in I} [-1, +1]$, $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ is locally compact at p.
(2) When p is the origin and I is infinite:

$$\text{Each } B(p, r) \text{ is not sequentially compact} \implies \text{Each } B(p, r) \text{ is not compact}$$

As $p \in \bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ has no compact neighbour, $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ is not locally compact at p.
(3) When p is not the origin:

$$\bigvee^{\xi}_{\lambda \in I} \mathbb{R} \text{ is locally homeomorphic to } \mathbb{R} \text{ at } p \implies \bigvee^{\xi}_{\lambda \in I} \mathbb{R} \text{ is locally compact at } p$$

Quod. Erat. Demonstrandum. □

Proposition 4.17. $\bigvee^{\xi}_{\lambda \in I} \mathbb{R}$ is path connected.

Proof. Each π_{μ} is continuous and π is continuous, so each $\pi \circ \pi_{\mu}$ is continuous.
$\mathbb{R}$ is path connected, so each $\pi \circ \pi_{\mu}(\mathbb{R})$ is path connected.

The origin is in each $\pi \circ \pi_\mu(\mathbb{R})$, so $\bigvee_{\lambda \in I}^{\xi} \mathbb{R} = \bigcup_{\lambda \in I} \pi \circ \pi_\mu(\mathbb{R})$ is path connected.
Quod. Erat. Demonstrandum. $\square$

Remark: *This implies $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ is connected.*

Proposition 4.18. $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ is locally path connected at every point.

Proof. For all $p \in \bigvee_{\lambda \in I}^{\xi} \mathbb{R}$:
Case 1: In this case, p is the origin.
As every open neighbour U of p contains a path connected open neighbour $V = \bigvee_{\lambda \in I}^{\xi} (-\epsilon, +\epsilon)$ of p, $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ is locally path connected at p.
Case 2: In this case, p is not the origin.
As $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ is locally homeomorphic to $\mathbb{R}$ at p, $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ is locally path connected at p.
Hence, $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ is locally connected at every point. Quod. Erat. Demonstrandum. $\square$

Remark: *This implies $\bigvee_{\lambda \in I}^{\xi} \mathbb{R}$ is locally connected at every point.*

4.3 A Special Class of Adjunction Space

We've investigated two ways of gluing two lines $X_1 = \mathbb{R}, X_2 = \mathbb{R}$ together:
(1) $U_1 = \mathbb{R} \backslash \{0\}, \sigma : x \mapsto x$.
(2) $U_1 = \{0\}, \sigma : 0 \mapsto 0$.
Notice that the second approach "seems more natural" as it is Hausdorff.
If U_1 is closed in X_1, and σ is a closed embedding, then $X_1 \cup_\sigma X_2$ "behaves well".

Proposition 4.19. Let X_1, X_2 be two topological spaces,
U_1 be a closed subset of X_1, and $\sigma : U_1 \to X_2$ be a closed embedding.
If X_1, X_2 are Hausdorff, then $X_1 \cup_\sigma X_2$ is Hausdorff.

Proof. For all distinct $p, p' \in X_1 \cup_\sigma X_2$:

Case 1: In this case, both of p, p' are in the identified part.
Step 1.1: Construct the following preimages.

$$\begin{aligned} &\exists! x_1 \in X_1 \text{ and } x_2 \in X_2, \quad p = \pi \circ \pi_1(x_1) = \pi \circ \pi_2(x_2) \\ &\exists! x_1' \in X_1 \text{ and } x_2' \in X_2, \quad p' = \pi \circ \pi_1(x_1') = \pi \circ \pi_2(x_2') \end{aligned}$$

Step 1.2: As X_1, X_2 are Hausdorff, construct the following open sets.

$$\begin{aligned} x_1 \neq x_1' &\implies \exists V_1, V_1' \in \mathcal{O}_{X_1}, \quad x_1 \in V_1 \text{ and } x_1' \in V_1' \text{ and } V_1 \cap V_1' = \emptyset \\ x_2 \neq x_2' &\implies \exists V_2, V_2' \in \mathcal{O}_{X_2}, \quad x_2 \in V_2 \text{ and } x_2' \in V_2' \text{ and } V_2 \cap V_2' = \emptyset \end{aligned}$$

Step 1.3: As $U_1 \cong \sigma(U_1) = U_2$, construct the following open sets.

$$\begin{aligned}
\sigma : U_1 \to U_2 \text{ is open} \quad &\implies \quad \exists W_2 \in \mathcal{O}_{X_2}, \quad \sigma(U_1 \cap V_1) = U_2 \cap W_2 \\
&\quad \text{and} \quad \exists W_2' \in \mathcal{O}_{X_2}, \quad \sigma(U_1 \cap V_1') = U_2 \cap W_2' \\
\sigma^{-1} : U_2 \to U_1 \text{ is open} \quad &\implies \quad \exists W_1 \in \mathcal{O}_{X_1}, \quad \sigma^{-1}(U_2 \cap V_2) = U_1 \cap W_1 \\
&\quad \text{and} \quad \exists W_1' \in \mathcal{O}_{X_1}, \quad \sigma^{-1}(U_2 \cap V_2') = U_1 \cap W_1'
\end{aligned}$$

Step 1.4: Construct the following two sets.

$$\begin{aligned}
P &= \pi([\pi_1(V_1) \cup \pi_2(W_2)] \cap [\pi_1(W_1) \cup \pi_2(V_2)]) \\
P' &= \pi([\pi_1(V_1') \cup \pi_2(W_2')] \cap [\pi_1(W_1') \cup \pi_2(V_2')])
\end{aligned}$$

Notice that:
(1) $(\pi \circ \pi_1)^{-1}(P) = V_1 \cap W_1$ is an open neighbour of x_1 and $(\pi \circ \pi_2)^{-1}(P) = W_2 \cap V_2$ is an open neighbour of x_2, so P is an open neighbour of p.
(2) $(\pi \circ \pi_1)^{-1}(P') = V_1' \cap W_1'$ is an open neighbour of x_1' and $(\pi \circ \pi_2)^{-1}(P') = W_2' \cap V_2'$ is an open neighbour of x_2', so P' is an open neighbour of p'.
(3) Assume to the contrary that $\exists q \in P \cap P'$.

$$\begin{aligned}
\exists (y, \nu) \in [\pi_1(V_1) \cup \pi_2(W_2)] \cap [\pi_1(W_1) \cup \pi_2(V_2)], \quad q = \pi(y, \nu) \\
\exists (y', \nu') \in [\pi_1(V_1') \cup \pi_2(W_2')] \cap [\pi_1(W_1') \cup \pi_2(V_2')], \quad q = \pi(y', \nu')
\end{aligned}$$

WLOG, assume that $\nu = 1$, then $y \in V_1 \cap W_1$.
If $\nu' = 1$, then $y \in V_1 \cap W_1$ and $y' \in V_1' \cap W_1'$,
a contradiction $y = y' \in V_1 \cap V_1' = \emptyset$ arises.
If $\nu' = 2$, then $y \in V_1 \cap W_1 \cap U_1$ and $y' \in W_2' \cap V_2' \cap U_2$,
a contradiction $\sigma(y) = y' \in V_2 \cap V_2' = \emptyset$ arises.
Hence, our assumption is false, and we've proven that $P \cap P' = \emptyset$.

Case 2: In this case, exactly one of p, p' is in the identified part.
WLOG, assume that p is in the identified part and p' is equal to some $\pi \circ \pi_1(x_1')$.
Step 2.1: Construct the following preimages.

$$\exists! x_1 \in X_1 \text{ and } x_2 \in X_2, \quad p = \pi \circ \pi_1(x_1) = \pi \circ \pi_2(x_2)$$

Step 2.2: As X_1 is Hausdorff, construct the following open sets.

$$x_1 \neq x_1' \quad \implies \quad \exists V_1, V_1' \in \mathcal{O}_{X_1}, \quad x_1 \in V_1 \text{ and } x_1' \in V_1' \text{ and } V_1 \cap V_1' = \emptyset$$

Step 2.3: As $U_1 \cong \sigma(U_1) = U_2$, construct the following open sets.

$$\sigma : U_1 \to U_2 \text{ is open} \quad \implies \quad \exists W_2 \in \mathcal{O}_{X_2}, \quad \sigma(U_1 \cap V_1) = U_2 \cap W_2$$

Step 2.4: Construct the following two sets.

$$P = \pi(\pi_1(V_1) \cup \pi_2(W_2))$$
$$P' = \pi(\pi_1(V_1' \backslash U_1))$$

Notice that:
(1) $(\pi \circ \pi_1)^{-1}(P) = V_1$ is an open neighbour of x_1 and $(\pi \circ \pi_2)^{-1}(P) = W_2$ is an open neighbour of x_2, so P is an open neighbour of p.
(2) $(\pi \circ \pi_1)^{-1}(P') = V_1' \backslash U_1$ is an open neighbour of x_1 and $(\pi \circ \pi_2)^{-1}(P') = \emptyset$, so P' is an open neighbour of p'.
(3) $P \cap P' \subseteq \pi([\pi_1(V_1) \cup \pi_2(W_2)] \cap \pi_1(V_1' \backslash U_1)) = \emptyset$, so $P \cap P' = \emptyset$.

Case 3: In this case, neither of p, p' is in the identified part.
Situation 3.1: If $\mu = \mu'$, then $x \neq x'$.
There exist $V \in \mathcal{O}_{X_\mu}$ and $V' \in \mathcal{O}_{X_{\mu'}}$,
such that $x \in V$ and $x' \in V'$ and $V \cap V' = \emptyset$.
There exist $W = \pi \circ \pi_\mu(V \backslash U_\mu), W' = \pi \circ \pi_{\mu'}(V' \backslash U_{\mu'}) \in \mathcal{O}_{X_1 \cup_\sigma X_2}$,
such that $p \in W$ and $p' \in W'$ and $W \cap W' = \emptyset$.
Situation 3.2: If $\mu \neq \mu'$, then:
There exist $W = \pi \circ \pi_\mu(X_\mu \backslash U_\mu), W' = \pi \circ \pi_{\mu'}(X_{\mu'} \backslash U_{\mu'}) \in \mathcal{O}_{X_1 \cup_\sigma X_2}$,
such that $p \in W$ and $p' \in W'$ and $W \cap W' = \emptyset$.

Hence, $X_1 \cup_\sigma X_2$ is Hausdorff. Quod. Erat. Demonstrandum. $\square$

Proposition 4.20. Let X_1, X_2, Y be three topological spaces,
U_1 be a closed subset of X_1, $\sigma : U_1 \to X_2$ be a closed embedding,
and $\tau_1 : X_1 \to Y, \tau_2 : X_2 \to Y$ be two functions with $\tau_1|_{U_1} = \tau_2 \circ \sigma$.
Define the adjunction function $\tau_1 \cup_\sigma \tau_2 : X_1 \cup_\sigma X_2 \to Y$ from τ_1 to τ_2 via σ by:

$$\tau_1 \cup_\sigma \tau_2(\pi(x, \lambda)) = \tau_\lambda(x)$$

$\tau_1 \cup_\sigma \tau_2$ is continuous.

Proof. As each $\pi \circ \pi_\lambda$ is injective, $\exists! \omega_\lambda : \pi \circ \pi_\lambda(X_\lambda) \to Y, \omega_\lambda \circ \pi \circ \pi_\lambda = \tau_\lambda$.
As each $\pi \circ \pi_\lambda(X_\lambda) \in \mathcal{C}_{X_1 \cup_\sigma X_2}$ and ω_1, ω_2 agree on common domain,
$\tau_1 \cup_\sigma \tau_2 = \omega_1 \cup \omega_2$ is continuous. Quod. Erat. Demonstrandum. $\square$

4.4 Embedding Adjunction Space

Theorem 4.21. Let X, Y be two topological spaces, $\sigma : X \to Y$ be a continuous surjection, and $\sim: X \to X, x \sim x'$ if $\sigma(x) = \sigma(x')$ be the equivalence relation induced by σ. If X is compact and Y is Hausdorff, then $\widetilde{X} \cong Y$.

Proof. It suffices to prove that $\widetilde{\sigma} : \widetilde{X} \to Y, \widetilde{\sigma}(\widetilde{x}) = y$ is a homeomorphism.
Step 1: For all $\widetilde{x}, \widetilde{x}' \in \widetilde{X}$:

$$\widetilde{x} = \widetilde{x}' \implies \widetilde{\sigma}(\widetilde{x}) = \sigma(x) = \sigma(x') = \widetilde{\sigma}(\widetilde{x}')$$

Hence, $\widetilde{\sigma}$ is well-defined.
Step 2: For all $V \in \mathcal{P}(Y)$:

$$V \in \mathcal{O}_Y \implies \widetilde{\sigma}^{-1}(V) = \pi(\sigma^{-1}(V)) \in \mathcal{O}_{\widetilde{X}}$$

Hence, $\widetilde{\sigma}$ is continuous.
Step 3: For all $U \in \mathcal{P}(X)$:

$$U \in \mathcal{C}_X \implies U \text{ is compact} \implies \sigma(U) \text{ is compact} \implies \sigma(U) \in \mathcal{C}_Y$$

Hence, σ is closed.
To conclude, σ is a homeomorphism. Quod. Erat. Demonstrandum. $\square$

Proposition 4.22. Construct the disjoint union of the following two functions:

$$\begin{aligned} \sigma_1 : \quad & \mathbb{R} \to \mathbb{R}^2, \quad \sigma_1(x) = (x, 0) \\ \sigma_2 : \quad & \mathbb{R} \to \mathbb{R}^2, \quad \sigma_2(x) = (0, x) \end{aligned}$$

Theorem 4.20. suggests that $\mathbb{R} \vee_\xi \mathbb{R} \cong \{(x, 0), (0, x)\}_{x \in \mathbb{R}}$.

Proposition 4.23. Construct the disjoint union of the following n functions:

$$\begin{array}{ccc} \sigma_0 : & \mathbb{S} \to \mathbb{C}, & \sigma_0(\mathrm{e}^{\mathrm{i}\theta}) = (1 - \cos\theta)\omega^0 \mathrm{e}^{\mathrm{i}\theta/n} \\ \sigma_1 : & \mathbb{S} \to \mathbb{C}, & \sigma_1(\mathrm{e}^{\mathrm{i}\theta}) = (1 - \cos\theta)\omega^1 \mathrm{e}^{\mathrm{i}\theta/n} \\ \sigma_2 : & \mathbb{S} \to \mathbb{C}, & \sigma_2(\mathrm{e}^{\mathrm{i}\theta}) = (1 - \cos\theta)\omega^2 \mathrm{e}^{\mathrm{i}\theta/n} \\ \vdots & \vdots & \vdots \\ \sigma_{n-1} : & \mathbb{S} \to \mathbb{C}, & \sigma_{n-1}(\mathrm{e}^{\mathrm{i}\theta}) = (1 - \cos\theta)\omega^{n-1} \mathrm{e}^{\mathrm{i}\theta/n} \end{array}$$

Theorem 4.20. suggests that $\bigvee_{0 \le k < n}^{\xi} \mathbb{S} \cong \{(1 - \cos n\phi)\mathrm{e}^{\mathrm{i}\phi}\}_{0 \le \phi < 2\pi}$.

Remark: *If the domain fails to be compact, then the theorem doesn't work. One famous counterexample for this is the Hawaii earring. The quotient map from the wedge sum of circles to the Hawaii earring fails to be open, thus not a homeomorphism.*

References

[1] H. Ren, "Template for math notes," 2021.

[2] J. R. Munkres, *Topology*, 2nd ed. Massachusetts Institute of Technology: Pearson, 2000.