## MATH3301 Tutorial 4

1. (a) Let  $G = \{f_1, f_2, f_3, f_4\}$  where  $f_1, f_2, f_3, f_4$  are real-valued functions on  $\mathbb{R}^{\times}$  defined as:

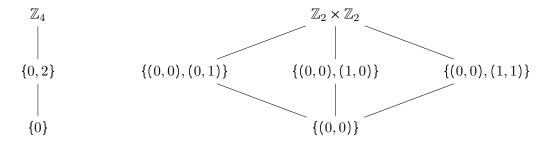
$$f_1(x) = x$$
,  $f_2(x) = \frac{1}{x}$ ,  $f_3(x) = -x$ ,  $f_4(x) = -\frac{1}{x}$   $(x \in \mathbb{R}^{\times})$ .

Is  $(G, \circ)$  a group under the function composite  $\circ$ ? Explain your answer. If it is a group, is it an abelian group, a cyclic group?

(b) Let 
$$GL_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \right\}.$$

Show that under matrix multiplication,  $GL_2(\mathbb{Z}_2)$  is a non-abelian group of order 6.

- (c) Compute the order of every element in  $GL_2(\mathbb{Z}_2)$ .
- (d) Is  $GL_2(\mathbb{Z}_2) \cong S_3$ ? Explain why if no and give an isomorphism if yes.
- 2. A lattice diagram for (subgroups of) a group is a diagram so that a line running downward from a subgroup H to a subgroup K means that K is a subgroup of H. e.g.



Lattice diagram for  $\mathbb{Z}_4$ 

Lattice diagram for  $\mathbb{Z}_2 \times \mathbb{Z}_2$ 

(a) Draw the lattice diagrams for the groups: (i)  $\mathbb{Z}_{12}$  and (ii)  $GL_2(\mathbb{Z}_2)$ .

[Hint: Use the Lagrange theorem to figure out the *possible* order of subgroups.]

- (b) Which subgroup(s) in  $\mathbb{Z}_{12}$  is isomorphic to  $\mathbb{Z}_4$ ? Answer the same question for  $GL_2(\mathbb{Z}_2)$ .
- 3. Consider the group  $G = \mathbb{Z}_8^{\times} \times \mathbb{Z}_{12}$ , and the subgroups  $H = \langle 3 \rangle \times \langle 4 \rangle$  and  $K = \langle 3 \rangle \times \langle 6 \rangle$ .
  - (a) Find all the elements in G whose orders are 6.
  - (b) Is  $\langle (3,4) \rangle = H$ ? Is  $\langle (3,6) \rangle = K$ ? [Hint: Evaluate ord((3,4)) and ord((3,6)).]
  - (c) Evaluate all (left-)cosets of H and [G:H]. Do the same for K.
  - (d) Draw the lattice diagram of G.

- 4. Let  $\phi: G_1 \to G_2$  be a homomorphism between groups. We abuse the notation with e for the identity elements in  $G_1$  and  $G_2$ . Prove the following statements:
  - (a) The image  $\phi(H)$  is a subgroup of  $G_2$ , for any subgroup H of  $G_1$ .
  - (b) The pre-image  $\phi^{-1}(K)$  is a subgroup of  $G_1$ , for any subgroup K of  $G_2$ .
  - (c) If one of  $G_1$  or  $G_2$  is a finite abelian group, then  $\phi(G_1)$  is finite abelian.
  - (d) If one of  $G_1$  or  $G_2$  is cyclic, then  $\phi(G_1)$  is cyclic.
- 5. Let  $\phi: G_1 \to G_2$  be a homomorphism between groups. We abuse the notation with e for the identity elements in  $G_1$  and  $G_2$ . Prove or disprove, with justification, the statements:
  - (a) If  $\psi: G_2 \to G_1$  is a function such that both  $\phi \circ \psi$  and  $\psi \circ \phi$  are the identity map, then  $\psi$  is a group isomorphism.
  - (b) If  $\phi$  is bijective, then  $\phi^{-1}(g) = \phi(g^{-1})$  for all  $g \in G_1$ .
  - (c) The pre-image  $\phi^{-1}(K)$  is abelian if K is an abelian subgroup of  $G_2$ .
  - (d) The pre-image  $\phi^{-1}(K)$  is cyclic if K is a cyclic subgroup of  $G_2$ .
  - (e) If  $|G_1| = 2024$  and  $K := \phi^{-1}(\{e\})$ , then |K| is not divisible by primes greater than 24.
- 6. Evaluate with explanation the number of nontrivial homomorphisms  $\phi$  in each case below:
  - (a)  $\phi: \mathbb{Z} \to \mathbb{Z}$ .
  - (b)  $\phi: \mathbb{Z} \to \mathbb{Z}$  is surjective.
  - (c)  $\phi: \mathbb{Z} \to \mathbb{Z}_2$ .
  - (d)  $\phi: \mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_5$ .
  - (e)  $\phi: \mathbb{Z}_{12} \to \mathbb{Z}_4$ .
  - (f)  $\phi: \mathbb{Z}_4 \to \mathbb{Z}_{12}$ .
  - (g)  $\phi: \mathbb{Z} \times \mathbb{Z} \to 2\mathbb{Z}$ .
  - (h)  $\phi: 2\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ .