Free product and amalgamated product of groups

Recall if $G = \langle S \rangle$ is a group generated by a set of elements S. (may be infinite)

We assume $e \notin S$

then any element of G can be represented as a (finite) word consisting of product of $x^{\pm 1}$ $x \in S$.

convention e = empty word.

! word presentation may not be unique

Example

1) $\mathbb{Z} = \langle a \rangle = \{ \dots a^{-1}, e, a, a^2, \dots \}$

$$a \cdot a^{-1} = e$$

2) $F_2 = \langle x, y \rangle = \{$ Any words in $x^{\pm 1}, y^{\pm 1} \}$

Defn G_1, G_2 two groups

$$G_1 * G_2 = \langle x_1 \cdots x_n \mid x_i \in G_1 \sqcup G_2 \rangle$$

where $x_i \neq e$ $\quad x_i x_{i+1} \notin G_1$ or G_2

$(x_1 \cdots x_n) \cdot (y_1 \cdots y_m)$ = reduction of

$(x_1 x_2)(x_3 x_4 x_5)(x_6)$ $\qquad x_1 \cdots x_n y_1 \cdots y_m$

Rmk 1) if $G_1 = \langle S_1 \rangle$ $\quad G_2 = \langle S_2 \rangle$

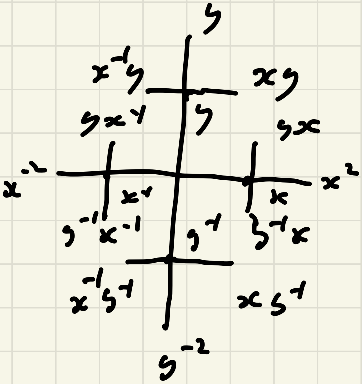
then

$$G_1 * G_2 = \langle \underbrace{x_1 \cdots x_n}_{\text{reduced}} \mid x_i \in S_1 \sqcup S_2 \rangle$$

2) $G_1 = \langle x \rangle$ $\qquad G_2 = \langle y \rangle$

$= \langle x^n \mid n \in \mathbb{Z} \rangle$ $\quad = \langle y^m \mid m \in \mathbb{Z} \rangle$

$G_1 * G_2 = F_2$ $\leftarrow$ free gp gen. by x, y



Cayley graph of F_2

3) $\mu_2 = \langle a \mid a^2 = 1 \rangle \qquad \mathbb{Z} = \langle b \rangle$

$$\mu_2 * \mathbb{Z} = \{ a^{0/1} b^{n_1} a b^{n_2} \cdots b^{n_k} a^{0/1} \}$$

$n_i \in \mathbb{Z}$

4) $G_1 \subset G_1 * G_2 \supset G_2$

as subgroups

$G_1 * G_2 \neq G_1 \times G_2$!!

Ex Compute $\mu_2 * \mu_3$.

Show that $\mu_2 * \mu_3 \cong PSL_2 \mathbb{Z} := SL_2\mathbb{Z} / \{\pm 1\}$

Def'n [Free product with amalgamation]

$$\phi_1 : A \to G_1 \qquad \phi_2 : A \to G_2$$

homomorphism of gps

$$G_1 *_A G_2 = G_1 * G_2 \big/ N$$

$$\begin{tikzcd}
 & G_1 \arrow[dr] & \\
A \arrow[ur] \arrow[dr] & & G_1 *_A G_2 \\
 & G_2 \arrow[ur] &
\end{tikzcd}$$

N is the normal sub group generated by

the word $\underset{\displaystyle G_1 * G_2}{\phi_1(a)\phi_2(a)^{-1}\ni}$ $\forall\, a \in A$

Prop (Universal property of $G_1 *_A G_2$)

Let H be a group and

$$\begin{tikzcd}
 & G_1 \arrow[dr, "f_1"] & \\
A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] \arrow[rr, phantom, "\shortparallel"] & & H \\
 & G_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

then $\exists\,!$

$$\begin{tikzcd}
 & G_1 \arrow[d] \arrow[drr, bend left, "f_1"] & & \\
A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & G_1 *_A G_2 \arrow[rr, "g"] & & H \\
 & G_2 \arrow[u] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

$$x_1 \cdots x_n N \longmapsto f_{i_1}(x_1) \cdots f_{i_n}(x_n)$$

$$i_j = 1, 2$$

since N is normal

left coset = right coset

e.f. if $x_1 \cdots x_n \in N$

$$g\left[\cdots \phi_1(a_1)\phi_2(a_1)^{-1}\phi_1(a_2)\phi_2(a_2)^{-1}\cdots\right]g^{-1}$$

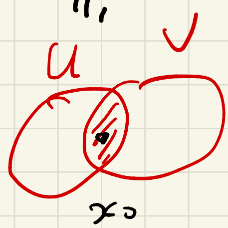
$$\downarrow$$

$$f(g)\left[\ e\ \right]f(g^{-1}) = f(e) = e$$

Van Kampen thm

$X = U \cup V \quad U \cap V \neq \emptyset$ path-conn.

then the following diagram of π_1 form a amalgamation product



$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr] & \\
\pi_1(U \cap V, x_0) \arrow[ur] \arrow[dr] & & \pi_1(X, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \\
 & \pi_1(V, x_0) \arrow[ur] &
\end{tikzcd}$$

proof: page 44 [Hatcher. Algebraic topology]

Applications

$x_0 \subset U \subset X$

$r : U \to x_0$ d.r.

$\tilde{r} : U \vee_{x_0 \sim y_0} Y \to Y$

1) Wedge sum

X, Y path connected
$\begin{matrix} \in & & \in \\ x_0 & & y_0 \end{matrix}$

$$\pi_1\Big(X \underset{x_0 \sim y_0}{\vee} Y \Big) = \pi_1(X, x_0) * \pi_1(Y, y_0)$$

if X, Y are locally contractible

$\pi_1\left(\overset{n}{\bigvee} S^1 \right) = F_n$ free group of n generators

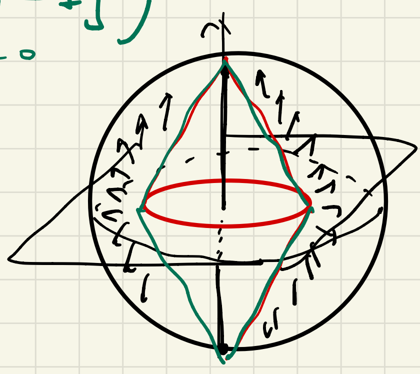
$\overset{\infty}{\bigvee} S^1$ = countable wedge sum of S^1

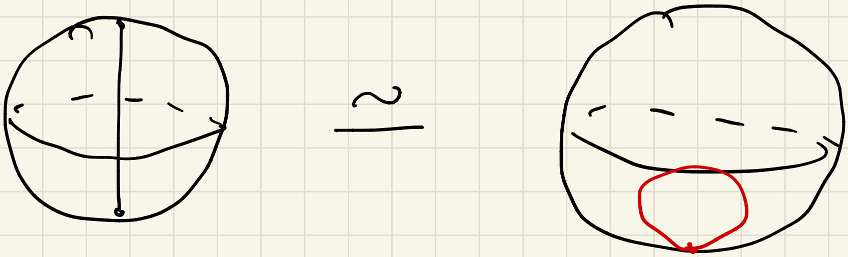
$\pi_1\left(\overset{\infty}{\bigvee} S^1 \right)$ = free group of countably many generators

2) Compute $\pi_1\left(\mathbb{R}^3 \setminus \{ x^2 + y^2 = 1, z = 0 \} \right)$

Claim $\mathbb{R}^3 \setminus \{ x^2 + y^2 = 1, z = 0 \}$

$\simeq S^2 \vee S^1$





$$\pi_1(S^2 \vee S^1) = \pi_1(S^2) * \pi_1(S^1) = \mathbb{Z}$$

3) Connected sum

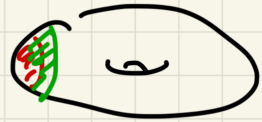
X, Y two path connected top. mfds

$$X \# Y = X \setminus D_1 \sqcup Y \setminus D_2 \Big/ \partial D_1 \cong \partial D_2$$

with canonical projection p

Example

$X = Y = T^2$

X  Y 

D_1^ε $\quad$ D_2^ε $\quad$ $\varepsilon < \varepsilon'$

$D_1^\varepsilon \subset D_1^{\varepsilon'}$ $\quad$ $D_2^\varepsilon \subset D_2^{\varepsilon'}$

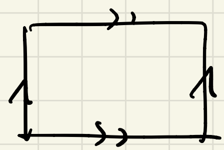
$U = p(X \setminus D_1^\varepsilon \sqcup D_2^{\varepsilon'} \setminus D_2^\varepsilon)$ is a cover of $X \# Y$

$V = p(Y \setminus D_2^\varepsilon \sqcup D_1^{\varepsilon'} \setminus D_1^\varepsilon)$

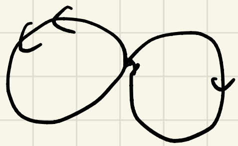
$U \cap V = p(D_1^{\varepsilon'} \setminus D_1^\varepsilon \sqcup D_2^{\varepsilon'} \setminus D_2^\varepsilon)$

$\simeq \partial D_1 \ \partial\, x_0$

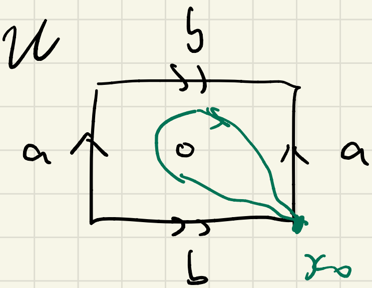
$U \cong X \setminus 1pt$ $\qquad$ $V \cong Y \setminus 1pt$

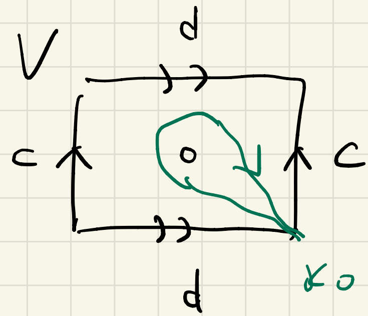
$U =$  $\xrightarrow{\text{deformation retract to}}$ 

||

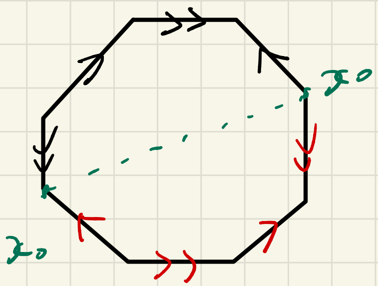
 $= S^1 \vee S^1$

$$\begin{tikzcd}
 & \langle a,b\rangle \arrow[d, phantom, "\shortparallel"] & \\
 & \pi_1(U) \cong F_2 \arrow[dr] & \\
\pi_1(U\cap V) \arrow[ur] \arrow[dr] \arrow[d, phantom, "\cong"] & & \pi_1(X\# Y) \\
\mathbb{Z} & \pi_1(V) \cong F_2 \arrow[ur] & \\
 & \langle c,d\rangle \arrow[u, phantom, "\shortparallel"] &
\end{tikzcd}$$





$$\begin{tikzcd}
 & a^{-1}bab^{-1} \arrow[dr, mapsto] & \\
1 \in \mathbb{Z} \arrow[ur, mapsto] \arrow[dr, mapsto] & & \langle a,b,c,d \mid a^{-1}bab^{-1}c^{-1}dcd^{-1} = 1 \rangle \\
 & c^{-1}dcd^{-1} \arrow[ur, mapsto] &
\end{tikzcd}$$



3) Fundamental gp of surfaces

Classification of surfaces Let Σ be

a compact orientable surface w/o boundary
2-dim'l. top mfd

Then Σ is homeomorphic to

i) $S^2 = \Sigma_0$

2) $\underbrace{T \# \cdots \# T}_{g} =: \Sigma_g \quad g \geq 1$

Lemma Let Σ, Σ' be two surfaces

if $\pi_1(\Sigma) \not\cong \pi_1(\Sigma')$ then

$\Sigma \not\approx \Sigma'$

Thm for $g_1 \neq g_2$ $\Sigma_{g_1} \not\cong \Sigma_{g_2}$

Pf. Applying Van Kampen thm inductively

$$\pi_1(\Sigma_g) = \left\langle a_i\ b_i \;\middle|\; \frac{i = 1, \dots g}{a_1^{-1} b_1 a_1 b_1^{-1} a_2^{-1} b_2 a_2 b_2^{-1} \cdots a_g^{-1} b_g a_g b_g^{-1} = 1} \right\rangle$$

G group

$[G, G]$ = normal sub gp generated by

$$g_1 g_2 g_1^{-1} g_2^{-1}$$

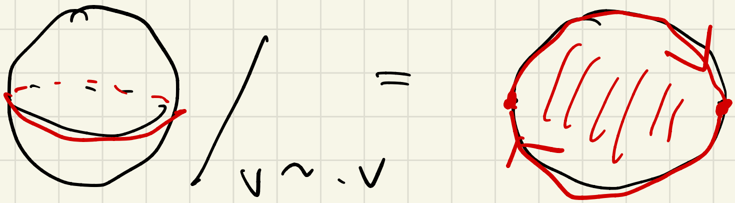
$G/[G,G] =: G^{ab}$ abelianization of G

$$\pi_1(\Sigma_g)^{ab} = \mathbb{Z}^{2g}$$

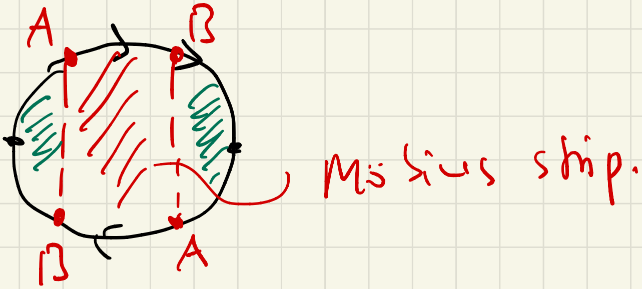
$$G \cong G' \Rightarrow G^{ab} \cong G'^{ab}$$

Fundamental gp of Klein bottle (Another way)

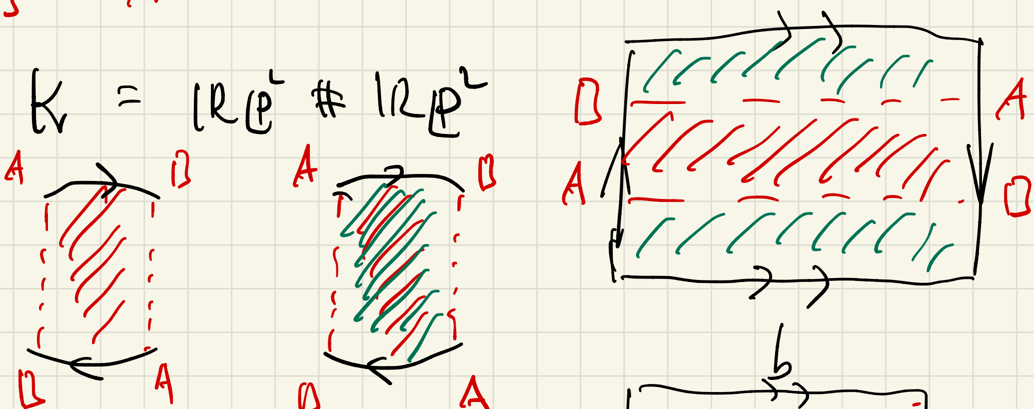
1) $\mathbb{RP}^2$ as adjunction space



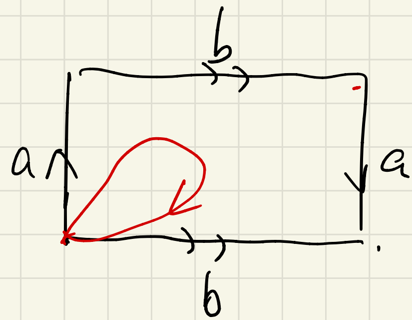
2) $\mathbb{RP}^2 = $ Möbius strip $\cup_\partial D^2$



3) $K = \mathbb{RP}^2 \# \mathbb{RP}^2$

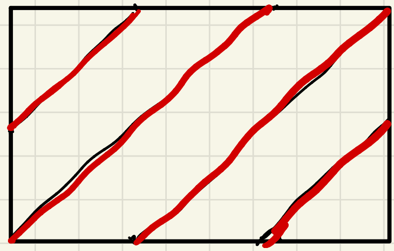


$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[dr] & \\
\mathbb{Z} \arrow[ur] \arrow[dr] & & \langle a,b \mid b^{-1}aba \rangle \\
 & \mathbb{Z} \arrow[ur] &
\end{tikzcd}
$$



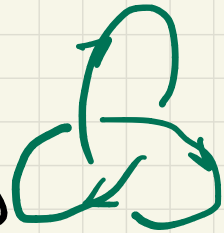
torus knot

$S^1 \times S^1$

K 

$K \subset \overset{''}{T} \subset \mathbb{R}^3 \subset S^3$

Compute $\pi_1(\mathbb{R}^3 \setminus K) = ?$



1) $\pi_1(\mathbb{R}^3 \setminus K) = \pi_1(S^3 \setminus K)$

*2) $S^3 \setminus K \simeq X_{3,2}$

$$X_{3,2} = S^1 \times I \Big/ \begin{matrix} (z,0) \sim (e^{2\pi i/3} z, 0) \\ (z,1) \sim (-z, 0) \end{matrix}$$

$$S^3 = \partial D^4 = \partial(D^2 \times D^2) = \underbrace{S^1 \times D^2} \cup \underbrace{D^2 \times S^1}$$

glue along $S^1 \times S^1$

3) $\pi_1(X_{3,2}) = \langle a, b \mid a^3 = b^2 \rangle =: G$

4) $\underset{\langle a^3 \rangle}{\mathbb{Z}} \longrightarrow G \longrightarrow \mathbb{Z}/2 * \mathbb{Z}/3$