THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH4302: Algebra II

May 9, 2023 9:30 a.m.-12:00 noon.

Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer ALL 5 questions

Note: You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

Q1	/20
Q2	/20
Q3	/25
Q4	/25
Q5	/10

1 True or False Problem [20 marks]

For the following list of statements, if you think it is correct please answer T and give a brief proof. If you think it is false please answer F. Then disprove it or give a counterexample. **Note that only answering T or F without justifications or counter examples earns zero mark.**

(a) The polynomial $x^5 + x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}[x]$.

(b) The ring of quadratic integers $\mathbb{Z}[\sqrt{-1}]$ is a UFD.

(c) The Frobenius map is surjective on any finite field.

(d) Let L/K be a finite field extension. Then the Galois group $\mathrm{Aut}(L/K)$ is finite.

(e) Let L/K be a finite field extension. Then there exists only finitely many intermediate fields $K \subset M \subset L$.

2 [20 marks]

Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(a)$ for $a = \sqrt{2} + \sqrt{5}$.

(1) Show that $\sqrt{2}, \sqrt{5} \in L$.

(2) Find the (monic) minimal polynomial of a in $K[x]$.

(3) Compute the Galois group $\mathrm{Aut}(L/K)$.

(4) Find all the intermediate fields $K \subset M \subset L$.

3 [25 marks]

Let L be a finite field.

(1) Show that the characteristic p of L is finite.

(2) Show that L contains $\mathbb{F}_p$ as a subfield.

(3) Suppose that $\deg(L/\mathbb{F}_p) = n$. Show that L is the splitting field of $x^{p^n} - x \in \mathbb{F}_p[x]$.

(4) Part (3) implies that there exists a unique field of order p^n up to isomorphism. We will denote it by $\mathbb{F}_{p^n}$. Show that for every positive integer $d|n$, there exists a unique subfield of $\mathbb{F}_{p^n}$ isomorphic to $\mathbb{F}_{p^d}$.

4 [25 marks]

Let R be a unital commutative ring and M be a finitely generated R-module. For $x \in M$, define $\mathrm{ann}(x) := \{r \in R \big| rx = 0\}$. We call M a *torsion* module if for every $x \in M$, $\mathrm{ann}(x) \neq 0$. We define $\mathrm{ann}(M) := \bigcap_{x \in M} \mathrm{ann}(x)$. The *support of* M is defined to be the set of maximal ideals of R that contains $\mathrm{ann}(M)$. Denote $\mathrm{supp}(M)$ the support of M.

Let $R = \mathbf{k}[x]$ be the polynomial ring over a field $\mathbf{k}$.

(1) Show that any nonzero prime ideal of R is maximal.

(2) Let S be an *infinite* set of *distinct* maximal ideals of R. Show that the intersection of all the ideals in S is zero. (**Hint:** Use the fact that R is a PID!)

(3) Let M be a finitely generated torsion R-module. Show that the support of M is a finite set.

(4) Classifiy all torsion modules M such that $\mathrm{supp}(M) = \{(x)\}$. (**Hint:** Use the classification theorem of finitely generated modules over PID!)

5 [10 marks]

This problem is a continuation of the previous one and you may use the results of the previous problem without giving a proof. Now let $R = \mathbb{R}[x]$ and M be a finitely generated **torsion** R-module. Let $\mathfrak{m}$ be a maximal ideal of R. We denote by $M_\mathfrak{m}$ the localization of M at $\mathfrak{m}$. Recall that $M_\mathfrak{m}$ consists of symbols $\frac{x}{r}$ where $x \in M$ and $r \in R \setminus \mathfrak{m}$. There is a canonical map $\phi_\mathfrak{m} : M \to M_\mathfrak{m}$ sending x to $\frac{x}{1}$.

(1) Classify all maximal ideals of R.

(2) Let $M = R/(x^4 + x^3 + x^2)$. Compute $\mathrm{supp}(M)$ and show that the map

$$M \to \prod_{\mathfrak{m} \in \mathrm{supp}(M)} M_\mathfrak{m}$$

is an isomorphism of rings. Here note that M is a quotient ring of R and $M_\mathfrak{m}$ is the localization of a ring, therefore also a ring.

END OF PAPER