MATH3301 Tutorial 8

1. Let H be a group of order n, and let p be a prime. Define $H^p = \underbrace{H \times \cdots \times H}_{p \text{ copies}}$ and

$$X = \{(a_0, a_1, \cdots, a_{p-1}) \in H^p : a_0 a_1 \cdots a_{p-1} = e\}.$$

Define for $i \in \mathbb{Z}_p$, $\ i * (a_0, a_1, \cdots, a_{p-1})) = (a_i, a_{i+1}, \cdots, a_p, a_0, \cdots, a_{i-1})$ where $0 \le i \le p-1$.

(a) Show that $*$ is a group action (of $\mathbb{Z}_p$ on X).

(b) Show that $I \neq \varnothing$ where I is the $\mathbb{Z}_p$-fixed point set.

(c) Let $K = \{a \in H : a^p = e\}$ where e denotes the identity of H. Show that there is an 1-1 correspondence between I and K.

2. This is a continuation of Qn 1. Suppose the prime p does not divide n. Show that

(a) $|I| = 1$, $\quad G = \mathbb{Z}_p \curvearrowright X$, $|X| = n^{p-1}$

(b) the cardinality of a non-singleton orbit is p, $\quad n^{p-1} = 1 + \sum_{\text{All Distinct Nonsingleton orbits}} |G*x|$

(c) $n^p \equiv n \bmod p$ for any $n \in \mathbb{N}$.

[The result (c) is due to Fermat, called **Fermat's Little Theorem**.]

3. (a) i. How many 2-Sylow (r_2) and 3-Sylow (r_3) subgroups may a group of order 24 have?

$r_2 \mid 3$ and $2 \mid r_2 - 1 \Rightarrow r_2 = 1$ or 3

$r_3 \mid 8$ and $3 \mid r_3 - 1 \Rightarrow r_3 = 1$ or 4

ii. Find all 3-Sylow subgroups of S_4.

$\{e, (1,2,3), (3,2,1)\}$, etc.

(b) i. Let G be a group and H, K be its subgroups. Suppose $HK = KH$ and $H \cap K$ is the trivial subgroup. Show that G contains a subgroup of order $|H||K|$.

ii. Construct a 2-Sylow subgroup of S_4. $\{0, (1,2)\} \langle 4$

4. (a) Let G be a group that contains a p-Sylow subgroup P and a q-Sylow subgroup Q where $p > q$ are any two primes.

(a) Show that $P \cap Q$ is the trivial subgroup.

(b) If $|G| = pq$, show that $G = PQ$.

(c) If $|G| = pq$ and $q \nmid (p-1)$, show that $G \cong \mathbb{Z}_{pq}$.

(b) Show that A_5 has no subgroup of order 15.

5. Let G be a group of order 10. $\quad$ 2-Sylow: H $\quad$ 5-Sylow: K

(a) Show that G must contain a non-trivial proper normal subgroup. $[G:K] = 2 \Rightarrow K \trianglelefteq G$

(b) Show that a group G of order 10 is either isomorphic to $\mathbb{Z}_{10}$ or D_5.

[Hint: Figure out what can yxy^{-1} be if $x, y \in G$ and $\text{ord}(x) = 5$ and $\text{ord}(y) = 2$?]

End

yxy^{-1} is of order 5, $\quad yxy^{-1} = x$ or x^4