Introduction

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Contents

1 Foundations of Complex Analysis

Let Ω be a domain that is open and connected, and consider $f : \Omega \to \mathbb{C}$ as a complex differentiable function.

There are three primary approaches to studying complex analytic (holomorphic) functions:

1. **Partial Differential Equations** (Cauchy-Riemann equations)
2. **Power Series**
3. **Integral Formulas**

1.1 The Cauchy-Riemann Approach

For a function $f : \Omega \to \mathbb{C}$, we define the Wirtinger derivatives:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \tag{1}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{2}$$

The Cauchy-Riemann equations are equivalent to the condition:

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ everywhere on } \Omega$$

1.2 Power Series Approach

For $a \in \mathbb{C}$, consider a convergent power series $\sum_{n\geq 0} c_n(z-a)^n$ with radius of convergence $R > 0$. We define:

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \text{ for } |z-a| < R$$

that is, for $z \in D(a; R)$ (the open disk centered at a with radius R).

Theorem 1.1. *If f is holomorphic in $D(a; R)$, then:*

$$f'(z) = \sum_{n\geq 1} nc_n(z-a)^{n-1}$$

1.3 Cauchy Integral Formula

Suppose $f : D(a; R) \to \mathbb{C}$ is holomorphic. For any $0 < r < R$ and $z \in D(a; r)$, we have:

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(a;r)} \frac{f(\xi)}{z-\xi} d\xi$$

Theorem 1.2 (Taylor's Theorem for Holomorphic Functions)**.** *Let $f : D(a; R) \to \mathbb{C}$ be holomorphic on the open disk $D(a; R) = \{z \in \mathbb{C} : |z-a| < R\}$, where $R > 0$. Then f has a convergent power series expansion around a:*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n, \qquad |z-a| < R,$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Moreover, the radius of convergence of this series is at least R.

2 Meromorpphic Functions