THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 4 Solution

Problem 1.

(i) The equation is hyperbolic because $\mathcal{D} = (\frac{1}{2})^2 - (1)(-2) = \frac{9}{4} > 0$.

(ii) The equation (1) can be rewritten as

$$(\partial_t + 2\partial_x)(\partial_t - \partial_x)u = 0.$$

To find the general solution to (1), we need to solve

$$\begin{cases} \partial_t v + 2\partial_x v = 0 \\ \partial_t u - \partial_x u = v \end{cases}.$$

Note that the general solution of $\partial_t v + 2\partial_x v = 0$ is

$$v = f(x - 2t),$$

for arbitrary function f because v is a constant along the characteristic curve

$$C_{x_0} = \{(t, x) : x = 2t + x_0\}.$$

To solve $\partial_t u - \partial_x u = v = f(x - 2t)$, find the characteristic curves,

$$\begin{cases} \dfrac{dt}{ds} = 1, \; t(0) = 0 \\ \dfrac{dx}{ds} = -1, \; x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} t = s \\ x = -s + x_0 \end{cases}$$

Then $x = -t + x_0$, and the characteristic curves can be parametrized by x_0:

$$\tilde{C}_{x_0} = \{(t,\, x) : x = -t + x_0\}.$$

Let $W(s) := u(t(s),\ x(s))$. Then

$$\frac{dW}{ds} = v(t(s),\ x(s)) = f(x(s) - 2t(s)) = f(-s + x_0 - 2s) = f(-3s + x_0).$$

Integrate $\frac{dW}{ds}$ along the characteristic curve $\tilde{C}_{x_0}$ from $(0, x_0)$ to (t, x), we have

$$\begin{aligned} u(t,x) - u(0,x_0) &= \int_0^t f(-3s + x_0)\ ds = -\frac{1}{3}\int_{x_0}^{-3t+x_0} f(\tilde{s})\ d\tilde{s}\ (\tilde{s} = -3s + x_0) \\ &= -\frac{1}{3}\int_{x_0}^{x-2t} f(\tilde{s})\ d\tilde{s} = F(x - 2t), \end{aligned}$$

where $F' = -\dfrac{f}{3}$ with $F(x_0) = 0$. Hence

$$u(t,x) = u(0,x_0) + F(x-2t) = u(0,x+t) + F(x-2t) = g(x+t) + F(x-2t),$$

for arbitrary functions g, F.

Problem 2.

(i) By product rule, $(\partial_x x)v = \partial_x(xv) = v + (x\partial_x)v = (1 + x\partial_x)v$. Similarly, $(\partial_y y)v = (1 + y\partial_y)v$. Thus,

$$\begin{aligned} (\partial_x - 2xy\partial_y)^2 u &= (\partial_x - 2xy\partial_y)(\partial_x - 2xy\partial_y)u \\ &= \partial_{xx}u - 2y(\partial_x x)\partial_y u - 2xy\partial_{xy}u + 4x^2 y(\partial_y y)\partial_y u \\ &= \partial_{xx}u - 2y(1 + x\partial_x)\partial_y u - 2xy\partial_{xy}u + 4x^2 y(1 + y\partial_y)\partial_y u \\ &= \partial_{xx}u - 4xy\partial_{xy}u + 4x^2y^2\partial_{yy}u + 2y(2x^2 - 1)\partial_y u. \end{aligned}$$

(ii) The first equation follows from (i) and the second equation holds by definition.

(iii) To find the characteristic curves,

$$\begin{cases} \dfrac{dx}{ds} = 1,\ x(0) = x_0 \\ \dfrac{dy}{ds} = -2xy,\ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0 + s \\ \dfrac{dy}{y} = -2(x_0 + s)ds,\ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0 + s \\ \ln(\dfrac{y}{y_0}) = -2x_0 s - s^2 \end{cases}$$

Then $y = y_0 e^{-s^2-2x_0 s} = y_0 e^{x_0^2 - x^2}$. Choose $x_0 = 0$, the characteristic curves can be parametrized by y_0:

$$C_{y_0} = \{(x,\, y) : y = y_0 e^{-x^2}\}.$$

Note that $v(x,y)$ remains unchanged along each characteristic curves and hence for all $(x,\, y) \in C_{y_0}$,

$$v(x,\, y) = v(0,\, y_0) = v(0,\, ye^{x^2}) = g(ye^{x^2}), \text{ where } g \text{ is arbitrary.}$$

(iv) Let $W(s) = u(x(s), y(s))$. Then $\frac{dW(s)}{ds} = v$ implies $dW = v\, ds$. By integrating along C_{y_0} from $(0,\, y_0)$ to $(x,\, y)$, we have

$$u(x,y) - u(0,y_0) = s\, v(0,\, y_0) = x\, g(ye^{x^2})$$

and hence

$$u(x,y) = u(0, ye^{x^2}) + x\, g(ye^{x^2}) = f(ye^{x^2}) + x\, g(ye^{x^2}),$$

where f, g are arbitrary.

Problem 3. Note that

$$\begin{aligned}
(-y\partial_x + x\partial_y)^2 u &= (-y\partial_x + x\partial_y)(-y\partial_x + x\partial_y)u \\
&= [y^2\partial_{xx} - y(\partial_x x)\partial_y - x(\partial_y y)\partial_x + x^2\partial_{yy}]u \\
&= [y^2\partial_{xx} - y(x\partial_x + 1)\partial_y - x(y\partial_y + 1)\partial_x + x^2\partial_{yy}]u \\
&= [y^2\partial_{xx} - 2xy\partial_{xy} + x^2\partial_{yy} - x\partial_x - y\partial_y]u.
\end{aligned}$$

To find the characteristic curves,

$$\begin{cases} \dfrac{dx}{ds} = -y, \ x(0) = x_0 \\ \dfrac{dy}{ds} = x, \ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} \dfrac{d^2x}{ds^2} = -x, \ x(0) = x_0 \\ y = -\dfrac{dx}{ds}, \ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0\cos s - y_0 \sin s \\ y = x_0 \sin s + y_0 \cos s \end{cases} \Longrightarrow x^2 + y^2 = x_0^2 + y_0^2.$$

Choose $y_0 = 0$, the characteristic curves can be parametrized by x_0:

$$C_{x_0} = \{(x,\, y) : x^2 + y^2 = x_0^2\}.$$

Note that $v(x,y)$ remains unchanged along each characteristic curves and hence for all $(x,\,y)\in C_{x_0}$,

$$v(x,\,y) = v(x_0,\,0) = g(\sqrt{x^2+y^2}),\ \text{where } g \text{ is arbitrary.}$$

Let $W(s) = u(x(s),y(s))$. Then $\frac{dW(s)}{ds} = v$ implies $dW = v\,ds$. By integrating along C_{x_0} from $(x_0,\,0)$ to $(x,\,y)$, we have

$$u(x,\,y) - u(x_0,\,0) = s\,v(x_0,\,0) = \tan^{-1}(\frac{y}{x})g(\sqrt{x^2+y^2})$$

and hence

$$u(x,y) = u(x_0,0)+\tan^{-1}(\frac{y}{x})g(\sqrt{x^2+y^2}) = f(\sqrt{x^2+y^2})+\tan^{-1}(\frac{y}{x})g(\sqrt{x^2+y^2}),$$

where f, g are arbitrary.

Remark. This problem can be solved by using polar coordinate, instead of the method of characteristics.

Problem 4.

(i) By the d'Alembert formula,

$$u(t,x) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds.$$

Then

$$\begin{aligned}
u(t,\,-x) &= \frac{1}{2}[\phi(-x+ct)+\phi(-x-ct)] + \frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(s)\,ds \\
&= \frac{1}{2}[\phi(x-ct)+\phi(x+ct)] - \frac{1}{2c}\int_{x+ct}^{x-ct}\psi(\tilde{s})\,d\tilde{s} \quad (\because\ \phi,\psi \text{ are even and let } \tilde{s}=-s) \\
&= \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\tilde{s})\,d\tilde{s} \\
&= u(t,\,x).
\end{aligned}$$

(ii)

$$\begin{aligned}
u(t,\,x) &= \frac{1}{2}[(x+ct)^4+(x-ct)^4] + \frac{1}{2c}\int_{x-ct}^{x+ct}\cos s\,ds \\
&= x^4+6c^2t^2x^2+c^4t^4+\frac{1}{2c}[\sin(x+ct)-\sin(x-ct)] \\
&= x^4+6c^2t^2x^2+c^4t^4+\frac{\cos x\sin ct}{c}.
\end{aligned}$$

Problem 5. For

$$\partial_{tt} u_h - \partial_{xx} u_h = 0,$$

the general solution is

$$u_h(t,x) = f(x+t) + g(x-t).$$

Now $0 = u_h|_{x=0} = f(t) + g(-t)$ implies

$$u(t,x) = f(x+t) - f(t-x).$$

Then $0 = u_h|_{x=1} = f(1+t) - f(t-1)$ implies $f(t+1) = f(t-1)$ and hence $f(t+1) = f(t-1)$ for all t. So f is a periodic function with period 2.

For

$$\partial_{xx} u_p = e^x,$$

by integrating the equation two times, $u_p(x) = e^x + Ax + B$. Moreover, $u_p(0) = e^0 + A(0) + B = 0$ implies $B = -1$. And $u_p(1) = e + A - 1 = 0$ implies $A = 1 - e$. So $u_p(x) = e^x + (1-e)x - 1$. Thus,

$$u(t,x) = u_h(t,x) + u_p(t,x) = f(x+t) - f(t-x) + e^x + (1-e)x - 1,$$

where f is any periodic function with period 2.