20241101 MATH3541 NOTE 8[1]

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Contents

1 Introduction

How to treat a function $f : X \to Y$ as a point in Y^X.
(1) First, we discuss metrics on Y^X.
(2) Second, we discuss topologies on Y^X.
(3) Third, we discuss homotopy relation on Y^X.

2 Metrics on Y^X

2.1 Uniform Norm

Definition 2.1. (**Uniform Norm**)
Let $\mathcal{B}(X, \mathbb{R})$ be the set of all bounded functions from a nonempty set X to $\mathbb{R}$.
Define $\| \bullet \| : \mathcal{B}(X, \mathbb{R}) \to \mathbb{R}, f \mapsto \sup_{x \in X} |f(x)|$ as the uniform norm on $\mathcal{B}(X, \mathbb{R})$.

Proposition 2.2. The uniform norm $\| \bullet \|$ is a well-defined norm on $B(X, \mathbb{R})$.

Proof. We may divide our proof into four parts.
Part 1: For all $f \in \mathcal{B}(X, \mathbb{R})$, construct the following set.

$$\mathrm{Im}(|f|) = \{|f(x)| \in \mathbb{R} : x \in X\}$$

$\mathrm{Im}(|f|)$ is bounded above, so a unique supremum $\|f\| = \sup_{x \in X} |f(x)|$ exists.
Hence, $\| \bullet \|$ is well-defined.
Part 2: For all $f \in \mathcal{B}(X, \mathbb{R})$:

$$\left[\|f\| = \sup_{x \in X} |f(x)| \geq 0\right] \text{ and } \left[\|f\| = 0 \implies \sup_{x \in X} |f(x)| = 0 \implies f = 0\right]$$

Hence, $\| \bullet \|$ is positive definite.
Part 3: For all $\lambda \in \mathbb{R}$ and $f \in \mathcal{B}(X, \mathbb{R})$:

$$\|\lambda f\| = \sup_{x \in X} |\lambda f(x)| = |\lambda| \sup_{x \in X} |f(x)| = |\lambda| \|f\|$$

Hence, $\| \bullet \|$ is absolute homogeneous.
Part 4: For all $f, f' \in \mathcal{B}(X, \mathbb{R})$:

$$\begin{aligned} \|f + f'\| &= \sup_{x \in X} |f(x) + f'(x)| \leq \sup_{x \in X} (|f(x)| + |f'(x)|) \\ &\leq \sup_{x \in X} |f(x)| + \sup_{x' \in X} |f'(x')| = \|f\| + \|f'\| \end{aligned}$$

Hence, $\| \bullet \|$ is subadditive.
Combine the four parts above, we've proven that $\|\bullet\|$ is a well-defined norm on $B(X, \mathbb{R})$.
Quod. Erat. Demonstrandum. $\square$

Proposition 2.3. Continuity is preserved under uniform limit.

Proof. Take a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{B}(X,\mathbb{R})$, all of which are continuous at some $x \in X$. Assume that $(f_n)_{n\in\mathbb{N}}$ converges to some $f_* \in \mathcal{B}(X,\mathbb{R})$ uniformly.
For all $\epsilon > 0$, we wish to find $\delta > 0$, such that for all $x' \in B(x,\delta)$:

$$|f_*(x) - f_*(x')| < \epsilon$$

Step 1: As $(f_n)_{n\in\mathbb{N}}$ converges to f_* uniformly, there exists $N \in \mathbb{N}$, such that:

$$\sup_{x\in X}|f_N(x) - f_*(x)| = \|f_N - f_*\| < \frac{\epsilon}{3}$$

Step 2: As f_N is continuous at x, there exists $\delta > 0$, such that for all $x' \in B(x,\delta)$:

$$|f_N(x) - f_N(x')| < \frac{\epsilon}{3}$$

Step 3: For this $\epsilon > 0$, there exists $\delta > 0$, such that for all $x' \in B(x,\delta)$:

$$|f_*(x) - f_*(x')| \leq |f_*(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f_*(x')| < \epsilon$$

Hence, f_* is continuous at this $x \in X$,
continuity is preserved under uniform limit.
Quod. Erat. Demonstrandum. $\square$

Remark: *Continuity is not preserved under pointwise limit, consider the sequence* $(x^n)_{n\in\mathbb{N}}$ *of continuous functions in* $\mathcal{B}([0,1],\mathbb{R})$*. The pointwise limit* f *is discontinuous.*

$$f : [0,1] \to \mathbb{R}, f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1; \\ 1 & \text{if} \quad \quad x = 1; \end{cases}$$

Proposition 2.4. Uniform continuity is preserved under uniform limit.

Proof. Take a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{B}(X,\mathbb{R})$, all of which are uniformly continuous on X. Assume that $(f_n)_{n\in\mathbb{N}}$ converges to some $f_* \in \mathcal{B}(X,\mathbb{R})$ uniformly.
For all $\epsilon > 0$, we wish to find $\delta > 0$, such that for all $x, x' \in X$ within distance δ:

$$|f_*(x) - f_*(x')| < \epsilon$$

Step 1: As $(f_n)_{n\in\mathbb{N}}$ converges to f_* uniformly, there exists $N \in \mathbb{N}$, such that:

$$\sup_{x\in X}|f_N(x) - f_*(x)| = \|f_N - f_*\| < \frac{\epsilon}{3}$$

Step 2: As f_N is uniformly continuous on X, there exists $\delta > 0$, such that for all $x, x' \in X$ within distance δ:

$$|f_N(x) - f_N(x')| < \frac{\epsilon}{3}$$

Step 3: For this $\epsilon > 0$, there exists $\delta > 0$, such that for all $x, x' \in X$ within distance δ:

$$|f_*(x) - f_*(x')| \leq |f_*(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f_*(x')| < \epsilon$$

Hence, f_* is uniformly continuous on X,
uniform continuity is preserved under uniform limit.
Quod. Erat. Demonstrandum. □

Proposition 2.5. k-Lipschitz continuity is preserved under uniform limit.

Proof. Take a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{B}(X, \mathbb{R})$, all of which are $k-$Lipschitz continuous on X. Assume that $(f_n)_{n\in\mathbb{N}}$ converges to some $f_* \in \mathcal{B}(X, \mathbb{R})$ uniformly.
For all $\epsilon > 0$, for all $x, x' \in X$, we wish to prove that:

$$|f_*(x) - f_*(x')| < kd_X(x, x') + \epsilon$$

Step 1: As $(f_n)_{n\in\mathbb{N}}$ converges to f_* uniformly, there exists $N \in \mathbb{N}$, such that:

$$\sup_{x\in X}|f_N(x) - f_*(x)| = \|f_N - f_*\| < \frac{\epsilon}{3}$$

Step 2: As f_N is k-Lipschitz continuous on X, for all $x, x' \in X$:

$$|f_N(x) - f_N(x')| < kd_X(x, x') + \frac{\epsilon}{3}$$

Step 3: For this $\epsilon > 0$, for all $x, x' \in X$:

$$|f_*(x) - f_*(x')| \leq |f_*(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f_*(x')| < kd_X(x, x') + \epsilon$$

Hence, f_* is k-Lipschitz continuous on X,
k-Lipschitz continuity is preserved under uniform limit.
Quod. Erat. Demonstrandum. □

2.2 p-norm

Definition 2.6. (p**-norm**)
Let $p \geq 1$ be a number, $\ell^p(K, \mathbb{R})$ be the set of all sequence a
from a nonempty subset K of $\mathbb{Z}$ to $\mathbb{R}$ such that $\sum_K |a_k|^p$ converges.
Define $\| \bullet \|_p : \ell^p(K, \mathbb{R}) \to \mathbb{R}, a \mapsto (\sum_K |a_k|^p)^{1/p}$ as the p-norm on $\ell^p(K, \mathbb{R})$

Definition 2.7. (**p-norm**)
Let $p \geq 1$ be a number, $L^p(I, \mathbb{R})$ be the set of all continuous function f from a nonempty interval I to $\mathbb{R}$ such that $\int_I |f(x)|^p \mathrm{d}x$ converges.
Define $\| \bullet \|_p : L^p(I, \mathbb{R}) \to \mathbb{R}, f \mapsto \left(\int_I |f(x)|^p \mathrm{d}x\right)^{1/p}$ as the p-norm on $L^p(I, \mathbb{R})$.

Theorem 2.8. (**Young's Inequality**)
Let A, B, p, q be four nonnegative numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. We have:

$$\frac{A^p}{p} + \frac{B^q}{q} \geq AB, \text{ with equality iff } A^p = B^q$$

Proof. WLOG, assume that $A, B > 0$. As ln is strictly concave, we have:

$$\ln\left(\frac{A^p}{p} + \frac{B^q}{q}\right) \geq \frac{\ln(A^p)}{p} + \frac{\ln(B^q)}{q}, \text{ with equality iff } A^p = B^q$$

It suffices to apply the exponential map $x \mapsto \mathrm{e}^x$. Quod. Erat. Demonstrandum. $\square$

Theorem 2.9. (**Hölder's Inequality**)
Let p, q be two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$,
and a, b be two sequences in $\ell^p(K, \mathbb{R}), \ell^q(K, \mathbb{R})$ respectively. We have:

$$\|ab\|_1 \leq \|a\|_p \|b\|_q, \text{ with equality iff } \mathrm{Rank}(|a|^p, |b|^q) < 2$$

Proof. WLOG, assume that $\|a\|_p, \|b\|_q > 0$. For each $k \in K$, define:

$$A_k = \frac{|a_k|}{\|a\|_p}, B_k = \frac{|b_k|}{\|b\|_q} \geq 0$$

Apply **Young's Inequality**, and we get:

$$\frac{|a_k|^p}{p\|a\|_p^p} + \frac{|b_k|^q}{q\|b\|_q^q} \geq \frac{|a_k b_k|}{\|a\|_p \|b\|_q}, \text{ with equality iff } \frac{|a_k|^p}{\|a\|_p^p} = \frac{|b_k|^q}{\|b\|_q^q}$$

Sum both sides over K, we get:

$$1 \geq \frac{\|ab\|_1}{\|a\|_p \|b\|_q}, \text{ with equality iff } \mathrm{Rank}(|a|^p, |b|^q) < 2$$

It suffices to multiply both sides by $\|a\|_p \|b\|_q$. Quod. Erat. Demonstrandum. $\square$

Theorem 2.10. (**Hölder's Inequality**)
Let p, q be two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$,
and f, g be two functions in $L^p(I, \mathbb{R}), L^q(I, \mathbb{R})$ respectively. We have:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \text{ with equality iff } \mathrm{Rank}(|f|^p, |g|^q) < 2$$

Proof. WLOG, assume that $\|f\|_p, \|g\|_q > 0$. For each $x \in I$, define:

$$A(x) = \frac{|f(x)|}{\|f\|_p}, B(x) = \frac{|g(x)|}{\|g\|_q} \geq 0$$

Apply **Young's Inequality**, and we get:

$$\frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q} \geq \frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}, \text{ with equality iff } \frac{|f(x)|^p}{\|f\|_p^p} = \frac{|g(x)|^q}{\|g\|_q^q}$$

Integrate both sides over I, we get:

$$1 \geq \frac{\|fg\|_1}{\|f\|_p\|g\|_q}, \text{ with equality iff } \mathrm{Rank}(|f|^p, |g|^q) < 2$$

It suffices to multiply both sides by $\|a\|_p\|b\|_q$. Quod. Erat. Demonstrandum. ☐

Theorem 2.11. (**Minkowski's Inequality**)
Let $p \geq 1$ be a number, and a, a' be two sequences in $\ell^p(K, \mathbb{R})$. We have:

$$\|a + a'\|_p \leq \|a\|_p + \|a'\|_p$$

Proof. WLOG, assume that $\|a + a'\|_p > 0$.

$$\begin{aligned}
\|a + a'\|_p^p = \sum_K |a_k + a'_k|^p &\leq \sum_K (|a_k| + |b_k|)|a_k + a'_k|^{p-1} \\
&= \sum_K |a_k||a_k + a'_k|^{p-1} + \sum_K |a'_k||a_k + a'_k|^{p-1} \\
&= \|a|a + a'|^{p-1}\|_1 + \|a'|a + a'|^{p-1}\|_1 \\
&\leq \|a\|_p\||a + a'|^{p-1}\|_{\frac{p}{p-1}} + \|a'\|_p\||a + a'|^{p-1}\|_{\frac{p}{p-1}} \\
&= (\|a\|_p + \|a'\|_p)\|a + a'\|_p^{p-1}
\end{aligned}$$

It suffices to cancel $\|a + a'\|_p^{p-1}$. Quod. Erat. Demonstrandum. ☐

Remark: *As a direct consequence, the p-norm is a norm on* $\ell^p(K, \mathbb{R})$.

Theorem 2.12. (**Minkowski's Inequality**)
Let $p \geq 1$ be a number, and f, f' be two functions in $L^p(I, \mathbb{R})$. We have:

$$\|f + f'\|_p \leq \|f\|_p + \|f'\|_p$$

Proof. WLOG, assume that $\|f+f'\|>0$.

$$\begin{aligned}\|f+f'\|_p^p &= \int_I |f(x)+f'(x)|^p \mathrm{d}x \le \int_I [|f(x)|+|f'(x)|]|f(x)+f'(x)|^{p-1}\mathrm{d}x \\ &= \int_I |f(x)||f(x)+f'(x)|^{p-1}\mathrm{d}x + \int_I |f'(x)||f(x)+f'(x)|^{p-1}\mathrm{d}x \\ &= \|f|f+f'|^{p-1}\|_1 + \|f'|f+f'|^{p-1}\|_1 \\ &\le \|f\|_p\||f+f'|^{p-1}\|_{\frac{p}{p-1}} + \|f'\|_p\||f+f'|^{p-1}\|_{\frac{p}{p-1}} \\ &= (\|f\|_p+\|f'\|_p)\|f+f'\|_p^{p-1}\end{aligned}$$

It suffices to cancel $\|f+f'\|_p^{p-1}$. Quod. Erat. Demonstrandum. ☐

Remark: *As a direct consequence, the p-norm is a norm on $L^p(I,\mathbb{R})$.*

2.3 Operator Norm

Definition 2.13. (**Operator Norm**)
Let U,V be two nontrivial normed vector spaces,
and $\mathcal{L}(U,V)$ be the set of all continuous linear operators from U to V.
Define $\|\bullet\|:\mathcal{L}(U,V)\to\mathbb{R}, A\mapsto \sup\limits_{\mathbf{u}\neq\mathbf{0}}\frac{\|A\mathbf{u}\|_V}{\|\mathbf{u}\|_U}$ as the operator norm on $\mathcal{L}(U,V)$.

Proposition 2.14. The operator norm $\|\bullet\|$ is a well-defined norm on $\mathcal{L}(U,V)$.

Proof. We may divide our proof into four parts.
Part 1: For all $A\in\mathcal{L}(U,V)$, since A is continuous, $\|A\mathbf{u}\|_V$ has some upperbound β on some hypersphere S centred at $\mathbf{0}\in U$ with radius $r>0$. This implies:

$$\forall \mathbf{u}\neq\mathbf{0}, \exists \mathbf{u}_0 = \frac{r}{\|\mathbf{u}\|_U}\mathbf{u}\neq\mathbf{0}, \frac{\|A\mathbf{u}\|_V}{\|\mathbf{u}\|_U} = \frac{\|A\mathbf{u}_0\|_V}{\|\mathbf{u}_0\|_U} \le \frac{\beta}{r}$$

Hence, $\|\bullet\|$ is well-defined.
Part 2: For all $\mathbf{u}\neq\mathbf{0}$:

$$\|A\| = \sup_{\mathbf{u}\neq\mathbf{0}}\frac{\|A\mathbf{u}\|_V}{\|\mathbf{u}\|_U} \ge 0$$

Hence, $\|\bullet\|$ is positive definite.
Part 3: For all $\lambda\in\mathbb{R}$ and $A\in\mathcal{L}(U,V)$:

$$\|\lambda A\| = \sup_{\mathbf{u}\neq\mathbf{0}}\frac{\|\lambda A\mathbf{u}\|_V}{\|\mathbf{u}\|_U} = |\lambda|\sup_{\mathbf{u}\neq\mathbf{0}}\frac{\|A\mathbf{u}\|_V}{\|\mathbf{u}\|_U} = |\lambda|\|A\|$$

Hence, $\|\bullet\|$ is absolute homogeneous.

Part 4: For all $A, A' \in \mathcal{L}(U, V)$:

$$\begin{aligned}\|A + A'\| = \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\|(A + A')\mathbf{u}\|_V}{\|\mathbf{u}\|_U} &\leq \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\|A\mathbf{u}\|_V + \|A'\mathbf{u}\|_V}{\|\mathbf{u}\|_U} \\ &\leq \sup_{\mathbf{u}\neq\mathbf{0}} \frac{\|A\mathbf{u}\|_V}{\|\mathbf{u}\|_U} + \sup_{\mathbf{u}'\neq\mathbf{0}} \frac{\|A'\mathbf{u}'\|_V}{\|\mathbf{u}'\|_U} = \|A\| + \|A'\|\end{aligned}$$

Hence, $\| \bullet \|$ is subadditive.
Combine the four parts above, we've proven that $\|\cdot\|$ is a well-defined norm on $\mathcal{L}(U, V)$.
Quod. Erat. Demonstrandum. □

Proposition 2.15. The operator norm $\| \bullet \|$ is also submultiplicative.

Proof. For all $A \in \mathcal{L}(U, V)$ and $B \in \mathcal{L}(V, W)$, as topology and linear structure are preserved under composition, $BA \in \mathcal{L}(U, W)$. Besides, for all $\mathbf{u} \in U$:

$$\|BA\mathbf{u}\|_W \leq \|B\|\|A\mathbf{u}\|_V \leq \|B\|\|A\|\|\mathbf{u}\|_U$$

Hence, $\|BA\| \leq \|B\|\|A\|$, $\| \bullet \|$ is submultiplicative. Quod. Erat. Demonstrandum. □

2.4 Examples

Lemma 2.16. In a series RC circuit, assume that the resistance R and the capacitance C are positive. The solution to the following Kirchhoff's equation:

$$\frac{Q_{\text{out}}(t)}{C} + R\dot{Q}_{\text{out}}(t) = V_{\text{in}}(t)$$

is:

$$Q_{\text{out}}(t) = Q_{\text{out}}(0)\mathrm{e}^{-\frac{t}{RC}} + \frac{1}{R}\int_0^t V_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\,\mathrm{d}u$$

Proof.

$$\begin{aligned}\frac{Q_{\text{out}}(t)}{C} + R\dot{Q}_{\text{out}}(t) = V_{\text{in}}(t) &\iff \dot{Q}_{\text{out}}(t) + \frac{Q_{\text{out}}(t)}{RC} = \frac{V_{\text{in}}(t)}{R} \\ &\iff \frac{\mathrm{d}}{\mathrm{d}t}\left[Q_{\text{out}}(t)\mathrm{e}^{\frac{t}{RC}}\right] = \frac{1}{R}V_{\text{in}}(t)\mathrm{e}^{\frac{t}{RC}} \\ &\iff Q_{\text{out}}(t)\mathrm{e}^{\frac{t}{RC}} - Q_{\text{out}}(0) = \frac{1}{R}\int_0^t V_{\text{in}}(u)\mathrm{e}^{\frac{u}{RC}}\,\mathrm{d}u \\ &\iff Q_{\text{out}}(t) = Q_{\text{out}}(0)\mathrm{e}^{-\frac{t}{RC}} + \frac{1}{R}\int_0^t V_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\,\mathrm{d}u\end{aligned}$$

Quod. Erat. Demonstrandum. □

Proposition 2.17. If the input signal V_{in} is k-Lipschitz continuous, then:

$$\overline{\lim_{t\to+\infty}} |Q_{\text{out}}(t) - CV_{\text{in}}(t)| \leq kRC^2$$

Proof. As $t \to +\infty$, we have:

$$\begin{aligned}
|Q_{\text{out}}(t) - CV_{\text{in}}(t)| &= \left| Q_{\text{out}}(0)\mathrm{e}^{-\frac{t}{RC}} + \frac{1}{R}\int_0^t V_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u - \frac{1}{R}\int_{-\infty}^t V_{\text{in}}(t)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u \right| \\
&= \left| \frac{1}{R}\int_0^t V_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u - \frac{1}{R}\int_0^t V_{\text{in}}(t)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u \right| + \mathcal{O}(1) \\
&\leq \frac{1}{R}\int_0^t |V_{\text{in}}(u) - V_{\text{in}}(t)|\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u + \mathcal{O}(1) \\
&\leq \frac{1}{R}\int_0^t k(t-u)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u + \mathcal{O}(1) \\
&\leq \frac{1}{R}\int_{-\infty}^t k(t-u)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u + \mathcal{O}(1) = kRC^2 + \mathcal{O}(1)
\end{aligned}$$

It suffices to take upper limit $t \to +\infty$. Quod. Erat. Demonstrandum. □

Remark: *Notice that we have a uniform upper bound kRC^2 for the error induced in the approximation $\hat{Q}_{\text{out}}(t) = CV_{\text{in}}(t)$, which is proportional to the resistance R.*

Proposition 2.18. Let ω be a positive constant. For certain input signal:

$$V_{\text{in}}(t) = V_{\text{in}}(0)\cos(\omega^2 t^2)$$

We have:

$$\lim_{t\to+\infty} |Q_{\text{out}}(t)| = 0$$

Proof. As $t \to +\infty$, we have:

$$\begin{aligned}
|Q_{\text{out}}(t)| &= \left| Q_{\text{out}}(0)\mathrm{e}^{-\frac{t}{RC}} + \frac{1}{R}\int_0^t V_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u \right| \\
&= \frac{1}{R}\left| \int_0^t \dot{\Phi}_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u \right| + \mathcal{O}(1) \\
&= \frac{1}{R}\left| \Phi_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\Big|_0^t - \int_0^t \Phi_{\text{in}}(u)\frac{\mathrm{d}\mathrm{e}^{\frac{u-t}{RC}}}{\mathrm{d}u}\mathrm{d}u \right| + \mathcal{O}(1) \\
&= \frac{1}{R^2C}\left| \int_0^t \Phi_{\text{in}}(u)\mathrm{e}^{\frac{u-t}{RC}}\mathrm{d}u \right| + \mathcal{O}(1)
\end{aligned}$$

Here, $\Phi_{\text{in}}(t) = \int_{+\infty}^t V_{\text{in}}(u)\mathrm{d}u$ is the integrated input signal with $\lim_{t\to+\infty}\Phi_{\text{in}}(t) = 0$. According to **Silverman-Toeplitz Theorem**, the last integral converges to 0. Quod. Erat. Demonstrandum. □

Remark: *Notice that if $V_{\text{in}}(0) \neq 0$, then we no longer have the upperbound kRC^2 because the input signal $V_{\text{in}}(t) = V_{\text{in}}(0)\cos(\omega^2 t^2)$ is not k-Lipschitz continuous.*

Theorem 2.19. (**Cantor's Lemma**)
Let X be a compact metric space, Y be an arbitrary metric space, and $f : X \to Y$ be a function. f is pointwisely continuous implies f is uniformly continuous.

Proof. Assume to the contrary that f is not uniformly continuous, that is, for some $\epsilon_0 > 0$, there exist two sequences $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$, such that:

$$\lim_{n\to+\infty} d_X(x_n, x'_n) = 0 \text{ and } \inf_{n\in\mathbb{N}} d_Y(f(x_n), f(x'_n)) \geq \epsilon_0$$

As X is (sequentially) compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit $x_* \in X$, and $\lim\limits_{n\to+\infty} d_X(x_n, x'_n) = 0$ suggests $\lim\limits_{k\to+\infty} x'_{n_k} = \lim\limits_{k\to+\infty} x_{n_k} = x_*$.
As f is continuous, we have:

$$\lim_{k\to+\infty} x'_{n_k} = \lim_{k\to+\infty} x_{n_k} = x_* \implies \lim_{k\to+\infty} f(x'_{n_k}) = \lim_{k\to+\infty} f(x_{n_k}) = f(x_*)$$

This contradicts to $\inf\limits_{n\in\mathbb{N}} d_Y(f(x_n), f(x'_n)) \geq \epsilon_0$. Quod. Erat. Demonstrandum. ☐

Theorem 2.20. (**Banach Fixed Point Theorem**)
Let X be a complete metric space, $0 \leq k < 1$ be number,
and $T : X \to X$ be a k-Lipschitz continuous function.
(1) T has a unique fixed point $x_* \in X$.
(2) For all $x \in X$, the sequence $(x_n)_{n=0}^{+\infty}$ defined below converges to x_*.

$$x_n = \begin{cases} x & \text{if} \quad n = 0; \\ T(x_{n-1}) & \text{if} \quad n > 0; \end{cases}$$

Proof. We may divide our proof into four steps.
Step 1: For all $x \in X$, we wish to prove that $(x_n)_{n=0}^{+\infty}$ is Cauchy.
The recurrence relation $d_X(x_{n+2}, x_{n+1}) \leq k d_X(x_{n+1}, x_n)$ suggests that:

$$d_X(x_{n+1}, x_n) \leq k^n d_X(x_1, x_0)$$

where the convention $0^0 = 1$ is made. For all $\epsilon > 0$, choose $N \in \mathbb{N}$ such that:

$$\frac{k^N d_X(x_1, x_0)}{1-k} < \epsilon$$

For this $N \in \mathbb{N}$, for all $n, m \geq N$, WLOG, assume that $n > m$, then:

$$\begin{aligned}
d_X(x_n, x_m) &\leq d_X(x_n, x_{n-1}) + d_X(x_{n-1}, x_{n-2}) + \cdots + d_X(x_{m+1}, x_m) \\
&\leq k^{n-1} d_X(x_1, x_0) + k^{n-2} d_X(x_1, x_0) + \cdots + k^m d_X(x_1, x_0) \\
&= \frac{k^m - k^n}{1-k} d_X(x_1, x_0) \leq \frac{k^N}{1-k} d_X(x_1, x_0) < \epsilon
\end{aligned}$$

Step 2: For all $x, x' \in X$, we wish to prove that $\lim_{n \to +\infty} d_X(x_n, x'_n) = 0$.
The recurrence relation $d_X(x_{n+1}, x'_{n+1}) \leq k d_X(x_n, x'_n)$ suggests that:

$$d_X(x_n, x'_n) \leq k^n d_X(x_0, x'_0)$$

For all $\epsilon > 0$, choose $N \in \mathbb{N}$ such that:

$$k^N d_X(x_0, x'_0) < \epsilon$$

For this $N \in \mathbb{N}$, for all $n \geq N$:

$$d_X(x_n, x'_n) \leq k^n d_X(x_0, x'_0) \leq k^N d_X(x_0, x'_0) < \epsilon$$

Step 3: Assume that the common limit is x_*.
As $\lim_{n \to +\infty} x_n = x_*$, its subsequence $(T(x_n) = x_{n+1})_{n \in \mathbb{N}}$ converges to x_* as well, so:

$$x_* = \lim_{n \to +\infty} T(x_n) = T\left(\lim_{n \to +\infty} x_n\right) = T(x_*)$$

That is, x_* is fixed under T.
Step 4: For all fixed points x_*, x^* under T:

$$d_X(x_*, x^*) = d_X(T(x_*), T(x^*)) \leq k d_X(x_*, x^*)$$

As $0 \leq k < 1$, the only real number $d_X(x_*, x^*)$ satisfying this is 0, and the positive definiteness of d_X suggests that $x_* = x^*$, so the fixed point is unique.
Quod. Erat. Demonstrandum. $\square$

3 Topologies on Y^X

3.1 Product Space Topology

Definition 3.1. (**Product Space Topology**)
Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces, and X be the Cartesian product of $(X_\lambda)_{\lambda \in I}$. Define the product space topology $\mathcal{O}_X$ of $(X_\lambda)_{\lambda \in I}$ on X as the topology generated by the subbasis $\mathcal{B}_X$, where $\mathcal{B}_X$ is the union of each initial topology $\mathcal{O}_X^{(\lambda)}$ of X_λ on X via $\pi_\lambda : X \to X_\lambda, x \mapsto x(\lambda)$.

Proposition 3.2. Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces, and X be the product space of $(X_\lambda)_{\lambda \in I}$. Each map $\pi_\mu : X \to X_\mu, x \mapsto x(\mu)$ is open.

Proof. For each π_μ, for each open subset U of X,
we wish to show that $\pi_\mu(U)$ is open in X_μ.
As X has a subbasis, we may shrink U to $\bigcap_{k=1}^m \pi_{\lambda_k}^{-1}(U_{\lambda_k})$,
where each U_{λ_k} is a nonempty open subset of X_{λ_k}.
Case 1: If μ is equal to some λ_k, then $\pi_\mu(U) = U_\mu$ is open in X_μ.
Case 2: If μ is equal to no λ_k, then $\pi_\mu(U) = X_\mu$ is open in X_μ.
Hence, π_μ is open. Quod. Erat. Demonstrandum. $\square$

Remark: *In $\mathbb{R}^2$, consider the map $\pi_1 : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1$.*

$$U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 1\} \text{ is closed in } \mathbb{R}^2 \text{ and } \pi_1(U) = \{0\}^c \text{ is not closed in } \mathbb{R}$$

$$V = [0, 1] \text{ is compact in } \mathbb{R} \text{ and } \pi_1^{-1}(V) = [0, 1] \times \mathbb{R} \text{ is not compact in } \mathbb{R}^2$$

Hence, π_μ is neither closed nor proper in general.

Proposition 3.3. Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces, and X be the product space of $(X_\lambda)_{\lambda \in I}$. X is Hausdorff iff each X_μ is Hausdorff.

Proof. We may divide our proof into two parts.
"if" direction: Assume that each X_μ is Hausdorff.
For all distinct $x, x' \in X$, some $x(\lambda) \neq x'(\lambda)$.
As X_λ is Hausdorff, there exist open subsets U_λ, U'_λ of X_λ, such that:

$$x(\lambda) \in U_\lambda \text{ and } x'(\lambda) \in U'_\lambda \text{ and } U_\lambda \cap U'_\lambda = \emptyset$$

Hence, there exist open subsets $\pi_\lambda^{-1}(U_\lambda), \pi_\lambda^{-1}(U'_\lambda)$ of X, such that:

$$x \in \pi_\lambda^{-1}(U_\lambda) \text{ and } x' \in \pi_\lambda^{-1}(U'_\lambda) \text{ and } \pi_\lambda^{-1}(U_\lambda) \cap \pi_\lambda^{-1}(U'_\lambda) = \emptyset$$

To conclude, X is Hausdorff.
"only if" direction: Assume that X is Hausdorff.
Fix an arbitrary element ξ in X.
For each X_μ, for all distinct $x_\mu, x'_\mu \in X_\mu$,
construct the following two distinct elements of X:

$$x, x' : I \to \bigcup_{\lambda \in I} X_\lambda, x, x'(\lambda) \begin{cases} = & x_\mu, x'_\mu & \text{if} & \lambda = \mu; \\ = & \xi(\lambda) & \text{if} & \lambda \neq \mu; \end{cases}$$

As X is Hausdorff, there exists open subsets U, U' of X, such that:

$$x \in U \text{ and } x' \in U' \text{ and } U \cap U' = \emptyset$$

As X has a subbasis, we may shrink U, U' to $\bigcap_{k=1}^{m} U_{\lambda_k}, \bigcap_{k=1}^{m} U'_{\lambda_k}$,
where each $U_{\lambda_k}, U'_{\lambda_k}$ are nonempty open subsets of X_{λ_k}.
As the only different component of x, x' is the μ^{th} component,
μ is equal to some λ_k and $U_\mu \cap U'_\mu = \emptyset$,
so there exist open subsets U_μ, U'_μ of X_μ, such that:

$$x_\mu \in U_\mu \text{ and } x'_\mu \in U'_\mu \text{ and } U_\mu \cap U'_\mu = \emptyset$$

To conclude, X_μ is Hausdorff.
Combine the two parts above, we've proven the logical equivalence.
Quod. Erat. Demonstrandum. $\square$

Proposition 3.4. Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces, and X be the product space of $(X_\lambda)_{\lambda \in I}$. X is regular iff each X_μ is regular.

Proof. We may divide our proof into two parts.
"if" direction: Assume that each X_μ is regular.
For all closed subset V of X and element x of V^c.
As X has a subbasis, we may shrink V^c to $\bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(U_{\lambda_k})$,
where each U_{λ_k} is a nonempty open subset of X_{λ_k}.
As each X_{λ_k} is regular, there exist open subsets $W_{\lambda_k}, W'_{\lambda_k}$ of X_{λ_k}, such that:

$$x(\lambda_k) \in W_{\lambda_k} \text{ and } \pi_{\lambda_k}^{-1}(U_{\lambda_k}^c) \subseteq W'_{\lambda_k} \text{ and } W_{\lambda_k} \cap W'_{\lambda_k} = \emptyset$$

Hence, there exist open subsets $W = \bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(W_{\lambda_k})$,
$W' = \bigcup_{k=1}^{m} \pi_{\lambda_k}^{-1}(W'_{\lambda_k})$ of X, such that:

$$x \in W \text{ and } \bigcup_{k=1}^{m} \pi_{\lambda_k}^{-1}(U_{\lambda_k}^c) \subseteq W' \text{ and } W \cap W' = \emptyset$$

To conclude, X is regular.
"only if" direction: Assume that X is regular.
Fix an arbitrary element ξ in X.
For each X_μ, for all closed subset V_μ of X_μ and an element x_μ of V_μ^c,
construct the following element of X and the following subset of X:

$$x : I \to \bigcup_{\lambda \in I} X_\lambda, x(\lambda) \begin{cases} = & x_\mu & \text{if} & \lambda = \mu; \\ = & \xi(\lambda) & \text{if} & \lambda \neq \mu; \end{cases}$$

$$V = \left\{ x' : I \to \bigcup_{\lambda \in I} X_\lambda, x'(\lambda) \begin{cases} \in & V_\mu & \text{if} & \lambda = \mu; \\ = & \xi(\lambda) & \text{if} & \lambda \neq \mu; \end{cases} \right\}$$

As X is regular, there exist open subsets W, W' of X, such that:

$$x \in W \text{ and } V \subseteq W' \text{ and } W \cap W' = \emptyset$$

As X has a subbasis, we may shrink W to $\bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(W_{\lambda_k})$,
where each U_{λ_k} is a nonempty open subset of X_{λ_k}.
As X has a subbasis, we may shrink W' to a union of such cuboids.
As the only different component of x, V is the μ^{th} component,
μ is equal to some λ_k and $W_\mu \cap \pi_\mu(W') = \emptyset$,
so there exist open subsets $W_\mu, \pi_\mu(W')$ of X_μ, such that:

$$x_\mu \in W_\mu \text{ and } V_\mu \subseteq \pi_\mu(W') \text{ and } W_\mu \cap \pi_\mu(W') = \emptyset$$

To conclude, X_μ is regular.
Combine the two parts above, we've proven the logical equivalence.
Quod. Erat. Demonstrandum. □

Remark: *The result proven by Dowker in 1951 suggests that the Cartesian product of normal spaces may not be normal even if each component space is normal.*

Lemma 3.5. Let X be a metrizable space with metric $d_X : X \times X \to \mathbb{R}$, and $f : [0, +\infty) \to [0 + \infty)$ be an embedding. If $\lim\limits_{\alpha \to 0^+} \frac{f(\alpha)}{\alpha} > 0$ and f is concave, then:
(1) $d'_X = f \circ d_X$ is a well-defined metric on X.
(2) d'_X generates the same topology as d_X.

Proof. We may divide our proof into four steps.
Step 1: We prove that f is positive definite, so d'_X is positive definite.
f is an embedding and $\lim\limits_{\alpha \to 0^+} \frac{f(\alpha)}{\alpha} > 0$ suggests that $f(0) = 0$.
Assume to the contrary that for some $\beta > 0$, $f(\beta) = 0$.
As $\lim\limits_{\alpha \to 0^+} \frac{f(\alpha)}{\alpha} > 0$, choose $0 < \alpha < \beta$, such that $f(\alpha) > 0$.
As f is concave, $f(2\beta - \alpha) < 0$, contradicting to $f(2\beta - \alpha) \in [0, +\infty)$.
Hence, our assumption is false, and we've proven that f is positive definite.
Step 2: We prove that d'_X is symmetric.
For all $x_1, x_2 \in X$:

$$d'_X(x_1, x_2) = f \circ d_X(x_1, x_2) = f \circ d_X(x_2, x_1) = d'_X(x_2, x_1)$$

Hence, f is symmetric.
Step 3: We prove that f is subadditive and increasing, so d'_X is subadditive.
On the open set $(0, +\infty)$, f is concave implies f is continuous.
At $\alpha = 0$, $\lim\limits_{\alpha \to 0^+} \frac{f(\alpha)}{\alpha} > 0$ implies f is continuous.
For all $0 < \alpha < \beta$, construct a parallelogram by the following vertices:

$$(0, 0), (\alpha, f(\alpha)), (\beta, f(\beta)), (\alpha + \beta, f(\alpha) + f(\beta))$$

The concavity of f suggests that:

$$\frac{f(\alpha)}{\alpha} \geq \frac{f(\alpha) + f(\beta)}{\alpha + \beta} \geq \frac{f(\beta)}{\beta}$$

The intermediate value theorem suggests the existence of $\alpha < \xi < \beta$, such that:

$$\frac{f(\xi)}{\xi} = \frac{f(\alpha) + f(\beta)}{\alpha + \beta}$$

The concavity of f suggests that:

$$\frac{f(\alpha) + f(\beta)}{\alpha + \beta} = \frac{f(\xi)}{\xi} \geq \frac{f(\alpha + \beta)}{\alpha + \beta}$$

Hence, f is subadditive.
Assume to the contrary that for some $0 < \alpha < \beta$, $f(\alpha) > f(\beta)$.
As f is concave, $\lim\limits_{\gamma \to +\infty} f(\gamma) = -\infty < 0$, contradiction to $f(\gamma) \geq 0$.
Hence, our assumption is false, and we've proven that f is increasing.
Step 4: As there exist $c, c' > 0$, such that $cx < f(x) < c'x$ near $x = 0$, U is an open ball with respect to d_X iff U is an open ball with respect to d'_X when radii are small.
Hence, d'_X generates the same topology as d_X.
Quod. Erat. Demonstrandum. $\square$

Remark: *Notice that it is always possible to choose a bounded embedding* $f : [0, +\infty) \to [0, +\infty), f(x) = \frac{x}{1+x}$*, so when discussing metric spaces, we can always assume that* X *is bounded without loss of generality.*

Definition 3.6. (**Pointwise Metric**)
Let Y be a bounded metric space, Y^K be the set of all sequence y from a nonempty subset K of $\mathbb{Z}$ to Y, and a be a sequence in $\ell^1(K, (0, +\infty))$. Define:

$$d_{Y^K} : Y^K \times Y^K \to \mathbb{R}, d_{Y^K}(y, y') = \sum_K a_k d_Y(y_k, y'_k)$$

as the pointwise metric on Y^K induced by a.

Proposition 3.7. Let Y be a bounded metric space, Y^K be the set of all sequence y from a nonempty subset K of $\mathbb{Z}$ to Y, and a be a sequence in $\ell^1(K, (0, +\infty))$. The pointwise metric d_{Y^K} is a metric on Y^K, which generates the product space topology on Y^K.

Proof. We may divide our proof into four parts.
Part 1: We prove that d_{Y^K} is positive definite.
For all $y, y' \in Y^K$:

$$d_{Y^K}(y, y') = \sum_K a_k d_Y(y_k, y'_k) \geq 0$$

$$\begin{aligned} d_{Y^K}(y, y') = \sum_K a_k d_Y(y_k, y'_k) = 0 &\implies \text{Each } d_Y(y_k, y'_k) = 0 \\ &\implies \text{Each } y_k = y'_k \implies y = y' \end{aligned}$$

Hence, d_{Y^K} is positive definite.
Part 2: We prove that d_{Y^K} is symmetric.
For all $y, y' \in Y^K$:

$$d_{Y^K}(y, y') = \sum_K a_k d_Y(y_k, y'_k) = \sum_K a_k d_Y(y'_k, y_k) = d_{Y^K}(y', y)$$

Hence, d_{Y^K} is symmetric.
Part 3: We prove that d_{Y^K} is subadditive.
For all $y, y', y'' \in Y^K$:

$$\begin{aligned} d_{Y^K}(y, y'') &= \sum_K a_k d_Y(y_k, y''_k) \\ &\leq \sum_K a_k d_Y(y_k, y'_k) + \sum_K a_k d_Y(y'_k, y''_k) \\ &= d_{Y^K}(y, y') + d_{Y^K}(y', y'') \end{aligned}$$

Part 4: We prove that d_{Y^K} generates the product space topology on Y^K.
WLOG, assume that $K = \mathbb{N}$ and $\sum_{k=1}^{+\infty} a_k = 1$.
For all basis element $U = \bigcap_{k=1}^{m} \pi_k^{-1}(B(y_k, r_k))$ of the product space topology on Y^K:
For all $y' \in U$, choose the following radius:

$$r' = \min_{1 \leq k \leq m} a_k(r_k - d_Y(y_k, y'_k))$$

Notice that $V' = B(y', r') \subseteq U$,
so $U = \bigcup_{y' \in U} V'$ is in the metric space topology of Y^K.
For all basis element $V = B(y, r)$ of the metric space topology on Y^K:
For all $y' \in V$, choose the following sequence of radii:

$$r'_k \begin{cases} = & r/2 & \text{if} \quad y \neq y'; \\ = & [r/d_{Y^K}(y, y') - 1][d_Y(y_k, y'_k)/2] & \text{if} \quad y \neq y'; \end{cases}$$

Notice that $\exists m \in \mathbb{N}, U' = \bigcap_{k=1}^{m} B(y'_k, r'_k) \subseteq V$,
so $V = \bigcup_{y' \in V} U'$ is in the product space topology of Y^K.
Hence, d_{Y^K} generates the product space topology on Y^K.
Quod. Erat. Demonstrandum. □

Remark: *This implies Y^K is always metrizable when Y is metrizable and $K \subseteq \mathbb{Z}$.*

Theorem 3.8. (**Alexander's Subbasis Theorem**[2])
Let X be a topological space, and $\mathcal{B}_X$ be a subbasis of X. X is compact if and only if every open cover $\mathcal{V} \subseteq \mathcal{B}_X$ of X has a finite subcover.

Proof. It suffices to prove **"if" direction**.
Assume to the contrary that X is not compact.
Step 1: Define $\mathbf{\Phi}$ as the set of all open cover $\mathcal{U}$ of X with no finite subcover.

Define a partial order $\leq: \mathbf{\Phi} \to \mathbf{\Phi}, \mathcal{U}_1 \leq \mathcal{U}_2$ if $\mathcal{U}_1 \subseteq \mathcal{U}_2$ on $\mathbf{\Phi}$.
For all nonempty totally ordered subset $\mathbf{\Psi}$ of $\mathbf{\Phi}$:
Property 1.1: $\forall (U_k)_{k=1}^m$ in $\bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U}, \exists (\mathcal{U}_k)_{k=1}^m$ in $\mathbf{\Psi},$ each $U_k \in \mathcal{U}_k$.
Without loss of generality, assume that $(\mathcal{U}_k)_{k=1}^m$ is ascending.
This implies $(U_k)_{k=1}^m$ in $\mathcal{U}_m$, so $(U_k)_{k=1}^m$ doesn't cover X. Hence, $\bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U} \in \mathbf{\Phi}$.
Property 1.2: $\forall \mathcal{V} \in \mathbf{\Psi}, \mathcal{V} \leq \bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U}$. Hence, $\bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U}$ is an upper bound of $\mathbf{\Psi}$.
According to **Zorn's Lemma**, $\mathbf{\Phi}$ has a maximal element $\mathcal{V}$.
Step 2: Assume to the contrary that $\mathcal{V} \cap \mathcal{B}_X$ is an open cover of X.
$\mathcal{V}$ has no finite subcover, neither does $\mathcal{V} \cap \mathcal{B}_X$.
However, $\mathcal{V} \cap \mathcal{B}_X \subseteq \mathcal{B}_X$, which has a finite subcover, a contradiction.
Hence, our assumption is false, and we've proven $\mathcal{V} \cap \mathcal{B}_X$ is not an open cover of X.
Step 3: Assume that $\mathcal{V} \cap \mathcal{B}_X = (V_\lambda)_{\lambda \in J}$, where $J \subset I$, and fix $x_0 \in \bigcup_{\lambda \in I \setminus J} V_\lambda$.
As $\mathcal{B}_X$ is a subbasis of X, $x_0 \in \bigcap_{k=1}^m W_k \subseteq V_{\lambda_*}$, where each $W_k \in \mathcal{B}_X, \lambda_* \in I$.
Assume to the contrary that some $W_k \in \mathcal{V}$.
As $W_k \in \mathcal{B}_X$, $x_0 \in W_k \in \mathcal{V} \cap \mathcal{B}_X$, but $\mathcal{V} \cap \mathcal{B}_X$ doesn't cover x_0, a contradiction.
Hence, our assumption is false, and we've proven each $W_k \notin \mathcal{V}$.
Step 4: For each W_k, define $\mathcal{V}_k$ as a finite subcover of $\mathcal{V} \cup \{W_k\}$.
Assume to the contrary that $W_k \notin \mathcal{V}_k$.
This implies $\mathcal{V}$ has a finite subcover $\mathcal{V}_k$, a contradiction.
Hence, each $\mathcal{V}_k$ is in the form $(W_k, V_{\lambda_{kl_k}})_{l_k=1}^{n_k}$. This implies:

$$X = W_k \cup \bigcup_{l_k=1}^{n_k} V_{\lambda_{kl_k}} \implies \bigcap_{l_k=1}^{n_k} V_{\lambda_{kl_k}}^c \subseteq W_k \implies \bigcap_{k=1}^{m} \bigcap_{l_k=1}^{n_k} V_{\lambda_{kl_k}}^c \subseteq \bigcap_{k=1}^{m} W_k \subseteq V_{\lambda_*}$$

To conclude, our assumption is false, as we've constructed a finite subcover $(V_{\lambda_*}, V_{\lambda_{kl_k}})$ of $\mathcal{V}$. Quod. Erat. Demonstrandum. □

Theorem 3.9. (**Tychonoff Theorem**[2])
Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces,
and X be the product space of $(X_\lambda)_{\lambda \in I}$.
If each X_λ is compact, then X is compact.

Proof. For all $\lambda \in I$, define $\mathcal{U}_\lambda$ as the initial topology of X_λ on X via π_λ.
It suffices to show that each open cover $\mathcal{V} \subseteq \bigcup_{\lambda \in I} \mathcal{U}_\lambda$ of X has a finite subcover.
Step 1: Assume to the contrary that no $\pi_\lambda(\mathcal{V} \cap \mathcal{U}_\lambda)$ covers X_λ.
For all $\lambda \in I$, the assumption above guarantees the existence of $\xi_\lambda \in (\pi(\mathcal{V} \cap \mathcal{U}_\lambda))^c$.
Construct $x \in X, x(\lambda) = \xi_\lambda$.
As each $\mathcal{V} \cap \mathcal{U}_\lambda$ doesn't cover x, neither does $\mathcal{V} = \bigcup_{\lambda \in I} (\mathcal{V} \cap \mathcal{U}_\lambda)$, a contradiction.
Hence, our assumption is wrong, and we've proven that some $\pi_\lambda(\mathcal{V} \cap \mathcal{U}_\lambda)$ covers X_λ.
Step 2: As some $\pi_\lambda(\mathcal{V} \cap \mathcal{U}_\lambda)$ covers X_λ, a finite subcover $\pi_\lambda(\mathcal{W})$ exists.
Hence, we've reduced our original open cover $\mathcal{V}$ to a finite subcover $\mathcal{W}$.
Quod. Erat. Demonstrandum. □

Remark: *Product space inherits compactness.*

Lemma 3.10. Let X_1, X_2 be two topological spaces,
and X be the product space of X_1, X_2.
If X_1, X_2 are connected, then X is connected.[3]

Proof. It suffices to notice the following identity:

$$X = \bigcup_{(x_1,x_2)\in X} (X_1 \times \{x_2\}) \cup (\{x_1\} \times X_2)$$

Quod. Erat. Demonstrandum. □

Lemma 3.11. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces,
X be the product space of $(X_\lambda)_{\lambda\in I}$, and ξ be an element of X.
For all $J \subseteq I$, define $X_J = \{x \in X : \forall\lambda \in J^c, x(\lambda) = \xi(\lambda)\}$.
$X' = \bigcup_{|J|<+\infty} X_J$ is dense in X.[3]

Proof. Assume to the contrary that some nonempty open subset U of X doesn't intersect X'. As X has a subbasis, we may shrink U to $\bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(U_{\lambda_k})$, where each $U_{\lambda_k} \in \mathcal{O}_{X_{\lambda_k}}$ is a nonempty open subset of X_{λ_k}. Fix $x \in U$, and construct $x' \in X$ by:

$$x'(\lambda) = \begin{cases} x(\lambda_k) & \text{if} \quad \lambda \text{ is equal to some } \lambda_k; \\ \xi(\lambda) & \text{if} \quad \lambda \text{ is equal to no } \lambda_k; \end{cases}$$

As each $x'(\lambda_k) = x(\lambda_k) \in U_{\lambda_k}$, $x' \in U$.
As $J = \{\lambda_k\}_{k=1}^m$ is finite and $\forall\lambda \in J^c, x'(\lambda) = x(\lambda)$, $x' \in X_J \subseteq X'$.
This contradicts to $U \cap X' = \emptyset$.
Hence, our assumption is false, and we've proven that X' is dense in X.
Quod. Erat. Demonstrandum. □

Proposition 3.12. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces,
and X be the product space of $(X_\lambda)_{\lambda\in I}$.
If each X_λ is connected, then X is connected.[3]

Proof. The $X' = \bigcup_{|J|<+\infty} X_J$ constructed in **Lemma 3.11.** satisfies:

$$\bigcap_{|J|<+\infty} X_J = \{x\} \neq \emptyset$$

Hence, the union X' is connected, and the closure X of X' is connected.
Quod. Erat. Demonstrandum. □

Remark: *Product space inherits connectedness.*

Proposition 3.13. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces, and X be the product space of $(X_\lambda)_{\lambda\in I}$.
If each X_λ is path connected, then X is path connected.

Proof. For all $x_0, x_1 \in X$, for each $\lambda \in I$:
As X_λ is path connected, there exists a path γ_λ from $x_0(\lambda)$ to $x_1(\lambda)$ in X_λ.
Hence, there exists a path $\gamma : [0,1] \to X, t \mapsto (\gamma_\lambda(t))_{\lambda\in I}$ in X from x_0 to x_1.
To conclude, X is path connected. Quod. Erat. Demonstrandum. □

Remark: *Product space inherits path connectedness.*

3.2 Uniform Topology

Proposition 3.14. In $\mathcal{B}([0,1],\mathbb{R})$ under uniform metric $\|\bullet\|$, the closed ball $B(0,1)$ is not compact, so the whole space $\mathcal{B}([0,1],\mathbb{R})$ is not compact.

Proof. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{R},\mathbb{R})$ defined below:

$$f_n(x) = \begin{cases} |\sin\frac{\pi}{x}| & \text{if} \quad x \in [\frac{1}{n+1}, \frac{1}{n}]; \\ 0 & \text{if} \quad x \notin [\frac{1}{n+1}, \frac{1}{n}]; \end{cases}$$

As $\|f_n - f_m\| = \delta_{nm}$, every subsequence of $(f_n)_{n\in\mathbb{N}}$ is not Cauchy, so every subsequence of $(f_n)_{n\in\mathbb{N}}$ is not convergent, $B(0,1)$ is not sequentially compact, $B(0,1)$ is not compact.
Quod. Erat. Demonstrandum. □

Proposition 3.15. In $\mathcal{B}(X,\mathbb{R})$ under uniform metric $\|\bullet\|$, the whole space $\mathcal{B}(X,\mathbb{R})$ is complete.

Proof. For all Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(X,\mathbb{R})$,
each $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.
As $\mathbb{R}$ is complete, $(f_n(x))_{n\in\mathbb{N}}$ converges to some $f(x) \in \mathbb{R}$,
which gives a limit function $f : X \to \mathbb{R}$. Note that:

$$\|f\| \leq \|f - f_n\| + \|f_n\| < +\infty$$

This implies $f \in \mathcal{B}(X,\mathbb{R})$, so X is complete.
Quod. Erat. Demonstrandum. □

Proposition 3.16. In $\mathcal{B}(X,\mathbb{R})$ under uniform metric $\|\bullet\|$, the whole space $\mathcal{B}(X,\mathbb{R})$ and every open ball $B(f,r)$ are convex, so $\mathcal{B}(X,\mathbb{R})$ is path connected and locally path connected.

Proof. It suffices to notice that $\|(1-\lambda)f + \lambda f'\| \leq \max\{\|f\|, \|f'\|\} < +\infty$.
Quod. Erat. Demonstrandum. □

4 Homotopy Relation on Y^X

4.1 Path Connectedness Relation and Homotopy Relation

Recall that a path is a continuous function from $[0,1]$ *to a topological space* X*.*
What about "a path in Y^X*"? A homotopy!*

Definition 4.1. (**Homotopy Relation**)
Let X, Y be two topological spaces, f, f' be two continuous functions from X to Y, and H be a continuous function from $X \times [0,1]$ to Y. If for all $x \in X$:

$$H(x,0) = f(x) \text{ and } H(x,1) = f'(x)$$

then $f \sim f'$, i.e., f is homotopic to f'.

Remark: *Notice that it is not enough to force all component functions* $H(x,\bullet), H(\bullet,t)$ *to be continuous. For the following function:*

$$H : [0,1]^2 \to [0,1], H(x,t) = \begin{cases} 0 & \text{if} \quad (x,t) = (0,0) \\ \frac{2xt}{x^2+t^2} & \text{if} \quad (x,t) \neq (0,0) \end{cases}$$

Although all component functions $H(x,\bullet), H(\bullet,t)$ *are continuous, the bivariate function* $H(x,t)$ *fails to be continuous at* $(x,t) = (0,0)$*.*

Proposition 4.2. Homotopic relation $\sim$ is an equivalence relation on $\mathcal{C}(X,Y)$.

Proof. We may divide our proof into three parts.
Part 1: For all $f \in \mathcal{C}(X,Y)$, define:

$$H : X \times [0,1] \to Y, H(x,t) = f(x)$$

For all open subset V of Y, $H^{-1}(V) = f^{-1}(V) \times [0,1]$ is open in $X \times [0,1]$.
Hence, H is continuous, and $f(x) = H(x,0) \sim H(x,1) = f(x)$.
Part 2: For all $f, f' \in \mathcal{C}(X,Y)$, if $f \sim f'$,
then there exists a continuous function H from $X \times [0,1]$ to Y,
such that for all $x \in X$:

$$H(x,0) = f(x) \text{ and } H(x,1) = f'(x)$$

Define:

$$H' : X \times [0,1] \to Y, H'(x,t) = H(x, 1-t)$$

As $e_X : X \to X, x \mapsto x$ and $\tau_{[0,1]} : t \to 1-t$ are continuous,
the map $(e_X, \tau_{[0,1]})$ is continuous.
Hence, $H' = H \circ (e_X, \tau_{[0,1]})$ is continuous, and $f'(x) = H'(x,0) \sim H'(x,1) = f(x)$.
Part 3: For all $f, f', f'' \in \mathcal{C}(X,Y)$, if $f \sim f'$ and $f' \sim f''$,

then there exist continuous functions H, H' from $X \times [0,1]$ to Y,
such that for all $x \in X$:

$$H(x,0) = f(x) \text{ and } H(x,1) = H'(x,0) = f'(x) \text{ and } H'(x,1) = f''(x)$$

Choose a constant $0 < c < 1$, and define:

$$H'' : X \times [0,1], H''(x,t) = \begin{cases} H\left(x, \frac{t-0}{c-0}\right) & \text{if} \quad 0 \le t \le c; \\ H'\left(x, \frac{t-c}{1-c}\right) & \text{if} \quad c \le t \le 1; \end{cases}$$

According to **The Gluing Lemma**, $H'' = H \cup H'$ is continuous.
Hence, $H'' = H \cup H'$ is continuous, and $f(x) = H''(x,0) \sim H''(x,1) = f''(x)$.
Combine the three parts above, $\sim$ is an equivalence relation on $\mathcal{C}(X,Y)$.
Quod. Erat. Demonstrandum. □

Remark: *We shall develop a method to prove that certain map $H : X \times [0,1] \to Y$ is a homotopy, and here is the crucial lemma.*

Proposition 4.3. Let X, Y, Z be three metric spaces,
$H : X \times Y \to Z$ be a bounded function, and x_*, y_* be points in X, Y respectively.
Equip $X \times Y$ with the product metric:

$$d_{X\times Y}(x_1, y_1, x_2, y_2) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

Equip $\mathcal{B}(Y,Z)$ with the uniform metric:

$$d(f, f') = \sup_{y\in Y} d_Z(f(y), f'(y))$$

If the following conditions hold, then H is continuous:
(1) $f_{x_*}(y) = H(x_*, y)$ is continuous at y_*.
(2) $g(x) = f_x$ is a continuous at x_*.

Proof. For all $\epsilon > 0$, we wish to find $\delta > 0$, such that for all $(x,y) \in B(x_*, y_*, \delta)$:

$$d_Z(H(x_*, y_*), H(x,y)) < \epsilon$$

Step 1: As f_* is continuous at y_*, there exists $\delta_1 > 0$, such that for all $y \in B(y_*, \delta_1)$:

$$d_Z(f_{x_*}(y), f_{x_*}(y_*)) < \frac{\epsilon}{2}$$

Step 2: As g is continuous at x_*, there exists $\delta_2 > 0$, such that for all $x \in B(x_*, \delta_2)$:

$$d(f_{x_*}, f_x) < \delta_2$$

Step 3: There exists $\delta = \min\{\delta_1, \delta_2\} > 0$, such that for all $(x, y) \in B(x_*, y_*, \delta)$:

$$\begin{aligned} d_Z(H(x_*, y_*), H(x, y)) &\leq d_Z(H(x_*, y_*), H(x_*, y)) + d_Z(H(x_*, y), H(x, y)) \\ &= d_Z(f_{x_*}(y_*), f_{x_*}(y)) + d_Z(f_{x_*}(y), f_x(y)) \\ &\leq d_Z(f_{x_*}(y_*), f_{x_*}(y)) + d(f_{x_*}, f_x) < \epsilon \end{aligned}$$

Hence, H is continuous at (x_*, y_*). Quod. Erat. Demonstrandum. □

Remark: *Homotopy is compatible with homeomorphism.*

Proposition 4.4. Let X_1, X_2, Y_1, Y_2 be four topological spaces,
$f_1, f_1' : X_1 \to Y_1$ be two homotopic functions,
and $\sigma : X_1 \to X_2, \tau : Y_1 \to Y_2$ be two homeomorphisms.
$\tau \circ f_1 \circ \sigma^{-1}$ is homotopic to $\tau \circ f_1' \circ \sigma^{-1}$.

Proof. Assume that $H_1 : X_1 \times [0, 1] \to Y_1$ is a homotopy from f_1 to f_1'.
We can construct a homotopy $(\tau, e) \circ H_1 \circ (\sigma^{-1}, e)$ from $\tau \circ f_1 \circ \sigma^{-1}$ to $\tau \circ f_1' \circ \sigma^{-1}$,
so $\tau \circ f_1 \circ \sigma^{-1}$ is homotopic to $\tau \circ f_1' \circ \sigma^{-1}$. Quod. Erat. Demonstrandum. □

Remark: *Homotopy is compatible with composition.*

Proposition 4.5. Let X, Y, Z be three topological spaces,
$f, f' : X \to Y$ be two homotopic functions,
and $g, g' : Y \to Z$ be two homotopic functions.
$g \circ f$ is homotopic to $g' \circ f'$.

Proof. Assume that H is a homotopy from f to f', and I is a homotopy from g to g'.
We can construct a homotopy $I \circ (H, e)$ from $g \circ f$ to $g' \circ f'$,
so $g \circ f$ is homotopic to $g' \circ f'$. Quod. Erat. Demonstrandum. □

4.2 The Fundamental Group

First, we restrict homotopy to relative homotopy.

Definition 4.6. (**Relative Homotopy**)
Let X, Y be two topological spaces, A be a subset of X,
and $H : X \times [0, 1] \to Y$ be a homotopy.
If $\forall x \in A$ and $t \in [0, 1], H(a, 0) = H(a, t) = H(a, 1)$,
then H is a homotopy relative to A.

Remark: *Similarly, we may prove that relative homotopy is an equivalence relation.*
Second, we define concatenation.

Definition 4.7. (**Concatenation**)
Let X be a topological space, x, x', x'' be three points in X,
$\gamma : [0,1] \to X$ be a path from x to x',
and $\gamma' : [0,1] \to X$ be a path from x' to x''.
Fix $0 < c < 1$, define:

$$\gamma \star_c \gamma' : [0,1] \to X, \gamma \star_c \gamma'(t) = \begin{cases} \gamma\left(\frac{t-0}{c-0}\right) & \text{if} \quad 0 \leq t \leq c; \\ \gamma'\left(\frac{t-c}{1-c}\right) & \text{if} \quad c \leq t \leq 1; \end{cases}$$

as the concatenation from γ to γ' at c.

Remark: *The gluing lemma suggests that $\gamma \star_c \gamma'$ is continuous. Homotopy relation is compatible with homeomorphism suggests that every $\gamma \star_{c_1} \gamma'$ and $\gamma \star_{c_2} \gamma'$ are homotopic relative to $\{x, x''\}$. Homotopy relation is compatible with composition suggests that $\star : [\mathcal{C}]([0,1], X, x, x') \times [\mathcal{C}]([0,1], X, x', x'') \to [\mathcal{C}]([0,1], X, x, x'')$ is well-defined:*

$$[\gamma] \star [\gamma'] \mapsto [\gamma \star_c \gamma']$$

In general, let $\gamma^{(0)}, \cdots, \gamma^{(n-1)} : [0,1] \to X$ be n paths connected from tip to tail.
Fix $0 = c_0 < c_1 < \cdots < c_{n-1} < c_n = 1$, define:

$$\gamma^{(0)} \star_{c_1} \cdots \star_{c_{n-1}} \gamma^{(n-1)} : [0,1] \to X$$

$$\gamma^{(0)} \star_{c_1} \cdots \star_{c_{n-1}} \gamma^{(n-1)}(t) = \begin{cases} \gamma^{(0)}\left(\frac{t-c_0}{c_1-c_0}\right) & \text{if} \quad c_0 \leq t \leq c_1; \\ & \vdots \\ \gamma^{(n-1)}\left(\frac{t-c_{n-1}}{c_n-c_{n-1}}\right) & \text{if} \quad c_{n-1} \leq t \leq c_n; \end{cases}$$

As the concatenation of $\gamma^{(0)}, \cdots, \gamma^{(n-1)}$ at $c_0, c_1, \cdots, c_{n-1}, c_n$.
The relative homotopy class of $\gamma^{(0)} \star_{c_1} \cdots \star_{c_{n-1}} \gamma^{(n-1)}$ doesn't depend on the choice of $c_0, c_1, \cdots, c_{n-1}, c_n$, so we can use this construction to prove the associativity of $\star$.

$$\begin{aligned} ([\gamma] \star [\gamma']) \star [\gamma''] &= [\gamma \star_c \gamma'] \star [\gamma''] = [(\gamma \star_c \gamma') \star_{c'} \gamma''] \\ &= [\gamma \star_{0+(c-0)(c'-0)} \gamma' \star_{c'} \gamma''] \\ &= [\gamma \star_c \gamma' \star_{1-(1-c')(1-c)} \gamma''] \\ &= [\gamma \star_c (\gamma' \star_{c'} \gamma'')] = [\gamma] \star [\gamma' \star_{c'} \gamma''] = [\gamma] \star ([\gamma'] \star [\gamma'']) \end{aligned}$$

It follows directly that the set $[\mathcal{C}]([0,1], X, x, x)$ forms a group under concatenation.

Definition 4.8. (**The Fundamental Group**)
Let X be a topological space, x be a point in X.
Define $[\mathcal{C}]([0,1], X, x, x)$ as the fundamental group of X at x.

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