1. (1) *Proof.* Assume to the contrary that q is a real root. Then $\mathrm{Im}(f(q)) = 2q = 0 \Rightarrow q = 0$. However, $f(0) = -\sqrt[5]{17} \neq 0$. $\square$

   (2) *Proof.* Let α be one of its root. By (1), we have $\alpha \notin \mathbb{R}$. Let

$$K = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt[4]{7}, \sqrt[5]{17}, i).$$

   Since α is a root of a function $f(x) \in K[x]$ of degree 11, we have $[K(\alpha) : K] \leq 11$. Note that $\sqrt{2}, \sqrt{5}, \sqrt[4]{7}, \sqrt[5]{17}, i$ are all algebraic in $\mathbb{Q}$, K is a finite extension of $\mathbb{Q}$. Thus by the tower theorem, $K(\alpha)$ is a finite extension of $\mathbb{Q}$. Thus $\alpha \in K(\alpha)$ is algebraic over $\mathbb{Q}$. $\square$

   (3) *Proof.* Note that

$$\begin{aligned} \sqrt{2} &\text{ is a root for } x^2 - 2 \\ \sqrt{5} &\text{ is a root for } x^2 - 5 \\ \sqrt[4]{7} &\text{ is a root for } x^4 - 7 \\ \sqrt[5]{17} &\text{ is a root for } x^5 - 17 \\ i &\text{ is a root for } x^2 + 1, \end{aligned}$$

   thus we have

$$\begin{aligned} &[K(\alpha) : \mathbb{Q}] \\ =&[K(\alpha) : K] \cdot [K : \mathbb{Q}] \\ \leq& 11 \cdot [K : \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt[4]{7}, \sqrt[5]{17})] \cdot [\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt[4]{7}, \sqrt[5]{17}) : \mathbb{Q}] \\ \leq& 11 \cdot 2 \cdot [\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt[4]{7})(\sqrt[5]{17}) : \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt[4]{7})] \cdot [\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt[4]{7}) : \mathbb{Q}] \\ \leq& \cdots \\ \leq& 11 \cdot 2 \cdot 2 \cdot 4 \cdot 5 \cdot 2 \\ =& 1760. \end{aligned}$$

   $\square$

2. *Proof.* Since these two conditions are symmetric, we only need to prove one side. Assume that f is irreducible over $K(\beta)$. Let $\deg(f) = m$. If g is reducible over $K(\alpha)$, there exists $g_1(x), g_2(x) \in K(\alpha)[x]$ such that $g(x) = g_1(x)g_2(x)$. Since $g(\beta) = 0$, at least one of $g_1(\beta)$, $g_2(\beta)$. Without loss of generality, let $g_1(\beta) = 0$. However $\deg(g_1) < \deg(g)$, in order not to contradict the minimal-

ism, we have $g_1(x) \notin K[x]$. Write

$$g_1(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Since $g_1(x) \in K(\alpha)[x]$, we can write $a_i = \sum_{j=0}^{t_i} c_{ij}\alpha^j$ and $t_i < m$ for $i = 0, 1, ..., n$. And $g_1(x) \notin K[x]$ shows that at least one of t_i is positive. Let $t = \max\{t_0, t_1, ..., t_n\}$, then $0 < t < m$. And we define

$$b_{ij} = \begin{cases} c_{ij} & \text{if } j \leq t_i; \\ 0 & \text{otherwise} \end{cases}$$

for all $0 \leq i \leq n$ and $0 \leq j \leq t$. Then

$$\begin{aligned} g_1(\beta) &= \sum_{i=0}^{n} a_i \beta^i = \sum_{i=0}^{n}\sum_{j=0}^{t_i} c_{ij}\alpha^j\beta^i = \sum_{i=0}^{n}\sum_{j=0}^{t} b_{ij}\alpha^j\beta^i \\ &= \sum_{j=0}^{t}\sum_{i=0}^{n} b_{ij}\beta^i\alpha^j = \sum_{j=0}^{t} b_j\alpha^j = f_1(\alpha), \end{aligned}$$

where $b_j = \sum_{i=0}^{n} b_{ij}\beta^i$ and $f_1(x) = \sum_{j=0}^{t} b_j x^j \in K(\beta)[x]$. Since $g_1(\beta) = 0$, $f_1(\alpha) = 0$, and thus contradicts the minimalism of f as $t < m$. □

3. (1) *Sol.* It is not constructible. The minimal polynomial for $\sqrt[3]{7}$ is $x^3 - 7$. Thus $[\mathbb{Q}[\sqrt[3]{7}] : \mathbb{Q}] = 3$, and is not a power of 2.

   (2) *Sol.* It is not constructible. The minimal polynomial for $\sqrt[3]{3}$ is $x^3 - 3$. Thus $[\mathbb{Q}[\sqrt[3]{3}] : \mathbb{Q}] = 3$, and is not a power of 2.

4. a) *Sol.* $f(x) = x^3 - 2 = (x - \sqrt[3]{2})(x - \sqrt[3]{2}\omega)(x - \sqrt[3]{2}\omega^2)$ where $\omega = e^{2\pi i/3}$. Thus the splitting field for f over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2) = \mathbb{Q}(\sqrt[3]{2}, \omega)$. Since $\sqrt[3]{2}$ is a root for $x^3 - 2$ which is irreducible in $\mathbb{Q}$, and ω is a root for $x^2 + x + 1$ which is irreducible in $\mathbb{Q}[\sqrt[3]{2}]$, so by the tower theorem

$$[\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}(\sqrt[3]{2})] \cdot [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3 \cdot 2 = 6.$$

   b) *Sol.* $f(x) = (x-1)(x+1)(x+i)(x-i)$. So the splitting field for f over $\mathbb{Q}$ is $\mathbb{Q}(-1, 1, i, -i) = \mathbb{Q}(i)$. Since i is a root for the function $x^2 + 1$ which is irreducible in $\mathbb{Q}$, we have

$$[\mathbb{Q}(i) : \mathbb{Q}] = 2.$$

c) *Sol.* $f(x) = (x - \sqrt{2})(x + \sqrt{2})(x^3 - 2)$. So the splitting field of f over $\mathbb{Q}$ is the splitting field of $x^2 - 2$ over $\mathbb{Q}(\sqrt[3]{2}, \omega)$. We claim that $x^2 - 2$ is irreducible in $\mathbb{Q}(\sqrt[3]{2}, \omega)$, otherwise

$$\sqrt{2} = a + b\sqrt[3]{2} + c\sqrt[3]{2}^2 + d\omega$$

for rational number a, b, c and d. Then square both sides, we have

$$2 = a^2 + 4bc + (2ab + 2c^2)\sqrt[3]{2} + (2ac + b^2)\sqrt[3]{2}^2 + 2(a + b\sqrt[3]{2} + c\sqrt[3]{2}^2)d\omega + d^2\omega^2.$$

So $ab + c^2 = 2ac + b^2 = d = 0$. If $a \neq 0$, then $b = -c^2/a$. So $2ac + c^4/a^2 = 0$, $2a^3 + c^3 = 0$, which is impossible. Thus $a = 0$, and $b = c = 0$, which is also impossible. Therefore $x^2 - 2$ is irreducible in $\mathbb{Q}(\sqrt[3]{2}, \omega)$, and by the tower theorem

$$[\mathbb{Q}(\sqrt[3]{2}, \omega, \sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{2}, \omega, \sqrt{2}) : \mathbb{Q}(\sqrt[3]{2}, \omega)] \cdot [\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}] = 2 \cdot 6 = 12.$$

5. *Proof.* Let $p(x)$ be an irreducible polynomial in $M[x]$ with a root α in L. Since L is algebraic over K, there exists a minimal polynomial $q(x) \in K[x]$ such that α is one of its roots. Since $K[x] \subset M[x]$, by the minimalism we have $q(x) | p(x)$. And by the irreducibility of $p(x)$, we have $p(x) = kq(x)$ for some constant $k \in M$. Since $K \subset L$ is normal and q has a root $\alpha \in L$, q splits over L, and so does p. □

6. *Proof.* Assume to the contrary that $\mathbb{Q}(\sqrt[3]{2})$ is a splitting field of some polynomial in $\mathbb{Q}[x]$. Then $\mathbb{Q}(\sqrt[3]{2})$ is a finite and normal extension of $\mathbb{Q}$. However, $x^3 - 2$, being an irreducible polynomial in $\mathbb{Q}[x]$ that has a root $\sqrt[3]{2}$ in $\mathbb{Q}(\sqrt[3]{2})$, does not split over $\mathbb{Q}(\sqrt[3]{2})$ ($e^{2\pi i/3} \notin \mathbb{Q}(\sqrt[3]{2})$). Thus $\mathbb{Q}(\sqrt[3]{2})$ is not a normal extension of $\mathbb{Q}$, and we arrive at a contradiction. □