Chapter 0. Introduction

MATH4406 Introduction to Partial Differential Equations

The University of Hong Kong

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Basic Information

Instructor	Teaching Assistant
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First Tutorial

Our FIRST tutorial will begin in the **SECOND** week, namely in the week of *September 9th*.

All homework assignments, supplementary materials, and other relevant information will be posted on the Moodle page for our course.

Course Description and Textbook

MATH4406 Introduction to Partial Differential Equations.

Prerequisite(s): Pass in MATH2101, MATH2102, MATH2241; and Pass in MATH3405, or already enrolled in this course.
This course introduces students to the basic techniques for solving partial differential equations as well as the underlying theories.

This semester will be an introduction to the mathematics of partial differential equations (PDE). The course will be at a level of generality which makes it suitable for students in a wide variety of economics, engineering and science majors. Knowledge of basic physical principles may be helpful at times but is not required.

Textbook

Strauss, Walter A., *Partial Differential Equations: An Introduction*, <u>2nd ed.</u>, ©2008, Wiley.

Solving PDE

The fundamental technical (philosophical?) problem which will be ever-present this semester is the problem of finding and understanding solutions to any given PDE. It turns out that this problem is infinitely more complicated than any other equation/solution problem you have encountered in calculus so far.

Thought Experiment

Suppose $F(x, y)$ is a function of two variables. What is the solution of the equation

$$\frac{\partial F}{\partial x}(x, y) = 0?$$

Real-World Problems Giving Rise to PDE

Wave Phenomena

Representative Example: The Wave Equation electromagnetism, transmission of signals along wires, water waves, general relativity, structural vibration, fluid dynamics

Diffusion Phenomena

Representative Example: The Heat Equation
heat flow, diffusion across biological membranes, the Black-Scholes derivative pricing model, reaction-diffusion chemical interactions

Equilibrium Phenomena

Representative Example: Laplace's Equation
soap bubbles, drum heads, steady-state behavior, minimal surfaces

More/Other Phenomena

weather prediction, quantum mechanics, gas dynamics,...

Goals for the Semester

1. Get to know to the fundamental PDE: the heat equation, the wave equation, and the Laplace equation
    - Identify how to correctly pose questions for each of these fundamental PDE and why well-posedness matters.
    - Apply powerful methods to solve well-posed PDE.
    - Predict and describe qualitative properties of solutions without resorting to elaborate computations.
    - Identify and implement a variety of successful methods for approximating solutions of PDE.
2. Get to know to the tools of the trade: method of characteristics, maximum principle, energy methods, Fourier series, separation of variables, Green's functions, etc.
    - Develop computational proficiency with these tools.
    - Successfully apply the tools and ideas in novel contexts.

Grading Procedures

Raw Score Weighting System

Homework	10%	due weekly at 3:30pm each Friday no late work, two lowest scores dropped
Test 1	20%	Tuesday, October 22nd, in class
Test 2	20%	Tuesday, November 19th, in class
Final Exam	50%	**centralized**, TBA make-ups subject to HKU policies on final exams

In truly exceptional circumstances, alternate weights or other arrangements may be made.

Our Expectations

- We expect students to arrive promptly to lectures and tutorials and to make reasonable efforts to minimize distractions due to cell phones, computers, and so on.
- We expect students to come to lectures and tutorials prepared. This includes having already completed assigned reading and homework.
- We expect students to fully participate in the learning process. This includes willingness to ask and answer questions and to voluntarily join in activities in lectures and tutorials.
- We expect submitted work to be of high quality. Among other things, this means it should be easy to read and easy to follow.
- We expect students to adhere to high moral and ethical standards while completing coursework. We will not tolerate cheating or any form of inappropriate reliance on classmates or other sources (electronic or otherwise) on any assignments.

Complex Arithmetic Review

We expand the real numbers by adjoining an element i which has the property that $i^2 = -1$.

Arithmetic Operations Defined:

$$(a+bi)+(c+di) := (a+c)+(b+d)i$$
$$(a+bi)(c+di) := (ac-bd)+(ad+bc)i$$

Let $z := a + bi$.

- The real part of z, $\mathrm{Re}(z)$, equals a.
- The imaginary part of z, $\mathrm{Im}(z)$, equals b (it's real!).
- The complex conjugate of z is written $\overline{z}$ and equals $a - bi$.
- The length of z, $|z|$, equals $\sqrt{z\overline{z}}$ or $\sqrt{a^2+b^2}$ (taking the positive square root).

The ratio of two complex numbers can be simplified by multiplying numerator and denominator by the conjugate of the denominator.

Complex Analysis Review

Almost any function you can think of can be evaluated on complex numbers by using the Taylor series. An important example is:

Euler's Formula

$$e^{a+bi} = e^a \cos b + i\ e^a \sin b$$

Using Euler's formula and complex arithmetic, any trigonometric function can be evaluated on the complex numbers.

Sine and Cosine as Complex Exponentials

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Vector Calculus Review

The Gradient: ∇f

Input	scalar functions of several variables (any number)
Output	a vector field (dim. matches # of variables)
Physical Intutition	the gradient points in the direction of fastest increase; its length is prop. to the rate of increase
$=0$?	the gradient vanishes at critical points
Formula	$\nabla f(x_1,\ldots,x_d) = \left\langle \frac{\partial f}{\partial x_1}(x_1,\ldots,x_d),\ldots,\frac{\partial f}{\partial x_d}(x_1,\ldots,x_d)\right\rangle$
Integral Theorem	$\int_\gamma \nabla f \cdot d\vec{r} = f(\text{end point}) - f(\text{start point})$

The Divergence: $\nabla \cdot \vec{F}$

Input	a vector field of several variables (any number)
Output	scalar field (dim. matches # of variables)
Physical Intutition	if the vector field represents fluid flow, divergence measures the tendency of molecules to spread out
$= 0$?	div vanishes at points of "incompressiblility"
Formula	$\nabla \cdot \vec{F}(x_1, \ldots, x_d) = \frac{\partial F_1}{\partial x_1}(x_1, \ldots, x_d) + \ldots + \frac{\partial F_d}{\partial x_d}(x_1, \ldots, x_d)$
Integral Theorem	$\oiint_{\partial V} \vec{F} \cdot d\vec{n} = \iiint_V \nabla \cdot \vec{F} dV$

The Laplacian: $\nabla^2 f$ or Δf

Input	a scalar field of several variables (any number)
Output	scalar field
Physical Intutition	positive Laplacian of f at point p means that f is larger than average at p compared to nearby points
$= 0$?	vanishes at points with a special kind of "flatness"
Formula	$\nabla^2 f(\vec{x}) = \nabla \cdot \nabla f(\vec{x}) = \frac{\partial^2 f}{\partial x_1^2}(x_1, \ldots, x_d) + \ldots + \frac{\partial^2 f}{\partial x_d^2}(x_1, \ldots, x_d)$
Typical Equality	$f(x) = \frac{1}{4\pi R^2}\left[\oiint_{\partial B_R(x)} f\ dS - \iiint_{B_R(x)} \left(\frac{1}{\lvert\lvert\vec{r}\rvert\rvert} - \frac{1}{R}\right) \nabla^2 f\ dV\right]$