20250207 MATH4302 NOTE 2[1]

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1 Field of Fractions

Example 1.1. We have the following matrix identity:

$$\begin{pmatrix} p' & -q' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q & q' \\ p & p' \end{pmatrix} = \begin{pmatrix} p'q - pq' & 0 \\ p & p' \end{pmatrix}$$

Hence, we have the following logical equivalence when $p, p' \neq 0$:

$$\text{Rank} \begin{pmatrix} q & q' \\ p & p' \end{pmatrix} = 1 \iff p'q - pq' = 0$$

Proposition 1.2. Let R be an integral domain.
The following relation on $R \times (R \backslash \{0\})$ is an equivalence relation:

$$\frac{q}{p} = \frac{q'}{p'} \iff \text{Rank} \begin{pmatrix} q & q' \\ p & p' \end{pmatrix} = 1$$

Proof. We may divide our proof into three parts.
Part 1: The following equation suggests that this relation is reflexive:

$$\begin{pmatrix} q & 0 \\ p & 0 \end{pmatrix} = \begin{pmatrix} q & q \\ p & p \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Part 2: The following equation suggests that this relation is symmetric:

$$\begin{pmatrix} q' & q \\ p' & p \end{pmatrix} = \begin{pmatrix} q & q' \\ p & p' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Part 3: The following equation suggests that this relation is transitive:

$$p'(p''q - pq'') = p(p''q' - p'q'') + p''(p'q - pq'), \text{where } p' \neq 0$$

Quod. Erat. Demonstrandum. $\square$

Example 1.3. We have the following matrix identity:

$$\begin{pmatrix} |P'|I & -Q'\underline{P'} \\ O & I \end{pmatrix} \begin{pmatrix} Q & Q' \\ P & P' \end{pmatrix} \begin{pmatrix} \underline{P} & O \\ O & \underline{P'} \end{pmatrix} = \begin{pmatrix} |P'|Q\underline{P} - |P|Q'\underline{P'} & O \\ |P|I & |P'|I \end{pmatrix}$$

Hence, we have the following logical equivalence when $|P|, |P'| \neq 0$:

$$\text{Rank} \begin{pmatrix} Q & Q' \\ P & P' \end{pmatrix} = n \iff |P'|Q\underline{P} - |P|Q'\underline{P'} = O$$

Proposition 1.4. Let R be an integral domain.
The following relation on $M_{n,n}(R) \times M^n_{n,n}(R)$ is an equivalence relation:

$$\frac{Q}{P} = \frac{Q'}{P'} \iff \text{Rank}\begin{pmatrix} Q & Q' \\ P & P' \end{pmatrix} = n$$

Proof. We may divide our proof into three parts.
Part 1: The following equation suggests that this relation is reflexive:

$$\begin{pmatrix} Q & O \\ P & O \end{pmatrix} = \begin{pmatrix} Q & Q \\ P & P \end{pmatrix} \begin{pmatrix} I & -I \\ O & I \end{pmatrix}$$

Part 2: The following equation suggests that this relation is symmetric:

$$\begin{pmatrix} Q' & Q \\ P' & P \end{pmatrix} = \begin{pmatrix} Q & Q' \\ P & P' \end{pmatrix} \begin{pmatrix} O & I \\ I & O \end{pmatrix}$$

Part 3: The following equation suggests that this relation is transitive:

$$\begin{aligned} |P'|(|P''|Q\underline{P} - |P|Q''\underline{P''}) = |P|(|P''|Q'\underline{P'} - |P'|Q''\underline{P''}) \\ + |P''|(|P'|Q\underline{P} - |P|Q'\underline{P'}), \text{ where } |P'| \neq 0 \end{aligned}$$

Quod. Erat. Demonstrandum. □

Proposition 1.5. The quotient Frac R of $R \times (R\backslash\{0\})$ under $\frac{q}{p} = \frac{q'}{p'}$ is a field.

Proof. We may divide our proof into two parts.
Part 1: Define addition as $\frac{q}{p} + \frac{q'}{p'} = \frac{p'q+pq'}{pp'}$. This operation is well-defined:

$$\begin{aligned} &p_2p_2'(p_1'q_1 + p_1q_1') - p_1p_1'(p_2'q_2 + p_2q_2') = \\ &p_1'p_2'(p_2q_1 - p_1q_2) + p_1p_2(p_2'q_1' - p_1'q_2') = 0 \end{aligned}$$

This operation satisfies four axioms:

$$\begin{aligned} \left(\frac{q}{p} + \frac{q'}{p'}\right) + \frac{q''}{p''} = \frac{p'p''q + p''pq' + pp'q''}{pp'p''} = \frac{q}{p} + \left(\frac{q'}{p'} + \frac{q''}{p''}\right) \\ \frac{q}{p} + \frac{q'}{p'} = \frac{p'q + pq'}{pp'} = \frac{q'}{p'} + \frac{q}{p} \\ \frac{0}{1} + \frac{q}{p} = \frac{p0 + 1q}{1p} = \frac{q}{p} \\ \frac{-q}{p} + \frac{q}{p} = \frac{-pq + pq}{p^2} = \frac{0}{1} \end{aligned}$$

Part 2: Define multiplication as $\frac{q}{p}\frac{q'}{p'} = \frac{qq'}{pp'}$. This operation is well-defined:

$$(p_2p_2')(q_1q_1') = (p_2q_1)(p_2'q_1') = (p_1q_2)(p_1'q_2') = (p_1p_1')(q_2q_2')$$

This operation satisfies five axioms:

$$\begin{aligned}
\left(\frac{q}{p}\frac{q'}{p'}\right)\frac{q''}{p''} &= \frac{qq'q''}{pp'p''} = \frac{q}{p}\left(\frac{q'}{p'}\frac{q''}{p''}\right) \\
\frac{q}{p}\frac{q'}{p'} &= \frac{q'q}{p'p} = \frac{q'}{p'}\frac{q}{p} \\
\frac{1}{1}\frac{q}{p} &= \frac{1q}{1p} = \frac{q}{p} \\
\frac{p}{q}\frac{q}{p} &= \frac{qp}{pq} = \frac{1}{1} \\
\frac{u}{l}\left(\frac{q}{p} + \frac{q'}{p'}\right) &= \frac{up'q + upq'}{lpp'} = \frac{u}{l}\frac{q}{p} + \frac{u}{l}\frac{q'}{p'}
\end{aligned}$$

Hence, Frac R is a field. Quod. Erat. Demonstrandum. $\square$

Proposition 1.6. The quotient $\text{Frac}_n R$ of $M_{n,n}(R) \times M_{n,n}^n(R)$ under $\frac{Q}{P} = \frac{Q'}{P'}$ is a ring, where every nonzero element is either a zero divisor or a unit.

Proof. We may divide our proof into three parts.
Part 1: Define addition as $\frac{Q}{P} + \frac{Q'}{P'} = \frac{|P'|Q\underline{P} + |P|Q'\underline{P'}}{|P||P'|I}$. This operation is well-defined:

$$\begin{aligned}
&|P_2||P_2'|I(|P_1'|Q_1\underline{P_1} + |P_1|Q_1'\underline{P_1'}) - |P_1||P_1'|I(|P_2'|Q_2\underline{P_2} + |P_2|Q_2'\underline{P_2'}) = \\
&|P_1'||P_2'|I(|P_2|Q_1\underline{P_1} - |P_1|Q_2\underline{P_2}) + |P_1||P_2|I(|P_2'|Q_1'\underline{P_1'} - |P_1'|Q_2'\underline{P_2'}) = O
\end{aligned}$$

This operation satisfies four axioms:

$$\begin{aligned}
\left(\frac{Q}{P} + \frac{Q'}{P'}\right) + \frac{Q''}{P''} &= \frac{|P'||P''|Q\underline{P} + |P''||P|Q'\underline{P'} + |P||P'|Q''\underline{P''}}{|P||P'||P''|} = \frac{Q}{P} + \left(\frac{Q'}{P'} + \frac{Q''}{P''}\right) \\
\frac{Q}{P} + \frac{Q'}{P'} &= \frac{|P'|Q\underline{P} + |P|Q'\underline{P'}}{|P||P'|I} = \frac{Q'}{P'} + \frac{Q}{P} \\
\frac{O}{I} + \frac{Q}{P} &= \frac{|P|O\underline{I} + |I|Q\underline{P}}{|I||P|I} = \frac{Q}{P} \\
\frac{-Q}{P} + \frac{Q}{P} &= \frac{-|P|Q\underline{P} + |P|Q\underline{P}}{|P|^2 I} = \frac{O}{I}
\end{aligned}$$

Part 2: Define multiplication as $\frac{Q}{P}\frac{Q'}{P'} = \frac{Q\underline{P}Q'\underline{P'}}{|P||P'|I}$. This operation is well-defined:

$$\begin{aligned}
\big||P_2||P_2'|I\big|\big(Q_1\underline{P_1}Q_1'\underline{P_1'}\big)\underline{|P_1||P_1'|I} &= \big(|P_1||P_1'||P_2||P_2'|\big)^{n-1}\big(|P_2|Q_1\underline{P_1}\big)\big(|P_2'|Q_1'\underline{P_1'}\big) \\
&= \big(|P_1||P_1'||P_2||P_2'|\big)^{n-1}\big(|P_1|Q_2\underline{P_2}\big)\big(|P_1'|Q_2'\underline{P_2'}\big) \\
&= \big||P_1||P_1'|I\big|\big(Q_2\underline{P_2}Q_2'\underline{P_2'}\big)\underline{|P_2||P_2'|I}
\end{aligned}$$

This operation satisfies four axioms:

$$\left(\frac{Q}{P}\frac{Q'}{P'}\right)\frac{Q''}{P''} = \frac{Q\underline{P}Q'\underline{P'}Q''\underline{P''}}{|P||P'||P''|I} = \frac{Q}{P}\left(\frac{Q'}{P'}\frac{Q''}{P''}\right)$$
$$\frac{I}{I}\frac{Q}{P} = \frac{I\underline{I}Q\underline{P}}{|I||P|I} = \frac{Q}{P}$$
$$\frac{U}{L}\left(\frac{Q}{P}+\frac{Q'}{P'}\right) = \frac{|P'|U\underline{L}Q\underline{P}+|P|U\underline{L}Q'\underline{P'}}{|L||P||P'|I} = \frac{U}{L}\frac{Q}{P}+\frac{U}{L}\frac{Q'}{P'}$$
$$\left(\frac{Q}{P}+\frac{Q'}{P'}\right)\frac{U}{L} = \frac{|P'|Q\underline{P}U\underline{L}+|P|Q'\underline{P'}U\underline{L}}{|P||P'||L|I} = \frac{Q}{P}\frac{U}{L}+\frac{Q'}{P'}\frac{U}{L}$$

Part 3: Assume that the nonzero element $\frac{Q}{P}$ is not a zero divisor.
This implies $Q \in M_{n,n}^n(R)$, so $\frac{Q}{P}$ has an multiplicative inverse $\frac{P}{Q}$:

$$\frac{P}{Q}\frac{Q}{P} = \frac{P\underline{Q}Q\underline{P}}{|Q||P|I} = \frac{I}{I}$$

Quod. Erat. Demonstrandum. □

Proposition 1.7. Let R be an integral domain.
All field F containing R contains Frac R.

Proof. F contains R means there exists an embedding $i : R \to F$. Define a map by:

$$I : \text{Frac } R \to F, \frac{q}{p} \mapsto \frac{i(q)}{i(p)}$$

(1) As i is an embedding, I is well-defined and injective.
(2) As i is an embedding, I is a ring homomorphism.
Hence, I is an embedding, so F contains Frac R. Quod. Erat. Demonstrandum. □

Proposition 1.8. Let R be an integral domain, and $F = \text{Frac } R$. Frac $R[x]$ is isomorphic to Frac $F[x]$, namely, the field of Laurent polynomials $F[x, x^{-1}]$.

Proof. Define a map by:

$$I : \text{Frac } R[x] \to \text{Frac } F[x], \frac{\sum_{j=0}^{n} q_j x^j}{\sum_{j=0}^{m} p_j x^j} \mapsto \frac{\sum_{i=0}^{n} \frac{q_i}{1} x^i}{\sum_{j=0}^{m} \frac{p_j}{1} x^j}$$

(1) For all element of Frac $F[x]$, as there are finitely many terms upstairs and downstairs, it is possible to reduce the fraction, which proves that I is surjective.
(2) As $R \to F, a_i \mapsto \frac{a_i}{1}$ is an embedding, I is a field homomorphism, which is injective.
This implies I is a field isomorphism. Quod. Erat. Demonstrandum. □

Proposition 1.9.

$$\text{Frac } \mathbb{Z}[\![x]\!] \neq \text{Frac } \mathbb{Q}[\![x]\!]$$

Proof. It suffices to prove that $\mathrm{e}^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \notin \operatorname{Frac} \mathbb{Z}[\![x]\!]$.
Assume to the contrary that:

$$\exists \sum_{n=0}^{+\infty} a_n x^n \in \mathbb{Z}[\![x]\!], \exists \sum_{n=0}^{+\infty} b_n x^n \in \mathbb{Z}[\![x]\!] \backslash \{0\}, \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} b_n x^n \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

As $\sum_{n=0}^{+\infty} b_n x^n \neq 0$, there exists a unique $b_m \neq 0$ with minimal $m \geq 0$.
As $b_m \neq 0$, there exists a positive number k, such that $k \nmid b_m$.
Consider the coefficient a_{m+k}:

$$\begin{aligned}
a_{m+k} &= \frac{b_{m+k}}{0!} + \cdots + \frac{b_{m+1}}{(k-1)!} + \frac{b_m}{k!} + \frac{b_{m-1}}{(k+1)!} + \cdots + \frac{b_0}{(m+k)!} \\
(k-1)! a_{m+k} (\text{integer}) &= (k-1)! \left[\frac{b_{m+k}}{0!} + \cdots + \frac{b_{m+1}}{(k-1)!} \right] (\text{integer}) \\
&+ \frac{b_m}{k} (\text{not integer}) \\
&+ (k-1)! \left[\frac{b_{m-1}}{(k+1)!} + \cdots + \frac{b_0}{(m+k)!} \right] (\text{zero, as } m \text{ is minimal})
\end{aligned}$$

This contradicts to $\mathbb{Z}$ is a group. Quod. Erat. Demonstrandum. □

2 Local Ring

Proposition 2.1. Let R be a commutative ring.
Any proper ideal $\mathfrak{a}$ of R is contained in some maximal ideal $\mathfrak{p}$ of $\mathfrak{p}$.

Proof. Let $\mathcal{A}$ be the set of all proper ideal containning $\mathfrak{a}$.
As $\mathfrak{a} \in \mathcal{A}$, and every chain $\mathcal{C} \subseteq \mathcal{A}$ has an upper bound $\bigcup_{\mathfrak{c} \in \mathcal{C}} \mathfrak{c} \not\ni 1$,
$\mathcal{A}$ contains a maximal element. Quod. Erat. Demonstrandum. □

Definition 2.2. (**Local Ring**)
Let R be a commutative ring.
If R has a unique maximal ideal, then R is local.

Example 2.3. Every field F has a unique maximal ideal $\{0\}$, so F is local.

Example 2.4. $\mathbb{Z}$ has two maximal ideals $2\mathbb{Z}, 3\mathbb{Z}$, so $\mathbb{Z}$ is not local.

Example 2.5. $\mathbb{Z}[\mathrm{i}]$ has two maximal ideals $(1 \pm 2\mathrm{i})\mathbb{Z}[\mathrm{i}]$, so $\mathbb{Z}[\mathrm{i}]$ is not local.

Example 2.6. $\mathbb{Z}_4$ has a unique maximal ideal $2\mathbb{Z}_4$, so $\mathbb{Z}_4$ is local.

Example 2.7. $\mathbb{Z}_6$ has two maximal ideals $2\mathbb{Z}_6, 3\mathbb{Z}_6$, so $\mathbb{Z}_6$ is not local.

Example 2.8. Every polynomial ring $F[x]$ over field F has two maximal ideals $xF[x], (x+1)F[x]$, so $F[x]$ is not local.

Proposition 2.9. Let R be a commutative ring.

$$R \text{ is local} \iff R \neq \{0\} \text{ and } R \backslash R^\times \text{ is an ideal of } R$$

Proof. We may divide our proof into two parts.
"if" direction: Assume that $R \neq \{0\}$ and $R\backslash R^\times$ is an ideal of R.
As $R \neq \{0\}$, $R\backslash R^\times$ is a nonempty set containing all proper ideals of R.
As $R\backslash R^\times$ is an ideal, it is the unique maximal ideal of R, so R is local.
"only if" direction: Assume that R is local.
As R has at least one maximal ideal, $R \neq \{0\}$.
As R has at most one maximal ideal, this ideal is $R\backslash R^\times$.
Quod. Erat. Demonstrandum. □

Proposition 2.10. Let R be an integral domain, but not a field. R is a local principal ideal domain iff for some nonzero, nonunit element r, $R\backslash\{0\} = R^\times r^{\mathbb{Z}_{\geq 0}}$.

Proof. We may divide our proof into two parts.
"if" direction: Assume that for some nonzero, nonunit element r, $R\backslash\{0\} = R^\times r^{\mathbb{Z}_{\geq 0}}$.
Now $R \neq \{0\}$ and $R\backslash R^\times = \{0\} \cup R^\times r^{\mathbb{Z}_{\geq 0}} = Rr$ is an ideal of R.
"only if" direction: Assume that R is a local and principal ideal domain.
Take the generator r of $R\backslash R^\times$. For every $a \neq 0$, divide it by r whenever a is nonunit.
According to ascending chain property for ideals, this process terminates.
Hence, $r \in R^\times r^{\mathbb{Z}_{\geq 0}}$, so $R\backslash\{0\} = R^\times r^{\mathbb{Z}_{\geq 0}}$ as r is not a zero-divisor.
Quod. Erat. Demonstrandum. □

Example 2.11. As $F[\![x]\!]\backslash\{0\} = F[\![x]\!]^\times x^{\mathbb{Z}_{\geq 0}}$, every formal power series ring $F[\![x]\!]$ over field F is local.

3 Localization of Ring

Definition 3.1. (**Multiplicative Subset**)
Let R be an integral domain, and D be a subset of R.
If $0 \notin D$ and $D^2 \subseteq D$, then D is multiplicative.

Example 3.2. Let R be an integral domain, and r be a nonzero element of R.
(1) If r is not a zero divisor, then its nonzero multiple $rR\backslash\{0\}$ is multiplicative.
(2) If r is not a nilpotent, then its nonnegative power $r^{\mathbb{Z}_{\geq 0}}$ is multiplicative.

Definition 3.3. (**Localization of Ring**)
Let R be an integral domain, and D be a multiplicative subset of R.
Define the localization $D^{-1}R$ of R at D as the quotient of $R \times D$ under $\frac{q}{p} \sim \frac{q'}{p'}$.

Example 3.4. Let R be an integral domain, and r be a nonzero element of R.
(1) If r is not a zero divisor, then define the localization of R at r as:

$$R_r = (rR\backslash\{0\})^{-1}R$$

(2) If r is not a nilpotent, then define the localization of R at r as:

$$R_r = r^{\mathbb{Z}_{\leq 0}}R$$

Example 3.5. Let R be an integral domain. The proper ideal $\mathfrak{p}$ is prime iff $\mathfrak{p}^c$ is multiplicative. In case that $\mathfrak{p}$ is prime, define the localization of R at $\mathfrak{p}$ as:

$$R_{\mathfrak{p}} = (\mathfrak{p}^c)^{-1}R$$

Theorem 3.6. (**Production of Local Principal Ideal Domain**)
Let R be a unique factorization domain with fraction field F.
If $D = (pR)^c$ for some prime element p, then R is a local principal ideal domain.

Proof. As $D^{-1}R\backslash\{0\} = (D^{-1}R)^{\times}p^{\mathbb{Z}_{\geq 0}}$, $D^{-1}R$ is a local principal ideal domain.
Quod. Erat. Demonstrandum. □

References

[1] H. Ren, "Template for math notes," 2021.