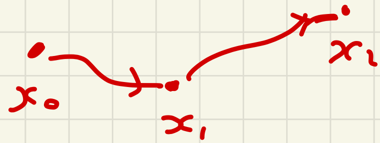
Fundamental gp



$\alpha, \beta \quad [0,1] \longrightarrow X$ s.t.

$x_0 = \alpha(0) \qquad x_1 = \alpha(1) = \beta(0) \qquad x_2 = \beta(1)$

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & t \in [0, \tfrac{1}{2}] \\ \beta(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

Lemma

$$\begin{tikzcd} x_0 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] \arrow[r, phantom, "\Downarrow A"] & x_1 \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\beta'"'] \arrow[r, phantom, "\Downarrow B"] & x_2 \end{tikzcd}$$

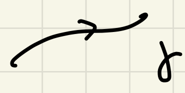
Then $\alpha * \beta \sim \alpha' * \beta'$

p.f. by horizontal composition.

A constant path is $e_x : I \longrightarrow X$

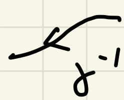
$e_x(t) = x \quad \forall t \in I$

$\gamma : [0,1] \longrightarrow X$ a path



the inverse path of γ is

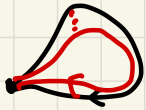
$\gamma^{-1} : [0,1] \longrightarrow X \qquad \gamma^{-1}(t) = \gamma(1-t)$



Fix $x_0 \in X$

A loop based at x_0 is $\gamma : [0,1] \to X$

$\gamma(0) = \gamma(1) = x_0$



two loops γ_1, γ_0 based at x_0

are homotopic iff $\exists$ $\gamma_t : I \times I \to X$

s.t $\gamma(s,0) = \gamma(s,1) = x_0$

$\gamma(0,t) = \gamma_0 \quad \gamma(1,t) = \gamma_1$

$[\gamma]$: homotopy class of γ

i.e the equivalence class of γ under homotopy equivalences.

Thm / Def'n Define $\pi_1(X, x_0)$ to be the set of homotopy classes of loops in X based at x_0. Define

$$[\alpha] \cdot [\beta] = [\alpha * \beta]$$

Then $\pi_1(X, x_0)$ is a group with product $\cdot$ and unit $[e_{x_0}]$.

p.f. We need to check:

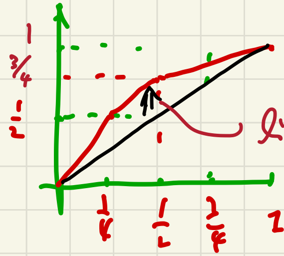
(!) 1) $\big([\alpha] * [\beta]\big) * [\gamma] = [\alpha] * \big([\beta] * [\gamma]\big)$

2) $[\gamma] * [\gamma^{-1}] = [e_x] = [\gamma^{-1}] * [\gamma]$

3) $[e_x] * [\gamma] = [\gamma] * [e_x] = [\gamma]$

$$I \xrightarrow{h} I$$

$$h(t) = \begin{cases} 2t & [0, \tfrac{1}{4}] \\ \tfrac{1}{4} + t & [\tfrac{1}{4}, \tfrac{1}{2}] \\ \tfrac{1}{2} + \tfrac{1}{2}t & [\tfrac{1}{2}, 1] \end{cases}$$



reparametrization of I.

linear homotopy on $I \times I$

P.f. $(\alpha * \beta) * \gamma = \begin{cases} \alpha(4t) & [0, \frac{1}{4}] \\ \beta(4t-1) & [\frac{1}{4}, \frac{1}{2}] \\ \gamma(2t-1) & [\frac{1}{2}, 1) \end{cases}$

$\alpha * (\beta * \gamma) = \begin{cases} \alpha(2t) & [0, \frac{1}{2}] \\ \beta(4t-2) & [\frac{1}{2}, \frac{3}{4}] \\ \gamma(4t-3) & [\frac{3}{4}, 1] \end{cases}$

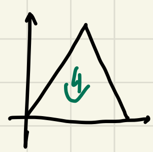
$\alpha * (\beta * \gamma) \circ h(t) = \begin{cases} \alpha(4t) & [0, \frac{1}{4}] \\ \beta(4t-1) & [\frac{1}{4}, \frac{1}{2}] \\ \gamma(2t-1) & [\frac{1}{2}, 1) \end{cases}$

$F : I \times I \longrightarrow I$

$F(s, t) = (1-s)t + s\,h(t)$

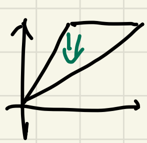
$\alpha * (\beta * \gamma) \circ F(s, t) = \begin{cases} \alpha * (\beta * \gamma) & s = 0 \\ (\alpha * \beta) * \gamma & s = 1 \end{cases}$

$\gamma * \gamma^{-1} = \begin{cases} \gamma(2t) & [0, \frac{1}{2}] \\ \gamma(1-2t) & [\frac{1}{2}, 1] \end{cases}$



$\gamma * \gamma^{-1} \circ (s\,t) = \begin{cases} e_{x_0} & s = 0 \\ \gamma * \gamma^{-1} & s = 1 \end{cases}$

$\gamma * e_{x_0} = \begin{cases} \gamma(2t) & [0, \frac{1}{2}] \\ x_0 & [\frac{1}{2}, 1] \end{cases}$

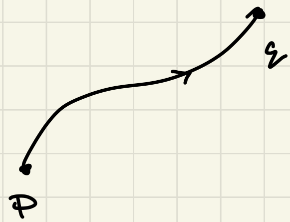


$h(t) = \begin{cases} 2t & [0, \frac{1}{2}] \\ 1 & [\frac{1}{2}, 1] \end{cases}$

$\gamma \circ (s \cdot h(t) + (1-s)t)$

Homotopy via reparametrization

$\gamma : I \longrightarrow \mathbb{R}^3$ a "curve)



$\gamma(0) = p \quad \gamma(1) = q$

An (orientation-preserving) reparametrization is

a homeo. $h : I \to I$ s.t $h(0) = 0 \quad h(1) = 1$

$\underbrace{\gamma \circ h} : I \to \mathbb{R}^3$ has the same "geometry"

with γ but with different "physics"

$$\text{clearly} \quad \overset{\gamma \circ F}{\gamma \circ h \sim \gamma}$$

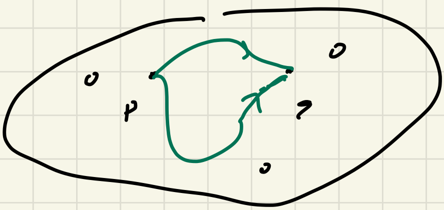
via $F : I \times I \longrightarrow I$

$$F(s, t) = (1-s)t + s\,h$$

Homotopy invariance in physics

$U \overset{\text{open}}{\subset} \mathbb{R}^3$ F a conservative vector field on U $\quad \mathcal{U}$

$\gamma_1, \gamma_2 : I \longrightarrow U \qquad \gamma_1 \sim \gamma_2$



Then $\displaystyle \int_{\gamma_1} F\,ds = \int_{\gamma_2} F\,ds$

i.e. $\displaystyle \int_{\gamma} F\,ds$ only depends on $[\gamma]$