Elliptic Functions, Part 2

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> You have to ask many times before you get to the right question.
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Contents

Recall the universal properties of elliptic functions:

Theorem 0.1 (General Properties of Elliptic Functions)**.** *Let Π be a fundamental domain for $X = \mathbb{C}/L$. Suppose f is an elliptic function such that f has no zeros nor poles on $\partial\Pi$. Then the following holds true:*

(a) $\displaystyle\sum_{a_k \in P(f|_\Pi)} \operatorname{Res}(f; a_k) = 0.$

(b) $\displaystyle\sum_{a_k \in Z(f|_\Pi)} \operatorname{ord}_{a_k}(f) + \sum_{b_l \in P(f|_\Pi)} \operatorname{ord}_{b_l}(f) = 0.$

(c) $\displaystyle\sideset{}{'}\sum_{a \in \Pi} \operatorname{ord}_a(f) \cdot a \equiv 0 \pmod{L}.$

From (a), it follows that there cannot exist $f \in M(X)$ such that f has a simple pole at some $x \in X$ and no other poles; otherwise,

$$\sideset{}{'}\sum_{x \in \Pi} \operatorname{Res}(f; x) = \operatorname{Res}(f; a) \neq 0,$$

contradicting (a).

1 Weierstrass $\wp$-function

Recall the *Eisenstein series*:

Corollary 1.1 (Eisenstein Series Construction)**.** *When $k \geq 3$, we can define the* Eisenstein series *by*

$$E_k(z) = \sum_{w \in L} \frac{1}{(z+w)^k},$$

where E stands for Eisenstein.
The function E_k has a pole of order k at lattice points and no other poles. Moreover, E_k is doubly periodic with respect to L, and hence descends to a meromorphic function on the elliptic curve $X = \mathbb{C}/L$.

When $k \geq 3$, we have $\mathrm{ord}_0(E_k) = -k$. The Weierstrass $\wp$-function satisfies the fundamental relation

$$\wp'(z) = -2E_3(z).$$

Observation: If f is elliptic with respect to L, then f' is also elliptic with respect to L.

Question: What about the converse?

Suppose f is meromorphic on $\mathbb{C}$ and f' is elliptic with respect to L.
Question 1. Is f elliptic?
Consider $\omega \in L$, and define

$$h_\omega(z) = f(z+\omega) - f(z), \quad h'_\omega(z) = f'(z+\omega) - f'(z) = 0.$$

By hypothesis, we have $h_\omega(z) = C_\omega$ for some constant $C_\omega \in \mathbb{C}$. Moreover,

$$C_{\omega+\omega'} = C_\omega + C_{\omega'}.$$

Question 2. Let g be elliptic with respect to L, i.e.

$$g(z+\omega) = g(z), \quad \forall\, z \in \mathbb{C}, \forall\, \omega \in L.$$

Consider

$$\begin{cases} \exists\, f \in \mathcal{M}(X) \text{ such that } f' \equiv g? \\ \text{If such } f \text{ exists, is it elliptic with respect to } L? \end{cases}$$

For the first part, we can try to integrate. Take $z_0 \in \mathbb{C}$ where g is holomorphic, and let γ be a piecewise C^1 path joining z_0 to z. Then we "define"

$$f(z) = \int_\gamma g(\xi)\, d\xi.$$

Without assuming the ellipticity of g, here is the complete answer. Let $\{a_k\}$ be the poles of g. Let Γ_k^0 be a small loop around a_k, and set $a_k \to a_k + \varepsilon e^{i\theta}$, where $\theta \in [0, 2\pi]$.

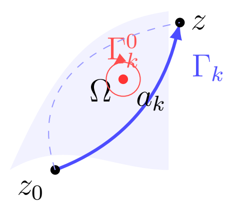
If there exists f such that $f' \equiv g$, then we must have

$$\int_{\Gamma_k^0} g(\xi)\, d\xi = 0,$$

where Γ_k^0 is an anticlockwise loop encircling a_k.

Now, let $\Gamma_k - \Gamma_k^0$ bound a domain Ω, i.e.

$$\partial\Omega = \Gamma_k - \Gamma_k^0.$$



Theorem 1.2 (Stokes' Theorem). *Let M be an oriented smooth compact surface with boundary ∂M, and let ω be a smooth differential 1-form defined on an open subset containing M. Then*

$$\int_{\partial M} \omega = \int_M d\omega,$$

where $d\omega$ denotes the exterior derivative of ω.

In particular, in the complex plane, if $\Omega \subset \mathbb{C}$ and $A(z), B(z)$ are smooth functions on a neighborhood of Ω, then

$$\int_{\partial\Omega} \big(A(z)\, dz + B(z)\, d\bar{z}\big) = \iint_\Omega \left(\frac{\partial B}{\partial z} - \frac{\partial A}{\partial \bar{z}}\right) dz \wedge d\bar{z}.$$

Applying this to our situation, let $\Omega \subset \mathbb{C}$ be a domain bounded by the curves Γ_k and Γ_k^0, that is,

$$\partial\Omega = \Gamma_k - \Gamma_k^0.$$

For the differential form $\omega = g(\xi)\, d\xi$, Stokes' Theorem gives

$$\int_{\Gamma_k - \Gamma_k^0} g(\xi)\, d\xi = \iint_\Omega d\big(g(\xi)\, d\xi\big).$$

Since g is holomorphic on Ω and smooth up to the boundary, Stokes' theorem further implies

$$\iint_\Omega d\big(g(\xi)\, d\xi\big) = \iint_\Omega \left(\frac{\partial g}{\partial \xi} d\xi \wedge d\xi + \frac{\partial g}{\partial \bar{\xi}} d\bar{\xi} \wedge d\xi\right) = -\iint_\Omega \frac{\partial g}{\partial \bar{\xi}} d\xi \wedge d\bar{\xi}.$$

Because g is holomorphic, we have $\dfrac{\partial g}{\partial \bar{\xi}} = 0$, and therefore

$$\iint_\Omega d\big(g(\xi)\, d\xi\big) = 0.$$

Hence, the existence of a holomorphic function f such that $f' \equiv g$ implies, by the Residue Theorem, that

$$0 = \int_{\partial\Omega} g(\xi)\, d\xi = 2\pi i \sum_{a_k \in \Omega} \operatorname{Res}(g; a_k),$$

and therefore

$$\operatorname{Res}(g; a_k) = 0 \quad \text{for all poles } a_k \text{ of } g. \tag{$*$}$$

Conversely, using some basic facts from topology (essentially that integration of a holomorphic differential form defines a closed 1-form), the condition $(*)$ implies that the 1-form

$$\omega = g(\xi)\, d\xi$$

is "closed" on the domain of definition of g. If the domain is simply connected, every closed 1-form is exact, hence there exists a holomorphic function f such that

$$df = \omega, \qquad \text{i.e.} \quad f' = g.$$

Thus, condition $(*)$—that the sum of residues of g in every bounded region vanishes—ensures the global existence of a primitive f on each simply connected component of the domain.

Now we come to another question. Suppose that g is elliptic and that there exists a function f such that $f' \equiv g$. Is f necessarily elliptic?

This question depends on the particular choice of g.

When $k \geq 3$, recall that

$$E_k'(z) = \sum_{w \in L} \frac{-k}{(z+w)^{k+1}} = -k\, E_{k+1}(z).$$

Hence, if we take $g = E_{k+1}$, then

$$f = -\frac{1}{k}\, E_k + \text{constant}.$$

In particular, for $k = 3$, we have

$$E_3(z) = \sum_{w \in L} \frac{1}{(z+w)^3}.$$

Since $\operatorname{Res}(E_3; w) = 0$ for all $w \in L$, we know (by the existence criterion discussed earlier) that there exists a function f such that $f' \equiv E_3$.

Moreover, near the origin we may write

$$f(z) = \frac{1}{z^2} + \cdots,$$

and we can normalize it so that

$$f(z) = \frac{1}{z^2} + 0 + 0 + \cdots.$$

This leads naturally to the definition of the *Weierstrass elliptic function*.

Theorem 1.3 (Weierstrass $\wp$-function)**.** *The Weierstrass $\wp$-function associated with the lattice L is defined by*

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in L^*} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right),$$

where $L^* = L \setminus \{0\}$.

The function $\wp(z)$ is elliptic with respect to the lattice L, and satisfies

$$\wp'(z) = -2E_3(z).$$

One first needs to justify that the function is actually convergent in an appropriate sense.

$$\frac{1}{(z+w)^2} - \frac{1}{w^2} = \frac{-z^2 - 2zw}{(z+w)^2 w^2}.$$

Take any $R > 0$ and consider $|z| \leq R$. Convergence then reduces to checking that $\forall R > 0, \forall z \in \overline{D(R)}$,

$$\sum_{|w| \geq 2R} \frac{-z^2 - 2zw}{(z+w)^2 w^2} < \infty.$$

since there are finitely many points in L where $|w| < 2R$. Now

$$\sum_{|w| \geq 2R} \left| \frac{-z^2 - 2zw}{(z+w)^2 w^2} \right| \leq \sum_{|w| \geq 2R} \left(\frac{|z|^2}{|(z+w)^2 w^2|} + \frac{|2zw|}{|(z+w)^2 w^2|} \right) \leq \sum_{|w| \geq 2R} \left(\frac{4R^2}{|w|^4} + \frac{8R}{|w|^3} \right),$$

since $|w + z| \geq |w|/2$.

Since

$$\sum_{w \in L^*} \frac{1}{|w|^3} < \infty \quad \forall\, p \geq 3,$$

we are done. Thus $\wp$ is convergent.

Now, is $\wp$ elliptic?

Theorem 1.4 (Weierstrass)**.** *Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice. Then*

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in L^*} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right)$$

is an elliptic function. Moreover, it has double poles at every lattice point $w \in L$ and no other poles.

Proof. It is obvious that $\wp$ has double poles on L and no other poles. What remains is to justify that $\wp$ is elliptic, i.e.,

$$\begin{cases} \wp(z + \omega_1) = \wp(z), \\ \wp(z + \omega_2) = \wp(z). \end{cases}$$

We have observed that there exist constants $C_{\omega_1}, C_{\omega_2} \in \mathbb{C}$ such that

$$\wp(z + \omega_1) - \wp(z) \equiv C_{\omega_1}, \qquad \wp(z + \omega_2) - \wp(z) \equiv C_{\omega_2}.$$

Take $i = 1, 2$ and substitute $z = -\omega_i/2$. Then

$$\wp(-\omega_i/2 + \omega_i) - \wp(-\omega_i/2) = C_{\omega_i}.$$

Note that $\wp$ is an even function, hence

$$C_{\omega_i} = \wp(\omega_i/2) - \wp(-\omega_i/2) = 0.$$

Therefore,

$$\wp(z + \omega_i) = \wp(z) \quad \text{for } i = 1, 2,$$

and thus $\wp$ is elliptic. □

Remark 1.5. We have actually proved a more general statement: if g is elliptic and odd, and if there exists a function f such that $f' = g$, then f is even and elliptic.

A general lookahead:

$$\begin{cases} \wp'(z) = -2E_3(z), \quad E_3'(z) = -3E_4(z), \\ \zeta'(z) = -\wp(z), \quad \zeta \text{ solves the Mittag–Leffler problem on } \mathbb{C}/L, \\ \big(\log \sigma(z)\big)' = \zeta(z), \quad \sigma \text{ solves the Weierstrass problem on } \mathbb{C}/L \end{cases}$$

2 The Zeta Function ζ

From the previous discussion, there exists a meromorphic function ζ on $\mathbb{C}$ such that

$$\zeta'(z) = -\wp(z).$$

We normalize the choice of ζ by requiring that the constant term in its Laurent expansion at $z = 0$ be 0. This gives the explicit representation

$$\zeta(z) = \frac{1}{z} + \sum_{w \in L^*} \left(\frac{1}{z+w} + \frac{z}{w^2} - \frac{1}{w} \right),$$

where $L^* = L \setminus \{0\}$ and $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is the underlying lattice.

Proof. We must show that the series defining $\zeta(z)$ converges normally on compact subsets of $\mathbb{C}$.

Let $K \subset \mathbb{C}$ be compact, say $K \subset \overline{D(0,R)} = \{z \in \mathbb{C} : |z| \leq R\}$. We separate finitely many lattice points $w \in L$ with $|w| \leq 2R$. For the remaining $w \in L$ with $|w| > 2R$ and for all $z \in K$, we have by Taylor expansion:

$$\frac{1}{z+w} = \frac{1}{w} \cdot \frac{1}{1 + z/w} = \frac{1}{w}\left(1 - \frac{z}{w} + \frac{z^2}{w^2} - \cdots\right).$$

Hence,

$$\frac{1}{z+w} + \frac{z}{w^2} - \frac{1}{w} = O\left(\frac{1}{|w|^3}\right) \quad \text{uniformly for } z \in K.$$

Since the number of lattice points with $|w| \le 2R$ is finite and $\sum_{w \in L^*} |w|^{-3}$ converges absolutely (because L is discrete in $\mathbb{C}$ and $\sum |w|^{-p}$ converges for $p > 2$), the series converges uniformly on K after removing those finitely many terms.

Thus $\zeta(z)$ converges normally on compact subsets of $\mathbb{C}$, defining a meromorphic function with poles at the lattice points.

Next, differentiating term by term, which is justified by uniform convergence on compacta away from the poles, gives

$$\zeta'(z) = -\frac{1}{z^2} - \sum_{w \in L^*} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right) = -\wp(z).$$

Finally, to show that ζ is not elliptic, suppose by contradiction that ζ were elliptic. Then

$$\mathrm{Res}(\zeta, w) = \mathrm{Res}(\zeta, 0)$$

for each $w \in L$, because $\zeta(z+w)$ would be the same function. But $\mathrm{Res}(\zeta, 0) = 1$ and ζ has simple poles at all lattice points $w \in L$. If ζ were elliptic, the sum of all residues in a fundamental parallelogram would have to vanish:

$$\sum_{a \in P} \mathrm{Res}(\zeta; a) = 0,$$

a general property of elliptic functions. Since there is exactly one pole (mod L) at 0 with residue 1, this is impossible. Hence ζ cannot be elliptic.

□

3 The Mittag–Leffler Problem on $\mathbb{C}/L$

Theorem 3.1 (Solving the Mittag–Leffler Problem on $\mathbb{C}/L$)**.** *Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$, and let $X = \mathbb{C}/L$ denote the associated elliptic curve.*

Let $\{x_1, x_2, \ldots, x_m\} \subset X$ be m distinct points. For each $k \in \{1, \ldots, m\}$, let p_k denote the prescribed principal part of a meromorphic function at x_k, written in the local coordinate z on $\mathbb{C}$ as

$$p_k(z) = \sum_{i=1}^{s_k} \frac{c_k^i}{(z - a_k)^i},$$

where $\pi(a_k) = x_k$ for the covering projection $\pi : \mathbb{C} \to X$.

Then the Mittag–Leffler problem for the given data

$$\{(x_k, p_k) \mid 1 \le k \le m\}$$

has a solution if and only if the following necessary and sufficient condition holds:

$$\sum_{k=1}^{m} c_k^1 = 0.$$

Proof. We prove both directions of the statement.

($\Rightarrow$) Suppose there exists $f \in \mathcal{M}(X)$ that solves the Mittag–Leffler problem for the given data $\{(x_k, p_k)\}$. Choose a fundamental parallelogram Π' such that f has no poles on $\partial\Pi'$.

By the properties of elliptic functions,

$$\sum_{a_k \in \Pi'} \operatorname{Res}(f; a_k) = 0.$$

Since the principal parts of f at a_k coincide with p_k, we have

$$0 = \sum_{k=1}^{m} \operatorname{Res}(f; a_k) = \sum_{k=1}^{m} \operatorname{Res}(p_k; a_k) = \sum_{k=1}^{m} c_k^1.$$

Hence, the necessary condition $\sum_{k=1}^{m} c_k^1 = 0$ holds.

($\Leftarrow$) Conversely, assume that $\sum_{k=1}^{m} c_k^1 = 0$. We now construct explicitly a meromorphic function f on $\mathbb{C}$ which is periodic with respect to L and descends to a meromorphic function on $X = \mathbb{C}/L$.

Define

$$f(z) = \sum_{k=1}^{m} c_k^1\, \zeta(z - a_k) + \sum_{k=1}^{m} c_k^2\, \wp(z - a_k) + \sum_{p=3}^{s} \sum_{k=1}^{m} c_k^p\, E_p(z - a_k),$$

where

- ζ is the Weierstrass zeta function (simple poles),
- $\wp$ is the Weierstrass elliptic function (double poles),
- and $s = \max(s_1, \ldots, s_m)$ with $c_k^p = 0$ for all $p > s_k$.

Then f has exactly the prescribed principal parts p_k at the points a_k, i.e.

$$\operatorname{pp}(f; a_k) = p_k, \qquad 1 \le k \le m,$$

since $\zeta, \wp, E_p$ each reproduce the appropriate pole order and coefficients.

Next, check the periodicity condition. For any lattice vector $\omega \in L$,

$$f(z + \omega) - f(z) = \sum_{k=1}^{m} c_k^1 \big(\zeta(z + \omega - a_k) - \zeta(z - a_k)\big),$$

because $\wp$ and E_p ($p \ge 3$) are elliptic, hence strictly periodic.

Recall the quasi-periodicity relation for the zeta-function:

$$\zeta(z + \omega) - \zeta(z) = A_\omega,$$

where $A_\omega \in \mathbb{C}$ depends only on the period ω. Thus,

$$f(z + \omega) - f(z) = \sum_{k=1}^{m} c_k^1 A_\omega = \Big(\sum_{k=1}^{m} c_k^1\Big) A_\omega = 0 \cdot A_\omega = 0.$$

Hence f is invariant under translations by L, i.e. f is elliptic and descends to a meromorphic function on $X = \mathbb{C}/L$.

Therefore, a meromorphic function f exists solving the Mittag–Leffler problem, completing the proof. $\square$

4 The Weierstrass Problem on $\mathbb{C}/L$

Theorem 4.1 (Solving the Weierstrass Problem on $\mathbb{C}/L$)**.** *Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$, and let $X = \mathbb{C}/L$ denote the associated elliptic curve.*

Let $\{x_1, x_2, \ldots, x_s\} \subset X$ be a finite set of distinct points. To each k, with $1 \leq k \leq s$, let an integer $n_k \in \mathbb{Z}$ be given. We call the collection

$$\{(x_k, n_k) \mid 1 \leq k \leq s\}$$

the Weierstrass data*.*

Then the Weierstrass problem for this data is solvable, i.e. there exists a meromorphic (elliptic) function f on X whose divisor satisfies

$$(f) = \sum_{k=1}^{s} n_k[x_k],$$

if and only if the following two conditions hold:

$$\text{(1)} \quad \sum_{k=1}^{s} n_k = 0,$$

$$\text{(2)} \quad \sum_{k=1}^{s} n_k \cdot x_k = 0 \quad \text{in } X.$$

Remark 4.2. Condition (2) can be expressed in terms of representatives in $\mathbb{C}$. Choose $a_k \in \mathbb{C}$ such that $\pi(a_k) = x_k$, where $\pi : \mathbb{C} \to X$ is the projection map. Then (2) is equivalent to:

$$\text{(2')} \quad \sum_{k=1}^{s} n_k a_k \equiv 0 \pmod{L}.$$

Preparation for the Proof: The Weierstrass σ-Function

The Weierstrass σ-function is an entire holomorphic function on $\mathbb{C}$ having a simple zero precisely at each lattice point of L. It is defined implicitly by the differential relation

$$(\log \sigma)' = \zeta, \qquad \text{that is,} \quad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z),$$

where $\zeta(z)$ denotes the Weierstrass zeta-function associated with the lattice L.

We require the normalization

$$\sigma(0) = 0, \qquad \sigma'(0) \neq 0, \qquad \text{and} \qquad \lim_{z \to 0} \frac{\sigma'(z)}{z} = 1.$$

Construction of σ. To construct such a function, we first seek a (possibly multivalued) function h satisfying

$$h'(z) = \zeta(z).$$

We will then define $\sigma = e^h$, chosen so that the exponential eliminates any ambiguity coming from the multivaluedness of h.

Formally, for a fixed base point $z_0 \in \mathbb{C}$, define

$$h(z) = \int_\gamma \zeta(w)\, dw,$$

where γ is any smooth path from z_0 to z that avoids the lattice points. Because $\zeta(w)$ has simple poles at every lattice point (each with residue 1), the value of this integral depends on the path chosen: if one deforms the path so that it winds once around a lattice point $w \in L$, the integral increases by

$$\int_{\Gamma_w^0} \zeta(w)\, dw = 2\pi i,$$

where Γ_w^0 is a small positively oriented loop around w,

$$\Gamma_w^0 : \theta \mapsto w + \varepsilon e^{i\theta}, \qquad \theta \in [0, 2\pi].$$

Multivaluedness and branches. Thus, the value of $h(z)$ depends on the *homotopy class* of the path: two different integration paths γ_1 and γ_2 from z_0 to z that differ by winding n_{12} times around lattice points produce values differing by integer multiples of $2\pi i$:

$$h_1(z) - h_2(z) = \int_{\gamma_1} \zeta(w)\, dw - \int_{\gamma_2} \zeta(w)\, dw = 2\pi i\, n_{12}, \qquad n_{12} \in \mathbb{Z}.$$

Hence, h is a *multivalued function*, well defined only modulo $2\pi i\mathbb{Z}$, and each "branch" of h corresponds to a specific choice of integration path.

Nevertheless, if we exponentiate h, the ambiguity disappears: for any two branches h_1, h_2,

$$e^{h_1(z)} = e^{h_2(z)} e^{2\pi i n_{12}} = e^{h_2(z)}.$$

Therefore, the function

$$\sigma(z) = e^{h(z)}$$

is *single valued* on $\mathbb{C}$, even though its logarithm h is not.

Behavior near the origin. Near $z = 0$, the zeta-function has the Laurent expansion

$$\zeta(z) = \frac{1}{z} + (\text{holomorphic terms}).$$

Integrating this local expansion gives

$$h(z) = \log z + \lambda(z),$$

where $\lambda(z)$ is holomorphic near 0. Hence,

$$e^{h(z)} = e^{\log z} e^{\lambda(z)} = z\, e^{\lambda(z)}.$$

By choosing the additive constant of integration so that $\lambda(0) = 0$, we obtain

$$\sigma(z) = z\, e^{\lambda(z)}, \qquad \sigma(0) = 0, \quad \sigma'(0) = 1.$$

Transformation under lattice translations. Because $\wp(z)$ is elliptic, its derivative $\zeta'(z) = -\wp(z)$ is doubly periodic, and we have

$$\zeta(z+\omega) - \zeta(z) = \eta_\omega, \qquad \text{for all } \ \omega \in L,$$

where η_ω is a constant (the *quasi-period*) depending only on ω and the lattice. Integrating this relation gives

$$\log \sigma(z+\omega) - \log \sigma(z) = \int_0^{z+\omega} \zeta(w)\, dw - \int_0^{z} \zeta(w)\, dw = \eta_\omega z + C_\omega,$$

for some constant C_ω independent of z. Exponentiating yields the *quasi-periodicity law* of the σ-function:

$$\sigma(z+\omega) = \exp(\eta_\omega z + C_\omega)\, \sigma(z), \qquad \forall \omega \in L.$$

Writing, more generally,

$$\sigma(z+\omega) = \exp(A_\omega z + B_\omega)\sigma(z).$$

Conclusion. The Weierstrass σ-function is therefore an *entire* function on $\mathbb{C}$ having simple zeros exactly at the lattice points L, normalized by $\sigma'(0) = 1$, and satisfying the fundamental identities

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z), \qquad \sigma(z+\omega) = \exp(A_\omega z + B_\omega)\sigma(z), \quad \forall \omega \in L.$$