THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 6

Date: Oct 21, 2024.

Instruction: *In order to have a better preparation of the coming test, you are recommended to try the problems below **BEFORE** the coming tutorial. Due to the time limit, the tutorial will NOT be able to go through all of the problems. Therefore, you should actively tell our TA which problems you wish to discuss first.*

Information: This collection of problems is intended to give you practice problems that are comparable in format and difficulty to those which will appear in the coming test. The questions in the actual exam will be **DIFFERENT**.

Problem 1. Let

$$A := \begin{pmatrix} 0 & -1 & 0 \\ 4 & 4 & 0 \\ 2 & -1 & 2 \end{pmatrix},$$

and complete the following parts.

(i) Find all eigenvalues and eigenvectors of A.

(ii) Find an invertible 3×3 real matrix P such that

$$A = P \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} P^{-1}.$$

(iii) Solve the following system of linear ordinary differential equations (ODE):

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x},$$

where the vector

$$\mathbf{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

is the unknown function of t.

Problem 2 (Invariant Transformations/Symmetries)**.** Let $u \in C^7$ be a solution to the wave equation

$$\partial_{tt}u - 4\partial_{xx}u = 0, \tag{1}$$

Show that the v defined in each of the cases below is also a solution to (1).

i (Translation)

$$v(t,x) := u(t-1, x-2).$$

ii (Differentiation) $v := \partial_t^2 \partial_x^3 u$.

iii (Convolution) For $g(y) = e^{-y^2}$,

$$v(t,x) := (u * g)(t,x) := \int_{-\infty}^{\infty} u(t, x-y)g(y)\, dy.$$

iv (Dilation/Scaling)

$$v(t,x) := u(4t, 4x).$$

Problem 3. Solve the following second-order linear PDE

$$3\partial_x u - \partial_{xy} u = 0. \tag{2}$$

in two different approaches as follows.

(i) Let $v := \partial_x u$. Verify that v satisfies

$$3v - \partial_y v = 0. \tag{3}$$

Solve for v in (3), then obtain u by integrating v with respect to x.

(ii) Rewrite the PDE (2) as

$$\partial_x(3u - \partial_y u) = 0. \tag{4}$$

Integrating (4) with respect to x, you can verify that u satisfies

$$3u - \partial_y u = g(y), \tag{5}$$

where g is any arbitrary function. Solve for u in (5) by using the method of integrating factors.

(iii) Are the solutions that you obtained in Part (i) and (ii) the SAME? Explain your answer briefly.

Problem 4 (Instability of Backward Heat Equation)**.** Consider the following PDE

$$\begin{cases} \partial_t u + 2\partial_{xx} u = 0 & \text{for } t > 0 \text{ and } -\pi/2 < x < \pi/2 \\ u|_{x=-\pi/2} = u|_{x=\pi/2} = 0 \\ u|_{t=0} = f, \end{cases} \tag{6}$$

where $f : [-\pi/2, \pi/2] \to \mathbb{R}$ is the initial data. Show that it is unstable with respect to the sup norm

$$\|g\|_{\sup} := \sup_{-\pi/2 \le x \le \pi/2} |g(x)|$$

by completing the following steps:

(i) Verify that for any positive odd integer n, the function

$$u_n(t,x) := \frac{1}{n} e^{2n^2 t} \cos nx$$

is a solution to the initial and boundary value problem (6) with the initial data

$$f_n(x) := \frac{1}{n}\cos nx.$$

(ii) Prove that

$$\lim_{n\to+\infty} \|f_n\|_{\sup} = 0.$$

(iii) Show that for any $T > 0$,

$$\lim_{n\to+\infty} \|u_n(T,\cdot)\|_{\sup} = +\infty.$$

Problem 5. Let $u := u(x,y)$ be a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$. Verify that the Laplace's equation $\Delta u := \partial_{xx}u + \partial_{yy}u = 0$ is invariant under rotations, that is,

$$\partial_{x'x'}u + \partial_{y'y'}u = \partial_{xx}u + \partial_{yy}u = 0,$$

where $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (θ is the angle of rotation).

Problem 6. Consider

$$\begin{cases} 3t\partial_t u - \partial_x u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi(x). \end{cases} \tag{7}$$

(i) Prove that if $\phi(x) := 2 + \cos x$, then (7) has no solution.

(ii) Assume that $\phi(x) \equiv 1$. Find all solutions to (7).

Problem 7. Solve the following boundary value problem

$$\begin{cases} y\partial_x u - x\partial_y u = x^2 + y^2 & \text{for } x, y > 0 \\ u|_{y=0} = x \end{cases}$$

by using the coordinate method.

(i) Rewrite the PDE in terms of the polar coordinates $r := \sqrt{x^2 + y^2}$ and $\theta := \tan^{-1}\left(\frac{y}{x}\right)$.

(ii) Express the boundary condition $u|_{y=0} = x$ in terms of r and θ.

(iii) Solve u in terms of r and θ.

(iv) Express your final answer u in terms of x and y.

Problem 8. Use the method of characteristics to find a solution for

$$\begin{cases} u_t(t,x) + u(t,x)u_x(t,x) = 0, & t > 0,\ x \in \mathbb{R}, \\ \lim\limits_{t\to 0^+} u(t,x) = x, & x \in \mathbb{R}. \end{cases}$$

Problem 9. Consider the second order equation

$$\partial_{tt}u + 2\partial_{tx}u - 3\partial_{xx}u = 0, \tag{8}$$

and complete the following parts.

(i) Verify that Equation (8) is hyperbolic by computing its discriminant $\mathcal{D}$.

(ii) Find the general solution to (8) by the method of characteristics.

(iii) Solve (8) with the following initial data:

$$\begin{cases} u|_{t=0} = 8x^2 \\ \partial_t u|_{t=0} = 24x. \end{cases}$$

Problem 10. In this problem we will solve the following second order equation

$$\partial_{tt}u + 6\partial_{tx}u + 9\partial_{xx}u = 0 \tag{9}$$

by the method of characteristics.

(i) Verify that the equation (9) can be written as

$$(\partial_t + 3\partial_x)^2 u = 0.$$

(ii) Let $v := (\partial_t + 3\partial_x)u = \partial_t u + 3\partial_x u$. Verify that u and v satisfy the following system:

$$\begin{cases} \partial_t v + 3\partial_x v = 0 \\ \partial_t u + 3\partial_x u = v. \end{cases}$$

(iii) Find the general solution to

$$\partial_t v + 3\partial_x v = 0.$$

(iv) Apply the method of characteristics to solve

$$\partial_t u + 3\partial_x u = v,$$

and prove that the general solution u to (9) has the form

$$u(t,x) = f(x-3t) + t\, g(x-3t)$$

where f and g are arbitrary functions.

Problem 11. In this problem answer the questions below regarding the following initial value problem for the wave equation:

$$\begin{cases} \partial_{tt} u - 4\partial_{xx} u = 0 \qquad \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi \\ \partial_t u|_{t=0} = \psi, \end{cases} \tag{10}$$

the functions ϕ and ψ are given initial data, and u is the unknown.

(i) For any given constant $\alpha \in (-\infty, \infty)$, we define

$$v(t,x) := u(t, x+\alpha).$$

Verify that v satisfies exactly the same wave equation:

$$\partial_{tt} v - 4\partial_{xx} v = 0.$$

Furthermore, find the initial conditions that v satisfies.

(ii) If both ϕ and ψ are odd around $x = 2$, that is, for any $x \in (-\infty, \infty)$,

$$\begin{cases} \phi(2-x) = -\phi(2+x) \\ \psi(2-x) = -\psi(2+x), \end{cases}$$

then what can you say about the solution u to the initial value problem (10)?