# Compact Riemann surfaces of genus $g \geq 2$, Part 2

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Let $X$ be a compact Riemann surface of genus $g \geq 2$.

Recall Poincaré Series. Let $h \in H^\infty(\mathbb{D})$ and let $k \geq 2$ be an integer. Define the Poincaré series of weight $k$ by

$$P_\Gamma^k(h)(z) = \sum_{\gamma \in \Gamma} h(\gamma z) \, (\gamma'(z))^k.$$

For sufficiently large $k$, the series converges, and we obtain a holomorphic function $f = P_\Gamma^k(h)$ on $\mathbb{D}$ satisfying

$$f(\gamma z) \, (\gamma'(z))^k = f(z), \qquad \forall \gamma \in \Gamma.$$

Thus $f(z)(dz)^k$ defines a holomorphic $k$-differential on $X = \mathbb{D}\backslash\Gamma$.

**Purpose.** We wish to prove that there exists an integer $k \gg 0$ such that the space of holomorphic $k$-differentials on $X$ provides a projective embedding

$$\Phi : X \hookrightarrow \mathbb{P}^N, \qquad \Phi = [\, s_0, s_1, \ldots, s_N \,].$$

That is, there exist holomorphic $k$-differentials

$$s_j(z) = f_j(z)(dz)^k, \qquad 0 \leq j \leq N,$$

such that $\Phi$ is an embedding.

Locally, we may also write

$$s_j(z) = f_j(z) \, \lambda(z) \, (dz)^k,$$

where $\lambda(z)$ is a nowhere-vanishing holomorphic function. Then

$$[s_0(z) : s_1(z) : \cdots : s_N(z)] = [\, \lambda(z)s_0(z) : \lambda(z)s_1(z) : \cdots : \lambda(z)s_N(z) \,],$$

so the projective point $\Phi(z)$ is independent of the local trivialization. Hence $\Phi$ is well defined on $X$.

*Example* 0.1. It is possible to obtain an embedding of

$$X = \mathbb{C}/L \longrightarrow \mathbb{P}^2$$

by means of *theta functions.* Define

$$\tilde{\Phi}(z) = [\,\theta_0(z) : \theta_1(z) : \theta_2(z)\,],$$

where, for $k = 0, 1, 2$, the functions $\theta_k$ are classical theta functions satisfying

$$\theta_k(z + \omega) = \exp(A_\omega z + B_\omega)\,\theta_k(z), \qquad \forall\, \omega \in L.$$

Then for all $\omega \in L$ and $z \in \mathbb{C}$,

$$\tilde{\Phi}(z + \omega) = \tilde{\Phi}(z).$$

Hence $\tilde{\Phi}$ descends to a well-defined holomorphic map

$$\Phi : X \longrightarrow \mathbb{P}^2,$$

which provides an embedding of $X$ into projective space via theta functions.

We have proved previously that if

$$X = \mathbb{D}/\Gamma,$$

where $\Gamma$ is a discrete torsion-free subgroup of $\mathrm{Aut}(\mathbb{D})$, then for any point $z_0 \in \mathbb{D}$, there exists a sufficiently large integer $k$ such that there exists a holomorphic $k$-differential

$$\phi(z) = f(z)(dz)^k,$$

invariant under $\Gamma$ and satisfying

$$\phi(z_0) \neq 0.$$

We begin by decomposing

$$\Gamma = \Gamma' \sqcup \Gamma'', \qquad \Gamma' = \{\,\gamma \in \Gamma : |\gamma'(z)| \geq \tfrac{1}{2}\,\}.$$

<u>Claim:</u> We can choose a function $h$ of the form $h = P|_{\mathbb{D}}$, where $P$ is a polynomial $P \in \mathbb{C}[z]$ satisfying

$$P(z_0) = 1, \qquad P(\gamma z_0) = 0 \quad \text{for all } \gamma \in \Gamma' \setminus \{\mathrm{id}\}.$$

Then we form the Poincaré series of weight $k$:

$$f(z) = P_\Gamma^k(h) = \sum_{\gamma \in \Gamma} h(\gamma z)\,(\gamma'(z))^k.$$

This sum can be split as

$$f(z) = \underbrace{\sum_{\gamma \in \Gamma'} h(\gamma z)\,(\gamma'(z))^k}_{f(z_0)=1} \;+\; \underbrace{\sum_{\gamma \in \Gamma''} h(\gamma z)\,(\gamma'(z))^k}_{\leq M}.$$

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*Proof of the Claim.* In general, let $z_1, z_2, \ldots, z_m \in \mathbb{C}$ be distinct points, and let $a_1, a_2, \ldots, a_m \in \mathbb{C}$ be prescribed values. We seek a polynomial $P \in \mathbb{C}[z]$ such that

$$P(z_k) = a_k, \qquad \text{for } k = 1, 2, \ldots, m.$$

This classical interpolation problem is solved by the *Lagrange interpolation formula.*

Define

$$Q(z) = \prod_{k=1}^{m}(z - z_k), \qquad R_k(z) = \frac{Q(z)}{(z - z_k)}.$$

Then

$$R_k(z_k) \neq 0, \qquad R_\ell(z_k) = 0 \quad \text{for all } \ell \neq k.$$

Hence the required polynomial is given by

$$P(z) = \sum_{k=1}^{m} a_k \frac{R_k(z)}{R_k(z_k)} = \sum_{k=1}^{m} a_k \frac{\prod_{\ell=1}^{m}(z - z_\ell)}{(z - z_k) \prod_{\ell \neq k}(z_k - z_\ell)}.$$

This polynomial satisfies $P(z_k) = a_k$ for all $k = 1, \ldots, m$. $\qquad\square$

By the compactness of $X$, we may choose $k \gg 0$ and finitely many holomorphic $k$-differentials on $X$,

$$\phi_0, \phi_1, \ldots, \phi_N,$$

such that for every $x \in X$, there exists some index $i \in \{0, 1, \ldots, N\}$ with

$$\phi_i(x) \neq 0.$$

For instance, suppose there exist points $x_1, x_2 \in X$ and differentials $\phi_1, \phi_2$ satisfying

$$\begin{cases} \phi_1(x_1) \neq 0, \\ \phi_1(x_2) = 0, \end{cases} \qquad \begin{cases} \phi_2(x_1) = 0, \\ \phi_2(x_2) \neq 0. \end{cases}$$

By scaling and small perturbation, we may assume

$$\begin{cases} \phi_1(x_1) = 1, & |\phi_1(x_2)| < \varepsilon, \\ |\phi_2(x_1)| < \varepsilon, & \phi_2(x_2) = 1, \end{cases}$$

for some small $\varepsilon > 0$. Then the map

$$\Phi : X \to \mathbb{P}^N, \qquad \Phi(x) = [\,\phi_0(x) : \phi_1(x) : \cdots : \phi_N(x)\,],$$

is non-constant, since the images of $x_1$ and $x_2$ differ:

$$\Phi(x_1) \neq \Phi(x_2).$$

**Alternative argument.** Fix a point $x_0 \in X$ and choose a lift $z_0 \in \mathbb{D}$ so that $\pi(z_0) = x_0$ (i.e. $x_0 = z_0 \bmod \Gamma$). Let $\phi_0$ be a holomorphic $k$-differential on $X$ for $k \gg 0$. Because $X$ is compact, $\phi_0$ must have a zero somewhere on $X$; that is, there exists $x_1 \in X$, $x_1 \neq x_0$, such that

$$\phi_0(x_1) = 0, \phi_0(x_0) \neq 0$$

(When $g = 1$, i.e. $X$ is an elliptic curve, there exists a nowhere-vanishing holomorphic $k$-differential for some $k$, and this distinction characterizes $g = 1$. )

Now choose another holomorphic $k$-differential $\phi_1$ such that $\phi_1(x_1) \neq 0$. Then the pairs of projective coordinates

$$(\phi_0(x_0), \phi_1(x_0)) \quad \text{and} \quad (\phi_0(x_1), \phi_1(x_1)) = (0, \phi_1(x_1))$$

are not proportional. Hence

$$\Phi(x_0) \neq \Phi(x_1),$$

showing that $\Phi : X \to \mathbb{P}^N$ is a non-constant holomorphic map.

Now we aim to conduct the next three steps:

**Step 1.**

$$\Phi = [\, \phi_0, \ldots, \phi_N \,] : X \hookrightarrow \mathbb{P}^N \quad \text{holomorphic map (actually non-constant).}$$

This part is done.

**Step 2.**
$$\Phi \text{ is a holomorphic immersion for some } k \gg 0.$$

**Step 3.** To prove that

$$\Phi \text{ separates points (for some } k \gg 0).$$

**Step 2.** To prove that $\Phi$ is a holomorphic immersion for $k \gg 0$, it suffices to show the following local statement:

Given $z_0 \in \mathbb{D}$, find two holomorphic $k$-differentials $f$ and $g$ such that

$$\text{(a) } f = P_\Gamma^k(t) \text{ satisfies } f(z_0) \neq 0, \qquad \text{(b) } g = P_\Gamma^k(s) \text{ satisfies } \left(\frac{g}{f}\right)'(z_0) \neq 0.$$

Indeed, in the inhomogeneous coordinates

$$\Phi = [t, s_1, \ldots, s_m] = \left(\frac{s_1}{t}, \ldots, \frac{s_m}{t}\right),$$

the immersion condition requires $\left(\frac{s_j}{t}\right)'(z_0) \neq 0$ for some $j$, i.e. the derivative does not vanish at $z_0$.

*Proof.* We first construct a pair of polynomials realizing prescribed value and derivative data under the $\Gamma$–relations.

We wish to prove that for given complex numbers $a, b \in \mathbb{C}$, there exists a polynomial $R(z)$ such that

$$R(z_0) = a, \qquad R'(z_0) = b,$$

and moreover

$$R(\gamma z_0) = 0, \qquad R'(\gamma z_0) = 0 \quad \text{for all } \gamma \in \Gamma \setminus \{\text{id}\}.$$

Let

$$P(z) = \prod_{i=2}^{N} (z - \gamma_i z_0)^2, \qquad \Gamma_0 = \{\gamma_1 = \text{id}, \gamma_2, \ldots, \gamma_N\}.$$

Define

$$Q(z) = (z - z_0) \prod_{i=2}^{N} (z - \gamma_i z_0)^2 = (z - z_0) P(z).$$

Then

$$P(z_0) = a \neq 0, \qquad P'(z_0) = b,$$

and hence

$$Q(z_0) = 0, \qquad Q'(z_0) = P(z_0) \neq 0.$$

Therefore, at $z_0$ we have

$$(P(z_0), P'(z_0)) = (a, b), \qquad (Q(z_0), Q'(z_0)) = (0, a), \quad a \neq 0.$$

Since these pairs are linearly independent, for any prescribed $(\tilde{a}, \tilde{b}) \in \mathbb{C}^2$ there exist $\alpha, \beta \in \mathbb{C}$ such that

$$R(z) = \alpha P(z) + \beta Q(z)$$

satisfies

$$R(z_0) = \tilde{a}, \qquad R'(z_0) = \tilde{b}.$$

We may take in particular

$$(\tilde{a}, \tilde{b}) = (1, 0) \quad \text{(Polynomial A)}, \qquad (\tilde{a}, \tilde{b}) = (0, 1) \quad \text{(Polynomial B)}.$$

Observe that

$$P(\gamma_i z_0) = P'(\gamma_i z_0) = 0, \qquad Q(\gamma_i z_0) = Q'(\gamma_i z_0) = 0, \quad \forall 2 \leq i \leq N.$$

Now let

$$t = A|_{\mathbb{D}}, \qquad s = B|_{\mathbb{D}},$$

and for $k \gg 0$ define

$$f = P_\Gamma^k(t), \qquad g = P_\Gamma^k(s),$$

where $P_\Gamma^k$ denotes the Poincaré series operator.

**Claim.** For sufficiently large $k$, the pair $(f, g)$ satisfies the required conditions in the proposition:

$$f(z_0) \neq 0, \qquad \left(\frac{g}{f}\right)'(z_0) \neq 0,$$

where
$$f(z) = \sum_{\gamma \in \Gamma_0} t(\gamma z) \, (\gamma'(z))^k, \qquad g(z) = \sum_{\gamma \in \Gamma \backslash \Gamma_0} s(\gamma z) \, (\gamma'(z))^k.$$

Thus $\Phi$ is a holomorphic immersion when $k \gg 0$. □

**Step 3.** To complete the proof of the theorem that

$$X \hookrightarrow \mathbb{P}^N$$

is an embedding for sufficiently large $k$, we must show that for the pluricanonical map

$$\Phi = [s_0, s_1, \ldots, s_N],$$

we have

$$\Phi(x_0) \neq \Phi(x_1) \quad \text{for all distinct } x_0, x_1 \in X, \quad \text{when } k \gg 0.$$

For two distinct points $x_0, x_1 \in X$, it suffices to find two holomorphic $k$-differentials $s_0, s_1$ such that

$$\begin{cases} s_0(x_0) \neq 0, & s_1(x_0) = 0, \\ s_0(x_1) = 0, & s_1(x_1) \neq 0. \end{cases}$$

Then

$$\Phi(x_0) = [1 : 0], \qquad \Phi(x_1) = [0 : 1],$$

so the image points are distinct. Hence, $\Phi$ separates $x_0$ and $x_1$.

Subtlety (Heine–Borel type issue). At first, one can solve the separation problem for each *individual pair* $(x_0, x_1)$ of distinct points by constructing corresponding sections $s_0, s_1$. However, the degree $k$ needed may depend on the chosen pair:

$$k = k(x_0, x_1).$$

To obtain a single embedding map $\Phi_k$ valid for all pairs, we must ensure that one sufficiently large $k$ works *uniformly* for every pair $(x_0, x_1)$.

Let

$$S = \{(x_0, x_1) \in X \times X : x_0 \neq x_1\} = (X \times X) \backslash \mathrm{Diag}(X).$$

Although $X \times X$ is compact, $S$ is open (the complement of the diagonal), hence *not compact*. Therefore, we cannot directly argue by compactness to obtain a uniform $k$.

Suppose now that for some large $k$, the map

$$\Phi = [\phi_0, \ldots, \phi_N] : X \hookrightarrow \mathbb{P}^N$$

is already a holomorphic immersion (by Step 2). Then for each $x \in X$, there exists a neighborhood $U(x)$ such that for all $x_1, x_2 \in U(x)$, $x_1 \neq x_2$, we have

$$\Phi(x_1) \neq \Phi(x_2).$$

Thus, the image points are separated locally.

Consequently, there exists a symmetric open set $T \subset X \times X$ (a *tubular neighborhood* of the diagonal) such that

$$(x_1, x_2) \in T \quad \Rightarrow \quad \Phi(x_1) \neq \Phi(x_2), \qquad \text{and} \qquad (x_1, x_2) \in T \iff (x_2, x_1) \in T.$$

The complement

$$(X \times X) \setminus T$$

is closed in $X \times X$, hence compact (since $X \times X$ is compact).

**Further subtlety** Start with an integer $k_0 \gg 0$ chosen so that the map $\Phi_{k_0}$ satisfies the immersion property (Step 1 and Step 2). To separate points, we work with pairs $(x_1, x_2) \in U = X \times X \setminus T$, where $T \subset X \times X$ is the symmetric tubular neighborhood of the diagonal, introduced previously.

For such pairs, we may need a larger degree $k = k(x_1, x_2) > k_0$ to achieve point separation. This suggests a potential further subtlety: the degree required might depend on the pair of points, preventing a single, uniform choice of $k$.

**Conceptual explanation: why we can avoid this subtlety.** Suppose we already have a map

$$\Phi = [s_0, s_1, s_2] : X \longrightarrow \mathbb{P}^2$$

that is a holomorphic immersion and separates points $(x_1, x_2) \in T \subset X \times X$. To increase its separating power globally, we can apply the *Veronese embedding* of projective space.

Define a new map

$$\nu_2 \circ \Phi : X \longrightarrow \mathbb{P}^{N'}$$

given by all quadratic monomials in the coordinates of $\Phi$:

$$[s_0, s_1, s_2] \longmapsto [s_0^2, \ s_1^2, \ s_2^2, \ s_0 s_1, \ s_0 s_2, \ s_1 s_2].$$

This realizes the composition

$$X \hookrightarrow \mathbb{P}^N \hookrightarrow \mathbb{P}^{N'},$$

where

$$N' = \frac{(N+1)(N+2)}{2} - 1.$$

For instance, when $N = 2$, we have the classical Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

Explicitly, if $(x, y, z)$ are homogeneous coordinates on $\mathbb{P}^2$, then

$$(x, y, z) \longmapsto (x, y, z)(x, y, z)^T = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix},$$

whose entries (up to order) yield the six homogeneous coordinates of $\mathbb{P}^5$. The target space here, $\mathbb{P}(S^2 \mathbb{C}^3)$, is the projectivization of the space of symmetric $3 \times 3$ matrices, which has dimension 6.

In general, for any $N$, the *Veronese embedding*

$$\nu_m : \mathbb{P}^N \hookrightarrow \mathbb{P}^{N_m}, \qquad N_m = \binom{N+m}{m} - 1,$$

is holomorphic and injective. Composing $\Phi_k$ with $\nu_m$ corresponds to replacing the sections $s_i$ by all degree-$m$ monomials in them, i.e. by sections of $(K_X^{\otimes k})^{\otimes m} = K_X^{\otimes mk}$.

Thus, after fixing a symmetric tubular neighborhood $T = T_k \subset X \times X$, the higher pluricanonical map

$$\Phi_{mk} : X \longrightarrow \mathbb{P}^{N_{mk}}$$

is still a holomorphic immersion and now separates all points $(x_1, x_2) \in X \times X \setminus T$.

**Final argument.** Given distinct points $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$, we want to find $k_0'$ large enough such that for every $k \geq k_0'$, there exist holomorphic $k$-differentials $s_1, s_2 \in H^0(X, K_X^{\otimes k})$ satisfying

$$\begin{cases} s_1(x_1) \neq 0, & s_1(x_2) = 0, \\ s_2(x_1) = 0, & s_2(x_2) \neq 0. \end{cases}$$

Then their corresponding evaluation points in projective space satisfy

$$[s_1(x_1) : s_2(x_1)] \neq [s_1(x_2) : s_2(x_2)].$$

Hence the pluricanonical map

$$\Phi_k = [s_0, s_1, \ldots, s_N] : X \longrightarrow \mathbb{P}^N$$

separates the distinct points $x_1$ and $x_2$.

*Proof.* Let $\Gamma$ be the covering (Fuchsian) group acting on the unit disk $D$, with quotient $X = D/\Gamma$. Fix two lifts $z_0, z_1 \in D$ corresponding to distinct points $x_0, x_1 \in X$.

Define

$$\Gamma' = \left\{ \gamma \in \Gamma : |\gamma'(z_0)| \geq \tfrac{1}{2} \text{ or } |\gamma'(z_1)| \leq \tfrac{1}{2} \right\}.$$

The set $\Gamma'$ is finite, because $\sum_{\gamma \in \Gamma} |\gamma'(z_0)|^2 < \infty$ implies that only finitely many $\gamma$ have derivatives not exponentially small. Denote its complement by

$$\Gamma'' = \Gamma \setminus \Gamma' = \left\{ \gamma \in \Gamma : |\gamma'(z_0)| < \tfrac{1}{2} \text{ and } |\gamma'(z_1)| > \tfrac{1}{2} \right\}.$$

We construct holomorphic polynomials $P_1, P_2$ (by Lagrange interpolation) satisfying for all $\gamma \in \Gamma'$:

$$\begin{cases} P_1(z_0) = 1, & P_1(\gamma z_0) = 0 \text{ for } \gamma \neq \mathrm{id}, \\ P_1(\gamma z_1) = 0, \\ P_2(z_1) = 1, & P_2(\gamma z_1) = 0 \text{ for } \gamma \neq \mathrm{id}, \\ P_2(\gamma z_0) = 0. \end{cases}$$

The finite nature of $\Gamma'$ guarantees that these interpolation conditions determine $P_1$ and $P_2$.

Let $f_j = P_j|_D$ for $j = 1, 2$.

For each integer $k > 0$, define

$$s_j(z) = \sum_{\gamma \in \Gamma} f_j(\gamma z) \, (\gamma'(z))^k, \qquad j = 1, 2.$$

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These functions are $\Gamma$-automorphic $k$-differentials since for all $\delta \in \Gamma$,

$$s_j(\delta z)\,(\delta'(z))^k = s_j(z).$$

We split the sum as

$$s_j = \underbrace{\sum_{\gamma \in \Gamma'} f_j(\gamma z)(\gamma'(z))^k}_{\text{main part}} + \underbrace{\sum_{\gamma \in \Gamma''} f_j(\gamma z)(\gamma'(z))^k}_{\text{tail}}.$$

For $\gamma \in \Gamma''$, we have $|\gamma'(z_0)| < \frac{1}{2}$, thus $|(\gamma'(z))^k| \leq 2^{-k}$, so the tail decays exponentially with $k$ and vanishes uniformly as $k \to \infty$.

By interpolation properties:

$$f_1(z_0) = 1, \quad f_1(\gamma z_0) = 0 \ (\gamma \neq \mathrm{id}), \quad f_1(\gamma z_1) = 0,$$

and analogously

$$f_2(z_1) = 1, \quad f_2(\gamma z_1) = 0 \ (\gamma \neq \mathrm{id}), \quad f_2(\gamma z_0) = 0.$$

Hence for the constructed series,

$$s_1(z_0) = f_1(z_0) + O(2^{-k}) = 1 + O(2^{-k}), \quad s_1(z_1) = 0,$$
$$s_2(z_0) = 0, \qquad\qquad\qquad\qquad\qquad s_2(z_1) = f_2(z_1) + O(2^{-k}) = 1 + O(2^{-k}).$$

Therefore, for sufficiently large $k$,

$$[s_1(z_0) : s_2(z_0)] = [1 : 0], \qquad [s_1(z_1) : s_2(z_1)] = [0 : 1].$$

So as $k \to \infty$, $[s_1(z_0) : s_2(z_0)] \neq [s_1(z_1) : s_2(z_1)]$. Thus the differentials $s_1, s_2$ separate the pair of points $x_0, x_1$ on $X$.

Therefore, for every distinct pair $(x_0, x_1) \in X \times X$, there exists $k_0'$ such that for all $k \geq k_0'$ we can construct holomorphic $k$-differentials $s_1, s_2$ with

$$[s_1(x_0) : s_2(x_0)] \neq [s_1(x_1) : s_2(x_1)].$$

Hence the map $\Phi_k$ separates all points for $k \gg 0$, and therefore defines a holomorphic embedding. $\qquad\square$