Department of Mathematics, The University of Hong Kong

MATH4302 (Algebra II), May 21, 2020
2:30 pm - 5:00 pm
Answer **ALL SEVEN** questions

This is a **closed-book exam**. Except for Problem 1, you should always give precise and adequate explanations to support your conclusions. Please do not forget to put your University ID on the top of **every page** of your answer sheet.

Problem 1 (18 points). Answer "True" or "False" to each of the following six statements. For this problem only, you do not need to explain your answers.

1) In any integral domain R, the ideal $\{0\}$ of R is prime;
2) For any integer $n \geq 2$, the ring $(\mathbb{Z}/n\mathbb{Z})[x_1, x_2]$ is a unique factorization domain.
3) The field extension $\mathbb{Q}(\sqrt[5]{2})$ of $\mathbb{Q}$ is normal;
4) The polynomial $f(x) = x^9 - 2 \in \mathbb{F}_3[x]$ is separable;
5) The splitting field of $(x^3 + 5)^3(x^8 - 9)^2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$ is a Galois extension of $\mathbb{Q}$;
6) No polynomial in $\mathbb{Q}[x]$ of order 5 is solvable by radicals.

Problem 2 (12 points). 1) If R is a unique factorization domain, and if $a, b \in R$ are both non-zero, define a *greatest common divisor* of a and b. Is it unique?

2) Determine the greatest common divisors of $a(x) = x^3 + 2$ and $b(x) = x + 1$ in $\mathbb{Z}[x]$ as well as in $\mathbb{F}_2[x]$.

Problem 3 (10 points). Let K be a field and L a field extension of K. Can two different irreducible monic polynomials $p(x)$ and $q(x)$ in $K[x]$ have a common root in L? Why?

Problem 4 (10 points). Let α be a real root of the polynomial $p(x) = x^3 + 9x + 1$. Can α be constructed by a ruler and compass? Explain your answer.

Problem 5 (15 points). Show that there are infinitely many prime elements in $K[x]$ for any field K.

Problem 6 (15 points). Let K be any field. State the definition of a splitting field of a non-constant polynomial $f \in K[x]$ over K. Give at least **three** facts about splitting fields to show that they are interesting and useful.

Problem 7 (20 points). Let $\omega = e^{\pi i/4} = \frac{\sqrt{2}}{2}(1 + i) \in \mathbb{C}$ and let $L = \mathbb{Q}(\omega)$.

(a) Is L a Galois extension of $\mathbb{Q}$? Why? What is $[L : \mathbb{Q}]$?

(b) Compute the Galois group $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$;

(b) Describe all the intermediate fields between $\mathbb{Q}$ and L.

******* **END OF PAPER** *******