Compact Riemann surfaces of genus $g \geq 2$, Part 1

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By the **Uniformization Theorem**, any simply connected Riemann surface is biholomorphic to exactly one of the following:

$$\widehat{\mathbb{C}}, \qquad \mathbb{C}, \qquad \text{or} \qquad \mathbb{D} = \{\, z \in \mathbb{C} : |z| < 1 \,\}.$$

In particular, if X is a compact Riemann surface of genus $g \geq 2$, its universal covering surface is the unit disk $\mathbb{D}$, and hence

$$X \;\cong\; \mathbb{D}/\Gamma,$$

where $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ is a discrete subgroup of Möbius transformations acting properly discontinuously (a *Fuchsian group*).

The unit disk $\mathbb{D}$ and the upper half-plane

$$\mathbb{H} = \{\, \tau \in \mathbb{C} \mid \Im(\tau) > 0 \,\}$$

are conformally equivalent. The equivalence is given by the **Cayley transform**

$$z = \frac{\tau - i}{\tau + i}, \qquad \tau = i\,\frac{1+z}{1-z}.$$

Thus, one can work equally well with the model $\mathbb{D}/\Gamma \subset \mathrm{Aut}(\mathbb{D})$ or with the equivalent model $\mathbb{H}/\Gamma' \subset \mathrm{PSL}_2(\mathbb{R})$, where $\Gamma' = C^{-1}\Gamma C$ under the Cayley conjugation C.

Poincaré Series

Let $X = \mathbb{D}/\Gamma$ be a compact Riemann surface of genus $g \geq 2$, where $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ is a discrete group (of Möbius transformations acting properly discontinuously on $\mathbb{D}$).

Let $h \in H^\infty(\mathbb{D})$, i.e. a bounded holomorphic function on $\mathbb{D}$, and let $k \geq 2$ be an integer. Define the **Poincaré series** of weight k associated to h by

$$\boxed{P_k^\Gamma(h)(z) \;=\; \sum_{\gamma \in \Gamma} h(\gamma z)\,(\gamma'(z))^k, \qquad z \in \mathbb{D}.}$$

Then the series

$$f(z) = P_k^\Gamma(h)(z)$$

converges *uniformly on compact subsets* of $\mathbb{D}$; hence f defines a holomorphic function on $\mathbb{D}$. Moreover, for every $\gamma \in \Gamma$, we have

$$f(\gamma z) = \frac{1}{(\gamma'(z))^k}\, f(z).$$

Thus f is a Γ-automorphic holomorphic function of weight k.

Preparation Let $X = \mathbb{D}/\Gamma$ be a compact Riemann surface of genus $g \geq 2$, where $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ is a discrete group of Möbius transformations.

A **holomorphic 1-form** on X can be lifted to a holomorphic 1-form on the disk $\mathbb{D}$ of the form $f(z)\,dz$. It descends to a well-defined form on X if and only if it is Γ-invariant, i.e.

$$f(\gamma z)\, d(\gamma z) = f(z)\, dz \qquad \forall\, \gamma \in \Gamma,\ z \in \mathbb{D}.$$

Since

$$\frac{d(\gamma z)}{dz} = \gamma'(z) \quad \Rightarrow \quad d(\gamma z) = \gamma'(z)\, dz,$$

the invariance condition becomes

$$f(\gamma z)\, \gamma'(z)\, dz = f(z)\, dz \quad \Longleftrightarrow \quad f(\gamma z) = \frac{1}{\gamma'(z)}\, f(z).$$

A holomorphic 1-form is also called a *holomorphic* 1*-differential*. More generally, for an integer $k \geq 1$, a **holomorphic k-differential** is a local expression of the form

$$f(z)\,(dz)^k,$$

subject to the following transformation rule: if we make a holomorphic change of coordinate $z \mapsto w(z)$, then

$$dw = \frac{dw}{dz}\, dz \quad \text{and hence} \quad (dw)^k = \left(\frac{dw}{dz}\right)^k (dz)^k.$$

Definition 0.1 (Holomorphic k-differential on $X = \mathbb{D}/\Gamma$)**.** Let $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ act properly discontinuously on $\mathbb{D}$. A *holomorphic k-differential* on $X = \mathbb{D}/\Gamma$ is a Γ-invariant tensor on $\mathbb{D}$ of the form

$$g(z) = f(z)\,(dz)^k,$$

where f is holomorphic on $\mathbb{D}$ and satisfies the transformation law

$$\boxed{f(\gamma z) = \frac{1}{(\gamma'(z))^k}\, f(z) \qquad \forall\, \gamma \in \Gamma,\ \forall\, z \in \mathbb{D}.}$$

Equivalently,

$$\gamma^* g = g \quad \text{for all } \gamma \in \Gamma,$$

since

$$\gamma^*\big(f(z)(dz)^k\big) = f(\gamma z)\,(d(\gamma z))^k = f(\gamma z)\,(\gamma'(z))^k\,(dz)^k.$$

In particular, when $f = P_k^\Gamma(h)$ is a Poincaré series constructed from a bounded holomorphic function $h \in H^\infty(\mathbb{D})$, the expression

$$f(z)\,(dz)^k$$

is Γ-invariant and hence defines a holomorphic k-differential on the quotient Riemann surface X.

Remark 0.2.

1. Even if $h \in H^\infty(\mathbb{D})$ (i.e. h is bounded on the disk), the associated Poincaré series $f = P_k^\Gamma(h)$ need not be bounded on $\mathbb{D}$.
2. There exist linearly independent holomorphic 1-forms on X which *cannot* be obtained as Poincaré series of the type $P_k^\Gamma(h)$.

Proof. We first check **Γ-invariance.**

Assume that

$$f = P_k^\Gamma(h)(z) = \sum_{\gamma\in\Gamma} h(\gamma z)\,(\gamma'(z))^k$$

is absolutely convergent and uniformly convergent on compact subsets of $\mathbb{D}$. Then $f(z)$ defines a holomorphic function on $\mathbb{D}$, i.e. $f \in H(\mathbb{D})$.

Let $\mu \in \Gamma$. We compute $f(\mu z)$:

$$f(\mu z) = \sum_{\gamma\in\Gamma} h(\gamma(\mu z))\,(\gamma'(\mu z))^k.$$

By the chain rule,

$$(\gamma \circ \mu)'(z) = \gamma'(\mu z)\,\mu'(z), \quad \text{so} \quad \gamma'(\mu z) = \frac{(\gamma\mu)'(z)}{\mu'(z)}.$$

Substituting this into the series, we get

$$f(\mu z) = \sum_{\gamma\in\Gamma} h((\gamma\mu)(z))\left(\frac{(\gamma\mu)'(z)}{\mu'(z)}\right)^k.$$

Since Γ is a group, the set $\{\gamma\mu : \gamma \in \Gamma\}$ is again all of Γ. Write $\delta = \gamma\mu$. Then the sum becomes

$$f(\mu z) = \frac{1}{(\mu'(z))^k} \sum_{\delta\in\Gamma} h(\delta z)\,(\delta'(z))^k.$$

Hence

$$f(\mu z) = \frac{1}{(\mu'(z))^k}\, f(z), \quad \forall\, \mu \in \Gamma,\ z \in \mathbb{D}.$$

Multiplying both sides by $(\mu'(z))^k (dz)^k$, we find

$$f(\mu z)\,(\mu'(z))^k\,(dz)^k = f(z)\,(dz)^k,$$

or, in invariant form,

$$f(\mu z)\,(d(\mu z))^k = f(z)\,(dz)^k.$$

Thus $f(z)(dz)^k$ is a Γ-invariant holomorphic k-differential on $X = \mathbb{D}/\Gamma$.

It remains to verify that, for $k \geq 0$, the series

$$P_k^\Gamma(h)(z) = \sum_{\gamma\in\Gamma} h(\gamma z)\,(\gamma'(z))^k$$

is **absolutely and uniformly convergent** on compact subsets of $\mathbb{D}$.

(a) Preliminary estimate. For any finite subset $\Gamma_0 \subset \Gamma$,

$$|f(z)| = \Big|\sum_{\gamma\in\Gamma_0} h(\gamma z)\,(\gamma'(z))^k\Big| \leq \sum_{\gamma\in\Gamma} |h(\gamma z)|\,|\gamma'(z)|^k.$$

Since $h \in H^\infty(\mathbb{D})$, there exists $M > 0$ with $|h(z)| \leq M$ for all $z \in \mathbb{D}$. Hence,

$$|f(z)| \leq M \sum_{\gamma\in\Gamma} |\gamma'(z)|^k.$$

We will show that

$$\sum_{\gamma\in\Gamma} |\gamma'(z)|^2 \leq C_K \quad \text{for all } z \in K,$$

for each compact $K \subset \mathbb{D}$ and some constant $C_K > 0$. This implies absolute and uniform convergence on K.

(b) Integral estimate via disjointness of fundamental domains.

Let $U \subset \mathbb{D}$ be a relatively compact open set such that $\gamma(U) \cap U = \emptyset$ for all $\gamma \neq \mathrm{id}$. Then, for each $\gamma \in \Gamma$, by letting $w = \gamma(z)$ (so that $dw = \gamma'(z)\,dz$),

$$\int_U |\gamma'(z)|^2\,|dz|^2 = \int_{\gamma(U)} |dw|^2 = \mathrm{Area}(\gamma(U)).$$

Summing over all γ,

$$\int_U \sum_{\gamma\in\Gamma} |\gamma'(z)|^2\,|dz|^2 \;\leq\; \sum_{\gamma\in\Gamma} \mathrm{Area}\big(\gamma(U)\big) = \mathrm{Area}\left(\bigcup_{\gamma\in\Gamma} \gamma(U)\right) = \mathrm{Area}(\mathbb{D}) < \infty.$$

Thus the function $\sum_{\gamma\in\Gamma} |\gamma'(z)|^2$ is locally integrable and bounded in average.

(c) From integral to pointwise estimates.

Let $S(z)$ be any holomorphic function on a disk $D(a,R)$. By the L^2 mean-value property,

$$|S(a)|^2 \leq \frac{1}{\pi R^2} \int_{D(a,R)} |S(z)|^2\,dx\,dy.$$

Indeed, if $S(z) = \sum_{n=0}^\infty a_n (z-a)^n$, then

$$\int_{D(a,R)} |S(z)|^2\,dx\,dy = \int_0^R \int_0^{2\pi} \Big|\sum_n a_n r^n e^{in\theta}\Big|^2 r\,d\theta\,dr = \pi \sum_n |a_n|^2 \frac{R^{2n+2}}{n+1},$$

and the orthogonality of $e^{in\theta}$ implies $|a_0|^2 \le \frac{1}{\pi R^2} \int_{D(a,R)} |S(z)|^2 \, dx \, dy$.

Applying this to the (partial sums of the) function $S(z) = P_k^\Gamma(h)(z)$ and the integral estimate from part (b), we obtain for every compact $K \subset \mathbb{D}$ that there exists a constant $C_K > 0$ with

$$\sup_{z \in K} \sum_{\gamma \in \Gamma} |\gamma'(z)|^2 \le C_K.$$

Therefore the series

$$\sum_{\gamma \in \Gamma} h(\gamma z)(\gamma'(z))^k$$

converges absolutely and uniformly on compact subsets of $\mathbb{D}$.

This completes the proof of convergence. $\square$