THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 11

Problem 1. Consider the following initial and boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2} - u & \text{for } 0 < x < \pi \text{ and } t > 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 0 \\ \dfrac{\partial u}{\partial t}(x,0) = 3\sin 3x. \end{cases} \tag{1}$$

Answer the following questions:

(i) Solve the initial and boundary value problem (1). To obtain full credit, you must show all details (including deriving ordinary differential equations, solving eigenvalue problem, $\cdots$, etc.) and express your final solution u without any undetermined coefficients.

(ii) Prove or disprove

$$\lim_{t\to\infty} u(x,t) = 0.$$

Problem 2. Solve the Laplace's equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

inside a quarter-circle with radius 1 $\left(0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\right)$ subject to the boundary conditions

$$\frac{\partial u}{\partial \theta}(r,0) = u\left(r, \frac{\pi}{2}\right) = 0 \quad \text{and} \quad u(1,\theta) = 4\cos\theta$$

by the *method of separation of variables*. In order to obtain full credit, you must show all steps (including deriving ordinary differential equations, solving eigenvalue problem, $\cdots$, etc.) and express your final answer without any undetermined coefficients.

Problem 3. Consider the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = 5\dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < 2,\ t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(2,t) = 0 & \\ u(x,0) = \begin{cases} 0 & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}, & \end{cases} \tag{2}$$

and answer the following questions:

(i) Solve the initial and boundary value problem (2). To obtain full credit, you must show all details (including deriving ordinary differential equations, solving eigenvalue problem, $\cdots$, etc.) and express your final solution u without any undetermined coefficients.

(ii) Prove or disprove

$$\lim_{t\to\infty} u(x,t) = 0.$$