MATH3541 INTRODUCTION TO TOPOLOGY ASSIGNMENT IV

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG

Due: 12:00 noon, 29th October 2024.

Instructions: Submit solutions to the problems in **Section B** for credit. Problems in Section A should be attempted and may be optionally submitted for feedback.

Guidelines on Writing: *You should write in complete sentences. Do not just give the solution in fragmentary bits and pieces. Clarity of presentation of your argument counts, so explain the meaning of every symbol that you introduce and avoid starting a sentence with a symbol.*

Section A

Problem 1. Show that the circle $C = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$ and the circle with a spike

$$S = C \cup \{(x, 0) \mid x \in [1, 2]\} \subset \mathbb{R}^2$$

are not homeomorphic.

Problem 2. If X has only a finite number of connected components, show that each connected component of X is both open and closed.

Problem 3. Give an example of a topological space with infinitely many connected components.

Problem 4. Show that path-connectedness is a topological invariant.

Problem 5. Show that the topologist's sine curve is connected but not path-connected.

Problem 6. If G is a topological group and H is a topological subgroup of G, show that $\overline{H}$ is also a topological subgroup of G.

Problem 7. Let G be a topological group; let G_0 be the connected component of G containing the identity element e. Show that G_0 is a normal subgroup of G.

Section B

Problem 8 (2 marks). Let $M = \big([0,1] \times (0,1)\big)/\!\sim$ be the Möbius strip without boundary. Let $S \cong S^1$ be the circle $\big([0,1] \times \{\frac{1}{2}\}\big)/\!\sim$ embedded in M. Explicitly construct a homeomorphism from $M \setminus S$ to the cylinder $S^1 \times (0,1)$.

Problem 9 (5 marks). Let $T^2 = S^1 \times S^1$ denote the 2-dimensional torus. Give actions of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on T^2 such that the quotient space is homeomorphic to:

(a) the torus T^2;
(b) the Klein bottle K;
(c) the real projective plane $\mathbb{RP}^2$.

Briefly justify why the quotient spaces are homeomorphic to the desired spaces. (HINT: Recall/check that $\mathbb{RP}^2 \cong S^2/\!\sim$ where $x \sim y$ iff $x = -y$. Taking the upper hemisphere, this is equivalent to $\mathbb{RP}^2 \cong D^2/\!\sim$ where $x \sim y$ iff $x = -y$ and $\|x\| = 1$.)

Problem 10 (6 marks). The *Borsuk—Ulam theorem* states that for any continuous map $f : S^n \to \mathbb{R}^n$ there exists x such that $f(x) = f(-x)$.

(a) Prove Borsuk—Ulam in the case of $n = 1$.

For the rest of this question, you may use Borsuk—Ulam for general n.

(b) Prove there does not exist a continuous map $f : S^n \to S^{n-1}$ which is odd. That is, a function satisfying $f(-x) = -f(x)$ for all $x \in S^n$.

Let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed n-dimensional disk and suppose $f : D^n \to D^n$ is a continuous function with no fixed points. That is, no $x \in D^n$ such that $f(x) = x$.

(c) Let L_x be the ray starting at $f(x)$ and going through x. Define a map $g : D^n \to S^{n-1}$ where $\{g(x)\} = L_x \cap \partial D^n$. Prove this function is well-defined and explain informally why it is continuous.
(d) Prove that the restriction of g to the boundary $g|_{\partial D^n} : S^{n-1} \to S^{n-1}$ is odd.
(e) From this, contradict part (b) and conclude that no such $f : D^n \to D^n$ exists.

Problem 11 (7 marks)**.** Consider the topological group $G = \mathrm{GL}_n(\mathbb{R})$ and the space $M = M_n(\mathbb{R})$ of $n \times n$ of real matrices with the conjugation action

$$G \times M \to M, (g, A) \mapsto gAg^{-1}.$$

Let $X = M/G$ be the orbit space with the quotient topology, and let q denote the quotient map $q : M \to X, A \mapsto [A]$ where $[A]$ is the equivalence class of A.

First consider the case $n = 2$.

(a) Describe the set of points of X. Specifically, use the rational canonical form of a matrix to show that

$$X = \left\{ \left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right] \;\middle|\; \lambda_1 \leq \lambda_2 \right\} \cup \left\{ \left[\begin{pmatrix} 0 & -\gamma \\ 1 & \beta \end{pmatrix} \right] \;\middle|\; \beta^2 - 4\gamma \leq 0 \right\}.$$

(b) Using part (a) and considering the trace and the determinant, find the closure of $\{[A]\}$ for each point $[A] \in X$

Now for general n, prove that

(c) X is not Hausdorff;

(d) X is connected.