THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 9 Solution

Problem 1. Recall the explicit formula

$$u(x,t) = \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,\mathrm{d}\,y + \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s,$$

where $S(t,x-y)$ is the heat kernel given by

$$S(t,x-y) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}.$$

(i) Let $p = \frac{y-x}{\sqrt{4t}}$, then the first term becomes

$$\begin{aligned}\int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,\mathrm{d}\,y &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y\,\mathrm{d}\,y \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}(\sqrt{4t}p + x)\,\mathrm{d}\,p = x.\end{aligned}$$

Let $p = \frac{y-x}{\sqrt{4(t-s)}}$, then the second term becomes

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}}(2s)\,\mathrm{d}\,y\,\mathrm{d}\,s \\ &= \int_0^t \frac{2s}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\,\mathrm{d}\,p\,\mathrm{d}\,s = t^2.\end{aligned}$$

So we have

$$u(x,t) = x + t^2.$$

(ii) Let us deal with the inhomogeneous term first

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}}(-4ys)\,\mathrm{d}\,y\,\mathrm{d}\,s \\ &= \int_0^t \frac{-4s}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}(p\sqrt{4(t-s)} + x)\,\mathrm{d}\,p\,\mathrm{d}\,s \\ &= \int_0^t -4sx\,\mathrm{d}\,s = -2xt^2.\end{aligned}$$

Next, we deal with the source term, i.e.,

$$\int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,\mathrm{d}\,y = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^3\,\mathrm{d}\,y =: v(x,t).$$

Consider

$$w = \partial_x^4 v,$$

then v and w satisfy the following initial value problems respectively

$$\begin{cases} \partial_t v - \partial_{xx} v = 0 \\ \qquad v|_{t=0} = x^3 \end{cases} \text{and} \begin{cases} \partial_t w - \partial_{xx} w = 0 \\ \qquad w|_{t=0} = 0, \end{cases} \tag{1}$$

By the uniqueness of the solution, $w \equiv 0$. And hence we can integrate with respect to x four times,

$$v(x,t) = A_3(t)x^3 + A_2(t)x^2 + A_1(t)x + A_0(t),$$

Plugging into (1), we obtain

$$A_3'(t) = A_2'(t) = A_1'(t) - 6A_3(t) = A_0'(t) - 2A_2(t) = 0.$$

Also $v(x,0) = x^3$,

$$A_3 = 1,\ A_2 = 0,\ A_1 = 6t,\ A_0 = 0.$$

So

$$v(x,t) = \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,\mathrm{d}\,y = x^3 + 6tx.$$

We conclude that

$$u(x,t) = x^3 + 6tx - 2xt^2.$$

(iii) Let us deal with the inhomogeneous term first

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-s)}}(3ye^s)\,\mathrm{d}\,y\,\mathrm{d}\,s \\ &= \int_0^t \frac{3e^s}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}(\sqrt{4(t-s)}p + x)\,\mathrm{d}\,p\,\mathrm{d}\,s \\ &= 3x\int_0^t e^s\,\mathrm{d}\,s \\ &= 3x(e^t - 1).\end{aligned}$$

Next, we deal with the source term, i.e.,

$$\int_{-\infty}^{\infty} S(t, x-y)\phi(y)\,\mathrm{d}\,y = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \cos y\,\mathrm{d}\,y := v(x,t)$$

Consider

$$\tilde{v} = -\partial_x^2 v,$$

then $v(x,t)$ and $\tilde{v}(x,t)$ satisfy the following initial value problems

$$\begin{cases} \partial_t w - \partial_{xx} w = 0 \\ \qquad w|_{t=0} = \cos x \end{cases} \tag{2}$$

By the uniqueness of the solution, $\tilde{v} \equiv v$. And hence by solving $\partial_x^2 v + v = 0$, we have

$$v(x,t) = A(t)\cos x + B(t)\sin x.$$

Plugging into (2), we obtain $A' + A = B' + B = 0$. As $v|_{t=0} = \cos x$

$$A = e^{-t}, B = 0.$$

So

$$v(x,t) = \int_{-\infty}^{\infty} S(t, x-y)\phi(y)\,dy = e^{-t}\cos x.$$

We conclude that

$$u(x,t) = e^{-t}\cos x + 3x(e^t - 1).$$

Problem 2. Recall the explicit formula

$$u(x,t) = \int_{-\infty}^{\infty} S(t, x-y)\phi(y)\,\mathrm{d}\,y + \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y) f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s,$$

where $S(t, x-y)$ is the heat kernel given by

$$S(t, x-y) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}}.$$

(i) Let $p = \frac{y-x}{\sqrt{4kt}}$, we get

$$\begin{aligned}\int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,\mathrm{d}\,y &= \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} y\,\mathrm{d}\,y \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}(\sqrt{4t}p + x)\,\mathrm{d}\,p = x.\end{aligned}$$

Let $p = \frac{y-x}{\sqrt{4k(t-s)}}$, we get

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty} S(t-s,x-y)f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s &= \int_0^t \frac{5}{\sqrt{4k\pi(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}}\,\mathrm{d}\,y\,\mathrm{d}\,s \\ &= \int_0^t \frac{5}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\,\mathrm{d}\,p\,\mathrm{d}\,s = 5t.\end{aligned}$$

So we conclude that

$$u(x,t) = x + 5t.$$

(ii) Let us deal with the inhomogeneous term first,

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty} S(t-s,x-y)f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s &= \int_0^t \frac{1}{\sqrt{4k\pi(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}}\sin s\,\mathrm{d}\,y\,\mathrm{d}\,s \\ &= \int_0^t \frac{\sin s}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\,\mathrm{d}\,p\,\mathrm{d}\,s = 1-\cos t.\end{aligned}$$

Next, we deal with the source term, i.e.,

$$\int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,\mathrm{d}\,y = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} y^4\,\mathrm{d}\,y := v(x,t).$$

Consider

$$w = \partial_x^5 v,$$

then $v(x,t)$ and $w(x,t)$ satisfy the following initial value problems respectively,

$$\begin{cases}\partial_t v - k\partial_{xx} v = 0 \\ v|_{t=0} = x^4\end{cases} \text{ and } \begin{cases}\partial_t w - k\partial_{xx} w = 0 \\ w|_{t=0} = 0,\end{cases} \tag{3}$$

By the uniqueness of the solution, $w \equiv 0$. And hence integrating with respect to x five times, we have

$$v(x,t) = A_4(t)x^4 + A_3(t)x^3 + A_2(t)x^2 + A_1(t)x + A_0(t).$$

Plugging into (3), we obtain

$$A_4'(t) = A_3'(t) = A_2'(t) - 12kA_4(t) = A_1'(t) - 6kA_3(t) = A_0'(t) - 2kA_2(t) = 0.$$

As $v|_{t=0} = x^4$,

$$A_4 = 1,\ A_3 = 0,\ A_2 = 12kt,\ A_1 = 0,\ A_0 = 12k^2t^2.$$

We have

$$v(x,t) = \int_{-\infty}^{\infty} S(t, x-y)\phi(y)\,\mathrm{d}\,y = x^4 + 12kx^2t + 12k^2t^2.$$

We can then conclude that

$$u(x,t) = x^4 + 12kx^2t + 12k^2t^2 + 1 - \cos t.$$

(iii) The goal is to express our integral in term of the Gauss error function. Let $p = \frac{y-x}{\sqrt{4kt}}$, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} S(t, x-y)\phi(y)\,\mathrm{d}\,y &= \frac{4}{\sqrt{4k\pi t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4kt}}\,\mathrm{d}\,y \\
&= \frac{4}{\sqrt{\pi}} \int_{\frac{-1-x}{\sqrt{4kt}}}^{\frac{1-x}{\sqrt{4kt}}} e^{-p^2}\,\mathrm{d}\,p \\
&= \frac{4}{\sqrt{\pi}} \int_0^{\frac{1-x}{\sqrt{4kt}}} e^{-p^2}\,\mathrm{d}\,p - \frac{4}{\sqrt{\pi}} \int_0^{\frac{-1-x}{\sqrt{4kt}}} e^{-p^2}\,\mathrm{d}\,p \\
&= 2\,\mathrm{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) - 2\,\mathrm{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right).
\end{aligned}$$

Let $p = \frac{y-x}{\sqrt{4k(t-s)}}$, we get

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} S(t-s,x-y) f(y,s)\,\mathrm{d}\,y\,\mathrm{d}\,s &= \int_0^t \frac{1}{\sqrt{4k\pi(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}} (y\ln(1+s))\,\mathrm{d}\,y\,\mathrm{d}\,s\\ &= \int_0^t \frac{\ln(1+s)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}(\sqrt{4k(t-s)}p + x)\,\mathrm{d}\,p\,\mathrm{d}\,s\\ &= x\int_0^t \ln(1+s)\,\mathrm{d}\,s\\ &= x(1+t)\ln(1+t) - xt.\end{aligned}$$

So we conclude that

$$u(x,t) = 2\,\mathrm{erf}\left(\frac{1-x}{\sqrt{4kt}}\right) - 2\,\mathrm{erf}\left(\frac{-1-x}{\sqrt{4kt}}\right) + x(1+t)\ln(1+t) - xt.$$

Problem 3.

$$u(t,x) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{4}$$

(i) Using (4) for $c = 1$, $\phi(x) = \cos x$, $\psi(x) = x^2$, and $f(x,t) = \sin x$, we have

$$\begin{aligned} u(t,x) &= \frac{1}{2}[\cos(x+t) + \cos(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} s^2\,ds + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \sin y\,dy\,ds\\ &= \cos x\cos t + \frac{1}{6}[(x+t)^3 - (x-t)^3] - \frac{1}{2}\int_0^t \cos(x+t-s) - \cos(x-t+s)\,ds\\ &= \cos x\cos t + x^2 t + \frac{t^3}{3} + \int_0^t \sin x \sin(t-s) ds\\ &= \cos x\cos t + x^2 t - \frac{t^3}{3} + \sin x\cos(t-s)\Big|_{s=0}^{t}\\ &= \cos x\cos t + x^2 t + \frac{t^3}{3} + \sin x - \sin x\cos t.\end{aligned}$$

(ii) True. It follows from (4) for $c = 2$, $\phi(x) = \psi(x) = x^2$, and $f(x,t) = t(t+1)$ that

$$u(x,t) \geq 0$$

because $\phi(x), \psi(x), f(x,t) \geq 0$ for $-\infty < x < \infty$ and $t > 0$.