Algebra II Assignment 4

Due Friday 8th April 2022

Please attempt all six problems in this assignment and submit your answers (before midnight on Friday 8th April 2022) by uploading your work to the Moodle page. If you have any questions, feel free to email me at adsg@hku.hk.

Problem 1. Consider $f(x) = x^{11} - (\sqrt{2} + \sqrt{5})x^5 + 3\sqrt[4]{7}x^3 + (1+2i)x - \sqrt[5]{17}$.

1. Show that f has no real roots.
2. Show that every root of f is algebraic over $\mathbb{Q}$.
3. Let α be any root of f. Show that the minimal polynomial of α over $\mathbb{Q}$ has degree at most 1760.

Problem 2 (Reciprocity theorem)**.** Suppose that α and β are algebraic over a field K, with minimal polynomials $f \in K[x]$ and $g \in K[x]$ respectively. Show that f is irreducible over $K(\beta)$ if and only if g is irreducible over $K(\alpha)$.

Problem 3 (Ruler and compass constructions)**.** Determine whether the following points are constructible using a ruler and compass:

1. $P = (\sqrt[3]{7}, 0) \in \mathbb{R}^2$,
2. $P = (\sqrt{2}, \sqrt[3]{3}) \in \mathbb{R}^2$.

Problem 4 (Splitting fields)**.** Find the degree over $\mathbb{Q}$ of a splitting field over $\mathbb{Q}$ of the following polynomials in $\mathbb{Q}[x]$:

$$a)\ f(x) = x^3 - 2, \quad b)\ f(x) = x^4 - 1, \quad c)\ f(x) = (x^2 - 2)(x^3 - 2).$$

Problem 5 (Normal extensions)**.** Let $K \subset L$ be a field extension, and let M be an intermediate field of $K \subset L$. Show that if the extension $K \subset L$ is normal, then the extension $M \subset L$ is also normal.

Problem 6. Show that $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field of any polynomial in $\mathbb{Q}[x]$.