THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 6

Date: Oct 21, 2024.

Instruction: *In order to have a better preparation of the coming test, you are recommended to try the problems below **BEFORE** the coming tutorial. Due to the time limit, the tutorial will NOT be able to go through all of the problems. Therefore, you should actively tell our TA which problems you wish to discuss first.*

Information: This collection of problems is intended to give you practice problems that are comparable in format and difficulty to those which will appear in the coming test. The questions in the actual exam will be **DIFFERENT**.

Problem 1. Let

$$A := \begin{pmatrix} 0 & -1 & 0 \\ 4 & 4 & 0 \\ 2 & -1 & 2 \end{pmatrix},$$

and complete the following parts.

(i) Find all eigenvalues and eigenvectors of A.

(ii) Find an invertible 3×3 real matrix P such that

$$A = P \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} P^{-1}.$$

(iii) Solve the following system of linear ordinary differential equations (ODE):

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x},$$

where the vector

$$\mathbf{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

is the unknown function of t.

Solution. (i)

$$\begin{aligned}\det(A - \lambda I) &= \det\begin{pmatrix} -\lambda & -1 & 0 \\ 4 & 4-\lambda & 0 \\ 2 & -1 & 2-\lambda \end{pmatrix} \\ &= -\lambda(4-\lambda)(2-\lambda) - (-1)(4)(2-\lambda) \\ &= (\lambda - 2)[\lambda(4-\lambda) - 4] = -(\lambda-2)^3.\end{aligned}$$

So the eigenvalue is 2.

To get $E_2 = \mathrm{Nul}(A - 2I)$, consider $(A - 2I)\mathbf{x} = \mathbf{0}$ with the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ t \text{ is free.}$$

So we get

$$E_2 = \mathrm{Nul}(A - 2I) = \mathrm{Span}\{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T\}.$$

Hence an eigenvector in the basis for E_2 are given by $\hat{\mathbf{p}}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

(ii) Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}$. Note that $A = P\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}P^{-1}$ is equivalent to

$$\begin{cases} A\mathbf{p}_1 & = 2\mathbf{p}_1 \\ A\mathbf{p}_2 & = \mathbf{p}_1 + 2\mathbf{p}_2 \\ A\mathbf{p}_3 & = \mathbf{p}_2 + 2\mathbf{p}_3 \end{cases}, \text{ i.e., } \begin{cases} (A-2I)\mathbf{p}_1 & = 0 \\ (A-2I)\mathbf{p}_2 & = \mathbf{p}_1 \\ (A-2I)\mathbf{p}_3 & = \mathbf{p}_2 \end{cases}.$$

By (i), we can choose $\mathbf{p}_1 = \hat{\mathbf{p}}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. To find $\mathbf{p}_2$, consider the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$\mathbf{p}_2 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}, \text{ where } t \text{ is free.}$$

Now we can take $t = 0$ to get a particular solution $\mathbf{p}_2 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$.

To find $\mathbf{p}_3$, consider the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & -1 & 0 & \frac{1}{4} \\ 4 & 2 & 0 & -\frac{1}{2} \\ 2 & -1 & 0 & 0 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{16} \\ 0 & 1 & 0 & -\frac{1}{8} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$\mathbf{p}_3 = \begin{bmatrix} -\frac{1}{16} \\ -\frac{1}{8} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}, \text{ where } t \text{ is free.}$$

Now we can take $t = 0$ to get a particular solution $\mathbf{p}_3 = \begin{bmatrix} -\frac{1}{16} \\ -\frac{1}{8} \\ 0 \end{bmatrix}$. Thus,

$$P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \frac{1}{16}\begin{pmatrix} 0 & 4 & -1 \\ 0 & -8 & -2 \\ 16 & 0 & 0 \end{pmatrix}.$$

(iii) Note that $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$, where $\mathbf{x}_0 = \mathbf{x}(0)$. It follows from (ii) that

$$e^{tA} = \exp\left(tP\begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}P^{-1}\right) = P\exp\left(t\begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}\right)P^{-1}.$$

As

$$\begin{aligned}\exp\left(t\begin{pmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}\right) &= \exp\left(2tI + t\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\right) = e^{2t}\exp\left(t\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\right)\\ &= e^{2t}\left(I + t\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} + \frac{t^2}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + 0\right)\\ &= e^{2t}\begin{pmatrix}1 & t & \frac{t^2}{2}\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix},\end{aligned}$$

the solution is

$$\begin{aligned}e^{tA}\mathbf{x}_0 &= e^{2t}P\begin{pmatrix}1 & t & \frac{t^2}{2}\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}P^{-1}\mathbf{x}_0 = e^{2t}\begin{pmatrix}0 & 4 & -1\\ 0 & -8 & -2\\ 16 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & t & \frac{t^2}{2}\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}\\ &= e^{2t}\begin{pmatrix}0 & 4 & 4t-1\\ 0 & -8 & -8t-2\\ 16 & 16t & 8t^2\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}\\ &= e^{2t}\left[16y_1\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + 4y_2\begin{pmatrix}1\\ -2\\ 4t\end{pmatrix} + y_3\begin{pmatrix}4t-1\\ -8t-2\\ 8t^2\end{pmatrix}\right],\end{aligned}$$

where $\begin{pmatrix}y_1 & y_2 & y_3\end{pmatrix}^T = \frac{1}{16}P^{-1}\mathbf{x}_0$ is arbitrary because $\mathbf{x}_0$ can be any vector in $\mathbb{R}^3$.

□

Problem 2 (Invariant Transformations/Symmetries)**.** Let $u \in C^7$ be a solution to the wave equation

$$\partial_{tt}u - 4\partial_{xx}u = 0, \tag{1}$$

Show that the v defined in each of the cases below is also a solution to (1).

i (Translation)

$$v(t,x) := u(t-1, x-2).$$

ii (Differentiation) $v := \partial_t^2 \partial_x^3 u$.

iii (Convolution) For $g(y) = e^{-y^2}$,

$$v(t,x) := (u * g)(t,x) := \int_{-\infty}^{\infty} u(t, x-y) g(y)\, dy.$$

iv (Dilation/Scaling)

$$v(t,x) := u(4t, 4x).$$

Solution. Note that

$$[\partial_{tt} u - 4\partial_{xx} u](t,x) = 0 \text{ for all } x, t \in \mathbb{R}.$$

i (Translation) For any x, $t \in \mathbb{R}$,

$$\begin{aligned} [\partial_{tt} v - 4\partial_{xx} v](t,x) &= \partial_{tt} v(t,x) - 4\partial_{xx} v(t,x) \\ &= \partial_{tt} u(t-1, x-2) - 4\partial_{xx} u(t-1, x-2) \\ &= [\partial_{tt} u - 4\partial_{xx} u](t-1, x-2) = 0. \end{aligned}$$

ii (Differentiation) For any x, $t \in \mathbb{R}$,

$$\begin{aligned} [\partial_{tt} v - 4\partial_{xx} v](t,x) &= \partial_{tt} v(t,x) - 4\partial_{xx} v(t,x) \\ &= \partial_{tt} \partial_t^2 \partial_x^3 u(t,x) - 4\partial_{xx} \partial_t^2 \partial_x^3 u(t,x) \\ &= \partial_t^2 \partial_x^3 \{[\partial_{tt} u - 4\partial_{xx} u](t,x)\} = 0. \end{aligned}$$

iii (Convolution) For any x, $t \in \mathbb{R}$,

$$\begin{aligned} [\partial_{tt} v - 4\partial_{xx} v](t,x) &= [\partial_{tt} - 4\partial_{xx}] v(t,x) \\ &= [\partial_{tt} - 4\partial_{xx}] \int_{-\infty}^{\infty} u(t, x-y) e^{-y^2}\, dy \\ &= \int_{-\infty}^{\infty} [\partial_{tt} - 4\partial_{xx}][u(t, x-y) e^{-y^2}]\, dy \\ &= \int_{-\infty}^{\infty} \{[\partial_{tt} u - 4\partial_{xx} u](t, x-y)\}\, e^{-y^2}\, dy \\ &= 0. \end{aligned}$$

iv (Dilation/Scaling) For any x, $t \in \mathbb{R}$,

$$\begin{aligned}
[\partial_{tt} v - 4\partial_{xx} v](t,x) &= \partial_{tt} v(t,x) - 4\partial_{xx} v(t,x) \\
&= \partial_{tt}[u(4t,4x)] - 4\partial_{xx}[u(4t,4x)] \\
&= 16(\partial_{tt} u)(4t,4x) - 64(\partial_{xx} u)(4t,4x) \\
&= 16\{[\partial_{tt} u - 4\partial_{xx} u](4t,4x)\} = 0.
\end{aligned}$$

□

Problem 3. Solve the following second-order linear PDE

$$3\partial_x u - \partial_{xy} u = 0. \tag{2}$$

in two different approaches as follows.

(i) Let $v := \partial_x u$. Verify that v satisfies

$$3v - \partial_y v = 0. \tag{3}$$

Solve for v in (3), then obtain u by integrating v with respect to x.

(ii) Rewrite the PDE (2) as

$$\partial_x (3u - \partial_y u) = 0. \tag{4}$$

Integrating (4) with respect to x, you can verify that u satisfies

$$3u - \partial_y u = g(y), \tag{5}$$

where g is any arbitrary function. Solve for u in (5) by using the method of integrating factors.

(iii) Are the solutions that you obtained in Part (i) and (ii) the SAME? Explain your answer briefly.

Solution.

(i) Let $v := \partial_x u$. Then

$$3v - \partial_y v = 3\partial_x u - \partial_{xy} u = 0.$$

The integrating factor is $\mu(y) = e^{\int -3dy} = e^{-3y}$ and hence

$$\partial_y[e^{-3y}v] = e^{-3y}(-3v + \partial_y v) = 0.$$

After integration with respect to y, we get

$$e^{-3y}v = F(x) \Longrightarrow v(x,y) = e^{3y}F(x),$$

where F is any arbitrary function. Now integrating v respect to x, we obtain

$$u(x,y) = e^{3y}\int F(x)dx + G(y),$$

where F, G are any arbitrary functions.

(ii) Note that $3u - \partial_y u = g(y)$, where g is any arbitrary function. The integrating factor is $\mu(y) = e^{\int -3dy} = e^{-3y}$ and hence

$$\partial_y[e^{-3y}u] = e^{-3y}(-3u + \partial_y u) = -e^{-3y}g(y).$$

After integration with respect to y, we get

$$e^{-3y}u = f(x) - \int e^{-3y}g(y)dy \Longrightarrow u(x,y) = e^{3y}f(x) - e^{3y}\int e^{-3y}g(y)dy,$$

where f is any arbitrary function.

(iii) As the functions $\int F(x)dx$ in (i) and $-e^{3y}\int e^{-3y}g(y)dy$ in (ii) are also arbitrary, we can obtain the same solution

$$u(x,y) = e^{3y}f(x) + G(y),$$

where f, G are arbitrary.

□

Problem 4 (Instability of Backward Heat Equation)**.** Consider the following PDE

$$\begin{cases} \partial_t u + 2\partial_{xx} u = 0 & \text{for } t > 0 \text{ and } -\pi/2 < x < \pi/2 \\ u|_{x=-\pi/2} = u|_{x=\pi/2} = 0 \\ u|_{t=0} = f, \end{cases} \tag{6}$$

where $f : [-\pi/2, \pi/2] \to \mathbb{R}$ is the initial data. Show that it is unstable with respect to the sup norm

$$\|g\|_{\sup} := \sup_{-\pi/2 \le x \le \pi/2} |g(x)|$$

by completing the following steps:

(i) Verify that for any positive odd integer n, the function

$$u_n(t,x) := \frac{1}{n} e^{2n^2 t} \cos nx$$

is a solution to the initial and boundary value problem (6) with the initial data

$$f_n(x) := \frac{1}{n} \cos nx.$$

(ii) Prove that

$$\lim_{n \to +\infty} \|f_n\|_{\sup} = 0.$$

(iii) Show that for any $T > 0$,

$$\lim_{n \to +\infty} \|u_n(T, \cdot)\|_{\sup} = +\infty.$$

Solution.

(i) Let $u_n(t,x) := \frac{1}{n} e^{2n^2 t} \cos nx$. Then

$$\partial_t u_n + 2\partial_{xx} u_n = 2n e^{2n^2 t} \cos nx - 2n e^{2n^2 t} \cos nx = 0.$$

The initial and boundary conditions

$$\begin{cases} u_n\big|_{x=-\pi/2} = u_n\big|_{x=\pi/2} = 0 \\ \quad u_n\big|_{t=0} = f_n(x) = \dfrac{1}{n}\cos(nx) \end{cases}$$

follow from $\cos(-n\pi/2) = \cos(n\pi/2) = 0$ and $e^{2n^2(0)} = 1$ for any odd integer n.

(ii) Since $\sup\limits_{-\pi/2\le x\le\pi/2} |\cos(nx)| = \max\limits_{-\pi/2\le x\le\pi/2} |\cos(nx)| = 1$,

$$\|f_n\|_{\sup} = 1/n \to 0 \text{ as } n \to \infty.$$

(iii) For fixed $T > 0$, since $\sup\limits_{-\pi/2\le x\le\pi/2} |\cos(nx)| = \max\limits_{-\pi/2\le x\le\pi/2} |\cos(nx)| = 1$, it follows from L'Hôpital's rule that

$$\lim_{n\to+\infty} \|u_n(T,\cdot)\|_{\sup} = \lim_{n\to+\infty} \frac{e^{2n^2T}}{n} = 4T \lim_{n\to+\infty} ne^{2n^2T} = +\infty.$$

$\square$

Problem 5. Let $u := u(x,y)$ be a smooth function from $\mathbb{R}^2$ to $\mathbb{R}$. Verify that the Laplace's equation $\Delta u := \partial_{xx}u + \partial_{yy}u = 0$ is invariant under rotations, that is,

$$\partial_{x'x'}u + \partial_{y'y'}u = \partial_{xx}u + \partial_{yy}u = 0,$$

where $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (θ is the angle of rotation).

Solution. Note that $x' = x\cos\theta - y\sin\theta$ and $y' = x\sin\theta + y\cos\theta$. By the chain rule,

$$\partial_x u = \partial_{x'}u \cdot \partial_x x' + \partial_{y'}u \cdot \partial_x y' = \cos\theta\,\partial_{x'}u + \sin\theta\,\partial_{y'}u$$

and

$$\partial_y u = \partial_{x'}u \cdot \partial_y x' + \partial_{y'}u \cdot \partial_y y' = -\sin\theta\,\partial_{x'}u + \cos\theta\,\partial_{y'}u.$$

By the chain rule again,

$$\partial_{xx}u = \cos\theta\,\partial_{x'}\partial_x u + \sin\theta\,\partial_{y'}\partial_x u = \cos^2\theta\,\partial_{x'x'}u + 2\cos\theta\sin\theta\,\partial_{x'y'}u + \sin^2\theta\,\partial_{y'y'}u$$

and

$$\partial_{yy}u = -\sin\theta\,\partial_{x'}\partial_y u + \cos\theta\,\partial_{y'}\partial_y u = \sin^2\theta\,\partial_{x'x'}u - 2\cos\theta\sin\theta\,\partial_{x'y'}u + \cos^2\theta\,\partial_{y'y'}u.$$

Thus,

$$\partial_{xx}u + \partial_{yy}u = \partial_{x'x'}u + \partial_{y'y'}u = 0.$$

□

Problem 6. Consider

$$\begin{cases} 3t\partial_t u - \partial_x u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi(x). \end{cases} \tag{7}$$

(i) Prove that if $\phi(x) := 2 + \cos x$, then (7) has no solution.

(ii) Assume that $\phi(x) \equiv 1$. Find all solutions to (7).

Solution.

$$\begin{cases} \dfrac{dt}{ds} = 3t,\ t(0) = t_0 \\ \dfrac{dx}{ds} = -1,\ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} \ln\dfrac{t}{t_0} = 3s \\ x = -s + x_0 \end{cases} \Longrightarrow \begin{cases} t = t_0 e^{3s} \\ x = -s + x_0 \end{cases}$$

Then $t = t_0 e^{3(x_0 - x)}$. Choose $x_0 = 0$, the characteristic curves can be parametrized by $t_0 > 0$:

$$C_{t_0} = \{(t,\, x) : t = t_0 e^{-3x}\}.$$

Now suppose that the problem has a solution $u(t,x)$. Note that $u(t,x)$ is continuous on the closed half plane $\{(t,x) : t \geq 0 \text{ and } x \in \mathbb{R}\}$. Since it remains unchanged along each characteristic curves, for all $x \in \mathbb{R}$ and $t_0 > 0$,

$$u(t_0 e^{-3x}, x) = u(t_0, 0) < \infty.$$

(i) However,

$$\lim_{x\to+\infty} u(t_0e^{-3x}, x) = \lim_{x\to+\infty} u(0,x) = \lim_{x\to+\infty} \cos x + 2,$$

which does not exist and hence get a contradiction.

(ii) Moreover,

$$\lim_{x\to+\infty} u(t_0e^{-3x}, x) = \lim_{x\to+\infty} u(0,x) = 1.$$

Thus for all $x \in \mathbb{R}$ and all $t_0 > 0$, $u(t_0e^{-3x}, x) = 1$ and hence $u \equiv 1$.

$\square$

Problem 7. Solve the following boundary value problem

$$\begin{cases} y\partial_x u - x\partial_y u = x^2 + y^2 & \text{for } x, y > 0 \\ u|_{y=0} = x \end{cases}$$

by using the coordinate method.

(i) Rewrite the PDE in terms of the polar coordinates $r := \sqrt{x^2+y^2}$ and $\theta := \tan^{-1}\left(\frac{y}{x}\right)$.

(ii) Express the boundary condition $u|_{y=0} = x$ in terms of r and θ.

(iii) Solve u in terms of r and θ.

(iv) Express your final answer u in terms of x and y.

Solution.

By the chain rule, $\partial_x u = \partial_r u \cdot \partial_x r + \partial_\theta u \cdot \partial_x \theta$ and $\partial_y u = \partial_r u \cdot \partial_y r + \partial_\theta u \cdot \partial_y \theta$. Now we compute $\partial_x r$, $\partial_y r$, $\partial_x \theta$ and $\partial_y \theta$ as follows:

$$\partial_x r = \partial_x (x^2+y^2)^{1/2} = \frac{x}{(x^2+y^2)^{1/2}} = \frac{r\cos\theta}{r} = \cos\theta;$$

$$\partial_y r = \partial_y (x^2+y^2)^{1/2} = \frac{y}{(x^2+y^2)^{1/2}} = \frac{r\sin\theta}{r} = \sin\theta;$$

$$\partial_x \tan\theta = \partial_x(\frac{y}{x}) \Longrightarrow \sec^2\theta\,\partial_x\theta = -\frac{y}{x^2} \Longrightarrow \partial_x\theta = -\frac{y\cos^2\theta}{x^2} = -\frac{r\sin\theta\cos^2\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r};$$

$$\partial_y \tan\theta = \partial_y(\frac{y}{x}) \Longrightarrow \sec^2\theta\ \partial_y\theta = \frac{1}{x} \Longrightarrow \partial_y\theta = \frac{\cos^2\theta}{x} = \frac{\cos^2\theta}{r\cos\theta} = \frac{\cos\theta}{r}.$$

Thus, we have

$$\partial_x u = \cos\theta\,\partial_r u - \frac{\sin\theta}{r}\partial_\theta u \text{ and } \partial_y u = \sin\theta\,\partial_r u + \frac{\cos\theta}{r}\partial_\theta u.$$

Hence

$$\begin{aligned} y\partial_x u - x\partial_y u &= r\sin\theta(\cos\theta\,\partial_r u - \frac{\sin\theta}{r}\partial_\theta u) - r\cos\theta(\sin\theta\,\partial_r u + \frac{\cos\theta}{r}\partial_\theta u) \\ &= -\partial_\theta u \end{aligned}$$

So the PDE can be rewritten as $\boxed{\partial_\theta u = -r^2}$.

(ii) The boundary condition becomes $u|_{\theta=0} = r\cos\theta|_{\theta=0} = r$.

(iii) By (i), it is easy to see that $u = -r^2\theta + f(r)$. It follows from (ii) that $f(r) = r$. So $\boxed{u = -r^2\theta + r}$.

(iv) By (iii), $\boxed{u = -(x^2+y^2)\tan^{-1}(y/x) + (x^2+y^2)^{1/2}}$.

□

Problem 8. Use the method of characteristics to find a solution for

$$\begin{cases} u_t(t,x) + u(t,x)u_x(t,x) = 0, & t>0,\ x\in\mathbb{R}, \\ \lim\limits_{t\to 0^+} u(t,x) = x, & x\in\mathbb{R}. \end{cases}$$

Solution.

$$\begin{cases} \dfrac{dt}{ds} = 1, \ \lim\limits_{s\to 0^+} t(s) = 0 \\ \dfrac{dx}{ds} = u(t(s), x(s)), \ \lim\limits_{s\to 0^+} x(s) = x_0 \\ \dfrac{dW}{ds} = 0, \ W(s) := u(t(s), x(s)) \end{cases}$$

$$\implies \begin{cases} t = s \\ \dfrac{dx}{ds} = W(s), \ \lim\limits_{s\to 0^+} x(s) = x_0 \\ W(s) = \lim\limits_{s\to 0^+} u(t(s), x(s)) = \lim\limits_{s\to 0^+} u(s, x(s)) = \lim\limits_{s\to 0^+} x(s) = x_0 \end{cases}$$

$$\implies \begin{cases} x = x_0 t + x_0 = x_0(t+1) \\ u(t,x) = x_0 \end{cases}$$

So the solution is

$$u(t,x) = \frac{x}{t+1} \text{ for } t > 0, x \in \mathbb{R}$$

with $u(0,x) = x$ for $x \in \mathbb{R}$. □

Problem 9. Consider the second order equation

$$\partial_{tt} u + 2\partial_{tx} u - 3\partial_{xx} u = 0, \tag{8}$$

and complete the following parts.

(i) Verify that Equation (8) is hyperbolic by computing its discriminant $\mathcal{D}$.

(ii) Find the general solution to (8) by the method of characteristics.

(iii) Solve (8) with the following initial data:

$$\begin{cases} u|_{t=0} = 8x^2 \\ \partial_t u|_{t=0} = 24x. \end{cases}$$

Solution. (i) The equation is hyperbolic because

$$\mathcal{D} = (1)^2 - (1)(-3) = 4 > 0.$$

(ii) The equation (3) can be rewritten as

$$(\partial_t - \partial_x)(\partial_t + 3\partial_x)u = 0.$$

To find the general solution to (1), we need to solve

$$\begin{cases} \partial_t v - \partial_x v = 0 \\ \partial_t u + 3\partial_x u = v \end{cases}.$$

Note that the general solution of $\partial_t v - \partial_x v = 0$ is

$$v = f(x+t),$$

for arbitrary function f because v is a constant along the characteristic curve

$$C_{x_0} = \{(t,x) : x = -t + x_0\}.$$

To solve $\partial_t u + \partial_x u = v = f(x+t)$, find the characteristic curves,

$$\begin{cases} \dfrac{dt}{ds} = 1,\ t(0) = 0 \\ \dfrac{dx}{ds} = 3,\ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} t = s \\ x = 3s + x_0 \end{cases}$$

Then $x = 3t + x_0$, and the characteristic curves can be parametrized by x_0:

$$\tilde{C}_{x_0} = \{(t,\ x) : x = 3t + x_0\}.$$

Let $W(s) := u(t(s),\ x(s))$. Then

$$\frac{dW}{ds} = v(t(s),\ x(s)) = f(x(s) + t(s)) = f(3s + x_0 + s) = f(4s + x_0).$$

Integrate $\frac{dW}{ds}$ along the characteristic curve $\tilde{C}_{x_0}$ from $(0, x_0)$ to (t, x), we have

$$\begin{aligned} u(t,x) - u(0,x_0) &= \int_0^t f(4s + x_0)\ ds = \frac{1}{4}\int_{x_0}^{4t+x_0} f(\tilde{s})\ d\tilde{s}\ (\tilde{s} = 4s + x_0) \\ &= \frac{1}{4}\int_{x_0}^{x+t} f(\tilde{s})\ d\tilde{s} = F(x+t), \end{aligned}$$

where $F' = \dfrac{f}{4}$ with $F(x_0) = 0$. Hence

$u(t,x) = u(0,x_0) + F(x+t) = u(0,x-3t) + F(x+t) = g(x-3t) + F(x+t),$

for arbitrary functions g, F.

(iii) As $u(0,x) = 8x^2$, $F(x) = 8x^2 - g(x)$ and hence

$$u(t,x) = g(x-3t) + F(x+t) = g(x-3t) + 8(x+t)^2 - g(x+t).$$

It follows from $u_t(0,x) = 24x$ that

$$\Big[-3g'(x-3t) + 16(x+t) - g'(x+t)\Big]_{t=0} = 24x,$$

thus $g'(x) = -2x$ and $g(x) = -x^2 + C$ for some constant C. Therefore

$$\begin{aligned} u(t,x) &= g(x-3t) + 8(x+t)^2 - g(x+t) \\ &= -(x-3t)^2 + 9(x+t)^2 \\ &= 8x^2 + 24xt. \end{aligned}$$

□

Problem 10. In this problem we will solve the following second order equation

$$\partial_{tt}u + 6\partial_{tx}u + 9\partial_{xx}u = 0 \tag{9}$$

by the method of characteristics.

(i) Verify that the equation (9) can be written as

$$(\partial_t + 3\partial_x)^2 u = 0.$$

(ii) Let $v := (\partial_t + 3\partial_x)u = \partial_t u + 3\partial_x u$. Verify that u and v satisfy the following system:

$$\begin{cases} \partial_t v + 3\partial_x v = 0 \\ \partial_t u + 3\partial_x u = v. \end{cases}$$

(iii) Find the general solution to

$$\partial_t v + 3\partial_x v = 0.$$

(iv) Apply the method of characteristics to solve

$$\partial_t u + 3\partial_x u = v,$$

and prove that the general solution u to (9) has the form

$$u(t,x) = f(x-3t) + t\, g(x-3t)$$

where f and g are arbitrary functions.

Solution.

(i) $(\partial_t + 3\partial_x)^2 u = (\partial_t + 3\partial_x)(\partial_t + 3\partial_x)u = u_{tt} + 6u_{tx} + 9u_{xx} = 0.$

(ii) The first equation follows from (i) and the second equation holds by definition.

(iii) To find the characteristic curves,

$$\begin{cases} \dfrac{dt}{ds} = 1, \ t(0) = t_0 \\ \dfrac{dx}{ds} = 3, \ x(0) = x_0 \end{cases} \implies \begin{cases} t = s + t_0 \\ x = 3s + x_0 \end{cases}$$

Then $x = 3(t - t_0) + x_0 = 3t - 3t_0 + x_0$. Choose $t_0 = 0$, the characteristic curves can be parametrized by x_0:

$$C_{x_0} = \{(t,\, x) : x = 3t + x_0\}.$$

Note that $v(t,x)$ remains unchanged along each characteristic curves and hence for all $(t,\, x) \in C_{x_0}$,

$$v(t,\, x) = v(0,\, x_0) = v(0,\, x + 3t) = g(x + 3t), \text{ where } g \text{ is arbitrary.}$$

(iv) Let $W(s) = u(t(s), x(s))$. Then $\frac{dW(s)}{ds} = v$ implies $dW = v\,ds$. By integrating along C_{x_0} from $(0, x_0)$ to (t, x), we have

$$u(t,x) - u(0,x_0) = s\,v(0,\,x_0) = t\,g(x-3t)$$

and hence

$$u(t,x) = u(0,x-3t) + t\,g(x-3t) = f(x-3t) + t\,g(x-3t),$$

where f, g are arbitrary.

$\square$

Problem 11. In this problem answer the questions below regarding the following initial value problem for the wave equation:

$$\begin{cases} \partial_{tt}u - 4\partial_{xx}u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi \\ \partial_t u|_{t=0} = \psi, \end{cases} \tag{10}$$

the functions ϕ and ψ are given initial data, and u is the unknown.

(i) For any given constant $\alpha \in (-\infty, \infty)$, we define

$$v(t,x) := u(t, x+\alpha).$$

Verify that v satisfies exactly the same wave equation:

$$\partial_{tt}v - 4\partial_{xx}v = 0.$$

Furthermore, find the initial conditions that v satisfies.

(ii) If both ϕ and ψ are odd around $x = 2$, that is, for any $x \in (-\infty, \infty)$,

$$\begin{cases} \phi(2-x) = -\phi(2+x) \\ \psi(2-x) = -\psi(2+x), \end{cases}$$

then what can you say about the solution u to the initial value problem (10)?

Solution. (i) $\partial_{tt}v(t,x) - 4\partial_{xx}v(t,x) = \partial_{tt}u(t,x+\alpha) - 4\partial_{xx}u(t,x+\alpha) = 0$ and the initial conditions are

$$\begin{cases} v|_{t=0} = u(0,x+\alpha) = \phi(x+\alpha) \\ \partial_t v|_{t=0} = \partial_t u(0,x+\alpha) = \psi(x+\alpha). \end{cases} \tag{11}$$

(ii) By the d'Alembert formula,

$$u(t,x) = \frac{1}{2}[\phi(x+2t) + \phi(x-2t)] + \frac{1}{4}\int_{x-2t}^{x+2t} \psi(s)ds.$$

Then u is also odd around $x = 2$ because

$$\begin{aligned}
& u(t,\,2-x) = \frac{1}{2}[\phi(2-x+2t) + \phi(2-x-2t)] + \frac{1}{4}\int_{2-x-2t}^{2-x+2t} \psi(s)\,ds \\
= \;& -\frac{1}{2}[\phi(2+x-2t) + \phi(2+x+2t)] + \frac{1}{4}\int_{x+2t}^{x-2t} \psi(2+\tilde{s})\,d\tilde{s} \\
& (\because \phi, \psi \text{ are odd around } x = 2 \text{ and let } \tilde{s} = 2-s) \\
= \;& -\frac{1}{2}[\phi(2+x+2t) + \phi(2+x-2t)] + \frac{1}{4}\int_{2+x+2t}^{2+x-2t} \psi(\hat{s})\,d\hat{s} \quad (\because \text{ let } \hat{s} = 2+\tilde{s}) \\
= \;& -\left[\frac{1}{2}[\phi(2+x+2t) + \phi(2+x-2t)] + \frac{1}{4}\int_{2+x-2t}^{2+x+2t} \psi(s)\,ds\right] \\
= \;& -u(t,\,2+x).
\end{aligned}$$

$\square$