THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 7 Solution

Problem 1.

The heat kernel is given by

$$S(t,x-y)=\sqrt{\frac{3}{2\pi t}}e^{-\frac{3(x-y)^2}{2t}}$$

(i) The solution is given by

$$u(x,t)=\sqrt{\frac{3}{2\pi t}}\int_{-\infty}^{\infty}e^{-\frac{3(x-y)^2}{2t}}e^{-4y}\,\mathrm{d}\,y.$$

Completing the square in y, we have

$$-\frac{3(y-x)^2+8ty}{2t}=-\frac{3(y-\frac{3x-4t}{3})^2-\frac{16}{3}t^2+8tx}{2t}.$$

Then the solution becomes

$$u(x,t)=e^{\frac{8}{3}t-4x}.$$

(ii) The solution is given by

$$u(t,x)=\sqrt{\frac{3}{2\pi t}}\int_{-\infty}^{\infty}e^{-\frac{3(x-y)^2}{2t}}\sin\frac{8y}{3}\,\mathrm{d}\,y.$$

It satisfies

$$\partial_{xx}u+\frac{64}{9}u=0,$$

which has the general solution of the form

$$u(t,x)=A(t)\cos\frac{8}{3}x+B(t)\sin\frac{8}{3}x$$

for $A(t), B(t)$ to be determined. Plugging into the original PDE for $u(t,x)$, we have

$$\Big(A'(t) + \frac{32}{27}A(t)\Big)\cos\frac{8}{3}x + \Big(B'(t) + \frac{32}{27}B(t)\Big)\sin\frac{8}{3}x = 0.$$

The initial data says that $A(0) = 0$ and $B(0) = 1$. Hence, we have

$$u(t,x) = e^{-\frac{32}{27}t}\sin\frac{8}{3}x.$$

(iii) The solution is given by

$$u(t,x) = \sqrt{\frac{3}{2\pi t}}\int_5^\infty e^{-\frac{3(x-y)^2}{2t}}e^{7y}\,\mathrm{d}\,y.$$

Completing the square in y

$$-\frac{3(x-y)^2 - 14yt}{2t} = -\frac{3(y-x-\frac{7}{3}t)^2 - \frac{49}{3}t^2 - 14tx}{2t}.$$

Thus,

$$\begin{aligned} u(t,x) &= e^{\frac{49}{6}t+7x}\sqrt{\frac{3}{2\pi t}}\int_5^\infty e^{-\frac{3(y-x-\frac{7}{3}t)^2}{2t}}\,\mathrm{d}\,y \\ &= e^{\frac{49}{6}t+7x}\sqrt{\frac{3}{2\pi t}}\int_{\frac{\sqrt{3}(5-x-\frac{7}{3}t)}{\sqrt{2t}}}^\infty e^{-q^2}\sqrt{\frac{2t}{3}}\,\mathrm{d}\,q \\ &= \frac{e^{\frac{49}{6}t+7x}}{2}\Big(1 - \mathrm{erf}\big(\frac{\sqrt{3}(5-x-\frac{7}{3}t)}{\sqrt{2t}}\big)\Big). \end{aligned}$$

(iv) The solution is given by

$$\begin{aligned}
u(t,x) &= \int_{|y|<2} S(t,x-y)\, 9 \,\mathrm{d}\, y + \int_{|y|\geq 2} S(t,x-y)\,(-1)\,\mathrm{d}\, y \\
&= -\int_{-\infty}^{\infty} S(t,x-y)\,\mathrm{d}\, y + 10\int_{|y|<2} S(t,x-y)\,\mathrm{d}\, y \\
&= -1 + 10\sqrt{\frac{3}{2\pi t}}\int_{-2}^{2} e^{-\frac{3(x-y)^2}{2t}}\,\mathrm{d}\, y \\
&= -1 + 10\sqrt{\frac{3}{2\pi t}}\int_{-\frac{\sqrt{3}(2+x)}{\sqrt{2t}}}^{\frac{\sqrt{3}(2-x)}{\sqrt{2t}}} e^{-p^2}\sqrt{\frac{2t}{3}}\,\mathrm{d}\, p \\
&= -1 + 5\left(\operatorname{erf}\Big(\frac{\sqrt{3}(2-x)}{\sqrt{2t}}\Big) - \operatorname{erf}\Big(-\frac{\sqrt{3}(2+x)}{\sqrt{2t}}\Big)\right)
\end{aligned}$$

Problem 2.

(i) Change of variables

$$\begin{aligned}
\partial_t u &= \partial_y u\ \partial_t y + \partial_\tau u\ \partial_t \tau = -\alpha \partial_y u + \partial_\tau u, \\
\partial_x u &= \partial_y u\ \partial_x y + \partial_\tau u\ \partial_x \tau = \partial_y u.
\end{aligned}$$

Then we have

$$\begin{aligned}
0 &= \partial_t u - k\partial_{xx} u + \alpha \partial_x u + \beta u \\
&= \partial_\tau u - k\partial_{yy} u + \beta u.
\end{aligned}$$

The initial condition changes to

$$u(y,\tau)\Big|_{\tau=0} = u(y,0) = \phi(y).$$

Hence,

$$\begin{cases} \partial_\tau u - k\partial_{yy} u + \beta u = 0 \\ u\Big|_{\tau=0}(y) = \phi(y). \end{cases}$$

(ii) Change of variables

$$\begin{aligned}
\partial_\tau v &= \beta e^{\beta\tau} u + e^{\beta\tau}\partial_\tau u, \\
\partial_{yy} v &= e^{\beta\tau}\partial_{yy} u.
\end{aligned}$$

Then we have

$$\begin{aligned}\partial_\tau v(y,\tau) - k\partial_{yy} v(y,\tau) &= \beta e^{\beta\tau} u(y,\tau) + e^{\beta\tau}\partial_\tau u(y,\tau) - ke^{\beta\tau}\partial_{yy} u(y,\tau)\\ &= e^{\beta\tau}(\partial_\tau u(y,\tau) - k\partial_{yy} u(y,\tau) + \beta u(y,\tau))\\ &= 0.\end{aligned}$$

The initial condition follows from that

$$v(y,0) = e^{0\beta} u(y,0) = u(y,0) = \phi(y).$$

(iii) The solution to equation (6.1) follows from the explicit formula that

$$v(y,\tau) = \int_{-\infty}^{\infty} S(\tau, y-\xi)\phi(\xi)\,\mathrm{d}\xi.$$

Then

$$u(y,\tau) = e^{-\beta\tau}\int_{-\infty}^{\infty} S(\tau, y-\xi)\phi(\xi)\,\mathrm{d}\xi.$$

Substitute back $y = x - \alpha t$ and $\tau = t$, and thus we have

$$u(x,t) = e^{-\beta t}\int_{-\infty}^{\infty} S(t, x-\alpha t-\xi)\phi(\xi)\,\mathrm{d}\xi,$$

where $S(t,z)$ is the heat kernel given by

$$S(t,z) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{z^2}{4kt}}.$$

Problem 3.

(i) Plugging the ansatz (3) into equation (2), we have

$$\begin{aligned}0 = \partial_t u(t,x) - k\partial_{xx} u(t,x) &= \sum_{n=0}^{N} A_n'(t)x^n - k\sum_{n=2}^{N} n(n-1)A_n(t)x^{n-2}\\ &= \sum_{n=0}^{N} A_n'(t)x^n - k\sum_{n=0}^{N-2}(n+2)(n+1)A_{n+2}(t)x^n\\ &= A_N'(t)x^N + A_{N-1}'(t)x^{N-1} + \sum_{n=0}^{N-2}\left[A_n'(t) - k(n+2)(n+1)A_{n+2}(t)\right]x^n.\end{aligned}$$

By the linear independence of functions x^n, we can compare coefficients and get

$$\begin{cases} A_n'(t) = k(n+1)(n+2)A_{n+2}(t) & \text{for } n = 0, 1, \cdots, N-2 \\ A_{N-1}'(t) = 0 \\ A_N'(t) = 0 \end{cases}$$

The initial condition $t = 0$ gives that

$$u(x, 0) = \phi(x) = \sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} A_n(0) x^n.$$

Again by the linear independence of functions x^n, we can compare coefficients and get

$$A_n(0) = a_n \quad \text{for } n = 0, 1, \cdots, N$$

(ii) Solve the differential equations iteratively,

$$A_N'(t) = 0 \text{ with } A_N(0) = a_N \Rightarrow A_N(t) = a_N,$$
$$A_{N-1}'(t) = 0 \text{ with } A_{N-1}(0) = a_{N-1} \Rightarrow A_{N-1}(t) = a_{N-1}.$$

Next, we have

$$\begin{aligned} A_{N-2}'(t) = k(N-1)Na_N \text{ with } A_{N-2}(0) = a_{N-2} \\ \Rightarrow A_{N-2}(t) = k(N-1)Na_N t + a_{N-2}, \end{aligned}$$

$$\begin{aligned} A_{N-3}'(t) = k(N-2)(N-1)a_{N-1} \text{ with } A_{N-3}(0) = a_{N-3} \\ \Rightarrow A_{N-3}(t) = k(N-2)(N-1)a_{N-1} t + a_{N-3}. \end{aligned}$$

Last,

$$\begin{aligned} A_{N-4}'(t) &= k(N-3)(N-2)A_{N-2} \\ &= k(N-3)(N-2)(k(N-1)Na_N t + a_{N-2}) \\ &= k^2(N-3)(N-2)(N-1)Na_N t + k(N-3)(N-2)a_{N-2} \end{aligned}$$

with $A_{N-4}(0) = a_{N-4}$. Then

$$A_{N-4}(t) = \frac{(kt)^2}{2}\frac{N!}{(N-4)!}a_N + (kt)\frac{(N-2)!}{(N-4)!}a_{N-2} + a_{N-4}.$$

Similarily,

$$\begin{aligned} A'_{N-5}(t) &= k(N-4)(N-3)A_{N-3} \\ &= k(N-4)(N-3)(k(N-2)(N-1)a_{N-1}t + a_{N-3}) \\ &= k^2(N-4)(N-3)(N-2)(N-1)a_{N-1}t + k(N-4)(N-3)a_{N-3} \end{aligned}$$

with $A_{N-5}(0) = a_{N-5}$. Then

$$A_{N-5}(t) = \frac{(kt)^2}{2}\frac{(N-1)!}{(N-5)!}a_{N-1} + (kt)\frac{(N-3)!}{(N-5)!}a_{N-3} + a_{N-5}.$$

(iii) Note that $N = 6$ and plug in the coefficients when

$$a_6 = 1,\ a_5 = -7,\ a_4 = 0,\ a_3 = 0,\ a_2 = 0,\ a_1 = 0.$$

Then we have

$$\begin{gathered} A_6(t) = 1,\ A_5(t) = -7,\ A_4(t) = 30kt, \\ A_3(t) = -140kt,\ A_2(t) = 180k^2t^2,\ A_1(t) = -420k^2t^2. \end{gathered}$$

It remains to solve

$$A'_0(t) = 2kA_2(t) = 360k^3t^2 \text{ with } A_0(0) = a_0 = 8.$$

Then we have $A_0(t) = 120k^3t^3 + 8$. So,

$$u(t,x) = x^6 - 7x^5 + 30ktx^4 - 140ktx^3 + 180k^2t^2x^2 - 420k^2t^2x + 120k^3t^3 + 8.$$

Problem 4.

(a) (i) It follows from Duhamel's formula (with $c = 3$), as well as the given source term and initial data, that the solution u is given by

$$\begin{aligned}
u(t,x) &= \frac{1}{2}\{\phi(x+ct)+\phi(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds \\
&\quad + \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\,dy\,ds \\
&= \frac{1}{2}\{\phi(x+3t)+\phi(x-3t)\} + \frac{1}{6}\int_{x-3t}^{x+3t}\psi(s)\,ds \\
&\quad + \frac{1}{6}\int_0^t\int_{x-3(t-s)}^{x+3(t-s)} f(s,y)\,dy\,ds \\
&= \frac{1}{2}\{\cos(x+3t)+\cos(x-3t)\} + \frac{1}{6}\int_{x-3t}^{x+3t} s^2\,ds \\
&\quad + \frac{1}{6}\int_0^t\int_{x-3(t-s)}^{x+3(t-s)} sy\,dy\,ds.
\end{aligned}$$

Using the sum-and-product formula, we have

$$\begin{aligned}
&\frac{1}{2}\{\cos(x+3t)+\cos(x-3t)\} \\
= {}& \cos\left(\frac{(x+3t)+(x-3t)}{2}\right)\cos\left(\frac{(x+3t)-(x-3t)}{2}\right) \\
= {}& \cos x\cos 3t.
\end{aligned}$$

Direct computations yield

$$\frac{1}{6}\int_{x-3t}^{x+3t} s^2\,ds = \left[\frac{1}{18}s^3\right]_{s=x-3t}^{x+3t} = \frac{1}{18}\left((x+3t)^3-(x-3t)^3\right) = tx^2+3t^3$$

and

$$\begin{aligned}
\frac{1}{6}\int_0^t\int_{x-3(t-s)}^{x+3(t-s)} sy\,dy\,ds &= \frac{1}{6}\int_0^t s\int_{x-3(t-s)}^{x+3(t-s)} y\,dy\,ds \\
&= x\int_0^t s(t-s)\,ds = \frac{1}{6}xt^3.
\end{aligned}$$

As a result, the final answer is

$$u(t,x) = \cos x\cos 3t + tx^2 + 3t^3 + \frac{1}{6}xt^3.$$

(ii) No. The counter-example is

$$u(t,x) := \cos x \cos 3t + \cos \pi x \sin 3\pi t,$$

which is a solution to $\partial_{tt} u - 9\partial_{xx} u = 0$, but not periodic in x. However, both

$$u(0,x) = \cos x \quad \text{and} \quad \partial_t u(0,x) = 3\pi \cos \pi x$$

are periodic in x.

(b) (i) It follows from Duhamel's formula that the bounded solution u is given by

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y) f(s,y)\,dy\,ds \\ &= \int_{-\infty}^{\infty} S(t,x-y)e^{-y^2}\,dy + \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)\sin y\,dy\,ds, \end{aligned}$$

where the heat kernel

$$S(t,x-y) := \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} = \frac{1}{\sqrt{\pi t}} e^{-\frac{(x-y)^2}{t}},$$

since the diffusivity constant $k := \frac{1}{4}$ in this problem.

Let us compute the integrals one by one as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} S(t,x-y)e^{-y^2}\,dy &= \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{t}} e^{-y^2}\,dy \\ &= \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{1+t}} \int_{-\infty}^{\infty} e^{-\frac{1+t}{t}\left(y-\frac{x}{1+t}\right)^2}\,dy \\ &= \frac{1}{\sqrt{1+t}} e^{-\frac{x^2}{1+t}}, \end{aligned}$$

since

$$-\frac{(x-y)^2}{t} - y^2 = -\frac{1+t}{t}\left(y - \frac{x}{1+t}\right)^2 - \frac{x^2}{1+t}$$

and

$$\int_{-\infty}^{\infty} e^{-\frac{1+t}{t}\left(y-\frac{x}{1+t}\right)^2}\,dy = \sqrt{\frac{t}{1+t}} \int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\frac{\pi t}{1+t}}.$$

On the other hand, since

$$(\partial_\tau - \frac{1}{4}\partial_{yy})S(\tau, x-y) = (\partial_\tau - \frac{1}{4}\partial_{xx})S(\tau, x-y) = 0,$$

we know that

$$I(\tau, x) := \int_{-\infty}^{\infty} S(\tau, x-y)\sin y \, dy$$

satisfies

$$\begin{aligned}
\partial_\tau I &= \int_{-\infty}^{\infty} \partial_\tau S(\tau, x-y)\sin y \, dy = \frac{1}{4}\int_{-\infty}^{\infty} \partial_{yy}\left(S(\tau, x-y)\right)\sin y \, dy \\
&= -\frac{1}{4}\int_{-\infty}^{\infty} \partial_y\left(S(\tau, x-y)\right)\cos y \, dy = -\frac{1}{4}\int_{-\infty}^{\infty} S(\tau, x-y)\sin y \, dy \\
&= -\frac{1}{4}I.
\end{aligned}$$

Solving

$$\begin{cases} \partial_\tau I = -\frac{1}{4}I, \\ I(0, x) = \sin x, \end{cases}$$

we actually have

$$I(\tau, x) = e^{-\frac{1}{4}\tau}\sin x.$$

Hence,

$$\begin{aligned}
\int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)\sin y \, dy \, ds &= \int_0^t I(t-s, x) \, ds \\
&= \left(\int_0^t e^{\frac{1}{4}s} \, ds\right) e^{-\frac{1}{4}t}\sin x \\
&= 4\left(e^{\frac{1}{4}t} - 1\right) e^{-\frac{1}{4}t}\sin x \\
&= 4\left(1 - e^{-\frac{1}{4}t}\right)\sin x.
\end{aligned}$$

Therefore, the final answer is

$$u(t, x) = \frac{1}{\sqrt{1+t}} e^{-\frac{x^2}{1+t}} + 4\left(1 - e^{-\frac{1}{4}t}\right)\sin x.$$

(ii) Yes. The proof is as follows: it follows from Duhamel's formula and the explicitly given form of f that the bounded solution u is given by

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)f(s,y)\,dy\,ds \\ &= \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,dy + \int_0^t \frac{1}{1+s^2}\int_{-\infty}^{\infty} S(t-s,x-y)\,dy\,ds. \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} S(t-s,x-y)\,dy \equiv 1 \quad \text{and} \quad \int_0^t \frac{1}{1+s^2}\,ds = \tan^{-1} t,$$

we actually have

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,dy + \int_0^t \frac{1}{1+s^2}\,ds \\ &= \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,dy + \tan^{-1} t. \end{aligned}$$

Therefore, by the triangle inequality and the fact that

$$\sup_{-\infty<z<\infty} |S(t,z)| = \frac{1}{\sqrt{\pi t}},$$

we have the following estimate:

$$\begin{aligned} \left|u(t,x) - \frac{\pi}{2}\right| &\le \left|\int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,dy\right| + \left|\tan^{-1} t - \frac{\pi}{2}\right| \\ &\le \left(\sup_{-\infty<z<\infty}|S(t,z)|\right)\int_{-\infty}^{\infty}|\phi(y)|\,dy + \left|\tan^{-1} t - \frac{\pi}{2}\right| \\ &= \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty}|\phi(y)|\,dy + \left|\tan^{-1} t - \frac{\pi}{2}\right| \\ &\to 0^+ \end{aligned}$$

as $t \to \infty$. Hence,

$$\lim_{t\to\infty} u(t,x) = \frac{\pi}{2}.$$

Problem 5.

(i) Direct computation yields that

$$\begin{aligned}\partial_{xx}v(x,y) &= a^2\partial_{xx}u(ax,ay),\\ \partial_{yy}v(x,y) &= a^2\partial_{yy}u(ax,ay).\end{aligned}$$

Thus,

$$\partial_{xx}v(x,y) + \partial_{yy}v(x,y) = a^2(\partial_{xx}u(ax,ay) + \partial_{yy}u(ax,ay)) = 0.$$

(ii) Let $p = \frac{x}{y}$. Then

$$\begin{aligned}\partial_x u &= \frac{g'(p)}{y},\\ \partial_y u &= -\frac{p^2 g'(p)}{x},\\ \partial_{xx}u &= \frac{g''(p)}{y^2},\\ \partial_{yy}u &= \frac{p^2 g''(p) + 2pg'(p)}{y^2}.\end{aligned}$$

Plug into $\partial_{xx}u + \partial_{yy}u = 0$, and then we have

$$(p^2+1)g''(p) + 2pg'(p) = 0.$$

(iii) Solve the equation for $g(p)$,

$$\frac{\mathrm{d}}{\mathrm{d}\,p}((p^2+1)g'(p)) = 0.$$

Then direct integration yields

$$g(p) = \int \frac{C_1}{p^2+1}\,\mathrm{d}\,p + C_2$$

for some constants C_1, C_2. Let $p = \tan\theta$. Then

$$\int \frac{1}{p^2+1}\,\mathrm{d}\,p = \int \frac{1}{\tan^2\theta + 1}\,\mathrm{d}(\tan\theta) = \theta + C_3 = \frac{1}{2}\tan^{-1}p + C_3.$$

(iv) And hence,

$$u(x,y) = g(\frac{x}{y}) = C_4 \tan^{-1}(\frac{x}{y}) + C_5.$$

(v) For any fixed $x > 0$,

$$\lim_{y\to 0^+} u(x,y) = C_4 \lim_{y\to 0^+} \tan^{-1}(\frac{x}{y}) + C_5 = C_4\frac{\pi}{2} + C_5.$$

(vi) For any fixed $x < 0$,

$$\lim_{y\to 0^+} u(x,y) = C_4 \lim_{y\to 0^+} \tan^{-1}(\frac{x}{y}) + C_5 = -C_4\frac{\pi}{2} + C_5.$$

(vii) The initial data say that

$$\begin{cases} C_4\frac{\pi}{2} + C_5 = 1, \\ -C_4\frac{\pi}{2} + C_5 = 0. \end{cases}$$

It follows that

$$C_4 = \frac{1}{\pi} \text{ and } C_5 = \frac{1}{2}.$$

Then the solution to the given Dirichlet problem is

$$u(x,y) = \frac{1}{\pi}\tan^{-1}(\frac{x}{y}) + \frac{1}{2}.$$

(viii) Direct computation gives that

$$v(x,y) = \partial_x u(x,y) = \frac{1}{\pi}\frac{1}{y}\frac{1}{1+(\frac{x}{y})^2} = \frac{y}{\pi(x^2+y^2)}.$$

Yes. This is because

$$\partial_{xx}\partial_x u + \partial_{yy}\partial_x u = \partial_x(\partial_{xx}u + \partial_{yy}u) = 0.$$

So, $v(x,y)$ is also a solution to (1).

(ix) Consider the following convolution

$$u(x,y) = (v * \phi)(x,y) = \int_{-\infty}^{\infty} v(x-\xi,y)\phi(\xi)\,\mathrm{d}\,\xi.$$

Then

$$\partial_{xx}u + \partial_{yy}u = \int_{-\infty}^{\infty} \left(\partial_{xx}v(x-\xi,y) + \partial_{yy}v(x-\xi,y)\right)\phi(\xi)\,\mathrm{d}\,\xi = 0.$$

This is because by part (viii) v is also a solution. Moreover,

$$u(x,0) = \int_{-\infty}^{\infty} v(x-\xi,0)\phi(\xi)\,\mathrm{d}\,\xi = \int_{-\infty}^{\infty} \delta(x-\xi)\phi(\xi)\,\mathrm{d}\,\xi = \phi(x).$$

Food for Thought. The solution formula is

$$u(x,y) = \int_{-\infty}^{\infty} \frac{y\phi(s)}{\pi((x-s)^2+y^2)}ds.$$

Problem 6. By the computation of problem 2 with

$$k = 4,\ \alpha = -9,\ \beta = 5 \text{ and } \phi(x) = e^{6x} + x^3 - 8x^2 + 7,$$

It follows that

$$u(x,t) = e^{-5t}\int_{-\infty}^{\infty} \frac{1}{\sqrt{16\pi t}} e^{-\frac{(x+9t-y)^2}{16t}} (e^{6y} + y^3 - 8y^2 + 7)\,\mathrm{d}\,y.$$

Now the question becomes how to evaluate the above integral efficiently. However, the computation is quite tedious. It would be much convenient that we follow the steps in 2. Consider the change of variables

$$v(y,\tau) = e^{5\tau}u(y,\tau), \quad \text{for } y = x+9t, \tau = t.$$

It satisfies that

$$\begin{cases} \partial_\tau v - 4\partial_{yy}v = 0 \\ v(y,0) = e^{6y} + y^3 - 8y^2 + 7. \end{cases}$$

The general solution to $v(y,\tau)$ is given by

$$v(y,\tau) = \frac{1}{\sqrt{16\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-\xi)^2}{16\tau}} (e^{6\xi} + \xi^3 - 8\xi^2 + 7)\,\mathrm{d}\,\xi = v_1(y,\tau) + v_2(y,\tau).$$

Deal with the integral term by term

$$\begin{aligned} v_1(y,\tau) &= \frac{1}{\sqrt{16\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-\xi)^2}{16\tau}+6\xi}\,\mathrm{d}\,\xi \\ &= \frac{e^{6y+144\tau}}{\sqrt{16\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(\xi-y-48\tau)^2}{16\tau}}\,\mathrm{d}\,\xi \\ &= e^{6y+144\tau}. \end{aligned}$$

And

$$v_2(y,\tau) = \frac{1}{\sqrt{16\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(y-\xi)^2}{16\tau}} (\xi^3 - 8\xi^2 + 7)\,\mathrm{d}\,\xi$$

has the form

$$v_2(y,\tau) = A_3(\tau)y^3 + A_2(\tau)y^2 + A_1(\tau)y + A_0(\tau)$$

with $A_3(0) = 1, A_2(0) = -8, A_1(0) = 0, A_0(0) = 7$, and

$$A_3'(\tau) = 0,\ A_2'(\tau) = 0,\ A_1'(\tau) = 24A_3(\tau),\ A_0'(\tau) = 8A_2(\tau).$$

It follows that

$$A_3 = 1,\ A_2 = -8,\ A_1 = 24\tau,\ A_0 = -64\tau.$$

Then

$$v(y,\tau) = e^{6y+144\tau} + y^3 - 8y^2 + 24\tau y - 64\tau.$$

And thus,

$$u(x,t) = e^{-5t}v(x+9t,t) = e^{6x+198t} + e^{-5t}((x+9t)^3 - 8(x+9t)^2 + 24t(x+9t) - 64t).$$

Problem 7. The general solution is given by

$$u(t,x) = \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,\mathrm{d}\,y + \int_0^t \int_{-\infty}^{\infty} S(t-s,x-y)f(s,y)\,\mathrm{d}\,y\,\mathrm{d}\,s.$$

(i) First, let

$$v(t,x) := \int_{-\infty}^{\infty} S(t,x-y)y^5 \,\mathrm{d}\, y \text{ and } w := \partial_x^6 v$$

then we can find w satisfy

$$\begin{cases} \partial_t w - k\partial_{xx} w = 0 \\ w\big|_{t=0} = 0 \end{cases}$$

The uniqueness shows that $w \equiv 0$. Integrate with respect to w 6 times, and then we have

$$v(t,x) = A_5(t)x^5 + A_4(t)x^4 + A_3(t)x^3 + A_2(t)x^2 + A_1(t)x + A_0(t)$$

for some functions $A_i(t)$. Since v satisfies

$$\begin{cases} \partial_t v - k\partial_{xx} v = 0 \\ v\big|_{t=0} = x^5 \end{cases}$$

Plug in the ansatz of v into this equation, we can solve these $A_i(t)$

$$A_5 = 1,\ A_4 = A_2 = A_0 = 0,\ A_3 = 20kt,\ A_1 = 60k^2t^2.$$

The second term is given by

$$3\int_0^t \int_{-\infty}^{\infty} S(t-s,x-y) \,\mathrm{d}\, y \,\mathrm{d}\, s = 3t.$$

Then we have

$$u(t,x) = x^5 + 20ktx^3 + 60k^2t^2x + 3t.$$

(ii) First, let

$$v(t,x) := \int_{-\infty}^{\infty} S(t,x-y)y^2 \,\mathrm{d}\, y \text{ and } w := \partial_x^3 v$$

then we can find w satisfy

$$\begin{cases} \partial_t w - k\partial_{xx} w = 0 \\ w\big|_{t=0} = 0 \end{cases}$$

The uniqueness says that $w \equiv 0$. Integrate with respect to w 3 times, and then we get

$$v(t,x) = A_2(t)x^2 + A_1(t)x + A_0(t)$$

for some functions $A_i(t)$. Since v satisfies

$$\begin{cases} \partial_t v - k\partial_{xx} v = 0 \\ v\big|_{t=0} = x^2 \end{cases}$$

Plug in the ansatz of v into this equation, we can solve these $A_i(t)$

$$A_2 = 1,\ \ A_1 = 0,\ \ A_0 = 2kt.$$

Then we have $v(t,x) = x^2 + 2kt$.

The second term is computed by

$$\begin{aligned} \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y) f(s,y) \,\mathrm{d}\,y \,\mathrm{d}\,s &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cos s \,\mathrm{d}\,y \,\mathrm{d}\,s \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} \cos s \,\mathrm{d}\,p \,\mathrm{d}\,s \\ &= \int_0^t \cos s \,\mathrm{d}\,s \\ &= \sin t \end{aligned}$$

Then we have,

$$u(t,x) = x^2 + 2kt + \sin t.$$

(iii) The first term is given by

$$\begin{aligned} \int_{-\infty}^{\infty} S(t, x-y)\phi(y) \,\mathrm{d}\,y &= \int_{-4}^{4} \frac{6}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}} \,\mathrm{d}\,y \\ &= \int_{\frac{-4-x}{\sqrt{4kt}}}^{\frac{4-x}{\sqrt{4kt}}} \frac{6}{\sqrt{\pi}} e^{-p^2} \,\mathrm{d}\,p \\ &= 3\,\mathrm{erf}\Big(\frac{4-x}{\sqrt{4kt}}\Big) - 3\,\mathrm{erf}\Big(\frac{-4-x}{\sqrt{4kt}}\Big). \end{aligned}$$

The second term is computed by

$$
\begin{aligned}
\int_0^t \int_{-\infty}^{\infty} S(t-s, x-y) f(s,y)\,\mathrm{d}\,y\,\mathrm{d}\,s &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} y e^{-2s}\,\mathrm{d}\,y\,\mathrm{d}\,s \\
&= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2} \left[x + p\sqrt{4k(t-s)}\right] e^{-2s}\,\mathrm{d}\,p\,\mathrm{d}\,s \\
&= \int_0^t x e^{-2s}\,\mathrm{d}\,s \\
&= \frac{x}{2}(1 - e^{-2t}).
\end{aligned}
$$

Then,

$$
u(t,x) = 3\,\mathrm{erf}\left(\frac{4-x}{\sqrt{4kt}}\right) - 3\,\mathrm{erf}\left(\frac{-4-x}{\sqrt{4kt}}\right) + \frac{x}{2}(1 - e^{-2t}).
$$