Algebra II Assignment 4

Due Friday 8th April 2022

Please attempt all six problems in this assignment and submit your answers (before midnight on Friday 8th April 2022) by uploading your work to the Moodle page. If you have any questions, feel free to email me at adsg@hku.hk.

Problem 1. Consider $f(x) = x^{11} - (\sqrt{2} + \sqrt{5})x^5 + 3\sqrt[4]{7}x^3 + (1+2i)x - \sqrt[5]{17}$.

1. Show that f has no real roots.

2. Show that every root of f is algebraic over $\mathbb{Q}$.

3. Let α be any root of f. Show that the minimal polynomial of α over $\mathbb{Q}$ has degree at most 1760.

Solution. 1. Suppose that $\alpha \in \mathbb{R}$ is a root of $f(x)$. Then, $\alpha^{11} - (\sqrt{2} + \sqrt{5})\alpha^5 + 3\sqrt[4]{7}\alpha^3 + 1\alpha - \sqrt[5]{17} = -2i\alpha$. The left-hand side is a real number and the left hand side a purely imaginary number; both sides are equal if and only if they are both zero. In particular, $\alpha = 0$. Yet 0 is not a root of f.
2. Set $K = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt[4]{7}, i, \sqrt[5]{17})$, and let α be a root of $f \in K[x]$. Then, $\alpha \in K(\alpha)$. By assumption, α is algebraic over K, so $[K(\alpha) : K] < \infty$. Furthermore, K is a finite extension of $\mathbb{Q}$. By the tower theorem, $[K(\alpha) : \mathbb{Q}] = [K(\alpha) : K][K : \mathbb{Q}] < \infty$ and so α belongs to a finite (and hence algebraic) extension of $\mathbb{Q}$.
3. Since α is algebraic over $\mathbb{Q}$, α has a unique minimal polynomial over $\mathbb{Q}$, whose degree is equal to $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. Notice that $\mathbb{Q} \subset \mathbb{Q}(\alpha) \subset K(\alpha)$, and so $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq [K(\alpha), \mathbb{Q}]$ (in fact, $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ divides $[K(\alpha), \mathbb{Q}]$). By finding polynomials over $\mathbb{Q}$ with roots the elements $\sqrt{2}, \sqrt{5}, \sqrt[4]{7}, i$ and $\sqrt[5]{17}$, it is clear that $[K : \mathbb{Q}] \leq 2 \cdot 2 \cdot 4 \cdot 2 \cdot 5$, and so $[K(\alpha), \mathbb{Q}] \leq 1760$.

Problem 2 (Reciprocity theorem)**.** Suppose that α and β are algebraic over a field K, with minimal polynomials $f \in K[x]$ and $g \in K[x]$ respectively. Show that f is irreducible over $K(\beta)$ if and only if g is irreducible over $K(\alpha)$.

Solution. Suppose that f is irreducible over $K(\beta)$. Then, $[K(\alpha, \beta) : K(\beta)] = \deg(f)$. By definition, $[K(\beta), K] = \deg(g)$, and so $[K(\alpha, \beta) : K] = \deg(f)\deg(g)$. On the other hand, $[K(\alpha, \beta) : K] = [K(\alpha, \beta), K(\alpha)][K(\alpha) : K] = [K(\alpha, \beta), K(\alpha)]\deg(f)$. This implies that $[K(\alpha, \beta), K(\alpha)] = \deg(g)$. For a contradiction, suppose that g is reducible over $K(\alpha)$. Then, g has a proper factorisation over $K(\alpha)[x]$, and so $[K(\alpha, \beta), K(\alpha)] < \deg(g)$, which is a contradiction. The converse is proven *mutatis mutandis*.

Problem 3 (Ruler and compass constructions)**.** Determine whether the following points are constructible using a ruler and compass:

1. $P = (\sqrt[3]{7}, 0) \in \mathbb{R}^2$,

2. $P = (\sqrt{2}, \sqrt[3]{3}) \in \mathbb{R}^2$.

Solution. 1. Suppose that the point is constructible. Then, $[\mathbb{Q}(\sqrt[3]{7}) \colon \mathbb{Q}]$ is a power of 2, which is a contradiction because the minimal polynomial of $\sqrt[3]{7}$ over $\mathbb{Q}$ is $x^3 - 7$.
2. Similar to 1.

Problem 4 (Splitting fields)**.** Find the degree over $\mathbb{Q}$ of a splitting field over $\mathbb{Q}$ of the following polynomials in $\mathbb{Q}[x]$:

1. $f(x) = x^3 - 2$
2. $f(x) = x^4 - 1$
3. $f(x) = (x^2 - 2)(x^3 - 2)$.

Solution. 1. The roots of f in $\mathbb{C}$ are $\sqrt[3]{2}$, $\sqrt[3]{2}\zeta_3$, $\sqrt[3]{2}\zeta_3^2$, where ζ_3 is a 3rd primitive root of unity. Thus, a splitting field is $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$. Since $\mathbb{Q}(\zeta_3)$ is a quadratic extension over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[3]{2})$ is cubic, one sees that a splitting field of f in $\mathbb{C}$, i.e., $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ is of degree 6.
2. It is clear that a splitting field of f in $\mathbb{C}$ is $\mathbb{Q}(i)$. Then, it is quadratic over $\mathbb{Q}$.
3. A splitting field of f in $\mathbb{C}$ is $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \zeta_3)$. A key observation is that $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$. Then, let me show you that $[\mathbb{Q}(\sqrt{2}, \sqrt{-3}) \colon \mathbb{Q}] = 4$. It suffices to show that $\sqrt{-3} = a + b\sqrt{2}$ is not solvable over $\mathbb{Q}$. From the equation, one has $-3 = a^2 + 2b^2 + 2\sqrt{2}ab$. Thus, either a or b must be 0. Then, the equation is unsolvable by the fundamental theorem of arithmetic. Hence, a splitting field is of degree 12.

Problem 5 (Normal extensions)**.** Let $K \subset L$ be a field extension, and let M be an intermediate field of $K \subset L$. Show that if the extension $K \subset L$ is normal, then the extension $M \subset L$ is also normal.

Solution. Let f be an irreducible polynomial over L with root $\alpha \in M$. Then, consider the minimal polynomial $m(x)$ of α in K. Since $K \subset M$ is normal, $m(x)$ splits completely in M. On the other hand, since $K \subset L$, $f(x)$ divides $m(x)$ in L. The roots of $f(x)$ are thus a subset of the roots of $m(x)$, all of which lie in M: f splits completely in M.

Problem 6. Show that $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field of any polynomial in $\mathbb{Q}[x]$.

Solution. Consider $f(x) = x^3 - 2$ which is irreducible over $\mathbb{Q}$ with a root $\sqrt[3]{2} \in \mathbb{Q}(\sqrt[3]{2})$. Notice that $f(x)$ does not split in $\mathbb{Q}(\sqrt[3]{2})$, since the two remaining roots of $f(x)$ are not real, but $\mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{R}$. This shows that $\mathbb{Q}(\sqrt[3]{2})$ is not normal over $\mathbb{Q}$, and therefore not the splitting field of any polynomial in $\mathbb{Q}[x]$.