THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH3301: Algebra I

December 16, 2022 9:30 am - 12:00 noon

No calculator is allowed in this examination.

Answer ALL FIVE questions

Note: You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

1. (15%) Determine with explanation whether the following statements are true.

   (a) The automorphism group $\mathrm{Aut}(\mathbb{Z}_8)$ is non-cyclic and non-simple.

   (b) All abelian groups that are of order 48 and have no element of order 4 are isomorphic to each other.

   (c) There is a non-abelian group G whose subgroups are either the trivial subgroup or G itself.

2. (20%) Let H be a subgroup of the group G and N a normal subgroup of G.

   (a) Show that HN is subgroup of G.

   (b) Suppose M is a normal subgroup of G. Show that

   $$M/(M \cap N) \times N/(M \cap N) \cong MN/(M \cap N).$$

   [Suggestion: Use the First Isomorphism Theorem.]

3. (20%) For any finite set S, we denote its cardinality by $|S|$.

   (a) Let G be a finite group acting on the finite set X. Show the following two equalities:

   $$\sum_{g \in G} |X_g| = |G| \sum_{x \in X} \frac{1}{|Gx|} = |G| \cdot |X/G|$$

   where $X_g := \{x \in X : gx = x\}$, Gx denotes the orbit of x, and X/G is the set of all distinct orbits in X.

   [Suggestion: Recall the definitions of orbits and stabilizers in the theory of group actions.]

   (b) Let $p > q$ be primes such that q does not divide $p-1$. If G is a group of order $|G| = pq$, show that G is cyclic.

   [Suggestion: Use Sylow's third theorem and the result in Q.2 (b).]

4. (20%) Let R be a non-zero commutative ring with unity, and $R[t]$ be the polynomial ring over R.

   (a) Give an example with brief justification to demonstrate that $R[t]$ may be an integral domain but not a principal ideal domain (PID).

   (b) Show that $R[t]$ is never a field.

   (c) If $R[t]$ is a PID, show that R is a field.

5. (25%) Let K be a finite field and K^K be the ring of all functions from K to K whose addition and multiplication are defined respectively by

$$(f+g)(x) = f(x) + g(x) \text{ and } (f \cdot g)(x) = f(x)g(x),$$

   for any $f, g \in K^K$ and $x \in K$. Let φ be the map sending the polynomial $f \in K[t]$ to the associated polynomial function $f_K \in K^K$.

   (a) Show that φ is surjective and not injective.

   (b) If K has q elements, show that $K^K \cong K[t]/\langle t^q - t\rangle$.

   (c) Hence or otherwise, show that K^K is a principal ideal ring (PIR) but not a principal ideal domain (PID).

   (d) Show that K^K/J is a field for any prime ideal J of K^K.

   [Warning: A non-zero prime ideal in PIR may not be maximal.]

******* **END OF PAPER** *******