MATH6101/MATH7101/EX1/NM/ML/25-26

THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH6101/MATH7101 Intermediate Complex Analysis

Assignment 1

Due Date: October 23, 2025

(Send scanned copies of your solutions to Mimi Lui at mimi@hku.hk.)

1. (a) Let $\{x_1, \ldots, x_p\}$ be a set of p distinct points, $p > 0$, on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \amalg \{\infty\}$ and let $n_1, \ldots, n_p$ be non-zero integers such that $n_1 + \cdots + n_p = 0$. Show that there exists a nonconstant meromorphic function f on $\mathbb{P}^1$ such that $\mathrm{ord}_{x_k}(f) = n_k$ for $1 \le k \le p$ and $\mathrm{ord}_x(f) = 0$ for every point x on $\mathbb{P}^1$ not belonging to $\{x_1, \ldots, x_p\}$. $\Big[$Here $\mathrm{ord}_a(f) = s$ is the zero order at a if $s > 0$, $-\mathrm{ord}_a(f) = -s$ is the pole order at a if $s < 0$, and $\mathrm{ord}_a(f) = 0$ if and only f is holomorphic and non-zero at a.$\Big]$.

   (b) Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice and define $X := \mathbb{C}/L$. Let $\{x_1, \ldots, x_p\}$, be p distinct points on $X, p > 0$, such that $x_k + x_\ell \neq 0$ on X for $1 \le k, \ell \le p$ (when $X = \mathbb{C}/L$ is regarded as a commutative group). For $1 \le k \le p$ let n_k be non-zero integers such that $n_1 + \cdots + n_p = 0$. Prove using the Weierstrass $\wp$-function that there exists a meromorphic function f on X such that $\mathrm{ord}_{x_k}(f) = \mathrm{ord}_{-x_k}(f) = n_k$, and $\mathrm{ord}_x(f) = 0$ for any $x \in X$ such that $x \notin \{x_1, \ldots, x_p; -x_1, \ldots, -x_p\}$.

2. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and write $L = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$ for the lattice generated by ω_1 and ω_2. Write $L' := L - \{0\}$.

   (a) Show that for $k \ge 3$ we have $\displaystyle\sum_{\omega \in L'} \frac{1}{|\omega|^k} < \infty$.

   (b) Show that $\displaystyle\sum_{\omega \in L} \frac{z + \omega}{(z + \omega)^6 - 1}$ converges in an appropriate sense to an elliptic function. Describe the nature of the convergence and explain why the limiting function is indeed doubly periodic with respect to L.

   (c) Suppose $\omega_1 = 2$ and $\omega_2 = 2i$ and write $f(z) := \displaystyle\sum_{\omega \in L} \frac{z + \omega}{(z + \omega)^6 - 1}$. Determine the poles of f and the pole order at each of the poles. Regarding f equivalently as a meromorphic function h on $X = \mathbb{C}/L$, counting multiplicities how many zeros of h are there on X? Explain why.

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Solution to Assignment 1

1. (a) Assume first that $\infty \notin \{x_1, \ldots, x_p\}$. Consider the function $f(z) = \prod_{k=1}^{p} (z - x_k)^{n_k}$. Then, for $1 \le k \le p$, $\mathrm{ord}_{x_k}(f) = n_k$ by construction. Moreover, f is holomorphic and non-zero whenever $x \in \mathbb{C}$ and $x \notin \{x_1, \ldots, x_p\}$. To prove that f satisfies the required properties it suffices to show that f has a removable singularity at $z = \infty$ and it is nonzero there after extension. Since $\sum_{k=1}^{p} n_k = 0$ we have $\prod_{k=1}^{p} z^{n_k} = 1$ and

$$f(z) = \frac{\prod_{k=1}^{p} (z - x_k)^{n_k}}{\prod_{k=1}^{p} z^{n_k}} = \prod_{k=1}^{p} \left(\frac{z - x_k}{z}\right)^{n_k} = \prod_{k=1}^{p} \left(1 - \frac{x_k}{z}\right)^{n_k} .$$

Taking limits as $z \to \infty$ we have $\lim_{z\to\infty} f(z) = 1$. In particular, f is locally bounded at $z = \infty$ and it has a removable singularity at $z = \infty$, such that $f(\infty) = 1 \neq 0$ after extension, as desired.

Suppose now $\infty \in \{x_1, \cdots, x_p\}$. Without loss of generality we may assume $x_p = \infty$. Define now $f(z) = \prod_{k=1}^{p-1} (z - x_k)^{n_k}$. Then, as in the former case we have $\mathrm{ord}_{x_k}(f) = n_k$ for $1 \le k \le p-1$ and for $x \in \mathbb{C} - \{x_1, \cdots, x_p\}$ we have $\mathrm{ord}_x(f) = 0$. It remains to verify that $\mathrm{ord}_\infty(f) = n_p$. Since $n_1 + \cdots + n_{p-1} = (n_1 + \cdots + n_p) - n_p = -n_p$ we have

$$f(z) = z^{n_1+\cdots+n_{p-1}} \prod_{k=1}^{p-1} \left(1 - \frac{x_k}{z}\right)^{n_k} = z^{-n_p} \prod_{k=1}^{p-1} \left(1 - \frac{x_k}{z}\right)^{n_k} = w^{n_p} \prod_{k=1}^{p-1} (1 - x_k w)^{n_k},$$

where $w = \dfrac{1}{z}$, showing that $\mathrm{ord}_\infty(f) = n_p$, as desired.

<u>An alternative solution for the second case</u>

If $\infty \in \{x_1, \ldots, x_p\}$ we choose $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ so that $\infty \notin \{\gamma x_1, \ldots, \gamma x_p\}$. By the above there exists some f meromorphic on $\mathbb{P}^1$ such that $\mathrm{ord}_{\gamma x_k}(f) = n_k$ for $1 \le k \le \infty$ and there are no other zeros nor poles on $\mathbb{P}^1$. Then, $f \circ \gamma$ gives the solution of the original problem.

3. Given a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, and writing $L^* = L - \{0\}$ define

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in L^*} \left(\frac{1}{z+\omega} + \frac{z}{\omega^2} - \frac{1}{\omega} \right).$$

Check using 2(a) that the following holds true: For any $R > 0$, on $D(R)$ one can decompose $\zeta(z)$ as a sum of a finite number of meromorphic functions and an infinite sum of holomorphic functions such that the latter sum converges uniformly on $D(R)$ to a holomorphic function.

4. For a given lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, L' := L - \{0\}$, it is assumed known that $\wp(z) := \dfrac{1}{z^2} + \sum_{\omega \in L'} \left(\dfrac{1}{(z+\omega)^2} - \dfrac{1}{\omega^2} \right)$ converges in an appropriate sense to a meromorphic function on $\mathbb{C}$.

   (a) Prove that $\wp$ is indeed an elliptic function with respect to L.

   (b) Define

$$\begin{cases} f(z) = \left(\wp'(z)\right)^2 \\ g(z) = \left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right) \left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right) \left(\wp(z) - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right). \end{cases}$$

   Show that, counting multiplicities, f and g have the same zeros and the same poles. Hence deduce that

$$\left(\wp'(z)\right)^2 = 4 \left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right) \left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right) \left(\wp(z) - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right).$$

   (c) Assume <u>known</u> that $\wp$ satisfies the equation $\left(\wp'\right)^2 = 4\wp^3 + a\wp + b$ for some complex numbers a, b (depending on L). Prove that

$$\wp\left(\frac{\omega_1}{2}\right) + \wp\left(\frac{\omega_2}{2}\right) + \wp\left(\frac{\omega_1+\omega_2}{2}\right) = 0.$$

(b) Denote by $\pi : \mathbb{C} \to X = \mathbb{C}/L$ the universal covering map. Write $\wp(z) = \dfrac{1}{z^2} + \sum_{\omega \in L^*} \left(\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2} \right)$, $L = L^* - \{0\}$, for the Weierstrass $\wp$-function associated to the lattice L. Then, for every integer $k, 1 \le k \le p$, choosing $y_k \in \pi^{-1}(x_k) \in \mathbb{C}$, the elliptic function $f_k(z) := \wp(z) - \wp(y_k)$ has a zero at any $z = \widetilde{x}_k$ whenever $\pi(\widetilde{x}_k) = x_k$, i.e., $\widetilde{x}_k = y_k + \omega$ for some $\omega \in L$. Regarded as a meromorphic function on X, f_k has a zero at x_k. Since $\wp$ is an even function on X, $f_k(-\widetilde{x}_k) = f_k(\widetilde{x}_k) = 0$, so that as a meromorphic function on X, $f_k(-x_k) = 0$. Now f_k has a double pole at $0 \in X$. Since $2x_k \ne 0$ on X (as a commutative group), f has two distinct zeros at x_k and $-x_k$. Since the total number of zeros of f_k agree with the number of poles, f_k must have a simple zero at x_k and at $-x_k$. By assumption $x_k + x_\ell \ne 0$ on x for $1 \le k, \ell \le n$, hence $\{x_1, \cdots, x_p; -x_1, \cdots, -x_p\}$ consist of $2p$ distinct points. Define now $f = f_1^{n_1} \cdots f_p^{n_p}$. Then,

$$\text{ord}_{x_k} f = \text{ord}_{-x_k} f = n_k \text{ for } 1 \le k \le p.$$

For $x \notin \{x_1, \cdots, x_p; -x_1, \cdots, -x_p; 0\}$, it is clear from the definition of f that $\text{ord}_x f = 0$. It remains to show that $\text{ord}_0 f = 0$. But now $f_k(z) = \wp(z) - \wp(y_k)$ has a double pole at 0 for each $k, 1 \le k \le p$. Hence,

$$\text{ord}_0 f = \sum_{k=1}^{p} \text{ord}_0\left(f_k^{n_k}\right) = 2\sum_{k=1}^{p} n_k = 0$$

by assumption. As a conclusion, f solves the Weierstrass Problem for the given data $\{(x_k; n_k) : 1 \le k \le p\}$.

2. (a) For an integer $n \ge 1$ define $S_n = \{n_1\omega_1 + n_2\omega_2 : -n \le n_1, n_2 \le n\}$, and define $T_0 = \{0\}, T_n := S_n - S_{n-1}$ for $n \ge 1$. We have $\text{Card}(S_n) = (2n+1)^2$, so that

$$\text{Card}(T_n) = (2n+1)^2 - (2n-1)^2 = (4n^2+4n+1) - (4n^2-4n+1) = 8n.$$

Then, S_n is the disjoint union of $T_0, T_1, \cdots, T_n$, and

$$\sum_{\omega \in L'} \frac{1}{|\omega|^k} = \sum_{n=1}^{\infty} \left(\sum_{\omega \in T_n} \frac{1}{|\omega|^k} \right).$$

Obviously there exists a constant $c > 0$ such that $|\omega| \geq cn$ for any $\omega \in T_n$. Hence, for $k \geq 3$ we have

$$\sum_{\omega \in L'} \frac{1}{|\omega|^k} \leq \sum_{n=1}^{\infty} \frac{8n}{(cn)^k} = \frac{8}{c^k} \sum_{n=1}^{\infty} \frac{1}{n^{k-1}} < \infty$$

since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^\alpha} < \infty$ whenever $\alpha > 1$.

(b) Let $K = \overline{D(R)}$. Formally

$$\sum_{\omega \in L} \frac{z+\omega}{(z+\omega)^6 - 1} = \sum_{|\omega| \leq 2R} \frac{z+\omega}{(z+\omega)^6 - 1} + \sum_{|\omega| > 2R} \frac{z+\omega}{(z+\omega)^6 - 1}$$

There are only finitely many terms in the first summation $\sum'$. For the second summation $\sum''$, $|z+\omega| \geq |\omega| - |z| \geq |\omega| - R > \dfrac{|\omega|}{2}$, so that

$$\sum_{|\omega| > 2R} \left| \frac{z+\omega}{(z+\omega)^6 - 1} \right| < \sum_{\omega > 2R} \frac{R + |\omega|}{(\frac{|\omega|}{2})^6 - 1}.$$

Choosing $R \geq 1$,

$$\sum_{|\omega| > 2R} \left| \frac{z+\omega}{(z+\omega)^6 - 1} \right| < \sum_{\omega > 2R} \frac{R + |\omega|}{\frac{1}{2}(\frac{|\omega|}{2})^6} < \sum_{\omega \in L'} \frac{2^7 \cdot R}{|\omega|^6} + \sum_{\omega \in L'} \frac{2^7}{|\omega|^5}.$$

By (a), the second summation $\sum''$ is absolutely and uniformly convergent on $\overline{D(R)}$. From absolute convergence of $\sum''$ it follows readily that the order of summation is immaterial, hence for any $\nu \in L$

$$\begin{aligned} f(z) := & \sum_{\omega \in L} \frac{z+\omega}{(z+\omega)^6 - 1} = \sum_{\omega \in L} \frac{z+(\nu+\omega)}{(z+(\nu+\omega))^6 - 1} = \sum_{\omega \in L} \frac{(z+\nu)+\omega}{((z+\nu)+\omega)^6 - 1} \\ = & \ f(z+\nu). \end{aligned}$$

Thus, f is an elliptic function.

(c) Suppose z_0 is a pole of f. Then, z_0 must be a pole of $\dfrac{z_0+\omega}{(z_0+\omega)^6-1}$ for some $\omega \in L$. Hence $(z_0+\omega)^6 = 1$, i.e.,

$$z_0 = \pm 1 + 2n_1 + 2n_2 i, \pm e^{\frac{\pi i}{3}} + 2n_1 + 2n_2 i, \pm e^{\frac{2\pi i}{3}} + 2n_1 + 2n_2 i$$

for some integers n_1, n_2. Thus, $z_0 = \pm 1, \pm\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right), \pm\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)$ and their translates by $L = 2\mathbb{Z}+2\mathbb{Z}i$ are all the possible poles. By inspection any two distinct entities of the six listed potential poles cannot be congruent to each other with the exception of the pair 1 and -1, where $1-(-1) = 2 \in L$. Thus, $\pm\left(\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i\right) + 2n_1 + 2n_2 i$ are poles of f, and they are simple poles since $\big((z_0+\omega)^6 - 1\big)' = 6(z+\omega)^5 \neq 0$ whenever $(z+\omega)^6 = 1$. On the other hand, at $z_0 = 1$, two summands $\dfrac{z}{z^6-1}$ and $\dfrac{z-2}{(z-2)^6-1}$ have simple poles at 1. The sum has either a simple pole or a removable singularity at 1. To determine this we have to determine the residues. We have $\big((z+\omega)^6-1\big)' = 6(z+\omega)^5$ and hence

$$\operatorname{Res}\left(\frac{z}{z^6-1};1\right) = \frac{1}{6}, \operatorname{Res}\left(\frac{z-2}{(z-2)^6-1};1\right) = \frac{1-2}{6(-1)^5} = \frac{-1}{-6} = \frac{1}{6}.$$

Hence the residues do not cancel each other, and there are exactly 5 poles, all simple, modulo L, given by $1+L$, $\pm\left(\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i\right) + L$. Finally, writing $f(z) = h\big(\pi(z)\big)$ for the uniformizing map $\pi : \mathbb{C} \to \mathbb{C}/L = X$, by the Residue Theorem, counting multiplicities, the number of zeros of h or X agrees with the number of poles. There are hence 5 zeros of h, counting multiplicities.

3. Fix $R > 0$. Let $A \subset L^*$ be the subset of all points ω in L^* such that $|\omega| < 2R$. Then on $D(R)$ we have

$$\begin{aligned}\zeta(z) = &\left\{\frac{1}{z} + \sum_{\omega\in A}\left(\frac{1}{z+\omega} + \frac{z}{\omega^2} - \frac{1}{\omega}\right)\right\} \\ &+ \sum_{\omega \in L^*-A}\left(\frac{1}{z+\omega} + \frac{z}{\omega^2} - \frac{1}{\omega}\right).\end{aligned}$$

Write f_R for the sum inside $\{\cdots\}$, and ζ_R for the infinite sum. Then f_R is a meromorphic function on $D(R)$. For the infinite sum, write a summand as

$$\frac{1}{z+\omega}+\frac{z}{\omega^2}-\frac{1}{\omega}=\frac{1}{(z+\omega)\omega^2}\left(\omega^2+z(z+\omega)-(z+\omega)\omega\right)$$

$$=\frac{1}{(z+\omega)\omega^2}\left(\omega^2+z^2+z\omega-z\omega-\omega^2\right)=\frac{z^2}{(z+\omega)\omega^2}.$$

Hence, on $D(R)$ we have

$$\left|\frac{1}{z+\omega}+\frac{z}{\omega^2}-\frac{1}{\omega}\right|$$

$$<\left|\frac{R^2}{(\omega-R)\omega^2}\right|\leq\frac{2R^2}{|\omega|^3}.$$

Thus

$$\sum_{\omega\in L^*-A}\left|\left(\frac{1}{z+\omega}+\frac{z}{\omega^2}-\frac{1}{\omega}\right)\right|\leq\sum_{|\omega|\geq 2R}\frac{2R^2}{|\omega|^3}<\infty$$

by 2(a), and hence the infinite sum under study converges absolutely and uniformly on $D(R)$, as desired.

4. (a) The convergence being absolute and uniform on compact subsets (after removing a finite number of terms with possible poles on a given compact subset), the order of summation is unimportant. We have $\wp'(z)=\frac{-2}{z^3}+\sum_{\omega\in L}\frac{-2}{(z+\omega)^3}=-2E_3$. which is elliptic. It follows that for any $\omega\in L$, $h_\omega(z):=\wp(z+\omega)-\wp(z)$ satisfies $h'_\omega(z)=-2E_3(z+\omega)+2E_3(z)=0$, hence $h_\omega(z)=C_\omega$ for some complex number C_ω. From $\wp(z+\omega)-\wp(z)=C_\omega$, substituting $z=-\frac{\omega_i}{2},\omega=\omega_i,i=1,2$, we have $C_{\omega_i}=\wp\left(-\frac{\omega_i}{2}+\omega_i\right)-\wp\left(-\frac{\omega_i}{2}\right)=\wp\left(\frac{\omega_i}{2}\right)-\wp\left(-\frac{\omega_i}{2}\right)$ noting that $\wp\left(\frac{\omega_i}{2}\right)$ and $\wp\left(-\frac{\omega_i}{2}\right)$ are finite. From the definition $\wp$ is an even function, i.e., $\wp(z)=\wp(-z)$. It follows that $C_{\omega_i}=\wp\left(\frac{\omega_i}{2}\right)-\wp\left(\frac{\omega_i}{2}\right)=0$ for $i=1,2$. Hence $\wp(z+\omega)=\wp(z)$ for $z\in\mathbb{C},\omega\in L$, i.e., $\wp$ is an elliptic function.

   (b) $f(z)=\left(\wp'(z)\right)^2$ has a pole of order 6 at $z=\omega$, $\omega\in L$, and no other poles. Consider $z=\frac{\omega_1}{2},\frac{\omega_2}{2}$ or $\frac{\omega_3}{2},\omega_3=\omega_1+\omega_2$. From $\wp'(z)=-\wp'(-z)$ since $\wp'$

is odd, for $1 \le i \le 3$, we have $\wp'\left(\frac{\omega_i}{2}\right) = -\wp'\left(-\frac{\omega_i}{2}\right) = -\wp'\left(\frac{\omega_i}{2}\right)$ since $\frac{\omega_i}{2} \equiv -\frac{\omega_i}{2} \bmod L$. It follows that $\wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1+\omega_2}{2}\right)$ $= 0$. Since $\wp$ has only a triple pole at lattice points $\omega \in L$ and no other poles modulo L, there are exactly 3 zeros of $\wp'$ counting multiplicities, so that $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ are simple zeros of $\wp'$ and hence double zeros for $f = (\wp')^2$. For the elliptic function $g(z) = \left(\wp - \wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp - \wp\left(\frac{\omega_2}{2}\right)\right)$ $\left(\wp - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right)$, g has a pole of order 6 at $\omega \in L$ and no other poles. Moreover g has zeros at $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. Write $\omega_3 := \omega_1 + \omega_2$. Now $\left(\wp - \wp\left(\frac{\omega_1}{2}\right)\right)' = \wp'$ and $\wp'\left(\frac{\omega_i}{2}\right) = 0$ for $i = 1, 2, 3$, so that $\wp - \wp\left(\frac{\omega_i}{2}\right)$ must have at least a double zero at $z = \frac{\omega_i}{2}, i = 1, 2, 3$. Counting multiplicities, the number of zeros modulo L agrees with the number of poles modulo L for the elliptic function g, which is equal to 6. Hence, g must have a double zero at $z = \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. Consequently f and g have exactly the same zeros and poles counting multiplicities. By the Maximum Principle $h := \frac{f}{g}$ is a non-zero constant λ. Expanding both sides using $\wp(z) = \frac{1}{z^2} + \cdots \wp'(z) = \frac{-2}{z^3} + \cdots$ it follows that $\lambda = 4$, hence $\left(\wp'(z)\right)^2 = 4\left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right)$, as desired.

(c) By (b) we have

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - 4\left(\wp\left(\frac{\omega_1}{2}\right) + \wp\left(\frac{\omega_2}{2}\right) + \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right)\wp(z)^2 + \cdots$$

so that $\wp$ satisfies the differential equation

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 + \alpha\wp(z)^2 + \beta\wp(z) + \gamma \text{ for some}$$

$\alpha, \beta, \gamma \in \mathbb{C}$, where $\alpha = \wp\left(\frac{\omega_1}{2}\right) + \wp\left(\frac{\omega_2}{2}\right) + \wp\left(\frac{\omega_1+\omega_2}{2}\right)$. Given that $\left(\wp'(z)\right)^2 = 4\wp(z)^3 + a\wp(z)^2 + b$ for some $a, b \in \mathbb{C}$, it follows that

$$\alpha\wp(z)^2 + (\beta - a)\wp(z) + (\gamma - b) = 0\,.$$

Considering the Laurent series expansion at $z = 0$ we have $\dfrac{\alpha}{z^4} + \cdots = 0$ which forces $\alpha = 0$. Thus $\wp\left(\dfrac{\omega_1}{2}\right) + \wp\left(\dfrac{\omega_2}{2}\right) + \wp\left(\dfrac{\omega_1 + \omega_2}{2}\right) = \alpha = 0$, as desired.