Local PIDs

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1. Localization and local PIDs.

$$\frac{1}{1-z} = 1+z+z^2+\cdots$$

Goal: More examples of PIDs

$$F((x)) = \left\{ \sum_{n=-N}^{\infty} a_n x^n : a_i \in F \right\}$$

Example. Consider the ring

$$F[[x]] = \left\{ \sum_{n=0}^{\infty} a_n x^n : a_0, a_1, \ldots \in F \right\}$$

$F(x)$

of formal power series in x with coefficients in a field F.

$F[x]$

Main properties of $F[[x]]$:

$F[[x]]$

non-zero

- $F[[x]]$ is a very special PID: every ideal is of the form

$$I_n = x^n F[[x]]$$

  for some integer $n \geq 0$;

- $\mathfrak{m} = xF[[x]]$ is the unique maximal ideal.

$F((x)) \stackrel{\text{def}}{=}$ The fraction field of $F[[x]]$

$$= \left\{ \frac{f(x)}{g(x)} : \begin{array}{l} f(x), g(x) \in F[[x]] \\ g(x) \neq 0 \end{array} \right\}$$

For $g(x) \in F[[x]]$, $g(x) \neq 0$, write

$$g(x) = \sum_{n=N}^{\infty} a_n x^n \qquad \text{where } \begin{array}{l} N \geq 0 \\ a_N \neq 0 \end{array}$$

Then $g(x) = a_N x^N h(x)$, $h(x) \in F[[x]]^\times$

Thus $\dfrac{1}{g(x)} = \dfrac{1}{a_N x^N} h(x)^{-1}$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{1}{x^N} \tilde{f}(x), \quad \text{where } \tilde{f}(x) \in F[[x]]$$

suppose you have a ring of functions. All functions that vanishes at a single point forms a maximal ideal, so geometrically, maximal ideals are points.
Localize means there is only one point in this space.

Definition A non-zero commutative ring R is said to be local if it has a unique maximal ideal.

Let $R^\times$ be the set of all units of R.

Lemma: A non-zero commutative ring R is local if and only if $R \backslash R^\times$ is an ideal, in which case $R \backslash R^\times$ is the unique maximal ideal of R.

Proof.

- Assume first that R is local and let $\mathfrak{m}$ be its unique maximal ideal.
- As $\mathfrak{m} \neq R$ by definition, $\mathfrak{m}$ does not contain any unit, so $\mathfrak{m} \subset R \backslash R^\times$.
- Conversely, let $a \in R \backslash R^\times$ be arbitrary.
- There is a maximal ideal $\mathfrak{m}'$ of R containing a. ✓ Zorn's lemma
- So $\mathfrak{m}' = \mathfrak{m}$, thus $a \in \mathfrak{m}$. Hence $\mathfrak{m} = R \backslash R^\times$.

Def: A PID that is also ~~ea~~ local is called a local PID.

Q.E.D.

Lemma: For integral domain R not a field, the following are equivalent:

1. R is a local PID; $x^n \cancel{u = x^{n'} u'}$ $\Rightarrow R \setminus R^\times = xR$
2. there exists a non-unit $x \in R$ such that every non-zero element $a \in R$ is of the form $a = x^n u$ for some $n \in \mathbb{N}$ and some unit u in R.

let $I \subset R$ be a non-zero ideal. let $n = \min\{n' : x^{n'} \in I\}$ Then $I = x^n R$

Proof. Clearly (2) implies (1).

- Assume that R is a local PID with a unique maximal ideal $\mathfrak{m}$.
- Then $\mathfrak{m} = xR$ for some non-unit $x \in R$. Let $a \in R \setminus \{0\}$.
- If a is a unit, $a = x^0 a$. Assume a is not a unit.
- Then $a \in \mathfrak{m}$, so $a = xa_1$ for some $a_1 \in R$. If a_1 is a unit, we are done.
- Otherwise, $a_1 = xa_2$ for some $a_2 \in R$, so $a = x^2 a_2$.
- If a_2 is a unit, we are done. Otherwise, continue.
- The sequence $a_1R \subset a_2R \subset a_3R \subset \cdots$ must stabilize. So $a = x^n u$ for some $n \geq 0$ and $u \in R^\times$.

Question: What are examples of local PIDs?

Q.E.D.

$p\mathbb{Z} \subset \mathbb{Z}$ is a prime ideal

More examples of local PIDs.

Example. For a prime number p, let

$(a, b) = 1$

$$\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : \ a, \ b \in \mathbb{Z}, \ b \neq 0, \ p \nmid b\}. = \{p^n \frac{\alpha}{\beta} : \ n \geq 0, \ p \nmid \alpha, \ p \nmid \beta$$

- Being a sub-ring of $\mathbb{Q}$, $\mathbb{Z}_{(p)}$ is an integral domain;
- Every non-zero $r \in \mathbb{Z}_{(p)}$ is uniquely of form

$$r = p^n \frac{a}{b}$$

  with $a, b \in \mathbb{Z} \backslash \{0\}$ and $p \nmid a$ and $p \nmid b$, so $\frac{a}{b}$ is a unit in $\mathbb{Z}_{(p)}$.
- Thus $\mathbb{Z}_{(p)}$ is a local PID with unique maximal ideal $p\mathbb{Z}_{(p)}$.

~~Pr~~ Exercise: $\mathbb{Z}_{(p)} \not\cong \mathbb{Q}[[x]]$ as rings

Answer: No bijection as $\mathbb{Z}_{(p)}$ is countable w/ $\mathbb{Q}[[x]]$ is not

$x\,K[x]$ is a prime ideal

Example: Let K be a field and let

$$K[x]_{(x)} = \{f/g :\ f,\, g \in K[x],\, \underline{g(0) \neq 0}\} \subset K(x).$$

- $K[x]_{(x)}$, a sub-ring of $K(x) = \mathrm{Frac}(K[x])$, is an integral domain;
- Every non-zero element in $K[x]_{(x)}$ is of the form

$$\phi = x^n \frac{f}{g} \qquad \frac{g}{f}$$

  where $f, g \in K[x]$ and $\underline{f(0) \neq 0,\ g(0) \neq 0}$, so $\frac{f}{g}$ is a unit in $K[x]_{(x)}$.
- Thus $K[x]_{(x)}$ is a local PID with the unique maximal ideal $xK[x]_{(x)}$.

More generally:

Lemma

Let R be any UFD with fraction field F, and let $p \in R$ be a prime element. The sub-ring

$$R_{(p)} \stackrel{\text{def}}{=} \left\{ p^n \frac{a}{b} : \ n \in \mathbb{N}, \, a, b \in R, b \neq 0, \, p \nmid a, \, p \nmid b \right\}$$

of F is a local PID with unique maximal ideal $pR_{(p)}$.

Proof. Exactly same as previous two examples.

let $P = pR$. Then P is a prime ideal of R

$D \stackrel{\text{def}}{=} R \setminus P$. Then $d \in D \iff d \notin P$

$\iff p \nmid d$.

Q.E.D.

§1.5.2: Localization and local PIDs

Localization: Let R be any commutative ring.

$$\frac{r_1}{d_1} = \frac{r_2}{d_2}$$

Definition. A subset D of $R\backslash\{0\}$ is said to be multiplicatively closed if $1 \in D$ and if $ab \in D$ for all $a, b \in D$.

Lemma-Definition. Let $D \subset R\backslash\{0\}$ be multiplicatively closed.

- One has the equivalence relation on $R \times D$ defined by

$$(r_1, d_1) \sim (r_2, d_2) \quad \text{if} \quad d(r_1 d_2 - r_2 d_1) = 0 \quad \text{for some } d \in D.$$

- Denote by $\frac{r}{d}$ the equivalence class of (r, d). The set $D^{-1}R$ of all equivalence classes in $R \times D$ is a ring with the operations

$$\frac{r_1}{d_1} + \frac{r_1}{d_1} = \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \qquad \frac{r_1}{d_1} \cdot \frac{r_1}{d_1} = \frac{r_1 r_2}{d_1 d_2}, \qquad (r_1, d_1), (r_2, d_2) \in R \times D.$$

- The map $R \longrightarrow D^{-1}R, r \longmapsto \frac{r}{1}$, is a ring homomorphism, which is injective if D ~~has~~ contains no zero divisor.
- The ring $D^{-1}R$ is called called the localization of R at D.

Ex: R is an integral domain, $D = R\backslash\{0\}$

Example. Let R be integral domain and F its fractions field.

- For any multiplicatively closed $D \subset R \backslash \{0\}$, one has injective ring homomorphism

$$\phi: \quad D^{-1}R \longrightarrow F, \quad \frac{r}{d} \longmapsto \frac{r}{d}.$$

- Image of ϕ is the sub-ring of F generated by $D^{-1} = \{d^{-1} : d \in D\}$ and R.

- As a sub-ring of F, the localization $D^{-1}R$ is also an integral domain.

Question: How to understand ideals of $D^{-1}R$?

Digression on extensions and contractions of ideals

This part omitted in class

Definition. Let R and Q be any commutative rings and let $\phi : R \to Q$ be a ring homomorphism.

1. For any ideal I of R, the ideal $\phi(I)Q$ of Q is called the extension of I to Q by ϕ, and we write 注意，对于交换环，phi(I)乘Q以后就是理想
$$I^e = \phi(I)Q \subset Q;$$
2. For any ideal J of Q, the ideal $\phi^{-1}(J)$ of R is called the contraction of J in R by ϕ, and we write
$$J^c = \phi^{-1}(J) \subset R.$$
3. Note that when R is a sub-ring of Q and $\phi : R \to Q$ is the inclusion, have $J^c = J \cap R$.

Lemma. Let R be any commutative ring and $D \subset R\backslash\{0\}$ multiplicatively closed. Consider extension and contractions of ideals by

$$\phi : \quad R \longrightarrow D^{-1}R.$$

$f(f^{-1}(J)) \subseteq J$
with eq if f is surj

- For any ideal J of $D^{-1}R$, we have

$$J = (J^c)^e.$$

Consequently, every ideal of $D^{-1}R$ is the extension of some ideal of R; Distinct ideals of $D^{-1}R$ have distinct contractions in R;

- For any ideal I of R, we have

$(I^e)^c = (D^{-1}I) \cap R$

$$(I^e)^c = \{r \in R : dr \in I \text{ for some } d \in D\}.$$

Moreover, $I^e = D^{-1}R$ if and only if $I \cap D \neq \emptyset$.

- Extension and contraction give a bijection between prime ideals I of R such that $I \cap D = \emptyset$ and prime ideals of $D^{-1}R$.

Most impotant example: localization at prime ideals

Lemma-Definition: Let R be an commutative ring and $P \subset R$ a prime ideal. Then

$$D = R \backslash P \subset R \backslash \{0\}$$

$d_1 \in D$ means $d_1 \notin P$.
$d_2 \in D \Rightarrow d_2 \notin P$
$\Rightarrow d_1 d_2 \notin P$

is multiplicatively closed, and the localization $D^{-1}R$ is called the localization of R at P and is denoted as R_P.

Lemma. Let R be an commutative ring and $P \subset R$ a prime ideal. Then by extension and contraction of ideals by

$$R \longrightarrow R_P, \quad r \longmapsto \frac{r}{1},$$

$I \cap P^c = \emptyset$

one has bijections

$$\{\text{prime ideals of } R_P\} \longleftrightarrow \{\text{prime ideals } I \subset R \text{ such that } I \subset P\}.$$

Remarks:

- Localization of a UFD at an arbitrary prime ideal is not necessarily a PID;
- Localization of a Dedekind domain at any prime ideal is a local PID.
- A local PID that is not a field is also called a Discrete Valuation Ring (D.V.R.).