**Algebra II Assignment 3**

*Due Friday 18th March 2022*

Please attempt all six problems in this assignment and submit your answers (before midnight on Friday 18th March 2022) by uploading your work to the Moodle page. If you have any questions, feel free to email me at adsg@hku.hk.

**Problem 1.** Show that every element in the subfield $\mathbb{Q}(\sqrt[3]{2})$ of $\mathbb{R}$ is of the form

$$\alpha = a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$$

for a unique $(a, b, c) \in \mathbb{Q}^3$. For $(a, b, c) \neq 0 \in \mathbb{Q}^3$, let

$$\frac{1}{\alpha} = a_1 + b_1\sqrt[3]{2} + c_1(\sqrt[3]{2})^2 \in \mathbb{Q}(\sqrt[3]{2})$$

with $(a_1, b_1, c_1) \in \mathbb{Q}^3$. Express $(a_1, b_1, c_1)$ in terms of $(a, b, c)$.

**Problem 2.** Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$.

1. Show that $\alpha$ is algebraic over $\mathbb{Q}$ and find the minimal polynomial of $\alpha$ over $\mathbb{Q}$;
2. Let $\beta = \sqrt{2 - \sqrt{2}} \in \mathbb{R}$. Show that $\beta \in \mathbb{Q}(\alpha)$ and write $\beta$ as a polynomial in $\alpha$ with coefficients in $\mathbb{Q}$.

**Problem 3.** Suppose that $K \subset L$ is a finite field extension, and consider $\alpha \in L$ and $\alpha \notin K$.

1. Show that there exists a minimal polynomial $p(x) \in K[x]$ of $\alpha$ over $K$.
2. If $\beta$ is another root of $p(x)$ in $L$, show that $\beta \notin K$ and $p(x)$ is also the minimal polynomial of $\beta$ over $K$.

**Problem 4.** For the following $\alpha \in \mathbb{R}$, find the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$:

$$1)\ \alpha = \sqrt{1 + \sqrt{3}};\quad 2)\ \alpha = \sqrt{3 - \sqrt{6}};\quad 3)\ \alpha = \sqrt{3 + 2\sqrt{2}}.$$

**Problem 5.** For which values of $p, q$ (both prime numbers) do we have $\mathbb{Q}(\sqrt{p}) \subset \mathbb{Q}(\sqrt[3]{q})$.

**Problem 6.** Let $f(x) = x^3 - 6x^2 + 4xi + (1 + 3i)$ in $R[x]$, where $R = \mathbb{Z}[i]$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ be the three roots of $f$.

1. Show that $f$ is irreducible over $R$ (Hint: you may use the fact that Eisenstein's criterion works over $R$ any UFD, with $p$ a prime element in $R$).
2. Deduce that the field extension $\mathbb{Q}(i, \alpha_1, \alpha_2, \alpha_3)$ has degree 6.