THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Test 1 Solution

Problem 1.

(i) By the method of characteristics, we solve the following system of ODEs,

$$\begin{cases} \dfrac{dx}{ds} = 2x,\ \ x(0) = x_0 \\ \dfrac{dy}{ds} = 1 + 2y,\ \ y(0) = y_0 \\ \dfrac{dW}{ds} = 3W,\ \ W(0) = u(x(0), y(0)) \end{cases} \Longrightarrow \begin{cases} x(s) = x_0 e^{2s} \\ y(s) = \Big(\dfrac{1}{2} + y_0\Big)e^{2s} - \dfrac{1}{2} \\ W(s) = W(0)e^{3s} \end{cases}$$

Choose $y_0 = 0$, then $y(s) = \frac{1}{2}e^{2s} - \frac{1}{2}$, so the characteristic curves can be parametrized by x_0:

$$C_{x_0} = \Big\{(x, y) : y = \frac{1}{2x_0}x - \frac{1}{2},\ \text{or}\ \frac{x}{2y+1} = x_0\Big\}.$$

From the relationship $W(s) = u(x(s), y(s))$ in C_{x_0}, we have

$$u(x, y) = u(x_0, y_0)e^{3s} = u\Big(\frac{x}{2y+1}, 0\Big)x^{\frac{3}{2}} = f\Big(\frac{x}{2y+1}\Big)x^{\frac{3}{2}}$$

where f is an arbitrary function.

From the initial condition $u(x, 0) = x^2$, we can then further deduce

$$x^2 = u(x, 0) = f(x)x^{\frac{3}{2}} \Longrightarrow f(x) = \sqrt{x}.$$

Therefore the solution to the initial value problem is

$$u(x, y) = \frac{x^2}{\sqrt{2y+1}}.$$

(ii) The limit indeed exists. When $y \to \infty$, x may be viewed as fixed, thus

$$\lim_{y\to\infty} u(x,y) = \lim_{y\to\infty} \frac{x^2}{\sqrt{2y+1}} = x^2 \lim_{y\to\infty} \frac{1}{\sqrt{2y+1}} = 0.$$

Problem 2.

(i) The equation (2) has

$$a_1 = 1,\ a_{12} = a_{21} = \frac{-8}{2} = -4,\ a_{22} = 16.$$

Therefore, the discriminant is given by

$$\mathcal{D} = a_{12}^2 - a_{11}a_{22} = (-4)^2 - (1)(16) = 0.$$

Hence, the equation is parabolic.

(ii) The equation (2) can be rewritten as

$$(\partial_t - 4\partial_x)(\partial_t - 4\partial_x)u = (\partial_t - 4\partial_x)^2 u = 0.$$

To find the general solution to (2), we need to solve

$$\begin{cases} \partial_t v - 4\partial_x v = 0 \\ \partial_t u - 4\partial_x u = v \end{cases}.$$

It is worth to notice that this system of equations is partially decoupled, so we can find a solution by solving the first equation and then the second.

We first solve for v. To find the characteristic curves,

$$\begin{cases} \dfrac{dt}{ds} = 1,\ t(0) = t_0 \\ \dfrac{dx}{ds} = -4,\ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} t = s + t_0 \\ x = -4s + x_0 \end{cases}$$

Then $x = -4(t-t_0) + x_0 = -4t + 4t_0 + x_0$. Choose $t_0 = 0$, the characteristic curves can be parametrized by x_0:

$$C_{x_0} = \{(t,\, x) : x = -4t + x_0\}.$$

Note that $v(t,x)$ remains unchanged along each characteristic curves and hence for all $(t,\, x) \in C_{x_0}$,

$$v(t,\, x) = v(0,\, x_0) = v(0,\, x+4t) = g(x+4t), \text{ where } g \text{ is arbitrary.}$$

Let $W(s) = u(t(s), x(s))$. Then $\frac{dW(s)}{ds} = v$ (where v is constant along C_{x_0}) implies $dW = v\,ds$. By integrating along C_{x_0} from $(0,\, x_0)$ to $(t,\, x)$, we have

$$u(t,x) - u(0,x_0) = \int_0^s g(x+4t)\, ds = \int_0^s g(x_0)\, ds = s\, v(0,\, x_0) = t\, g(x+4t),$$

and hence

$$u(t,x) = u(0, x+4t) + t\, g(x+4t) = f(x+4t) + t\, g(x+4t),$$

where f, g are arbitrary. Now use the initial data, we have

$$\begin{cases} u|_{t=0} = f(x) = x^3, \\ \partial_t u|_{t=0} = 4f'(x) + g(x) = 6x^2. \end{cases}$$

Thus we can deduce $g(x) = 6x^2 - 4f'(x) = -6x^2$, and substituting f and g into the formulation of u,

$$u(t,x) = (x+4t)^3 - 6t(x+4t)^2 = (x+4t)^2(x-2t) = x^3 + 6tx^2 - 32t^3.$$