MATH3541 INTRODUCTION TO TOPOLOGY SAMPLE SOLUTIONS FOR THE MIDTERM

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Q1 [50 marks]

(a) Let U and V be open sets of $\mathbb{R}^{Zar}$ such that $U \ni 0$ and $V \ni 1$. Nonempty open sets in $\mathbb{R}^{zar}$ are cofinite sets of the form $U = \mathbb{R} \setminus \{x_1, \dots, x_m\}$ and $V = \mathbb{R} \setminus \{y_1, \dots, y_n\}$. Since the cardinality of $\mathbb{R}$ is infinite, we can pick some $c \in \mathbb{R} \setminus \{x_1, \dots, x_m, y_1, \dots, y_n\}$ so that $c \in U \cap V$. Hence any two open sets around 0 and 1 intersect, and $\mathbb{R}^{Zar}$ is not Hausdorff.

(b) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $\mathbb{R}^{Zar}$. Since $0 \in \mathbb{R}^{Zar}$, there exists $i_0 \in I$ such that $0 \in U_{i_0}$. Open sets are the complement of finitely many points, so we write $U_{i_0} = \mathbb{R} \setminus \{a_1, \dots, a_n\}$ for some $n \in \mathbb{N}$. Then since $\mathcal{U}$ is an open cover, there exist $U_{i_k} \in \mathcal{U}$ such that $a_k \in U_{i_k}$ for all $k = 1, \dots, n$. Now $\{U_{i_0}, U_{i_1}, \dots, U_{i_n}\}$ is a finite subcover of $\mathcal{U}$, so $\mathbb{R}^{Zar}$ is compact.

(c) A subset $Z \subset \mathbb{R}^2$ is closed in the Zariski topology on $\mathbb{R}^2$ iff there exists a subset $T \subset \mathbb{R}[x, y]$ such that

$$Z = V(f) = \{(x, y) \in \mathbb{R}^2 \mid \forall f \in T, f(x, y) = 0\}.$$

Note that this already includes $\mathbb{R}^2 = V(\{0\})$ and $\emptyset = V(\{1\})$.

(d) The product topology on $\mathbb{R}^{Zar} \times \mathbb{R}^{Zar}$ is not the same as the Zariski topology on $\mathbb{R}^2$. We know that Hausdorffness of $\mathbb{R}^{Zar}$ is equivalent to the diagonal $\{(x, x)\} \subset \mathbb{R}^{Zar} \times \mathbb{R}^{Zar}$ being closed. Then by (a), the diagonal is not closed, but $V(\{x - y\})$ is closed in $\mathbb{R}^2$ with the Zariski topology.

More explicitly, recall that the product topology on $\mathbb{R}^{Zar} \times \mathbb{R}^{Zar}$ is the coarsest topology such that the projection maps $\pi_i : \mathbb{R}^{Zar} \times \mathbb{R}^{Zar} \to \mathbb{R}^{Zar}$ for $i = 1, 2$ are continuous, so the product topology has a basis of open sets given by sets of the form $U \times V$ for two open sets $U, V \subset \mathbb{R}^{Zar}$. Then by the argument in (a), there exists $x \in \mathbb{R}$ such that $(x, x) \in U \times V$. Hence the complement of $V(\{x - y\}) = \{(x, x) \mid x \in \mathbb{R}\}$ is not open in the product topology on $\mathbb{R}^{Zar} \times \mathbb{R}^{Zar}$.

(e) If U_1, U_2 are Zariski open sets in $\mathbb{R}^2$, then $U_1 = \mathbb{R}^2 \setminus V(T_1)$ and $U_2 = \mathbb{R}^2 \setminus V(T_2)$ for subsets of polynomials $T_1, T_2 \in \mathbb{R}[x, y]$. Observe that we may assume that $0 \in T_i \neq \emptyset$ as it doesn't change the vanishing set $V(T_i)$. Then note that an open set $U = \mathbb{R}^2 \setminus V(T)$ is non empty iff $T \neq \{0\}$, so we know that neither of T_1, T_2 are equal to $\{0\}$. Then

$$\begin{aligned} U_1 \cap U_2 &= \left(\mathbb{R}^2 \setminus V(T_1)\right) \cap \left(\mathbb{R}^2 \setminus V(T_1)\right) \\ &= \mathbb{R}^2 \setminus (V(T_1) \cup V(T_2)) \\ &= \mathbb{R}^2 \setminus (V(T_1 T_2)), \end{aligned}$$

where $T_1 T_2 = \{fg \mid f \in T_1, g \in T_2\}$. Since the product of two nonzero polynomials is nonzero, the subset $T_1 T_2 \neq \{0\}$, and $U_1 \cap U_2$ is nonempty.

(f) The closure $\overline{I}$ of I in $\mathbb{R}^2$ with the Zariski topology is the smallest Zariski closed subset of $\mathbb{R}^2$ containing I. Note that $I \subset V(\{y\})$, and so $\overline{I} \subset V(\{y\})$. Now let $\overline{I} = V(T)$ for some $T \subset \mathbb{R}[x, y]$ so that $f((x, 0)) = 0$ for all $f \in T$ and $0 < x < 1$. For any such $f \in T$, consider polynomial $f(x, 0) \in \mathbb{R}[x]$ as a polynomial in one variable. This has infinitely many roots, and hence must be equal to the zero polynomial so $f(x, 0) = 0$ for all $x \in \mathbb{R}$, $f \in T$. Thus, the polynomial y divides every $f \in T \subset \mathbb{R}[x, y]$ and $V(y) \subseteq V(T)$. This implies $V(y) = \overline{I}$.

(g) Consider the preimage of the closed set $\{0\}$ under the addition map. This is the set $\{(x, y) \in \mathbb{R}^{Zar} \times \mathbb{R}^{Zar} \mid x + y = 0\} = \{(x, -x) \mid x \in \mathbb{R}\}$. We show, by an argument similar to part (d), that this set is not closed. Suppose U and V are open sets of $\mathbb{R}^{Zar}$ and note that $-V = \{-x \mid x \in V\}$ is then also an open set of $\mathbb{R}^{Zar}$. By the same arugment as in part (a), there exists $x \in \mathbb{R}$ such that $(x, x) \in U \cap (-V)$, and so $(x, -x) \in U \cap V$. Then the complement of $\{(x, -x) \mid x \in \mathbb{R}\}$ is not open, and hence $(+)^{-1}(0)$ is not closed in the product topology on $\mathbb{R}^{Zar} \times \mathbb{R}^{Zar}$.

There exists a closed set of the target space whose preimage is not closed, which shows the addition map $+ : \mathbb{R}^{Zar} \times \mathbb{R}^{Zar} \to \mathbb{R}^{Zar}$ is not continuous.

(h) We need to check that the preimage of a closed set $Z \subset \mathbb{R}^{Zar}$ under the addition map $+ : \mathbb{R}^2 \to \mathbb{R}^{Zar}$ is closed. If Z is the empty set, its preimage is empty, which is closed. If Z is the whole space $\mathbb{R}$, then its preimage is the whole space $\mathbb{R}^2$, which is closed. If $Z = \{a_1, \ldots, a_n\}$ is a finite set of real numbers, then its preimage

$$\begin{aligned} (+)^{-1}(Z) &= \{(x, y) \in \mathbb{R}^2 \mid x + y = a_i \text{ for some } i = 1, \ldots, n\} \\ &= \bigcup_{i=1}^{n} \{(x, y) \in \mathbb{R}^2 \mid x + y = a_i\} \\ &= \bigcup_{i=1}^{n} V(x + y - a_i) \end{aligned}$$

is a finite union of closed sets, and hence closed.

Q2 [50 marks]

(a) As we identify $M_{n,n}$ with the metric space $\mathbb{R}^{n^2}$, we know that $M_{n,n}$ is Hausdorff. As an open subset, $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is then Hausdorff with the subset topology. The product of invertible matrices is once again invertible, and the inverse of a matrix is invertible, so $\mathrm{GL}_n(\mathbb{R})$ is closed under matrix multiplication and inversion.

The product of two matrices AB has (i,j)-th entry $\sum k = 1^n a_{ik} b_{kj}$, hence product is a continuous maps as each entry is are determined by polynomials of the coordinate functions.

The inverse of a matrix A has (i,j)-th entry $\frac{C_{ji}}{\det(A)}$ where C_{ji} is the (j,i)-th cofactor, i.e. the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by removing the j-th row and i-th column. This is a continuous map as each entry is a rational function in the coordinate functions where the denominator doesn't vanish on the domain.

(b) The set $\mathbb{R} \setminus \{0\} = \mathbb{R}_{<0} \cup \mathbb{R}_{>0}$ is disconnected as union of disjoint, nonempty, open subsets. Then $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is the preimage of a disconnected set under a continuous map, and is hence disconnected as it can be covered by $\det^{-1}(\mathbb{R}_{<0})$ and $\det^{-1}(\mathbb{R}_{>0})$.

(c) Note that if $(x,y,z) \in V = \{(x,y,z) \mid xy - z > 0)\}$, we cannot simultaneously have $x = z = 0$ as otherwise $0 > 0$. Hence we can cover V by two open sets $V \cap \{x \neq 0\}$ and $V \cap \{z \neq 0\}$. We can refine this open cover to a cover of three sets which we denote

$$V_{x>0} = V \cap \{x > 0\}, V_{x<0} = V \cap \{x < 0\}, \text{ and } V_{z<0} = V \cap \{z < 0\}.$$

Observe that $V_{x>0}$ is homeomorphic to $\mathbb{R} \times (\mathbb{R}_{>0})^2$ which is connected as a product of connected sets. A homeomorphism given explicitly by $(x,y,z) \mapsto (y; x, xy - z)$ and inverse $(s; t, u) \mapsto (t, s, st - u)$. Similarly, $V_{x<0}$ is homeomorphic to $\mathbb{R} \times (\mathbb{R}_{>0})^2$ with a homeomorphism given explicitly by $(x,y,z) \mapsto (y; -x, xy - z)$ and inverse $(s; t, u) \mapsto (-t, s, st - u)$. Finally, $V_{z<0}$ is homeomorphic to $\mathbb{R} \times (\mathbb{R}_{>0})^2$ with a homeomorphism given explicitly by $(x,y,z) \mapsto (y; -x, xy - z)$ and inverse $(s; t, u) \mapsto (-t, s, st - u)$.

Then the intersections$(1,1,-1) \in V_{x>0} \cap V_{z<0}$ and $(-1,-1,-1) \in V_{x>0} \cap V_{z<0}$ are nonempty, and so $V_{x>0} \cup V_{z<0}$ and $V_{x<0} \cap V_{z<0}$ are connected as the union of connected open sets with nonempty intersection. These have nonempty intersection, namely $V_{z<0}$, and hence their union V is connected.

[**Alternative solution using path connectedness.**] Alternatively, we show that V is path connected by constructing a path from any $(x,y,z) \in V$ to the point $(0,0,-1) \in V$.

First suppose that $z > 0$. Then for any $z' < z$, we have $xy - z' > xy - z > 0$ and hence $(x,y,z') \in V$. Then

$$\gamma : [0,1] \to V, \quad t \mapsto (x, y, (1-t)z - t)$$

is a path within V connecting (x,y,z) to $(x,y,-1)$ inside V.

Next, if $z < 0$, we have that for any $t \in [0,1]$, we have $txy - z > tz - z > (t-1)z \geq 0$. So $t \mapsto (tx, y, z)$ is a path connecting (x, y, z) to $(0, y, z)$ inside V. Similarly, $t \mapsto (0, ty, z)$ is a path connecting $(0, y, z)$ to $(0, 0, z)$.

At this stage, if $z = -1$ we are done, but if $z < 0$ is not equal to -1, we can use one final path $t \mapsto (0, 0, (1-t)z-t)$ to connect $(0, 0, z)$ with $(0, 0, -1)$. As the third coordinate is negative throughout, we check $0 \times 0 - ((1-t)z - t) = t + (1-t)z > 0$ to see that the path stays in V. Hence any point is path connected to $(0, 0, -1)$, and V is path connected.

(d) A connected component is a maximal (by inclusion) connected subset. Since the continuous image of a connected component is connected, if U is a connected component of $GL_2(k)$ we must have $\det(U) \subset \mathbb{R}_{>0}$ or $\det(U) \subset \mathbb{R}_{<0}$, so $U \subseteq \mathrm{GL}_2^-(\mathbb{R})$ or $\mathrm{GL}_2^+(\mathbb{R})$. It remains to show that $\mathrm{GL}_2^-(\mathbb{R})$ and $\mathrm{GL}_2^+(\mathbb{R})$ are connected. Note that these two spaces are homeomorphic under a map which negates one column/row. Explicitly this can given by left multiplication by a $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and multiplying again gives its continuous inverse, so this is a homeomorphism. Hence it suffices to prove connectedness of one of these sets, and we will do $\mathrm{GL}_2^+(\mathbb{R})$. Note that we can cover

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc > 0 \right\}$$

by four open sets $U_b^+ = \{b > 0\}$, $U_b^- = \{b < 0\}$, $U_d^+ = \{d > 0\}$, and $U_d^- = \{d < 0\}$. From part (c), we have that $V = \{(x, y, z) \in \mathbb{R}^3 \mid xy - z > 0\}$ is connected and hence $V \times \mathbb{R}_{>0}$ is connected as the product of two connected spaces. Observe that,

$$\phi_b^+ : V \times \mathbb{R}_{>0} \to U_b^+, \quad (x, y, z; t) \mapsto \begin{pmatrix} x & t \\ z & yt \end{pmatrix}$$

is a continuous surjective map onto U. Indeed, $xyt - zt = t(xy - z) > 0$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is mapped onto by $(a, d/t, c; b)$. So U_b^+ is connected as the image of a connected set under a continuous map. Similarly, we have continuous surjections onto U_b^-, U_d^+, and U_d^- by the maps

$$\begin{aligned} \phi_b^- &: V \times \mathbb{R}_{>0} \to U_b^-, \quad (x, y, z; t) \mapsto \begin{pmatrix} x & -t \\ -z & ty \end{pmatrix}, \\ \phi_d^+ &: V \times \mathbb{R}_{>0} \to U_d^+, \quad (x, y, z; t) \mapsto \begin{pmatrix} -z & -xt \\ y & t \end{pmatrix} \\ \phi_d^- &: V \times \mathbb{R}_{>0} \to U_d^-, \quad (x, y, z; t) \mapsto \begin{pmatrix} z & -xt \\ y & -t \end{pmatrix} \end{aligned}$$

Then notice that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in U_b^+ \cap U_d^+ \neq \emptyset$, so $U_b^+ \cup U_d^+$ is connected. Similarly, $\left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right) \in U_b^- \cap U_d^- \neq \emptyset$, so $U_b^{+-} \cup U_d^-$ is connected.

Finally, $\mathrm{GL}_2^+(\mathbb{R})$ is covered by both $(U_b^+ \cap U_d^+)$ and $(U_b^- \cap U_d^-)$, which have nonempty intersection (for example $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$), and thus we conclude that $\mathrm{GL}_2^+(\mathbb{R})$ is connected.

(e) Consider the matrices $\mathbf{0} = \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 1&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right)$, and denote their equivalence classes in $M_{2,4}/\mathrm{GL}_2(\mathbb{R})$ by $[\mathbf{0}]$ and $[A]$. Let $p : M_{2,4} \to M_{2,4}/\mathrm{GL}_2(\mathbb{R})$ denote the quotient map. Let U be an open set of $[\mathbf{0}]$, then $p^{-1}(U)$ is an open set around $\mathbf{0}$. So there exists an $\varepsilon > 0$ such that the open ball $B(\mathbf{0}, \varepsilon) \subset U$. Now note that $\begin{pmatrix} 1/t & 0 \\ 0 & 1/t \end{pmatrix} A = \begin{pmatrix} 1/t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is a representative of $[A]$ which for $t > 1/\varepsilon$ will be in $B(\mathbf{0}, \varepsilon)$. Hence, any open set of $[\mathbf{0}]$ contains $[A]$, so these points cannot be separated by disjoint open sets and $M_{2,4}/\mathrm{GL}_2(\mathbb{R})$ is not Hausdorff.

(f) The space $X = M_{2,4}^1/GL_2(\mathbb{R})$ is neither open nor closed inside $M_{2,4}/\mathrm{GL}_2(\mathbb{R})$. It is not closed as in part (e) we showed that any open set of $\mathbf{0}$ intersects X, and yet $\mathbf{0}$ is of rank 0 so $\mathbf{0} \notin X$. Next, we show that X is not open in $M_{2,4}/GL_2(\mathbb{R})$. Again consider the point $[A] \in X$ where $A = \left(\begin{smallmatrix} 1&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right)$. Let $U \subset M_{2,4}/\mathrm{GL}_2(\mathbb{R})$ be an open set containing $[A]$. Then since U is open, its preimage $p^{-1}(U)$ is open in $M_{2,4}$. Let $\varepsilon > 0$ be such that $B(A, \varepsilon) \subset p^{-1}(U)$. Then observe that $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon/2 & 0 & 0 \end{pmatrix} \in B(A, \varepsilon)$, and so $[B] \in U$. But B is a matrix of rank 2, and so $[B] \notin X$. Hence X contains non-interior points, and is not open.

(g) We construct a map $\varphi : M_{2,4}^1/\mathrm{GL}_2(\mathbb{R}) \to \mathbb{RP}^3$ recalling that for 3 dimensional projective space we have homeomorphisms

$$\mathbb{RP}^3 \cong \left(\mathbb{R}^4 \setminus \{0\}\right)/(x \sim \lambda x) \cong S^3/\{\pm 1\}.$$

Let $A \in M_{2,4}^1$ be of rank 1, then the row vectors $w_1, w_2 \in \mathbb{R}^4$ are proportionate, and their span $\langle w_1, w_2 \rangle$ is a one dimensional subspace of $\mathbb{R}^4$, i,e, a point in $\mathbb{RP}^3$.

Note that $\overline{\varphi}$ is invariant under the $\mathrm{GL}_2(\mathbb{R})$ action on $M_{2,4}^1$ as row operations don't change the span of the rows. Hence, $\overline{\varphi}$ induces a well defined function $\varphi : X \to \mathbb{RP}^3$ such that $\overline{\varphi}(A) = \varphi(p(A))$.

Let U be an open set of $\mathbb{RP}^3$. Up to taking a smaller open set, we may assume that one of the coordinates doesn't vanish on U. Without loss of generality, we then have

$$U \subset \left\{[x_1{:}x_2{:}x_3{:}x_4] \in \mathbb{RP}^3 \mid x_4 \neq 0\right\} \cong \mathbb{R}^4 \cap \{x_4 = 1\} \cong \mathbb{R}^3.$$

Then U is homeomorphic to some open subset $\overline{U}$ of $\mathbb{R}^3$ under the identification above.

Now consider the preimage U under $\overline{\varphi}$. This is the set matrices in $M_{2,4}^1$ whose row span intersects with $\overline{U} \subset \mathbb{R}^4 \cap \{x_4 = 1\}$, i.e.

$$\overline{\varphi}^{-1}(U) = \left\{ \begin{pmatrix} t_1 v_1 & t_1 v_2 & t_1 v_3 & t_1 \\ t_2 v_1 & t_2 v_2 & t_2 v_3 & t_2 \end{pmatrix} \in M_{2,4}^1 \;\middle|\; \begin{array}{l} (t_1, t_2) \neq (0,0), \\ (v_1, v_2, v_3, 1) \in \overline{U} \end{array} \right\}.$$

To see this is an open set of $M_{2,4}^1$, fix

$$A_0 = \begin{pmatrix} s_0w_1 & s_0w_2 & s_0w_3 & s_0 \\ t_0w_1 & t_0w_2 & t_0w_3 & t_0 \end{pmatrix} \in \overline{\varphi}^{-1}(U).$$

Here, at least one of s_0 or t_0 is nonzero. Let's assume WLOG that $s_0 \neq 0$, the case of $t_0 \neq 0$ is identical with swapped letters. Note that as $\overline{U} \subset \mathbb{R}^3$ is open, there is an $\varepsilon > 0$ such that $(v_1, v_2, v_3, 1) \in \overline{U}$ if $|v_i - w_i| < \varepsilon$ for each $i = 1, 2, 3$.

Now take some $A' = \begin{pmatrix} sv_1 & sv_2 & sv_3 & s \\ tv_1 & tv_2 & tv_3 & t \end{pmatrix}$ which differs from A_0 by at most some $\delta > 0$ in each entry, so we have bounds $|sv_i - s_0w_i| < \delta$ and $|tv_i - t_0w_i| < \delta$ for $i = 1, 2, 3$, and $|s - s_0| < \delta$, $|t - t_0| < \delta$. Note that the reverse triangle inequality gives that $\big||s| - |s_0|\big| < \delta$, and so for $\delta < \frac{|s_0|}{2}$ we have $s \neq 0$. Then for each $i = 1, 2, 3$ we have that

$$\begin{aligned} |v_i - w_i| &= |v_i - \tfrac{s_0}{s}w_i + \tfrac{s_0}{s}w_i - w_i|, \\ &\leq \tfrac{1}{|s|}\left(|sv_i - s_0w_i| + |s_0w_i - sw_i|\right), \\ &\leq \tfrac{1}{|s_0|-\delta}\left(\delta + \delta|w_i|\right) \leq \tfrac{\delta}{|s_0|-\delta}\left(1 + |w_i|\right) < \varepsilon \end{aligned}$$

where the last inequality can be attained for sufficiently small δ. One can check that we need $\delta < \frac{\varepsilon|s_0|}{\varepsilon+1+|w_i|}$ for each i.

Thus, the map $\overline{\varphi}$ is continuous. Further, as $\overline{\varphi} = \varphi \circ p$, we have $\overline{\varphi}^{-1}(U) = p^{-1}(\varphi^{-1}(U))$. Hence, $\varphi^{-1}(U)$ is open by the definition of the quotient topology and we conclude that $\varphi : X \to \mathbb{RP}^3$ is continuous.

Next, suppose $\overline{\varphi}(A) = [w] = \overline{\varphi}(A')$ for two matrices $A \in M_{2,4}^1$, then picking a representative $w \in S^3$, we can write $A = (t_1w, t_2w)^T$ and $A' = (t_1'w, t_2'w)^T$ for some constants $t_1, t_2, t_1', t_2' \in \mathbb{R}^4$. As not both of t_1, t_2 can be zero, if $t_1 = 0$, we can left multiply A by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ to get $A = (t_2w, t_1w)^T$ and swap t_1 and t_2. Now assuming WLOG that $t_1 \neq 0$, we have $A' = \left(\begin{smallmatrix} t_1'/t_1 & 0 \\ (t_2'-t_2)/t_1 & 1 \end{smallmatrix}\right) A$. So the map $\varphi : X \to \mathbb{RP}^3$ is injective.

Also, $\varphi : X \to \mathbb{RP}^3$ is surjective as for any point $[w] \in \mathbb{RP}^3$, it is mapped onto by the matrix $(w, 0)^T \in M_{2,4}^1$ under the map $\overline{\varphi}$ and hence mapped onto by its equivalence class under φ. Thus, we have a continuous bijection between X and $\mathbb{RP}^3$. It remains to see that this inverse $[w] \mapsto \left[(w, 0)^T\right]$ is continuous.

Let $\overline{\psi} : S^3 \to M_{2,4} \cong \mathbb{R}^8$ be given by $w \mapsto (w, 0)^T$. This is a continuous map as each entry is given by coordinate functions (alternatively, it's a composition of the embeddings $S^3 \hookrightarrow \mathbb{R}^4 \hookrightarrow \mathbb{R}^8$). As $w \neq 0$ for $w \in S^3$ and only one row is nonzero $(w, 0)^T$, the image is contained inside $M_{2,4}^1$. So we have $\overline{\psi} : S^3 \to M_{2,4}^1$. Observe that the matrices $(w, 0)^T$ and $(w', 0)^T$ are row transformation equivalent if and only if $w = \pm w'$, i.e. either the same point or antipodal points on the sphere S^3. Therefore, we composing with p, we get a continuous map

$$\psi = p \circ \overline{\psi} : S^3 \to X, \quad w \mapsto \left[(w, 0)^T\right],$$

which maps w and $-w$ to the same point in X. Then by the quotient property for $\mathbb{RP}^3 \cong S^3/\{\pm 1\}$, we get a well defined continuous map

$$\overline{\overline{\psi}} : \mathbb{RP}^3 \to X, \quad [w] \mapsto \left[(w, 0)^T\right].$$