20240913 MATH3541 NOTE 4[1]

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Contents

1 Introduction

Continuity preserves topology under preimage. If we want to preserve image as well, then homeomorphism should be introduced. While identifying two spaces with a homeomorphism is difficult in general, we lower our requirement and seek local identification with analysis techniques. Open and closed functions appear in the middle way as we want to verify certain sets are open or closed.

How to distinguish two topological spaces, i.e., disprove the existence of a homeomorphism between them? Well, we investigate certain properties that are unchanged under it. These properties are called topological invariants. Hausdorffness is the simplest topological invariant. Upon its importance, we ask whether we can construct new Hausdorff space from old one, and that's the end of today's lecture.

2 Identification of Topological Spaces

2.1 Continuity

Definition 2.1. (**Continuous Function**)
Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a function.
If $\forall V \in \mathcal{P}(Y), V \in \mathcal{O}_Y \implies \sigma^{-1}(V) \in \mathcal{O}_X$, then σ is continuous.

Definition 2.2. (**Continuous Function**)
Let $(X, \mathcal{C}_X), (Y, \mathcal{C}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a function.
If $\forall V \in \mathcal{P}(Y), V \in \mathcal{C}_Y \implies \sigma^{-1}(V) \in \mathcal{C}_X$, then σ is continuous.

Remark: *As* $\forall V \in \mathcal{P}(Y), \sigma^{-1}(V^c) = \sigma^{-1}(V)^c$, *the two definitions above are equivalent.*

Definition 2.3. (**Initial Topology**)
Let X be a set, $(Y, \mathcal{O}_Y)$ be a topological space, and $\sigma : X \to Y$ be a function.
Define $\mathcal{O}_X = \{\sigma^{-1}(V)\}_{V \in \mathcal{O}_Y}$ as the initial topology of $(Y, \mathcal{O}_Y)$ on X via σ.

Remark: *The initial topology of* $(Y, \mathcal{O}_Y)$ *on* X *via* σ *is the coarsest topology on* X *such that* σ *is continuous.*

Definition 2.4. (**Final Topology**)
Let $(X, \mathcal{O}_X)$ be a topological space, Y be a set, and $\sigma : X \to Y$ be a function.
Define $\mathcal{O}_Y = \{V\}_{\sigma^{-1}(V) \in \mathcal{O}_X}$ as the final topology of $(X, \mathcal{O}_X)$ on Y via σ.

Remark: *The final topology of* $(X, \mathcal{O}_X)$ *on* Y *via* σ *is the finest topology on* Y *such that* σ *is continuous.*

Definition 2.5. (**Subspace Topology**)
Let $(X, \mathcal{O}_X)$ be a topological space, and X' be a subset of X.
Define the subspace topology of $(X, \mathcal{O}_X)$ on X'
as the initial topology of $(X, \mathcal{O}_X)$ on X' via $\pi : X' \to X, \pi(x) = x$.

Remark: *The subspace topology of $(X, \mathcal{O}_X)$ on X' is the coarsest topology on X' such that π is continuous, and since $\forall V \in \mathcal{O}_X, \pi^{-1}(V) = V \cap X'$, we can explicitly construct this subspace topology by $\mathcal{O}_{X'} = \{V \cap X'\}_{V \in \mathcal{O}_X}$.*

Definition 2.6. (**Product Space Topology**)
Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ be an indexed family of topological spaces.
Define the product space topology of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ on $X = \prod_{\lambda \in I} X_\lambda$
as the topology generated by the subbasis $\mathcal{B}_X$, which is the union of each initial topology of $(X_\lambda, \mathcal{O}_{X_\lambda})$ on X via $\pi_\lambda : X \to X_\lambda, \pi_\lambda(x) = x(\lambda)$.

Remark: *The product space topology of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ on X is the coarsest topology on X such that each π_λ is continuous. To achieve this, we shall not assume that a box with infinitely many restrictions $\prod_{\lambda \in I} V_\lambda$ is open. Hence, product space topology is coarser than box topology. For all $V \in \mathcal{O}_X$, V is a union of blocks, where each block is in the form $\bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(V_{\lambda_k})$. This suggests we can reduce our open set V to this simplest form without loss of generality.*

Definition 2.7. (**Coproduct Space Topology**)
Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ be an indexed family of topological spaces.
Define the coproduct space topology of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ on $X = \coprod_{\lambda \in I} X_\lambda$
as the intersection of each final topology of $(X_\lambda, \mathcal{O}_{X_\lambda})$ on X via $\pi_\lambda : X_\lambda \to X, \pi_\lambda(x) = (x, \lambda)$.

Remark: *The coproduct space topology of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ on X is the finest topology such that each π_λ is continuous. This is like "arranging an indexed family of spaces in a sequence, and knit these fibres into a larger topological space".*

Definition 2.8. (**Quotient Space Topology**)
Let $(X, \mathcal{O}_X)$ be a topological space, and $\sim : X \to X$ be an equivalence relation on X. Define the quotient space topology of $(X, \mathcal{O}_X)$ on $X/\sim$
as the final topology of $(X, \mathcal{O}_X)$ on $X/\sim$ via $\pi : X \to X/\sim, \pi(x) = [x]_\sim$.

Remark: *The quotient space topology of $(X, \mathcal{O}_X)$ on $X/\sim$ is the finest topology on $X/\sim$ such that π is continuous, and since $\forall V \in \mathcal{O}_{X/\sim}, \pi^{-1}(V) = \bigcup_{[x]_\sim \in V} [x]_\sim$, we can explicitly construct this quotient space topology by $\mathcal{O}_{X/\sim} = \{V : \bigcup_{[x]_\sim \in V} [x]_\sim \in \mathcal{O}_X\}$.*

Proposition 2.9. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z)$ be three topological spaces, and $\sigma : X \to Y, \tau : Y \to Z$ be two functions.
σ, τ are continuous implies $\tau \circ \sigma$ is continuous.

Proof. Assume that σ, τ are continuous.

$$\forall W \in \mathcal{P}(Z), W \in \mathcal{O}_Z \implies \tau^{-1}(W) \in \mathcal{O}_Y \implies (\tau \circ \sigma)^{-1}(W) = \sigma^{-1}(\tau^{-1}(W)) \in \mathcal{O}_X$$

Hence, $\tau \circ \sigma$ is continuous. Quod. Erat. Demonstrandum. □

Proposition 2.10. Let $(X, \mathcal{O}_X)$ be a topological space. id_X is continuous.

Proof.

$$\forall V \in \mathcal{P}(X), V \in \mathcal{O}_X \implies id_X^{-1}(V) = V \in \mathcal{O}_X$$

Hence, id_X is continuous. Quod. Erat. Demonstrandum. □

Proposition 2.11. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, $\mathcal{B}_Y$ be a basis of $(Y, \mathcal{O}_Y)$, and $\sigma : X \to Y$ be a function.
σ is continuous if and only if $\forall V \in \mathcal{P}(Y), V \in \mathcal{B}_Y \implies \sigma^{-1}(V) \in \mathcal{O}_X$.

Proof. We may divide our proof into two parts.
"if" direction: Assume that $\forall V \in \mathcal{P}(Y), V \in \mathcal{B}_Y \implies \sigma^{-1}(V) \in \mathcal{O}_X$.
For all $V \in \mathcal{P}(Y)$, $V \in \mathcal{O}_Y$ implies $V = \bigcup_{\lambda \in I} V_\lambda$, where $(V_\lambda)_{\lambda \in I}$ is in $\mathcal{B}_Y$.
Hence, $\sigma^{-1}(V) = \sigma^{-1}\left(\bigcup_{\lambda \in I} V_\lambda\right) = \bigcup_{\lambda \in I} \sigma^{-1}(V_\lambda) \in \mathcal{O}_X$.
"only if" direction: Assume that σ is continuous.
For all $V \in \mathcal{P}(Y)$, $V \in \mathcal{B}_Y$ implies $V \in \mathcal{O}_Y$, so $\sigma^{-1}(V) \in \mathcal{O}_X$.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. □

Proposition 2.12. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, $\mathcal{B}_Y$ be a subbasis of $(Y, \mathcal{O}_Y)$, and $\sigma : X \to Y$ be a function.
σ is continuous if and only if $\forall V \in \mathcal{P}(Y), V \in \mathcal{B}_Y \implies \sigma^{-1}(V) \in \mathcal{O}_X$.

Proof. We may divide our proof into two parts.
"if" direction: Assume that $\forall V \in \mathcal{P}(Y), V \in \mathcal{B}_Y \implies \sigma^{-1}(V) \in \mathcal{O}_X$.
For all $V \in \mathcal{P}(Y)$, V is in basis implies $V = \bigcap_{k=1}^{m} V_k$, where $(V_k)_{k=1}^{m}$ is in $\mathcal{B}_Y$.
Hence $\sigma^{-1}(V) = \sigma^{-1}\left(\bigcap_{k=1}^{m} V_k\right) = \bigcap_{k=1}^{m} \sigma^{-1}(V_k) \in \mathcal{O}_X$.
"only if" direction: Assume that σ is continuous.
For all $V \in \mathcal{P}(Y)$, $V \in \mathcal{B}_Y$ implies V is in basis, so $\sigma^{-1}(V) \in \mathcal{O}_X$.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. □

Proposition 2.13. Let $(X, d_X), (Y, d_Y)$ be two metric spaces, and $\sigma : X \to Y$ be a function. σ is continuous in metric sense if and only if σ is continuous in topological sense if the topologies are inherited from metric spaces.

Proof. We may divide our proof into two parts.
Part 1: Assume that σ is continuous in metric sense.
For all $V \in \mathcal{P}(Y)$, assume that $V \in \mathcal{O}_Y$, so for all $y \in V$, there exists $s > 0$, such that $B(y, s) \subseteq V$. We would like to show $\sigma^{-1}(V) \in \mathcal{O}_X$.
For all $x_* \in \sigma^{-1}(V)$, $f(x_*) \in V$. Since σ is continuous in metric sense, for certain $s > 0$, there exists $r > 0$, such that $d_Y(\sigma(x), \sigma(x_*)) < s$ whenever $d_X(x, x_*) < r$. This implies the existence of $r > 0$, such that $B(x_*, r) \subseteq \sigma^{-1}(V)$, thus $\sigma^{-1}(V) \in \mathcal{O}_X$.
Part 2: Assume that σ is continuous in topological sense.
For all $x_* \in X$, for all $s > 0$, we would like to show the existence of $r > 0$, such that $d_Y(\sigma(x), \sigma(x_*)) < s$ whenever $d_X(x, x_*) < r$.
For $B(\sigma(x_*), s) \in \mathcal{O}_Y$, $x_* \in \sigma^{-1}(B(\sigma(x_*), s)) \in \mathcal{O}_X$, so there exists $r > 0$, such that $B(x_*, r) \subseteq \sigma^{-1}(B(\sigma(x_*), s))$. This implies the existence of $r > 0$, such that $d_Y(\sigma(x), \sigma(x_*)) < s$ whenever $d_X(x, x_*)$.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. ☐

2.2 Homeomorphism

Definition 2.14. (**Homeomorphism**)
Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a function.
If σ is bijective, and $\forall U \in \mathcal{P}(X), U \in \mathcal{O}_X \iff \sigma(U) \in \mathcal{O}_Y$,
then σ is a homeomorphism, and $(X, \mathcal{O}_X)$ is homeomorphic to $(Y, \mathcal{O}_Y)$.

Remark: *σ is a homeomorphism if and only if it is bijective, and σ, σ^{-1} are continuous.*

Proposition 2.15. The homeomorphic relation $\cong$ on the set of all topological spaces is an equivalence relation.

Proof. We may divide our proof into three parts.
If unspecified, X means a topological space and $\sigma : X \to Y$ means a homeomorphism.
Part 1: In this part, we prove that $\cong$ is reflexive.
For all topological space $(X, \mathcal{O}_X)$:

$$\exists id_X : X \to X \implies X \cong X$$

Part 2: In this part, we prove that $\cong$ is symmetric.
For all topological spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$:

$$X \cong Y \implies \exists \sigma : X \to Y \implies \exists \sigma^{-1} : Y \to X \implies Y \cong X$$

Part 3: In this part, we prove that $\cong$ is transitive.
For all topological spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z)$:

$$X \cong Y \text{ and } Y \cong Z \implies \exists \sigma : X \to Y, \mu : Y \to Z \implies \exists \tau \circ \sigma : X \to Z \implies X \cong Z$$

Combine the three parts above, we've proven that $\cong$ is an equivalence relation.
Quod. Erat. Demonstrandum. $\square$

Proposition 2.16. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, $[0, 2\pi)$ as a subspace of $\mathbb{R}$, $\mathbb{R}^2$ as the product space of $\mathbb{R}, \mathbb{R}$, and $\mathbb{S} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ as a subspace of $\mathbb{R}^2$, then the function $\sigma : [0, 2\pi) \to \mathbb{S}, \sigma(\theta) = (\cos\theta, \sin\theta)$ is a continuous bijection, but it is not a homeomorphism.

Proof. Notice that the following two functions are continuous:

$$\begin{aligned} \pi_1 \circ \sigma : \quad & [0, 2\pi) \to \mathbb{R}, \quad \pi_1 \circ \sigma(\theta) = \cos\theta \\ \pi_2 \circ \sigma : \quad & [0, 2\pi) \to \mathbb{R}, \quad \pi_2 \circ \sigma(\theta) = \sin\theta \end{aligned}$$

Hence, the injective function $\sigma : [0, 2\pi) \to \mathbb{R}^2$ is continuous, so is its bijective restriction $\sigma : [0, 2\pi) \to \sigma([0, 2\pi)) = \mathbb{S}^2$.
Since the open set $[0, \pi) = B_\pi(0)$ has a nonopen image $\sigma([0, \pi)) = \{(1, 0)\} \cup B_{\sqrt{2}}(0, 1)$, σ is not a homeomorphism. Quod. Erat. Demonstrandum. $\square$

2.3 Open and Closed Functions

Definition 2.17. (**Open Function**)
Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a function.
If $\forall U \in \mathcal{P}(X), U \in \mathcal{O}_X \implies \sigma(U) \in \mathcal{O}_Y$, then σ is open.

Definition 2.18. (**Closed Function**)
Let $(X, \mathcal{C}_X), (Y, \mathcal{C}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a function.
If $\forall U \in \mathcal{P}(X), U \in \mathcal{C}_X \implies \sigma(U) \in \mathcal{C}_Y$, then σ is closed.

Remark: *Ensuring that a function is open is far more easy than ensuring that a function is closed, we will see the reason in our further discussion.*

Proposition 2.19. Let $(X, \mathcal{O}_X)$ be a topological space, and $(X', \mathcal{O}_{X'})$ be a subspace of $(X, \mathcal{O}_X)$. If $X' \in \mathcal{O}_X$, then $\pi : X' \to X, \pi(x) = x$ is open.

Proof. For all $U' \in \mathcal{O}_{X'}$, there exists $U \in \mathcal{O}_X$, such that $U' = X' \cap U$, so $\pi(U') = U' = X' \cap U \in \mathcal{O}_X$, which implies π is open. Quod. Erat. Demonstrandum. $\square$

Proposition 2.20. Let $(X, \mathcal{C}_X)$ be a topological space, and $(X', \mathcal{C}_{X'})$ be a subspace of $(X, \mathcal{C}_X)$. If $X' \in \mathcal{C}_X$, then $\pi : X' \to X, \pi(x) = x$ is closed.

Proof. For all $U' \in \mathcal{C}_{X'}$, there exists $U \in \mathcal{C}_X$, such that $U' = X' \cap U$, so $\pi(U') = U' = X' \cap U \in \mathcal{C}_X$, which implies π is closed. Quod. Erat. Demonstrandum. □

Proposition 2.21. Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ be an indexed family of topological spaces, and $(X, \mathcal{O}_X)$ be the product space of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$.
Each $\pi_\lambda : X \to X_\lambda, \pi_\lambda(x) = x(\lambda)$ is open.

Proof. For all $U \in \mathcal{O}_X$, without loss of generality, assume that $U = \bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(U_{\lambda_k})$, where each $U_{\lambda_k} \in \mathcal{O}_{X_{\lambda_k}}$ is nonempty. There are two cases to consider:
Case 1: If $\lambda = \lambda_k$ for some λ_k, then $\pi_\lambda(U) = \pi_\lambda(\pi_\lambda^{-1}(U_\lambda)) = U_\lambda \in \mathcal{O}_{X_\lambda}$;
Case 2: If $\lambda \neq \lambda_k$ for all λ_k, then $\pi_\lambda(U) = \pi_\lambda(X) = X_\lambda \in \mathcal{O}_{X_\lambda}$.
In both cases, $\pi_\lambda(U) \in \mathcal{O}_{X_\lambda}$, so π_λ is open. Quod. Erat. Demonstrandum. □

Proposition 2.22. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, and $\mathbb{R}^2$ as the product space of $\mathbb{R}, \mathbb{R}$, then $\pi_1 : \mathbb{R}^2 \to \mathbb{R}, \pi_1(x_1, x_2) = x_1$ sends the closed set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 1\}$ to the nonclosed set $\mathbb{R} \backslash \{0\}$.

Proposition 2.23. Let $((X_\lambda, \mathcal{O}_\lambda))_{\lambda \in I}$ be an indexed family of topological spaces, and $(X, \mathcal{O}_X)$ be the coproduct space of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$.
Each $\pi_\lambda : X_\lambda \to X, \pi_\lambda(x) = (x, \lambda)$ is open.

Proof. For all $U_\lambda \in \mathcal{O}_{X_\lambda}$, for all $\mu \in I$:
Case 1: If $\mu = \lambda$, then $\pi_\lambda^{-1}(\pi_\lambda(U_\lambda)) = U_\lambda \in \mathcal{O}_{X_\lambda}$;
Case 2: If $\mu \neq \lambda$, then $\pi_\mu^{-1}(\pi_\lambda(U_\lambda)) = \emptyset \in \mathcal{O}_{X_{mu}}$.
In both cases, $\pi_\mu^{-1}(\pi_\lambda(U)) \in \mathcal{O}_{X_\mu}$, so $\pi_\lambda(U_\lambda) \in \mathcal{O}_X$, which implies π_λ is open.
Quod. Erat. Demonstrandum. □

Proposition 2.24. Let $((X_\lambda, \mathcal{C}_\lambda))_{\lambda \in I}$ be an indexed family of topological spaces, and $(X, \mathcal{C}_X)$ be the coproduct space of $((X_\lambda, \mathcal{C}_{X_\lambda}))_{\lambda \in I}$.
Each $\pi_\lambda : X_\lambda \to X, \pi_\lambda(x) = (x, \lambda)$ is closed.

Proof. For all $U_\lambda \in \mathcal{C}_{X_\lambda}$, for all $\mu \in I$:
Case 1: If $\mu = \lambda$, then $\pi_\lambda^{-1}(\pi_\lambda(U_\lambda)) = U_\lambda \in \mathcal{C}_{X_\lambda}$;
Case 2: If $\mu \neq \lambda$, then $\pi_\mu^{-1}(\pi_\lambda(U_\lambda)) = \emptyset \in \mathcal{C}_{X_{mu}}$.
In both cases, $\pi_\mu^{-1}(\pi_\lambda(U)) \in \mathcal{C}_{X_\mu}$, so $\pi_\lambda(U_\lambda) \in \mathcal{C}_X$, which implies π_λ is closed.
Quod. Erat. Demonstrandum. □

Proposition 2.25. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, and define partition $\mathbb{R}/\sim=\{\{x\}\}_{x\in[0,1]^c}\cup\{[0,1]\}$, then $\pi:\mathbb{R}\to\mathbb{R}/\sim,\pi(x)=[x]_\sim$ sends the open set $(0,1)$ to the nonopen set $\{[0,1]\}$.

Proposition 2.26. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, and define partition $\mathbb{R}/\sim=\{\{x\}\}_{x\in(0,1)^c}\cup\{(0,1)\}$, then $\pi:\mathbb{R}\to\mathbb{R}/\sim,\pi(x)=[x]_\sim$ sends the closed set $\{1/2\}$ to the nonclosed set $\{(0,1)\}$.

2.4 Local Homeomorphism

Definition 2.27. (**Local Homeomorphism**)
Let $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ be two topological spaces, and $\sigma:X\to Y$ be a function. If σ is open, and for all x, there exists an open neighbour U of x, such that $f(U)$ is open in Y, and the restricted function $\sigma:U\to\sigma(U)$ is a homeomorphism, then σ is a local homeomorphism, and $(X,\mathcal{O}_X)$ is locally homeomorphic to $(Y,\mathcal{O}_Y)$.

Theorem 2.28. (**Banach Fixed-point Theorem**)
Let (X,d_X) be a complete metric space, and $T:X\to X$ be a $\lambda-$Lipschitz continuous function with $\lambda\in[0,1)$. There exists a unique $\xi\in X$, such that:

$$\forall x\in X,\lim_{n\to+\infty}T^n(x)=\xi$$

Proof. We may divide our proof into two parts.
We accept the conventions $\lambda^0=1$ and $T^0=id_X$, and define $x_n=T^n(x),y_n=T^n(y)$. Here, if a statement involves n, then it holds for all $n\geq 0$ if no clarification is made.
Step 1: For all $x\in X$, we prove that $(x_n)_{n=0}^{+\infty}$ converges.

$$\begin{aligned}
d_X(x_{n+2},x_{n+1})\leq\lambda d_X(x_{n+1},x_n)&\implies d_X(x_{n+1},x_n)\leq\lambda^n d_X(x_1,x_0)\\
&\implies d_X(x_n,x_m)\leq\frac{\lambda^{\min\{n,m\}}}{1-\lambda}d_X(x_1,x_0)\\
&\implies (x_n)_{n=0}^{+\infty}\text{ is Cauchy}\\
&\implies (x_n)_{n=0}^{+\infty}\text{ converges}
\end{aligned}$$

Step 2: For all $x,y\in X$, we prove that $(x_n)_{n=1}^{+\infty},(y_n)_{n=1}^{+\infty}$ have the same limit.

$$\begin{aligned}
d_X(x_{n+1},y_{n+1})\leq\lambda d_X(x_n,y_n)&\implies d_X(x_n,y_n)\leq\lambda^n d_X(x_n,y_n)\\
&\implies \lim_{n\to+\infty}d_X(x_n,y_n)=0\\
&\implies \lim_{n\to+\infty}x_n=\lim_{n\to+\infty}y_n
\end{aligned}$$

Combine the two parts above, we've proven the theorem.

Quod. Erat. Demonstrandum. $\square$

Remark: *As $(T(T^n(x)))_{n=0}^{+\infty}$ is a subsequence of the convergent sequence $(T^n(x))_{n=0}^{+\infty}$, it follows that $T(\xi) = \xi$, which helps us locate the solution of certain equation.*

Theorem 2.29. (**Inverse Function Theorem**)[2][3]
Let $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1 with $\mathbf{f}(\boldsymbol{\xi}) = \boldsymbol{\eta}$ and $\det D\mathbf{f}(\boldsymbol{\xi}) \neq 0$.
1. There exist open sets $U \ni \boldsymbol{\xi}, V \ni \boldsymbol{\eta}$,
such that the restricted function $\mathbf{f}: U \to V$ is bijective;
2. The inverse function $\mathbf{g}: V \to U$ is of class C^1,
and for all $\mathbf{y}_0 \in V$, $D\mathbf{g}(\mathbf{y}_0) = D\mathbf{f}(\mathbf{x_0})^{-1}$, where $\mathbf{x}_0 = \mathbf{g}(\mathbf{y}_0)$.

The logic of the proof for **Theorem 2.29.** will be as follows:
Step 1: Find $\delta > 0$, such that for all $\mathbf{x} \in B(\mathbf{0}, \delta), \mathbf{h} \in \partial B(\mathbf{0}, 1)$, $\|[D\mathbf{f}(\mathbf{x}) - I]\mathbf{h}\| \leq \frac{1}{2}$;
Step 2: Define $U = B(\mathbf{0}, \delta), V = \mathbf{f}(U)$. For all $\mathbf{y}_0 \in V$, define $\mathbf{T}_{\mathbf{y}_0}(\mathbf{x}) = \mathbf{x} - (\mathbf{f}(\mathbf{x}) - \mathbf{y}_0)$, and prove that $\mathbf{T}_{\mathbf{y}_0}$ is $\frac{1}{2}-$Lipschitz continuous on U;
Step 3: Prove that the restricted function $\mathbf{f}: U \to V$ is bijective;
Step 4: Prove that the restricted function $\mathbf{f}: U \to V$ is open;
Step 5: Prove that the restricted inverse function $\mathbf{g}: V \to U$ is $2-$Lipschitz continuous;
Step 6: Prove that the restricted inverse function $\mathbf{g}: V \to U$ is of class C^1, and for all $\mathbf{y}_0 \in V$, $D\mathbf{g}(\mathbf{y}_0) = D\mathbf{f}(\mathbf{x}_0)^{-1}$, where $\mathbf{x}_0 = \mathbf{g}(\mathbf{y}_0)$.

Proof. Without loss of generality, we may assume that $\boldsymbol{\xi} = \boldsymbol{\eta} = \mathbf{0}$ and $D\mathbf{f}(\boldsymbol{\xi}) = I$
Step 1: Find $\delta > 0$, such that for all $\mathbf{x} \in B(\mathbf{0}, \delta), \mathbf{h} \in \partial B(\mathbf{0}, 1)$, $\|[D\mathbf{f}(\mathbf{x}) - I]\mathbf{h}\| \leq \frac{1}{2}$;
Since $\mathbf{f}$ is of class C^1, the function $\mathbf{x} \mapsto \sup\limits_{\mathbf{h} \in \partial B(\mathbf{0},1)} \|[D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{0})]\mathbf{h}\|$ tends to 0 as $\mathbf{x}$ tends to $\mathbf{0}$. Hence, there exists $\delta > 0$, such that for all $\mathbf{x} \in B(\mathbf{0}, \delta)$:

$$\sup_{\mathbf{h} \in \partial B(\mathbf{0},1)} \|[D\mathbf{f}(\mathbf{x}) - I]\mathbf{h}\| = \sup_{\mathbf{h} \in \partial B(\mathbf{0},1)} \|[D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{0})]\mathbf{h}\| \leq \frac{1}{2}$$

Hence, the statement is proven if we apply the definition of supremum.
Step 2: Define $U = B(\mathbf{0}, \delta), V = \mathbf{f}(U)$. For all $\mathbf{y}_0 \in V$, define $\mathbf{T}_{\mathbf{y}_0}(\mathbf{x}) = \mathbf{x} - (\mathbf{f}(\mathbf{x}) - \mathbf{y}_0)$, and prove that $\mathbf{T}_{\mathbf{y}_0}$ is $\frac{1}{2}-$Lipschitz continuous on U;
From **Step 1**, we get:

$$\forall \mathbf{x} \in U, \mathbf{h} \in \partial B(\mathbf{0}, 1), \|D\mathbf{T}_{\mathbf{y}_0}(\mathbf{x})\mathbf{h}\| = \|[I - D\mathbf{f}(\mathbf{x})]\mathbf{h}\| \leq \frac{1}{2}$$

For all $\mathbf{x}_1, \mathbf{x}_2 \in U$ with $\mathbf{x}_1 \neq \mathbf{x}_2$:

$$\begin{aligned}
\|\mathbf{T}_{\mathbf{y}_0}(\mathbf{x}_2) - \mathbf{T}_{\mathbf{y}_0}(\mathbf{x}_1)\| &= \left\| \int_0^1 D\mathbf{T}_{\mathbf{y}_0}((1-t)\mathbf{x}_1 + t\mathbf{x}_2)(\mathbf{x}_2 - \mathbf{x}_1)\mathrm{d}t \right\| \\
&\leq \int_0^1 \|D\mathbf{T}_{\mathbf{y}_0}((1-t)\mathbf{x}_1 + t\mathbf{x}_2)(\mathbf{x}_2 - \mathbf{x}_1)\|\mathrm{d}t \\
&\leq \frac{1}{2} \int_0^1 \|\mathbf{x}_2 - \mathbf{x}_1\|\mathrm{d}t = \frac{1}{2}\|\mathbf{x}_2 - \mathbf{x}_1\|
\end{aligned}$$

Hence, $\mathbf{T}_{\mathbf{y}_0} : U \to \mathbb{R}^n$ is $\frac{1}{2}-$Lipschitz continuous.
Step 3: Prove that the restricted function $\mathbf{f} : U \to V$ is bijective;
For all $\mathbf{y}_0 \in V$, there exists $\mathbf{x}_0 \in U$, such that $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$, so $\mathbf{y}_0$ has one preimage $\mathbf{x}_0$;
For this $\mathbf{y}_0$, since $\mathbf{T}_{\mathbf{y}_0}$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2} \in [0, 1)$, it has only one fixed point $\mathbf{x}_0$, so $\mathbf{y}_0$ has only one preimage $\mathbf{x}_0$.
Hence, the restricted function $\mathbf{f} : U \to V$ is bijective.
Step 4: Prove that the restricted function $\mathbf{f} : U \to V$ is open;
For all $S \in \mathcal{P}(U)$, assume that $S \in \mathcal{O}_U$, let's prove that $T = \mathbf{f}(S) \in \mathcal{O}_V$.
For all $\mathbf{y}_0 \in T$, assume that its unique preimage is $\mathbf{x}_0$.
Since $S \in \mathcal{O}_U$, there exists $r > 0$, such that the closed ball $A = \overline{B}_r(\mathbf{x}_0) \subseteq S$.
Let's prove that $B = \overline{B}_{r/2}(\mathbf{y}_0) \subseteq T$.
For all $\mathbf{y} \in B$, consider the function $\mathbf{T}_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} - (\mathbf{f}(\mathbf{x}) - \mathbf{y})$. For all $\mathbf{x} \in A$:

$$\begin{aligned}
\|\mathbf{T}_{\mathbf{y}}(\mathbf{x}) - \mathbf{x}_0\| &= \|[\mathbf{T}_{\mathbf{y}}(\mathbf{x}) - \mathbf{T}_{\mathbf{y}}(\mathbf{x}_0)] + [\mathbf{T}_{\mathbf{y}}(\mathbf{x}_0) - \mathbf{x}_0]\| \\
&\leq \|\mathbf{T}_{\mathbf{y}}(\mathbf{x}) - \mathbf{T}_{\mathbf{y}}(\mathbf{x}_0)\| + \|\mathbf{T}_{\mathbf{y}}(\mathbf{x}_0) - \mathbf{x}_0\| \\
&\leq \|\mathbf{x} - \mathbf{x}_0\|/2 + \|\mathbf{y} - \mathbf{y}_0\| \leq r/2 + r/2 = r, \mathbf{T}_{\mathbf{y}}(\mathbf{x}) \in A
\end{aligned}$$

Hence, we may apply **Theorem 2.28.** and show that there is a unique fixed-point $\mathbf{x}$ of $\mathbf{T}_{\mathbf{y}}$. This implies $\mathbf{y} \in T$, so $B \subseteq T$. To conclude, $T \in \mathcal{O}_V$.
Step 5: Prove that the restricted inverse function $\mathbf{g} : V \to U$ is $2-$Lipschitz continuous;
For all $\mathbf{y}_1, \mathbf{y}_2 \in V$, assume that $\mathbf{x}_1 = \mathbf{g}(\mathbf{y}_1), \mathbf{x}_2 = \mathbf{g}(\mathbf{y}_2) \in U$, then:

$$\|\mathbf{x}_2 - \mathbf{x}_1\| - \|\mathbf{f}(\mathbf{x}_2) - \mathbf{f}(\mathbf{x}_1)\| \leq \|[\mathbf{x}_2 - \mathbf{f}(\mathbf{x}_2)] - [\mathbf{x}_2 - \mathbf{f}(\mathbf{x}_1)]\| = \|\mathbf{T}_{\mathbf{y}_1}(\mathbf{x}_2) - \mathbf{T}_{\mathbf{y}_1}(\mathbf{x}_1)\| \leq \frac{\|\mathbf{x}_2 - \mathbf{x}_1\|}{2}$$

$$\|\mathbf{g}(\mathbf{y}_2) - \mathbf{g}(\mathbf{y}_1)\| = \|\mathbf{x}_2 - \mathbf{x}_1\| \leq 2\|\mathbf{f}(\mathbf{x}_2) - \mathbf{f}(\mathbf{x}_1)| = 2\|\mathbf{y}_2 - \mathbf{y}_1\|$$

Hence, $\mathbf{g}$ is Lipschitz continuous with Lipschitz constant 2.
Step 6: Prove that the restricted inverse function $\mathbf{g} : V \to U$ is of class C^1, and for all $\mathbf{y}_0 \in V$, $D\mathbf{g}(\mathbf{y}_0) = D\mathbf{f}(\mathbf{x}_0)^{-1}$, where $\mathbf{x}_0 = \mathbf{g}(\mathbf{y}_0)$.
For all $\mathbf{y}_0 \in V$, consider the following limit:

$$\lim_{\mathbf{k} \to \mathbf{0}} \frac{\|\mathbf{g}(\mathbf{y}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{y}_0) - D\mathbf{f}(\mathbf{x}_0)^{-1}\mathbf{k}\|}{\|\mathbf{k}\|}$$

Define $\mathbf{h} = \mathbf{g}(\mathbf{y_0} + \mathbf{k}) - \mathbf{g}(\mathbf{y_0})$. Since $\mathbf{g}$ is a continuous bijection, we have:

$$\mathbf{k} = \mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0)$$

$$\lim_{\mathbf{k} \to \mathbf{0}} \frac{\|\mathbf{g}(\mathbf{y}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{y}_0) - D\mathbf{f}(\mathbf{x}_0)^{-1}\mathbf{k}\|}{\|\mathbf{k}\|} = \lim_{\mathbf{h} \to \mathbf{0}} \frac{\|\mathbf{h} - D\mathbf{f}(\mathbf{x}_0)^{-1}[\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0)]\|}{\|\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0)\|}$$

Here, we may manipulate the numerator and denominator respectively:

$$\begin{aligned}\|\mathbf{h} - D\mathbf{f}(\mathbf{x}_0)^{-1}[\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0)]\| &= \|D\mathbf{f}(\mathbf{x}_0)^{-1}D\mathbf{f}(\mathbf{x}_0)\mathbf{h} - D\mathbf{f}(\mathbf{x}_0)^{-1}[\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0)]\| \\ &= \|D\mathbf{f}(\mathbf{x}_0)^{-1}[\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}]\| \\ &\leq M\|[\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\|\end{aligned}$$

One may verify that $M = \sup\limits_{\mathbf{k}\in\partial B(\mathbf{0},1)} \|D\mathbf{f}(\mathbf{x}_0)^{-1}\mathbf{k}\|$ is a positive number.

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0)\| &= \|D\mathbf{f}(\mathbf{x}_0)\mathbf{h} + [\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}]\| \\ &\geq \|D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\| - \|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\| \\ &\geq N\|\mathbf{h}\| - \|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\|\end{aligned}$$

One may verify that $N = \inf_{\|\mathbf{k}\|=1} \|D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\|$ is a positive number.

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{h} - D\mathbf{f}(\mathbf{x}_0)^{-1}[\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0)]\|}{\|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0)\|} \leq \lim_{\mathbf{h}\to\mathbf{0}} \frac{M\|[\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\|}{N\|\mathbf{h}\| - \|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\|} = 0$$

Hence, we've proven that $\mathbf{g}$ is differentiable, and for all $\mathbf{y}_0 \in V$, $D\mathbf{g}(\mathbf{y}_0) = D\mathbf{f}(\mathbf{x}_0)^{-1}$, where $\mathbf{x}_0 = \mathbf{g}(\mathbf{y}_0)$. $D\mathbf{g}$ is continuous as matrix inversion, $D\mathbf{f}$ and $\mathbf{g}$ are all continuous. Combine the six parts above, we've proven the theorem. Quod. Erat. Demonstrandum.

□

Theorem 2.30. (**Implicit Function Theorem**)[2]
Let $\mathbf{f}: \mathbb{R}^{n+m} \to \mathbb{R}^m$ be of class C^1 with $\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \mathbf{0}$ and $\det D_{\mathbf{y}}\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) \neq 0$.
1. There exist open sets $U \ni \boldsymbol{\xi}, V \ni \boldsymbol{\eta}$,
such that for all $\mathbf{x} \in U$, there exists a unique $\mathbf{y} \in V$, such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$;
2. The function $\mathbf{Y}: U \to V$ defined by $\mathbf{f}(\mathbf{x}, \mathbf{Y}(\mathbf{x})) = \mathbf{0}$ is of class C^1,
and for all $\mathbf{x}_0 \in U$, $D\mathbf{Y}(\mathbf{x}_0) = -D_{\mathbf{y}}\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)$, where $\mathbf{y}_0 = \mathbf{Y}(\mathbf{x}_0)$.

The logic of the proof for **Theorem 2.30.** will be as follows:
Step 1: Construct $\mathbf{F}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{f}(\mathbf{x}, \mathbf{y}))$, and check the existence of $S \ni (\boldsymbol{\xi}, \boldsymbol{\eta}), T \ni (\boldsymbol{\xi}, \mathbf{0})$, such that the restricted function $\mathbf{F}: S \to T$ is bijective.
Step 2: Define $\mathbf{G}: T \to S$ as the restricted inverse function.
For all $(\mathbf{x}_0, \mathbf{c}_0) \in T$, calculate $D\mathbf{G}(\mathbf{x}_0, \mathbf{c}_0)$ explicitly.
Step 3: Define $U = \pi_{\mathbf{x}}(T), V = \pi_{\mathbf{y}}(S)$.
Verify that the second component function of $\mathbf{G}$ evaluated at $(\mathbf{x_0}, \mathbf{0})$ gives $\mathbf{Y}(\mathbf{x_0})$.

Proof. We may divide our proof into three steps.
Step 1: Construct $\mathbf{F}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{f}(\mathbf{x}, \mathbf{y}))$, and check the existence of $S \ni (\boldsymbol{\xi}, \boldsymbol{\eta}), T \ni (\boldsymbol{\xi}, \mathbf{0})$, such that the restricted function $\mathbf{F}: S \to T$ is bijective.
First, $\mathbf{f}$ is of class C^1 implies $\mathbf{F}$ is of class C^1.
Second, $\det D\mathbf{F}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \det \begin{pmatrix} I & O \\ D_{\mathbf{x}}\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) & D_{\mathbf{y}}\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) \end{pmatrix} = \det D_{\mathbf{y}}\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta}) \neq 0$.

According to **Theorem 2.29.**, there exist open sets $S \ni (\boldsymbol{\xi}, \boldsymbol{\eta}), T \ni (\boldsymbol{\xi}, \mathbf{0})$, such that the restricted function $\mathbf{F} : S \to T$ is bijective.
Step 2: Define $\mathbf{G} : T \to S$ as the restricted inverse function.
For all $(\mathbf{x}_0, \mathbf{c}_0) \in T$, calculate $D\mathbf{G}(\mathbf{x}_0, \mathbf{c}_0)$ explicitly.

$$\begin{pmatrix} I & O & I & O \\ D_{\mathbf{x}}\mathbf{f} & D_{\mathbf{y}}\mathbf{f} & O & I \end{pmatrix} \xrightarrow{\mathbf{R}_2 - D_{\mathbf{x}}\mathbf{f}\mathbf{R}_1} \begin{pmatrix} I & O & I & 0 \\ O & D_{\mathbf{y}}\mathbf{f} & -D_{\mathbf{x}}\mathbf{f} & I \end{pmatrix}$$

$$\xrightarrow{D_{\mathbf{y}}\mathbf{f}^{-1}\mathbf{R}_2} \begin{pmatrix} I & O & I & 0 \\ O & I & -D_{\mathbf{y}}\mathbf{f}^{-1}D_{\mathbf{x}}\mathbf{f} & D_{\mathbf{y}}\mathbf{f}^{-1} \end{pmatrix}$$

$$D\mathbf{G}(\mathbf{x}_0, \mathbf{c}_0) = D\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)^{-1} = \begin{pmatrix} I & O \\ -D_{\mathbf{y}}\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} D_{\mathbf{x}}\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) & D_{\mathbf{y}}\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \end{pmatrix}$$

Here, $(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{G}(\mathbf{x}_0, \mathbf{c}_0)$.
Step 3: Define $U = \pi_{\mathbf{x}}(T), V = \pi_{\mathbf{y}}(S)$.
Verify that the second component function of $\mathbf{G}$ evaluated at $(\mathbf{x_0}, \mathbf{0})$ gives $\mathbf{Y}(\mathbf{x_0})$.
For all $\mathbf{x}_0 \in U$, since $(\mathbf{x}_0, \mathbf{0}) \in T$, there exists a unique $(\mathbf{x}_0, \mathbf{y}_0) \in S$, such that $(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{G}(\mathbf{x}_0, \mathbf{0})$, i.e., $\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$. Hence, we've proven the existence and uniqueness of $\mathbf{Y}(\mathbf{x}_0)$.
Combine the three parts above, we've proven the theorem.
Quod. Erat. Demonstrandum. $\square$

Remark: *Inverse function theorem gives local homeomorphisms from a "nondegenerate subspace" of $\mathbb{R}^n$ to another "nondegenerate subspace" of $\mathbb{R}^n$. For example, $(1,2) \times \mathbb{R} \to \{(x_1, x_2) \in \mathbb{R}^2 : 1 < x_1^2 + x_2^2 < 4\}, (r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. Implicit function theorem gives local homeomorphisms from a "nondegenerate subspace" of $\mathbb{R}^n$ to a "degenerate subspace" of $\mathbb{R}^{n+m}$. For example, the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ is locally a plane.*

Proposition 2.31. If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then for all open set $U \not\ni 0$, its image under $\sigma : \mathbb{C} \to \mathbb{C}, z \mapsto z^n (n \in \mathbb{Z})$ is open.

Proof. Since σ is a continuously differentiable function, and for all $z \in \mathbb{C}\backslash\{0\}$, $\det D\sigma(z) = n^2|z|^{2n-2} \neq 0$, according to **Theorem 2.29.**, for each $z \in \mathbb{C}\backslash\{0\}$, there exists an open subset $U_z \ni z$ of U, such that $\sigma(U_z)$ is open. This implies $\sigma(U) = \sigma\left(\bigcup_{z\in U} U_z\right) = \bigcup_{z \in U} \sigma(U_z)$ is open. Quod. Erat. Demonstrandum. $\square$

Proposition 2.32. If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then for all open set $U \ni 0$, its image under $\sigma : \mathbb{C} \to \mathbb{C}, z \mapsto z^n (n \in \mathbb{Z}_{\geq 2})$ is open.

Proof. It suffices to show $\sigma(0) = 0 \in \sigma(U)^\circ$.
As $0 \in U$ and U is open, there exists $r > 0$, such that $B(0, r) \subseteq U$.
For all $w \in B(0, r^n)$, there exists a root $z \in B(0, r)$, such that $w = \sigma(z)$, so $w \in \sigma(U)$.
This implies $B(0, r^n) \subseteq \sigma(U)$, and we are done. Quod. Erat. Demonstrandum. $\square$

2.5 Topological Invariant

Definition 2.33. (**Topological Invariant**)
Let p be an open statement on the set of all topological spaces.
If $\forall (X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$, $(X, \mathcal{O}_X) \cong (Y, \mathcal{O}_Y) \implies p(X, \mathcal{O}_X) = p(Y, \mathcal{O}_Y)$,
then p is a topological invariant.

3 Hausdorffness as a Topological Invariant

3.1 Definition and Properties of Hausdorff Space

Definition 3.1. (**Hausdorff Space**)
Let $(X, \mathcal{O}_X)$ be a topological space.
If $\forall$ distinct $x_1, x_2 \in X, \exists O_1, O_2 \in \mathcal{O}_X, x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$,
then $(X, \mathcal{O}_X)$ is Hausdorff.

Proposition 3.2. Let $(X, \mathcal{O}_X)$ be a topological space.
$(X, \mathcal{O}_X)$ is Hausdorff if and only if the image of $\Delta : X \to X \times X, \Delta(x) = (x, x)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$.

Proof. We may divide our proof into two parts.
"if" direction: Assume that $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$.
For all $x_1, x_2 \in X$:

$$\begin{aligned} x_1 \neq x_2 &\implies (x_1, x_2) \in \Delta(X)^c \\ &\implies \exists O_1 \times O_2 \in \mathcal{O}_{X \times X} \text{ with } O_1 \times O_2 \subseteq \Delta(X)^c, (x_1, x_2) \in O_1 \times O_2 \\ &\implies \exists O_1, O_2 \in \mathcal{O}_X, x_1 \in O_1 \text{ and } x_2 \in O_2 \text{ and } O_1 \cap O_2 = \emptyset \end{aligned}$$

Hence, $(X, \mathcal{O}_X)$ is Hausdorff.
"only if" direction: Assume that $(X, \mathcal{O}_X)$ is Hausdorff.
For all $(x_1, x_2) \in X \times X$:

$$\begin{aligned} (x_1, x_2) \in \Delta(X)^c &\implies x_1 \neq x_2 \\ &\implies \exists O_1, O_2 \in \mathcal{O}_X, x_1 \in O_1 \text{ and } x_2 \in O_2 \text{ and } O_1 \cap O_2 = \emptyset \\ &\implies \exists O_1 \times O_2 \in \mathcal{O}_{X \times X} \text{ with } O_1 \times O_2 \subseteq \Delta(X)^c, (x_1, x_2) \in O_1 \times O_2 \end{aligned}$$

Hence, $\Delta(X)$ is closed in $(X \times X, \mathcal{O}_{X \times X})$.
Quod. Erat. Demonstrandum. $\square$

Proposition 3.3. Hausdorffness is a topological invariant.

Proof. For all $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$, assume that there exists a homeomorphism $\sigma : X \to Y$. Assume that $(X, \mathcal{O}_X)$ is Hausdorff.
For all $y_1, y_2 \in Y$:

$$\begin{aligned} y_1 \neq y_2 &\implies \sigma^{-1}(y_1) \neq \sigma^{-1}(y_2) \\ &\implies \exists U_1, U_2 \in \mathcal{O}_X, \sigma^{-1}(y_1) \in U_1 \text{ and } \sigma^{-1}(y_2) \in U_2 \text{ and } U_1 \cap U_2 = \emptyset \\ &\implies \exists \sigma(U_1), \sigma(U_2) \in \mathcal{O}_X, y_1 \in \sigma(U_1) \text{ and } y_2 \in \sigma(U_2) \text{ and } \sigma(U_1) \cap \sigma(U_2) = \emptyset \end{aligned}$$

Hence, $(Y, \mathcal{O}_Y)$ is Hausdorff.
As σ^{-1} is also a homeomorphism, the reverse implication is also true.
Quod. Erat. Demonstrandum. □

Proposition 3.4. Let $(X, \mathcal{O}_X)$ be a topological space.
If $(X, \mathcal{O}_X)$ is Hausdorff, the $(X, \mathcal{O}_X)$ is finer than cofinite topological space.

Proof. For all $x \in X$, for all $y \in \{x\}^c$, there exist $U_y, V_y \in \mathcal{O}_X$ with $x \in U_y$ and $y \in V_y$ and $U_y \cap V_y = \emptyset$, so $V_y \subseteq \{x\}^c$, which implies $\{x\}^c = \bigcup_{y \in \{x\}^c} V_y \in \mathcal{O}_X, \{x\} \in \mathcal{C}_X$.
Quod. Erat. Demonstrandum. □

Proposition 3.5. Let $(X, \mathcal{O}_X)$ be a topological space.
If $(X, \mathcal{O}_X)$ is Hausdorff, then every sequence $(x_n)_{n=1}^{+\infty}$ in X has at most one limit.

Proof. Assume to the contrary that $(x_n)_{n=1}^{+\infty}$ has two distinct limits x_*, x^*.
As $(X, \mathcal{O}_X)$ is Hausdorff, there exist $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. However, there exist $N_1, N_2 \in \mathbb{N}$, such that $\{x_n\}_{n=N_1}^{+\infty} \subseteq O_1$ and $\{x_n\}_{n=N_2}^{+\infty} \subseteq O_2$, which leaves $x_{\max\{N_1,N_2\}}$ no where to go.
Quod. Erat. Demonstrandum. □

3.2 Construct New Hausdorff Space from Old One

Proposition 3.6. Let $(X, \mathcal{O}_X)$ be a topological space, and $(X', \mathcal{O}_{X'})$ be a subspace of $(X, \mathcal{O}_X)$. $(X, \mathcal{O}_X)$ is Hausdorff implies $(X', \mathcal{O}_{X'})$ is Hausdorff.

Proof. For all distinct $x_1, x_2 \in X'$, $x_1, x_2 \in X$, so there exist $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. Hence, there exist $X' \cap O_1, X' \cap O_2 \in \mathcal{O}_{X'}$, such that $x_1 \in X' \cap O_1$ and $x_2 \in X' \cap O_2$ and $(X' \cap O_1) \cap (X' \cap O_2) = \emptyset$, which implies $(X', \mathcal{O}_{X'})$ is Hausdorff. Quod. Erat. Demonstrandum. □

Proposition 3.7. If we regard $\mathbb{R}^n$ as a metric space with Euclidean metric, then $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$ is Hausdorff.

Proposition 3.8. Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ be an indexed family of topological spaces, and $(X, \mathcal{O}_X)$ be the product space of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$.
$(X, \mathcal{O}_X)$ is Hausdorff if and only if each $(X_\lambda, \mathcal{O}_{X_\lambda})$ is Hausdorff.

Proof. We may divide our proof into two parts.
"if" direction: Assume that each $(X_\lambda, \mathcal{O}_{X_\lambda})$ is Hausdorff.
For all distinct $x_1, x_2 \in X$, there exists $\lambda \in I$, such that $x_1(\lambda) \neq x_2(\lambda)$. There exist $O_{1,\lambda}, O_{2,\lambda} \in \mathcal{O}_{X_\lambda}$, such that $x_1(\lambda) \in O_{1,\lambda}$ and $x_2(\lambda) \in O_{2,\lambda}$ and $O_{1,\lambda} \cap O_{2,\lambda} = \emptyset$.
This implies the existence of $\pi_\lambda^{-1}(O_{1,\lambda}), \pi_\lambda^{-1}(O_{2,\lambda}) \in \mathcal{O}_X$, such that $x_1 \in \pi_\lambda^{-1}(O_{1,\lambda})$ and $x_2 \in \pi_\lambda^{-1}(O_{2,\lambda})$ and $\pi_\lambda^{-1}(O_{1,\lambda}) \cap \pi_\lambda^{-1}(O_{2,\lambda}) = \emptyset$. Hence, $(X, \mathcal{O}_X)$ is Hausdorff.
"only if" direction: Assume that $(X, \mathcal{O}_X)$ is Hausdorff.
For all $\lambda \in I$, choose $\xi_\lambda \in X_\lambda$.
For all $\lambda \in I$, for all distinct $\xi_{1,\lambda}, \xi_{2,\lambda} \in X_\lambda$, construct:

$$x_1 : I \to X, x_1(\mu) = \begin{cases} \xi_{1,\lambda} & \text{if} \quad \mu = \lambda; \\ \xi_\mu & \text{if} \quad \mu \neq \lambda; \end{cases}, x_2 : I \to X, x_2(\mu) = \begin{cases} \xi_{2,\lambda} & \text{if} \quad \mu = \lambda; \\ \xi_\mu & \text{if} \quad \mu \neq \lambda; \end{cases}$$

There exist $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.
Without loss of generality, assume that $O_1 = \bigcup_{k=1}^m \pi_{\lambda_k}^{-1}(O_{1,\lambda_k}), O_2 = \bigcup_{k=1}^m \pi_{\lambda_k}^{-1}(O_{2,\lambda_k})$, where each nonempty $O_{1,\lambda_k}, O_{2,\lambda_k} \in \mathcal{O}_{X_{\lambda_k}}$. As $O_1 \cap O_2 \neq \emptyset$, and each $\lambda_k \neq \lambda$ gives $O_{1,\lambda_k} \cap O_{2,\lambda_k} \neq \emptyset$, it must be true that some $\lambda_k = \lambda$ and $O_{1,\lambda} \cap O_{2,\lambda} = \emptyset$.
This implies the existence of $O_{1,\lambda}, O_{2,\lambda} \in \mathcal{O}_{X_\lambda}$, such that $\xi_{1,\lambda} \in O_{1,\lambda}$ and $\xi_{2,\lambda} \in O_{2,\lambda}$ and $O_{1,\lambda} \cap O_{2,\lambda} = \emptyset$. Hence, each $(X_\lambda, \mathcal{O}_{X_\lambda})$ is Hausdorff.
Combine the two parts together, we've proven the biconditional.
Quod. Erat. Demonstrandum. □

Proposition 3.9. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then the set of all real-valued sequences $\mathbb{R}^\mathbb{N}$ is Hausdorff.
Especially, the set of all convergent sequences is Hausdorff.

Proposition 3.10. Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$ be an indexed family of topological spaces, and $(X, \mathcal{O}_X)$ be the coproduct space of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda \in I}$.
$(X, \mathcal{O}_X)$ is Hausdorff if and only if each $(X_\lambda, \mathcal{O}_{X_\lambda})$ is Hausdorff.

Proof. We may divide our proof into two parts.
"if" direction: Assume that each $(X_\lambda, \mathcal{O}_{X_\lambda})$ is Hausdorff.
For all distinct $(x_1, \lambda_1), (x_2, \lambda_2) \in X$:
Case 1: If $\lambda_1 \neq \lambda_2$, then there exist $X_1 \times \{\lambda_1\}, X_2 \times \{\lambda_2\} \in \mathcal{O}_X$, such that $(x_1, \lambda_1) \in X_1 \times \{\lambda_1\}$ and $(x_2, \lambda_2) \in X_2 \times \{\lambda_2\}$ and $(X_1 \times \{\lambda_1\}) \cap (X_2 \times \{\lambda_2\}) = \emptyset$.
Case 2: If $\lambda_1 = \lambda_2 = \lambda$, then there exist $U_{1,\lambda}, U_{2,\lambda} \in \mathcal{X}_\lambda$, such that $x_1 \in U_{1,\lambda}$ and $x_2 \in U_{2,\lambda}$ and $U_{1,\lambda} \cap U_{2,\lambda} = \emptyset$. This implies the existence of $U_{1,\lambda} \times \{\lambda\}, U_{2,\lambda} \times \{\lambda\} \in \mathcal{O}_{X_\lambda}$, such that $(x_1, \lambda) \in U_{1,\lambda} \times \{\lambda\}$ and $(x_2, \lambda) \in U_{2,\lambda} \times \{\lambda\}$ and $(U_{1,\lambda} \times \{\lambda\}) \cap (U_{2,\lambda} \times \{\lambda\}) = \emptyset$.

Hence, $(X, \mathcal{O}_X)$ is Hausdorff.
"only if" direction: Assume that $(X, \mathcal{O}_X)$ is Hausdorff.
For all $\lambda \in I$, for all distinct $x_1, x_2 \in X_\lambda$, construct $(x_1, \lambda), (x_2, \lambda) \in X$.
There exist $U_1, U_2 \in \mathcal{O}_X$, such that $(x_1, \lambda) \in U_1$ and $(x_2, \lambda) \in U_2$ and $U_1 \cap U_2 = \emptyset$.
This implies the existence of $\pi_\lambda^{-1}(U_1), \pi_\lambda^{-1}(U_2) \in \mathcal{O}_{X_\lambda}$, such that $x_1 \in \pi_\lambda^{-1}(U_1)$ and $x_2 \in \pi_\lambda^{-1}(U_2)$ and $\pi_\lambda^{-1}(U_1) \cap \pi_\lambda^{-1}(U_2) = \emptyset$. Hence, each $(X_\lambda, \mathcal{O}_{X_\lambda})$ is Hausdorff.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. $\square$

Proposition 3.11. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then $\bigsqcup_{\lambda \in \mathbb{R}} \mathbb{R}$ is Hausdorff. But $\bigsqcup_{\lambda \in \mathbb{R}} \mathbb{R}$ is different from $\mathbb{R}^2$.

Definition 3.12. (**Saturated Open Set**)
Let $(X, \mathcal{O}_X)$ be a topological space, $\sim: X \to X$ be an equivalence relation on X, $(X/\sim, \mathcal{O}_{X/\sim})$ be a quotient space of $(X, \mathcal{O}_X)$, and U be a subset of X.
If $U \in \mathcal{O}_X$ and $\pi^{-1}(\pi(U)) = U$, then U is a saturated open set in $(X, \mathcal{O}_X)$.

Proposition 3.13. Let $(X, \mathcal{O}_X)$ be a topological space, $\sim: X \to X$ be an equivalence relation on X, and $(X/\sim, \mathcal{O}_{X/\sim})$ be a quotient space of $(X, \mathcal{O}_X)$.
$(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff if and only if $\forall$ distinct $[x_1]_\sim, [x_2]_\sim \in X/\sim$,
$\exists$ saturated $U_1, U_2 \in \mathcal{O}_X, [x_1]_\sim \subseteq U_1$ and $[x_2]_\sim \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$.

Proof. We may divide our proof into two parts.
"if" direction: Assume that $\forall$ distinct $[x_1]_\sim, [x_2]_\sim \in X/\sim, \exists$ saturated $U_1, U_2 \in \mathcal{O}_X, [x_1]_\sim \subseteq U_1$ and $[x_2]_\sim \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$.
As π is an open mapping, $\pi(U_1), \pi(U_2) \in \mathcal{O}_{X/\sim}$.
As U_1, U_2 are saturated, $\pi(U_1) \cap \pi(U_2) = \emptyset$.
This implies the existence of $\pi(U_1), \pi(U_2) \in \mathcal{O}_{X/\sim}$, such that $[x_1]_\sim \in \pi(U_1)$ and $[x_2]_\sim \in \pi(U_2)$. Hence, $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff.
"only if" direction: Assume that $(X/\sim, \mathcal{O}_{X/\sim})$ is Hausdorff.
For all distinct $[x_1]_\sim, [x_2]_\sim \in X/\sim$, there exist $V_1, V_2 \in \mathcal{O}_{X/\sim}$, such that $[x_1]_\sim \in V_1$ and $[x_2]_\sim \in V_2$ and $V_1 \cap V_2 = \emptyset$.
This implies the existence of saturated $\pi^{-1}(V_1), \pi^{-1}(V_2) \in \mathcal{O}_X$, such that $[x_1]_\sim \subseteq \pi^{-1}(V_1)$ and $[x_2]_\sim \subseteq \pi^{-1}(V_2)$ and $\pi^{-1}(V_1) \cap \pi^{-1}(V_2)$.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. $\square$

Proposition 3.14. If we regard $\mathbb{R}^n$ as a metric space with Euclidean metric, $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$ as a subspace of $\mathbb{R}^n$,
then $\mathbb{S}^{n-1}/\sim = \{\{\mathbf{x}, -\mathbf{x}\}\}_\mathbf{x}$ is Hausdorff.

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