Algebra II: Tutorial 7

March 30, 2022

Throughout this tutorial, assume that S is a set consisting of the points $P_0 = (0, 0)$ and $P_1 = (1, 0)$, and identify the paper with $\mathbb{R}^2$. We say that a number $\alpha \in \mathbb{R}$ is constructible if there exists two constructible points whose distance is $|\alpha|$.

Problem 1. Suppose that $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are constructible. Show that $a + b, -a, ab$ and $\frac{1}{a}$ are constructible. Deduce that the set of all constructible points forms a field containing $\mathbb{Q}$.

Problem 2. Show that if $a \in \mathbb{R}$ is constructible, then $\sqrt{a}$ is constructible.

Problem 3. Show that a number a is constructible if there is a tower of field extensions $\mathbb{Q} \subset F_1 \subset F_2 \subset \cdots \subset F_n$ such that $a \in F_n$, and each of the degrees $[F_i : f_{i-1}] = 2$.