THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Test 2 Solution

Problem 1.

(i) Given that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$-5\partial_{xx}u - 4\partial_{yy}u - 3\partial_{zz}u + 2\partial_x u - \partial_y u < 0, \tag{1}$$

since u is a continuous function on the compact set $\bar{\Omega}$, it then follows from the extreme value theorem that there exists a point $(x_0, y_0, z_0) \in \bar{\Omega}$ such that

$$u(x_0, y_0, z_0) = \max_{\bar{\Omega}} u.$$

Seeking for a contradiction, we assume that $(x_0, y_0, z_0) \in \Omega^\circ = (0,1)^3$. By the first and second order tests for local/interior maximum in elementary calculus, we know that

$$\partial_x u(x_0, y_0, z_0) = \partial_y u(x_0, y_0, z_0) = 0, \quad \text{and}$$
$$\partial_{xx}u(x_0, y_0, z_0),\ \partial_{yy}u(x_0, y_0, z_0),\ \partial_{zz}u(x_0, y_0, z_0) \le 0.$$

Hence, at this (x_0, y_0, z_0), we actually have

$$-5\partial_{xx}u - 4\partial_{yy}u - 3\partial_{zz}u + 2\partial_x u - \partial_y u = -5\partial_{xx}u - 4\partial_{yy}u - 3\partial_{zz}u \ge 0,$$

which contradicts with inequality (1). This means that the assumption "$(x_0, y_0, z_0) \in (0,1)^3$" is wrong, which implies

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

(ii) Given $v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$-5\partial_{xx}v - 4\partial_{yy}v - 3\partial_{zz}v + 2\partial_x v - \partial_y v \le 0.$$

For any $\epsilon > 0$, we define

$$v_\epsilon(x,y,z) := v(x,y,z) + \epsilon z^2.$$

Then

$$\begin{aligned} &-5\partial_{xx}v_\epsilon - 4\partial_{yy}u_\epsilon - 3\partial_{zz}v_\epsilon + 2\partial_x v_\epsilon - \partial_y v_\epsilon \\ =&-5\partial_{xx}v - 4\partial_{yy}v - 3\left(\partial_{zz}v + 2\epsilon\right) + 2\partial_x v - \partial_y v \le -6\epsilon < 0. \end{aligned}$$

Therefore, applying the result in (i), we have

$$\max_{\bar{\Omega}} v_\epsilon = \max_{\partial\Omega} v_\epsilon,$$

which implies

$$\max_{\bar{\Omega}} v \le \max_{\bar{\Omega}} v_\epsilon = \max_{\partial\Omega} v_\epsilon = \max_{\partial\Omega} v + \max_{\partial\Omega} \epsilon z^2 = \max_{\partial\Omega} v + \epsilon.$$

Passing to the limit as $\epsilon \to 0^+$ in the above inequality, we obtain

$$\max_{\bar{\Omega}} v \le \max_{\partial\Omega} v.$$

Since $\partial\Omega \subset \bar{\Omega}$, it follows from the definition of maximum that

$$\max_{\bar{\Omega}} v \ge \max_{\partial\Omega} v.$$

and hence,

$$\max_{\bar{\Omega}} v = \max_{\partial\Omega} v.$$

(iii) To prove uniqueness, let v_1 and v_2 be solutions to the initial and boundary value problem

$$\begin{cases} -5\partial_{xx}v - 4\partial_{yy}v - 3\partial_{zz}v + 2\partial_x v - \partial_y v = 0 & \text{in } \Omega \\ \hfill v|_{\partial\Omega} = g. & \end{cases}$$

where the given function g are the SAME for both v_1 and v_2. Define $\tilde{v} := v_1 - v_2$. Then $\tilde{u}$ satisfies the Dirichlet problem

$$\begin{cases} -5\partial_{xx}\tilde{v} - 4\partial_{yy}\tilde{v} - 3\partial_{zz}\tilde{v} + 2\partial_x\tilde{v} - \partial_y\tilde{v} = 0 \quad \text{in } \Omega \\ \tilde{v}|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

Applying part (ii) to $\tilde{v}$ and using the initial and boundary conditions for $\tilde{v}$, we have

$$\max_{\bar{\Omega}} \tilde{v} = \max_{\partial\Omega} \tilde{v} = 0, \tag{3}$$

Indeed, $-\tilde{v}$ also satisfies the same Dirichlet problem (2), and hence, applying part (ii) again and using the boundary condition $-\tilde{v}|_{\partial\Omega} \equiv 0$, we obtain

$$\max_{\bar{\Omega}} (-\tilde{v}) = \max_{\partial\Omega} (-\tilde{u}) = 0,$$

which implies

$$\min_{\bar{\Omega}} \tilde{v} = 0. \tag{4}$$

Combining (3) and (4), we finally obtain

$$\tilde{v} \equiv 0 \quad \text{in } \bar{\Omega},$$

and this proves the uniqueness.

(iv) Let $v := w^2$. Then

$$\begin{aligned} \partial_x v = 2w\partial_x w,\ \partial_{xx} v = \partial_x (2w\partial_x w) = 2w\partial_{xx} w + 2\left|\partial_x w\right|^2 \\ \partial_y v = 2w\partial_x w,\ \partial_{yy} v = \partial_y (2w\partial_y w) = 2w\partial_{yy} w + 2\left|\partial_y w\right|^2 \\ \partial_z v = 2w\partial_z w,\ \partial_{zz} v = \partial_z (2w\partial_z w) = 2w\partial_{zz} w + 2\left|\partial_z w\right|^2. \end{aligned}$$

and hence, using the given equation

$$5\partial_{xx} w + 4\partial_{yy} w + 3\partial_{zz} w - 2\partial_x w + \partial_y w = w^5,$$

we have

$$\begin{aligned}
&-5\partial_{xx}v-4\partial_{yy}v-3\partial_{zz}v+2\partial_x v-\partial_y v\\
=&-5\left(2w\partial_{xx}w+2\left|\partial_x w\right|^2\right)-4\left(2w\partial_{yy}w+2\left|\partial_y w\right|^2\right)-3\left(2w\partial_{zz}w+2\left|\partial_z w\right|^2\right)\\
&+2\left(2w\partial_x w\right)-\left(2w\partial_y w\right)\\
=&-2w\left(5\partial_{xx}w+4\partial_{yy}w+3\partial_{zz}w-2\partial_x w+\partial_y w\right)-10\left|\partial_x w\right|^2-8\left|\partial_y w\right|^2-6\left|\partial_z w\right|^2\\
=&-2w^6-10\left|\partial_x w\right|^2-8\left|\partial_y w\right|^2-6\left|\partial_z w\right|^2\le 0.
\end{aligned}$$

Applying part (ii) to v, we have

$$\max_{\bar{\Omega}} v=\max_{\partial\Omega} v.$$

which is equivalent to

$$\max_{\bar{\Omega}} w^2=\max_{\partial\Omega} w^2,$$

or alternatively,

$$\max_{\bar{\Omega}} |w|=\max_{\partial\Omega} |w|,$$

Problem 2.

(i) To prove uniqueness, let u_1 and u_2 be two solutions to

$$\begin{cases}
\partial_{tt}u-\partial_{xx}u+4u=f-\partial_t u & \text{for } 0\le x\le 2 \text{ and } t>0,\\
u|_{x=0}=u|_{x=2}=0, & \text{for } t>0,\\
u|_{t=0}=\phi & \text{for } 0\le x\le 2,\\
\partial_t u|_{t=0}=\psi & \text{for } 0\le x\le 2.
\end{cases}$$

Define $\tilde{u}:=u_1-u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases}
\partial_{tt}\tilde{u}-\partial_{xx}\tilde{u}+4\tilde{u}=-\partial_t\tilde{u} & \text{for } 0\le x\le 2 \text{ and } t>0,\\
\tilde{u}|_{x=0}=\tilde{u}|_{x=2}=0, & \text{for } t>0,\\
\tilde{u}|_{t=0}=0 & \text{for } 0\le x\le 2,\\
\partial_t\tilde{u}|_{t=0}=0 & \text{for } 0\le x\le 2.
\end{cases}$$

Observe that $\partial_{tt}\tilde{u}$ is the highest order time derivative. By multiplying $\partial_{tt}\tilde{u} - \partial_{xx}\tilde{u} + 4\tilde{u} = -\partial_t\tilde{u}$ by $\partial_t\tilde{u}$, and then integrating with respect to x, we have

$$\int_0^2 \partial_t\tilde{u}\left(\partial_{tt}\tilde{u} - \partial_{xx}\tilde{u} + 4\tilde{u}\right)\ dx = -\int_0^2 |\partial_t\tilde{u}|^2\ dx.$$

Integrating by parts on the left hand side and using the boundary conditions for $\tilde{u}$, we obtain

$$\begin{aligned}
&\int_0^2 \partial_t\tilde{u}\left(\partial_{tt}\tilde{u} - \partial_{xx}\tilde{u} + 4\tilde{u}\right)\ dx \\
=&\int_0^2 \partial_t\tilde{u}\partial_{tt}\tilde{u}\ dx - \int_0^2 \partial_t\tilde{u}\partial_{xx}\tilde{u}\ dx + \int_0^2 4\tilde{u}\partial_t\tilde{u}\ dx \\
=&\frac{1}{2}\frac{d}{dt}\left(\int_0^2 |\partial_t\tilde{u}|^2\ dx\right) - \left[\partial_t\tilde{u}\partial_x\tilde{u}\right]_{x=0}^2 + \int_0^2 \partial_{tx}\tilde{u}\partial_x\tilde{u}\ dx + \frac{1}{2}\frac{d}{dt}\left(\int_0^2 4|\tilde{u}|^2\ dx\right) \\
=&\frac{1}{2}\frac{d}{dt}\left(\int_0^2 |\partial_t\tilde{u}|^2 + 4|\tilde{u}|^2\ dx\right) + \frac{1}{2}\frac{d}{dt}\left(\int_0^2 |\partial_x\tilde{u}|^2\ dx\right) \\
=&\frac{1}{2}\frac{d}{dt}\left(\int_0^2 |\partial_t\tilde{u}|^2 + 4|\tilde{u}|^2 + |\partial_x\tilde{u}|^2\ dx\right).
\end{aligned}$$

Notice that we have used the following fact,

$$\tilde{u}|_{x=0} = \tilde{u}|_{x=2} = 0 \implies \partial_t\tilde{u}|_{x=0} = \partial_t\tilde{u}|_{x=2} = 0.$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^2 |\partial_t\tilde{u}|^2 + 4|\tilde{u}|^2 + |\partial_x\tilde{u}|^2\ dx\right) = -\int_0^2 |\partial_t\tilde{u}|^2\ dx \le 0.$$

Define

$$E(t) \coloneqq \int_0^2 |\partial_t\tilde{u}(t,x)|^2 + 4|\tilde{u}(t,x)|^2 + |\partial_x\tilde{u}(t,x)|^2\ dx.$$

The above inequality can then be written as

$$\frac{d}{dt}E(t) \le 0.$$

A direct integration yields, for any $t \ge 0$,

$$E(t) \le E(0).$$

Using the initial conditions $\tilde{u}|_{t=0} = 0$ and $\partial_t \tilde{u}|_{t=0} = 0$, we have $\partial_x \tilde{u}|_{t=0} = 0$ and thus

$$E(0) = \int_0^2 |\partial_t \tilde{u}(0,x)|^2 + 4\,|\tilde{u}(0,x)|^2 + |\partial_x \tilde{u}(0,x)|^2 \; dx = 0.$$

It follows from the definition of E that $E(t) \geq 0$, so

$$0 \leq E(t) \leq E(0) = 0,$$

which implies $E(t) \equiv 0$, and hence,

$$|\partial_t \tilde{u}|^2 + 4\,|\tilde{u}|^2 + |\partial_x \tilde{u}|^2 \equiv 0.$$

This directly implies

$$\tilde{u} \equiv 0.$$

That is, $u_1 \equiv u_2$, so the solution is unique by the energy method.

(ii) To prove uniqueness, let u_1 and u_2 be two solutions to

$$\begin{cases} \partial_t u = \partial_{xx} u + \partial_{yy} u & \text{for } 0 < x < 1,\ 0 < y < 1 \text{ and } t > 0, \\ \partial_x u|_{x=0} = \partial_x u|_{x=1} = 0, & \text{for } 0 < y < 1 \text{ and } t > 0, \\ u|_{y=0} = u|_{y=1} = 0, & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u|_{t=0} = \phi & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1. \end{cases}$$

Define $\tilde{u} := u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} = \partial_{xx} \tilde{u} + \partial_{yy} \tilde{u} & \text{for } 0 < x < 1,\ 0 < y < 1 \text{ and } t > 0, \\ \partial_x \tilde{u}|_{x=0} = \partial_x \tilde{u}|_{x=1} = 0, & \text{for } 0 < y < 1 \text{ and } t > 0, \\ \tilde{u}|_{y=0} = \tilde{u}|_{y=1} = 0, & \text{for } 0 < x < 1 \text{ and } t > 0, \\ \tilde{u}|_{t=0} = 0 & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1. \end{cases}$$

Observe that $\partial_t \tilde{u}$ is the highest order time derivative. By multiplying $\partial_t \tilde{u} = \partial_{xx} \tilde{u} + \partial_{yy} \tilde{u}$ by $\tilde{u}$, and then integrating with respect to x and y, we have

$$\int_0^1 \int_0^1 \tilde{u} \partial_t \tilde{u} \; dx dy = \int_0^1 \int_0^1 \tilde{u} \left(\partial_{xx} \tilde{u} + \partial_{yy} \tilde{u} \right) \; dx dy.$$

For the left hand side, we have

$$\int_0^1 \int_0^1 \tilde{u}\partial_t \tilde{u}\ dxdy = \frac{1}{2}\frac{d}{dt}\left(\int_0^1 \int_0^1 |\tilde{u}|^2\ dxdy\right).$$

For the right hand side, apply integration by parts and using the boundary conditions for $\tilde{u}$, we obtain

$$\begin{aligned}
&\int_0^1 \int_0^1 \tilde{u}\left(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}\right)\ dxdy \\
&= \int_0^1 \int_0^1 \tilde{u}\partial_{xx}\tilde{u}\ dxdy + \int_0^1 \int_0^1 \tilde{u}\partial_{yy}\tilde{u}\ dydx \\
&= \int_0^1 \left([\tilde{u}\partial_x \tilde{u}]_{x=0}^1 - \int_0^1 |\partial_x \tilde{u}|^2\ dx\right) dy + \int_0^1 \left([\tilde{u}\partial_y \tilde{u}]_{y=0}^1 - \int_0^1 |\partial_y \tilde{u}|^2\ dy\right) dx \\
&= -\int_0^1 \int_0^1 |\partial_x \tilde{u}|^2\ dxdy - \int_0^1 \int_0^1 |\partial_y \tilde{u}|^2\ dxdy \\
&= -\int_0^1 \int_0^1 |\partial_x \tilde{u}|^2 + |\partial_y \tilde{u}|^2\ dxdy \le 0.
\end{aligned}$$

Define

$$E(t) := \int_0^1 \int_0^1 |\tilde{u}(t,x,y)|^2\ dxdy.$$

The above inequality can then be written as

$$\frac{d}{dt}E(t) \le 0.$$

A direct integration yields, for any $t \ge 0$,

$$E(t) \le E(0).$$

Using the initial condition $\tilde{u}|_{t=0} = 0$, we have

$$E(0) = 0.$$

It follows from the definition of E that $E(t) \ge 0$, so

$$0 \le E(t) \le E(0) = 0,$$

which implies $E(t) \equiv 0$, and hence,

$$|\tilde{u}|^2 \equiv 0.$$

This implies

$$\tilde{u} \equiv 0.$$

That is, $u_1 \equiv u_2$. Hence the solution is unique.