K. Chandrasekharan

Elliptic Functions

With 14 Figures

Springer-Verlag
Berlin Heidelberg New York Tokyo

Chapter IV

The zeta-function and the sigma-function of Weierstrass

§ 1. The function $\zeta(z)$. Weierstrass's ζ-function is a meromorphic function, which has *simple* poles, with residues equal to one, at all points which correspond to the periods of Weierstrass's $\wp$-function. It is *not* elliptic. But every elliptic function can be expressed in terms of ζ and its derivatives; in fact $\zeta'(z) = -\wp(z)$.

As in the case of $\wp(z)$, we shall define $\zeta(z)$ by an infinite series. Let ω_1, ω_2 be two complex numbers, with $\operatorname{Im} \dfrac{\omega_2}{\omega_1} > 0$. Then the series

$$\sum_{|\omega| > 2R > 0} \left\{ \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\}, \qquad \omega = m\omega_1 + n\omega_2,$$

where m and n run through all integers, converges uniformly in the circle $|z| \leqq R$, since for $|\omega| > 2R \geqq 2|z|$, we have

$$\left| \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right| \leqq \frac{|z|^2}{|\omega|^3 \left(1 - \dfrac{|z|}{|\omega|}\right)} \leqq \frac{2|z|^2}{|\omega|^3},$$

and the series $\sum_{\omega \neq 0} |\omega|^{-3}$ converges (Theorem 1, Ch. III). It follows that the series

$$\frac{1}{z} + \sum_{\omega \neq 0} \left\{ \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\}$$

converges *absolutely* for $z \neq \omega$, so that the sum is independent of the order of the terms, and that for every finite $R > 0$, the series converges uniformly in the circle $|z| \leqq R$, after the omission of a certain number of initial terms. If we *define*

$$\begin{aligned} \zeta(z) &\equiv \zeta(z;\omega) \equiv \zeta(z;\omega_1,\omega_2) \\ &= \frac{1}{z} + \sum_{\omega \neq 0} \left\{ \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\}, \end{aligned} \tag{1.1}$$

for complex z, with $\omega = m\omega_1 + n\omega_2$; $\operatorname{Im} \dfrac{\omega_2}{\omega_1} > 0$; and $m, n = 0, \pm 1, \pm 2, \ldots$, then $\zeta(z)$ is holomorphic except for the points $z = \omega$, where it has simple poles with residue 1. It is *not* an elliptic function with periods ω_1, ω_2, since in a fundamental period-parallelogram, it has only one simple pole with residue $+1$

(Theorem 2, Chapter II). Since ω can be replaced by $-\omega$ in the defining series without altering the sum, $\zeta(z)$ is an odd function of z. By differentiation, we see that

$$\zeta'(z) = -\wp(z). \tag{1.2}$$

In a neighbourhood of the origin, we have the expansion

$$\zeta(z) = \frac{1}{z} - \frac{b_1}{3} z^3 - \frac{b_2}{5} z^5 - \ldots, \tag{1.3}$$

since, by Theorem 4 of Chapter III, $\wp(z) = \dfrac{1}{z^2} + \sum_{n=1}^{\infty} b_n z^{2n}$ near $z = 0$.

If λ is any complex number, $\lambda \neq 0$, then

$$\zeta(\lambda z; \lambda\omega_1, \lambda\omega_2) = \frac{1}{\lambda} \zeta(z;\omega_1,\omega_2). \tag{1.4}$$

By integrating the relation $\wp(z+\omega_1) = \wp(z)$, we obtain

$$\zeta(z+\omega_1) = \zeta(z) + 2\eta_1,$$

say, where $2\eta_1$ is *independent* of z. On setting $z = -\dfrac{\omega_1}{2}$, we obtain

$$\begin{aligned} \zeta\left(\frac{\omega_1}{2}\right) &= \zeta\left(-\frac{\omega_1}{2} + \omega_1\right) = \zeta\left(-\frac{\omega_1}{2}\right) + 2\eta_1 \\ &= -\zeta\left(\frac{\omega_1}{2}\right) + 2\eta_1, \end{aligned}$$

since $\zeta(z)$ is an odd function of z. Hence $\eta_1 = \zeta\left(\dfrac{\omega_1}{2}\right)$. If η_2 is defined by the relation: $\zeta(z+\omega_2) = \zeta(z) + 2\eta_2$, we similarly obtain: $\eta_2 = \zeta\left(\dfrac{\omega_2}{2}\right)$. If we then set $\eta_3 = \eta_2 + \eta_1$, and $\omega_3 = \omega_1 + \omega_2$, we have $\zeta(z+\omega_3) = \zeta(z) + 2(\eta_1+\eta_2) = \zeta(z) + 2\eta_3$, hence $\eta_3 = \zeta\left(\dfrac{\omega_1+\omega_2}{2}\right)$. We summarize these observations in the following

Theorem 1. *The zeta-function of Weierstrass, which is denoted by $\zeta(z)$, and defined in (1.1), is an odd function of z, with the properties*

$$\begin{aligned} &\zeta'(z) = -\wp(z); \\ &\zeta(z+\omega_1) = \zeta(z) + 2\eta_1, \qquad \eta_1 = \zeta\left(\frac{\omega_1}{2}\right); \\ &\zeta(z+\omega_2) = \zeta(z) + 2\eta_2, \qquad \eta_2 = \zeta\left(\frac{\omega_2}{2}\right); \\ &\zeta(z+\omega_3) = \zeta(z) + 2\eta_3, \qquad \eta_3 = \zeta\left(\frac{\omega_3}{2}\right), \end{aligned}$$

where $\omega_3=\omega_1+\omega_2$, $\eta_3=\eta_1+\eta_2$, *and* (ω_1, ω_2) *is a pair of basic periods of Weierstrass's elliptic function* $\wp(z)$, *with* $\operatorname{Im}\frac{\omega_2}{\omega_1}>0$. *It is holomorphic except for the points* $z=\omega$, *where it has simple poles with residue 1.*

Remarks. Since $\zeta(z)$ is *not* an elliptic function, it cannot happen that $\eta_1=\eta_2=0$.

If m and n are integers, we have obviously

$$\zeta(z+m\omega_1+n\omega_2)=\zeta(z)+2m\eta_1+2n\eta_2\,.$$

The relation between the constants η_1 and η_2, and the periods ω_1 and ω_2, is given by

Theorem 2 (Legendre). *If* (ω_1, ω_2) *is a pair of basic periods of Weierstrass's* $\wp$*-function, so chosen that* $\operatorname{Im}\frac{\omega_2}{\omega_1}>0$, *then*

$$\eta_1\omega_2-\eta_2\omega_1=\pi i\,. \tag{1.5}$$

Proof. We choose a period-parallelogram of $\wp(z)$ with vertices A, B, C, D, in such a way that (the pole of $\wp(z)$ at) the origin is located at its centre (Fig. 6).

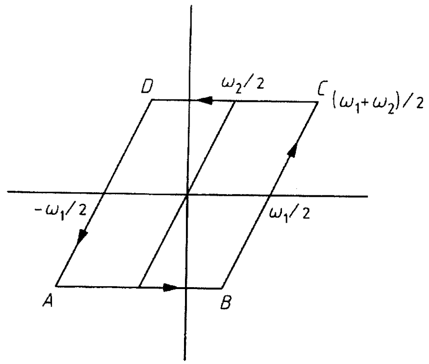

Fig. 6

By Cauchy's theorem of residues, we have

$$\int_{ABCDA}\zeta(z)dz=2\pi i\,. \tag{1.6}$$

On the other hand, we have, by Theorem 1,

$$\int_{CD}\zeta(z)dz=\int_{BA}\zeta(z+\omega_2)dz=\int_{BA}\zeta(z)dz+\int_{BA}2\eta_2 dz$$
$$=\int_{BA}\zeta(z)dz-2\eta_2\omega_1\,,$$

so that

$$\int_{AB}\zeta(z)dz+\int_{CD}\zeta(z)dz=-2\eta_2\omega_1\,,$$

and similarly

$$\int_{BC}\zeta(z)dz+\int_{DA}\zeta(z)dz=2\eta_1\omega_2\,,$$

which taken together with (1.6) yields (1.5).

§ 2. The function $\sigma(z)$. Weierstrass's σ-function is an entire function, with its zeros, all of which are *simple*, located precisely at those points which correspond to the periods of the $\wp$-function. It is an *odd* function, which is *not* elliptic. But every elliptic function can be expressed in terms of the σ-function. To define it we need some preliminary analysis.

Lemma. *If* $E(z)=(1-z)e^{z+\frac{1}{2}z^2}$, *where* z *is complex, then*

$$|E(z)-1|\leqq 2|z|^3, \quad \textit{for} \quad |z|\leqq\tfrac{1}{2}. \tag{2.1}$$

Proof. For $|z|<1$, we have

$$1-z=e^{\operatorname{Log}(1-z)}=e^{-\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\dots\right)};$$

hence

$$E(z)=e^{-\sum\limits_{k=3}^{\infty}\frac{z^k}{k}}=e^{g(z)}, \qquad g(z)=-\sum_{k=3}^{\infty}\frac{z^k}{k}. \tag{2.2}$$

If, in addition, $|z|\leqq\frac{1}{2}$, then

$$|g(z)|\leqq\sum_{k=3}^{\infty}\frac{|z|^k}{k}\leqq\frac{1}{3}\sum_{k=3}^{\infty}|z|^k$$
$$=\frac{|z|^3}{3}\cdot\frac{1}{1-|z|}\leqq\frac{2}{3}|z|^3. \tag{2.3}$$

To estimate $|E(z)-1|=|e^{g(z)}-1|$, we need the two inequalities:

(a) $|e^z-1|\leqq e^{|z|}-1$, which follows from the expansion of e^z; and

(b) $e^x-1\leqq xe^x$, for any real x. This follows from the inequality

$$1+x\leqq e^x, \quad \text{for every real } x; \tag{2.4}$$

which is obvious if $x\geqq 0$; and if $x<0$, the function $e^x-(1+x)$ has a negative derivative, hence decreases in $[-\infty, 0]$ as x increases, and equals zero for $x=0$, so that it is positive for $x<0$. Replacing x by $-x$ in (2.4), we get (b).

Now

$$|E(z)-1|=|e^{g(z)}-1|\leqq e^{|g(z)|}-1\leqq|g(z)|\cdot e^{|g(z)|}$$
$$\leqq\frac{2}{3}|z|^3e^{|z|^3}\leqq 2|z|^3,$$

since $e^{|z|^3}<e<3$, for $|z|\leqq\frac{1}{2}$, and (2.1) follows.

We shall define $\sigma(z)$ by an infinite product, for the absolute convergence of which we shall use the above lemma.

If (ω_1, ω_2) is a pair of complex numbers, with $\operatorname{Im}\frac{\omega_2}{\omega_1}>0$, and $\omega=m\omega_1+n\omega_2$, with $m,n=0, \pm 1, \pm 2, \ldots$, then for $|z|\leqq R$, we have

$$\sum_{|\omega|>2R>0}\left|E\left(\frac{z}{\omega}\right)-1\right|\leqq 2\sum_{|\omega|>2R>0}\left|\frac{z}{\omega}\right|^3,$$

by the lemma, and the last series converges uniformly in the circle $|z|\leqq R$, for every finite $R>0$. Hence the infinite product

$$z\prod_{\omega\neq 0}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

converges absolutely, and uniformly in the circle $|z|\leqq R$, for every finite $R>0$, and its value does not depend on the order of the factors. If we *define*

$$\sigma(z)\equiv\sigma(z;\omega)\equiv\sigma(z;\omega_1,\omega_2)=$$

$$=z\prod_{\omega\neq 0}\left(1-\frac{z}{\omega}\right)e^{\frac{z}{\omega}+\frac{z^2}{2\omega^2}}, \tag{2.5}$$

then $\sigma(z)$ is an *entire* function of z, which is *not* constant, hence *not* elliptic. Since we may replace ω by $-\omega$, without altering the product, we have $\sigma(-z)=-\sigma(z)$, so that $\sigma(z)$ is an *odd* function of z. It has *simple zeros* precisely at the points $z=\omega$.

If λ is any complex number, with $\lambda\neq 0$, then

$$\sigma(\lambda z;\lambda\omega_1,\lambda\omega_2)=\lambda\sigma(z;\omega_1,\omega_2)\,. \tag{2.6}$$

We note that

$$\zeta(z)=\frac{d}{dz}(\log\sigma(z))=\frac{\sigma'(z)}{\sigma(z)}, \tag{2.7}$$

the many-valuedness of $\log\sigma(z)$ being removed by the differentiation.

By integrating the relation $\zeta(z+\omega_1)=\zeta(z)+2\eta_1$, we have $\sigma(z+\omega_1)=ce^{2\eta_1 z}\sigma(z)$, where c is a constant. On setting $z=-\frac{\omega_1}{2}$, we obtain $\sigma\left(\frac{\omega_1}{2}\right)=-ce^{-\eta_1\omega_1}\sigma\left(\frac{\omega_1}{2}\right)$, since $\sigma(z)$ is an *odd* function of z, hence $c=-e^{\eta_1\omega_1}$, and

$$\sigma(z+\omega_1)=-\sigma(z)\cdot e^{2\eta_1\left(z+\frac{\omega_1}{2}\right)};$$

similarly also

$$\sigma(z+\omega_2)=-\sigma(z)\cdot e^{2\eta_2\left(z+\frac{\omega_2}{2}\right)}; \tag{2.8}$$

and

$$\sigma(z+\omega_3)=-\sigma(z)\cdot e^{2\eta_3\left(z+\frac{\omega_3}{2}\right)},$$

where $\omega_3=\omega_1+\omega_2$, $\eta_3=\eta_1+\eta_2$.

Since $\sigma(z)$ is an *entire* function of z, which is *odd*, with simple zeros at $z=m\omega_1+n\omega_2$; $m,n=0, \pm 1, \pm 2, \ldots$, we have for $\sigma(z)$ an expansion of the form

$$\sigma(z)=a_1 z+a_3 z^3+\ldots,$$

valid in the whole z-plane, and by differentiation, a corresponding one for $\sigma'(z)$. On the other hand, $\zeta(z)$ has the expansion (1.3) near the origin:

$$\zeta(z)=\frac{1}{z}-\frac{b_1}{3}z^3-\frac{b_2}{5}z^5-\ldots,$$

where the b's are those given by the expansion

$$\wp(z)=\frac{1}{z^2}+\sum_{n=1}^{\infty}b_n z^{2n},$$

near the origin. We have seen that the coefficients b_n satisfy a recurrence relation, which enables one to calculate them, and that they are, in fact, polynomials in g_2, g_3, the invariants of $\wp(z)$, with positive, rational numbers as coefficients (Theorem 4, Chapter III). Hence we deduce that

$$\sigma(z)=z-\frac{g_2\cdot z^5}{2^4\cdot 3\cdot 5}-\frac{g_3 z^7}{2^3\cdot 3\cdot 5\cdot 7}-\frac{g_2^2 z^9}{2^9\cdot 3^2\cdot 5\cdot 7}-\frac{g_2\cdot g_3 z^{11}}{2^7\cdot 3^2\cdot 5^2\cdot 7\cdot 11}-\ldots \tag{2.9}$$

in the whole z-plane. We have thus proved

Theorem 3. *The sigma-function of Weierstrass, which is denoted by $\sigma(z)$, and defined in* (2.5), *is an entire, odd function of z, with zeros precisely at the points $z=\omega$, which correspond to the periods of $\wp(z)$, and has the properties*

$$\zeta(z)=\frac{\sigma'(z)}{\sigma(z)};$$

$$\sigma(z+\omega_1)=-\sigma(z)e^{2\eta_1\left(z+\frac{1}{2}\omega_1\right)};$$

$$\sigma(z+\omega_2)=-\sigma(z)e^{2\eta_2\left(z+\frac{1}{2}\omega_2\right)};$$

$$\sigma(z+\omega_3)=-\sigma(z)e^{2\eta_3\left(z+\frac{1}{2}\omega_3\right)},$$

where $\omega_3=\omega_1+\omega_2$, $\eta_3=\eta_1+\eta_2$, and (ω_1,ω_2) is a pair of basic periods of $\wp(z)$, with $\operatorname{Im}\frac{\omega_2}{\omega_1}>0$, while η_1, η_2 are defined as in Theorem 1.

Remarks. The infinite product for $\sigma(z)$ is similar to that of $\sin z$ given by

$$\sin z=z\prod_{m\neq 0}\left\{\left(1-\frac{z}{m\pi}\right)e^{z/m\pi}\right\}.$$

Now $\frac{d}{dz}(\log \sin z) = \cot z$, and $\cot z = \frac{1}{z} + \sum_{m \neq 0} \left\{ \frac{1}{z - m\pi} + \frac{1}{m\pi} \right\}$. As $\frac{d}{dz}(\cot z) = -\operatorname{cosec}^2 z$, so also we have $\frac{d}{dz}(\zeta(z)) = -\wp(z)$.

§ 3. An expression for elliptic functions. Every elliptic function can be expressed in terms of the σ-function. We can thereby furnish an alternative proof of Theorem 7 of Chapter III.

Theorem 4. *Let $f(z)$ be an elliptic function with (ω_1, ω_2) as a pair of primitive (or, reduced) periods, with $\operatorname{Im} \frac{\omega_2}{\omega_1} > 0$. Let $a_1, a_2, \ldots, a_n$ be the zeros, and $b_1, b_2, \ldots, b_n$ the poles of $f(z)$, repeated according to their degree of multiplicity, which lie in a fundamental period-parallelogram P, so that the difference between the sum of the zeros and the sum of the poles, namely*

$$\Omega = (a_1 + a_2 + \ldots + a_n) - (b_1 + b_2 + \ldots + b_n) \tag{3.1}$$

is a period of $f(z)$. Then

$$f(z) = c \cdot \frac{\sigma(z - a_1) \cdots \sigma(z - a_n)}{\sigma(z - b_1) \cdots \sigma(z - b_n - \Omega)}, \tag{3.2}$$

where c is a constant.

Conversely every function of the form (3.2), where the a's and b's are complex numbers satisfying condition (3.1), is an elliptic function.

Proof. The meromorphic function φ defined by

$$\varphi(z) = \frac{\sigma(z - a_1) \cdots \sigma(z - a_n)}{\sigma(z - b_1) \cdots \sigma(z - b_n - \Omega)}$$

has precisely the same zeros and poles as $f(z)$ inside the fundamental parallelogram P, for the only zero of the factor $\sigma(z - a_k)$ in P is at $z = a_k$ (and it has no poles, since it is entire). Further $\varphi(z)$ has ω_1 and ω_2 as periods, since

$$\begin{aligned}\varphi(z + \omega_1) &= \frac{(-1)^n e^{2\eta_1\left(nz + n\frac{\omega_1}{2} - a_1 - a_2 - \ldots - a_n\right)} \cdot \sigma(z - a_1) \cdots \sigma(z - a_n)}{(-1)^n e^{2\eta_1\left(nz + n\frac{\omega_1}{2} - b_1 - b_2 - \ldots - b_n - \Omega\right)} \cdot \sigma(z - b_1) \cdots \sigma(z - b_n - \Omega)} \\ &= \varphi(z)\end{aligned} \tag{3.3}$$

by (2.8), and similarly also $\varphi(z + \omega_2) = \varphi(z)$. The quotient $\frac{f(z)}{\varphi(z)}$ is therefore doubly periodic, and has no poles or zeros, hence a constant c.

Conversely, if the a's and b's satisfy (3.1), then (3.3) shows that the function given by (3.2) is elliptic.

Corollary. *We have the formulae*

$$\wp(u) - \wp(v) = -\frac{\sigma(u + v)\sigma(u - v)}{\sigma^2(u)\sigma^2(v)}, \tag{3.4}$$

$$-\frac{\wp'(v)}{\wp(u) - \wp(v)} = \zeta(u + v) - \zeta(u - v) - 2\zeta(v), \tag{3.5}$$

and

$$\frac{1}{2} \cdot \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} = \zeta(u + v) - \zeta(u) - \zeta(v). \tag{3.6}$$

Proof. Consider the elliptic function $\wp(u) - \wp(v)$ as a function of u, with a fixed v, which is not equal to a period of $\wp$.

If we assume, for the moment, that v is not equal to a half-period of $\wp$, then, as in Theorem 4, with $a_1 = v$, $a_2 = -v$, $b_1 = 0$, $b_2 = 0$, we see that

$$\frac{\sigma(u + v)\sigma(u - v)}{\sigma^2(u)}$$

is an elliptic function, with the same zeros and poles as $\wp(u) - \wp(v)$, and with the same periods, so that

$$\wp(u) - \wp(v) = c \cdot \frac{\sigma(u + v)\sigma(u - v)}{\sigma^2(u)},$$

where c is a constant independent of u. By considering the principal parts of both sides of this formula, we deduce that $1 = -c\sigma^2(v)$, hence (3.4). By continuity this holds also when v is a half-period. By logarithmic differentiation of (3.4), relative to u, we obtain

$$\frac{\wp'(u)}{\wp(u) - \wp(v)} = \zeta(u + v) + \zeta(u - v) - 2\zeta(u);$$

and by interchanging u with v, we obtain

$$-\frac{\wp'(v)}{\wp(u) - \wp(v)} = \zeta(u + v) - \zeta(u - v) - 2\zeta(v),$$

which is (3.5). By adding the last two formulae we get (3.6).

If u or v is a period of $\wp$, then both sides of (3.4) are infinite.

Remarks

(i) On setting $u = v + h$ in (3.4), multiplying by $\frac{1}{h}$, $h \neq 0$, and then letting $|h| \to 0$, we obtain the formula

$$\wp'(v) = -\frac{\sigma(2v)}{\{\sigma(v)\}^4}. \tag{3.7}$$

(ii) By differentiating formula (3.6) with respect to u, and then interchanging u and v, we get two new formulae. By adding them together, and using the relation $\wp''=6\wp^2-\frac{g_2}{2}$, one deduces the addition-theorem for $\wp$.

(iii) We have seen in Theorem 4 that every elliptic function can be expressed in terms of the sigma-function alone. In Theorem 8, of Chapter III, it was shown that every elliptic function can be expressed in terms of $\wp$ and $\wp'$. After Hermite, one can express every elliptic function as a linear combination of ζ and its derivatives.

(iv) Let Ω be *any* period of the $\wp$-function. By formula (3.4), we have

$$\wp(u+\Omega)-\wp(u)=0=-\frac{\sigma(2u+\Omega)\sigma(\Omega)}{\sigma^2(u+\Omega)\cdot\sigma^2(u)},$$

for all u, hence $\sigma(\Omega)=0$, so that Ω is a zero of $\sigma(u)=\sigma(u;\omega_1,\omega_2)$, which implies that $\Omega=m\omega_1+n\omega_2$, where m and n are integers (cf. Theorem 2, Chapter I).

Notes on Chapter IV

§§ 1–2. The zeta-function of Weierstrass can be defined by the equation

$$\frac{d\zeta(z)}{dz}=-\wp(z),$$

together with the condition $\lim_{z\to 0}\{\zeta(z)-z^{-1}\}=0$. See, for instance, Whittaker and Watson (cited in the Notes on Ch. I, § 4), p. 445–447. One can then introduce the sigma-function by the equation

$$\frac{d}{dz}\{\log\sigma(z)\}=\zeta(z),$$

together with the condition $\lim_{z\to 0}\{\sigma(z)/z\}=1$.

The definition of $\sigma(z)$ by an infinite product is of interest, since it is an instance of Weierstrass's general theorem on the representation of an entire function by an infinite product. See H. A. Schwarz (cited in the Notes on Ch. III), § 6; Saks and Zygmund (cited in the Notes on Ch. I), § 2.

"The first mathematician to discover that the elliptic functions could be got at starting from the infinite product $\prod\left(1-\frac{x}{n}\right)$ was Eisenstein, *Crelle's J.* Bd. 27, where he has two articles on the subject.... In *Crelle*, Bd. 35, he gives a long discussion also on the convergence of these last products. The necessity of a factor of convergence seems not to have occurred to him." So writes A. L. Daniels, *American J. Math.* 6 (1884), 253–269. See also Jacobi's remarks, *Crelle's J.* 30 (1845), 183–184; *Werke*, II, 85–86. Of interest therefore are the emendations by André Weil, *Elliptic functions according to Eisenstein and Kronecker*, *Ergebnisse der Math.* 88 (Springer Verlag, 1976). See also the Notes on Chapter VII, § 5.

§ 3. For a number of formulae involving the σ-function, see the book by H. A. Schwarz (cited in the Notes on Ch. III), §§ 5–8, 13–16.

Every elliptic function can be expressed as a linear combination of zeta-functions and their derivatives. See Ch. Hermite, *Crelle's J.* 82, p. 343; *Oeuvres*, III, 420–424; Whittaker and Watson (cited in the Notes on Ch. I, § 4), § 20.52; Saks and Zygmund (cited in the Notes on Ch. I, § 1), Chapter VIII, § 8.

It is known that if $x+y+z=0$, then

$$\{\zeta(x)+\zeta(y)+\zeta(z)\}^2+\zeta'(x)+\zeta'(y)+\zeta'(z)=0,$$

a result due to G. Frobenius and L. Stickelberger, *J. für Math.* 83 (1877), 175–179. This is not considered an addition-theorem, since $\zeta'(x)$, $\zeta'(y)$, $\zeta'(z)$ are not algebraic functions of $\zeta(x)$, $\zeta(y)$, $\zeta(z)$. See Whittaker and Watson (loc. cit.), § 20.41.

Corresponding to the *limiting cases* $\varphi(z;\omega_1)$ and $\varphi(z)$ of $\wp(z)$ (see the Notes on Ch. III, § 5) we have the *limiting cases* of $\zeta(z)$ and $\sigma(z)$, namely

$$\zeta(z;\omega_1,\omega_2\to\infty)=\frac{1}{3}\left(\frac{\pi}{\omega_1}\right)^2 z+\frac{\pi}{\omega_1}\cot\left(\frac{\pi z}{\omega_1}\right),$$

$$\zeta(z;\omega_1\to\infty,\omega_2\to\infty)=\frac{1}{z},$$

and

$$\sigma(z;\omega_1,\omega_2\to\infty)=\frac{\omega_1}{\pi}\sin\left(\frac{\pi z}{\omega_1}\right)\cdot e^{\frac{1}{6}\left(\frac{\pi z}{\omega_1}\right)^2},$$

$$\sigma(z;\omega_1\to\infty,\omega_2\to\infty)=z.$$

If ω is a primitive period of a $\wp$-function, with *algebraic* invariants g_2, g_3, and $\eta=2\zeta\left(\frac{\omega}{2}\right)$, then any linear combination $\alpha\omega+\beta\eta$, where α and β are *algebraic*, not both zero, is *transcendental*. It follows that the circumference of an ellipse with *algebraic* axes-lengths is transcendental. These are consequences of a general theorem of Th. Schneider *Math. Annalen*, 121 (1949), 131–140, and are so described in Chapter 6 of the book by Alan Baker (cited in the Notes on Ch. III, § 5).