THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 10

Date: Nov 18, 2024.

Instruction: *In order to have a better preparation of the coming test, you are recommended to try the problems below **BEFORE** the coming tutorial. Due to the time limit, the tutorial will NOT be able to go through all of the problems. Therefore, you should actively tell our TA which problems you wish to discuss first.*

Information: This collection of problems is intended to give you practice problems that are comparable in format and difficulty to those which will appear in the coming test. The questions in the actual exam will be **DIFFERENT**.

Problem 1. Do Problem 4 of Dec 2021 Final Exam.

Problem 2. Let u_1 and u_2 be two solutions to the same Laplace equation

$$\Delta u := \partial_{xx} u + \partial_{yy} u = 0 \qquad \text{on } \Omega = [-4,4] \times [-4,4],$$

but u_1 and u_2 satisfy different boundary conditions: for $i = 1$, 2,

$$\begin{cases} u_i|_{x=-4} = g_i \\ u_i|_{x=4} = h_i \\ u_i|_{y=-4} = \phi_i \\ u_i|_{y=4} = \psi_i \end{cases}$$

where g_i, h_i, ϕ_i and ψ_i, are given data. Prove that if

$$\begin{cases} g_1 \leq g_2 \\ h_1 \leq h_2, \\ \phi_1 \leq \phi_2 \\ \psi_1 \leq \psi_2 \end{cases}$$

then

$$u_1 \leq u_2.$$

Problem 3. (i) Let $u := u(t,x) \in C^2([0,T] \times [0,L])$ be a solution to

$$\partial_t u - \partial_{xx} u = |\partial_x u|^2 + 3,$$

for $0 < x < L$ and $0 < t < T$. Show that

$$\min_{\substack{0 \leq x \leq L \\ 0 \leq t \leq T}} u(t,x) \geq \min \left\{ \min_{0 \leq x \leq L} u(0,x), \min_{0 \leq t \leq T} u(t,0), \min_{0 \leq t \leq T} u(t,L) \right\}.$$

(ii) Let

$$D := \left\{ (x,y) \in \mathbb{R}^2;\ x^2 + y^2 < 4 \right\},$$

and $u \in C(\bar{D}) \cap C^2(D)$ be a solution to

$$3\partial_{xx} u + 8\, \partial_{yy} u = u^2\, \partial_y u.$$

Show that

$$\max_{\bar{D}} |u| = \max_{\partial D} |u|.$$

Problem 4. Let u be a solution to the boundary value problem

$$\begin{cases} \Delta u := \partial_{xx} u + \partial_{yy} u = 0 \qquad \text{on } \Omega = [-1,1] \times [-1,1] \\ u|_{x=-1} = u|_{x=1} \equiv 0 \\ u|_{y=-1} = u|_{y=1} \equiv \phi(x), \end{cases}$$

Prove the following statements:

(i) If ϕ is even, that is,
$$\phi(-x) = \phi(x),$$
then u is also even in x, that is,
$$u(-x, y) = u(x, y).$$
(Hint: let $v(x, y) := u(-x, y)$, then what is the initial and boundary value problem that v satisfies?).

(ii) If ϕ is odd, that is,
$$\phi(-x) = -\phi(x),$$
then u is also odd in x, that is,
$$u(-x, y) = -u(x, y).$$

Problem 5. Do Problem 1 of Dec 2020 Final Exam.

Problem 6. Let u satisfy the following PDE

$$\partial_{tt}u - c^2\partial_{xx}u = -\alpha u \qquad \text{for } -\infty < x < \infty \text{ and } \alpha, c, t > 0. \tag{1}$$

Given finite interval (a, b), we define the local energy by

$$E(t) := \int_{a+Mt}^{b-Mt} e(t,x)\, dx, \text{ where } e(t,x) := \tfrac{1}{2}|\partial_t u|^2 + \tfrac{c^2}{2}|\partial_x u|^2 + \tfrac{\alpha}{2}|u^2|.$$

(i) Let $p(t, x) := \partial_t u\, \partial_x u$. Prove that $\partial_t e = c^2\partial_x p$.

(ii) Show that $e \pm cp = \frac{1}{2}(\partial_t u \pm c\partial_x u)^2 + \frac{\alpha}{2}u^2$.

(iii) Using part (i), verify via a direct differentiation that

$$\frac{dE}{dt}(t) = c^2[p(t, b - Mt) - p(t, a + Mt)] - M[e(t, b - Mt) + e(t, a + Mt)].$$

(Hint: Use $\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\, dx = \int_{a(t)}^{b(t)} \partial_t f(t,x)\, dx + f(t, b(t))b'(t) - f(t, a(t))a'(t)$.)

(iv) Suppose $M \geq c$. Using (ii) and (iii), show that $\dfrac{dE}{dt} \leq 0$. Hence if $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), show that $u \equiv 0$ in $\Delta := \{(x,t) \in (-\infty,\infty) \times [0,\infty) : a + Mt \leq x \leq b - Mt\}$.

Problem 7. Apply the **energy method** to show the uniqueness for the following problems:

(i)

$$\begin{cases} \partial_t u - \partial_{xx} u = -9u & \text{for } 0 < x < L,\ t > 0 \\ u|_{t=0} = \phi \\ u|_{x=0} = g \\ u|_{x=L} = h \end{cases}$$

where ϕ, g and h are given data.

(ii)

$$\begin{cases} \partial_{tt} u - 4\partial_{xx} u = -u - \partial_t u & \text{for } -1 < x < 1,\ t > 0 \\ u|_{t=0} = \phi \\ \partial_t u|_{t=0} = \psi \\ u|_{x=-1} = g \\ u|_{x=1} = h \end{cases}$$

where ϕ, ψ, g and h are given data.

Problem 8.

Let the concentration $u := u(t,x)$ satisfy

$$\begin{cases} \partial_t u - \partial_{xx} u = 3t^2 x^2 & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(0,x) = e^x + 1 \\ \partial_x u(t,0) = 1 \\ \partial_x u(t,1) = 4. \end{cases}$$

Compute the total mass $M(t) := \int_0^1 u(t,x)\,dx$.

Problem 9. Let u be a solution to the following initial value problem

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = 0 & \\ \partial_t u|_{t=0} = 3x^2. & \end{cases}$$

(i) Find $u(t,x)$.

(ii) Let the local energy $E(t)$ be defined by $E(t) := \dfrac{1}{2}\displaystyle\int_0^1 |\partial_t u|^2 + |\partial_x u|^2\,dx$. Find $E'(t)$.

Problem 10. Do Problem 3 of Dec 2019 Final Exam.