THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH4302: Algebra II

May 18, 2024 2:30pm. – 5:00pm

No calculators are allowed in the examination.

Answer ALL EIGHT questions

Note: You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

1. (10 points). Answer "True" or "False" to each of the following six statements. For this problem only, you do not need to explain your answers.

   1) The principal ideal $I = \langle x-1 \rangle$ of $\mathbb{Z}[x]$ is prime but not maximal;

   2) The quotient ring $\mathbb{Q}[x]/\langle x^5 - 5 \rangle$ is a field;

   3) The $\mathbb{Z}[x]$-module $M = \mathbb{Z}[x]/\langle x-1 \rangle \oplus \mathbb{Z}[x]/\langle (x+15)^3 \rangle$ is a torsion module;

   4) For any field K and any non-constant $f(x) \in K[x]$, if $f(x)$ is irreducible in $K[x]$, then $f(x^2)$ is irreducible in $K[x]$;

   5) For any field K and any non-constant $f(x) \in K[x]$, if $f(x^2)$ is irreducible in $K[x]$, then $f(x)$ is irreducible in $K[x]$.

2. (10 points) Consider the $\mathbb{R}[x]$-module $V = \mathbb{R}[x]/\langle (x-3)^2(x+5)^2 \rangle$ as a vector space over $\mathbb{R}$, and let $T : V \to V$ be the $\mathbb{R}$-linear map defined by the multiplication by $x \in \mathbb{R}[x]$.

   1) Find a basis of V with respect to which T is in Jordan canonical form;

   2) Find a basis of V with respect to which T is in rational canonical form.

3. 1) (5 points) Let $f(x) = x^5 + 2x^4 + 4x^3 - 6x + 2 \in \mathbb{Q}[x]$ and let α be a root of f in $\mathbb{C}$. Determine whether or not $\sqrt[3]{2} \in \mathbb{Q}(\alpha)$.

   2) (5 points) Let β be any root of

$$g(x) = -x^{19} + i\sqrt[5]{11}x^4 + \frac{\sqrt[3]{5} + 191}{\sqrt{37} + 1}x^3 + 1 \in \mathbb{C}[x]$$

   in $\mathbb{C}$. Show that β is a root of a polynomial with coefficients in $\mathbb{Q}$.

4. (10 points) Let $p > 2$ be a prime number. Show that if the angle $\frac{2\pi}{p}$ is constructable by a ruler and a compass, then $p-1$ must be a power of 2.

5. 1) (5 points) State the elementary divisor form of the classification theorem of finitely generated modules over a PID;

2) (5 points) Use the classification theorem of finitely generated modules over a PID to prove the classification theorem of finite abelian groups;

3) (10 points) Let K be an arbitrary field and consider $K\backslash\{0\}$ as an abelian group under multiplication in K. Use the classification theorem of finite abelian groups to show that every finite subgroup of $K\backslash\{0\}$ is cyclic;

4) (5 points) Use the result in 3) to show that every finite extension of every finite field is simple.

6. (10 points) Let p be a prime number and let $g(x)$ be any irreducible polynomial in $\mathbb{F}_p[x]$. Show that $\mathbb{F}_p[x]/\langle g(x)\rangle$ is a splitting field of $g(x)$ over $\mathbb{F}_p$.

7. (10 points) Let $K \subset L$ be a finite Galois field extension with Galois group $G = \mathrm{Gal}(L/K)$, and let $\alpha \in L$. Let $G\alpha = \{\sigma(\alpha) : \sigma \in G\} = \{\alpha = \alpha_1, \alpha_2, \ldots, \alpha_r\}$, where $\alpha_i \neq \alpha_j$ for $i, j \in [1, r]$ and $i \neq j$. Show that

$$q(x) = (x - \alpha)(x - \alpha_2) \cdots (x - \alpha_r)$$

is the minimal polynomial of α in $K[x]$.

8. 1) (5 points) Recall that a *real quadratic extension of* $\mathbb{Q}$ is, by definition, a sub-field M of $\mathbb{R}$ such that $|M : \mathbb{Q}| = 2$. Show that every real quadratic extension of $\mathbb{Q}$ is of the form $\mathbb{Q}(\sqrt{n})$, where n is a square-free positive integer, i.e., $n > 1$ and n is a product of pairwise distinct prime numbers;

2) (10 points) Show that any sub-field L of $\mathbb{C}$ which is a degree four Galois extension of $\mathbb{Q}$ must contain a real quadratic extension of $\mathbb{Q}$.

******* **END OF PAPER** *******