Algebra II: Tutorial 3

February 21, 2022

Problem 1. Determine whether or not the following polynomials are irreducible over $\mathbb{Q}$:

1. $f(x) = 2x^9 + 12x^4 + 36x^3 + 27x + 6$,
2. $f(x) = x^4 + 25x + 7$.

Problem 2. Use Eisenstein's criterion to show that $x^{p-1} + x^{p-2} + \cdots + 1$ is irreducible over $\mathbb{Q}$.

Problem 3 (Follow-up question from tutorial 2: ACCPI domains)**.** Let R be an integral domain, and suppose that R satisfies the ACCPI condition. For this question, we will call such rings ACCPI domains.

1. Show that any non-zero element in $R \backslash R^\times$ admits a factorisation into a product of finitely many irreducible elements.

Consider the ring of algebraic integers $\overline{\mathbb{Z}}$.

1. Show that $2^{\frac{1}{2^n}} \in \overline{\mathbb{Z}}$.
2. Show that there are no irreducible elements in $\overline{\mathbb{Z}}$. (Hint: consider the square root of an algebraic integer).
3. Show that, $\forall n \mathbb{N}, (2^{\frac{1}{2^n}}) \subset (2^{\frac{1}{2^{n+1}}})$. Does this chain terminate? Deduce that $\overline{\mathbb{Z}}$ does not have the ACCPI property.

Problem 4. Let K be a field, and denote by 1 the multiplicative identity in K. Show that the subfield of K generated by $\{1\}$ is the prime subfield of K. Deduce that the prime subfield of K is $\mathbb{Q}$ (resp. is $\mathbb{F}_p$) if and only if K has characteristic zero (resp. has characteristic p).