Holomorphic Line Bundles

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Contents

Backgroud: Let X be a compact Riemann surface and let $\mathcal{M}(X)$ denote the field of meromorphic functions on X.

- For $g = 0$, we have $X \cong \mathbb{P}^1$ and

$$\mathcal{M}(\mathbb{P}^1) = \{\text{rational functions}\}.$$

- For $g = 1$, we have $X = \mathbb{C}/L$ (a complex torus) and

$$\mathcal{M}(X) \cong \{\text{elliptic functions}\}.$$

  In this case, $\mathcal{M}(X)$ is generated by the Weierstrass functions $\wp$ and $\wp'$ satisfying the relation

$$(\wp')^2 = 4\wp^3 + g_2\wp + g_3,$$

  for some constants g_2, g_3.

- For $g \geq 2$, we have $X = D/\Gamma$, where D is the unit disk and Γ is a discrete subgroup of $\mathrm{Aut}(D)$. In this case, the space of automorphic forms

$$P_k^\Gamma(h), \quad h \in H^\infty(D),$$

  is used, and for sufficiently large k, the corresponding Poincaré series define an embedding

$$X \hookrightarrow \mathbb{P}^N.$$

To obtain more holomorphic or meromorphic functions, we consider holomorphic line bundles. Sections of holomorphic line bundles behave like *twisted holomorphic functions*. When $g = 0$: We can write

$$\mathbb{P}^1 = \{[t_0 : t_1] \mid t_0, t_1 \in \mathbb{C}\},$$

so $\mathbb{C} \subset \mathbb{P}^1$ via $z = t_1/t_0$. Rational functions on $\mathbb{P}^1$ are of the form

$$f = \frac{P(t_0, t_1)}{Q(t_0, t_1)},$$

where $P, Q \in \mathbb{C}[t_0, t_1]$ are homogeneous polynomials of the same degree ≥ 0, with $Q \neq 0$.

For example,

$$f([t_0 : t_1]) = \frac{t_0^2 + 2t_0t_1}{3t_1^2 + 5t_0^2} = \frac{1 + 2z}{3z^2 + 5}, \quad \text{where } z = \frac{t_1}{t_0}.$$

Why is $\dfrac{P}{Q}$ a well-defined function on $\mathbb{P}^1$? Because homogeneity makes it independent of the representative $t = (t_0, t_1)$ of a point $[t_0 : t_1]$:

$$P(\lambda t) = P(\lambda t_0, \lambda t_1) = \lambda^d P(t_0, t_1), \quad Q(\lambda t) = \lambda^d Q(t_0, t_1),$$

so

$$\frac{P(\lambda t)}{Q(\lambda t)} = \frac{P(t)}{Q(t)}, \quad \forall\, \lambda \in \mathbb{C}^*.$$

Therefore, the quotient $\frac{P}{Q}$ depends only on the point $[t_0 : t_1] \in \mathbb{P}^1$, not on its coordinates.

When $g = 1$, we have $X = \mathbb{C}/L$, where L is a lattice in $\mathbb{C}$. A *system of factors of automorphy* is a family of holomorphic functions

$$\{\exp(P_\omega z + Q_\omega)\}_{\omega \in L},$$

such that these functions satisfy the compatibility (or cocycle) condition described below.

A theta function θ with respect to the system $\{\exp(P_\omega z + Q_\omega)\}$ satisfies

$$\theta(z + \omega) = \exp(P_\omega z + Q_\omega)\, \theta(z), \qquad \forall\, \omega \in L.$$

Applying this twice gives the requirement for compatibility:

$$\begin{aligned}
\theta(z + \omega + \omega') &= \exp(P_{\omega'}(z + \omega) + Q_{\omega'})\, \theta(z + \omega) \\
&= \exp(P_{\omega'}(z + \omega) + Q_{\omega'})\, \exp(P_\omega z + Q_\omega)\, \theta(z) \\
&= \exp(P_{\omega+\omega'} z + Q_{\omega+\omega'})\, \theta(z).
\end{aligned}$$

This must hold for all $\omega, \omega' \in L$.

Thus, the *compatibility (or cocycle) condition* for the factors of automorphy is

$$\exp(P_{\omega+\omega'} z + Q_{\omega+\omega'}) = \exp(P_{\omega'}(z + \omega) + Q_{\omega'})\ \exp(P_\omega z + Q_\omega).$$

That is,

$$\boxed{e^{P_{\omega+\omega'}z+Q_{\omega+\omega'}} = e^{P_{\omega'}(z+\omega)+Q_{\omega'}}e^{P_\omega z+Q_\omega}.}$$

In general, for a complex manifold X with an open cover

$$\mathcal{U} = \{\, U_\alpha \,\}_{\alpha\in A}, \qquad X = \bigcup_{\alpha\in A} U_\alpha,$$

a system of *transition functions*

$$\phi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$

encodes a holomorphic line bundle if the following compatibility (cocycle) relations hold:

$$\begin{cases} \text{(a) } \phi_{\alpha\alpha} = 1 & \text{on } U_\alpha, \\ \text{(b) } \phi_{\alpha\beta} = (\phi_{\beta\alpha})^{-1} & \text{on } U_{\alpha\beta}, \\ \text{(c) } \phi_{\alpha\beta}\,\phi_{\beta\gamma}\,\phi_{\gamma\alpha} = 1 & \text{on } U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma. \end{cases}$$

Here each $\phi_{\alpha\beta}$ is a nowhere-vanishing holomorphic function (on its domain of definition). In particular, for the elliptic case, we may denote

$$\phi_{\alpha\beta}(x) = \exp(P_{\omega_{\alpha\beta}}(z) + Q_{\omega_{\alpha\beta}}), \quad x \in X,$$

and the same cocycle relations (a)–(c) must be satisfied.

1 Holomorphic Line Bundles

Definition 1.1 (Holomorphic line bundle on a Riemann surface)**.** Let X be a Riemann surface. A *holomorphic line bundle* L on X consists of the following data:

1. An open covering $\{U_\alpha\}_{\alpha\in A}$ of X, that is,

$$U_\alpha \subset X \text{ open}, \qquad X = \bigcup_{\alpha\in A} U_\alpha.$$

2. A system of nowhere-vanishing holomorphic functions

$$\phi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$

   satisfying the following axioms:

   (a) $\phi_{\alpha\alpha} \equiv 1$ on U_α;

   (b) $\phi_{\alpha\beta} = \dfrac{1}{\phi_{\beta\alpha}}$ on $U_{\alpha\beta}$;

   (c) $\phi_{\alpha\beta}\,\phi_{\beta\gamma}\,\phi_{\gamma\alpha} \equiv 1$ on $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$.

Remark 1.2. From condition (c) we may equivalently write

$$\phi_{\alpha\beta}\,\phi_{\beta\gamma} = \frac{1}{\phi_{\gamma\alpha}} = \phi_{\alpha\gamma},$$

by using (b).

The functions $\phi_{\alpha\beta}$ are called *holomorphic transition functions* (for example, factors of holomorphic powers of the Jacobians $(\gamma')^k$).
Geometric meaning of the symbol L: A holomorphic line bundle L is a geometric object defined on a complex manifold X, which may be one of the familiar spaces such as

$$\mathbb{C}^n, \quad \mathbb{P}^N, \quad \mathbb{C}^n/\Lambda,$$

where Λ is a lattice. Changes of local coordinates are holomorphic maps

$$w = (w_1, w_2) = \Phi(z) = \Phi(z_1, z_2),$$

where Φ is (bi)holomorphic.

From a compatible system of transition functions $\{\phi_{\alpha\beta}\}$ satisfying conditions (a)–(c), we are going to construct a 2-dimensional complex manifold L that will serve as the *total space* of a holomorphic line bundle over X.

Define

$$\mathscr{O} = \bigsqcup_{\alpha} \big(\{\alpha\} \times U_\alpha \times \mathbb{C}\big),$$

that is, the disjoint union of sets $\{\alpha\} \times U_\alpha \times \mathbb{C}$. Each piece $\{\alpha\} \times U_\alpha \times \mathbb{C}$ is considered distinct from $\{\beta\} \times U_\beta \times \mathbb{C}$ even if $U_\alpha \cap U_\beta \neq \varnothing$.

Next, we introduce a relation $\mathcal{E} \subset \mathscr{O} \times \mathscr{O}$ and write $(A, B) \in \mathcal{E}$ as $A \sim B$.

Definition 1.3 (The relation $\sim = \sim_{\mathcal{E}}$)**.** We declare that

$$(\alpha, x, v^{(\alpha)}) \sim (\beta, y, v^{(\beta)}) \quad \text{if and only if} \quad x = y \in U_{\alpha\beta} \text{ and } v^{(\alpha)} = \phi_{\alpha\beta}(x)\, v^{(\beta)}.$$

Lemma 1.4. *The relation $\sim$ is an equivalence relation on $\mathscr{O}$, hence we can define $L = \mathscr{O}/\sim$ as a set.*

Now that $L = \mathscr{O}/\sim$ is defined as a set, we will endow L with the structure of a 2-dimensional complex manifold, equipped with a natural projection map

$$\pi : L \longrightarrow X.$$

Define

$$\rho : \bigsqcup_{\alpha} \big(\{\alpha\} \times U_\alpha \times \mathbb{C}\big) \longrightarrow X, \qquad \rho(\alpha, x, v^{(\alpha)}) = x.$$

If $(\alpha, x, v^{(\alpha)}) \sim (\beta, y, v^{(\beta)})$, then by definition $x = y$. Hence, the map ρ descends to a well-defined map on the quotient:

$$\pi : L = \mathscr{O}/\sim \longrightarrow X, \qquad \pi\big([\,\alpha, x, v^{(\alpha)}\,]\big) = x.$$

Consider an open set $U_\alpha \subset X$. Recall that on the component $\{\alpha\} \times U_\alpha \times \mathbb{C}$, we have the projection

$$\rho : \{\alpha\} \times U_\alpha \times \mathbb{C} \longrightarrow U_\alpha, \qquad (\alpha, x, v^{(\alpha)}) \mapsto x.$$

On overlaps $U_\alpha \cap U_\beta = U_{\alpha\beta}$, we also have

$$\rho : \{\beta\} \times U_\beta \times \mathbb{C} \longrightarrow U_\beta.$$

Using the identification given by $(\alpha, x, v^{(\alpha)}) \sim (\beta, x, v^{(\beta)})$ where $v^{(\alpha)} = \phi_{\alpha\beta}(x)v^{(\beta)}$, we see that

$$\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}.$$

We denote this by

$$L|_{U_\alpha} := \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C},$$

and call this isomorphism the *local trivialization* of the holomorphic line bundle on U_α.

By construction, we have for each α:

$$L|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{C},$$

and similarly,

$$L|_{U_\beta} \xrightarrow{\cong} U_\beta \times \mathbb{C}.$$

The same point $v \in L$ projects under π to $\pi(v) = x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$.

As a set, we may write:

$$L = \bigcup_{\alpha \in A} (U_\alpha \times \mathbb{C}),$$

so that a point in L is represented by $(x, v^{(\alpha)})$ with respect to the local trivialization

$$L|_{U_\alpha} := \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}.$$

Likewise, over U_β the same point is represented as $(x, v^{(\beta)})$. On the overlap $U_{\alpha\beta}$, the two descriptions are related by

$$(x, v^{(\alpha)}) \longleftrightarrow (x, v^{(\beta)}), \qquad v^{(\alpha)} = \phi_{\alpha\beta}(x)\, v^{(\beta)}, \qquad z^{(\alpha)} = z^{(\beta)} = x.$$

That is,

$$U_\alpha \times \mathbb{C} \xleftrightarrow{\phi_{\alpha\beta}} U_\beta \times \mathbb{C}.$$

Choose holomorphic coordinate functions

$$z^{(\alpha)} : U_\alpha \xrightarrow{\cong} D_\alpha \subset \mathbb{C}, \qquad z^{(\beta)} : U_\beta \xrightarrow{\cong} D_\beta \subset \mathbb{C}.$$

Then each local chart of L can be expressed as

$$U_\alpha \times \mathbb{C} \cong D_\alpha \times \mathbb{C} \subset \mathbb{C}^2.$$

With this identification, the change of variables between overlapping charts is

$$(z^{(\alpha)}, v^{(\alpha)}) \longleftrightarrow (z^{(\beta)}, v^{(\beta)}), \qquad v^{(\alpha)} = \phi_{\alpha\beta}(x)\, v^{(\beta)}.$$

Thus, transitions are holomorphic mappings between open subsets of $\mathbb{C}^2$.

Proposition 1.5. *Let $L = \mathscr{O}/\sim$ with the projection map $\pi : L \to X$ as constructed above. Then L carries a natural structure of a 2-dimensional complex manifold, such that for every $x \in X$ there exists an open neighborhood $U_x \subset X$ containing x with*

$$\pi^{-1}(U_x) \cong U_x \times \mathbb{C}.$$

Taking $U_x = U_\alpha$ for some α with $x \in U_\alpha$, we have the local trivialization

$$L|_{U_\alpha} \cong U_\alpha \times \mathbb{C}.$$

Moreover, on the overlap $U_{\alpha\beta}$, the change of coordinates between the two trivializations

$$(z^{(\alpha)}, v^{(\alpha)}) \leftrightarrow (z^{(\beta)}, v^{(\beta)})$$

is given by

$$v^{(\alpha)} = \phi_{\alpha\beta}(x)\, v^{(\beta)},$$

which, for each $x \in U_{\alpha\beta}$, defines an isomorphism of one-dimensional complex vector spaces.

From now on, we denote by

$$\pi : L \longrightarrow X$$

the *holomorphic line bundle* constructed above.

- The 2-dimensional complex manifold L is called the *total space* of the holomorphic line bundle.
- The map π is called the *canonical projection*.

Definition 1.6 (Holomorphic sections)**.** Let $\pi : L \to X$ be a holomorphic line bundle over a Riemann surface X.

A *holomorphic section* of π is a map

$$s : X \longrightarrow L$$

satisfying:

(a) s is holomorphic as a map between complex manifolds;

(b) $\pi \circ s = \mathrm{id}_X$.

Theorem 1.7. *Let X be a compact Riemann surface, and $\pi : E \to X$ a holomorphic line bundle. Then the space of holomorphic sections $\Gamma(X, E)$ is finite-dimensional.*

Proof. Assume $\Gamma(X, E)$ is nonzero. Pick a section $s_0 \in \Gamma(X, E)$ such that $s_0 \not\equiv 0$. Choose a point $x_0 \in X$ where $s_0(x_0) \neq 0$.

For any other section $s \in \Gamma(X, E)$, consider the pointwise ratio

$$\frac{s}{s_0}.$$

This is a meromorphic function on X: it is holomorphic wherever s_0 is nonzero, and may have poles exactly at the zeros of s_0.

Choose a holomorphic coordinate z on a neighborhood U of x_0 such that $z(x_0) = 0$. Since $s_0(x_0) \neq 0$, the quotient $\frac{s}{s_0}$ is holomorphic near x_0 and has a Taylor expansion

$$\frac{s}{s_0}(z) = \sum_{n=0}^{\infty} c_n(s)\, z^n.$$

For each integer $m \geq 0$, define the linear map

$$\Phi_m : \Gamma(X, E) \longrightarrow \mathbb{C}^{m+1}, \qquad \Phi_m(s) = (c_0(s), c_1(s), \ldots, c_m(s)).$$

Clearly, Φ_m is complex-linear.

We will prove that there exists m such that Φ_m is injective. This implies that $\Gamma(X, E)$ is finite-dimensional, because it injects into the finite-dimensional space $\mathbb{C}^{m+1}$.

Let N be the total number of zeros of s_0, counted with multiplicity. Since X is compact and $s_0 \not\equiv 0$, N is finite.

Now take any $s \in \Gamma(X, E)$ with $s \not\equiv 0$. Then $\frac{s}{s_0}$ is a nonzero meromorphic function on X. Its poles can occur only where s_0 vanishes, so the total number of poles (counted with multiplicity) satisfies

$$\#\text{poles}\left(\frac{s}{s_0}\right) \leq N.$$

Because X is compact, any nonzero meromorphic function has the same total number of zeros and poles (each counted with multiplicity). Therefore,

$$\#\text{zeros}\left(\frac{s}{s_0}\right) = \#\text{poles}\left(\frac{s}{s_0}\right) \leq N.$$

This means that $\frac{s}{s_0}$ can have at most N zeros on X.

Suppose now that $s \in \ker(\Phi_m)$. Then the first $m+1$ coefficients $c_0(s), \ldots, c_m(s)$ vanish, so

$$\frac{s}{s_0}(z) = z^{m+1} g(z)$$

for some holomorphic function $g(z)$ near $z = 0$. Hence, $\frac{s}{s_0}$ has a zero of order at least $m+1$ at the point x_0.

If $s \not\equiv 0$, this gives at least $m+1$ zeros of $\frac{s}{s_0}$, all at the same point x_0. But from the argument above, the total number of zeros of $\frac{s}{s_0}$ cannot exceed N. Therefore, if $m+1 > N$, such an s cannot exist, forcing $s \equiv 0$.

Thus, for all m with $m + 1 > N$,

$$\ker(\Phi_m) = \{0\}.$$

That is, Φ_m is injective.

Since $\Phi_m : \Gamma(X, E) \to \mathbb{C}^{m+1}$ is an injective linear map into a finite-dimensional space, it follows that

$$\dim_{\mathbb{C}} \Gamma(X, E) < \infty.$$

□

2 The Line Bundle Associated to a Divisor

Definition 2.1 (Divisors on X)**.** Let X be a compact Riemann surface. A *divisor* on X is a formal finite sum

$$D = \sum_{k=1}^{s} n_k x_k,$$

where $x_k \in X$ and $n_k \in \mathbb{Z}$ for $1 \leq k \leq s$.

We say that D is an *effective divisor* if and only if

$$n_k \geq 0 \quad \text{for all } k = 1, \ldots, s.$$

Definition 2.2 (Divisor of a meromorphic function)**.** Let $f \in \mathcal{M}(X)$ be a nonzero meromorphic function on X. Let $x_1, \ldots, x_s \in X$ be the points where f has either a zero or a pole, so that

$$\operatorname{ord}_x(f) = 0 \quad \text{for all } x \notin \{x_1, \ldots, x_s\}.$$

Write $n_k = \operatorname{ord}_{x_k}(f)$ for $1 \leq k \leq s$. The *divisor of* f is defined by

$$\operatorname{div}(f) = \sum_{k=1}^{s} n_k x_k.$$

We now associate to each divisor D on a compact Riemann surface X a holomorphic line bundle, denoted by $[D]$.

Consider the complex vector space

$$V_D := \big\{ f \in \mathcal{M}(X) : f \neq 0,\ \operatorname{div}(f) \geq -D \big\} \ \sqcup\ \{0\}.$$

It is clear that V_D is a complex vector space under pointwise addition and scalar multiplication.

Equivalently, $f \in V_D$ if and only if: f is holomorphic on $X \setminus \operatorname{supp}(D)$, where $\operatorname{supp}(D) = \{x_1, \ldots, x_s\}$; and at each x_k (for $1 \leq k \leq s$) we have $\operatorname{ord}_{x_k}(f) \geq -n_k$.

We write

$$D_1 \geq D_2 \iff D_1 - D_2 \text{ is an effective divisor.}$$

Example 2.3. Let $X = \mathbb{P}^1$ and take $D = m \cdot \infty$, so that $\operatorname{supp}(D) = \{\infty\}$, and the point at infinity corresponds to $w = 1/z = 0$. Then

$$V_D = V_{m\cdot\infty} = \{P \in \mathbb{C}[z] : P = 0 \text{ or } \deg P \leq m\}.$$

Hence,

$$\dim_{\mathbb{C}} V_{m\cdot\infty} = m + 1 < \infty.$$

(This finiteness property holds for every divisor D on any compact Riemann surface X.)

Moreover, on each open set U_α, there exists a holomorphic function f_α such that

$$\operatorname{div}(f_\alpha) = D|_{U_\alpha} = {\sum}'_{x \in D \cap U_\alpha} n_x \cdot x,$$

where $n_x = 0$ whenever $x \notin \{x_1, \dots, x_s\}$.

We want to choose divisors D such that $V_D \neq \{0\}$, and in fact so that V_D is high-dimensional.

Suppose $h \in V_D$ with $h \neq 0$. Then by definition,

$$\operatorname{div}(h) \geq -D.$$

Let

$$D|_{U_\alpha} = \operatorname{div}(f_\alpha)$$

be a local defining function of the divisor D on each open set U_α of a covering $\{U_\alpha\}$ of X. Then on U_α,

$$\operatorname{div}(h|_{U_\alpha}) \geq -\operatorname{div}(f_\alpha).$$

Hence the product

$$s_\alpha := h \cdot f_\alpha$$

is holomorphic on U_α: it can have at worst removable singularities at points of $\operatorname{supp}(D) \cap U_\alpha$, because for any such point x,

$$\operatorname{ord}_x(s_\alpha) = \operatorname{ord}_x(h) + \operatorname{ord}_x(f_\alpha) \geq -\operatorname{ord}_x(f_\alpha) + \operatorname{ord}_x(f_\alpha) = 0.$$

Thus,

$$h|_{U_\alpha} = \frac{s_\alpha}{f_\alpha}, \qquad s_\alpha \in \mathcal{O}(U_\alpha),$$

where $\mathcal{O}(U_\alpha)$ denotes the holomorphic functions on U_α.

On an overlap $U_\alpha \cap U_\beta$, we have

$$h = \frac{s_\alpha}{f_\alpha} = \frac{s_\beta}{f_\beta},$$

and hence

$$s_\alpha = \frac{f_\alpha}{f_\beta} s_\beta.$$

Define

$$\phi_{\alpha\beta} := \frac{f_\alpha}{f_\beta}.$$

Then clearly

$$\phi_{\alpha\alpha} = 1, \qquad \phi_{\alpha\beta}\phi_{\beta\alpha} = 1, \qquad \phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1.$$

Moreover,

$$\operatorname{div}(\phi_{\alpha\beta}) = \operatorname{div}(f_\alpha) - \operatorname{div}(f_\beta) = D|_{U_\alpha \cap U_\beta} - D|_{U_\alpha \cap U_\beta} = 0,$$

so each $\phi_{\alpha\beta}$ is a nowhere-vanishing holomorphic function on $U_\alpha \cap U_\beta$. Thus, the family $\{\phi_{\alpha\beta}\}$ defines a compatible system of transition functions.

Over each open set U_α, introduce a local trivialization

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C},$$

and define the local holomorphic frame (or "unit section")

$$e_\alpha : U_\alpha \longrightarrow L, \qquad e_\alpha(x) = \Phi_\alpha^{-1}(x, 1).$$

On overlaps $U_\alpha \cap U_\beta$, the change of trivialization is given by

$$(x, v^{(\alpha)}) \sim (x, v^{(\beta)}) \quad \text{whenever} \quad v^{(\alpha)} = \phi_{\alpha\beta}(x)\, v^{(\beta)}.$$

A holomorphic section $t \in \Gamma(X, L)$ is represented locally by holomorphic functions $t_\alpha \in \Gamma(U_\alpha, \mathcal{O})$ satisfying

$$t_\alpha = \phi_{\alpha\beta}\, t_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

That is, on each U_α we can write

$$t|_{U_\alpha} = t_\alpha e_\alpha,$$

and the compatibility condition above ensures that these local expressions glue into a global section.

For any $h \in V_D$ (with $h \neq 0$), we can write locally

$$h|_{U_\alpha} = \frac{s_\alpha}{f_\alpha},$$

where each s_α is holomorphic and satisfies

$$s_\alpha = \phi_{\alpha\beta} s_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

If we define

$$t_\alpha := s_\alpha,$$

then the family $\{t_\alpha\}$ obeys the same transition rule, so it defines a global holomorphic section $t = \{t_\alpha\} \in \Gamma(X, L)$.

Conversely, given a holomorphic section $t = \{t_\alpha\}$, we can reconstruct h by setting

$$h|_{U_\alpha} := \frac{t_\alpha}{f_\alpha}.$$

This construction provides a natural one-to-one correspondence

$$V_D \longleftrightarrow \Gamma(X, L).$$

Definition 2.4 (Divisor Line Bundle)**.** Given a divisor D on X, we denote by

$$[D] := L$$

the holomorphic line bundle constructed above. It is called the *divisor line bundle associated to D*.

Theorem 2.5. *For any divisor D on a compact Riemann surface X, there is a natural vector-space isomorphism*

$$V_D \cong \Gamma(X, [D]).$$

Remark 2.6.

1. Although we assumed $D \geq 0$ (i.e. that D is effective) in the discussion above, the construction of the line bundle $[D]$ from local data f_α satisfying $\operatorname{div}(f_\alpha) = D|_{U_\alpha}$ works for any divisor D on X.

2. If $D \geq 0$, then under the natural isomorphism

$$V_D \;\cong\; \Gamma(X, [D]),$$

the constant function $1 \in V_D$ corresponds to a distinguished global holomorphic section

$$s_D \in \Gamma(X, [D]),$$

called the *canonical section* of the divisor line bundle $[D]$.

Indeed, for $h = 1$, the correspondence $h \longleftrightarrow \{t_\alpha = hf_\alpha\}$ gives

$$t_\alpha = f_\alpha.$$

These local functions satisfy the transition rule $f_\alpha = \frac{f_\alpha}{f_\beta} f_\beta = \phi_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta$, so they *glue* to a well-defined global section

$$s_D = \{s_{D,\alpha} = f_\alpha\}.$$

The section s_D vanishes precisely along the divisor D; in particular, $\operatorname{div}(s_D) = D$.

Observation:

$$1 \in V_D \quad\Longleftrightarrow\quad D \geq 0.$$

In this case, $1 \in V_D$ corresponds to the canonical section $s_D \in \Gamma(X, [D])$ defined locally by $s_D|_{U_\alpha} = f_\alpha$.

3 Classification of Holomorphic Line Bundles up to Isomorphism

Definition 3.1 (Isomorphism of holomorphic line bundles)**.** Let

$$\pi_1 : E \longrightarrow X, \qquad \pi_2 : E' \longrightarrow X$$

be holomorphic line bundles over the same complex manifold X.

A *holomorphic line bundle isomorphism* from E to E' is a biholomorphic map of total spaces

$$\Phi : E \longrightarrow E'$$

such that the following two conditions hold:

(i) The projection maps are preserved; i.e. the diagram

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E' \arrow[d, "\pi_2"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

commutes. In other words, $\pi_2 \circ \Phi = \pi_1$.

(ii) For each $x \in X$, the induced map on the fiber

$$\Phi_x := \Phi|_{E_x} : E_x \longrightarrow E'_x$$

is a *complex-linear isomorphism* between the one-dimensional complex vector spaces E_x and E'_x.

If such a map Φ exists, we say that E and E' are *isomorphic holomorphic line bundles*, and we write

$$E \cong E'.$$

Let

$$\pi_i : E_i \longrightarrow X, \qquad i = 1, 2$$

be holomorphic line bundles over a complex manifold X. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X such that each restricted bundle is holomorphically trivial:

$$E_i|_{U_\alpha} \cong U_\alpha \times \mathbb{C}, \qquad i = 1, 2.$$

On overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the corresponding transition functions are

$$\phi^i_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathbb{C}^*, \qquad i = 1, 2,$$

which satisfy the cocycle (compatibility) conditions:

$$\phi^i_{\alpha\beta}\, \phi^i_{\beta\gamma}\, \phi^i_{\gamma\alpha} = 1 \quad \text{on } U_{\alpha\beta\gamma}.$$

(1) Tensor product. We define the tensor product line bundle $E_1 \otimes E_2$ by taking the same open cover $\{U_\alpha\}$ and using new transition functions

$$\psi_{\alpha\beta} := \phi^1_{\alpha\beta}\, \phi^2_{\alpha\beta} \quad \text{on } U_{\alpha\beta}.$$

Since the product of two cocycles is again a cocycle, $\{\psi_{\alpha\beta}\}$ satisfies the compatibility conditions:

$$\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = (\phi^1_{\alpha\beta}\phi^2_{\alpha\beta})(\phi^1_{\beta\gamma}\phi^2_{\beta\gamma})(\phi^1_{\gamma\alpha}\phi^2_{\gamma\alpha}) = 1.$$

Hence $\{\psi_{\alpha\beta}\}$ defines a holomorphic line bundle, denoted $E_1 \otimes E_2$.

If E_1 and E_2 are described on possibly different open covers $\{U^1_a\}_{a \in A_1}$ and $\{U^2_A\}_{A \in A_2}$, we can pass to a common refinement as follows.

For indices $\alpha = (a, A)$ and $\beta = (b, B)$, define

$$U_{\alpha\beta} := U^1_a \cap U^1_b \cap U^2_A \cap U^2_B.$$

The collection

$$\{\, U^1_a \cap U^2_A \mid a \in A_1,\, A \in A_2 \,\}$$

is an open cover of X, because

$$\bigcup_{a,A} (U^1_a \cap U^2_A) = \left(\bigcup_a U^1_a \right) \cap \left(\bigcup_A U^2_A \right) = X \cap X = X.$$

Hence this refinement is still an open cover of X, and we can define the transition functions for the tensor product bundle on these intersections by

$$\psi_{\alpha\beta} := \phi^1_{ab}\,\phi^2_{AB} \quad \text{on } U_{\alpha\beta}.$$

(2) Dual bundle. Given a holomorphic line bundle $\pi : E \to X$ with transition functions $\{\phi_{\alpha\beta}\}$, define the *dual line bundle* E^{-1} (or E^*) by taking the same open cover and setting

$$\psi_{\alpha\beta} := \frac{1}{\phi_{\alpha\beta}}.$$

Since $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1$, it follows that $\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = 1$. Hence $\{\psi_{\alpha\beta}\}$ defines a holomorphic line bundle E^{-1}.

The tensor and inverse constructions yield operations:

$$(E_1, E_2) \longmapsto E_1 \otimes E_2, \qquad E \longmapsto E^{-1},$$

which behave analogously to multiplication and inverse in an abelian group.

(3) Canonical isomorphisms.

$$E_1 \otimes E_2 \cong E_2 \otimes E_1, \qquad E \otimes \mathcal{O}_X \cong E,$$

where the *trivial line bundle* $\mathcal{O}_X$ (or $\mathbb{1}$) is defined by the constant transition functions $\phi_{\alpha\beta} \equiv 1$.

Theorem 3.2. *The set of isomorphism classes of holomorphic line bundles over X, denoted by*

$$\mathrm{Pic}(X),$$

forms an abelian group under the operation

$$[E_1], [E_2] \longmapsto [E_1 \otimes E_2],$$

called the Picard group *of X. The identity element is the class of the trivial bundle $\mathcal{O}_X$, and the inverse of $[E]$ is $[E^{-1}]$.*

Preparation: What does it mean to say $E \cong E'$ as holomorphic line bundles over X, in terms of their transition functions?

Let

$$\Psi : E \xrightarrow{\cong} E'$$

be an isomorphism of holomorphic line bundles over X. Choose an open covering $\mathcal{U} = \{U_\alpha\}$ that works for both E and E', i.e.

$$E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}, \qquad E'|_{U_\alpha} \cong U_\alpha \times \mathbb{C}.$$

On each U_α, Ψ restricts to a biholomorphic map making the diagram commute:

$$\begin{tikzcd} E|_{U_\alpha} \arrow[r, "\Psi|_{U_\alpha}"] \arrow[d, "\cong"'] & E'|_{U_\alpha} \arrow[d, "\cong"] \\ U_\alpha \times \mathbb{C} \arrow[r, "(\mathrm{id}, \psi_\alpha)"'] & U_\alpha \times \mathbb{C}, \end{tikzcd}$$

where $\psi_\alpha : U_\alpha \to \mathbb{C}^*$ is a nowhere-vanishing holomorphic function such that locally

$$\Psi(x, v^{(\alpha)}) = \big(x,\, \psi_\alpha(x)\, v^{(\alpha)}\big).$$

On an overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have similar expressions:

$$\Psi(x, v^{(\alpha)}) = (x,\, \psi_\alpha(x)\, v^{(\alpha)}), \qquad \Psi(x, v^{(\beta)}) = (x,\, \psi_\beta(x)\, v^{(\beta)}).$$

On E, the local trivialisations relate via transition functions

$$v^{(\alpha)} = \phi_{\alpha\beta}(x)\, v^{(\beta)},$$

and on E',

$$\Psi(x, v^{(\alpha)}) = \phi'_{\alpha\beta}(x)\, \Psi(x, v^{(\beta)}).$$

Compatibility of these descriptions under Ψ requires:

$$\psi_\alpha(x)\, v^{(\alpha)} = \phi'_{\alpha\beta}(x)\, \psi_\beta(x)\, v^{(\beta)}.$$

Comparing coefficients of $v^{(\beta)}$, we obtain

$$\psi_\alpha(x)\, \phi_{\alpha\beta}(x) = \phi'_{\alpha\beta}(x)\, \psi_\beta(x),$$

or equivalently,

$$\boxed{\psi_\alpha^{-1}\, \phi'_{\alpha\beta}\, \psi_\beta = \phi_{\alpha\beta}, \quad \text{i.e.} \quad \frac{\phi'_{\alpha\beta}}{\phi_{\alpha\beta}} = \frac{\psi_\alpha}{\psi_\beta}.}$$

Proof. We must verify that

$$E_1 \cong \tilde{E}_1, \quad E_2 \cong \tilde{E}_2 \quad \Longrightarrow \quad E_1 \otimes E_2 \cong \tilde{E}_1 \otimes \tilde{E}_2.$$

Let E_1 and E_2 be holomorphic line bundles on X defined by their transition functions $\{\phi^1_{\alpha\beta}\}$ and $\{\phi^2_{\alpha\beta}\}$, respectively, and let $\{\tilde{\phi}^1_{\alpha\beta}\}$ and $\{\tilde{\phi}^2_{\alpha\beta}\}$ define $\tilde{E}_1$ and $\tilde{E}_2$.

The assumption $E_1 \cong \tilde{E}_1$ means that there exist nowhere-vanishing holomorphic functions $\{\psi^1_\alpha\}$ such that

$$\frac{\tilde{\phi}^1_{\alpha\beta}}{\phi^1_{\alpha\beta}} = \frac{\psi^1_\alpha}{\psi^1_\beta}.$$

Similarly, from $E_2 \cong \tilde{E}_2$, there exist $\{\psi^2_\alpha\}$ such that

$$\frac{\tilde{\phi}^2_{\alpha\beta}}{\phi^2_{\alpha\beta}} = \frac{\psi^2_\alpha}{\psi^2_\beta}.$$

The tensor product bundles $E_1 \otimes E_2$ and $\tilde{E}_1 \otimes \tilde{E}_2$ are defined by the products of their transition functions:

$$\phi^1_{\alpha\beta}\phi^2_{\alpha\beta} \quad \text{and} \quad \tilde{\phi}^1_{\alpha\beta}\tilde{\phi}^2_{\alpha\beta},$$

respectively.

Then we compute:

$$\frac{\tilde{\phi}^1_{\alpha\beta}\tilde{\phi}^2_{\alpha\beta}}{\phi^1_{\alpha\beta}\phi^2_{\alpha\beta}} = \frac{\psi^1_\alpha\psi^2_\alpha}{\psi^1_\beta\psi^2_\beta} := \frac{\lambda_\alpha}{\lambda_\beta}, \quad \text{where} \quad \lambda_\alpha := \psi^1_\alpha\psi^2_\alpha.$$

Hence the bundles $E_1 \otimes E_2$ and $\tilde{E}_1 \otimes \tilde{E}_2$ are isomorphic.

Similarly, suppose $E \cong E'$. Then there exist functions $\{\psi_\alpha\}$ such that, if $\{\phi_{\alpha\beta}\}$ and $\{\phi'_{\alpha\beta}\}$ define E and E', we have

$$\frac{\phi'_{\alpha\beta}}{\phi_{\alpha\beta}} = \frac{\psi_\alpha}{\psi_\beta}.$$

The inverse bundle E^{-1} is defined by the transition functions $\phi^{-1}_{\alpha\beta} = \phi_{\beta\alpha}$. Thus,

$$\frac{(\phi'_{\alpha\beta})^{-1}}{(\phi_{\alpha\beta})^{-1}} = \frac{\psi_\beta}{\psi_\alpha} = \frac{1/\psi_\alpha}{1/\psi_\beta},$$

which shows that

$$E^{-1} \cong (E')^{-1}.$$

□

Remark 3.3. The Picard group $\mathrm{Pic}(X)$ provides an example of a *classification (or moduli) problem*: it classifies holomorphic line bundles on X up to isomorphism.

Question: Can one determine $\mathrm{Pic}(X)$ for a given complex manifold X?

Theorem 3.4.

$$\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}.$$

In fact, every holomorphic line bundle $\pi : E \to \mathbb{P}^1$ is isomorphic to a divisor line bundle $[m \cdot \infty]$ for some integer $m \in \mathbb{Z}$.

Preparation:
Fact: Any holomorphic line bundle over a disk $D(a; r)$ is holomorphically trivial, i.e.

$$\pi : E \longrightarrow D(a; r) \quad \text{is isomorphic to} \quad \pi_0 : D(a; r) \times \mathbb{C} \longrightarrow D(a; r).$$

That is, every holomorphic line bundle over a simply-connected domain in $\mathbb{C}$ is trivial.

Lemma 3.5. *Let f be a nowhere–vanishing holomorphic function on $\mathbb{C}^*$. Then there exists an integer $k \in \mathbb{Z}$ and a holomorphic function h on $\mathbb{C}^*$ such that*

$$f(z) = z^k e^{h(z)}.$$

Proof. Since f is nowhere zero, the logarithmic derivative $\dfrac{f'(z)}{f(z)}$ is holomorphic on $\mathbb{C}^*$. Consider a positively oriented circle $\gamma(t) = re^{it}$, $t \in [0, 2\pi]$, for some fixed $r > 0$.

Define

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma d(\log f).$$

By the argument principle, this integral equals the winding number of $f(\gamma)$ around the origin, which is an integer:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = k \in \mathbb{Z}.$$

Now set

$$g(z) := \frac{f(z)}{z^k}.$$

Then

$$\frac{g'(z)}{g(z)} = \frac{f'(z)}{f(z)} - \frac{k}{z}.$$

The integral of this differential around any closed loop γ in $\mathbb{C}^*$ vanishes, because

$$\oint_\gamma \frac{g'(z)}{g(z)}\,dz = \oint_\gamma \frac{f'(z)}{f(z)}\,dz - k\oint_\gamma \frac{dz}{z} = 2\pi ik - 2\pi ik = 0.$$

Hence the 1–form $\dfrac{g'(z)}{g(z)}dz$ is exact on $\mathbb{C}^*$, which means we can define a holomorphic function

$$h(z) := \int_{\gamma_z} \frac{g'(w)}{g(w)}\,dw,$$

where the integral is independent of the path γ_z from a fixed basepoint to z.

Then $g = e^h$, so

$$f(z) = z^k e^{h(z)}.$$

□

Proof of Theorem 3.4. Let

$$\mathcal{U} = \{U_0, U_1\}, \quad U_0 = \{[t_0 : t_1] \in \mathbb{P}^1 \mid t_0 \neq 0\}, \quad U_1 = \{[t_0 : t_1] \in \mathbb{P}^1 \mid t_1 \neq 0\}.$$

In homogeneous coordinates we set

$$z = \frac{t_1}{t_0} \quad \text{on } U_0 \cong \mathbb{C}, \qquad w = \frac{t_0}{t_1} = \frac{1}{z} \quad \text{on } U_1 \cong \mathbb{C}.$$

Then their intersection $U_{01} = U_0 \cap U_1 \cong \mathbb{C}^*$, with coordinates related by $z = 1/w$.

By the preceding fact, every holomorphic line bundle over a disk is holomorphically trivial. Hence, over the open sets U_0 and U_1, the restrictions $E|_{U_0}$ and $E|_{U_1}$ are both trivial:

$$E|_{U_i} \cong U_i \times \mathbb{C} \qquad (i = 0, 1).$$

Therefore, E is determined by a single transition function $\phi_{01} \in \Gamma(U_{01}, \mathcal{O}^*) = \Gamma(\mathbb{C}^*, \mathcal{O}^*)$ relating the local trivialisations:

$$(z, v^{(0)}) \leftrightarrow (w, v^{(1)}), \qquad v^{(0)} = \phi_{01}(z)\, v^{(1)}.$$

By the lemma above, there exist $k \in \mathbb{Z}$ and $h \in \Gamma(\mathbb{C}^*, \mathcal{O})$ such that

$$\phi_{01}(z) = z^k e^{h(z)}.$$

Let A denote the line bundle defined by transition function z^k, and B the bundle defined by e^h. Then $E \cong A \otimes B$.

Define local frames:

$$e_0 \equiv 1 \text{ on } U_0, \qquad e_1 \equiv w^k \text{ on } U_1.$$

Since on U_{01} we have $z = 1/w$, these satisfy

$$e_0 = z^k e_1.$$

Hence A is exactly the standard line bundle $\mathcal{O}_{\mathbb{P}^1}(k) = [k \cdot \infty]$.

We claim there exist holomorphic functions $s_0 \in \Gamma(U_0, \mathcal{O})$, $s_1 \in \Gamma(U_1, \mathcal{O})$ such that

$$h = s_0 - s_1 \quad \text{on } U_{01} \cong \mathbb{C}^*.$$

If so, then $e^h = e^{s_0}/e^{s_1}$, and by redefining the local frames $\tilde{e}_i := e^{s_i} e_i$, the new transition function becomes $\tilde{\phi}_{01} = 1$, showing B is holomorphically trivial.

To prove the claim, expand h as a Laurent series on $\mathbb{C}^*$:

$$h(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Split this series into nonnegative and negative parts:

$$h(z) = \underbrace{\sum_{n\geq 0} c_n z^n}_{=:s_0(z) \text{ holomorphic on } U_0 \cong \mathbb{C}} - \underbrace{\sum_{m\geq 1} (-c_{-m}) z^{-m}}_{=:s_1(w) \text{ holomorphic on } U_1 \cong \mathbb{C}_w}.$$

Rewriting the negative part in terms of $w = 1/z$, we have

$$s_1(w) = -\sum_{m\geq 1} c_{-m} w^m,$$

which is holomorphic on $U_1 \cong \mathbb{C}$. Thus $h = s_0 - s_1$ on U_{01}, proving the claim. Therefore, B is trivial.

Combining these, we obtain

$$E \cong A \otimes B \cong [k \cdot \infty].$$

Consequently, every holomorphic line bundle on $\mathbb{P}^1$ is of the form $[k \cdot \infty]$, and distinct integers k give non-isomorphic bundles. Hence

$$\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}.$$

$\square$