MATH3301 Tutorial 5

1. Let H and K be subgroups of a group G. Define $HK = \{hk : h \in H,\ k \in K\} \subset G$. Give an example to illustrate that HK may not be a subgroup. Show that HK is a subgroup of G if and only if $HK = KH$. Give examples to demonstrate the existence of G, H, K for (i) $HK \cong H \times K$, (ii) $HK \ncong H \times K$ respectively.

2. Let G be a group and define $[G,G]$ to be the subgroup generated by all elements $xyx^{-1}y^{-1}$ with $x, y \in G$, i.e. $[G,G] = \langle\{xyx^{-1}y^{-1} : x, y \in G\}\rangle$. If H is a subgroup of G, show that $[H,H]$ is a subgroup of $[G,G]$. For simplicity write G' for $[G,G]$. Compute S_3' and S_3''.

3. Let $G = \left\{\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R}\right\}$. Show that G is a group under matrix multiplication. Show that the center $Z(G)$ of G equals its commutator subgroup. Find a group law $*$ on $\mathbb{R}^3$ such that $G \cong (\mathbb{R}^3, *)$.

$(G(1), G(3)) = (1,3)$

4. Give a formula for the cycle $\sigma(i_1, i_2, \cdots, i_k)\sigma^{-1}$ where $\sigma \in S_k$.

   Consider $\alpha = (1,3)$, $\beta = (1,3)(2,4)$ and $\gamma = (1,2,3)$ in S_4. List all the elements in centralizers (i) $C(\alpha)$, (ii) $C(\beta)$, (iii) $C(\gamma)$ and (iv) it center $Z(S_4)$. Explain your calculation.

   [Remark. We defined the centralizer of a subgroup, but in fact, the definition makes sense when H is just a subset. Here $C(\alpha)$ means $C(\{\alpha\})$.]

5. Describe the elements in (i) $\mathrm{Aut}(\mathbb{Z}_4)$ and (ii) $\mathrm{Aut}(\mathbb{Z}_8)$. Verify that $\mathrm{Aut}(G)$ is a group under the operation of function composition, for any group G. Show that $\mathrm{Aut}(\mathbb{Z}_4) \cong \cancel{\mathbb{Z}_3} \cong \mathbb{Z}_4^\times$ and $\mathrm{Aut}(\mathbb{Z}_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_8^\times$.

   $\mathbb{Z}_2$

6. Let G be a group, H and K be its subgroups such that $H \subset K \subset G$. (a) Show that $H \lhd G$ implies $H \lhd K$. (b) Give an example that $H \lhd K$ but $H \ntriangleleft G$ (meaning H is not normal in G). (c) Give an example that $H \lhd K$ and $K \lhd G$ but $H \ntriangleleft G$.

End