Connected sum

M, N are two top mfds of dim n with boundary

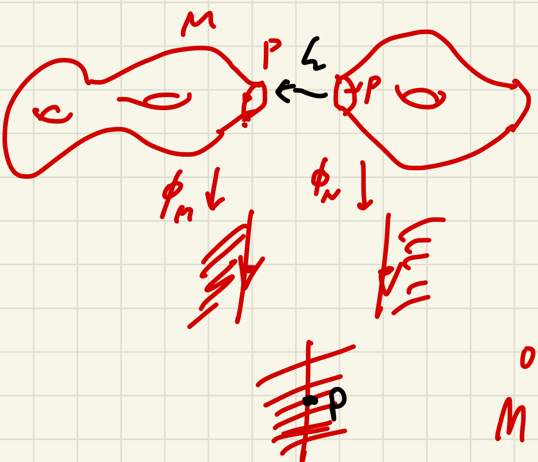
Let $h : \partial N \xrightarrow{\cong} \partial M \subset M$, $\quad \partial N \subset N$

Define the gluing of M and N along boundary to be $M \cup_h N$

Thm $M \cup_h N$ is a top mfd of dim n w/o boundary. (i.e. $\partial = \phi$)

pf. (Sketch)

it suffices to construct chart at $p \in \partial M \cong \partial N$



$$
\begin{array}{ccc}
\mathcal{U} \sqcup V & \longrightarrow & \phi_M(\mathcal{U}) \sqcup \phi_N(V) \\
\downarrow & & \downarrow \\
\mathcal{U} \cup_h V & \overset{\tilde{\phi}}{\dashrightarrow} & \underbrace{\phi_M(\mathcal{U}) \sqcup \phi_N(V)}_{\phi_M \, h \, \phi_N^{-1}}
\end{array}
$$

open $\subset M \cup_h N$ $\qquad$ $\mathbb{R}^2 \supset$

$\tilde{\phi}$ is a closed and open map
by the property of quotient top.

Connected sum

$\tilde{M}_1$, $\tilde{M}_2$ top mfd of dim n
$\ni p_1$, $\ni p_2$ w/o boundary

$\phi_1 : U_1 \longrightarrow \mathbb{R}^n$ $\qquad$ $\phi_2 : U_2 \longrightarrow \mathbb{R}^n$

$p_1 \longrightarrow \phi_1(p_1) = 0$ $\qquad$ $p_2 \longrightarrow \phi_2(p_2) = 0$

choose $\varepsilon_1, \varepsilon_2 > 0$ s.t.

$$\phi_i^{-1}(B(0, \varepsilon)) \subset U_i$$

$$M_i = \tilde{M}_i \setminus \phi_i^{-1}(B(0, \varepsilon))$$

$$\phi_i : \partial M_i \cong S^{n-1}$$

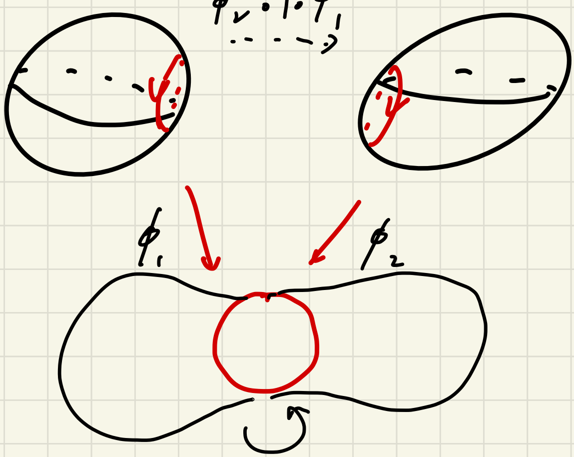
$$\partial M_1 \xrightarrow{h = \phi_2^{-1} \circ \phi_1} \partial M_2 \subset M_2$$

$$M_1 \underset{\phi_1, \phi_2}{\#} M_2 := M_1 \cup_h M_2$$

Example

Connected sum with "twist"

$\phi_2^{-1} \circ T \circ \phi_1$



$T : S^{n-1} \xrightarrow{\cong} S^{n-1}$

Question: does $M_1 \underset{\phi_1, \phi_2}{\#} M_2$ depends on ϕ_1, ϕ_2 up to homeo.?

This is a subtle question!

for higher dim'l case the answer is no!

for dim two. yes. but it's a hard thm!

Homotopy

Def'n $f, g : X \to Y$ cont. maps

A homotopy from f to g is a cont. map

$$F : X \times I \longrightarrow Y \quad \text{s.t} \quad \begin{array}{l} F(x,0) = f \\ F(x,1) = g \end{array}$$

We write $f \sim_F g$ (or $f \sim g$)

and say f is homotopic to g.

Def'n $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ $\quad A \subset X$ s.t $f|_A = g|_A$

A homotopy relative to A, denote by $f \sim g \text{ rel } A$

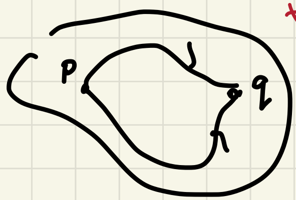
is a homotopy $F : X \times I \to Y$ s.t $F(x,0) = f$

$F(x,1) = g$

$F(x,t) = f(x) = g(x)$

$\forall t \quad x \in A.$

Examples and properties



1) [Path homotopy]

$\gamma(s,t) : I \times I \longrightarrow X \qquad \gamma(s,0) = p \quad \gamma(s,1) = q \ \forall s$

2) $Y \subset \mathbb{R}^n$ is a convex set

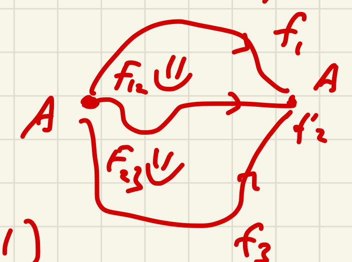
Any two cont. maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ are homotopic

$$F(x,t) := (1-t)f(x) + t\,g(x)$$

3) [Vertical composition]

$\sim$ ($\sim$ rel A) is a equivalence relation of homotopy

$$\underbrace{F_{23} \overset{v}{\circ} F_{12}}_{F_{13}} : X \times I \longrightarrow Y$$



$t \in [0, \frac{1}{2}]$ $\quad F_{13}(x,t) := F_{12}(x, 2t)$

$t \in [\frac{1}{2}, 1]$ $\quad F_{13}(x,t) := F_{23}(x, 2t-1)$

4) [Horizontal composition]

$$\begin{matrix} X & \underset{g}{\overset{f}{\rightrightarrows}} & Y & \underset{k}{\overset{h}{\rightrightarrows}} & Z \\ \cup & & \cup & & \\ A & & B & & \end{matrix}$$

if $f \sim g$ rel A

$h \sim k$ rel B

then $hf \sim hg$ rel A

$hf \sim kf$ rel $f^{-1}B$

$f|_A = g|_A \qquad h|_B = k|_B$

horizontal composition

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "F\Downarrow"] & Y \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow G"] & Z \end{tikzcd}$$

$h \circ F : X \times I \to Z$

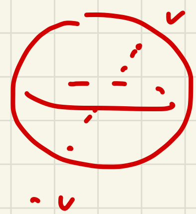
$G \circ g : X \times I \to Z$

$h \circ F \qquad G \circ g$

$h \circ f \sim h \circ g \sim k \circ g$

5) Any non surjective cont. map $f : X \to S^2$ is homotopy to a const. map.

suppose $p \notin \operatorname{im} f$



Consider: $F : X \times I \to S^2$

$$F(x,t) = \frac{(1-t) f(x) - t \cdot p}{\|(1-t) f(x) - t \cdot p\|}$$

$F(x,0) = f(x)$

$F(x,1) = p$

Def'n X, Y two top spaces

we say X is homotopic to Y if

$\exists$ $f : X \to Y$ $\quad g : Y \to X$
cont. map

$gf \sim \mathrm{id}_X \qquad fg \sim \mathrm{id}_Y$

write $X \sim Y$

properties

← prove it using horizontal composition.

1) $\sim$ is an equivalence relation

2) if $X \cong Y$ then $X \sim Y$

3) A convex set $X \subset \mathbb{R}^n$ is homotopic to a pt] such space are called contractible space.

$p \in X$ $\quad p \xhookrightarrow{i} X \xrightarrow{c} p \xhookrightarrow{i} X$

$$i \circ c \sim id_X$$

$$F(x,t) = (1-t)\cdot p + t \cdot x$$

Def'n $A \subset X$ A cont. map $r : X \longrightarrow A$ is called a **retraction** if $r(x) = x$ $x \in A$

We call A a **retract of X.**

A **deformation retract of X to A** is

a cont. map $F : X \times I \longrightarrow X$

$F(x,0) = id_X$ $\quad f(x,1) : X \longrightarrow A \subset X$ is a retract.

Properties & examples

0) $F: X \times I \to X$ is a deformation retract from X to A

then $X \sim A$

$i_A : A \hookrightarrow X \xrightarrow{r} A \xhookrightarrow{i_A} X$

$r \circ i_A = id_A \qquad i_A \circ r \underset{F}{\sim} id_X$

1) Composition of deformation retract is a d.r.

$F : X \times I \to X \qquad F(x,1) \subset A \subset X \qquad B \subset A \subset X$

$G : A \times I \to A \qquad G(x,1) \subset B$

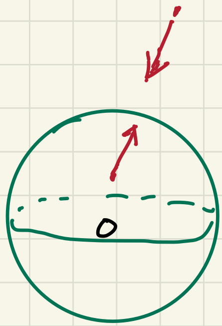
$G \overset{v}{\circ} F : X \times I \to X$

$$G \circ F(x, t) = \begin{cases} F(x, 2t) & t \in [0, \frac{1}{2}] \\ G(x, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

2) $S^n = \{ v \mid |v| = 1 \} \qquad \mathbb{R}^{n+1} \setminus 0 \sim S^n$

$F : \mathbb{R}^{n+1} \setminus 0 \times I \to \mathbb{R}^{n+1} \setminus 0$ is a d.r.

$$F(v, t) = (1-t) \cdot v + t \frac{v}{\|v\|}$$



3) There exists a homotopy from $GL_2\mathbb{R}$ to $O(2)$ (A proof for QR decomposition)

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2\mathbb{R}$$

$= (v_1, v_2)$ v_1, v_2 linearly independent.

$$GL_2\mathbb{R} \times I \longrightarrow O(2)$$

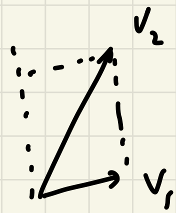
step 1

$$((v_1, v_2), t) \mapsto (\|v_1\|^{-t} \cdot v_1, v_2)$$

step 2

Assume $\|v_1\| = 1$

$$((v_1, v_2), t) \mapsto \left(v_1, v_2 - t \cdot pr_{v_1} v_2\right)$$



$$\det [v_1, v_2 - t\, pr_{v_1} v_2] \neq 0$$

step 3

Assume $\|v_1\| = 1$ $v_1 \perp v_2$

$$((v_1, v_2), t) \mapsto (v_1, \|v_2\|^{-t} \cdot v_2)$$

$t = 1$ image $\subset O(2)$

Integral curve of vector field.

continuous

$X \subset \mathbb{R}^2$ open subset $\quad V : X \to \mathbb{R}^n$ a vector field

$V(x) = (V_1(x), \dots, V_n(x))$

An integral curve of V starting at x

is a curve $\gamma : [0, \delta] \longrightarrow X$

s.t. $\gamma'(t) = V(\gamma(t))$

$\gamma(0) = x$

Assume that $\forall x \in X \quad \exists \delta > 0$ s.t.

integral curve $\gamma_x : [0, \delta] \to X \quad \gamma(0) = x$ exists

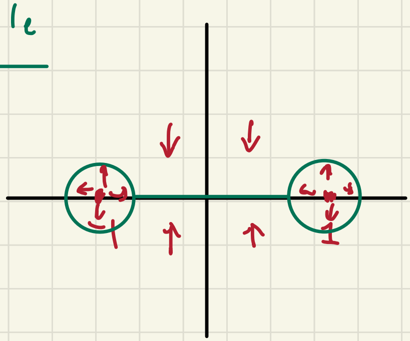
WLOG set $\delta = 1$ e.g. may set $\bar{\gamma}(x, t)$

then we have a differentiable map $= \gamma(x, t/\delta)$

$\Gamma : X \times I \longrightarrow X$ ↑ by the theory of ODE.

$\Gamma(x, t) = \gamma_x(t)$ We may use Γ to construct d.r.

Example



V vanishes on