MATH3301 Tutorial 7

1. Let $n \geq 3$.

   (a) Show that the alternating group A_n contains all 3-cycles.

   (b) Let S be the set of all 3-cycles in A_n. Show that $A_n = \langle S \rangle$, i.e. S generates A_n. [Hint: Show that the product of 2-cycles $(a, b)(c, d)$ can be expressed as a product of two 3-cylces.]

   (c) Given $r, s \in \{1, 2, \cdots, n\}$, and $R = \{(r, s, i) : 1 \leq i \leq n\}$. Show that $A_n = \langle R \rangle$.

   (d) Suppose N is a normal subgroup of A_n and N contains a 3-cycle. Show that $N = A_n$.

2. Let $n \geq 5$ and μ be a cycle in A_n. Suppose N is a normal subgroup of A_n.

   (a) Suppose $(a_1, \cdots, a_r)$ with $r > 3$ and μ are disjoint cycles.
   If $\sigma := \mu(a_1, \cdots, a_r) \in N$, show that $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1} \in N$ is a 3-cycle.

   (b) Suppose the cycles (a_1, a_2, a_3), (a_4, a_5, a_6) and μ are mutually disjoint.
   If $\sigma := \mu(a_4, a_5, a_6)(a_1, a_2, a_3) \in N$, show that $\sigma^{-1}(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1} \in N$ is a cycle of length greater than 3 and disjoint with μ.

   (c) Suppose (a_1, a_2, a_3) and μ are disjoint cycles, and μ is a product of disjoint 2-cycles.
   If $\sigma := \mu(a_1, a_2, a_3) \in N$, show that $\sigma^2 \in N$ is a 3-cycle.

   (d) Suppose the cycles (a_1, a_2), (a_3, a_4) and μ are mutually disjoint, and μ is a product of an even number of disjoint 2-cycles.
   If $\sigma := \mu(a_3, a_4)(a_1, a_2) \in N$, show that $\alpha := \sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1} \in N$ is a product of two 2-cycles. Let $\beta = (a_1, a_3, i)$ where $i \notin \{a_1, a_2, a_3, a_4\}$. Show that $\beta^{-1}\alpha\beta\alpha = \beta$.

3. (a) Show that A_3 and A_4 has a non-trivial proper normal subgroup.

   (b) Show that for $n \geq 5$, A_n has no non-trivial proper normal subgroup.
   [Hint: Use the results in the above questions.]

4. (a) Give a complete list of all non-isomorphic cyclic groups of order 40.
   (b) Give a complete list of all non-isomorphic abelian groups of order 40 that are not cyclic.
   (c) Give a complete list of all non-isomorphic abelian groups of order 40 that have no elements of order 8.

5. Prove or disprove the following with justification or counterexample.
   (a) A simple group has no non-trivial proper subgroup.
   (b) A group that has no non-trivial proper subgroup is a simple group.
   (c) A simple finite abelian group is cyclic of prime order.
   (d) A cyclic group of order 40 must have an element of order 8.
   (e) An abelian group of order 40 must have an element of order 10.

6. Below n, d denote positive integers.

   Recall: If G is a cyclic group of order n, then for any $d|n$, G has *at least one element* in G of order d and G has *exactly one subgroup* H of order d.

   Suppose G is an abelian group of order n.
   (a) For any $d|n$, show that G has a subgroup H of order d.
   (b) Give an example to show that G may have more than more subgroup of the same order.
   (c) For any composite divisor d of n (i.e. $d|n$ and d is not a prime), show that there is a group G for which G have no element of order d.

End