THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3301
Assignment 3

Due Date: Oct 24, 2024, 23:59.

Submission Guidelines

(i) Write your solution on papers of about A4 size.

(ii) Scan your work properly and save it as **one** PDF file.

Warning: Please make sure that your work is properly scanned. Oversized, blurred or upside-down images will NOT be accepted.

(iii) While you can upload and save draft in moodle, you MUST click the "Submit" button to declare your final submission before the due date. Otherwise, you will be considered late.

Preparation Guidelines

(i) Your solution should be well written and organized. It is good to work out a draft for each question on a separate paper, polish/rewrite/reorganize your answer suitably and then write it (the final form) on the paper to be scanned.

(ii) You may imagine that you are teaching this course and writing a solution to demonstrate the answer. Hence, especially for proof-type questions, you have to convince everyone that your solution (proof) is correct, without any oral explanation from you. i.e. Another student should be able to understand the answer (proof) completely via your written word, and/or diagrams or tables you create in your solution.

(iii) Follow HKU's regulations on academic honesty. Plagiarism is inacceptable and may have severe consequences for your record. See `https://tl.hku.hk/plagiarism/` for "What is plagiarism?". *If you have used AI tools to explore, check or refine your work, please acknowledge and clearly identify the parts of your work that involve AI output to avoid plagiarism or related academic dishonesty. Indicate the extent to which the AI output is used (e.g. directly copied or paraphrased/modified or checked for errors or reorganized the presentation).*

To be handed in

1. View $\mathbb{Z}$ as a subgroup of $(\mathbb{Q}, +)$. Check $\mathbb{Z}$ is a normal subgroup of $\mathbb{Q}$.

   (a) Write down the cosets of $\frac{1}{3}$ and $\frac{1}{4}$ in $\mathbb{Q}$. Are these two cosets disjoint or identical?

   (b) Show that the quotient group $\mathbb{Q}/\mathbb{Z}$ is infinite.

   (c) Evaluate the order of $\overline{\frac{2}{3}}$ (where $\overline{\frac{2}{3}} = \frac{2}{3} + \mathbb{Z}$ is an element in $\mathbb{Q}/\mathbb{Z}$).

   (d) Show that for any $\overline{x} \in \mathbb{Q}/\mathbb{Z}$, there exists an integer $n \geq 1$ such that $n\overline{x} = \overline{0}$.

   (e) Show that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the subgroup $\{e^{2\pi i\theta} : \theta \in \mathbb{Q}\}$ of $\mathbb{C}$.

2. Must a group G be finite if all its elements are of finite order? Justify your answer.

3. Compute the quotient group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(2,3)\rangle$.

4. Let H and K be subgroups of a group G. Define $HK = \{hk : h \in H,\ k \in K\} \subset G$. [Recall that HK is a subgroup if and only if $HK = KH$.]

   Suppose H is any subgroup of G and N is any *normal* subgroup of G.

   (a) Explain why the two quotient groups $H/(H \cap N)$ and HN/N make sense (i.e. why they are well-defined).

   (b) Prove the **Second Isomorphism Theorem** – which says that

   for any subgroup H of G and any normal subgroup N of G, we have

   $$H/(H \cap N) \cong HN/N,$$

   i.e. the factor group $H/(H \cap N)$ is isomorphic to the factor group HN/N.

5. Let N and H be normal subgroups in the group G, and $N \subset H$. Show that

   (a) H/N is a normal subgroup of G/N.

   (b) Prove the **Third Isomorphism Theorem** – which says that

   for any normal subgroups N and H of a group G and $N \subset H$, we have

   $$G/H \cong (G/N)/(H/N).$$

   [Remark. We do not include $N \triangleleft H$ as a condition because $N \triangleleft G$ and $N \subset H$ imply $N \triangleleft H$, for any subgroup H of G. Notationally the formula seems to cancel the two denominators N. Of course now you know the underlying deep mathematical content.]

6. Let G be a group.
   (a) Let H be a normal subgroup of G. Show that G/H is abelian if and only if $[G,G] \subset H$.
   (b) Show that $[G,G]$ is the *smallest normal* subgroup N of G such that G/N is *abelian.*†

7. Show that if G is nonabelian, then the quotient group $G/Z(G)$ is not cyclic where $Z(G)$ is the center of G. [Recall that the center of a group is always a normal subgroup.]

   Give an example of nonabelian group G such that $G/Z(G)$ is abelian.

8. Show that D_4 solvable.

9. (a) Let $\sigma = (i, j, k)$ and $\tau = (k, a, b)$ be two 3-cycles. Evaluate $\sigma\tau\sigma^{-1}\tau^{-1}$.
   (b) Let H be a subgroup of the permutation group S_n. If H contains all 3-cycles, show that its commutator subgroup $[H,H]$ contains all 3-cycles as well.
   (c) Using (b) and the definition of solvable groups to show that S_5 is not solvable.

End

† Here "smallest" means if N' is a normal subgroup of G such that G/N' is abelian, then $N' \supset [G,G]$.