Contents

1. Cauchy Integral Formula & Residue Theorem

1.1. Cauchy integral formula.

Theorem 1 (Cauchy Integral Formula 1^{st} form). *Let f be a holomorphic function defined on a neighborhood of $\overline{\Delta(a;r)}$. For any $z \in \Delta(a;r)$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Delta(a;r)} \frac{f(\zeta)}{\zeta - z} d\zeta$$

More generally, if $\Omega \subset \mathbb{C}$ is a domain with piecewise C^1 boundary, then

Corollary 2 (Generalized Cauchy Integral Formula). *([Ahlfors] p.119)*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Proof. Let $\Omega_\epsilon = \Omega - \delta(z,\epsilon)$ where ϵ is sufficiently small such that $\overline{\delta(z,\epsilon)} \cap \partial\Omega = \emptyset$. Since $f(\zeta)/(\zeta - z)$ is holomorphic as a function of ζ on Ω_ϵ. Using Stoke's Theorem and the above theorem yields the result. □

1.2. Residue theorem.

Let Ω be defined as above and f be a meromorphic function on $\overline{\Omega}$ without pole on $\partial\Omega$. It automatically implies that f has only finite number of poles. Then, we have

Theorem 3 (Residue Theorem). *([Ahlfors] p.150)*

$$\int_{\partial\Omega} f(z)dz = 2\pi i \sum_j \mathrm{Res}(f; a_j)$$

Proof. For f meromorphic, let $\{a_j\}$ be the finite set of poles of f in Ω. Consider the function

$$g = f - \sum_j P_j(f)$$

where $P_j(f)$ is the principal part of f at a_j. Then, apply the generalized Cauchy integral formula yields the result. □

We have a generalization for the residue theorem. Take $\overline{\Omega} \subset \Omega'$, consider a discrete set of points $E \subset \Omega$ such that $E \cap \partial\Omega = \emptyset$. If $f \in \mathcal{O}(\Omega' - E)$, then the Residue theorem still holds for f. The proof follows the same argument as above while the principal parts

$$P_j(f) = \sum_{n=-\infty}^{-1} c_n^{(j)} (z - a_j)^n$$

1.3. **Winding number.**

Definition 4. Let $\Gamma : [a,b] \to \mathbb{C}$ be a closed piecewise C^1 curve. For $z \notin \mathrm{Im}(\Gamma)$. Define

$$n(\Gamma, z) = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta - z}$$

$n(\Gamma, z)$ is called the *winding number* of Γ at z.

Fact. $n(\Gamma, z)$ is an integer and hence is constant on each connected component of $\mathbb{C} - \mathrm{Im}(\Gamma)$ by continuity.

1.4. **Cauchy theorem with winding numbers.** Using the winding number, we can generalize the Cauchy integral formula by allowing the domain Ω be multi-connected open subset of $\mathbb{C}$.

Theorem 5 (Homotopy form of Cauchy integral theorem). *([Conway] p.84) Let γ be a closed piecewise C^1 curve. Assume that γ is homotopic to a constant in Ω. Then, for any $z \in \Omega - \mathrm{Im}(\gamma)$,*

$$n(\gamma, z) f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

We say that a closed continuous curve γ *is homotopic to* Γ if and only if $\exists F : [0,1] \times [0,1] \to \Omega$ such that

$$\begin{cases} F(0,s) = \gamma(s) \\ F(1,s) = \Gamma(s) \end{cases}$$

For our purpose, taking γ, Γ to be piecewise C^1. We may assume that $\gamma_t(s) = F(t,s)$ is piecewise C^1 in s.

If $\Gamma(s) = p \in \Omega$, then γ is said to be *homotopic to a constant.* In this case, one may assume that $F(t,0) = p$, $\forall t \in [0,1]$ which means that the end point of each curve for each time t is fixed at p.

Proof. Take homotopy $F : [0,1] \times [0,1] \to \Omega$,

$$\begin{cases} F(0,s) = \gamma(s), \quad \gamma(0) = p = \gamma(1) \\ F(1,s) = p \end{cases}$$

Consider

$$h(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

which is a holomorphic function in ζ defined on $\Omega - \{z\}$. Then, $\text{Res}(h, z) = f(z)$ and the principal part is $f(z)/(\zeta - z)$. Define

$$\begin{aligned} g(\zeta) &= h(\zeta) - \text{principal part at } z \\ &= \frac{f(\zeta)}{\zeta - z} - \frac{f(z)}{\zeta - z} \\ &= \frac{f(\zeta) - f(z)}{\zeta - z} \end{aligned}$$

which has a removable singularity at z. Therefore,

$$\begin{aligned} \int_{\gamma_t} g(\zeta) d\zeta &= \int_{\gamma_1} g(\zeta) d\zeta \\ &= 0 \end{aligned}$$

$\square$

References

[Ahlfors] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill Book Company, New York, 1979.
[Conway] J. B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1978.