Otto Forster

Lectures on Riemann Surfaces

Translated by Bruce Gilligan

With 6 Figures

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CHAPTER 1

Covering Spaces

Riemann surfaces originated in complex analysis as a means of dealing with the problem of multi-valued functions. Such multi-valued functions occur because the analytic continuation of a given holomorphic function element along different paths leads in general to different branches of that function. It was the idea of Riemann to replace the domain of the function with a many sheeted covering of the complex plane. If the covering is constructed so that it has as many points lying over any given point in the plane as there are function elements at that point, then on this "covering surface" the analytic function becomes single-valued. Now, forgetting the fact that these surfaces are "spread out" over the complex plane (or the Riemann sphere), we get the notion of an abstract Riemann surface and these may be considered as the natural domain of definition of analytic functions in one complex variable.

We begin this chapter by discussing the general notion of a Riemann surface. Next we consider covering spaces, both from the topological and analytic points of view. Finally, the theory of covering spaces is applied to the problem of analytic continuation, to the construction of Riemann surfaces of algebraic functions, to the integration of differential forms and to finding the solutions of linear differential equations.

§1. The Definition of Riemann Surfaces

In this section we define Riemann surfaces, holomorphic and meromorphic functions on them and also holomorphic maps between Riemann surfaces.

Riemann surfaces are two-dimensional manifolds together with an additional structure which we are about to define. As is well known, an

n-dimensional manifold is a Hausdorff topological space X such that every point $a \in X$ has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

1.1. Definition. Let X be a two-dimensional manifold. A *complex chart* on X is a homeomorphism $\varphi\colon U \to V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$. Two complex charts $\varphi_i\colon U_i \to V_i$, $i = 1, 2$ are said to be *holomorphically compatible* if the map

$$\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is biholomorphic (see Fig. 1).

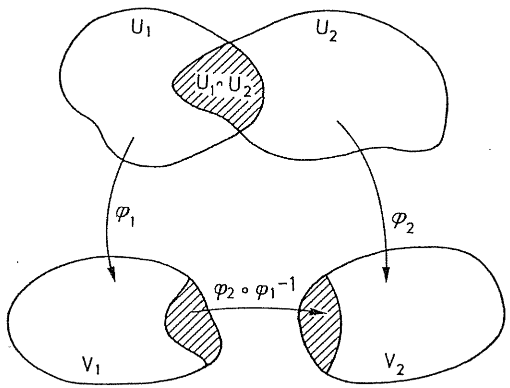

Figure 1

A *complex atlas* on X is a system $\mathfrak{A} = \{\varphi_i\colon U_i \to V_i, i \in I\}$ of charts which are holomorphically compatible and which cover X, i.e., $\bigcup_{i \in I} U_i = X$.

Two complex atlases $\mathfrak{A}$ and $\mathfrak{A}'$ on X are called *analytically equivalent* if every chart of $\mathfrak{A}$ is holomorphically compatible with every chart of $\mathfrak{A}'$.

1.2. Remarks

(a) If $\varphi\colon U \to V$ is a complex chart, U_1 is open in U and $V_1 := \varphi(U_1)$, then $\varphi \mid U_1 \to V_1$ is a chart which is holomorphically compatible with $\varphi\colon U \to V$.

(b) Since the composition of biholomorphic mappings is again biholomorphic, one easily sees that the notion of analytic equivalence of complex atlases is an equivalence relation.

1.3. Definition. By a *complex structure* on a two-dimensional manifold X we mean an equivalence class of analytically equivalent atlases on X.

Thus a complex structure on X can be given by the choice of a complex atlas. Every complex structure Σ on X contains a unique maximal atlas $\mathfrak{A}^*$. If $\mathfrak{A}$ is an arbitrary atlas in Σ, then $\mathfrak{A}^*$ consists of all complex charts on X which are holomorphically compatible with every chart of $\mathfrak{A}$.

1.4. Definition. A *Riemann surface* is a pair (X, Σ), where X is a connected two-dimensional manifold and Σ is a complex structure on X.

One usually writes X instead of (X, Σ) whenever it is clear which complex structure Σ is meant. Sometimes one also writes $(X, \mathfrak{A})$ where $\mathfrak{A}$ is a representative of Σ.

Convention. If X is a Riemann surface, then by a chart on X we always mean a complex chart belonging to the maximal atlas of the complex structure on X.

Remark. Locally a Riemann surface X is nothing but an open set in the complex plane. For, if $\varphi\colon U \to V \subset \mathbb{C}$ is a chart on X, then φ maps the open set $U \subset X$ bijectively onto V. However, any given point of X is contained in many different charts and no one of these is distinguished from the others. For this reason we may only carry over to Riemann surfaces those notions from complex analysis in the plane which remain invariant under biholomorphic mappings, i.e., those notions which do not depend on the choice of a particular chart.

1.5. Examples of Riemann Surfaces

(a) *The Complex Plane* $\mathbb{C}$. Its complex structure is defined by the atlas whose only chart is the identity map $\mathbb{C} \to \mathbb{C}$.

(b) *Domains.* Suppose X is a Riemann surface and $Y \subset X$ is a domain, i.e., a connected open subset. Then Y has a natural complex structure which makes it a Riemann surface. Namely, one takes as its atlas all those complex charts $\varphi\colon U \to V$ on X, where $U \subset Y$. In particular, every domain $Y \subset \mathbb{C}$ is a Riemann surface.

(c) *The Riemann sphere* $\mathbb{P}^1$. Let $\mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$, where ∞ is a symbol not contained in $\mathbb{C}$. Introduce the following topology on $\mathbb{P}^1$. The open sets are the usual open sets $U \subset \mathbb{C}$ together with sets of the form $V \cup \{\infty\}$, where $V \subset \mathbb{C}$ is the complement of a compact set $K \subset \mathbb{C}$. With this topology $\mathbb{P}^1$ is a compact Hausdorff topological space, homeomorphic to the 2-sphere S^2. Set

$$U_1 := \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$$
$$U_2 := \mathbb{P}^1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}.$$

Define maps $\varphi_i\colon U_i \to \mathbb{C}$, $i = 1, 2$, as follows. φ_1 is the identity map and

$$\varphi_2(z) := \begin{cases} 1/z & \text{for } z \in \mathbb{C}^* \\ 0 & \text{for } z = \infty. \end{cases}$$

Clearly these maps are homeomorphisms and thus $\mathbb{P}^1$ is a two-dimensional manifold. Since U_1 and U_2 are connected and have non-empty intersection, $\mathbb{P}^1$ is also connected.

The complex structure on $\mathbb{P}^1$ is now defined by the atlas consisting of the charts $\varphi_i\colon U_i \to \mathbb{C}$, $i = 1, 2$. We must show that the two charts are holomorphically compatible. But $\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{C}^*$ and

$$\varphi_2 \circ \varphi_1^{-1}\colon \mathbb{C}^* \to \mathbb{C}^*, \qquad z \mapsto 1/z,$$

is biholomorphic.

Remark. The notation $\mathbb{P}^1$ comes from the fact that one may consider $\mathbb{P}^1$ as the 1-dimensional projective space over the field of complex numbers.

(d) *Tori.* Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. Define

$$\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n\omega_1 + m\omega_2\colon n, m \in \mathbb{Z}\}.$$

Γ is called the lattice spanned by ω_1 and ω_2 (Fig. 2). Two complex numbers $z, z' \in \mathbb{C}$ are called equivalent mod Γ if $z - z' \in \Gamma$. The set of all equivalence classes is denoted by $\mathbb{C}/\Gamma$. Let $\pi\colon \mathbb{C} \to \mathbb{C}/\Gamma$ be the canonical projection, i.e., the map which associates to each point $z \in \mathbb{C}$ its equivalence class mod Γ.

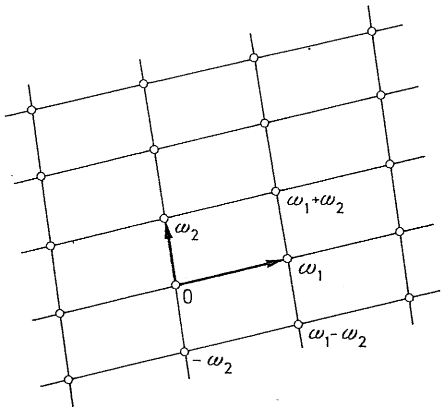

Figure 2

Introduce the following topology (the quotient topology) on $\mathbb{C}/\Gamma$. A subset $U \subset \mathbb{C}/\Gamma$ is open precisely if $\pi^{-1}(U) \subset \mathbb{C}$ is open. With this topology $\mathbb{C}/\Gamma$ is a Hausdorff topological space and the quotient map $\pi\colon \mathbb{C} \to \mathbb{C}/\Gamma$ is continuous. Since $\mathbb{C}$ is connected, $\mathbb{C}/\Gamma$ is also connected. As well $\mathbb{C}/\Gamma$ is compact, for it is covered by the image under π of the compact parallelogram

$$P := \{\lambda\omega_1 + \mu\omega_2\colon \lambda, \mu \in [0, 1]\}.$$

The map π is open, i.e., the image of every open set $V \subset \mathbb{C}$ is open. To see this one has to show that $\hat{V} := \pi^{-1}(\pi(V))$ is open. But

$$\hat{V} = \bigcup_{\omega \in \Gamma} (\omega + V).$$

Since every set $\omega + V$ is open, so is $\hat{V}$.

The complex structure on $\mathbb{C}/\Gamma$ is defined in the following way. Let $V \subset \mathbb{C}$ be an open set such that no two points in V are equivalent under Γ. Then $U := \pi(V)$ is open and $\pi | V \to U$ is a homeomorphism. Its inverse $\varphi\colon U \to V$ is a complex chart on $\mathbb{C}/\Gamma$. Let $\mathfrak{A}$ be the set of all charts obtained in this fashion. We have to show that any two charts $\varphi_i\colon U_i \to V_i$, $i = 1, 2$, belonging to $\mathfrak{A}$ are holomorphically compatible. Consider the map

$$\psi := \varphi_2 \circ \varphi_1^{-1}\colon \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2).$$

For every $z \in \varphi_1(U_1 \cap U_2)$ one has $\pi(\psi(z)) = \varphi_1^{-1}(z) = \pi(z)$ and thus $\psi(z) - z \in \Gamma$. Since Γ is discrete and ψ is continuous, this implies that $\psi(z) - z$ is constant on every connected component of $\varphi_1(U_1 \cap U_2)$. Thus ψ is holomorphic. Similarly ψ^{-1} is also holomorphic.

Now let $\mathbb{C}/\Gamma$ have the complex structure defined by the complex atlas $\mathfrak{A}$.

Remark. Let $S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$ be the unit circle. The map which associates to the point of $\mathbb{C}/\Gamma$ represented by $\lambda\omega_1 + \mu\omega_2$, $(\lambda, \mu \in \mathbb{R})$, the point

$$(e^{2\pi i\lambda}, e^{2\pi i\mu}) \in S^1 \times S^1,$$

is a homeomorphism of $\mathbb{C}/\Gamma$ onto the torus $S^1 \times S^1$.

1.6. Definition. Let X be a Riemann surface and $Y \subset X$ an open subset. A function $f\colon Y \to \mathbb{C}$ is called *holomorphic*, if for every chart $\psi\colon U \to V$ on X the function

$$f \circ \psi^{-1}\colon \psi(U \cap Y) \to \mathbb{C}$$

is holomorphic in the usual sense on the open set $\psi(U \cap Y) \subset \mathbb{C}$. The set of all functions holomorphic on Y will be denoted by $\mathcal{O}(Y)$.

1.7. Remarks

(a) The sum and product of holomorphic functions are again holomorphic. Also constant functions are holomorphic. Thus $\mathcal{O}(Y)$ is a $\mathbb{C}$-algebra.

(b) Of course the condition in the definition does not have to be verified for all charts in a maximal atlas on X, just for any family of charts covering Y. Then it is automatically fulfilled for all other charts.

(c) Every chart $\psi\colon U \to V$ on X is, in particular, a complex-valued function on U. Trivially it is holomorphic. One also calls ψ a local coordinate or a uniformizing parameter and (U, ψ) a *coordinate neighborhood* of any point $a \in U$. In this context one generally uses the letter z instead of ψ.

1.8. Theorem (Riemann's Removable Singularities Theorem). *Let U be an open subset of a Riemann surface and let $a \in U$. Suppose the function $f \in \mathcal{O}(U\backslash\{a\})$ is bounded in some neighborhood of a. Then f can be extended uniquely to a function $\tilde{f} \in \mathcal{O}(U)$.*

This follows directly from Riemann's Removable Singularities Theorem in the complex plane.

We now define holomorphic mappings between Riemann surfaces.

1.9. Definition. Suppose X and Y are Riemann surfaces. A continuous mapping $f\colon X \to Y$ is called *holomorphic*, if for every pair of charts $\psi_1\colon U_1 \to V_1$ on X and $\psi_2\colon U_2 \to V_2$ on Y with $f(U_1) \subset U_2$, the mapping

$$\psi_2 \circ f \circ \psi_1^{-1}\colon V_1 \to V_2$$

is holomorphic in the usual sense.

A mapping $f\colon X \to Y$ is called *biholomorphic* if it is bijective and both $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ are holomorphic. Two Riemann surfaces X and Y are called *isomorphic* if there exists a biholomorphic mapping $f\colon X \to Y$.

1.10. Remarks

(a) In the special case $Y = \mathbb{C}$, holomorphic mappings $f\colon X \to \mathbb{C}$ are clearly the same as holomorphic functions.

(b) If X, Y and Z are Riemann surfaces and $f\colon X \to Y$ and $g\colon Y \to Z$ are holomorphic mappings, then the composition $g \circ f\colon X \to Z$ is also holomorphic.

(c) A continuous mapping $f\colon X \to Y$ between two Riemann surfaces is holomorphic precisely if for every open set $V \subset Y$ and every holomorphic function $\psi \in \mathcal{O}(V)$, the "pull-back" function $\psi \circ f\colon f^{-1}(V) \to \mathbb{C}$ is contained in $\mathcal{O}(f^{-1}(V))$. This follows directly from the definitions and the remarks (1.7.c) and (1.10.b).

In this way a holomorphic mapping $f\colon X \to Y$ induces a mapping

$$f^*\colon \mathcal{O}(V) \to \mathcal{O}(f^{-1}(V)), \qquad f^*(\psi) = \psi \circ f.$$

One can easily check that f^* is a ring homomorphism. If $g\colon Y \to Z$ is another holomorphic mapping, W is open in Z, $V := g^{-1}(W)$ and $U := f^{-1}(V)$, then $(g \circ f)^*\colon \mathcal{O}(W) \to \mathcal{O}(U)$ is the composition of the mappings $g^*\colon \mathcal{O}(W) \to \mathcal{O}(V)$ and $f^*\colon \mathcal{O}(V) \to \mathcal{O}(U)$, i.e., $(g \circ f)^* = f^* \circ g^*$.

1.11. Theorem (Identity Theorem). *Suppose X and Y are Riemann surfaces and $f_1, f_2\colon X \to Y$ are two holomorphic mappings which coincide on a set $A \subset X$ having a limit point $a \in X$. Then f_1 and f_2 are identically equal.*

PROOF. Let G be the set of all points $x \in X$ having an open neighborhood W such that $f_1 \mid W = f_2 \mid W$. By definition G is open. We claim that G is also closed. For, suppose b is a boundary point of G. Then $f_1(b) = f_2(b)$ since f_1 and f_2 are continuous. Choose charts $\varphi\colon U \to V$ on X and $\psi\colon U' \to V'$ on Y with $b \in U$ and $f_i(U) \subset U'$. We may also assume that U is connected. The mappings

$$g_i := \psi \circ f_i \circ \varphi^{-1}\colon V \to V' \subset \mathbb{C}$$

are holomorphic. Since $U \cap G \neq \varnothing$, the Identity Theorem for holomorphic functions on domains in $\mathbb{C}$ implies g_1 and g_2 are identically equal. Thus $f_1 \mid U = f_2 \mid U$. Hence $b \in G$ and thus G is closed. Now since X is connected either $G = \varnothing$ or $G = X$. But the first case is excluded since $a \in G$ (using the Identity Theorem in the plane again). Hence f_1 and f_2 coincide on all of X. □

1.12. Definition. Let X be a Riemann surface and Y be an open subset of X. By a *meromorphic function* on Y we mean a holomorphic function $f\colon Y' \to \mathbb{C}$, where $Y' \subset Y$ is an open subset, such that the following hold:

(i) $Y \backslash Y'$ contains only isolated points.

(ii) For every point $p \in Y \backslash Y'$ one has

$$\lim_{x \to p} |f(x)| = \infty.$$

The points of $Y \backslash Y'$ are called the *poles* of f. The set of all meromorphic functions on Y is denoted by $\mathcal{M}(Y)$.

1.13. Remarks

(a) Let (U, z) be a coordinate neighborhood of a pole p of f with $z(p) = 0$. Then f may be expanded in a Laurent series

$$f = \sum_{\nu=-k}^{\infty} c_\nu z^\nu$$

in a neighborhood of p.

(b) $\mathcal{M}(Y)$ has the natural structure of a $\mathbb{C}$-algebra. First of all the sum and the product of two meromorphic functions $f, g \in \mathcal{M}(Y)$ are holomorphic functions at those points where both f and g are holomorphic. Then one holomorphically extends, using Riemann's Removable Singularities Theorem, $f + g$ (resp. fg) across any singularities which are removable.

1.14. Example. Suppose $n \geq 1$ and let

$$F(z) = z^n + c_1 z^{n-1} + \cdots + c_n, \qquad c_k \in \mathbb{C},$$

be a polynomial. Then F defines a holomorphic mapping $F\colon \mathbb{C} \to \mathbb{C}$. If one thinks of $\mathbb{C}$ as a subset of $\mathbb{P}^1$, then $\lim_{z\to\infty} |F(z)| = \infty$. Thus $F \in \mathcal{M}(\mathbb{P}^1)$.

We now interpret meromorphic functions as holomorphic mappings into the Riemann sphere.

1.15. Theorem. *Suppose X is a Riemann surface and $f \in \mathcal{M}(X)$. For each pole p of f, define $f(p) := \infty$. Then $f\colon X \to \mathbb{P}^1$ is a holomorphic mapping. Conversely, if $f\colon X \to \mathbb{P}^1$ is a holomorphic mapping, then f is either identically equal to ∞ or else $f^{-1}(\infty)$ consists of isolated points and $f\colon X \backslash f^{-1}(\infty) \to \mathbb{C}$ is a meromorphic function on X.*

From now on we will identify a meromorphic function $f \in \mathcal{M}(X)$ with the corresponding holomorphic mapping $f: X \to \mathbb{P}^1$.

PROOF

(a) Let $f \in \mathcal{M}(X)$ and let P be the set of poles of f. Then f induces a mapping $f: X \to \mathbb{P}^1$ which is clearly continuous. Suppose $\varphi: U \to V$ and $\psi: U' \to V'$ are charts on X and $\mathbb{P}^1$ resp. with $f(U) \subset U'$. We have to show that

$$g := \psi \circ f \circ \varphi^{-1}: V \to V'$$

is holomorphic. Since f is holomorphic on $X \backslash P$, it follows that g is holomorphic on $V \backslash \varphi(P)$. Hence by Riemann's Removable Singularities Theorem, g is holomorphic on all of V.

(b) The converse follows from the Identity Theorem (1.11). □

1.16. Remark. From (1.11) and (1.15) it follows that the Identity Theorem also holds for meromorphic functions on a Riemann surface. Thus any function $f \in \mathcal{M}(X)$ which is not identically zero has only isolated zeros. This implies that $\mathcal{M}(X)$ is a field.