THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3301
Assignment 4

Due Date: Nov 7, 2024, 23:59.

Submission Guidelines

(i) Write your solution on papers of about A4 size.

(ii) Scan your work properly and save it as **one** PDF file.

Warning: Please make sure that your work is properly scanned. Oversized, blurred or upside-down images will NOT be accepted.

(iii) While you can upload and save draft in moodle, you MUST click the "Submit" button to declare your final submission before the due date. Otherwise, you will be considered late.

Preparation Guidelines

(i) Your solution should be well written and organized. It is good to work out a draft for each question on a separate paper, polish/rewrite/reorganize your answer suitably and then write it (the final form) on the paper to be scanned.

(ii) You may imagine that you are teaching this course and writing a solution to demonstrate the answer. Hence, especially for proof-type questions, you have to convince everyone that your solution (proof) is correct, without any oral explanation from you. i.e. Another student should be able to understand the answer (proof) completely via your written word, and/or diagrams or tables you create in your solution.

(iii) Follow HKU's regulations on academic honesty. Plagiarism is inacceptable and may have severe consequences for your record. See `https://tl.hku.hk/plagiarism/` for "What is plagiarism?". *If you have used AI tools to explore, check or refine your work, please acknowledge and clearly identify the parts of your work that involve AI output to avoid plagiarism or related academic dishonesty. Indicate the extent to which the AI output is used (e.g. directly copied or paraphrased/modified or checked for errors or reorganized the presentation).*

To be handed in

1. Suppose G is a group of order pq^m where p, q are distinct primes. If the center of G contains a q-Sylow subgroup, show that G is abelian.

2. Let G be a finite group acting on the finite set X. Define $X_g = \{x \in X : gx = x\}$. Show that
$$\sum_{g \in G} |X_g| = \sum_{x \in X} |G_x|. \qquad \text{(Recall } G_x := \{g \in G : gx = x\}.\text{)}$$
(Think about the meaning on each side of the equation.)

3. Let G be a group with center Z. If G/Z is cyclic, show that G must be abelian.

4. Let G be a finite group acting on the finite set X.

   (a) Show that $\displaystyle\sum_{y \in Gx} \frac{1}{|Gy|} = 1$ for any $x \in X$.

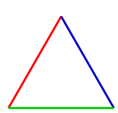
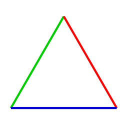
   (b) Show that the number of (distinct) orbits is $\displaystyle\sum_{x \in X} \frac{1}{|Gx|}$.

   (c) Prove the following result

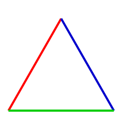
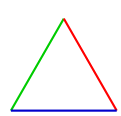
   Burnside's lemma : The number of orbits of G in X $= \dfrac{1}{|G|} \displaystyle\sum_{g \in G} |X_g|$.

5. Given 4 colors to paint the edges of an equilateral triangle. A single edge is painted by one color and the same color may be used on different edges.

   (a) How many ways can you paint the edges in all?

   In this case, the figures  ,  are regarded as different.

   (b) Now we are allowed to move the triangle by rotation or reflection.
   How many distinguishable ways can you paint the edges?

   In this case, the figures  ,  are of the same painting way.

   [Hint: Use Burnside's Lemma.]

6. Let G be a finite group of order n. We say G satisfies the property (*) if

$$(*)\ \forall\ N \lhd G,\ \forall \text{ prime } p \big| |G/N|,\ \exists\ a \in G/N \text{ with } \mathrm{ord}(a) = p \text{ and } \langle a \rangle \lhd G/N.$$

In (a)-(c) below, we assume G satisfies the property (*).

(a) Show that for any prime $p \big| n$, G has a normal subgroup of order p.

(b) Let $N \lhd G$. Define $G' := G/N$. Show that G' satisfies the property (*).

(c) Show that for any $d \big| |G|$, there exists a subgroup H of G with $|H| = d$.

(d) Show (i) all abelian groups and (ii) all p-groups satisfy the property (*).

7. (a) Let G be a group with $|G| = 60$, and consider the product group $G \times \mathbb{Z}_{30}$. Define $N := \{e\} \times \mathbb{Z}_{30}$ ($\lhd G \times \mathbb{Z}_{30}$ by Part I Qn 4). Assume H is a subgroup of $G \times \mathbb{Z}_{30}$ and $|H| = 450$. Show that $|H/(H \cap N)|$ is either 15 or 30. [Hint: Use Tutorial 8 Qn 2.]

(b) Show that $A_5 \times \mathbb{Z}_{30}$ does not have a subgroup of order 450.

(c) Give an example to illustrate the following: there exists a finite group G of order n such that for all prime $p \big| n$, G has an element a of order p and $\langle a \rangle \lhd N$ but for some $d \big| n$, G has no subgroup of order d.

8. (a) Find all units and zero divisors in $\mathbb{Z}_3 \times \mathbb{Z}_4$.

(b) For any integral domain R_1 and R_2, show that $R_1 \times R_2$ is not an integral domain (whose addition and multiplication are defined componentwisely).

(c) Show that $I \times J$ is an ideal of the product ring $R_1 \times R_2$ if I and J are ideals of the rings R_1 and R_2 respectively. Is every ideal of the product ring $R_1 \times R_2$ of the form $I \times J$ for some ideals $I \subset R_1$ and $J \subset R_2$? Justify your answer.

End