Date / /

20241218 MATH3541 Sample Test 2017.

1.(a). True.

The sequence $a_n = \frac{\sin n}{n}$ has limit 0, so its range $\{\frac{\sin n}{n}\}_{n\in\mathbb{N}}$ has a unique limit point 0. $\overline{\{\frac{\sin n}{n}\}_{n\in\mathbb{N}}} = \{\frac{\sin n}{n}\}_{n\in\mathbb{N}} \cup \{\frac{\sin n}{n}\}'_{n\in\mathbb{N}} = \{\frac{\sin n}{n}\}_{n\in\mathbb{N}} \cup \{0\}$ is closed in the metric space $\mathbb{R}$.

(b) False.

We've mentioned in class without proof that an open half ball $\setminus\{\vec{0}\}$ is contractible, while an open ball $\setminus\{\vec{0}\}$ is not contractible.

Assume to the contrary that $\overline{\mathbb{H}}^3 = \{(x_1,x_2,x_3)^T \in \mathbb{R}^3 : x_3 \geq 0\}$ is homeomorphic to $\mathbb{R}^3$. WLOG, assume that a homeomorphism sends an open neighbour U of $\vec{0}$ in $\overline{\mathbb{H}}^3$ to an open ball V centred at $\vec{0}$ in $\mathbb{R}^3$. After restriction, we get a homeomorphism from a non contractible space $V\setminus\{\vec{0}\}$ to a contractible space $U\setminus\{\vec{0}\}$, which is a contradiction.

(c) True. Consider two compact subsets S, T of a Hausdorff space X.

Step 1: Fix $t\in T$. For all $s\in S$, as $S\cap T=\phi$, there exists $U_{s,t}, V_{s,t}$, such that $U_{s,t}\ni s$, $V_{s,t}\ni t$, $U_{s,t}\cap V_{s,t}=\phi$.

As s is arbitrary, $(U_{s,t})_{s\in S}$ covers the compact subset of X.

Reduce the open cover $(U_{s,t})_{s\in S}$ to a finite subcover $(U_{s_k,t})_{k=1}^m$

Now $M_t = \bigcup_{k=1}^m U_{s_k,t}$, $N_t = \bigcap_{k=1}^m V_{s_k,t}$ separates S, t.

Step 2: As t is arbitrary, $(N_t)_{t\in T}$ covers the compact subset T of X

Do the same argument to find $(N_{t_\ell})_{\ell=1}^m$, so $P = \bigcap_{\ell=1}^m M_{t_\ell}$, $Q = \bigcup_{\ell=1}^m N_{t_\ell}$ separates S, T.

Date / /

(d) False. Consider $X=[0,2]$, $Y_1=[0,1]$, $Y_2=\{0\}\cup(1,2]$

$$f_1: Y_1\to\mathbb{R},\ f_1(x)=0$$

$$f_2: Y_2\to\mathbb{R},\ f_2(x)=\begin{cases}0 & \text{if } x=0;\\ 1 & \text{if } 1<x\le 2;\end{cases}$$

On $Y_1\cap Y_2=\{0\}$, $f_1(x)=f_2(x)=0$, f_1, f_2 agrees well.

However, the gluing: $f=f_1\cup f_2: X=Y_1\cup Y_2\to\mathbb{R}$, $f(x)=\begin{cases}0 \text{ if } 0\le x\le 1;\\ 1 \text{ if } 1<x\le 2;\end{cases}$

is discontinuous at $x=1$ as $\lim\limits_{x\to 1} f(x)$ fails to exist.

(e) False. Consider the topologist's sine curve.

$$T=\overline{S}, \text{ where } S=\{(x,\sin\tfrac{\pi}{x})\in\mathbb{R}^2: 0<x\le 1\}.$$

As $(0,0)=\lim\limits_{n\to+\infty}\left(\frac{1}{n},\sin\frac{\pi}{1/n}\right)$, $(0,0)\in T=\overline{S}$

As $(1,0)=(1,\sin\frac{\pi}{1})\in S$, $(1,0)\in S$.

Assume to the contrary that there exists a path $\vec{\gamma}=(\gamma_x,\gamma_y)$ from $(0,0)$ to $(1,0)$ along T.

As $\gamma_x^{-1}(\{0\})$ is nonempty, proper and compact in $[0,1]$,

it has a unique maximum $0\le\xi<1$.

For all $0<\varepsilon<1-\xi$, there exists $x_1, x_2\in(\xi,\varepsilon+\xi)$,

such that $\gamma_x(x_1)=\frac{1}{2n-\frac{1}{2}}$, $\vec{\gamma}(x_1)=\left(\frac{1}{2n-\frac{1}{2}},-1\right)$,

$\gamma_x(x_2)=\frac{1}{2n+\frac{1}{2}}$, $\vec{\gamma}(x_2)=\left(\frac{1}{2n+\frac{1}{2}},+1\right)$, where $n\in\mathbb{N}$.

This implies $\lim\limits_{t\to\xi^+}\vec{\gamma}(t)$ fails to exist, contradict to the continuity.

However, S is the continuous image of a connected set $[0,1]$,

so S is connected, and it follows that $T=\overline{S}$ is connected.

(f) True. WLOG, assume that $\overline{Z}=X$.

For all clopen subset U of $X=\overline{Z}$, assume that $U\neq\phi$, we wish to show $U=X$.

U is clopen in $X \Rightarrow U\cap Z$ is clopen in Z

$\Rightarrow U\cap Z=\phi$ or $U\cap Z=Z$, as Z is connected,

$\Rightarrow U\cap Z=Z$, as U is the open neighbourhood of some $x\in X=\overline{Z}$, $U\cap Z\neq\phi$.

$\Rightarrow Z\subseteq U$

$\Rightarrow X=\overline{Z}\subseteq\overline{U}=U$, as U is closed in X

$\Rightarrow U=X$.

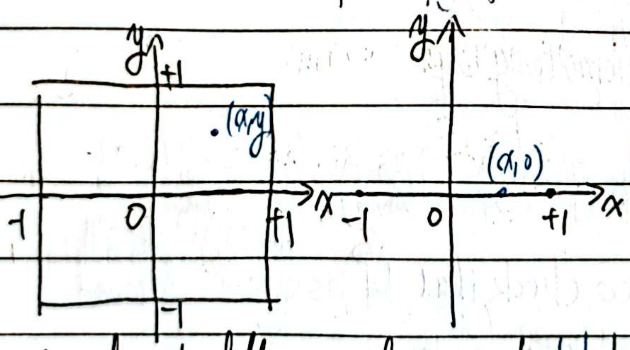
Hence, $X=\overline{Z}$ is connected.

(g) True. Because any point $x\in X$ has a contractible neighbourhood X of x.

2. Proof: We may divide our proof into two parts.

Part 1: We show that the closed Möbius strip is homotopic to a closed cylinder.

First, we construct a deformation retraction from $[-1,+1]\times[-1,+1]$ to $[-1,+1]\times\{0\}$.



$H((x,y),t)=(x,(1-t)y)$

Second, do the following gluings, and H becomes the corresponding homotopies.

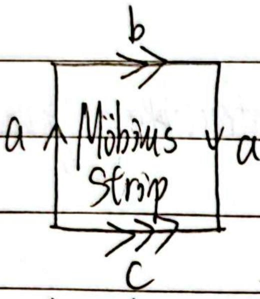
(1) Identify every $(-1,+y)$ with $(+1,+y)$, and H becomes a homotopy from the cylinder $S^1\times[-1,+1]$ to the circle $S^1\times\{0\}$.

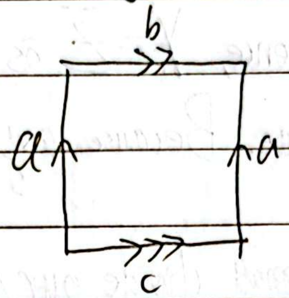
(2) Identify every $(-1,-y)$ with $(+1,+y)$, and H becomes a homotopy from the Möbius band M^2 to the circle $S \times \{0\}$.

As homotopy is transitive and symmetric, $S\times[-1,1] \sim M^2$.

Part 2: We show that the closed Möbius strip is not homeomorphic to a closed cylinder.

As the closed Möbius strip is non orientable, and the closed cylinder is orientable, they are not homeomorphic.





$aba c^{-1}$ $\qquad\qquad$ $aba^{-1}c^{-1}$

3.(a) Proof: Assume that $\vec f: S^n \to S^n$ is a continuous map without fixed point.

Construct a homotopy by:

$$\vec H: S^n\times[0,1] \to S^n,\ \vec H(\vec x,t) = \frac{(1-t)\vec f(\vec x)+t(-\vec x)}{\|(1-t)\vec f(\vec x)+t(-\vec x)\|}$$

I would like to check that $\vec H$ is well-defined.

For all $(\vec x,t)\in S^n\times[0,1]$,

$$(1-t)\vec f(\vec x)+t(-\vec x)=\vec 0 \Rightarrow \overset{\geq 0}{(1-t)}\overset{\|\cdot\|=1}{\vec f(\vec x)} = \overset{\geq 0}{t}\overset{\|\cdot\|=1}{\vec x}$$

$$\Rightarrow 1-t=t=0 \text{ or } \left(1-t=t=\tfrac{1}{2} \text{ and } \vec f(\vec x)=\vec x\right) \Rightarrow \text{Contradiction.}$$

Hence, the denominator is nonzero, $\vec H$ is continuous, and $\vec f(\vec x)=\vec H(\vec x,0)$ is homotopic to $-1\vec x=\vec H(\vec x,1)$ via $\vec H$.

(b) Proof: When n is odd, $n+1$ is even, and -1 can be extended to the following linear transformation from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$:

$$-1: \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_{\frac{n+1}{2}} \\ y_{\frac{n+1}{2}} \end{pmatrix} \mapsto \begin{pmatrix} \cos\pi & -\sin\pi & 0 & 0 & \cdots & 0 & 0 \\ \sin\pi & \cos\pi & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos\pi & -\sin\pi & \cdots & 0 & 0 \\ 0 & 0 & \sin\pi & \cos\pi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cos\pi & -\sin\pi \\ 0 & 0 & 0 & 0 & \cdots & \sin\pi & \cos\pi \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_{\frac{n+1}{2}} \\ y_{\frac{n+1}{2}} \end{pmatrix}$$

Hence, there exists a homotopy from $+1$ to -1:

$$\vec{H}: \left(\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_{\frac{n+1}{2}} \\ y_{\frac{n+1}{2}} \end{pmatrix}, t \right) \mapsto \begin{pmatrix} \cos\pi t & -\sin\pi t & 0 & 0 & \cdots & 0 & 0 \\ \sin\pi t & \cos\pi t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos\pi t & -\sin\pi t & \cdots & 0 & 0 \\ 0 & 0 & \sin\pi t & \cos\pi t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cos\pi t & -\sin\pi t \\ 0 & 0 & 0 & 0 & \cdots & \sin\pi t & \cos\pi t \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_{\frac{n+1}{2}} \\ y_{\frac{n+1}{2}} \end{pmatrix}$$

Solution:

4. (a) We would like to apply Van Kampen's Theorem.

Step1: At $\vec{p} \in \Sigma$, as Σ is a two dimensional topological manifold, there exists an open neighbour U of $\vec{p}$ on Σ, such that U is homeomorphic to the two dimensional open disk $\mathbb{D}^2$.

Similarly, choose an open neighbour U' of $\vec{p}'$ on Σ', such that $U' \cong \mathbb{D}^2$

Step2: Define $Y = \Sigma \vee \Sigma'$, $X = \Sigma \vee U'$, $X' = U \vee \Sigma'$, $S = U \vee U'$

(1) $Y = X \cup X'$, and we are going to compute $\pi_1(Y)$.

(2) Σ is path connected, U' is path connected, so $X = \Sigma \vee U'$ is path connected

U' is open in Σ', so $X = \Sigma \vee U'$ is open in $Y = \Sigma \vee \Sigma'$

Date / /

(3) Similarly, $X' = U \vee \Sigma'$ is path connected, and $X' = U \vee \Sigma'$ is open in $Y = \Sigma \vee \Sigma'$

(4) The intersection $S = X \cap X' = U \vee U'$ is homotopic to a singleton, so S is path connected and nonempty.

Now choose a base point $\{p, p'\}$ in the wedge sum $U \vee U'$.

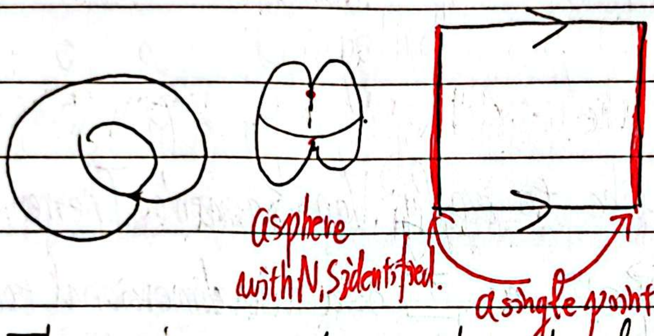
$$\pi_1(\Sigma \vee \Sigma') \cong \pi_1(Y) \cong \pi_1(X) *_{\pi_1(S)} \pi_1(X')$$

$$\cong \pi_1(\Sigma \vee U') *_{\pi_1(U \vee U')} \pi_1(U \vee \Sigma')$$

$$\cong \pi_1(\Sigma) *_{\pi_1(pt)} \pi_1(\Sigma')$$

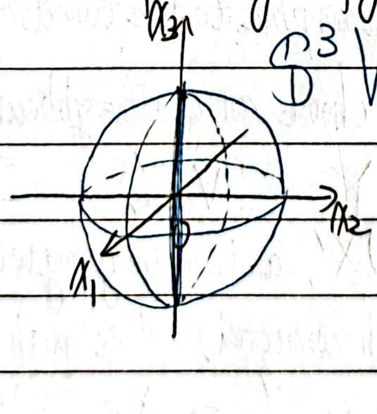
$$\cong \pi_1(\Sigma) *_{\{e\}} \pi_1(\Sigma') \cong \pi_1(\Sigma) * \pi_2(\Sigma').$$

(b) Solution:



(c) Solution: The space is a sphere with north pole, south pole identified. It is homotopic to the following configuration:

$$S^3 \vee_{north,south} \{(x,y,z) \in \mathbb{R}^3 : x = y = 0, -1 \le z \le 1\}.$$



I would like to define:

$Y = \mathbb{S}^3 \setminus_{\text{north, south}} \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0, -1 \leq x_3 \leq +1\}$.

$X_{strip} = Y \setminus \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 < 0, x_2 = 0, x_3 \in \mathbb{R}\}$, open and path connected.

By the homotopy $H((r\cos\theta, r\sin\theta, x_3), t)$

$= (r\cos(1-t)\theta, r\sin(1-t)\theta, x_3)$ $(r \geq 0, -\pi < \theta < +\pi, 0 \leq t \leq 1, x_3 \in \mathbb{R})$.

one may shrink X_{strip} to a circle, so $\pi_1(X_{strip}) \cong \mathbb{Z}$.

$X_{ext} = Y \setminus \{(0,0,0)\}$, open and path connected.

By the homotopy $H((x_1,x_2,x_3),t) = \begin{cases} (x_1,x_2,x_3) & \text{if } x_1^2+x_2^2 \neq 0 \\ (0,0,(1-t)x_3+t) & \text{if } x_1^2+x_2^2=0, x_3>0. \\ (0,0,(1-t)x_3-t) & \text{if } x_1^2+x_2^2=0, x_3<0, \end{cases}$

one may shrink X_{ext} to a sphere, so $\pi_1(X_{ext}) \cong \{e\}$.

$S = X_{strip} \cap X_{ext}$, nonempty and path connected.

By the homotopy on X_{strip}, one may shrink S to a circle with a point removed, so $\pi_1(S) \cong \{e\}$.

Hence $\pi_1(Y) \cong \pi_1(X_{strip} \cup X_{ext})$

$\cong \pi_1(X_{strip}) *_{\pi_1(S)} \pi_1(X_{ext})$

$\cong \mathbb{Z} *_{\{e\}} \{e\} \cong \mathbb{Z}$.

5.(a). Solution: Define two functions on $GL_2(\mathbb{C})$:

$$\text{Comp}: \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1}b_{1,1}+a_{1,2}b_{2,1} & a_{1,1}b_{1,2}+a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1}+a_{2,2}b_{2,1} & a_{2,1}b_{1,2}+a_{2,2}b_{2,2} \end{pmatrix}$$

$$\text{Inv}: \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}^{-1} = \frac{1}{a_{1,1}a_{2,2}-a_{1,2}a_{2,1}}\begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}$$

As polynomials $a_{1,1}b_{1,1}+a_{1,2}b_{2,1}$, $a_{1,1}b_{1,2}+a_{1,2}b_{2,2}$,

$a_{2,1}b_{1,1}+a_{2,2}b_{2,1}$, $a_{2,1}b_{1,2}+a_{2,2}b_{2,2}$,

$a_{2,2}$, $-a_{1,2}$

$-a_{2,1}$, $a_{1,1}$

$a_{1,1}a_{2,2}-a_{1,2}a_{2,1}$ are continuous,

and $a_{1,1}a_{2,2}-a_{1,2}a_{2,1} \neq 0$ over $GL_2(\mathbb{C})$,

the two functions Comp, Inv are continuous,

so $GL_2(\mathbb{C})$ is a topological group.

(b) Solution: It suffices to prove that each elementary matrix induces a path from the identity matrix to the elementary matrix itself.

Scaling a row: $\begin{pmatrix} 1 & 0 \\ 0 & e^{a+ib} \end{pmatrix}$ induces a path $\begin{pmatrix} 1 & 0 \\ 0 & e^{(a+ib)t} \end{pmatrix}$ from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & e^{a+ib} \end{pmatrix}$.

Here, every nonzero complex number has at least one complex logarithm $a+ib$, so it is appropriate to assume $k=e^{a+ib}$.

And, as $\text{Det}\begin{pmatrix} 1 & 0 \\ 0 & e^{(a+ib)t} \end{pmatrix} = e^{(a+ib)t} \neq 0$ such path is well-defined.

Swapping two rows:

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ induces a path $\begin{pmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2} e^{\pi i t} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} e^{\pi i t} \end{pmatrix}$ from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Here, $\mathrm{Det}\begin{pmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2}e^{\pi i t} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2}e^{\pi i t} \end{pmatrix} = e^{\pi i t} \neq 0$, so such path is well-defined.

Adding a scalar multiple of row 2 to row 1:

$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ induces a path $\begin{pmatrix} 1 & kt \\ 0 & 1 \end{pmatrix}$ from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

Here, $\mathrm{Det}\begin{pmatrix} 1 & kt \\ 0 & 1 \end{pmatrix} = 1 \neq 0$, so such path is well-defined.

As every matrix is a finite product of the above three types, we may composite and concatenate the above mentioned paths to connect I to any matrix $A \in GL_2(\mathbb{C})$, this implies $GL_2(\mathbb{C})$ is path connected, thus connected.

(c) Proof: Take a square root k of $\frac{c'}{c}$, and define the following map: $\sigma: \mathrm{Det}^{-1}(\{c\}) \to \mathrm{Det}^{-1}(\{c'\})$, $\sigma(A) = kA$.

For all $A_1, A_2 \in \mathrm{Det}^{-1}(\{c\})$,

$\sigma(A_1) = \sigma(A_2) \Rightarrow kA_1 = kA_2 \Rightarrow A_1 = A_2$.

For all $A' \in \mathrm{Det}^{-1}(\{c'\})$, for some $\frac{1}{k}A'$:

$\sigma(\frac{1}{k}A') = k\frac{1}{k}A' = A'$.

Hence, σ is bijective.

$\|\sigma(A_1) - \sigma(A_2)\|_{spec} = \|kA_1 - kA_2\|_{spec} = |k|\,\|A_1 - A_2\|_{spec}$

$\|\sigma^{-1}(A_1') - \sigma^{-1}(A_2')\|_{spec} = \|\frac{1}{k}A_1' - \frac{1}{k}A_2'\|_{spec} = \frac{1}{|k|}\|A_1' - A_2'\|_{spec}$

Hence, σ, σ^{-1} are Lipschitz continuous, thus continuous.

This implies σ is a homeomorphism.

(d) Proof: Note that $(a,b)=(a,0)$, $Re(a)=0$

and $(a,b)=(0,b)$, $Re(b)=0$

together implies $(a,b)=(0,0)$,

contradicting to $(a,b)\neq(0,0)$.

Hence, $U=\mathbb{C}^2\setminus\{(a,0)\in\mathbb{C}^2 : Re(a)=0\}$

$V=\mathbb{C}^2\setminus\{(0,b)\in\mathbb{C}^2 : Re(b)=0\}$ forms an open cover of $\mathbb{C}^2\setminus\{(0,0)\}$

(1) $Y=\mathbb{C}^2\setminus\{(0,0)\}=U\cup V$, and we are going to compute $\pi_1(Y)$.

(2) $U=\mathbb{C}^2\setminus\{(a,0)\in\mathbb{C}^2 : Re(a)=0\}$

$\cong \mathbb{R}^4\setminus$ a straight line $\mathbb{R}$

$\sim \mathbb{R}^3\setminus$ a point $\{0\}$ by projection along $\mathbb{R}$.

$\sim S^2$ by dilation $\vec{x}\mapsto\vec{x}/\|\vec{x}\|$, which has $\pi_1(S^2)\cong\{e\}$.

This U is path connected. As $\mathbb{R}$ is closed in $\mathbb{R}^4$, $U\cong\mathbb{R}^4\setminus\mathbb{R}$ is open in $\mathbb{C}^2$, thus open in $\mathbb{C}^2\setminus\{(0,0)\}$.

(3) $V=\mathbb{C}^2\setminus\{(0,b)\in\mathbb{C}^2 : Re(b)=0\}$

is open in $\mathbb{C}^2\setminus\{0\}$ and path connected for similar reason.

(4) $S=U\cap V=\mathbb{C}^2\setminus(\mathbb{R}\vee\mathbb{R})$

$\cong \mathbb{R}^4\setminus(\mathbb{R}\vee\mathbb{R})$

$\sim \mathbb{R}^3\setminus\mathbb{R}\vee\{pt\}$

$\sim \mathbb{R}^2\setminus\{pt\}\vee\{pt\}\sim S$, $\pi_1(S)=\mathbb{Z}$

It follows that $\pi_1(Det^{-1}(\{1\}))\cong\pi_1(Y)\cong\{e\}$.

S is nonempty and path connected.

Hence $\pi_1(Y)\cong\{e\}*_{\mathbb{Z}}\{e\}\cong\{e\}$.

As $Det^{-1}(\{1\})\cap U\sim\left\{\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix} : Re(a)=0\right\}$ and $Det^{-1}(\{1\})\cap V\sim\left\{\begin{pmatrix}0&b\\-b^{-1}&0\end{pmatrix} : Re(b)=0\right\}$