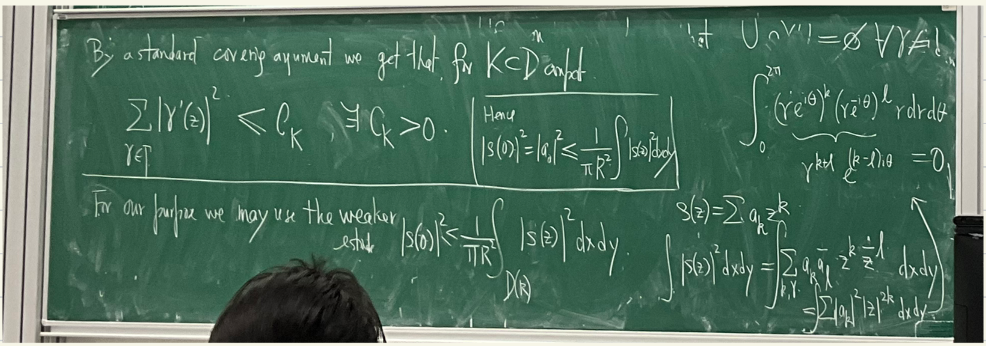
Rem For our purpose, we may use the weaker estimate:

$$|S(0)|^2 \leq \frac{1}{\pi R^2}\int_{D(R)} |S(z)|^2 \, dx\,dy.$$

$$S(z) = \sum_{k \geq 0} a_k z^k, \quad |a_0|^2 \leq \frac{1}{\pi R}\int_{D(R)} |S(z)|^2 \, dx\,dy.$$



$$\int \sum_{k,l} a_k \bar{a}_l z^k \bar{z}^l \, dx\,dy = \int \sum |a_k|^2 |z|^{2k} \, dx\,dy.$$

because $\displaystyle\int_0^R\int_0^{2\pi} (re^{i\theta})^k (re^{-i\theta})^l \, r\,dr\,d\theta$ $\quad (k \neq l)$

$$= \int_0^{2\pi} r^{k+l+1} e^{i(k-l)\theta} \, dr\,d\theta$$

$$= \left(\int_0^{2\pi} e^{i(k-l)\theta} \, d\theta\right)\left(\int_0^R r^{k+l+1} \, dr\right) = 0.$$

Prop: Let $z_0 \in D$ be arbitrary. There exists $h \in H^\infty(D)$ and $k \geq 2$ such that $f(z_0) = P_\Gamma^k(h)(z_0) \neq 0$.

Fix $z_0 \in D$, write

$$\Gamma = \Gamma' \uplus \Gamma'', \text{ where}$$

$$\begin{cases} \Gamma' = \{ \gamma \in \Gamma : |\gamma'(z_0)| \geq 1/2 \}. & , e \in \Gamma' \\ \Gamma'' = \{ \gamma \in \Gamma : |\gamma'(z_0)| < 1/2 \}. \end{cases}$$

Clearly Γ' is finite and Γ'' is infinite, as.

$$\sum_\Gamma |\gamma'(z_0)|^2 = \sum_{\Gamma'} + \sum_{\Gamma''} < \infty.$$

Say $|\Gamma'| = m$. Pick a polynomial Q (by interpolation) of degree at most $m-1$. s.t.

$$Q(z_0) = 1 \quad \text{and} \quad Q(\gamma z_0) = 0 \quad \text{for all } \gamma \in \Gamma' \setminus id.$$

Pick k large, and

$$f(z) = P_\Gamma^k(Q)(z) = \sum_{\gamma \in \Gamma} Q(\gamma z) (\gamma'(z))^k.$$

$$= \sum_{\gamma \in \Gamma'} Q(\gamma(z)) (\gamma'(z))^k + \sum_{\gamma \in \Gamma''} Q(\gamma z) (\gamma'(z))^k.$$

I want to pick k so that. $f(z_0) \neq 0$.

Note $\sum_{\Gamma'} Q(\gamma z_0) (\gamma'(z_0))^k = 1$.

WTS. $\exists\, k >> 0$, $\left| \sum_{\Gamma''} Q(\gamma z_0) (\gamma'(z_0))^k \right| \leq 1/2$. and

this suffices to complete the proof.

$\exists\, C > 0$, s.t. $|Q(z)| < C \quad \forall\, z \in \overline{D}$

$$\Rightarrow \sum_{\gamma \in \Gamma''} Q(\gamma z_0)(\gamma'(z_0))^k \leq C \sum_{\gamma \in \Gamma''} |\gamma'(z_0)|^k.$$

$$= C \sum_{\gamma \in \Gamma''} |\gamma'(z_0)|^2 |\gamma'(z_0)|^{k-2}$$

$$< \left(\tfrac{1}{2}\right)^{k-2} \underbrace{\left(\sum_{\gamma \in \Gamma''} |\gamma'(z_0)|^2\right)}_{\text{uniformly bounded}} C. \quad \circledast$$

$\Rightarrow \exists\, k \gg 0$, s.t. $\circledast \leq \frac{1}{2}$.

//.