Lecture notes: Introduction to Topology, HKU, Spring 2022

Based on previous lecture notes by Dr. Tak Wing Ching and Prof. Jiang-Hua Lu

MATH3541, Introduction to Topology, HKU, Fall 2022

- References: M. A. Armstrong, “Basic Topology”, Springer, 1983 (e-book available at HKU library).
- Topics: We aim at covering Chapters 1-7 of Armstrong’s book.

About the lecture notes

These notes are based on the lecture notes by Dr. Tak Wing Ching and Prof. Jiang-Hua Lu from a previous year but will be updated continuously. The notes are to facilitate the lectures, and they are not sufficient to master the course contents without further reading and practices. Students are supposed to read the reference book or any other books on basic topology **and** do enough exercises.

Contents

Chapter 1 — Review of Basic Concepts

1.1 Set Theory

The following is a list the definitions and notations that we shall use.

- $\mathbb{N}$: the set of all natural numbers (positive integers);
- $\mathbb{Z}$: the set of all integers;
- $\mathbb{Q}$: the set of all rational numbers;
- $\mathbb{R}$: the set of all real numbers;
- $\mathbb{C}$: the set of all complex numbers;
- (a, b): the open interval $\{x \in \mathbb{R} : a < x < b\}$;
- $[a, b]$: the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$;
- $[a, b)$: the half-open interval $\{x \in \mathbb{R} : a \leqslant x < b\}$;
- (a, ∞): the interval $\{x \in \mathbb{R} : x > a\}$;
- $(-\infty, b]$: the interval $\{x \in \mathbb{R} : x \leqslant b\}$;
- $(a, b], (-\infty, b), [a, \infty)$: similarly defined;
- $\{a : P(a)\}$: the set containing all elements a such that $P(a)$ is satisfied;
- $a \in S$: the element a belongs to the set S;
- $a \notin S$: the element a does not belong to the set S;
- $\varnothing$: the empty set;
- $|S|$: the cardinality of the set S;
- finite set: a set containing finitely many elements;

- infinite set: a set containing infinitely many elements;
- denumerable set: a set having the same cardinality as $\mathbb{N}$, i.e. there exists a bijective function from $\mathbb{N}$ to the set;
- countable set: a finite set or a denumerable set;
- uncountable set: a set which is not countable;
- family of sets: a set whose every element is a set;
- index set: a set whose elements are used to label another set;
- $X \subset Y$: X is a subset of Y, i.e. every element of X belongs to Y;
- $X \subsetneq Y$: X is a proper subset of Y, i.e. $X \subset Y$ and $X \neq Y$;
- $\mathcal{P}(S)$: the power set of S, i.e. the family of all subsets of S;
- $X \cup Y$: the union of X and Y, i.e. the set $\{a : a \in X \text{ or } a \in Y\}$;
- $\bigcup\limits_{\lambda \in \Lambda} S_\lambda$: the set $\{a : a \in S_\lambda \text{ for some } \lambda \in \Lambda\}$;
- $X \cap Y$: the intersection of X and Y, i.e. the set $\{a : a \in X \text{ and } a \in Y\}$;
- $\bigcap\limits_{\lambda \in \Lambda} S_\lambda$: the set $\{a : a \in S_\lambda \text{ for all } \lambda \in \Lambda\}$;
- disjoint sets: two sets whose intersection is the empty set;
- $X - Y$: the complement of Y in X, i.e. the set $\{a : a \in X \text{ and } a \notin Y\}$; We will denote $X - Y$ as $X \backslash Y$.
- $S_1 \times S_2 \times \cdots \times S_n$: the Cartesian product of $S_1, S_2, \ldots, S_n$, i.e. the set

$$\{(a_1, a_2, \ldots, a_n) : a_k \in S_k\}$$

- ordered pair: an element of the form (a_1, a_2);
- ordered n-tuple: an element of the form $(a_1, a_2, \ldots, a_n)$;
- $\forall$: for all;
- $\exists$: there exists.

The following is a list some basic facts in set theory.

Proposition 1.1.1. Let X and Y be two sets. Then $X = Y$ if and only if $X \subset Y$ and $Y \subset X$.

Proposition 1.1.2. Let X, Y, Z and Y_λ $(\lambda \in \Lambda)$ be sets where Λ is an index set. Then

1. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$;
2. $X \cap \left(\bigcup_{\lambda \in \Lambda} Y_\lambda \right) = \bigcup_{\lambda \in \Lambda} (X \cap Y_\lambda)$;
3. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$;
4. $X \cup \left(\bigcap_{\lambda \in \Lambda} Y_\lambda \right) = \bigcap_{\lambda \in \Lambda} (X \cup Y_\lambda)$;
5. if $X \subset Y$, then $(X \cap Z) \subset (Y \cap Z)$.

Proposition 1.1.3. (De Morgan's laws) Let X, Y and X_λ $(\lambda \in \Lambda)$ be subsets of Z where Λ is an index set. Then

1. $Z - (X \cup Y) = (Z - X) \cap (Z - Y)$;
2. $Z - \left(\bigcup_{\lambda \in \Lambda} X_\lambda \right) = \bigcap_{\lambda \in \Lambda} (Z - X_\lambda)$;
3. $Z - (X \cap Y) = (Z - X) \cup (Z - Y)$;
4. $Z - \left(\bigcap_{\lambda \in \Lambda} X_\lambda \right) = \bigcup_{\lambda \in \Lambda} (Z - X_\lambda)$.

Proposition 1.1.4. Let $X_1, X_2, Y_1, Y_2, X_\lambda$ and Y_λ $(\lambda \in \Lambda)$ be sets where Λ is an index set. Then

1. $(X_1 \times Y_1) \cap (X_2 \times Y_2) = (X_1 \cap X_2) \times (Y_1 \cap Y_2)$;
2. $\bigcap_{\lambda \in \Lambda} (X_\lambda \times Y_\lambda) = \left(\bigcap_{\lambda \in \Lambda} X_\lambda \right) \times \left(\bigcap_{\lambda \in \Lambda} Y_\lambda \right)$.

1.2 Relations and Orders

A relation R from X to Y is a subset of $X \times Y$. A relation on X is a relation from X to X. If $(x, y) \in R$, then we write xRy.

An equivalence relation on S is a relation $\sim$ satisfying the following conditions.

1. (reflexive) for any $a \in S$, we have $a \sim a$
2. (symmetric) for any $a, b \in S$, if $a \sim b$, then $b \sim a$
3. (transitive) for any $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$

Let $\sim$ be an equivalence relation on S. For each $a \in S$, the equivalence class of a with respect to $\sim$ is the set

$$[a] = \{b \in S : b \sim a\}.$$

For any $a, b \in S$, $[a] = [b]$ if and only if $a \sim b$. So the set of all equivalence classes forms a partition of S, i.e. a collection of pairwise disjoint sets whose union is S.

A total order on S is a relation $\leqslant$ satisfying the following conditions.

1. (comparability) for any $a, b \in S$, we have $a \leqslant b$ or $b \leqslant a$
2. (antisymmetry) for any $a, b \in S$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$
3. (transitivity) for any $a, b, c \in S$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$

In that case, we say that S is a totally ordered set. We write $a < b$ if $a \leqslant b$ and $a \neq b$. Notations like $a \geqslant b$ and $a > b$ can be defined in the obvious way. Also, if $a < b$, we can define the open interval

$$(a, b) = \{x \in S : a < x < b\}$$

and other intervals $[a, b], [a, b), (a, b]$ analogous to the case of real numbers.

Let X and Y be two sets with total orders $\leqslant_X$ and $\leqslant_Y$ respectively. The lexicographical order (or dictionary order) on $X \times Y$ is the relation $\leqslant$ such that $(a, b) \leqslant (c, d)$ if one of the following holds:

1. $a <_X c$; or
2. $a = c$ and $b \leqslant_Y d$.

This gives a total order on $X \times Y$.

1.3 Maps

A map from X to Y, denoted by $f : X \to Y$, is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$. We write $y = f(x)$ in that case. We call y the image of x, and call x a preimage of y. We also say that x is mapped to y, and denote this by $x \mapsto y$. The set X is called the domain of f, and Y is called the co-domain of f. The range of f is the set

$$f(X) = \{y \in Y : f(x) = y \text{ for some } x \in X\}.$$

For any subset A of X, the image of A is the set

$$f(A) = \{f(a) : a \in A\}.$$

For any subset B of Y, the preimage of B is the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Proposition 1.3.1. Let $f : X \to Y$ be a map where X and Y are nonempty sets. Let A, A_1, A_2 be subsets of X and let B, B_1, B_2, B_λ $(\lambda \in \Lambda)$ be subsets of Y where λ is an index set. Then

1. $A \subset f^{-1}(f(A))$;
2. $f(f^{-1}(B)) \subset B$;
3. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$;
4. $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$;
5. $f(A_1) - f(A_2) \subset f(A_1 - A_2)$;
6. $f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B)$;
7. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$;
8. $f^{-1}\Big(\bigcup_{\lambda \in \Lambda} B_\lambda\Big) = \bigcup_{\lambda \in \Lambda} f^{-1}(B_\lambda)$;
9. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;
10. $f^{-1}\Big(\bigcap_{\lambda \in \Lambda} B_\lambda\Big) = \bigcap_{\lambda \in \Lambda} f^{-1}(B_\lambda)$.

The composition of the maps $f : X \to Y$ and $g : Y \to Z$, denoted by $g \circ f$, is the map from X to Z such that

$$(g \circ f)(x) = g(f(x))$$

for any $x \in X$.

A map $f : X \to Y$ is said to be injective if for any distinct elements $a, b \in X$, we have $f(a) \neq f(b)$. Equivalently, f is injective if and only if for any $a, b \in X$, $f(a) = f(b)$ implies $a = b$.

A map $f : X \to Y$ is said to be surjective if for any $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Equivalently, f is surjective if and only if $f(X) = Y$.

A map $f : X \to Y$ is bijective if it is both injective and surjective. This holds if and only if the inverse map of f exists, i.e. there exists a map $f^{-1} : Y \to X$ such that

$$(f^{-1} \circ f)(x) = x \quad \text{and} \quad (f \circ f^{-1})(y) = y$$

for any $x \in X$ and $y \in Y$. The inverse map f^{-1} is uniquely determined and $(f^{-1})^{-1} = f$.

Let $f : X \to Y$ and $g : Y \to Z$ be bijective maps. Then $g \circ f$ has an inverse map and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

$\mathcal{O} \sim$ open

$\mathcal{C} \sim$ closed

Chapter 2 — Topological Spaces

定义什么是开集 $(X, \mathcal{O})$

2.1 Topological spaces and examples

Definition 2.1.1. A *topology* on a set X is a collection $\mathcal{O}$ of subsets of X such that the following conditions are satisfied:

1. The empty set and X belong to $\mathcal{O}$.
2. Any arbitrary union of subsets in $\mathcal{O}$ belongs to $\mathcal{O}$. **任意并和有限交**
3. Any finite intersection of subsets in $\mathcal{O}$ belongs to $\mathcal{O}$.

可以说是 closed under finite intersection and arbitrary union

A *topological space* is a pair $(X, \mathcal{O})$, where X is a set and $\mathcal{O}$ is a topology on X.

Given a topological space $(X, \mathcal{O})$, any subset of X belonging to $\mathcal{O}$ is called an *open subset* of X, and the complement of an open subset is called a *closed subset* of X. An element $x \in X$ is called a *point*.

一个集合上可以定义不同**的拓扑**

Alternatively, one can specify a topology on X by specifying the collection $\mathcal{C}$ of closed subsets. The proof of the following statement is left as an exercise.

Proposition 2.1.1. Let X be a set and let $\mathcal{C}$ be a collection of subsets in X. Let $\mathcal{O} = \{X - A : A \in \mathcal{C}\}$. Then $\mathcal{O}$ is a topology on X if and only if $\mathcal{C}$ satisfies the following:
1. $\varnothing \in \mathcal{C}$ and $X \in \mathcal{C}$;
2. An arbitrary intersection of subsets in $\mathcal{C}$ belongs to $\mathcal{C}$.
3. Any finite union of subsets in $\mathcal{C}$ belongs to $\mathcal{C}$.

We remark that the same set X can be given more than one topology. In other words, the same set X can be made into a topological space in more than one way, as we will see in the next few examples.

Example 2.1.1. Let X be a nonempty set and let $\mathcal{O} = \mathcal{P}(X)$. Then $\mathcal{O}$ is a topology on X, called the *discrete topology*.

拓扑也可以被等价的定义为闭集的集合，这就好像良序原理既可以是最小元存在，**也可以是最大元存在**

对拓地，$\mathcal{O}=\mathcal{P}(X)$是最大的拓扑

平凡拓扑，trivial topology**，**这是一个集合能获得的最小的拓扑

Example 2.1.2. Let X be a nonempty set and let $\mathcal{O} = \{\varnothing, X\}$. Then $\mathcal{O}$ is a topology on X, called the *indiscrete topology* (or *trivial topology*).

co-finite

去看

Example 2.1.3. Let X be a nonempty set. Let $\mathcal{O}$ be the collection of the empty set and all subsets U of X such that $X - U$ is a finite set. Then $\mathcal{O}$ is a topology on X, called the *finite complement topology* (or *co-finite topology*).

这里直接用闭集来定义会比较简单，**就把**X的所有有限子集构成的集合作为闭集的集合，**很不幸，**这不是豪斯多夫的

Example 2.1.4. Let $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$ and consider $X = \mathbf{k}^n$. Let $A = \mathbf{k}[x_1, \ldots, x_n]$ be the space of all polynomial functions on X. For any subset $T \subset A$, let

域K上的n元多项式环

$$Z(T) = \{x \in \mathbf{k}^n : f(x) = 0 \ \forall \ f \in T\}.$$

(a) $Z(T) = Z(\mathrm{gen}(T))$

One checks that $Z(T_1) \cup Z(T_2) = Z(T_1T_2)$, where $T_1T_2 = \{f_1f_2 : f_1 \in T_1, f_2 \in T_2\}$, and

$Z(T_1) \cup Z(T_2) = Z(T_1T_2)$

(b) I_α ideal, $\bigcap Z(I_\alpha) = Z(\sum I_\alpha)$

$$\bigcap_{\alpha \in \mathcal{I}} Z(T_\alpha) = Z(\bigcup_{\alpha \in \mathcal{I}} T_\alpha)$$

zariski topology**中，**
闭集是多项式方程的公共解集

(c) $\bigcup_{\alpha=1}^{N} Z(I_\alpha) = Z(\prod_{\alpha=1}^{N} I_\alpha)$ 有限并

for any index set $\mathcal{I}$. Note also that $Z(\{1\}) = \varnothing$ and $Z(\{0\}) = \mathbf{k}^n$. Thus one has a topology on $\mathbf{k}^n$ in which a subset is closed if and only if it is of the form $Z(T)$ for a subset T of polynomial functions. This topology is called the *Zariski topology* of $\mathbf{k}^n$, which is very important in algebraic geometry. When $n = 1$, using the fact that the algebra $\mathbf{k}[x]$ is a Principal Ideal Domain, one sees that the Zariski topology on $\mathbf{k}^1$ is nothing but the co-finite topology.

$k[x_1, \ldots, x_n]$: a commutative ring with unit

Example 2.1.5. Consider the set $\mathbb{R}$. Define a subset A of $\mathbb{R}$ to be open if either $A = \varnothing$ or $A \neq \varnothing$ and has the property that for any $a \in A$ there is an open interval containing a that lies in A. The collection $\mathcal{O}$ of all such open subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$, which is sometimes called the *classical topology* or the *Euclidean topology* of $\mathbb{R}$. This example is a special case of the topology induced by metrics.

这里的思想可以用来证明闭集的有限并仍然是闭集

Definition 2.1.2. A *metric space* is a nonempty set X together with a distance function $d : X \times X \to \mathbb{R}$, called a *metric* on X, such that the following conditions are satisfied.

$\subseteq$ (direct)

1. $d(x, y) \geqslant 0$ for any $x, y \in X$, and equality holds if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for any $x, y \in X$;
3. (triangle inequality) $d(x, y) \leqslant d(x, z) + d(z, y)$ for any $x, y, z \in X$;

$+$ $\supseteq$ (contra positive)

Sometimes we may use (X, d) to denote the metric space to emphasize the metric.

这里讲了理想ideal，详细请参考20240902_MATH3301_Note_1

$X \in \mathcal{O} \Leftrightarrow \forall\, x \in X, \exists\, r>0, B(x,r) \subseteq X$

这里复习一个点，**就是在后面的“存在”**会依赖于在前面的内容，存在任意不能交换

co-finite topology and metric topology on \mathbb{R}^{n} are different in sense that co-finite topology is larger!

Definition 2.1.3. Let (X, d) be a metric space. For any $a \in X$ and $r > 0$, we define the *open ball* with centre a and radius r to be the set

$$B(a,r) = \{x \in X : d(a,x) < r\}.$$

A subset A of X is said to be open if either $A = \varnothing$ or $A \neq \varnothing$ and has the property that for any $a \in A$ there is an open ball containing a that lies in A. The collection $\mathcal{O}$ of all such open subsets is a topology on X called the *metric topology* defined by d.

As an example of metric space, let $X = C[0,1]$, the space of all continuous functions on the closed interval $[0,1]$, and for $f, g \in X$, define

$$d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}.$$

Then (X, d) is a metric space. The induced metric topology for such examples are very important in functional analysis.

Definition 2.1.4. Let X be a topological space with two topologies $\mathcal{O}$ and $\mathcal{O}'$. We say that $\mathcal{O}'$ is *larger than* $\mathcal{O}$, or $\mathcal{O}'$ is *larger than* $\mathcal{O}$, if

$$\mathcal{O} \subset \mathcal{O}'.$$

coarser 粗糙!

In this case, we also say that $\mathcal{O}$ is *coarser than* $\mathcal{O}'$, or $\mathcal{O}$ is *smaller than* $\mathcal{O}'$

coarser, finer

It is a good exercise to compare the various topologies on $\mathbb{R}$ given as above.

2.2 Bases and sub-bases of topologies

Sometimes a topology is determined by a smaller sub-collection of open subsets.

Definition 2.2.1. Let $(X, \mathcal{O})$ be a topological space. A sub-collection $\mathcal{B}$ of $\mathcal{O}$ is called a *basis* of the topology $\mathcal{O}$ if every member of $\mathcal{O}$ is a union of some members of $\mathcal{B}$.

Example 2.2.1. Let (X, d) be a metric space and let $\mathcal{B}$ be the collection of all open balls $B(a,r)$ where $a \in X$ and $r > 0$. Then $\mathcal{B}$ is a basis of the metric topology on X defined by d.

Note that two necessary conditions for a sub-collection $\mathcal{B}$ of $\mathcal{O}$ to be a basis of $\mathcal{O}$ are:
1) the union of all members of $\mathcal{B}$ is X;
2) for any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is a union of members in $\mathcal{B}$.

这里的basis**似乎只有**span**的属性，而没有**linearly independent**的属性**

Find an subasis for Zariski Topology (is it all prime ideals?)

Lemma 2.2.1. Let X be a set and let $\mathcal{B}$ be a collection of subsets of X. If $\mathcal{B}$ satisfies

1) the union of all members of $\mathcal{B}$ is X;

2) for any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is a union of members in $\mathcal{B}$,

then the collection $\mathcal{O}$ consisting of all arbitrary unions of members of $\mathcal{B}$ is a topology on X, and $\mathcal{B}$ is a basis of $\mathcal{O}$.

Proof. Straightforward using the definitions. □

Definition 2.2.2. Let $(X, \mathcal{O})$ be a topological space. A sub-collection of members of $\mathcal{O}$ is called a *sub-basis* of $\mathcal{O}$ if every element of $\mathcal{O}$ is a union of finite intersections of members of $\mathcal{B}_0$.

Example 2.2.2. The collection of all intervals of the form $(-\infty, b)$ or (a, ∞), where $a, b \in \mathbb{R}$, is a sub-basis of the classical topology on $\mathbb{R}$.

Note that one necessary condition for a sub-collection $\mathcal{B}_0$ of $\mathcal{O}$ to be a sub-basis of $\mathcal{O}$ is that the union of all members of $\mathcal{B}_0$ is X.

Lemma-Definition 2.2.1. Let X be a set and let $\mathcal{B}_0$ be a collection of subsets of X such that the union of all members of $\mathcal{B}_0$ is X. Then the collection $\mathcal{O}$ consisting of all arbitrary unions of finite intersections of elements in $\mathcal{B}_0$ is a topology on X, called the *topology on X generated by* $\mathcal{B}_0$, and $\mathcal{B}_0$ is a sub-basis of $\mathcal{O}$.

Note that if $\mathcal{B}_0$ is a sub-basis of a topology $\mathcal{O}$ on X, then $\mathcal{B}$, the collection of all finite intersections of members of $\mathcal{B}_0$, is a basis of $\mathcal{O}$.

2.3 Constructing new topology from old

In this section, we will learn how to construct new topological spaces from old ones in three ways, namely we will introduce subspace topology, product topology, and quotient topology. The constructions are all based on the following fundamental concept in topology, namely *continuity of maps from one topological space to another.* Although we will focus more on continuity later, we give the definition first.

Definition 2.3.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A map $F : X \to Y$ is said to be *continuous* if $F^{-1}(U) \subset \mathcal{O}_X$ for every $U \in \mathcal{O}_Y$.

The constructions of subspace topology, product topology, and quotient topology are all guided by principle that we want certain natural maps associated to the spaces to be continuous.

From now on, when we say "let X be a topological space", we understand that a topology $\mathcal{O}_X$ is specified.

2.3.1 Subspace topology

Lemma-Definition 2.3.1. Given a topological space $(X, \mathcal{O})$ and a subset Y of X, define $\mathcal{O}_Y = \{A \cap Y : A \in \mathcal{O}\}$. Then $\mathcal{O}_Y$ is a topology on Y, called the subspace topology from $\mathcal{O}$, and $(Y, \mathcal{O}_Y)$ is called a subspace of $(X, \mathcal{O})$.

Proof. We need to prove that $\mathcal{O}_Y$ is indeed a topology on Y. First, we have $\varnothing \in \mathcal{O}_Y$ and $Y \in \mathcal{O}_Y$. For any collection $\{O_\alpha\}_{\alpha \in I}$ of elements in $\mathcal{O}$, we have

$$\bigcup_{\alpha \in I} (Y \cap O_\alpha) = Y \cup \bigcup_{\alpha \in I} O_\alpha \quad \text{and} \quad \bigcap_{\alpha \in I} (Y \cap O_\alpha) = Y \cap \bigcap_{\alpha \in I} O_\alpha,$$

so $\mathcal{O}_Y$ is closed under arbitrary union and finite intersection. Thus $\mathcal{O}_Y$ is a topology on Y. □

Thus if X is a topological space, every subset Y of X automatically becomes a topological space. For example, consider $X = \mathbb{R}^n$ with the Euclidean topology. Then every subset of $\mathbb{R}^n$ becomes a topological space with the subspace topology.

Exercise 2.3.1. In the special case of $Y = \mathbb{Z}^n \subset \mathbb{R}^n$, where $\mathbb{R}^n$ has the Euclidean topology, prove that the subspace topology on $\mathbb{Z}^n$ is the discrete topology on $\mathbb{Z}^n$.

Lemma 2.3.1. Let $(X, \mathcal{O}_X)$ be a topological space and $Y \subset X$. Then the subspace topology $\mathcal{O}_Y$ is the smallest topology such that the inclusion map $I : Y \hookrightarrow X$ is continuous, i..e, $I : (Y, \mathcal{O}_Y) \hookrightarrow (X, \mathcal{O}_X)$ is continuous, and if $\mathcal{O}'_Y$ is another topology such that $I : (Y, \mathcal{O}'_Y) \hookrightarrow (X, \mathcal{O}_X)$ is continuous, then $\mathcal{O}_Y \subset \mathcal{O}'_Y$.

Proof. Note that for any $A \subset X$, $I^{-1}(A) = A \cap Y$. By the definition of $\mathcal{O}_Y$, if $A \in \mathcal{O}_X$, then $I^{-1}(A) = A \cap Y \in \mathcal{O}_Y$. Thus $I : (Y, \mathcal{O}_Y) \hookrightarrow (X, \mathcal{O}_X)$ is continuous. Suppose now that $\mathcal{O}'_Y$ is another topology such that $I : (Y, \mathcal{O}'_Y) \hookrightarrow (X, \mathcal{O}_X)$ is continuous. Then for any $A \in \mathcal{O}_X$, $A \cap Y = I^{-1}(A) \in \mathcal{O}'_Y$. This shows that $\mathcal{O}_Y \subset \mathcal{O}'_Y$. □

Example 2.3.1. Consider the topological space $\mathbb{R}$ and the subspace $Y = [-1, 1]$. Since $(0, 1] = Y \cap (0, 2)$, the set $(0, 1]$ is open in Y.

Example 2.3.2. The subspace $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ of the topological space $\mathbb{R}^{n+1}$ is called the *unit n-sphere*.

We already know that for a subspace Y in a topological space X, a subset A of Y is open in Y if and only if $A = Y \cap O$ for some open subset O of X (this is the definition of the subspace topology on Y). A similar statement holds for closed subsets in Y.

Lemma 2.3.2. Let X be a topological space and let Y be a subspace of X. A subset A of Y is closed in Y if and only if $A = Y \cap Z$ for some closed subset Z of X.

Proof. This follows from the fact that for any subset Z of X,

$$A = Y \cap Z \iff Y \backslash A = Y \cap Z^c,$$

where $Z^c = X \backslash Z$, the complement of Z in X. $\square$

Example 2.3.3. Consider $\mathbb{R}$ with the classical topology and $\mathbb{Q} \subset \mathbb{R}$ with the subspace topology. Then $(-\pi, \pi) \cap \mathbb{Q} = [-\pi, \pi] \cap \mathbb{Q}$ is both closed and open in $\mathbb{Q}$.

Example 2.3.4. For integers $m, n \geq 1$, equip $\mathbb{R}^{n+m}$ with the Euclidean topology. Identify

$$\mathbb{R}^n \cong \{(x_1, \ldots, x_n, 0, \ldots, 0) : (x_1, \ldots, x_n) \in \mathbb{R}^n\} \subset \mathbb{R}^{n+m}.$$

Denote by $B_k = \{x \in \mathbb{R}^k : d(a,x) < r\}$ for integer $k \geq 1$, $a \in \mathbb{R}^k$ and $r > 0$. Using that fact that $B_{n+m}(a,r) \cap \mathbb{R}^n = B_n(a,r)$ for $a \in \mathbb{R}^n$ and $r > 0$, one sees that the subspace topology on $\mathbb{R}^n$ is the Euclidean topology

2.3.2 Product topology

Let X and Y be topological spaces. Let $\mathcal{B}$ be the collection of all subsets of $X \times Y$ of the form $U \times V$ where U is open in X and V is open in Y Note that $X \times Y$ is in $\mathcal{B}$.

Definition 2.3.2. The topology $\mathcal{O}$ of $X \times Y$ generated by $\mathcal{B}$ is called the *product topology*, and we also call $(X \times Y, \mathcal{O})$ the *product* of the topological spaces X and Y.

Note that the collection $\mathcal{B}$ satisfies the two conditions in Lemma 2.2.1, so $\mathcal{B}$ is a basis for the product topology on $X \times Y$. In general, we can define a product space $X_1 \times X_2 \times \cdots \times X_n$ analogously.

Example 2.3.5. The classical topology on $\mathbb{R}^n$ is the product topology. This is left as an exercise.

Note that in the product topology on $X \times Y$, an open subset of $X \times Y$ *is not necessarily* of the form $U \times V$ for an open subset U of X and an open subset V of Y. For example, is every open subset of $\mathbb{R}^2$ for the classical topology of the form $U \times V$, where $U \subset \mathbb{R}$ and $V \in \mathbb{R}$ are open in the classical topology of $\mathbb{R}^1$?

Example 2.3.6. The product space $S^1 \times [0,1]$ is called a *cylinder*. The n-fold product space $T^n = S^1 \times S^1 \times \cdots \times S^1$ is called the n-torus.

Lemma 2.3.3. The product topology of $X \times Y$ is the smallest (or finest) topology on $X \times Y$ such that both projections

$$p_X : X \times Y \to X, \ (x,y) \mapsto x, \ \text{ and } \ p_Y : X \times Y \to Y, \ (x,y) \mapsto y,$$

are continuous.

Proof. For any $U \in \mathcal{O}_X$ and $V \in \mathcal{O}_Y$,

$$U \times V = (U \times Y) \cap (X \times V) = p_X^{-1}(U) \cap p_Y^{-1}(V)$$

must be open in any topology on $X \times Y$ such that both p_X and p_Y are continuous. $\square$

Example 2.3.7. Under the identification $\mathbb{R}^{m+n} \cong \mathbb{R}^n \times \mathbb{R}^m$, the Euclidean topology on $\mathbb{R}^{m+n}$ is the product topology, but the Zariski topology is not.

2.3.3 Quotient topology

Recall that if X is a non-empty set and $\sim$ an equivalence relation on X, the set $X/\sim$ is the set of all equivalence classes of $\sim$ in X. set

$$[X] = X/\sim,$$

and for $x \in X$, let $[x] \in [X]$ denote the equivalence class of x, so we have the map

$$p: \ X \longrightarrow [X], \ \ p(x) = [x],$$

called the *projection map* or the *quotient map*.

Definition 2.3.3. Let X be a topological space and let $\sim$ be an equivalence relation on X. The *quotient topology* on $[X]$ is the collection $\mathcal{O}_{[X]}$ of all subsets A of $[X]$ such that $p^{-1}(A)$ is an open subset of X.

We need to check that $\mathcal{O}_{[X]}$ is indeed a topology on $[X]$: as $p^{-1}([X]) = X$ and $p^{-1}(\varnothing) = \varnothing$, we see that $[X] \in \mathcal{O}_{[X]}$ and $\varnothing \in \mathcal{O}_{[X]}$. Recall that $\{A_\alpha\}_{\alpha \in I}$ is any collection of subsets in $[X]$, one has

$$p^{-1}(\bigcup_{\alpha\in I} A_\alpha) = \bigcup_{\alpha\in I} p^{-1}(A_\alpha) \quad \text{and} \quad p^{-1}(\bigcap_{\alpha\in I} A_\alpha) = \bigcap_{\alpha\in I} p^{-1}(A_\alpha).$$

It follows that $\mathcal{O}_{[X]}$ is a topology on $[X]$.

Lemma 2.3.4. Let X be a topological space and let $\sim$ be an equivalence relation on X. Then the quotient topology $\mathcal{O}_{[X]}$ on $[X]$ is the largest topology such that the quotient map $p: X \to [X]$ is continuous.

Proof. Let $\mathcal{O}$ be any topology on $[X]$ such that $p: (X, \mathcal{O}_X) \to ([X], \mathcal{O})$ is continuous. Then for any $A \in \mathcal{O}$, $p^{-1}(X) \in \mathcal{O}_X$, so $A \in \mathcal{O}_{[X]}$. Thus $\mathcal{O} \subset \mathcal{O}_{[X]}$. $\square$

ini: coarsest

fin: finest

2.3.4 Initial topology and final topology

We will now explain that subspace topology and product topology are special cases of the so-called *initial topology*, and quotient topology is an example of the so-called *final topology*.

Suppose that X is any set and that $\{(Y_\alpha, \mathcal{O}_\alpha)\}_{\alpha\in\mathcal{A}}$ is any family of topological spaces, together with a map

$$f_\alpha : \ X \longrightarrow Y_\alpha$$

for every $\alpha \in \mathcal{A}$. Consider the collection $\mathcal{B}_0$ of subsets of X given by

$$\mathcal{B}_0 = \{f_\alpha^{-1}(U_\alpha) : \ \alpha \in \mathcal{A}, \ U_\alpha \in \mathcal{O}_\alpha\}.$$

Note that X is in $\mathcal{B}_0$. Let $\mathcal{O}_X$ be the topology on X generated by $\mathcal{B}_0$, i.e., $\mathcal{O}_X$ consists of arbitrary unions of finite intersections of elements in $\mathcal{B}_0$.

Lemma-Definition 2.3.2. The topology $\mathcal{O}_X$ on X generated by $\mathcal{B}_0$ is the *smallest* topology on X such that

$$f_\alpha : \ (X, \mathcal{O}_X) \longrightarrow (Y_\alpha, \mathcal{O}_\alpha)$$

is continuous for every $\alpha \in \mathcal{A}$. We call $\mathcal{O}_X$ the *initial topology on X* defined by the family $\{(f_\alpha : X \to Y_\alpha)\}_{\alpha\in\mathcal{A}}$.

Proof. Exercise. □

Exercise 2.3.2. Given a family $\{(f_\alpha : X \to Y_\alpha)\}_{\alpha\in\mathcal{A}}$ of continuous maps, and equip X with the initial topology. Then for any topological space Z, a map $h : Z \to X$ is continuous iff $f_\alpha \circ h : Z \to Y_\alpha$ is continuous for every $\alpha \in \mathcal{A}$.

Example 2.3.8. Both the subspace topology and the product topology are examples of initial topology.

Similarly, given a set X and a family of topological spaces $\{(Y_\alpha, \mathcal{O}_\alpha)\}_{\alpha\in\mathcal{A}}$ and a map

$$g_\alpha : \ Y_\alpha \longrightarrow X$$

for each $\alpha \in \mathcal{A}$, let $\mathcal{O}_X$ be the collection of all subsets U of X such that $g_\alpha^{-1}(U) \in \mathcal{O}_\alpha$ for every $\alpha \in \mathcal{A}$.

Lemma-Definition 2.3.3. As defined above, $\mathcal{O}_X$ is a topology on X, called the *final topology on X* defined by the family $\{(g_\alpha : Y_\alpha \to X)\}_{\alpha\in\mathcal{A}}$. Furthermore, $\mathcal{O}_X$ is the *largest* topology on X such that

$$g_\alpha : \ (Y_\alpha, \mathcal{O}_\alpha) \longrightarrow (X, \mathcal{O}_X)$$

is continuous for every $\alpha \in \mathcal{A}$.

this example is new

$X_-=(-\infty,0)$, $X_+=[0,+\infty)$

$\mathbb{R}=(-\infty,0)\sqcup[0,+\infty)$

1:1 final topology (not the same top)

Proof. Exercise. □

Exercise 2.3.3. Given a family $\{(g_\alpha : Y_\alpha \to X)\}_{\alpha\in\mathcal{A}}$ of continuous maps, and equip X with the final topology. Then for any topological space Z, a map $h : X \to Z$ is continuous iff $h \circ g_\alpha : Y_\alpha \to Z$ is continuous for every $\alpha \in \mathcal{A}$.

Example 2.3.9. The quotient topology is an example of a final topology.

Example 2.3.10. The *disjoint union* of a family $\{X_\alpha : \alpha \in \mathcal{A}\}$ of sets is the set

$$X \overset{\text{denote}}{=} \bigsqcup_{\alpha\in\mathcal{A}} X_\alpha \overset{\text{def}}{=} \bigsqcup_{\alpha\in\mathcal{A}} \{(x,\alpha) : x \in X_\alpha\}.$$

Note that each $\alpha \in \mathcal{A}$, we have the injective map

$$I_{X_\alpha} : \ X_\alpha \longrightarrow X, \ \ x \longmapsto (x,\alpha),$$

and if $\alpha,\beta \in \mathcal{A}$ and $\alpha \neq \beta$, the images of X_α and X_β in X do not intersect in X. We often identify X_α with its image in X, so that X is the disjoint union of all the X_α's.

Suppose now that each X_α is a topological space. Then we can equip X with the final topology defined by the injective maps $I_{X_\alpha} : X_\alpha \to X$ for all $\alpha \in \mathcal{A}$. In other words, a $U \subset X$ is open iff $I_{X_\alpha}^{-1}(U)$ is open in X_α for every $\alpha \in \mathcal{A}$. Equivalently, $U \subset X$ is open iff $U \cap X_\alpha$ is open in X_α for each α. The set-theoretic disjoint union X with this final topology is called the disjoint union of the family $\{X_\alpha : \alpha \in \mathcal{A}\}$ of topological spaces. As a special property of the final topology, for any topological space Y, a map $f : X \to Y$ is continuous iff

$$f|_{X_\alpha} = f \circ I_{X_\alpha} : \ X_\alpha \longrightarrow Y$$

is continuous.

2.4 Other basic Notions

2.4.1 Interiors of subsets and neighborhoods of points

We now give precise mathematical formulations of what our intuition tells us about *interiors* and *neighborhoods*.

Let X be a topological space.

Definition 2.4.1. For $A \subset X$, the *interior* of A, denoted by $\mathring{A}$ or A°, is the union of all open subsets of X that are contained in A.

We have the following easy facts from definitions:

- $\mathring{A}$ is always open, and A is open iff $A = \mathring{A}$.

$$A^\circ = \bigcup_{O\in\mathcal{O}_X,\, O\subseteq A} O$$

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假[illegible]

- $\mathring{A}$ is the *largest* open subset of X contained A, i.e.,
  1) $\mathring{A}$ is open and is contained in A;
  2) if O is an open subset of X and $O \subset A$, then $O \subset \mathring{A}$.

Definition 2.4.2. Let $x \in X$. A subset $N \subset X$ is called a *neighborhood* of x if $x \in \mathring{N}$.

Note that a neighborhood N of x in X contains an open neighborhood of x, namely $\mathring{N}$. We denote the collection of all neighborhoods of x by $\mathcal{N}_x$.

Lemma 2.4.1. Let X be a topological space. For any two subsets A and B of X, if $A \subset B$, then $\mathring{A} \subset \mathring{B}$.

Proof. If U is an open subset of X such that $U \subset A$, then $U \subset B$. Thus $\mathring{A} \subset \mathring{B}$. □

Proposition 2.4.1. Let X be a topological space and let $A, B \subset X$. Then
1) $(A^\circ)^\circ = \mathring{A}$;
2) $\mathring{A} \cup \mathring{B} \subset (A \cup B)^\circ$;
3) $\mathring{A} \cap \mathring{B} = (A \cap B)^\circ$.

Proof. 1) holds because $\mathring{A}$ is open. As $A, B \subset A \cup B$, by Lemma 2.4.1, one has

$$\mathring{A} \subset (A \cup B)^\circ \quad \text{and} \quad \mathring{B} \subset (A \cup B)^\circ.$$

Thus 2) holds. To prove 3), note first that $(A \cap B)^\circ \subset \mathring{A} \cap \mathring{B}$ because $A \cap B \subset A$ and $A \cap B \subset B$. Conversely, $\mathring{A} \cap \mathring{B}$ is open and is contained in $A \cap B$, so $\mathring{A} \cap \mathring{B} \subset (A \cap B)^\circ$. □

Here is a statement whose proof requires understanding of product topology.

Lemma 2.4.2. For any topological spaces X and Y and subsets $A \subset X$ and $B \subset Y$, one has $(A \times B)^\circ = \mathring{A} \times \mathring{B}$ as subsets of $X \times Y$ with the product topology.

Proof. Note first that

$$\begin{aligned}
(x, y) \in (A \times B)^\circ &\Rightarrow (x, y) \in O \subset A \times B \text{ for some open } O \subset X \times Y \\
&\Rightarrow (x, y) \in U \times V \text{ for some open } U \subset X, V \subset Y \\
&\Rightarrow x \in \mathring{A},\ y \in \mathring{B}, \\
&\Rightarrow (x, y) \in \mathring{A} \times \mathring{B}.
\end{aligned}$$

Thus $(A \times B)^\circ \subset \mathring{A} \times \mathring{B}$. On the other hand, $\mathring{A} \times \mathring{B}$ is open and is contained in $A \times B$. Thus $\mathring{A} \times \mathring{B} \subset (A \times B)^\circ$. □

Example 2.4.1. For $\mathbb{R}$ with the Euclidean topology, one has $\mathring{\mathbb{Q}} = \varnothing$.

Example 2.4.2. Consider $\mathbb{R}^2$ with the Euclidean topology and let

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}.$$

We now prove that $\mathring{A} = D := \{p = (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. First of all, since D is open and $D \subset A$, we know that $D \subset \mathring{A}$. We now show that $\mathring{A} \subset D$. Suppose that $p = (x,y) \in \mathring{A} \subset A$. If $p \notin D$, then $x^2 + y^2 = 1$. Then for any $r > 0$, the open ball $B(p,r)$ contains points not in A, so A is not a neighborhood of p and thus $p \notin \mathring{A}$, a contradiction. Thus $D = \mathring{A}$.

Note that the same arguments show that for any $(a,b) \in \mathbb{R}^2$ and $r > 0$, one has

$$\mathring{A} = \{(x,y) \in \mathbb{R}^2 : \ (x-a)^2 + (y-b)^2 < r^2\}$$

for $A = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 \le r^2\}$. The result holds if $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$.

2.4.2 Closures of subsets

Definition 2.4.3. Let X be a topological space and let $A \subset X$. The *closure* of A, denoted by $\overline{A}$, is the intersection of all closed subsets in X containing A.

As with the case of the interiors of subsets, we have the following *easy facts from the definition.*

- $\overline{A}$ is always closed, and A is closed iff $A = \overline{A}$.
- $\overline{A}$ is the *smallest* closed subset of X containing A, i.e.,
  1. $\overline{A}$ is closed and contains A;
  2. if B is a closed subset of X and $A \subset B$, then $\overline{A} \subset B$.

重新抽象地把这些点集拓扑学知识研究一下

The proofs of the following two lemmas are left as as exercises.

Lemma 2.4.3. Let X be a topological space and let $A, B \subset X$. Then

$$\overline{\overline{A}} = \overline{A}, \quad \overline{A \cup B} = \overline{A} \cup \overline{B}, \quad \overline{A \cap B} \subset \overline{A} \cap \overline{B}.$$

Furthermore, if $A \subset B$, then $\overline{A} \subset \overline{B}$. Do we always have $\overline{A \cap B} \supset \overline{A} \cap \overline{B}$?

2.4.3 Limit points and boundaries of subsets

Definition 2.4.4. Let X be a topological space and $A \subset X$. A point $x \in X$ is called a *limit point* of A if every neighborhood of x contains a point in A different from x.

Since, a neighborhood of x must contain an open neiborhood, a point $x \in X$ is a limit point of A iff every open neighborhood of x contains a point in A different from x.

Note also that a limit point x may or may not belong to A.

$x \in A'$ if $x \in \overline{A - \{x\}}$

$(A \times B)^\circ = A^\circ \times B^\circ$, but not $\left(\prod_{\alpha \in I} A_\alpha\right)^\circ = \prod_{\alpha \in I} A_\alpha^\circ$ ☆

closure

差的 interior

Example 2.4.3. The set of limit points of $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is precisely the closed ball $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. In this case, every point in A is also a limit point. On the other hand, for $A = \{1/n : n \in \mathbb{Z}_{\ge 1}\}$ as a subset of $\mathbb{R}$ with the Euclidean topology, only 0 is a limit point of A.

Denote by $\mathrm{Lim}(A)$ the set of all limit points of a subset A of X.

Lemma 2.4.4. One has $\mathrm{Lim}(A) \subset \overline{A}$ for any $A \subset X$.

Proof. Let $x \in \mathrm{Lim}(A)$. If $x \notin \overline{A}$, then $X \backslash \overline{A}$ is a neighborhood of x containing no points of A, a contradiction. $\square$

Proposition 2.4.2. One has $\overline{A} = A \cup \mathrm{Lim}(A)$ for any $A \subset X$.

Proof. We already know that $A \cup \mathrm{Lim}(A) \subset \overline{A}$. Now we show that $\overline{A} \subset A \cup \mathrm{Lim}(A)$. Suppose that $x \in \overline{A} \backslash A$. Need to show $x \in \mathrm{Lim}(A)$. Let U an open neighborhood of x. If $A \cap U = \varnothing$, then $A \subset X \backslash U$, the latter being closed, so $x \in \overline{A} \subset X \backslash U$, a contradiction. Thus $A \cap U \neq \varnothing$. Since $x \notin A \cap U$, $A \cap U$ contains $x' \neq x$. Thus $x \in \mathrm{Lim}(A)$. $\square$

Corollary 2.4.1. For any $x \in X$, $x \in \overline{A}$ iff every neighborhood of x in X intersects with A, or, equivalently, iff every open neighborhood of x in X intersects with A.

Proposition 2.4.3. For any topological spaces X and Y and subsets $A \subset X$ and $B \subset Y$, one has $\overline{A \times B} = \overline{A} \times \overline{B}$ in the product topology on $X \times Y$.

Proof. As the completement of $\overline{A} \times \overline{B}$ in $X \times Y$ is $((X \backslash \overline{A}) \times Y) \cup (X \times (Y \backslash \overline{B}))$, we know that $\overline{A} \times \overline{B}$ is closed in $X \times Y$. Thus $\overline{A \times B} \subset \overline{A} \times \overline{B}$. To show that $\overline{A} \times \overline{B} \subset \overline{A \times B}$, let $(x,y) \in \overline{A} \times \overline{B}$, and let O be any open neighborhood of (x,y) in $X \times Y$. By Corollary 2.4.1, it is enough to show that $O \cap (A \times B) \neq \varnothing$. By the definition of the product topology on $X \times Y$, there exist open subset U of X and open subset V of Y such that $(x,y) \subset U \times V \subset O$. Since $x \in \overline{A}$, by Corollary 2.4.1 again, $U \cap A \neq \varnothing$. Similarly, $V \cap B \neq \varnothing$. Then $(U \times V) \cap (A \times B) \neq \varnothing$. Hence $O \cap (A \times B) \neq \varnothing$. $\square$

Definition 2.4.5. Let X be a topological space and let $A \subset X$. We say that A is *dense* in X if $\overline{A} = X$.

Example 2.4.4. For $\mathbb{R}$ with the Euclidean topology, the subset $\mathbb{Q}$ is dense in $\mathbb{R}$. On the other hand, for $\mathbb{R}$ with the Zariski topology, any infinite subset is dense because the only infinite closed subset is $\mathbb{R}$ itself. In particular, every non-empty open subset is dense!

Definition 2.4.6. Let X be a topological space and let $A \subset X$. The *boundary*, also called the *frontier* of A is the set

$$\partial A = \overline{A} \backslash \mathring{A}.$$

$$\overline{\prod_{\alpha \in I} A_\alpha} = \prod_{\alpha \in I} \overline{A_\alpha}$$

Cantor Set, length 0, not countable 复习.

Lemma 2.4.5. For any $A \subset X$, $\partial A = \overline{A} \cap \overline{X \backslash A}$. Thus $\partial A = \partial(X \backslash A)$;

Proof. Exercise. □

Example 2.4.5. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
1) With the Euclidean topology on $\mathbb{R}^2$, $\partial D = C$;
2) If D is regarded as a subset of itself, $\partial D = \varnothing$;
3) If $D = \{(x,y,0) : x^2 + y^2 \le 1\} \subset \mathbb{R}^3$, then $\mathring{D} = \varnothing$, so $\partial D = D$.

Exercise 2.4.1. Prove the following statements for any topological space.

- For $x \in X$, $x \in \partial A$ if every neighborhood of x in X contains a point in A and a point not in A.
- A set is closed if and only if it contains its boundary, and open if and only if it is disjoint from its boundary;
- The closure of a set equals the union of the set with its boundary;
- The boundary of a set is empty if and only if the set is both closed and open (that is, a clopen set). The interior of the boundary of the closure of a set is the empty set.
- Write out the above statements in formulas, e.g. $\overline{A} = A \cup \partial A$.

2.4.4 Hausdorff topological spaces

从这里开始讲分离性

Definition 2.4.7. Let X be a topological space. We say that X is a *Hausdorff space* (or a T_2 *space*) if for any distinct points $x, y \in X$, there exist disjoint open subsets U and V of X such that $x \in U$ and $y \in V$.

Example 2.4.6. Any metric space (X,d) is Hausdorff. Thus $\mathbb{R}^n$ with the Euclidean topology is Hausdorff. Indeed, if $x, y \in X$ and $x \neq y$, then $B(x, d_0/3)) \cap B(y, d_0/3) = \varnothing$, where $d_0 = d(x,y)$.

Example 2.4.7. While the Euclidean topology on $\mathbb{R}^n$ is Hausdorff, the Zariski topology on $\mathbb{R}^n$ is famously not Hausdorff. Consider, for example, $\mathbb{R}^1 = \mathbb{R}$ with the Zariski topology. For any non-empty open subsets A and B, $A \cap B \neq \varnothing$ (otherwise $A \subset \mathbb{R} \backslash B$ must be finite so $A = \varnothing$). Thus the Zariski topology on $\mathbb{R}$ is not Hausdorff.

Proposition 2.4.4. Let X be a Hausdorff space. Then
1) any finite subset of X is closed;
2) any subspace of X is Hausdorff.

$\forall$ distinct $x_1, x_2 \in X$, $\exists\, O_1, O_2 \in \mathcal{O}_X$,

$x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \varnothing$

X is Haus iff $\Delta : X \to X \times X$, $\Delta(x) = (x,x)$ has closed image

$\forall (x_1, x_2) \in X \times X$, $\exists U_1, U_2 \in \mathcal{O}_X$, $x_1 \in U_1, x_2 \in U_2$, $U_1 \cap U_2 = \emptyset$

Proof. To prove 1) it is enough to show that any set consisting of a single element is closed, because any finite set is a finite union of such. For any $x \in X$, the Hausdorff property implies that the complement of $\{x\}$ is open as it contains an open subset around each point in the complement.

To prove 2), let Y be any no-empty subspace of X, and let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Let U_i, $i = 1, 2$, be two disjoint open subsets of X such that $y_i \in U_i$. Then $Y \cap U_1$ and $Y \cap U_2$ are two disjoint open subsets in Y containing y_1 and y_2 respectively. Thus Y with the subspace topology is Hausdorff. □

Lemma 2.4.6. Let X and Y be nonempty topological spaces. Then $X \times Y$ is Hausdorff if and only if X and Y are Hausdorff.

Proof. Suppose first that $X \times Y$ is Hausdorff, and we show that X is Hausdorff. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Since $Y \neq \emptyset$, choosing any $y \in Y$, then (x_1, y) and (x_2, y) are two distinct points in $X \times Y$, so there exist disjoint open subsets O_1 and O_2 of $X \times Y$ such that $(x_1, y) \in O_1$ and $(x_2, y) \in O_2$. By the definition of the product topology on $X \times Y$, we can replace O_1 by $U_1 \times V_1 \subset O_1$ and O_2 by $U_2 \times V_2 \subset O_2$, where U_1, U_2 are open subsets of X and V_1, V_2 are open subsets of Y. If $U_1 \cap U_2 \neq \emptyset$, since $y \in V_1 \cap V_2$, for any $x \in U_1 \cap U_2$ we would have

$$(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2) \subset O_1 \cap O_2 = \emptyset,$$

a contradiction. Thus $U_1 \cap U_2 = \emptyset$. Since $x_1 \in U_1$ and $x_2 \in U_2$, it follows that X is Hausdorff. The proof that Y is Hausdorff is similar.

Assume now that X and Y are Hausdorff. Suppose that $(x_1, y_1), (x_2, y_2) \in X \times Y$ are distinct. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. Assume, without loss of generality, that $x_1 \neq x_2$. Then there exist disjoint open subsets U_1 and U_2 of X such that $x_1 \in U_1$ and $x_2 \in U_2$. Let $O_1 = U_1 \times Y$ and $O_2 = U_2 \times Y$. Then O_1 and O_2 are disjoint open subsets of $X \times Y$ with $(x_1, y_1) \in O_1$ and $(x_2, y_2) \in O_2$. Thus $X \times Y$ is Hausdorff. □

Lemma 2.4.7. A topological space X is Hausdorff iff $X_{\text{diag}} := \{(x, x) :\in X\}$ is a closed subset of $X \times X$ with the product topology.

Proof. Exercise. □

We now see how the Hausdorff condition is natural when we consider limits of sequences in topological spaces.

Definition 2.4.8. — Let X be a topological space. A sequence $\{p_n\}$ in X is said to converge to $p \in X$ if for any neighborhood U of p there exists integer $N \geq 1$ such that $p_n \in U$ for all $n \geq N$. In such a case we say that the sequence $\{p_n\}$ is convergent and we call p a limit of $\{p_n\}$, and we write $\lim_{n \to \infty} p_n = p$ or simply $p_n \to p$ as $n \to \infty$.

Example 2.4.8. Let X be equipped with the trivial topology. Then every sequence in X converges to every point in X.

subspace of T_2 is T_2, $X_1 \times X_2$ T_2 iff X_1, X_2 T_2

Unique limit in T_2

Lemma 2.4.8. In a Hausdorff topological space, the limit of a convergent sequence is unique.

Proof. Assume that $p_n \to p$ and $p_n \to q$ as $n \to \infty$. If $p \neq q$, let U and V be disjoint neighborhoods of p and q respectively. Then there exists $N > 0$ such that $p_n \in U \cap V$ for all $n > N$, contradicting the fact that $U \cap V = \varnothing$. $\square$

We now discuss when a quotient topology is Hausdorff. Let $\sim$ be an equivalence relation on a topological space. Equip

$$[X] = X/\sim$$

with the quotient topology, and let $q : X \to [X]$ be the quotient map. Note that fibers of q are precisely the equivalence classes in X.

A subset A of X is said to be *saturated* if A is a union of fibers of q. For any $A \subset X$, it follows from definition that

$$\begin{aligned} A \text{ is saturated} &\iff A = q^{-1}(A') \text{ for some } A' \subset [X], \\ &\iff A = q^{-1}(q(A)). \end{aligned}$$

One thus has a $1-1$ correspondence

$$[X] \supset U \longmapsto q^{-1}(U) \subset X$$

between subsets of $[X]$ and saturated subsets of X, which also gives a $1-1$ correspondence between open subsets of $[X]$ and saturated open subsets of X. We thus have the following statement on the quotient topology.

Proposition 2.4.5. A quotient space $[X]$ is Hausdorff if and only if two different equivalence classes in X can be separated by *disjoint saturated open subsets*, i.e., if C_1 and C_2 are distinct equivalence classes in X, then there exist open subsets O_1 and O_2 of X such that

$$C_1 \subset O_1, \quad C_2 \subset O_2, \quad O_1 \cap O_2 = \varnothing.$$

Example 2.4.9. ("line with two origins") Consider

$$X = \{(x,0) : x \in \mathbb{R}\} \cup \{(x,1) : x \in \mathbb{R}\}.$$

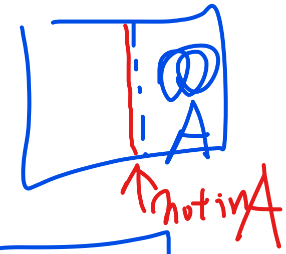
Define an equivalence relation $\sim$ on X with the non-trivial (i.e., other then $a \sim a$ for all $a \in X$) equivalences given by

$$(x,0) \sim (x,1) \text{ for } x \neq 0.$$

Then $[X] = X/\sim = \{[x,0] : x \in \mathbb{R}\} \cup \{(0,1)\}$ with the quotient topology is called the *line with two origins*. Let $p : X \to [X], a \to [a]$, be the projection map. For any open subsets $U_0 \subset [X]$ containing $[0,0]$ and $U_1 \subset [X]$ containing $[0,1]$, as $p^{-1}(U_0)$ and $p^{-1}(U_1)$ must contain open intervals in the two respective lines, $U_0 \cap U_1 \neq \varnothing$. Thus $[X]$ with the quotient topology is not Hausdorff.

These are "nice quotients"!

Ex →



Example 2.4.10. For any topological space X and any $A \subset X$, define

$$x \sim x' \text{ iff either } x, x' \in A \text{ or } x = x' \notin A.$$

Denote by $X/\sim$ by X/A and call it *the collapsing of $A \subset X$ to a point.* Let $q : X \to X/A$ be the projection and $p_0 = q(A) \in X/A$. Since $q^{-1}(p_0) = A$, $\{p_0\} \subset X/A$ is closed iff $A \subset X$ is closed. Note that the assignment $U \mapsto q^{-1}(U)$ is a $1-1$ correspondence between

1) open $U \subset X/A$, $p_0 \in U$, and open $O \subset X$, $O \supset A$;
2) open $U \subset X/A$, $p_0 \notin U$, and open $O \subset X$, $O \cap A = \varnothing$.

Assume that X is Hausdorff. We thus conclude that X/A is Hausdorff iff the following two conditions hold:

1) A is closed;
2) for every $y \in X \backslash A$, $\exists$ open subsets O and O_y of X such that

This is T3, go and check it!

$$A \subset O, \quad y \in O_y, \quad O \cap O_y = \varnothing. \tag{2.1}$$

A topological space X is called a T_3*-space* or *regular Hausdorff* if it is Hausdorff and if for any closed $A \subset X$ and any $y \in X \backslash A$, $\exists$ open subsets O and O_y of X satisfying (2.1). Thus, if X is T_3 and $A \subset X$ is closed, then X/A is Hausdorff. Moreover, X/A is Hausdorff for every closed subset A of X iff X is T_3.

Proposition 2.4.6. Let Z be a topological space, $X \subset Z$ and $Y \subset Z$ both closed, and $Z = X \cup Y$. If X and Y are Hausdorff, so is Z.

Proof. Exercise. □

2.5 Continuity

2.5.1 Definition and immediate examples

The notion of continuity is the second most important one in topology, the first one being that of the definition of a topological space. We have introduced the definition before. Here it is again.

Definition 2.5.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A map $f : X \to Y$ is said to be *continuous* if

$$f^{-1}(U) \in \mathcal{O}_X \qquad \forall \; U \in \mathcal{O}_Y. \tag{2.2}$$

Warning on terminology. In Armstrong's book, he calls a map $X \to Y$ a "function", and he calls a "continuous function" a "map", which it very non-standard.

If $\mathcal{O}_Y$ has a basis $\mathcal{B}$, only need to check condition (2.2) on open subsets in $\mathcal{B}$.

Lemma 2.5.1. Let X and Y be two topological spaces and let $\mathcal{B}$ be a basis of the topology on Y. Then a map $f : X \to Y$ is continuous if and only if $f^{-1}(U)$ is open in X for every $U \in \mathcal{B}$.

Proof. This is because every open subset of Y is of the form $\bigcup_\alpha U_\alpha$ where $U_\alpha \in \mathcal{B}$, and

$$f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha).$$

□

Definition 2.5.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces, and let $x \in X$. A map $f : X \to Y$ is said to be continuous at x if for every $U \in \mathcal{O}_Y$ containing $f(x)$, $f^{-1}(U)$ contains an open subset of X that contains x.

Lemma 2.5.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces, and let $f : X \to Y$. Then f is continuous if and only if f is continuous at every $x \in X$.

Proof. Clearly f is continuous implies that it is continuous at every $x \in X$. Assume that $f : X \to Y$ is continuous at every $x \in A$. Let U be any open subset of Y such that $f^{-1}(U) \neq \varnothing$. For every $x \in f^{-1}(U)$, since U is an open subset of Y containing $f(x)$, by definition of f being continuous at x, there is an open subset U_x of X containing x such that $U_x \subset f^{-1}(U)$. It follows that $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} U_x$ is open. Thus f is continuous. □

We want to see that the definition of continuous functions we learned in Calculus coincides with our new abstract definition.

Definition 2.5.3 (Calculus)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at $a \in \mathbb{R}$ if

$$\lim_{x \to a} f(x) = f(a),$$

i.e., if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever} \quad |x - a| < \delta.$$

The function f is said to be continuous if it is continuous at every $a \in \mathbb{R}$.

These two definitions are indeed the same: the $\epsilon - \delta$ condition says that for any $a \in \mathbb{R}$, the pre-image under f of any open interval around $f(a)$ contains an open interval around a, which is exactly what it means for f to be continuous at a according to the abstract definition. From now on we assume that we have proved in *Calculus*, or in *Introduction to Mathematical Analysis*, or in *Analysis I*, that all the polynomials functions on $\mathbb{R}$, exponential functions, trigonometric functions such as $\sin x$ and $\cos x$, are all continuous, so they will be our examples of continuous functions/maps according to the abstract definition.

Note that the above discussion works for any metric space: let (X,d) be a metric space, and recall that the metric topology defined by d is such that a subset U of X is open if and only if U contains an open ball centered at a for every $a \in U$.

Lemma 2.5.3. For two metric spaces (X, d_X) and (Y, d_Y) with the metric topology, a map $f : X \to Y$ is continuous if and only if for any $a \in X$ and any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(f(x), f(a)) < \epsilon \quad \text{whenever} \quad d_X(x,a) < \delta.$$

Most of the continuous functions in calculus are given in nice formulas. Here is a new one.

Example 2.5.1. Let (X,d) be any metric space and let A be any subset of X. For $x \in X$, define

$$d(x,A) = \inf\{d(x,a) : a \in A\}.$$

The $f : X \to \mathbb{R}, f(x) = d(x,A)$, is a continuous function on X. To see this, let $x \in X$ be arbitrary. Then for any $\epsilon > 0$, need to find $\delta > 0$ such that

$$-\epsilon < d(x',A) - d(x,A) \le \epsilon \quad \text{whenever} \quad d(x',x) < \delta. \tag{2.3}$$

We claim that $\delta = \epsilon/2$ would work. Indeed, by the definition of infimum, there exists $a \in A$ such that $d(x,a) < d(x,A) + \epsilon/2$. Assume $d(x',x) < \epsilon/2$. Then

$$d(x',A) \le d(x',a) \le d(x',x) + d(x,a) < \frac{\epsilon}{2} + d(x,A) + \frac{\epsilon}{2} = \epsilon + d(x,A).$$

Switching x and x', we also have $d(x',A) < \epsilon + d(x,A)$. Thus (2.3) holds.

We now give an equivalent definition of continuous maps.

Lemma 2.5.4. Let X and Y be two topological spaces and let $f : X \to Y$. Then f is continuous if and only if $f^{-1}(C)$ is closed in X for every closed subset C of Y.

Proof. The statement follows from the fact that for any $C \subset Y$,

$$f^{-1}(Y \backslash C) = \{x \in X : f(x) \notin C\} = \{x : x \notin f^{-1}(C)\} = X \backslash (f^{-1}(C)).$$

□

As an immediate application of Lemma 2.5.4, we have the following *Gluing Lemma*:

Lemma 2.5.5. Let X and Y be two topological spaces, and let X_1 and X_2 be two closed subsets of X and $X_1 \cup X_2 = X$. If $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are two continuous maps such that $f_1|_{X_1 \cap X_2} = f_2|_{X_1 \cap X_2}$, then the map $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X_1, \\ f_2(x), & \text{if } x \in X_2, \end{cases}$$

is continuous.

Proof. Let C be any closed subset of Y. Then

$$f^{-1}(C) = (X_1 \cap f^{-1}(C)) \cup (X_2 \cap f^{-1}(C)).$$

Note that $X_1 \cap f^{-1}(C) = f_1^{-1}(C)$. Since $f_1 : X_1 \to Y$ is continuous, $f_1^{-1}(C)$ is closed in X_1. Thus there exists a closed subset C_1 of X such that

$$X_1 \cap f^{-1}(C) = f_1^{-1}(C) = X_1 \cap C_1.$$

Since X_1 is closed in X, $X_1 \cap C_1$ is closed in X, and thus $X_1 \cap f^{-1}(C)$ is closed in X. Similarly, $X_2 \cap f^{-1}(C)$ is closed in X. Thus $f^{-1}(C)$ is closed in X. By Lemma 2.5.4, $f : X \to Y$ is continuous. □

Note now that by replacing all the closed subsets in the proof of Lemma 2.5.4 by open subsets and using the original definition of continuous maps, we also have the following version of the Gluing Lemma.

Lemma 2.5.6. Let X and Y be two topological spaces, and let X_1 and X_2 be two *open subsets* of X and $X_1 \cup X_2 = X$. If $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are two continuous maps such that $f_1|_{X_1 \cap X_2} = f_2|_{X_1 \cap X_2}$, then the map $f : X \to Y$ defined by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X_1, \\ f_2(x), & \text{if } x \in X_2, \end{cases}$$

is continuous.

For further examples, let us recap what we did when we were constructing new topology from old ones in §2.3.

1. For a topological space X and a subset $X_1 \subset X$, the *subspace topology* on X_1 is the *smallest* topology on X_1 such that the inclusion map

   $$I : \ X_1 \longrightarrow X, \ \ x_1 \longmapsto x_1$$

   is continuous.

2. For two topological spaces X and Y, the *product topology* on $X \times Y$ is the *smallest* topology on $X \times Y$ such that both projections

   $$p_X : \ X \times Y \longrightarrow X, \ (x, y) \longmapsto x,$$
   $$p_Y : \ X \times Y \longrightarrow Y, \ (x, y) \longmapsto y,$$

   are continuous.

3. For a topological space X with an equivalence relation $\sim$, the *quotient topology* on the quotient space $[X] = X/\sim$ is the *largest* topology on $[X]$ such that the quotient map
$$q: X \longrightarrow [X], \ q(x) = [x]$$
is continuous.

4. More generally, assume that X is any set and $\{(Y_\alpha, \mathcal{O}_\alpha)\}_{\alpha\in I}$ is any family of topological spaces together with a map
$$f_\alpha: X \longrightarrow Y_\alpha$$
for each $\alpha \in I$. Then the *initial topology on* X is the *smallest* topology $\mathcal{O}_X$ on X such that
$$f_\alpha: (X, \mathcal{O}_X) \longrightarrow (Y_\alpha, \mathcal{O}_\alpha)$$
is continuous for every $\alpha \in I$.

5. Similarly, for any set X and any family $\{(Y_\alpha, \mathcal{O}_\alpha)\}_{\alpha\in I}$ of topological spaces together with a map
$$g_\alpha: Y_\alpha \longrightarrow X$$
for every $\alpha \in I$, the *finial topology on* X defined by the family $\{(g_\alpha : Y_\alpha \to X)\}_{\alpha\in I}$ is the *largest* topology $\mathcal{O}_X$ on X such that
$$g_\alpha: (Y_\alpha, \mathcal{O}_\alpha) \longrightarrow (X, \mathcal{O}_X)$$
is continuous for every $\alpha \in I$.

2.5.2 Basic properties

We establish some simple facts that will help to determine quickly that some given maps are continuous.

Lemma 2.5.7. If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is their composition $g \circ f : X \to Z$.

Proof. For any $C \subset Z$,
$$\begin{aligned}(g\circ f)^{-1}(C) &= \{x \in X : g(f(x)) \in C\} = \{x \in X : f(x) \in g^{-1}(C)\}\\ &= \{x \in X : x \in f^{-1}(g^{-1}(C))\} = f^{-1}(g^{-1}(C)).\end{aligned}$$
If $f : X \to Y$ and $g : Y \to Z$ are continuous, then for any $C \subset Z$ is open, $g^{-1}(C) \subset Y$ is open and $f^{-1}(g^{-1}(C)) \subset X$ is open. Thus $g \circ f : X \to Z$ is continuous. $\square$

Lemma 2.5.8. Let X and Y be topological spaces, and let $f : X \to Y$ be any map.

1) If $f : X \to Y$ is continuous and $X_1 \subset X$ has the subspace topology, then

$$f|_{X_1} : X_1 \longrightarrow Y, \; f|_{X_1}(x_1) = f(x), \quad x_1 \in X_1,$$

is also continuous.

2) Suppose that $Y_1 \subset Y$ is such that $f(X) \subset Y_1$ and regard f as a map from X to Y_1, i.e., define

$$\tilde{f} : X \longrightarrow Y_1, \; \tilde{f}(x) = f(x),$$

and equip Y_1 with the subspace topology of Y. Then $f : X \to Y$ is continuous if and only if $\tilde{f} : X \to Y_1$ is continuous.

Proof. 1) This is because $f|_{X_1} = f \circ I$, where $I : X_1 \to X$ is the inclusion map.

2) For any open subset U of Y. Then

$$\tilde{f}^{-1}(Y_1 \cap U) = \{x \in X : \tilde{f}(x) \in Y_1 \cap U\} = \{x \in X : f(x) \in U\} = f^{-1}(U).$$

Thus $f : X \to Y$ is continuous if and only if $\tilde{f} : X \to Y_1$ is continuous. □

Lemma 2.5.8 is very useful to determine whether or not a map $f : X \to Y$ is continuous if both X and Y are subspaces of bigger topological spaces.

Example 2.5.2. Consider the map $f : [0, 1) \to S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ given by

$$f(t) = e^{2\pi i t}, \quad t \in [0, 1).$$

Equip $[1, 0)$ with the subspace topology of $\mathbb{R}^1$ with the Euclidean topology, and S^1 the subspace of $\mathbb{R}^2$, also with the Euclidean topology. Since $f : \mathbb{R} \to \mathbb{R}^2, f(t) = e^{2\pi i t}$ is continuous, by Lemma 2.5.8, $f : [0, 1) \to S^1$ is continuous. We can also prove that $f : [0, 1) \to S^1$ is continuous by definition: take any open arc A on S^1 and look at its preimage under f. If the point $(1, 0) \notin A$, $f^{-1}(A)$ is an open interval contained in $[0, 1)$. If $(1, 0) \in A$, $f^{-1}(A) = [0, a) \cup (b, 1)$ for $0 < a < b < 1$, which is still open in $[0, 1)$ with the subspace topology.

2.5.3 Homeomorphisms

Now we introduce the third most important notion in topology.

Definition 2.5.4. Let X and Y be topological spaces and let $f : X \to Y$ be a bijective map. If $f : X \to Y$ and its inverse $f^{-1} : Y \to X$ are both continuous, then f is called a *homeomorphism*. The topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism from X to Y.

If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism, then we have a one-to-one correspondence between $\mathcal{O}_X$ and $\mathcal{O}_Y$ given by

$$\mathcal{O}_X \ni U \longleftrightarrow f(U) \in \mathcal{O}_Y,$$

so the two topological spaces are regarded as the same. Since compositions of continuous maps are again continuous, the relation of being homeomorphic among topological spaces is an equivalence relation.

Exercise 2.5.1. Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. For any subspace Z of X, the restriction $f|_Z : Z \to f(Z)$ is a homeomorphism.

Definition 2.5.5. A *topological property* (or *topological invariant*) is a property of a topological space that is preserved under homeomorphisms.

Example 2.5.3. The Hausdorff condition is a topological property. In other word, if two topological spaces X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff.

The main objective of topology is to identify topological properties of topological spaces!

We now give various examples of homeomorphisms.

Example 2.5.4. For $r_1, r_2 > 0$, the map

$$\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1} : \quad x \longmapsto \frac{r_2}{r_1} x$$

is a homeomorphism. By restrictions, the two spheres

$$S^n_{r_i} = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = r_i^2\}, \qquad i = 1, 2,$$

are homeomorphic. Thus the radius of a sphere is NOT a topological invariant.

Example 2.5.5. Consider $\mathbb{R}^1$ with the Euclidean topology and any of its subsets with the subspace topology.

1) The function $f(x) = \arctan x$ is a homeomorphism from $\mathbb{R}$ to $(-\pi/2, \pi/2)$;

2) The function $f(x) = e^x$ is a homeomorphism from $\mathbb{R}^1$ to $(0, \infty)$.

3) Any finite open interval (a, b) is homeomorphic to $(0, 1)$ by

$$f(x) = \frac{1}{b-a}(x - a), \qquad x \in (a, b). \tag{2.4}$$

4) Any two open intervals in $\mathbb{R}^1$, being both homeomorphic to $(0, 1)$, are homeomorphic.

5) The same function in (2.4) also defines a homeomorphism from $[a, b]$ to $[0, 1]$.

6) Since every continuous function on $[a, b]$ must be bounded, $[a, b]$ can not be homeomorphic to $\mathbb{R}$ and thus not to $(0, 1)$.

Thus the "length" of an open interval in $\mathbb{R}^1$ is not a topological property.

Example 2.5.6. Consider the subspaces

$$X = S^2 \backslash \{(0,0,1)\} \quad \text{and} \quad Y = \{(u,v,0) : u, v \in \mathbb{R}\}$$

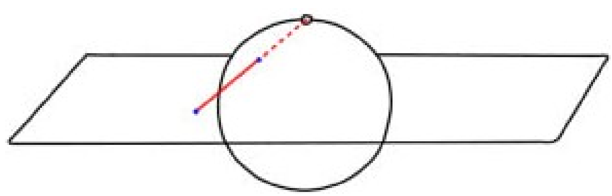
of $\mathbb{R}^3$. Define $f : X \to Y$ where for each $(x,y,z) \in X$, $f(x,y,z)$ is the point in Y such that $(0,0,1), (x,y,z), f(x,y,z)$ lie on the same straight line. Explicit formula for f is

$$f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right), \qquad (x,y,z) \in S^2 \backslash \{(0,0,1)\},$$

and its inverse $f^{-1} : Y \to X$ is

$$f^{-1}(u,v,0) = \left(\frac{\overset{u}{2\cancel{x}}}{1+u^2+v^2}, \frac{\overset{v}{2\cancel{y}}}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

The map f is called the *stereographic projection*, and it induces a natural homeomorphism between $S^2 \backslash \{(0,0,1)\}$ and $\mathbb{R}^2$.



Example 2.5.7. The annulus

$$A = \{(x,y) \in \mathbb{R}^2 : 1 \leqslant x^2 + y^2 \leqslant 4\} = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$$

and the cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ 0 \leqslant z \leqslant 1\} = S^1 \times [0,1] = \{z \in \mathbb{C} : |z| = 1\} \times [0,1]$$

are homeomorphic. An explicit homeomorphism is given by

$$f : A \longrightarrow C, \quad f(z) = (z/|z|,\ |z| - 1).$$

Example 2.5.8. Let X be any set with two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$, and consider the identity map $I_X : X \to X, I(x) = x$. Then

1) $I_X : (X, \mathcal{O}_1) \to (X, \mathcal{O}_2)$ is continuous if and only if $\mathcal{O}_2 \subset \mathcal{O}_1$;
2) $I_X : (X, \mathcal{O}_2) \to (X, \mathcal{O}_1)$ is continuous if and only if $\mathcal{O}_1 \subset \mathcal{O}_2$;
3) $I_X : (X, \mathcal{O}_1) \to (X, \mathcal{O}_2)$ is a homeomorphism if and only if $\mathcal{O}_1 = \mathcal{O}_2$.

For example, if $\mathcal{O}_{\text{discrete}}$ is the discrete topology on X, and $\mathcal{O}_X$ any other topology on X. Then the identity map $I_X : (X, \mathcal{O}_{\text{discrete}}) \to (X, \mathcal{O}_X)$ is always continuous, but unless $\mathcal{O}_X = \mathcal{O}_{\text{discrete}}$, the inverse $I_X : (X, \mathcal{O}_X) \to (X, \mathcal{O}_{\text{discrete}})$ is not continuous.

Example 2.5.9. Consider $\mathbb{R}^2$ with two metrics: for $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$,

$$d_2(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \qquad d_\infty(p_1, p_2) = \max(|x_1 - x_2|, |y_1 - y_2|).$$

Then $\mathbb{R}^2$ has two induced metric topologies. On the other hand, note that the "open disc" $Q_{(a,b)}(r)$ centered at (a, b) and of radius r in $\mathbb{R}^2$ with respect to the metric d_∞ is the open square

$$Q_{(a,b)}(r) = (a - r, a + r) \times (b - r, b + r).$$

As every open disc B defined by d_2 contains an open square (which is an open ball defined by d_∞) and vice versa, the two topologies are the same.

Example 2.5.10. The square $[-1, 1]^2$ and the disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ are homeomorphic. To write down an explicit homeomorphism, consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = \begin{cases} (0, 0), & \text{if } (x, y) = (0, 0), \\ \frac{\|(x,y)\|_2}{\|(x,y)\|_\infty}(x, y), & \text{if } (x, y) \neq (0, 0), \end{cases}$$

where $\|(x, y)\|_2 = \sqrt{x^2 + y^2}$ and $\|(x, y)\|_\infty = \max(|x|, |y|)$. We claim that f is a homeomorphism, and it inverse is given by

$$f^{-1}(x, y) = \begin{cases} (0, 0), & \text{if } (x, y) = (0, 0), \\ \frac{\|(x,y)\|_\infty}{\|(x,y)\|_2}(x, y), & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Indeed, note that f has the property that for any $(x, y) \in \mathbb{R}^2$, $f(x, y)$ and (x, y) lie in the same ray that starts from the origin and passes through (x, y), and

$$\|f(x, y)\|_\infty = \|(x, y)\|_2, \quad \forall\, (x, y) \in \mathbb{R}^2. \tag{2.5}$$

It follows that f is bijective and with inverse map as given. Clearly f is continuous at every $(a, b) \neq (0, 0)$. By (2.5), f is continuous at $(0, 0)$. Thus f is continuous. Similarly, f^{-1} is continuous. Thus f is a homeomorphism.

Note that f maps the (closed) unit disk D_1 centered at $(0, 0)$ to the (closed) unique square Q_1 centered at $(0, 0)$. Thus D_1 and Q_1 are homeomorphic.

We now give introdue the notion of *topological embeddings*.

Definition 2.5.6. A continuous map $f : X \to Y$ is called a *topological embedding* if f is injective and if $f : X \to f(X)$ is homeomorphism, where $f(X)$ has the subspace topology from Y.

Example 2.5.11. As a trivial example, for any subspace X_1 of X, the inclusion map

$$X_1 \hookrightarrow X$$

is a topological embedding. If X has two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$, with $\mathcal{O}_2 \subset \mathcal{O}_1$ but $\mathcal{O}_2 \neq \mathcal{O}_1$, then $I_X : X \to X, x \mapsto x$, is injective and continuous but not a topological embedding.

去掉

local homeomorphism 一定open但不一定closed

2.5.4 Local homeomorphisms

We now introduce the notion of local homeomorphisms which is an essential concept in the theory of manifolds.

Let X and Y be two topological spaces. 这里，函数本身是否连续不重要

Definition 2.5.7. A map $f : X \to Y$ is called a *local homeomorphism* if for each $x \in X$ there exists an open neighborhood U of x in X such that $f(U)$ is open in Y and $f|_U : U \to f(U)$ is a homeomorphism.

Example 2.5.12. By the *inverse function theorem*, if U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is differentiable with invertible differential $D_x f : \mathbb{R}^n \to \mathbb{R}^n$ at every $x \in U$, then f is a local homeomorphism.

如果每个点都可逆，那么就不用连续

Example 2.5.13. The map

$$\mathbb{R} \longrightarrow S^1,\ x \longmapsto e^{2\pi i x},$$

is a local homeomorphism but not a homeomorphism. Indeed, as any continuous function on S^1 must be bounded, there can be no surjective continuous map from S^1 to $\mathbb{R}$.

Lemma 2.5.9. A local homeomorphism is continuous.

Proof. Let $f : X \to Y$ be a local homeomorphism. For each $x \in X$ choose open subsets U_x of X such that $x \in U_x$ and $f|_{U_x} : U_x \longrightarrow f(U_x)$ is continuous. Then

$$X = \bigcup_{x \in X} U_x.$$

Let $V \subset Y$ be any open subset. Then

$$f^{-1}(V) = \bigcup_{x \in X} f^{-1}(V) \cap U_x.$$

For each $x \in X$, one has

$$f^{-1}(V) \cap U_x = \{x' \in U_x : f(x') \in V\} = \left(f|_{U_x}\right)^{-1}(V \cap f(U_x)),$$

so $f^{-1}(V) \cap U_x$ is open in X. Thus $f^{-1}(V)$ is open in X. It follows that f is continuous. $\square$

Corollary 2.5.1. A bijective local homeomorphism is a homeomorphism.

Definition 2.5.8. A topological space X is said to be *locally homeomorphic* to Y if every $x \in X$ has an open neighborhood homeomorphic to an open subset of Y.

Note that if there is a local homeomorphism $f : X \to Y$, then X and Y are locally homeomorphic, but two topological spaces can be locally homeomorphic without there being a local homeomorphism.

We now give more examples.

Example 2.5.14. For any $n \geq 2$, the map $S^1 \to S^1, z \mapsto z^n$ is a local homeomorphism. The number n is called the *winding number* of the map.

Example 2.5.15. For any $n \geq 1$,

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = 1\}$$

is locally homeomorphic to $\mathbb{R}^n$.

2.5.5 Open and closed maps

Definition 2.5.9. Let $f : X \to Y$ be a map between topological spaces. We say that f is *open* if $f(U) \subset Y$ is open for every open $U \subset X$; We say that f is *closed* if $f(U) \subset Y$ is closed for every closed $U \subset X$;

Warnings: Open and closed maps are not necessarily continuous; Continuous maps may or may not be open or closed; Although seemingly natural, the notion of open or closed maps is far less useful in topology then that for continuous maps. We illustrate these by examples.

Example 2.5.16. (1) The map $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ is continuous, closed, but not open: the image of f is $[0, \infty)$, not an open subset of $\mathbb{R}$.

(2) The projection $p : \mathbb{R}^2 \to \mathbb{R}^1, (x, y) \mapsto x$, is continuous, open but not closed: the subset $A = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ of $\mathbb{R}^2$ is closed but $p(A) = \mathbb{R}^1 \backslash \{0\} \subset \mathbb{R}^1$ is not closed.

(3) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two topologies on a set X, and $f = I_X : X \to X$ the identity map. Then I_X is open iff I_X is closed, which is in turn equivalent to $\mathcal{O}_1 \subset \mathcal{O}_2$.

(4) *Open mapping theorem in complex analysis*: every non-constant holomorphic function on a connected open subset U of $\mathbb{C}$ is an open map from U to $\mathbb{C}$;

(5) *Open mapping theorem in functional analysis*: every surjective continuous linear operator between Banach spaces is open.

The following statements can be proved directly and their proofs are left as exercises.

(1) A bijective map is open iff it is closed;

(2) A bijective continuous $f : X \to Y$ is a homeomorphism iff it is open;

(3) A bijective continuous $f : X \to Y$ is a homeomorphism iff it is closed;

(4) Every local homeomorphism is open;

(5) For a product space $X \times Y$, both projections $X \times Y \to X$ and $X \times Y \to Y$ are open;

现在讲这里

Remark. For a quotient space $[X] = X/\sim$, the quotient map $q : X \to [X], x \mapsto [x]$, is in general not an open map, but q maps saturated open subsets of X to open subsets of $[X]$: indeed, this is so because a subset A of X is saturated iff $q^{-1}(q(A)) = A$.

Let $f : X \to Y$ be a continuous map. Define equivalence relation $\underset{f}{\sim}$ on X by

$$x_1 \underset{f}{\sim} x_2 \quad \text{iff} \quad f(x_1) = f(x_2).$$

Let $X/\underset{f}{\sim}$ be the quotient space. We then have well-defined continuous map

$$[f] : \ X/\underset{f}{\sim} \longrightarrow Y, \ \ [x] \longmapsto f(x).$$

If f is surjective, $[f]$ is surjective.

Definition 2.5.10. A surjective continuous map $f : X \to Y$ is called a *quotient map* if $[f] : X/\underset{f}{\sim} \to Y$ is a homeomorphism.

有一个例子 手机、上拍

Lemma 2.5.10. A surjective continuous map $f : X \to Y$ is a *quotient map* if and only if f maps every saturated open subset of X to an open subset of Y.

Proof. Since $[f] : X/\underset{f}{\sim} \to Y$ is bijective and continuous, it is a homeomorphism iff it is open. Since open subset $A \subset X/\underset{f}{\sim}$ are in one to one correspondence with saturated subsets $\widetilde{A}$ of X via

$$A = [f](\widetilde{A}),$$

and $[f](A) = f(\widetilde{A})$, we see that $[f]$ is open iff f maps very saturated open subset of X to an open subset of Y. □

2.6 Compactness

2.6.1 Definition and first examples

We now introduce another topological invariant of a topological space, namely the notion of compactness. You should have seen the term from *Analysis*.

Definition 2.6.1. Let X be a topological space. A collection $\mathcal{F} = \{O_i : i \in \mathcal{I}\}$ of subsets of X is called a *cover* of X if

$$\bigcup_{i \in \mathcal{I}} O_i = X,$$

and we also say that $\mathcal{F}$ covers X and call $\mathcal{F}$ an *open cover* if every $O_i \in \mathcal{F}$ is open in X.

Definition 2.6.2. Let $\mathcal{F}$ be a cover of X. A *subcover* of $\mathcal{F}$ is a subset of $\mathcal{F}$ which also covers X.

$(\mathbb{R}, \mathrm{Zar})$ is compact

discrete topological space is compact if and only if the space is finite

In Zariski Topology, is compact?

Definition 2.6.3. A topological space X is said to be *compact* if every open cover of X has a finite subcover.

Definition 2.6.4. Let Y be any subset of a topological space X.
1) Y is said to be a *compact subset* if Y is compact with the subspace topology.
2) A collection $\mathcal{F} = \{O_i : i \in \mathcal{I}\}$ of open subsets of X such that

$$Y \subset \bigcup_{i \in \mathcal{I}} O_i$$

is called *a cover of Y by open subsets of X*.

If we trace the definition, we see immediate the following equivalent definition of a subset being compact.

Lemma 2.6.1. Let X be a topological space and Y a subset of X. Then Y is compact subset if and only if every cover of Y by open subsets of X has a finite sub-cover.

Example 2.6.1. Any non-empty set X with the trivial topology is compact because there are only two open subsets $\varnothing$ and X.

Example 2.6.2. Let X be a non-empty set equipped with the discrete topology. Then every singleton subset is open and the collection of all singleton subsets is a cover of X. Thus X is compact iff X is a finite set.

Example 2.6.3. It is evident from the definition that every singleton set $\{x\}$ in any topological space X is compact.

Example 2.6.4. Consider $\mathbb{R}$ with the co-finite topology (i.e., the Zariski topology). We claim that every subset A of $\mathbb{R}$ is compact. Indeed, suppose that $\mathcal{F} = \{O_i : i \in \mathcal{I}\}$ is an open cover of A. Pick any O_{i_0} in this cover. Since O_{i_0} is open, it contains all of $\mathbb{R}$ except for finitely many points. Thus it contains all of A except for finitely points, say $\{a_1, \dots, a_k\}$. Since $\mathcal{F}$ covers A, each a_j is contained in some O_{i_j}. Thus $A \subset O_{i_0} \bigcup O_{i_1} \bigcup \cdots \bigcup O_{i_k}$, so $\{O_{i_0}, O_{i_1}, \dots, O_{i_k}\}$ is a finite sub-cover of $\mathcal{F}$.

Lemma 2.6.2. If (X, d) is a metric space, then any compact subset Y of X must be bounded, i.e., there exist $M > 0$ and $x_0 \in X$ such that

$$d(y, x_0) < M, \quad \forall \ y \in Y.$$

Proof. Take any $x_0 \in X$. Then the collection of open balls

$$B_M = \{x \in X : d(x, x_0) < M\}, \quad M \geq 0$$

covers Y, so $Y \subset B_M$ for some M big enough. In particular $\mathbb{R}^n$ is not compact. □

Whenever we talk about $\mathbb{R}^n$ as a topological space, the topology is always understood to be the Euclidean topology unless otherwise specified.

Theorem 2.6.1. The closed interval $[0,1]$ is compact.

Proof. Let $\mathcal{F} = \{O_i : i \in \mathcal{I}\}$ be a family of open subsets of $\mathbb{R}$ covering $[0,1]$. Let

$$X = \{x \in (a,b] : [a,x] \text{ is covered by finitely many members of } \mathcal{F}\}.$$

Since $a \in O_i$ for some i and since O_i is open, there exists $x \in (a,b]$ such that $[a,x] \in O_i$. Thus $X \neq \varnothing$. Let $c = \mathrm{Sup}(X)$. Then $c \leq b$. We claim that $c = b$. Suppose $c < b$. Since $c \in [a,b]$, $c \in O_j$ for some $j \in \mathcal{I}$. Since O_j is open, there exists $\epsilon > 0$ such that

$$c + \epsilon < b \quad \text{and} \quad (c - \epsilon, c + \epsilon] \in O_j.$$

On the other hand, since $c = \mathrm{Sup}(X)$, there exists $x \in X$ such that $c - \epsilon < x < c$. By the assumption that $x \in X$, there exists finitely many $i_1, \ldots, i_n \in \mathcal{I}$ such that

$$[a,x] \subset O_{i_1} \cup \cdots \cup O_{i_n}.$$

Then $[a, c+\epsilon] \subset O_j \cup O_{i_1} \cup \cdots \cup O_{i_n}$, contradicting the assumption that $c = \mathrm{Sup}(X)$. Thus $c = b \in X$. □

The following Haine-Borel theorem for $\mathbb{R}^n$ will be proved in Theorem 2.6.5.

Theorem 2.6.2 (Haine-Borel Theorem)**.** A subset C of $\mathbb{R}^n$ is compact if and only if C is closed and bounded.

Assume Haine-Borel Theorem for $\mathbb{R}^1$. You have the Extreme Value Theorem.

Theorem 2.6.3. (Extreme Value Theorem) A continuous real-valued function f on any compact space X is bounded and attains its bounds, i.e., there exist $a, b \in X$ such that

$$f(a) \leqslant f(x) \leqslant f(b)$$

for any $x \in X$.

Proof. As X is compact and f is continuous, $f(X)$ is a compact subset of $\mathbb{R}$, so $f(X)$ is bounded and closed. Thus f is bounded. Let

$$m = \inf f(X) \quad \text{and} \quad M = \sup f(X).$$

Since $f(X)$ is closed, both m and M are in $f(X)$. Thus there exist $a, b \in X$ such that $m = f(a)$ and $M = f(b)$. □

Example 2.6.5. Assume the Haine-Borel theorem for $\mathbb{R}^n$. Then all closed and bounded subsets of $\mathbb{R}^n$ are compact with respect to the subspace topology. For example, suppose that we have a continuous map

$$f(x) = (f_1(x), \dots, f_m(x)) : \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

Then for any $y_0 \in \mathbb{R}^m$, since the one point set $\{y_0\}$ of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$, the level set

$$f^{-1}(y_0) = \{x \in \mathbb{R}^n : f(x) = y_0\}$$

is a closed subset of $\mathbb{R}^n$. If $f^{-1}(y_0)$ is also bounded, then it is compact. As an example, we know that the spheres in $\mathbb{R}^n$ are all compact.

Example 2.6.6. As another example, consider the set of all $n \times n$ real orthonormal matrices

$$O(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) : A^t A = I_n\},$$

try to do this with matrix norm

where I_n is the $n \times n$ identity matrix. Then $O(\mathbb{R}) \subset \mathbb{R}^{n^2}$ is bounded and closed, so it is compact. Note that $O(n, \mathbb{R})$ is also a group under matrix multiplication. It is an example of a "compact topological group". The same is true for the group $U(n)$ of all $n \times n$ unitary matrices.

2.6.2 Compactness as a topological invariant

We now prove that compactness is a topological invariant.

Proposition 2.6.1. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. If X is compact, then $f(X)$ is compact.

Proof. We already know that f, regarded as a map from X to $f(X)$, is also continuous, where $f(X)$ has the subspace topology from Y. Thus we may assume, without loss of generality, that $f(X) = Y$.

Let $\mathcal{F} = \{O_i : i \in \mathcal{I}\}$ be an open cover of Y. Then for every $x \in X$, $f(x) \in O_i$ for some $i \in \mathcal{I}$. Thus $x \in f^{-1}(O_i)$, and note that $f^{-1}(O_i) \subset X$ is open as f is continuous. Thus $\{f^{-1}(O_i) : i \in \mathcal{I}\}$ is an open cover of X. Since X is compact, there exists $i_1, \dots, i_n \in \mathcal{I}$ such that $\{f^{-1}(O_{i_j}) : j = 1, \dots, n\}$ is a cover of X. Then every $y \in Y$, being $f(x)$ for some $x \in X$, lies in some O_{i_j}. Thus $\{O_{i_j} : j = 1, \dots, n\}$ is a finite sub-cover of $\mathcal{F}$. □

box fails!

Lemma 2.6.3. For any compact topological space X and any equivalence relation $\sim$ on X, the quotient space $X/\sim$ is compact.

Proof. This is a direct consequence of the fact that the image of a compact space under a continuous map is compact. □

example

Example 2.6.7. The projective space $\mathbb{R}P^n = S^n/\mathbb{Z}_2$ is compact.

difficult!

$(X_\lambda)_{\lambda \in I}$ comp $\Rightarrow \overline{\prod_{\lambda \in I}} X_\lambda$ comp

Compact space $\Rightarrow$ Compact subset

Corollary 2.6.1. Compactness is a topological property, i.e., if two topological spaces X and Y are homeomorphic, then X is compact if and only if Y is compact.

Example 2.6.8. The sphere S^n can not be homeomorphic to $\mathbb{R}^n$ because S^n is compact while $\mathbb{R}^n$ is not.

Example 2.6.9. The half closed half open interval $[0,1)$ can not be homeomorphic to the circle S^1 as the former is not compact while the latter is compact.

2.6.3 Products of compact spaces are compact

In this section, we prove that the product of two compact spaces is compact. We first introduce some terms and prepare a lemma.

Let X and Y be two topological spaces and equip $X \times Y$ with the product topology. Call of a subset A of $X \times Y$ *a basic open subset* if $A = U \times V$, where U is an open subset of X and V is an open subset of Y. Recall that a subset A of $X \times Y$ is open if and only if A is a union of basic open subsets.

We now prove the so-called *Tube Lemma.* We explain the word *tube*.

Definition 2.6.5. Let X and Y be two topological spaces. For a subset U of X, $U \times Y$ is called a Y*-tube* in $X \times Y$ with base U. If U is open (resp. closed), $U \times Y$ is called an *open* Y*-tube* (resp. *closed* Y*-tube*) in $X \times Y$. When $U = \{x\}$ for $x \in X$, For $x \in X$, the tube $\{x\} \times Y$ is also called a Y*-slice* in $X \times Y$.

The following Tube Lemma says that when Y is compact, any open subset in $X \times Y$ containing a Y-slice contains an open Y-tube.

Lemma 2.6.4 (Tube Lemma)**.** Let X be a topological space and Y a compact topological space. Let $x \in X$. If O is an open subset of $X \times Y$ containing the slice $\{x\} \times Y$, then there exists an open neighborhood U of x in X such that $U \times Y \subset O$.

The Tube Lemma is a special case of the following *Generalized Tube Lemma.*

Lemma 2.6.5 (Generalized Tube Lemma)**.** Let X and Y be two topological spaces and let $A \subset X$ and $B \subset Y$ be both compact. If O is an open subset of $X \times Y$ containing $A \times B$, then there exists a basic open subset $U \times V$ of $X \times Y$ such that $A \times B \subset U \times V \subset O$.

Proof. Fix $a \in A$. Then for any $b \in B$, since $(a,b) \in O$, there a basic open subset $U_{a,b} \times V_{a,b}$ of $X \times Y$ such that $(a,b) \in U_{a,b} \times V_{a,b} \subset O$. We then have an open cover

$$\mathcal{F}^a = \{V_{a,b} : \ b \in B\}$$

of B. Since B is compact, there exists a finite subset $B(a)$ of B such that $\{V_{a,b} : b \in B(a)\}$ is a cover of B. Let

$$U'_a = \bigcap_{b \in B(a)} U_{a,b} \subset X \quad \text{and} \quad V'_a = \bigcup_{b \in B(a)} V_{a,b} \subset Y.$$

Then U'_a is open in X, V'_a is open in Y, and $B \subset V'_a$. It follows that

$$U'_a \times B \subset U'_a \times V'_a.$$

Consider now the open cover $\mathcal{F}_A = \{U'_a : a \in A\}$ of A. Since A is compact, $\mathcal{F}_A$ contains a finite subcover $\{U'_{a_1}, \dots, U'_{a_m}\}$ of A. Let

$$U = \bigcup_{i=1}^{m} U'_{a_i} \subset X \quad \text{and} \quad V = \bigcap_{i=1}^{m} V'_{a_i} \subset Y.$$

Then U is open in X with $A \subset U$, and V is open in Y with $B \subset V$. Thus $A \times B \subset U \times V$. Furthermore, let $u \in U$ and $v \in V$. Then there exists $1 \le i \le m$ such that $u \in U'_{a_i}$. As

$$v \in V \subset V'_{a_i} = \bigcup_{b \in B(a)} V_{a_i,b},$$

there exists $b_i \in B(a_i)$ such that $v \in V_{a_i,b_i}$. Since $U'_{a_i} \subset U_{a_i,b_i}$, one has

$$(u,v) \in U_{a_i,b_i} \times V_{a_i,b_i} \subset O.$$

We have thus proved that

$$A \times B \subset U \times V \subset O.$$

□

Remark. Note that the Tube Lemma is a special case of the Generalized Tube Lemma, but if we only want to prove the Tube Lemma, one only needs to use the first half of the arguments in the proof of the Generalized Tube Lemma.

Theorem 2.6.4. If X and Y are two compact topological space, the product space $X \times Y$ is also compact.

Proof. Let $\mathcal{F} = \{O_\alpha : \alpha \in I\}$ be an open cover of $X \times Y$. Let $x \in X$. Note that the map $i_x : Y \to \{x\} \times Y, i_x(y) = (x,y)$, is continuous with image $\{x\} \times Y$, so $\{x\} \times Y$ is a compact subset of $X \times Y$. Regarding $\mathcal{F}$ as a cover of $\{x\} \times Y$, one then knows that there exists a finite subset $\mathcal{I}(x)$ of $\mathcal{I}$ such that

$$\{x\} \times Y \subset O^x \overset{\text{def}}{=} \bigcup_{\alpha \in I(x)} O_\alpha.$$

By the Tube Lemma, there exists an open neighborhood U_x of x in X such that $U_x \times Y \subset O^x$. Consider now the open cover $\{U_x : x \in X\}$ of X. Since X is compact, there exist $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n U_{x_i}$. One then has

$$X \times Y = (\bigcup_{i=1}^{n} U_{x_i}) \times Y = \bigcup_{i=1}^{n} (U_{x_i} \times Y) \subset \bigcup_{i=1}^{n} O^{x_i} = \bigcup_{i=1}^{n} \bigcup_{\alpha \in I(x_i)} O_\alpha.$$

Let $I' = \bigcup_{i=1}^n I(x_i)$. Then I' is a finite subset of I and

$$X \times Y \subset \bigcup_{\alpha \in I'} O_\alpha,$$

so $X \times Y = \bigcup_{\alpha \in I'} O_\alpha$, and thus $\{O_\alpha : \alpha \in I'\}$ is a finite subcover of $\mathcal{F}$. □

2.6.4 Further properties of compact spaces

In this section, we first discuss some relations between compactness and closedness of subsets. We will then prove the so called Closed Map Lemma, so named as in Lemma 4.50 of John Lee's book *Introduction to topological manifolds*, 2nd edition.

Lemma 2.6.6. A closed subset of a compact topological space is compact.

Proof. Let X be a compact topological space and let $Y \subset X$ be a closed subset. To show that Y is compact, let $\mathcal{F} = \{O_i : i \in \mathcal{I}\}$ be an open cover of Y, i.e., each O_i is an open subset of X and $Y \subset \bigcup_{i \in \mathcal{I}} O_i$. Adding the open subset $X \backslash Y$ of X to $\mathcal{F}$, we get an open cover of X. Since X is compact, there exists $i_1, \ldots, i_n$ such that

$$X = (X \backslash Y) \cup \bigcup_{j=1}^{n} O_{i_j}.$$

It follows that $Y \subset \bigcup_{j=1}^{n} O_{i_j}$. $\square$

Proposition 2.6.2. Let X be a Hausdorff space and let Y be a compact subset of X. Then for any $x \in X \backslash Y$, there exist open subsets U and V of X such that $U \cap V = \varnothing$, $Y \subset U$ and $x \in V$.

Proof. Let $x \in X \backslash Y$. For every $y \in Y$, there exists an open subset U_y of X containing y and an open subset V_y of X containing x such that $U_y \cap V_y = \varnothing$. Now $\{U_y : y \in Y\}$ is an open cover of Y, so it has a finite subcover $\{U_{y_1}, \ldots, U_{y_n}\}$. Take $U = \bigcup_{i=1}^{n} U_{y_i}$ and $V = \bigcap_{i=1}^{n} V_{y_i}$. Then U and V are open in X, $U \cap V = \varnothing$, $Y \subset U$ and $x \in V$, as required. $\square$

Corollary 2.6.2. A compact subset of a Hausdorff space is closed.

Proof. Let X be a Hausdorff space and let Y be a compact subspace of X. Proposition 2.6.2 implies that $X \backslash Y$ is open in X, so Y is closed in X. $\square$

Note how the Hausdorff property is used in the proof of the proposition. Can we remove the Hausdorff condition? The answer is "no", as we see from Example 2.6.4: in $\mathbb{R}$ with the co-finite topology, every subset of $\mathbb{R}$ is compact.

We can now give a proof of the Haine-Borel theorem for $\mathbb{R}^n$.

Theorem 2.6.5 (Haine-Borel Theorem for $\mathbb{R}^n$)**.** A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Let $A \subset \mathbb{R}^n$ be compact. Since $\mathbb{R}^n$ is Hausdorff, by Corollary 2.6.2, A is closed. The collection of all open balls of radius n for all integers $n \geq 1$ is a cover of A, so it has a finite subcover, so A is bounded. See also Lemma 2.6.2.

Assume now that $A \subset \mathbb{R}^n$ is closed and bounded. Then there exists a closed interval $[a, b]$ in $\mathbb{R}$ such that $A \subset [a, b]^n$. By Theorem 2.6.4, $[a, b]^n$ is compact. Since A is now a closed subset of the compact set $[a, b]^n$, A is compact by Lemma 2.6.6. $\square$

Theorem 2.6.6. (Closed Map Lemma). Suppose $f : X \to Y$ is continuous map, where X is compact and Y is Hausdorff space. Then

(a) f is a closed map.
(b) If f is injective, it is a closed topological embedding.
(c) If f is bijective, it is a homeomorphism.

Proof. (a) Let $C \subset X$ be closed. As X is compact, we know by Lemma 2.6.6 that C is compact. By Proposition 2.6.1, $f(C)$ is compact in Y. Since Y is Hausdorff, by Corollary 2.6.2, $f(C) \subset Y$ is closed. Thus f is a closed map.

(b) follows from (a) and the definition of closed topological embedding.

(d) follows from the definitions. □

One important application of the Closed Map Lemma is that it can be used to define embeddings of quotient spaces into known spaces such as $\mathbb{R}^n$.

Recall again that if $f : X \to Y$ is a continuous map, and if $X/\underset{f}{\sim}$ is the quotient space defined by the equivalence relation

$$x_1 \underset{f}{\sim} x_2 \quad \text{iff} \quad f(x_1) = f(x_2),$$

then we have well-defined continuous map

$$[f] : \; X/\underset{f}{\sim} \longrightarrow Y, \; [x] \longmapsto f(x),$$

and f is called a quotient map if f is surjective and if the bijective map $[f] : X/\underset{f}{\sim} \to Y$ is a homeomorphism.

Proposition 2.6.3. Let $f : X \to Y$ be a continuous map, and suppose that X is compact and Y is Hausdorff.

1) $[f] : X/\underset{f}{\sim} \longrightarrow Y$ is an embedding.
2) If f is surjective then f is a quotient map.

Proof. 1) The map $[f] : X/\underset{f}{\sim} \to Y$, being an injective continuous map from a compact space to a Hausdorff space, is an embedding by the Closed Map Lemma.

2) follows from 1). □

Example 2.6.10. By Proposition 2.6.3., the map $f : S^2 \to \mathbb{R}^4$ given by

$$f(x, y, z) = (x^2 - y^2, \, xy, \, xz, \, yz)$$

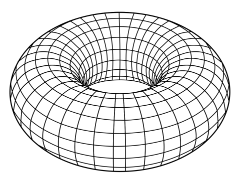
induces an embedding of the projective space $\mathbb{RP}^2$ to $\mathbb{R}^4$.

2.6.5 The torus, the Mobius strip, and the Klein bottle

We now give more examples of Proposition 2.6.3.

Example 2.6.11. The *torus* is defined as the quotient space of $X = [0,1] \times [0,1]$ by the partition consisting of

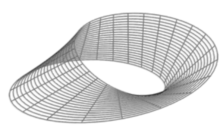
1) the four point set $\{(0,0),(0,1),(1,0),(1,1)\}$;
2) the two point set $\{(x,0),(x,1)\}$ for each $0 < x < 1$;
3) the two point set $\{(0,y),(1,y)\}$ for each $0 < y < 1$;
4) a singleton $\{(x,y)\}$ for each (x,y) such that $0 < x, y < 1$.



By Proposition 2.6.3, the map $f : [0,1] \times [0,1] \to \mathbb{C}^2 : (x,y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$ is an embedding of the torus.

Example 2.6.12. The *Mobius strip* is defined as the quotient space of $[0,1] \times [0,1]$ by the partition of X consisting of

1) the two point set $\{(0,y),(1,1-y)\}$ for each $0 \leqslant y \leqslant 1$;
2) the singleton $\{(x,y)\}$ for each (x,y) such that $0 < x < 1$ and $0 \leqslant y \leqslant 1$.



By Proposition 2.6.3, the map $f : [0,2\pi] \times [-1,1] \to \mathbb{R}^3$ given by

$$f(s,t) = \left(\left(1 + \frac{t}{2}\cos\left(\frac{s}{2}\right)\right)\cos s,\ \left(1 + \frac{t}{2}\cos\left(\frac{s}{2}\right)\right)\sin s,\ \frac{t}{2}\sin\left(\frac{s}{2}\right)\right).$$

induces an embedding of the Mobius strip to $\mathbb{R}^3$.

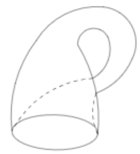
Example 2.6.13. The *Klein bottle* is defined as the quotient space of $[0,1] \times [0,1]$ by the partition of X consisting of

1) the 4 point set $\{(0,0),(0,1),(1,0),(1,1)\}$;
2) the two point set $\{(x,0),(x,1)\}$ for each $0 < x < 1$;
3) the two point set $\{(0,y),(1,1-y)\}$ for each $0 < y < 1$;
4) the singleton $\{(x,y)\}$ for each (x,y) such that $0 < x, y < 1$.

By Proposition 2.6.3, the map $f : [0,2\pi] \times [0,\pi] \to \mathbb{R}^4$ given by

$$f(s,t) = ((2 + \cos s)\cos(2t),\ (2 + \cos s)\sin(2t),\ \sin s \cos t,\ \sin s \sin t)$$

induces an embedding of the Klein bottle to $\mathbb{R}^4$.



2.6.6 Other compactness

Limit point compactness and sequential compactness. In Calculus or in Analysis, we speak a lot about convergent sequences. We can define this notion in any topological space.

Definition 2.6.6. A topological space X is said to be *limit point compact* if every infinite subset of X has a limit point.

Lemma 2.6.7. A compact space is limit point compact.

Proof. Let A be an infinite subset of a compact space X. Suppose that A has no limit point. Then for every $x \in X$ there exists an open neighborhood U_x of x in X such that $|A \cap U_x| \le 1$. Since X is compact, there are exist $x_1, \dots, x_n$ such that $X = \bigcup_{i=1}^n U_{x_i}$. It follows that $|A| \le n$, a contradiction. Thus A must have a limit point. $\square$

Recall the following definition.

Definition 2.6.7. Let X be a topological space and let $\{x_n\}_{n \geqslant 0}$ be a sequence of points in X. We say that x is a *limit* of $\{x_n\}$ or the sequence *converges* to x if for any open set U containing x, there exists $N \in \mathbb{N}$ such that $x_k \in U$ for all $k \geqslant N$.

Example 2.6.14. If X is a Hausdorff space, then every sequence in X can have at most one limit point.

Definition 2.6.8. Let X be a topological space. If every sequence in X has a convergent sub-sequence, then X is said to be *sequentially compact.*

If (X, d) is a metric space and $\{x_n : n \in \mathbb{N}\} \subset X$, one sees from the definition that

$$\lim_{n \to \infty} x_n = x_0 \quad \text{iff} \quad \lim_{n \to \infty} d(x_n, x_0) = 0.$$

Proposition 2.6.4. Let X be a metric space. Then the following are equivalent:
1) X is compact;
2) X is limit point compact;
3) X is sequentially compact.

We have proved that 1) implies 2) for any compact spaces (not necessarily metric spaces). The proof that 3) implies 1) is a bit involved and we will not discuss here. In the following, we prove that 2) implies 3).

Lemma 2.6.8. A metric space that is limit point compact is also sequentially compact.

Proof. Let (X, d) be a metric space that is limit point compact. Let $s : \mathbb{N} \to X, n \to x_n$, be a sequence in X. If there exists $x \in X$ such that

$$\{n \in \mathbb{N} : x_n = x\}$$

is an infinite set, then we clearly have a sub-sequence of s, namely a constant function $\mathbb{N} \to X$ taking the value x only, that converges to x. Otherwise,

$$A := s(\mathbb{N}) = \{x_n : n \in \mathbb{N}\}$$

must be an infinite set, so by assumption it has a limit point $x_0 \in X$. For $\epsilon > 0$, let

$$B(\epsilon) = \{x \in X : d(x, x_0) < \epsilon\} \quad \text{and} \quad A(\epsilon) = B(\epsilon) \cap (A \backslash \{x_0\}).$$

By the definition of limit point, $A(\epsilon) \neq \varnothing$ for every $\epsilon > 0$. If there exists $\epsilon > 0$ such that $A(\epsilon)$ is a finite set, then for any $\epsilon' > 0$ such that $\epsilon' < \epsilon$ and

$$\epsilon' < \min\{d(x_0, x) : x \in A(\epsilon)\},$$

the set $A(\epsilon') = \varnothing$, which is a contradiction. Thus $A(\epsilon)$ is a infinite set for every ϵ. Take now $\epsilon = 1$. Then there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} \in A(1)$. Once x_{n_1} is chosen, as $A(1/2)$ is an infinite set, there exists $n_2 > n_1$ such that $x_{n_2} \in A(1/2)$. In general, suppose that $k \in \mathbb{N}$, $k \geq 1$, and $n_1 < n_2 < \cdots < n_k$ have been chosen such that $x_{n_j} \in A(1/j)$ for $1 \leq j \leq k$, since $A(1/(k+1))$ is an infinite set, there exists $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in A(1/(k+1))$. The sub-sequence $\{x_{n_k} : k \in \mathbb{N}\}$ chosen this way clearly converges to x_0. We have thus proved X is sequentially compact. □

Recall that in a metric space X, the diameter of a subset A of X is

$$d(A) = \sup\{d(x, y) : x, y \in A\}.$$

The following Lebesgue's lemma will be used in later sections when study fundamental groups.

Lemma 2.6.9. [Lebesgue's Lemma] Let X be a compact metric space, and let $\mathcal{F}$ be an open cover of X. Then there exists a real number $\delta > 0$, called a Lebesgue number of $\mathcal{F}$, such that any subset of X of diameter less than δ is contained in some member of $\mathcal{F}$.

Proof. We prove by contradiction. Suppose not. Then taking $\delta = 1/n$ for each positive integer n, there exists a subset A_n of X with $d(A_n) < 1/n$ and A_n is not contained in any member of $\mathcal{F}$. Choose $x_n \in A_n$ for each n and consider the set $\mathbf{x} = \{x_n : n \in \mathbb{Z}_{>0}\}$. If $\mathbf{x}$ is a finite set, let p be such that $x_n = p$ for infinitely many values of n. If $\mathbf{x}$ is an infinite set, it must have a limit point since X is compact and thus limit point compact, and let p be such a limit point. Let $O \in \mathcal{F}$ be such that $p \in O$. Choose $\epsilon > 0$ such that $B(p, \epsilon) \subset O$ and choose N big enough such that $d(A_N) < \epsilon/2$ and $d(x_N, p) < \epsilon$. Then for any $x \in A_n$,

$$d(x, p) \leq d(x, x_N) + d(x_N, p) \leq d(A_N) + \epsilon < \epsilon,$$

and thus $x \in B(p, \epsilon) \subset O$. Hence $A_N \subset O$, contradicting to the choice of A_N. □

Local compactness. Recall that if X is a topological space and $x \in X$, a neighborhood of x in X is by definition any subset$N$ of X whose interior contains x.

Definition 2.6.9. Let X be any topological space and let $x \in X$. By a *compact neighborhood* of x in X we mean a neighborhood of x in X that is compact, i.e., a compact subset K of X such that there exists an open subset U of X satisfying $x \in U \subset K$.

Definition 2.6.10. A topological space X is said to be *locally compact* if every $x \in X$ has a compact neighborhood.

Clearly a compact space is locally compact.

Lemma 2.6.10. If X is a Hausdorff space, then X is locally compact if and only if every x in X has an open neighborhood U such that $\overline{U}$ is compact.

Proof. If U is an open neighborhood of x in X such that $\overline{U}$ is compact, then $\overline{U}$ is a compact neighborhood o x.

Assume that X is locally compact, and let $x \in X$. Then there exists a compact subset K of X such that $x \in \mathring{K}$. Let $U = \mathring{K}$. Then U is an open neighborhood of x in X. Since X is Hausdorff and K is compact, we know that $K \subset X$ is closed. Since $U \subset K$, we have $\overline{U} \subset \overline{K} = K$. Being a closed subset of a compact set, $\overline{U}$ is compact. □

2.6.7 Proper maps

Recall that the Closed Map Lemma says that if X is compact and Y is Hausdorff, then every continuous map $f : X \to Y$ is a closed map. A consequence of this statement then says that an injective continuous map from a compact space to a Hausdorff space is a (closed) embedding.

In many applications, we can not assume that X is compact, but we would still want to know whether a map $f : X \to Y$ is an embedding. We now see that the compactness of X can be replaced some other conditions.

Definition 2.6.11. For any two topological spaces X and Y, a map $f : X \to Y$ is said to be *proper* if for every compact subset K of Y, $f^{-1}(K)$ is a compact subset of X.

Lemma 2.6.11. If X is compact and Y is Hausdorff, then every continuous map $f : X \to Y$ is proper.

Proof. Let $K \subset Y$ be compact. Since Y is Hausdorff, $K \subset Y$ is closed. Since $f : X \to Y$ is continuous, $f^{-1}(K) \subset X$ is closed. Since X is compact, $f^{-1}(K)$ is compact. □

Proposition 2.6.5. (The Proper Map Lemma) Let X be any topological space and let Y be a locally compact Hausdorff space. If $f : X \to Y$ is continuous and proper, then f is a closed map.

Proof. Let $C \subset X$ be closed. We need to show that $f(C)$ is closed in Y. Let $y \in Y \backslash f(C)$ be arbitrary. Since Y is Hausdorff, by Lemma 2.6.10 there exists an open neighborhood U of y in Y such that $\overline{U}$ is compact. Since f is proper, $f^{-1}(\overline{U})$ is a compact subset of X. Let $C_1 = C \cap f^{-1}(\overline{U})$. Then C_1 is a closed subset of $f^{-1}(\overline{U})$. Since closed subsets of compact sets are compact, C_1 is compact. Since f is continuous, $f(C_1) \subset Y$ is compact. Since compact subsets of Hausdorff spaces are closed, $f(C_1) \subset Y$ is closed. Let $V = U \backslash f(C_1) \subset Y$. Then V is an open neighborhood of y in Y. Note, on the other hand, that

$$f(C_1) = f(C \cap f^{-1}(\overline{U}) = \{f(c) : c \in C, f(c) \in \overline{U}\} = f(C) \cap \overline{U},$$

so $V = U \backslash (f(C) \cap \overline{U}) = U \backslash f(C)$. Thus $V \cap f(C) = \varnothing$. □

Corollary 2.6.3. An injective proper continuous map from any topological space to a locally compact Hausdorff space is a closed embedding.

2.7 Connectedness

2.7.1 Connected spaces and examples

Intuitively, a topological space is said to be connected *if it does not have more than one piece*. Here is the precise definition.

Definition 2.7.1. A topological space is said to be *connected* if it is not the disjoint union of two non-empty open subsets. In other words, if $X = U \cup V$, where both U and V are open and $U \cap V = \varnothing$, then either $U = \varnothing$ or $V = \varnothing$;

By definition, the empty set is connected.

Let's re-formulate. Given a topological space and a subset A of X, we set

$$A^c = X \backslash A.$$

Proposition 2.7.1. Let X be a topological space. Then the following are equivalent.

1) X is connected;
2) The only subsets in X which are both open and closed are $\varnothing$ and X;
3) For any nonempty subsets A and B of X

$$A \cup B = X \implies \text{either } \overline{A} \cap B \neq \varnothing \text{ or } A \cap \overline{B} \neq \varnothing.$$

Proof. Since $X = U \cup U^c$ for any subset U of X, 1) and 2) are equivalent.

2) $\implies$ 3): Assume that $A \subset X$ and $B \subset X$ are such that $A \cup B = X$. Then

$$A^c \subset B \quad \text{and} \quad B^c \subset A.$$

If $\overline{A} \cap B = \varnothing$, then $B \subset \overline{A}^c$, and it follows from $A^c \subset B$ that

$$A^c \subset B \subset \overline{A}^c,$$

so $A^c = B = \overline{A}^c$ because $\overline{A}^c \subset A^c$, and thus B is open and A is closed. Thus if both $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$, then B is both open and closed. If $B \neq \varnothing$, then $B = X$ so $A = B^c = \varnothing$.

3) $\implies$ 2): Suppose that $A \subset X$ is both open and closed and $A \neq \varnothing$ and $A \neq X$. Let $B = A^c$. Then $B \neq \varnothing$ and

$$\overline{A} \cap B = A \cap B = \varnothing \quad \text{and} \quad A \cap \overline{B} = A \cap B = \varnothing,$$

contradicting 3). So either $A = \varnothing$ or $A = X$. □

As always, we make the following definition of connected subsets.

Definition 2.7.2. A subset X_1 of a topological space is said to be connected if X_1 is connected with the subspace topology.

Example 2.7.1. The subset $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$ is not connected.

Example 2.7.2. The subset $\mathbb{Q}$ of $\mathbb{R}$ is not connected:

$$\mathbb{Q} = (\mathbb{Q} \cap (-\infty, \pi)) \cup (\mathbb{Q} \cap (\pi, \infty)).$$

Exercise 2.7.1. Let X be a topological space and let Y be a connected subspace of X. Suppose U and V are disjoint nonempty open sets such that $U \cup V = X$. Then $Y \subset U$ or $Y \subset V$.

We now prove that connectedness is a topological property.

Lemma 2.7.1. The continuous image of a connected space is connected.

Proof. Let $f : X \to Y$ be a continuous map and assume that X is connected. We need to show that $f(X)$ is a connected subset of Y. As f regarded as a map from X to $f(X)$ is also continuous, where $f(X)$ has the subspace topology form Y, we may assume that $f(X) = Y$.

Let $A \subset Y$ be both open and closed. Then $f^{-1}(A) \subset X$ is both open and closed, so either $f^{-1}(A) = \varnothing$ which implies that $A = \varnothing$, or $f^{-1}(A) = X$ which implies that $A = Y$. □

Corollary 2.7.1. Connectedness is a topological property.

We prove some more lemmas before looking at examples.

Lemma 2.7.2. Let X be a topological space and $Z \subset X$. If Z is connected and dense in X, then X is connected.

Proof. Let A be a non-empty subset of X that is both open and closed. As Z is dense in X, $Z \cap A \neq \varnothing$, so $Z \cap A$ is non-empty subset of Z that is both open and closed in Z. Since Z is connected, $Z \cap A = Z$, i.e., $Z \subset A$, and thus

$$X = \overline{Z} \subset \overline{A} = A,$$

so $X = A$. □

Corollary 2.7.2. If Z is connected subset of a topological space X, then any subset Y of X satisfying $Z \subset Y \subset \overline{Z}$ is connected. In particular, $\overline{Z}$ is connected.

Proof. One first check that for any $A \subset B \subset X$, the closure of A in B is equal to $B \cap \overline{A}$, where $\overline{A}$ is the closure of A in X. Applying to the case of $Z \subset Y$, the closure of Z in Y is $Y \cap \overline{Z} = Y$, so Z is dense in Y. Thus Y is connected by Lemma 2.7.2. □

We now want to prove that non-empty connected subsets of $\mathbb{R}$ are precisely all the intervals. Recall that a non-empty subset I of $\mathbb{R}$ is called a *finite interval* if there exist $a, b \in \mathbb{R}$ and $a < b$ such that I is one of

$$(a, b), \quad [a, b), \quad [a, b], \quad (a, b],$$

and I is called an infinite interval in $\mathbb{R}$ if there exist $a \in \mathbb{R}$ such that I is one of

$$(-\infty, a), \quad (-\infty, a], \quad [a, +\infty), \quad (a, +\infty).$$

A non-empty subset I is called an interval in $\mathbb{R}$ if it is either a finite interval or an infinite interval. Note that the closure of an interval I in $\mathbb{R}$ is the interval obtained by adding to I its infimum and supremum.

We will use the following characterization of intervals.

Lemma 2.7.3. A non-empty subset I of $\mathbb{R}$ is an interval in $\mathbb{R}$ if and only if

$$x, y \in I \text{ and } x < y \implies [x, y] \subset I. \tag{2.6}$$

Proof. It is clear from the definitions of intervals that they all have the property in (2.6). Conversely, suppose that $I \subset \mathbb{R}$ has the property in (2.6), and let

$$a = \mathrm{Inf}(I) \quad \text{and} \quad b = \mathrm{Sup}(I).$$

Then $I \subset \overline{(a,b)}$. Conversely, if $z \in (a,b)$, then by the definitions of infimum and supremum, there exist $x, y \in I$ such that $a < x < z < y < b$ so $z \in I$. Thus

$$(a,b) \subset I \subset \overline{(a,b)},$$

and hence I is an interval. $\square$

Lemma 2.7.4. If a non-empty $X \subset \mathbb{R}$ is not an interval, then X is not connected.

Proof. By Lemma 2.7.3, there exist $x, y \in X$ and $z \in \mathbb{R}$ such that $x < z < y$ and $z \notin X$. Let

$$A = X \cap (-\infty, z) \quad \text{and} \quad B = X \cap (z, +\infty).$$

Then both A and B are non-empty open subsets of X with $A \cap B = \varnothing$ and $A \cup B = X$, so X is not connected. $\square$

Lemma 2.7.5. Intervals in $\mathbb{R}$ are connected.

Proof. By Corollary 2.7.2, we just need to show that any open interval (finite or infinite) is connected. Since every open interval is homeomorphic to $\mathbb{R}$, we only need to show that $\mathbb{R}$ is connected.

Assume that $\mathbb{R} = A \cup B$, where both A and B are both non-empty open subsets of $\mathbb{R}$ and $A \cap B = \varnothing$. Let $a \in A$ and $b \in B$ and we may assume that $a < b$. Let

$$X = A \cap (-\infty, b),$$

so $X \neq \varnothing$ and bounded. Let $s = \mathrm{Sup}(X)$. By the definition of supremum, $s \leq b$ and $s \in \overline{A} = A$. As $b \notin A$, one has $s < b$ and thus $(s,b) \subset B$, which then implies that $s \in \overline{B} = B$. This shows that $s \in A \cap B$, a contradiction. $\square$

Corollary 2.7.3. A non-empty subset of $\mathbb{R}$ is connected if and only if it is an interval.

Example 2.7.3. The two intervals $(0,1]$ and $(0,1)$ of $\mathbb{R}$ are not homeomorphic (each with the subspace topology). Indeed, suppose that $f : (0,1] \to (0,1)$ is a homeomorphism and let $a = f(1)$. Then

$$f_1 : (0,1) \longrightarrow (0,1) \backslash \{a\} = (0,a) \cup (a,1)$$

is a homeomorphism, which is not possible, as connectedness is preserved under homeomorphisms and we know that $(0,1)$ is connected while $(0,a) \cup (a,1)$ is not.

Theorem 2.7.1. (Intermediate value theorem) Let X be a topological connected space and $f : X \to \mathbb{R}$ a continuous function on X. If $a, b \in X$ and $r \in \mathbb{R}$ such that $f(a) < r < f(b)$, then there exists $c \in X$ such that $f(c) = r$.

Proof. By Lemma 2.7.1, $f(X)$ is a connected subset of $\mathbb{R}$, so $f(X)$ is an interval. By Lemma 2.7.5, $r \in f(X)$. $\square$

internal $(-\infty, 0) \cup (0, +\infty)$ is connected in Zariski Topology

2.7.2 Further properties and examples

Lemma 2.7.6. Let X be a topological space and let $\{Y_\alpha : \alpha \in \mathcal{I}\}$ be a family of connected subsets of X such that
1) $X = \bigcup_{\alpha \in \mathcal{I}} Y_\alpha$;
2) $Y_\alpha \cap Y_{\alpha'} \neq \varnothing$ for all $\alpha, \alpha' \in \mathcal{I}$.

Then X is connected.

Proof. Let A be a subset of X that is both open and closed. We prove that $A = \varnothing$ or $A = X$. Suppose that $A \neq \varnothing$, and let $x \in A$. By Condition 1), $x \in Y_\alpha$ for some $\alpha \in \mathcal{I}$, so $Y_\alpha \cap A$ is a non-empty subset of Y_α which is both open and closed in Y_α. Since Y_α is connected, $Y_\alpha \cap A = Y_\alpha$, so $Y_\alpha \subset A$. Similarly, if $A \neq X$, then $A^c \neq \varnothing$, so there exists $\alpha' \in \mathcal{I}$ such that $Y_{\alpha'} \subset A^c$. It follows that $Y_\alpha \cap Y_{\alpha'} = \varnothing$, contradicting Condition 2). Thus $A = \varnothing$ or $A = X$. It follows that X is connected. □

Theorem 2.7.2. Let X and Y be two topological spaces. Then $X \times Y$ is connected if and only if both X and Y are connected.

Proof. If $X \times Y$ is connected, then X is connected because it is the image of the projection map from $X \times Y$ which is continuous. Similarly Y is connected.

Assume now both X and Y are connected. For $(x, y) \in X \times Y$, let

$$C_{x,y} = (X \times \{y\}) \cup (\{x\} \times Y) \subset X \times Y.$$

Note that $X \times \{y\}$ is connected because it is homeomorphic to X. Similarly, $\{x\} \times Y$ is connected. Furthermore, $(X \times \{y\}) \cap (\{x\} \times Y) \neq \varnothing$ as it contains the point (x, y). By Lemma 2.7.6, $C_{x,y}$ is connected. Moreover, for any $(x', y') \in (X \times Y, C_{x,y} \cap C_{x',y'}$ is non-empty since it contains (x, y'). By Lemma 2.7.6 again,

$$X \times Y = \bigcup_{(x,y) \in X \times Y} C_{x,y}$$

is connected. □

Example 2.7.4. By Theorem 2.7.2, $\mathbb{R}^n$ is connected. Consider the unit sphere S^n in $\mathbb{R}^{n+1}$. Since $S^n \backslash \{N\}$ is homeomorphic to $\mathbb{R}^n$ via the stereographic projection, where $N = (0, \ldots, 0, 1)$, we know that $S^n \backslash \{N\}$ is connected. Since $S^n \backslash \{N\}$ is dense in S^n, we know that S^n is connected. Furthermore, the n-torus

$$T^n = S^1 \times S^1 \times \cdots \times S^1$$

is connected, again by Theorem 2.7.2.

Definition 2.7.3. Let X be a topological space. A *connected component* of X is a maximal connected subset of X, i.e., a connected subset A such that if $A \subset B$ and B is connected, then $A = B$.

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Proposition 2.7.2. Let X be any topological space.

1) Two connected components of X are either identical or are disjoint;
2) For each $x \in X$ there is a unique connected component of X containing x.
3) All connected components of X are closed in X.
4) If X has finitely many connected components, then every connected component is open.

Proof. 1) If C_1 and C_2 are two connected components and $C_1 \cap C_2 \neq \varnothing$, then $C_1 \cup C_2$ is again connected by Lemma 2.7.6, the $C_1 = C_2 = C_1 \cup C_2$.

2) Let $x \in X$. Obviously the singleton set $\{x\}$ is a connected subset in X. Let $\mathcal{C}_x$ be the collection of all connected subsets containing x, and let

$$C_x = \bigcup_{A \in \mathcal{C}_x} A.$$

By Lemma 2.7.6, C_x is connected. If C is any connected subset of X such that $C_x \subset C$, then $x \in C$, so $C \subset C_x$, so $C = C_x$. This shows that C_x is a maximal connected subset of X.

3) Suppose now that C is an arbitrary connected component of X. By Corollary 2.7.2, $\overline{C}$ is connected. As $C \subset \overline{C}$ and C is maximal, $C = \overline{C}$.

4) Suppose that X has finitely many connected components $C_1, \ldots, c_n$. Then $X = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_n$ by 2). Thus $C_i = X \backslash (C_1 \sqcup \cdots \sqcup C_{i=1} \sqcup C_{i+1} \sqcup C_n)$ is open. ☐

Example 2.7.5. Consider $\mathbb{Q}$ with the subspace topology of $\mathbb{R}$. If A is any subset of $\mathbb{Q}$ containing more than two points, by picking $a, b \in \mathbb{Q}$, $a < b$ and any irrational number c such that $a < c < b$, we have

$$A = (A \cap (-\infty, c)) \cup (A \cap (c, +\infty))$$

so A is not connected. Thus the connected components of $\mathbb{Q}$ are exactly the singletons.

A topological space in which every singleton is a connected component is said to be *totally disconnected*

Definition 2.7.4. A topological space X is said to be *locally connected* if every $x \in X$ has a neighborhood U containing a connected neighborhood V of x.

Example 2.7.6. The proof of the following statements are straightforward.

1) Connected spaces are locally connected;
2) Locally connectedness is a topological property;
3) $\mathbb{R}^n$ is locally connected;
4) Every topological manifold is locally connected;
5) $A = \{0\} \cup \{1/n : n = 1, 2, \ldots\}$ with subspace topology from $\mathbb{R}$ is not locally connected.

Sim, locally path connected.

$\forall x \in X, \forall U_x \in \mathcal{U}_x, \exists$ connected $V_x \subseteq U_x$

2.7.3 Path-connectedness

Definition 2.7.5. Let X be a topological space and let $x, y \in X$. A *path* from x to y is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We call x the *initial point* and y the *terminal point* of the path.

Definition 2.7.6. Let X be a topological space. We say that X is *path-connected* if for any $x, y \in X$, there exists a path from x to y.

Example 2.7.7. Consider the topological space $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$ and $r > 0$. Then the open ball $B(a, r)$ is path-connected. Recall that $X \subset \mathbb{R}^n$ is said to be convex if for any $x, y \in X$, the line segment

$$\{x + \lambda(y - x) : \lambda \in [0,1]\}$$

connecting x and y is contained in X. Thus any convex subset of $\mathbb{R}^n$ is path-connected.

Example 2.7.8. The sphere S^n is path-connected.

Exercise 2.7.2. Path-connectedness is a topological property.

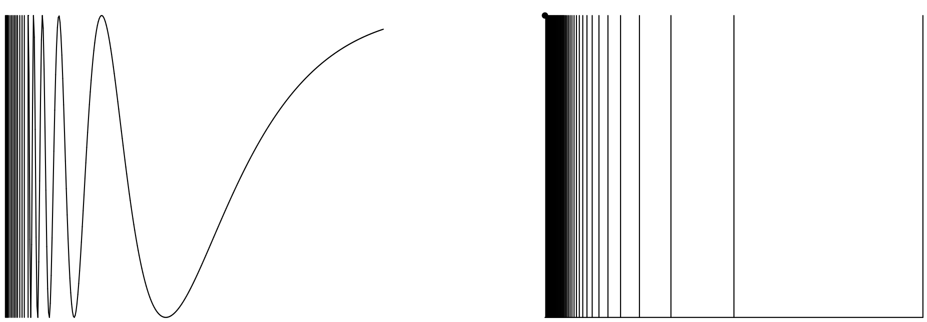
Proposition 2.7.3. Let X be a topological space. If X is path-connected, then it is connected.

Proof. Let X be path-connected, and let A be a non-empty subset of X that is both open and closed in X. Suppose that $A \neq X$. Choose $x \in A$ and $y \in X \backslash A$. Then there is a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(A)$ is a non-empty subset of $[0,1]$ which, by the continuity of γ, is both open and closed in $[0,1]$. Thus $\gamma^{-1}(A) = [0,1]$, contradicting that fact that $\gamma(1) = y \notin A$. Thus $A = X$. Hence X is connected. $\square$

Example 2.7.9. Consider the topological space $\mathbb{R}^2$. The *topologist's sine curve*

$$\left\{\left(x, \sin\frac{\pi}{x}\right) : 0 < x \leqslant 1\right\} \cup \{(0, y) : -1 \leqslant y \leqslant 1\}$$

is a connected subspace but not path-connected. See Armstrong's book Page 62 for a proof.



locally $\mathbb{R}^n$ —— $\forall x, \exists U_x, U_x \cong \mathbb{R}^n$

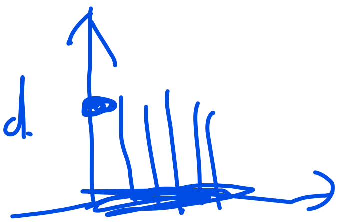
locally $\mathbb{R}^n$ $\Longrightarrow$ locally path connected.

topological manifold.

Example 2.7.10. Consider the topological space $\mathbb{R}^2$. The *deleted comb space*

$$\{(x,y):x=0,y=1\}\cup\{(x,y):0\leqslant x\leqslant 1,y=0\}\cup\left\{(x,y):x=\frac{1}{n},0\leqslant y\leqslant 1,n\in\mathbb{N}\right\}$$

is a connected subspace but not path-connected.



connected + connected $\Rightarrow$ connected

2.8 Adjunction spaces

2.8.1 Definition and examples

Recall that the *disjoint union* of a family of topological spaces $\{X_\alpha : \alpha\in\mathcal{A}\}$ is the (set-theoretical) disjoint union

$$X=\bigsqcup_{\alpha\in\mathcal{A}} X_\alpha$$

together with the final topology defined by the injective maps

$$I_{X_\alpha}:\ X_\alpha\longrightarrow X,$$

or π_α in my notation.

i.e., the largest topology making each I_{X_α} continuous. In other words, a subset $U\subset X$ is open iff $I_{X_\alpha}^{-1}(U)$ is open in X_α for every $\alpha\in\mathcal{A}$. Equivalently, $U\subset X$ is open iff $U\cap X_\alpha$ is open in X_α for each α. For each $\alpha\in\mathcal{A}$, the inclusion map

$$I_{X_\alpha}:\ X_\alpha\longrightarrow X$$

is then a topological embedding, which is also both open and closed. Moreover, by the characterizing property of the final topology, for any topological space Y, a map $f:X\to Y$ is continuous iff

$$f|_{X_\alpha}=f\circ I_{X_\alpha}:\ X_\alpha\longrightarrow Y$$

is continuous.

Definition 2.8.1. Let X and Y be two topological spaces and let $A\subset Y$ equipped with the subspace topology. Let

$$f:\ A\longrightarrow X$$

be a continuous map. The *gluing of X and Y along f* is the quotient space

$$X\cup_f Y=(X\sqcup Y)/\sim$$

where the equivalence relation $\sim$ is generated by $a\sim f(a)$ for $a\in A$, i.e., the equivalence relations are $x\sim x$ for all $x\in X$, $y\sim y$ for all $y\in Y\backslash A$, and $f(a)\sim a$ for all $a\in A$. Furthermore, f is called the *gluing map*, or the *attaching map*, and $X\cup_f Y$ is also called the *adjunction space of X and Y by f*.

X and Y by f

怎么把两个拓扑空间沿着某条边粘合起来？
想要粘，先别粘。
先把这两个拓扑空间“没有额外约束地放在一起”，**那就是无交并；**
再把这两个拓扑空间“沿着边界粘起来”，那就是商空间

identification $X_a \cong X_a \times \{a\}$

Let $X \cup_f Y$ be the adjunction space by $f : A \to X$. Let

$$q: \quad X \sqcup Y \longrightarrow X \cup_f Y$$

be the quotient map. Identify X and Y with their images in $X \sqcup Y$. Note that we have two continuous maps:

$$q|_X: \quad X \longrightarrow X \cup_f Y \quad \text{ and } \quad q|_Y: \quad Y \longrightarrow X \cup_f Y.$$

Every $x \in f(A)$ is identified with the whole fiber $f^{-1}(x) \subset A$. Moreover,
1) $q|_X$ is injective;
2) $q|_{Y \setminus A} : Y \setminus A \to X \cup_f Y$ is injective.
3) We have disjoint union

$$X \cup_f Y = q(X) \sqcup q(Y \setminus A).$$

Proposition 2.8.1. Assume that A is closed in Y. Then
1) $q|_X : X \to X \cup_f Y$ is a closed embedding;
2) $q|_{Y \setminus A} : Y \setminus A \to X \cup_f Y$ is an open embedding.

Proof. Recall that an injective closed map is a closed embedding. It is thus enough to show that $q|_X : X \to X \cup_f Y$ is a closed map.

Let $C \subset X$ be closed. We need to show that $q(C) \subset X \cup_f Y$ is closed. Note that

$$q^{-1}(q(C)) = \left(q^{-1}(q(C)) \cap X\right) \sqcup \left(q^{-1}(q(C)) \cap Y\right).$$

One checks directly from definition that

$$q^{-1}(q(C)) \cap X = C \quad \text{ and } \quad q^{-1}(q(C)) \cap Y = f^{-1}(C).$$

As $A \subset Y$ is closed and $f : A \to X$ is continuous, $f^{-1}(C)$ is closed in A and thus closed in Y. Thus $q^{-1}(q(C))$ is closed in $X \sqcup Y$. Hence $q(C)$ is closed in $X \cup_f Y$. This shows that $q|_X : X \to X \cup_f Y$ is a closed map and is thus a closed embedding.

We now show that $q|_{Y \setminus A} : Y \setminus A \to X \cup_f Y$ is an open embedding. Again it is enough to show that $q|_{Y \setminus A} : Y \setminus A \to X \cup_f Y$ is an open map.

Let $O \subset Y \setminus A$ be open. We need to show $q^{-1}(q(O)) \subset X \sqcup Y$ is open. Since $A \subset Y$ is closed, O is open in Y and $A \cap O = \varnothing$. Now

$$q^{-1}(q(O)) = \left(q^{-1}(q(O)) \cap X\right) \sqcup \left(q^{-1}(q(O)) \cap Y\right).$$

By definition,

$$q^{-1}(q(O)) \cap X = \varnothing \quad \text{ and } \quad q^{-1}(q(O)) \cap Y = O.$$

Thus $q^{-1}(q(O))$ is open in $X \sqcup Y$, so $q(O)$ is open in $X \cup_f Y$. This shows that $q|_{Y \setminus A} : Y \setminus A \to X \cup_f Y$ is an open map, so it is an open embedding. $\square$

Example 2.8.1. Let Z be any topological space and suppose that $X \subset Z$ and $Y \subset Z$ are two closed subspaces such that $A = X \cap Y \neq \emptyset$. Regard A as a subspace of Y, and let

$$f : A \longrightarrow X, \ \ x \longmapsto x.$$

Then $X \cup_f Y$ is obtained by *cutting Z at A and glue it back together.* We claim that $X \cup_f Y$ is homeomorphic to Z. Indeed, we first have the well-defined injective map

$$f : \ \ Z \longrightarrow X \cup_f Y$$

whose restriction to X and Y are the projections to $X \cup_f Y$. By the gluing lemma, f is continuous. The continuous map

$$X \sqcup Y \to Z, \ \ x \longmapsto x, \ y \longmapsto y,$$

descends to the continuous $X \cup_f Y \to Z$ which is the inverse of f. Thus f is a homeomorphism.

Example 2.8.2. The sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

is homeomorphic to the gluing of the upper hemisphere and the lower hemisphere along the identity map on the equator.

Example 2.8.3. (Collapsing a subset to a point): When X is a one point space, we get

$$X \cup_f Y = Y/A$$

the *collapsing of $A \subset Y$ to a point* that we looked at in Example 2.4.10. Recall from Example 2.4.10 that a topological space Y is called a *T_3-space* or *regular Hausdorff* if it is Hausdorff and if for any closed $A \subset Y$ and any $y \in Y \backslash A$, $\exists$ open subsets O and O_y of Y satisfying (2.1). Thus, if Y is T_3 and $A \subset Y$ is closed, then Y/A is Hausdorff. Moreover, Y/A is Hausdorff for every closed subset A of Y iff Y is T_3.

Example 2.8.4. For any topological space X, let

$$Y = X \times [0, 1],$$

called *the cylinder with base X*. Let $A = X \times \{1\}$. Then

$$CX := Y/A$$

is called *the cone* with base X.

Example 2.8.5. (Wedge sums): Given topological spaces X and Y, $x_0 \in X$ and $y_0 \in Y$, let $A = \{y_0\}$ and

$$f: \ A \longrightarrow X, \ \ y_0 \longmapsto x_0.$$

The resulting $X \cup_f Y \overset{\text{def}}{=} X \vee Y$ is called a *wedge sum* of X and Y.

Lemma 2.8.1. If X and Y are Hausdorff, so is any $X \vee Y$.

Proof. Let $x_0 \in X$, $y_0 \in Y$, and $X \vee Y = X \cup_f Y$ with $f(y_0) = x_0$. Let $q : X \sqcup Y \to X \vee Y$ be the projection, and let $p_0 = q(x_0) = q(y_0)$. Then $q|_X : X \to X \vee Y$ and $q|_Y : Y \to X \vee Y$ are closed embeddings; We have $1-1$ correspondence

$$U \underset{\text{open}}{\subset} X \vee Y, \ p_0 \notin U$$

$$\Longleftrightarrow$$

$$q^{-1}(U) = O_X \sqcup O_Y, \text{ where } x_0 \notin O_X \underset{\text{open}}{\subset} X, \ \ y_0 \notin O_Y \underset{\text{open}}{\subset} Y,$$

and $1-1$ correspondence

$$U \underset{\text{open}}{\subset} X \vee Y, \ p_0 \in U$$

$$\Longleftrightarrow$$

$$q^{-1}(U) = O_X \sqcup O_Y, \text{ where } x_0 \in O_X \underset{\text{open}}{\subset} X, \ \ y_0 \in O_Y \underset{\text{open}}{\subset} X.$$

It follows that if both X and Y are Hausdorff, then so is $X \vee Y$. □

2.8.2 A special class of adjunction spaces

In this section, we consider the case when X and Y are two topological spaces, $A \subset Y$ a closed subspace, and

$$f: \ A \longrightarrow X$$

a closed embedding. Let $Z = X \cup_f Y$. Let $q : X \sqcup Y \to X \cup_f Y$ be the quotient map. Then both $q|_X : X \to Z$ and $q|_Y : Y \to Z$ are closed embeddings, and we have

$$Z = q(X) \cup q(Y) \quad \text{ and } \quad q(X) \cap q(Y) = q(f(A)) = q(A).$$

An alternative way of thinking about $Z = X \cup_f Y$ is thus as the union of the two closed subspaces $q(X)$ and $q(Y)$.

Conversely, we have

Lemma 2.8.2. If Z is a topological space and $X \subset Z$ and $Y \subset Z$ are two closed subsets such that

$$Z = X \cup Y.$$

Then Z is homeomorphic to $X \cup_f Y$, where

$$f: \ X \cap Y \longrightarrow X, \ \ x \longmapsto x.$$

Proof. We first have the well-defined injective map

$$f:\ Z \longrightarrow X \cup_f Y$$

whose restriction to X and Y are the projections to $X \cup_f Y$. By the gluing lemma, f is continuous. The continuous map

$$X \sqcup Y \to Z,\ \ x \longmapsto x,\ y \longmapsto y,$$

descends to the continuous $X \cup_f Y \to Z$ which is the inverse of f. Thus f is a homeomorphism. □

Thus adjunction spaces $Z = X \cup_f Y$ formed by closed $A \subset Y$ and closed embeddings $f : A \to X$ are equivalent to topological spaces Z with closed subsets X and Y such that

$$Z = X \cup Y.$$

Recall Proposition 2.4.6 that if Z is a topological space, and $X \subset Z$ and $Y \subset Z$ are closed subspaces of Z such that $Z = X \cup Y$ and both X and Y are Hausdorff, then Z is Hausdorff.

Corollary 2.8.1. If X and Y are Hausdorff, then any adjunction space $X \cup_f Y$ formed by closed $A \subset Y$ and closed embedding $f : A \to X$ is Hausdorff.

Recall now that the *Gluing Lemma* says that if Z is a topological space, $X \subset Z$ and $Y \subset Z$ are two closed subsets, and $Z = X \cup Y$, then for any topological space W, if

$$\alpha:\ X \longrightarrow W \quad \text{an} \quad \beta:\ Y \longrightarrow W$$

are continuous and $\alpha|_{X\cap Y} = \beta|_{X \cap Y}$, then

$$\gamma:\ Z \longrightarrow W,\ \ \gamma(z) = \begin{cases} \alpha(x), & x \in X, \\ \beta(y), & y \in Y, \end{cases}$$

is well-defined and continuous.

As a consequence of the Gluing Lemma, we have the following

Proposition 2.8.2. (Adjunction of maps): Let $Z = X \cup_f Y$ with $A \subset Y$ closed and $f : A \to X$ a closed embedding. For any topological space W, if

$$\alpha:\ X \longrightarrow W \quad \text{and} \quad \beta:\ Y \longrightarrow W$$

are continuous and if $\alpha(f(y)) = \beta(y)$ for every $y \in A$, then

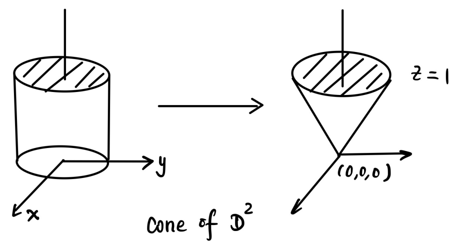
$$\gamma:\ X \cup_f Y \longrightarrow W,\ \ \gamma(z) = \begin{cases} \alpha(x), & z = q(x) \text{ with } x \in X, \\ \beta(y), & z = q(y) \text{ with } y \in Y, \end{cases}$$

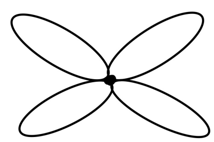
is well-defined and continuous.

We call γ the *gluing* or the *adjunction* of α and β.

2.8.3 Embedding adjunction spaces

Often we want to show the abstractly defined $X \cup_f Y$ is homeomorphic to known spaces, such as a subspace of $\mathbb{R}^n$.





Rose

Example 2.8.6. For example, for $X = D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, we want to show the cone CX is homeomorphic to

$$C = \{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 \leq z^2, \; 0 \leq z \leq 1\},$$

where C has the subspace topology from $\mathbb{R}^3$. Similarly, wedge sums of S^1 are *bouquets of circles*, including *roses*.

We have the following very useful theorem.

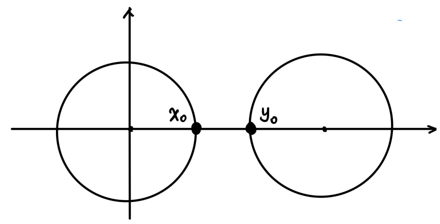
Theorem 2.8.1. Suppose that Z is a compact topological space, W is a Hausdorff space, and $f : Z \to W$ is a continuous map. Let $\sim$ be the equivalence relation on Z generated by

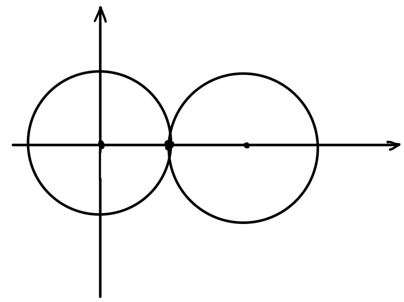
$$z_1 \sim z_2 \quad iff \quad f(z_1) = f(z_2),$$

and let $[Z] = Z/\sim$ be the quotient space. Then

$$[f] : \; [Z] \longrightarrow W, \; [z] \longmapsto f(z),$$

is a topological embedding, so $[Z]$ is homeomorphic to $f(Z)$ with the subspace topology from W.





Proof. We know that $[Z]$ is compact since Z is and $[f] : [Z] \to W$ is well-defined, continuous, and injective; Moreover, W is Hausdorff. Since an injective continuous map from a compact space to a Hausdorff space is an embedding, $[f]$ is an embedding. $\square$

Example 2.8.7. The wedge sum $S^1 \vee S^1$ is homeomorphic to $C_1 \cup C_2$, where

$$C_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \quad C_2 = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 = 1\},$$

and $C_1 \cup C_2$ has subspace topology from $\mathbb{R}^2$. Indeed, take $Z = C_1 \sqcup C_2'$, where

$$C_2' = \{(x,y) \in \mathbb{R}^2 : (x-5)^2 + y^2 = 1\},$$

and equip Z with subspace topology from $\mathbb{R}^2$. Define $f : Z_2 \to C_1 \cup C_2$ by $f(z) = z$ for $z \in C_1$, and

$$f(x,y) = (x-3,y), \ (x,y) \in C_2.$$

Then we have the homeomorphisms $[f] : S^1 \vee S^1 \to C_1 \cup C_2$.

Example 2.8.8. For $X = D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, the cone CX is homeomorphic to

$$C = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 \leq z^2, \ 0 \leq z \leq 1\},$$

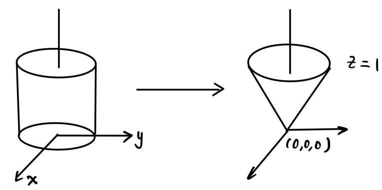
where C has the subspace topology from $\mathbb{R}^3$. Indeed, consider the cylinder

$$Z = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 \leq 1, 1 \leq z \leq 1\}$$

with the subspace topology, and define

$$f : \ Z \longrightarrow \mathbb{R}^3 : \ (x, \ y, \ z) \longmapsto (xz, \ yz, \ z).$$

Then f is an embedding of $CX \cong Z/\sim$ into $\mathbb{R}^3$with image C.



2.9 Topological groups and orbit spaces

2.9.1 Topological groups

We first give a quick review on group theory without proofs.

Definition 2.9.1. A *group* is a nonempty set G together with a binary operation $*$ on G such that the following conditions are satisfied.
1) (associativity) $(a * b) * c = a * (b * c)$ for any $a, b, c \in G$;
2) (existence of identity) there exists $e \in G$ such that $a * e = e * a = a$ for any $a \in G$;
3) (existence of inverse) for any $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$.

The element e is called the *identity* of the group G. For each $a \in G$, we call the element b satisfying $a * b = b * a = e$ the *inverse* of a. This element is usually denoted by a^{-1}. The identity and the inverse of each element are unique. If the binary operation is clear from the the context, we may write ab instead of $a * b$. Also, we write a^2, a^3 to mean $a * a$ and $(a * a) * a$, etc.

Example 2.9.1. The singleton set $\{e\}$ is a group under the operation $e * e = e$. This is called the *trivial group*.

Example 2.9.2. The sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are groups under the usual addition.

Example 2.9.3. The set $\mathbb{Z}_2 = \{0, 1\}$ with the binary operation $+$ defined by

$$0 + 0 = 1 + 1 = 0 \quad \text{and} \quad 0 + 1 = 1 + 0 = 1$$

is a group.

Example 2.9.4. If X is a topological space, the set of all homeomorphisms from X to itself is a group under composition.

Example 2.9.5. An *infinite cyclic group* is the group $\{\ldots, a^{-2}, a^{-1}, e, a, a^2, \ldots\}$ with the binary operation $*$ defined by

$$a^m * a^n = a^{m+n}$$

for $m, n \in \mathbb{Z}$, where $a^0 = e$.

Example 2.9.6. Let G and G' be two groups. We define a binary operation $*$ on $G \times G'$ by

$$(a, a') * (b, b') = (ab, a'b').$$

Then this makes $G \times G'$ a group. We call this group the *direct product* of G and G'.

Definition 2.9.2. Let G and G' be two groups and let $\phi : G \to G'$ be a function. We say that ϕ is a *group homomorphism* if

$$\phi(ab) = \phi(a)\phi(b), \quad \forall \ a, b \in G.$$

Definition 2.9.3. Let G and G' be two groups and let $\phi : G \to G'$ be a map. We say that ϕ is a *group isomorphism* if ϕ is a bijective homomorphism. In that case, we say that G is *isomorphic* to G', and denote this by $G \cong G'$.

Example 2.9.7. Any infinite cyclic group is isomorphic to $\mathbb{Z}$.

We now turn to topological groups. 我们总是假定拓扑群是豪斯多夫空间

Definition 2.9.4. A *topological group* is both a Hausdorff topological space and a group, such that the two structures are compatible in the sense that the group multiplication

$$m : \ G \times G \longrightarrow G, \ (a, b) \longmapsto ab$$

is continuous, and that the inverse map $i : G \to G, a \to a^{-1}$, is also continuous.

The following fact is checked directly from the definitions.

Lemma 2.9.1. 1) If G_1 and G_2 are topological groups, so is the product group $G_1 \times G_2$;

2) If H is a subgroup of a topological group G, then H, with the subspace topology, is a topological group, called a *topological subgroup* of G.

Definition 2.9.5. If the underlying topology of a topological group G is compact, we call G a compact topological group.

We have a vast list of examples of topological groups. In the following, for any integer $n \geq 1$, $\mathbb{R}^n$ will be equipped with the Euclidean topology and any subset of $\mathbb{R}^n$ will have the subspace topology.

Example 2.9.8. 1) For any integer $n \geq 1$, $\mathbb{R}^n$ with the group structure being addition;

2) $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a compact topological group, with group structure being multiplication of complex numbers;

3) For any integer $n \geq 1$, the torus $T^n = S^1 \times \cdots \times S^1$ (n-copies) is a compact topological group.

4) Any group G with the discrete topology is a topological group, called a discrete topological group;

5) The group $GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ of all $n \times n$ real invertible matrices with the subspace topology of $\mathbb{R}^{n^2}$ and the matrix multiplication as the group structure. This is called the *general linear group* (see Theorem 4.12 of Armstrong's book).

6) The group $O(n, \mathbb{R})$ of $n \times n$ orthogonal matrices is a compact topological subgroup of $GL(n, \mathbb{R})$. The group

$$SO(n, \mathbb{R}) = \{g \in O(n, \mathbb{R}) : \det(g) = 1\}$$

is a topological subgroup of $O(n, \mathbb{R})$, called the *special orthogonal group*. See Theorem 4.13 of Armstrong's book.

Definition 2.9.6. If G_1 and G_2 are two topological groups and if $\phi : G_1 \to G_2$ is both a group homomorphism and a continuous map, then ϕ is called a homomorphism of topological groups. If ϕ is also a homeomorphism, then ϕ is called an *isomorphism of topological groups.*

Example 2.9.9. The map $\phi : \mathbb{R} \to S^1, \phi(x) = e^{2\pi i}$ is a surjective homomorphism of topological group.

2.9.2 Orbit spaces

If G is a group and X is a set, recall that a *left action* of G on X is a map, called the *action map*,

$$\sigma : \ G \times X \longrightarrow X, \ \sigma(g, x) = gx,$$

such that

1) $g(h(x)) = (gh)x$ for all $g, h \in G$ and $x \in X$;

2) $ex = x$ for all $x \in X$, where e is the identity element of G. Given such an action, for each $x \in X$, the subset

$$Gx = \{gx : g \in G\}$$

of X is called the G-*orbit* of x, and we have a partition of X into the G-orbits in X. If there is only one G-orbit in X, we say that the action σ is *transitive.*

Definition 2.9.7. If G is a topological group and X is a topological space, by a *left topological action* of G on X we mean an action map

$$\sigma : \ G \times X \longrightarrow X, \ \sigma(g, x) = gx,$$

which is continuous, where $G \times X$ has the product topology.

这里的 G-orbit of x **很像** coset

Lemma 2.9.2. If $\sigma : G \times X \to X$ is a left topological action, then for each $g \in G$, the map

$$\sigma_g : \ X \longrightarrow X, \ \ \sigma_g(x) = gx, \qquad x \in X,$$

is a homeomorphism of X to itself.

Proof. For $g \in G$, the map $\sigma_g : X \to X$ is the composition of the continuous map

$$i_g : \ X \longrightarrow \{g\} \times X, \ \ i_g(x) = (g, x)$$

and $\sigma|_{\{g\}\times X} : \{g\} \times X \to X$. Thus σ_g is continuous. As $\sigma_g^{-1} = \sigma_{g^{-1}}$ is also continuous, σ_g is a homeomorphism. $\square$

Definition 2.9.8. Given a topological action $\sigma : G \times X \to X$, the set of G-orbits in X, denoted as

$$G\backslash X := \{Gx : x \in X\}$$

and equipped with the quotient topology of G, is called the *orbit space* of the topological action σ.

$\begin{pmatrix}\frac{1}{n} & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots\end{pmatrix}\begin{pmatrix}1\\0\\0\\ \vdots\end{pmatrix} \to \begin{pmatrix}0\\0\\0\\ \vdots\end{pmatrix}$

Exercise 2.9.1. Show that the orbit space of the $GL(n, \mathbb{R})$-action on $\mathbb{R}^n$ has two points and it not Hausdorff.

$\{\vec{0}\}$ and $\mathbb{R}^n \setminus \{0\}$

so $\vec{0}$ has no open neighbour.

Exercise 2.9.2. Let $\sigma : G \times X \to X$ be an action of a topological group G, and let

$$\pi : G \longrightarrow G\backslash X$$

be the quotient map. Show that π is an open map. Show that the orbit space is Hausdorff if and only if the set

$$\{(x, gx) : x \in X, g \in G\} \subset X \times X$$

is closed.

Recall that in general the quotient map $X \to X/\sim$ for an arbitrary quotient space is not necessarily an open map.

Lemma 2.9.3. For any topological group action of G on X, the quotient map

$$\pi : \ X \longrightarrow G\backslash X, \ \ x \longmapsto Gx,$$

is an open map.

Proof. For any open subset $A \subset X$, one has

$$\pi^{-1}(\pi(A)) = \bigcup_{g \in G} gA.$$

Since each $g \in G$ acts on X by homeomorphisms, gA is again open in X, so $\pi^{-1}(\pi(A))$ is open in X. Thus $\pi(A)$ is open in $G\backslash X$. $\square$

Lemma 2.9.4. Let $G \times X \to X$ be a topological action.

1) If X is compact so is $G\backslash X$;
2) If X is connected, so is $G\backslash X$.

Proof. The statements are true for any quotient spaces: continuous images of compact (resp. connected) spaces are compact (resp. connected). □

We now give some remarks on *right group actions*. Given a topological group G and a topological space X, a *right action* of G on X is a continuous map, again called the action map,

$$\sigma : \ X \times G \longrightarrow X, \ \sigma(xg) = xg,$$

where $G \times X$ has the product topology, such that

1) $(xg)h = x(gh)$ for all $g, h \in G$ and $x \in X$;
2) $xe = x$ for all $x \in X$, where e is again the identity element of G.

Given such an action and $x \in X$, the set $xG := \{xg : g \in G\}$ is called the orbit of G through x and the orbits space is denoted as

$$X/G := \{xG : x \in X\}.$$

Equipped with the quotient topology of G, X/G is again called the *orbit space* of the topological action σ. Note that $\sigma : X \times G \to X, (x, g) \to xg$ is a right action if and only if

$$G \times X \longrightarrow X, \ \ (g, x) \longmapsto xg^{-1}$$

is a left action.

<u>Remark</u>. It is very common to also use X/G to denote the orbit space of a left action of G on X, which we will adopt from now on.

We now look at many examples of quotient spaces.

Example 2.9.10. Consider $G = \mathbb{Z}$ as a topological group with the discrete topology and addition as group multiplication. Then $\mathbb{Z}$ acts on $\mathbb{R}$ by

$$\sigma : \ \mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}, \ \ (n, x) \longmapsto n + x.$$

For a given $x \in \mathbb{R}^1$, the $\mathbb{Z}$-orbit of x is

$$x + \mathbb{Z} = \{x + n : n \in \mathbb{Z}\}.$$

Consider the continuous map

$$\phi : \ \mathbb{R}^1 \longrightarrow S^1, \ \phi(x) = e^{2\pi i x}.$$

It is clear that the fibers of ϕ are precisely the orbits of the $\mathbb{Z}$-action on $\mathbb{R}$. Since ϕ also maps open subsets of $\mathbb{R}^1$ to open subsets of S^1, ϕ is a quotient map. We conclude

$$\mathbb{R}^1/\mathbb{Z} \cong S^1,$$

i.e., they are homeomorphic, where S^1 has the subspace topology from $\mathbb{C} \cong \mathbb{R}^2$.

Example 2.9.11. Generalizing Example 2.9.10, for any integer $n \geq 1$, the n-torus $T^n = S^1 \times \cdots \times S^1$ homeomorphic to the orbits space $\mathbb{R}^n/\mathbb{Z}^n$ for the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by

$$\sigma: \ \mathbb{Z}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ \ \sigma(v, x) = v + x.$$

Example 2.9.12. Consider the left action of $O(n, \mathbb{R})$ on $S^{n-1} = \{x \in \mathbb{R}^n : |x|^2 = 1\}$ by

$$\sigma: \ O(n, \mathbb{R}) \times S^{n-1} \longrightarrow S^{n-1}, \ \ \sigma(g, x) = gx,$$

where $x \in \mathbb{R}^n$ is regarded as a column vector and gx stands for matrix multiplication. Note every $x \in S^{n-1}$ is in the same $O(n, \mathbb{R})$-orbit with the unit vector $e_1 = (1, 0, \ldots, 0)^{\text{transpose}}$, we see that the orbit space for the action is just one point. In other words, the action σ is transitive.

Example 2.9.13. Consider $\mathbb{Z}_2 = \{0, 1\}$ act on S^2 where 1 acts on S^2 by $x \to -x$. Then $S^2/\mathbb{Z}_2$ is the projective plane $\mathbb{RP}^2$. Replacing S^2 by $S^n \subset \mathbb{R}^{n+1}$ for any integer $n \geq 1$, we get the real projective space

$$\mathbb{RP}^n = S^n/\mathbb{Z}_2.$$

The rest of section is devoted to the Klein bottle being an orbit space.

Consider first the so-called Euclidean group $E(2) = O(2) \times \mathbb{R}^2$ with the group multiplication

$$(\theta, v)(\phi, w) = (\theta\phi, \theta w + v), \qquad \theta, \phi \in O(2), \ v, w \in \mathbb{R}^2,$$

where again $w \in \mathbb{R}^2$ is a column vector and θw is matrix multiplication. Then $E(2)$ acts on $\mathbb{R}^2$ by

$$\sigma: \ E(2) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ \ \sigma((\theta, v), x) = \theta x + v.$$

Let G be the subgroup of $E(2)$ generated by

$$a = (I_2, e_1) \in E(2), \ \ b = (J_2, e_2) \in E(2),$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in O(2), e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and

$$I_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in O(2), \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2.$$

One checks directly that $aba = b$ as elements in G. Using the relation $aba = b$, one can write an element in G uniquely as $a^m b^n$ for $m, n \in \mathbb{Z}$, and the group multiplication in G becomes

$$a^m b^n a^{m'} b^{n'} = a^{m+(-1)^n m'} b^{n+n'},$$

so we can also identify $G \cong \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z} \times \mathbb{Z}$ has the group structure

$$(m, n) * (m', n') = (m + (-1)^n m', \ n + n'), \qquad n, m, m', n' \in \mathbb{Z}.$$

In this presentation of G, the action of G on $\mathbb{R}^2$ becomes

$$(m,n)\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-1)^n x + m \\ y+n \end{pmatrix}, \quad (m,n)\in\mathbb{Z}\times\mathbb{Z}, \begin{pmatrix} x \\ y \end{pmatrix}\in\mathbb{R}^2.$$

If we write an element in $\mathbb{R}^2$ as a row vector (x,y), the action of G on $\mathbb{R}^2$ is given by

$$a\cdot(x,y) = (x+1,y), \quad b\cdot(x,y) = (-x,y+1),$$

or more generally

$$a^m b^n\cdot(x,y) = ((-1)^n x + m,\ y+n), \quad m,n\in\mathbb{Z}, (x,y)\in\mathbb{R}^2.$$

We now restrict the action of $E(2)$ on $\mathbb{R}^2$ to an action of G on $\mathbb{R}^2$, and consider the orbit space $\mathbb{R}^2/G$.

Lemma 2.9.5. The orbit space $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle.

Outline of the proof. Recall that the Klein bottle K is defined to be quotient space of the square $S=[0,1]\times[0,1]$ by the equivalence relation $\sim$ whose equivalence classes are
1) the 4 point set $\{(0,0),(0,1),(1,0),(1,1)\}$;
2) the two point set $\{(x,0),(1-x,1)\}$ for each $0<x<1$;
3) the two point set $\{(0,y),(1,y)\}$ for each $0<y<1$;
4) the singleton $\{(x,y)\}$ for each (x,y) such that $0<x,y<1$.

For $(x,y)\in S$, let $[(x,y)]\in K$ be the image of (x,y) in K. It is easy to see that

$$\phi:\ K\longrightarrow\mathbb{R}^2/G,\ \ [(x,y)]\longmapsto G(x,y)$$

is a bijective continuous map. On the other hand, one shows that $\mathbb{R}^2/G$ is Hausdorff (see Exercise 2.9.2). As K is compact, and we know (Closed Map Lemma) that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, we see that ϕ is an isomorphism.

Example 2.9.14. Let p and q be relatively prime positive integers, and let the cyclic group $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ acts on the 3-sphere $S^3=\{(z_1,z_2)\in\mathbb{C}\times\mathbb{C} : \|z_1\|^2+\|z_2\|^2=1\}$ by

$$g\cdot(z_1,z_2) = (e^{2\pi i/p}z_1,\ e^{2\pi qi/p}z_2),$$

where g is a generator of $\mathbb{Z}_p$. The quotient space $S^3/\mathbb{Z}_p$ is called a *lens space* and is denoted as $L(p,q)$.

We now consider a class of orbit spaces called *homogeneous spaces*.

Let G be a topological group and $H\subset G$ a topological subgroup. Let H act on G from the right by right translation, i.e.,

$$G\times H\longrightarrow G,\ \ (g,h)\longmapsto gh.$$

The H-orbits are then precisely all the right cosets gH for $g \in G$. The quotient space G/H is thus the set of all right H-cosets. Note that G acts on G/H from the left by

$$G \times (G/H) \longrightarrow G/H, \ \ (g_1, g_2H) \longmapsto g_1g_2H.$$

The action is topological and is transitive.

Definition 2.9.9. The quotient space with the left action of G above is called a *homogeneous space* of G.

Proposition 2.9.1. Let G be any topological group. If $H \subset G$ is a closed subgroup, then G/H is Hausdorff.

Proof. Consider the map

$$f: \ G \times G \longrightarrow G, \ \ (g_1, g_2) \longmapsto g_1^{-1}g_2.$$

Let $\pi : G \to G/H$ be the quotient map. Since f is continuous, $f^{-1}(H)$ is a closed subset of $G \times G$. Let $g_1, g_2 \in G$ be such that $g_1H \neq g_2H$. Then $g_1^{-1}g_2 \notin H$, so $(g_1, g_2) \notin f^{-1}(H)$. BY the definition of the product topology on $G \times G$, there exist open subsets $O_1, O_2 \subset G$ such that

$$g_1 \in O_1, \ \ g_2 \in O_2, \ \ (O_1 \times O_2) \cap f^{-1}(H) = \varnothing.$$

Since $\pi : G \to H$ is open, $\pi(O_1)$ and $\pi(O_2)$ are open in G/H. Note now that $g_1H \in \pi(O_1)$ and $g_2H \in \pi(O_2)$. Moreover, $(O_1 \times O_2) \cap f^{-1}(H) = \varnothing$ implies that $\pi(O_1) \cap \pi(O_2) = \varnothing$. Thus the two distinct points g_1H and g_2H in G/H are separated by the two open subsets $\pi(O_1)$ and $\pi(O_2)$ of G/H. □

We now look at orbits of topological group actions.

Let $G \times X \to X$ be a topological group action. For each $x \in X$, the set

$$G_x = \{g \in G : \ gx = x\} \subset G$$

is a subgroup of G, called the *stabilizer* of G at x. The action is said to be *free* if $G_x = \{e\}$ for every $x \in X$.

Lemma 2.9.6. If X is Hausdorff, then $G_x \subset G$ is closed for every $x \in X$.

Proof. Consider the map $f : G \to X, g \mapsto gx$. Then $G_x = f^{-1}(x)$. Since X is Hausdorff, the singleton $\{x\}$ is a closed subset of X. Since f is continuous, $G_x = f^{-1}(x)$ is closed in G. □

Lemma 2.9.7. Let $G \times X \to X$ be a topological group action. If G is compact and X is Hausdorff, then for every $x \in X$,

$$G/G_x \longrightarrow Gx : \ \ gG_x \longmapsto gx,$$

is a homeomorphism, where G/G_x has the quotient topology and $Gx \subset X$ has the subspace topology.

Proof. Consider $f : G/G_x \to X$, $gG_x \mapsto gx$. Note that f is well-defined, continuous, and injective. Moreover, the image of f is Gx. Since X is Hausdorff and G/G_x is compact, we know by the Closed Map Lemma, f is an embedding. Thus f induces a homeomorphism from G/G_x to Gx. □

Example 2.9.15. Let $n \geq 2$ and consider again the action of $SO(n, \mathbb{R})$ on $\mathbb{R}^n$. The orbit of the action through any unit vector in $\mathbb{R}^n$ given a homeomorphism

$$SO(n, \mathbb{R})/SO(n-1, \mathbb{R}) \cong S^{n-1}.$$

Second countable cannot be removed

2.10 Topological manifolds

e.g. $\bigsqcup_{x\in\mathbb{R}} \mathbb{R}$ is not a manifold.

2.10.1 Topological manifolds (without boundary)

Definition 2.10.1. A topological space X is said to be *second countable* if it has a countable base.

B should contain ϕ, X, but not necessarily X.

We now show that second countability is inherited by subspaces.

Lemma 2.10.1. A subspace of a second countable space is second countable.

Proof. If $\mathcal{B}$ is a basis for X and $A \subset X$, then

$$\{A \cap B : B \in \mathcal{B}\}$$

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is a basis for the subspace topology on A. □

Our basic example is $\mathbb{R}^n$ with the Euclidean topology. Any subspace of $\mathbb{R}^n$ is thus also second countable.

Recall that a topological space X is said to be locally homeomorphic to Y if every $x \in X$ has an open neighborhood in X that is homeomorphic to an open subspace of Y.

Definition 2.10.2. A *topological manifold of dimension n* is a topological space that is Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$.

Thus $\mathbb{R}^n$ is a topological manifold of dimension n.

can be relaxed, but if the domains cover the whole space, then it is usually good enough

chart at x

atlas

Definition 2.10.3. Let M be an n-dimensional topological manifold. A *coordinate chart* on M is a pair (U, ϕ), where U is an open subset of M and $\phi : U \to V \subset \mathbb{R}^n$ a homeomorphism, where V is an open subset of $\mathbb{R}^n$. An *atlas* on M is a collection of $\{(U_\alpha, \phi_\alpha)\}$ of coordinate charts on U such that $\cup_\alpha U_\alpha = M$.

To show a topological space M is an n-dimensional manifold, need to show that M is Hausdorff, second countable, and has an atlas.

Note that both the Hausdorff condition and the second countability condition are *hereditary*, i.e., if X is Hausdorff and second countable, so is every $S \subset X$ with the

$\forall\, m \in M$, $\exists$ open neighbour U_m of m and $\sigma : U_m \to \mathbb{R}^n$, s.t. σ is a homeomorphism.

dimension is well defined, because $\boxed{\mathbb{R}^n \ncong \mathbb{R}^m \text{ (n,m distinct)}}$

dimension is invariant

dimension is the same in linear algebra if field is $\mathbb{R}$

double ... if field is $\mathbb{C}$.

subspace topology; Furthermore, if X is locally homeomorphic to $\mathbb{R}^n$, so is any open subset of X.

It is also clear from the definition that if M is an n-dimensional topological manifold, and is M' is a topological space homeomorphic to M, then M' is an n-dimensional topological manifold.

If M is an n-dimensional topological manifold, so is every open subset of M with the subspace topology; Thus all open subsets of $\mathbb{R}^n$ are n-dimensional topological manifolds.

Lemma 2.10.2. If M is Hausdorff and has countably many open subsets $\{U_\alpha : \alpha \in \mathcal{A}\}$ s.t.

$$\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M,$$

and each U_α is homeomorphic to an open subset of $\mathbb{R}^n$, then M is a topological manifold of dim n.

Proof. If $\{(U_\alpha, \phi_\alpha) : \alpha \in \mathcal{A}\}$ is a countable atlas on M, then each U_α has a countable basis, the union of which is a countable basis of M. □

Example 2.10.1. Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a continuous function. The *graph of f* is defined by

$$\text{graph}(f) = \{(x, f(x)) : x \in U\} \subset \mathbb{R}^{n+1}$$

and with the subspace topology of $\mathbb{R}^{n+1}$. Since graph(f) is homeomorphic to U via the map $(x, f(x)) \mapsto x$, we see that graph(f) is an n-dimensional topological manifold.

Example 2.10.2. The *circle* $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, with the subspace topology from $\mathbb{R}^2$, is a 1-dimensional topological manifold.

More generally, for any integer $n \geq 1$, the *n-sphere*

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

with the subspace topology from $\mathbb{R}^{n+1}$, is an n-dimensional topological manifold.

Example 2.10.3. The *real projective space* $\mathbb{R}P^n = S^n/\mathbb{Z}_2$, with the quotient topology, is an n-dimensional topological manifold. Indeed, let $\pi : S^n \to \mathbb{R}P^n$ be the projection. Note first that if U is open in S^n, then $\pi^{-1}(\pi(U)) = U \cup (-U)$, so $\pi(U)$ is an open subset of $\mathbb{R}P^n$.

Suppose that $x, y \in S^n$ are such that $[x] \neq [y]$, then $\{x, -x\} \cap \{y, -y\} = \varnothing$. Find charts (U, ϕ) on S^n around x and (V, ψ) on around y such that $U \cap V = \varnothing$. Make them small if necessary, we may assume that $U, -U, V, -V$ are pairwise disjoint. Then $\pi(U) \cap \pi(V) = \varnothing$. Thus $\mathbb{R}P^n$ is Hausdorff.

Note now that if (U, ϕ) is an open subset of S^n such that $U \cap (-U) = \varnothing$, then $\pi|_U : U \to \pi(U)$ is a homeomorphism. Thus if we use the the $n+1$ charts (U_i^+, ϕ_i) on S^n

Important example: $\mathbb{R}P^2 = \mathbb{R}^3 \setminus \{\vec{0}\} / \vec{v} \sim \lambda\vec{v}$

$U_0 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : x_0 \neq 0\}, \pi(U_0) \cong \mathbb{R}^2$

given by the hemispheres, where $i = 1, \dots, n+1$, we get an open cover of $\mathbb{R}P^n$ by the $n+1$ open subsets $\{V_i = \pi(U_i^+) : i = 1, \dots, n+1\}$, and we have homeomorphisms

$$\psi_i = \phi_i^+ \circ (\pi|_{U_i^+})^{-1} : \ V_i \longrightarrow B^n, \qquad i = 1, \dots, n+1.$$

Thus $\mathbb{R}P^n$ is an n-dimensional topological manifold

We now turn to product manifolds.

Lemma 2.10.3. If M_1 and M_2 are topological manifolds of respective dimensions m_1 and m_2, then $M_1 \times M_2$, with the product topology, is a topological manifold of dimension $m_1 + m_2$.

Proof. Exercise. □

Example 2.10.4. For an integer $n \geq 1$, define

$$T^n = S^1 \times S^1 \times \cdots \times S^1$$

(n times) and equip it with the product topology. Then T^n is an n-dimensional topological manifold, called the *n-dimensional compact torus.*

For $n = 2$, the map $F : T^2 \to \mathbb{R}^3$,

$$F(x_1, y_1, x_2, y_2) = ((2 + x_1)x_2, (2 + x_1)y_2, y_1),$$

is an embedding of T^2 into $\mathbb{R}^3$. Its image is the surface obtained by revolving the circle $(y-2)^2 + z^2 = 1$ in the (y, z)-plane about the z-axis.

It is very natural to ask whether a topological manifold could simultaneously have two different dimension.

Theorem 2.10.1. (Theorem on Invariance of Domain) *A topological manifold can not be both m and n dimensional for $m \neq n$.*

Proof. Assume $n = 0$. If M is also of dimension $m > 0$, then a singleton subset of M would be homeomorphic to an open subset V of $\mathbb{R}^m$, not possible because V is uncountable.

For $n > 0$, the proof uses tools from algebraic topology. □

2.10.2 Topological manifolds with boundary

$\mathcal{H}$

We now turn to the definition of *manifolds with boundary.*

For $n \geq 1$, the *closed upper-half space of* $\mathbb{R}^n$ is

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \ x_n \geq 0\}$$

with the subspace topology of $\mathbb{R}^n$. Set

$$\partial\,\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \ x_n = 0\},$$
$$\operatorname{Int}\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \ x_n > 0\}.$$

Definition 2.10.4. An *n-dimensional manifold with boundary* is a second countable Hausdorff topological space M locally homeomorphic to $\mathbb{H}^n$.

Definition 2.10.5. Let M be an n-dimensional manifold with boundary.

1) *An interior chart* on M is a pair (U, ϕ) of an open $U \subset M$ and a

$$\text{homeomorphism } \phi\colon\; U \longrightarrow V \underset{\text{open}}{\subset} \operatorname{Int}\mathbb{H}^n.$$

2) *A boundary chart* on M is a pair (U, ϕ) of an open $U \subset M$ and a

$$\text{homeomorphism } \phi\colon\; U \longrightarrow V \underset{\text{open}}{\subset} \mathbb{H}^n, \quad V \cap \partial\,\mathbb{H}^n \neq \varnothing.$$

3) A point $p \in M$ is an *interior point* if p is in the domain of an interior chart;

4) A point $p \in M$ is a *boundary point* if p is in the domain of a boundary chart that maps p to $\partial\,\mathbb{H}^n$.

5) Set

$$\text{Int}M = \text{set of all interior points}, \qquad \partial M = \text{set of all boundary points},$$

and call Int$M$ the *interior of M* and ∂M the *boundary of M*.

Lemma 2.10.4. For any topological manifolds with boundary, one has

$$\text{Int}M \cup \partial M = M.$$

Proof. Let $p \in M$ and let (U, ϕ) be a coordinate chart such that $p \in U$. If $\phi(p) \notin \partial\,\mathbb{H}^n$, by choosing a smaller U, we may assume that $\phi(U) \subset \operatorname{Int}\mathbb{H}^n$, so $p \in \operatorname{Int} M$. If $p \in \partial\,\mathbb{H}^n$, then $p \in \partial M$ by definition. □

It is again natural to ask whether a point p in M can be both an interior and a boundary point.

Theorem 2.10.2. (Theorem on Invariance of Boundary) For any topological manifolds with boundary,

$$\text{Int}M \cap \partial M = \varnothing.$$

Example 2.10.5. $\mathbb{H}^n$ is an n-dimensional manifold with boundary $\partial\mathbb{H}^n = \mathbb{R}^{n-1}$. The closed ball

$$B^n = \{x \in \mathbb{R}^n : \|x\|^2 \le 1\}$$

is an n-dimensional topological manifold with $\partial B^n = S^{n-1}$.

Remark. We make some remarks on the terminology. An n-dimensional topological manifold defined earlier (locally homeomorphic to $\mathbb{R}^n$) is also called an n-dimensional topological manifold without boundary. In other words, a *topological manifold without boundary* is topological manifold M with boundary such that $\partial M = \varnothing$! For a topological manifold M with $\partial M \neq \varnothing$, the boundary ∂M here is not the same as boundary of M as a subspace of a larger space.

Lemma 2.10.5. For an n-dimensional manifold M with boundary,

1) $\mathrm{Int} M$ is an open subset of M and is an n-dimensional manifold without boundary.
2) ∂M is closed in M and is an $(n-1)$-dimensional manifold without boundary.

Proof. Exercise. □

Note that every topological manifold is both locally Hausdorff and locally compact.

Lemma 2.10.6. Every topological manifold (with or without boundary) is locally compact.

Proof. Let M be an n-dimensional manifold, and let $p \in M$. Suppose first that $p \in \mathrm{Int}\, M$. Let (U, ϕ) be an interior coordinate chart with $\phi(p) = 0$, and we may assume that $\phi(U)$ is an open ball $B(0; R)$ in $\mathbb{R}^n$ of radius R for some $R > 0$. Then $\phi^{-1}(\overline{B(0; R/2)})$ is a compact neighborhood of p in M. Suppose now that $p \in \partial M$. Let (U, ϕ) be a boundary coordinate chart with $\phi(p) = 0$, and we may assume that $\phi(U) = B(0; R) \cap \mathbb{H}^n$. Then $\phi^{-1}(\overline{B(0; R/2)} \cap \mathbb{H}^n)$ is a compact neighborhood of p in M. □

2.10.3 Connected sums of manifolds

We now turn to connected sums of topological manifolds. References for this section are Theorem 3.79, Exercises 4-18, 4-19 and Chapter 6 of Lee's book *Introduction to topological manifolds*, second edition.

Definition 2.10.6. Let M and N be two n-dimensional manifolds with boundary, and assume that

$$h:\ \partial N \longrightarrow \partial M$$

is a homeomorphism. Regard h as a map from ∂N to M, the adjunction space

$$M \cup_h N$$

is called the *the gluing of M and N along their boundaries.*

Note that $M \cup_h N = M \sqcup N$ when $\partial M = \varnothing = \partial N$.

Note that also since ∂N is closed in N a∂M is closed in M, h is a closed embedding, and $M \cap_h N$ is of the special class of adjunction spaces discussed in §2.8.2. In particular, we have closed embeddings

$$e:\ M \longrightarrow M \cup_h N \quad \text{and} \quad f:\ N \longrightarrow M \cup_f N$$

and the decomposition

$$M \cup_h N = e(M) \cup f(N) \quad \text{and} \quad e(M) \cap f(N) = q(\partial M) = q(partial N),$$

where $q : M \sqcup N \to M \cup_h N$ is the quotient map.

Theorem 2.10.3. For any two n-dimensional manifolds M and N with boundary and for any homeomorphism $h : \partial N \to \partial M$, the adjunction space $M \cup_h N$ is an n-dimensional without boundary.

Proof. We given an outline of the proof. For details see Theorem 3.79 of John Lee's book.

Since $f : \partial N \to M$ is a closed embedding, $M \cup_h N$ is Hausdorff by Corollary 2.8.1.

Assume that we have proved that $M \cup_h N$ is locally $\mathbb{R}^n$. Then $M \cup_h N$ is second countable by the following fact to be covered in tutorial: If a quotient space P of a second countable space X is locally Euclidean, then P is second countable. It remains to prove that $M \cup_f N$ is locally $\mathbb{R}^n$.

Let $q : M \sqcup N \to M \cup_f N$, and let

$$S = q(\partial M \sqcup \partial N) = e(\partial M) = f(\partial N) = e(M) \cap f(N).$$

We then have the homeomorphism

$$q|_{\operatorname{Int} M \sqcup \operatorname{Int} N} : \quad \operatorname{Int} M \sqcup \operatorname{Int} N \longrightarrow (M \cup_h N) \backslash S.$$

So $(M \cup_h N)\backslash S$ is locally $\mathbb{R}^n$. We thus only need to show that every $s \in S$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Let $s_0 \in S$, and let

$$y_0 \in \partial N \qquad x_0 = h(y_0) \in \partial M, \qquad s_0 = q(x_0) = q(y_0).$$

Choose boundary charts

$$x_0 \in U \xrightarrow{\phi} \widehat{U} = \phi(U) \subset \mathbb{H}^n, \qquad U \subset M,\ \phi(x_0) \in \partial\,\mathbb{H}^n,$$
$$y_0 \in V \xrightarrow{\psi} \widehat{V} = \psi(V) \subset \mathbb{H}^n, \qquad V \subset N,\ \psi(y_0) \in \partial\,\mathbb{H}^n.$$

such that $h(V \cap \partial N) = U \cap \partial M$. Then $q(U \sqcup V)$ is an open neighborhood of s_0 in $M \cup_h N$, and $q(U \sqcup V) = U \cup_{h'} V$, where

$$h' = h|_{V \cap \partial N} : \ V \cap \partial N \longrightarrow U \cap \partial M.$$

One then shows that adjunction of $\phi|_{U \cap \partial M} : U \cap \partial M \to \mathbb{H}^n \subset \mathbb{R}^n$ and a certain

$$\widetilde{\psi} : \ V \cap \partial N : \ V \cap \partial N \longrightarrow -\mathbb{H}^n \subset \mathbb{R}^n$$

gives a homeomorphism from $q(U \sqcup V)$ to an open subset of $\mathbb{R}^n$. □

Definition 2.10.7. For any n-dimensional topological manifold M with boundary, by taking $h : \partial M \to \partial M$ to be the identity map, we have the adjunction space $M \cup_h M$, which is called the *double of M*.

Corollary 2.10.1. Any n-dimensional topological manifold M with boundary can be embedded into an n-dimensional topological manifold M without boundary as a closed subset.

We now show that one can produce a manifold with boundary from any manifold at any interior point. We fix the following notation.

Notation 2.10.1. For integer $n \geq 1$ and $r > 0$, set

$$\begin{aligned} D^n(0;r) &= \{x \in \mathbb{R}^n : \|x\| \leq 1\}, \\ B^n(0;r) &= \{x \in \mathbb{R}^n : \|x\| < 1\}, \\ S^{n-1} &= \{x \in \mathbb{R}^n : \|x\| = 1\}. \end{aligned}$$

Call $D^n(0;r)$ and $B^n(0;r)$ respectively a closed ball and an open ball around 0 in $\mathbb{R}^n$.

Definition 2.10.8. Let M be an n-dimensional manifold without boundary, and let $p \in M$. If (U, ϕ) is a coordinate chart (U, ϕ) on M such that $\phi(p) = 0$ and

$$\phi(U) = B^n(0;R),$$

we call U a *coordinate ball* on M around p, and for $0 < r < R$, we call $B = \phi^{-1}(B(0;r))$ a *regular coordinate ball* on M around p.

Using the definitions, one checks that if B is a coordinate ball on M around p, then $M \backslash B$ is an n-dimensional manifold with

$$\partial M \cong S^{n-1}$$

and M is homeomorphic to $M \cup_h D^n$ for some homeomorphism $h : S^{n-1} = \partial D^n \to \partial M$.

Definition 2.10.9. For $i = 1, 1$, let M_i be an n-dimensional manifold without boundary, and let $p_i \in M_i$. Let B_i be a regular coordinate ball on M_i around p_i, and let

$$h : \partial M_1 \longrightarrow \partial M_2$$

be any homeomorphism via homeomorphisms $\partial M_1 \cong S^{n-1} \cong \partial M_2$. The adjunction space, which we know by Theorem 2.10.3 to be an n-dimensional manifold without boundary, is called a *connected sum* of M_1 and M_2 and is denoted by

$$M_1 \# M_2.$$

We we refer to Exercises 4-18, 4-19, and the discussions in Chapter 6 in John Lee's book for a discussion on connected sums of manifolds.

Chapter 3 — The fundamental group and covering spaces

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3.1 Homotopy

3.1.1 Homotopy between maps

How to tell apart the open unit disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ from $D \backslash \{(0,0)\}$ as topological spaces?

We have been using certain topological properties such as compactness and connectedness to prove that certain topological spaces can not be homeomorphic, but In this chapter, we will introduce the next level of topological invariants, namely the fundamental group of a topological space, which can be used to tell apart D from $D \backslash \{0, 0\}$.

The new idea is that of *homotopy*, which is a weaker equivalence relation among topological spaces and is used in the construction of the fundamental group. If X and Y are two topological spaces, two continuous maps $f_0, f_1 : X \to Y$ are said to be homotopic if they can be connected by a continuous family of continuous maps $f_t : X \to Y$, where $t \in [0,1]$. Here is the precise definition.

Definition 3.1.1. Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous maps. A *homotopy* from f to g is a continuous map $F : X \times [0,1] \to Y$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x)$$

for all $x \in X$. In that case, we say that f and g are *homotopic*, and we write

$$f \sim g \quad \text{or} \quad f \sim_F g.$$

Consider paths in a given topological space Y connecting two given points y_0 and y_1, i.e., continuous maps

$$\gamma : \ [0,1] \longrightarrow Y \quad \text{such that } \ \gamma(0) = y_0, \ \ \gamma(1) = y_1.$$

It is then natural to look at only those homotopies from one such path to another path such that the inbetween maps are also paths from y_0 to y_1. This motivates the following definition.

Definition 3.1.2. Let X and Y be two topological spaces and let $A \subset X$. Consider two continuous maps $f, g : X \to Y$ such that $f|_A = g|_A$, i.e.,

$$f(a) = g(a), \quad \forall\, a \in A.$$

By a *homotopy from f to g relative to A* we mean a continuous map

$$F : \; X \times [0,1] \longrightarrow Y$$

such that $F(x,0) = f(x), F(x,1) = g(x)$ and $F(a,t) = f(a)$ for all $a \in A$ and $t \in [0,1]$. In such a case, we also write

$$f \sim g \ \text{ rel } A \quad \text{ or } \quad f \sim_F g \ \text{ rel } A.$$

Of course Definition 3.1.2 recovers Definition 3.1.1 when $A = \varnothing$.

Notation 3.1.1. For a homotopy $F : X \times [0,1] \to Y$ and for $t \in [0,1]$, we will sometimes use the notation

$$F(\bullet, t) : \; X \longrightarrow Y, \; x \longmapsto F(x,t).$$

Thus if F is a homotopy from f to g, then $F(\bullet,0) = f$ and $F(\bullet,1) = g$.

Example 3.1.1. Given a topological space Y and $y_0, y_1 \in Y$, let

$$\mathcal{P}_{y_0,y_1} = \text{the set of all continuous maps } \gamma : \; [0,1] \longrightarrow Y \text{ such that} \gamma(0) = y_0, \gamma(1) = y_1.$$

Given $\alpha, \beta \in \mathcal{P}_{y_0,y_1}$, a homotopy relative to $\{0,1\}$ from α to β is called a *path-homotopy* from α to β, which, by definition, a continuous map

$$F : \; [0,1] \times [0,1] \longrightarrow Y$$

such that $F(\bullet,0) = \alpha$, $F(\bullet,1) = \beta$ and $F(0,t) = y_0$ and $F(1,t) = y_1$ for all $t \in [0,1]$.

Example 3.1.2. If Y is a convex subspace of $\mathbb{R}^n$, then for any topological space X and for any two continuous maps $f, g : X \to Y$, the map

$$F : \; X \times [0,1] \longrightarrow Y, \; F(x,t) = (1-t)f(x) + tg(x)$$

is a homotopy from f to g. Such an F is called a *straight-line homotopy.*

Example 3.1.3. Let X be any topological space and $f, g : X \to S^n$ any two continuous maps such that

$$f(x) + g(x) \neq 0, \quad \forall\, x \in X.$$

Then for any $x \in X$, the line segment connecting $f(x)$ and $g(x)$ does not pass through $0 \in \mathbb{R}^{n+1}$, i.e., $(1-t)f(x) + tg(x) \neq 0$ for any $x \in X$ and $t \in [0,1]$. Thus

$$F : \; X \times [0,1] \longrightarrow S^n, \; F(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a homotopy from f to g. For example, if $f : X \to S^n$ is not onto, i.e., there exists $p \in S^n$ such that $f(x) \neq p$ for any $x \in X$, then f is homotopic to the constant map $g(x) = -p$ for all $x \in X$.

We now prove two properties of homotopies.

Lemma 3.1.1. For any two topological spaces X and Y and for any subset A of X, the relation of "homotopy relative to A" is an equivalence relation among all continuous maps from X to Y that agree on A.

Proof. For any continuous map $f : X \to Y$, clearly

$$F: \ X \times [0,1] \longrightarrow Y, \ \ F(x,t) = f(x)$$

is a homotopy of f to itself. If $f \sim_F g$ relative to A, define

$$G: \ X \times [0,1] \longrightarrow Y, \ G(x,t) = F(x, 1-t).$$

Then $g \sim_G f$ relative to A. Finally, if $f \sim_F g$ relative to A and $g \sim_G h$ relative to A, then

$$H: \ X \times [0,1] \longrightarrow Y, \ \ H(x,t) = \begin{cases} F(x,2t), & t \in [0,1/2], \\ G(x,2t-1), & t \in [1/2,1], \end{cases}$$

is a homotopy from f to h relative to A. Note that continuity of H is guaranteed by the gluing lemma. $\square$

The next lemma expresses how the homotopy relation respects compositions of continuous maps.

Lemma 3.1.2. Let X, Y, Z be three topological spaces, and let

$$f, g: \ X \longrightarrow Y \quad \text{ and } \quad h, k: \ Y \longrightarrow Z$$

be continuous maps. Let $A \subset X$ and $B \subset Y$. Suppose that

$$f \sim_F g \ \text{ rel } A \quad \text{ and } \quad h \sim_G k \ \text{ rel } B.$$

Then for $h \circ f, h \circ g : X \to Z$ and $h \circ f, k \circ f : X \to Z$, one has

$$(h \circ f) \sim_{h \circ F} (h \circ g) \ \text{ rel } A \quad \text{ and } \quad (h \circ f) \sim_H (k \circ f) \ \text{ rel } f^{-1}(B),$$

where $H : X \times [0,1] \to Z, H(x,t) = G(f(x),t)$.

Proof. The proof is a straightforward checking. $\square$

3.1.2 Homotopy types

We now define homotopies between topological spaces.

Definition 3.1.3. Two topological spaces X and Y are said to be *of the same homotopy type* or *homotopic to each other* if there exist continuous maps

$$f: \ X \longrightarrow Y \quad \text{and} \quad g: \ Y \longrightarrow X$$

such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$. In such a case, we call f a *homotopy* from X to Y and g a *homotopy inverse* of f. We write $X \sim Y$ when X and Y have the same homotopy type.

Lemma 3.1.3. The relation $X \sim Y$ is an equivalence relation among all topological spaces.

Proof. The relation is clearly reflexive and symmetric. To show that it is transitive, assume that

$$f, \ X \longrightarrow Y, \quad g: \ Y \longrightarrow X, \quad h: \ Y \longrightarrow Z \quad \text{and} \quad k: \ Z \longrightarrow Y$$

are such that

$$g \circ f \sim \mathrm{Id}_X, \quad f \circ g \sim \mathrm{Id}_Y, \quad k \circ h \sim \mathrm{Id}_Y, \quad h \circ k \sim \mathrm{Id}_Z.$$

Then by Lemma 3.1.2,

$$\begin{aligned}(g \circ k) \circ (h \circ f) \sim g \circ \mathrm{Id}_Y \circ f = g \circ f \sim \mathrm{Id}_X, \\ (h \circ f) \circ (g \circ k) \sim h \circ \mathrm{Id}_Y \circ k = h \circ k \sim \mathrm{Id}_Z.\end{aligned}$$

Thus X and Z are homotopic. □

Corollary 3.1.1. Homeomorphic spaces are homotopic.

The next example shows that being homotopic is a weaker condition than being homeomorphic.

Lemma 3.1.4. Any convex subset of $\mathbb{R}^n$ is homotopic to a one point space. In particular, $\mathbb{R}^n$ is homotopic to a one point set.

Proof. Let X be a convex subset of $\mathbb{R}^n$ and choose any $x_0 \in X$. Let $f : X \to \{x_0\}$ be the constant map and $g : \{x_0\} \to X$ by $g(x_0) = x_0$. Then $g \circ f = f$ and $f \circ g = \mathrm{Id}_{\{x_0\}}$. We have seen in Example 3.1.2 that $f \sim \mathrm{Id}_X$. Thus $X \sim \{x_0\}$. □

Note that $\mathbb{R}^n$ is not homeomorphic to a one point space, as the former is non-compact while the latter is compact.

Definition 3.1.4. A topological space that is homotopic to a one point space is said to be *contractible.*

Thus any convex subset of $\mathbb{R}^n$ is contractible.

Definition 3.1.5. Let X be a topological space and $A \subset X$.

1) A *retraction from X to A* is continuous map $f : X \to A$ such that $f(a) = a$ for all $a \in A$, and the subset A is called a *retract* if there exists a retraction from X to A.

2) A *deformation retraction from X to A* is a homotopy $G : X \times [0,1] \longrightarrow X$ such that

$$G(x,0) = x \quad \text{and} \quad G(x,1) \in A \quad \forall\ x \in X.$$

The subset A is called a *deformation retract* of X if there exists a deformation retraction from X to A.

Note that if $G : X \to [0,1]$ is a deformation retraction from X to A, then

$$f:\ A \longrightarrow X,\ \ f(a) = a \quad \text{and} \quad g:\ X \longrightarrow A,\ g(x) = G(x,1)$$

are homotopy inverses of each other, so A and X have the same homotopy type.

Example 3.1.4. The n-sphere $S^n \subset \mathbb{R}^{n+1}$ is a deformation retract of $\mathbb{R}^{n+1}\backslash\{0\}$. Indeed, the map

$$G:\ \mathbb{R}^{n+1}\backslash\{0\} \times [0,1] \longrightarrow \mathbb{R}^{n+1}\backslash\{0\}, \quad G(x,t) = (1-t)x + t\frac{x}{\|x\|},$$

is a deformation retraction from $\mathbb{R}^{n+1}\backslash\{0\}$ to S^n. In particular, $\mathbb{R}^{n+1}\backslash\{0\}$ has the same homotopy type as S^n.

Example 3.1.5. For any convex subset of $\mathbb{R}^n$ and any $x_0 \in X$, the one point set $\{x_0\}$ is a deformation retract of X.

3.2 The Fundamental Group

3.2.1 Construction and first properties

$[\qquad\qquad]$
$0 \qquad\qquad 1$

We are now ready to construct the fundamental group of a topological space.

Recall that for a topological space X and two points $x_0, x_1 \in X$, a path in X from x_0 to x_1 is a continuous map

$$\gamma:\ [0,1] \longrightarrow X$$

such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. For such a path, we denote by $[\gamma]$ the path-homotopy equivalence class of γ.

Prove that $(f*g)_{t_0} = \begin{cases} f\left(\frac{t-0}{t_0-0}\right) & \text{if } 0 \le t \le t_0 \\ g\left(\frac{t-t_0}{1-t_0}\right) & \text{if } t_0 \le t \le 1 \end{cases}$ $\quad \frac{1}{2}$ $\quad$ uniformly conv

$\sim (f*g)_{t_1}$ $\qquad$ Pf: Define $H(s,t) = (f*g)_{\frac{s-1}{0-1}t_0 + \frac{s-0}{1-0}t_1}$

$s \to 1$

source homotopy

composable if the starting point of beta equals to the ending point of alpha

Preserves homotopy class

但其实中间那个点怎么选不重要

Let α be a path in X from x_0 to x_1, and β a path in X from x_1 to x_2. The *composition*, also called the *concatenation*, of the paths α and β is denoted by $\alpha * \beta$ and is defined as the path from x_0 to x_2 given by

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$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } \quad t \in [0, 1/2], \\ \beta(2t - 1) & \text{if } \quad t \in [1/2, 1]. \end{cases}$$

Note again that $\alpha * \beta$ is continuous by the gluing lemma. For each $x \in X$, let $e_x : [0, 1] \to X$ be the constant path defined by

$$e_x(t) = x, \qquad t \in [0, 1].$$

Definition 3.2.1. For a path α from x_0 to x_1 and a path β from x_1 and x_2, we define the operation $*$ on the path-homotopy classes by

$$[\alpha] * [\beta] = [\alpha * \beta].$$

Theorem 3.2.1. [The fundamental groupoid] The operation $*$ is well-defined and the following hold.

1. For any paths α, β, γ in X, if $([\alpha] * [\beta]) * [\gamma]$ is defined, then $[\alpha] * ([\beta] * [\gamma])$ is defined and
$$([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma]).$$
2. For any path γ from x_0 to x_1, we have
$$[\gamma] * [e_{x_1}] = [\gamma] \quad \text{and} \quad [e_{x_0}] * [\gamma] = [\gamma].$$
3. For any path γ from x_0 to x_1, let γ^{-1} be the path defined by $\gamma^{-1}(t) = \gamma(1 - t)$ for $0 \leqslant t \leqslant 1$. Then
$$[\gamma] * [\gamma^{-1}] = [e_{x_0}] \quad \text{and} \quad [\gamma^{-1}] * [\gamma] = [e_{x_1}].$$

Proof. Let's first prove that $*$ is well-defined. Assume thus that

$$\alpha' \sim_F \alpha \ \text{ rel } \{0, 1\} \quad \text{ and } \quad \beta' \sim_G \beta \ \text{ rel } \{0, 1\}.$$

Then it is easy to see that $\alpha' * \beta' \sim_H \alpha * \beta$ rel $\{0, 1\}$, where

$$H(s, t) = \begin{cases} F(2s, t), & s \in [0, 1/2], \\ G(2s - 1, t), & s \in [1/2, 1]. \end{cases}$$

Thus $[\alpha' * \beta'] = [\alpha * \beta]$.

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Please check the details of category

We now check that the partially defined multiplication is associative, i.e., assuming that α is a path from x_0 to x_1, β a path from x_1 to x_2,and γ a path from x_2 to x_3, we need to show that the two paths $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ from x_0 to x_3 are homotopic relative to $\{0, 1\}$. One proves this by directly checking that

$$(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ f : \ [0,1] \longrightarrow X,$$

where $f : [0, 1] \to [0, 1]$ is given by

$$f(s) = \begin{cases} 2s, & s \in [1, \frac{1}{4}], \\ s + \frac{1}{4}, & s \in [\frac{1}{4}, \frac{1}{2}], \\ \frac{s+1}{2}, & s \in [\frac{1}{2}, 1]. \end{cases}$$

Since $[0, 1]$ is convex and $f(0) = 0$ and $f(1) = 1$, there is a homotopy from f to the identity map of $[0, 1]$ relative to $\{0, 1\}$. It follows from Lemma 3.1.2 that

$$(\alpha * \beta) * \gamma = (\alpha * (\beta * \gamma)) \circ f \sim (\alpha * (\beta * \gamma)) \circ \mathrm{Id}_{[0,1]} = \alpha * (\beta * \gamma).$$

The rest of the statements are proved similarly and we omit them. See § 5.2 of Armstrong's book. □

Definition 3.2.2. Let X be a topological space and let $x_0 \in X$. A *loop* with *base point* x_0, or simply a *loop at* x_0, is a path γ in X from x_0 to x_0.

Corollary 3.2.1. Let X be a topological space and let $x_0 \in X$. The set of all homotopy classes of loops with base point x_0 forms a group under the operation $*$. We call this the *fundamental group* (or *first homotopy group*) with base point x_0, and denote this by $\pi_1(X, x_0)$.

Theorem 3.2.2. Let X be a path-connected space and let $x_0, x_1 \in X$. For any path γ in X from x_0 to x_1, the map

$$\gamma_* : \ \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1), \ \ [\alpha] \longmapsto [(\gamma^{-1} * \alpha) * \gamma)],$$

is a group isomorphism.

Proof. If α is a loop at x_0, then $(\gamma^{-1} * \alpha) * \gamma)$ is a loop at x_1, and by Theorem 3.2.1, γ_* is a well-defined group homomorphism. Moreover, γ_* has $(\gamma^{-1})_*$ as its inverse. Thus γ_* is a group isomorphism. □

Therefore, for a path-connected space X, we may define the *fundamental group* of X to be a group isomorphic to any $\pi_1(X, x_0)$. We denote this by $\pi_1(X)$.

Definition 3.2.3. Let X be a topological space. We say that X is *simply connected* if X is path-connected and $\pi_1(X)$ is the trivial group.

Example 3.2.1. Let Y be any convex subset of $\mathbb{R}^n$. Then Y is path-connected. Any loop in Y is homotopic to a constant loop, so Y is simply connected. In particular, $\mathbb{R}^n$ is simply connected.

We now turn to properties of fundamental groups. The properties stated in the following Proposition 3.2.1 and Proposition 3.2.2 are also referred to as the *functoriality* of the fundamental group.

Proposition 3.2.1. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. For any $x \in X$ and $y = f(x)$, the map

$$f_* : \ \pi_1(X, x) \longrightarrow \pi_1(Y, y), \ \ f_*([\gamma]) = [f \circ \gamma],$$

is a well-defined group homomorphism. We call f_* the homomorphism induced by f.

Proof. It is clear that for any loop γ at x, $f \circ \gamma$ is a loop at y. By Lemma 3.1.2, if $[\gamma] = [\gamma']$ then $[f \circ \gamma] = [f \circ \gamma']$. Thus f_* is well-defined. Furthermore, since for any loop α, β at x,

$$f \circ (\alpha * \beta) = (f \circ \alpha) * (f \circ \beta),$$

we see that f_* is a group homomorphism. □

The following fact also straightforward to prove.

Proposition 3.2.2. Let X, Y and Z be topological spaces and $f : X \to Y$ and $g : Y \to Z$ be continuous maps. For any $x \in X$ and $y = f(x) \in Y$ and $z = g(y) \in Z$, one has

$$(g \circ f)_* = g_* \circ f_* : \ \pi_1(X, x) \longrightarrow \pi_1(Z, z).$$

Corollary 3.2.2. Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. Then for any $x \in X$ and $y = f(x) \in Y$,

$$f_* : \ \pi_1(X, x) \longrightarrow \pi_1(Y, y)$$

is an isomorphism of groups.

Thus if X and Y are path-connected spaces which are homeomorphic, then they have isomorphic fundamental groups. We formulate this fact in the following corollary.

Corollary 3.2.3. The fundamental group is a topological invariant for path-connected topological spaces.

The following proposition tells us how to compute the fundamental groups for product spaces.

Proposition 3.2.3. Let X and Y be path-connected spaces. Then $\pi_1(X \times Y)$ is isomorphic to the product group and $\pi_1(X) \times \pi_1(Y)$.

Proof. Fix $x \in X$ and $y \in Y$ and we set

$$\pi_1(X \times Y) = \pi_1(X \times Y, (x,y)), \quad \pi_1(X) = \pi_1(X,x), \quad \pi_1(Y) = \pi_1(Y,y).$$

Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the two projections. One then has

$$\phi := p_{1*} \times p_{2*} : \ \pi_1(X \times Y) \longrightarrow \pi_1(X) \times \pi_1(Y), \ [\alpha] \longmapsto (p_{1*}([\alpha]), \ p_{2*}([\alpha])).$$

Equip $\pi_1(X) \times \pi_2(Y)$ the product group structure. By Proposition 3.2.1, both p_{1*} and p_{2*} are group homomorphisms, so ϕ is also a group homomorphism. We now show that ϕ is an isomorphism by proving that ϕ is bijective.

To show that ϕ is injective, since it is a group homomorphism, we only need to show that its kernel is trivial. Suppose that α is a loop at (x,y) such that

$$p_1 \circ \alpha \sim_F e_x, \quad p_2 \circ \alpha \sim_G e_y.$$

Then it is straightforward to see that $\alpha \sim_H e_{(x,y)}$, where

$$H : \ [0,1] \times [0,1] \longrightarrow X \times Y, \ H(s,t) = (F(s,t), \ G(s,t)).$$

Thus ϕ is injective. To show that ϕ is surjective, for any loop β in X at x and any loop γ in Y at y, the loop α in $X \times Y$ at (x,y) defined by

$$\alpha(s) = (\beta(s), \ \gamma(s)), \quad s \in [0,1],$$

satisfies $\phi([\alpha]) = ([\beta],[\gamma])$. Thus ϕ is surjective. 这个要看 □

Recall that we have proved that the fundamental group is a topological invariance. We now prove a stronger result, namely the fundamental group is a homotopy invariant. More precisely, we prove that two path-connected spaces of the same homotopy type have isomorphic fundamental groups.

Consider first two homotopic maps $f \sim_F g : X \to Y$. Let $x \in X$ and consider

$$\gamma : \ [0,1] \longrightarrow Y, \ \gamma(t) = F(x,t), \quad t \in [0,1],$$

which is a path from $\gamma(0) = f(x)$ to $\gamma(1) = g(x)$ in Y. Define again

$$\gamma_* : \ \pi_1(Y, f(x)) \longrightarrow \pi_1(Y, g(x)), \ \gamma_*([\alpha]) = [\gamma^{-1} * \alpha * \gamma].$$

By Theorem 3.2.2, γ_* is a group isomorphism.

Lemma 3.2.1. With the notation as above, one has

$$g_* = \gamma_* \circ f_* : \ \pi_1(X,x) \longrightarrow \pi_1(Y, g(x)).$$

也看一下

Proof. Let α be any loop at $x \in X$. We need to show that the two loops $g \circ \alpha$ and $\gamma^{-1} * (f \circ \alpha) * \gamma$ at $g(x)$ are homotopic relative to $\{0,1\}$. One checks directly that the following $H : [0,1] \times [0,1] \to Y$ is an explicit homotopy from $g \circ \alpha$ to $\gamma^{-1} * (f \circ \alpha) * \gamma$:

$$H(s,t) = \begin{cases} \gamma(1-4s), & 0 \le s \le \frac{1-t}{4}, \\ F\left(\alpha\left(\frac{4s+t-1}{3t+1}\right), t\right), & \frac{1-t}{4} \le s \le \frac{1+t}{2}, \\ \gamma(2s-1), & \frac{1+t}{2} \le s \le 1. \end{cases}$$

□

Corollary 3.2.4. Path-connected topological spaces of the same homotopy type have isomorphic fundamental groups.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be such that

$$\mathrm{Id}_X \sim_F (g \circ f) \quad \text{and} \quad \mathrm{Id}_Y \sim_G (f \circ g).$$

Fix $y \in Y$ and let $x = g(y)$, We now prove that the group homomorphism

$$f_* : \ \pi_1(X,x) \longrightarrow \pi_1(Y, f(x))$$

is bijective so it is an isomorphism. Let $\gamma : [0,1] \to X, \gamma(t) = F(x,t)$. Then $\gamma(0) = x$ and $\gamma(1) = g(f(x))$. By Lemma 3.2.1,

$$g_* \circ f_* = (g \circ f)_* = \gamma_* : \ \pi_1(X,x) \longrightarrow \pi_1(Y, g(f(x)))$$

is an isomorphism. Thus f_* is injective and g_* is surjective. Switching the role of f and g, one sees that f_* is also surjective. Thus f_* is bijective. □

3.2.2 The Brouwer fixed-point theorem

In this section, we assume that $\pi_1(S^1) \cong \mathbb{Z}$, and we will use it is prove the Brouwer fixed-point theorem for $n = 1$.

For any integer $n \ge 1$, let

$$D^n = \{x \in \mathbb{R}^n : \ \|x\} \le 1\}$$

and call it the n-dimensional unit ball. Its boundary is the $(n-1)$-dimensional sphere S^{n-1}.

Theorem 3.2.3 (The Brouwer fixed-point theorem)**.** For any integer $n \ge 1$, any continuous map $f : D^n \to D^n$ must have a fixed point, i.e., there exists $x_0 \in D^n$ such that $f(x_0) = x_0$.

The proof of the Brouwer fixed-point theorem is a beautiful application of the topological invariants. Interestingly, for different n, we need to use different topological invariants. As we will see, for $n = 1$, we use connectedness, and for $n = 2$, we use the fundamental group. For higher n, we need to use higher homology groups, a topic in more advanced topology courses.

Proof of the Brouwer fixed-point theorem for $n = 1$. For $n = 1$, $D^1 = [-1, 1]$. Suppose that $f : [-1, 1] \to [-1, 1]$. If $f(-1) = -1$ or $f(1) = 1$, we are done. Assume that

$$f(-1) > -1 \quad \text{and} \quad f(1) < 1,$$

and consider $g(x) = f(x) - x$. Then $g(-1) > 0$ and $g(1) < 0$. Since $[-1, 1]$ is connected, by the Intermediate Value Theorem, there exists $x_0 \in [-1,]$ such that $g(x_0) = 0$, i.e., $f(x_0) = x_0$. This finishes the proof of the Brouwer fixed-point theorem for $n = 1$.

Note that to apply the Intermediate Value Theorem, we rely in the crucial property of the interval $[-1, 1]$ being connected.

To prove the Brouwer fixed-point theorem for the case of $n = 2$, we recall that for a topological space X and a subset A of X, a retraction from X to A is a continuous map $g : X \to A$ such that $g(a) = a$ for all $a \in A$.

Lemma 3.2.2. If $g : X \to A$ is a retraction, then for any $a \in A$, the group homomorphism $g_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

Proof. Let $i : A \to X$ be the inclusion map. By the definition of $g : X \to A$ being a retraction, $g \circ i = \mathrm{Id}_A$. One thus has

$$g_* \circ i_* = (g \circ i)_* = (\mathrm{Id}_A)_* = \mathrm{Id}_{\pi_1(A,a)} : \ \pi_1(A, a) \longrightarrow \pi_1(A, a).$$

It follows that $g_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective. □

Proof of the Brouwer fixed-point theorem for $n = 2$. Suppose that $f : D^2 \to D^2$ is continuous but has not fixed point. Then for each $x \in D^2$, $f(x) \neq x$, so we can draw the half line L_x from $f(x)$ to x and denote by $g(x)$ the intersection of this half line with the boundary circle S^1 of D^2. Then we have a continuous map

$$g : \ D^2 \longrightarrow S^1 \quad \text{such that} \quad g(x) = x \quad \forall\, x \in S^1.$$

In other word, g is a retraction of D^2 to its subset S^1. Let $x_0 = (1, 0) \in S^1$. By Lemma 3.2.2, $g_* : \pi_1(D^2, x_0) \to \pi_1(S^1, x_0)$ is surjective, which is not possible because $\pi_1(D^2, x_0) = \{e\}$ and $\pi_1(S^1, x_0) \cong \mathbb{Z}$. This finishes the proof of the Brouwer fixed-point theorem for $n = 2$.

3.2.3 Examples of fundamental groups

We now give examples of fundamental groups. The proofs of some of the facts used in this section will be given in a later section.

We have seen our first examples: any convex subset of $\mathbb{R}^n$ has trivial fundamental group, i.e. are simply connected. Our other examples will be based on the following theorem on the fundamental groups of the unit n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

where n is any positive integer.

Theorem 3.2.4. 1) For $n = 1$, $\pi_1(S^1) \cong \mathbb{Z}$;

2) For $n \geq 2$, $\pi_1(S^n)$ is trivial, i.e., S^n is simply connected.

We prove Theorem 3.2.4 as special cases of some general facts.

Proposition 3.2.4. Let X be a topological space and let U, V be simply connected open sets of X such that $U \cup V = X$ and $U \cap V$ is nonempty and path-connected. Then X is simply connected.

Proof. We first show that X is path-connected. Indeed, for any $x_0, x_1 \in X$, if they both lie in U or in V, then there is a path in either U or V connecting them. Suppose that $x_0 \in U$ and $x_1 \in V$. Choose any $p \in U \cap V$. Then the product of a path in U from x_0 to p with a path in V from p to x_1 is a path connecting x_0 and x_1. Thus X is path-connected.

To show that X is simply connected, we fix any $p \in U \cap V$ and we only need to show that any loop $\alpha : [0,1] \to X$ at p must be homotopic to the trivial loop at p. For notational simplicity, set $I = [0,1]$.

For any $t \in I$, the point $\alpha(t) \in X$ is either in U or in V. As both U and V are open, by continuity of α, we know that there exists $\epsilon_t > 0$ such that

$$\alpha((t - 2\epsilon, t + 2\epsilon) \cap I) \subset U \quad \text{or} \quad \alpha((t - 2\epsilon, t + 2\epsilon) \cap I) \subset V.$$

The collection $\{(t - 2\epsilon, t + 2\epsilon) \cap I : t \in I\}$ is then an open cover of I. As I is compact, By Lebesgue's Lemma 2.6.9, there exist

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$$

such that for $k = 1, \ldots, n$, $\alpha([t_{k-1}, t_k])$ is contained either in U or in V. Let $k = 1, \ldots, n-1$, let $p_k = \alpha(t_k)$, and choose a path γ_k from p to p_k as follows:

1) If $p_k \in U \backslash V$, let γ_k be a paths in U from p to p_k;

2) If $p_k \in V \backslash U$, let γ_k be a paths in V from p to p_k;

3) If $p_k \in U \cap V$, let γ_k be a paths in $U \cap V$ from p to p_k;

Set also $p_0 = p_n = p$. For $k = 1, \ldots, n$, let also $\alpha_k : [0,1] \to X$ be the path from p_{k-1} to p_k given by

$$\alpha_k(s) = \alpha((t_k - t_{k-1})s + t_{k-1}), \quad s \in [0,1],$$

and define the loop β_k at p by

$$\beta_1 = \alpha_1 * \gamma_1^{-1}, \ \ \beta_2 = \gamma_1 * \alpha_2 * \gamma_2^{-1}, \quad \beta_3 = \gamma_2 * \alpha_3 * \gamma_3^{-1}, \quad \dots, \quad \beta_n = \gamma_{n-1} * \alpha_n.$$

Note that each β_k is loop at p contained either in U or in V. Using the fact that both U and V are simply connected, we know that each β_k is homotopic to the trivial loop at p. On the other hand, one knows by Theorem 3.2.1 that

$$[\alpha] = [\alpha_1 * \alpha_2 * \cdots * \alpha_n] = [\beta_1 * \beta_2 * \cdots * \beta_n] = [\beta_1] * [\beta_2] * \cdots * [\beta_n] = [e_p].$$

In other words, α is homotopic to the trivial loop at p. □

Corollary 3.2.5. The n-sphere S^n for $n \geqslant 2$ is simply connected.

Proof. Clearly S^n is clearly path connected. Taking any $p \in S^n$ and $U = S^n \backslash \{p\}$ and $V = S^n \backslash \{-p\}$, one applies Proposition 3.2.4 to conclude that S^n is simply connected. □

The case of the circle is a special case of the following fact on the fundamental groups of orbit spaces.

Theorem 3.2.5. *If a group G acts on a simply connected space X by homeomorphisms such that for each $x \in U$ there exists an open neighborhood U of x in X such that $U \cap gU = \varnothing$ for all $g \in G\backslash\{e\}$, then $\pi_1(X/G)$ is isomorphic to G.*

In the context of Theorem 3.2.5, the quotient map $p : X \to X/Q$ is an example of a *covering map*, and X is a *covering space* of X/G. We will study covering maps and covering spaces in the next chapter. For now we assume that Theorem 3.2.5 holds and we apply it to the case of the circle S^1, the real projective spaces, and the Klein bottle K.

Corollary 3.2.6. $\pi_1(S^1) \cong \mathbb{Z}$.

Proof. Recall that S^1 is homeomorphic to the orbit space $\mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation. Conditions of Theorem 3.2.5 are clearly met, so we see that $\pi_1(S^1) \cong \mathbb{Z}$. The homotopy class of the loop

$$\alpha : \ [0,1] \longrightarrow S^1, \ \ \alpha(t) = e^{2\pi i t},$$

is a generator for the infinite cyclic group $\pi_1(S^1)$. □

Example 3.2.2. For any integer $n \geq 1$, the n-torus T^n is homeomorphic to the orbit space $\mathbb{R}^n/\mathbb{Z}^n$, where $\mathbb{Z}^n$ acts by translation. Applying Theorem 3.2.5, we see that $\pi_1(T^n) \cong \mathbb{Z}^n$.

Example 3.2.3. Recall that the projective space $\mathbb{RP}^n = S^n/\mathbb{Z}_2$, so $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$.

Example 3.2.4. Recall from Lemma 2.9.5 that the Klein bottle is homeomorphic to the orbit space $\mathbb{R}^2/G$, where G is the group generated by the two elements a and b with the relation $aba = b$. In other words, G is the set $G = \{a^m b^n : m, n \in \mathbb{Z}\}$ with the group multiplication

$$a^m b^n a^{m'} b^{n'} = a^{m+(-1)^n m'} b^{n+n'},$$

and G acts on $\mathbb{R}^2$ via

$$a^m b^n \cdot (x, y) = ((-1)^n x + m,\ y + n), \qquad m, n \in \mathbb{Z}, (x, y) \in \mathbb{R}^2.$$

Conditions of Theorem 3.2.5 are met, and thus $\pi_1(K) \cong G$ as groups.

The homotopy invariance of the fundamental group also leads to conclusions on the fundamental groups of the cylinder, the Mobius strip, and $\mathbb{R}^n \backslash \{0\}$.

First recall from Definition 3.1.5 that for a topological space X and a subset A, we say that A is a deformation retract of X if there exists a deformation retraction from X to A, i.e., a homotopy $G : X \times [0, 1] \longrightarrow X$ such that

$$G(x, 0) = x \quad \text{and} \quad G(x, 1) \in A \quad \forall\ x \in X,$$

and we have shown that if A is a deformation retract of X, then A and X have the same homotopy type and thus isomorphic fundamental groups.

Corollary 3.2.7. 1) The 2-dimensional cylinder, the Mobius strip, as well as $\mathbb{R}^2 \backslash \{0\}$ all have fundamental group isomorphic to $\mathbb{Z}$;

2) For $n \geq 3$, the fundamental group of $\mathbb{R}^n \backslash \{0\}$ is trivial.

Proof. 1) The 2-dimensional cylinder, the Mobius strip, as well as $\mathbb{R}^2 \backslash \{0\}$ all have a circle as a deformation retract, and we know that $\pi_1(S^1) \cong \mathbb{Z}$.

2) For $n \geq 3$, $\mathbb{R}^n \backslash \{0\}$ has S^{n-1} as a deformation retract, and we know that $\pi_1(S^{n-1}) = \{e\}$ as $n - 1 \geq 2$. □

3.3 Covering spaces

3.3.1 Covering Spaces

Definition 3.3.1. Let X and $\widetilde{X}$ be topological spaces and $p : \widetilde{X} \to X$ a continuous surjective map.

1) An open set U in X is said to be *evenly covered* by p if $p^{-1}(U)$ is the union of the collection $\{\widetilde{U}_\lambda : \lambda \in \Lambda\}$ of pairwise disjoint open sets of $\widetilde{X}$ and such that for each $\lambda \in \Lambda$, the restriction

$$p|_{\widetilde{U}_\lambda} : \ \widetilde{U}_\lambda \longrightarrow U$$

is a homeomorphism,

2) The map p is called a *covering map* if for each point $x \in X$, there exists an open neighborhood of x which is evenly covered by p. In such a case, we also say that $\widetilde{X}$ is a *covering space* of X.

For each point $x \in X$, the subset $p^{-1}(\{x\})$ of $\widetilde{X}$ is called the *fiber* of p over x.

Example 3.3.1. Let X be a topological space and let Y be the discrete topological space $\{1, 2, \ldots, n\}$. Then $X \times Y$ is a covering space of X.

Example 3.3.2. Let n be a positive integer. The map $p : S^1 \to S^1$ defined by $p(z) = z^n$ is a covering map. The fiber over each point consists of n points.

Example 3.3.3. The topological space $\mathbb{R}$ is a covering space of S^1. A covering map $p : \mathbb{R} \to S^1$ is given by $p(x) = e^{2\pi i x}$. Each fiber has infinitely many points.

The proofs of the following two lemmas are straightforward and we omit the proofs.

Lemma 3.3.1. If $p_1 : \widetilde{X} \to X$ and $p_2 : \widetilde{Y} \to Y$ are covering maps, then the map

$$p : \ \widetilde{X} \times \widetilde{Y} \longrightarrow X \times Y, \ \ p((x, y)) = (p_1(x), p_2(y))$$

is a covering map.

Lemma 3.3.2. Let X, Y and Z be topological spaces. Let $p_1 : X \to Y$ and $p_2 : Y \to Z$ be covering maps. If $p_2^{-1}(\{z\})$ is finite for every $z \in Z$, then $p_2 \circ p_1$ is a covering map.

Exercise 3.3.1. Show that a covering map is an open map.

Lemma 3.3.3. Let $p : \widetilde{X} \to X$ be a covering map and let $U \subset X$ be an evenly covered open subset of X with

$$p^{-1}(U) = \bigcup_{\lambda \in \Lambda} \widetilde{U}_\lambda,$$

Then for each $x \in U$ and $\lambda \in \Lambda$, the set $p^{-1}(x) \cap \widetilde{U}_\lambda$ has exactly one point which is

$$\phi(x, \lambda) = \left(p|_{\widetilde{U}_\lambda}\right)^{-1}(x).$$

The map $\phi : U \times \Lambda \longrightarrow p^{-1}(U)$ is a homeomorphism, where U has the subspace topology from X, Λ has the discrete topology, and $U \times \Lambda$ has the product topology.

Proof. Let $\lambda \in \Lambda$. Since $p|_{\widetilde{U}_\lambda} : \widetilde{U}_\lambda \to U$ is a homeomorphism, for each $x \in U$, $\phi(x, \lambda) = \left(p|_{\widetilde{U}_\lambda}\right)^{-1}(x)$ is the unique point in $p^{-1}(x) \cap \widetilde{U}_\lambda$. By definition,

$$p(\phi(x, \lambda)) = x, \qquad \forall\ x \in U,\ \lambda \in \Lambda.$$

If $(x, \lambda), (x', \lambda') \in U \times \Lambda$ are such that $\phi(x, \lambda) = \phi(x', \lambda')$, by applying p, one gets $x = x'$, and since $\phi(x, \lambda) = \phi(x, \lambda') \in \widetilde{U}_\lambda \cap \widetilde{U}_{\lambda'}$, and since $\widetilde{U}_\lambda \cap \widetilde{U}_{\lambda'} = \varnothing$ if $\lambda \neq \lambda'$, we must have

$\lambda = \lambda'$. This shows that ϕ is injective. For any $\tilde{x} \in p^{-1}(U)$, one has $\tilde{x} \in \widetilde{U}_\lambda$ for some $\lambda \in \Lambda$, and thus $\phi(p(\tilde{x}), \lambda) = \tilde{x}$. This shows that ϕ is surjective. Thus ϕ is bijective.

With the subspace topology on U and the discrete topology on Λ, the open subsets of the product topology on $U \times \Lambda$ are precisely of the form $U' \times \Lambda'$, where U' is any open subset of X contained in U and Λ' is any subset of Λ. For any such $U' \subset U$ and $\Lambda' \subset \Lambda$, it follows from the definition of ϕ that

$$\phi(U' \times \Lambda') = \bigcup_{\lambda \in \Lambda'} \left(p|_{\widetilde{U}_\lambda}\right)^{-1}(U'),$$

so $\phi(U' \times \Lambda')$ is open in $\widetilde{X}$ and thus in $p^{-1}(U)$. Furthermore, for any open subset $\widetilde{U}'$ of $\widetilde{X}$ that is contained in $p^{-1}(U)$, $\phi^{-1}(\widetilde{U}) = U' \times \Lambda'$, where $U' = p(\widetilde{U})$ is open in X (see Exercise 3.3.1) and contained in U, and

$$\Lambda' = \{\lambda \in \Lambda : \widetilde{U} \cap \widetilde{U}_\lambda \neq \varnothing\}.$$

This shows that ϕ is a homeomorphism from $U \times \Lambda$ to $p^{-1}(U)$. □

Corollary 3.3.1. If $p : \widetilde{X} \to X$ is a covering map, then
1) for any $x \in X$, the fiber over x is a discrete subspace of $\widetilde{X}$;
2) if X is connected, then all the fibers of p have the same cardinality.

Proof. 1) follows directly from Lemma 3.3.3.

For 2), For any $x \in X$ and let W_x be the subset of X consisting of all $x' \in X$ such that $p^{-1}(x')$ has the same cardinality as $p^{-1}(x)$, i.e., there exists a bijection between $p^{-1}(x')$ and $p^{-1}(x)$. Then $W_x \neq \varnothing$ because $x \in W_x$. It clear that

$$\{W_x : x \in X\}$$

is a partition of X. Let $x \in X$. Suppose that $x' \in W_x$. Then $W_x = W_{x'}$. Let U be evenly covered open neighborhood of x' in X. By Lemma 3.3.3, $U \subset W_{x'} = W_x$. Thus W_x is an open subset of X. The complement of W_x in X, being either empty or a union of $W_{x'}$ for some collection $x' \in X$, is again open. Then W_x is both open and closed in X. Since $W_x \neq \varnothing$ and X is connection. One has $W_x = X$. Thus all the fibers of p have the same cardinality. □

Definition 3.3.2. For a covering map $p : \widetilde{X} \to X$ where X is connected, if all the fibers of p have n elements, we say that $\widetilde{X}$ is an *n-fold covering* of X.

3.3.2 Lifting lemmas and the monodromy action

Definition 3.3.3. Let $X, \widetilde{X}$ and Y be topological spaces. Let $p : \widetilde{X} \to X$ and $f : Y \to X$ be continuous maps. A *lifting* of f is a continuous function $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 3.3.1. [Path-lifting Lemma.] Let $p : \widetilde{X} \to X$ be a covering map. Let $\tilde{x}_0 \in \widetilde{X}$ and $p(\tilde{x}_0) = x_0$. Then any path $\gamma : [0,1] \to X$ starting at x_0 has a unique lifting $\widetilde{\gamma}$ in $\widetilde{X}$ starting at $\tilde{x}_0$. We also call $\tilde{\gamma}$ *the lifting of* γ *based at* $\tilde{x}_0$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

Proof. Let $\mathcal{F}$ be the collection of all open subsets of X that are evenly covered by p. Then $\mathcal{F}$ is an open cover of X, so $\{\gamma^{-1}(U) : U \in \mathcal{F}\}$ is an open cover of $[0,1]$. By Lebesgue's Lemma, there exists $\delta > 0$ such that every sub-interval of $[0,1]$ with length less than δ is totally contained in some member of $\{\gamma^{-1}(U) : U \in \mathcal{F}\}$. Let $U_0 \in \mathcal{F}$ be such that $\gamma([0,t_1]) \subset U_0$, and let $\widetilde{U}_0$ be the connected component of $p^{-1}(U_0)$ containing $\tilde{x}_0$. Using the homeomorphism $p|_{\widetilde{U}_0} : \widetilde{U}_0 \to U_0$, one can define

$$\tilde{\gamma} : \ [0,t_1] \longrightarrow \widetilde{X}, \ \ \tilde{g}(s) = \left(p|_{\widetilde{U}_0}\right)^{-1}(\gamma(s)).$$

Inductively, suppose that $1 \le k \le m-1$ and that we have lifted $g|_{[0,t_k]}$ to $\tilde{\gamma} : [0,t_k] \to \widetilde{X}$, let $U_k \in \mathcal{F}$ be such that $\gamma([t_k, t_{k+1}]) \subset U_k$, and let $\widetilde{U}_k$ be the connected component of $p^{-1}(U_k)$ containing $\tilde{\gamma}(t_k)$. Using the homeomorphism $p|_{\widetilde{U}_k} : \widetilde{U}_k \to U_k$, one can define

$$\tilde{\gamma} : \ [0,t_{k+1}] \longrightarrow \widetilde{X}, \ \ \tilde{g}(s) = \begin{cases} \tilde{\gamma}(s), & s \in [0,t_k], \\ \left(p|_{\widetilde{U}_k}\right)^{-1}(\gamma(s)), & s \in [t_k, t_{k+1}]. \end{cases}$$

Carrying on until $k = m-1$, we then have a lift $\tilde{\gamma}$ of γ. Note that the continuity of γ is guaranteed by the gluing lemma.

To show that the lifting $\tilde{\gamma}$ is unique, note that on each $[t_k, t_{k+1}]$ the lifting of γ is unique. Thus $\tilde{\gamma}$ is unique on $[0,1]$. $\square$

Example 3.3.4. Clearly the unique lifting of the trivial loop e_{x_0} in X at $x_0 \in X$ is the trivial loop $e_{\tilde{x}_0}$ in $\widetilde{X}$ at $\tilde{x}_0$.

To understand the effect of path-lifting on homotopy classes of paths, we need to also lift homotopies.

Theorem 3.3.2. [Homotopy-lifting Lemma]. Let $p : \widetilde{X} \to X$ be a covering map, and let $\gamma : [0,1] \to X$ be a path in X with a lifting $\tilde{\gamma} : [0,1] \to \widetilde{X}$. If $F : [0,1] \times [0,1] \to X$ is a homotopy such that $F(\bullet, 0) = \gamma$, there exists a unique lifting

$$\widetilde{F} : \ [0,1] \times [0,1] \longrightarrow \widetilde{X}$$

of F such that $\widetilde{F}(\bullet, 0) = \widetilde{\gamma}$ (see Notation 3.1.1). If F is a path-homotopy, so is $\widetilde{F}$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ {[0,1] \times [0,1]} \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

Proof. By Lebesgue's Lemma again, we can subdivide $[0.1] \times [0,1]$ into squares formed by horizontal and vertical lines such that the image of each square lies in an evenly covered open subset of X. Starting from the square at the lower left corner $S_{0,0}$, if $F(S_{0,0})$ lies entirely in an evenly covered open subset $U_{0,0}$ of X, then the image of the bottom side of $S_{0,0}$, being connected, must be contained in a unique connected component $\widetilde{U}_{0,0}$ of $p^{-1}(U_{0,0})$, and we can construct a unique lift $\widetilde{F}|_{S_{0,0}}$ of $F|_{S_{0,0}}$. If $S_{1,0}$ is the square to the right of $S_{0,0}$, and if $U_{1,0}$ is an evenly covered open subset of X containing $F(S_{1,0})$, we construct the unique lift $\widetilde{F}|_{S_{0,0} \cup S_{1,0}}$ of $F|_{S_{0,0} \cup S_{1,0}}$ using the homeomorphism defined by p from the unique connected component of $p^{-1}(U_{1,0})$ containing the image under $\widetilde{F}|_{S_{0,0}}$ of the left and the bottom sides of $S_{1,0}$. Working this way from the left to the right to first cover the squares at the bottom of the subdivision and then to the row second to the bottom and then up to cover all the squares, we construct the lift $\widetilde{F}$ as required. The lifting is also unique because the extension of $\widetilde{F}$ to each additional square is unique.

If the homotopy F is relative to $\{0,1\}$, i.e., $F(0,t) = \gamma(0)$ and $F(1,t) = \gamma(1)$ for all $t \in [0,1]$, then the continuous map

$$[0,1] \longrightarrow \widetilde{X}, \ \ t \longmapsto \widetilde{F}(0,t),$$

having its image lying in the fiber $p^{-1}(\gamma(0))$ with the discrete topology, must be a constant map. Hence $\widetilde{F}(0,t) = \tilde{\gamma}(0)$ for all $t \in [0,1]$. Similarly, $\widetilde{F}(1,t) = \tilde{\gamma}(1)$ for all $t \in [0,1]$. □

We now look at what the Homotopy-lifting Lemma says on lifting of loops.

Let $p : \widetilde{X} \to X$ be a covering map, and let $x \in X$ and $\tilde{x} \in p^{-1}(x)$. Suppose that $\alpha : [0,1] \to X$ is a loop in X at $x \in X$, and let $\tilde{\alpha} : [0,1] \to \widetilde{X}$ the unique lifting of α based at $\tilde{x} \in \widetilde{X}$. It follows from $p \circ \tilde{\alpha} = \alpha$ that $\tilde{\alpha}(1) \in p^{-1}(x)$. In general, we may not have $\tilde{\alpha}(1) = \tilde{x}$, i.e., $\tilde{\alpha}$ is not necessarily a loop at $\tilde{x}$.

Lemma 3.3.4. With the notation as above, let α' be another loop in X at x and let $\tilde{\alpha}'$ be the lifting of α' based at $\tilde{x}$. If $[\alpha] = [\alpha']$, then $\tilde{\alpha}'(1) = \tilde{\alpha}(1)$.

Proof. Assume that $[\alpha] = [\alpha']$. Let $F : [0,1] \times [0,1] \to X$ be a path-homotopy from α to α'. The Homotopy-lifting Lemma says that the unique lifting $\widetilde{F}$ of F with $\widetilde{F}(\bullet, 0) = \tilde{\alpha}$ is a path-homotopy. This means that for every $t \in [0,1]$, the map $F(\bullet, t) : [0,1] \to \widetilde{X}$ is a path in $\widetilde{X}$ with the same end points as $\tilde{\alpha}$, i.e., $F(\bullet, t)$ is a path in $\widetilde{X}$ from $\tilde{x}$ to $\tilde{\alpha}(1)$. Note also that $\widetilde{F}(\bullet, 1) = \tilde{\alpha}'$. Thus $\tilde{\alpha}'(1) = \tilde{\alpha}(1) \in \widetilde{X}$. □

Continuing with the discussion, for any $x \in X$, we thus have a well-defined map

$$\sigma : \ p^{-1}(x) \times \pi_1(X, x) \longrightarrow p^{-1}(x), \ \ (\tilde{x}, [\alpha]) \longmapsto \tilde{\alpha}(1),$$

where α is a loop in X at x, and $\tilde{\alpha}$ is the lifting of α based at $\tilde{x}$. The following lemma can be proved using the definitions and the Path-lifting Lemma and the Homotopy-lifting Lemma.

Theorem 3.3.3. [Monodromy action] For any covering map $p : \widetilde{X} \to X$ and any $x \in X$, the map σ is a right action of the group $\pi_1(X, x)$ on the fiber $p^{-1}(x)$.

We now look at more consequences of the Homotopy-lifting Lemma on fundamental groups. Let $p : \widetilde{X} \to X$ be a covering map. Let again $\tilde{x} \in \widetilde{X}$ and $x = p(\tilde{x}) \in X$. Consider the group homomorphism

$$p_* : \ \pi_1(\widetilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x), \ \ [\tilde{\alpha}] \longmapsto [p \circ \tilde{\alpha}].$$

Lemma 3.3.5. Suppose that $\tilde{\alpha}$ is a loop in $\widetilde{X}$ at $\tilde{x}$ such that $\alpha = p \circ \tilde{\alpha}$ is homotopic to the trivial loop e_x at x. Then $\tilde{\alpha}$ is homotopic to the trivial loop at $\tilde{x}$.

Proof. Let $F : [0,1] \times [0,1] \to X$ be a path-homotopy from e_x to α. By the Homotopy-lifting Lemma, there is a unique path-homotopy

$$\widetilde{F} : \ [0,1] \times [0,1] \longrightarrow \widetilde{X}$$

lifting F and such that $\widetilde{F}(\bullet, 0) : [0,1] \to \widetilde{X}$ is the unique lifting of the trivial loop e_x, so $\widetilde{F}(s, 0) = \tilde{x}$ for all $s \in [0,1]$. Let L be the union of the left, bottom and right edges of $[0,1] \times [0,1]$. Then F maps L to the single point $x \in X$. It follows from $p \circ \widetilde{F} = F$ that $\widetilde{F}(L) \subset p^{-1}(x)$. Since $p^{-1}(x)$ is a discrete subspace of $\widetilde{X}$ and L is connected, we have $\widetilde{F}(L) = \{\tilde{x}\}$. In other words, $\tilde{\alpha}$ is homotopic to the trivial loop at $\tilde{x}$. □

Theorem 3.3.4. Let $\widetilde{X}$ and X be path-connected and let $p : \widetilde{X} \to X$ be a covering map. Let $\tilde{x} \in \widetilde{X}$ and $x = p(\tilde{x}) \in X$.

1) The group homomorphism $p_* : \pi_1(\widetilde{X}, \tilde{x}) \to \pi_1(X, x)$ is injective.

2) A loop α in X at $x \in X$ lifts to a loop $\tilde{\alpha}$ in $\widetilde{X}$ at $\tilde{x}$ if and only if $[\alpha] \in p_*(\pi_1(\widetilde{X}, \tilde{x}))$;

3) For $\tilde{x}'$ also in the fiber of p over x, the two subgroups $p_*(\pi_1(\widetilde{X}, \tilde{x}))$ and $p_*(\pi_1(\widetilde{X}, \tilde{x}'))$ of $\pi_1(X, x)$ are conjugate to each other.

Proof. 1) By Lemma 3.3.5, the kernel of the group homomorphism p is trivial. Thus p is injective.

2) We only need to prove one direction: suppose that α is a loop in X at $x \in X$ and that $[\alpha] \in p_*(\pi_1(\widetilde{X}, \tilde{x}))$. Then there exists a loop β in $\widetilde{X}$ at $\tilde{x}$ such that $\alpha \sim p \circ \beta$. An argument similar to that in the proof of 1) shows that the lifting $\tilde{\alpha}$ of α with starting point $\tilde{x}$ must have the same end point as β, namely $\tilde{x}$. Thus $\tilde{\alpha}$ is a loop at $\tilde{x}$.

3) For $\tilde{x}'$ also in the fiber of p over x, let γ be a path in $\widetilde{X}$ from $\tilde{x}$ to $\tilde{x}'$. Then $p \circ \gamma$ is a loop at x, and it is easy to see that

$$p_*(\pi_1(\widetilde{X}, \tilde{x})) \longrightarrow p_*(\pi_1(\widetilde{X}, \tilde{x}')), \ \ [p \circ \tilde{\alpha}] \longmapsto [p \circ \gamma] * [p \circ \tilde{\alpha}] * [p \circ \gamma]^{-1},$$

is a group isomorphism. $\square$

<u>Remark</u>. Note that for a covering space $p : \widetilde{X} \to X$ and $x \in X$ and $\tilde{x} \in p^{-1}(x)$, a loop $\tilde{\alpha}$ at $\tilde{x} \in p^{-1}(x)$ gives rise to a loop $p \circ \tilde{\alpha}$ in X at x, but a path $\tilde{\alpha}$ in $\widetilde{X}$ connecting $\tilde{x}$ to any other point in $p^{-1}(x)$ also gives a loop $p \circ \tilde{\alpha}$ in X at x. 2) of Theorem 3.3.4 says that for a loop α in X at x, $[\alpha] \in \pi_1(X, x)$ lies in the subgroup $\pi_*(\pi_1(\widetilde{X}, \tilde{x}))$ if and only if α lifts to a loop at $\tilde{x}$.

Strengthening 3) of Theorem 3.3.4, one can show that as $\tilde{x}$ runs over all the points in the fiber $p^{-1}(x)$, the groups $p_*(\pi_1(\widetilde{X}, \tilde{x}))$ form a conjugacy class of subgroups of $\pi_1(X, x)$.

For the following Map-lifting Lemma, we need the notion of *locally path-connectedness*.

Definition 3.3.4. A topological space X is said to be *locally path-connected* if for every $x \in X$, every open neighborhood U of x in X contains a path-connected open neighborhood U' of x.

While connectedness plus locally path-connectedness imply path-connectedness (Problem 44 of Chapter 3 of Armstrong), path-connectedness does not imply locally path-connectedness (see Problem 43 of Chapter 3 of Armstrong).

Theorem 3.3.5. [Map-lifting Lemma] Let $p : \widetilde{X} \to X$ be a covering map, where both $\widetilde{X}$ and X are path-connected. Let Y be a path-connected and locally path-connected space and let $f : Y \to X$ be a continuous map. Let $y \in Y$, $x = f(y) \in X$, and let $\tilde{x} \in p^{-1}(x)$. Then there exists a lift $\tilde{f} : Y \to \widetilde{X}$ such that $\tilde{f}(y) = \tilde{x}$ if and only if

$$f_*(\pi_1(Y, y)) \subset p_*(\pi_1(\widetilde{X}, \tilde{x})).$$

Moreover, this lift is unique.

Proof. Given any $y \in Y$, draw a path γin Y from y_0 to y. Then $\alpha = f \circ \gamma$ is a path in X starting at x_0. Lift α to a path $\tilde{\alpha}$ in $\widetilde{X}$ stating at $\tilde{x}_0$. Define $\tilde{f}(y) = \tilde{\alpha}(1) \in \widetilde{X}$. Show that $\tilde{f}$ is as desired. For detail of the proof, see Theorem 10.16 of Armstrong. Pay attention to how the locally path-connectedness of Y is used in the proof. $\square$

3.3.3 Deck transformations and universal covers

Definition 3.3.5. Given a covering map $p : \widetilde{X} \to X$, a *deck transformation of p*, also called a *covering transformation* or an *automorphism of p*, is a homeomorphism $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$. We denote the set of all deck transformations of p by $D(p)$.

It is easy to see that $D(p)$ is a group under composition of maps. Furthermore, for each $x \in X$, clearly each $h \in D(p)$ maps $\tilde{x} \in p^{-1}(x)$ to a (possibly another) point in $p^{-1}(x)$. Moreover, if $h(\tilde{x}) = \tilde{x}$, then both h and $\mathrm{Id}_{\widetilde{X}}$ are lifts of $p : \widetilde{X} \to X$ and fixed $\tilde{x}$, so $h = \mathrm{Id}_{\widetilde{X}}$. In other words, the action of $D(p)$ on the fiber $p^{-1}(x)$ is free.

Definition 3.3.6. A covering map $p : \widetilde{X} \to X$ is said to be *regular*, or *normal*, or *Galois*, is $D(p)$ acts transitively on each fiber of p.

Definition 3.3.7. A covering map $p : \widetilde{X} \to X$ is said to be *universal* if $\widetilde{X}$ is simply-connected and locally path-connected. In this case we call $\widetilde{X}$ a *universal cover* of X.

The Map-lifting Lemma implies that universal covers, if exists, are all homeomorphic. The general theory on covering spaces studies relations between deck transformations, universal covers and the fundamental group of the base space X. We will only look at the special case when the covering map is given by a quotient map for a group action.

Recall that we have stated in Theorem 3.2.5 that if a group G acts on a simply connected space X by homeomorphisms such that for each $x \in X$ there exists an open neighborhood U of x in X such that $U \cap gU = \varnothing$ for all $g \in G\backslash\{e\}$, then $\pi_1(X/G)$ is isomorphic to G. For the remaining of this section, we will only look at this example. As we have been using X for the base space for a covering map, we adjust our notation and spell out our assumptions.

Assumption 1. Assume that G is a group acting on a simply connected space $\widetilde{X}$ by homeomorphisms such that for each $\tilde{x} \in \widetilde{X}$ there exists an open neighborhood $\widetilde{U}$ of $\tilde{x}$ in $\widetilde{X}$ such that $\widetilde{U} \cap g\widetilde{U} = \varnothing$ for all $g \in G\backslash\{e\}$.

Remark. In the literature, an action of G on X satisfying the above assumptions is sometimes called a *free and properly discontinuous* action of G on X, a not so good term, so we will not use it.

Proposition 3.3.1. In the setting of Assumption 1, the quotient map

$$p : \ \widetilde{X} \longrightarrow X := \widetilde{X}/G$$

is a covering map. If $\widetilde{X}$ is also locally path-connected, p is a universal covering map.

Proof. We first prove that p is a covering map without using the assumption that $\widetilde{X}$ is simply connected. When $\widetilde{X}$ is simply connected and locally path-connected, then, by definition p, becomes a universal covering map.

Let $\tilde{x} \in \widetilde{X}$ be arbitrary and let $x = G\tilde{x} \in X/G$. Let $\widetilde{U}$ be an open neighborhood of $\tilde{x}$ in $\widetilde{X}$ such that $\widetilde{U} \cap g\widetilde{U} = \varnothing$ for all $g \in G\backslash\{e\}$, and let $U = p(\widetilde{U})$. Then

$$p^{-1}(U) = \bigcup_{g \in G} g\widetilde{U}.$$

As each $g \in G$ acts on $\widetilde{X}$ by homeomorphism, $g\widetilde{U}$ is open in $\widetilde{X}$. It also follows from the fact that $\widetilde{U} \cap g\widetilde{U} = \varnothing$ for all $g \in G\backslash\{e\}$ that the open subsets of $\widetilde{X}$ in the collection $\{g\widetilde{U} : g \in G\}$ are pairwise disjoint. Moreover, for each $g \in G$, the map

$$p|_{g\widetilde{U}} : \ g\widetilde{U} \longrightarrow U$$

is continuous and bijective. Since $p : \widetilde{X} \to X$ is an open map (see Exercise 2.9.2), $p|_{g\widetilde{U}} : g\widetilde{U} \longrightarrow U$ is a homeomorphism. Thus U is an evenly covered open neighborhood of x in X. This shows that $p : \widetilde{X} \to X = \widetilde{X}/G$ is a covering map. $\square$

Let the setting be as in Assumption 1. Fix $\tilde{x} \in \widetilde{X}$ and let $x = p(\tilde{x}) \in X$. For $g \in G$, let $\tilde{\alpha}_g$ be any path in $\widetilde{X}$ from $\tilde{x}$ to $g\tilde{x}$ and let $\alpha_g = p \circ \tilde{\alpha}_g$, which is a loop in X at x. If we choose a different path $\tilde{\alpha}'_g$ in $\widetilde{X}$ from $\tilde{x}$ to $g\tilde{x}$, since $\widetilde{X}$ is simply connected, the two paths $\tilde{\alpha}_g$ and $\tilde{\alpha}'_g$ in $\widetilde{X}$ are path-homotopic, so the two loops $\alpha_g = p(\tilde{\alpha})$ and $\alpha'_g = p(\tilde{\alpha}'_g)$ in X at x are also path-homotopic. This shows that we have a well-defined map

$$\phi_{\tilde{x}} : \ G \longrightarrow \pi_1(X, x), \ g \longmapsto [p \circ \tilde{\alpha}_g], \quad \tilde{\alpha}_g \text{ is a path in } \widetilde{X} \text{ from } \tilde{x} \text{ to } g\tilde{x}.$$

Theorem 3.3.6. For any $\tilde{x} \in \widetilde{X}$ and $x = p(\tilde{x}) \in X$, the map $\phi_{\tilde{x}} : G \to \pi_1(X, x)$ is a group isomorphism.

Proof. Let $g, h \in G$. If $\tilde{\alpha}_h$ is a path in $\widetilde{X}$ from $\tilde{x}$ to $h\tilde{x}$, then

$$g\tilde{\alpha}_h : \ [0,1] \longrightarrow \widetilde{X}, \ s \longmapsto g\tilde{\alpha}_h(s), \quad s \in [0,1],$$

is a path in $\widetilde{X}$ from $g\tilde{x}$ to $(gh)\tilde{x}$. Let $\tilde{\alpha}_g$ be a path in $\widetilde{X}$ from $\tilde{x}$ to $g\tilde{x}$ and let $\alpha_g = p \circ \tilde{\alpha}_g$. Then

$$\tilde{\alpha}_{gh} := \tilde{\alpha}_g * (g\tilde{\alpha}_h)$$

is a path in $\widetilde{X}$ from $\tilde{x}$ to $(gh)\tilde{x}$. Let $\alpha_h = p \circ \tilde{\alpha}_h$. Then

$$\begin{aligned}\phi_{\tilde{x}}(gh) &= [p \circ (\tilde{\alpha}_{gh})] = [p \circ (\tilde{\alpha}_g * (g\tilde{\alpha}_h))] = [\alpha_g * \alpha_h] = [\alpha_g] * [\alpha_h] \\ &= \phi_{\tilde{x}}(g)\phi_{\tilde{x}}(h).\end{aligned}$$

This shows that $\phi_{\tilde{x}}$ is a group homomorphism. It remains to show that $\phi_{\tilde{x}}$ is bijective.

Suppose that $g \in G$ is such that $\phi_{\tilde{x}}(g) = e \in \pi_1(X, x)$, and let $\tilde{\alpha}_g$ be a path in $\widetilde{X}$ from $\tilde{x}$ to $g\tilde{x}$. Then $p \circ \alpha_g$ is path-homotopic to the trivial loop e_x at x. Now $\tilde{\alpha}_g$ is the lifting of $p \circ \alpha_g$ based at $\tilde{x}$, and the trivial loop $e_{\tilde{x}}$ is the lifting of e_x based at $\tilde{x}$. By Lemma 3.3.4, $g\tilde{x} = \tilde{x}$, so $g = e$. This shows that $\phi_{\tilde{x}}$ is injective.

To show that $\phi_{\tilde{x}}$ is surjective, let α be any loop in X at x and let $\tilde{\alpha}$ be the lifting of α based at $\tilde{x}$. Then $\tilde{\alpha}(1) \in p^{-1}(x)$, so there exists $g \in G$ such that $\tilde{\alpha}(1) = gx$. By definition, $\phi_{\tilde{x}}(g) = [\alpha]$. This shows that $\phi_{\tilde{x}}$ is surjective. □

Example 3.3.5. 1) For any integer $n \geq 1$, $\mathbb{R}^n$ is a universal cover of the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and $\pi_1(T^n) \cong \mathbb{Z}^n$;

2) For $n \geq 2$, S^n is a universal cover of the real projective space $\mathbb{RP}^n = S^n/\mathbb{Z}_2$, and $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$;

3) The Klein bottle also has $\mathbb{R}^2$ as a universal cover.

4) Recall that for p and q relatively prime positive integers, the lens space $L(p,q)$ is the quotient space of $S^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : \|z_1\|^2 + \|z_2\|^2 = 1\}$ by the action of the cyclic group $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ given by

$$g \cdot (z_1, z_2) = (e^{2\pi i/p} z_1,\ e^{2\pi q i/p} z_2),$$

where g is a generator of $\mathbb{Z}_p$. Since the quotient map $S^3 \to L(p,q)$ is a universal covering map, we see that $\pi_1(L(p,q)) \cong \mathbb{Z}_p$.

Chapter 4 — Classification of surfaces

Our references for this chapter are

1) S. W. Massey, Chapter 1, *A basic course on Algebraic Topology*, 1991;
2) C. Kinsey, Chapter 4, *Topological surfaces*, available online.
3) N. Hitchin, Chapter 2, Oxford lecture notes *Geometry of Surfaces*, available online.

4.1 Statement of the classification theorem

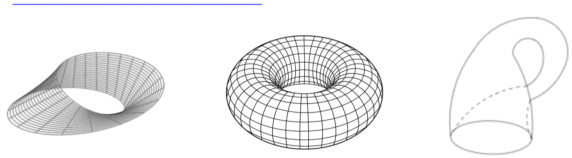
4.1.1 Examples and connected sums of surfaces

We recall from §2.10 that

1) an n-dimensional manifold without boundary is a topological space M that is Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$;
2) an n-dimensional manifold with boundary is a topological space M that is Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$ or to $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ and such that $\partial M \neq \emptyset$.
3) for two n-dim manifolds M and N with boundary and for any homeomorphism $h : \partial N \to \partial M$, the adjunction space $M \cup_h N$ is an n-dim without boundary.

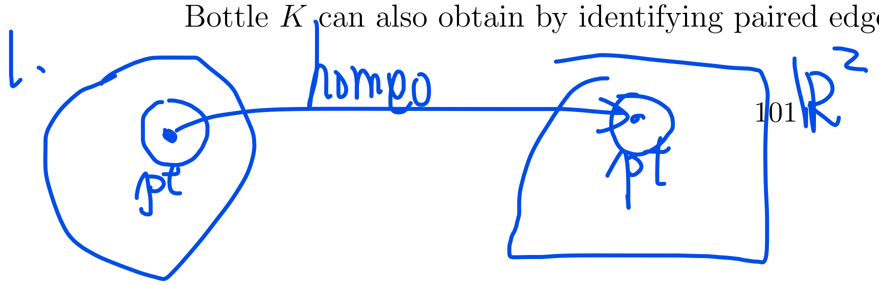
Definition 4.1.1. A *surface* is a compact and connected 2-dimensional topological manifold without boundary.

The Mobius strip is a compact and connected 2-dimensional topological manifold *with* boundary, so it is a *surface with boundary*. Standard examples of surfaces are the sphere S^2, the torus T^2, the real projective space $\mathbb{RP}^2$, and the Klein Bottle K.

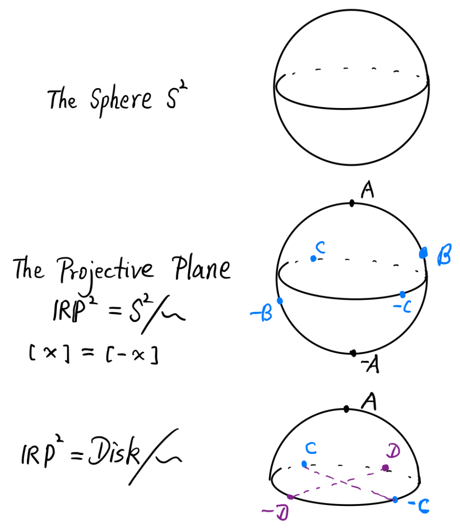


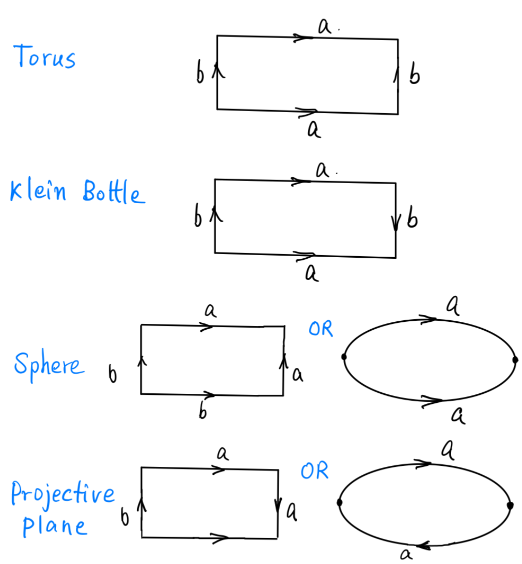
Recall that sphere S^2, the torus T^2, the real projective space $\mathbb{RP}^2$, and the Klein Bottle K can also obtain by identifying paired edges of a rectangle.

1.



2. Hausdorff
3. Linde





We now turn to the construction of connected sums.

Given two surfaces S_1 and S_2, one can construct another surface as follows: choose subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ such that D_1 ad D_2 are both homeomorphic to

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Let $\mathring{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $\mathring{D}_1 \subset D_1$ and $\mathring{D}_2 \subset D_2$ be the images of $\mathring{D}$ in D_1

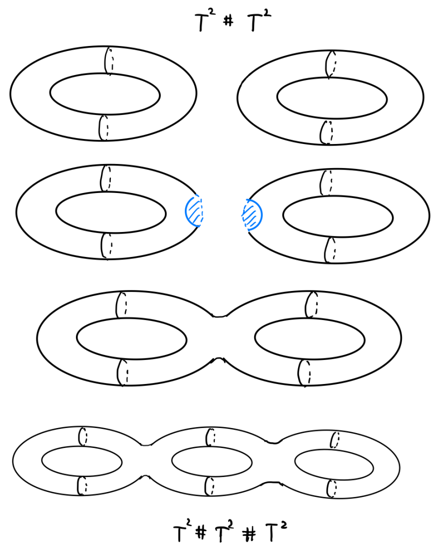
and D_2 under the homemorphisms. Then

$$S_1' = S_1 \backslash \mathring{D}_1 \quad \text{and} \quad S_2' = S_2 \backslash \mathring{D}_2$$

are both 2-dimensional topological manifolds with boundaries, and $\partial S_1' \cong \partial S_2' \cong S^1$. Choose any homeomorphism $h : \partial S_1' \to \partial S_2'$, and form the adjunction space

$$S_1' \cup_h S_2'$$

which is a surface. One can show that any two such surfaces, i.e., for different choices of $D_1 \subset S_1$ and $D_2 \subset S_2$ and different choices of h, are homeomorphic to each other. We denote any such a surface by $S_1 \# S_2$ and call it the *connected sum* of S_1 and S_2.



4.1.2 Statement of the classification theorem

We can now state the classification theorem on surfaces, which is one of the major theorems in topology.

Theorem 4.1.1. Every surface is homeomorphic to *one and only one* of the following:

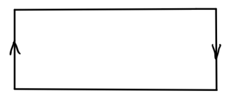
1) the 2-sphere S^2;
2) n-fold connected sums of the 2-torus with itself:

$$nT^2 = T^2 \# T^2 \# \cdots \# T^2;$$

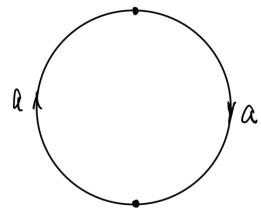
3) n-fold connected sums of the projective plane $\mathbb{RP}^2$ with itself:

$$n\,\mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2.$$

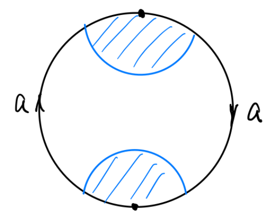
Mobius Strip



Projective plane $\mathbb{RP}^2$



Cutting out a disk in $\mathbb{RP}^2$ to get a Mobius Strip



Example 4.1.1. One has Disc $\cup_f$ Mobius strip $\cong \mathbb{RP}^2$.

Example 4.1.2. One has $\mathbb{RP}^2 \# \mathbb{RP}^2 \cong K$. Equivalently,

$$K \cong \text{Mobius} \cup_f \text{Mobius}.$$

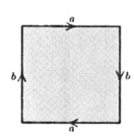


FIGURE 1.4. Construction of a Klein bottle from a square.

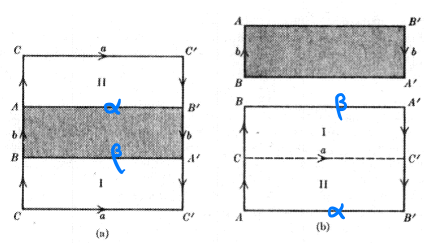


FIGURE 1.5. The Klein bottle is the union of two Möbius strips.

4.2 Triangulations of surfaces

4.2.1 Definition and examples

Let S be a surface (compact and connected 2-manifold).

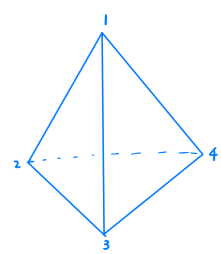
Definition 4.2.1. A *triangulation of S* consists of

1) a finite family $K = \{T_1, \dots, T_n\}$ of closed subsets of S such that $S = \cup_{i=1}^n T_i$;
2) a family $\{T_1', \dots, T_n'\}$ of pairwise disjoint triangles in $\mathbb{R}^2$ and homeomorphisms

$$\phi_i : T_i' \longrightarrow T_i, \quad i = 1, \dots, n,$$

such that for any $i \neq j$, T_i and T_j are either disjoint, or have a single vertex in common, or an entire edge in common, where by *vertices and edges of T_i* we mean the ϕ_i-images of vertices and edges of T_i'. Each T_i is called a *triangle of the triangulation*. We also just refer to K as the triangulation.

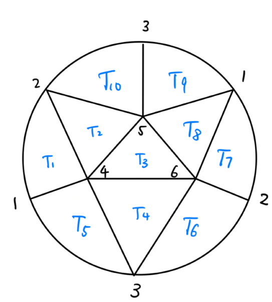
Example 4.2.1. The sphere S^2 has a triangulation with 4 triangles dividing the upper hemisphere and 4 dividing the lower hemisphere. Here is another triangulation of S^2:



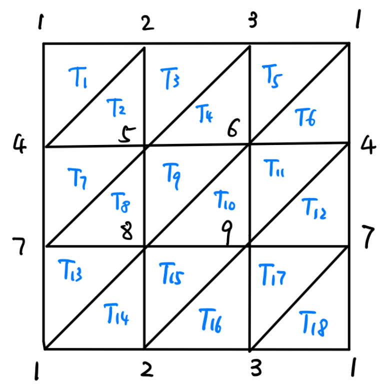
Example 4.2.2. An example of a triangulation of $\mathbb{RP}^2$ with 10 triangles listed as

$$124, \quad 245, \quad 456, \quad 346, \quad 134, \quad 236, \quad 126, \quad 156, \quad 135, \quad 235.$$

A triangulation of $\mathbb{RP}^2$ with 6 vertices, 15 edges, and 10 triangles.



Example 4.2.3. Here is an example of a triangulation of T^2 with 18 triangles:

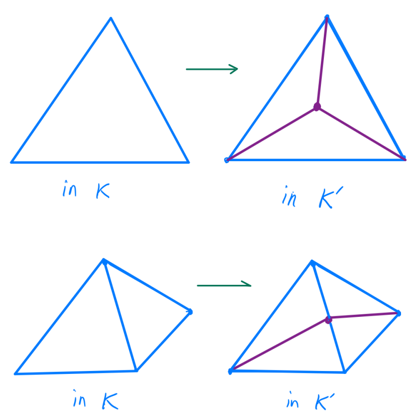


A triangulation of T^2 with 18 triangles,
27 edges, and 9 vertices

4.2.2 The triangulation theorem

The triangulation theorem says that every surface (even with boundary) has a triangulation. There is also a stronger version of the triangulation theorem. To state the stronger version, we make some definitions.

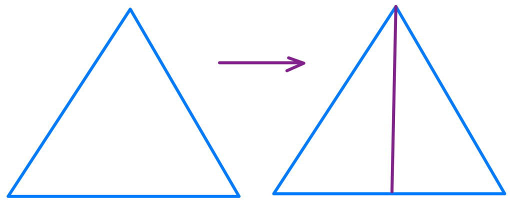
Definition 4.2.2. Given a triangulation K of a surface S, one defines two types of *stellar moves* to the triangles in K to get a sub-triangulation K' as follows:



Stellar moves of triangulations of surface

If S is a surface with boundary, one also allows the third type of stellar moves:

Theorem 4.2.1 (Alexander's Theorem on stellar moves)**.** For any two triangulations K and L of a surface S, there exist triangulations K' and L' such that



1) K' is obtained from K by finite steps of stellar moves;
2) L' is obtained from L by finite steps of stellar moves;
3) K' and L' are *simplicially isomorphic*.

Here *simplicially isomorphic* means that there is a homeomorphism from S to itself that maps bijectively the vertices of K' to that of L' and extends linearly on the triangles.

Theorem 4.2.2 (Strong Triangulation Theorem)**.** 1) Every surface S (with or without boundary) has a triangulation;

2) Any two triangulations of S are equivalent in the sense of Alexander's Theorem.

The proofs of the Alexander theorem and the strong triangulation theorem require a lot of detailed arguments. Here are some references:

1. J. W. Alexander, The combinatorics of complexes, *Ann. Math* 1930, **31** (2), 292 - 320.
2. §8 of E. Moise, Geometric topology in dimensions 2 and 3, 1977
3. Chapter 3 of E. Bloch: A first course in geometric topology and differential geometry.
4. §2.7 of V. Buchstaber and T. Panov, Toric topology, Mathematical Surveys and Monographs, **204** (2015).

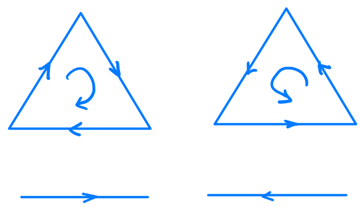
4.3 Proof of the classification theorem, Part I

In this section, we will prove half of the classification theorem, namely that the surfaces given in the list in the classification theorem are pair-wise non-homeomorphism. This is done through two topological invariants of surfaces: orientability and Euler characteristics .

4.3.1 Orientability of surfaces

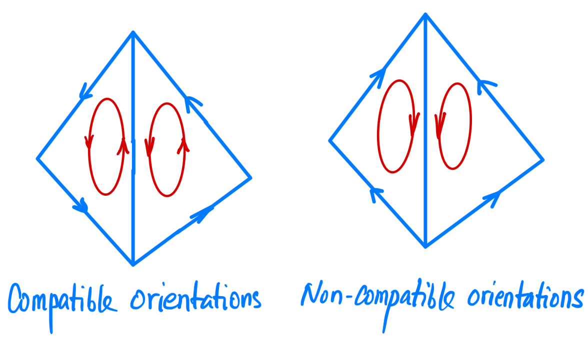
Definition 4.3.1. For a triangle T in a vector space V, an ordering of its three vertices is called an *orientation* of T; Two orderings are said to define the *same orientation* if they differ by a cyclic permutation.

Note that there are precisely two orientations of each triangle, and an orientation of a triangle induces an orientation, i.e., an order of the two vertices, of every one of its three edges.



We now introduce orientability of triangulations.

Definition 4.3.2. A triangulation K of a surface is said to be *orientable* if there is an assignment of an orientation to each triangle in K such that the orientations on any two triangles intersecting at an edge are *compatible* in the sense tthat hey induce opposite orientations on the common edge:



The following lemma is crucial to us.

Lemma 4.3.1. If K is a triangulation of a surface, and K' is obtained from K through finitely many steps of stellar moves, then K is orientable if and only if K' is.

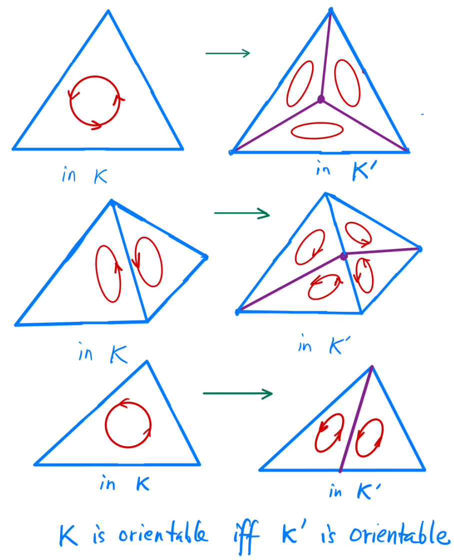
Proof. The lemma is proved by considering each case.

□

Corollary 4.3.1. For any surface, the following two statements are equivalent:
1) There exists a triangulation of S that is orientable;
2) Every triangulation of S is orientable.

We thus have the following well-defined notion of orientability of surfaces.

Definition 4.3.3. A surface S is said to be *orientable* if it has an orientable triangulation.

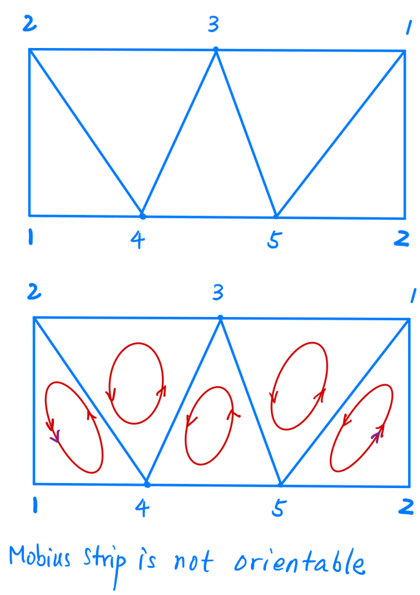


Theorem 4.3.1. Orientability of a surface is a topological property.

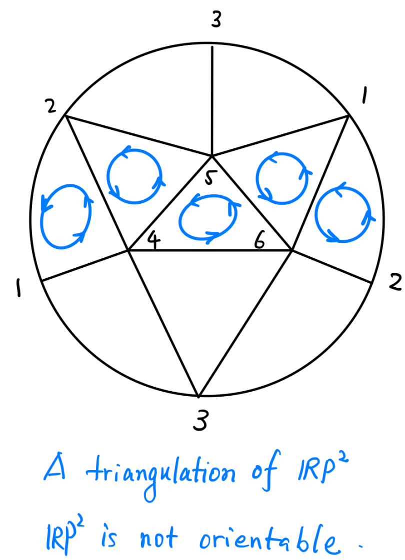
Proof. If $\phi : S \to S'$ is a homeomorphism, K is a triangulation of S iff $\phi(K)$ is a triangulation of S', and K is orientable iff $\phi(K)$ is orientable. $\square$

We now come to our famous example of non-orientable surfaces.

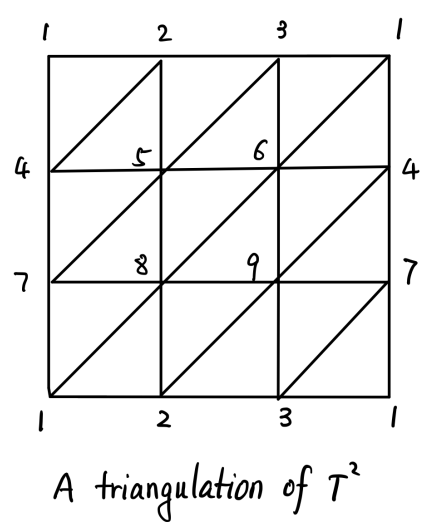
Example 4.3.1. The Mobius strip is not orientable because of the following non-orientable triangulation:



Example 4.3.2. $\mathbb{RP}^2$ is not orientable because of the following non-orientable triangulation.



Similarly S^2 is orientable because one can easily find an orientable triangulation. T^2 is orientable because he following triangulation is orientable.



The proof of the following lemma can be found in §7.2 of Armstrong.

Lemma 4.3.2. A surface is *not orientable* if and only if it contains a closed subset homeomorphic to the Mobius strip.

Theorem 4.3.2. If two surfaces S_1 and S_2 are orientable, so is $S_1 \# S_2$.

Proof. Pick orientable triangulations K_1 for S_1 and K_2 for S_2 and form $S_1 \# S_2$ by guing along a triangle in K_1 and a triangle in K_2 in opposite orientation. □

Corollary 4.3.2. Among the list of standard surfaces

$$S^2, \quad mT^2, \quad n\,\mathbb{RP}^2, \quad m, n \geq 1,$$

$n\,\mathbb{RP}^2$ is not orientable for every $n \geq 1$. Others are orientable.

Corollary 4.3.3. No surface from the list

$$S^2, \quad mT^2, \quad m \geq 1$$

is homeomorphic to any surface from the list

$$n\mathbb{RP}^2, \quad n \geq 1.$$

The above corollary is already part of the classification theorem. We will need the notion of Euler characteristics to tell apart the surfaces from the same lists.

4.3.2 Euler characteristics of surfaces

Recall that surfaces are compact connected 2-manifolds without boundary. We now introduce one of the most important numerical topological invariants of a surface.

Definition 4.3.4. For a surface S with a triangulation K, let

$$\begin{aligned} V(K) &= \text{number of vertices of } K, \\ E(K) &= \text{number of edges of } K, \\ F(K) &= \text{number of triangles of } K. \end{aligned}$$

Theorem 4.3.3. For any surface S, the integer

$$\chi(S) = V(K) - E(K) + F(K),$$

is independent of the triangulation K of S and it called *the Euler characteristic of S*.

Proof. Let K be any triangulation of S. Recall the two types of stellar moves.

Case 1. Changing K to K' by one stellar move of the first type, one has

$$V(K') = V(K) + 1, \;\; E(K') = E(K) + 3, \;\; F(K') = F(K) + 2.$$

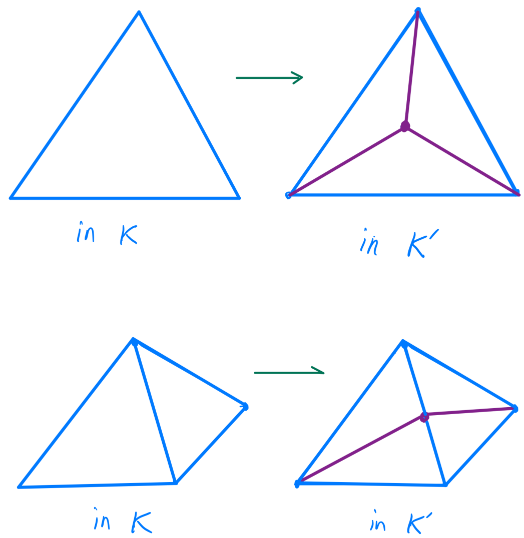
Thus $V(K') - E(K') + F(K') = V(K) - E(K) + F(K)$.

Case 2. Changing K to K' by one stellar move of the second type, one has

$$V(K') = V(K) + 1, \;\; E(K') = E(K) + 3, \;\; F(K') = F(K) + 2.$$

Again $V(K') - E(K') + F(K') = V(K) - E(K) + F(K)$.

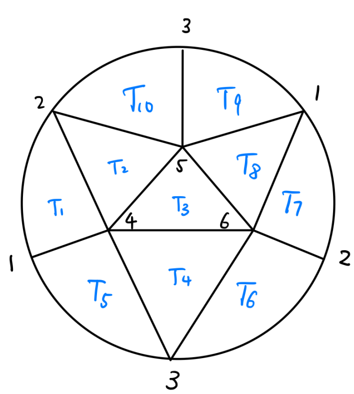
Theorem now follows from the Strong Triangulation Theorem. □



Example 4.3.3. One has

$$\chi(S^2) = \chi(\text{shallow tetrahedron}) = 2.$$

Using the triangulation for $\mathbb{RP}^2$, we have $\chi(\mathbb{RP}^2) = 6 - 15 + 10 = 1$:

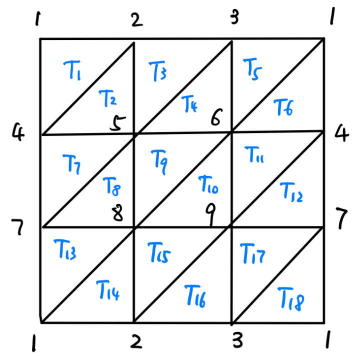


A triangulation of $\mathbb{RP}^2$ with 6 vertices, 15 edges, and 10 triangles

Using the triangulation for T^2, we get $\chi(T^2) = 9 - 27 + 18 = 0$:

Theorem 4.3.4. For any two surfaces S_1 and S_2,

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$



A triangulation of T^2 with 18 triangles, 27 edges, and 9 vertices

Proof. Let K_i be a triangulation of S_i for $i = 1, 2$. Pick a triangle T_1 in K_1 and T_2 in K_2; Glue $S_1 \backslash \mathring{T}_1$ and $S_2 \backslash \mathring{T}_2$ along ∂T_1 and ∂T_2 to get $S_1 \# S_2$. We then have a triangulation K of $S_1 \# S_2$ with

$$V(K) = V(K_1) + V(K_2) - 3, \quad E(K) = E)K_1) + E(K_2) - 3,$$
$$F(K) = F(K_1) + F(K_2) - 2.$$

One thus has

$$\begin{aligned}\chi(S_1 \# S_2) &= \chi(K) = V(K) - E(K) + F(K) \\ &= \chi(K_1) + \chi(K_2) - 3 + 3 - 2 \\ &= \chi(S_1) + \chi(S_2) - 2.\end{aligned}$$

□

The following lemma is proved by induction on n.

Lemma 4.3.3. One has $\chi(S^2) = 2$, and for any integer $n \geq 1$,

$$\chi\left(n\, T^2\right) = 2 - 2n,$$
$$\chi\left(n\, \mathbb{RP}^2\right) = 2 - n.$$

4.3.3 Proof of one half of the classification theorem

Theorem 4.3.5. The following surfaces are pair-wise non-homeomorphic:

$$S^2, \quad nT^2 = T^2 \# T^2 \# \cdots \# T^2, \quad m\, \mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2, \quad n, m \geq 1.$$

Proof. By orientability, neither S^2 nor any nT^2 is homeomorphic to any $m\,\mathbb{RP}^2$. Comparing Euler characteristics, S^2 is not homeomorphic to any nT^2, and nT^2 is not homeomorphicto $n'T^2$ if $n \neq n'$. Similarly, $m\,\mathbb{RP}^2$ is not homeomorphicto $m'\mathbb{RP}^2$ if $m \neq m'$. $\square$

Example 4.3.4. Recall that is Klein Bottle is homeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$, so

$$\chi(T^2) = \chi(\text{Klein Bottle}) = 0$$

As T^2 is orientable while the Klein Bottle is not, these two surfaces are not homeomorphic.

4.3.4 Classification of 2-dimensional regular polyhedra

We now consier a very classical problem.

Definition 4.3.5. For fixed integers $m, n \geq 3$, a *regular (2-d) polyhedron of type* (m, n) is the sphere S^2 with a division into n-gons such that exactly two n-gons meet at each edge and exactly m-edges at each vertex.

Theorem 4.3.6. There are exactly *five* cases of (m, n), given by

$$(3,\,3), \quad (3,\,4), \quad (4,\,3), \quad (3,\,5), \quad (5,\,3),$$

total number of faces being $4, 6, 8, 12$, and 20 respectively, and called

- the regular tetrahedron;
- the regular cube;
- the regular octahedron;
- the regular dodecahedron;
- the regular icosahedron.

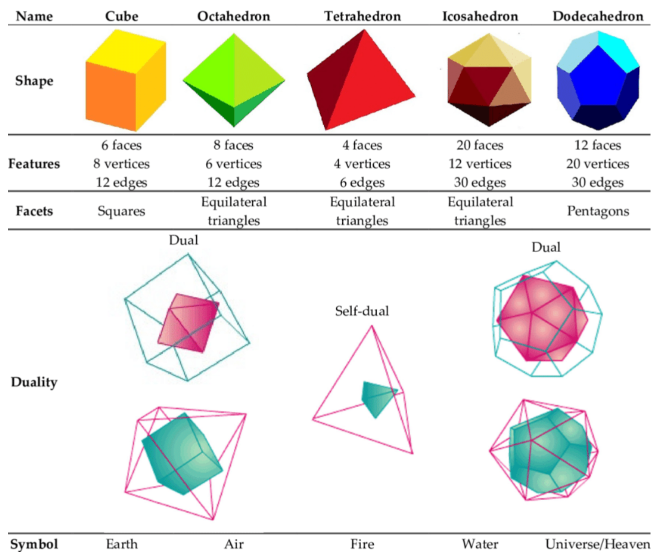
Proof. Let N be the total number of n-gons. Connect the center of each n-gon with its n vertices to divide the n-gon into n triangles. We thus get a triangulation K of S^2 with

$$V(K) = N + \frac{nN}{m}, \quad E(K) = nN + \frac{nN}{2}, \quad F(K) = nN.$$

It follows from $V(K) - E(K) + F(K) = 2$ that $N\left(1 + \frac{n}{m} - \frac{n}{2}\right) = 2$. Now $1 + \frac{n}{m} - \frac{n}{2} > 0$ implies that $2m + 2n - mn > 0$, so

$$(m-2)(n-2) < 4.$$

Thus $(m, n) = (3,\,3),\ (3,\,4),\ (4,\,3),\ (3,\,5),\ (5,\,3)$. $\square$



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Lemma 4.3.4. If S is glued together by polygons instead of triangles, we still have

$$\chi(S) = V(K) - E(K) - F(K),$$

where $F(K)$ is the number of polygonal faces in K.

Proof. Suppose first there is only one n-gons in K and the rest are triangles. Divide the unique n -gon into triangles by adding a vertex at its center and connecting the center to each vertex of the n-gon, and denote the resulting triangulation of S by K'. Then

$$V(K') = V(K) + 1, \quad E(K') = E(K) + n, \quad F(K') = F(K) + n - 1.$$

So $V(K) - E(K) - F(K) = V(K') - E(K') - F(K') - 1 + n - n + 1 = V(K') - E(K') - F(K') = \chi(S)$. Carrying out the above procedure for each polygon in K, one proves the claim. $\square$

For example, for the five regular polyhedra, we respectively have

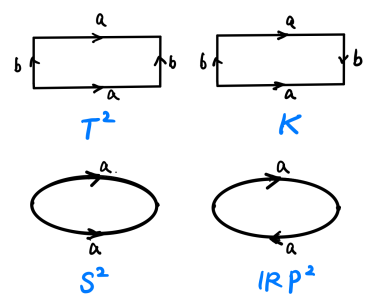
$$V(K) - E(K) - F(K) = \begin{cases} 8 - 12 + 6 = 2, & \text{cube}, \\ 4 - 6 + 4 = 2, & \text{tetrahedon}, \\ 6 - 12 + 8 = 2, & \text{octahedron}, \\ 20 - 30 + 12 = 2, & \text{dodecahedron}, \\ 30 - 12 + 20 = 2, & \text{icosahedron}. \end{cases}$$

4.4 Proof of the classification theorem, Part II

In this section, we finish the proof of the classification theorem of surfaces by using planar models.

4.4.1 Planar models of surfaces

We first recall the torus T^2, the Klein bottle K, the sphere S^2, and the real projective space $\mathbb{RP}^2$ are all quotient spaces obtained by identifying edges of a rectangle or the two semi-circles of the boundary of a disc as follows:



It turns out that one can construct more general surfaces using a similar construction, called quotients of planar models.

Definition 4.4.1. A *planar model* is a $2n$-gon D together with

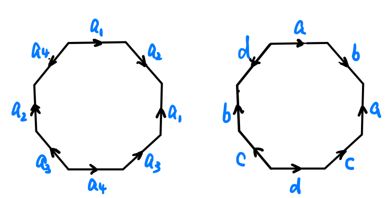
1) an arrow on every edge (called an *orientation* of D);

2) a division of the edges of D into pairs, and a labeling of each pair of the by a different letter from

$$\{a_1,\ a_2,\ \ldots,\ a_n\}.$$

We remark that in the definition of a planar model we allow arbitrary divisions of the edges of D into pairs and arbitrary directions of arrows on the edges.

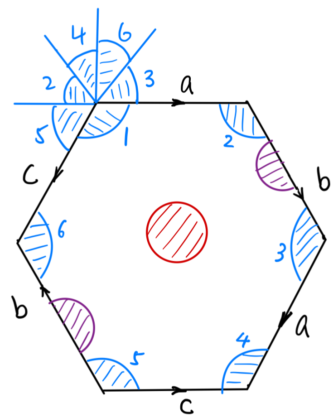
An example of a planar model with $n = 4$ is given as follows:



An Example of a planar model

Theorem 4.4.1. The quotient space of any planar model D by identifying its paired edges using the given orientation is a compact surface, which we denote as $S(D)$ and we call $S(D)$ the *quotient surface* of D.

Proof. We also ready know that $S(D)$ is compact and connected. It is also easy to show that it is second countable. It remians to that $S(D)$ is locally homeomorphic to $\mathbb{R}^2$. We do this by an example as in the picture:



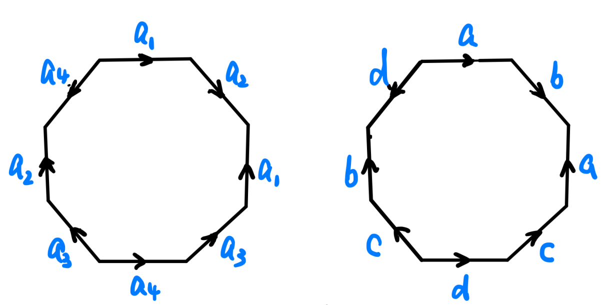
$\square$

Definition 4.4.2. *A planar model for a surface* S is a planar model D such that

$$S(D) \text{ is homeomorphic to } S.$$

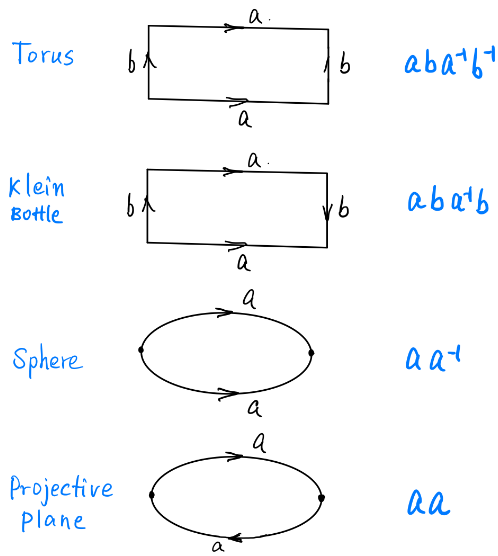
A natural question is whether or not every surface have a planar model. We also need to know when two planar models have homeomorphic quotient surfaces. We answer the first question in this §4.4.1. The second question will be answered later.

Given a planar model D, insread of drawing a picture for D as a $2n$-gon, we can express a planar model as an *edge word* as follows: start with any vertex and trace the $2n$-gon clockwise; For an edge with label a_i and clockwise arrow, write a_i; For an edge with label a_i and counter-clockwise arrow, write a_i^{-1}; This way we get a word in the letters $\{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$, called the*edge word* of the planar model.



$$a_1 a_2 a_1^{-1} a_3^{-1} a_4^{-1} a_3 a_2 a_4^{-1} = d^{-1} c b d^{-1} a b a^{-1} c^{-1}$$

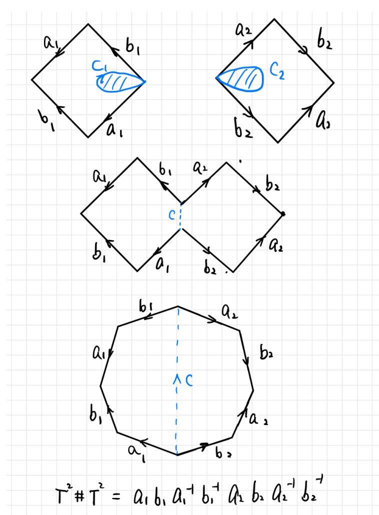
Example 4.4.1. We have the following edge ords for the planar models

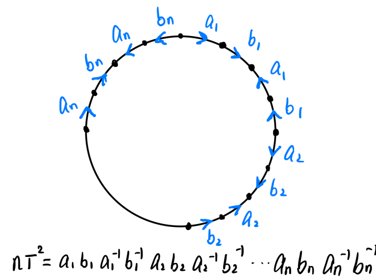


Proposition 4.4.1. Given two surfaces S_1 and S_2, the concatenation of an edge word for a planar model for S_1 following by an edge word for a planar model for S_2 is an edge word for a planar model for $S_1 \# S_2$.

We illustrate the proposition by examples.

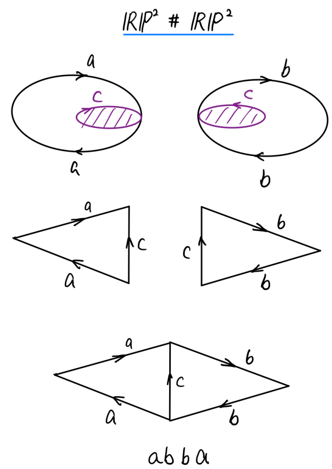
Example 4.4.2. Planar model for $T^2 \# T^2$:



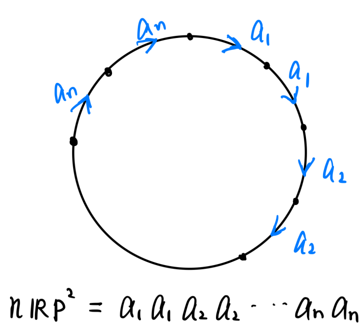


Example 4.4.3. Planar model for nT^2:

Example 4.4.4. Planar model for $\mathbb{RP}^2 \# \mathbb{RP}^2$



Example 4.4.5. Planar model for $n\,\mathbb{RP}^2$:



Definition 4.4.3. The following are called *standard edge-words (planar models)*:

1) The sphere: $S^2 = aa^{-1}$;

2) The n-fold connected sum of T^2:

$$nT^2 = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1}.$$

3) The n-fold connected sum of the projection plane:

$$n\,\mathbb{RP}^2 = a_1 a_1 a_2 a_2 \cdots a_n a_n.$$

In §4.4.2, we will show that every surface has a planar model.

Definition 4.4.4. We say that two planar models D and D' are *equivalent* if

$$S(D) \text{ is homeomorphic to } S(D').$$

We will show in §4.4.3 that every planar model is equivalent to one of the standard planar models.

4.4.2 From triangulations to planar models of surfaces

In this section, we show that a triangulation of S gives a way of constructing S via a step-by-step gluing of triangles along their edges. *Gluing* here means forming adjunction spaces. Recall the following fact.

Lemma 4.4.1. The following are equivalent:

1) an adjunction space $Z = X \cup_f Y$ formed by a closed $A \subset Y$ and a closed embedding $f : A \to X$;

2) a topological space Z with closed subsets X and Y such that

$$Z = X \cup Y \quad \text{and} \quad X \cap Y = A.$$

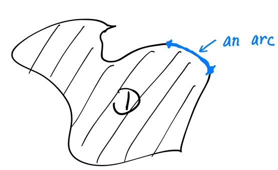
We first make precise the definition of polygons.

Definition 4.4.5. 1) A topological space D homeomorphic to D^2 is called a *disc*, where

$$D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Thus every disc D is a 2-dimensional manifold with $\partial D \cong S^1$.

2) Any $A \subset \partial D$ homeomorphic to $[0, 1]$ is called an *arc* of the disc D;



3) A disc D with n marked point on ∂D is called an *n-gon*.

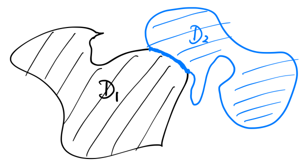
4) The marked points on an n-gon D are called the *vertices*;

5) The arcs between successive vertices, going clock-wise, are called *edges* of D.

6) An *orientation* of an n-gon is an assignment of an arrow to an edge.

We quote the following lemma from Armstrong's book on gluing of discs.

Lemma 4.4.2 (Lemma 2.11 of Armstrong)**.** : If $D = D_1 \cup D_2$, where D_1 and D_2 are two discs and $D_1 \cap D_2$ is an arc of both D_1 and D_2, then D is itself a disc.



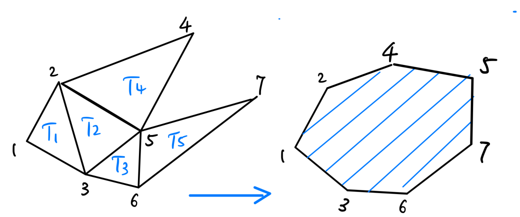
Gluing two discs along an arc.

Corollary 4.4.1. 1) Gluing (i.e., adjunction) of an n-*gon* with an m-*gon* along an edge of each is an $(n + m - 2)$-*gon*.

2) Step-by-step gluing of triangles along edges gives a polygon.

Example 4.4.6. The gluing of the five triangles in $\mathbb{R}^2$ along an edge in the order

$$123, \quad 235, \quad 356, \quad 245, \quad 567:$$



Theorem 4.4.2. Every surface S has a planar model.

Proof. We give an outline of the proof.

First choose any triangulation $K = \{\phi_i : T_i' \to T_i\}_{i=1}^n$ of S. By definition, each edge of K is an edge of exactly two triangles, and that for each vertex of K, we can arrange the set of all triangles with v as a vertex in a cyclic order

$$T_0,\ T_1, ; \dots,\ T_{m-1},\ T_m = T_0,$$

such that T_i and T_{i+1} has one edge in common for $0 \le i \le m-1$. It follows that the triangles in K can be listed as $T_1, T_2, \dots, T_n$ such that for each $i = 2, \dots, n$,

$$T_i \text{ has an edge } e_i \text{ in common with some } T_j \text{ for } 1 \le j \le i-1.$$

Correspondingly list the triangles $T_1', T_2', \dots, T_n'$ in $\mathbb{R}^2$.

If e_1 is the edge of T_1' and e_2 the edge of T_2' such that $\phi_1(e_1) = \phi_2(e_2)$, let $f_2 = \phi_1^{-1}\phi_2|_{e_2} : e_2 \to T_1'$ and let $D_1 = T_1' \cup_{f_2} T_2'$. By Lemma 2.11 of Armstrong, D_1 is a disc. By Lemma 2.11 of Armstrong, $D \overset{\text{def}}{=} T_1 \cup T_2 \cup \cdots \cup T_n$ is a disc; Leftover edges of the triangles form ∂D and divided into pairs because $\partial S = \varnothing$. Thus D is a planar model for S. $\square$

4.4.3 Equivalence of planar models

Recall that we say that two planar models D and D' are euivalent if they have homeomorphic quotient surfaces.

The following cutting and pasting procedure produces equivalent planar models. Let D be planar model.

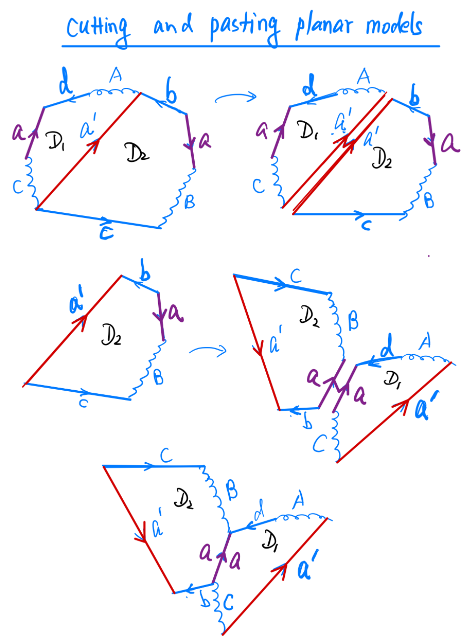
1) Cut D into two parts D_1 and D_2 along the line segment connecting two non-adjacent vertices such that there is an edge labeled by a in D_1 and an edge also labeled by a inD_2.

2) Label the line segment a'.

3) Glue D_1 and D_2 along the edge a of D_1 and the edge a of D_2.

4) We have thus created a new planar model D' with no edges labeled a but a pair of new edges labeled a'.

Here is an example of cutting and pasting.



Theorem 4.4.3. Any planar model D' obtained from D by cutting and pasting as above is equivalent to D.

Proof. The theorem follows from the characterization of adjunction spaces in Lemma 4.4.1. $\square$

Example 4.4.7. Using cutting and pasting, one can also show that $\mathbb{RP}^2 \# \mathbb{RP}^2 \cong K$.

From *Topology of surfaces* by Christine Kinsey:

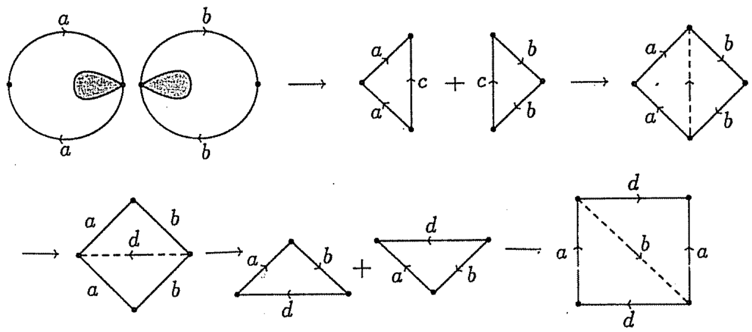


Fig. 4.30. $\mathbb{P}^2 \# \mathbb{P}^2 = \mathbb{K}^2$

Example 4.4.8. Using cutting and pasting, one can also show that

$$T^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \cong K \# \mathbb{RP}^2.$$

From *Topology of surfaces* by Christine Kinsey:

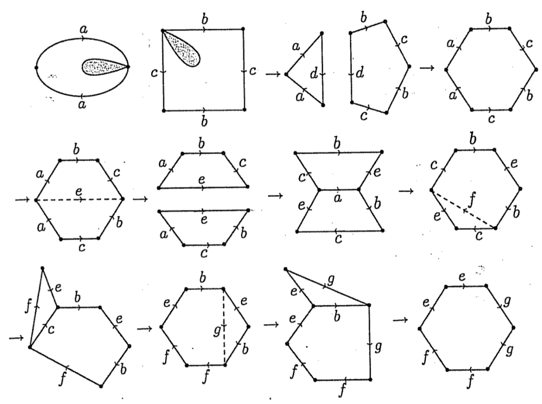


Fig. 4.58. $\mathbb{T}^2 \# \mathbb{P}^2 = \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$

Theorem 4.4.4. Every planar model is equivalent to a standard one.

Proof. We give an outline of the proof. For details, see Massey's book Chpater 1.

Let D be a planar model. If there are only two edges, then $D = aa^{-1}$ or aa. Assume there are at least 4 edges; We perform the following steps.

1) Eliminate adjacent edges labeled as aa^{-1}.

2) By a sequence of cutting and pasting operations, transform D to a D' in which all vertices are identified;

3) Make any pairs appearing as $\cdots a \cdots a \cdots$ adjacent;

4) Make $\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$ into $\cdots cdc^{-1}d^{-1} \cdots$.

We thus see that $D \sim a_1a_1\cdots a_ma_mb_1c_1b_1^{-1}c_1^{-1}\cdots b_nc_nb_n^{-1}c_n^{-1}$.

If $m = 0$, we get $D \sim b_1c_1b_1^{-1}c_1^{-1}\cdots b_nc_nb_n^{-1}c_n^{-1}$.

If $m > 0$, using $aabcb^{-1}c^{-1} \sim aabbcc$, we get $D \sim a_1a_1\cdots a_ma_ma_{m+1}a_{m+1}\cdots a_{m+2n}a_{m+2n}$. □

4.4.4 Proof of the classification theorem of surfaces

We can now finally prove the Classification Theorem of Surfaces.

Theorem 4.4.5. Every surface is homeomorphic to *one of and only one* of the following:

1) the 2-sphere S^2;

2) n-fold connected sums of the 2-torus with itself:

$$nT^2 = T^2 \# T^2 \# \cdots \# T^2;$$

3) n-fold connected sums of the projective plane $\mathbb{RP}^2$ with itself:

$$n\,\mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2.$$

Proof. The theorem is proved in the following steps. Let S be a surface.

1) Triangulation Theorem says that S has a triangulation.

2) Using a triangulation of S, we have proved that S has a planar model D;

3) Useing cutting and pasting, we have proved that D is equivalent to a standard one;

4) Use Strong Triangulation Theorem, we have defined orientability and Euler characteristics of surfaces and shown them to be topological invariants;

5) Using orientability and Euler characteristics, we have shown that the surfaces in the lists are pair-wise non-homeomorphic. □

4.5 The van Kampen theorem and fundamental groups of compact surfaces

4.5.1 Preparation of group theory

To explain van Kampen's theorem, we need some preparation on group theory. Let G_1 and G_2 be two groups.

Definition 4.5.1. (a) The *free product* of G_1 and G_2, denoted as $G_1 * G_2$, is the set of reduced words $x_1x_2\cdots x_n$ in the $G_1 \sqcup G_2$, where

1) each $x_i \in G_1 \sqcup G_2$, and $x_i \neq e$;

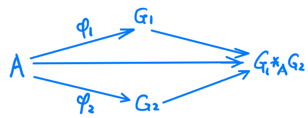
2) x_i and x_{i+1} not in the same group,

and the group multiplication is concatenation followed by reducing, with the empty word as the identity element.

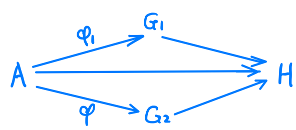
(b) If $\phi_1 : A \to G_1$ and $\phi_2 : A \to G_2$ are group homomorphisms, the *amalgamation group* $G_1 *_A G_2$ is the quotient group $(G_1 * G_2)/N$, where $N \subset G_1 * G_2$ the normal subgroup generated by the words $\phi_1(a)\phi_2(a)^{-1}$ for all $a \in A$.

We have the following *universal property of* $G_1 *_A G_2$:

Proposition 4.5.1. One has the commutative diagram of group homomorphisms:



Every commutative diagram

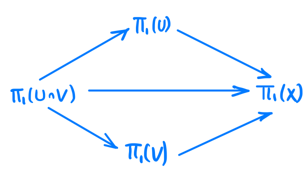


induces a unique group homomorphism $G_1 *_A G_2 \to H$ through which the homomorphisms from G_1, G_2, and A to H factor.

4.5.2 The van Kampen Theorem

Let X be any topological space.

Theorem 4.5.1 (The van Kampen Theorem)**.** Suppose that $X = U \cup V$, where U, V, and $U \cap V \neq \varnothing$ are open and path-connected. Then the diagram



gives a group isomorphism

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \longrightarrow \pi_1(X),$$

where all fundamental groups are based at some $x_0 \in U \cap V$.

We now look at several special cases of van Kampen Theorem.
We have already seen the following first special case.

Proposition 4.5.2. Suppose that $X = U \cup V$, where U, V, and $U \cap V \neq \varnothing$ are open and path-connected. *If U and V are simply connected*, then X is simply connected.

Proof. One has $\pi_1(U) = \pi_1(V) = \{e\}$, so $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \{e\}$. $\square$

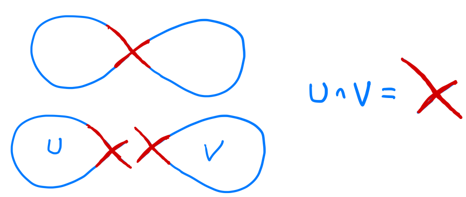
Consider now the second special case.

Proposition 4.5.3. Suppose that $X = U \cup V$, where U, V, and $U \cap V \neq \varnothing$ are open and path-connected. *If $U \cap V$ is contractible,* then

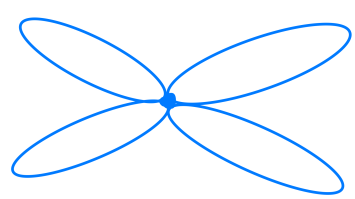
$$\pi_1(X) \cong \pi_1(U) * \pi_1(V).$$

Proof. One has $\pi_1(U\cap V) = \{e\}$, so $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \pi_1(U) * \pi_1(V)$. $\square$

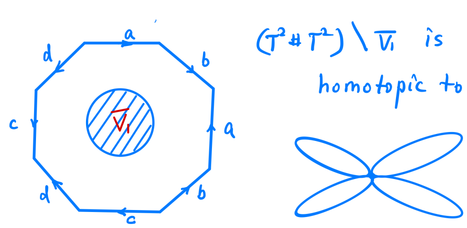
Example 4.5.1. For E the figure 8, one has $\pi_1(E) \cong \mathbb{Z} * \mathbb{Z}$.



Example 4.5.2. For $X = S^1 \vee S^1 \vee S^1 \vee S^1$, one has $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.



Example 4.5.3. For $U = (T^2 \# T^2)\backslash$(a closed disc), one has $\pi_1(U) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, free group with generators a, b, c, d.



We have the following third special case .

Proposition 4.5.4. Suppose that $X = U \cup V$, where U, V, $U \cap V \neq \varnothing$ are open and path-connected. If V *simply connected*, then

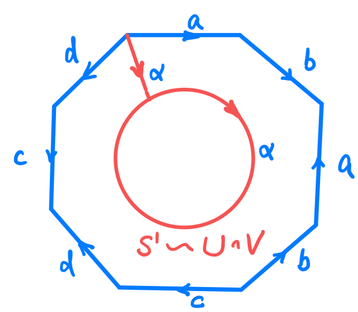
$$\pi_1(X) \cong \pi_1(U)/N$$

where N is the normal subgroup of $\pi_1(U)$ generated by the image of $\pi_1(U \cap V) \to \pi_1(U)$.

Proof. One has $\pi_1(V) = \{e\}$, so $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \cong \pi_1(U)/N$ by definition. $\square$

Example 4.5.4. Let $X = T^2 \# T^2$: Take $V \subset X$ a disc, so $\pi_1(V) = \{e\}$,

$$V_1 \subset V \text{ smaller disc}, \quad U = (T^2 \# T^2) \backslash \overline{V_1}.$$



[α] = aba⁻¹b⁻¹cdc⁻¹d⁻¹

We have $\pi_1(U) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, with generators a, b, c, d, and $U \cap V \sim S^1$ with

$$\pi_1(U \cap V) = \langle aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle \subset \pi_1(U).$$

Thus $\pi_1(T^2 \# T^2) \cong \{a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1\}$.

Similar arguments no give the following conclusions on the fundamental groups of surfaces.

Theorem 4.5.2. For any integer $n \geq 1$, one has

$$\pi_1(nT^2) \cong \{a_1, b_1, \ldots, a_n, b_n \,|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = e\},$$
$$\pi_1(n\,\mathbb{RP}^2) \cong \{a_1, a_2, \ldots, a_n \,|\, a_1^2 a_2^2 \cdots a_n^2 = e\}.$$