ASSIGNMENT V, ALGEBRA II, HKU, SPRING 2025

DUE AT 11:59PM ON FRIDAY APRIL 11, 2025

(1) Recall that $\overline{\mathbb{Q}} \subset \mathbb{C}$ is the field of all algebraic numbers. Let $\alpha \in \mathbb{C}$.

  (a) Show that $\alpha \in \overline{\mathbb{Q}}$ if and only if there exists a non-zero finite dimensional $\mathbb{Q}$-vector subspace V of $\mathbb{C}$ such that $\alpha V \subset V$.

  (b) Assume that $\alpha \in \overline{\mathbb{Q}}$. What is the minimal dimension of a non-zero finite dimensional $\mathbb{Q}$-vector subspace V of $\mathbb{C}$ such that $\alpha V \subset V$?

(2) An element $\alpha \in \mathbb{C}$ is called an *algebraic integer* if it is the root of some monic $f(x) \in \mathbb{Z}[x]$. Denote the set of all algebraic integers in $\mathbb{C}$ by $\overline{\mathbb{Z}}$. Clearly $\overline{\mathbb{Z}} \subset \overline{\mathbb{Q}}$.

  (a) Show that $\overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$;

  (b) Show that if $\alpha \in \overline{\mathbb{Q}}$, then there exists $n \in \mathbb{Z}$ such that $n\alpha \in \overline{\mathbb{Z}}$;

  (c) Show that $\alpha \in \mathbb{C}$ is in $\overline{\mathbb{Z}}$ if and only if $\alpha \in \overline{\mathbb{Q}}$ and its minimal polynomial over $\mathbb{Q}$ lies in $\mathbb{Z}[x]$;

  (d) Show that $\overline{\mathbb{Z}}$ is a sub-ring of $\overline{\mathbb{Q}}$;

  (e) Show that $\overline{\mathbb{Q}}$ is the fraction field of $\overline{\mathbb{Z}}$.

(3) Consider the cyclotomic field $C_n = \mathbb{Q}(\omega_n)$, where $n \geq 2$ and $\omega_n = e^{\frac{2\pi i}{n}}$. Recall that the degree of C_n as a field extension of $\mathbb{Q}$ is $\phi(n)$, the number of all integers $1 \leq k \leq n$ such that k is relatively prime to n.

  1) Show that ω_n is a root of the polynomial $x^2 - 2\cos(\frac{2\pi}{n})x + 1$ in $\mathbb{C}$

  2) Show that if the angle $\frac{2\pi}{n}$ is constructable by a ruler and a compass, then $\phi(n)$ must be a power of 2.

(4) Find the degree over $\mathbb{Q}$ of a splitting field over $\mathbb{Q}$ of the following polynomials in $\mathbb{Q}[x]$:

  1) $f(x) = x^3 - 2$, 2) $f(x) = x^4 - 1$, 3) $f(x) = (x^2 - 2)(x^3 - 2)$.

(5) Let $f(x) = x^7 - 1 \in \mathbb{Q}[x]$, and let L be the splitting field in $\mathbb{C}$ of f over $\mathbb{Q}$. Find $[L : \mathbb{Q}]$ and find a basis of L over $\mathbb{Q}$.

(6) Consider $f(x) = x^7 - 1$ as an element in $\mathbb{F}_7[x]$, and let M be the splitting field of f over $\mathbb{F}_7$. What is $[M : \mathbb{F}_7]$?