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DEPARTMENT OF MATHEMATICS

MATH6101 / MATH7101: Intermediate Complex Analysis
Holomorphic Immersion via Poincaré Series

Proposition. *Let $X = \Delta/\Gamma$ be a compact Riemann surface of genus $g \geq 2$. Let $x_0 \in \Delta$. Then there exists an integer $k \geq 2$, and bounded holomorphic functions $t, s \in H^\infty(\Delta)$ such that*

(a) *$f := P^k_\Gamma(t)$ satisfies $f(x_0) \neq 0$*

(b) *$g := P^k_\Gamma(s)$ satisfies $\left(\dfrac{g}{f}\right)'(x_0) \neq 0$*

Proof. Define $\Gamma_0 \subset \Gamma$ by $\Gamma_0 := \left\{\gamma \in \Gamma : |\gamma'(x_0)| \geq \dfrac{1}{2}\right\}$. By the proof of the convergence of Poincaré series for $k = 2$, we have $S := \sum_{\gamma\in\Gamma} \left|\gamma'(x_0)\right|^2 < \infty$. Hence, the subset $\Gamma_0 \subset \Gamma$ is finite. Enumerate Γ_0 as $\Gamma_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_N\}$ where $\gamma_1 = \mathrm{id}$. Consider the following two polynomials.

$$P(z) = \prod_{i=2}^{N} \Big(z - \gamma_i x_0\Big)^2$$

$$Q(z) = (z - x_0) \prod_{i=2}^{N} \Big(z - \gamma_i x_0\Big)^2 .$$

Since $\gamma x_0 \neq x_0$ for any $\gamma \in \Gamma - \{\mathrm{id}\}$, we have $P(x_0) \neq 0$. Clearly $Q(x_0) = 0$. Moreover, $Q(z) = (z - x_0)P(z)$ hence

$$Q'(z) = P(z) + (z - x_0)P'(z),$$

so that $Q'(x_0) = P(x_0) \neq 0$. Thus, $\big(P(x_0), P'(x_0)\big)$, and $\big(Q(x_0), Q'(x_0)\big)$, are linearly independent, from which we deduce readily

($\sharp$) Given any complex numbers $a, b \in \mathbb{C}$, there exists a polynomial $R(z) = \alpha P(z) + \beta Q(z)$ for some $\alpha, \beta \in \mathbb{C}$, such that $R(x_0) = a, R'(x_0) = b$.

We consider special cases of ($\sharp$). Let $A(z)$ be the polynomial given in ($\sharp$) such that $A(x_0) = 1, A'(x_0) = 0$, and let $B(z)$ be the polynomial given in ($\sharp$) satisfying $B(x_0) = 0$, $B'(x_0) = 1$. Denote now by $t := A\,|_\Delta \in H^\infty(\Delta), s := B\,|_\Delta \in H^\infty(\Delta)$. We claim that for k sufficiently large, $f := P^k_\Gamma(t), g := P^k_\Gamma(s)$ satisfy the requirements in the proposition.

Noting that $t(\gamma_i x_0) = 0$ for $2 \le i \le N$, we have

$$\begin{aligned} f(x_0) &= \sum_{\gamma\in\Gamma_0} t(\gamma x_0)\big(\gamma'(x_0)\big)^k + \sum_{\gamma\in\Gamma-\Gamma_0} t(\gamma x_0)\big(\gamma'(x_0)\big)^k, \\ &= 1 + \sum_{\gamma\in\Gamma-\Gamma_0} t(\gamma x_0)\big(\gamma'(x_0)\big)^k, \end{aligned}$$

so that

$$|f(x_0) - 1| \le \sum_{\gamma\in\Gamma-\Gamma_0} \big|t(\gamma x_0)\big| \, \big|\gamma'(x_0)\big|^k.$$

Since t is bounded on Δ, there exists $M > 0$ such that $\big|t(\gamma x_0)\big| \le M$ for any $\gamma \in \Gamma$. Recall that $S := \sum\limits_{\gamma\in\Gamma} |\gamma'(x_0)|^2 < \infty$. Hence,

$$\begin{aligned} |f(x_0) - 1| &\le M \sum_{\gamma\in\Gamma-\Gamma_0} \big|\gamma'(x_0)\big|^k = M \sum_{\gamma\in\Gamma-\Gamma_0} \big|\gamma'(x_0)\big|^2 \, \big|\gamma'(x_0)\big|^{k-2} \\ &\le M \left(\sum_{\gamma\in\Gamma} \big|\gamma'(x_0)\big|^2\right) \left(\sup_{\gamma\in\Gamma-\Gamma_0} \big|\gamma'(x_0)\big|\right)^{k-2} \le MS\left(\frac{1}{2}\right)^{k-2}. \end{aligned}$$

Choose k_0 such that $MS\left(\frac{1}{2}\right)^{k_0-2} < \frac{1}{2}$. Then, for any $k \ge k_0$, we have $|f(x_0) - 1| < \frac{1}{2}$, hence $|f(x_0)| > \frac{1}{2}$. In particular, $f(x_0) \neq 0$.

For the function g we want to estimate $\left(\frac{g}{f}\right)'(x_0)$. Now

$$\left(\frac{g}{f}\right)'(x_0) = \frac{1}{\big(f(x_0)\big)^2}\big(f(x_0)g'(x_0) - g(x_0)f'(x_0)\big).$$

Hence,

$$\begin{aligned} f'(x_0) = &\sum_{\gamma\in\Gamma_0} t'(\gamma x_0)\big(\gamma'(x_0)\big)^k + \sum_{\gamma\in\Gamma_0} kt(\gamma x_0)\big(\gamma'(x_0)\big)^{k-1}\gamma''(x_0) \\ &+ \sum_{\gamma\in\Gamma-\Gamma_0} t'(\gamma x_0)\big(\gamma'(x_0)\big)^k + \sum_{\gamma\in\Gamma-\Gamma_0} kt(\gamma x_0)\big(\gamma'(x_0)\big)^{k-1}\gamma''(x_0). \end{aligned}$$

Note that by Cauchy estimates there exists a constant C such that $|\gamma''(x_0)| \le C$ for any $\gamma \in \Gamma$. By our choice of $t = A|_\Delta$ we have $t(\gamma_i x_0) = t'(\gamma_i x_0) = 0$ for $2 \le i \le N$, while $t(x_0) = 1, t'(x_0) = 0$. Moreover for $\gamma = \mathrm{id}$, $\gamma'' = 0$. Hence,

$$f'(x_0) = \sum_{\gamma\in\Gamma-\Gamma_0} t'(\gamma x_0)\big(\gamma'(x_0)\big)^k + k\sum_{\gamma\in\Gamma-\Gamma_0} t(\gamma x_0)\big(\gamma'(x_0)\big)^{k-1}\gamma''(x_0).$$

Writing $\big(\gamma'(x_0)\big)^k = \big(\gamma'(x_0)\big)^2\big(\gamma'(x_0)\big)^{k-2}$ and $\big(\gamma'(x_0)\big)^{k-1} = \big(\gamma'(x_0)\big)^2\big(\gamma'(x_0)\big)^{k-3}$ we have

$$\big|f'(x_0)\big| \le \sup_{\Delta}\big|t'(z)\big| \cdot S\Big(\frac{1}{2}\Big)^{k-2} + k\sup_{\Delta}\big|t(z)\big| \cdot S\Big(\frac{1}{2}\Big)^{k-3} \cdot C\,. \tag{1}$$

Our task is to prove that $\Big(\dfrac{g}{f}\Big)'(x_0) \neq 0$ for k sufficiently large. Given that $s(x_0) = 0, s'(x_0) = 1, s(\gamma_i x_0) = s'(\gamma_i x_0) = 0$ for $2 \le i \le N$, we have

$$\begin{aligned} g(x_0) &= \sum_{\gamma\in\Gamma_0} s(\gamma x_0)\big(\gamma'(x_0)\big)^k + \sum_{\gamma\in\Gamma-\Gamma_0} s(\gamma x_0)\big(\gamma'(x_0)\big)^k \\ &= \sum_{\gamma\in\Gamma-\Gamma_0} s(\gamma x_0)\big(\gamma'(x_0)\big)^k, \end{aligned}$$

so that

$$|g(x_0)| \le \sup_{\Delta}|s(z)| \cdot S\Big(\frac{1}{2}\Big)^{k-2}. \tag{2}$$

Moreover,

$$\begin{aligned} g'(x_0) = &\sum_{\gamma\in\Gamma_0} s'(\gamma x_0)\big(\gamma'(x_0)\big)^k + \sum_{\gamma\in\Gamma_0} ks(\gamma x_0)\big(\gamma'(x_0)\big)^{k-1}\gamma''(x_0) \\ &+ \sum_{\gamma\in\Gamma-\Gamma_0} s'(\gamma x_0)\big(\gamma'(x_0)\big)^k + \sum_{\gamma\in\Gamma-\Gamma_0} ks(\gamma x_0)\big(\gamma'(x_0)\big)^{k-1}\gamma''(x_0) \\ = &\ 1 + \sum_{\gamma\in\Gamma-\Gamma_0} s'(\gamma x_0)\big(\gamma'(x_0)\big)^k + k\sum_{\gamma\in\Gamma-\Gamma_0} s(\gamma x_0)\big(\gamma'(x_0)\big)^{k-1}\gamma''(x_0) \end{aligned}$$

so that

$$|g'(x_0) - 1| \le \sup_{\Delta}|s'(z)| \cdot S\Big(\frac{1}{2}\Big)^{k-2} + k\sup_{\Delta}|s(z)| \cdot S\Big(\frac{1}{2}\Big)^{k-3} \cdot C\,. \tag{3}$$

From (1), (2) and (3), noting that $\lim\limits_{k\to\infty}\Big(\dfrac{1}{2}\Big)^k = \lim\limits_{k\to\infty} k\Big(\dfrac{1}{2}\Big)^{k-3} = 0$, we deduce that there exists an integer $k_1 \ge k_0$ such that for any $k \ge k_1$ we have

$$|g'(x_0)| \ge \frac{1}{2}, |g(x_0)| \le \frac{1}{4}, |f'(x_0)| \le \frac{1}{4}\,.$$

Since $|f(x_0)| \ge \dfrac{1}{2}$ for $k \ge k_0$, whenever $k \ge k_1$ we have

$$\big|f(x_0)g'(x_0) - g(x_0)f'(x_0)\big| \ge \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{4}\cdot\frac{1}{4} > 0,$$

so that

$$\Big(\frac{g}{f}\Big)'(x_0) = \frac{1}{\big(f(x_0)\big)^2}\big(f(x_0)g'(x_0) - g(x_0)f'(x_0)\big) \neq 0,$$

as desired. The proof of the proposition is complete. $\square$