MATH3541 INTRODUCTION TO TOPOLOGY
EXTRA PRACTICE PROBLEMS

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG

Due: 12:00 noon, 26th November 2024.

Instructions: Submit solutions to the problems in **Section B** for credit. Problems in Section A should be attempted and may be optionally submitted for feedback.

Section A

Problem 1. Let X be a path-connected topological space. Let $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ denote the fundamental groups based at x_0 and x_1 respectively.

(a) Use the existence of a path γ from x_0 to x_1 to construct an isomorphism of groups $\phi_\gamma : \pi_1(X, x_0) \to \pi_1(X, x_1)$.
(b) Show that ϕ_γ depends only on $[\gamma]$, the path-homotopy class (i.e. endpoint preserving) of γ. That is, if γ' is another path from x_0 to x_1, prove that $\phi_\gamma = \phi_{\gamma'}$.

Problem 2. Determine whether the following statements are true or false by a short proof or a counterexample.

(a) Any two continuous maps of the circle into the plane are homotopic.
(b) If X is contractible, then every map from X to itself has a fixed point.
(c) If $f : (X, x_0) \to (Y, y_0)$ is an injective continuous map, then the induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is injective.
(d) If $f : (X, x_0) \to (Y, y_0)$ is a surjective continuous map, then the induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective.
(e) The fundamental group of a topological space is abelian.
(f) For a pointed topological space (X, x_0), if two path connected subsets A, B each contain x_0, then $\pi_1(A \cup B, x_0) = \pi_1(A, x_0) \times \pi_1(B, x_0)$.
(g) There is a deformation retract from the disk to the circle.
(h) The Möbius strip and the cylinder are homotopy equivalent.
(i) Homotopic maps between spaces induce the same homomorphism on fundamental groups.

Problem 3. Show that the unit sphere, the Klein bottle, and the real projective spaces are pairwise non-homotopic.

Problem 4. Is there a retraction from the unit sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ to its equator $E = \{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$? Explain your answer.

Problem 5. Let X be a topological space and let $\gamma\colon [0,1] \to X$ be a path. Let $t_1 \in [0,1]$ and define two paths $\gamma_1, \gamma_2\colon [0,1] \to X$ by

$$\gamma_1(s) = \gamma(t_1 s),\ \ \gamma_2(s) = \gamma((1-t_1)s + t_1),\ \ s \in [0,1],$$

so γ_1 is a path from $\gamma(0)$ to $\gamma(t_1)$ and γ_2 is a path from $\gamma(t_1)$ to $\gamma(1)$. Prove that $[\gamma] = [\gamma_1 * \gamma_2]$.

Problem 6. Show that a covering map is an open map.

Problem 7. Show that the map $f\colon \mathbb{C} \to \mathbb{C} - \{0\}$ defined as $f(z) = e^z$, is a covering map.

Problem 8. Recall that quotient map $p\colon S^2 \to\cong S^2/\{\pm 1\} \cong \mathbb{RP}^2$ is a covering map.

(a) Determine the deck transformation group of p.
(b) Draw picture(s) to illustrate the generator(s) and relation(s) of the fundamental group $\pi(\mathbb{RP}^2, x_0)$

Problem 9. Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two copies of S^1. Draw/construct the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$. Compute its deck tranformation group to check that it is the correct one.

Problem 10. This question is about some basic properties of covering maps.

(a) For a covering map $p : X \to Y$ and a subset $A \subset Y$, show that $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is a covering map.
(b) For covering maps $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$, show that the product $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also a covering map.

Section B

Problem 11 (6 marks)**.** In this problem, we will show that the twice punctured plane is homotopy equivalent to the connected sum of two circles. Let $P = \mathbb{R}^2 \setminus \{p, q\}$ where $p = (-1, 0)$ and $q = (1, 0)$.

(a) Construct a closed embedding $\phi : S^1 \vee S^1 \to \mathbb{R}^2$ whose image is the vanishing set $V\left(((x-1)^2 + y^2 - 1)((x+1)^2 + y^2 - 1)\right)$.

(b) Determine an explicit deformation retraction from P to

$$P' := \mathbb{R}^2 \setminus (B(p, 1) \cup B(q, 1)),$$

where $B(c, 1) = \{x \in \mathbb{R}^2 \mid \|x - c\| < 1\}$ is an open ball centred on c.

(c) For any point $(x, y) \in P' \setminus \{(0, 0)\}$, show that the line segment through x and the origin intersects $S^1 \vee S^1$ at exactly two points, and determine a formula for the point which is not the origin.

(d) Use part (c) to construct a deformation retration from P' to $S^1 \vee S^1$, making sure to explicitly check continuity.

(e) Combine the previous parts to write $S^1 \vee S^1$ as a deformation retract of the twice punctured plane P.

Problem 12 (7 marks)**.** Let X, Y, and Z be path-connected and locally path-connected. For continuous maps $q : X \to Y$ and $r : Y \to Z$, let $p = r \circ q$ be their composition.

(a) Show that if p and q are covering maps, then r is a covering map.

(b) Show that if p and r are covering maps, then q is a covering map. [Hint: Use the map-lifting lemma]

(c) Suppose that $r^{-1}(z)$ is finite for each $z \in Z$. Show that if q and r are covering maps, then p is a covering map.

(d) Recall the definition of a universal cover (Definition 3.3.7.). Prove that if Z has a universal cover $u : \tilde{Z} \to Z$ and q and r are covering maps, then p is a covering map.

Problem 13 (7 marks)**.** Let G be a group acting on a Hausdorff topological space X such that if $g \cdot x = x$ for some $x \in X$, then $g = e$ is the identity.

(a) Show that if G is finite, then the action of G on X is discontinuous.

Now, additionally suppose that X is locally compact and that for each compact subspace C of X, there are finitely many $g \in G$ such that the intersection $C \cap g(C) \neq \emptyset$.

(b) For each compact $C \subset X$, prove the orbit $\bigcup_{g \in G} g(C)$ is closed in X.

(c) Show that X/G is Hausdorff

(d) Show that the action of G is discontinuous.

(e) Show that X/G is locally compact