2) $\pi : X \to X/G$ is open since X/G is equipped with quotient top.

$U \subset X$ open it suffices to show $\pi^{-1}(\pi(U))$ is open

$$\pi^{-1}\pi(U) = GU = \{ gx \mid x \in U \ g \in G \}$$
$$= \bigcup_{g \in G} gU \text{ open.}$$

$\Gamma = \bigcup \Gamma_g$

Assume X is Hausdorff

3) X/G is Hausdorff $\Leftarrow$ $\{(x, gx) \mid x \in X, g \in G\} \subset X \times X$ is closed

in particular if G is finite X/G is T_2

"$\Rightarrow$" X/G is Hausdorff $\Leftrightarrow$ $Gx \neq Gy$ can be separated

$$(x,y) \in \Gamma^c \Leftrightarrow y \notin Gx \text{ i.e. } Gx \neq Gy$$

$\exists\, U, V \subset X$ s.t. $\quad U \times V \cap \Gamma = \emptyset$
$x \in U$, $y \in V$

$$\Leftrightarrow \forall \ x_1 \in U \ x_2 \in V, g \in G$$
$$g x_1 \neq x_2$$
$$\Leftrightarrow \pi U \cap \pi V = \emptyset$$

"$\Leftarrow$" Ex

4) if X is compact (resp. conn.) then so is X/G

Examples

1) $\underset{G}{GL_n(\mathbb{R})} \times \underset{V}{\mathbb{R}^n} \longrightarrow \underset{V}{\mathbb{R}^n}$

$(g, v) \longmapsto gv$

V/G is not Hausdorff.

$[v] = Gv$

$$V/G = \left\{ [\vec{0}], [\vec{e}_1] = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$g_n = \begin{pmatrix} \frac{1}{n} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad g_n e_1 = \begin{pmatrix} \frac{1}{n} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{n \to \infty} \vec{0}$$

Generalization

$$\mathbb{R}^{n \cdot m} = \underset{v_1}{\mathbb{R}^n} \times \underset{v_2}{\mathbb{R}^n} \times \cdots \times \underset{v_m}{\mathbb{R}^m} = \underset{[v_1, v_2, \ldots, v_m]}{Mat_{n \times m}}$$

$GL_n \times Mat_{n \times m} \longrightarrow Mat_{n \times m}$

$(g, [v_1 \ldots v_m]) \longmapsto [gv_1, \ldots, gv_m]$

From the case $n=1$ the orbit space $\mathrm{Mat}_{n\times m}/GL_n$ is not $T2$.

Claim $X = \{A \in \mathrm{Mat}_{n\times m} \mid \mathrm{rk}\, A = \max\{n, m\}\}$

1) GL_n acts on M

2) the orbit space X/GL_n is $T2$

3) X/GL_n is compact [Difficult] Assume $n < m$

a) show that $X/GL_n \overset{\text{as sets}}{=} \{n\text{-dim'l sub space of } \mathbb{R}^m\}$

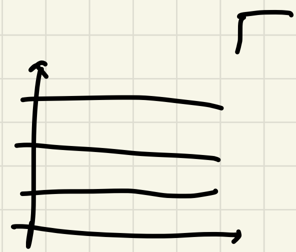
b) $X/GL_n = \{S^{n-1} \subset S^{m-1} \subset \mathbb{R}^m\}$ $\quad S^{n-1} = S^{m-1} \cap \mathbb{R}^{n-1}$

c) show that $X/GL_n = Y/O(n)$

$$Y = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \quad \{w_1, \dots w_n\} \text{ is orthonormal}$$

2) $G = \mathbb{Z}$ $\quad X = \mathbb{R}$ $\quad \mathbb{R}/\mathbb{Z} \cong S^1$

$$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(n, x) \longmapsto x + n \cdot 2\pi$$



$$\mathbb{R} \longrightarrow S^1$$
$$\theta \longrightarrow \theta \bmod 2\pi$$

$\mathbb{R}/\mathbb{Z} \longrightarrow S^1$

cont. surjection.

it suffices to show $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is open

but q is a local homeomorphism.

3) Let V be a $\mathbb{R}$-vector space of dim n

and $v_1 \dots v_n$ be n linearly independent

vectors $\quad \mathbb{Z}^n \ni e_i := \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ i \\ \\ \\ \end{array}$

$$6: \mathbb{Z}^n \times V \longrightarrow V$$

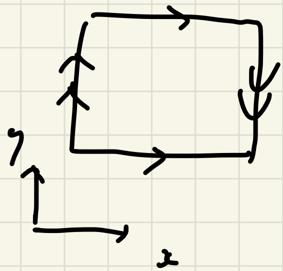
$$(e_i, v) \longmapsto v + v_i$$

$$\mathbb{R}^n/\mathbb{Z}^n \cong \underbrace{S^1 \times S^1 \cdots \times S^1}_{n \text{ copies}}$$

4) [Klein bottle]

$\boxed{\text{Adjunction space construction}}$

$[0,1] \times [0,1]$



$$(x,0) \sim (x,1)$$

$$(0,y) \sim (1, 1-y)$$

$X := [0,1]\times[0,1] \big/ \sim$ with quotient top.

Orbit space construction

$\mathbb{Z}\times\mathbb{Z}$ has a different gp structure

check this is a group

$$(m,n)\cdot(m',n') = (m+(-1)^n m', n+n')$$

We denote this group (with discrete top)

$$\mathbb{Z}\times'\mathbb{Z}$$

Claim $X \cong \mathbb{R}^2 / \mathbb{Z}\times'\mathbb{Z}$ $\quad \sigma((m,n),(x,y))$

$$= (m+(-1)^n x, n+y)$$

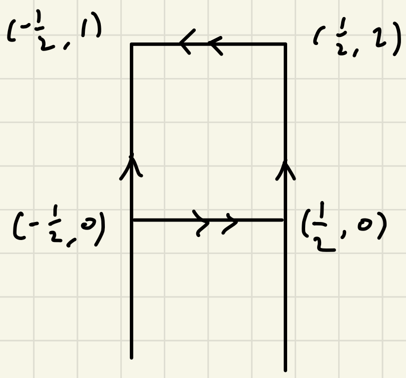
$a=(1,0)$ $\quad b=(0,1) \in \mathbb{Z}\times'\mathbb{Z}$

$a\cdot b = (1,1)$ $\quad aba = (1,0)\cdot(-1,1)$

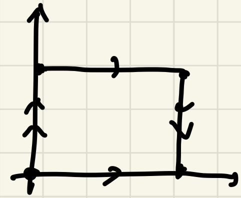
$b\cdot a = (-1,1)$ $\quad = (1-1, 1) = (0,1) = b$

Show that $\mathbb{Z}\times'\mathbb{Z} = \langle a,b \mid aba=b\rangle$

$$a\cdot(x,y) = (1+x, y)$$
$$b(x,y) = (-x, 1+y)$$



the action on boundary on $[0,1]\times[0,1]$ coincides with eq-relations defining X.



$$\begin{tikzcd} {[0,1]\times[0,1]} \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{R}^2 \arrow[d] \\ {[0,1]^2/\sim} \arrow[r, hook] & \mathbb{R}^2/\mathbb{Z}\times'\mathbb{Z} \end{tikzcd}$$

hausdorff

compact

bijective, therefore a homeomorphism.

A matrix representation of $\mathbb{Z}\times'\mathbb{Z}$

Affine transformation on $\mathbb{R}^2$

$v=(a,b)\in\mathbb{R}^2 \qquad \theta\in O(2)$

$w=(x,y)\in\mathbb{R}^2$

$$(\theta,v)\cdot w = \theta w+v$$

$O(2)\times\mathbb{R}^2 =: E(2)$

with this group str.

is called (2D) Euclidean group.

$$\underbrace{(\theta_1,v_1)*(\theta_2,v_2)}_{=\ (\theta_1\theta_2,\ \theta_1 v_2+v_1)}\cdot w = (\theta_1,v_1)(\theta_2 w+v_2)$$

$$= \theta_1(\theta_2 w+v_2)+v_1 = \theta_1\theta_2 w+\theta_1 v_2+v_1$$