THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 4

Due 3:30pm[1], October 4th (Friday), **in-class**.

Aim of this Homework: *In this assignment you will classify linear second-order PDE, and study first-order and second-order PDE by using the method of characteristics, as well as the coordinate method.*

Reading Assignment: Read the following material(s):

(i) Section 2.2-2.3 of the textbook.

Instruction: Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the "Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

Problem 1. What is the type (e.g., elliptic, parabolic, hyperbolic) of each of the following equations? Justify your answers.

(i) $9\partial_{xx}u - 12\partial_{xy}u + 16\partial_{yy}u = \tan^{-1}\left(\frac{xy}{1+x^4+y^4}\right)$.

[1] You are expected to submit your homework before the beginning of Friday lecture **in-class**.

(ii) $\sqrt{2}\partial_{xx}u + \sqrt{2}\partial_{yy}u + \sqrt{2}\partial_{zz}u = 2\partial_{xy}u + 2\partial_{yz}u$.

(iii) $x\partial_{xx}u + 2\sqrt{1+x^2+y^2}\partial_{xy}u + y\partial_{yy}u + 27\partial_x u + 9\partial_y u + 2024u = 0$.

(iv) $\partial_{x_0}^2 u - 5\sum_{i=1}^{3}\partial_{x_i}^2 u = 6\sum_{i=2}^{3}\partial_{x_{i-1}}\partial_{x_i}u$.

Remark. In physics literature, particularly in the special relativity, the variable x_0 is commonly used to represent the time t. This notation enables a consistent representation of spatial and temporal variables within the four-dimensional spacetime framework, so that the corresponding mathematical expressions of equations are streamlined.

Problem 2. In this problem, we consider the following initial and boundary value problem:

$$\begin{cases} \partial_t u + a(t,x)\partial_x u = f(u) & \text{for } x > 0 \text{ and } t > 0 \\ u|_{x=0} = g(t) & \\ u|_{t=0} = \phi(x), & \end{cases} \tag{1}$$

where ϕ and g are given initial and boundary data respectively. The coefficient a and the non-homogeneous term f will be given differently in different parts below. Complete the following parts.

(i) Let $a(t,x) := 2x - 7$ and $f(u) := 1$. Find the compatibility condition(s) of g and ϕ for the existence of (1). Do not solve the problem (1).

(ii) Let $a(t,x) := t^2 + x$ and $f(u) := \sqrt{1+u^2}$. Under the compatibility condition $g(0) = \phi(0) = 0$, solve the initial and boundary value problem (1). Express your final answer u in terms of t, x, g and ϕ only.

(iii) Let $a(t,x) := -27x - 9t - 2024$, $f(u) := 3\sin u + 4\cos u + 5$, $g(t) := -20\sinh 24t$ and $\phi(x) := 10 - 4\tanh x$. Either solve the problem (1) and express your final answer u in terms of t and x only, or showing that the problem (1) is NOT solvable.

Problem 3. Consider the second-order equation

$$\partial_{tt}u - \partial_{tx}u - 12\partial_{xx}u = 0, \tag{2}$$

and complete the following parts.

(i) Verify that Equation (2) is hyperbolic by computing its discriminant $\mathcal{D}$.

(ii) Find the general solution to (2) by the method of characteristics.

(iii) Solve (2) subject to the initial data

$$\begin{cases} u|_{t=0} = 6x^3 \\ \partial_t u|_{t=0} = 59x^2. \end{cases}$$

Problem 4. Do Problem 1 of Dec 2019 Final Exam.

Remark. It is worth noting that all of the available past Final Exam papers of this course are uploaded in our Moodle site. More precisely, you can access the past papers via the pdf file "*Collection of Past Final Exams (2003-2023)*" under the module "*Other Materials*" in our Moodle site.

The following problem(s) is/are *optional*:

Problem 5. In this problem we will solve the following second order equation

$$9\partial_{tt}u - 12\partial_{tx}u + 4\partial_{xx}u = 0 \tag{3}$$

by the method of characteristics. Complete the following parts.

(i) Determine the type (e.g., elliptic, parabolic, hyperbolic) of Equation (3) by computing its discriminant $\mathcal{D}$.

(ii) Verify that Equation (3) can be written as

$$(3\partial_t - 2\partial_x)^2 u = 0.$$

(iii) Let $v := (3\partial_t - 2\partial_x)u = 3\partial_t u - 2\partial_x u$. Verify that u and v satisfy the following system:

$$\begin{cases} 3\partial_t v - 2\partial_x v = 0 \\ 3\partial_t u - 2\partial_x u = v. \end{cases}$$

(iv) Find the general solution to the first-order equation

$$3\partial_t v - 2\partial_x v = 0.$$

(v) Apply the method of characteristics to solve

$$3\partial_t u - 2\partial_x u = v,$$

and prove that the general solution u to (3) has the form

$$u(t,x) = f(3x+2t) + t\, g(3x+2t),$$

where both f and g are arbitrary functions.

Food for Thought. Are you able to solve the initial-value problem for Equation (3)?

Remark. It is worth noting that the extra power "t" appears (in front of the arbitrary function g) in the general solution (see Part (iv) of Problem 5 for instance) to Equation (3) is due to the resonance.

Problem 6. Consider the initial-value problem for the wave equation:

$$\begin{cases} \partial_{tt} u - c^2 \partial_{xx} u = 0 \qquad \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi \\ \partial_t u|_{t=0} = \psi, \end{cases} \tag{4}$$

where the wave speed $c > 0$ is a given constant, the functions ϕ and ψ are given initial data, and the function $u := u(t,x)$ is the unknown; and complete the following parts.

(i) Using the d'Alembert's formula or any other method, verify that if both ϕ and ψ are odd functions, that is, for any $x \in (-\infty, \infty)$,

$$\begin{cases} \phi(-x) = -\phi(x) \\ \psi(-x) = -\psi(x), \end{cases}$$

then for any $t \geq 0$,

$$u(t, 0) = 0.$$

(ii) If both ϕ and ψ have compact supports, that is, there exists a constant $M > 0$ such that

$$\begin{cases} \phi(x) = 0 \quad \text{for all } |x| > M \\ \psi(x) = 0 \quad \text{for all } |x| > M, \end{cases}$$

then what can you say about the solution u to the initial-value problem (4)?

(iii) If both ϕ and ψ are periodic functions with periods T_ϕ and T_ψ respectively, then under what conditions on T_ϕ and T_ψ, must the solution u be periodic in x? State the weakest sufficient condition that you can obtain, and justify your answer by providing a rigorous proof.