Chapter 7. Duhamel's Principle

MATH4406 Introduction to Partial Differential Equations

The University of Hong Kong

Table of Contents

This chapter is related to the materials in Section 3.3 and 3.4 of the Textbook.

7.1 Duhamel's Principle for Ordinary Differential Equations

What Is Duhamel's Principle?

Duhamel's principle

If we know how to solve a (linear) homogeneous problem, then we will be able to solve the corresponding (linear) non-homogeneous problem.

Philosophy of Duhamel's principle

Use homogeneous solutions to construct non-homogeneous solutions.

Duhamel's principle for Ordinary Differential Equations (ODEs)

Roughly speaking, it is just the method of integrating factors.

To illustrate the idea, we will start with

- the scalar ODE, and
- the system of ODEs.

Scalar ODE

Non-Homogeneous Problem

$$\begin{cases} \dfrac{du}{dt} + au = f \\ \qquad u\big|_{t=0} = u_0. \end{cases}$$

where the constants a and u_0 are scalars, $f : [0, \infty) \to \mathbb{R}$ is the given source term.

The Corresponding Homogeneous Problem

$$\begin{cases} \dfrac{dv}{dt} + av = 0 \\ \qquad v\big|_{t=0} = u_0. \end{cases}$$

Solution to the Homogeneous Problem

$$v(t) = e^{-at} u_0 =: \phi(t) u_0.$$

Remark

Here, we should think: for any fix $t > 0$, $\phi(t)$ is a **mapping**, namely

$$\begin{aligned} \phi(t) : \text{scalar} &\mapsto \text{scalar} \\ u_0 &\mapsto e^{-at}u_0. \end{aligned}$$

In other words, $\phi(t)$ is NOT just a scalar e^{-at}.

Now, applying the method of integrating factors, we have

Solution to the Non-Homogeneous Problem

$$\begin{aligned} u(t) &= e^{-at}u_0 + \int_0^t e^{-a(t-s)} f(s)\, ds \\ &= \phi(t)u_0 + \int_0^t \phi(t-s)f(s)\, ds, \end{aligned}$$

where $\phi(t) := e^{-at}$.

System of ODEs

Non-Homogeneous Problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F \\ \quad U|_{t=0} = U_0, \end{cases}$$

where A is a $d \times d$ constant matrix, $F : [0, \infty) \to \mathbb{R}^d$ is the given source term, the initial data $U_0 \in \mathbb{R}^d$.

The Corresponding Homogeneous Problem

$$\begin{cases} \dfrac{dV}{dt} + AV = 0 \\ \quad V|_{t=0} = U_0. \end{cases}$$

Solution to the Homogeneous Problem

$$V(t) = e^{-At} U_0 =: \Phi(t) U_0.$$

Remark

The mapping $\Phi(t) := e^{-At}$ is called the "fundamental matrix", and can be defined by using a Taylor series expansion.

Now, applying the method of variation of parameters, we have

Solution to the Non-Homogeneous Problem

$$\begin{aligned} u(t) &= e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(s)\, ds \\ &= \Phi(t)u_0 + \int_0^t \Phi(t-s) f(s)\, ds, \end{aligned}$$

where $\Phi(t) := e^{-At}$.

Moral (Lesson that We Learn from These Two Examples)

The explicit form of A is NOT important; the crucial point is that $\Phi(t)$ maps initial data to the homogeneous solution at the time t.

7.2 Duhamel's Principle for Partial Differential Equations

Duhamel's Principle/Operator Method

Theorem (Duhamel's Principle/Operator Method)

Let $\mathcal{A}$: (function of x) $\mapsto$ (function of x) be a given linear operator, $f := f(t,x)$ be a given non-homogeneous source term, and $u_0 := u_0(x)$ be a given initial data. Then the solution to the non-homogeneous problem

$$\begin{cases} \partial_t u + \mathcal{A}u = f \\ \quad u\big|_{t=0} = u_0 \end{cases}$$

is

$$u(t) := \Phi(t)u_0 + \int_0^t \Phi(t-s)f(s)\,ds, \tag{Duh}$$

provide that $v := \Phi(t)u_0$ is the unique solution to the corresponding homogeneous problem:

$$\begin{cases} \partial_t v + \mathcal{A}v = 0 \\ \quad v\big|_{t=0} = u_0. \end{cases}$$

Formal Proof of the Theorem

Using the solution formula (Duh), we have

$$
\begin{aligned}
\partial_t u &= \partial_t \left(\Phi(t)u_0 + \int_0^t \Phi(t-s)f(s)\,ds \right) \\
&= \underbrace{\partial_t\left(\Phi(t)u_0\right)}_{=-\mathcal{A}(\Phi(t)u_0)} + \underbrace{\Phi(t-s)f(s)\Big|_{s=t}}_{=\Phi(0)f(t)=f(t)} + \int_0^t \underbrace{\partial_t\left(\Phi(t-s)f(s)\right)}_{=-\mathcal{A}(\Phi(t-s)f(s))}\,ds \\
&= -\mathcal{A}\underbrace{\left(\Phi(t)u_0 + \int_0^t \Phi(t-s)f(s)\,ds \right)}_{=u} + f(t),
\end{aligned}
$$

because $v := \Phi(t)g$ satisfies the homogeneous equation $\partial_t v + \mathcal{A}v = 0$, $\Phi(0) = Id$, $\mathcal{A}$ is a linear operator, and u is defined via (Duh). In other words, the u defined by (Duh) satisfies

$$\partial_t u + \mathcal{A}u = f.$$

Formal Proof of the Theorem (Continued)

Finally, evaluating

$$u(t) := \Phi(t)u_0 + \int_0^t \Phi(t-s)f(s)\,ds$$

at $t = 0$, we have

$$u(0) = \underbrace{\Phi(0)u_0}_{=u_0} + \underbrace{\int_0^0 \Phi(0-s)f(s)\,ds}_{=0} = u_0$$

because $\Phi(0) = Id$, and both the upper and lower limits of the last integral are the SAME. This verifies the initial condition as well. $\square$

Exercise

The uniqueness of the non-homogeneous problem follows directly from that of the corresponding homogeneous problem.

7.3 Duhamel's Principle for Non-Homogeneous Heat Equations

Non-Homogeneous Heat Equations

When the operator $\mathcal{A} := -k\partial_{xx}$ where $k > 0$ is a given constant, the non-homogeneous problem

$$\begin{cases} \partial_t u + \mathcal{A}u = f \\ \quad u\big|_{t=0} = u_0 \end{cases}$$

becomes

$$\begin{cases} \partial_t u - k\partial_{xx} u = f \\ \quad u\big|_{t=0} = u_0. \end{cases} \tag{NonP}$$

The corresponding homogeneous problem

$$\begin{cases} \partial_t v - k\partial_{xx} v = 0 \\ \quad v\big|_{t=0} = u_0 \end{cases}$$

has a solution

$$v(t,x) = \int_{-\infty}^{\infty} S(t, x-y) u_0 \, dy =: \Phi(t) u_0.$$

Remark

- Let us recall that the heat kernel $S(t, x-y) := \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}}$.
- For any fix $t \geq 0$,

$$\begin{array}{rcl} \Phi(t) : (\text{function of } x) & \mapsto & (\text{function of } x) \\ u_0 & \mapsto & u(t, \cdot). \end{array}$$

According to Duhamel's principle, the solution to (NonP) is

$$u(t,x) = \Phi(t)u_0 + \int_0^t \Phi(t-s)f(s)\, ds$$
$$= \int_{-\infty}^{\infty} S(t, x-y)u_0(y)\, dy + \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)f(s,y)\, dy\, ds,$$

because

$$\Phi(t)g := \int_{-\infty}^{\infty} S(t, x-y)g(y)\, dy.$$

Remark

One can also verify the explicit solution formula

$$u(t,x) = \int_{-\infty}^{\infty} S(t, x-y)u_0(y)\, dy + \int_0^t \int_{-\infty}^{\infty} S(t-s, x-y)f(s,y)\, dy\, ds$$

via a direct differentiation; see Page 69 of the textbook for instance.

Example

Question: Solve $\partial_t u - k\partial_{xx} u = 1$ and $u|_{t=0} = x^2$.
Solution: It follows from the explicit formula (with $f(t,x) \equiv 1$) that

$$\begin{aligned} u(t,x) &= \underbrace{\int_{-\infty}^{\infty} S(t, x-y)y^2\, dy}_{=x^2+2kt} + \int_0^t \underbrace{\int_{-\infty}^{\infty} S(t-s, x-y)1\, dy}_{\equiv 1}\, ds \\ &= x^2 + (2k+1)t. \end{aligned}$$

7.4 Duhamel's Principle for Non-Homogeneous Wave Equations

Non-Homogeneous Wave Equations

Consider the Cauchy problem for non-homogeneous wave equation

$$\begin{cases} \partial_{tt} u - c^2 \partial_{xx} u = f & \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ u|_{t=0} = \phi \\ \partial_t u|_{t=0} = \psi, \end{cases} \tag{NonWave}$$

where the propagation speed $c > 0$ is a given constant, the source term $f := f(t, x)$ and initial data $\phi := \phi(x)$ and $\psi := \psi(x)$ are given functions.

Question

How to solve (NonWave)?

Observation

Due to the linearity and solvability of homogeneous wave equation (i.e., we can solve (NonWave) when $f \equiv 0$), it suffices to solve (NonWave) when $\phi \equiv \psi \equiv 0$.

Non-homogeneous Wave Equation with Trivial Initial Data

Consider

$$\begin{cases} \partial_{tt} U - c^2 \partial_{xx} U = f & \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ U|_{t=0} = \partial_t U|_{t=0} \equiv 0. \end{cases} \tag{SimNW}$$

To solve (SimNW), one can apply either the method of characteristics, or coordinate method. Here, we will solve the problem by using the *method of characteristics* as follows.

Method of Characteristics

Let $V := \partial_t U + c\partial_x U$. Then U and V satisfy

$$\begin{cases} \partial_t U + c\partial_x U = V, & U|_{t=0} \equiv 0, \\ \partial_t V - c\partial_x V = f, & V|_{t=0} \equiv 0. \end{cases}$$

Since $f := f(t, x)$ is given, the system for U and V is partially decoupled. We will solve for V first, and then U.

Solving for V

It follows from the chain rule that

$$\frac{d}{ds}V(s, x_0 - cs) = (\partial_t V - c\partial_x V)\Big|_{(t,x)=(s,x_0-cs)} = f(s, x_0 - cs).$$

Integrating the above equation with respect to s from 0 to t, we have

$$V(t, x_0 - ct) - \underbrace{V(0, x_0)}_{=0} = \int_0^t f(s, x_0 - cs)\, ds,$$

since $V|_{t=0} \equiv 0$. We cannot further simplify the last integral, unless f is given explicitly. Setting $x := x_0 - ct$, we know that $x_0 = x + ct$, and hence,

$$V(t, x) = \int_0^t f(s, x + ct - cs)\, ds.$$

Solving for U

It follows from the chain rule that

$$\frac{d}{d\tau}U(\tau, x_0 + c\tau) = (\partial_t U - c\partial_x U)\Big|_{(t,x)=(\tau,x_0+c\tau)} = V(\tau, x_0 + c\tau)$$
$$= \int_0^\tau f(s, (x_0 + c\tau) + c\tau - cs)\, ds = \int_0^\tau f(s, x_0 + 2c\tau - cs)\, ds.$$

Integrating the above equation with respect to τ from 0 to t, we have

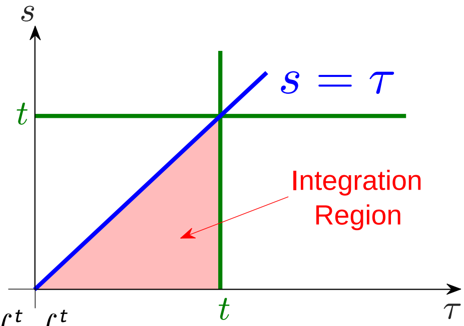
$$U(t, x_0 + ct) - \underbrace{U(0, x_0)}_{=0} = \int_0^t \int_0^\tau f(s, x_0 + 2c\tau - cs)\, ds\, d\tau,$$

since $V|_{t=0} \equiv 0$. Setting $x := x_0 + ct$, we have $x_0 = x - ct$, and hence,

$$U(t, x) = \int_0^t \int_0^\tau f(s, x - ct + 2c\tau - cs)\, ds\, d\tau.$$

Observation

Both arguments in $f(s, x - ct + 2c\tau - cs)$ depend on s, so in order to simplify the integral $\int_0^t \int_0^\tau f(s, x - ct + 2c\tau - cs)\, ds\, d\tau$, we want to interchange ds with $d\tau$.



Thus, $U(t, x) = \int_0^t \int_s^t f(s, x - ct + 2c\tau - cs)\, d\tau\, ds.$

Let $y := x - ct + 2c\tau - cs$. Then $d\tau = \dfrac{1}{2c} dy$, and

$$\begin{aligned} \tau = s \quad &\Longleftrightarrow \quad y = x - c(t-s), \\ \tau = t \quad &\Longleftrightarrow \quad y = x + c(t-s). \end{aligned}$$

Hence,

$$\begin{aligned} U(t,x) &= \int_0^t \int_s^t f(s, x - ct + 2c\tau - cs)\, d\tau\, ds \\ &= \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\, dy\, ds. \end{aligned}$$

Conclusion

The solution to the original non-homogeneous problem (NonWave) is

$$\begin{aligned} u(t,x) = \frac{1}{2}\{\phi(x+ct) + \phi(x-ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds \\ + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\, dy\, ds. \end{aligned}$$