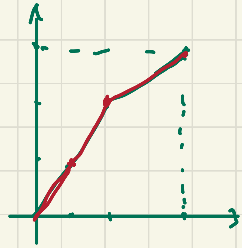
Fundamental gp of circle



Thm $\pi_1(S^1) = (\mathbb{Z}, +)$

$\pi : \mathbb{R} \longrightarrow S^1$ $\quad x_0 = 1$

$x \longmapsto e^{2\pi i x}$

$\alpha : [0,1] \longrightarrow S^1$ $\quad \alpha^n = \underbrace{\alpha * \cdots * \alpha}_{n}$ $\quad n \in \mathbb{Z}$

$\alpha(t) = e^{2\pi i t}$ $\quad \sim e^{2\pi i n t}$

$\phi : \mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$ is a group homomorphism

$n \longrightarrow [\alpha^n]$

Need to check

1) ϕ is inj. i.e. $\alpha^n \sim e_{x_0} \iff n = 0$

2) ϕ is surj. i.e. $\forall\, \gamma : [0,1] \to S^1$

s.t. $\gamma(0) = \gamma(1) = 1$ $\quad \exists\, n \in \mathbb{N}$ $\gamma \sim \alpha^n$.

Lemma Given $[\gamma] \in \pi_1(X, x_0)$ and

$\tilde{x}_0 \in \mathbb{R}$ s.t $\pi(\tilde{x}_0)$, $\exists$ path

$\tilde{\gamma} : [0,1] \to \mathbb{R}$ s.t $\tilde{\gamma}(0) = \tilde{x}_0$ $\pi \circ \tilde{\gamma} = \gamma$.

if $\gamma_1 \sim \gamma_2$ then $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) \Rightarrow \tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$

$(\Leftarrow)$

injectivity: if $\alpha^n \sim e_{x_0}$ $\tilde{x}_0 = 0$

$\tilde{\alpha} = nt$ $\tilde{e}_{x_0} = 0$

by lemma $\tilde{\alpha}(1) = \tilde{e}_{x_0}(1)$

$$\begin{array}{ccc} \tilde{\alpha}(1) & = & \tilde{e}_{x_0}(1) \\ \shortparallel & & \shortparallel \\ n & & 0 \end{array}$$

surjectivity $[\gamma] \in \pi_1$ $\tilde{\gamma}(1) \in \mathbb{N}$

Suppose $\tilde{\gamma}(1) = n$ $\tilde{\gamma}(0) = 0$

$\tilde{\alpha}^n = nt$ $\tilde{\alpha}^n(0) = 0$ $\tilde{\alpha}^n(1) = n$

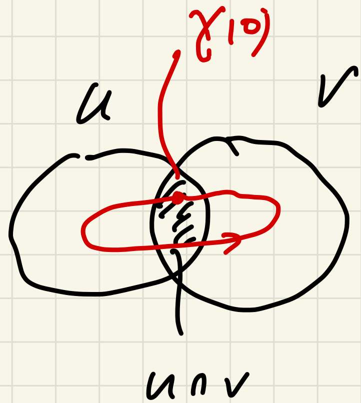
$F(s,t) = (1-s)\tilde{\gamma} + s(nt)$

$\gamma \overset{\pi \circ F}{\sim} \alpha^n$

$\pi_1(S^n)$ $n > 1$

Thm $\pi_1(S^n) = 1$ $n > 1$

Lemma X top space $U \cup V = X$ is an open cover s.t. U, V are simply conn (& path-conn) and $U \cap V$ is path conn. nonempty. Then X is simply conn



sublemma [Lebesgue]

Let X be a compact metric space and U_α an open cover of X

$\exists\, \delta > 0$ s.t. $\forall\ x \in X$ $B(x, \delta) \subset U_\alpha$ for some α.

Pf. of lemma Clearly X is path conn.

$\gamma : I \to X \qquad \gamma^{-1}(U) \cup \gamma^{-1}(V) = I$

by sublemma $\exists\ \delta > 0$ s.t

$\forall\ t \in I \quad (t-\delta, t+\delta) \subset \gamma^{-1}(U)$ or $\gamma^{-1}(V)$

by compactness, $\exists\, t_0, t_1, \dots t_n \in I$

$t_0 = 0, \quad t_n = 1$

s.t $\displaystyle\bigcup_{i=0}^{n} I_i = I \qquad I_i := (t_i - \delta, t_i + \delta)$

WLOG we may assume (by taking union)

$\gamma(I_0) \subset U, \gamma(I_1) \subset V, \gamma(I_2) \subset U \ \cdots$

choose $x_1 \in \gamma(I_0 \cap I_1) \,..\, x_i \in \gamma(I_{i-1} \cap I_i)$

$\gamma(I_0 \cap I_1) \subset U \cap V, \quad \gamma(I_{i-1} \cap I_i) \subset U \cap V$

$\beta_i : [0,1] \to U \cap V$

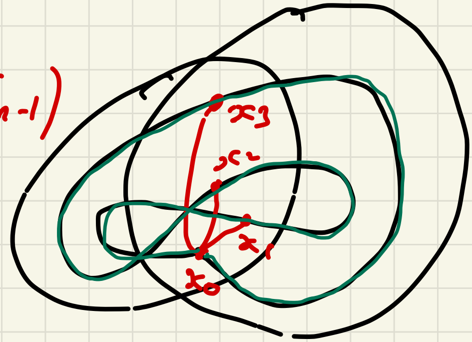
$\beta_i(0) = x_0 \quad \beta_i(1) = x_i$

$\gamma_i := \gamma|_{[x_i, x_{i+1}]}$

$$\gamma \sim (\gamma_0 \beta_1^{-1})(\beta_1 \gamma_1 \beta_2^{-1}) \cdots (\beta_n \gamma_{n-1})$$

$$\quad \cap \qquad\quad \cap \qquad\quad \cdots$$

$$\quad \mathcal{U} \qquad\quad V \qquad\quad \cdots$$



since U, V are simply conn.

$\gamma_0 \beta_1^{-1}$, $\beta_1 \gamma_1 \beta_2^{-1}$ ... are

homotopic to e_{x_0}

$$\Rightarrow [\gamma] = [e_{x_0}]$$

Pf. of Lebesgue lemma

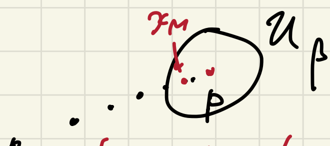
prove by contradiction

$X = \bigcup \mathcal{U}_\alpha$ $\forall \frac{1}{n}$ $\exists x_n$ s.t. $B(x_n, \frac{1}{n}) \not\subset \mathcal{U}_\alpha$ $\forall \alpha$

Let p be a limit pt of $\{x_1, x_2, \dots\}$

say $p \in \mathcal{U}_\beta$ choose $\varepsilon > 0$

s.t. $B(p, \varepsilon) \subset \mathcal{U}_\beta$



$\exists N$ s.t. $n > N \Rightarrow x_n \in B(p, \varepsilon)$

choose $M \gg N$ s.t. $|x_M - p| < \varepsilon - \frac{1}{M}$

$|y - x_M| < \frac{1}{M}$

$|y - p| < \varepsilon$ $\Rightarrow \in$

$\Rightarrow B(x_M, \frac{1}{M}) \subset \mathcal{U}_\beta$

Pf. of thm

$n > 1$ $\quad S^n = U \cup V$

$$U \cong \mathbb{R}^n \quad V \cong \mathbb{R}^n \quad U \cap V \cong \mathbb{R}^n \setminus 0$$

by stereographic projection. lemma applies.

π_1 of orbit space

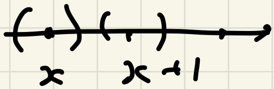
$\sigma : G \times X \to X$ action of top. group is called

discontinuous if $\forall\, x \in X$ $\exists$ nbhd $\mathcal{U}$ of x

$\forall\, g \in G$ s.t. $\mathcal{U} \cap g\mathcal{U} = \emptyset$

Examples 1) $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is discontinuous

$(n, x) \mapsto x + n$ $\qquad \varepsilon < \frac{1}{2}$



2) $\mu_2 = \{\pm 1\} \times S^2 \to S^n$

$v \longmapsto -v$

is discontinuous

3) $G = \langle a, b \mid aba = b \rangle$ is discont.

$G \times \mathbb{R}^2 \to \mathbb{R}^2$

$a \cdot (x, y) = (x+1, y)$ $\qquad b \cdot (x, y) = (-x, y+1)$

Thm Let $G \times X \to X$ be a discontinuous action

then $\pi_1(X/G) = G$.

pf of this thm & the unproved lemma in the calculation of $\pi_1(S^1)$ use the same technique of covering space, will be provided later.

Applications

1) $\pi_1(T^2 = \mathbb{R}^2/\mathbb{Z}^2) \cong \mathbb{Z}^2$

2) $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2 \quad n \geq 2$

3) $\pi_1(\text{Klein bottle}) = G = \langle a, b \mid aba = b \rangle$