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20250324 MATH4302 Algebra II

1. Solution: We may divide our solution into four steps.

Step1: Factorize 72.

$$72 = \underbrace{2^3}_{2\text{-part}} \cdot \underbrace{3^2}_{3\text{-part}}$$

Step2: As G is a finite module over the infinite principal ideal domain $\mathbb{Z}$, G is isomorphic to a finite product of finite cyclic groups with elementary divisors as their orders.

Step3: All nontrivial elementary divisors that divide 2^3 are $2, 4, 8$.

Hence, all possible choices of the 2-part of G are:

$G_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_8$

All nontrivial elementary divisors that divide 3^2 are $3, 9$.

Hence, all possible choices of the 3-part of G are:

$G_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9$.

Step4: All structures of G up to isomorphism are:

$$G \cong \underbrace{(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}_{2\text{-part}} \times \underbrace{(\mathbb{Z}_3 \times \mathbb{Z}_3)}_{3\text{-part}} \text{ or } \underbrace{(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)}_{2\text{-part}} \times \underbrace{\mathbb{Z}_9}_{3\text{-part}}$$

$$\underbrace{(\mathbb{Z}_2 \times \mathbb{Z}_4)}_{2\text{-part}} \times \underbrace{(\mathbb{Z}_3 \times \mathbb{Z}_3)}_{3\text{-part}} \text{ or } \underbrace{(\mathbb{Z}_2 \times \mathbb{Z}_4)}_{2\text{-part}} \times \underbrace{\mathbb{Z}_9}_{3\text{-part}}$$

$$\underbrace{\mathbb{Z}_8}_{2\text{-part}} \times \underbrace{(\mathbb{Z}_3 \times \mathbb{Z}_3)}_{3\text{-part}} \text{ or } \underbrace{\mathbb{Z}_8}_{2\text{-part}} \times \underbrace{\mathbb{Z}_9}_{3\text{-part}}.$$

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(2) Solution: To find the invariant factor form of this group G, define a map from $\mathbb{Z}^6$ to $\mathbb{Z}^6$, such that $\mathbb{Z}^6/\mathrm{Im}(6) \cong G$:

$$6 = \begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 4 & & & \\ & & & 3 & & \\ & & & & 9 & \\ & & & & & 9 \end{pmatrix} \quad \mathbb{Z}^6/\mathrm{Im}(6) \cong G.$$

Now $\binom{6}{0} = 1$, $m_0 = 1$

$d_1 = \frac{m_1}{m_0} = 1$ $\quad \binom{6}{1} = 6$, $m_1 = \gcd(2,2,4,3,9,9) = 1$

$d_2 = \frac{m_2}{m_1} = 1$ $\quad \binom{6}{2} = 15$, $m_2 = \gcd(4,8,6,18,18,\ 8,6,18,18,\ 12,36,36,\ 27,27,\ 81) = 1$

$d_3 = \frac{m_3}{m_2} = 1$ $\quad \binom{6}{3} = 20$, $m_3 = \gcd(16,12,36,36,24,72,72,54,54,162,\ 24,72,72,54,54,162,\ 108,108,324,\ 243) = 1$

$d_4 = \frac{m_4}{m_3} = 6$ $\quad \binom{6}{4} = 15$, $m_4 = \gcd(48,144,144,108,108,324,\ 216,216,648,486,216,216,\ 648,486,972) = 6$

$d_5 = \frac{m_5}{m_4} = 18$ $\quad \binom{6}{5} = 6$, $m_5 = \gcd(432,432,1296,972,1944,1944) = 108$

$d_6 = \frac{m_6}{m_5} = 36$ $\quad \binom{6}{6} = 1$, $m_6 = 3888$

$$G \cong \mathbb{Z}^6/\mathrm{Im}(6) \cong \mathbb{Z}/\mathbb{Z} \times \mathbb{Z}/\mathbb{Z} \times \mathbb{Z}/\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}.$$

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To find the elementary divisor form, we don't need to do anything at all:

$$G \cong \underbrace{\left(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}\right)}_{2\text{-part}} \times \underbrace{\left(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}\right)}_{3\text{-part}}$$

(3) Solution: Assume that $\bar{x}$ is the equivalence class of x in $\mathbb{R}[x]/\langle x(x-2)^3\rangle$.

Define $f_{0,0}(x) = \left(\frac{x-2}{0-2}\right)^3 (x-0)^0$; $\therefore$ $\bar{x} f_{0,0}(\bar{x}) = 0 f_{0,0}(\bar{x})$

$f_{2,0}(x) = \left(\frac{x-0}{2-0}\right)^1 (x-2)^0$; $\therefore$ $\bar{x} f_{2,0}(\bar{x}) = 2 f_{2,0}(\bar{x}) + f_{2,1}(\bar{x})$

$f_{2,1}(x) = \left(\frac{x-0}{2-0}\right)^1 (x-2)^1$ $\qquad \bar{x} f_{2,1}(\bar{x}) = 2 f_{2,1}(\bar{x}) + f_{2,2}(\bar{x})$

$f_{2,2}(x) = \left(\frac{x-0}{2-0}\right)^1 (x-2)^2$ $\qquad \bar{x} f_{2,2}(\bar{x}) = 2 f_{2,2}(\bar{x})$

On one hand, $\dim\left[\mathbb{R}[x]/\langle x(x-2)^3\rangle\right] = 4$.

On the other hand, for all linear combination $c_{0,0} f_{0,0}(x) + c_{2,0} f_{2,0}(x) + c_{2,1} f_{2,1}(x) + c_{2,2} f_{2,2}(x)$ of $f_{0,0}(x), f_{2,0}(x), f_{2,1}(x), f_{2,2}(x)$:

The evaluation of $\frac{\text{it}}{(x-0)}$ at $x=0$ is $0 \Rightarrow c_{0,0} = 0$

The evaluation of $\frac{\text{it}}{(x-2)}$ at $x=2$ is $0 \Rightarrow c_{2,0} = 0$

The evaluation of $\frac{\text{it}}{(x-2)}$ at $x=2$ is $0 \Rightarrow c_{2,1} = 0$

The evaluation of $\frac{\text{it}}{(x-2)^2}$ at $x=2$ is $0 \Rightarrow c_{2,2} = 0$

$\Rightarrow$ $f_{0,0}(x), f_{2,0}(x), f_{2,1}(x), f_{2,2}(x)$ are linearly independent

Hence, we've found a basis $f_{0,0}(x), f_{2,0}(x), f_{2,1}(x), f_{2,2}(x)$, such that the linear operator T is in its Jordan canonical form

$$\begin{array}{cl} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} & \begin{array}{l} \leftarrow c_{0,0} \\ \leftarrow c_{2,0} \\ \leftarrow c_{2,1} \\ \leftarrow c_{2,2} \end{array} \end{array}.$$

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(4) Solution: $A=\begin{pmatrix}4&0&0\\0&2&0\\0&0&6\end{pmatrix}$.

$$\left.\begin{aligned}
A\vec{e_1}&=\begin{pmatrix}4&0&0\\0&2&0\\0&0&6\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}4\\0\\0\end{pmatrix}=4\vec{e_1}\\
A\vec{e_2}&=\begin{pmatrix}4&0&0\\0&2&0\\0&0&6\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}=2\vec{e_2}\\
A\vec{e_3}&=\begin{pmatrix}4&0&0\\0&2&0\\0&0&6\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\6\end{pmatrix}=6\vec{e_3}
\end{aligned}\right\}$$

$\Rightarrow$ A is already in its rational canonical form and Jordan canonical form

$$B=\begin{pmatrix}3&3&0\\2&1&4\\3&4&2\end{pmatrix}$$

To find the rational canonical form and Jordan canonical form of B, define a map from $K[x]^3$ to $K[x]^3$, such that $K[x]^3/Im_{K[x]}(6)\cong K[B]^3$

$$6=\begin{pmatrix}3-x&3&0\\2&1-x&4\\3&4&2-x\end{pmatrix},\ K[x]^3/Im_{K[x]}(6)\cong K[B]^3$$

$$2\cdot(3-x)+x\cdot2=\underline{6},\ \overset{\text{unit}}{}\begin{pmatrix}2&x&0\\-2&3-x&0\\0&0&1\end{pmatrix}\begin{pmatrix}3-x&3&0\\2&1-x&4\\3&4&2-x\end{pmatrix}$$

$$=\begin{pmatrix}6&6+x-x^2&4x\\0&-3-4x+x^2&12-4x\\3&4&2-x\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}6&6+x-x^2&4x\\0&-3-4x+x^2&12-4x\\3&4&2-x\end{pmatrix}=\begin{pmatrix}3&4&2-x\\0&-3-4x+x^2&12-4x\\6&6+x-x^2&4x\end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 & 2-x \\ 0 & -3-4x+x^2 & 12-4x \\ 6 & 6+x-x^2 & 4x \end{pmatrix} = \begin{pmatrix} 3 & 4 & 2-x \\ 0 & -3-4x+x^2 & 12-4x \\ 0 & -2+x-x^2 & -4+6x \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 & 2-x \\ 0 & -3-4x+x^2 & 12-4x \\ 0 & -2+x-x^2 & -4+6x \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{-4}{3} & \frac{-2+x}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3-4x+x^2 & 12-4x \\ 0 & -2+x-x^2 & -4+6x \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2-3x & 6-2x \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3-4x+x^2 & 12-4x \\ 0 & -2+x-x^2 & -4+6x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -13-10x+2x^2 & 28 \\ 0 & -18+11x+6x^2-x^3 & 0 \end{pmatrix}$$

$$3\cdot(12-4x)+2\cdot(-4+6x) = \overset{\text{unit}}{\underline{28}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -13-10x+2x^2 & 28 \\ 0 & -18+11x+6x^2-x^3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{28} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -13-10x+2x^2 \\ 0 & 0 & -18+11x+6x^2-x^3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -13-10x+2x^2 \\ 0 & 0 & -18+11x+6x^2-x^3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 13+10x-2x^2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -18+11x+6x^2-x^3 \end{pmatrix}$$

$m_0(x)=1,\ m_1(x)=1,\ m_2(x)=1,\ m_3(x)=-18+11x+6x^2-x^3$

$d_1(x)=1,\ d_2(x)=1,\ d_3(x)=-18+11x+6x^2-x^3$

$$K[B]^3 \cong K[x]^3/\mathrm{Im}_{K[x]}(6) \cong K[x]/K[x] \times K[x]/K[x]$$

$$\times\, K[x]/(-18+11x+6x^2-x^3)K[x]$$

The rational canonical form of B is $\begin{pmatrix} 0 & 0 & -18 \\ 1 & 0 & 11 \\ 0 & 1 & 6 \end{pmatrix}$

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As $(-18+11x+6x^2-x^3,\ 11+12x-3x^2)$

$=(-54+33x+18x^2-3x^3,\ 11+12x-3x^2)$

$=(-54+33x+18x^2-3x^3-11x+12x^2+3x^3,\ 11+12x-3x^2)$

$=(-54+22x+30x^2,\ 11+12x-3x^2)$

$=(-54+22x+30x^2+110+120x-30x^2,\ 11+12x-3x^2)$

root is rational $\quad \Delta=12$ is not a perfect square, roots are irrational

$=(\underline{56+142x},\ \underline{11+12x-3x^2})=1$,

the roots of $-18+11x+6x^2-x^3$ are distinct.

Hence, B is diagonalizable with Jordan canonical form $\begin{pmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{pmatrix}$.

Here, $(\alpha-x)(\beta-x)(\gamma-x)=-18+11x+6x^2-x^3$.

(5) Solution:

Let L/K be a field extension. If for some $\theta\in L$, $L=K(\theta)$

$=\left\{\dfrac{\sum_{k=0}^{m}a_k\theta^k}{\sum_{l=0}^{n}b_l\theta^l}:\sum_{l=0}^{n}b_l\theta^l\neq 0;\ (a_k)_{k=0}^{m},(b_l)_{l=0}^{n}\text{ are in }K\right\}$

$\forall n\geq 0$, $\{x^0,\dots,x^n\}$ is linearly independent.

then L/K is simple. For some simple field extension $K(x)/K$, it is infinite.

(6) Solution:

$\forall$ prime p, $e^{\frac{0\pi i}{p}}, e^{\frac{2\pi i}{p}}, \dots, e^{\frac{2(p-1)\pi i}{p}}$ are linearly independent

Let L/K be a field extension. If for all $\theta\in L$,

for some nonzero polynomial $f(x)\in K[x]$, $f(\theta)=0$, then L/K is algebraic.

For some algebraic field extension $\mathbb{Q}(\text{all } e^{2\pi i r}, \text{where } r\in\mathbb{Q})/\mathbb{Q}$, it is infinite.

(7) Proof$_{(a)}$: We may divide our proof into two parts.

"if" direction: If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a constant polynomial,

then $f'(x) = \overset{0 \text{ in } K}{\overset{\|}{a_1}} + \cdots + n\overset{0 \text{ in } K}{\overset{\|}{a_n}}x^{n-1} \overset{\text{in } K[x]}{=} 0$.

"only if" direction: If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is not a constant polynomial,

then $n > 0$ and $a_n \neq 0$. As $\mathrm{Char}(K) = 0$, $\underset{}{n} \overset{\text{in } K}{\neq} 0$, so:

$$f'(x) = a_1 + \cdots + \overbrace{\overset{\text{in } \mathbb{Z}}{n}a_n}^{0 \text{ in } K \atop \nparallel} x^{n-1} \overset{\text{in } K[x]}{\neq} 0$$

Proof$_{(b)}$: We may divide our proof into two parts.

"if" direction: If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_px^p + \cdots$
$+ a_{kp+1}x^{kp+1} + a_{kp+2}x^{kp+2} + \cdots + a_{kp+p}x^{kp+p} + \cdots$

factors through $x \mapsto x^p$,

then $f'(x) = \overset{0 \text{ in } K}{\overset{\|}{a_1}} + 2\overset{0 \text{ in } K}{\overset{\|}{a_2}}x + \cdots + \overbrace{pa_p}^{0 \text{ in } K \atop \|}x^{p-1} + \cdots$
$+ (kp+1)\overset{0 \text{ in } K}{\overset{\|}{a_{kp+1}}}x^{kp} + (kp+2)\overset{0 \text{ in } K}{\overset{\|}{a_{kp+2}}}x^{kp+1} + \cdots + \overbrace{(kp+p)a_{kp+p}}^{0 \text{ in } K \atop \|}x^{kp+p-1} + \cdots$
$= 0$ in $K[x]$

"only if" direction: If $f(x) = a_0 + \cdots + a_rx^r + \cdots + a_nx^n$
doesn't factor through $x \mapsto x^p$, then for some $r \in (\mathbb{Z}/p\mathbb{Z})^\times$, $a_r \neq 0$, so:

$$f'(x) = \cdots + \overbrace{ra_r}^{0 \text{ in } K \atop \nparallel}x^{r-1} + \cdots + na_nx^{n-1} \overset{\text{in } K[x].}{\neq} 0$$

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(8) Proof: Assume to the contrary that there exists a field isomorphism $\sigma: \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{3}]$,

~~let's prove that the image $s = \sigma(\sqrt{2})$ is not well-defined.~~

$$s^2 = \sigma(\sqrt{2})^2 = \sigma(\sqrt{2}^2) = \sigma(2) = 2$$

$(a+b\sqrt{3})^2 = 2$, $\sqrt{3} \mapsto -\sqrt{3}$: $(a-b\sqrt{3})^2 = 2$

$$ab = \frac{(a+b\sqrt{3})^2 - (a-b\sqrt{3})^2}{2\sqrt{3}} = 0, \quad a = 0 \text{ or } b = 0.$$

If $a = 0$, then $b^2 = \frac{2}{3}$, $b = \frac{\sqrt{6}}{3} \notin \mathbb{Q}$.

If $b = 0$, then $a^2 = 2$, $a = \sqrt{2} \notin \mathbb{Q}$.

In both cases, $s = \sigma(\sqrt{2})$ is not well-defined, so such σ fails to exist.

(9) Proof: We may divide our proof into three steps.

Step 1: For all $a_0 \in K$, identify the followings:

(1) The element a_0 of K

(2) The constant polynomial a_0 in $K[x]$

(3) The evaluation of a_0 at $a \in L$

This means $a_0 \in K(a)$, $K \subseteq K(a)$

The degree of field extension is the degree of the polynomial $f(x)$

Step 2: For all $s_0, t_0 \in L$:

$s_0, t_0 \in K(a) \Rightarrow$ For some $s(x), t(x) \in K[x]$, $s_0 = s(a)$, $t_0 = t(a)$

$\Rightarrow$ For some $s(x)+t(x)$, $s(x)t(x) \in K[x]$,

$s_0 + t_0 = s(a) + t(a)$, $s_0 t_0 = s(a)t(a) \Rightarrow s_0 + t_0, s_0 t_0 \in K(a)$

Step 3: Assume that a has minimal polynomial $f(x) \in K[x]$,

for all $s_0 \in L \setminus \{0\}$, for some $s(x) \in [K[x]/f(x)] \setminus \{0\}$, $s_0 = s(a)$.

As $f(x) \in K[x]$ is minimal, it is irreducible, so $K[x]/f(x)$ is a field,

$[K[x]/f(x)] \setminus \{0\}$ is equal to $[K[x]/f(x)]^\times$, $s(x)^{-1} \equiv t(x) \pmod{f(x)}$.

This implies $s(a)^{-1} = t(a) \in K(a)$, $s_0^{-1} = t_0 \in K(a)$.

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(10)

(1) We find a polynomial in $\mathbb{Q}[x]$, such that it has root $\sqrt{1+\sqrt{3}}$:

$(x-\sqrt{1+\sqrt{3}})(x+\sqrt{1+\sqrt{3}}) = x^2-1-\sqrt{3}$

$(x-\sqrt{1-\sqrt{3}})(x+\sqrt{1-\sqrt{3}}) = x^2-1+\sqrt{3}$

$(x^2-1-\sqrt{3})(x^2-1+\sqrt{3}) = (x^2-1)^2-3 = x^4-2x^2-2$

For some prime number $p=2$,

$p \nmid a_4=1, p \mid a_3=0, p \mid a_2=-2, p \mid a_1=0, p \mid a_0=-2, p^2 \nmid a_0=-2$

Hence, x^4-2x^2-2 is irreducible in $\mathbb{Z}[x]$, thus irreducible in $\mathbb{Q}[x]$.

x^4-2x^2-2 is the minimal polynomial of $\sqrt{1+\sqrt{3}}$ up to associates.

$[\mathbb{Q}(\sqrt{1+\sqrt{3}}):\mathbb{Q}] = \deg(x^4-2x^2-2) = 4$.

(2) We find a polynomial in $\mathbb{Q}[x]$, such that it has root $\sqrt{3-\sqrt{6}}$:

$(x-\sqrt{3-\sqrt{6}})(x+\sqrt{3-\sqrt{6}}) = x^2-3+\sqrt{6}$

$(x-\sqrt{3+\sqrt{6}})(x+\sqrt{3+\sqrt{6}}) = x^2-3-\sqrt{6}$.

$(x^2-3+\sqrt{6})(x^2-3-\sqrt{6}) = (x^2-3)^2-6 = x^4-6x^2+3$

For some prime number $p=3$,

$p \nmid a_4=1, p \mid a_3=0, p \mid a_2=-6, p \mid a_1=0, p \mid a_0=3, p^2 \nmid a_0=3$.

Hence, x^4-6x^2+3 is irreducible in $\mathbb{Z}[x]$, thus irreducible in $\mathbb{Q}[x]$.

x^4-6x^2+3 is the minimal polynomial of $\sqrt{3-\sqrt{6}}$ up to associates.

$[\mathbb{Q}(\sqrt{3-\sqrt{6}}):\mathbb{Q}] = \deg(x^4-6x^2+3) = 4$.

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(3) We find a polynomial in $\mathbb{Q}[x]$, such that it has root $\sqrt{3+2\sqrt{2}}=\sqrt{2}+1$.

$(x-\sqrt{2}-1)(x+\sqrt{2}-1)=(x-1)^2-2=x^2-2x-1$

As $\Delta=(-2)^2-4\cdot1\cdot(-1)=8$ is not a perfect square,

x^2-2x-1 is irreducible in $\mathbb{Q}[x]$, x^2-2x-1 is the minimal polynomial of $\sqrt{2}+1$

up to associates. $[\mathbb{Q}(\sqrt{2}+1):\mathbb{Q}]=\deg(x^2-2x-1)=2$.