Algebra II: Tutorial 8

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Problem 1. Let K be field, and $a_1, a_2, \cdots, a_n$ be algebraic over K. Let $L = K(a_1, a_2, \cdots, a_n)$, and suppose that $K \subset M \subset L$.

1. Show that $M(a_1, a_2, \cdots, a_n) = L$. Deduce that if L is a splitting field of f over K, then L is a splitting field of f over M.

2. Give an example of polynomial f over K and a field extension M of K such that the splitting fields of f over K and M are not isomorphic.

Solution. 1. By definition, L is the smallest field containing K and $a_1, a_2, \cdots, a_n$. Since $K \subset M$, $L \subset M$. If $\alpha \in M$, then $\alpha \in L(a_1, a_2, \cdots, a_n) = L$ by definition. Therefore, $M = L$. Since $f \in K[x]$, f is a polynomial over M. To show that L is the splitting field of f over M, we need to show that f splits in L, and that $L = M(a_1, a_2, \cdots, a_n)$. This is now obvious.
2. The polynomial $x^2 + 1$ has splitting field $\mathbb{Q}(i)$ over $\mathbb{Q}$ but $\mathbb{C}$ over $\mathbb{R}$. ■

Problem 2. Give an example of a normal algebraic extension of $\mathbb{Q}$ which is not finite.

Solution. The algebraic extension $\overline{\mathbb{Q}}$ is normal over $\mathbb{Q}$, but not finite. This is because $\overline{\mathbb{Q}}$ is the relative algebraic closure of $\mathbb{Q}$ in the algebraically closed field $\mathbb{C}$, and therefore algebraically closed itself. Then, if a field extension is algebraically closed, it is normal. ■

Problem 3. Show that an algebraic extension $K \subset L$ is normal if and only if for every $\alpha \in L$, the minimal polynomial of α over K splits completely over L.

Solution. Suppose that L is normal over K. Take $\alpha \in L$, and consider the minimal polynomial $m_\alpha(x)$ for α over K. By definition, m_α is irreducible over K, and $m_\alpha(\alpha) = 0$; by normality this implies that m_α splits completely over L. Conversely, let $f \in K[x]$ be an irreducible polynomial over K, and suppose that f has a root α in L. Then, up to rescaling by a non-zero unit, f is the minimal polynomial of α over K, and splits completely, by assumption. ■

Problem 4. Show that any quadratic extension is normal.

Solution. By definition, a quadratic extension $K \subset L$ is a finite extension of degree $[L : K] = 2$. By the tower theorem, L is a simple extension of K; without loss of generality say $L = K(\alpha)$ for some α a root of an irreducible polynomial $p(x) = x^2 + a_1 x + a_2$ of degree 2. By assumption, $p(\alpha) = 0$, and so $p(x) = (x - \alpha)(x - \beta)$ for some $\beta \in L$. Thus f splits completely in L, and $L = K(\alpha, \beta)$, so L is a splitting field of f over K; in particular L is normal over K. ■

Problem 5. Let $K \subset L \subset M$ with $K \subset L$ normal and $L \subset M$ normal. Does this imply that $K \subset M$ is normal?

Solution. No: take $K = \mathbb{Q}, L = \mathbb{Q}(\sqrt{2})$ and $M = \mathbb{Q}(\sqrt[4]{2})$. The extensions $K \subset L$ and $L \subset M$ are quadratic (and hence normal), but the $\mathbb{Q}$-irreducible polynomial $f(x) = x^4 - 2$ has roots $\pm\sqrt[4]{2}$ in M without splitting completely in M - $f(x)$ has two complex roots yet $M \subset \mathbb{R}$. Notice that this does not contradict the normality of M over L, since $f(x)$ is irreducible over K but reducible over L: $f(x) = (x^2 - \sqrt{2})(x^2 + \sqrt{2})$ in $L[x]$. ■