ASSIGNMENT I, ALGEBRA II, HKU, SPRING 2025

DUE AT 11:59PM ON MONDAY FEBRUARY 7, 2025

(1) Prove the following statements:

1) If R is an integral domain, then $\text{char}(R) = 0$ or is a prime number;

2) A non-zero commutative ring with finitely many elements is a field if and only if it is an integral domain.

3) If R is an integral domain and $a, b \in R\backslash\{0\}$, then $aR = bR$ iff a and b are associates.

(2) Show that a non-zero sub-ring of an integral domain is again an integral domain, but a non-zero quotient ring of an integral domain is not necessarily an integral domain.

(3) Show that $(\mathbb{Z}[\sqrt{-1}], v)$ is an Euclidean domain, where

$$\mathbb{Z}[\sqrt{-1}] = \{m + n\sqrt{-1} : \ m, n \in \mathbb{Z}\}$$

is the ring of Gauss integers and $v : \mathbb{Z}[\sqrt{-1}]\backslash\{0\} \longrightarrow \mathbb{N}$ is given by

$$v(m + n\sqrt{-1}) = m^2 + n^2, \quad m, n \in \mathbb{Z}, \ m + n\sqrt{-1} \neq 0.$$

(4) Describe all the irreducible elements in $\mathbb{R}[x]$; Classify all prime ideals and all maximal ideals of $\mathbb{R}[x]$.

(5) Suppose that R is a PID but is not a field. Show that $R[x]$ is not a PID.

(6) Find all irreducible quadratic and cubic polynomials in $F_3[x]$, where F_3 is the field with three elements

(7) Show that if R is a UFD, then the intersection of two principal ideals of R is again principal.

(8) Prove Kaplansky's criterion on UFDs: An integral domain R is a UFD if and only if every non-zero prime ideal in R contains a prime element,

(9) Show that in an integral domain R, gcds for a set of non-zero elements B, if exist, are unique up to associates.

(10) Compute a greatest common divisor in $\mathbb{Z}[x]$ of $f(x) = 2x^2 + 2$ and $g(x) = x^6 - 1$.