THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 5 Solution

Problem 1.

(i) Direct computation yields that

$$\begin{aligned} \partial_t e &= \partial_t u \cdot \partial_{tt} u + 4\partial_x u \cdot \partial_{tx} u + 2u\partial_t u \\ &= \partial_t u(\partial_{tt} u + 2u) + 4\partial_x u \cdot \partial_{tx} u \\ &= 4[\partial_t u \cdot \partial_{xx} u + \partial_x u \cdot \partial_{tx} u] \quad \text{by (1)} \\ &= 4\partial_x(\partial_x u \partial_t u) = 4\partial_x p. \end{aligned}$$

(ii) Direct computation yields that

$$\begin{aligned} \frac{1}{2}(\partial_t u \pm 2\partial_x u)^2 + u^2 &= \frac{1}{2}|\partial_t u|^2 + 2|\partial_x u|^2 + |u|^2 \pm 2\partial_t u \partial_x u \\ &= e \pm 2p. \end{aligned}$$

(iii) Recall the Leibniz integral rule: let $f(x,t)$ be a continuous function depending on x then

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}\,t\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \partial_x f(x,t)\,\mathrm{d}\,t.$$

For $t \geq 0$, we then have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,t}E(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{a+4t}^{b-4t} e(t,x)\,\mathrm{d}\,x \\ &= \int_{a+4t}^{b-4t} \partial_t e(t,x)\,\mathrm{d}\,x - 4e(t,b-4t) - 4e(t,a+4t) \\ &= 4\left[\int_{a+4t}^{b-4t} \partial_x p(t,x)\,dx - e(t,b-4t) - e(t,a+4t))\right] \quad \text{by (i)} \\ &= 4[p(t,b-4t) - p(t,a+4t) - e(t,b-4t) - e(t,a+4t)]. \end{aligned}$$

(iv) For $t \geq 0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}E(t) &= 4[p(t,b-4t) - p(t,a+4t) - e(t,b-4t) - e(t,a+4t)] \quad \text{by (iii)} \\
&= -2[e(t,b-4t) - 2p(t,b-4t)] - 2[e(t,a+4t) + 2p(t,a+4t)] \\
&\qquad - 2e(t,b-4t) - 2e(t,a+4t) \\
&\leq -[\partial_t u(t,b-4t) - 2\partial_x u(t,b-4t)]^2 - 2u^2(t,b-4t) \\
&\qquad - [\partial_t u(t,a+4t) + 2\partial_x u(t,a+4t)]^2 - 2u^2(t,a+4t) \quad \text{by (ii) and } e(t,x) \geq 0 \\
&\leq 0.
\end{aligned}$$

If $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), then $\partial_x u(0,x) \equiv 0$ on (a,b). Thus,

$$E(0) = \int_a^b \frac{|\partial_t u(0,x)|^2}{2} + 2|\partial_x u(0,x)|^2 + |u(0,x)|^2 dx = 0.$$

On the other hand, for $0 \leq t \leq (b-a)/8$,

$$0 \leq E(t) \leq E(0) = 0 \Rightarrow E(t) = 0.$$

So

$$\int_{a+4t}^{b-4t} |u(t,x)|^2 dx \leq E(t) = 0$$

and hence $u(t,x) = 0$ for $a + 4t \leq x \leq b - 4t$.

Problem 2.

(i) Let u_1 and u_2 be two solutions of the problem. Define $\tilde{u} := u_1 - u_2$. Then we have

$$\begin{cases} \partial_t \tilde{u} - 4\partial_{xx}\tilde{u} = -4\tilde{u} & \text{for } 0 < x < L,\ t > 0 \\ \tilde{u}|_{t=0} = \tilde{u}|_{x=0} = \partial_x \tilde{u}|_{x=L} = 0 \end{cases}$$

Multiplying by $\tilde{u}$, and then integrating with respect to x over $(0,L)$, we have

$$\int_0^L \tilde{u}\partial_t\tilde{u}\,\mathrm{d}\,x = 4\int_0^L \tilde{u}\partial_{xx}\tilde{u}\,\mathrm{d}\,x - 4\int_0^L |\tilde{u}|^2\,\mathrm{d}\,x$$

Then it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\int_0^L |\tilde{u}|^2\,\mathrm{d}\,x &= 4\big[\tilde{u}\partial_x\tilde{u}\big]_{x=0}^L - 4\int_0^L |\partial_x\tilde{u}|^2\,\mathrm{d}\,x - 4\int_0^L |\tilde{u}|^2\,\mathrm{d}\,x \\ &= -4\int_0^L |\partial_x\tilde{u}|^2\,\mathrm{d}\,x - 4\int_0^L |\tilde{u}|^2\,\mathrm{d}\,x \quad \text{by (2)} \\ &\le 0.\end{aligned}$$

Thus,

$$\int_0^L |\tilde{u}(t,x)|^2 dx \le \int_0^L |\tilde{u}(0,x)|^2 dx = 0 \quad \text{by (2)}$$

and hence $\tilde{u} = 0$.

(ii) Let u_1 and u_2 be two solutions of the problem. Define $\tilde{u} := u_1 - u_2$. Then we have

$$\begin{cases}\partial_{tt}\tilde{u} - 4\partial_{xx}\tilde{u} = -\tilde{u} - \partial_t\tilde{u} & \text{for } 0 < x < L,\ t > 0\\ \tilde{u}|_{t=0} = \partial_t\tilde{u}|_{t=0} = \partial_x\tilde{u}|_{x=0} = \tilde{u}|_{x=L} = 0.\end{cases}$$

Note that

$$\tilde{u}(0,x) = 0 \Rightarrow \partial_x\tilde{u}(0,x) \equiv 0$$

and

$$\tilde{u}(t,L) = 0 \Rightarrow \partial_t\tilde{u}(t,L) \equiv 0.$$

Multiplying the equation by $\partial_t\tilde{u}$, and then integrating with respect to x over $(0,L)$, we have

$$\int_0^L \partial_{tt}\tilde{u}\cdot\partial_t\tilde{u} + \tilde{u}\partial_t\tilde{u}\ \mathrm{d}\,x = 4\int_0^L \partial_t\tilde{u}\partial_{xx}\tilde{u}\,\mathrm{d}\,x - \int_0^L |\partial_t\tilde{u}|^2\ \mathrm{d}\,x.$$

$$\Rightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\int_0^L |\partial_t\tilde{u}|^2 + |\tilde{u}|^2\ \mathrm{d}\,x = 4[\partial_t\tilde{u}\partial_x\tilde{u}]_{x=0}^L - 4\int_0^L \partial_{tx}\tilde{u}\cdot\partial_x\tilde{u}\,\mathrm{d}\,x - \int_0^L |\partial_t\tilde{u}|^2\,\mathrm{d}\,x.$$

$$\Rightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\int_0^L |\partial_t\tilde{u}|^2 + 4|\partial_x\tilde{u}|^2 + |\tilde{u}|^2\,\mathrm{d}\,x = -\int_0^L |\partial_t\tilde{u}|^2\,\mathrm{d}\,x \le 0.$$

Thus,

$$\begin{aligned}\int_0^L |\tilde{u}(t,x)|^2 dx &\le \int_0^L |\partial_t\tilde{u}(t,x)|^2 + 4|\partial_x\tilde{u}(t,x)|^2 + |\tilde{u}(t,x)|^2\,\mathrm{d}\,x \\ &\le \int_0^L |\partial_t\tilde{u}(0,x)|^2 + 4|\partial_x\tilde{u}(0,x)|^2 + |\tilde{u}(0,x)|^2\,\mathrm{d}\,x = 0.\end{aligned}$$

And hence $\tilde{u} = 0$.

Problem 3. Direct computation yields that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}M(t) = \int_0^1 \partial_t u \,\mathrm{d}\,x = \int_0^1 \partial_{xx}u + 2tx \,\mathrm{d}\,x = \partial_x u(t,1) - \partial_x u(t,0) + tx^2\Big|_{x=0}^1 = 2 + t.$$

And hence

$$\begin{aligned} M(t) &= \int_0^t 2 + t \,\mathrm{d}\,t + M(0) \\ &= 2t + \frac{t^2}{2} + \int_0^1 u(0,x)\,\mathrm{d}\,x \\ &= 2t + \frac{t^2}{2} + \int_0^1 3x^2 + 1 \,\mathrm{d}\,x \\ &= 2t + \frac{t^2}{2} + (x^3 + x)\Big|_{x=0}^1 \\ &= \frac{t^2}{2} + 2t + 2. \end{aligned}$$

Problem 4. By D' Alembert's formula, we have

$$u(x,t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)).$$

(i) Correct. For $-\infty < x < \infty$ and $t \geq 0$,

$$|u(x,t)| \leq \frac{|\phi(x+t)| + |\phi(x-t)|}{2} \leq \max_{x\in(-\infty,\infty)} |\phi(x)|.$$

So

$$\max_{\substack{-\infty<x<\infty \\ t\geq 0}} |u(t,x)| \leq \max_{x\in(-\infty,\infty)} |\phi(x)|.$$

(ii) Incorrect. Take $\phi(x) = x^2$, then $u(x,t) = x^2 + t^2$. Thus

$$E(t) = 2\int_0^1 t^2 + x^2 dx = 2t^2 + \frac{2}{3}.$$