THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 10

Date: Nov 18, 2024.

Instruction: *In order to have a better preparation of the coming test, you are recommended to try the problems below **BEFORE** the coming tutorial. Due to the time limit, the tutorial will NOT be able to go through all of the problems. Therefore, you should actively tell our TA which problems you wish to discuss first.*

Information: This collection of problems is intended to give you practice problems that are comparable in format and difficulty to those which will appear in the coming test. The questions in the actual exam will be **DIFFERENT**.

Problem 1. Do Problem 4 of Dec 2021 Final Exam.

Solution. (a) The proof can be divided into two steps, in which the first step will show the maximum principle, and the second step will apply the maximum principle (that we will prove in the first step) to obtain the comparison principle.

Step 1. (Maximum Principle). Let $w \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\begin{cases} -\sum_{j=1}^{d} j\, \partial_{x_j x_j} w < 0 & \text{in } \Omega \\ w|_{\partial\Omega} = g, \end{cases} \tag{1}$$

where g is a given boundary condition. Since w is a continuous function defined on a compact set $\bar{\Omega}$, it follows from the extreme value theorem that there exists a point $(x_0, y_0) \in \bar{\Omega}$ such that

$$w(x_0, y_0) = \max_{\bar{\Omega}} w.$$

Seeking for a contradiction, we assume that $(x_0, y_0) \in \Omega$. By the second-order tests for local/interior maximum in elementary calculus, we know that for any $j = 1, 2, \cdots, d$,

$$\partial_{x_j x_j} w(x_0, y_0) \leq 0,$$

and hence, at $(x, y) = (x_0, y_0)$, we actually have

$$-\sum_{j=1}^{d} j\ \partial_{x_j x_j} w(x_0, y_0) \geq 0,$$

which contradicts with Inequality $(1)_1$. This means that the assumption "$(x_0, y_0) \in \Omega$" is wrong, so $(x_0, y_0) \in \partial\Omega$, which implies

$$\max_{\bar{\Omega}} w = \max_{\partial\Omega} w.$$

Step 2. (Comparison Principle). Denote by $\tilde{u} := u_1 - u_2$, $\tilde{f} := f_1 - f_2$ and $\tilde{g} := g_1 - g_2$. If follows from the assumptions stated in the problem that $\tilde{u}$ satisfies

$$\begin{cases} -\displaystyle\sum_{j=1}^{d} j\ \partial_{x_j x_j} \tilde{u} = \tilde{f} < 0 & \text{in } \Omega \\ \qquad\quad \tilde{u}|_{\partial\Omega} = \tilde{g} \leq 0, \end{cases}$$

Therefore, we can apply the result in Step 1 to $w := \tilde{u}$, and obtain

$$\max_{\bar{\Omega}} \tilde{u} = \max_{\partial\Omega} \tilde{u} = \max_{\partial\Omega} \tilde{g} \leq 0,$$

which implies

$$u_1 \leq u_2 \quad \text{in } \Omega.$$

(b) We will first show a maximum principle, and then apply it to compute the desired supremum.

Step 1. (Maximum Principle). For any $T > 0$, let w be a $C^2((0,T) \times (0,1)) \cap C([0,T] \times [0,1])$ solution to

$$\partial_t w - w\partial_x w - 2\partial_{xx} w < 0. \tag{2}$$

Then we want to show

$$\max_{\substack{0 \le x \le 1 \\ 0 \le t \ge T}} w(t,x) = \max\left\{\max_{0\le x\le 1} w(0,x), \max_{0\le t\le T} w(t,0), \max_{0\le t\le T} w(t,1)\right\} \tag{3}$$

as follows. First of all, since $w \in C([0,T] \times [0,1])$, it follows from the extreme value theorem that there exists $(t_0, x_0) \in [0,T] \times [0,1]$ such that

$$w(t_0, x_0) = \max_{\substack{0 \le x \le 1 \\ 0 \le t \ge T}} w(t,x).$$

Seeking for a contradiction, we assume that $(t_0, x_0) \in (0,T] \times (0,1)$.

Case 1: $(t_0, x_0) \in (0,T) \times (0,1)$. It follows from the first and second-order derivative test for local/interior maximum that

$$\partial_t w(t_0,x_0) = \partial_x w(t_0,x_0) = 0 \quad \text{and} \quad \partial_{xx} w(t_0,x_0) \le 0,$$

so

$$\partial_t w(t_0,x_0) - w(t_0,x_0)\partial_x w(t_0,x_0) - 2\partial_{xx} w(t_0,x_0) \ge 0,$$

which contradicts with (2). Thus, $(t_0, x_0) \notin (0,T) \times (0,1)$.

Case 2: $(t_0, x_0) \in \{T\} \times (0,1)$. Applying the first and second-order derivative test for local/interior maximum to $w(T, \cdot)$, we have

$$\partial_x w(T,x_0) = 0 \quad \text{and} \quad \partial_{xx} w(T,x_0) \le 0.$$

On the other hand, applying the first-order derivative test for a boundary maximum to $u(\cdot, x_0)$, we also have

$$\partial_t w(T,x_0) \ge 0.$$

Therefore,

$$\partial_t w(T, x_0) - w(T, x_0)\partial_x w(T, x_0) - 2\partial_{xx} w(T, x_0) \geq 0,$$

which contradicts with (2) again. Thus, $(t_0, x_0) \notin \{T\} \times (0, 1)$.

Combining two cases, we conclude the maximum principle (3).

Step 2. (Computing the Supremum). Since $-e^u < 0$, the solution u indeed satisfies Inequality (2). Now, for any $(t_1, x_1) \in [0, \infty) \times [0, 1]$, we can always find a constant $T > t_1$, so we can apply the maximum principle (3) proved in Step 1, and obtain

$$\begin{aligned} u(t_1, x_1) &\leq \max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \geq T}} u(t, x) \\ &= \max\left\{\max_{0 \leq x \leq 1} w(0, x), \max_{0 \leq t \leq T} w(t, 0), \max_{0 \leq t \leq T} w(t, 1)\right\} \\ &\leq \max\left\{\max_{0 \leq x \leq 1} w(0, x), \sup_{t \geq 0} w(t, 0), \sup_{t \geq 0} w(t, 1)\right\} \end{aligned}$$

Taking supremum over $(t_1, x_1) \in [0, \infty) \times [0, 1]$, we have

$$\sup_{\substack{0 \leq x \leq 1 \\ t \geq 0}} u(t, x) \leq \max\left\{\max_{0 \leq x \leq 1} w(0, x), \sup_{t \geq 0} w(t, 0), \sup_{t \geq 0} w(t, 1)\right\}.$$

In addition, since the set $\{t = 0\} \cup \{x = 0\} \cup \{x = 1\}$ is a subset of $[0, \infty) \times [0, 1]$, so we actually have

$$\sup_{\substack{0 \leq x \leq 1 \\ t \geq 0}} u(t, x) = \max\left\{\max_{0 \leq x \leq 1} w(0, x), \sup_{t \geq 0} w(t, 0), \sup_{t \geq 0} w(t, 1)\right\}.$$

Finally, it follows from elementary calculus that

$$\begin{aligned} \max_{0 \leq x \leq 1} w(0, x) &= \max_{0 \leq x \leq 1} -3x^2 + 4x + 1 = \frac{7}{3} \\ \sup_{t \geq 0} w(t, 0) &= \sup_{t \geq 0} \frac{1}{1 + t} = 1 \\ \sup_{t \geq 0} w(t, 1) &= \sup_{t \geq 0} 2e^{-t} = 2, \end{aligned}$$

so

$$\sup_{\substack{0\le x\le 1\\ t\ge 0}} u(t,x) = \max\left\{\frac{7}{3}, 1, 2\right\} = \frac{7}{3}.$$

□

Problem 2. Let u_1 and u_2 be two solutions to the same Laplace equation

$$\Delta u := \partial_{xx}u + \partial_{yy}u = 0 \qquad \text{on } \Omega = [-4,4]\times[-4,4],$$

but u_1 and u_2 satisfy different boundary conditions: for $i = 1$, 2,

$$\begin{cases} u_i|_{x=-4} = g_i \\ u_i|_{x=4} = h_i \\ u_i|_{y=-4} = \phi_i \\ u_i|_{y=4} = \psi_i \end{cases}$$

where g_i, h_i, ϕ_i and ψ_i, are given data. Prove that if

$$\begin{cases} g_1 \le g_2 \\ h_1 \le h_2, \\ \phi_1 \le \phi_2 \\ \psi_1 \le \psi_2 \end{cases}$$

then

$$u_1 \le u_2.$$

Solution. Let $v := u_1 - u_2$. Then v satisfies the Laplace equation

$$\Delta u := \partial_{xx}u + \partial_{yy}u = 0 \qquad \text{on } \Omega = [-4,4]\times[-4,4]$$

subject to the boundary conditions:

$$\begin{cases} v|_{x=-4} = g_1 - g_2 \le 0 \\ v|_{x=4} = h_1 - h_2 \le 0 \\ v|_{y=-4} = \phi_1 - \phi_2 \le 0 \\ v|_{y=4} = \psi_1 - \psi_2 \le 0. \end{cases}$$

Then it follows from the maximum principle that for all $(x, y) \in \bar{\Omega}$,

$$u_1(x, y) - u_2(x, y) = v(x, y) \le \max_{\bar{\Omega}} v = \max_{\Gamma} v \le 0,$$

and hence $u_1 \le u_2$ on $\bar{\Omega}$. □

Problem 3. (i) Let $u := u(t,x) \in C^2([0,T] \times [0,L])$ be a solution to

$$\partial_t u - \partial_{xx} u = |\partial_x u|^2 + 3,$$

for $0 < x < L$ and $0 < t < T$. Show that

$$\min_{\substack{0 \le x \le L \\ 0 \le t \le T}} u(t,x) \ge \min\left\{ \min_{0 \le x \le L} u(0,x), \min_{0 \le t \le T} u(t,0), \min_{0 \le t \le T} u(t,L) \right\}.$$

(ii) Let

$$D := \left\{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 < 4\right\},$$

and $u \in C(\bar{D}) \cap C^2(D)$ be a solution to

$$3\partial_{xx} u + 8\,\partial_{yy} u = u^2\,\partial_y u.$$

Show that

$$\max_{\bar{D}} |u| = \max_{\partial D} |u|.$$

Solution. (i) Let $\Omega = (0,T)\times(0,L)$ and $\Gamma := \left\{(t,x) \in \Omega;\ t = 0 \text{ or } x = 0 \text{ or } L\right\}$. As u is continuous over $\overline{\Omega}$, it follows from the extreme value theorem that the minimum exists on $\overline{\Omega}$, say $u(t_0, x_0) = \min_{\bar{\Omega}} u$. Now we will show that $(t_0, x_0) \notin \overline{\Omega} \setminus \Gamma$. Assume on the contrary that $(t_0, x_0) \in \bar{\Omega} \setminus \Gamma$.

If $\boxed{(t_0, x_0) \in (0,\, T) \times (0,\, L)}$, then $\partial_x u(t_0, x_0) = \partial_t u(t_0, x_0) = 0$ and $\partial_{xx} u(t_0, x_0) \ge 0$. Thus

$$LHS = -\partial_{xx} u(t_0, x_0) \le 0 < 3 = RHS$$

which give a contradiction.

If $\boxed{t_0 = T}$ and $\boxed{x_0 \in (0, L)}$, then $\partial_x u(t_0, x_0) = 0$, $\partial_t u(t_0, x_0) \le 0$ and $\partial_{xx} u(t_0, x_0) \ge 0$. Thus

$$LHS = \partial_t u(t_0, x_0) - \partial_{xx} u(t_0, x_0) \le 0 < 3 = RHS$$

which also give a contradiction.

Therefore, $(t_0, x_0) \in \Gamma$ and hence $\min_{\bar{\Omega}} u = u(t_0, x_0) = \min_{\Gamma} u$.

(ii) Step 1: Show that if $3\partial_{xx} v + 8\partial_{yy} v - u^2 \partial_y v > 0$, then $\max_{\bar{D}} v = \max_{\partial D} v$.

As v is continuous over $\overline{D}$, it follows from the extreme value theorem that the maximum exists on $\overline{D}$, say

$$v(x_0, y_0) = \max_{\bar{D}} v.$$

Assume on the contrary that $(x_0, y_0) \in D$. Then $\partial_y v(x_0, y_0) = 0$ and $\partial_{xx} v(x_0, y_0)$, $\partial_{yy} v(x_0, y_0) \le 0$. Thus

$$3\partial_{xx} v(x_0, y_0) + 8\partial_{yy} v(x_0, y_0) - u^2 \partial_y v(x_0, y_0) = 3\partial_{xx} v(x_0, y_0) + 8\partial_{yy} v(x_0, y_0) \le 0,$$

which give a contradiction.

Step 2: Show that if $3\partial_{xx} v + 8\partial_{yy} v - u^2 \partial_y v \ge 0$, then $\max_{\bar{D}} v = \max_{\partial D} v$.

Given any $\epsilon > 0$, we define, for any $(x, y) \in \bar{D}$,

$$v_\epsilon(x,\, y) := v(x,\, y) + \epsilon x^2.$$

Then

$$3\partial_{xx} v_\epsilon + 8\partial_{yy} v_\epsilon - u^2 \partial_y v_\epsilon = 3\partial_{xx} v + 8\partial_{yy} v + 6\epsilon - u^2 \partial_y v > 0.$$

By Step 1,

$$\max_{\bar{D}} v \le \max_{\bar{D}} (v + \epsilon x^2) = \max_{\bar{D}} v_\epsilon = \max_{\partial D} v_\epsilon = \max_{\partial D} (v + \epsilon x^2) \le \max_{\partial D} v + 4\epsilon$$

and hence taking $\epsilon \to 0^+$, we get

$$\max_{\bar{D}} v \leq \max_{\partial D} v.$$

On the other hand, as $\partial D \subset \bar{D}$, $\max_{\bar{D}} v \geq \max_{\partial D} v$. Thus, $\max_{\bar{D}} v = \max_{\partial D} v$.

Step 3: Show that if $\partial_{xx} u + 2\partial_{yy} u - u^4 \partial_y u \equiv 0$, then $\max_{\bar{D}} |u| = \max_{\partial D} |u|$.
By Step 2, we obtain $\max_{\bar{D}} u = \max_{\partial D} u$. On the other hand, as

$$\partial_{xx}(-u) + 2\partial_{yy}(-u) - (-u)^4 \partial_y(-u) \equiv 0 \text{ on } D,$$

it follows from Step 2 that

$$\min_{\bar{D}} u = -\max_{\bar{D}}(-u) = -\max_{\partial D}(-u) = \min_{\partial D} u.$$

Thus,

$$\max_{\bar{D}} |u| = \max(|\max_{\bar{D}} u|, |\min_{\bar{D}} u|) = \max(|\max_{\partial D} u|, |\min_{\partial D} u|) = \max_{\partial D} |u|.$$

□

Problem 4. Let u be a solution to the boundary value problem

$$\begin{cases} \Delta u := \partial_{xx} u + \partial_{yy} u = 0 & \text{on } \Omega = [-1,1] \times [-1,1] \\ u|_{x=-1} = u|_{x=1} \equiv 0 \\ u|_{y=-1} = u|_{y=1} \equiv \phi(x), \end{cases}$$

Prove the following statements:

(i) If ϕ is even, that is,

$$\phi(-x) = \phi(x),$$

then u is also even in x, that is,

$$u(-x,y) = u(x,y).$$

(Hint: let $v(x,y) := u(-x,y)$, then what is the initial and boundary value problem that v satisfies?).

(ii) If ϕ is odd, that is,

$$\phi(-x) = -\phi(x),$$

then u is also odd in x, that is,

$$u(-x,y) = -u(x,y).$$

Solution. (i) Let $v(x,y) := u(-x,y)$. Then v is a solution to the same the boundary value problem

$$\begin{cases} \Delta v := \partial_{xx}v + \partial_{yy}v = 0 \qquad \text{on } \Omega = [-1,1]\times[-1,1] \\ v|_{x=-1} = v|_{x=1} \equiv 0 \\ v|_{y=-1} = v|_{y=1} \equiv \phi(-x) = \phi(x), \end{cases}$$

Then $w := u - v$ satisfies the same Laplace equation $\Delta w = 0$ on Ω subject to $w|_{x=-1} = w|_{x=1} = w|_{y=-1} = w|_{y=1} = 0$. By the maximum principle, we obtain

$$\max_{\bar{\Omega}} |w| = \max_{\Gamma} |w| = 0$$

which implies $w = 0$ on $\bar{\Omega}$ and hence $v = u$ on $\bar{\Omega}$, that is, $u(-x,y) = u(x,y)$ for any $(x,y) \in \bar{\Omega}$.

(ii) Let $\tilde{v}(x,y) := -u(-x,y)$. Then $\tilde{v}$ is a solution to the same the boundary value problem

$$\begin{cases} \Delta \tilde{v} := \partial_{xx}\tilde{v} + \partial_{yy}\tilde{v} = 0 \qquad \text{on } \Omega = [-1,1]\times[-1,1] \\ \tilde{v}|_{x=-1} = \tilde{v}|_{x=1} \equiv 0 \\ \tilde{v}|_{y=-1} = \tilde{v}|_{y=1} \equiv -\phi(-x) = \phi(x), \end{cases}$$

Then $w := u - \tilde{v}$ satisfies the same Laplace equation $\Delta w = 0$ on Ω subject to $w|_{x=-1} = w|_{x=1} = w|_{y=-1} = w|_{y=1} = 0$. By the maximum principle, we obtain

$$\max_{\bar{\Omega}} |w| = \max_{\Gamma} |w| = 0$$

which implies $w = 0$ on $\bar{\Omega}$ and hence $\tilde{v} = u$ on $\bar{\Omega}$, that is, $u(-x, y) = -u(x, y)$ for any $(x, y) \in \bar{\Omega}$.

□

Problem 5. Do Problem 1 of Dec 2020 Final Exam.

Solution. (a) The proof can be divided into two steps, in which the first step will show the conclusion by a stronger assumption, and the second step will weaken the assumption used in the first step.

Step 1. (Maximum Principle for Strict Inequality). Let $w \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$-\Delta w + \vec{b} \cdot \nabla w < 0. \tag{4}$$

Since w is a continuous function defined on a compact set $\bar{\Omega}$, it follows from the extreme value theorem that there exists a point $(x_0, y_0) \in \bar{\Omega}$ such that

$$w(x_0, y_0) = \max_{\bar{\Omega}} w.$$

Seeking for a contradiction, we assume that $(x_0, y_0) \in \Omega$. By the first and second order tests for local/interior maximum in elementary calculus, we know that

$$\nabla w(x_0, y_0) = 0, \quad \text{and} \quad \Delta w(x_0, y_0) \leq 0,$$

and hence, at $(x, y) = (x_0, y_0)$, we actually have

$$-\Delta w(x_0, y_0) + \vec{b} \cdot \nabla w(x_0, y_0) \geq 0,$$

which contradicts with Inequality (4). This means that the assumption "$(x_0, y_0) \in \Omega$" is wrong, so $(x_0, y_0) \in \partial\Omega$, which implies

$$\max_{\bar{\Omega}} w = \max_{\partial\Omega} w.$$

Step 2. (Maximum Principle for Weak Inequality). Let us recall that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$-\Delta u + \vec{b} \cdot \nabla u \leq 0,$$

where $\vec{b} := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is a given constant vector. Let $\lambda > |b_1| + 1 > 0$ be a positive constant. For any $\epsilon > 0$, we define

$$w_\epsilon(x,y) := u(x,y) + \epsilon e^{\lambda x}.$$

Then

$$\begin{aligned} -\Delta w_\epsilon + \vec{b} \cdot \nabla w_\epsilon &= -\Delta u + \vec{b} \cdot \nabla u - \epsilon \lambda^2 e^{\lambda x} + \epsilon \lambda b_1 e^{\lambda x} \\ &\le -\epsilon\lambda \left(\lambda + b_1\right) e^{\lambda x} < 0. \end{aligned}$$

Therefore, applying the result in Step 1, we have

$$\max_{\bar{\Omega}} w_\epsilon = \max_{\partial\Omega} w_\epsilon,$$

which implies

$$\max_{\bar{\Omega}} u \le \max_{\bar{\Omega}} w_\epsilon = \max_{\partial\Omega} w_\epsilon \le \left(\max_{\partial\Omega} u \right) + \epsilon e^{\lambda L},$$

where the constant $L := \max\left\{x; (x,y) \in \bar{\Omega}\right\} < \infty$. Passing to the limit as $\epsilon \to 0^+$ in the above inequality, we obtain

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u.$$

Since $\partial\Omega \subset \bar{\Omega}$, by the definition of maximum, we also have

$$\max_{\bar{\Omega}} u \ge \max_{\partial\Omega} u,$$

and hence,

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

(b) Considering $w := -u$, one can apply part (a) to w and obtain the following assertion: if u satisfies

$$-\Delta u + \vec{b} \cdot \nabla u \ge 0,$$

then

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u.$$

Combining this result and part (a), we finally have

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{5}$$

Now, let u_1 and u_2 be solutions to the boundary-value problem

$$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u = 0, & \text{in } \Omega, \\ \qquad\qquad u|_{\partial\Omega} = g, \end{cases}$$

where the given data g is the SAME for both u_1 and u_2. Define $\tilde{u} := u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} -\Delta \tilde{u} + \vec{b} \cdot \nabla \tilde{u} = 0, & \text{in } \Omega, \\ \qquad\qquad \tilde{u}|_{\partial\Omega} = 0. \end{cases}$$

Applying (5) to $\tilde{u}$, we have

$$\max_{\bar{\Omega}} |\tilde{u}| = \max_{\partial\Omega} |\tilde{u}| = 0,$$

which implies

$$u_1 \equiv u_2.$$

This shows the uniqueness.

(c) Let $u := v^2$. Then

$$-\Delta u = -2v\Delta v - 2\left|\nabla v\right|^2 = -2v^4 - 2\left|\nabla v\right|^2 \le 0.$$

Applying part (a) to u with $\vec{b} := \vec{0}$, we have

$$\max_{\bar{\Omega}} v^2 = \max_{\bar{\Omega}} u = \max_{\partial\Omega} u = \max_{\partial\Omega} v^2,$$

which is equivalent to

$$\max_{\bar{\Omega}} |v| = \max_{\partial\Omega} |v|.$$

$\square$

Problem 6. Let u satisfy the following PDE

$$\partial_{tt}u - c^2\partial_{xx}u = -\alpha u \qquad \text{for } -\infty < x < \infty \text{ and } \alpha, c, t > 0. \tag{6}$$

Given finite interval (a,b), we define the local energy by

$$E(t) := \int_{a+Mt}^{b-Mt} e(t,x)\,dx, \text{ where } e(t,x) := \tfrac{1}{2}|\partial_t u|^2 + \tfrac{c^2}{2}|\partial_x u|^2 + \tfrac{\alpha}{2}|u^2|.$$

(i) Let $p(t,x) := \partial_t u\,\partial_x u$. Prove that $\partial_t e = c^2\partial_x p$.

(ii) Show that $e \pm cp = \frac{1}{2}(\partial_t u \pm c\partial_x u)^2 + \frac{\alpha}{2}u^2$.

(iii) Using part (i), verify via a direct differentiation that

$$\frac{dE}{dt}(t) = c^2[p(t,b-Mt) - p(t,a+Mt)] - M[e(t,b-Mt) + e(t,a+Mt)].$$

(Hint: Use $\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\,dx = \int_{a(t)}^{b(t)} \partial_t f(t,x)\,dx + f(t,b(t))b'(t) - f(t,a(t))a'(t)$.)

(iv) Suppose $M \ge c$. Using (ii) and (iii), show that $\frac{dE}{dt} \le 0$. Hence if $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), show that $u \equiv 0$ in $\Delta := \{(x,t) \in (-\infty,\infty)\times[0,\infty) : a + Mt \le x \le b - Mt\}$.

Solution. Let u satisfy the following PDE

$$\partial_{tt}u - c^2\partial_{xx}u + \alpha u = 0 \qquad \text{for } -\infty < x < \infty \text{ and } c, \alpha, t > 0. \tag{7}$$

Given finite interval (a,b), we define the local energy by

$$E(t) := \int_{a+Mt}^{b-Mt} e(t,x)\,dx, \text{ where } e(t,x) := \tfrac{1}{2}|\partial_t u|^2 + \tfrac{c^2}{2}|\partial_x u|^2 + \tfrac{\alpha}{2}|u^2|.$$

(i)

$$\begin{aligned}
\partial_t e &= \partial_t u\cdot\partial_{tt}u + c^2\partial_x u\cdot\partial_{tx}u + \alpha u\partial_t u \\
&= \partial_t u(\partial_{tt}u + \alpha u) + c^2\partial_x u\cdot\partial_{tx}u \\
&= c^2[\partial_t u\cdot\partial_{xx}u + \partial_x u\cdot\partial_{tx}u] \ \text{ (by (1))} \\
&= c^2\partial_x(\partial_x u\partial_t u) = c^2\partial_x p.
\end{aligned}$$

(ii)

$$\begin{aligned}\frac{1}{2}(\partial_t u \pm c\partial_x u)^2 + \frac{\alpha}{2}u^2 &= \frac{1}{2}|\partial_t u|^2 + \frac{c^2}{2}|\partial_x u|^2 + \frac{\alpha}{2}|u|^2 \pm c\partial_t u\partial_x u \\ &= e \pm cp.\end{aligned}$$

(iii) For $t \geq 0$,

$$\begin{aligned}\frac{dE}{dt}(t) &= \frac{d}{dt}\int_{a+Mt}^{b-Mt} e(t,x)\,dx \\ &= \int_{a+Mt}^{b-Mt} \partial_t e(t,x)\,dx - Me(t,b-Mt) - Me(t,a+Mt)) \quad \text{(by hint)} \\ &= c^2\int_{a+Mt}^{b-Mt} \partial_x p(t,x)\,dx - M[e(t,b-Mt) + e(t,a+Mt)] \quad \text{(by (i))} \\ &= c^2[p(t,b-Mt) - p(t,a+Mt)] - M[e(t,b-Mt) + e(t,a+Mt)].\end{aligned}$$

(iv) For $t \geq 0$,

$$\begin{aligned}&\frac{dE}{dt}(t) = c^2[p(t,b-Mt) - p(t,a+Mt)] - M[e(t,b-Mt) + e(t,a+Mt)] \quad \text{(by (iii))} \\ \leq\ & c[cp(t,b-Mt) - cp(t,a+Mt) - e(t,b-Mt) - e(t,a+Mt)] \\ \leq\ & -c[e(t,b-Mt) - cp(t,b-Mt) + e(t,a+Mt) + cp(t,a+Mt)] \\ \leq\ & 0 \text{ (by (ii))}.\end{aligned}$$

If $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), then $\partial_x u(0,x) \equiv 0$ on (a,b). Thus,

$$E(0) = \int_a^b \frac{|\partial_t u(0,x)|^2}{2} + \frac{c^2|\partial_x u(0,x)|^2}{2} + \frac{\alpha|u(0,x)|^2}{2}dx = 0.$$

On the other hand, for $0 \leq t \leq (b-a)/2M$,

$$0 \leq E(t) \leq E(0) = 0 \Longrightarrow E(t) = 0.$$

So

$$\int_{a+Mt}^{b-Mt} |u(t,x)|^2 dx \leq E(t) = 0$$

and hence $u(t,x) = 0$ for $a + Mt \leq x \leq b - Mt$, that is, $u \equiv 0$ in $\Delta := \{(x,t) \in (-\infty,\infty) \times [0,\infty) : a + Mt \leq x \leq b - Mt\}$.

$\square$

Problem 7. Apply the **energy method** to show the uniqueness for the following problems:

(i)

$$\begin{cases} \partial_t u - \partial_{xx} u = -9u & \text{for } 0 < x < L,\ t > 0 \\ u|_{t=0} = \phi \\ u|_{x=0} = g \\ u|_{x=L} = h \end{cases}$$

where ϕ, g and h are given data.

(ii)

$$\begin{cases} \partial_{tt} u - 4\partial_{xx} u = -u - \partial_t u & \text{for } -1 < x < 1,\ t > 0 \\ u|_{t=0} = \phi \\ \partial_t u|_{t=0} = \psi \\ u|_{x=-1} = g \\ u|_{x=1} = h \end{cases}$$

where ϕ, ψ, g and h are given data.

Solution. (i) Let u_1 and u_2 be two solutions of the problem. Define $\tilde{u} := u_1 - u_2$. Then we have

$$\begin{cases} \partial_t \tilde{u} - \partial_{xx} \tilde{u} = -9\tilde{u} & \text{for } 0 < x < L,\ t > 0 \\ \tilde{u}|_{t=0} = \tilde{u}|_{x=0} = \tilde{u}|_{x=L} = 0 \end{cases} \tag{8}$$

Multiplying the PDE $\partial_t \tilde{u} = \partial_{xx} \tilde{u} - 9\tilde{u}$ by $\tilde{u}$, and then integrating with respect to x over $(0, L)$, we have

$$\int_0^L \tilde{u} \partial_t \tilde{u} dx = \int_0^L \tilde{u} \partial_{xx} \tilde{u} dx - 9 \int_0^L |\tilde{u}|^2 dx$$

$$\begin{aligned}
\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_0^L |\tilde{u}|^2 dx &= \left[\tilde{u}\partial_x\tilde{u}\right]_{x=0}^L - \int_0^L |\partial_x\tilde{u}|^2 dx - 9\int_0^L |\tilde{u}|^2 dx \\
&= -\int_0^L |\partial_x\tilde{u}|^2 dx - 9\int_0^L |\tilde{u}|^2 dx \quad \text{(by (6))} \\
&\le 0.
\end{aligned}$$

Thus,

$$\int_0^L |\tilde{u}(t,x)|^2 dx \le \int_0^L |\tilde{u}(0,x)|^2 dx = 0 \quad \text{(by (6))}$$

and hence $\tilde{u} = 0$, that is, $u_1 = u_2$.

(ii) Let u_1 and u_2 be two solutions of the problem. Define $\tilde{u} \coloneqq u_1 - u_2$. Then we have

$$\begin{cases} \partial_{tt}\tilde{u} - 4\partial_{xx}\tilde{u} = -\tilde{u} - \partial_t\tilde{u} & \text{for } -1 < x < 1,\ t > 0 \\ \tilde{u}|_{t=0} = \partial_t\tilde{u}|_{t=0} = \tilde{u}|_{x=-1} = \tilde{u}|_{x=1} = 0 \end{cases} \tag{9}$$

Note that

$$\tilde{u}|_{t=0} = 0 \Longrightarrow \partial_x\tilde{u}(0,x) \equiv 0 \tag{10}$$

and

$$\tilde{u}|_{x=-1} = \tilde{u}|_{x=1} = 0 \Longrightarrow \partial_t\tilde{u}(t,-1) = \partial_t\tilde{u}(t,1) \equiv 0 \tag{11}$$

Multiplying the PDE $\partial_{tt}\tilde{u} + \tilde{u} = 4\partial_{xx}\tilde{u} - \partial_t\tilde{u}$ by $\partial_t\tilde{u}$, and then integrating with respect to x over $(-1,1)$, we have

$$\int_{-1}^1 \partial_{tt}\tilde{u}\cdot\partial_t\tilde{u} + \tilde{u}\partial_t\tilde{u}\ dx = 4\int_{-1}^1 \partial_t\tilde{u}\partial_{xx}\tilde{u}dx - \int_{-1}^1 |\partial_t\tilde{u}|^2\ dx$$

$$\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_{-1}^1 |\partial_t\tilde{u}|^2 + |\tilde{u}|^2\ dx = 4[\partial_t\tilde{u}\partial_x\tilde{u}]_{x=0}^L - 4\int_{-1}^1 \partial_{tx}\tilde{u}\cdot\partial_x\tilde{u}dx - \int_{-1}^1 |\partial_t\tilde{u}|^2 dx$$

$$\begin{aligned}
\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_{-1}^1 |\partial_t\tilde{u}|^2 + 4|\partial_x\tilde{u}|^2 + |\tilde{u}|^2 dx &= -\int_{-1}^1 |\partial_t\tilde{u}|^2 dx \quad \text{(by (3) and (5))} \\
&\le 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{-1}^1 |\tilde{u}(t,x)|^2 dx &\le \int_{-1}^1 |\partial_t\tilde{u}(t,x)|^2 + 4|\partial_x\tilde{u}(t,x)|^2 + |\tilde{u}(t,x)|^2 dx \\
&\le \int_{-1}^{1} \tilde{|}\partial_t\tilde{u}(0,x)|^2 + 4|\partial_x\tilde{u}(0,x)|^2 + |\tilde{u}(0,x)|^2 dx \\
&= 0 \quad \text{(by (3) and (4))}
\end{aligned}$$

and hence $\tilde{u} = 0$, that is, $u_1 = u_2$.

$\square$

Problem 8.

Let the concentration $u := u(t,x)$ satisfy

$$\begin{cases} \partial_t u - \partial_{xx} u = 3t^2x^2 & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u(0,x) = e^x + 1 \\ \partial_x u(t,0) = 1 \\ \partial_x u(t,1) = 4. \end{cases}$$

Compute the total mass $M(t) := \int_0^1 u(t,x)\,dx$.

Solution.

$$\frac{d}{dt}M(t) = \int_0^1 \partial_t u\,dx = \int_0^1 \partial_{xx}u + 3t^2x^2\,dx = \partial_x u(t,1) - \partial_x u(t,0) + t^2x^3\Big|_{x=0}^1 = 3 + t^2$$

and hence

$$\begin{aligned} M(t) &= \int_0^t 3 + t^2\,dt + M(0) = 3t + \frac{t^3}{3} + \int_0^1 u(0,x)dx \\ &= 3t + \frac{t^3}{3} + \int_0^1 e^x + 1\,dx = 3t + \frac{t^3}{3} + (e^x + x)\Big|_{x=0}^1 \\ &= 3t + \frac{t^3}{3} + e. \end{aligned}$$

$\square$

Problem 9. Let u be a solution to the following initial value problem

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = 0 \\ \partial_t u|_{t=0} = 3x^2. \end{cases}$$

(i) Find $u(t,x)$.

(ii) Let the local energy $E(t)$ be defined by $E(t) \coloneqq \dfrac{1}{2}\displaystyle\int_0^1 |\partial_t u|^2 + |\partial_x u|^2\, dx$. Find $E'(t)$.

Solution. (i) By D' Alembert's formula, we have

$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_{x-t}^{x+t} 3s^2 ds \\ &= \frac{(x+t)^3 - (x-t)^3}{2} = 3x^2 t + t^3. \end{aligned}$$

(ii) Multiplying the PDE $\partial_{tt} u = \partial_{xx} u$ by $\partial_t u$, and then integrating with respect to x over $(0,1)$, we have

$$\begin{aligned} &\int_0^1 \partial_{tt} u \cdot \partial_t u \ dx = \int_0^1 \partial_t u \partial_{xx} u \ dx \\ \Longrightarrow\ &\frac{1}{2}\frac{d}{dt}\int_0^1 |\partial_t u|^2 \ dx = [\partial_t u \partial_x u]_{x=0}^1 - \int_0^1 \partial_{tx} u \cdot \partial_x u \, dx \\ \Longrightarrow\ &E'(t) = \partial_t u \partial_x u\Big|_{x=0}^1 = \Big[(3x^2+3t^2)(6xt)\Big]_{x=0}^1 = 18t(1+t^2). \end{aligned}$$

$\square$

Problem 10. Do Problem 3 of Dec 2019 Final Exam.

Solution. (a) Let u_1 and u_2 be two solutions to

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ u|_{\partial\Omega} = g. \end{cases}$$

Define $\tilde{u} \coloneqq u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \Delta \tilde{u} = \tilde{u} & \text{in } \Omega, \\ \tilde{u}|_{\partial\Omega} = 0. \end{cases}$$

Multiplying $\Delta\tilde{u} = \tilde{u}$ by $\tilde{u}$, and then integrating over Ω, we have

$$\iint_\Omega \tilde{u}\Delta\tilde{u} \ dxdy = \iint_\Omega \tilde{u}^2 \ dxdy.$$

Integrating by parts on the left hand side, and using the boundary condition $\tilde{u}|_{\partial\Omega} = 0$, we obtain

$$-\iint_\Omega |\nabla \tilde{u}|^2 \, dxdy = \iint_\Omega \tilde{u}^2 \, dxdy,$$

or equivalently,

$$\iint_\Omega |\nabla \tilde{u}|^2 + \tilde{u}^2 \, dxdy = 0.$$

Since $|\nabla \tilde{u}|^2 + \tilde{u}^2 \geq 0$, it follows from the first vanishing theorem that

$$|\nabla \tilde{u}|^2 + \tilde{u}^2 \equiv 0,$$

which implies

$$\tilde{u} \equiv 0.$$

That is, $u_1 \equiv u_2$.

(b) Let u_1 and u_2 be two solutions to

$$\begin{cases} \partial_{tt} u + x^2 \partial_t u = \partial_{xx} u + \partial_{yy} u & \text{for } 0 < x, y < 1 \text{ and } t > 0, \\ u(0,x,y) = \phi(x,y), & \dfrac{\partial u}{\partial t}(0,x,y) = \psi(t,y), \\ u(t,0,y) = g_0(t,y), & u(t,1,y) = g_1(t,y), \\ u(t,x,0) = h_0(t,x), & u(t,x,1) = h_1(t,x). \end{cases}$$

Define $\tilde{u} := u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \partial_{tt} \tilde{u} + x^2 \partial_t \tilde{u} = \partial_{xx} \tilde{u} + \partial_{yy} \tilde{u} & \text{for } 0 < x, y < 1 \text{ and } t > 0, \\ \tilde{u}(0,x,y) = 0, & \dfrac{\partial \tilde{u}}{\partial t}(0,x,y) = 0, \\ \tilde{u}(t,0,y) = 0, & \tilde{u}(t,1,y) = 0, \\ \tilde{u}(t,x,0) = 0, & \tilde{u}(t,x,1) = 0. \end{cases}$$

Multiplying $\partial_{tt}\tilde{u} + x^2 \partial_t \tilde{u} = \partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}$ by $\partial_t \tilde{u}$, and then integrating with respect to x and y, we have

$$\int_0^1 \int_0^1 \partial_t \tilde{u} \left(\partial_{tt}\tilde{u} + x^2 \partial_t \tilde{u} \right) \, dxdy = \int_0^1 \int_0^1 \partial_t \tilde{u} \left(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u} \right) \, dxdy.$$

Integrating by parts on the right hand side and using the boundary conditions for $\tilde{u}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1 |\partial_t\tilde{u}|^2 + |\partial_x\tilde{u}|^2 + |\partial_y\tilde{u}|^2 \; dxdy = -\int_0^1\int_0^1 x^2 \, |\partial_t\tilde{u}|^2 \; dxdy \leq 0.$$

Define $E(t) := \int_0^1\int_0^1 |\partial_t\tilde{u}(t,x,y)|^2 + |\partial_x\tilde{u}(t,x,y)|^2 + |\partial_y\tilde{u}(t,x,y)|^2 \; dxdy$. The above inequality can be written as

$$\frac{d}{dt}E(t) \leq 0.$$

A direct integration yields, for any $t \geq 0$,

$$E(t) \leq E(0).$$

Using the initial conditions $\tilde{u}(0,x,y) = 0$ and $\frac{\partial\tilde{u}}{\partial t}(0,x,y) = 0$, we have

$$E(0) = 0.$$

It follows from the definition of E that $E(t) \geq 0$, so

$$0 \leq E(t) \leq E(0) = 0,$$

which implies $E(t) \equiv 0$, and hence,

$$|\partial_t\tilde{u}|^2 + |\partial_x\tilde{u}|^2 + |\partial_y\tilde{u}|^2 \equiv 0.$$

This implies

$$\tilde{u} \equiv C,$$

where C is a constant. To determine the constant C, we use the initial condition $\tilde{u}(0,x,y) = 0$ again, and obtain $C = 0$. Thus,

$$\tilde{u} \equiv 0.$$

That is, $u_1 \equiv u_2$.

$\square$