MATH3541 INTRODUCTION TO TOPOLOGY
ASSIGNMENT I

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG

Due: 12:00 noon, Tuesday 17th September 2024.

Instructions: Submit solutions to the problems in **Section B** for credit. Problems in Section A should be attempted and may be optionally submitted for feedback.

Guidelines on Writing: *You should write in complete sentences. Do not just give the solution in fragmentary bits and pieces. Clarity of presentation of your argument counts, so explain the meaning of every symbol that you introduce and avoid starting a sentence with a symbol.*

Section A

Problem 1. Let X be a set. Let $\mathcal{C}$ be the collection of all subsets A of X such that $A = X$ or A is countable. Show that $\mathcal{O} = \{X - A : A \in \mathcal{C}\}$ is a topology on X. When $X = \mathbb{R}$, is the topology $\mathcal{O}$ coarser than the classical topology on $\mathbb{R}$?

not $\{\frac{1}{n}\}_{n=1}^{+\infty}$ in former but not latter

$B(a_k)_{k=1}^{n}$

$\cap \mathbb{Z}^n = \{(a_k)_{k=1}^{n}\}$

Problem 2. Consider $\mathbb{R}^n$ with the Euclidean topology. Prove that the induced subspace topology on $\mathbb{Z}^n$ is the discrete topology on $\mathbb{Z}^n$.

Problem 3. Let X be a topological space and let Y be any subspace. Let $U \subset Y$. If U is open in Y and Y is open in X, prove that U is open in X. If U is closed in Y and Y is closed in X, prove that U is closed in X.

Problem 4. Let $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ and consider $X = \mathbf{k}^n$. For a polynomial $f \in \mathbf{k}[x_1, \ldots, x_n]$, define $D(f)$ to be the complement of $Z(\{f\})$. Show that the collection of all $D(f)$ is a basis for the Zariski topology.

Problem 5. Prove that for $n \geq 2$ the classical topology on $\mathbb{R}^n$ is the product topology.

Problem 6. Let X and Y be two topological spaces and equip $X \times Y$ with the product topology. Let $X_1 \subset X$ and $Y_1 \subset Y$, and consider two topologies $\mathcal{O}$ and $\mathcal{O}'$ on $X_1 \times Y_1$, where $\mathcal{O}$ is the subspace topology by regarding $X_1 \times Y_1$ as a subspace of $X \times Y$, and $\mathcal{O}'$ is the product topology of the subspace topology on X_1 and the subspace topology on Y_1. Prove that $\mathcal{O} = \mathcal{O}'$.

Problem 7. Let Y be a subspace of X. Given $A \subset Y$, show that $A^\circ_X \subset A^\circ_Y$ where the former one is the interior of A by regarding A as a subset of X and the latter one is the interior of A by regarding A as a subset of Y. Show by an example that the two may not be equal.

A $\quad X = \mathbb{R}$

$Y = \{1, 2, 3\}$, $P(Y)$

$A_Y^o = \{1\}$,

$A_X^o = \emptyset$

Problem 8. Let X be a topological space and let $A, B \subset X$. Prove

(1) $\overline{\overline{A}} = \overline{A}$;
(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;
(3) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

Problem 9. Prove that the metric topology on any metric space is Hausdorff.

Problem 10. Prove that any subspace of a Hausdorff topological space is again Hausdorff.

Problem 11. Let $f : X \to Y$ be a homeomorphism of topological spaces. Show that $f|_U : U \to f(U)$ is a homeomorphism for any subset $U \subseteq X$ where we equip U and $f(U)$ with their respective subspace topologies.

Problem 12. Prove that a topological space X is Hausdorff if and only if

$$X_{\text{diag}} := \{(x, x) \mid x \in X\}$$

is a closed subset of $X \times X$ with the product topology.

Section B

Problem 13. (3 marks) Let Λ be a (not necessarily finite) index set and $\{X_\lambda \mid \lambda \in \Lambda\}$ a family of topological spaces with subsets $A_\lambda \subset X_\lambda$ for each $\lambda \in \Lambda$.

(a) In the product space $\prod_{\lambda \in \Lambda} X_\lambda$ with the product topology, prove that

$$\overline{\prod_{\lambda \in \Lambda} A_\lambda} = \prod_{\lambda \in \Lambda} \overline{A_\lambda}.$$

That is, the closure of the product is the product of the closures.

(b) In the same product space, show that if Λ is finite, we also have

$$\left(\prod_{\lambda \in \Lambda} A_\lambda \right)^\circ = \prod_{\lambda \in \Lambda} \left(A_\lambda^\circ \right).$$

That is, the interior of the product is the product of the interiors.

(c) Explain, with use of a counterexample, why the statement in part (b) cannot be extended to infinite products.

Problem 14. (3 marks) Order following topologies on $X = [0,1]$ by coarser/finer. If two topologies cannot be compared, give examples of open sets which are contained in exactly one of the two.
Which of these topologies are Hausdorff?

(A) The trivial (or indiscrete) topology
(B) The discrete topology
(C) The Euclidean (or metric) topology
(D) The co-finite topology
(E) The co-countable topology: U open iff $U = \emptyset$ or $X - U$ is countable.
(F) U open iff $U = X$ or $0 \notin U$.
(G) U open iff $U = X$, $U = (0,1)$, or $U = \emptyset$.

Problem 15. (6 marks) Let X be a topological space and $\sim$ an equivalence relation on X. Denote by $p : X \to Y$ the quotient map, where $Y = X/\sim$ denotes the quotient space with the quotient topology.

(a) Prove that a map $f : Y \to Z$ is continuous if and only if $f \circ p : X \to Y$ is continuous.
(b) Let $X = S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ and $x \sim -x$ for all $x \in S^1$. Show that the quotient map $p : X \to X/\sim$ is an open map, i.e. it maps open sets to open sets. (You saw in the lectures that this is not always the case.)
(c) Let $X = \mathbb{R}^2$ and $(x_1, x_2) \sim (y_1, y_2)$ iff $(y_1, y_2) = (ax_1, a^{-1}x_2)$ for some $a \in \mathbb{R} - \{0\}$. Prove that this quotient space Y is not Hausdorff.
(d) Prove that the disk $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ is homeomorphic to $D^2/\sim$, where $x \sim -x$ for all $x \in D^2$.

Problem 16. (8 marks)
Recall that as a set

$$\operatorname{Spec} \mathbb{Z} = \{(0)\} \cup \{(p) \mid p \text{ is prime}\}$$

is the set of all prime ideals of $\mathbb{Z}$. We equip $X = \operatorname{Spec} \mathbb{Z}$ with the Zariski topology, which has closed sets given by $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z} \mid \mathfrak{a} \subset \mathfrak{p}\}$ for any ideal $\mathfrak{a} \subset \mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain (PID), the ideals of $\mathbb{Z}$ are of the form $\mathfrak{a} = (a)$ for some $a \in \mathbb{Z}$. You may find the Fundamental Theorem of Arithmetic useful.

(a) Show that $V((a)) = \{(p) \mid p \text{ divides } a\}$.
(b) Verify that the above collection of closed sets defines a topology on $\operatorname{Spec} \mathbb{Z}$.
(c) Show that the closure of $\{(0)\}$ in X is the whole space X. Equivalently, show that (0) is contained in every non-empty open set.
(d) Is the Zariski topology on X finer than the co-finite topology on X?
(e) For $a \in \mathbb{Z}$, let $D(a) = X - V((a))$. Show that the collection $\{D(a) \mid a \in \mathbb{Z}\}$ forms a basis of the Zariski topology. Show that the collection $\{D(p) \mid p \text{ is prime}\}$ forms a sub-basis.
(f) Find the closure of the diagonal $\overline{X_{\text{diag}}}$ inside $X \times X$.
(g) Explain briefly how part (f) shows that $X = \operatorname{Spec} \mathbb{Z}$ is not Hausdorff..