Algebra II: Tutorial 6

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Problem 1. Factorise the polynomial $x^{24}-1$ over $\mathbb{Q}$. Hence, find the minimal polynomial of $\exp(\frac{2\pi i}{24})$.

Solution.

$$\begin{aligned}
x^{24}-1 &= (x^{12}-1)(x^{12}+1) \\
&= (x^6-1)(x^6+1)(x^{12}+1) \\
&= (x^3-1)(x^3+1)(x^6+1)(x^{12}+1) \\
&= (x-1)(x^2+x+1)(x^3+1)(x^6+1)(x^{12}+1),
\end{aligned}$$

where the first two factors are irreducible over $\mathbb{Q}$. Notice that the last three factors are all of the form X^3+1 for $X = x, x^2, x^4$. By a direct computation, $X^3+1 = (X+1)(X^2-X+1)$, where both factors on the right are irreducible in $\mathbb{Q}[X]$. Hence, $x^3+1 = (x+1)(x^2-x+1), x^6+1 = (x^2+1)(x^4-x+1)$ and $x^{12}+1 = (x^4+1)(x^8-x^4+1)$. It is clear that this is a decomposition of x^3+1 and x^6+1 into irreducibles over $\mathbb{Q}$. It remains to show that x^8-x^4+1 is irreducible. This can be checked directly using Sage. All in all, we get:

$$x^{24}-1 = (x-1)(x^2+x+1)(x+1)(x^2-x+1)(x^2+1)(x^4-x+1)(x^4+1)(x^8-x^4+1).$$

Set $\alpha = \exp(\frac{2\pi i}{24})$, then notice that $\alpha^8 - \alpha^4 = -1$, and so α is a root of x^8-x^4+1. By definition, x^8-x^4+1 is then the minimal polynomial of α. ■

Problem 2 (Algebraically closed fields are infinite)**.** Show that every algebraically closed field is infinite.

Solution. Let F be algebraically closed (i.e. every polynomial in $F[x]$ has a root in F), and suppose that F is finite, say $F = \{a_1, a_2, \cdots, a_n\}$. Note that $n > 1$, and so F has at least one non-zero element; WLOG say $a_1 \neq 0$. Then, define the degree n polynomial $f(x) = \prod_{i=1}^{n}(x-a_i) + a_1$ in $F[x]$. Then, $f(a) = a_1 \neq 0$ for all $a \in F$, and therefore f has no root in F. This contradicts the assumption that F is algebraically closed. ■

Problem 3 (Extensions of algebraically closed fields)**.** Suppose that K is algebraically closed, and let $K \subset L$ be an algebraic extension. Show that $K = L$.

Solution. The inclusion $K \subset L$ is obvious; we show that $L \subset K$. Suppose that L is an algebraic extension of K. Take $a \in L$; a is algebraic over K, i.e. there exists a polynomial $f \in K[x]$ such that $f(a) = 0$. Since K is a field, $K[x]$ is a UFD, therefore we can assume without loss of generality that f is monic irreducible over K. Since K is algebraically closed, f has a root in K, and therefore $f = x - a$. Since $f \in K[x]$, we have $a \in K$, and so $L \subset K$. ■

Problem 4 (Degree of splitting fields: upper bound)**.** Let K be any field, and suppose that $f \in K[x]$ is a polynomial of degree n. Let L be a splitting field of f over K. Show that $[L : K] \leq n!$ (Hint: use induction on n).

Solution. If $n = 1$, then $f = ax + b$ with $a, b \in K$, so $L = K$ and $[L : K] = 1 \leq 1!$. Suppose now that the claim is true for $n - 1 \in \mathbb{N}$ fixed. Take $f \in K[x]$ is a polynomial of degree n with roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. Over $K(\alpha_n)$, f has a root, and so $f(x) = (x - \alpha_n)h(x)$ for some $h(x) \in K(\alpha_n)[x]$. By comparing degrees, $h(x)$ has degree $n - 1$, and by inductive hypothesis the splitting field $L_h = K(\alpha_n)(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ of h has degree at most $(n-1)!$ over $K(\alpha_n)$. Therefore, $[L : K(\alpha_n)] \leq (n - 1)!$. Furthermore, $[K(\alpha_n) : K] \leq n$, so by the Tower theorem $[L : K] \leq n!$.