Chapter 4. Maximum Principles

MATH4406 Introduction to Partial Differential Equations

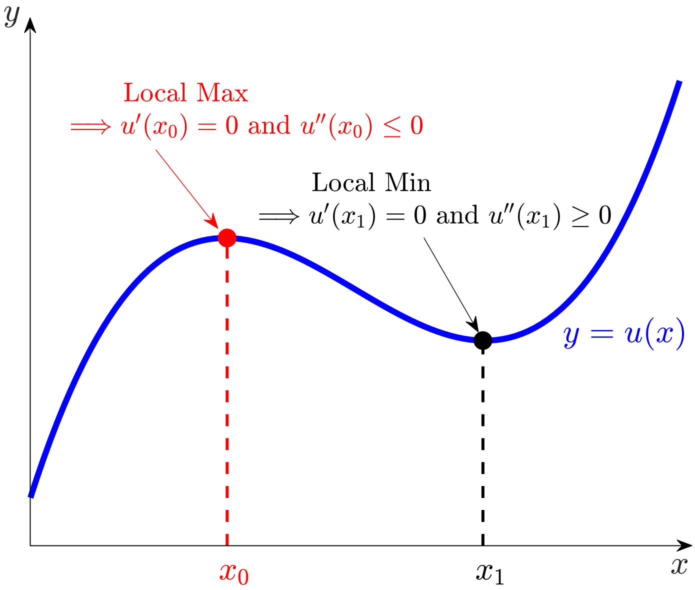
The University of Hong Kong

Table of Contents

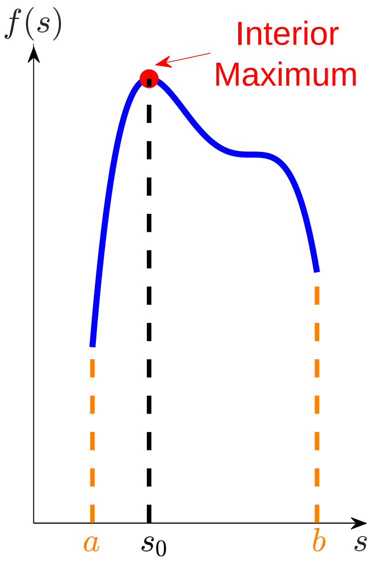
This chapter is related to the materials in Section 2.3 and 6.1 of the Textbook.

4.1 Calculus Review

What Have We Learned in Calculus of a Single Variable?



Interior Maximum in a Closed and Bounded Interval



Let $f : [a, b] \to \mathbb{R}$ be C^2. By the *extreme value theorem*, there exists a point $s = s_0 \in [a, b]$ such that

$$f(s_0) = \max_{[a,b]} f < \infty.$$

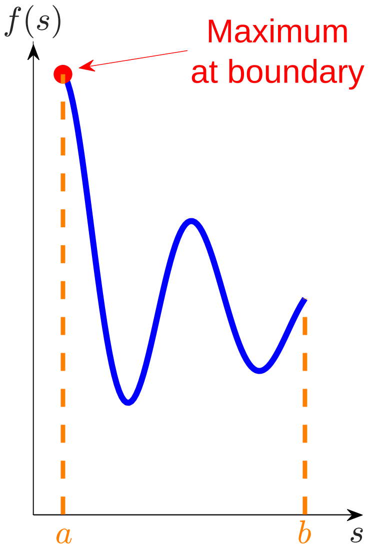
If $a < s_0 < b$, then we call this as an interior maximum.

What do we know at $s = s_0$?

$$f'(s_0) = 0$$
$$f''(s_0) \leq 0.$$

It is worth noting that the extreme value theorem is NOT able to tell you the precise location of s_0.

Maximum at the Left Endpoint



Now, if the maximum attains at the left endpoint (i.e., $s_0 = a$), then

$$f(a) = \max_{[a,b]} f < \infty.$$

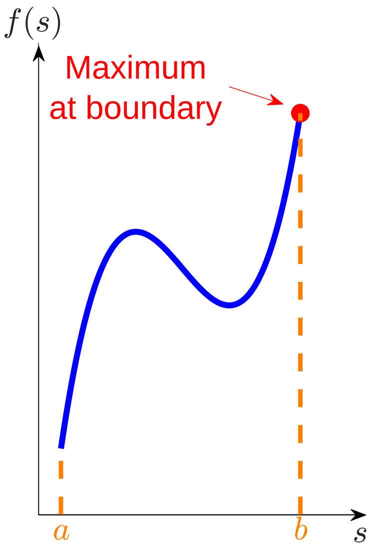
What do we know at $s = a$?

$$f'(a) \leq 0.$$

It is worth noting that the derivative $f'(a)$ above is only defined as a **right derivative**, *namely*

$$f'(a) := \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}.$$

Maximum at the Right Endpoint



Finally, if the maximum attains at the right endpoint (i.e., $s_0 = b$), then

$$f(b) = \max_{[a,b]} f < \infty.$$

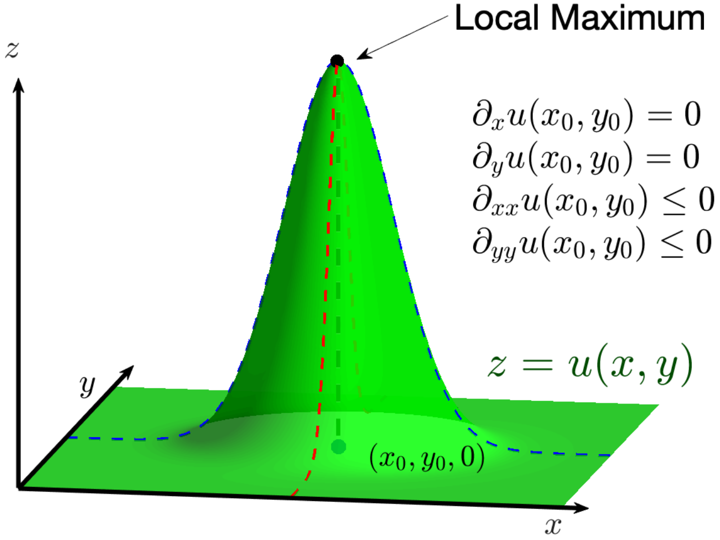
What do we know at $s = b$?

$$f'(b) \geq 0.$$

It is worth noting that the derivative $f'(b)$ above is only defined as a **left derivative**, *namely*

$$f'(b) := \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}.$$

What Have We Learned in Calculus of Two Variables?



What Should We Know About the Local Maximum?

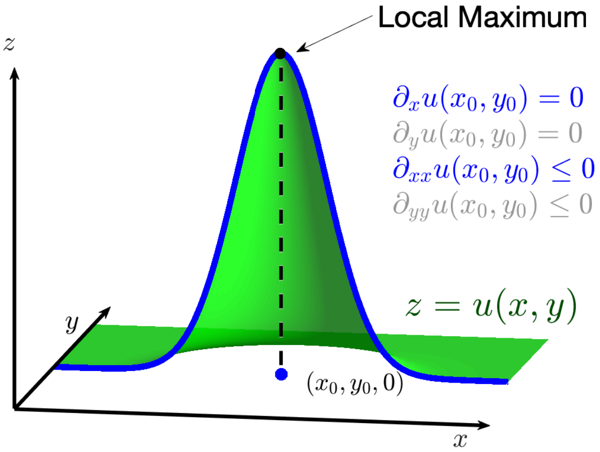


Figure: The Cross Section along $y = y_0$, which is parallel to the x-axis.

What Should We Know About the Local Maximum?

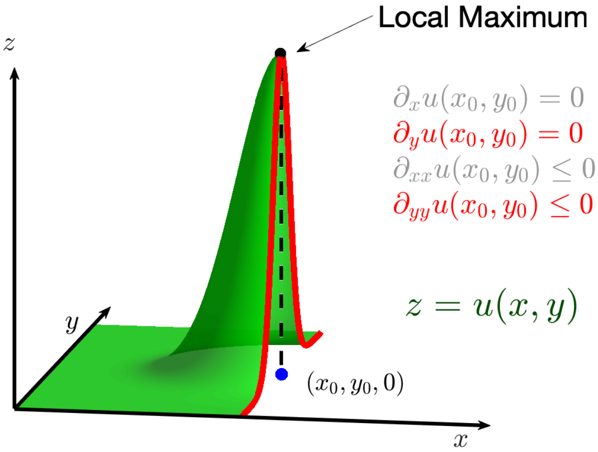


Figure: The Cross Section along $x = x_0$, which is parallel to the y-axis

More Direct Consequences from Calculus of Two Variables

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function, and attain a local maximum at $(x, y) = (x_0, y_0)$.

- The local maximum $(x, y) = (x_0, y_0)$ must be a critical point, namely

$$\nabla u(x_0, y_0) = 0,$$

  or equivalently,

$$\partial_x u(x_0, y_0) = 0 \quad \text{and} \quad \partial_y u(x_0, y_0) = 0.$$

- The Hessian matrix $H(u) := \begin{pmatrix} \partial_{xx} u & \partial_{xy} u \\ \partial_{xy} u & \partial_{yy} u \end{pmatrix}$ is negative semi-definite at $(x, y) = (x_0, y_0)$, which means ALL of the eigenvalues of $H(u)|_{(x,y)=(x_0,y_0)}$ are non-positive.

When Will the Hessian Matrix be Negative Semi-Definite?

It follows from the quadratic formula that the eigenvalues of the 2×2 symmetric matrix $A := H(u)|_{(x,y)=(x_0,y_0)}$ are

$$\lambda_\pm = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4 \det A}}{2}.$$

Thus,

$$\lambda_\pm \le 0 \iff \begin{cases} \operatorname{tr} A \le 0 \\ \det A \ge 0 \end{cases} \iff \begin{cases} (\partial_{xx} u + \partial_{yy} u)|_{(x,y)=(x_0,y_0)} \le 0 \\ \left(\partial_{xx} u \partial_{yy} u - (\partial_{xy} u)^2\right)|_{(x,y)=(x_0,y_0)} \ge 0. \end{cases}$$

After some more algebraic manipulations, one may finally obtain

$$\lambda_\pm \le 0 \iff \begin{cases} \partial_{xx} u|_{(x,y)=(x_0,y_0)} \le 0 \\ \partial_{yy} u|_{(x,y)=(x_0,y_0)} \le 0 \\ \left(\partial_{xx} u \partial_{yy} u - (\partial_{xy} u)^2\right)|_{(x,y)=(x_0,y_0)} \ge 0. \end{cases}$$

Further Remarks

If you are not familiar with these properties/facts, then please have a look at the following websites:

- (PlanetMath.Org) `https://planetmath.org/RelationsBetweenHessianMatrixAndLocalExtrema`
- (Khan Academy) `https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/optimizing-multivariable-functions/a/second-partial-derivative-test`

Exercise

Are you able to state the corresponding results for a local minimum?

Exercise

Are you able to state the corresponding results for higher dimensional cases?

4.2 Maximum Principles for Laplace's Equations

Maximum/Minimum Principles for the 2D Laplace's Equation

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set. Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\partial_{xx} u + \partial_{yy} u = 0.$$

Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u, \tag{MaxP}$$

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u, \tag{MinP}$$

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{MaxP$|u|$}$$

Food for Thought (for Students Who Learned Complex Analysis)

Can you prove (MaxP$|u|$) by using techniques in complex analysis?

Remark

In the statement, we assume that $u := u(x, y) \in C^2(\Omega)$, which means all of the u, $\partial_x u$, $\partial_y u$, $\partial_{xx} u$, $\partial_{xy} u$ and $\partial_{yy} u$ exist and are continuous in Ω.

Remark

Since u is continuous in the compact set $\bar{\Omega}$, it follows from the extreme value theorem that all of the

$$\max_{\bar{\Omega}} u, \quad \min_{\bar{\Omega}} u \quad \text{and} \quad \max_{\bar{\Omega}} |u|$$

exist and finite.

Why is the Maximum Principle Important?

The maximum principle implies the uniqueness and stability of the boundary-value problem for the Laplace's equation.

Maximum Principle for Differential Inequality

Proposition 1

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\partial_{xx} u + \partial_{yy} u > 0. \qquad (\Delta u > 0)$$

Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u. \qquad \text{(MaxP)}$$

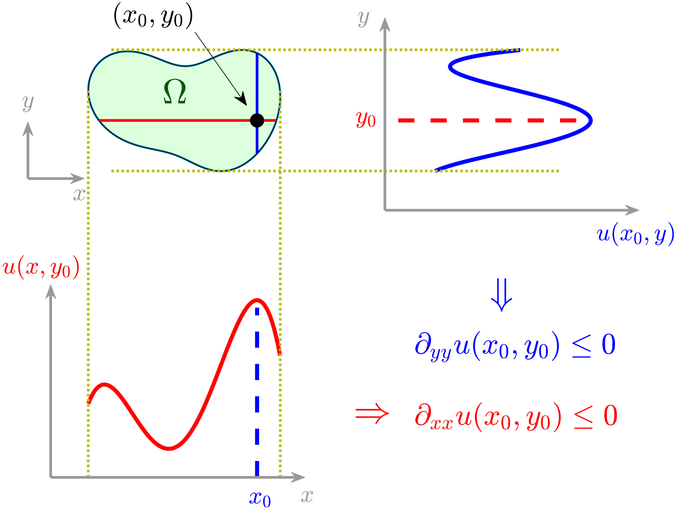
Proof of Proposition 1

It follows from the extreme value theorem that there exists a point $(x_0, y_0) \in \bar{\Omega}$ such that

$$u(x_0, y_0) = \max_{\bar{\Omega}} u.$$

It suffices to show that this $(x_0, y_0) \notin \Omega$. *Seeking for a contradiction*, we assume $(x_0, y_0) \in \Omega$.

Proof of Proposition 1 (Continued).



$$\Downarrow$$

$$\partial_{yy} u(x_0, y_0) \le 0$$

$$\Rightarrow \quad \partial_{xx} u(x_0, y_0) \le 0$$

Thus, $\partial_{xx} u(x_0, y_0) + \partial_{yy} u(x_0, y_0) \le 0$, which contradicts with $(\Delta u > 0)$.

Proposition 2 (Maximum Principle for Subharmonic Functions)

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\partial_{xx} u + \partial_{yy} u \geq 0.$$

Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u. \tag{MaxP}$$

Moral

Idea: make "$\geq$" to be "$>$".
Trick: consider $v_\epsilon(x, y) := u(x, y) + \epsilon(x^2 + y^2)$.

Proof of Proposition 2

For any $\epsilon > 0$, we define, for any $(x, y) \in \bar{\Omega}$,

$$v_\epsilon(x, y) := u(x, y) + \epsilon(x^2 + y^2).$$

Proof of Proposition 2 (Continued)

A direct differentiation on $v_\epsilon(x,y) := u(x,y) + \epsilon(x^2+y^2)$ yields

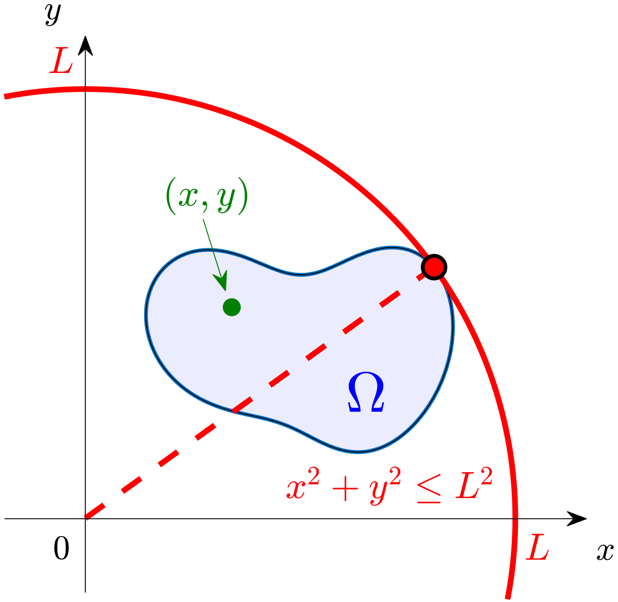
$$\begin{aligned}\partial_{xx} v_\epsilon + \partial_{yy} v_\epsilon &= \partial_{xx}\left(u + \epsilon(x^2+y^2)\right) + \partial_{yy}\left(u + \epsilon(x^2+y^2)\right)\\ &= \underbrace{(\partial_{xx} u + \partial_{yy} u)}_{\geq 0} + \underbrace{4\epsilon}_{>0} > 0.\end{aligned}$$

Applying Proposition 1 to v_ϵ, we have

$$\begin{aligned}\max_{\bar\Omega} u \leq \max_{\bar\Omega}\left(u + \underbrace{\epsilon(x^2+y^2)}_{\geq 0}\right) &=: \max_{\bar\Omega} v_\epsilon = \max_{\partial\Omega} v_\epsilon\\ &:= \max_{\partial\Omega}\left(u + \epsilon(x^2+y^2)\right) \leq \left(\max_{\partial\Omega} u\right) + \epsilon L^2,\end{aligned}$$

where the positive constant $L := \max\left\{\sqrt{x^2+y^2};\ (x,y)\in\bar\Omega\right\} < \infty$.

An Upper Bound for $x^2 + y^2$



Proof of Proposition 2 (Continued).

Passing to the limit $\epsilon \to 0^+$ in $\max_{\bar{\Omega}} u \leq \left(\max_{\partial\Omega} u\right) + \epsilon L^2$, we obtain

$$\max_{\bar{\Omega}} u \leq \left(\max_{\partial\Omega} u\right) + \underbrace{\lim_{\epsilon\to 0^+} \epsilon L^2}_{=0} = \max_{\partial\Omega} u.$$

Hence, due to $\partial\Omega \subset \bar{\Omega}$,

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u \leq \max_{\bar{\Omega}} u \implies \max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

This completes the proof. □

Question

What will we be able to obtain in the case of $\leq$ sign?

Corollary 3 (Minimum Principle for Superharmonic Functions)

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\partial_{xx} u + \partial_{yy} u \leq 0.$$

Then

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u. \tag{MinP}$$

Main Ideas

- $\max_{\bar{\Omega}}(-u) = -\min_{\bar{\Omega}} u$, and
- $(\partial_{xx} + \partial_{yy})(-u) \geq 0$.

Proof of Corollary 3

Define $w(x,y) := -u(x,y)$, for any $(x,y) \in \bar{\Omega}$. Then

$$\partial_{xx} w + \partial_{yy} w = -(\partial_{xx} u + \partial_{yy} u) \geq 0.$$

Proof of Corollary 3 (Continued).

Applying Proposition 2 to w, we have

$$-\min_{\bar{\Omega}} u = \max_{\bar{\Omega}} w = \max_{\partial\Omega} w = -\min_{\partial\Omega} u,$$

because $w = -u$. This implies

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u.$$

□

Summary

In Proposition 1 to Corollary 3, we have showed

- **Prop 1:** $\partial_{xx} u + \partial_{yy} u > 0 \implies \max_{\bar{\Omega}} u = \max_{\Gamma} u.$
- **Prop 2:** $\partial_{xx} u + \partial_{yy} u \geq 0 \implies \max_{\bar{\Omega}} u = \max_{\Gamma} u.$
- **Cor 3:** $\partial_{xx} u + \partial_{yy} u \leq 0 \implies \min_{\bar{\Omega}} u = \min_{\Gamma} u.$

Maximum Principles for Laplace's Equation

Theorem (Maximum Principles for Laplace's Equation)

Let $\Omega \subset \mathbb{R}^2$ *be an open and bounded set, and* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfy*

$$\partial_{xx} u + \partial_{yy} u = 0.$$

Then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u \tag{MaxP}$$

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u \tag{MinP}$$

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|. \tag{MaxP$|u|$}$$

Proof.

Inequality (MaxP) and (MinP) follows directly from Proposition 2 and Corollary 3, respectively. Combining (MaxP) and (MinP), we finally obtain (MaxP$|u|$). □

4.3 Uniqueness and Stability for Laplace's Equation

Application: Uniqueness and Stability for the Dirichlet Problem of Poisson's Equation

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set, and f be a given source term. Suppose that u_1 and u_2 belong to $C^2(\Omega) \cap C(\bar{\Omega})$, and satisfy the following Boundary-Value Problem (BVP): for $i = 1, 2$,

$$\begin{cases} -\Delta u_i = f & \text{in } \Omega \\ u_i\big|_{\partial\Omega} = g_i, \end{cases} \tag{DPPE}$$

where g_1 and g_2 are given boundary data that are continuous on $\partial\Omega$.

Question 1 (Uniqueness)

If $g_1 \equiv g_2$, then must $u_1 \equiv u_2$?

Question 2 (Stability)

If we know that g_1 and g_2 are close in a certain sense (that one has to describe more precisely), then will we be able to estimate the difference between the solutions u_1 and u_2?

How to Answer These Questions?

Denote $\tilde{u} := u_1 - u_2$ and $\tilde{g} := g_1 - g_2$. Then by the linearity, $\tilde{u}$ satisfies

$$\begin{cases} -\Delta \tilde{u} = 0 & \text{in } \Omega \\ \tilde{u}|_{\partial\Omega} = \tilde{g}. \end{cases}$$

By the Maximum Principle,

$$\|\tilde{u}\|_{\sup,\bar{\Omega}} := \max_{\bar{\Omega}} |\tilde{u}| = \max_{\partial\Omega} |\tilde{g}| =: \|\tilde{g}\|_{\sup,\partial\Omega}.$$

In other words, the difference between u_1 and u_2 can be estimated by using the sup-norm as follows:

$$\|u_1 - u_2\|_{\sup,\bar{\Omega}} = \|g_1 - g_2\|_{\sup,\partial\Omega},$$

which is the *stability* in the sup-norm $\|\cdot\|_{\sup}$. In particular, if $g_1 \equiv g_2$, then

$$\|u_1 - u_2\|_{\sup,\bar{\Omega}} = \|g_1 - g_2\|_{\sup,\partial\Omega} = 0,$$

which implies $u_1 \equiv u_2$. The *uniqueness* holds!!

4.4 Maximum Principles for the Heat Equations

Maximum/Minimum Principles for the 1D Heat Equation

Let $u := u(t,x) \in C^2((0,T] \times (0,L)) \cap C([0,T] \times [0,L])$ satisfy

$$\partial_t u = k \partial_{xx} u \quad \text{in } (0,T] \times (0,L),$$

where $k > 0$ is a given constant. Then

$$\max_{\substack{0 \le x \le L \\ 0 \le t \le T}} u(t,x) = \max \left\{ \max_{0 \le x \le L} u(0,x), \max_{0 \le t \le T} u(t,0), \max_{0 \le t \le T} u(t,L) \right\}, \tag{MaxP}$$

and

$$\min_{\substack{0 \le x \le L \\ 0 \le t \le T}} u(t,x) = \min \left\{ \min_{0 \le x \le L} u(0,x), \min_{0 \le t \le T} u(t,0), \min_{0 \le t \le T} u(t,L) \right\}. \tag{MinP}$$

Remark

In the statement, we assume that $u := u(t,x) \in C^2((0,T] \times (0,L))$, which means all of the u, $\partial_t u$, $\partial_x u$, $\partial_{tt} u$, $\partial_{tx} u$ and $\partial_{xx} u$ exist and are continuous in $(0,T] \times (0,L)$.

Remark

Since u is continuous in the compact set $[0,T] \times [0,L]$, it follows from the extreme value theorem that both

$$\max_{\substack{0 \le x \le L \\ 0 \le t \le T}} u(t,x) \quad \text{and} \quad \min_{\substack{0 \le x \le L \\ 0 \le t \le T}} u(t,x)$$

exist and finite.

Moral

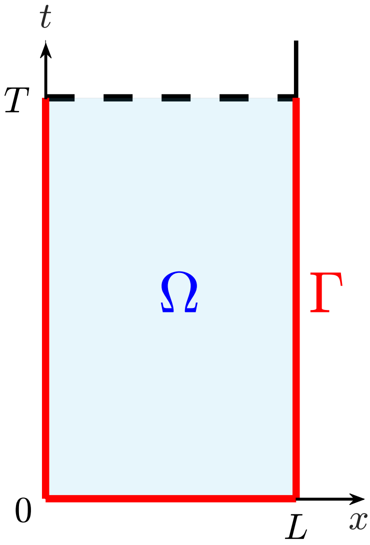
At a possible location of maximum/minimum, there are some special differential structures (coming from the Calculus facts) that may not be compatible with the underlying PDE.

Philosophy for Maximum Principles

1. Extreme values (i.e., maxima and minima) **ONLY** attain at the parabolic boundary Γ.

2. $$\max_{\bar{\Omega}} u = \max_{\Gamma} u,$$
$$\min_{\bar{\Omega}} u = \min_{\Gamma} u.$$

3. Hence,
$$\max_{\bar{\Omega}} |u| = \max_{\Gamma} |u|.$$



Maximum Principle for Differential Inequality

Notation

Define $\Omega := (0, T) \times (0, L)$ and its parabolic boundary

$$\Gamma := \left\{(t,x) \in \bar{\Omega};\ t = 0\right\} \cup \left\{(t,x) \in \bar{\Omega};\ x = 0\right\} \cup \left\{(t,x) \in \bar{\Omega};\ x = L\right\}.$$

Proposition 1

Let $u \in C^2\left(\bar{\Omega} \setminus \Gamma\right) \cap C\left(\bar{\Omega}\right)$ satisfy

$$\partial_t u - k\partial_{xx} u < 0, \qquad \text{(Heat} < 0\text{)}$$

where the constant $k > 0$. Then

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u.$$

Proof of Proposition 1

It suffices to show that the global maximum does not attain in $\bar{\Omega} \setminus \Gamma$. Since u is continuous on a compact set $\bar{\Omega}$, it follows from the extreme value theorem that there exists a point $(t_0, x_0) \in \bar{\Omega}$ such that

$$u(t_0, x_0) = \max_{\bar{\Omega}} u.$$

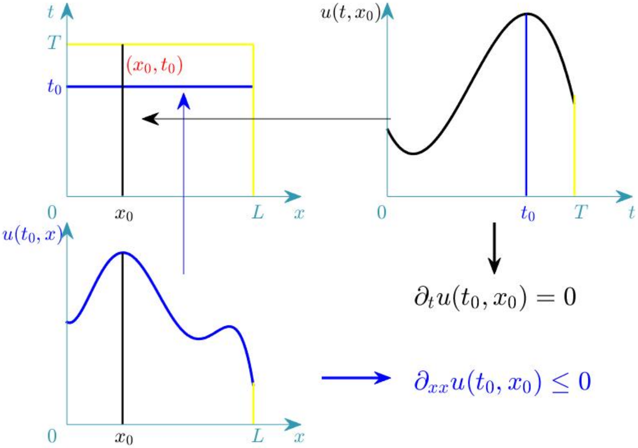
Seeking for a contradiction, we assume $(t_0, x_0) \in \bar{\Omega} \setminus \Gamma$. Either

- Case (i): $(t_0, x_0) \in (0, T) \times (0, L)$, or
- Case (ii): $t_0 = T$ and $0 < x_0 < L$.

Exercise

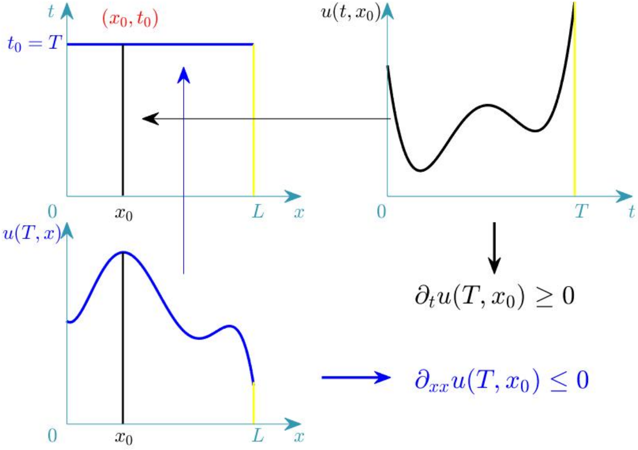
The graphical proofs for the non-existence of (t_0, x_0) in Case (i) and (ii) will be given in the following slides. Students are asked to write down your own proof rigorously.

Case (i): $(t_0, x_0) \in (0, T) \times (0, L)$.



Thus, $\partial_t u(t_0, x_0) - k\partial_{xx} u(t_0, x_0) \geq 0$, which contradicts with (Heat< 0).

Case (ii): $t_0 = T$ and $0 < x_0 < L$.



Thus, $\partial_t u(t_0, x_0) - k\partial_{xx} u(t_0, x_0) \geq 0$, which contradicts with (Heat< 0).

Proof of Proposition 1 (Continued).

Combining the analysis in both Case (i) and (ii), we conclude that $(t_0, x_0) \notin \bar{\Omega} \setminus \Gamma$, and hence, $(t_0, x_0) \in \Gamma$. Thus,

$$\max_{\bar{\Omega}} u = u(t_0, x_0) = \max_{\Gamma} u.$$

This completes the proof. $\square$

Moral

Differential structures (coming from the PDE or partial differential inequality) avoid the maximum of u to occur outside the parabolic Γ.

Remark

One may guess that "$\leq$" as the limit of "$<$", so one may try to "upgrade" the maximum principle for (Heat< 0) to that for

$$\partial_t u - k\partial_{xx} u \leq 0.$$

Proposition 2

Let $u \in C^2\left(\bar{\Omega} \setminus \Gamma\right) \cap C\left(\bar{\Omega}\right)$ satisfy

$$\partial_t u - k\partial_{xx} u \leq 0,$$

where the constant $k > 0$. Then

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u.$$

Moral

Idea: make "$\leq$" to be "$<$".
Trick: consider $v_\epsilon(t,x) := u(t,x) + \epsilon x^2$.

Proof of Proposition 2

For any $\epsilon > 0$, we define, for any $(t,x) \in \bar{\Omega}$,

$$v_\epsilon(t,x) := u(t,x) + \epsilon x^2.$$

Proof of Proposition 2 (Continued)

A direct differentiation on $v_\epsilon(t,x) := u(t,x) + \epsilon x^2$ yields

$$\partial_t v_\epsilon - k\partial_{xx} v_\epsilon = \partial_t\left(u + \epsilon x^2\right) - k\partial_{xx}\left(u + \epsilon x^2\right) = \underbrace{\partial_t u - k\partial_{xx} u}_{\leq 0} \underbrace{-2k\epsilon}_{<0} < 0.$$

Applying Proposition 1 to v_ϵ, we have

$$\begin{aligned}\max_{\bar{\Omega}} u &\leq \max_{\bar{\Omega}} (u + \underbrace{\epsilon x^2}_{\geq 0}) =: \max_{\bar{\Omega}} v_\epsilon = \max_{\Gamma} v_\epsilon := \max_{\Gamma}\left(u + \epsilon x^2\right) \\ &\leq \left(\max_{\Gamma} u\right) + \epsilon L^2.\end{aligned}$$

Passing to the limit $\epsilon \to 0^+$ in the above inequality, we obtain

$$\max_{\bar{\Omega}} u \leq \left(\max_{\Gamma} u\right) + \underbrace{\lim_{\epsilon \to 0^+} \epsilon L^2}_{=0} = \max_{\Gamma} u.$$

Proof of Proposition 2 (Continued).

Hence, due to $\Gamma \subset \bar{\Omega}$,

$$\max_{\bar{\Omega}} u \le \max_{\Gamma} u \le \max_{\bar{\Omega}} u \implies \max_{\bar{\Omega}} u = \max_{\Gamma} u.$$

This completes the proof. □

Moral

Viewing "$\le$" as the limit of "$<$", one may conjecture that

some fact holds for "$<$" may also hold for "$\le$".

Question

What can we conclude in the case of $\ge$ sign?

Corollary 3 (Minimum Principle)

Let $u \in C^2\left(\bar{\Omega} \setminus \Gamma\right) \cap C\left(\bar{\Omega}\right)$ satisfy

$$\partial_t u - k\partial_{xx} u \geq 0,$$

where the constant $k > 0$. Then

$$\min_{\bar{\Omega}} u = \min_{\Gamma} u.$$

Main Idea

- $\max_{\bar{\Omega}}(-u) = -\min_{\bar{\Omega}} u$, and
- $(\partial_t - k\partial_{xx})(-u) \leq 0$.

Proof of Corollary 3

Define $w(t,x) := -u(t,x)$, for any $(t,x) \in \bar{\Omega}$. Then

$$\partial_t w - k\partial_{xx} w = -(\partial_t u - k\partial_{xx} u) \leq 0.$$

Proof of Corollary 3 (Continued).

Applying Proposition 2 to w, we have

$$-\min_{\bar{\Omega}} u = \max_{\bar{\Omega}} w = \max_{\Gamma} w = -\min_{\Gamma} u,$$

because $w = -u$. This implies

$$\min_{\bar{\Omega}} u = \min_{\Gamma} u.$$

□

Summary

In Proposition 1 to Corollary 3, we have showed

- **Prop 1:** $\partial_t u - k\partial_{xx} u < 0 \implies \max_{\bar{\Omega}} u = \max_{\Gamma} u$.
- **Prop 2:** $\partial_t u - k\partial_{xx} u \leq 0 \implies \max_{\bar{\Omega}} u = \max_{\Gamma} u$.
- **Cor 3:** $\partial_t u - k\partial_{xx} u \geq 0 \implies \min_{\bar{\Omega}} u = \min_{\Gamma} u$.

Theorem

Let $u \in C^2\left(\bar{\Omega} \setminus \Gamma\right) \cap C\left(\bar{\Omega}\right)$ *satisfy*

$$\partial_t u - k\partial_{xx} u = 0,$$

where the constant $k > 0$. *Then*

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u, \tag{MaxP}$$

$$\min_{\bar{\Omega}} u = \min_{\Gamma} u, \tag{MinP}$$

$$\max_{\bar{\Omega}} |u| = \max_{\Gamma} |u|. \tag{MaxP$|u|$}$$

Proof.

Inequality (MaxP) and (MinP) follows directly from Proposition 2 and Corollary 3, respectively. Combining (MaxP) and (MinP), we finally obtain (MaxP$|u|$). □

4.5 Uniqueness and Stability for the Heat Equations

Application: Uniqueness and Stability for the IBVP of the Heat Equation

Define $\Omega := (0, T) \times (0, L)$, and its parabolic boundary

$$\Gamma := \left\{(t,x) \in \bar{\Omega};\ t = 0\right\} \cup \left\{(t,x) \in \bar{\Omega};\ x = 0\right\} \cup \left\{(t,x) \in \bar{\Omega};\ x = L\right\}.$$

Let the parameter $k > 0$ and the source term $f := f(t,x)$ be given. For any $i = 1$, 2, let $u_i \in C^2\left(\bar{\Omega} \setminus \Gamma\right) \cap C\left(\bar{\Omega}\right)$ satisfy

$$\begin{cases} \partial_t u_i - k\partial_{xx} u_i = f \\ \qquad\qquad u_i\big|_{t=0} = \phi_i(x) \\ \qquad\qquad u_i\big|_{x=0} = g_i(t) \\ \qquad\qquad u_i\big|_{x=L} = h_i(t), \end{cases}$$

where the initial and boundary data ϕ_i, g_i and h_i are given.

Question

How does u_i depend on ϕ_i, g_i and h_i?

Denote $\tilde{u} := u_1 - u_2$, $\tilde{\phi} := \phi_1 - \phi_2$, $\tilde{g} := g_1 - g_2$ and $\tilde{h} := h_1 - h_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} - k\partial_{xx}\tilde{u} = 0 \\ \qquad \tilde{u}|_{t=0} = \tilde{\phi}(x) \\ \qquad \tilde{u}|_{x=0} = \tilde{g}(t) \\ \qquad \tilde{u}|_{x=L} = \tilde{h}(t). \end{cases}$$

By the Maximum Principle,

$$\max_{\bar{\Omega}} |\tilde{u}| = \max \left\{ \max_{0 \le x \le L} |\tilde{\phi}(x)|, \max_{0 \le t \le T} |\tilde{g}(t)|, \max_{0 \le t \le T} |\tilde{h}(t)| \right\},$$

or equivalently,

$$\|\tilde{u}\|_{\sup,\bar{\Omega}} = \max \left\{ \|\tilde{\phi}\|_{\sup,[0,L]}, \|\tilde{g}\|_{\sup,[0,T]}, \|\tilde{h}\|_{\sup,[0,T]} \right\},$$

which is the *stability* in the sup-norm $\| \cdot \|_{\sup}$. For the *uniqueness*, if $\tilde{\phi} \equiv \tilde{g} \equiv \tilde{h} \equiv 0$, then

$$\|\tilde{u}\|_{\sup,\bar{\Omega}} = \max \left\{ \|\tilde{\phi}\|_{\sup,[0,L]}, \|\tilde{g}\|_{\sup,[0,T]}, \|\tilde{h}\|_{\sup,[0,T]} \right\} = 0,$$

and hence, $u_1 \equiv u_2$.