THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 1 Solution

Problem 1.

(i) When $m \neq n$, the integral is zero. When $m = n$,

$$\int_{-L}^{L} \sin^2 \frac{m\pi x}{L} \,\mathrm{d}\, x = \int_{-L}^{L} \frac{1}{2}(1 - \cos \frac{2m\pi x}{L}) \,\mathrm{d}\, x = L.$$

(ii) Note that

$$\frac{L}{2} - \left|\frac{L}{2} - x\right| = \begin{cases} x, & \text{for } x \in [0, \frac{L}{2}]; \\ L - x, & \text{for } x \in (\frac{L}{2}, L]. \end{cases}$$

Then the integral equals to

$$\int_{0}^{\frac{L}{2}} x \cos \frac{m\pi x}{L} \,\mathrm{d}\, x + \int_{\frac{L}{2}}^{L} (L - x) \cos \frac{m\pi x}{L} \,\mathrm{d}\, x.$$

The first integral yields

$$\int_{0}^{\frac{L}{2}} x \cos \frac{m\pi x}{L} \,\mathrm{d}\, x = \frac{L}{m\pi} \left(\frac{L}{2} \sin \frac{m\pi}{2} + \frac{L}{m\pi} \cos \frac{m\pi}{2} - \frac{L}{m\pi} \right).$$

The second integral yields

$$\int_{\frac{L}{2}}^{L} (L - x) \cos \frac{m\pi x}{L} \,\mathrm{d}\, x = \frac{L}{m\pi} \left(-\frac{L}{2} \sin \frac{m\pi}{2} - \frac{L}{m\pi} \cos m\pi + \frac{L}{m\pi} \cos \frac{m\pi}{2} \right).$$

It follows that

$$\int_{0}^{L} \left(\frac{L}{2} - \left|\frac{L}{2} - x\right| \right) \cos \frac{m\pi x}{L} \,\mathrm{d}\, x = \left(\frac{L}{m\pi} \right)^2 \left(-1 - \cos m\pi + 2 \cos \frac{m\pi}{2} \right).$$

(iii) By the Euler's formula, we have

$$\int_{-\pi}^{\pi} e^{\mathrm{i}mx}\sin mx \,\mathrm{d}\,x = \int_{-\pi}^{\pi}\sin mx\cos mx \,\mathrm{d}\,x + \mathrm{i}\int_{-\pi}^{\pi}\sin^2 mx \,\mathrm{d}\,x = i\pi$$

Problem 2. Our goal is to prove $f \equiv 0$.

We first prove $\nabla f \equiv 0$ on $[0,1]\times[0,1]$ by contradiction. Assume on the contrary that there exists $(x_0, y_0) \in [0,1]\times[0,1]$ such that $\nabla f(x_0,y_0) \neq 0$, that is, $|\nabla f(x_0,y_0)| > 0$. Since $f\colon [0,1]\times[0,1] \to \mathbb{R}$ is C^1, so $\nabla f : [0,1]\times[0,1] \to \mathbb{R}\times\mathbb{R}$ is continuous, and so as $|\nabla f|$. We can find a $\delta > 0$ such that for all $x \in B_\delta(x_0)$,

$$||\nabla f(x,y)| - |\nabla f(x_0,y_0)|| < \frac{|\nabla f(x_0,y_0)|}{2}.$$

It follows that for $(x,y) \in B_\delta(x_0,y_0)$,

$$|\nabla f(x,y)| > |\nabla f(x_0,y_0)| - \frac{|\nabla f(x_0,y_0)|}{2} = \frac{|\nabla f(x_0,y_0)|}{2} > 0,$$

yielding

$$\begin{aligned}\int_{[0,1]\times[0,1]} |\nabla f|^2 \,\mathrm{d}\,x\,\mathrm{d}\,y &\geq \int_{B_\delta(x_0)} |\nabla f|^2 \,\mathrm{d}\,x\,\mathrm{d}\,y \\ &> \int_{B_\delta(x_0)} \frac{|\nabla f(x_0,y_0)|^2}{4}\,\mathrm{d}\,x\,\mathrm{d}\,y > 0,\end{aligned}$$

Contradiction. So we deduced $\nabla f \equiv 0$ on $[0,1]\times[0,1]$, i.e. $f \equiv$ constant. By the given pointwise condition $f(0,0) = 0$, we can further conclude $f \equiv 0$.

Problem 3. By separation of variables,

$$\frac{du}{u^{1+\epsilon}} = dt,$$

thus integration on both sides gives the general solution

$$-\frac{u^{-\epsilon}}{\epsilon} = t + C.$$

Substituting the initial condition $u(0) = 1$ deduces $C = -\frac{1}{\epsilon}$. Hence,

$$u(t) = \frac{1}{(1-\epsilon t)^{\frac{1}{\epsilon}}}.$$

By setting $T = \frac{1}{\epsilon}$, we have $1 - \epsilon T = 0$, and thus $\lim\limits_{t \to T^-} u(t) = \infty$.

Problem 4.

(i) Find the root of the characteristic polynomial of A,

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 2-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0.$$

So the eigenvalues are $\lambda = 1,\ 2$ with algebraic multiplicities 2 and 1 respectively. To find their corresponding eigenfunctions, we solve

$$(A - I)x = 0,\ (A - 2I)y = 0$$

for x and y. It yields that

$$x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} s,\ y = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} t \quad \text{for } s,\ t \in \mathbb{R} \backslash \{0\}.$$

Then it suffice to take $(0,\ 1,\ 1)^T$ and $(1,\ 2,\ 1)^T$ as eigenvectors for $\lambda = 1$ and $\lambda = 2$ respectively.

(ii) The computation of the Jordan canonical form. We first write

$$P = (p_1,\ p_2,\ p_3).$$

Then we have

$$(Ap_1,\ Ap_2,\ Ap_3) = (2p_1,\ p_2,\ p_2 + p_3).$$

That is to say,

$$\begin{cases} Ap_1 = 2p_1, \\ Ap_2 = p_2, \\ (A - I)p_3 = p_2. \end{cases}$$

Take p_1 and p_2 to be the eigenvector of 2 and 1 respectively, and solve the linear system consecutively. Then we have

$$p_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 \\ t \\ t \end{pmatrix} \quad \text{for } t \in \mathbb{R} \backslash \{0\}.$$

Then it suffice to take $p_3 = (1,\ 0,\ 0)^T$, so we have the required $P = (p_1,\ p_2,\ p_3)$.

(iii) Denote the Jordan form of A by J, i.e.,

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then the general solution to the system of ODE has the form

$$\mathbf{x}(t) = e^{tA}\mathbf{x}(0),$$

where $\mathbf{x}(0)$ represents the initial condition with $t = 0$, and

$$e^{tA} = e^{tPJP^{-1}} = Pe^{tJ}P^{-1}.$$

The exponential of the Jordan from can be directly given as follows: by $e^{tI} = e^tI$,

$$e^{tJ} = \exp\left[I + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right] = e^t \exp\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right] = e^t \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

So the desired general solution is obtained. One may further explicitly write

$$e^{tA} = e^t \begin{pmatrix} 1 & e^t - 1 & -e^t + 1 \\ t & 2e^t - t - 1 & -2e^t + t + 2 \\ t & 2e^t - t - 1 & -e^t + t + 2 \end{pmatrix}.$$

Food for Thought. Find the root of the characteristic polynomial of A,

$$\det(A-\lambda I)=\begin{vmatrix}-\lambda & 1 & 0\\ -1 & 2-\lambda & 1\\ 4 & -3 & 4-\lambda\end{vmatrix}=-(\lambda-2)^3=0$$

So $\lambda=2$ is the only eigenvalue of algebraic multiplicity 2. To find its corresponding eigenfunction, we solve $(A-2I)x=0$ for x. It yields that

$$x=\begin{pmatrix}1\\2\\1\end{pmatrix}t \quad \text{for } t\in\mathbb{R}\backslash\{0\}.$$

Then it suffice to take $(1,\ 2,\ 1)^T$ as the eigenvector for $\lambda=2$.

Now compute the Jordan canonical form. We first write

$$P=(p_1,\ p_2,\ p_3).$$

Then we have

$$(Ap_1,\ Ap_2,\ Ap_3)=(2p_1,\ p_1+2p_2,\ p_2+2p_3).$$

That is to say,

$$\begin{cases}Ap_1=2p_1,\\(A-2I)p_2=p_1,\\(A-2I)p_3=p_2.\end{cases}$$

So we take p_1 to be the eigenvector of 2, and solve the linear system consecutively. Then we have

$$p_1=\begin{pmatrix}1\\2\\1\end{pmatrix},\quad p_2=\begin{pmatrix}1\\3\\3\end{pmatrix},\quad p_3=\begin{pmatrix}1\\3\\4\end{pmatrix}.$$

So we have the required $P=(p_1,\ p_2,\ p_3)$. Then the general solution to the system of ODE has the form

$$x(t)=e^{tA}x(0),$$

where

$$e^{tA} = e^{tPJP^{-1}} = Pe^{tJ}P^{-1}.$$

The exponential of the Jordan from can be directly given as follows

$$e^{tJ} = e^{2t}\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

So the desired general solution is obtained.

Problem 5.

(i) We compute the following partial derivatives

$$\begin{aligned} \partial_x u(x,y) &= \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}, \\ \partial_y u(x,y) &= -\frac{-2y^3 + 6yx^2}{(x^2+y^2)^3}, \\ \partial_{xx} u(x,y) &= \frac{6x^4 + 6y^4 - 36x^2y^2}{(x^2+y^2)^4}, \\ \partial_{yy} u(x,y) &= -\frac{6y^4 + 6^4 - 36x^2y^2}{(x^2+y^2)^4}. \end{aligned}$$

Then it is straightforward to verify that $\Delta u = 0$ and $\partial_y u|_{y=0} = 0$.

(ii) We compute the following partial derivatives

$$\begin{aligned} \partial_t w &= -16\,\mathrm{sech}^2(x-4t)\tanh(x-4t), \\ \partial_x w &= 4\,\mathrm{sech}^2(x-4t)\tanh(x-4t), \\ \partial_{xx} w &= 4\,\mathrm{sech}^4(x-4t) - 8\,\mathrm{sech}^2(x-4t)\tanh^2(x-4t), \\ \partial_{xxx} w &= -32\,\mathrm{sech}^4(x-4t)\tanh(x-4t) + 16\,\mathrm{sech}^2(x-4t)\tanh^3(x-4t). \end{aligned}$$

Then it is straightforward to verify that w satisfies the KdV equation:

$$\partial_t w = 6w\partial_x w - \partial_{xxx} w.$$

(iii) We denote the concerned part by

$$h(t,x,y) := \frac{1}{t} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}},$$

and compute its partial derivatives

$$\begin{aligned}
\partial_t h &= -\frac{1}{t^2} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} + \frac{1}{4t^3} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} \left[(x-\tilde{x})^2 + (y-\tilde{y})^2\right],\\
\partial_x h &= -\frac{1}{2t^2} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} (x-\tilde{x}),\\
\partial_{xx} h &= -\frac{1}{2t^2} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} + \frac{1}{4t^3} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} (x-\tilde{x})^2,\\
\partial_{yy} h &= -\frac{1}{2t^2} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} + \frac{1}{4t^3} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} (y-\tilde{y})^2.
\end{aligned}$$

And thus,

$$\Delta h = -\frac{1}{t^2} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} + \frac{1}{4t^3} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} \left[(x-\tilde{x})^2 + (y-\tilde{y})^2\right] = \partial_t h.$$

Then the original $v(t,x,y)$ satisfies

$$\begin{aligned}
\partial_t v &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \operatorname{erf} \tilde{y} \int_{-\infty}^{\infty} \frac{1}{1+\tilde{x}^4} (\partial_t h) \, \mathrm{d}\,\tilde{x} \, \mathrm{d}\,\tilde{y}\\
&= \frac{1}{4\pi} \int_{-\infty}^{\infty} \operatorname{erf} \tilde{y} \int_{-\infty}^{\infty} \frac{1}{1+\tilde{x}^4} (\Delta h) \, \mathrm{d}\,\tilde{x} \, \mathrm{d}\,\tilde{y} = \Delta v
\end{aligned}$$

as suggested.

Problem 6. Prove by contradiction. Assume on the contrary that there exists $x_0 \in \Omega$ such that $f(x_0) \neq 0$. Without loss of generality, we assume further that $f(x_0) > 0$. Since $f\colon \Omega \to \mathbb{R}$ is continuous, we can find a $\delta > 0$ such that for all $x \in B_\delta(x_0)$,

$$|f(x) - f(x_0)| < \frac{f(x_0)}{2}.$$

It follows that

$$f(x) > f(x_0) - \frac{f(x_0)}{2} > \frac{f(x_0)}{2} = 0 \quad \text{for } x \in B_\delta(x_0).$$

Then since $p > 0$ we have

$$\int_{\Omega} |f(x)|^p \,\mathrm{d}\,x \geq \int_{B_\delta(x_0)} |f(x)|^p \,\mathrm{d}\,x > \int_{B_\delta(x_0)} \frac{|f(x_0)|^p}{2^p} \,\mathrm{d}\,x > 0,$$

Contradiction. So we conclude that $f \equiv 0$ on Ω.

Food for Thought. The assertion in general may not be true if f is only integrable. Consider the case when $p = d = 1$, $\Omega = (-1, 1)$, and $f : \Omega \mapsto \mathbb{R}$ defined as follows

$$f(x) := \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise.} \end{cases}$$

We have $\int_{\Omega} f(x)\, dx = 0$, but $f \not\equiv 0$.

Problem 7.

(i) Given $(x_1, x_2) \in \mathbb{R}^2$, let $F = F(x_1, x_2) = \big(f_1(x_1, x_2), f_2(x_1, x_2)\big)$ and $g = g(x_1, x_2)$. Then

$$\begin{aligned} \nabla \cdot (Fg) &= \partial_{x_1}(f_1 g) + \partial_{x_2}(f_2 g) = g\partial_{x_1} f_1 + f_1 \partial_{x_1} g + g\partial_{x_2} f_2 + f_2 \partial_{x_2} g \\ &= g(\partial_{x_1} f_1 + \partial_{x_2} f_2) + (f_1 \partial_{x_1} g + f_2 \partial_{x_2} g) \\ &= g\nabla \cdot F + (f_1, f_2) \cdot (\partial_{x_1} g, \partial_{x_2} g) = g\nabla \cdot F + F \cdot \nabla g. \end{aligned}$$

(ii) Given $(x, y) \in \mathbb{R}^2$, let $F = F(x, y)$ and $g = g(x, y)$. Then

$$\begin{aligned} \oint_{\partial\Omega} gF \cdot n \, d\sigma &= \iint_{\Omega} \nabla \cdot (Fg) \, dxdy \quad \text{(by the divergence theorem)} \\ &= \iint_{\Omega} (g\nabla \cdot F + F \cdot \nabla g) \, dxdy \text{ (by (i)).} \end{aligned}$$

Thus,

$$\iint_{\Omega} g\nabla \cdot F \, dxdy = \oint_{\partial\Omega} gF \cdot n \, d\sigma - \iint_{\Omega} F \cdot \nabla g \, dxdy.$$

(iii) If $F(x,y) = \big(f(x,y), 0\big)$ and $n = (n_1, n_2)$, then

$$\nabla \cdot F = \partial_x f + \partial_y(0) = \partial_x f, \ \ F \cdot n = f n_1 \text{ and } F \cdot \nabla g = f\partial_x g + (0)\partial_y g = f\partial_x g.$$

By (ii), we have

$$\iint_\Omega g\partial_x f \, dxdy = \oint_{\partial\Omega} g f n_1 \, d\sigma - \iint_\Omega f\partial_x g \, dxdy.$$

Food for Thought. The generalized result in Part (iii) of Problem 7 to higher dimensional cases can be found by repeating the proofs in Parts (i) and (ii) on $\mathbb{R}^n$. Here we provide the general proof of this problem.

Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $F = F(x) = \big(f_1(x), \ldots, f_n(x)\big)$ be a smooth vector field and $g = g(x)$ be a smooth scalar-valued function. Then

$$\begin{aligned} \nabla \cdot (Fg) &= \sum_{i=1}^n \partial_{x_i}(f_i g) = \sum_{i=1}^n g\partial_{x_i} f_i + f_i \partial_{x_i} g = g\sum_{i=1}^n \partial_{x_i} f_i + \sum_{i=1}^n f_i \partial_{x_i} g \\ &= g\nabla \cdot F + (f_1, \ldots, f_n) \cdot (\partial_{x_1} g, \ldots, \partial_{x_n} g) = g\nabla \cdot F + F \cdot \nabla g. \end{aligned}$$

Then by the divergence theorem, for any smooth and bounded region $\Omega \subseteq \mathbb{R}^n$,

$$\oint_{\partial\Omega} gF \cdot n \, d\sigma = \iint_\Omega \nabla \cdot (Fg) \, dx = \iint_\Omega (g\nabla \cdot F + F \cdot \nabla g) \, dx.$$

Thus,

$$\iint_\Omega g\nabla \cdot F \, dx = \oint_{\partial\Omega} gF \cdot n \, d\sigma - \iint_\Omega F \cdot \nabla g \, dx.$$

If we let $F = \nabla\psi$, where $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ are smooth scalar-valued functions, then we can further obtain the Green's first identity,

$$\oint_{\partial\Omega} g(\nabla\psi \cdot n) \, d\sigma = \iint_\Omega (g\nabla \cdot (\nabla\psi) + \nabla\psi \cdot \nabla g) \, dx = \iint_\Omega (g\Delta\psi + \nabla\psi \cdot \nabla g) \, dx.$$