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19.8/20

Q1. When the ideal $\langle x-1\rangle$ is prime in $R[x]$, $R[x]/\langle x-1\rangle$ is $\boxed{\text{an integral domain}}$.

As $\langle x-1\rangle \neq \{0\}$, this is equivalent to $x-1$ is a prime element in $R[x]$.

Express every polynomial $f(x)=\sum_{k=0}^{n} a_k(x-1)^k$, so this is equivalent to $a_0 b_0=0 \Rightarrow a_0=0$ or $b_0=0$

Hence, $R[x]/\langle x-1\rangle$ is an integral domain iff $\boxed{R \text{ is an integral domain}}$.

When the ideal $\langle x-1\rangle$ is maximal in $R[x]$, $R[x]/\langle x-1\rangle$ is a $\boxed{\text{field}}$.

Actually this is equivalent to R is a field, because:

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$\langle x-1\rangle$ is maximal in $R[x] \Leftrightarrow (\forall$ ideal J of $R[x]$, $J \supsetneq \langle x-1\rangle \Rightarrow J=R[x])$

$\Leftrightarrow \forall$ nonzero $r\in R$, $\langle r, x-1\rangle = R[x]$

$\Leftrightarrow \forall$ nonzero $r\in R$, r is a unit $\Leftrightarrow \boxed{R \text{ is a field}}$.

~~If R is a field, then R[x] is Euclidean, thus PID and UFD.~~

~~If R[x]~~

Define $\pi: R[x]\to R$, $f(x)\mapsto f(1)$

As $\pi(1)=1$, $\pi(f(x)+g(x)) = \pi\left(\sum_{i=0}^{\max\{m,n\}}(a_i+b_i)x^i\right) = \sum_{i=0}^{\max\{m,n\}}(a_i+b_i) = f(1)+g(1)$

$$\pi(f(x)g(x)) = \pi\left(\sum_{i,j=0,0}^{m,n} a_i b_j x^{i+j}\right) = \sum_{i,j=0,0}^{m,n} a_i b_j = f(1)g(1),$$

π is a ring homomorphism.

π fixes $R\subseteq R[x]$, so $\mathrm{Im}(\pi)=R$.

$\mathrm{Ker}(\pi) = \{f(x)\in R[x] : f(1)=0\} = \langle x-1\rangle$.

Hence, $R[x]/\langle x-1\rangle \cong R$, so $R[x]/\langle x-1\rangle$ is integral/PID/UFD/field iff R is.

if we have another $s_1 s_2 \cdots s_m$, then $n=m$, and for some $\sigma \in S_n$, ~~$r_1 = s_{\sigma(1)}$~~ $r_1 = s_{\sigma(1)}$
$r_2 = s_{\sigma(2)}$
$\vdots$
$r_n = s_{\sigma(m)}$.

Q2 (a) Let R be an integral domain.

If every nonzero, nonunit element r of R is a unique product ~~$r_1 r$~~ $r_1 \cdot r_2 \cdots r_n$ of irreducible elements $r_1, r_2, \cdots, r_n$ of R up to association relation ($r_1 \sim r_2$ iff $\exists u \in R^\times$, $r_1 = r_2 u$), then R is a UFD.

unique in what sense

(b) Proof: Assume that $R[x]$ is a UFD.

For all nonzero, nonunit element r of R, identify it with a constant polynomial in $R[x]$

By assumption, $r = r_1(x) r_2(x) \cdots r_n(x)$ is a unique product of irreducible polynomials in $R[x]$ up to associates.

Now $0 = \underline{\mathrm{Deg}\, r = \mathrm{Deg}\, r_1(x) + \mathrm{Deg}\, r_2(x) + \cdots + \mathrm{Deg}\, r_n(x)}$; This is correct because UFD $\subseteq$ Integral Domain

which implies $r_1(x), r_2(x), \cdots, r_n(x)$ are irreducibles in R, and we're done.

Q3: Let R be a UFD. If a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ is nonzero, and the greatest common divisor of $(a_i)_{i=0}^{n}$ is 1 (up to associates), then $f(x)$ is primitive. Gauss Lemma says, a product $f(x)g(x)$ of primitive polynomials $f(x), g(x) \in R[x]$ is primitive.

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Proof: Assume to the contrary that $h(x) = f(x)g(x)$ is not primitive.

As R is a UFD, irreducibles = primes, for some prime element $p \in R$, p divides all coefficients of $f(x)g(x)$. Consider the following ring homomorphism: $\pi_p: R[x] \to R[x]/\langle p \rangle$, $\pi_p(f(x)) = \bar{f}_p(x)$.

$$h(x) = f(x)g(x) \Rightarrow 0 = \pi_p h(x) = \pi_p(f(x)g(x)) = \pi_p(f(x))\pi_p(g(x))$$

$$\Rightarrow \pi_p(f(x)) = 0 \text{ or } \pi_p(g(x)) = 0, \text{ contradicting to primitivety!}$$

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Q4. (4) Proof: As $f(x)=x^3+2x+1$ is a cubic polynomial over field $\mathbb{F}_5=\{0,1,2,3,4\}$, it suffices to show that it has no root. (content is always a unit).

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x	$f(x)$
0	$0^3+2\cdot0+1=1\neq0$
1	$1^3+2\cdot1+1=4\neq0$.
2	$2^3+2\cdot2+1=13=3\neq0$.
3	$3^3+2\cdot3+1=34=4\neq0$.
4	$4^3+2\cdot4+1=73=3\neq0$.

In this case, $\mathbb{F}_5[x]/\langle f(x)\rangle$ is a 3-dimensional $\mathbb{F}_5$-vector space,

$$\left|\mathbb{F}_5[x]/\langle f(x)\rangle\right|=|\mathbb{F}_5|^3=125$$

Q5. (a) Let R be a PID, and $A=(a_{ij})$ be a n by m R-valued matrix.

There exists a unique chain $d_1\leq d_2\leq\cdots\leq d_k\leq0\leq0\leq\cdots$

up to associates | $a\leq b$ iff $a\mid b$ OK ? ?

-0.2 K unique?

such that for some $P\in GL_n(R)$, $Q\in GL_m(R)$:

$$PAQ=\begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & d_2 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & d_3 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_k & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \end{pmatrix}$$

(b) $\begin{pmatrix} -7 & 0 & -6 \\ 6 & 3 & 0 \\ 6 & 0 & 6 \end{pmatrix} \xrightarrow[\text{Column}①+\text{Column}②\times(-2)]{\text{Column}①+\text{Column}③\times(-1)} \begin{pmatrix} -1 & 0 & -6 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$\xrightarrow{\text{Column}③+\text{Column}①\times(-6)} \begin{pmatrix} \underset{d_1}{-1} & 0 & 0 \\ 0 & \underset{d_2}{3} & 0 \\ 0 & 0 & \underset{d_3}{6} \end{pmatrix}$ $d_1\leq d_2\leq d_3$

if you like, $d_1\mid d_2\mid d_3$.

3 (c) Solution: As column transformation preserves column space.

$$N = Col\begin{pmatrix} -7 & 0 & -6 \\ 6 & 3 & 0 \\ 6 & 0 & 6 \end{pmatrix} = Col\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \mathbb{Z} \times (3\mathbb{Z}) \times (6\mathbb{Z}).$$

$$\mathbb{Z}^3/N = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}) / [\mathbb{Z} \times (3\mathbb{Z}) \times (6\mathbb{Z})]$$

$$\cong (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$$