THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 2 Solution

Problem 1.

(i) Assume a solution is of the form $u(x) = e^{\alpha x}$. Then

$$\alpha^2 e^{\alpha x} - 2\alpha e^{\alpha x} + 2e^{\alpha x} = 0 \Longrightarrow \alpha^2 - 2\alpha + 2 = 0$$

and hence $\alpha = 1 + i$ or $1 - i$. Thus, the general solution is of the form

$$u(x) = Ae^{(1+i)x} + Be^{(1-i)x} = Ce^x \cos x + De^x \sin x, \ \ A, B, C, D \in \mathbb{R}$$

(ii) As $u(0) = 0$, we have $C = 0$ and hence $u(x) = De^x \sin x$. Moreover, $u'(L) = 0$ implies

$$D(\sin L + \cos L) = 0$$

$(\Leftarrow)$: If $L \neq \frac{(4k-1)\pi}{4}$, then $\sin L + \cos L \neq 0$ and hence $D = 0$. So $u \equiv 0$ is the unique solution.
$(\Rightarrow)$: If $L = \frac{(4k-1)\pi}{4}$, then $\sin L + \cos L = 0$ and hence D is arbitrary. So the solution $u(x) = De^x \sin x$ is not unique.

Problem 2.

(i) Note that $2u + \partial_y u = g(y)$, where g is any arbitrary function. The integrating factor is $\mu(y) = e^{\int 2dy} = e^{2y}$ and hence

$$\partial_y \left[e^{2y} u\right] = e^{2y} \left(2u + \partial_y u\right) = e^{2y} g(y).$$

After integration with respect to y, we get

$$\begin{aligned} e^{2y} u &= f(x) + \int e^{2y} g(y) dy \\ \Longrightarrow u(x, y) &= e^{-2y} f(x) + e^{-2y} \int e^{2y} g(y) dy = e^{-2y} f(x) + G(y). \end{aligned}$$

where f is any arbitrary function and $G(y) = e^{-2y} \int e^{2y} g(y) dy$.

(a) False. Take $f(x) = x + 1$ and $g(y) = 0$, then $u(x,y) = (x+1)e^{-2y}$ and $v(x,y) = xe^{-4y}$. But

$$2\partial_x v + \partial_{xy} v = 2e^{-4y} - 4e^{-4y} \neq 0.$$

(b) False. Take $f(x) = x^2$ and $g(y) = 0$, then $u(x,y) = x^2 e^{-2y}$ and $v(x,y) = 2xye^{-2y}$. But

$$2\partial_x v + \partial_{xy} v = 4ye^{-2y} + 2e^{-2y} - 4ye^{-2y} \neq 0.$$

(c) True. For any $g : \mathbb{R} \to \mathbb{R}$ and for any $x, y \in \mathbb{R}$,

$$\begin{aligned}
[2\partial_x v + \partial_{xy} v](x,y) &= [2\partial_x + \partial_{xy}]\, v(x,y) \\
&= [2\partial_x + \partial_{xy}] \int_{-\infty}^{\infty} u(x-t,y) g(t) dt \\
&= \int_{-\infty}^{\infty} [2\partial_x + \partial_{xy}] [u(x-t,y)g(t)] dt \\
&= \int_{-\infty}^{\infty} \{[2\partial_x + \partial_{xy}]\, u(x-t,y)\}\, g(t) dt \\
&= 0.
\end{aligned}$$

Problem 3. Let $v := \partial_x u$. Then

$$\frac{2}{y+1} v + \partial_y v = \frac{e^y}{(y+1)^2}.$$

The integrating factor is $\mu(y) = e^{\int \frac{2}{y+1} dy} = (y+1)^2$ and hence

$$\partial_y \left[(y+1)^2 v\right] = 2(y+1)v + (y+1)^2 \partial_y v = e^y.$$

After integration with respect to y, we get

$$(y+1)^2 v = e^y + f(x) \Longrightarrow v = \frac{e^y + f(x)}{(y+1)^2},$$

where f is any arbitrary function. Now integrating v respect to x, we obtain

$$u(x,y) = \frac{[e^y x + F(x)]}{(y+1)^2} + g(y),$$

where $F(x) = \int f(x)\,dx$ is any arbitrary function. Then

$$u\big|_{x=0} = u\big|_{y=0} = 0 \Longrightarrow \begin{cases} x + F(x) + g(0) = 0 \\ \frac{F(0)}{(y+1)^2} + g(y) = 0 \end{cases} \Longrightarrow \begin{cases} F(x) = -x - g(0) \\ g(y) = -\frac{F(0)}{(y+1)^2} \end{cases}.$$

Moreover, $F(0) + g(0) = u(0,0) = 0$. Thus

$$u(x,y) = \frac{e^y x - x - g(0)}{(y+1)^2} - \frac{F(0)}{(y+1)^2} = \frac{(e^y - 1)\,x}{(y+1)^2}.$$

Problem 4. Let $u_n(t,x) \coloneqq \frac{1}{n}\cosh(n\pi t)\sin(n\pi x)$.

(i) As $\partial_{xx}\sin(n\pi x) = -n^2\pi^2\sin(n\pi x)$ and $\partial_{tt}\cosh(n\pi t) = n^2\pi^2\cosh(n\pi t)$, we have

$$\partial_{xx}u_n + \partial_{tt}u_n = -n\pi^2\sin(n\pi x)\cosh(n\pi t) + n\pi^2\sin(n\pi x)\cosh(n\pi t) = 0$$

The given initial and boundary conditions follow from

$$\begin{cases} \sin(0) = \sin(n\pi) = 0 \\ \cosh(0) = 1 \text{ and } \partial_t\cosh(0) = \sinh(0) = 0. \end{cases}$$

(ii) Since $\sup\limits_{x\in[0,1]}|\sin(n\pi x)| = \max\limits_{x\in[0,1]}|\sin(n\pi x)| = 1$,

$$\|f_n\|_{\sup} = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

and for fixed $T > 0$, it follows from L'Hôpital's rule that

$$\lim_{n\to+\infty}\|u_n(T,\cdot)\|_{\sup} = \lim_{n\to+\infty}\frac{\cosh(n\pi T)}{n} = \pi T\lim_{n\to+\infty}\sinh(n\pi T) = +\infty.$$

Problem 5. (i) Recall that $u_k(t,x) \coloneqq u(k^2t, kx)$ then

$$\partial_t u_k - \partial_{xx}u_k = \partial_t u(k^2t,kx) - \partial_{xx}u(k^2t,kx) = k^2(\partial_t u(k^2t,kx) - \partial_{xx}u(k^2t,kx)) = 0.$$

(ii) Recall that u_1, u_2 are solutions to the equation. Then it follows that

$$\partial_t(\alpha u_1 + \beta u_2) - \partial_{xx}(\alpha u_1 + \beta u_2) = \alpha(\partial_t u_1 - \partial_{xx}u_1) + \beta(\partial_t u_2 - \partial_{xx}u_2) = 0.$$

Problem 6.

(i) Using the chain rule, for $i = 1, 2$,

$$\partial_{x_i} u = \partial_{y_1} u \cdot \partial_{x_i} y_1 + \partial_{y_2} u \cdot \partial_{x_i} y_2.$$

Thus

$$\begin{aligned}\partial_{x_i}^2 u &= \partial_{x_i}\left(\partial_{y_1} u\right) \cdot \partial_{x_i} y_1 + \partial_{y_1} u \cdot \partial_{x_i}^2 y_1 + \partial_{x_i}\left(\partial_{y_2} u\right) \cdot \partial_{x_i} y_2 + \partial_{y_2} u \cdot \partial_{x_i}^2 y_2 \\ &= \partial_{y_1}^2 u \cdot \left(\partial_{x_i} y_1\right)^2 + \partial_{y_1} u \cdot \partial_{x_i}^2 y_1 + 2\partial_{y_1 y_2} u \cdot \partial_{x_i} y_1 \cdot \partial_{x_i} y_2 \\ &\quad + \partial_{y_2}^2 u \cdot \left(\partial_{x_i} y_2\right)^2 + \partial_{y_2} u \cdot \partial_{x_i}^2 y_2,\end{aligned}$$

where the last equality holds because for $j = 1, 2$, by the chain rule,

$$\partial_{x_i}\left(\partial_{y_j} u\right) = \partial_{y_1 y_j} u \cdot \partial_{x_i} y_1 + \partial_{y_2 y_j} u \cdot \partial_{x_i} y_2.$$

(ii) It follows from $y_1 = x_1 - x_2$ and $y_2 = x_1 + x_2$ that

$$\partial_{x_1}^2 y_1 = \partial_{x_1}^2 y_2 = \partial_{x_2}^2 y_1 = \partial_{x_2}^2 y_2 = 0$$

and

$$\partial_{x_1} y_1 = \partial_{x_1} y_2 = \partial_{x_2} y_2 = 1; \partial_{x_2} y_1 = -1.$$

Using (i),

$$\partial_{x_1}^2 u + \partial_{x_2}^2 u = \left(\partial_{y_1}^2 u + 2\partial_{y_1 y_2} u + \partial_{y_2}^2 u\right) + \left(\partial_{y_1}^2 u - 2\partial_{y_1 y_2} u + \partial_{y_2}^2 u\right) = 2\left(\partial_{y_1}^2 u + \partial_{y_2}^2 u\right) = 2.$$