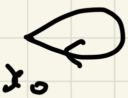
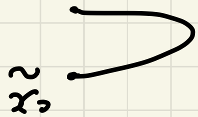
More properties of covering map

1) $p_*:\pi_1(\tilde{X},\tilde{x}_0)\longrightarrow \pi_1(X, x_0)$, $x_0 = p(\tilde{x}_0)$

is injective.

2) α  $\tilde{\alpha}$ 

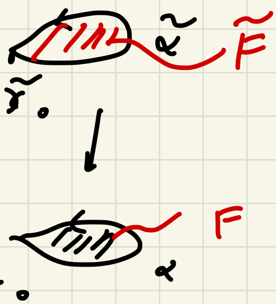
$\tilde{\alpha}(1)=\tilde{x}_0 \iff [\alpha]\in p_*(\pi_1(\tilde{X},\tilde{x}_0))$

3) for $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$

$p_*\pi_1(\tilde{X},\tilde{x}_0)$ and $p_*\pi_1(\tilde{X},\tilde{x}_1)$

are conjugate subgroups if $\tilde{X}$ is path conn.

Pf.



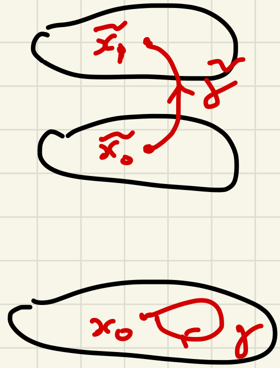
1) $\alpha \overset{F}{\sim} e_{x_0}$

$\tilde{\alpha} \overset{\tilde{F}}{\sim} e_{\tilde{x}_0}$

by homotopy lifting.

2) if $\tilde{\alpha}(1)=\tilde{x}_0$ since $\alpha=p\circ\tilde{\alpha}$ $[\alpha]\in \operatorname{im} p_*$

if $\alpha = p\circ\tilde{\tilde{\alpha}}$ by uniqueness of lifting $\tilde{\alpha}=\tilde{\tilde{\alpha}}$

3) 

$$\pi_1(\tilde X, \tilde x_0) \longrightarrow \pi_1(\tilde X, \tilde x_1)$$
$$\tilde\alpha \longmapsto \tilde\gamma^{-1}\cdot\tilde\alpha\cdot\tilde\gamma$$

set $\gamma = p\circ\tilde\gamma$

$$[\alpha] \mapsto [\gamma]\cdot[\alpha]\,[\gamma]^{-1}$$

Note that $[\gamma]$ can be taken to be any element of $\pi_1(X, x_0)$ by varying $\tilde x_1$.

4) [Universal property of covering map]

$p : \tilde X \to X$ covering map $\tilde X, X$ path conn.

Y path conn. & locally path conn.

$\forall$ continuous map $f : Y \to X$

$y_0 \in Y$ $\quad x_0 = f(y_0)$ $\quad \tilde x_0 \in p^{-1}(x_0)$

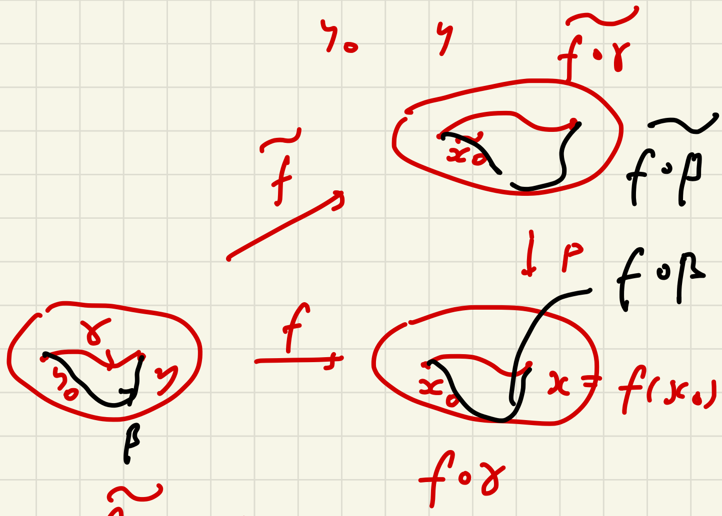
$\exists!$ lifting $\tilde f : Y \to \tilde X$ s.t

$p\circ\tilde f = f$

$$\begin{tikzcd} & \tilde X \arrow[d, "p"] & \tilde x_0 \arrow[ddl, bend left] \\ Y \arrow[ur, "\exists!\,\tilde f"] \arrow[r, "f"] & X & \\ y_0 \arrow[r, mapsto] & x_0 & \end{tikzcd}$$

iff $\quad f_*\pi_1(Y, y_0) \subset p_*\pi_1(\tilde X, \tilde x_0)$

Pf. $y \in Y$ let $y_0 \overset{\gamma}{\sim} y$



$\tilde{f}(y) := \widetilde{f\circ\gamma}(1)$

Need to show

1) $\tilde{f}(y)$ is independent of choice of γ

2) $\tilde{f}$ is continuous (Need to use locally path conn.)

Def A covering map $p: \tilde{X} \to X$ is called a universal covering map if $\tilde{X}$ is simply conn & locally path conn.

Deck transformation

Monodromy action can be "generalized"

$p : \tilde{X} \longrightarrow X$ covering

$$D(p) = \left\{ \begin{tikzcd} \tilde{X} \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & \circlearrowright & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd} \right\}$$ is a group.

Def'n $p : \tilde{X} \to X$ is called *normal*

if $D(p) \times \tilde{X} \longrightarrow \tilde{X}$ is transitive,

at every fiber.

Examples 1) $e^{2\pi i\theta}$ $\mathbb{R} \to \mathbb{Z}$ is a universal covering

2) Universal cover if exists

is unique up to homeomorphism.

(prove using universal property)

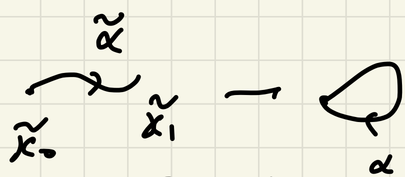
Suppose $\begin{array}{ccc} \tilde{x}_0 & \tilde{X} & \\ \downarrow & \downarrow p & \\ x_0 & X & \end{array}$ is path conn. locally path conn. covering map

Thm $H := p_* \pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$

a) p is normal $\iff$ $H \triangleleft \pi_1(X, x_0)$

b) $D(p) \cong N(H)/H$ $\quad N(H) =$ normalizer

p.f. a) "$\Rightarrow$" $\forall\ \tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$



$$[\alpha]\, p_*(\pi_1(\tilde{X}, x_0))\, [\alpha]^{-1} = p_* \pi_1(\tilde{X}, \tilde{x}_1)$$

take $h : \tilde{X} \to \tilde{X}$ s.t $h(\tilde{x}_0) = \tilde{x}_1$ since p is normal

$$p_* h_* \pi_1(\tilde{X}, \tilde{x}_0) = p_* \pi_1(\tilde{X}, \tilde{x}_1) = p_* \pi_1(\tilde{X}, \tilde{x}_0)$$

$\Rightarrow H \triangleleft \pi_1(X, x_0)$

"$\Leftarrow$" $H \triangleleft \pi_1(X, x_0) \Rightarrow \forall\ \tilde{x}_1, \tilde{x}_0 \in p^{-1}(x_0)$

$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ $\quad \exists\, h$, s.t $h(x_0) = x_1$

by lifting property

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, dashed, "\exists h", "\cong"'] \arrow[r, "p"'] & X \end{tikzcd}$$

b) $h: N(H) \longrightarrow D(p)$

$\alpha \in N(H)$ $\quad \tilde{\alpha}(0) = \tilde{x}_0 \quad \tilde{\alpha}(1) = \tilde{x}_1$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] & \tilde{x}_1 \arrow[d, bend left] \\ \tilde{X} \arrow[ur, "\exists! h"] \arrow[r, "p"] & X & \\ \tilde{x}_0 \arrow[r] & x_0 & \end{tikzcd}$$

by part (a) $\alpha \in N(H) \Rightarrow p_* \pi_1(\tilde{X}, \tilde{x}_0) = p_* \pi_1(\tilde{X}, \tilde{x}_1)$

by univ. lifting property $\exists!\ h_\alpha : \tilde{X} \to \tilde{X}$ $\quad h_\alpha(\tilde{x}_0) = \tilde{x}_1$

$\tilde{x}_0 \sim \tilde{x}_1 \iff [\alpha] \in H$

$h \in D(p) \quad h(\tilde{x}_0) = \tilde{x}_1 \quad h = h_\alpha .$

Cor if $p: \tilde{X} \to X$ is a universal cover

then $D(p) = \pi_1(X)$

Thm $G \curvearrowright \tilde{X}$ is a properly discontinuous action.

then $p: \tilde{X} \longrightarrow \tilde{X}/G$ is a covering map

If G is locally path-conn. simply conn.

then this is a universal cover.

pf. $\forall\, \tilde{x} \in \tilde{X} \quad \exists\ \tilde{x} \in \overset{\text{open}}{\tilde{U}} \subset \tilde{X}$ s.t.

$g\tilde{U} \cap \tilde{U} = \emptyset \quad \forall g \neq e$

Since $p : \tilde{X} \to \tilde{X}/G$ is open

$U := p(\tilde{u})$ is open nbhd of $p(\tilde{x})$

$p^{-1}U = \bigsqcup_{g \in G} g\tilde{u}$ $\quad p|_{\tilde{u}} : \tilde{u} \xrightarrow{\cong} U$.

∎

Thm if G acts on $\tilde{X}$ properly discontinuously and $\tilde{X}$ is path conn. loc. path conn.

then $D(p) \cong G$, if $\tilde{X}$ is simply conn. $\pi_1(\tilde{X}/G) \cong G$

p.f.

$$\phi : G \hookrightarrow D(p)$$

$$g \longmapsto \left\{ \begin{tikzcd} \tilde{X} \arrow[rr, "g\cdot"] \arrow[dr] & & \tilde{X} \arrow[dl] \\ & \tilde{X}/G & \end{tikzcd} \right\}$$

Need to show ϕ is surjective

Given
$$\begin{tikzcd} \tilde{X} \arrow[rr, "\tilde{f}"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{X} \arrow[dl, "p"] \\ & \tilde{X}/G & \end{tikzcd}$$

$\tilde{f}(\tilde{x}_0) \in p^{-1}(\tilde{x}_0) = G\tilde{x}_0$

Since G action is free $\exists!\, g$ s.t. $\tilde{f}(\tilde{x}_0) = g\,\tilde{x}_0$

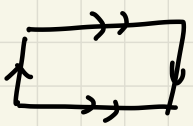
Need to check this is independent of $\tilde{x}_0$.

Set $\bar{f} = g^{-1} \circ f \in D(p)$ $\quad \bar{f}(\tilde{x}_0) = \tilde{x}_0$

by universal property $\bar{f} = id$. ∎

Applications

1) $\pi_1(\mathbb{RP}^n) = \mu_2 \quad n > 1$

2) $K =$  $= \mathbb{R}^2 / G \quad G = \langle a, b \mid abab^{-1} = 1 \rangle$

$$\pi_1(K) = G$$

3) $\pi_1(L(p,q)) = \mu_p$

$$L(p,q) = S^3 \big/ (z_1, z_2) \sim (\zeta_p z_1, \zeta_p^q z_2)$$

4) $T = S^1 \times S^1$

we will construct a covering map

$p : T \to K$.

Recall $K = \mathbb{R}^2/G$ $\quad G = \langle a, b \mid aba = b \rangle$

$$ab^2 = ba^{-1}b = b^2 a$$

$H = \langle a, b^2 \rangle \subset G$ $\qquad H \cong \mathbb{Z} \times \mathbb{Z}$

Show that $\mathbb{R}^2/H \to \mathbb{R}^2/G$ is a covering map

$$(e^{i\theta_1}, e^{i\theta_2})$$
$$\Downarrow$$
$$(-e^{i\theta_1}, e^{-i\theta_2})$$