ASSIGNMENT I, ALGEBRA II, HKU, SPRING 2021
DUE AT 10PM ON MONDAY FEBRUARY 01, 2021

(1) Show a non-zero commutative ring with finitely many elements is a field if and only if it is an integral domain.

(2) Let $n \geq 1$ be any integer. Describe all the units in the ring $\mathbb{Z}/n\mathbb{Z}$, and show that every non-zero element in $\mathbb{Z}/n\mathbb{Z}$ is either a unit or a zero divisor.

(3) Show that $(\mathbb{Z}[\sqrt{-1}], v)$ is an Euclidean domain, where

$$\mathbb{Z}[\sqrt{-1}] = \{m + n\sqrt{-1} : \; m, n \in \mathbb{Z}\}$$

is the ring of Gauss integers and $v : \mathbb{Z}[\sqrt{-1}]\backslash\{0\} \longrightarrow \mathbb{N}$ is given by

$$v(m + n\sqrt{-1}) = m^2 + n^2, \quad m + n\sqrt{-1} \in \mathbb{Z}[\sqrt{-1}]\backslash\{0\}.$$

(4) Consider the sub-ring $D = \mathbb{Z}[\sqrt{2}]$ of $\mathbb{R}$, where

$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : \; a, b \in \mathbb{Z}\}$$

with the function $v : D\backslash\{0\} \to \mathbb{N}$ given by $v(a + b\sqrt{2}) = |a^2 - 2b^2|$. Show that (D, v) is an Euclidean domain.

(5) For a commutative ring R, denote by $R^\star$ the group of all units in R. Show that the group $(\mathbb{Z}/7\mathbb{Z})^\star$ is cyclic and find one of its generators.

(6) Describe all the irreducible elements in $\mathbb{Z}[\sqrt{-1}]$; Classify all prime ideals and all maximal ideals of $\mathbb{Z}[\sqrt{-1}]$.

(7) Describe all the irreducible elements in $\mathbb{R}[x]$; Classify all prime ideals and all maximal ideals of $\mathbb{R}[x]$.

(8) Suppose that R is a PID but is not a field. Show that $R[x]$ is not a PID.

(9) Describe all ideals, prime ideals, and maximal ideals of the ring $\mathbb{Z}/n\mathbb{Z}$ for any integer $n \geq 2$.

(10) Compute a greatest common divisor in $\mathbb{Z}[\sqrt{-1}]$ of $14 + 2i$ and $21 + 26i$.