THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 8

This is our **LAST** Homework.

Due 3:30pm[1], November 29th (Friday), **in-class**.

Remark: Based on the feedback from *mid-term course survey*, it has been brought to our attention that some students may need more exercises to consolidate their understanding of the course material in anticipation of the upcoming final exam. In response to this situation, we have decided to increase the number of optional problems in this assignment. Furthermore, we would like to extend our gratitude to our TA, who has generously offered to provide solutions for ALL problems. We acknowledge that this will definitely increase his workload, and we really appreciate his dedication to teaching.

Aim of this Homework: *In this assignment you will practice the* method of separation of variables*. More precisely, you will apply this method to solve different IBVP and BVP.*

Reading Assignment: Read the following material(s):

(i) Chapter 4 of the textbook; and

(ii) Chapter 2 of Richard Haberman's "*Applied Partial Differential Equations: with Fourier Series and Boundary Value Problems*".

[1]You are expected to submit your homework before the beginning of Friday lecture **in-class**.

Instruction: Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the "Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

Problem 1. Do Problem 1 of Dec 2022 Final Exam.

Problem 2. Solve the boundary value problem

$$\begin{cases} 4\partial_{xx}u + \partial_{yy}u = 0 & \text{for } 0 < x < 2 \text{ and } 0 < y < 5 \\ u|_{x=0} = 0 & \text{for } 0 < y < 5 \\ u|_{x=2} = \dfrac{1}{6}\sin 3\pi y & \text{for } 0 < y < 5 \\ u|_{y=0} = x^2 - 2x & \text{for } 0 < x < 2 \\ u|_{y=5} = 0 & \text{for } 0 < x < 2 \end{cases}$$

by the *method of separation of variables.* In order to obtain full credit, you must show all steps (including deriving ordinary differential equations, solving eigenvalue problem, ⋯, etc.) and express your final answer without any undetermined coefficients.

Problem 3. Solve the Laplace's equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

inside a semi-circle with radius 3 $(0 \le r \le 3,\ 0 \le \theta \le \pi)$ subject to the boundary conditions

$$u(r,0) = u(r,\pi) = 0 \quad \text{and} \quad u(3,\theta) = 4\sin 5\theta + 6\sin 7\theta$$

by the *method of separation of variables.* In order to obtain full credit, you must show all steps (including deriving ordinary differential equations, solving eigenvalue problem, ⋯, etc.) and express your final answer without any undetermined coefficients.

Problem 4. Let $u := u(x,t)$ be the vertical displacement of a vibrating string and satisfy

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2} - 5u & \text{for } 0 < x < 2 \text{ and } t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = u(2,t) = 0 & \text{for } t > 0 \\ u(x,0) = x^2 - 4 & \text{for } 0 < x < 2 \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{for } 0 < x < 2. \end{cases} \tag{1}$$

Solve the initial and boundary value problem (1). To obtain full credit, you must show all details (including deriving ordinary differential equations, solving eigenvalue problem, $\cdots$, etc.) and express your final solution u without any undetermined coefficients.

Remark. You may have already noticed that the *method of separation of variables* has been a recurring theme in recent final exams. Have you already been equipped to tackle various problems centered around this hot topic? To aid in your preparation, we are providing more optional problems in this homework assignment. You are highly recommended attempting them for optimal understanding.

The following problem(s) is/are *optional*:

Problem 5. Do Problem 4 of Dec 2019 Final Exam.

Problem 6. Do Part (a) of Problem 4 of Dec 2020 Final Exam.

Problem 7. Do Part (a) and (b) of Problem 3 of Dec 2021 Final Exam.

Food for Thought. Are you able to complete Part (b) of Problem 4 of Dec 2020 Final Exam, and Part (c) of Problem 3 of Dec 2021 Final Exam?

Problem 8. Solve the Laplace's equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

inside a circular annulus ($1 < r < 2$) subject to the boundary conditions

$$u(1,\theta) = 3\sin 3\theta \quad \text{and} \quad u(2,\theta) = 15\cos 2\theta - \frac{15}{2}\sin 3\theta$$

by the *method of separation of variables*. In order to obtain full credit, you must show all steps (including deriving ordinary differential equations, solving eigenvalue problem, ⋯, etc.) and express your final answer without any undetermined coefficients.

Food for Thought. Are you able to solve the Laplace's equation in any other domains?

Problem 9. Consider the following initial and boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = 5\dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < 2,\ t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(2,t) = 0 & \\ u(x,0) = x, & \end{cases} \tag{2}$$

and answer the following questions:

(i) Solve the initial and boundary value problem (2). To obtain full credit, you must show all details (including deriving ordinary differential equations, solving eigenvalue problem, ⋯, etc.) and express your final solution u without any undetermined coefficients.

(ii) Prove or disprove

$$\lim_{t\to\infty} u(x,t) = 0.$$

Problem 10. Let L and k be given positive constants. Consider the following IBVP for the heat equation in a one-dimensional rod ($0 < x < L$):

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < L \text{ and } t > 0 \\ u\big|_{x=0} = \dfrac{\partial u}{\partial x}\Big|_{x=L} = 0 & \text{for } t > 0 \\ u\big|_{t=0} = x^2 - 2Lx & \text{for } 0 < x < L, \end{cases} \tag{3}$$

and complete the parts below.

(i) What is the physical meaning of the boundary condition at the left endpoint (i.e., $x = 0$)?

(ii) What is the physical meaning of the boundary condition at the right endpoint (i.e., $x = L$)?

(iii) Solve the initial and boundary value problem (3). To obtain full credit, you must show all details (including deriving ordinary differential equations, solving eigenvalue problem, ⋯, etc.) and express your final solution u without any undetermined coefficients.

Problem 11. Let L, c and β be given positive constants. Let $u : [0,\infty) \times [0,L] \to \mathbb{R}$ be the vertical displacement of a vibrating string and satisfy

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} - \beta u & \text{for } 0 < x < L \text{ and } t > 0 \\ u|_{x=0} = u|_{x=L} = 0 & \text{for } t > 0 \\ u|_{t=0} = x(x-L) & \text{for } 0 < x < L \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 & \text{for } 0 < x < L. \end{cases} \tag{4}$$

Solve the initial and boundary value problem (4). To obtain full credit, you must show all details (including deriving ordinary differential equations, solving eigenvalue problem, ⋯, etc.) and express your final solution u without any undetermined coefficients.

Problem 12. Solve the Laplace's equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

inside a quarter-circle with radius $1/2$ $\left(0 \le r \le 1/2,\ 0 \le \theta \le \frac{\pi}{2}\right)$ subject to the boundary conditions

$$u(r,0) = u\left(r,\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \frac{\partial u}{\partial r}\left(\frac{1}{2},\theta\right) = \frac{7}{2}\sin 4\theta + \frac{15}{16}\sin 6\theta$$

by the *method of separation of variables.* In order to obtain full credit, you must show all steps (including deriving ordinary differential equations, solving eigenvalue problem, $\cdots$, etc.) and express your final answer without any undetermined coefficients.

Food for Thought. Do we have any compatibility condition(s) for boundary value problem in Problem 12?