6.(a) $+\infty$. For each $n \in \mathbb{Z}_{>0}$, $m \mapsto nm$ gives a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$.

(b) 2. If the homomorphism $\phi: \mathbb{Z} \to \mathbb{Z}$ is surjective, then $|\phi(1)| = 1$, so $\phi(1) = 1, \phi(n) = n$ and $\phi(1) = -1, \phi(n) = -n$ are all the homomorphisms we can find.

(c) 1. As the homomorphism $\phi: \mathbb{Z} \to \mathbb{Z}$ is nontrivial, $\phi(1) = 1$, so $\phi(n) = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even} \end{cases}$ is the unique homomorphism.

(d) 1. Construct projection maps $\pi_1: \mathbb{Z}_2 \times \mathbb{Z}_5 \to \mathbb{Z}_2$, $\pi_1(x,y) = x$; $\pi_2: \mathbb{Z}_2 \times \mathbb{Z}_5 \to \mathbb{Z}_5$, $\pi_2(x,y) = y$. The projection maps are homomorphisms, so $\pi_2 \circ \phi$ is a homomorphism. According to the First Isomorphism Theorem:

$$(\mathbb{Z}_2 \times \mathbb{Z}_4)/\mathrm{Ker}(\pi_2 \circ \phi) \cong \mathrm{Im}(\pi_2 \circ \phi).$$

Hence, $8 = |\mathbb{Z}_2 \times \mathbb{Z}_4| = |(\mathbb{Z}_2 \times \mathbb{Z}_4)/\mathrm{Ker}(\pi_2 \circ \phi)|\,|\mathrm{Ker}(\pi_2 \circ \phi)|$
$= |\mathrm{Im}(\pi_2 \circ \phi)|\,|\mathrm{Ker}(\pi_2 \circ \phi)|$

As 8 is not divisible by 5, its factor $|\mathrm{Im}(\pi_2 \circ \phi)|$ is not divisible by 5.

As $\mathrm{Im}(\pi_2 \circ \phi) \leq \mathbb{Z}_5$ and $|\mathbb{Z}_5| = 5$ is prime, $\mathrm{Im}(\pi_2 \circ \phi)$ is the trivial group.

Hence, $\pi_2 \circ \phi(x,y) = 0$ for all $(x,y) \in \mathbb{Z}_2 \times \mathbb{Z}_4$.

As ϕ is nontrivial, $\phi(1,0) = (1,0)$, so $\phi(x,y) = (x,0)$ is the unique homomorphism.

(e) 3. As ϕ is nontrivial, $\phi(1) = 1, 2,$ or 3.

$\phi(1) = 1 \Rightarrow \phi(0) = \phi(4) = \phi(8) = 0$
$\phi(1) = \phi(5) = \phi(9) = 1$
$\phi(2) = \phi(6) = \phi(10) = 2$
$\phi(3) = \phi(7) = \phi(11) = 3$

$\phi(1) = 2 \Rightarrow \phi(0) = \phi(2) = \phi(4) = \phi(6) = \phi(8) = \phi(10) = 0$
$\phi(1) = \phi(3) = \phi(5) = \phi(7) = \phi(9) = \phi(11) = 2$

$\phi(1)=3 \Rightarrow \phi(0)=\phi(4)=\phi(8)=0$

$\phi(1)=\phi(5)=\phi(9)=3$

$\phi(2)=\phi(6)=\phi(10)=2$

$\phi(3)=\phi(7)=\phi(11)=1$

The three functions give three distinct homomorphisms.

(f) 3. There are two cases to consider as:

(1) ϕ is nontrivial $\Rightarrow |\phi(\mathbb{Z}_4)| \geq 2$

(2) $\phi(\mathbb{Z}_4) \cong \mathbb{Z}_4/\ker(\phi) \Rightarrow |\phi(\mathbb{Z}_4)| \mid |\mathbb{Z}_4|=4$

Case1: $|\phi(\mathbb{Z}_4)|=2$. In this case, $\phi(\mathbb{Z}_4)=\{0,6\}$.

$\phi(2)=\phi(1)+\phi(1)=0$, so $\phi(1)=6$ as ϕ is nontrivial.

This implies. $\phi(0)=\phi(2)=0$, $\phi(1)=\phi(3)=6$, which gives a homomorphism. ✓

Case2: $|\phi(\mathbb{Z}_4)|=4$. In this case, $\phi(\mathbb{Z}_4)=\{0,3,6,9\}$

$\phi(1)=0 \Rightarrow \phi(\mathbb{Z}_4)=\{0\} \Rightarrow$ F

$\phi(1)=3 \Rightarrow \phi(0)=0, \phi(1)=3, \phi(2)=6, \phi(3)=9$, a homomorphism. ✓

$\phi(1)=6 \Rightarrow \phi(\mathbb{Z}_4)=\{0,6\} \Rightarrow$ F

$\phi(1)=9 \Rightarrow \phi(0)=0, \phi(1)=9, \phi(2)=6, \phi(3)=3$, a homomorphism. ✓

To conclude, we get three nontrivial homomorphisms in total.

(g) $+\infty$. For each $n \in \mathbb{Z}_{>0}$, $(x,y) \mapsto 2nx$ gives a homomorphism from $\mathbb{Z}^2$ to $2\mathbb{Z}$.

(h) $+\infty$. For each $n \in \mathbb{Z}_{>0}$, $2m \mapsto (nm, 0)$ gives a homomorphism from $2\mathbb{Z}$ to $\mathbb{Z}^2$