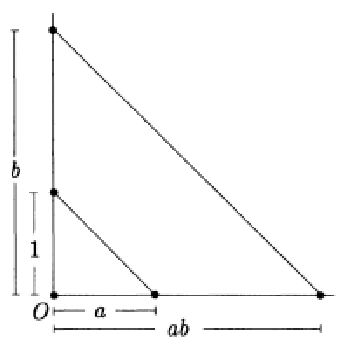
Algebra II: Tutorial 7

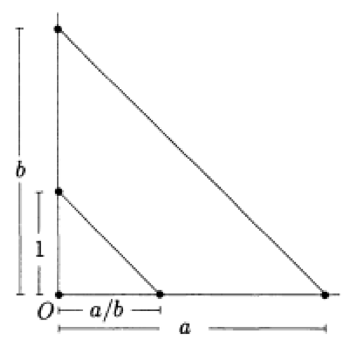
April 9, 2022

Throughout this tutorial, assume that S is a set consisting of the points $P_0 = (0, 0)$ and $P_1 = (1, 0)$, and identify the paper with $\mathbb{R}^2$. We say that a number $\alpha \in \mathbb{R}$ is constructible if there exists two constructible points whose distance is $|\alpha|$.

Problem 1. Suppose that $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are constructible. Show that $a + b, -a, ab$ and $\frac{1}{a}$ are constructible. Deduce that the set of all constructible numbers forms a field containing $\mathbb{Q}$.

Solution. It is easy to construct a sum and a difference of two constructible numbers. Thus, it suffices to deal with a product and a quotient. The ways to construct a quotient and a product of two constructible numbers are shown in the following figures.





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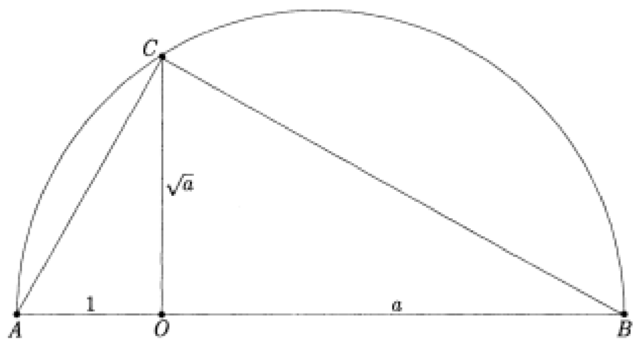
Problem 2. Show that if $a \in \mathbb{R}$ is constructible, then $\sqrt{a}$ is constructible.

Solution. The way to construct a square root of a number is shown in the following figure.



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Problem 3. Show that a number a is constructible if there is a tower of field extensions $\mathbb{Q} \subset F_1 \subset F_2 \subset \cdots \subset F_n$ such that $a \in F_n$, and each of the degrees $[F_i : F_{i-1}] = 2$.

Solution. You know that a quadratic extension over F is of the form $F(\sqrt{a})$ for $a \in F$. Thus, $F_1 = \mathbb{Q}(\sqrt{a})$ for $a \in \mathbb{Q}$. Then, any $\alpha \in F_1$ is a linear combination of $1, \sqrt{a}$ with coefficients from $\mathbb{Q}$. Since arithmetic operations, square root of a number, and any rational numbers are constructible. You can construct α. Thus, repeatly applying these arguments, you should be able to construct any number in F_n. ■