MATH3301 Tutorial 1

1. Explain the mathematical meaning of the underlined word(s) below: (i) a map from the set S to the set T, (ii) the composition $\psi \circ \phi$ of two maps ϕ and ψ, (iii) an injective map, (iv) a surjective map and a bijective map.

2. Let X, Y be two sets and $f : X \to Y$ be a function.

   (a) Suppose f is surjective. Show that for any two functions $g : Y \to Z$ and $h : Y \to Z$, $g \circ f = h \circ f$ implies $g = h$.

   (b) Suppose for any two functions $g : Y \to Z$ and $h : Y \to Z$, $g \circ f = h \circ f$ implies $g = h$. Show that f is surjective.

   (c) Suppose for any set W and any function $g : W \to Y$, there exists a function h such that the following diagram commutes:

$$\begin{tikzcd}
 & W \arrow[d, "g"] \\
X \arrow[ur, dashed, "h"] \arrow[r, "f"] & Y
\end{tikzcd}$$

   Must the function f be surjective? Justify your answer.

3. Let X, Y be two sets and $f : X \to Y$ be a function.

   (a) Give an example of functions f, g, h such that f is injective, $g \circ f = h \circ f$ but $g \neq h$.

   (b) Suppose for any two functions $g : Z \to X$ and $h : Z \to X$, $f \circ g = f \circ h$ implies $g = h$. Show that f is injective. Is the converse true?

4. (a) Explain the mathematical meaning of an equivalence relation.

(b) Explain whether the following relations are equivalence relations.

(i) Consider the set $\mathbb{Z}$ of all integers and define the relation $\sim$ on $\mathbb{Z}$ as follows:

for any $a, b \in \mathbb{Z}$, $a \sim b$ if $ab \geq 0$.

(ii) Consider the set C of all composite numbers (i.e. positive integers that are not primes) and define the relation $\sim$ on C as follows:

for any $a, b \in C$, $a \sim b$ if $\gcd(a, b) > 1$.

5. Let $\{S_\alpha : \alpha \in \mathcal{A}\}$ be a family of non-empty sets, and $X = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$. Assume $S_\alpha \cap S_\beta = \varnothing$ for any $\alpha \neq \beta \in \mathcal{A}$. Define the relation on X by

for any $x, y \in X$, $x \sim y$ if both x, y are belonged to the same S_α, for some $\alpha \in \mathcal{A}$.

(a) Show that $\sim$ is an equivalence relation on X. Describe, in terms of S_α, the equivalence classes $[x]$ of elements $x \in X$.

(b) Could we obtain the same result if the union is not disjoint? Justify your answer.

(c) Write $X/\sim$ for the quotient set of the equivalence relation, i.e. the set of all equivalence classes.

(i) Show that $\pi : X \to X/\sim$, $x \mapsto [x]$, is a well-defined surjective function.

(ii) If $f : X \to Z$ is a function that is constant on every S_α, i.e. for all $\alpha \in \mathcal{A}$, there exists $z_\alpha \in Z$ such that $f(x) = z_\alpha$ for all $x \in S_\alpha$, show that there is a unique function $\bar{f} : X/\sim \to Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Z \\ X/\sim \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

(iii) Does the converse of Part (ii) hold?

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