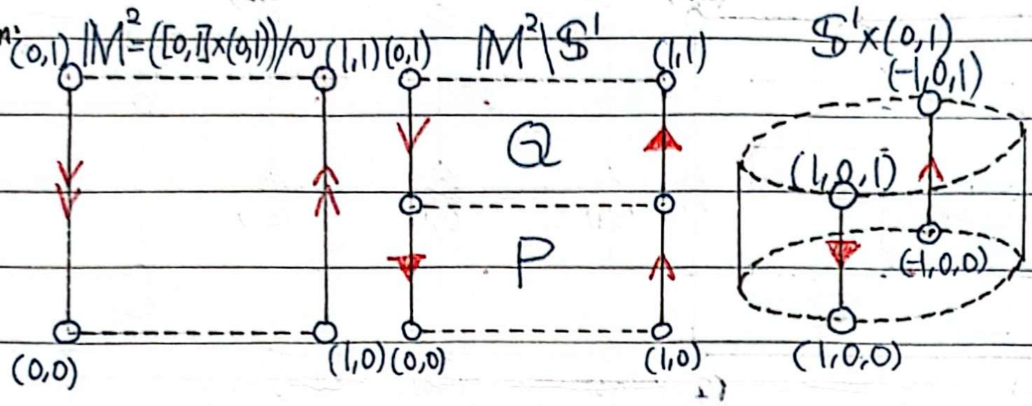
20241021 MATH3541 Assignment 4. Section B.

Problem 8:

Solution:



Define $p: [0,1]\times(0,\frac{1}{2}) \to S^1\times(0,1)$

$p(x,y) = (e^{\pi i x}, 2y)$, where the domain $P=[0,1]\times(0,\frac{1}{2})$ is closed in $[0,1]\times((0,\frac{1}{2})\cup(\frac{1}{2},1))$

and $q: [0,1]\times(\frac{1}{2},1) \to S^1\times(0,1)$

$q(x,y) = (-e^{\pi i x}, 2-2y)$, where the domain $Q=[0,1]\times(\frac{1}{2},1)$ is closed in $[0,1]\times((0,\frac{1}{2})\cup(\frac{1}{2},1))$.

The two functions are continuous because exp, id, const are continuous.

Construct a gluing map:

$\sigma: \{0,1\}\times(0,\frac{1}{2}) \to [0,1]\times(\frac{1}{2},1)$

$\sigma(0,y) = (1,1-y)$ (▼), $\sigma(1,y) = (0,1-y)$ (∧)

This is a closed embedding, satisfying $\forall (x,y)\in\{0,1\}\times(0,\frac{1}{2})$, $q\circ\sigma(x,y) = p(x,y)$.

Hence, the map $p\cup_\sigma q: M^2\setminus S^1 = ([0,1]\times(0,\frac{1}{2}))\cup_\sigma([0,1]\times(\frac{1}{2},1)) \to S^1\times(0,1)$

$$p\cup_\sigma q(\pi((x,y),\lambda)) = \begin{cases} p(x,y), & \text{if } \lambda=1 \\ q(x,y), & \text{if } \lambda=2 \end{cases}$$ is continuous.

Notice that p has a continuous inverse

$p^{-1}(e^{\pi i x}, 2y) = (x,y)$, or $p^{-1}(z,w) = (\frac{1}{\pi}\arg z, \frac{1}{2}w)$ explicitly

and q has a continuous inverse

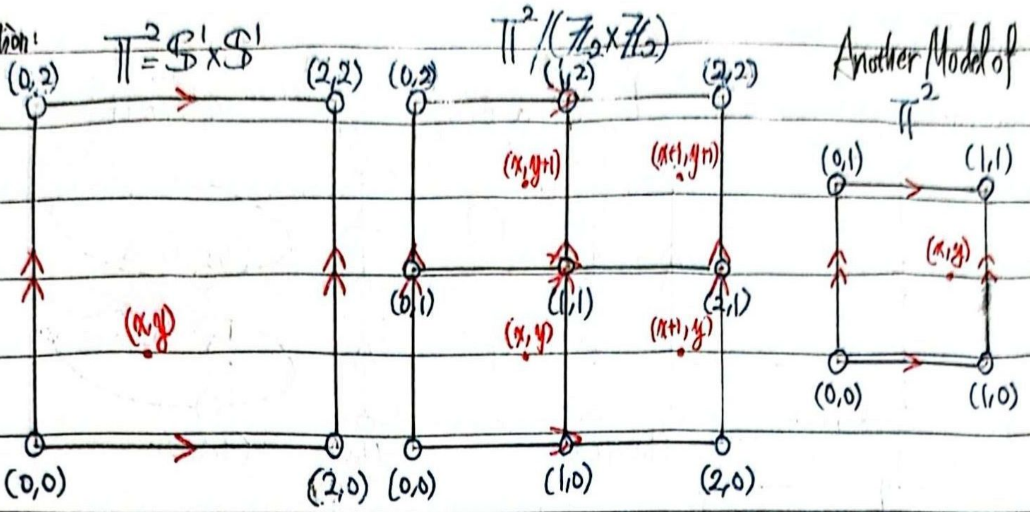
$q^{-1}(-e^{\pi i x}, 2-2y) = (x,y)$, or $q^{-1}(z,w) = (\frac{1}{\pi}\arg(-z), 1-\frac{1}{2}w)$ explicitly

By a similar gluing process, we can show that $(p\cup_\sigma q)^{-1} = p^{-1}\cup_\tau q^{-1}$ is continuous.

To conclude, $p\cup_\sigma q: M^1\setminus S^1 \to S^1\times(0,1)$ is a homeomorphism.

Problem 9:

(a) Solution:



Define Left: $(\mathbb{Z}_2\times\mathbb{Z}_2)\times\mathbb{T}^2 \to \mathbb{T}^2$,

$$\text{Left}((n,m),(x,y)) = (n+x, m+y).$$

(i) For all $(n,m)\in\mathbb{Z}_2\times\mathbb{Z}_2$, $(n',m')\in\mathbb{Z}_2\times\mathbb{Z}_2$ and $(x,y)\in\mathbb{T}^2$,

$$(n,m)+[(n',m')+(x,y)] = (n,m)+(n'+x, m'+y) = (n+(n'+x), m+(m'+y))$$

$$= ((n+n')+x, (m+m')+y) = (n+n', m+m')+(x,y) = ((n,n')+(m,m'))+(x,y)$$

(ii) For all $(x,y)\in\mathbb{T}^2$, $(0,0)+(x,y) = (0+x, 0+y) = (x,y)$.

(iii) Linear function is continuous, so Left is continuous.

Hence, Left is a well-defined left topological action.

Define $\tilde{\sigma}: \mathbb{T}^2/(\mathbb{Z}_2\times\mathbb{Z}_2) \to \mathbb{T}^2$, $\tilde{\sigma}((x,y)+\mathbb{Z}_2\times\mathbb{Z}_2) = (2x,2y)$.

This map is obtained by $\sigma: \mathbb{T}^2 \to \mathbb{T}^2$, $\sigma(x,y) = (2x,2y)$, where:

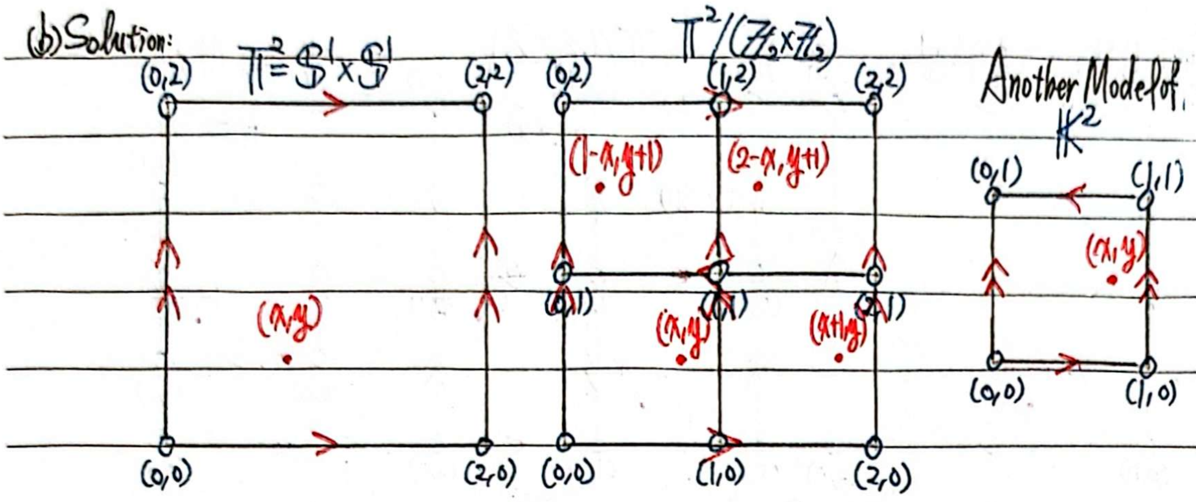
(i) The domain $\mathbb{T}^2$ is compact;

(ii) The codomain $\mathbb{T}^2$ is Hausdorff;

(iii) σ is surjective and continuous.

Hence, $\tilde{\sigma}$ is a homeomorphism, i.e., $\mathbb{T}^2/(\mathbb{Z}_2\times\mathbb{Z}_2) \cong \mathbb{T}^2$

(b) Solution:



Define Left: $(\mathbb{Z}_2\times\mathbb{Z}_2)\times\mathbb{T}^2\to\mathbb{T}^2$

$$\text{Left}((n,m),(x,y)) = (n+(-1)^m x,\ m+y)$$

(i) For all $(n,m)\in\mathbb{Z}_2\times\mathbb{Z}_2$, $(n',m')\in\mathbb{Z}_2\times\mathbb{Z}_2$ and $(x,y)\in\mathbb{T}^2$,

$$\begin{aligned}(n,m)[(n',m')(x,y)] &= (n,m)(n'+(-1)^{m'}x,\ m'+y)\\ &= (n+(-1)^m(n'+(-1)^{m'}x),\ m+m'+y)\\ &= (n+(-1)^m n'+(-1)^{m+m'}x,\ m+m'+y)\\ &= (n+(-1)^m n',\ m+m')(x,y) \qquad \text{In } \mathbb{Z}_2,\ (-1)^m=1\\ &= (n+n',\ m+m')(x,y) = [(n,m)+(n',m')](x,y)\end{aligned}$$

(ii) For all $(x,y)\in\mathbb{T}^2$, $(0,0)(x,y) = (0+(-1)^0x,\ 0+y) = (x,y)$

(iii) Linear function and $m\mapsto(-1)^m$ are continuous, so Left is continuous.

Define $\sigma:\mathbb{T}^2\to\mathbb{K}^2$ by $\forall(x,y)\in[0,1]\times[0,1]$, $\sigma(x,y)=\sigma(x+1,y)=\sigma(1-x,y+1)=\sigma(2-x,y+1)=(2x,2y)$

This map satisfies the following three properties:

(i) The domain $\mathbb{T}^2$ is compact;

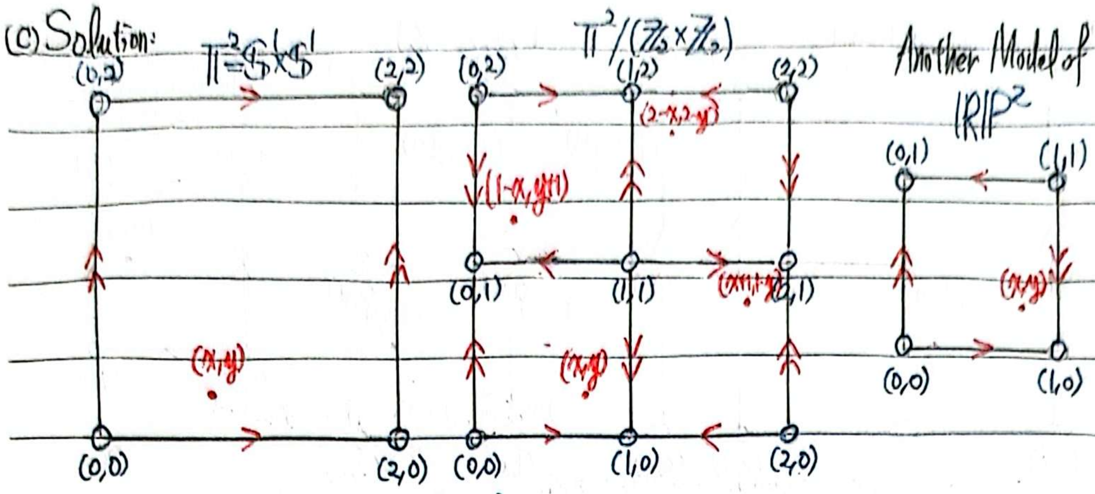
(ii) The codomain $\mathbb{K}^2$ is Hausdorff;

(iii) σ is surjective and continuous.

Hence, the quotient map $\tilde{\sigma}:\mathbb{T}^2/(\mathbb{Z}_2\times\mathbb{Z}_2)\to\mathbb{K}^2$ is a homeomorphism,

$\mathbb{T}^2/(\mathbb{Z}_2\times\mathbb{Z}_2)\cong\mathbb{K}^2$

(c) Solution:



Define Left: $(\mathbb{Z}_2\times\mathbb{Z}_2)\times\mathbb{T}^2\to\mathbb{T}^2$

$$Left((n,m),(x,y))=\left(n+\tfrac{1}{2}-(-1)^m(\tfrac{1}{2}-x),\ m+\tfrac{1}{2}-(-1)^n(\tfrac{1}{2}-y)\right)$$

(i) For all $(n,m)\in\mathbb{Z}_2\times\mathbb{Z}_2$, $(n',m')\in\mathbb{Z}_2\times\mathbb{Z}_2$ and $(x,y)\in\mathbb{T}^2$:

$$(n,m)[(n',m')(x,y)]=(n,m)\left(n'+\tfrac{1}{2}-(-1)^{m'}(\tfrac{1}{2}-x),\ m'+\tfrac{1}{2}-(-1)^{n'}(\tfrac{1}{2}-y)\right)$$

$$=\left(n+\tfrac{1}{2}-(-1)^m((-1)^{m'}(\tfrac{1}{2}-x)-n'),\ m+\tfrac{1}{2}-(-1)^n((-1)^{n'}(\tfrac{1}{2}-y)-m')\right)$$

$$=\left(n+(-1)^m n'+\tfrac{1}{2}-(-1)^{m+(-1)^m m'}(\tfrac{1}{2}-x),\ m+(-1)^n m'+\tfrac{1}{2}-(-1)^{n+(-1)^n n'}(\tfrac{1}{2}-y)\right)$$

$$=(n+(-1)^m n',\ m+(-1)^n m')(x,y) \quad \text{In } \mathbb{Z}_2,\ (-1)^m=(-1)^n=1$$

$$=(n+n',\ m+m')(x,y)=[(n,m)+(n',m')](x,y)$$

(ii) For all $(x,y)\in\mathbb{T}^2$, $(0,0)(x,y)=\left(0+\tfrac{1}{2}-(-1)^0(\tfrac{1}{2}-x),\ 0+\tfrac{1}{2}-(-1)^0(\tfrac{1}{2}-y)\right)=(x,y)$

(iii) Linear function, $n\mapsto(-1)^n$, and $m\mapsto(-1)^m$ are continuous, so Left is continuous.

Hence, Left is a well-defined left topological action.

Define $\sigma:\mathbb{T}^2\to\mathbb{RP}^2$ by $\forall(x,y)\in[0,1]\times[0,1]$, $\sigma(x,y)=\sigma(x+1,1-y)=\sigma(1-x,y+1)=\sigma(2-x,2-y)=(2x,2y)$

This map satisfies the following three properties:

(i) The domain $\mathbb{T}^2$ is compact;

(ii) The codomain $\mathbb{RP}^2$ is Hausdorff;

(iii) σ is surjective and continuous.

Hence, the quotient map $\tilde{\sigma}:\mathbb{T}^2/(\mathbb{Z}_2\times\mathbb{Z}_2)\to\mathbb{RP}^2$ is a homeomorphism,

$\mathbb{T}^2/(\mathbb{Z}_2\times\mathbb{Z}_2)\cong\mathbb{RP}^2$

Problem 10:

(a) Proof: Assume to the contrary that there exists a continuous map $f: \mathbb{S}^1 \to \mathbb{R}^1$,

such that for all $e^{i\theta} \in \mathbb{S}^1$, $f(e^{i\theta}) \neq f(e^{i(\theta+\pi)})$.

Define $g: \mathbb{S}^1 \to \mathbb{R}$, $g(e^{i\theta}) = f(e^{i\theta}) - f(e^{i(\theta+\pi)})$.

WLOG, assume that $f(1) > f(-1)$, so $g(1) > 0 > g(-1)$

As g is continuous and $\mathbb{S}^1$ is connected, $g(\mathbb{S}^1)$ is connected,

so $g(\mathbb{S}^1) \subseteq \mathbb{R}^1$ is convex, $0 = \frac{g(1)+g(-1)}{2} \in g(\mathbb{S}^1)$,

there exists $e^{i\theta} \in \mathbb{S}^1$, such that $g(e^{i\theta}) = 0$, $f(e^{i\theta}) = f(e^{i(\theta+\pi)})$, a contradiction.

Hence, our assumption is wrong, and we've proven the statement.

(b) Proof: Assume to the contrary that there exists a continuous map $\vec{f}: \mathbb{S}^n \to \mathbb{S}^{n-1}$,

such that for all $\vec{x} \in \mathbb{S}^n$, $\vec{f}(\vec{x}) = -\vec{f}(\vec{x})$.

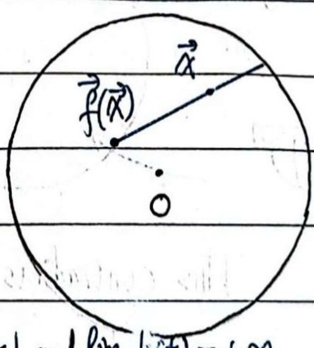
Extend the codomain $\mathbb{S}^{n-1}$ to $\mathbb{R}^n \supseteq \mathbb{S}^{n-1}$, $\vec{f}$ remains continuous.

According to the Borsuk-Ulam theorem, there exists $\vec{\alpha} \in \mathbb{S}^n$,

such that $\vec{f}(\vec{\alpha}) = \vec{f}(-\vec{\alpha})$. Note that $\vec{f}(-\vec{\alpha}) = \vec{f}(\vec{\alpha}) = -\vec{f}(\vec{\alpha}) \Rightarrow \vec{f}(\vec{\alpha}) = \vec{0} \notin \mathbb{S}^{n-1}$, a contradiction.

Hence, our assumption is wrong, and we've proven the statement.

(c) Proof: For all $\vec{\alpha} \in \mathbb{D}^n$ construct the following function:



$h(t) = \|\vec{\alpha} + t(\vec{\alpha} - \vec{f}(\vec{\alpha}))\|^2$

$= \|\vec{\alpha} - \vec{f}(\vec{\alpha})\|^2 t^2 + 2\vec{\alpha}\cdot(\vec{\alpha} - \vec{f}(\vec{\alpha}))t + \|\vec{\alpha}\|^2$

from $[-1, +\infty)$ to $\mathbb{R}$. The image of this continuous

function is $\{\vec{f}(\vec{\alpha})\} \cup L_{\vec{\alpha}}$.

As $h(-1) \leq 1$ and $h(0) \leq 1$ and $\forall t \in (-1, 0)$, $h(t) < 1$ and $\lim_{t \to +\infty} h(t) = +\infty$,

there exists $t_* \in [0, +\infty)$, such that (i) $h(t_*) = 1$; (ii) $\forall t \in (-1, +\infty)$, $h(t) = 1 \Rightarrow t = t_*$.

This implies $\vec{g}: \mathbb{D}^n \to \mathbb{S}^{n-1}$, $\vec{g}(\vec{\alpha}) = \vec{\alpha} + t_*(\vec{\alpha} - \vec{f}(\vec{\alpha}))$ is well-defined.

Moreover: $\vec{g}(\vec{\alpha}) = \vec{\alpha} + \frac{\pm(\vec{f}(\vec{\alpha}) - \vec{\alpha})\cdot\vec{\alpha} + \sqrt{[(\vec{f}(\vec{\alpha}) - \vec{\alpha})\cdot\vec{\alpha}]^2 - \|\vec{\alpha} - \vec{f}(\vec{\alpha})\|^2(\|\vec{\alpha}\|^2 - 1)}}{\|\vec{\alpha} - \vec{f}(\vec{\alpha})\|^2}(\vec{\alpha} - \vec{f}(\vec{\alpha}))$

so $\vec{g}$ is continuous as it has a formula where $\|\vec{\alpha} - \vec{f}(\vec{\alpha})\|^2$ is always nonzero.

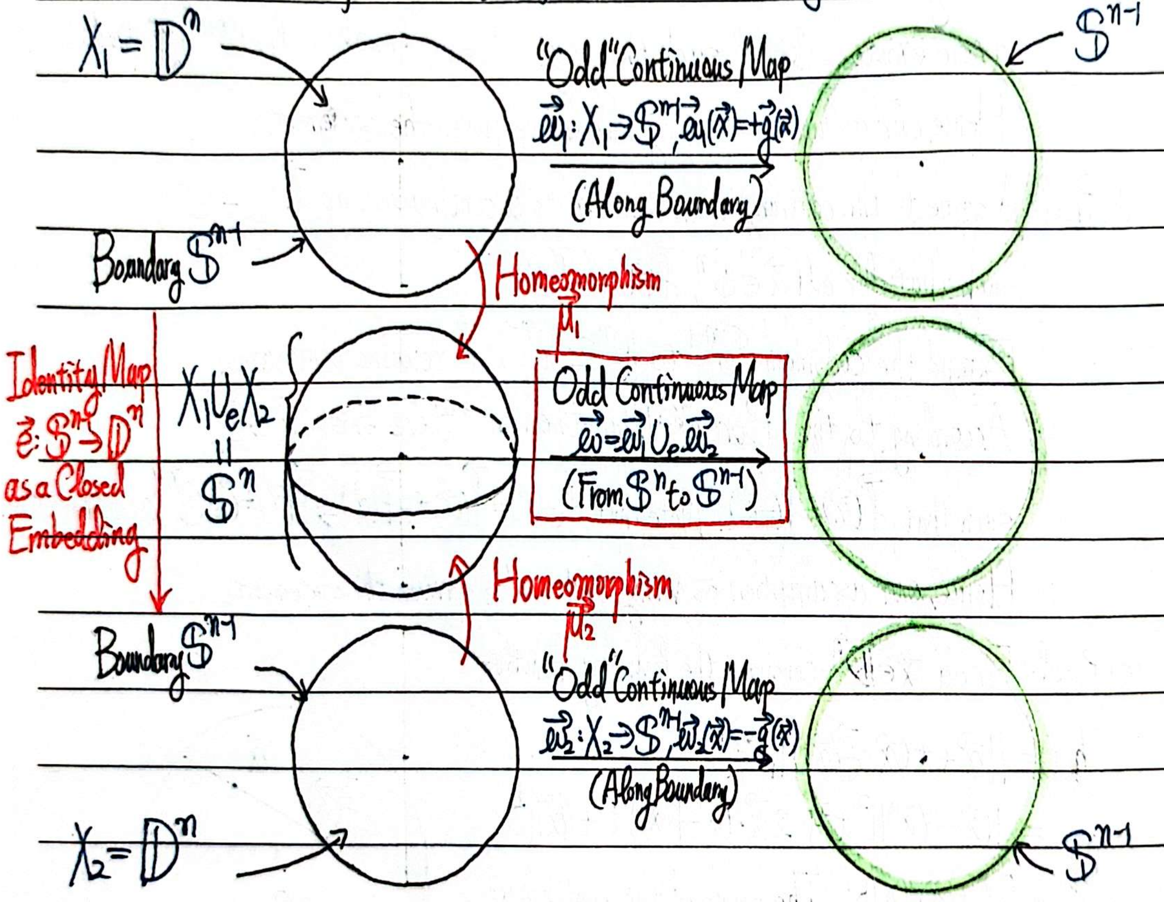
(d) Proof: For all $\vec{x} \in \mathbb{S}^{n-1}$, $\|\vec{x}\|^2 = \vec{x}\cdot\vec{x} = 1 \geq \vec{f}(\vec{x})\cdot\vec{x}$, so:

$$\vec{g}(\vec{x}) = \vec{x} + \frac{+(\vec{f}(\vec{x})-\vec{x})\cdot\vec{x} + \sqrt{[(\vec{f}(\vec{x})-\vec{x})\cdot\vec{x}]^2 - \|\vec{x}-\vec{f}(\vec{x})\|^2(\|\vec{x}\|^2-1)}}{\|\vec{x}-\vec{f}(\vec{x})\|^2}(\vec{x}-\vec{f}(\vec{x}))$$

$$= \vec{x} + \frac{\vec{f}(\vec{x})\cdot\vec{x} - 1 + \sqrt{(\vec{f}(\vec{x})\cdot\vec{x}-1)^2 - 0}}{\|\vec{x}-\vec{f}(\vec{x})\|^2}(\vec{x}-\vec{f}(\vec{x})) = \vec{x}.$$

$\vec{g}(-\vec{x}) = -\vec{x} = -\vec{g}(\vec{x})$, the restriction $\vec{g}|_{\partial\mathbb{D}^n}$ is odd.

(e) Proof: Assume to the contrary that there exists a continuous map $\vec{f}: \mathbb{D}^n \to \mathbb{D}^n$, such that for all $\vec{x} \in \mathbb{D}^n$, $\vec{f}(\vec{x}) \neq \vec{x}$. Construct $\vec{g}: \mathbb{D}^n \to \mathbb{S}^{n-1}$ as described in (c) (d).



This contradicts to (b). Hence, our assumption is false, and we've proven the famous Brouwer fixed point theorem.

Problem 11.

(a) Proof: Every point $\pi(A) \in M_n(\mathbb{R})/GL_n(\mathbb{R})$

is a set of linear transformations on $\mathbb{R}^n$,

for all $A' \in \pi(A)$, if a correct basis $P = (\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n)$ is chosen,

then A is the standard matrix of $\vec{x} \mapsto A'\vec{x}$ along basis P, i.e., $A = P^{-1}A'P$

I think that there is a typo. To be more specific, X should be:

To include αI

$$\left\{\pi\begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha-\beta \end{pmatrix} \in M_2(\mathbb{R}) : \alpha \in \mathbb{R} \text{ and } \beta \geq 0\right\} \cup$$

To include $\begin{pmatrix} 0 & -\alpha^2 \\ 1 & 2\alpha \end{pmatrix}$

$$\left\{\pi\begin{pmatrix} 0 & -\alpha^2-\beta^2 \\ 1 & 2\alpha \end{pmatrix} \in M_2(\mathbb{R}) : \alpha \in \mathbb{R} \text{ and } \beta \geq 0\right\}$$

For all $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, parametrize its characteristic polynomial.

▶ Case 1: $\lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - 2\alpha\lambda + (\alpha^2 - \beta^2)$ for some $\alpha \in \mathbb{R}$ and $\beta > 0$.

In this case, A has two (distinct) real eigenvalues $\lambda_+ = \alpha+\beta$, $\lambda_- = \alpha-\beta$, so:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}-a_{22}}{2}+\beta & a_{12} \\ a_{21} & \frac{a_{22}-a_{11}}{2}-\beta \end{pmatrix}\begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha-\beta \end{pmatrix}\begin{pmatrix} \frac{a_{11}-a_{22}}{2}+\beta & a_{12} \\ a_{21} & \frac{a_{22}-a_{11}}{2}-\beta \end{pmatrix}^{-1} \text{ or}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}-a_{22}}{2}-\beta & a_{12} \\ a_{21} & \frac{a_{22}-a_{11}}{2}+\beta \end{pmatrix}\begin{pmatrix} \alpha-\beta & 0 \\ 0 & \alpha+\beta \end{pmatrix}\begin{pmatrix} \frac{a_{11}-a_{22}}{2}-\beta & a_{12} \\ a_{21} & \frac{a_{22}-a_{11}}{2}+\beta \end{pmatrix}^{-1}$$

Notice that $\det\begin{pmatrix} \frac{a_{11}-a_{22}}{2}+\beta & a_{12} \\ a_{21} & \frac{a_{22}-a_{11}}{2}-\beta \end{pmatrix} = -\left(\frac{a_{11}-a_{22}}{2}+\beta\right)^2 - a_{12}a_{21}$

$\det\begin{pmatrix} \frac{a_{11}-a_{22}}{2}-\beta & a_{12} \\ a_{21} & \frac{a_{22}-a_{11}}{2}+\beta \end{pmatrix} = -\left(\frac{a_{11}-a_{22}}{2}-\beta\right)^2 - a_{12}a_{21}$

cannot be zero simultaneously, so at least one of the above diagonalization is valid.

Hence, $\pi(A) \in \left\{\pi\begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha-\beta \end{pmatrix} \in M_2(\mathbb{R}) : \alpha \in \mathbb{R} \text{ and } \beta \geq 0\right\}$

▷ Case 2: $\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})=\lambda^2-2\alpha\lambda+(\alpha^2+\beta^2)$ for some $\alpha\in\mathbb{R}$ and $\beta>0$

In this case, choose an arbitrary $\vec{p}=\begin{pmatrix}p_1\\p_2\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}(\in\mathbb{R}^2)$, it is automatically true that $A^2\vec{p}-2\alpha A\vec{p}+(\alpha^2+\beta^2)\vec{p}=\vec{0}$.

As $\vec{p}$ is not an eigenvector of A, $A\vec{p}\notin\mathrm{Span}(\vec{p})$, $\vec{p}, A\vec{p}$ forms a basis of $\mathbb{R}^2$

This implies $\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}=\begin{pmatrix}p_1&p_1a_{11}+p_2a_{12}\\p_2&p_1a_{21}+p_2a_{22}\end{pmatrix}\begin{pmatrix}0&-\alpha^2-\beta^2\\1&2\alpha\end{pmatrix}\begin{pmatrix}p_1&p_1a_{11}+p_2a_{12}\\p_2&p_1a_{21}+p_2a_{22}\end{pmatrix}^{-1}$

Hence, $\pi(A)\in\left\{\pi\begin{pmatrix}0&-\alpha^2-\beta^2\\1&2\alpha\end{pmatrix}\in M_2(\mathbb{R}):\alpha\in\mathbb{R}\text{ and }\beta\geq0\right\}$

▷ Case 3: $\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})=\lambda^2-2\alpha\lambda+\alpha^2$ for some $\alpha\in\mathbb{R}$.

In this case, consider the geometric multiplicity of $\lambda=\alpha$.

Situation 3.1: $gm(\alpha)=2$, so $A=\alpha I$, $\pi(A)\in\left\{\pi\begin{pmatrix}\alpha+\beta&0\\0&\alpha-\beta\end{pmatrix}\in M_2(\mathbb{R}):\alpha\in\mathbb{R}\text{ and }\beta\geq0\right\}$

Situation 3.2: $gm(\alpha)=1$, so $\mathrm{rank}(A-\alpha I)=2-1=1$. choose an arbitrary $\vec{p}=\begin{pmatrix}p_1\\p_2\end{pmatrix}\in E_\alpha^c$

it is automatically true that $A(A-\alpha I)\vec{p}=\alpha I(A-\alpha I)\vec{p}$, i.e., $(A-\alpha I)\vec{p}\in E_\alpha$.

This implies $\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}=\begin{pmatrix}p_1&p_1a_{11}+p_2a_{12}\\p_2&p_1a_{21}+p_2a_{22}\end{pmatrix}\begin{pmatrix}0&-\alpha^2\\1&2\alpha\end{pmatrix}\begin{pmatrix}p_1&p_1a_{11}+p_2a_{12}\\p_2&p_1a_{21}+p_2a_{22}\end{pmatrix}^{-1}$,

$$\pi(A)\in\left\{\pi\begin{pmatrix}0&-\alpha^2-\beta^2\\1&2\alpha\end{pmatrix}\in M_2(\mathbb{R}):\alpha\in\mathbb{R}\text{ and }\beta\geq0\right\}$$

To conclude, $M_2(\mathbb{R})/GL_2(\mathbb{R})\subseteq\left\{\pi\begin{pmatrix}\alpha+\beta&0\\0&\alpha-\beta\end{pmatrix}\in M_2(\mathbb{R}):\alpha\in\mathbb{R}\text{ and }\beta\geq0\right\}$

$\cup\left\{\pi\begin{pmatrix}0&-\alpha^2-\beta^2\\1&2\alpha\end{pmatrix}\in M_2(\mathbb{R}):\alpha\in\mathbb{R}\text{ and }\beta\geq0\right\}\subseteq M_2(\mathbb{R})/GL_2(\mathbb{R})$,

which implies they are equal.

(b) Solution: For all $\pi(A)\in M_n(\mathbb{R})/GL_n(\mathbb{R})$:

Case 1: $\pi(A)=\pi\begin{pmatrix}\alpha+\beta&0\\0&\alpha-\beta\end{pmatrix}$ for some $\alpha\in\mathbb{R}$ and $\beta>0$.

In this case, $\pi(B)\in\overline{\{\pi(A)\}}\Leftrightarrow\chi(B)=\chi(A)=\lambda^2-2\alpha\lambda+(\alpha^2-\beta^2)$

$\Leftrightarrow\pi(B)=\pi\begin{pmatrix}\alpha+\beta&0\\0&\alpha-\beta\end{pmatrix}$, so $\overline{\{\pi(A)\}}=\left\{\pi\begin{pmatrix}\alpha+\beta&0\\0&\alpha-\beta\end{pmatrix}\right\}$

Case 2: $\pi(A)=\pi\begin{pmatrix}0&-\alpha^2-\beta^2\\1&2\alpha\end{pmatrix}$ for some $\alpha\in\mathbb{R}$ and $\beta>0$.

In this case, $\pi(B)\in\overline{\{\pi(A)\}}\Leftrightarrow\chi(B)=\chi(A)=\lambda^2-2\alpha\lambda+(\alpha^2+\beta^2)$

$\Leftrightarrow\pi(B)=\pi\begin{pmatrix}0&-\alpha^2-\beta^2\\1&2\alpha\end{pmatrix}$, so $\overline{\{\pi(A)\}}=\left\{\pi\begin{pmatrix}0&-\alpha^2-\beta^2\\1&2\alpha\end{pmatrix}\right\}$

Case 3: $\pi(A)=\pi\begin{pmatrix}\alpha&0\\0&\alpha\end{pmatrix}$ or $\pi(A)=\pi\begin{pmatrix}0&-\alpha^2\\1&2\alpha\end{pmatrix}$ for some $\alpha\in\mathbb{R}$.

In this case, $\pi(B)\in\overline{\{\pi(A)\}}\Leftrightarrow\chi(B)=\chi(A)=\lambda^2-2\alpha\lambda+\alpha^2$

$\Leftrightarrow\pi(B)=\pi\begin{pmatrix}\alpha&0\\0&\alpha\end{pmatrix}$ or $\pi(B)=\pi\begin{pmatrix}0&-\alpha^2\\1&2\alpha\end{pmatrix}$, so $\overline{\{\pi(A)\}}=\left\{\pi\begin{pmatrix}0&-\alpha^2\\1&2\alpha\end{pmatrix}\right\}$,

(c) Proof: For some $\pi(O) \in M_n(\mathbb{R})/GL_n(\mathbb{R})$,

$\overline{\{\pi(O)\}}$ is the set of all classes obtained by nilpotent matrices,

which is not $\{\pi(O)\}$, so $M_n(\mathbb{R})/GL_n(\mathbb{R})$ contains a nonclosed singleton,

$M_n(\mathbb{R})/GL_n(\mathbb{R})$ is not Hausdorff.

(d) Proof:

The set $M_n(\mathbb{R})$ is convex, i.e., for all distinct $A, B \in M_n(\mathbb{R})$,

the line segment $\{(1-t)A + tB \in M_n(\mathbb{R}) : t \in [0,1]\}$ is contained in $M_n(\mathbb{R})$.

$M_n(\mathbb{R})$ is convex $\Rightarrow$ $M_n(\mathbb{R})$ is path connected $\Rightarrow$ $M_n(\mathbb{R})$ is connected.

As the projection map $\pi: M_n(\mathbb{R}) \to M_n(\mathbb{R})/GL_n(\mathbb{R})$, $\pi(A) = \mathrm{Left}(GL_n(\mathbb{R}), A)$

is continuous, $M_n(\mathbb{R})/GL_n(\mathbb{R}) = \pi(M_n(\mathbb{R}))$ is connected.