20240907 MATH3541 NOTE 2[1]

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Contents

1 Introduction

Yesterday, Prof. Hua generalized special Zariski topology on $\mathbb{F}^n$ to general Zariski topology on spec R. He introduced prime ideal to define such topology, which reminds me of the difference between prime element and irreducible element. Hence, the first purpose of this note is to tell their difference, and define general Zariski topology.

Besides, Prof. Hua defined product topology, which was different from the note I had taken. What I had wrote is the finite version of product topology, which is trivial. In infinite case, there seem to be two "identical" topology on the product set: The product topology and the box topology. However, the product topology is coarser than the box topology as certain counterexample from calculus exists. Hence, the second purpose of this note is to construct new topology from old topology.

2 Prime or Irreducible?

2.1 Preliminary Algebraic Structures

Definition 2.1. (**Rng**)
Let R be a set. If:
1. A binary operation $R \times R \to R, (r_1, r_1) \mapsto r_1 + r_2$ is defined to be the addition on R, satisfying the following 4 axioms:
1.1. Commutative Law of Addition:

$$\forall r_1, r_2 \in R, r_1 + r_2 = r_2 + r_1$$

1.2. Associative Law of Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$$

1.3. Identity Element of Addition:

$$\exists 0 \in R, \forall r \in R, 0 + r = r + 0 = r$$

1.4. Inverse Element of Addition:

$$\forall r \in R, \exists -r \in R, (-r) + r = r + (-r) = 0$$

2. An operation $R \times R \to R, (r_1, r_2) \mapsto r_1 \cdot r_2$ (the symbol "$\cdot$" is usually omitted) is defined to be the multiplication on R, satisfying the following 2 axioms:
2.1. Associative Law of Multiplication:

$$\forall r_1, r_2, r_3 \in R, (r_1 r_2) r_3 = r_1 (r_2 r_3)$$

2.2. The Distributive Law of Multiplication over Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) r_3 = r_1 r_3 + r_2 r_3 \text{ and } r_1 (r_2 + r_3) = r_1 r_2 + r_1 r_3$$

then $(R, +, \cdot)$ is a rng.

Proposition 2.2. $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ is a rng.

Ring is stricter than rng in sense that a multiplicative identity exists.

Definition 2.3. (**Ring**)
Let R be a set. If:
1. A binary operation $R \times R \to R, (r_1, r_1) \mapsto r_1 + r_2$ is defined to be the addition on R, satisfying the following 4 axioms:
1.1. Commutative Law of Addition:

$$\forall r_1, r_2 \in R, r_1 + r_2 = r_2 + r_1$$

1.2. Associative Law of Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$$

1.3. Identity Element of Addition:

$$\exists 0 \in R, \forall r \in R, 0 + r = r + 0 = r$$

1.4. Inverse Element of Addition:

$$\forall r \in R, \exists -r \in R, (-r) + r = r + (-r) = 0$$

2. An operation $R \times R \to R, (r_1, r_2) \mapsto r_1 \cdot r_2$ (the symbol "$\cdot$" is usually omitted) is defined to be the multiplication on R, satisfying the following 3 axioms:
2.1. Associative Law of Multiplication:

$$\forall r_1, r_2, r_3 \in R, (r_1 r_2) r_3 = r_1 (r_2 r_3)$$

2.2. The Distributive Law of Multiplication over Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) r_3 = r_1 r_3 + r_2 r_3 \text{ and } r_1 (r_2 + r_3) = r_1 r_2 + r_1 r_3$$

2.3. Identity Element of Multiplication:

$$\exists 1 \in R, \forall r \in R, 1r = r1 = r$$

then $(R, +, \cdot)$ is a ring.

Proposition 2.4. $\mathrm{Hom}(V, V)$ is a ring.

Commutative ring is stricter than ring in sense that multiplication is commutative.

Definition 2.5. (**Commutative Ring**)
Let R be a set. If:
1. A binary operation $R\times R \to R, (r_1, r_1) \mapsto r_1 + r_2$ is defined to be the addition on R, satisfying the following 4 axioms:
1.1. Commutative Law of Addition:

$$\forall r_1, r_2 \in R, r_1 + r_2 = r_2 + r_1$$

1.2. Associative Law of Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$$

1.3. Identity Element of Addition:

$$\exists 0 \in R, \forall r \in R, 0 + r = r + 0 = r$$

1.4. Inverse Element of Addition:

$$\forall r \in R, \exists -r \in R, (-r) + r = r + (-r) = 0$$

2. An operation $R \times R \to R, (r_1, r_2) \mapsto r_1 \cdot r_2$ (the symbol "$\cdot$" is usually omitted) is defined to be the multiplication on R, satisfying the following 4 axioms:
2.1. Associative Law of Multiplication:

$$\forall r_1, r_2, r_3 \in R, (r_1 r_2) r_3 = r_1 (r_2 r_3)$$

2.2. The Distributive Law of Multiplication over Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) r_3 = r_1 r_3 + r_2 r_3 \text{ and } r_1(r_2 + r_3) = r_1 r_2 + r_1 r_3$$

2.3. Identity Element of Multiplication:

$$\exists 1 \in R, \forall r \in R, 1r = r1 = r$$

2.4. Commutative Law of Multiplication:

$$\forall r_1, r_2 \in R, r_1 r_2 = r_2 r_1$$

then $(R, +, \cdot)$ is a commutative ring.

Proposition 2.6. Fix an indexed family of pairwisely commutative linear operators $(A_\lambda)_{\lambda \in I}$ in $\mathrm{Hom}(V, V)$. $\mathbb{F}[A_\lambda]_{\lambda \in I} = \mathrm{span}\ (\prod_{k=1}^{m} A_{\lambda_k}^{\alpha_k})_{(\lambda_k, \alpha_k)_{k=1}^{m} \text{ in } I \times \mathbb{Z}_{\geq 0}}$ is a commutative ring. Here, we accept the convention $A^0 = I$.

Integral domain is stricter than commutative ring in sense that identical factors can be cancelled out.

Definition 2.7. (**Integral Domain**)
Let R be a set. If:
1. A binary operation $R \times R \to R, (r_1, r_1) \mapsto r_1 + r_2$ is defined to be the addition on R, satisfying the following 4 axioms:
1.1. Commutative Law of Addition:

$$\forall r_1, r_2 \in R, r_1 + r_2 = r_2 + r_1$$

1.2. Associative Law of Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$$

1.3. Identity Element of Addition:

$$\exists 0 \in R, \forall r \in R, 0 + r = r + 0 = r$$

1.4. Inverse Element of Addition:

$$\forall r \in R, \exists -r \in R, (-r) + r = r + (-r) = 0$$

2. An operation $R \times R \to R, (r_1, r_2) \mapsto r_1 \cdot r_2$ (the symbol "$\cdot$" is usually omitted) is defined to be the multiplication on R, satisfying the following 5 axioms:
2.1. Associative Law of Multiplication:

$$\forall r_1, r_2, r_3 \in R, (r_1 r_2) r_3 = r_1 (r_2 r_3)$$

2.2. The Distributive Law of Multiplication over Addition:

$$\forall r_1, r_2, r_3 \in R, (r_1 + r_2) r_3 = r_1 r_3 + r_2 r_3 \text{ and } r_1 (r_2 + r_3) = r_1 r_2 + r_1 r_3$$

2.3. Identity Element of Multiplication:

$$\exists 1 \in R, \forall r \in R, 1r = r1 = r$$

2.4. Commutative Law of Multiplication:

$$\forall r_1, r_2 \in R, r_1 r_2 = r_2 r_1$$

2.5. Cancellation Law of Multiplication:

$$\forall \lambda \in R \backslash \{0\}, \forall r_1, r_2 \in R, \lambda r_1 = \lambda r_2 \text{ or } r_1 \lambda = r_2 \lambda \implies r_1 = r_2$$

then $(R, +, \cdot)$ is an integral domain.

Proposition 2.8. Fix an indexed family of indeterminates $(x_\lambda)_{\lambda\in I}$. If:
(1) Any polynomials are equal if and only if they have equal nonzero terms;
(2) Any indeterminate satisfies the recursive definition of power;
(3) Polynomial addition is commutative;
(4) Polynomial addition is associative;
(5) Polynomial multiplication is commutative;
(6) Polynomial multiplication is associative;
(7) Polynomial multiplication is distributive over polynomial addition,
then $\mathbb{F}[x_\lambda]_{\lambda\in I} = \operatorname{span}\left(\prod_{k=1}^{m} x_{\lambda_k}^{\alpha_k}\right)_{(\lambda_k,\alpha_k)_{k=1}^{m} \text{ in } I\times\mathbb{Z}_{\geq 0}}$ is an integral domain.
Here, we accept the convention $x^0 = 1$, so actually $\mathbb{F}$ is a subring of $\mathbb{F}[x_\lambda]_{\lambda\in I}$.

2.2 Prime Element and Prime Ideal

Definition 2.9. (**Unit Element**)
Let R be a commutative ring, and u be an element of R.
If there exists $v \in R$, such that $vu = uv = 1$, then u is a unit element.

Definition 2.10. (**Prime Element**)
Let R be a commutative ring, and p be an element of R. If:
(1) $p \neq 0$;
(2) p is not a unit element;
(3) $\forall$ elements $a_1, a_2, p|a_1a_2 \implies p|a_1$ or $p|a_2$,
then p is a prime element.

Proposition 2.11. $1 + 2\mathrm{i}$ is a prime element in integral domain $\mathbb{Z}[\mathrm{i}]$.

Proof. We may divide our proof into three parts.
Part 1: $1 + 2\mathrm{i} \neq 0$.
Part 2: For all element $a + b\mathrm{i}$, $|(a + b\mathrm{i})(1 + 2\mathrm{i})|^2 = |(1 + 2\mathrm{i})(a + b\mathrm{i})|^2 = 5(a^2 + b^2) \neq 1$, so $(a + b\mathrm{i})(1 + 2\mathrm{i}) = (1 + 2\mathrm{i})(a + b\mathrm{i}) \neq 1$, which implies $1 + 2\mathrm{i}$ is not a unit element.
Part 3: We prove the contrapositive.
Assume to the contrary that for some elements $a_1 + b_1\mathrm{i}, a_2 + b_2\mathrm{i}$, $1 + 2\mathrm{i} \nmid a_1 + b_1\mathrm{i}$ and $1 + 2\mathrm{i} \nmid a_2 + b_2\mathrm{i}$. So, all their possible remainders under $1 + 2\mathrm{i}$ are $1, 2, 1 + \mathrm{i}, 2 + \mathrm{i}$.
We compute all possible remainders of $(a_1 + b_1\mathrm{i})(a_2 + b_2\mathrm{i})$ under $1 + 2\mathrm{i}$.

	1	2	$1+\mathrm{i}$	$2+\mathrm{i}$
1	1	2	$1+\mathrm{i}$	$2+\mathrm{i}$
2	2	$2+\mathrm{i}$	1	$1+\mathrm{i}$
$1+\mathrm{i}$	$1+\mathrm{i}$	1	$2+\mathrm{i}$	2
$2+\mathrm{i}$	$2+\mathrm{i}$	$1+\mathrm{i}$	2	1

Hence, $1 + 2\mathrm{i} \nmid (a_1 + b_1\mathrm{i})(a_2 + b_2\mathrm{i})$.

Combine the three parts above, we've proven the statement.
Quod. Erat. Demonstrandum. ☐

Definition 2.12. (**Prime Ideal**)
Let R be a commutative ring, and P be an ideal of R. If:
(1) $P \neq \{0\}$;
(2) $P \neq R$;
(3) $\forall$ elements $a_1, a_2 \in R, a_1a_2 \in P \implies a_1 \in P$ or $a_2 \in P$,
then P is a prime ideal. We define spec R as the set of all prime ideals in R.

Proposition 2.13. Let R be a commutative ring. For all $p \in R$, p is a prime element if and only if gen $\{p\}$ is a prime ideal.

Proof. We may divide our proof into three parts.
Part 1: $p = 0 \iff$ gen $\{p\} = \{0\}$.
Part 2: We may divide our proof into two steps.
Step 2.1: Assume that $\forall$ element $a, ap = pa \neq 1$, so $1 \notin$ gen $\{p\}$, gen $\{p\} \neq R$.
Step 2.2: Assume to the contrary that $\exists$ element $a, ap = pa = 1$.
$\forall k \in R, k = k1 = k(ap) = (ka)p \in$ gen $\{p\}$, so $R \subseteq$ gen $\{p\}$, which implies gen $\{p\} = R$.
The two steps above prove that the second conditions are equivalent.
Part 3: Since $\forall$ element $a, p|a \iff a \in$ gen $\{p\}$, the third conditions are equivalent.
Combine the three parts above, we've proven that p is a prime element if and only if gen $\{p\}$ is a prime ideal. Quod. Erat. Demonstrandum. ☐

2.3 Prime Element and Irreducible Element

Definition 2.14. (**Association Relation**)
Let R be a commutative ring, and a_1, a_2 be two elements of R.
If $a_1|a_2$ and $a_2|a_1$, i.e., there exists a unit element u, such that $a_1 = ua_2 = a_2u$, then a_1, a_2 are associated. We write $a_1 \sim a_2$ to represent this relation.

Definition 2.15. (**Irreducible Element**)
Let R be a commutative ring, and p be an element of R. If:
(1) $p \neq 0$;
(2) p is not a unit element;
(3) $\forall$ elements $a_1, a_2, p \sim a_1a_2 \implies p \sim a_1$ or $p \sim a_2$,
then p is an irreducible element.

Proposition 2.16. Let R be an integral domain, and p be an element of R.
If p is a prime element, then p is an irreducible element.[3]

Proof. For all elements a_1, a_2, assume that $p \sim a_1 a_2$, i.e., there exists a unit $u \in R$, such that $p = u a_1 a_2$. This implies $p | u a_1 a_2$, so $p|u$ or $p|a_1$ or $p|a_2$. As $p|u$ is impossible, without loss of generality, assume that $p|a_1$, so there exists $q_1 \in R$, such that $a_1 = q_1 p, (u q_1 a_2) p = p = 1p, u q_1 a_2 = 1$. Hence, there exists $uq_1 \in R$, such that $(uq_1)a_2 = a_2(uq_1) = 1$, which implies a_2 is a unit and $p = a_1 a_2 \sim a_1$.
Quod. Erat. Demonstrandum. $\square$

Proposition 2.17. There exists an irreducible element 3 in integral domain $\mathbb{Z}[\sqrt{5}\mathrm{i}]$, which is not prime.[3]

Proof. We may divide our proof into four parts.
Part 1: $3 \neq 0$.
Part 2: For all element $a + b\sqrt{5}\mathrm{i}$, $|(a + b\sqrt{5}\mathrm{i})3|^2 = |3(a + b\sqrt{5}\mathrm{i})|^2 = 9(a^2 + 5b^2) \neq 1$, so $(a + b\sqrt{5}\mathrm{i})3 = 3(a + b\sqrt{5}\mathrm{i}) \neq 1$, which implies 3 is not a unit element.
Part 3: There exist elements $2 + \sqrt{5}\mathrm{i}, 2 - \sqrt{5}\mathrm{i}$, such that $3 | 9 = (2 + \sqrt{5}\mathrm{i})(2 - \sqrt{5}\mathrm{i})$ and $3 \nmid 2 + \sqrt{5}\mathrm{i}$ and $3 \nmid 2 - \sqrt{5}\mathrm{i}$, so 3 is not a prime element.
Part 4: For all elements $a_1 + b_1\sqrt{5}\mathrm{i}, a_2 + b_2\sqrt{5}\mathrm{i}$, assume that $3 \sim (a_1 + b_1\sqrt{5}\mathrm{i})(a_2 + b_2\sqrt{5}\mathrm{i})$. This implies $9 = |3|^2 = |(a_1 + b_1\sqrt{5}\mathrm{i})(a_2 + b_2\sqrt{5}\mathrm{i})|^2 = (a_1^2 + 5b_1^2)(a_2^2 + 5b_2^2)$, so:

$$\begin{cases} a_1^2 + 5b_1^2 & = & 1 \\ a_2^2 + 5b_2^2 & = & 9 \end{cases} \text{or} \begin{cases} a_1^2 + 5b_1^2 & = & 3 \\ a_2^2 + 5b_2^2 & = & 3 \end{cases} \text{or} \begin{cases} a_1^2 + 5b_1^2 & = & 1 \\ a_2^2 + 5b_2^2 & = & 9 \end{cases}$$

Solve this Diophantine system:

$$(a_1 + b_1\sqrt{5}\mathrm{i}, a_2 + b_2\sqrt{5}\mathrm{i}) \in \{(\pm 1, \pm 3), (\pm 1, \pm 2 \pm \sqrt{5}\mathrm{i}), (\pm 3, \pm 1), (\pm 2 \pm \sqrt{5}\mathrm{i}, \pm 1)\}$$

After checking these solutions, only $(\pm 1, \pm 3), (\pm 3, \pm 1)$ satisfy $3 \sim (a_1 + b_1\mathrm{i})(a_2 + b_2\mathrm{i})$, all of which gives $3 \sim a_1 + b_1\sqrt{5}\mathrm{i}$ or $3 \sim a_2 + b_2\sqrt{5}\mathrm{i}$, so 3 is an irreducible element.
Combine the four parts above, we've proven that 3 is an irreducible element which is not prime. Quod. Erat. Demonstrandum. $\square$

Definition 2.18. (**Unique Factorization Domain**)
Let R be an integral domain.
If for all element a with $a \neq 0$ and a is not a unit element:
(1) There exists $\{(p_k, \alpha_k)\}_{k=1}^m$, where each p_k is a irreducible element and each α_k is a positive integer, such that $a = \prod_{k=1}^m p_k^{\alpha_k}$;
(2) For all $\{(p_k, \alpha_k)\}_{k=1}^m, \{(q_l, \beta_l)\}_{l=1}^n$, where each p_k, q_l are irreducible elements and each α_k, β_l are positive integers, $a = \prod_{k=1}^m p_k^{\alpha_k}$ and $a = \prod_{l=1}^n q_l^{\beta_l}$ implies $\{([p_k]_\sim, \alpha_k)\}_{k=1}^m = \{([q_l]_\sim, \beta_l)\}_{l=1}^n$,
then R is a unique factorization domain.

Proposition 2.19. Let R be a unique factorization domain, and p be an element of R. If p is an irreducible element, then p is a prime element.

Proof. It suffices to prove the third condition:

$$\forall \text{ elements } a_1, a_2, p|a_1a_2 \implies p|a_1 \text{ or } p|a_2$$

Assume to the contrary that $p \nmid a_1$ and $p \nmid a_2$. Without loss of generality, we may assume that a_1, a_2 are not unit elements. We proceed to factorize them:

$$a_1 = \prod_{k_1=1}^{m_1} p_{1,k_1}^{\alpha_{1,k_1}} \text{ and } a_2 = \prod_{k_2=1}^{m_2} p_{2,k_2}^{\alpha_{2,k_2}}$$

According to **Definition 2.18.**, factorizations of a_1, a_2 never contain any factor in $[p]_\sim$.
According to **Definition 2.18.**, every factorization of every element in $[a_1a_2]_\sim$ is the product of the factorizations of some pair of elements in $[a_1]_\sim, [a_2]_\sim$, so factorizations of a_1a_2 never contain any factor in $[p]_\sim$, which implies $p \nmid a_1a_2$.
Quod. Erat. Demonstrandum. □

3 Zariski Topology

3.1 Special Zariski Topology

Definition 3.1. (**Special Zariski Topology**)
Let $\mathbb{F}$ be a field, $(x_l)_{l=1}^n$ be n independent indeterminates, and $\mathbb{F}[x_l]_{l=1}^n$ be the corresponding polynomial ring. We define:

$$\mathcal{C} = \{C \in \mathcal{P}(\mathbb{F}^n) : \exists T \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n), C \text{ is the solution set of } T\}$$

as the special Zariski topology on $\mathbb{F}^n$.

Proposition 3.2. The special Zariski topology induces a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset \in \mathcal{P}(\mathbb{F}^n)$ is the solution set of $\{1\}$, so $\emptyset \in \mathcal{C}$;
and $\mathbb{F}^n \in \mathcal{P}(\mathbb{F}^n)$ is the solution set of $\{0\}$, so $\mathbb{F}^n \in \mathcal{C}$.
Part 2: For all $(C_k)_{k=1}^m$ in $\mathcal{C}$, assume that each C_k is the solution set of $T_k \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n)$:
Step 2.1: We prove $\cup_{k=1}^m C_k$ is contained in the solution set of $\Pi_{k=1}^m T_k$.
Let's prove this directly.
For all $(\xi_l)_{l=1}^n$ in $\cup_{k=1}^m C_k$, $(\xi_l)_{l=1}^n$ is in some C_k, so all polynomial in T_k vanishes at $(\xi_l)_{l=1}^n$. Hence, all polynomial in $\Pi_{k=1}^m C_k$ also vanishes at $(\xi_l)_l^n$, this implies $\cup_{k=1}^m C_k$ is contained in the solution set of $\Pi_{k=1}^m T_k$.
Step 2.2: We prove the solution set of $\Pi_{k=1}^m T_k$ is contained in $\cup_{k=1}^m C_k$.
Let's prove the contrapositive.
For all $(\xi_l)_{l=1}^n$ not in $\cup_{k=1}^m C_k$, it is not in each C_k, so each C_k gives $f_k(x_l)_{l=1}^n$ which doesn't vanish at $(\xi_l)_{l=1}^n$. Hence, some polynomial $\Pi_{k=1}^m f_k(x_l)_{l=1}^n$ in $\Pi_{k=1}^m T_k$ doesn't vanish at $(\xi_l)_{l=1}^n$, this implies $(\xi_l)_{l=1}^n$ is not in the solution set of $\Pi_{k=1}^m T_k$.

The two steps above show $\cup_{k=1}^{m} C_k$ is indeed the solution set of $\Pi_{k=1}^{m} T_k$, which implies $\mathcal{C}$ is closed under finite union.

Part 3: For all $(C_\lambda)_{\lambda\in I}$ in $\mathcal{C}$, assume that each C_λ is the solution set of $T_\lambda \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^{n})$:

Step 3.1: We prove $\cap_{\lambda\in I} C_\lambda$ is contained in the solution set of $\cup_{\lambda\in I} T_\lambda$.

Let's prove this directly.

For all $(\xi_l)_{l=1}^{n}$ in $\cap_{\lambda\in I} C_\lambda$, $(\xi_l)_{l=1}^{n}$ is in each C_λ, so all polynomial in T_λ vanishes at $(\xi_l)_{l=1}^{n}$. Hence, all polynomial in $\cup_{\lambda\in I} T_\lambda$ vanishes at $(\xi_l)_{l=1}^{n}$, this implies $\cap_{\lambda\in I} C_\lambda$ is contained in the solution set of $\cup_{\lambda\in I} T_\lambda$.

Step 3.2: We prove the solution set of $\cup_{\lambda\in I} T_\lambda$ is contained in $\cap_{\lambda\in I} C_\lambda$.

Let's prove the contrapositive.

For all $(\xi_l)_{l=1}^{n}$ not in $\cap_{\lambda\in I} C_\lambda$, $(\xi_l)_{l=1}^{n}$ is not in some C_λ, so some polynomial $f_\lambda(x_l)_{l=1}^{n}$ in T_λ doesn't vanish at $(\xi_l)_{l=1}^{n}$. Hence, some polynomial $f_\lambda(x_l)_{l=1}^{n}$ in $\cup_{\lambda\in I} T_\lambda$ doesn't vanish at $(\xi_l)_{l=1}^{n}$, this implies $(\xi_l)_{l=1}^{n}$ is not in the solution set of $\cup_{\lambda\in I} T_\lambda$.

The two steps above show $\cap_{\lambda\in I} C_\lambda$ is indeed the solution set of $\cup_{\lambda\in I} T_\lambda$, which implies $\mathcal{C}$ is closed under arbitrary intersection.

Combine the three parts above, we've proven that the special Zariski topology induces a topological space. Quod. Erat. Demonstrandum. □

By the way, Prof. Hua also mentioned in class that for any $T \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^{n})$, the solution set of T is the same thing as the solution set of gen T. As ideal has better structure than an arbitrary set, without loss of generality, we may assume that T is an ideal when dealing with the Zariski topological space $(\mathbb{F}^n, \mathcal{C})$.

3.2 General Zariski Topology

Definition 3.3. (**General Zariski Topology**)
Let R be a commutative ring. We define:

$$\mathcal{C} = \{C \in \mathcal{P}(\text{spec } R) : \exists \text{ ideal } A, C \text{ is the set of all prime ideal } P \supseteq A\}$$

as the general Zariski topology on spec R.

Proposition 3.4. The general Zariski topology induces a topological space.

Proof. We may divide our proof into three parts.

Part 1: $\emptyset \in \mathcal{P}(\text{spec } R)$ is the set of all prime ideal $P \supseteq R$, so $\emptyset \in \mathcal{C}$;

and spec $R \in \mathcal{P}(\text{spec } R)$ is the set of all prime ideal $P \supseteq \{0\}$, so spec $R \in \mathcal{C}$.

Part 2: For all $(C_k)_{k=1}^{m}$ in $\mathcal{C}$, assume that each C_k is the set of all prime ideal $P \supseteq A_k$.

Step 2.1: We prove $\cup_{k=1}^{m} C_k$ is contained in the set of all prime ideal $P \supseteq \prod_{k=1}^{m} A_k$.

Let's prove this directly.

For all prime ideal $P \in \cup_{k=1}^{m} C_k$, P is in some C_k, so $P \supseteq A_k, P \supseteq \prod_{k=1}^{m} A_k$.

Step 2.2: We prove the set of all prime ideal $P \supseteq \prod_{k=1}^{m} A_k$ is contained in $\cup_{k=1}^{m} C_k$.

Let's prove the contrapositive.

For all $P \notin \cup_{k=1}^{m} C_k$, P is not in each C_k, so $P \not\supseteq A_k$, $P \not\supseteq \prod_{k=1}^{m} A_k$.
(Remark: The complement of P is closed under multiplication)
The two steps above show $\cup_{k=1}^{m} C_k$ is indeed the set of all prime ideal $P \supseteq \prod_{k=1}^{m} A_k$, which implies $\mathcal{C}$ is closed under finite union.
Part 3: For all $(C_\lambda)_{\lambda \in I}$ in $\mathcal{C}$, assume that each C_λ is the set of all prime ideal $P \supseteq A_\lambda$.
Step 3.1: We prove $\cap_{\lambda \in I} C_\lambda$ is contained in the set of all prime ideal $P \supseteq \sum_{\lambda \in I} A_\lambda$.
Let's prove this directly.
For all $P \in \cap_{\lambda \in I} C_\lambda$, P is in each C_λ, so $P \supseteq A_\lambda, P \supseteq \sum_{\lambda \in I} A_\lambda$.
Step 3.2: We prove the set of all prime ideal $P \supseteq \sum_{\lambda \in I} A_\lambda$ is contained in $\cap_{\lambda \in I} C_\lambda$.
Let's prove the contrapositive.
For all $P \notin \cap_{\lambda \in I} C_\lambda$, P is not in some C_λ, so $P \not\supseteq A_\lambda, P \not\supseteq \sum_{\lambda \in I} A_\lambda$.
The two steps above show $\cap_{\lambda \in I} C_\lambda$ is indeed the set of all prime ideal $P \supseteq \sum_{\lambda \in I} A_\lambda$, which implies $\mathcal{C}$ is closed under arbitrary intersection.
Combine the three parts above, we've proven that the general Zariski topology induces a topological space. Quod. Erat. Demonstrandum. □

4 Identification of Topological Space

4.1 Homeomorphism and Local Homeomorphism

In topology, we identify topological spaces by homeomorphism.

Definition 4.1. (**Homeomorphism**)
Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces, and $\sigma : X \to Y$ be a function. If:
1. σ is bijective;
2. $\forall U \in \mathcal{P}(X), U \in \mathcal{O}_X \iff \sigma(U) \in \mathcal{O}_Y$, or in brief, $\mathcal{O}_Y = \sigma(\mathcal{O}_X)$,
then σ is a homeomorphism, and $(X, \mathcal{O}_X)$ is homeomorphic to $(Y, \mathcal{O}_Y)$.

After proving several lemmas, we can prove that being homeomorphic is an equivalence relation on the set of all topological spaces.

Lemma 4.2. Let $(X, \mathcal{O}_X)$ be a topological space, and $id_X : X \to X, id_X(x) = x$ be the identity function on X. id_X is a homeomorphism.

Proof. We may divide our proof into two parts.
Part 1: id_X is bijective.
Part 2: $\forall U \in \mathcal{P}(X), U \in \mathcal{O}_X \iff id_X(U) = U \in \mathcal{O}_X$.
Combine the two parts above, we've proven that id_X is a homeomorphism.
Quod. Erat. Demonstrandum. □

Lemma 4.3. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a homeomorphism. $\sigma^{-1} : Y \to X$ is a homeomorphism.

Proof. We may divide our proof into two parts.
Part 1: σ is bijective implies σ^{-1} is bijective.
Part 2: $\forall V \in \mathcal{P}(Y), V \in \mathcal{O}_Y \iff \sigma(\sigma^{-1}(V)) \in \mathcal{O}_Y \iff \sigma^{-1}(V) \in \mathcal{O}_X$.
Combine the two parts above, we've proven that σ^{-1} is a homeomorphism.
Quod. Erat. Demonstrandum. □

Lemma 4.4. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y), (Z, \mathcal{O}_Z)$ be three topological spaces, and $\sigma : X \to Y, \tau : Y \to Z$ be two homeomorphisms. $\tau \circ \sigma : X \to Z$ is a homeomorphism.

Proof. We may divide our proof into two parts.
Part 1: σ, τ are bijective implies $\tau \circ \sigma$ is bijective.
Part 2: $\forall U \in \mathcal{P}(X), U \in \mathcal{O}_X \iff \sigma(U) \in \mathcal{O}_Y \iff \tau \circ \sigma(U) = \tau(\sigma(U)) \in \mathcal{O}_Z$.
Combine the two parts above, we've proven that $\tau \circ \sigma$ is a homeomorphism.
Quod. Erat. Demonstrandum. □

Proposition 4.5. The homeomorphic relation $\cong$ on the set of all topological spaces is an equivalence relation.

Notice that homomorphism is stronger than continuous bijection.

Definition 4.6. (**Continuous Function**)
Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a function. If $\forall V \in \mathcal{P}(Y), V \in \mathcal{O}_Y \implies \sigma^{-1}(V) \in \mathcal{O}_X$, then σ is continuous.

Proposition 4.7. Let $(X, d_X), (Y, d_Y)$ be two metric spaces, and $\sigma : X \to Y$ be a function. σ is continuous in metric sense if and only if σ is continuous in topological sense if the topologies are inherited from metric spaces.

Proof. We may divide our proof into two parts.
Part 1: Assume that σ is continuous in metric sense.
For all $V \in \mathcal{P}(Y)$, assume that $V \in \mathcal{O}_Y$, so for all $y \in V$, there exists $s > 0$, such that $B(y, s) \subseteq V$. We would like to show $\sigma^{-1}(V) \in \mathcal{O}_X$.
For all $x_* \in \sigma^{-1}(V)$, $f(x_*) \in V$. Since σ is continuous in metric sense, for certain $s > 0$, there exists $r > 0$, such that $d_Y(\sigma(x), \sigma(x_*)) < s$ whenever $d_X(x, x_*) < r$. This implies the existence of $r > 0$, such that $B(x_*, r) \subseteq \sigma^{-1}(V)$, thus $\sigma^{-1}(V) \in \mathcal{O}_X$.
To conclude, σ is continuous in topological sense.
Part 2: Assume that σ is continuous in topological sense.
For all $x_* \in X$, for all $s > 0$, we would like to show the existence of $r > 0$, such that $d_Y(\sigma(x), \sigma(x_*)) < s$ whenever $d_X(x, x_*) < r$.
For $B(\sigma(x_*), s) \in \mathcal{O}_Y$, $x_* \in \sigma^{-1}(B(\sigma(x_*), s)) \in \mathcal{O}_x$, so there exists $r > 0$, such that $B(x_*, r) \subseteq \sigma^{-1}(B(\sigma(x_*), s))$. This implies the existence of $r > 0$, such that $d_Y(\sigma(x), \sigma(x_*)) < s$ whenever $d_X(x, x_*)$.

To conclude, σ is continuous in metric sense.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. $\square$

Proposition 4.8. If we regard $[0, 2\pi), \mathbb{S}$ as metric spaces with Euclidean metric, then the function $\sigma : [0, 2\pi) \to \mathbb{S}, \sigma(\theta) = (\cos\theta, \sin\theta)$ is a continuous bijection, but it is not a homeomorphism.

Proof. Trigonometric functions are continuous, so with these functions as components, σ is continuous. The reason why σ is bijective is a direct consequence of the definition of 2π. Since the open set $[0, \pi) = B_\pi(0)$ has a nonopen image $\sigma([0, \pi)) = \{(1, 0)\} \cup B_{\sqrt{2}}(0, 1)$, σ is not a homeomorphism. Quod. Erat. Demonstrandum. $\square$

The problem is, even for metric spaces as simple as the unit circle $\mathbb{S}$, we cannot find any homeomorphism from $\mathbb{R}$ to $\mathbb{S}$.

Proposition 4.9. If we regard $\mathbb{R}, \mathbb{S}$ as metric spaces with Euclidean metric, then $\mathbb{R}$ is not homeomorphic to $\mathbb{S}$.

Proof. Assume to the contrary that there exists some homeomorphism $\sigma : \mathbb{R} \to \mathbb{S}$.
Since $\mathbb{S}$ is sequentially compact and σ^{-1} is continuous, $\mathbb{R} = \sigma^{-1}(\mathbb{S})$ should be sequentially compact, but $\mathbb{R}$ contains a divergent sequence $(n)_{n\in\mathbb{N}}$, contradiction!
Hence, our assumption is false, and we've proven that σ is not a homeomorphism.
Quod. Erat. Demonstrandum. $\square$

So, although homeomorphism is a powerful tool to classify topological spaces, it is not a good way to study them because many spaces are not homeomorphic to $\mathbb{R}^n$. To solve this problem, we introduce the idea of local homeomorphism.

Definition 4.10. (**Local Homeomorphism**)
Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be topological spaces, and $\sigma : X \to Y$ be a function.
If for all $x \in X$, there exists $U \in \mathcal{O}_X$, such that $x \in U$, $\sigma(U) \in \mathcal{O}_Y$, and the restricted map $\sigma : U \to \sigma(U)$ is a homeomorphism, then σ is a local homeomorphism, and $(X, \mathcal{O}_X)$ is locally homeomorphic to $(Y, \mathcal{O}_Y)$.

The reason why $U, f(U)$ "inherits" a topology from X, Y will be clarified in **Subsection 5.3.**. Now, let's try to show that $\mathbb{R}$ is locally homeomorphic to $\mathbb{S}$.

Proposition 4.11. If we regard $\mathbb{R}, \mathbb{S}$ as two metric spaces with Euclidean metric, then $\mathbb{R}$ is locally homeomorphic to $\mathbb{S}$.

Proof. Define $\sigma : \mathbb{R} \to \mathbb{S}, \sigma(\theta) = (\cos\theta, \sin\theta)$.
For all $\theta \in \mathbb{R}$, there exists an open set $(\theta - \pi/2, \theta + \pi/2) = B_{\pi/2}(\theta)$, such that $\theta \in (\theta - \pi/2, \theta + \pi/2)$, the image set $\sigma((\theta - \pi/2, \theta + \pi/2)) = B_{\sqrt{2}}(\cos\theta, \sin\theta)$ is open, and

the restricted map $\sigma : (\theta - \pi/2, \theta + \pi/2) \to \sigma((\theta - \pi/2, \theta + \pi/2))$ is a homeomorphism. Hence, $\mathbb{R}$ is locally homeomorphic to $\mathbb{S}$. Quod. Erat. Demonstrandum. $\square$

4.2 Basis and Subbasis

In topology, we define basis as a smaller set to characterize topological space.

Definition 4.12. (**Basis**)
Let $(X, \mathcal{O}_X)$ be a topological space, and $\mathcal{B}_X$ be a subset of $\mathcal{O}_X$.
If every $U \in \mathcal{O}_X$ is a union of elements in $\mathcal{B}_X$, then $\mathcal{B}_X$ is a basis of $(X, \mathcal{O}_X)$.

Every metric space has a basis, namely, the set of all open balls.

Proposition 4.13. In metric space (X, d_X), $\mathcal{B}_X = \{B_r(x)\}_{(r,x)\in\mathbb{R}_{>0}\times X}$ is a basis.

Homeomorphic spaces have "identical bases".

Proposition 4.14. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be two topological spaces, and $\sigma : X \to Y$ be a homeomorphism. If $\mathcal{B}_X$ is a basis of $(X, \mathcal{O}_X)$, then $\sigma(\mathcal{B}_X) = \{\sigma(U) \in \mathcal{O}_Y : U \in \mathcal{B}_X\}$ is a basis of $(Y, \mathcal{O}_Y)$.

Proof. For all $V \in \mathcal{O}_Y$, as σ is a homeomorphism, so does σ^{-1}, which implies $\sigma^{-1}(V) \in \mathcal{O}_X$. Since $\mathcal{B}_X$ is a basis of X, $\sigma^{-1}(V)$ is a union of elements in $\mathcal{B}_X$, so $V = \sigma(\sigma^{-1}(V))$ is a union of elements in $\sigma(\mathcal{B}_X)$. Quod. Erat. Demonstrandum. $\square$

One would like a basis to be "as small as possible", so the rest of this section discusses two ways to "reduce the cardinality of a basis".

The first way is to set up a countable basis, whose cardinality is "the smallest infinity".

Definition 4.15. (**Second-countable Topological Space**)
Let $(X, \mathcal{O}_X)$ be a topological space.
If $(X, \mathcal{O}_X)$ has a countable basis $\mathcal{B}_X$, then X is second-countable.

Our familiar topological space $\mathbb{R}$ is second-countable.

Proposition 4.16. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then $\mathbb{R}$ is second-countable.

Proof. It suffices to prove that $\mathcal{B} = \{(a, b) \text{ is open} : a, b \in \mathbb{Q} \text{ and } a < b\}$ is a basis of $\mathbb{R}$. We may divide our proof into two parts.
For all open set U, for all $y \in U$, there exist $s_y, t_y \in \mathbb{R}$, such that $y \in (s_y, t_y) \subseteq U$.
Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $u_y, v_y \in \mathbb{Q}$, such that $s_y < u_y < y < v_y < t_y$.
So there exists an indexed family of sets $((u_y, v_y))_{y\in U}$ in $\mathcal{B}$, such that $U = \cup_{y\in U}(u_y, v_y)$.

Combine the two parts above, we've proven that $\{(a,b) \subseteq \mathbb{R} : a,b \in \mathbb{Q} \wedge a < b\}$ is a basis of $\mathbb{R}$, so $\mathbb{R}$ is second-countable. Quod. Erat. Demonstrandum. $\square$

The second way is to allow intersection, which removes many "redundant" open sets from the basis.

Definition 4.17. (**Subbasis**)
Let $(X, \mathcal{O}_X)$ be a topological space, and $\mathcal{B}_X$ be a subset of $\mathcal{O}_X$. If $\{\cap_{k=1}^m U_k \in \mathcal{O}_X : (U_k)_{k=1}^m \text{ in } \mathcal{B}_X\}$ is a basis of $(X, \mathcal{O}_X)$, then $\mathcal{B}_X$ is a subbasis of $(X, \mathcal{O}_X)$.

Our familiar topological space $\mathbb{R}$ has a subbasis.

Proposition 4.18. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then $\mathcal{B} = \{(-\infty, a), (a, +\infty) \text{ are open} : a \in \mathbb{R}\}$ is a subbasis of $\mathbb{R}$.

Proof. $\{\cap_{k=1}^m U_k \in \mathcal{O}_X : (U_k)_{k=1}^m \text{ is a finite collection of sets in } \mathcal{B}\} \supseteq \{(a,b) \subseteq \mathbb{R} : a,b \in \mathbb{R} \wedge a < b\}$ is a basis of $\mathbb{R}$, so $\mathcal{B}$ is a subbasis of $\mathbb{R}$. Quod. Erat. Demonstrandum. $\square$

5 Construct New Topology from Old Topology

5.1 Initial Topology and Final Topology

Suppose that we have a function:

$$\sigma : X \to Y$$

(1) If σ is continuous and the topology $\mathcal{O}_Y$ on Y is known, then every valid topology $\mathcal{O}_X$ of X contains $\{\sigma^{-1}(V) \in \mathcal{P}(X) : V \in \mathcal{O}_Y\}$;

(2) If σ is continuous and the topology $\mathcal{O}_X$ on X is known, then $\{V \in \mathcal{P}(Y) : \sigma^{-1}(V) \in \mathcal{O}_X\}$ contains every valid topology $\mathcal{O}_Y$ of Y.

So every topology on the domain has a lower bound, and every topology on the codomain has an upper bound. We ask whether these bounds give global extrema, which gives rise to the ideal of initial topology and final topology.

Definition 5.1. (**Initial Topology**)
Let X be a set, $(Y, \mathcal{O}_Y)$ be a topological space, and $\sigma : X \to Y$ be a function. We define $\mathcal{O}_X = \{\sigma^{-1}(V) \in \mathcal{P}(X) : V \in \mathcal{O}_Y\}$ as the initial topology of $(Y, \mathcal{O}_Y)$ on X via σ.

Proposition 5.2. The initial topology inherits a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset, Y \in \mathcal{O}_Y \implies \emptyset = \sigma^{-1}(\emptyset), X = \sigma^{-1}(Y) \in \mathcal{O}_X$.
Part 2: $\forall (\sigma^{-1}(V_k))_{k=1}^m$ in $\mathcal{O}_X, \cap_{k=1}^m \sigma^{-1}(V_k) = \sigma^{-1}(\cap_{k=1}^m V_k) \in \mathcal{O}_X$.
Part 3: $\forall (\sigma^{-1}(V_\lambda))_{\lambda \in I}$ in $\mathcal{O}_X, \cup_{\lambda \in I} \sigma^{-1}(V_\lambda) = \sigma^{-1}(\cup_{\lambda \in I} V_\lambda) \in \mathcal{O}_X$.
Combine the three parts above, we've proven that initial topology inherits a topological space. Quod. Erat. Demonstrandum. □

Definition 5.3. (**Final Topology**)
Let $(X, \mathcal{O}_X)$ be a topological space, Y be a set, and $\sigma : X \to Y$ be a function.
We define $\mathcal{O}_Y = \{V \in \mathcal{P}(Y) : \sigma^{-1}(V) \in \mathcal{O}_X\}$ as the final topology of $(X, \mathcal{O}_X)$ on Y via σ.

Proposition 5.4. The final topology inherits a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\sigma^{-1}(\emptyset) = \emptyset, \sigma^{-1}(Y) = X \in \mathcal{O}_X \implies \emptyset, Y \in \mathcal{O}_Y$.
Part 2: $\forall (V_k)_{k=1}^m$ in $\mathcal{O}_Y, \sigma^{-1}(\cap_{k=1}^m V_k) = \cap_{k=1}^m \sigma^{-1}(V_k) \in \mathcal{O}_X \implies \cap_{k=1}^m V_k \in \mathcal{O}_Y$.
Part 3: $\forall (V_\lambda)_{\lambda \in I}$ in $\mathcal{O}_Y, \sigma^{-1}(\cup_{\lambda \in I} V_\lambda) = \cup_{\lambda \in I} \sigma^{-1}(V_\lambda) \in \mathcal{O}_X \implies \cup_{\lambda \in I} V_\lambda \in \mathcal{O}_Y$.
Combine the three parts above, we've proven that final topology inherits a topological space. Quod. Erat. Demonstrandum. □

5.2 Subspace Topology

Definition 5.5. (**Subspace Topology**)
Let $(X, \mathcal{O}_X)$ be a topological space, and X' be a subset of X.
We define the subspace topology $\mathcal{O}_{X'}$ of $(X, \mathcal{O}_X)$ on X' as define the initial topology of $(X, \mathcal{O}_X)$ on X' via $\pi : X' \to X : \pi(x) = x$.

Proposition 5.6. Let $(X, \mathcal{O}_X)$ be a topological space, X' be a subset of X.
If $\mathcal{O}_{X'}$ is the subspace topology of $(X, \mathcal{O}_X)$ on X', then $\mathcal{O}_{X'} = \{O \cap X' \in \mathcal{P}(X') : O \in \mathcal{O}_X\}$.

Proof. It suffices to notice that $\forall A \in \mathcal{P}(X), \pi^{-1}(A) = A \cap X'$.
Quod. Erat. Demonstrandum. □

Theorem 5.7. (**The Gluing Lemma**)
1. Let $(X, \mathcal{O}_X)$ be a topological space, $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ be an indexed family of open subspaces of $(X, \mathcal{O}_X)$, $(\sigma_\lambda : X_\lambda \to Y)_{\lambda\in I}$ be an indexed family of continuous functions. If $X = \cup_{\lambda\in I} X_\lambda$ and each $\sigma_{\lambda_1}, \sigma_{\lambda_2}$ agree on $X_{\lambda_1} \cap X_{\lambda_2}$, then the function $\sigma : X \to Y$ defined by $\sigma(x) = \sigma_\lambda(x)$ if $x \in X_\lambda$ is continuous.
2. Let $(X, \mathcal{C}_X)$ be a topological space, $((X_k, \mathcal{C}_{X_k}))_{k=1}^m$ be a finite collection of closed subspaces of $(X, \mathcal{C}_X)$, $(\sigma_k : X_k \to Y)_{k=1}^m$ be a finite collection of continuous functions. If $X = \cup_{k=1}^m X_k$, and each $\sigma_{k_1}, \sigma_{k_2}$ agree on $X_{k_1} \cap X_{k_2}$, then the function $\sigma : X \to Y$ defined by $\sigma(x) = \sigma_k(x)$ if $x \in X_k$ is continuous.

Proof. We may divide our proof into two parts.
Part 1: For all $V \in \mathcal{O}_Y$, each $\sigma_\lambda^{-1}(V) \in \mathcal{O}_{X_\lambda}$, so there exists $U_\lambda \in \mathcal{O}_{X_\lambda}$, such that $\sigma_\lambda^{-1}(V) = X_\lambda \cap U_\lambda$. This gives $\sigma^{-1}(V) = \cup_{\lambda\in I} \sigma_\lambda^{-1}(V) = \cup_{\lambda\in I}(X_\lambda \cap U_\lambda) \in \mathcal{O}_X$.
Part 2: For all $V \in \mathcal{C}_Y$, each $\sigma_k^{-1}(V) \in \mathcal{C}_{X_k}$, so there exists $U_k \in \mathcal{C}_{X_k}$, such that $\sigma_k^{-1}(V) = X_k \cap U_k$. This gives $\sigma^{-1}(V) = \cup_{k=1}^m f_k^{-1}(V) = \cup_{k=1}^m (X_k \cap U_k) \in \mathcal{C}_X$.
Quod. Erat. Demonstrandum. □

5.3 Product Space Topology and Box Topology[2]

The Cartesian product should be extended before we investigate the difference between product space topology and box topology.

Definition 5.8. (**Cartesian Product**)
Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of sets.
We define the Cartesian product $\prod_{\lambda\in I} X_\lambda$ of $(X_\lambda)_{\lambda\in I}$ as:

$$\{x : I \to \cup_{\lambda\in I} X_\lambda : \forall\lambda \in I, x(\lambda) \in X_\lambda\}$$

We introduce product space topology first.

Definition 5.9. (**Product Space Topology**)
Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ be an indexed family of topological space.
We define the product space topology $\mathcal{O}_X$ of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ on $X = \prod_{\lambda\in I} X_\lambda$ as the topology generated by the subbasis $\mathcal{B}_X$, which is the union of each initial topology of $(X_\lambda, \mathcal{O}_{X_\lambda})$ on X via $\pi_\lambda : X \to X_\lambda, \pi_\lambda(x) = x(\lambda)$.

Proposition 5.10. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then $\prod_{k=1}^{+\infty} \left(-\frac{1}{k}, \frac{1}{k}\right) \notin \mathcal{O}_X$, where $X = \mathbb{R}^{\mathbb{N}}$ is equipped with product space topology.

Proof. Assume to the contrary that $\prod_{k=1}^{+\infty} \left(-\frac{1}{k}, \frac{1}{k}\right) \in \mathcal{O}_X$, then $\Delta : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}, x \mapsto (x, x, x, \cdots)$ is discontinuous as $\Delta^{-1}\left(\prod_{k=1}^{+\infty} \left(-\frac{1}{k}, \frac{1}{k}\right)\right) = \{0\}$ is not open.
Since each projection map $\pi_\lambda : \mathbb{R}^{\mathbb{N}}$ is continuous, the composition map $\pi_\lambda \circ \Delta$ should

be discontinuous. However, $\pi_\lambda \circ \Delta = id_{\mathbb{R}}$, a contradiction!
Hence, our assumption is false, and we've proven that $\prod_{k=1}^{+\infty}\left(-\frac{1}{k}, \frac{1}{k}\right) \notin \mathcal{O}_X$.
Quod. Erat. Demonstrandum. $\square$

We introduce box topology next.

Definition 5.11. (**Box Topology**)
Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ be an indexed family of topological spaces.
We define the box topology $\mathcal{O}_X$ of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ on $X = \prod_{\lambda\in I} X_\lambda$ as the topology generated by the basis $\mathcal{B}_X = \{\prod_{\lambda\in I} O_{X_\lambda} \in \mathcal{P}(X) : \text{Each } O_{X_\lambda} \in \mathcal{O}_{X_\lambda}\}$.

Now we are ready to distinguish product space topology and box topology.

Proposition 5.12. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then $\prod_{k=1}^{+\infty}\left(-\frac{1}{k}, \frac{1}{k}\right) \in \mathcal{O}_X$, where $X = \mathbb{R}^{\mathbb{N}}$ is equipped with box topology.

Proposition 5.13. Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ be an indexed family of topological space, and X be the Cartesian product of $(X_\lambda)_{\lambda\in I}$.
The product space topology of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ on X is coarser than the box topology of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ on X.

Proof. It suffices to show that each O in the subbasis given in **Definition 5.8.** is in the basis given in **Definition 5.10.**.
Assume that O is in the subbasis given in **Definition 5.8.**.
O is in some initial topology of $(X_\lambda, \mathcal{O}_{X_\lambda})$ on X via $\pi_\lambda : X \to X_\lambda, \pi_\lambda(x) = x(\lambda)$.
This implies the existence of $O_{X_\lambda} \in \mathcal{O}_{X_\lambda}$, such that $O = \pi_\lambda^{-1}(O_{X_\lambda})$,
so there is no restriction on the output of x except $x(\lambda) = \pi_\lambda(x) \in O_{X_\lambda}$.
Define an indexed family of sets:

$$\forall \mu \in I, U_{X_\mu} = \begin{cases} O_{X_\lambda} & \text{if} \quad \mu = \lambda \\ X_\mu & \text{if} \quad \mu \neq \lambda \end{cases}$$

I claim that $O = \prod_{\mu\in I} U_{X_\mu}$ and each $U_{X_\mu} \in \mathcal{O}_{X_\mu}$,
so O is in the basis given in **Definition 5.10.**.
Quod. Erat. Demonstrandum. $\square$

The reason why the counter example described in **Proposition 5.9.** and **Proposition 5.11.** exists is we only allow finite intersection of elements in a subbasis, so:

(1) Each function x in certain open set O in product sense has only finitely many restrictions on function values;

(2) Each function x in certain open set O in box sense can have infinitely many restrictions on function values.

5.4 Quotient Space Topology

Definition 5.14. (**Quotient Space Topology**)
Let $(X, \mathcal{O}_X)$ be a topological space, and $\sim: X \to X$ be an equivalence relation on X. We define the quotient space topology $\mathcal{O}_{X/\sim}$ of $(X, \mathcal{O}_X)$ on $X/\sim$ as the final topology of $(X, \mathcal{O}_X)$ on $X/\sim$ via $\pi : X \to X/\sim, \pi(x) = [x]_\sim$.

Proposition 5.15. Let $(X, \mathcal{O}_X)$ be a topological space, and $\sim: X \to X$ be an equivalence relation on X. If $\mathcal{O}_{X/\sim}$ is the quotient space topology of $(X, \mathcal{O}_X)$ on $X/\sim$, then $\mathcal{O}_{X/\sim} = \{\{[o_\lambda]_\sim\}_{\lambda\in I} \in \mathcal{P}(X/\sim) : \cup_{\lambda\in I}[o_\lambda]_\sim \in \mathcal{O}_X\}$.

Proof. It suffices to notice that $\forall \{[a_\lambda]_\sim\}_{\lambda\in I} \in \mathcal{P}(X/\sim)$, $\pi^{-1}(\{[a_\lambda]_\sim\}_{\lambda\in I}) = \cup_{\lambda\in I}[a_\lambda]_\sim$. Quod. Erat. Demonstrandum. $\square$

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