Example

1) $(X_\alpha, \mathcal{O}_\alpha)$ $\qquad X = \prod_{\alpha \in \Lambda} X_\alpha$

the initial top. on $X \xrightarrow{p_\alpha} X_\alpha$

is called the product top.

2) $(X, \mathcal{O}_X)$ $\quad Y \subset X$

the subspace top on Y is the initial top.

3) $X \xrightarrow{p} [X]$

the quotient top on $[X]$ is the final top

4) $(X_\alpha, \mathcal{O}_\alpha)$ $\qquad X := \bigsqcup_{\alpha \in \Lambda} X_\alpha$ $\quad$ set theoretical disjoint union.

the final top of $X_\alpha \to X$ is called

the topological disjoint union.

X_α is both open & closed.

Example $\mathbb{R} = (-\infty, 0) \sqcup [0, +\infty)$

metric top on $\mathbb{R}$ is not the disjoint union top.

$[0, 1)$ is not open in $(\mathbb{R}, |\cdot|)$

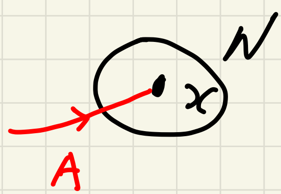
Def $(X, \mathcal{O}_X) \supset A$

$x \in X$ is called an interior pt if $\exists\, U \in \mathcal{O}_X$ $U \ni x$.

interior: $$A^\circ = \bigcup_{\substack{U \subset A \\ U \in \mathcal{O}_X}} U$$

neighborhood: $x \in X$ $N \in \mathcal{O}_X$ is called (open) nbhd if $N \ni x$

limit pt: $x \in X$ is a limit pt of A if $\forall$ nbhd N of x $(N \setminus x) \cap A \neq \emptyset$



closure: $\overline{A} = A \cup$ limit pts of A.

check: intersection of all closed sets $\supset A$.

boundary: $\partial A = \overline{A} \setminus A^\circ$

Properties

$\leftarrow (\quad] \quad) \rightarrow$

1) A° is open

$(A^\circ)^\circ = A^\circ \qquad A^\circ \cup B^\circ \subset (A\cup B)^\circ$

$(A\cap B)^\circ = A^\circ \cap B^\circ \qquad \underbrace{(A\times B)^\circ = A^\circ \times B^\circ}_{\text{fail for arbitrary prod.}}$

$\left(\prod [\frac{1}{n}, \frac{1}{n}]\right)^\circ = \prod (\frac{1}{n}, \frac{1}{n})$ in box top

but $\neq$ in product top.

2) $\overline{A}$ is closed

$\overline{\overline{A}} = \overline{A} \qquad \overline{A} \cup \overline{B} = \overline{A \cup B}$

$\overline{A} \cap \overline{B} \supset \overline{A \cap B} \qquad \overline{A\times B} = \overline{A} \times \overline{B}$

holds for arbitrary product.

$\prod \overline{A_\alpha} \supset \overline{\prod A_\alpha} \qquad \subset \quad \checkmark$

3) $A \subset X \quad \partial A = \partial(X \setminus A)$

$\partial A = \overline{A} \setminus A^{\circ} = \overline{A} \cap (X \setminus A^{\circ}]$

$= \bar{A} \cap \overline{X \setminus A}$

Examples

1) $\mathbb{Q} \subset (\mathbb{R}, |\cdot|)$ $\quad \mathbb{Q}^{\circ} = \emptyset$

i.e. $\mathbb{Q}$ is dense. $\quad \overline{\mathbb{Q}} = \mathbb{R}$

$\mathrm{Lim}\, \mathbb{Q} = \mathbb{R}$

$\partial \mathbb{Q} = \mathbb{R}$

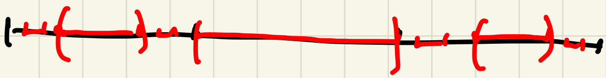
2) $A = \{ \frac{1}{n} \mid n \in \mathbb{N} \} \quad B = \{0\}$

$\overline{A} \cap \overline{B} = \{0\} \quad A \cap B = \emptyset$

3) A is closed $\Leftrightarrow A = \bar{A}$

A is open $\Leftrightarrow A = A^{\circ}$

4) $C \subset [0,1]$ Cantor set



C is closed, uncountable $\underbrace{C^\circ = \emptyset}$

check that!

5) $\bar{D}^n = \{v \in \mathbb{R}^n \mid |v| \leq 1\}$

$\partial \bar{D}^n = S^{n-1}$ $\quad (\bar{D}^n)^\circ = D^n = \{v \mid |v| < 1\}$

__Def__ X is called **Hausdorff (or T_2)**

if $\forall$ $x \neq y \in X$ $\exists$ nbhd U (of x)

and V (of y) s.t. $U \cap V = \emptyset$

Prop Any metric space $(X, |\cdot|)$ is Hausdorff

prove it!

Example Zariski top on $\mathbb{R}$ is not Hausdorff. $A = \mathbb{R}[x]$

$T \subset A$ $\quad I_T$: ideal generated by T

$Z(T) = Z(I_T)$ $\quad A$ is PID

$I_T = (g)$ $\quad Z(T) = Z(g) =$ real roots of g.

$U \subset \mathbb{R}$ is open in Zariski top $\iff$ $\mathbb{R} \setminus U$ is a finite set

Any finite set = real roots of a poly.

$\Rightarrow$ Any two nonempty open sets intersects.

Prop X is Hausdorff then $\{x\} \subset X$ is closed. so is any finite subset

Example Zariski top on $\operatorname{Spec}\mathbb{Z}$ is not Hausdorff

Recall $\operatorname{Spec}\mathbb{Z} = \{(p) \mid p \text{ prime}\} \cup (0)$

1) (p) is closed $\quad Z((p)) = \{\text{prime ideals containing } (p)\}$

$= (p)$

2) (0) is not closed

all closed sets are of the form $Z((n))$ $\quad n \in \mathbb{Z}$

but $Z((1)) \neq \{(0)\}$

3) $\operatorname{Spec}\mathbb{Z} = \overline{(0)} \implies$ non Hausdorff

i.e $\forall\ p \quad (p)$ is a limit pt of (0)

$\neq \operatorname{Spec}\mathbb{Z}$

Note: (0) is not contained in proper closed sets

$\iff (0) \in$ any non empty open sets ✓

Prop X is Hausdorff $\iff$ $\delta: X \to X \times X$ has closed image.

p.f. "$\Rightarrow$" $\Delta = (x, x) \subset X \times X$

$X \times X \setminus \Delta = \{(x, y) \mid x \neq y\}$ $\quad U \cap V = \emptyset$, $x \in U$, $y \in V$

"$\Leftarrow$" W nbhd of (x, y) disjoint from Δ

$\exists\ U \times V \subset W$

prop • subspace of Hausdorff space is Hausdorff.

• X, Y are Hausdorff $\iff$ $X \times Y$ is so.

p.f. 

Def $\{x_1, x_2, \dots\} \subset X$ $\quad x_\infty \in X$

we say $\lim_{i \to \infty} x_i = x_\infty$ if

$\forall$ nbhd U of x_∞ $\exists N$ s.t. $i \geq N$

$x_i \in N$.

if such x_∞ exists we say x_i converges to the limit pt x_∞.