Richard Haberman's "Applied Partial Differential Equations: with Fourier Series and Boundary Value Problems": Chapter 3. Fourier Series

3.1 Introduction

Introduction

What is a Fourier Series?

In chapter 2, we have learnt that in solving PDEs by the method of separation of variables, the initial condition, say $u(x,0) = f(x)$ could be satisfied only if $f(x)$ could be equated to an infinite linear combination of eigenfunctions of a given BVP. This infinite series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

is called a **Fourier series**[a].

[a] It is named in honour of Jean-Baptiste Joseph Fourier (1768-1830).

Two questions about Fourier series:

(1) Does this infinite series converge?

(2) If it converges, will it converge to $f(x)$?

Fourier Coefficients and a Fourier Series on $-L \leq x \leq L$

Fourier Series on the interval $-L \leq x \leq L$

We write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

to represent that $f(x)$ has the Fourier series (even if it diverges)

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \qquad (3.2.1)$$

where the Fourier coefficients is defined by (for any $n = 1, 2, \cdots$)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

Different Types of Fourier Series

Fourier Sine Series on $[0, L]$

$$f \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \qquad B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Fourier Cosine Series on $[0, L]$

$$f \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}, \qquad A_0 = \frac{1}{L} \int_0^L f(x) dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.$$

Fourier (Sine+Cosine) Series on $[-L, L]$

$$f \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \qquad A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx,$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.$$

We formulate Fourier (sine+cosine) series on the interval $[-L, L]$:

- because a Fourier cosine series on $[0, L]$ can be thought of as the Fourier Sine+Cosine series of the *even* periodic extension on $[-L, L]$ (i.e., for even periodic extensions of this type, the coefficients of sine will all vanish).
- because a Fourier sine series on $[0, L]$ can be thought of as the Fourier Sine+Cosine series of the *odd* periodic extension on $[-L, L]$ (i.e., for odd periodic extensions of this type, the coefficients of cosine will all vanish).

Definition

Given a function on $[-L, L]$, we define

$$f^{even}(x) = \frac{f(x) + f(-x)}{2} \text{ and } f^{odd}(x) = \frac{f(x) - f(-x)}{2}.$$

Fact: The "cosine part" of the Fourier series of f on $[-L, L]$ equals the Fourier cosine series of f^{even} on $[0, L]$. The "sine part" is the Fourier sine series of f^{odd} on $[0, L]$.

Periodic Extension

Each function in the Fourier series is periodic with period $2L$, but the function $f(x)$ does not need to be periodic. Hence, we need the **periodic extension** of $f(x)$. To sketch the periodic extension of $f(x)$, we

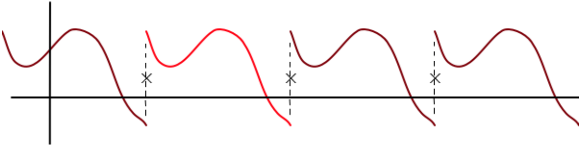
- sketch $f(x)$ for $-L \leq x \leq L$
- repeat the same pattern with period $2L$ by translating the original sketch for $-L \leq x \leq L$.

How to extend a function periodically?

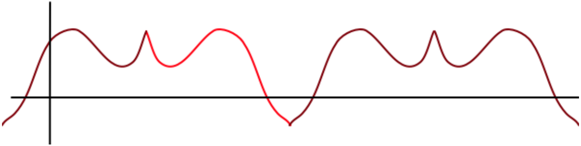
There are three natural methods to extend a given function periodically. We will illustrate them by the following three examples.

Three Natural Ways to Extend Functions Periodically

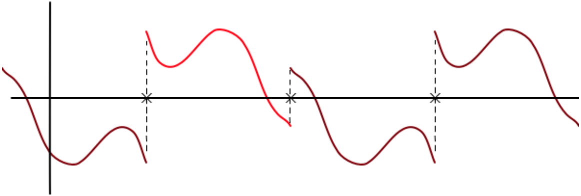
Simple Periodic (Fourier Sine+Cosine Series have this symmetry)



Even Periodic (Fourier Cosine Series have this symmetry)



Odd Periodic (Fourier Sine Series have this symmetry)



Mathematical Definitions

Let us introduce the following two important concepts:

Definition: (Piecewise smooth)

We say that a function $f(x)$ of one variable is **piecewise smooth** when its domain can be divided into at most finitely many subintervals so that (possibly excluding endpoints) the function is continuous, has a continuous derivative, and the limits of $f(x)$ and $f'(x)$ toward the left and right endpoints of each such interval exist and are finite.

Definition: (Jump discontinuity)

A function $f(x)$ has **a jump discontinuity** at a point $x = x_0$ if the limits from the left $[f(x_0^-)]$ and the limit from the right $[f(x_0^+)]$ both exist and are unequal.

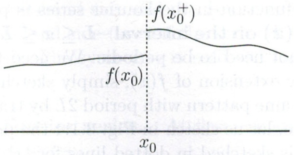


FIGURE 3.1.1 Jump discontinuity at $x = x_0$.

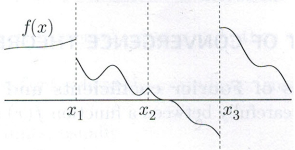


FIGURE 3.1.2 Example of a piecewise smooth function.

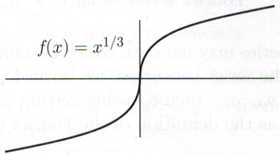


FIGURE 3.1.3 Example of a function that is not piecewise smooth.

3.2 Statement of Convergence Theorem

Convergence of Fourier series (Fourier's theorem)

If f is **piecewise smooth** on $[-L, L]$, then the Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

of f converges for all $x \in (-\infty, \infty)$. Furthermore, this Fourier series converges to

- the value of the periodic extension of f at the point x when the extension is continuous there.
- the average of the left and right limits of the periodic extension at x,
$$\frac{1}{2}[f(x+) + f(x-)],$$
where the extension has a jump discontinuity.

Mathematically, if $f(x)$ is piecewise smooth, then for $-L < x < L$,

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

At points where $f(x)$ is continuous, $f(x+) = f(x-)$, and hence

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Remarks on Fourier's theorem

- The Fourier series actually converges to $f(x)$ at points between $-L$ and L, where $f(x)$ is continuous.
- At end points, $x = L$ or $-L$, the series converges to the average of the two values of the periodic extension.
- Outside $-L \leq x \leq L$, the series converges to a value determined using periodicity of the series.

Sketching Fourier Series

Given a function $f(x)$ defined on $-L \leq x \leq L$. We wish to sketch its Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

Important Notice

The function may not equal to its Fourier series.

Algorithm of Sketching the Fourier Series:

1. Sketch $f(x)$ for $-L \leq x \leq L$.
2. Sketch the periodic extension of $f(x)$.
3. Mark an "x" at the average of the two values at any jump discontinuity of the periodic extension according to the Fourier's theorem.

Consider

$$f(x) = \begin{cases} 0 & x < \dfrac{L}{2} \\ 1 & x > \dfrac{L}{2}. \end{cases}$$

1. We begin by sketching $f(x)$ for $-L \leq x \leq L$ in Fig.3.2.1.

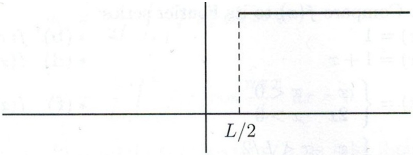


FIGURE 3.2.1 Sketch of $f(x)$.

Note:

$f(x)$ is piecewise smooth, so we can apply Fourier's theorem.

2. According to the Fourier's theorem, the Fourier series of $f(x)$ equals the periodic extension of $f(x)$ wherever the periodic extension is continuous. Hence, We then sketch the periodic extension of $f(x)$ at least three full periods, $-3L \leq x \leq 3L$ in Fig.3.2.2.

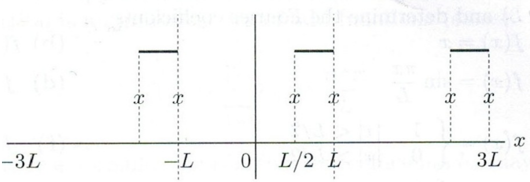


FIGURE 3.2.2 Fourier series of $f(x)$.

3. By the Fourier's theorem again, at the points of jump discontinuity, the Fourier series converge to the average, which is $\frac{1}{2}$ in this case. We mark the points of jump discontinuity, $x = \frac{L}{2} \pm 2nL$ and $x = L \pm 2nL$ with an "x" in Fig 3.2.2.

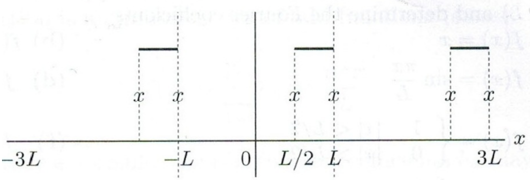


FIGURE 3.2.2 Fourier series of $f(x)$.

In summary, for this example,

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} \frac{1}{2} & x = -L \\ 0 & -L < x < \frac{L}{2} \\ \frac{1}{2} & x = \frac{L}{2} \\ 1 & \frac{L}{2} < x < L \\ \frac{1}{2} & x = L, \end{cases}$$

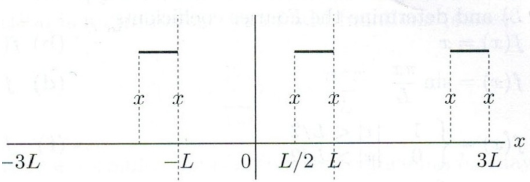


FIGURE 3.2.2 Fourier series of $f(x)$.

with Fourier coefficients are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{L/2}^{L} dx = \frac{1}{4}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= \frac{1}{n\pi}(\sin n\pi - \sin\frac{n\pi}{2})$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx = -\frac{1}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= \frac{1}{n\pi}(\cos\frac{n\pi}{2} - \cos n\pi)$$

Important Remark

In order to sketch the Fourier series, it is **NOT** necessary to calculate the Fourier coefficients indeed.

Question(s) for Further Discussion (Section 3.2)

Let $f(x) := x$ for $-5 < x < \infty$. Denote by $g(x)$ the Fourier series of $f(x)$ on the interval $-1 \leq x \leq 1$, that is,

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x =: g(x).$$

Answer the following questions:

1. What is the domain of $g(x)$?
2. Is $g(x)$ periodic?
3. Is $g(x)$ bounded?
4. Is $g(x)$ continuous?
5. What is $g(1)$?
6. What is a_3?

3.3 Fourier Cosine and Sine Series

3.3.1 Fourier Sine Series

Odd functions

A function f with the property $f(-x) = -f(x)$ is called an odd function.

Elementary property for odd functions

$$\int_{-L}^{L} f(x)\, dx = 0. \ (why?)$$

Let us compute the Fourier coefficients of an odd function f defined on $-L \leq x \leq L$:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx = 0$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = 0 \qquad \text{since } f(x) \cos \frac{n\pi x}{L} \text{ is odd.}$$

Fourier Series of Odd functions

Therefore, all $a_n = 0$, and hence,

Fourier series of odd functions

The Fourier series of an odd function is an infinite series of odd functions (sines):

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx,$$

because the integrand is even.

Odd Extension

Odd extension of $f(x)$

If $f(x)$ is given only for $0 \leq x \leq L$, then it can be *extended* as an odd function, (see Fig 3.3.2.) called the **odd extension of f(x)**. The odd extension of $f(x)$ is defined for $-L \leq x \leq L$.

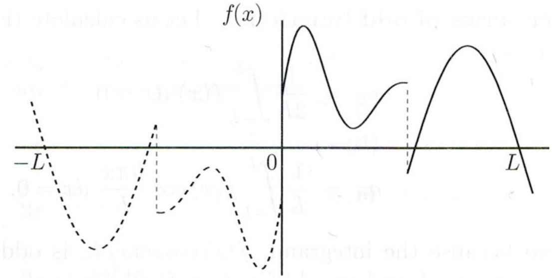


FIGURE 3.3.2 Odd extension of $f(x)$.

Convergence of Fourier sine series (Fourier's theorem)

If f is **piecewise smooth** on $[0, L]$, then the Fourier sine series

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

of f at x converges to

- the value of the *odd periodic extension* of f at the point x when the extension is continuous there.
- the average of the left and right limits of the odd periodic extension at x when the extension has a jump discontinuity.

Fourier series of the odd extension of $f(x)$

Since the odd extension of $f(x)$ is odd, its full Fourier series over the interval $-L \leq x \leq L$ involves only sines:

$$\text{odd extension of } f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \ -L \leq x \leq L.$$

Fourier sine series of $f(x)$

$f(x)$ is identical to its odd extension for $0 \leq x \leq L$. Hence,

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \qquad 0 \leq x \leq L,$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

Algorithm of Sketching the Fourier Sine Series:

1. Sketch $f(x)$ for $0 < x < L$.
2. Sketch its odd extension.
3. Extend as a periodic function (with period $2L$).
4. Mark an "x" at the average at points where the odd periodic extension of $f(x)$ has a jump discontinuity.

Example: Sketching the Fourier Sine Series of $f(x) \equiv 100$

Consider $f(x) \equiv 100$ only for $0 \leq x \leq L$. Let us sketch its Fourier sine series as follows. We begin by sketching its odd extension:

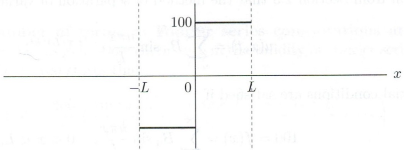


FIGURE 3.3.3 Odd extension of $f(x) = 100$.

Next, we extend the odd extension periodically with period $2L$:

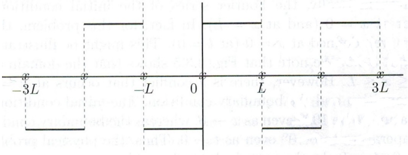


FIGURE 3.3.4 Fourier sine series of $f(x) = 100$.

Finally, at points of discontinuity, mark the average with an "x".

At $x = 0$ and L, Fig 3.3.4 shows that the Fourier sine series converges to 0 because the average of 100 and -100 is 0.

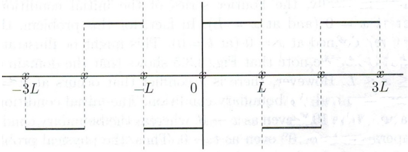


FIGURE 3.3.4 Fourier sine series of $f(x) = 100$.

Moreover, we can compute the Fourier coefficients B_n as follows:

$$
\begin{aligned}
B_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{200}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx \\
&= -\frac{200}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L = -\frac{200}{n\pi}\cos n\pi + \frac{200}{n\pi} \qquad (3.3.8)\\
&= -\frac{200}{n\pi}(-1)^n + \frac{200}{n\pi} = \begin{cases} 0 & n \text{ even} \\ \dfrac{400}{n\pi} & n \text{ odd.} \end{cases}
\end{aligned}
$$

Physical Example

Consider a 1D heat equation with zero BC and constant initial temperature, 100°:

$$\begin{aligned}
\text{PDE:} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,\ t > 0 \\
\text{BC1:} \quad & u(0,t) = 0 \\
\text{BC2:} \quad & u(L,t) = 0 \\
\text{IC:} \quad & u(x,0) = f(x) = 100^\circ.
\end{aligned}$$

Recall from Section 2.3 that the method of separation of variables implied that the solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 kt}. \tag{3.3.9}$$

The IC are satisfied if

$$100 = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

Fourier Sine Series and Coefficients

The Fourier coefficients B_n of the Fourier sine series of $f(x) = 100$ are already determined (see (3.3.8)):

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{200}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx$$
$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{400}{n\pi} & n \text{ odd.} \end{cases}$$

The solution u of the IBVP is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-(\frac{n\pi}{L})^2 kt}$$
$$= \sum_{j=0}^{\infty} B_{2j+1}\sin\frac{(2j+1)\pi x}{L} e^{-\left(\frac{(2j+1)\pi}{L}\right)^2 kt}$$
$$= \sum_{j=0}^{\infty} \frac{400}{(2j+1)\pi}\sin\frac{(2j+1)\pi x}{L} e^{-\left(\frac{(2j+1)\pi}{L}\right)^2 kt}.$$

Discontinuities between IC and BCs

Discontinuities

The IC prescribes the temperature to be 100° even as $x \to 0$, but the BCs (x = 0, L) prescribes the temperature to be 0° even as $t \to 0$ (see Fig. 3.3.5).

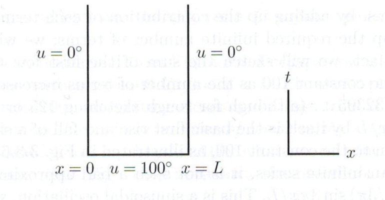


FIGURE 3.3.5 Boundary and initial conditions.

Hence, the physical problem has discontinuities at $(0, 0)$ and $(L, 0)$.

Reality

In actual physical world, the temperature cannot be discontinuous.

Comment on the IC and BCs

What does that mean?

- We introduced a discontinuity into our mathematical model by "instantaneously" transporting (at $t = 0$) the rod from a 100° bath to a 0° bath at $x = 0$.
- We introduce the temperature discontinuity to approximate the more complicated real physical situation.
- Hence, Fourier's convergence theorem illustrates how the physical discontinuity at the boundary is reproduced mathematically.

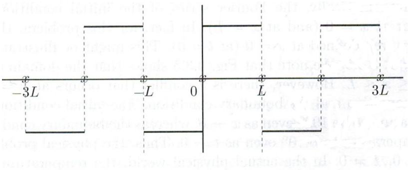


FIGURE 3.3.4 Fourier sine series of $f(x) = 100$.

Finite Number of Terms in Fourier Series

The Fourier sine series of $f(x) = 100$ states that

$$100 = \frac{400}{\pi}\left(\frac{\sin \pi x/L}{1} + \frac{\sin 3\pi x/L}{3} + \frac{\sin 5\pi x/L}{5} + \cdots\right). \tag{3.3.10}$$

where the equality holds for all $0 < x < L$.

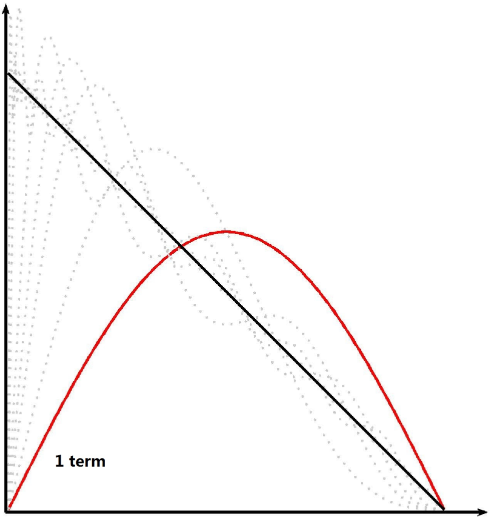
Example (Euler's formula of π)

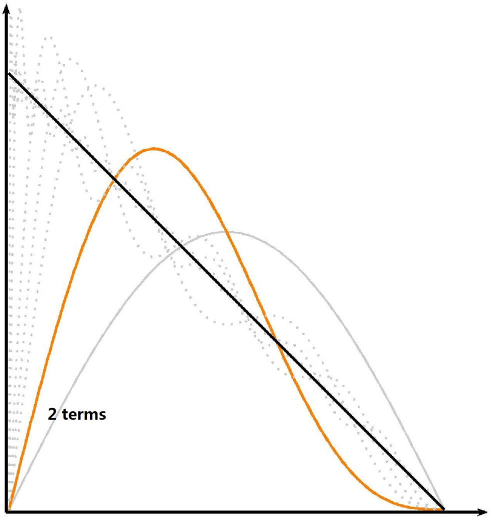
Substituting $x = \dfrac{L}{2}$ into it and then simplifying it, we have

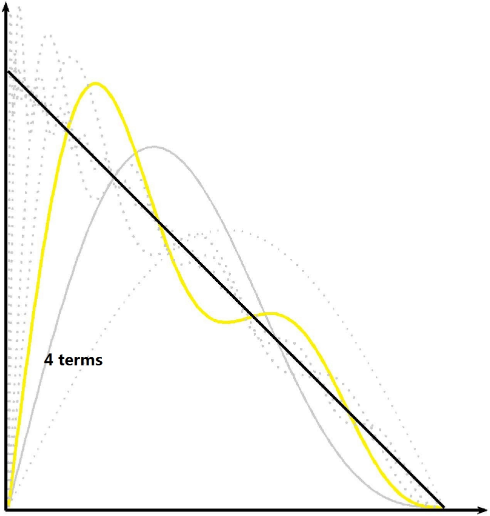
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

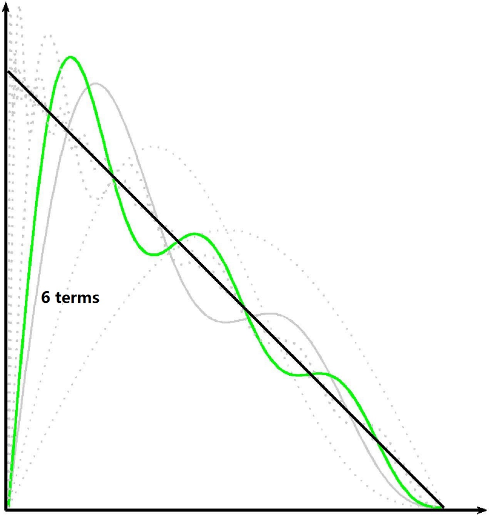
Example (Finite terms application)

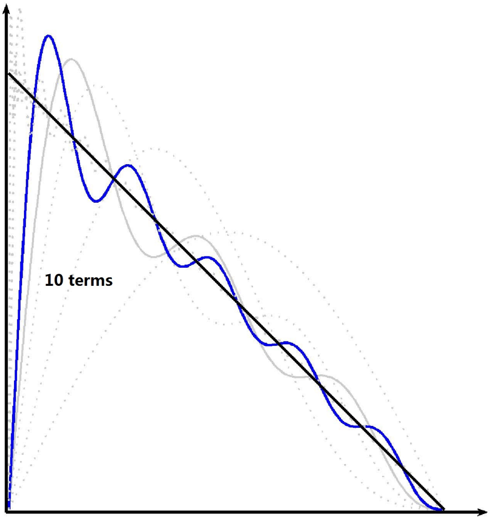
Let us approximate the function $f(x) = 1 - x$ by adding up the first few terms of its Fourier sine series as follows.

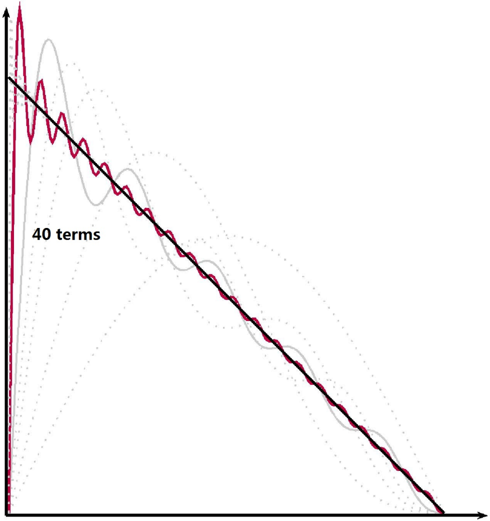

1 term


2 terms


4 terms


6 terms


10 terms


40 terms


200 terms

Gibbs Phenomenon

Overshoot

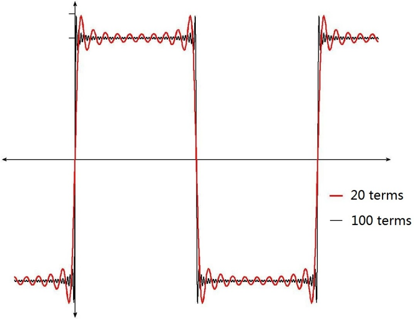
For a finite number of terms in the series, the solution shoots up beyond the curve. We call this overshoot.

Gibbs Phenomenon

- The series should become more and more accurate as the number of terms increases.
- The overshoot vanishes as $n \to \infty$, but put a straight edge on the points of maximum overshoot.
- This overshoot is called the **Gibbs phenomenon**.
- In general, there is an overshoot of approximately 9% of the jump discontinuity.
- The Gibbs phenomenon occurs only when a finite series of eigenfunctions approximates a discontinuous function.

Gibbs Phenomenon

Consider the Fourier sine series for $f(x) = 1$ on $[0, 1]$:



When there are jump discontinuities, the overshoot persists forever but gets narrower and narrower as you sum more terms.

3.3.2 Fourier Cosine Series

Even functions

A function f with the property $f(-x) = f(x)$ is called an even function.

Elementary property for even functions

$$\int_{-L}^{L} f(x)\, dx = 2\int_{0}^{L} f(x)\, dx.$$

Let us compute the Fourier coefficients of an even function f defined on $-L \leq x \leq L$:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx = 0$$

since $f(x) \sin\frac{n\pi x}{L}$ is odd.

Fourier Series of Even Functions

Fourier series of even functions

The Fourier series of an even function is an infinite series of even functions (cosines):

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}$$

where for $n \geq 1$,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx = \frac{1}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx,$$

because $f(x)$ and $f(x) \cos \frac{n\pi x}{L}$ are even.

Even Extension

Even extension of $f(x)$

If $f(x)$ is given only for $0 \le x \le L$, then it can be *extended* as an even function, (see Fig 3.3.12.) called the **even extension of f(x)**. The even extension of $f(x)$ is defined for $-L \le x \le L$.

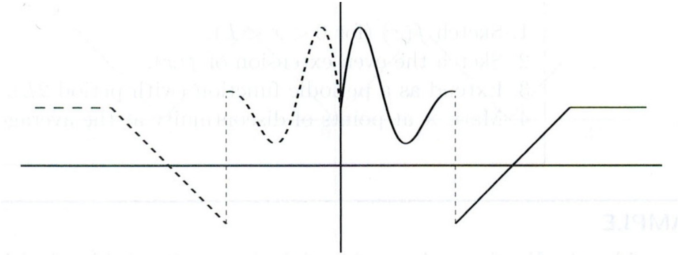


FIGURE 3.3.12 Even extension of $f(x)$.

Fourier Series of the Even Extension

Convergence of Fourier cosine series (Fourier's theorem)

If f is **piecewise smooth** on $[0, L]$, then the Fourier cosine series

$$\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$$

of f at x converges to

- the value of the *even periodic extension* of f at the point x when the extension is continuous there.
- the average of the left and right limits of the even periodic extension at x when the extension has a jump discontinuity.

Fourier series of the even extension of $f(x)$

Since the even extension of $f(x)$ is even, its full Fourier series over the interval $-L < x < L$ involves only cosines:

$$\text{even extension of } f(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}, \; -L \leq x \leq L.$$

Fourier Cosine Series of $f(x)$

Fourier cosine series of $f(x)$

$f(x)$ is identical to its even extension for $0 \leq x \leq L$. Hence,

$$f(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}, \qquad 0 \leq x \leq L,$$

where $A_0 = \dfrac{1}{L} \displaystyle\int_0^L f(x)\, dx$ and $A_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$.

Algorithm of Sketching the Fourier Cosine Series: (Similar algorithm as Fourier sine series)

1. Sketch $f(x)$ for $0 < x < L$.
2. Sketch its even extension.
3. Extend as a periodic function (with period $2L$).
4. Mark an "x" at points of discontinuity at the average.

Moral:

Any piecewise smooth function $f(x)$ may be represented both as a Fourier, Fourier sine and Fourier cosine series.

Let us consider the sketches of the Fourier, Fourier Sine and Fourier cosine series of

$$f(x) = \begin{cases} -\frac{L}{2}\sin\frac{\pi x}{L} & x < 0 \\ x & 0 < x < \frac{L}{2} \\ L - x & x > \frac{L}{2}. \end{cases}$$

as an example.

Graph of f

The graph of $f(x)$ is sketched for $-L < x < L$:

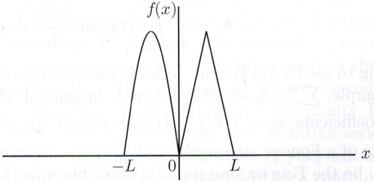
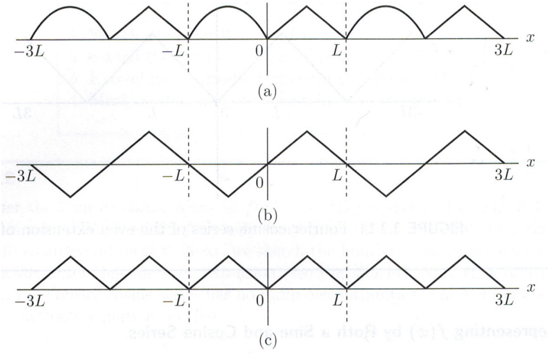


FIGURE 3.3.15 The graph of $f(x)$ for $-L < x < L$.

- Sketch the Fourier series of $f(x)$ by repeating this pattern with period $2L$.
- Sketch the Fourier sine (cosine) series by first sketching the odd (even) extension of $f(x)$ and then repeating the pattern.

Graphs of (a) Fourier Series, (b) Fourier Sine Series and (c) Fourier Cosine Series of $f(x)$.



- Note that only the Fourier series of $f(x)$ actually equals $f(x)$.
- However the series equals $f(x)$ over $0 \leq x \leq L$ for all three cases.

3.3.4 Even and Odd Parts

Even and odd parts

Any function can be written as the sum of an odd function and an even function:

$$f(x) = f_e(x) + f_o(x),$$

where
$$f_e(x) = \frac{1}{2}[f(x) + f(-x)] = \text{the even part of } f(x)$$
$$f_o(x) = \frac{1}{2}[f(x) - f(-x)] = \text{the odd part of } f(x).$$

For example,

$$f(x) = \frac{1}{1+x} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) + \frac{1}{2}\left(\frac{1}{1+x} - \frac{1}{1-x}\right)$$
$$= \underbrace{\frac{1}{1-x^2}}_{\text{even}} - \underbrace{\frac{x}{1-x^2}}_{\text{odd}}.$$

Decomposition of Fourier series

Let us recall that

$$\begin{aligned}\text{Fourier series of } f_o(x) &= \text{ Fourier sine series of } f_o(x)\\ \text{Fourier series of } f_e(x) &= \text{ Fourier cosine series of } f_e(x),\end{aligned}$$

so we have

$$\begin{aligned}&\text{Fourier series of } f(x)\\ =\ &\text{Fourier series of } f_o(x) + \text{ Fourier series of } f_e(x)\\ =\ &\text{Fourier sine series of } f_o(x) + \text{ Fourier cosine series of } f_e(x)\end{aligned}$$

where

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)] \text{ and } f_e(x) = \frac{1}{2}[f(x) + f(-x)].$$

3.3.5 Continuous Fourier Series

Let us recall:

- The Fourier series of $f(x)$ equals $f(x)$ **EXCEPT** at those few points where the periodic extension of $f(x)$ has a *jump discontinuity*.
- Fourier sine (cosine) series can be analyzed in the same way, where instead the odd (even) periodic extension must be considered.

Question

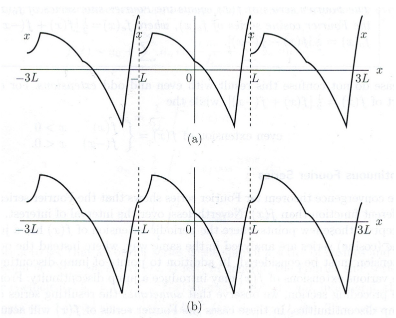
When will the Fourier series of $f(x)$ equal $f(x)$ for all x?

We wish to summarize the necessary and sufficient conditions under which a Fourier series is continuous.

Fourier series

For piecewise smooth $f(x)$, the Fourier series of $f(x)$ is continuous and converges to $f(x)$ for $-L \leq x \leq L$ if and only if

$$f(x) \text{ is continuous and } f(-L) = f(L).$$



Fourier Cosine Series

Fourier cosine series

For piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \leq x \leq L$ if and only if

$$f(x) \text{ is continuous.}$$

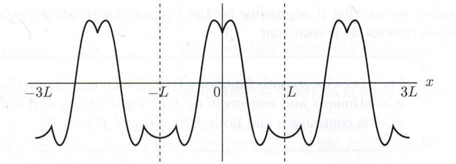


FIGURE 3.3.18 Fourier cosine series of a continuous function.

Note that there is no addition conditions on $f(x)$ as the even extension is the same at $\pm L$.

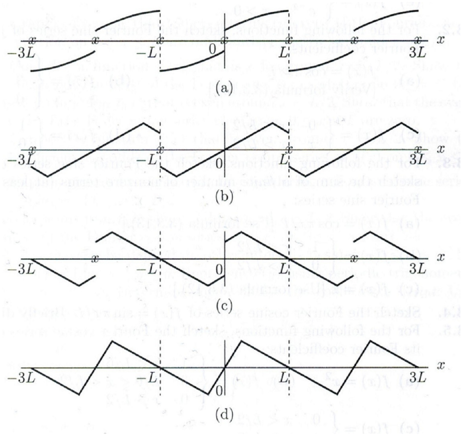
Fourier sine series

For piecewise smooth $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \leq x \leq L$ if and only if

$$f(x) \text{ is continuous and } f(0) = f(L) = 0.$$

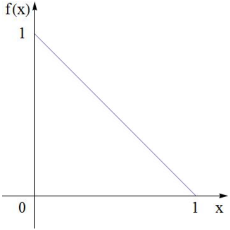
Note that

- If $f(0) \neq 0$, then the odd extension of $f(x)$ will have a jump discontinuity at $x = 0$. See Figs 3.3.19(a) and (c).
- If $f(L) \neq 0$, then the odd extension at $x = -L$ will be of opposite sign from $f(L)$. Hence, it will *NOT* be continuous at the endpoints if $f(L) \neq 0$. See Figs 3.3.19(a) and (b).
- If $f(0) = f(L) = 0$, then the odd extension of a continuous function must be continuous. See Figs 3.3.19(d).



(a) $f(0) \neq 0$ and $f(L) \neq 0$; (b) $f(0) = 0$ and $f(L) \neq 0$;
(c) $f(0) \neq 0$ and $f(L) = 0$; (d) $f(0) = 0$ and $f(L) = 0$.

We define the function $f(x)$ for $x \in [0,1]$ by $f(x) := 1 - x$.



Decide whether the following statements are true or false.

1. The sine series for f converges to 1 at $x = 0$. Y N
2. The cosine series for f converges to 1 at $x = 0$. Y N
3. The sine series for f defines an odd function for $-\infty < x < \infty$. Y N
4. The cosine series for f defines a 2-periodic function for $-\infty < x < \infty$. Y N

3.4 Term-by-Term Differentiation of Fourier Series

Term-by-Term Differentiation

Question of this section

Can we differentiate the Fourier series of a given function f term-by-term? For example,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\right)$$
$$\stackrel{?}{=} \sum_{n=1}^{\infty} A_n \frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \frac{d}{dx}\left(\sin\frac{n\pi x}{L}\right)?$$

Answer

Not always, but we can if

- the periodic extension of f is continuous and
- f' is piecewise smooth.

Counter Example

Ideally, we would like to be able to state that Fourier series can be differentiated term by term without any surprises, but unfortunately this is **not** always true that:

$$\frac{d}{dx}\sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n \frac{du_n}{dx}.$$

Counter example

Take the sine series of the constant function on $[0,1]$, for example:

$$1 = 2\sum_{m=1}^{\infty} \frac{1-(-1)^m}{m\pi} \sin m\pi x = \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin(2j+1)\pi x$$

for all $0 < x < 1$. However, we *cannot* differentiate it term-by-term because

$$0 \neq 4\sum_{j=0}^{\infty} \cos(2j+1)\pi x.$$

When Can We Differentiate Term-by-Term?

Theorem (term-by-term differentiation)

If f satisfies

(i) the simple periodic/odd periodic/even periodic extension of f is continuous, and

(ii) f' is piecewise smooth,

then its Fourier/Fourier sine/Fourier cosine series can be differentiated term by term.

f' being piecewise smooth is easy to be understood, but

Question

When will the simple periodic/odd periodic/even periodic extension of f be continuous?

Let us be more precise in each cases as follows.

Theorem (term-by-term differentiation of (full) Fourier series)

Let $f : [-L, L] \to (-\infty, \infty)$ have the Fourier series

$$\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

If

(i) f is continuous in $[-L, L]$ and $f(-L) = f(L)$, and

(ii) f' is piecewise smooth,

then its Fourier series can be differentiated term by term:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}\right)$$
$$= \sum_{n=1}^{\infty} A_n \frac{d}{dx}\left(\cos \frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \frac{d}{dx}\left(\sin \frac{n\pi x}{L}\right).$$

Theorem (term-by-term differentiation of Fourier sine series)

Let $f : [0, L] \to (-\infty, \infty)$ have the Fourier sine series

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

If

(i) f is continuous in $[0, L]$ and $f(0) = f(L) = 0$, and

(ii) f' is piecewise smooth,

then its Fourier sine series can be differentiated term by term:

$$\begin{aligned} \frac{d}{dx}\left(\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}\right) &= \sum_{n=1}^{\infty} B_n \frac{d}{dx}\left(\sin \frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) B_n \cos \frac{n\pi x}{L}. \end{aligned}$$

Theorem (term-by-term differentiation of Fourier cosine series)

Let $f : [0, L] \to (-\infty, \infty)$ have the Fourier cosine series

$$\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}.$$

If

(i) f is continuous in $[0, L]$, and

(ii) f' is piecewise smooth,

then its Fourier cosine series can be differentiated term by term:

$$\begin{aligned}\frac{d}{dx}\left(\sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}\right) &= \sum_{n=0}^{\infty} A_n \frac{d}{dx}\left(\cos \frac{n\pi x}{L}\right) \\ &= -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) A_n \sin \frac{n\pi x}{L}.\end{aligned}$$

Application of Term-by-Term Differentiation

Example (Fourier Cosine Series)

Consider the Fourier cosine series of $f(x) = x$ (according to (3.3.21)-(3.3.23)):

$$x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ is odd}} \frac{1}{n^2} \cos \frac{n\pi x}{L}, \qquad 0 \le x \le L. \tag{3.4.6}$$

Since f is continuous and f' is piecewise smooth, we can differentiate (3.4.6) term-by-term, and obtain

$$f'(x) = 1 \sim \frac{4}{\pi} \sum_{n \text{ is odd}} \frac{1}{n} \sin \frac{n\pi x}{L}, \tag{3.4.7}$$

which is a correct Fourier sine series of 1.

Moral

One can find the Fourier (sine, cosine, or sine+cosine) series by applying the term-by-term differentiation appropriately.

Method of Eigenfunction Expansion

The method of eigenfunction expansion is similar to separation of variables, but it often allows you to skip a bunch of steps. The basic idea is to **expand everything in terms of a Fourier series**, then equate the coefficients and solve.

Example: Solve the *inhomogeneous* heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$$

subject to boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ and initial condition $u(x,0) = 0$. Assume

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{L}.$$

We differentiate this series term-by-term (after full justifications!), expand the function 1 into a sine series, and match coefficients. (To learn the technique, try Exercise 3.4.9 and 3.4.12.)

3.5 Term-by-Term Integration of Fourier Series

Term-by-Term Integration of a Fourier Series

Question of this section

Can we integrate the Fourier series of a given function f term-by-term? For example,

$$\int \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, dx \stackrel{?}{=} \sum_{n=1}^{\infty} A_n \int \cos\frac{n\pi x}{L}\, dx + \sum_{n=1}^{\infty} B_n \int \sin\frac{n\pi x}{L}\, dx?$$

Answer

Yes, we can if f is piecewise smooth.

Term-by-term integration of a Fourier series

If f is piecewise smooth, then term-by-term integration of its Fourier series is always legal and the result is a convergent infinite series that always converges to the integral of $f(x)$ for $-L \leq x \leq L$ (even if the original Fourier series has jump discontinuities).

Term-by-term integration of a Fourier sine/cosine series

Analogous results also hold for the Fourier sine/cosine series.

Remarks

- Be aware that the term-by-term integral of a Fourier series may include new terms (i.e. linear terms) which technically do not belong in a Fourier series.
- One may apply the term-by-term integration to find the Fourier series. (See the example below.)

Example (Computing Fourier cosine series of x)

Recall that the constant function 1 has the Fourier sine series:

$$1 \sim \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} \sin \frac{(2j+1)\pi x}{L}. \tag{3.4.7}$$

Integrating (3.4.7) w.r.t. x, we have

$$x \sim c - \frac{4L}{\pi^2} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \cos \frac{(2j+1)\pi x}{L},$$

but the integration constant c here is **NOT** arbitrary; This c should be the constant term A_0 of the Fourier cosine series of x, so

$$c = \frac{1}{L} \int_0^L x \, dx = \frac{L}{2}.$$

Thus, the Fourier cosine series of x is

$$x \sim \frac{L}{2} - \frac{4L}{\pi^2} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} \cos \frac{(2j+1)\pi x}{L}.$$

3.6 Complex Form of Fourier Series

Complex Form of Fourier Series

When dealing with a full Fourier series (i.e., Sine+Cosine series), it is frequently simpler algebraically to use the "complex form" of the Fourier series.

Complex Fourier Series

For a function f defined on the interval $[-L, L]$,

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{n\pi x}{L}} \tag{3.6.6}$$

where $c_n = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x) e^{i\frac{n\pi x}{L}} dx$.

This is not a new series! It is merely a way of rewriting the Fourier Sine+Cosine series that exploits Euler's formula. Note that a small benefit for doing so is that there is no longer a distinction between the formulae for c_0 and c_n with $n \neq 0$.

Rewriting the (Full) Fourier Series into its Complex Form

Euler's formulae

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Using the Euler's formulae, the full Fourier series become

$$\begin{aligned}
f(x) &\sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \\
&\sim a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{\frac{in\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)e^{-\frac{in\pi x}{L}} \\
&\sim a_0 + \frac{1}{2}\sum_{n=-\infty}^{-1}(a_{(-n)} - ib_{(-n)})e^{-\frac{in\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)e^{-\frac{in\pi x}{L}},
\end{aligned}$$

where we replaced n by $-n$ in the first summation.

From the definitions of a_n and b_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{and } b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,$$

we have $a_{(-n)} = a_n$ and $b_{(-n)} = -b_n$. Hence,

$$\begin{aligned} f(x) &\sim a_0 + \frac{1}{2}\sum_{n=-\infty}^{-1}(a_{(-n)} - ib_{(-n)})e^{-\frac{in\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)e^{-\frac{in\pi x}{L}} \\ &\sim a_0 + \frac{1}{2}\sum_{n=-\infty}^{-1}(a_n + ib_n)e^{-\frac{in\pi x}{L}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)e^{-\frac{in\pi x}{L}} \\ &\sim \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}}, \end{aligned}$$

where $c_0 = a_0$ and $c_n = \dfrac{a_n + ib_n}{2}$. This the complex form of the Fourier series of $f(x)$, with the *complex Fourier coefficients*:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right)\,dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{in\pi x}{L}}\,dx.$$

Another Way to Derive the Complex Fourier Coefficients

Complex Orthogonality

A complex function ϕ is said to be orthogonal to a complex function ψ if

$$\int_a^b \bar{\phi}\psi \, dx = 0,$$

where $\bar{\phi}$ is the complex conjugate of ϕ.

Using this notion, the eigenfunctions $\left\{e^{-\frac{in\pi x}{L}}\right\}_{n=-\infty}^{\infty}$ form an orthogonal set because

$$\begin{aligned}\int_{-L}^{L} \overline{(e^{-im\pi x/L})} e^{-in\pi x/L} \, dx &= \int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L} \, dx \\ &= \int_{-L}^{L} e^{i(m-n)\pi x/L} \, dx = \begin{cases} 0 & n \neq m \\ 2L & n = m. \end{cases}\end{aligned}$$

To determine the complex Fourier coefficients c_n, we multiply

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{-\frac{in\pi x}{L}} \tag{3.6.6}$$

by $e^{im\pi x/L}$ and integrate from $-L$ to L:

$$\int_{-L}^{L} f(x) e^{im\pi x/L}\, dx = \sum_{-\infty}^{\infty} c_n \int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L}\, dx = 2Lc_m$$

because of the complex orthogonality:

$$\int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L}\, dx = \begin{cases} 0 & n \neq m \\ 2L & n = m. \end{cases}$$

Complex Fourier coefficients

$$c_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{im\pi x/L}\, dx. \tag{3.6.7}$$