20240919 MATH3541 NOTE 5[1]

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Contents

1 Introduction

This note aims at quotient map and compact topological space.

2 Quotient Map

2.1 Partition Induced by a Surjective Function

Definition 2.1. (**Partition**)
Let X be a set, and $\widetilde{X}$ be a subset of $\mathcal{P}(X)$. If:
(1) $\forall \widetilde{x} \in \widetilde{X}, \widetilde{x} \neq \emptyset$;
(2) $\forall \widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}, \widetilde{x}_1 \neq \widetilde{x}_2 \implies \widetilde{x}_1 \cap \widetilde{x}_2 = \emptyset$;
(3) $\forall x \in X, \exists \widetilde{x} \in X, x \in \widetilde{x}$,
then $\widetilde{X}$ partitions X.

Proposition 2.2. Let X, Y be two sets, and $\sigma : X \to Y$ be a surjective function. The set $\{\pi(y) = \sigma^{-1}(\{y\})\}_{y \in Y}$ of fibres of σ partitions X.

Proof. We may divide our proof into three parts.
Part 1: For all fibre $\sigma^{-1}(\{y\})$ of σ:

$$\begin{aligned} \exists x \in X, \sigma(x) = y &\implies \exists x \in X, x \in \sigma^{-1}(\{y\}) \\ &\implies \sigma^{-1}(\{y\}) \neq \emptyset \end{aligned}$$

Part 2: For all fibres $\sigma^{-1}(\{y_1\}), \sigma^{-1}(\{y_2\})$ of σ:

$$\begin{aligned} \sigma^{-1}(\{y_1\}) \neq \sigma^{-1}(\{y_2\}) &\implies y_1 \neq y_2 \\ &\implies \sigma^{-1}(\{y_1\}) \cap \sigma^{-1}(\{y_2\}) = \sigma^{-1}(\{y_1\} \cap \{y_2\}) = \sigma^{-1}(\emptyset) = \emptyset \end{aligned}$$

Part 3: For all $x \in X$:

$$x \in \sigma^{-1}(\{\sigma(x)\})$$

To conclude, $\{\sigma^{-1}(\{y\})\}_{y \in Y}$ partitions X. Quod. Erat. Demonstrandum. □

Remark: *Restrict the potential function V to a surjective function, then the set of all equipotential surfaces partitions the whole space.*

Proposition 2.3. Let X, Y be two sets, and $\sigma : X \to Y$ be a surjective function. The relation $\widetilde{\sigma} : \widetilde{X} \to Y, \widetilde{X}(\pi(y)) = y$ is a bijective function.

Remark: *This proposition sets up a one to one correspondence between the equipotential surfaces and the potentials. Is this map continuous? Is this map a homeomorphism? These questions will be answered in the next subsection.*

2.2 Quotient Map and Its Criterion

Definition 2.4. (**Quotient Map**)
Let X, Y be two topological spaces,
and $\sigma : X \to Y$ be a surjective continuous function.
If $\widetilde{\sigma} : \widetilde{X} \to Y, \widetilde{\sigma}(\pi(y)) = y$ is a homeomorphism,
then σ is a quotient map.

Definition 2.5. (**Saturated Set**)
Let X be a topological space,
$\sim: X \to X$ be an equivalence relation on X,
$\widetilde{X}$ be the quotient space of $\sim$ on X, and U be a subset of X.
If $\pi^{-1}(\pi(U)) = U$, then U is saturated in X.

Proposition 2.6. Let X, Y be two topological spaces,
and $\sigma : X \to Y$ be a surjective continuous function.
The following two statements are logically equivalent:
(1) σ is a quotient map.
(2) $\forall U \in \mathcal{P}(X), U$ is saturated and open in $X \implies \sigma(U)$ is open in Y.

Proof. We may divide our proof into two parts.
(1) $\implies$ (2) : For all $V \in \mathcal{P}(\widetilde{X})$:

$$\begin{aligned} V \text{ is open in } \widetilde{X} &\iff \pi^{-1}(V) \text{ is saturated and open in } X \\ &\iff \widetilde{\sigma}(V) = \sigma(\pi^{-1}(V)) \text{ is open in } Y \end{aligned}$$

(2) $\implies$ (1) : For all $U \in \mathcal{P}(X)$:

$$\begin{aligned} U \text{ is saturated and open in } X &\implies \pi(U) \text{ is open in } \widetilde{X} \\ &\implies \sigma(U) = \widetilde{\sigma}(\pi(U)) \text{ is open in } Y \end{aligned}$$

Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. □

Definition 2.7. (**Hawaii Earring** $\mathbb{H}_N$)
Define $\mathbb{S}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$.
Define the subspace $\bigcup_{n=N}^{+\infty} \mathbb{S}\left(\frac{\mathrm{i}}{3^n}, \frac{1}{3^n}\right)$ of $\mathbb{C}$ as the Hawaii earring $\mathbb{H}_N$.

Proposition 2.8. The Hawaii earring $\mathbb{H}_N$ is not a wedge sum.

Proof. Consider the point $0 \in \mathbb{H}_N$.

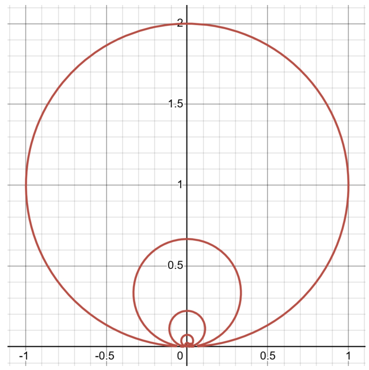

Figure 1: Hawaii Earring $\mathbb{H}_0$

For all $\epsilon > 0$, there exists $M \geq N$, such that for all $n \geq M$, the diameter $\frac{2}{3^n} < \epsilon$. This implies the partial union $\bigcup_{n=M}^{+\infty} \mathbb{S}\left(\frac{\mathrm{i}}{3^n}, \frac{1}{3^n}\right)$ is contained in $\mathbb{D}(0, \epsilon)$.
That is, every open neighbour of $0 \in \mathbb{H}_N$ contains infinitely many circles,
which is not true in the wedge sum case. Quod. Erat. Demonstrandum. $\square$

Proposition 2.9. The projection σ from $\coprod_{n=N}^{+\infty} \mathbb{S}$ to $\mathbb{H}_N$ is surjective and continuous. As $\mathbb{H}_N$ is strictly coarser than $\coprod_{n=N}^{+\infty} \mathbb{S}$, $\widetilde{\sigma}$ is not continuous.

3 Compactness in Topological space

3.1 Compact Topological Space

Definition 3.1. (**Open Cover**)
Let X be a topological space, and $(U_\lambda)_{\lambda \in I}$ be an indexed family of open sets of X. If $\bigcup_{\lambda \in I} U_\lambda = X$, then $(U_\lambda)_{\lambda \in I}$ is an open cover of X.

Definition 3.2. (**Subcover**)
Let X be a topological space, $(U_\lambda)_{\lambda \in I}$ be an open cover of X, and J be a subset of I. If $(U_\lambda)_{\lambda \in J}$ is also an open cover of X, then $(U_\lambda)_{\lambda \in J}$ is a subcover of $(U_\lambda)_{\lambda \in I}$. If J is finite, then the subcover $(U_\lambda)_{\lambda \in J}$ is finite.

Definition 3.3. (**Compact Topological Space**)
Let X be a topological space. If all open cover $(U_\lambda)_{\lambda \in I}$ of X has a finite subcover $(U_{\lambda_k})_{k=1}^{m}$, then X is compact.

3.2 Construct New Compact Spaces from Old Ones

Proposition 3.4. Let X, Y be two topological spaces,
and $\sigma : X \to Y$ be a surjective continuous function.
If X is compact, then Y is compact.

Proof. For all open cover $(V_\lambda)_{\lambda \in I}$ of Y, $(\sigma^{-1}(V_\lambda))_{\lambda \in I}$ is an open cover of X.
There exists $(\lambda_k)_{k=1}^m$ in I, such that $(\sigma^{-1}(V_{\lambda_k}))_{k=1}^m$ covers X.
There exists $(\lambda_k)_{k=1}^m$ in I, such that $(V_{\lambda_k} = \sigma(\sigma^{-1}(V_{\lambda_k})))_{k=1}^m$ covers Y.
Hence, Y is compact. Quod. Erat. Demonstrandum. □

Proposition 3.5. Let X be a topological space,
and X' be a closed subspace of X.
If X is compact, then X' is compact.

Proof. For all open cover $(U'_\lambda)_{\lambda \in I}$ of X', each $U'_\lambda = U_\lambda \cap X'$ for some $U_\lambda \in \mathcal{O}_X$.
This implies $(U_\lambda, X'^c)_{\lambda \in I}$ is an open cover of $(X, \mathcal{O}_\mathcal{X})$.
There exists $(\lambda_k)_{k=1}^m$ in I, such that $(U_{\lambda_k}, X'^c)_{k=1}^m$ covers X.
There exists $(\lambda_k)_{k=1}$ in I, such that $(U'_{\lambda_k})_{k=1}^m$ covers X.
Hence, X' is compact. Quod. Erat. Demonstrandum. □

3.3 Tychonoff's Theorem

Definition 3.6. (**Product Space Topology**)
Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces.
Define the product topology $\mathcal{O}_X$ of $(X_\lambda)_{\lambda \in I}$ on $X = \prod_{\lambda \in I} X_\lambda$ as the topology generated by the subbasis $\mathcal{B}_X$, which is the union of each initial topology of X_λ on X via $\pi_\lambda : X \to X_\lambda, \pi_\lambda(x) = x(\lambda)$.

Definition 3.7. (**Partial Order**)
Let X be a set, and $\leq : X \to X$ be a relation on X. If:
(1) $\forall x \in X, x \leq x$;
(2) $\forall x_1, x_2 \in X, x_1 \leq x_2$ and $x_2 \leq x_1 \implies x_1 = x_2$;
(3) $\forall x_1, x_2, x_3 \in X, x_1 \leq x_2$ and $x_2 \leq x_3 \implies x_1 \leq x_3$,
then $\leq$ is a partial order on X.
If $x_1 \leq x_2$ and $x_1 \neq x_2$, then $x_1 < x_2$.

Definition 3.8. (**Total Order**)
Let X be a set, and $\leq: X \to X$ be a relation on X. If:
(1) $\forall x \in X, x \leq x$;
(2) $\forall x_1, x_2 \in X, x_1 \leq x_2$ and $x_2 \leq x_1 \implies x_1 = x_2$;
(3) $\forall x_1, x_2, x_3 \in X, x_1 \leq x_2$ and $x_2 \leq x_3 \implies x_1 \leq x_3$;
(4) $\forall x_1, x_2 \in X, x_1 \leq x_2$ or $x_2 \leq x_1$,
then $\leq$ is a total order on X.

Definition 3.9. (**Upper Bound**)
Let $(X, \leq)$ be a partially ordered set,
A be a subset of X, and β be an element of X.
If $\forall x \in A, x \leq \beta$, then β is an upper bound of A.

Definition 3.10. (**Maximal Element**)
Let $(X, \leq)$ be a partially ordered set, and x be an element of X.
If $\forall x' \in X, x \leq x' \implies x = x'$, then x is maximal.

Lemma 3.11. (**Zorn's Lemma**[2])
Let $(X, \leq)$ be a nonempty partially ordered set,
If each totally ordered nonempty subset A of X has an upper bound $\beta \in X$, then X has a maximal element $x \in X$.

Remark: *In this note, we assume* ***Lemma 3.11.*** *without proof.*

Theorem 3.12. (**Alexander's Subbasis Theorem**[2])
Let X be a topological space, and $\mathcal{B}_X$ be a subbasis of X. X is compact if and only if every open cover $\mathcal{V} \subseteq \mathcal{B}_X$ of X has a finite subcover.

Proof. It suffices to prove **"if" direction**.
Assume to the contrary that X is not compact.
Step 1: Define $\mathbf{\Phi}$ as the set of all open cover $\mathcal{U}$ of X with no finite subcover.
Define a partial order $\leq: \mathbf{\Phi} \to \mathbf{\Phi}, \mathcal{U}_1 \leq \mathcal{U}_2$ if $\mathcal{U}_1 \subseteq \mathcal{U}_2$ on $\mathbf{\Phi}$.
For all nonempty totally ordered subset $\mathbf{\Psi}$ of $\mathbf{\Phi}$:
Property 1.1: $\forall (U_k)_{k=1}^m$ in $\bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U}, \exists (\mathcal{U}_k)_{k=1}^m$ in $\mathbf{\Psi}$, each $U_k \in \mathcal{U}_k$.
Without loss of generality, assume that $(\mathcal{U}_k)_{k=1}^m$ is ascending.
This implies $(U_k)_{k=1}^m$ in $\mathcal{U}_m$, so $(U_k)_{k=1}^m$ doesn't cover X. Hence, $\bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U} \in \mathbf{\Phi}$.
Property 1.2: $\forall \mathcal{V} \in \mathbf{\Psi}, \mathcal{V} \leq \bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U}$. Hence, $\bigcup_{\mathcal{U} \in \mathbf{\Psi}} \mathcal{U}$ is an upper bound of $\mathbf{\Psi}$.
According to **Lemma 3.11.**, $\mathbf{\Phi}$ has a maximal element $\mathcal{V}$.
Step 2: Assume to the contrary that $\mathcal{V} \cap \mathcal{B}_X$ is an open cover of X.
$\mathcal{V}$ has no finite subcover, neither does $\mathcal{V} \cap \mathcal{B}_X$.
However, $\mathcal{V} \cap \mathcal{B}_X \subseteq \mathcal{B}_X$, which has a finite subcover, a contradiction.

Hence, our assumption is false, and we've proven $\mathcal{V} \cap \mathcal{B}_X$ is not an open cover of X.
Step 3: Assume that $\mathcal{V} \cap \mathcal{B}_X = (V_\lambda)_{\lambda \in J}$, where $J \subset I$, and fix $x_0 \in \bigcup_{\lambda \in I \setminus J} V_\lambda$.
As $\mathcal{B}_X$ is a subbasis of X, $x_0 \in \bigcap_{k=1}^m W_k \subseteq V_{\lambda_*}$, where each $W_k \in \mathcal{B}_X, \lambda_* \in I$.
Assume to the contrary that some $W_k \in \mathcal{V}$.
As $W_k \in \mathcal{B}_X$, $x_0 \in W_k \in \mathcal{V} \cap \mathcal{B}_X$, but $\mathcal{V} \cap \mathcal{B}_X$ doesn't cover x_0, a contradiction.
Hence, our assumption is false, and we've proven each $W_k \notin \mathcal{V}$.
Step 4: For each W_k, define $\mathcal{V}_k$ as a finite subcover of $\mathcal{V} \cup \{W_k\}$.
Assume to the contrary that $W_k \notin \mathcal{V}_k$.
This implies $\mathcal{V}$ has a finite subcover $\mathcal{V}_k$, a contradiction.
Hence, each $\mathcal{V}_k$ is in the form $(W_k, V_{\lambda_{kl_k}})_{l_k=1}^{n_k}$. This implies:

$$X = W_k \cup \bigcup_{l_k=1}^{n_k} V_{\lambda_{kl_k}} \implies \bigcap_{l_k=1}^{n_k} V_{\lambda_{kl_k}}^c \subseteq W_k \implies \bigcap_{k=1}^{m} \bigcap_{l_k=1}^{n_k} V_{\lambda_{kl_k}}^c \subseteq \bigcap_{k=1}^{m} W_k \subseteq V_{\lambda_*}$$

To conclude, our assumption is false, as we've constructed a finite subcover $(V_{\lambda_*}, V_{\lambda_{kl_k}})$ of $\mathcal{V}$. Quod. Erat. Demonstrandum. □

Theorem 3.13. (**Tychonoff Theorem**[2])
Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces,
and X be the product space of $(X_\lambda)_{\lambda \in I}$.
If each X_λ is compact, then X is compact.

Proof. For all $\lambda \in I$, define $\mathcal{U}_\lambda$ as the initial topology of X_λ on X via π_λ.
It suffices to show that each open cover $\mathcal{V} \subseteq \bigcup_{\lambda \in I} \mathcal{U}_\lambda$ of X has a finite subcover.
Step 1: Assume to the contrary that no $\pi_\lambda(\mathcal{V} \cap \mathcal{U}_\lambda)$ covers X_λ.
For all $\lambda \in I$, the assumption above guarantees the existence of $\xi_\lambda \in (\pi(\mathcal{V} \cap \mathcal{U}_\lambda))^c$.
Construct $x \in X, x(\lambda) = \xi_\lambda$.
As each $\mathcal{V} \cap \mathcal{U}_\lambda$ doesn't cover x, neither does $\mathcal{V} = \bigcup_{\lambda \in I} (\mathcal{V} \cap \mathcal{U}_\lambda)$, a contradiction.
Hence, our assumption is wrong, and we've proven that some $\pi_\lambda(\mathcal{V} \cap \mathcal{U}_\lambda)$ covers X_λ.
Step 2: As some $\pi_\lambda(\mathcal{V} \cap \mathcal{U}_\lambda)$ covers X_λ, a finite subcover $\pi_\lambda(\mathcal{W})$ exists.
Hence, we've reduced our original open cover $\mathcal{V}$ to a finite subcover $\mathcal{W}$.
Quod. Erat. Demonstrandum. □

3.4 Compactness as a Topological Invariant

Proposition 3.14. Compactness is a topological invariant.

Proof. For all X, Y, assume that there exists a homeomorphism $\sigma : X \to Y$.
As σ is surjective and continuous, X is compact implies Y is compact.
As σ^{-1} is surjective and continuous, Y is compact implies X is compact.
Hence, we've proven that compactness is a topological invariant.
Quod. Erat. Demonstrandum. □

Proposition 3.15. Let X be a topological space, and X' be a subspace of X. If X is Hausdorff and X' is compact, then $X' \in \mathcal{C}_X$.

Proof. For all $x \in X'^c$ and $x' \in X'$, there exist $U_{xx'}, V_{xx'} \in \mathcal{O}_X$,
such that $x \in U_{xx'}$ and $x' \in V_{xx'}$ and $U_{xx'} \cap V_{xx'} = \emptyset$.
Fix $x \in X'^c$, and we get an open cover $(V_{xx'} \cap X')_{x' \in X'}$ of X'.
There exists $(x'_k)_{k=1}^m$ in X', such that $(V_{xx'_k} \cap X')_{k=1}^m$ covers X'.
There exist $U_x = \bigcap_{k=1}^m U_{xx'_k}, V_{X'} = \bigcup_{k=1}^m V_{xx'_k} \in \mathcal{O}_X$,
such that $x \in U_x$ and $X' \subseteq V_{X'}$ and $U_x \cap V_{X'} = \emptyset$.
Hence, $X'^c = \bigcup_{x \in X'^c} U_x \in \mathcal{O}_X$, which implies $X' \in \mathcal{C}_X$.
Quod. Erat. Demonstrandum. □

Lemma 3.16. (**Closed Map Lemma**[3])
Let X, Y be two topological spaces,
and $\sigma : X \to Y$ be a continuous function.
If X is compact and Y is Hausdorff, then σ is closed.

Proof. For all $X' \in \mathcal{C}_X$:
According to **Proposition 3.5.**, X' is compact.
According to **Proposition 3.4.**, $\sigma(X')$ is compact.
According to **Proposition 3.15.**, $\sigma(X') \in \mathcal{C}_X$.
Hence, σ is closed. Quod. Erat. Demonstrandum. □

Remark: *Prof. Hua said that closeness is a global property of certain function. This lemma offers us an insight into it. Notice that two topological invariants, i.e., compactness and Hausdorffness, are involved to ensure the closeness of a continuous function.*

Definition 3.17. (**Proper Map**)
Let X, Y be two topological spaces, and $\sigma : X \to Y$ be a function.
If $\forall V \in \mathcal{P}(Y), V$ is compact $\implies \sigma^{-1}(V)$ is compact, then σ is proper.

Proposition 3.18. Let X, Y be two topological spaces,
and $\sigma : X \to Y$ be a function.
If X is compact, Y is Hausdorff and σ is continuous, then σ is proper.

Proof. For all $V \in \mathcal{P}(Y)$, assume that V is compact.
According to **Proposition 3.15.**, Y is Hausdorff implies $V \in \mathcal{C}_Y$.
As σ is continuous, $V \in \mathcal{C}_Y$ implies $\sigma^{-1}(V) \in \mathcal{C}_X$.
According to **Proposition 3.5.**, $\sigma^{-1}(V) \in \mathcal{C}_X$ implies $\sigma^{-1}(V)$ is compact.
Hence, σ is proper. Quod. Erat. Demonstrandum. □

Definition 3.19. (**Locally Compact Topological Space**)
Let X be a Hausdorff topological space.
If $\forall x \in X, \exists U \in \mathcal{O}_X, x \in U$ and $\overline{U}$ is compact, then X is locally compact.

Lemma 3.20. (**The Proper Map Lemma**[3])
Let X, Y be two topological spaces, and $\sigma : X \to Y$ be a function.
If Y is locally compact, and σ is continuous and proper, then σ is closed.

Proof. For all $U \in \mathcal{P}(X)$, let's assume $U \in \mathcal{C}_X$, and prove $\sigma(U) \in \mathcal{C}_Y$.
For all $y \in \sigma(U)^c$, we try to find $W \in \mathcal{O}_Y$ with $W \subseteq \sigma(U)^c$, such that $y \in W$:
As Y is locally compact, there exists $V \in \mathcal{O}_Y$, such that $y \in V$ and $\overline{V}$ is compact.
As σ is proper, $\overline{V}$ is compact implies $\sigma^{-1}(\overline{V})$ is compact.
According to **Proposition 3.5.**, $U \cap \sigma^{-1}(\overline{V})$ is closed implies $U \cap \sigma^{-1}(\overline{V})$ is compact.
As σ is continuous, $\sigma(U \cap \sigma^{-1}(\overline{V})) = \sigma(U) \cap \overline{V}$ is compact.
According to **Proposition 3.15.**, $\sigma(U) \cap \overline{V}$ is compact implies $\sigma(U) \cap \overline{V}$ is closed.
Take $W = V \backslash [\sigma(U) \cap \overline{V}] \in \mathcal{O}_Y$.
Note that $y \in W = V \backslash \sigma(U) \subseteq \sigma(U)^c$, so $\sigma(U)^c \in \mathcal{O}_Y$.
This implies $\sigma(U) \in \mathcal{C}_X$, so σ is closed. Quod. Erat. Demonstrandum. □

4 Compactness in Metric Space

4.1 Completeness, Total Boundedness and Compactness

Definition 4.1. (**Metric Space**)
Let X be a set, and $d_X : X \times X \to \mathbb{R}$ be a function. If:
(1) $\forall x_1, x_2 \in X, d_X(x_1, x_2) \geq 0$ and $d_X(x_1, x_2) = 0 \iff x_1 = x_2$;
(2) $\forall x_1, x_2 \in X, d_X(x_1, x_2) = d_X(x_2, x_1)$;
(3) $\forall x_1, x_2, x_3 \in X, d_X(x_1, x_2) + d_X(x_2, x_3) \geq d_X(x_1, x_3)$,
then X is a metric space.

Definition 4.2. (**Metric Space Topology**)
Let X be a metric space.
Define metric space topology on X as the topology $\mathcal{O}_X$ generated by the basis $\mathcal{B}_X = \{B(x, r)\}_{(x,r) \in X \times \mathbb{R}_{>0}}$, where $B(x, r) = \{x'\}_{d(x,x') < r}$.

Definition 4.3. (**Convergent Sequence**)
Let X be a metric space, $(x_n)_{n \in \mathbb{N}}$ be a sequence in X, and x_* be a point in X. If $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}_{\geq N}, d(x, x_*) < \epsilon$, then $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in X with limit x_*.

Definition 4.4. (**Cauchy Sequence**)
Let X be a metric space, and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n_1, n_2 \in \mathbb{N}_{\geq N}, d(x_{n_1}, x_{n_2}) < \epsilon$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Proposition 4.5. Let X be a metric space, and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(x_n)_{n\in\mathbb{N}}$ is convergent in X, then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof. Assume that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in X with limit x_*.
$\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}_{\geq N}, d(x_n, x_*) < \frac{\epsilon}{2}$.
$\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n_1, n_2 \in \mathbb{N}_{\geq N}, d(x_{n_1}, x_{n_2}) \leq d(x_{n_1}, x_*) + d(x_{n_2}, x_*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
Hence, $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Quod. Erat. Demonstrandum. $\square$

Definition 4.6. (**Complete Metric Space**)
Let X be a metric space.
If every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X is convergent in X,
then X is complete.

Proposition 4.7. Let X be a metric space.
If X is compact, then X is complete.

Proof. Assume to the contrary that X is not complete.
There exists some Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that it has no limit.
For each $x \in X$, there exists $r_x > 0$, such that $B(x, r_x)$ only contains finitely many terms of $(x_n)_{n\in\mathbb{N}}$, so if the open cover $(B(x, r_x))_{x\in X}$ is shrinked to a finite subcover, we get a contradiction where $(x_n)_{n\in\mathbb{N}}$ should and shouldn't contain finitely many terms.
Hence, our assumption is false, and we've proven that X is complete.
Quod. Erat. Demonstrandum. $\square$

Definition 4.8. (**Totally Bounded Metric Space**)
Let X be a metric space.
If every $r > 0$ gives a finite cover $(B(x_k, r))_{k=1}^m$ of X,
then X is totally bounded.

Proposition 4.9. Let X be a metric space.
If X is compact, then X is totally bounded.

Proof. For all $r > 0$, construct an open cover $(B(x, r))_{x\in X}$ of X.
There exists a finite subcover $(B(x_k, r))_{k=1}^m$.
Hence, X is totally bounded. Quod. Erat. Demonstrandum. $\square$

Proposition 4.10. Let X be a metric space.
If X is complete and totally bounded, then X is compact.[4]

Proof. Assume to the contrary that X is not compact.
There exists an open cover $(U_\lambda)_{\lambda\in I}$ of X, which has no finite subcover.
Let's find a hole in X.
Step 1: Construct a sequence of open balls.
$\exists(x^1_{k_1})^{m_1}_{k_1=1}$ in $X, (B(x^1_{k_1}, 1))^{m_1}_{k_1=1}$ is an open cover of X,
so some $B(x^1_{k_1}, 1) \cap X$ can't be finitely covered by $(U_\lambda)_{\lambda\in I}$.
$\exists(x^{n+1}_{k_{n+1}})^{m_{n+1}}_{k_{n+1}=1}$ in $X, (B(x^{n+1}_{k_{n+1}}, 2^{-n-1}))^{m_{n+1}}_{k_{n+1}=1}$ is an open cover of $B(x^n_{k_n}, 2^{-n})$,
so some $B(x^{n+1}_{k_{n+1}}, 2^{-n-1}) \cap B(x^n_{k_n}, 2^{-n})$ can't be finitely covered by $(U_\lambda)_{\lambda\in I}$.
Step 2: State some key properties of this sequence of open balls.
Property 2.1: The radius sequence $(r_n = 2^{-n-1})^{+\infty}_{m=0}$ tends to 0;
Property 2.2: The centre sequence $(c_n = x^n_{k_n})_{n\in\mathbb{N}}$ is Cauchy;
Property 2.3: Each $B(c_n, r_n)$ cannot be finitely covered by $(U_\lambda)_{\lambda\in I}$.
As $(c_n)_{n\in\mathbb{N}}$ converges to some $c_* \in X$, there exists $U_{\lambda_*} \ni c_*$.
This implies some $B(c_n, r_n)$ is covered by U_{λ_*}, a contradiction.
Hence, our assumption is false, and we've proven that X is compact.
Quod. Erat. Demonstrandum. □

4.2 Compact Subsets of $\mathbb{R}$ and $M_m(\mathbb{C})$

Proposition 4.11. Let X be a metric space, and U be a subset of X.
If X is complete, then U is closed if and only if U is complete.

Proof. We may divide our proof into two parts.
Part 1: Assume that U is complete.
For all $x_* \in U^c$, construct the following set:

$$R = U^c \backslash \{x_*\}$$

Define the following number:

$$r = \begin{cases} 1 & \text{if} \quad R = \emptyset; \\ \inf\limits_{x\in R} d_X(x, x_*) & \text{if} \quad R \neq \emptyset; \end{cases}$$

There exists an open ball $B(x_*, r)$ with $B(x_*, r) \subseteq U^c$,
such that $B(x_*, r) \ni x_*$, which implies U is closed.
Part 2: Assume to the contrary that U is not complete.
There exists a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in U, which is not convergent in U.
As $(x_n)_{n\in\mathbb{N}}$ is in X, which is complete, it must converge to some $x_* \in U^c$.
This implies the existence of $n \in \mathbb{N}$, such that $x_n \in U^c$, a contradiction.
Hence, our assumption is false, and we've proven that U is complete.

Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. ☐

Definition 4.12. (**Bounded Metric Space**)
Let X be a metric space.
If $\exists x \in X$ and $r > 0, B(x, r) = X$, then X is bounded.

Proposition 4.13. Let X be a metric space.
If X is totally bounded, then X is bounded.

Proof. 1 gives a finite cover $(B(x_k, 1))_{k=1}^m$ of X.
If we take $r = \max\{d_X(x_1, x_k)\}_{k=1}^m + 1 > 0$, then $X = B(x_1, r)$,
which implies X is bounded. Quod. Erat. Demonstrandum. ☐

Proposition 4.14. In metric space $\mathbb{R}$,
if a subspace U is bounded, then U is totally bounded.

Proof. Assume that U is bounded, then U is contained in some open interval.
Without loss of generality, we may assume that it is $(0, 1)$.
For all $r > 0$, there exists $m \in \mathbb{N}$, such that $mr > 1$.
Hence, $r > 0$ gives a finite cover $(((k-1)r, (k+1)r))_{k=1}^m$ of U.
This implies U is totally bounded. Quod. Erat. Demonstrandum. ☐

Theorem 4.15. (**Heine-Borel Theorem**)
In metric space $\mathbb{R}$, every $U \subseteq \mathbb{R}$ is compact if and only if U is closed and bounded.

Proof. We may divide our proof into two parts.
"if" direction: Assume that U is closed and bounded.
According to **Proposition 4.11.**, U is closed implies U is complete.
According to **Proposition 4.14.**, U is bounded implies U is totally bounded.
According to **Proposition 4.10.**, U is compact.
"only if" direction: Assume that U is compact.
According to **Proposition 4.7.**, U is compact implies U is complete.
According to **Proposition 4.11.**, U is complete implies U is closed.
According to **Proposition 4.9.**, U is compact implies U is totally bounded.
According to **Proposition 4.13.**, U is totally bounded implies U is bounded.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. ☐

Definition 4.16. (**Normed Vector Space**)
Let Z be a vector space over field $\mathbb{C}$, and $\|\cdot\|_Z : Z \to \mathbb{R}$ be a function. If:
(1) $\forall \mathbf{z} \in Z, \|\mathbf{z}\|_Z \geq 0$ and $\|\mathbf{z}\|_Z = 0 \iff \mathbf{z} = \mathbf{0}$;
(2) $\forall \lambda \in \mathbb{C}, \|\lambda \mathbf{z}\|_Z = |\lambda| \|\mathbf{z}\|_Z$;
(3) $\forall \mathbf{z}_1, \mathbf{z}_2 \in Z, \|\mathbf{z}_1\|_Z + \|\mathbf{z}_2\|_Z \geq \|\mathbf{z}_1 + \mathbf{z}_2\|_Z$,
then $(Z, \|\cdot\|)$ is a normed vector space.

Proposition 4.17. In metric space $M_m(\mathbb{C})$, let A be a matrix. Define:

$$\|A\|_{\text{spectral}} = \sup_{\|\mathbf{z}\|=1} \|A\mathbf{z}\| \quad \|A\|_{\text{Frobenius}} = \sqrt{\sum_{k=1}^{m} \|A\mathbf{e}_k\|^2}$$

The two norms above induce the same topology on $M_m(\mathbb{C})$.

Proof.

$$\begin{aligned}
\|A\|_{\text{spectral}} &= \sup_{\|\mathbf{z}\|=1} \|A\mathbf{z}\| = \sup_{\|\mathbf{z}\|=1} \left\| \sum_{k=1}^{m} z_k A\mathbf{e}_k \right\| \leq \sup_{\|\mathbf{z}\|=1} \sum_{k=1}^{m} |z_k| \, \|A\mathbf{e}_k\| \\
&\leq \sup_{\|\mathbf{z}\|=1} \sqrt{\sum_{k=1}^{m} |z_k|^2} \sqrt{\sum_{k=1}^{m} \|A\mathbf{e}_k\|^2} = \|A\|_{\text{Frobenius}} \\
\|A\|_{\text{Frobenius}} &= \sqrt{\sum_{k=1}^{m} \|A\mathbf{e}_k\|^2} \leq \sqrt{\sum_{k=1}^{m} \left(\sup_{\|\mathbf{z}\|=1} \|A\mathbf{z}\| \right)^2} = \sqrt{m} \|A\|_{\text{spectral}}
\end{aligned}$$

Hence, the two norms induce the same topology on $M_m(\mathbb{C})$.
Quod. Erat. Demonstrandum. □

Proposition 4.18. Let $O_m(\mathbb{C})$ be a subspace of $M_m(\mathbb{C})$. $O_m(\mathbb{C})$ is compact.

Proof. With spectral norm, $O_m(\mathbb{C}) = \partial B(O, 1)$ is a closed bounded subspace of $M_m(\mathbb{C})$, so it is compact. Quod. Erat. Demonstrandum. □

Proposition 4.19. Let $\mathcal{A}$ be a compact subset of $M_m(\mathbb{C})$.
The function $\phi : \mathbb{C} \times \mathcal{A} \to \mathbb{C}, \phi(\lambda, A) = \det(\lambda I - A)$ is proper.

Proof. For all $V \in \mathcal{P}(\mathbb{C})$, assume that V is compact, let's prove that $\phi^{-1}(V)$ is compact.
According to **Theorem 4.15.**, V is compact implies V is closed and bounded.
As ϕ is continuous, V is closed implies $\phi^{-1}(V)$ is closed.
Assume to the contrary that $\pi_1(\phi^{-1}(V))$ is not bounded.
Fix $A \in \mathcal{A}$, find $(\lambda_n)_{n \in \mathbb{N}}$ in $\pi_1(\phi^{-1}(V))$, such that:

$$\lim_{n \to +\infty} \lambda_n = \infty \implies \lim_{n \to +\infty} \phi(\lambda_n, A) = \lim_{n \to +\infty} \lambda_n^m = \infty$$

Hence, our assumption is false, and we've proven that $\pi_1(\phi^{-1}(V))$ is bounded.
$\pi_1(\phi^{-1}(V)), \pi_2(\phi^{-1}(V))$ are bounded implies $\phi^{-1}(V)$ is bounded.
According to **Theorem 4.15.**, $\phi^{-1}(V)$ is closed and bounded implies $\phi^{-1}(V)$ is compact, so ϕ is proper. Quod. Erat. Demonstrandum. $\square$

Proposition 4.20. Let $\mathcal{A}$ be a compact subset of $M_m(\mathbb{C})$.
spec $(\mathcal{A}) = \pi_1(\phi^{-1}(\{0\}))$ is compact.

4.3 Limit Point Compactness and Sequential Compactness

Definition 4.21. (**Distance from Point to Subet**)
Let X be a metric space, x be a point of X, and X' be a nonempty subset of X.
Define $d_X(x, X') = \inf_{x' \in X'} d_X(x, x')$ as the distance from x to X'.

Definition 4.22. (**Distance from Subset to Subset**)
Let X be a metric space, and X', X'' be two nonempty subsets of X.
Define $d_X(X', X'') = \inf_{(x', x'') \in X' \times X''} d_X(x', x'')$ as the distance from X' to X''.

Proposition 4.23. Let X be a metric space, and X' be a nonempty subset of X. $\rho_{X'} : X \to \mathbb{R}, \rho_{X'}(x) = d_X(x, x')$ is 1-Lipschitz continuous.

Proof. For all $x_1, x_2 \in X$:

$$\begin{aligned}
\rho_{X'}(x_2) &= \inf_{x' \in X'} d_X(x_2, x') \\
&\leq \inf_{x' \in X'} [d_X(x_2, x_1) + d_X(x_1, x')] = d_X(x_2, x_1) + \rho_{X'}(x_1) \\
\rho_{X'}(x_1) &= \inf_{x' \in X'} d_X(x_1, x') \\
&\leq \inf_{x' \in X'} [d_X(x_1, x_2) + d_X(x_2, x')] = d_X(x_1, x_2) + \rho_{X'}(x_2) \\
|\rho_{X'}(x_2) - \rho_{X'}(x_1)| &\leq d_X(x_2, x_1)
\end{aligned}$$

Hence, $\rho_{X'}$ is 1-Lipschitz continuous. Quod. Erat. Demonstrandum. $\square$

Definition 4.24. (**Lebesgue Function**)
Let X be a metric space, U be a nonempty subset of X,
and $(U_\lambda)_{\lambda \in I}$ be a nontrivial open cover of U.
If $\forall x \in X, \{d_X(x, U_\lambda^c)\}_{\lambda \in I}$ is bounded above,
then define the Lebesgue function with respect to $U, (U_\lambda)_{\lambda \in I}$ as:

$$\ell_U : U \to \mathbb{R}, \ell(x) = \sup_{\lambda \in I} d_X(x, U_\lambda^c)$$

Proposition 4.25. Let X be a metric space,
U be a nonempty subset of X, and $(U_\lambda)_{\lambda \in I}$ be a nontrivial open cover of U.
If $\forall x \in U, \{d_X(x, U_\lambda^c)\}_{\lambda \in I}$ is bounded above, then ℓ_U is 1-Lipschitz continuous.

Proof. For all $x_1, x_2 \in X$:

$$\begin{aligned}
\ell_U(x_2) &= \sup_{\lambda \in I} d_X(x_2, U_\lambda^c) \\
&\leq \sup_{\lambda \in I} [d_X(x_2, x_1) + d_X(x_1, U_\lambda^c)] = d_X(x_2, x_1) + \ell_U(x_1) \\
\ell_U(x_1) &= \sup_{\lambda \in I} d_X(x_1, U_\lambda^c) \\
&\leq \sup_{\lambda \in I} [d_X(x_1, x_2) + d_X(x_2, U_\lambda^c)] \\
&= d_X(x_1, x_2) + \ell_U(x_2) |\ell_U(x_2) - \ell_U(x_1)| \leq d_X(x_2, x_1)
\end{aligned}$$

Hence, ℓ_U is 1-Lipschitz continuous. Quod. Erat. Demonstrandum. $\square$

Remark: *The two* 1*-Lipschitz continuous functions seem similar, but the second function is more interesting because it is positive definite.*

Proposition 4.26. Let X be a metric space,
U be a nonempty subset of X, and $(U_\lambda)_{\lambda \in I}$ be a nontrivial open cover of U.
If $\forall x \in U, \{d_X(x, U_\lambda^c)\}_{\lambda \in I}$ is bounded above, then $\forall x \in U, \ell_U(x) > 0$.

Proof. For all $x \in U$:
As $(U_\lambda)_{\lambda \in I}$ covers U, x is in some U_{λ_*}.
As $U_{\lambda_*} \in \mathcal{O}_X$, U_{λ_*} contains some $B(x, r_{\lambda_*})$.
As $(U_\lambda)_{\lambda \in I}$ is nontrivial, $U_{\lambda_*} \not\supseteq U$.
This implies $U_{\lambda_*}^c \neq \emptyset$, so:

$$\ell_U(x) = \sup_{\lambda \in I} d_X(x, U_\lambda^c) \geq d_X(x, U_{\lambda_*}^c) \geq r_* > 0$$

Quod. Erat. Demonstrandum. $\square$

Remark: *We would like to minimize this function.*

Definition 4.27. (**Lebesgue Number**)
Let X be a metric space, U be a nonempty subset of X,
and $(U_\lambda)_{\lambda \in I}$ be a nontrivial open cover of U.
If $\forall x \in U, \{d(x, U_\lambda^c)\}_{\lambda \in I}$ is bounded above and $\min_{x \in U} \ell_U(x)$ exists, then define:

$$L_U = \min_{x \in U} \ell_U(x)$$

as the Lebesgue number with respect to $U, (U_\lambda)_{\lambda \in I}$.

Definition 4.28. (**Limit Point Compact Set**)
Let X be a metric space, and U be a subset of X. If every infinite subset V of U has an limit point, then X is limit point compact.

Definition 4.29. (**Sequentially Compact Set**)
Let X be a metric space, and U be a subset of X. If every sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$, then U is sequentially compact.

Proposition 4.30. Let X be a metric space, and U be a subset of X.
U is sequentially compact if and only if U is limit point compact.

Proof. As long as an denumerable infinite set $\{x_n\}_{n\in\mathbb{N}}$ is identified with $(x_n)_{n\in\mathbb{N}}$, the logical equivalency will be clear. Quod. Erat. Demonstrandum. □

Lemma 4.31. Let $(X, d_X), (Y, d_Y)$ be two metric spaces,
U be a subset of X, and $\sigma : U \to V$ be a continuous function.
If U is sequentially compact, then $\sigma(U)$ is sequentially compact.

Proof. For all $(y_n)_{n\in\mathbb{N}}$ in $\sigma(U)$, each y_n is the image of some $x_n \in U$.
As U is sequentially compact, (x_n) has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$.
As σ is continuous, $(\sigma(x_n))_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$.
Hence, $\sigma(U)$ is sequentially compact. Quod. Erat. Demonstrandum. □

Remark: *The continuous image of a sequentially compact set is sequentially compact. Hence, any continuous function defined on a sequentially compact domain has minimum.*

Lemma 4.32. Let X be a metric space, U be a subset of X,
and $(U_\lambda)_{\lambda\in I}$ be a nontrivial open cover of U.
If U is sequentially compact, then $\forall x \in U, \{d(x, U_\lambda^c)\}_{x\in I}$ is bounded above.

Proof. Assume to the contrary that $\exists x_* \in U, \{d_X(x_*, U_\lambda^c)\}_{\lambda\in I}$ is not bounded above,
so there exists $(\lambda_k)_{k\in\mathbb{N}} \in I$, such that $\lim\limits_{k\to+\infty} d_X(x_*, U_{\lambda_k}^c) = +\infty$.
As $(U_\lambda)_{\lambda\in I}$ is nontrivial, each $U_{\lambda_k} \nsupseteq U$, we may choose x_k from $U \cap U_{\lambda_k}^c$ as $U \cap U_{\lambda_k}^c \neq \emptyset$.
As $d_X(x_*, x_k) \geq d_X(x_*, U_{\lambda_k}^c)$ and $\lim\limits_{k\to+\infty} d_X(x_*, U_{\lambda_k}^c) = +\infty$, $\lim\limits_{k\to+\infty} d_X(x_*, x_k) = +\infty$.
This implies $(x_k)_{k\in\mathbb{N}}$ has no convergent subsequence,
so U is not sequentially compact. Quod. Erat. Demonstrandum. □

Remark: *Sequential compactness bounds the set* $\{d_X(x, U_\lambda^c)\}_{\lambda\in I}$.

Lemma 4.33. Let X be a metric space, and U be a subset of X.
If U is sequentially compact, then U is totally bounded.

Proof. Assume to the contrary that U is not totally bounded,
so for some $r > 0$, for all finite sequence $(x_k)_{k=1}^m$ in U, $\bigcup_{k=1}^n B(x_k, r) \subset U$.
Step 1: Construct a sequence in U.
As $U \backslash \emptyset$ is not totally bounded,
$\exists x_1 \in U \backslash \emptyset, \bigcup_{k=1}^1 B(x_k, r) \subset U$.
As $U \backslash \bigcup_{k=1}^n B(x_k, r)$ is not totally bounded,
$\exists x_{n+1} \in U \backslash \bigcup_{k=1}^n B(x_k, r), \bigcup_{k=1}^{n+1} B(x_k, r) \subset U$.
Step 2: State some key properties of this sequence.
Property 2.1: Each $x_n \in U$;
Property 2.2: Each distinct x_n, x_m satisfies $d_X(x_n, x_m) > r$.
This implies some $(x_n)_{n\in\mathbb{N}}$ in U has no convergent subsequence,
so U is not sequentially compact. Quod. Erat. Demonstrandum. □

Remark: *Sequential compactness implies total boundedness.*

Lemma 4.34. Let X be a metric space, U be a nonempty subset of X,
and $(U_\lambda)_{\lambda\in I}$ be a nontrivial open cover of U.
If U is sequentially compact, then for all $x \in U$ and $r > 0$:

$$r < L_U \implies B(x, r) \text{ is contained in some } A_\lambda$$

Proof. Assume that $r < L_U$. According to **Definition 4.24.** and **Definition 4.27.**:

$$r < L_U \leq \ell_U(x) = \sup_{\lambda\in I} d_X(x, U_\lambda^c)$$

So there exists $\lambda \in I$, such that $d_X(x, U_\lambda^c) > r$. For all $x' \in B(x, r)$:

$$d_X(x, x') < r < d_X(x, U_\lambda^c) = \inf_{x''\in U_\lambda^c} d_X(x, x'') \implies x' \notin U_\lambda^c \implies x' \in U_\lambda$$

Hence, $B(x, r)$ is contained in some U_λ. Quod. Erat. Demonstrandum. □

Remark: *This important property helps us reduce the cardinality of an open cover.*

Proposition 4.35. Let X be a metric space, and U be a subset of X.
If U is sequentially compact, then U is compact.

Proof. Assume that U is sequentially compact.
For all nontrivial open cover $(U_\lambda)_{\lambda\in I}$ of U:
According to **Lemma 4.34.**, for all $r > 0$, for all $x \in U$:

$$r < L_U \implies \exists \lambda_* \in I, B_r(x) \subseteq U_{\lambda_*}$$

According to **Lemma 4.33.**, $\frac{1}{2}L_U > 0$ gives a finite cover $(B(x_k, \frac{1}{2}L_U))_{k=1}^m$ of U.
For each $B(x_k, r)$, choose one superset U_{λ_k} of it in $(U_\lambda)_{\lambda\in I}$.

This gives a finite subcover $(U_{\lambda_k})_{k=1}^{m}$ of U.
Hence, U is compact. Quod. Erat. Demonstrandum. $\square$

Proposition 4.36. Let X be a metric space, and U be a subset of X.
If U is compact, then U is sequentially compact.

Proof. Assume to the contrary that U is not sequentially compact,
so each $x \in U$ gives $V_x \in \mathcal{O}_X$ with $x \in V_x$, such that $V_x \cap \{x_n\}_{n\in\mathbb{N}}$ is finite.
As U is compact, its open cover $(V_x)_{x\in U}$ has a finite subcover $(V_{x_k})_{k=1}^{m}$.
This implies $U \subseteq \bigcup_{k=1}^{m} V_{x_k}$ contains finitely many terms in $(x_n)_{n\in\mathbb{N}}$, a contradiction.
Hence, our assumption is false, and we've proven that U is sequentially compact.
Quod. Erat. Demonstrandum. $\square$

Theorem 4.37. (**Cantor's Intersection Theorem**)
Let X be a metric space, and $(U_n)_{n\in\mathbb{N}}$ be a nested family of nonempty closed and totally bounded sets. If X is complete, then $\bigcap_{n=1}^{+\infty} U_n \neq \emptyset$.

Proof. Assume to the contrary that $\bigcap_{n=1}^{+\infty} U_n = \emptyset$.
Step 1: Construct a sequence in U.
For each U_n, as $U_n \neq \emptyset$, there exists $x_n \in U_n$.
As $(U_n)_{n\in\mathbb{N}}$ is a nested sequence with empty intersection,
there exists $m \in \mathbb{N}$, such that $x_n \notin U_k$ whenever $k \geq m$.
Step 2: State some key properties of this sequence.
As X is complete, U_m is closed implies U_m is complete.
As U_m is complete and totally bounded, U_m is compact.
As U_m is compact, U_m is sequentially compact.
As $(x_n)_{n\in\mathbb{N}}$ is a sequence in a sequentially compact set U_1,
it has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit $x_* \in U_1$.
For each U_m, by removing finitely many terms if necessary,
$(x_{n_k})_{k\in\mathbb{N}}$ becomes a sequence in the sequentially compact set U_m.
This implies $x_* \in \bigcap_{m=1}^{+\infty} U_m$, which is a contradiction.
Hence, our assumption is false, and we've proven that $\bigcap_{n=1}^{+\infty} U_n \neq \emptyset$.
Quod. Erat. Demonstrandum. $\square$

Proposition 4.38. Let $\theta : [0, +\infty) \to \mathbb{R}$ be a function of class C^2, where $\exists \beta > 0, \forall s \geq 0, \theta'(s) > 0$ and $\theta''(s) \geq \beta$, and $\widetilde{\alpha} : [0, +\infty) \to \mathbb{C}$ be a path defined by:

$$\widetilde{\alpha}(s) = \int_0^s \mathrm{e}^{\mathrm{i}\theta(u)} \mathrm{d}u$$

(1) The limit $\widetilde{\alpha}(+\infty) = \int_0^{+\infty} \mathrm{e}^{\mathrm{i}\theta(u)} \mathrm{d}u$ exists;
(2) The estimation $\widetilde{\alpha}(s) = \widetilde{\alpha}(+\infty) + \mathcal{O}\left(\frac{1}{\theta'(s)}\right)$ is valid as $s \to +\infty$. [5]

Proof. For each $s \in [0, +\infty)$, determine the curvature of $\widetilde{\alpha}$:

$$\widetilde{T}(s) = \widetilde{\alpha}'(s) = \mathrm{e}^{\mathrm{i}\theta(s)} \implies \kappa(s) = \|\widetilde{T}'(s)\| = \theta'(s)$$

Define $\mathcal{D}_s$ as the osculating disk at s. For all $s_1 \leq s_2$:

$$\begin{aligned}
\|\widetilde{\alpha}_0(s_2) - \widetilde{\alpha}_0(s_1)\| &= \left\| \widetilde{\alpha}(s_2) - \widetilde{\alpha}(s_1) + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta(s_2)}}{\theta'(s_2)} - \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta(s_1)}}{\theta'(s_1)} \right\| \\
&= \left\| \int_{s_1}^{s_2} \mathrm{e}^{\mathrm{i}\theta(u)} \mathrm{d}u + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta(u)}}{\theta'(u)} \bigg|_{s_1}^{s_2} \right\| = \left\| \int_{s_1}^{s_2} \frac{\theta''(u)\mathrm{e}^{\mathrm{i}\theta(u)}}{\theta'(u)^2} \mathrm{d}u \right\| \\
&\leq \int_{s_1}^{s_2} \left\| \frac{\theta''(u)\mathrm{e}^{\mathrm{i}\theta(u)}}{\theta'(u)^2} \right\| \mathrm{d}u = \int_{s_1}^{s_2} \frac{\theta''(u)}{\theta'(u)^2} \mathrm{d}u = \frac{1}{\theta'(s_1)} - \frac{1}{\theta'(s_2)}
\end{aligned}$$

So we've constructed a nested family of closed disks $(\mathcal{D}_s)_{s \geq 0}$ in $\mathbb{C}$
with radius $\frac{1}{\theta'(s)} \to 0$. There exists a unique $\widetilde{\xi} \in \bigcap_{s \geq 0} \mathcal{D}_s$, such that:
(1) $\forall s \geq 0, \widetilde{\alpha}(s) \in \mathcal{D}_s$ implies the existence of $\widetilde{\alpha}(+\infty) = \widetilde{\xi}$;
(2) $\forall s \geq 0, \|\widetilde{\alpha}(s) - \widetilde{\alpha}(+\infty)\| \leq \|\widetilde{\alpha}_0(s) - \widetilde{\alpha}(s)\| + \|\widetilde{\alpha}_0(s) - \widetilde{\alpha}(+\infty)\| \leq \frac{2}{\theta'(s)}$
implies $\widetilde{\alpha}(s) = \widetilde{\alpha}(+\infty) + \mathcal{O}\left(\frac{1}{\theta'(s)}\right)$ as $s \to +\infty$.
Quod. Erat. Demonstrandum. □

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