20241115 MATH3541 NOTE 9[1]

Author: Be $\sqrt{-1}$maginative, and nothing will be $\frac{\mathrm{d}}{\mathrm{d}x}$ifficult!
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Contents

1 Introduction

How to prove that a 2-dimensional sphere $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\}$ is not homeomorphic to the 2-fold product $\mathbb{S} \times \mathbb{S}$ of the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$?
Notice that neither compactness nor connectedness distinguishes the two spaces, so we indeed need to invent some new topological invariant, which is the fundamental group.

2 Homotopy

2.1 Homotopic Functions

Definition 2.1. (**Homotopy and Relative Homotopy**)
Let X, Y be two topological spaces, A be a subset of X,
and f, f' be two functions from X to Y.
(1) If there exists a continuous function $H : X \times [0,1] \to Y$,
such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = f'(x)$,
then $f \sim f'$, i.e., f is homotopic to f'.
(2) If there exists a continuous function $H : X \times [0,1] \to Y$,
such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = f'(x)$,
and for all $x \in A$ and $t \in [0,1]$, $f(x) = H(x,t) = f'(x)$,
then $f \sim f'$ rel A, i.e., f is homotopic to f' relative to A.

Remark: $f \sim f'$ rel $A \implies f \sim f' \implies f \sim f'$ rel $\emptyset$, $f \sim f' \implies f, f'$ are continuous.

Proposition 2.2. Let $\mathcal{C}(X,Y)$ be the set of all continuous functions from X to Y. Homotopy relation $\sim$ relative to A is an equivalence relation on $\mathcal{C}(X,Y)$.

Proof. We may divide our proof into three parts.
Part 1: For all $f \in \mathcal{C}(X,Y)$, define the following identity homotopy:

$$e_f : X \times [0,1] \to Y, e_f(x,t) = f(x)$$

(1) f is continuous implies e_f is continuous.
(2) For all $x \in X$ and $t \in [0,1]$:

$$e_f(x,0) = e_f(x,t) = e_f(x,1) = f(x)$$

Hence, e_f is a homotopy relative to A, $f \sim f$ rel A.
Part 2: For all $f, f' \in \mathcal{C}(X,Y)$, assume that a homotopy H from f to f' relative to A exists. Define the following inverse homotopy:

$$H^{-1} : X \times [0,1] \to Y, H^{-1}(x,t) = H(x, 1-t)$$

(1) H and $\tau : X \times [0,1] \to X \times [0,1], (x,t) \mapsto (x, 1-t)$ are continuous implies $H^{-1} = H \circ \tau$

is continuous.
(2) For all $x \in X$:

$$H^{-1}(x,0) = H(x,1) = f'(x) \text{ and } H^{-1}(x,1) = H(x,0) = f(x)$$

(3) For all $x \in A$ and $t \in [0,1]$:

$$f'(x) = H^{-1}(x,t) = H(x,1-t) = f(x)$$

Hence, H^{-1} is a homotopy relative to A, $f' \sim f$ rel A.
Part 3: For all $f, f', f'' \in \mathcal{C}(X,Y)$, assume that a homotopy H from f to f' relative to A and a homotopy H' from f' to f'' relative to A exist. Fix an arbitrary $x \in (0,1)$, define the following concatenate homotopy at c:

$$H \star_c H' : X \times [0,1] \to Y, H \star_c H'(x,t) = \begin{cases} H(x, \frac{t-0}{c-0}) & \text{if} \quad 0 \le t \le c; \\ H'(x, \frac{t-c}{1-c}) & \text{if} \quad c \le t \le 1; \end{cases}$$

(1) H, H' and $\ell_c : X \times [0,c] \to X \times [0,1], (x,t) \mapsto (x, \frac{t-0}{c-0})$, $\ell'_c : X \times [c,1] \to X \times [0,1], (x,t) \mapsto (x, \frac{x-c}{1-c})$ are continuous implies $H \star_c H' = (H \circ \ell_c) \cup (H' \circ \ell'_c)$ is continuous.
(2) For all $x \in X$:

$$H \star_c H'(x,0) = H(x,0) = f(x) \text{ and } H \star_c H'(x,1) = H'(x,1) = f'(x)$$

(3) For all $x \in A$ and $t \in [0,1]$:

$$H \star_c H'(x,t) = \begin{cases} H(x, \frac{t-0}{c-0}) = f(x) = f'(x) & \text{if} \quad 0 \le t \le c; \\ H'(x, \frac{t-c}{1-c}) = f'(x) = f''(x) & \text{if} \quad c \le t \le 1; \end{cases}$$

Hence, $H \star_c H'$ is a homotopy relative to A, $f \sim f''$ rel A.
Combine the three parts above, we've proven that $\sim$ rel A is an equivalence relation.
Quod. Erat. Demonstrandum. $\square$

Remark: *When X is a singleton, homotopy degenerates to path, so we've simultaneously proven that path connected components form a partition.*

Proposition 2.3. Let $f, f' : X \to Y$, $g, g' : Y \to Z$ be four continuous functions. If $f \sim f'$ rel A and $g \sim g'$ rel $f(A) = f'(A)$, then $g \circ f \sim g' \circ f'$ rel A.

Proof. Assume that a homotopy H from f to f' relative to A and a homotopy I from g to g' relative to $f(A) = f'(A)$ exist. Define the following composite homotopy:

$$I \diamond H : X \times [0,1] \to Z, I \diamond H(x,t) = I(H(x,t),t)$$

(1) I and $J : X \times [0,1] \to Y \times [0,1], (x,t) \mapsto (H(x,t),t)$ are continuous implies $I \diamond H = I \circ J$ is continuous.

(2) For all $x \in X$:

$$I \diamond H(x,0) = I(f(x),0) = g \circ f(x) \text{ and } I \diamond H(x,1) = I(f'(x),1) = g' \circ f'(x)$$

(3) For all $x \in A$ and $t \in [0,1]$:

$$g \circ f(x) = I \diamond H(x,t) = I(H(x,t),t) = g' \circ f'(x)$$

Hence, $I \diamond H$ is a homotopy relative to A, $g \circ f \sim g' \circ f'$ rel A.
Quod. Erat. Demonstrandum. ☐

Proposition 2.4. Let X be a topological space, Y be a normed vector space, B be a convex subset of Y, and $\mathbf{f} : X \to B$ be a continuous function.
$\mathbf{f}$ is null-homotopic, i.e., for all $\boldsymbol{\eta} \in B$, $\mathbf{f}' : X \to B, x \mapsto \boldsymbol{\eta}$ is homotopic to $\mathbf{f}$.

Proof. Define the following null-homotopy from $\mathbf{f}'$ to $\mathbf{f}$:

$$\mathbf{H} : X \times [0,1] \to Y, \mathbf{H}(x,t) = (1-t)\boldsymbol{\eta} + t\mathbf{f}(x)$$

(1) $t, \mathbf{f}(x) - \boldsymbol{\eta}$ are continuous implies $t[\mathbf{f}(x) - \boldsymbol{\eta}]$ is continuous, which further implies $\mathbf{H}(x,t) = \boldsymbol{\eta} + t[\mathbf{f}(x) - \boldsymbol{\eta}]$ is continuous.
(2) For all $x \in X$:

$$\mathbf{H}(x,0) = \boldsymbol{\eta} = \mathbf{f}'(x) \text{ and } \mathbf{H}(x,1) = \mathbf{f}(x)$$

Hence, $\mathbf{H}$ is a homotopy, $\mathbf{f}' \sim \mathbf{f}$. Quod. Erat. Demonstrandum. ☐

Proposition 2.5. Let $\mathbf{f}, \mathbf{f}'$ be two continuous functions from X to $\mathbb{S}^n$.
If $\forall x \in X$ and $t \in [0,1], (1-t)\mathbf{f}(x) + t\mathbf{f}'(x) \neq \mathbf{0}$, then $\mathbf{f} \sim \mathbf{f}'$.

Proof. Define the following homotopy from $\mathbf{f}$ to $\mathbf{f}'$:

$$\mathbf{H} : X \times [0,1] \to \mathbb{S}^n, \mathbf{H}(x,t) = \frac{(1-t)\mathbf{f}(x) + t\mathbf{f}'(x)}{\|(1-t)\mathbf{f}(x) + t\mathbf{f}'(x)\|}$$

(1) As $\forall x \in X$ and $t \in [0,1], (1-t)\mathbf{f}(x) + t\mathbf{f}'(x) \neq \mathbf{0}$, $\mathbf{H}$ is well-defined on $X \times [0,1]$.
(2) $\mathbf{f}(x), \mathbf{f}'(x)$ and $t, \|\mathbf{v}\|$ are continuous implies $\mathbf{H}(x,t) = \frac{(1-t)\mathbf{f}(x)+t\mathbf{f}'(x)}{\|(1-t)\mathbf{f}(x)+t\mathbf{f}'(x)\|}$ is continuous.
(3) For all $x \in X$:

$$\mathbf{H}(x,0) = \frac{\mathbf{f}(x)}{\|\mathbf{f}(x)\|} = \mathbf{f}(x) \text{ and } \mathbf{H}(x,1) = \frac{\mathbf{f}'(x)}{\|\mathbf{f}'(x)\|} = \mathbf{f}'(x)$$

Hence, $\mathbf{H}$ is a homotopy, $\mathbf{f} \sim \mathbf{f}'$. Quod. Erat. Demonstrandum. ☐

2.2 Homotopic Topological Spaces

Definition 2.6. (**Homotopy**)
Let X, X' be two topological spaces.
If there exist two continuous functions $f : X \to X', g : X' \to X$,
such that $g \circ f \sim e_X$ and $f \circ g \sim e_{X'}$ on X',
then X, X' are homotopic.

Proposition 2.7. If X, X' are homeomorphic, then X, X' are homotopic.

Proof. Assume that σ is a homeomorphism from X to X'.
There exist two continuous functions $f = \sigma : X \to X', g = \sigma^{-1} : X' \to X$,
such that $g \circ f = e_X \sim e_X$ and $f \circ g = e_{X'} \sim e_{X'}$.
Hence, $X \sim X'$. Quod. Erat. Demonstrandum. □

Remark: *Being homeomorphic is stronger than being homotopic.*

Proposition 2.8. Let Top be the set of all topological spaces.
Homotopy relation $\sim$ is an equivalence relation on Top.

Proof. We may divide our proof into three parts.
Part 1: For all $X \in \text{Top}$, there exists a continuous function $e_X : X \to X$, such that:

$$e_X \circ e_X = e_X \sim e_X$$

Hence, $X \sim X$.
Part 2: For all $X, X' \in \text{Top}$, assume that there exist two continuous functions $f : X \to X', g : X' \to X$, such that $g \circ f \sim e_X$ and $f \circ g \sim e_{X'}$.
There exist two continuous functions $g : X' \to X, f : X \to X'$, such that:

$$f \circ g \sim e_{X'} \text{ and } g \circ f \sim e_X$$

Hence, $X' \sim X$.
Part 3: For all $X, X', X'' \in \text{Top}$, assume that there exist four continuous functions $f : X \to X', g : X' \to X, f' : X' \to X'', g' : X'' \to X'$, such that $g \circ f \sim e_X$ and $f \circ g \sim e_{X'}$ and $g' \circ f' \sim e_{X'}$ and $f' \circ g' \sim e_{X''}$.
There exist two continuous functions $f' \circ f : X \to X''$, $g \circ g' : X'' \to X$, such that:

$$g \circ g' \circ f' \circ f \sim g \circ f \sim e_X \text{ and } f' \circ f \circ g \circ g' \sim f' \circ g' \sim e_{X''}$$

Hence, $X \sim X''$.
Combine the three parts above, we've proven that $\sim$ is an equivalence relation.
Quod. Erat. Demonstrandum. □

Proposition 2.9. Let X be a topological space,
Y be a normed vector space, B be a convex subset of Y.
B is contractible, i.e., for all $\boldsymbol{\eta} \in B$, $\{\boldsymbol{\eta}\}$ is homotopic to B.

Proof. There exist two continuous functions $\mathbf{f} : \{\boldsymbol{\eta}\} \to B, \boldsymbol{\eta} \mapsto \boldsymbol{\eta}$,
$\mathbf{f}' : B \to \{\boldsymbol{\eta}\}, \mathbf{x} \mapsto \boldsymbol{\eta}$, such that $\mathbf{f}' \circ \mathbf{f} = e_{\{\boldsymbol{\eta}\}} \sim e_{\{\boldsymbol{\eta}\}}$ and $\mathbf{f} \circ \mathbf{f}' = \mathbf{f}' \sim e_B$.
Hence, B is contractible. Quod. Erat. Demonstrandum. □

Proposition 2.10. Let $\boldsymbol{\eta}$ be a point on $\mathbb{S}^n$. $\mathbb{S}^n \backslash \{\boldsymbol{\eta}\}$ is contractible.

Proof. $\mathbb{R}^n$ is contractible implies $\mathbb{S}^n \backslash \{\boldsymbol{\eta}\}$ is contractible. Quod. Erat. Demonstrandum. □

2.3 Construction of Homotopy

Proposition 2.11. Let X, Y be two topological spaces, A be a subset of X,
B be a subset of Y, and f, f' be two continuous functions with domain X.
If $f \sim f'$ rel A with codomain B, then $f \sim f'$ rel A with codomain Y.

Proof. It suffices to notice that H is continuous with codomain B implies H is continuous with codomain Y. Quod. Erat. Demonstrandum. □

Remark: *However, if someone "dig a hole" in the codomain, then certain continuous functions can no longer be homotopic.*

Proposition 2.12. Let X be a topological space, A be a subset of X,
$(Y_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces with product $Y = \prod_{\lambda \in I} Y_\lambda$,
and $(f_\lambda)_{\lambda \in I}, (f'_\lambda)_{\lambda \in I}$ be two indexed families of continuous functions from X to Y with products $f = \prod_{\lambda \in I} f_\lambda, f' = \prod_{\lambda \in I} f'_\lambda$. $f \sim f'$ rel A iff each $f_\lambda \sim f'_\lambda$ rel A.

Proof. It suffices to notice that $H = \prod_{\lambda \in I} H_\lambda$ is continuous iff each H_λ is continuous.
Quod. Erat. Demonstrandum. □

Remark: *Recall the following two facts from multivariable calculus:*
(1) $H(x_1, x_2)$ *is continuous* $\implies$ $H(x_1, \xi_2), H(\xi_1, x_2)$ *are continuous.*
(2) $(H_1(x), H_2(x))$ *is continuous* $\iff$ $H_1(x), H_2(x)$ *are continuous.*

Proposition 2.13. Let X be a topological space, A be a subset of X,
$(Y_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces with coproduct $Y = \coprod_{\lambda \in I} Y_\lambda$,
and f_μ, f'_μ be two continuous functions from X to Y_μ.
If $\pi_\mu : Y_\mu \to Y, y \mapsto (y, \mu)$, then $\pi_\mu \circ f_\mu \sim \pi_\mu \circ f'_\mu$ rel A iff $f_\mu \sim f'_\mu$ rel A.

Proof. It suffices to notice that π_μ is an embedding, and a path in Y is restricted to move in one single slice Y_μ. Quod. Erat. Demonstrandum. $\square$

Proposition 2.14. Let X, Y be two topological spaces,
A be a subset of X, $[Y]$ be the quotient space of Y under $[\bullet]$,
and f, f' be two continuous functions from X to Y.
If $[\bullet] : Y \to [Y], y \mapsto [y]$, then $[f] \sim [f']$ rel A if $f \sim f'$ rel A.

Proof. It suffices to notice that $[H]$ is continuous if H is continuous.
Quod. Erat. Demonstrandum. $\square$

Remark: *Notice that the other implication is wrong.*

3 Elementary Category Theory

3.1 Category

Category is introduced to describe the structures of mathematical objects.

Definition 3.1. (**Category**)
Let $(\mathrm{Obj}, \mathrm{Mor})$ be a tuple of two sets. If:
(1) For all objects $A, B \in \mathrm{Obj}$, there exists a unique morphism class:

$$\mathrm{Mor}(A, B) \subseteq \mathrm{Mor}$$

(2) For all objects $A, B, C \in \mathrm{Obj}$, there exists a unique binary operation:

$$\circ : \mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C), (\sigma, \tau) \mapsto \tau \circ \sigma$$

(3) For all objects $A, B, C, D \in \mathrm{Obj}$:

$$\forall \mu \in \mathrm{Mor}(A, B), \forall \nu \in \mathrm{Mor}(B, C), \forall \sigma \in \mathrm{Mor}(C, D), \sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$$

(4) For all object $A \in \mathrm{Obj}$:

$$\begin{aligned} \exists e_A \in \mathrm{Mor}(A, A), \quad &\forall B \in \mathrm{Obj}, \quad \forall \sigma \in \mathrm{Mor}(A, B), \quad \sigma \circ e_A = \sigma; \\ &\forall B \in \mathrm{Obj}, \quad \forall \tau \in \mathrm{Mor}(B, A), \quad e_A \circ \tau = \tau; \end{aligned}$$

Then, $(\mathrm{Obj}, \mathrm{Mor})$ is a category.

Proposition 3.2. Define the followings:
(1) $\mathrm{Obj} = [\text{All sets}]$.
(2) $\mathrm{Mor} = [\text{All functions}]$.
$(\mathrm{Obj}, \mathrm{Mor})$ is a category.

Proof. We may divide our proof into four parts.
(1) For all sets $A, B \in \mathrm{Obj}$, there exists a unique function class:

$$\mathrm{Mor}(A, B) = [\text{All function } f \text{ from } A \text{ to } B] \subseteq \mathrm{Mor}$$

(2) For all sets $A, B, C \in \mathrm{Obj}$, there exists a unique binary operation:

$$\circ : \mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C), (\sigma, \tau) \mapsto \tau \circ \sigma$$

The following argument suggests that $\tau \circ \sigma \in \mathrm{Mor}(A, C)$ is well-defined:

$$\forall a \in A, \exists! \tau(\sigma(a)) \in C, \tau \circ \sigma(a) = \tau(\sigma(a))$$

(3) For all sets $A, B, C, D \in \mathrm{Obj}$:

$$\forall \mu \in \mathrm{Mor}(A, B), \forall \nu \in \mathrm{Mor}(B, C), \forall \sigma \in \mathrm{Mor}(C, D), \sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$$

The following argument suggests that $\sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$:

$$\forall a \in A, \sigma \circ (\nu \circ \mu)(a) = \sigma(\nu(\mu(a))) = (\sigma \circ \nu) \circ \mu(a)$$

(4) For all set $A \in \mathrm{Obj}$:

$$\begin{aligned} \exists e_A \in \mathrm{Mor}(A, A), \quad & \forall B \in \mathrm{Obj}, \quad \forall \sigma \in \mathrm{Mor}(A, B), \quad \sigma \circ e_A = \sigma; \\ & \forall B \in \mathrm{Obj}, \quad \forall \tau \in \mathrm{Mor}(B, A), \quad e_A \circ \tau = \tau; \end{aligned}$$

The following argument suggests that $e_A \in \mathrm{Mor}(A, A)$ is well-defined:

$$\forall a \in A, \exists! a \in A, e_A(a) = a$$

The following argument suggests that $\sigma \circ e_A = \sigma$:

$$\forall a \in A, \sigma \circ e_A(a) = \sigma(e_A(a)) = \sigma(a)$$

The following argument suggests that $e_A \circ \tau = \tau$:

$$\forall b \in B, e_A \circ \tau(b) = e_A(\tau(b)) = \tau(b)$$

Hence, $(\mathrm{Obj}, \mathrm{Mor})$ is a category. Quod. Erat. Demonstrandum. □

Remark: *"No structure" is a structure.*

Proposition 3.3. Define the followings:
(1) $\mathrm{Obj} = [\text{All groups}]$.
(2) $\mathrm{Mor} = [\text{All group homomorphisms}]$.
$(\mathrm{Obj}, \mathrm{Mor})$ is a category.

Proof. We may divide our proof into four parts.
(1) For all groups $A, B \in \text{Obj}$, there exists a unique group homomorphism class:

$$\text{Mor}(A, B) = [\text{All group homomorphism } f \text{ from } A \text{ to } B] \subseteq \text{Mor}$$

(2) For all groups $A, B, C \in \text{Obj}$, there exists a unique binary operation:

$$\circ : \text{Mor}(A, B) \times \text{Mor}(B, C) \to \text{Mor}(A, C), (\sigma, \tau) \mapsto \tau \circ \sigma$$

The following argument suggests that $\tau \circ \sigma \in \text{Mor}(A, C)$ is well-defined:

$$\begin{aligned} \forall a, a' \in A, \tau \circ \sigma(a_1 a_2) &= \tau(\sigma(a_1 a_2)) = \tau(\sigma(a_1)\sigma(a_2)) \\ &= \tau(\sigma(a_1))\tau(\sigma(a_2)) = \tau \circ \sigma(a_1)\tau \circ \sigma(a_2) \end{aligned}$$

(3) For all groups $A, B, C, D \in \text{Obj}$:

$$\forall \mu \in \text{Mor}(A, B), \forall \nu \in \text{Mor}(B, C), \forall \sigma \in \text{Mor}(C, D), \sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$$

(4) For all group $A \in \text{Obj}$:

$$\begin{aligned} \exists e_A \in \text{Mor}(A, A), \quad &\forall B \in \text{Obj}, \quad \forall \sigma \in \text{Mor}(A, B), \quad \sigma \circ e_A = \sigma; \\ &\forall B \in \text{Obj}, \quad \forall \tau \in \text{Mor}(B, A), \quad e_A \circ \tau = \tau; \end{aligned}$$

The following argument suggests that $e_A \in \text{Mor}(A, A)$ is well-defined:

$$\forall a_1, a_2 \in A, e_A(a_1 a_2) = a_1 a_2 = e_A(a_1) e_A(a_2)$$

Hence, (Obj, Mor) is a category. Quod. Erat. Demonstrandum. □

Proposition 3.4. Define the followings:
(1) $\text{Obj} = [\text{All vector space over field } \mathbb{F}]$.
(2) $\text{Mor} = [\text{All linear mappings}]$.
(Obj, Mor) is a category.

Proof. We may divide our proof into four parts.
(1) For all vector spaces $A, B \in \text{Obj}$, there exists a unique linear mapping class:

$$\text{Mor}(A, B) = [\text{All linear mapping } f \text{ from } A \text{ to } B] \subseteq \text{Mor}$$

(2) For all vector spaces $A, B, C \in \text{Obj}$, there exists a unique binary operation:

$$\circ : \text{Mor}(A, B) \times \text{Mor}(B, C) \to \text{Mor}(A, C), (\sigma, \tau) \mapsto \tau \circ \sigma$$

The following argument suggests that $\tau \circ \sigma \in \mathrm{Mor}(A, C)$ is well-defined:

$$\begin{aligned}
\forall \mathbf{a}_1, \mathbf{a}_2 \in A, \tau \circ \sigma(\mathbf{a}_1 + \mathbf{a}_2) &= \tau(\sigma(\mathbf{a}_1 + \mathbf{a}_2)) = \tau(\sigma(\mathbf{a}_1) + \sigma(\mathbf{a}_2)) \\
&= \tau(\sigma(\mathbf{a}_1)) + \tau(\sigma(\mathbf{a}_2)) = \tau \circ \sigma(\mathbf{a}_1) + \tau \circ \sigma(\mathbf{a}_2) \\
\forall \lambda \in \mathbb{F} \text{ and } \mathbf{a} \in A, \tau \circ \sigma(\lambda \mathbf{a}) &= \tau(\sigma(\lambda \mathbf{a})) = \tau(\lambda \sigma(\mathbf{a})) \\
&= \lambda \tau(\sigma(\mathbf{a})) = \lambda \tau \circ \sigma(\mathbf{a})
\end{aligned}$$

(3) For all vector spaces $A, B, C, D \in \mathrm{Obj}$:

$$\forall \mu \in \mathrm{Mor}(A, B), \forall \nu \in \mathrm{Mor}(B, C), \forall \sigma \in \mathrm{Mor}(C, D), \sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$$

(4) For all vector space $A \in \mathrm{Obj}$:

$$\begin{aligned}
\exists e_A \in \mathrm{Mor}(A, A), \quad &\forall B \in \mathrm{Obj}, \quad \forall \sigma \in \mathrm{Mor}(A, B), \quad \sigma \circ e_A = \sigma; \\
&\forall B \in \mathrm{Obj}, \quad \forall \tau \in \mathrm{Mor}(B, A), \quad e_A \circ \tau = \tau;
\end{aligned}$$

The following argument suggests that $e_A \in \mathrm{Mor}(A, A)$ is well-defined:

$$\begin{aligned}
\forall \mathbf{a}_1, \mathbf{a}_2 \in A, e_A(\mathbf{a}_1 + \mathbf{a}_2) &= \mathbf{a}_1 + \mathbf{a}_2 = e_A(\mathbf{a}_1) + e_A(\mathbf{a}_2) \\
\forall \lambda \in \mathbb{F} \text{ and } \mathbf{a} \in A, e_A(\lambda \mathbf{a}) &= \lambda \mathbf{a} = \lambda e_A(\mathbf{a})
\end{aligned}$$

Hence, $(\mathrm{Obj}, \mathrm{Mor})$ is a category. Quod. Erat. Demostrandum. □

Proposition 3.5. Define the followings:
(1) $\mathrm{Obj} =$ [All rings with unity].
(2) $\mathrm{Mor} =$ [All ring homomorphisms].
$(\mathrm{Obj}, \mathrm{Mor})$ is a category.

Proof. We may divide our proof into four parts.
(1) For all rings $A, B \in \mathrm{Obj}$, there exists a unique ring homomorphism class:

$$\mathrm{Mor}(A, B) = [\text{All ring homomorphism } f \text{ from } A \text{ to } B] \subseteq \mathrm{Mor}$$

(2) For all rings $A, B, C \in \mathrm{Obj}$, there exists a unique binary operation:

$$\circ : \mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C), (\sigma, \tau) \mapsto \tau \circ \sigma$$

The following argument suggests that $\tau \circ \sigma \in \mathrm{Mor}(A, C)$ is well-defined:

$$\begin{aligned}
\forall a_1, a_2 \in A, \tau \circ \sigma(a_1 + a_2) &= \tau(\sigma(a_1 + a_2)) = \tau(\sigma(a_1) + \sigma(a_2)) \\
&= \tau(\sigma(a_1)) + \tau(\sigma(a_2)) = \tau \circ \sigma(a_1) + \tau \circ \sigma(a_2) \\
\tau \circ \sigma(1_A) &= \tau(\sigma(1_A)) = \tau(1_B) = 1_C \\
\forall a_1, a_2 \in A, \tau \circ \sigma(a_1 a_2) &= \tau(\sigma(a_1 a_2)) = \tau(\sigma(a_1)\sigma(a_2)) \\
&= \tau(\sigma(a_1))\tau(\sigma(a_2)) = \tau \circ \sigma(a_1)\tau \circ \sigma(a_2)
\end{aligned}$$

(3) For all rings $A, B, C, D \in \mathrm{Obj}$:

$$\forall \mu \in \mathrm{Mor}(A,B), \forall \nu \in \mathrm{Mor}(B,C), \forall \sigma \in \mathrm{Mor}(C,D), \sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$$

(4) For all ring $A \in \mathrm{Obj}$:

$$\begin{aligned} \exists e_A \in \mathrm{Mor}(A,A), \quad & \forall B \in \mathrm{Obj}, \quad \forall \sigma \in \mathrm{Mor}(A,B), \quad \sigma \circ e_A = \sigma; \\ & \forall B \in \mathrm{Obj}, \quad \forall \tau \in \mathrm{Mor}(B,A), \quad e_A \circ \tau = \tau; \end{aligned}$$

The following argument suggests that $e_A \in \mathrm{Mor}(A,A)$ is well-defined:

$$\forall a_1, a_2 \in A, e_A(a_1 + a_2) = a_1 + a_2 = e_A(a_1) + e_A(a_2)$$
$$e_A(1_A) = 1_A$$
$$\forall a_1, a_2 \in A, e_A(a_1 a_2) = a_1 a_2 = e_A(a_1) e_A(a_2)$$

Hence, $(\mathrm{Obj}, \mathrm{Mor})$ is a category. Quod. Erat. Demostrandum. $\square$

Proposition 3.6. Define the followings:
(1) $\mathrm{Obj} =$ [All topological spaces].
(2) $\mathrm{Mor} =$ [All continuous maps].
$(\mathrm{Obj}, \mathrm{Mor})$ is a category.

Proof. We may divide our proof into four parts.
(1) For all topological spaces $A, B \in \mathrm{Obj}$, there exists a unique continuous map class:

$$\mathrm{Mor}(A,B) = [\text{All continuous map } f \text{ from } A \text{ to } B] \subseteq \mathrm{Mor}$$

(2) For all topological spaces $A, B, C \in \mathrm{Obj}$, there exists a unique binary operation:

$$\circ : \mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \to \mathrm{Mor}(A,C), (\sigma, \tau) \mapsto \tau \circ \sigma$$

The following argument suggests that $\tau \circ \sigma \in \mathrm{Mor}(A,C)$ is well-defined:

$$\forall W \in \mathcal{O}_C, (\tau \circ \sigma)^{-1}(W) = \sigma^{-1}(\tau^{-1}(W)) \in \mathcal{O}_A$$

(3) For all topological spaces $A, B, C, D \in \mathrm{Obj}$:

$$\forall \mu \in \mathrm{Mor}(A,B), \forall \nu \in \mathrm{Mor}(B,C), \forall \sigma \in \mathrm{Mor}(C,D), \sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$$

(4) For all topological space $A \in \mathrm{Obj}$:

$$\begin{aligned} \exists e_A \in \mathrm{Mor}(A,A), \quad & \forall B \in \mathrm{Obj}, \quad \forall \sigma \in \mathrm{Mor}(A,B), \quad \sigma \circ e_A = \sigma; \\ & \forall B \in \mathrm{Obj}, \quad \forall \tau \in \mathrm{Mor}(B,A), \quad e_A \circ \tau = \tau; \end{aligned}$$

The following argument suggests that $e_A \in \mathrm{Mor}(A, A)$ is well-defined:

$$\forall U \in \mathcal{O}_A, e_A^{-1}(U) = U \in \mathcal{O}_A$$

Hence, $(\mathrm{Obj}, \mathrm{Mor})$ is a category. Quod. Erat. Demonstrandum. $\square$

3.2 Functor

If $([\text{All categories}], \bullet)$ *is a category, then what should be* $\bullet$*?*
Well, for all categories $(\mathrm{Obj}, \mathrm{Mor}), (\mathrm{Obj}', \mathrm{Mor}')$*, a structure-preserving map should preserve both objects and morphisms, which gives rise to the idea of functor.*

Definition 3.7. (**Functor**)
Let $(\mathrm{Obj}, \mathrm{Mor}), (\mathrm{Obj}', \mathrm{Mor}')$ be two categories,
and $\sigma : \mathrm{Obj} \sqcup \mathrm{Mor} \to \mathrm{Obj}' \sqcup \mathrm{Mor}'$ be a map. If:
(1) For all object $A \in \mathrm{Obj}$:

$$\sigma(e_A) = e_{\sigma(A)}$$

(2) For all objects $A, B, C \in \mathrm{Obj}$:

$$\forall \mu \in \mathrm{Mor}(A, B), \forall \nu \in \mathrm{Mor}(B, C), \sigma(\nu\mu) = \sigma(\nu)\sigma(\mu)$$

Then σ is a functor from $(\mathrm{Obj}, \mathrm{Mor})$ to $(\mathrm{Obj}', \mathrm{Mor}')$.

Remark: *To define a ring homomorphism, it is necessary to require that the multiplicative identity is preserved because* $r^2 = r$ *doesn't imply* $r = 1_R$*. For the same reason, it is necessary to require that every identity map is preserved under functors.*

Proposition 3.8. Define the followings:
(1) $\mathbf{Obj} = [\text{All categories}]$.
(2) $\mathbf{Mor} = [\text{All functors}]$.
$(\mathbf{Obj}, \mathbf{Mor})$ is a category.

Proof. We may divide our proof into four parts.
(1) For all categories $\mathbf{A}, \mathbf{B} \in \mathbf{Obj}$, there exists a unique functor class:

$$\mathbf{Mor}(\mathbf{A}, \mathbf{B}) = [\text{All functor } \mathbf{f} \text{ from } \mathbf{A} \text{ to } \mathbf{B}] \subseteq \mathbf{Mor}$$

(2) For all categories $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{Obj}$, there exists a unique binary operation:

$$\circ : \mathbf{Mor}(\mathbf{A}, \mathbf{B}) \times \mathbf{Mor}(\mathbf{B}, \mathbf{C}) \to \mathbf{Mor}(\mathbf{A}, \mathbf{C}), (\boldsymbol{\sigma}, \boldsymbol{\tau}) \mapsto \boldsymbol{\tau} \circ \boldsymbol{\sigma}$$

The following argument suggests that $\boldsymbol{\tau} \circ \boldsymbol{\sigma} \in \mathbf{Mor}(\mathbf{A}, \mathbf{C})$ is well-defined.

For all object A of $\mathbf{A}$:

$$\boldsymbol{\tau} \circ \boldsymbol{\sigma}(e_A) = \boldsymbol{\tau}(\boldsymbol{\sigma}(e_A)) = \boldsymbol{\tau}(e_{\boldsymbol{\sigma}(A)}) = e_{\boldsymbol{\tau}(\boldsymbol{\sigma}(A))} = e_{\boldsymbol{\tau} \circ \boldsymbol{\sigma}(A)}$$

For all objects A, B, C of $\mathbf{A}$:

$$\begin{aligned} \forall \mu \in \mathrm{Mor}(A, B), \forall \nu \in \mathrm{Mor}(B, C), \boldsymbol{\tau} \circ \boldsymbol{\sigma}(\nu\mu) &= \boldsymbol{\tau}(\boldsymbol{\sigma}(\nu\mu)) = \boldsymbol{\tau}(\boldsymbol{\sigma}(\nu)\boldsymbol{\sigma}(\mu)) \\ &= \boldsymbol{\tau}(\boldsymbol{\sigma}(\nu))\boldsymbol{\tau}(\boldsymbol{\sigma}(\mu)) = \boldsymbol{\tau} \circ \boldsymbol{\sigma}(\nu)\boldsymbol{\tau} \circ \boldsymbol{\sigma}(\mu) \end{aligned}$$

(3) For all categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathbf{Obj}$:

$$\forall \boldsymbol{\mu} \in \mathbf{Mor}(\mathbf{A}, \mathbf{B}), \forall \boldsymbol{\nu} \in \mathbf{Mor}(\mathbf{B}, \mathbf{C}), \forall \boldsymbol{\sigma} \in \mathbf{Mor}(\mathbf{C}, \mathbf{D}), \boldsymbol{\sigma} \circ (\boldsymbol{\nu} \circ \boldsymbol{\mu}) = (\boldsymbol{\sigma} \circ \boldsymbol{\nu}) \circ \boldsymbol{\mu}$$

(4) For all category $\mathbf{A} \in \mathbf{Obj}$:

$$\begin{aligned} \exists \mathbf{e_A} \in \mathbf{Mor}(\mathbf{A}, \mathbf{A}), \quad &\forall \mathbf{B} \in \mathbf{Obj}, \quad \forall \boldsymbol{\sigma} \in \mathbf{Mor}(\mathbf{A}, \mathbf{B}), \quad \boldsymbol{\sigma} \circ \mathbf{e_A} = \boldsymbol{\sigma}; \\ &\forall \mathbf{B} \in \mathbf{Obj}, \quad \forall \boldsymbol{\tau} \in \mathbf{Mor}(\mathbf{B}, \mathbf{A}), \quad \mathbf{e_A} \circ \boldsymbol{\tau} = \boldsymbol{\tau}; \end{aligned}$$

The following argument suggests that $\mathbf{e_A} \in \mathbf{Mor}(\mathbf{A}, \mathbf{A})$ is well-defined:
For all object A of $\mathbf{A}$:

$$\mathbf{e_A}(e_A) = e_A = e_{\mathbf{e_A}(A)}$$

For all objects A, B, C of $\mathbf{A}$:

$$\forall \mu \in \mathrm{Mor}(A, B), \forall \nu \in \mathrm{Mor}(B, C), \mathbf{e_A}(\nu\mu) = \nu\mu = \mathbf{e_A}(\nu)\mathbf{e_A}(\mu)$$

Hence, $(\mathbf{Obj}, \mathbf{Mor})$ is a category. Quod. Erat. Demonstrandum. □

Definition 3.9. (**Dual Space of $\mathbb{F}^n$**)
Define the following subset $\mathbb{F}_n$ of $\mathbb{F}^n[x_1, x_2, \cdots, x_n]$ as the dual space of $\mathbb{F}^n$:

$$\mathbb{F}_n = \left\{ u(x_1, x_2, \cdots, x_n) = \tfrac{\partial u}{\partial x_1} x_1 + \tfrac{\partial u}{\partial x_2} x_2 + \cdots + \tfrac{\partial u}{\partial x_n} x_n : \tfrac{\partial u}{\partial x_1}, \tfrac{\partial u}{\partial x_2}, \cdots, \tfrac{\partial u}{\partial x_n} \in \mathbb{F} \right\}$$

Proposition 3.10. $\mathbb{F}_n$ is a vector space over field $\mathbb{F}$.

Proof. We may divide our proof into eight parts.
(1) For all $\mathbf{u}_1 \cdot \mathbf{x}, \mathbf{u}_2 \cdot \mathbf{x} \in \mathbb{F}_n$:

$$\mathbf{u}_1 \cdot \mathbf{x} + \mathbf{u}_2 \cdot \mathbf{x} = \mathbf{u}_2 \cdot \mathbf{x} + \mathbf{u}_1 \cdot \mathbf{x}$$

(2) For all $\mathbf{u}_1 \cdot \mathbf{x}, \mathbf{u}_2 \cdot \mathbf{x}, \mathbf{u}_3 \cdot \mathbf{x} \in \mathbb{F}_n$:

$$(\mathbf{u}_1 \cdot \mathbf{x} + \mathbf{u}_2 \cdot \mathbf{x}) + \mathbf{u}_3 \cdot \mathbf{x} = \mathbf{u}_1 \cdot \mathbf{x} + (\mathbf{u}_2 \cdot \mathbf{x} + \mathbf{u}_3 \cdot \mathbf{x})$$

(3) There exists $0 \in \mathbb{F}_n$, such that for all $\mathbf{u} \cdot \mathbf{x} \in \mathbb{F}_n$:

$$0 + \mathbf{u} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{x} + 0 = \mathbf{u} \cdot \mathbf{x}$$

(4) For all $\mathbf{u} \cdot \mathbf{x} \in \mathbb{F}_n$, there exists $-\mathbf{u} \cdot \mathbf{x} \in \mathbb{F}_n$, such that:

$$(-\mathbf{u} \cdot \mathbf{x}) + \mathbf{u} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{x} + (-\mathbf{u} \cdot \mathbf{x}) = 0$$

(5) For all $\lambda_1, \lambda_2 \in \mathbb{F}$ and $\mathbf{u} \cdot \mathbf{x} \in \mathbb{F}_n$:

$$(\lambda_1 \lambda_2)\mathbf{u} \cdot \mathbf{x} = \lambda_1(\lambda_2 \mathbf{u} \cdot \mathbf{x})$$

(6) For the unity $1 \in \mathbb{F}$, for all $\mathbf{u} \cdot \mathbf{x} \in \mathbb{F}_n$:

$$1\mathbf{u} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{x}$$

(7) For all $\lambda_1, \lambda_2 \in \mathbb{F}$ and $\mathbf{u} \cdot \mathbf{x} \in \mathbb{F}_n$:

$$(\lambda_1 + \lambda_2)\mathbf{u} \cdot \mathbf{x} = \lambda_1 \mathbf{u} \cdot \mathbf{x} + \lambda_2 \mathbf{u} \cdot \mathbf{x}$$

(8) For all $\lambda \in \mathbb{F}$ and $\mathbf{u}_1 \cdot \mathbf{x}, \mathbf{u}_2 \cdot \mathbf{x} \in \mathbb{F}_n$:

$$\lambda(\mathbf{u}_1 \cdot \mathbf{x} + \mathbf{u}_2 \cdot \mathbf{x}) = \lambda \mathbf{u}_1 \cdot \mathbf{x} + \lambda \mathbf{u}_2 \cdot \mathbf{x}$$

Hence, $\mathbb{F}_n$ is a vector space over field $\mathbb{F}$. Quod. Erat. Demonstrandum. □

Proposition 3.11. The polynomials $x_1, x_2, \cdots, x_n$ form a basis of $\mathbb{F}_n$.

Proof. We may divide our proof into two parts.
(1) For all $u(x_1, x_2, \cdots, x_n) \in \mathbb{F}_n$, there exists $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n} \in \mathbb{F}$, such that:

$$u(x_1, x_2, \cdots, x_n) = \frac{\partial u}{\partial x_1} x_1 + \frac{\partial u}{\partial x_2} x_2 + \cdots + \frac{\partial u}{\partial x_n} x_n$$

(2) For all $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n} \in \mathbb{F}$:

$$\frac{\partial u}{\partial x_1} x_1 + \frac{\partial u}{\partial x_2} x_2 + \cdots + \frac{\partial u}{\partial x_n} x_n = 0 \implies \frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_2} = \cdots = \frac{\partial u}{\partial x_n} = 0$$

Hence, the polynomials $x_1, x_2, \cdots, x_n$ form a basis of $\mathbb{F}_n$.
Quod. Erat. Demonstrandum. □

Proposition 3.12. Define the followings:
(1) $\text{Obj} = [\text{All subspace } U \text{ of } \mathbb{F}^n]$.
(2) $\text{Mor} = [\text{All linear mapping from some } U \in \text{Obj} \text{ to some } V \in \text{Obj}]$.
(3) $\text{Obj}' = [\text{All subspace } U' \text{ of } \mathbb{F}_n]$.
(4) $\text{Mor}' = [\text{All linear mapping from some } U' \in \text{Obj}' \text{ to some } V' \in \text{Obj}']$.
The following two functors are well-defined:
(1) $\mu : (\text{Obj}, \text{Mor}) \to (\text{Obj}', \text{Mor}'), \mathbf{u} \mapsto \mathbf{u} \cdot \mathbf{x}$.
(2) $\nu : (\text{Obj}', \text{Mor}') \to (\text{Obj}, \text{Mor}), u(\mathbf{x}) \mapsto \boldsymbol{\nabla} u$.

Proof. We may divide our proof into four parts.
(1) For all subspace $U \in \text{Obj}$:

$$\mu(e_U) = \mu(\mathbf{u} \mapsto \mathbf{u}) = \mathbf{u} \cdot \mathbf{x} \mapsto \mathbf{u} \cdot \mathbf{x} = e_{\mu(U)}$$

(2) For all subspace $U' \in \text{Obj}'$:

$$\nu(e_{U'}) = \nu(u(\mathbf{x}) \mapsto u(\mathbf{x})) = \boldsymbol{\nabla} u \mapsto \boldsymbol{\nabla} u = e_{\nu(U')}$$

(3) For all subspaces $U, V, W \in \text{Obj}$:

$$\begin{aligned} \forall \sigma \in \text{Mor}(U, V), \forall \tau \in \text{Mor}(V, W), \mu(\tau\sigma) &= \mu(\mathbf{u} \mapsto \mathbf{w}) \\ &= \mathbf{u} \cdot \mathbf{x} \mapsto \mathbf{w} \cdot \mathbf{x} = \mu(\tau)\mu(\sigma) \end{aligned}$$

(4) For all subspaces $U', V', W' \in \text{Obj}'$:

$$\begin{aligned} \forall \sigma \in \text{Mor}'(U', V'), \forall \tau \in \text{Mor}'(V', W'), \nu(\tau\sigma) &= \nu(u(\mathbf{x}) \mapsto w(\mathbf{x})) \\ &= \boldsymbol{\nabla} u \mapsto \boldsymbol{\nabla} w = \nu(\tau)\nu(\sigma) \end{aligned}$$

Hence, the functors μ, ν are well-defined. Quod. Erat. Demonstrandum. □

Remark: *As* $\nu \circ \mu = e_{\text{Mor} \sqcup \text{Obj}}$ *and* $\mu \circ \nu = e_{\text{Mor}' \sqcup \text{Obj}'}$, $(\text{Obj}, \text{Mor}) \cong (\text{Obj}', \text{Mor}')$.

Definition 3.13. (**Left Action**)
Let G be a group, and X be a set.
If a function $* : G \times X \to X$ satisfies the following two axioms:
(1) For the identity $e \in G$, for all $x \in X$, $e * x = x$.
(2) For all $g_1, g_2 \in G$ and $x \in X$, $(g_1 g_2) * x = g_1 * (g_2 * x)$.
Then $*$ is a left action of G on X.

Proposition 3.14. Define the followings:
(1) $\mathrm{Obj} = \{G\}$.
(2) $\mathrm{Mor} = G$.
(3) $\mathrm{Obj}' = \{X\}$.
(4) $\mathrm{Mor}' = \mathrm{Perm}(X)$.
The following two statements are logically equivalent:
(1) $* : G \times X \to X$ is a left action of G on X.
(2) $\sigma : G \mapsto X, g \mapsto (x \mapsto g * x)$ is a functor.

Proof. By omitting quantifiers, we can prove this statement directly.

$$\begin{aligned} * \text{ is a left action } &\iff e * x = x \text{ and } (g_1 g_2) * x = g_1 * (g_2 * x) \\ &\iff \sigma(e_G) = e_X \text{ and } \sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2) \iff \sigma \text{ is a functor} \end{aligned}$$

Quod. Erat. Demonstrandum. □

4 The Fundamental Group

4.1 The Fundamental Groupoid

To describe the set of all paths, we define groupoid.

Definition 4.1. (**Groupoid**)
Let $(\mathrm{Obj}, \mathrm{Mor})$ be a category.
$(\mathrm{Obj}, \mathrm{Mor})$ is a groupoid if for all objects $A, B \in \mathrm{Obj}$:

$$\forall \sigma \in \mathrm{Mor}(A, B), \exists \tau \in \mathrm{Mor}(B, A), \tau \circ \sigma = e_A \text{ and } \sigma \circ \tau = e_B$$

Definition 4.2. (**Concatenation**)
Let X be a topological space, $x_0, x_1, \cdots, x_n$ be a sequence of points,
$0 = c_0 < c_1 < \cdots < c_n = 1$ be a partition of $[0, 1]$,
and $\gamma_0, \gamma_1, \cdots, \gamma_{n-1} : [0, 1] \to X$ be a sequence of paths satisfying:

$$x_0 = \gamma_0(0), \gamma_0(1) = x_1 = \gamma_1(0), \cdots, \gamma_{n-1}(1) = x_n$$

Define the following path $\gamma = \gamma_0 \star_{c_1} \gamma_1 \star_{c_2} \cdots \star_{c_{n-1}} \gamma_{n-1} : [0, 1] \to X$
as the concatenation of $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ at $c_0, c_1, \cdots, c_n$:

$$\gamma(t) = \begin{cases} \gamma_0\left(\frac{t - c_0}{c_1 - c_0}\right) & \text{if} \quad c_0 \le t \le c_1; \\ \gamma_1\left(\frac{t - c_1}{c_2 - c_1}\right) & \text{if} \quad c_1 \le t \le c_2; \\ \vdots & \qquad \vdots \\ \gamma_{n-1}\left(\frac{t - c_{n-1}}{c_n - c_{n-1}}\right) & \text{if} \quad c_{n-1} \le t \le c_n; \end{cases}$$

Remark: *The continuity of γ follows from the gluing lemma.*

Definition 4.3. (**Path Homotopy**)
Let X be a topological space, and $\gamma, \gamma' : [0,1] \to X$ be two paths.
If $\gamma \sim \gamma'$ rel $\{0,1\}$, then $\gamma \approx \gamma'$, i.e., the path γ is homotopic to the path γ'.

Remark: *From now on, the notation $[\![\gamma]\!]$ means the path homotopy class of γ.*

Definition 4.4. (**The Fundamental Groupoid**)
Let X be a topological space. Define the followings:
(1) $\mathrm{Obj} = X$.
(2) $\mathrm{Mor} = $ [All path homotopy class $[\![\gamma_0]\!]$].
Define $\pi_1(X) = (\mathrm{Obj}, \mathrm{Mor})$ as the fundamental groupoid of X.

Proposition 4.5. $\pi_1(X)$ is a groupoid.

Proof. We may divide our proof into five parts.
(1) For all points $x_0, x_1 \in \mathrm{Obj}$, there exists a unique collection of path homotopy classes:

$$\mathrm{Mor}(x_0, x_1) = [\text{All path homotopy class } [\![\gamma_0]\!] \text{ from } x_0 \text{ to } x_1] \subseteq \mathrm{Mor}$$

(2) For all points $x_0, x_1, x_2 \in \mathrm{Obj}$, there exists a unique binary operation:

$$\star : \mathrm{Mor}(x_0, x_1) \times \mathrm{Mor}(x_1, x_2) \to \mathrm{Mor}(x_0, x_2), [\![\gamma_0]\!] \star [\![\gamma_1]\!] = [\![\gamma_0 \star_c \gamma_1]\!]$$

The following argument suggests that $[\![\gamma_0]\!] \star [\![\gamma_1]\!] \in \mathrm{Mor}(x_0, x_2)$ is well-defined:

$$[\![\gamma_0]\!] = [\![\gamma_0']\!] \text{ and } [\![\gamma_1]\!] = [\![\gamma_1']\!] \implies [\![\gamma_0]\!] \star [\![\gamma_1]\!] = [\![\gamma_0 \star_c \gamma_1]\!] = [\![\gamma_0' \star_{c'} \gamma_1']\!] = [\![\gamma_0']\!] \star [\![\gamma_1']\!]$$

(3) For all points $x_0, x_1, x_2, x_3 \in \mathrm{Obj}$:

$$\forall [\![\gamma_0]\!] \in \mathrm{Mor}(x_0, x_1), \forall [\![\gamma_1]\!] \in \mathrm{Mor}(x_1, x_2), \forall [\![\gamma_2]\!] \in \mathrm{Mor}(x_2, x_3)$$

$$\begin{aligned}
([\![\gamma_0]\!] \star [\![\gamma_1]\!]) \star [\![\gamma_2]\!] &= [\![\gamma_0 \star_{c_1} \gamma_1]\!] \star [\![\gamma_2]\!] = [\![(\gamma_0 \star_{c_1} \gamma_1) \star_{c_2} \gamma_2]\!] \\
&= [\![\gamma_0 \star_{0 + c_2(c_1 - 0)} \gamma_1 \star_{c_2} \gamma_2]\!] = [\![\gamma_0 \star_{c_1} \gamma_1 \star_{1 - c_1(1 - c_2)} \gamma_2]\!] \\
&= [\![\gamma_0 \star_{c_1} (\gamma_1 \star_{c_2} \gamma_2)]\!] = [\![\gamma_0]\!] \star [\![\gamma_1 \star_{c_2} \gamma_2]\!] = [\![\gamma_0]\!] \star ([\![\gamma_1]\!] \star [\![\gamma_2]\!])
\end{aligned}$$

(4) For all point $x_0 \in \mathrm{Obj}$:

$$\begin{aligned}
\exists [\![e_{x_0} : t \mapsto x_0]\!] \in \mathrm{Mor}(x_0, x_0), \quad &\forall x_1 \in \mathrm{Obj}, \quad \forall [\![\sigma]\!] \in \mathrm{Mor}(x_0, x_1), \quad [\![e_{x_0}]\!] \star [\![\sigma]\!] = [\![\sigma]\!]; \\
&\forall x_1 \in \mathrm{Obj}, \quad \forall [\![\tau]\!] \in \mathrm{Mor}(x_1, x_0), \quad [\![\tau]\!] \star [\![e_{x_0}]\!] = [\![\tau]\!];
\end{aligned}$$

(5) For all points $x_0, x_1 \in \mathrm{Obj}$:

$$\forall [\![\sigma]\!] \in \mathrm{Mor}(x_0, x_1), \exists [\![\tau : t \mapsto \sigma(1-t)]\!] \in \mathrm{Mor}(x_1, x_0)$$

$$[\![\sigma]\!] \star [\![\tau]\!] = [\![e_{x_0}]\!] \text{ and } [\![\tau]\!] \star [\![\sigma]\!] = [\![e_{x_1}]\!]$$

Hence, $(\mathrm{Obj}, \mathrm{Mor})$ is a groupoid. Quod. Erat. Demonstrandum. □

Proposition 4.6. Let $(\mathrm{Obj}, \mathrm{Mor})$ be a groupoid.
For all $A \in \mathrm{Obj}$, $\mathrm{Mor}(A, A)$ is a group.

Proof. We may divide our proof into four parts.
(1) For all $\sigma, \tau \in \mathrm{Mor}(A, A)$, there exists a unique $\tau \circ \sigma \in \mathrm{Mor}(A, A)$.
(2) For all $\mu, \nu, \sigma \in \mathrm{Mor}(A, A)$:

$$\sigma \circ (\nu \circ \mu) = (\sigma \circ \nu) \circ \mu$$

(3) There exists $e_A \in \mathrm{Mor}(A, A)$, such that for all $\sigma \in \mathrm{Mor}(A, A)$:

$$\sigma \circ e_A = e_A \circ \sigma = \sigma$$

(4) For all $\sigma \in \mathrm{Mor}(A, A)$, there exists $\tau \in \mathrm{Mor}(A, A)$, such that:

$$\sigma \circ \tau = \tau \circ \sigma = e_A$$

Hence, $\mathrm{Mor}(A, A)$ is a group. Quod. Erat. Demonstrandum. □

4.2 The Fundamental Functor

The fundamental groupoid $\pi_1(X)$ is an additional algebraic structure that "grows from the topological space X", just like a flower blossoms in a lush field of greenery.

Definition 4.7. (**The Fundamental Functor**)
Define the followings:
(1) **Obj** = [All topological space X].
(2) **Mor** = [All continuous map σ].
(3) **Obj′** = [All groupoid X'].
(4) **Mor′** = [All groupoid homomorphism σ'].
Define the fundamental functor π_1 as the functor that:
(1) Sends every topological space X
to the fundamental groupoid $X' = \pi_1(X)$.
(2) Sends every continuous function σ
to the groupoid homomorphism $\sigma' : x \mapsto \sigma(x), [\![\gamma]\!] \mapsto [\![\sigma \circ \gamma]\!]$.

Proposition 4.8. π_1 is a functor.

Proof. We may divide our proof into two parts.
(1) For all topological space $X \in \mathbf{Obj}$:

$$\pi_1(e_X) = [\![\gamma]\!] \mapsto [\![e_X \circ \gamma]\!] = e_{\pi_1(X)}$$

(2) For all topological spaces $X, Y, Z \in \mathbf{Obj}$:

$$\begin{aligned}\forall \mu \in \mathbf{Mor}(X,Y), \forall \nu \in \mathbf{Mor}(Y,Z), \pi_1(\nu \circ \mu) &= x \mapsto \nu \circ \mu(x), [\![\gamma]\!] \mapsto [\![\nu \circ \mu \circ \gamma]\!] \\ &= \pi_1(\nu) \circ \pi_1(\mu)\end{aligned}$$

Hence, π_1 is a functor. Quod. Erat. Demonstrandum. $\square$

Remark: *It follows directly that $X \cong Y \implies \pi_1(X) \cong \pi_1(Y)$.*
To make things even more interesting, a weakened hypothesis yields a similar result!

Proposition 4.9. Define the followings:
(1) $\mathbf{Obj}$ = [All topological space X with base point x_0].
(2) $\mathbf{Mor}$ = [All base point preserving continuous map σ].
(3) $\mathbf{Obj}'$ = [All group X'].
(4) $\mathbf{Mor}'$ = [All group homomorphism σ'].
For all $(X, x_0), (Y, y_0) \in \mathbf{Obj}$:

$$(X, x_0) \sim (Y, y_0) \implies \pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

Proof. It suffices to prove that for all base point preserving homotopic continuous functions $f \sim g$ from (X, x_0) to (Y, y_0), the group homomorphisms f', g' are equal.
For all group element $[\![\sigma]\!] \in \pi_1(X, x_0)$:

$$f'([\![\sigma]\!]) = [\![f \circ \sigma]\!] = [\![g \circ \sigma]\!] = g'([\![\sigma]\!])$$

Hence, $f' = g'$.
Now, if there exist two base point preserving continuous functions $f : X \to Y, g : Y \to X$, such that $g \circ f \sim e_X$ and $f \circ g \sim e_Y$, then the group homomorphisms f', g' satisfy $g' \circ f' = e_{X'}$ and $f' \circ g' = e_{Y'}$, so f', g' are isomorphisms, and $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.
Quod. Erat. Demonstrandum. $\square$

4.3 Construction of the Fundamental Groupoid

Proposition 4.10. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces, and $X = \coprod_{\lambda\in I} X_\lambda$ be the coproduct space of $(X_\lambda)_{\lambda\in I}$.

$$\pi_1(X) \cong \coprod_{\lambda\in I} \pi_1(X_\lambda)$$

Proof. Define σ from $\coprod_{\lambda\in I}\pi_1(X_\lambda)$ to $\pi_1(X)$ as a groupoid homomorphism that:
(1) Sends every object (x_0,λ) to the object (x_0,λ) .
(2) Sends every morphism $([\![t\mapsto\gamma(t)]\!],\lambda)$ to the morphism $[\![t\mapsto(\gamma(t),\lambda)]\!]$.
We need to prove that σ is indeed a bijective groupoid homomorphism.
(1) For all object (x_0,λ):

$$\sigma([\![e_{x_0}]\!],\lambda) = \sigma([\![t\mapsto x_0]\!],\lambda) = [\![t\mapsto(x_0,\lambda)]\!] = [\![e_{(x_0,\lambda)}]\!]$$

(2) For all objects $(x_0,\lambda),(x_1,\lambda),(x_2,\lambda)$ with the same subscript λ:

$$\forall[\![\gamma_0]\!]\in\mathrm{Mor}(x_0,x_1),\forall[\![\gamma_1]\!]\in\mathrm{Mor}(x_1,x_2)$$

$$\begin{aligned}\sigma([\![\gamma_0]\!]\star[\![\gamma_1]\!],\lambda) &= \sigma([\![t\mapsto\gamma_0\star_c\gamma_1(t)]\!],\lambda) = [\![t\mapsto(\gamma_0\star_c\gamma_1(t),\lambda)]\!]\\ &= [\![t\mapsto(\gamma_0(t),\lambda)]\!]\star[\![t\mapsto(\gamma_1(t),\lambda)]\!] = \sigma([\![\gamma_0]\!],\lambda)\star\sigma([\![\gamma_1]\!],\lambda)\end{aligned}$$

(3) For all objects $(x_0,\lambda),(x_1,\lambda),(x_0',\lambda'),(x_1',\lambda')$ with subscripts λ,λ':

$$\forall[\![\gamma]\!]\in\mathrm{Mor}(x_0,x_1),\forall[\![\gamma']\!]\in\mathrm{Mor}(x_0',x_1')$$

$$\begin{aligned}\sigma([\![\gamma]\!],\lambda) = \sigma([\![\gamma']\!],\lambda') &\implies [\![t\mapsto(\gamma(t),\lambda)]\!] = [\![t\mapsto(\gamma'(t),\lambda')]\!]\\ &\implies ([\![\gamma]\!],\lambda) = ([\![\gamma']\!],\lambda')\end{aligned}$$

(4) As the components of coproduct space are pairwisely not path connected, every morphism of $\pi_1(X)$ is in the form $[\![t\mapsto(\gamma(t),\lambda)]\!]$.
Hence, σ is bijective groupoid homomorphism, σ is a groupoid isomorphism, $\pi_1(X)\cong\coprod_{\lambda\in I}\pi_1(X_\lambda)$. Quod. Erat. Demonstrandum. $\square$

Proposition 4.11. Let $(X_\lambda)_{\lambda\in I}$ be an indexed family of topological spaces, and $X = \prod_{\lambda\in I} X_\lambda$ be the product space of $(X_\lambda)_{\lambda\in I}$.

$$\pi_1(X) \cong \prod_{\lambda\in I} \pi_1(X_\lambda)$$

Proof. Define σ from $\pi_1(X)$ to $\prod_{\lambda\in I}\pi_1(X_\lambda)$ as a groupoid homomorphism that:
(1) Sends every object x_0 to the object x_0.

(2) Sends every morphism $[\![t \mapsto (\gamma_\lambda(t))_{\lambda\in I}]\!]$ to the morphism $([\![t \mapsto \gamma_\lambda(t)]\!])_{\lambda\in I}$.
We need to prove that σ is indeed a bijective groupoid homomorphism.
(1) For all object x_0:

$$\sigma([\![e_{x_0}]\!]) = \sigma([\![t \mapsto (x_{0,\lambda})_{\lambda\in I}]\!]) = ([\![t \mapsto x_{0,\lambda}]\!])_{\lambda\in I} = (e_{x_{0,\lambda}})_{\lambda\in I}$$

(2) For all objects x_0, x_1, x_2:

$$\forall [\![\gamma_0]\!] \in \mathrm{Mor}(x_0, x_1), \forall [\![\gamma_1]\!] \in \mathrm{Mor}(x_1, x_2)$$

$$\begin{aligned}\sigma([\![\gamma_0]\!] \star [\![\gamma_1]\!]) &= \sigma([\![t \mapsto ((\gamma_0 \star_c \gamma_1)_\lambda(t))_{\lambda\in I}]\!]) = ([\![t \mapsto (\gamma_0 \star_c \gamma_1)_\lambda(t)]\!])_{\lambda\in I} \\ &= ([\![t \mapsto \gamma_{1,\lambda}(t)]\!])_{\lambda\in I} \star ([\![t \mapsto \gamma_{2,\lambda}(t)]\!])_\lambda = \sigma([\![\gamma_0]\!]) \star \sigma([\![\gamma_1]\!])\end{aligned}$$

(3) For all objects x_0, x_1, x_0', x_1':

$$\forall [\![\gamma]\!] \in \mathrm{Mor}(x_0, x_1), \forall [\![\gamma']\!] \in \mathrm{Mor}(x_0', x_1')$$

$$\begin{aligned}\sigma([\![\gamma_0]\!]) = \sigma([\![\gamma_1]\!]) &\implies ([\![t \mapsto \gamma_{0,\lambda}(t)]\!])_{\lambda\in I} = ([\![t \mapsto \gamma_{1,\lambda}(t)]\!])_{\lambda\in I} \\ &\implies [\![\gamma_0]\!] = [\![\gamma_1]\!]\end{aligned}$$

(4) By the definition of Cartesian product,
every morphism of $\pi_1(X)$ is in the form $[\![t \mapsto \gamma_{0,\lambda}(t)]\!]$.
Hence, σ is bijective groupoid homomorphism, σ is a groupoid isomorphism,
$\pi_1(X) \cong \prod_{\lambda\in I} \pi_1(\lambda)$. Quod. Erat. Demonstrandum. $\square$

Remark: *It follows that $\pi_1(\mathbb{S} \times \mathbb{S}) = \pi_1(\mathbb{S}) \times \pi_1(\mathbb{S})$.*

References

[1] H. Ren, "Template for math notes," 2021.