MATH3541 INTRODUCTION TO TOPOLOGY
ASSIGNMENT II

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG

Due: 12:00 noon, Tuesday 1st October 2024.

Instructions: Submit solutions to the problems in **Section B** for credit. Problems in Section A should be attempted and may be optionally submitted for feedback.

Guidelines on Writing: *You should write in complete sentences. Do not just give the solution in fragmentary bits and pieces. Clarity of presentation of your argument counts, so explain the meaning of every symbol that you introduce and avoid starting a sentence with a symbol.*

Section A

Problem 1. Let X and Y be two topological spaces and let $f\colon X \to Y$ be a map. Prove that the following statements are equivalent:

(1) f is continuous;
(2) $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X;
(3) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every subset B of Y.

Problem 2. Let X, Y, and Z be topological spaces, and equip $X \times Y$ with the product topology. Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be the the projection maps. Prove that a map $f\colon Z \to X \times Y$ is continuous if and only if both $p_1 \circ f\colon Z \to X$ and $p_2 \circ f\colon Z \to Y$ are continuous.

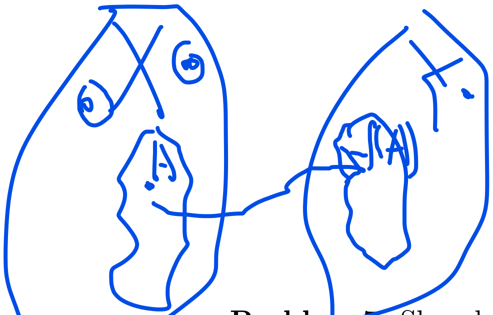
Problem 3. Let $f\colon X \to Y$ be a continuous map, let $\mathrm{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y$, and equip $\mathrm{Graph}(f)$ with the subspace topology of the product space $X \times Y$. Show that $\mathrm{Graph}(f)$ is homeomorphic to X.

Problem 4. Show that for any two disjoint closed subsets A and B of a metric space X, there is a continuous $\mathbb{R}$-valued function f on X such that $f|_A = 1$, $f|_B = -1$, and $f(x) \in (-1, 1)$ if $x \notin A \cup B$. Show that if A and B are disjoint, then there exist open subsets U, V of X such that $A \subset U, B \subset V$.

Problem 5. State the definition of a topological embedding. Prove that an injective continuous map that is either open or closed is a topological embedding.

Problem 6. Let X be a topological space, let $[X]$ be the quotient space of X associated to a partition of X, and let $p\colon X \to [X]$ be the quotient map. Let Y be any topological space. Prove that a map $f\colon [X] \to Y$ is continuous if and only if $f \circ p\colon X \to Y$ is continuous.

$X \sqcup Y/\sim$



$X \sqcup Y = \{(x,0)-(y,1)\}$
$\{(x,0), (f(x),1)\}$

Problem 7. Show by examples that open or closed maps are not necessarily continuous, and continuous maps do not have to be either open or closed. (What happens if the spaces have the discrete topology or the trivial topology?)

Problem 8. Suppose that X and Y are Hausdorff. Let A be a closed subset of Y and assume that $f\colon A \to X$ is a closed embedding. prove or disprove that the adjunction space $X \cup_f Y$ is Hausdorff.

Section B

Problem 9. (2 marks) Are the following pairs of spaces homeomorphic? Give a short justification for each.

(a) The circle S^1 and the closed interval $[0,1]$.
(b) The circle S^1 and the sphere S^2.
(c) The product spaces $[0,1) \times [0,1)$ and $[0,1] \times [0,1)$.
(d) The product spaces $[0,1) \times [0,1)$ and $[0,1] \times [0,1]$.

Problem 10. (4 marks) Let X be a non-compact, Hausdorff topological space with its collection of open sets denoted $\mathcal{O}$. Let $\mathcal{O}'$ be the collection of all subsets $U \subset X$ such that $U = \emptyset$ or $X - U$ is compact in $(X, \mathcal{O})$.

(a) Show that $\mathcal{O}' \subseteq \mathcal{O}$.
(b) Prove that $(X, \mathcal{O}')$ is a topological space.
(c) Prove that $(X, \mathcal{O}')$ is not Hausdorff.
(d) Prove that $(X, \mathcal{O}')$ is compact.

Problem 11. (8 marks) First prove some criteria for open maps:

(a) Let $f : X \to Y$ be a (not necessarily continuous) function between topological spaces which is locally open. That is, for any $x \in X$, there exists an open set $U \ni x$ such that $f|_U : U \to Y$ is open. Prove that f is open.
(b) If p is a local homeomorphism, and q is an open map, show that $q \circ p$ is an open map.

Now consider polynomial functions $p : \mathbb{C} \to \mathbb{C}$,

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{C}[z].$$

(c) If $p'(z_0) \neq 0$, show that p is locally a homeomorphism at $z_0 \in \mathbb{C}$. That is, there exists an open set $U \ni z_0$ such that $p|_U : U \to \mathbb{C}$ is a homeomorphism.
(d) For $d \in \mathbb{Z}$, $d > 0$, show that $q(z) = z^d$ is locally open at 0.
(e) Show that if $p'(z_0) = 0$, then there is a local homeomorphism $\psi : B(z_0, \delta) \to \mathbb{C}$ such that $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, and $p(z) - p(z_0) = (\psi(z))^d$ on $B(z_0, \delta)$ for some d.
(e) Combine the previous parts to show that every non-constant polynomial function $p : \mathbb{C} \to \mathbb{C}$ is an open map.

Problem 12. (6 marks) Let $(X, \mathcal{O})$ be a topological space. A point $x \in X$ is called *isolated* if and only if $\{x\}$ is open.

(a) If X is compact and $(A_n)_{n=1}^{\infty}$ is a sequence of nested ($A_{n+1} \subseteq A_n$), non-empty, closed subsets of X, show that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

(b) Suppose that X is non-empty and Hausdorff with no isolated points. Show that given any non-empty open set U and any $x \in X$, there is a non-empty open $V \subseteq U$ such that $x \notin \overline{V}$.

(c) Let X be non-empty, compact, Hausdorff with no isolated points and suppose we have a function $f : \mathbb{N} \to X$, $n \mapsto x_n$. Find a sequence of nested, non-empty, open sets $V_n \subseteq V_{n-1}$ such that $x_n \notin \overline{V_n}$. Apply part (a) to show that f cannot be surjective.

(This problem shows that a non-empty, compact, Hausdorff topological space with no isolated points must be uncountable.)