Algebra II: Tutorial 10

April 20, 2022

Problem 1 (Finite fields are normal)**.** Let $f(x)$ be a monic irreducible polynomial over $\mathbb{F}_p$, and let α be a root of f in some splitting field of f over $\mathbb{F}_p$. Show that $L = \mathbb{F}_p(\alpha)$ is a splitting field for f over $\mathbb{F}_p$.

Problem 2. Show that there are exactly two cubic irreducible polynomials in $\mathbb{F}_2[x]$, namely $f = x^3 + x + 1$ and $g = x^3 + x^2 + 1$. Write down the multiplication tables of the field extensions of $\mathbb{F}_2$ by adding a root of f and g, say $\mathbb{F}_8$ and $\mathbb{F}_8'$. Show that they are isomorphic.

Problem 3 (Recognising prime subfields of finite fields)**.** Let L be a field containing $\mathbb{F}_p$. For $\alpha \in L$, show that $\alpha \in \mathbb{F}_p$ if and only if $\alpha^p - \alpha = 0$.

Problem 4. Let $f(x) = x^9 - x + 1$ in $\mathbb{F}_3$.

1. Show that f has no roots in $\mathbb{F}_3$ and in $\mathbb{F}_9$.
2. Show that $\mathbb{F}_{27} \cong \frac{\mathbb{F}_3[x]}{(x^3-x-1)}$, and show that every root of $x^3 - x - 1$ is a root of f.
3. Determine all the roots of f over $\mathbb{F}_{27}$, and deduce a factorisation of f over $\mathbb{F}_3$.