THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3301
Assignment 2

Due Date: Oct 3, 2024, 23:59.

Submission Guidelines

(i) Write your solution on papers of about A4 size.

(ii) Scan your work properly and save it as **one** PDF file.

Warning: Please make sure that your work is properly scanned. Oversized, blurred or upside-down images will NOT be accepted.

(iii) While you can upload and save draft in moodle, you MUST click the "Submit" button to declare your final submission before the due date. Otherwise, you will be considered late.

Preparation Guidelines

(i) Your solution should be well written and organized. It is good to work out a draft for each question on a separate paper, polish/rewrite/reorganize your answer suitably and then write it (the final form) on the paper to be scanned.

(ii) You may imagine that you are teaching this course and writing a solution to demonstrate the answer. Hence, especially for proof-type questions, you have to convince everyone that your solution (proof) is correct, without any oral explanation from you. i.e. Another student should be able to understand the answer (proof) completely via your written word, and/or diagrams or tables you create in your solution.

(iii) Follow HKU's regulations on academic honesty. Plagiarism is inacceptable and may have severe consequences for your record. See `https://tl.hku.hk/plagiarism/` for "What is plagiarism?". *If you have used AI tools to explore, check or refine your work, please acknowledge and clearly identify the parts of your work that involve AI output to avoid plagiarism or related academic dishonesty. Indicate the extent to which the AI output is used (e.g. directly copied or paraphrased/modified or checked for errors or reorganized the presentation).*

Part I: Not to be handed in

1. Given that G is a group of order 8 having two generators x, y such that $x^4 = y^2 = e$ and $xy = yx^3$, where e is the identity of G.

   (i) For any $m, n \in \mathbb{Z}$, show that $y^m x^n = x^i y^j$ for some $i \in \{0,1,2,3\}$ and $j \in \{0,1\}$.

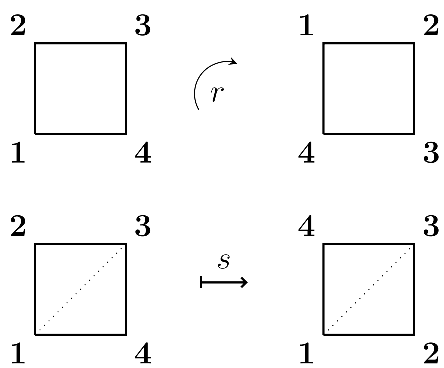
   (ii) For any $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, show that $x^{m_1} y^{n_1} x^{m_2} y^{n_2} = x^i y^j$ for some $i \in \{0,1,2,3\}$ and $j \in \{0,1\}$.

   (iii) Show that every element of G can be written in the form of $x^i y^j$ with $i \in \{0,1,2,3\}$ and $j \in \{0,1\}$. Verify that these elements are distinct.

   (ii) Complete the Cayley table below by expressing the products in the form $x^i y^j$ with $i = 0,1,2,3$ and $j = 0,1$.

$\cdot$	e	x	x^2	x^3	y	yx	yx^2	yx^3
e	x^0y^0	x^1y^0	x^2y^0	x^3y^0	x^0y^1	x^3y^1		
x								
x^2								
x^3								
y								
yx								
yx^2								
yx^3								

2. Consider the group D_4 of rigid motions of a square. Below two motions are shown.



$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Show that $D_4 = \{\text{id}, r, r^2, r^3, s, sr, sr^2, sr^3\}$ under the operation of composition is the "same" as (i.e. isomorphic to) the group G in Qn 1 by working out its Cayley table.

3. Let G_1, G_2, G_1', G_2' be groups. Suppose $G_1 \cong G_1'$ and $G_2 \cong G_2'$. Show that $G_1 \times G_2 \cong G_1' \times G_2'$.

Part II: To be handed in

1. Let G be a group. A subgroup H of G is said to be **normal** if $xHx^{-1} = H$ for *all* $x \in G$, where $aHb := \{ahb : h \in H\}$ for any $a, b \in G$.

   (a) Give an example to show that xhx^{-1} may not equal h even if $h \in H$ and H is a normal subgroup.

   (b) Show that if H is a normal subgroup, then $xH = Hx$ for all $x \in G$ and vice versa.

   (c) Show that H is a normal subgroup if $H < G$ and $xHx^{-1} \subset H$, $\forall\ x \in G$.

   (d) Show that H is a normal subgroup if $H < G$ and $[G : H] = 2$.

   (e) Show that $\phi^{-1}(H')$ is a normal subgroup of G if $\phi : G \to G'$ is a group homomorphism and H' is a normal subgroup of G'.

   (f) Show that $\phi(H)$ is a normal subgroup of $\phi(G)$ if $\phi : G \to G'$ is a group homomorphism and H is a normal subgroup of G.

2. Find all the isomorphisms from S_3 to S_3. Let X be the set of all isomorphisms from S_3 to S_3. What do you observe for $(X, \circ)$ where $\circ$ denotes the operation of function composition?

3. Let $x = \begin{pmatrix} x_1 & x_2 & \cdots & x_k \end{pmatrix}$ be a cycle of length k in a permutation group S_n. Show that for any $\sigma \in S_n$, the conjugate $\sigma x \sigma^{-1}$ is the cycle $\begin{pmatrix} \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_k) \end{pmatrix}$.

4. Consider cycles in the permutation group S_n. Show that two cycles are conjugate to each other if and only if the two cycles have the same length. [**Note**: Let $\sigma, \tau \in S_n$. We say that σ is **conjugate to** τ if there exists $\alpha \in S_n$ such that $\sigma = \alpha\tau\alpha^{-1}$.]

5. (a) List all the elements of S_4 according to its cycle patterns.
   [S_4 **has elements of 5 cycle patterns:** (i) **trivial element;** (ii) **cycles of length** 2**,** e.g. $(1,2)$**;** (iii) **products of two disjoint cycles of length** 2**,** e.g. $(1,2)(3,4)$**;** (iv) **cycles of length** 3**,** e.g. $(1,2,3)$**;** (v) **cycles of length** 4**,** e.g. $(1,2,3,4)$**.**]

   (b) Let V be the following set of four permutations:
   $$V = \{\,(1),\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\,\}.$$
   Show that V is a subgroup of S_4 and V is abelian.

   (c) List out all the left and right cosets of V in S_4.

   (d) By using Qn 4 and Part (b), show that V is an abelian normal subgroup of S_4.

End