THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH6101 / MATH7101 Intermediate Complex Analysis

**Remarks on holomorphic line bundles**

Let $Z$ be a Riemann surface, and $\Gamma \subset \mathrm{Aut}(Z)$ be a discrete group of automorphisms acting without fixed point. Define $X = Z/\Gamma$. $X$ inherits the structure of a quotient Riemann surface. In practice $Z$ is either the complex plane $\mathbb{C}$ or the unit disk $D$ (equivalently the upper half-plane $\mathcal{H}$), the canonical map $\rho : Z \to Z/\Gamma = X$ is the universal covering map and $\Gamma \cong \pi_1(X)$ is the group of covering transformations (also called Deck transformations).

For each $\gamma \in \Gamma$ let $\Phi(\gamma) : Z \times \mathbb{C} \to Z \times \mathbb{C}$ be given by $\Phi(\gamma)(z, w) = \big(\gamma(z), \varphi_\gamma(z)w\big)$, where $\varphi_\gamma : Z \to \mathbb{C}^*$ is a nowhere 0 holomorphic function. Suppose $\Phi$ satisfies the transformation rule

$$(\dagger) \qquad \Phi(\gamma_1\gamma_2)(z, w) = \Phi(\gamma_1)\big(\Phi(\gamma_2)(z, w)\big);\ \Phi(id)(z, w) = (z, w).$$

This is the case if and only if $\varphi_{id}(z) = 1$ and

$$\begin{aligned}
&\big((\gamma_1\gamma_2)(z), \varphi_{\gamma_1\gamma_2}(z)w\big) = \Phi(\gamma_1\gamma_2)(z, w) \\
=\ & \Phi(\gamma_1)\big(\Phi(\gamma_2)(z, w)\big) = \Phi(\gamma_1)\big(\gamma_2(z), \varphi_{\gamma_2}(z)w\big) \\
=&\big(\gamma_1\left(\gamma_2(z)\right), \varphi_{\gamma_1}\left(\gamma_2(z)\right)\varphi_{\gamma_2}(z)w\big).
\end{aligned}$$

Noting that $\gamma_1(\gamma_2(z)) = (\gamma_1\gamma_2)(z)$, $(\dagger)$ holds true for all $\gamma_1, \gamma_2 \in \Gamma$ and all $z \in Z$ if and only if we have the compatibility condition

$$(\dagger\dagger) \qquad \varphi_{\gamma_1\gamma_2}(z) = \varphi_{\gamma_1}\big(\gamma_2(z)\big)\varphi_{\gamma_2}(z); \quad \varphi_{id}(z) = 1$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and all $z \in Z$. Conversely, given a family of holomorphic functions $\varphi_\gamma : Z \to \mathbb{C}^*$ satisfying $(\dagger\dagger)$, defining $\Phi(\gamma) : Z \times \mathbb{C} \to Z \times \mathbb{C}$ by $\Phi(\gamma)(z, w) = \big(\gamma(z), \varphi_\gamma(z)w\big)$ we have the transformation rules $(\dagger)$.

Introduce now an equivalence relation $\sim$ on $Z \times \mathbb{C}$ by declaring $(z_2, w_2) \sim (z_1, w_1)$ if and only if there exists some $\gamma \in \Gamma$ such that $(z_2, w_2) = \Phi(\gamma)(z_1, w_1)$. Then, $\sim$ is an equivalence relation. In fact, we have

▶ **Reflexive property**

$(z, w) = \Phi(id)(z, w)$ by definition.

▶ **Symmetric property**

Given $(z_2, w_2) = \Phi(\gamma)(z_1, w_1)$, we have

$$\Phi(\gamma^{-1})(z_2, w_2) = \Phi(\gamma^{-1})\,(\Phi(\gamma)(z_1, w_1))$$

$$= \Phi(\gamma^{-1}\gamma)(z_1, w_1) = \Phi(id)(z_1, w_1) = (z_1, w_1).$$

Thus, $(z_2, w_2) \sim (z_1, w_1)$ implies $(z_1, w_1) \sim (z_2, w_2)$.

▶ **Transitive property**

$(z_3, w_3) \sim (z_2, w_2)$ and $(z_2, w_2) \sim (z_1, w_1)$ imply that there exists $\mu, \gamma \in \Gamma$ such that

$$(z_3, w_3) = \Phi(\mu)(z_2, w_2)$$

$$(z_2, w_2) = \Phi(\gamma)(z_1, w).$$

Hence,

$$(z_3, w_3) = \Phi(\mu)\big(\Phi(\gamma)(z_1, w_1)\big) = \Phi(\mu\gamma)(z_1, w_1), \text{ by } (\dagger),$$

proving that $\sim$ is an equivalence relation on $Z \times \mathbb{C}$. We may now define $E$ : $(Z \times \mathbb{C})/\!\sim\; = (Z \times \mathbb{C})/\Gamma$ as a 2-dimensional complex manifold, noting that $\Gamma$ acts discretely on $Z \times \mathbb{C}$ without fixed points. We have canonically a holomorphic map $\pi : E = (Z \times \mathbb{C})/\Gamma \to Z/\Gamma = X$. For each point $x_0 \in X$, and for any choice of $z_0 \in Z$ such that $\rho(z_0) = x_0$, there exists an open neighborhood $U$ of $x_0$ in $X$ and an open coordinate neighborhood $W_0$ of $z_0$ in $Z$ such that $\pi\big|_{W_0} : W \xrightarrow{\cong} U$ is a biholomorphism. Denoting by $\alpha : Z \times \mathbb{C} \to (Z \times \mathbb{C})/\!\sim\; = (Z \times \mathbb{C})/\Gamma = E$ the canonical map, $\alpha\big|_{W_0 \times \mathbb{C}} : W \times \mathbb{C} \to E$ is injective, with image the open subset $\pi^{-1}(U) \subset E$. Thus, $\alpha\big|_{W_0 \times \mathbb{C}} : W_0 \times \mathbb{C} \xrightarrow{\cong} \pi^{-1}(U)$ serves as a holomorphic coordinate chart on $E$. We have

**Lemma.** *For $E = (Z \times \mathbb{C})/\!\sim$, $\pi : E \to X($ i.e., $Z/\Gamma)$ realizes the 2-dimensional complex manifold as the total space of a holomorphic line bundle.*

Proof We have the holomorphic coordinate charts $\epsilon := \alpha\big|_{W\times\mathbb{C}} : W \times \mathbb{C} \xrightarrow{\cong} \pi^{-1}(U)$. Thus, $\epsilon^{-1} : \pi^{-1}(U) \xrightarrow{\cong} W \times \mathbb{C}$, and $\delta := (\rho\big|_W, id) : W \times \mathbb{C} \xrightarrow{\cong} U \times \mathbb{C}$, so that $\delta \circ \epsilon^{-1} : \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}$. Suppose $x \in U_\alpha \cap U_\beta$ and we have $\Psi_\alpha := \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{C}$, $\Psi_\beta := \pi^{-1}(U_\beta) \xrightarrow{\cong} U_\beta \times \mathbb{C}$ obtained as above (where $U_\alpha$ and $U_\beta$ are neighborhoods of $x$ both playing the role of $U$). Write $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Then, we have

$$\Phi_{\alpha\beta} := \Psi_\alpha \circ \Psi_\beta^{-1}\big|_{U_{\alpha\beta}\times\mathbb{C}} : U_{\alpha\beta} \times \mathbb{C} \xrightarrow{\cong} U_{\alpha\beta} \times \mathbb{C}.$$

Now, $\rho\big|_{w_\alpha} : W_\alpha \xrightarrow{\cong} U_\alpha$, $\rho\big|_{w_\beta} : W_\beta \xrightarrow{\cong} U_\beta$, and for a variable point $x \in U_{\alpha\beta}$, we have variable points $z^{(\alpha)} \in W_\alpha$, $z^{(\beta)} \in W_\beta$, $\pi(z^{(\alpha)}) = \pi(z^{(\beta)}) = x$, and there exists $\gamma_{\alpha\beta} \in \Gamma$ such that $z^{(\alpha)} = \gamma_{\alpha\beta}(z^{(\beta)})$. Then,

$$\Phi_{\alpha\beta}(x, w) = \Bigg(x, \varphi_\gamma(z^{(\beta)})w\Bigg)$$

is a transformation $\Phi_{\alpha\beta} : U_{\alpha\beta} \times \mathbb{C} \xrightarrow{\cong} U_{\alpha\beta} \times \mathbb{C}$ respecting the projections $U_{\alpha\beta} \times \mathbb{C} \to U_{\alpha\beta}$ which is holomorphic and linear in the fiber variable $w$. This realizes $\pi : E \to X$ as a holomorphic line bundle, noting that $\delta \circ \epsilon^{-1} : \pi^{-1}(U) \to U \times \mathbb{C}$ are trivializations of $E\big|_U$ over open subsets of $U \subset X$. $\square$

From general theory we know that the holomorphic line bundle is equivalently defined by (holomorphic) transition function $\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$. In the case at hand $\varphi_{\alpha\beta}(x) = \varphi_\gamma(z^{(\beta)}), \gamma = \gamma_{\alpha\beta}$, where $z^{(\beta)}$ is a variable point on $W_\beta, \theta_\beta := \rho\big|_{W_\beta} : W_\beta \xrightarrow{\cong} U_\beta$ and $\gamma \in \Gamma$ is the automorphism such that $z^{(\alpha)} = \gamma(z^{(\beta)})$, $z^{(\alpha)}$ being a variable point on $W_\alpha$, $\theta_\alpha := \rho\big|_{W_\beta} : W_\beta \xrightarrow{\cong} U_\alpha$. In other words, $\gamma$ is the covering transformation which induces $\theta_\beta^{-1}(U_{\alpha\beta}) \xrightarrow{\cong} \theta_\alpha^{-1}(U_{\alpha\beta})$, $z^{(\alpha)} = \gamma(z^{(\beta)})$, noting that $\rho(z^{(\alpha)}) = \rho(z^{(\beta)}) = x$. The compatibility condition

$$(\dagger\dagger) \qquad \varphi_{\gamma_1\gamma_2}(z) = \varphi_{\gamma_1}(\gamma_2(z))\varphi_{\gamma_2}(z)$$

then translates into the compatibility condition $\varphi_{\alpha\beta}\varphi_{\beta\alpha}\varphi_{\gamma\alpha} = 1$.