Chapter 3. Wave Equations on the Whole Real Line

MATH4406 Introduction to Partial Differential Equations

The University of Hong Kong

Table of Contents

This chapter is related to the materials in Section 2.1-2.2 of the Textbook.

3.1 Differential Operators

Notation of Differential Operators

Philosophy

In mathematical literature, the terminology "**operator**" usually refers to mapping from functions to functions, namely

$$\text{function} \xmapsto{\text{Operator}} \text{function}.$$

The adjective "**differential**" implies that the operator consists of some kind of differentiation.

Remark

One may define the mathematical object "**differential operators**" more rigorously, but this is not that necessary for understanding the materials in this course. Instead, we will illustrate some of their properties by providing examples as follows.

Examples of Differential Operators

Example

The operator $\partial_t + 2\partial_x$ means that for any "input" function u, the "output" is

$$(\partial_t + 2\partial_x)u := \partial_t u + 2\partial_x u.$$

Example (Product/Composition of Differential Operators)

One may also combine differential operators via a "composition". For example,

$$\begin{aligned}
(\partial_t + 3\partial_x)(\partial_t + 2\partial_x)u &:= (\partial_t + 3\partial_x)\left\{(\partial_t + 2\partial_x)u\right\} \\
&= (\partial_t + 3\partial_x)\left\{\partial_t u + 2\partial_x u\right\} \\
&= \partial_t\left\{\partial_t u + 2\partial_x u\right\} + 3\partial_x\left\{\partial_t u + 2\partial_x u\right\} \\
&= \left\{\partial_{tt} u + 2\partial_{tx} u\right\} + \left\{3\partial_{tx} u + 6\partial_{xx} u\right\} \\
&= \partial_{tt} u + 5\partial_{tx} u + 6\partial_{xx} u.
\end{aligned}$$

Exercise

Verify the following identity:

$$(\partial_t + 3\partial_x)^2 u = \partial_{tt} u + 6\partial_{tx} u + 9\partial_{xx} u.$$

[Recall: $(\partial_t + 3\partial_x)^2 u := (\partial_t + 3\partial_x)\{(\partial_t + 3\partial_x)u\}$.]

Example

Factorizing Differential Operators

1. $\partial_{xx} u + 6\partial_{xy} u + 9\partial_{yy} u = (\partial_x + 3\partial_y)^2 u$.
2. $\partial_{tt} u - 5\partial_{tx} u + 6\partial_{xx} u = (\partial_t - 2\partial_x)(\partial_t - 3\partial_x)u$.
   Remark: It is also equal to $(\partial_t - 3\partial_x)(\partial_t - 2\partial_x)u$.
3. $\partial_{tt} u - 16\partial_{xx} u = (\partial_t + 4\partial_x)(\partial_t - 4\partial_x)u$.

Question

Do ALL differential operators commute?

Example (Operators with Non-Constant Coefficients May NOT Commute.)

Let us consider the products of $\partial_t + x\partial_x$ and $2\partial_x$ as follows:

$$\begin{aligned}(\partial_t + x\partial_x)(2\partial_x)u &:= (\partial_t + x\partial_x)\left\{(2\partial_x)u\right\}\\ &= (\partial_t + x\partial_x)\left\{2\partial_x u\right\}\\ &= \partial_t\left\{2\partial_x u\right\} + x\partial_x\left\{2\partial_x u\right\}\\ &= 2\partial_{tx}u + 2x\partial_{xx}u,\end{aligned}$$

and

$$\begin{aligned}(2\partial_x)(\partial_t + x\partial_x)u &:= (2\partial_x)\left\{(\partial_t + x\partial_x)u\right\}\\ &= (2\partial_x)\left\{\partial_t u + x\partial_x u\right\}\\ &= 2\partial_x\left\{\partial_t u\right\} + 2\partial_x\left\{x\partial_x u\right\}\\ &= 2\partial_{tx}u + 2x\partial_{xx}u + 2\partial_x u.\end{aligned}$$

Therefore, in general,

$$(\partial_t + x\partial_x)(2\partial_x)u \neq (2\partial_x)(\partial_t + x\partial_x)u.$$

3.2 Solving the Wave Equations by Method of Characteristics

Wave Equations in One Spatial Dimension

The vertical displacement u of a vibrating string satisfies the wave equation in one spatial dimension:

$$\partial_{tt} u - c^2 \partial_{xx} u = 0, \tag{1DWave}$$

where the given constant $c > 0$ represents the *speed of propagation*.

Question

How to solve it?

Main Observation

$$0 = \partial_{tt} u - c^2 \partial_{xx} u = (\partial_t - c\partial_x)(\partial_t + c\partial_x) u.$$

Based on this main observation, one may solve (1DWave) by

- method of characteristics (will be seen in this section); or
- coordinate method (via using the "characteristic coordinates").

Main Idea

Rewrite the wave equation into a system of first-order equations.

Let $v := (\partial_t + c\partial_x)u = \partial_t u + c\partial_x u$. Then the wave equation

$$0 = \partial_{tt} u - c^2 \partial_{xx} u = (\partial_t - c\partial_x)(\partial_t + c\partial_x)\, u$$

can be written as the following system:

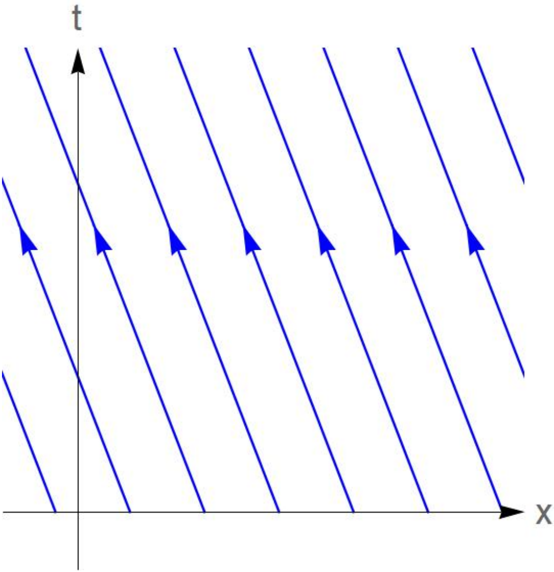
$$\begin{cases} \partial_t v - c\partial_x v = 0 \\ \partial_t u + c\partial_x u = v. \end{cases} \tag{1}$$

Applying the method of characteristics, we can solve $(1)_1$, and obtain

$$v(t,x) = h(x+ct), \tag{2}$$

for some arbitrary function h.

Characteristics for v – Traveling to the Left at Speed c



Substituting (2) into $(1)_2$, we have

$$\partial_t u + c\partial_x u = v = h(x+ct).$$

Question

How to solve $\partial_t u + c\partial_x u = h(x+ct)$?

Answer (at least two methods as follows)

1. Using the homogeneous solution u_h and particular solution u_p – Since the PDE is linear, their linear combination gives the solution.

$$u_h(t,x) := g(x-ct),$$
$$u_p(t,x) := f(x+ct), \quad \text{where } f' = \frac{1}{2c}h.$$

2. Using method of characteristics. (We will explain this in the next slide.)

Method of Characteristics

Recall from Chapter 2:

Algorithm for Solving $a(x,y)\partial_x u + b(x,y)\partial_y u = f(u,x,y)$

1. Solve for $X(s)$ and $Y(s)$ in
$$\begin{cases} \dfrac{dX}{ds} = a(X,Y) \\ \dfrac{dY}{ds} = b(X,Y). \end{cases}$$
2. Solve for $W(s)$ in
$$\frac{dW}{ds} = f(W,X,Y).$$
3. Try to find u from the relationship
$$W(s) = u(X(s),Y(s)).$$

Adjustment for $\partial_t u + c\partial_x u = h(x + ct)$

For the equation $\partial_t u + c\partial_x u = h(x + ct)$, we should adjust the algorithm as follows: (e.g., changing y to be t, setting $a \equiv c$, $b \equiv 1$, etc.)

Algorithm for Solving $\partial_t u + c\partial_x u = h(x + ct)$

1. Solve for $T(s)$ and $X(s)$ in

$$\frac{dT}{ds} = 1 \quad \text{and} \quad \frac{dX}{ds} = c.$$

2. Solve for $W(s) := u(T(s), X(s))$ in

$$\frac{dW}{ds} = h(X(s) + cT(s)).$$

3. Try to find u from the relationship

$$W(s) = u(T(s), X(s)).$$

Back to Solving the Wave Equation

Solving

$$\begin{cases} \dfrac{dT}{ds} = 1, \quad T(0) = 0, \\ \dfrac{dX}{ds} = c, \quad X(0) = x_0, \end{cases}$$

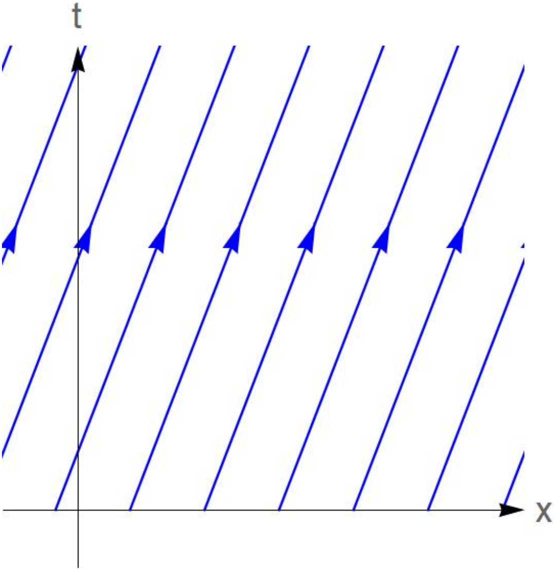
we obtain

$$T(s) = s \quad \text{and} \quad X(s) = x_0 + cs.$$

Substituting $T(s) = s$ and $X(s) = x_0 + cs$ into $\dfrac{dW}{ds} = h(X(s) + cT(s))$, we know that $W(s) := u(T(s), X(s)) = u(s, x_0 + cs)$ satisfies

$$\begin{cases} \dfrac{dW}{ds} = h(x_0 + 2cs) \\ W(0) = u(0, x_0). \end{cases}$$

Characteristics for u – Traveling to the Right at Speed c



Integrating $\dfrac{dW}{ds} = h(x_0 + 2cs)$ with respect to s over $[0, t]$, and using IC $W(0) = u(0, x_0)$, we have

$$
\begin{aligned}
W(t) - \underbrace{W(0)}_{=u(0,x_0)} &= \int_0^t h(x_0 + 2cs)\, ds \\
&= \frac{1}{2c}\int_{x_0}^{x_0+2ct} h(\tau)\, d\tau \qquad (\tau := x_0 + 2cs) \\
&= \frac{1}{2c} H(x_0 + 2ct),
\end{aligned}
$$

where the anti-derivative $H(\beta) := \displaystyle\int_{x_0}^{\beta} h(\tau)\, d\tau$ satisfies $H' = h$ and $H(x_0) = 0$. Indeed, h is an arbitrary function, so is the $f(\beta) := \dfrac{1}{2c} H(\beta)$. Furthermore, since we do not have any information for $g(x_0) := u(0, x_0)$, we should just see g as any arbitrary function. As a result, we finally have

$$u(t, x_0 + ct) = f(x_0 + 2ct) + g(x_0),$$

for some arbitrary functions f and g.

In order to express u in terms of t and x, we set

$$x := x_0 + ct,$$

which implies

$$x_0 = x - ct.$$

Hence, the identity

$$u(t, x_0 + ct) = f(x_0 + 2ct) + g(x_0)$$

becomes

$$\boxed{u(t, x) = f(x + ct) + g(x - ct),}$$

for some arbitrary function f and g.

Moral

The function $f(x + ct)$ is traveling to the left at speed c; meanwhile, the function $g(x - ct)$ is traveling to the right at speed c.

3.3 Solving the Wave Equations by Coordinate Method

Solving Wave Equations in One Spatial Dimension

For any given constant $c > 0$, the solution u to the wave equation in one spatial dimension:

$$\partial_{tt} u - c^2 \partial_{xx} u = 0 \tag{1DWave}$$

can be found via the following

Main Observation

$$0 = \partial_{tt} u - c^2 \partial_{xx} u = \left(\partial_t + c\partial_x\right)\left(\partial_t - c\partial_x\right) u.$$

Based on this main observation, one may solve (1DWave) by

- method of characteristics; or
- coordinate method (will be seen in this section).

Remark

The coordinate system that we will use to solve (1DWave) is also called "**characteristic coordinates**".

Philosophy of Using the Characteristic Coordinates

Main Idea

Want to find two independent variables ξ and η such that

$$\underbrace{(\partial_t + c\partial_x)}_{=A\partial_\xi}\underbrace{(\partial_t - c\partial_x)}_{=B\partial_\eta} u = 0,$$

for some non-zero constants A and B.

Moral

We are able to solve $\partial_\xi \partial_\eta u = 0$ via direct integrations:

$$\partial_\xi \partial_\eta u = 0 \xrightarrow{\text{Integrating with respect to } \xi \text{ and } \eta} u(\xi, \eta) = f(\xi) + g(\eta).$$

Question

How to find the characteristic coordinates ξ and η?

Searching for the Characteristic Coordinates

Guess

Let $M := (m_{ij})_{i,j=1}^2 \in M_{2\times 2}(\mathbb{R})$ be a constant 2×2 matrix that will be determined later. Define

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} := M \begin{pmatrix} x \\ t \end{pmatrix},$$

or equivalently,

$$\begin{cases} \xi := m_{11}x + m_{12}t \\ \eta := m_{21}x + m_{22}t. \end{cases}$$

Now, we are going to choose m_{ij} such that

$$\begin{cases} A\partial_\xi := \partial_t + c\partial_x \\ B\partial_\eta := \partial_t - c\partial_x. \end{cases}$$

Finding m_{ij}'s

By the chain rule,

$$\begin{cases} \xi := m_{11}x + m_{12}t \\ \eta := m_{21}x + m_{22}t. \end{cases} \implies \boxed{\begin{aligned} \partial_x &= (\partial_x \xi)\partial_\xi + (\partial_x \eta)\partial_\eta = m_{11}\partial_\xi + m_{21}\partial_\eta \\ \partial_t &= (\partial_t \xi)\partial_\xi + (\partial_t \eta)\partial_\eta = m_{12}\partial_\xi + m_{22}\partial_\eta. \end{aligned}}$$

Therefore, our desired identity

$$\begin{aligned} A\partial_\xi &= \partial_t + c\partial_x \\ &= (m_{12}\partial_\xi + m_{22}\partial_\eta) + c(m_{11}\partial_\xi + m_{21}\partial_\eta) \\ &= (m_{12} + cm_{11})\partial_\xi + (m_{22} + cm_{21})\partial_\eta \end{aligned}$$

implies

$$\begin{cases} A = m_{12} + cm_{11} \\ 0 = m_{22} + cm_{21}. \end{cases} \tag{3}$$

Similarly, our another desired identity

$$\begin{aligned} B\partial_\eta &= \partial_t - c\partial_x \\ &= (m_{12}\partial_\xi + m_{22}\partial_\eta) - c(m_{11}\partial_\xi + m_{21}\partial_\eta) \\ &= (m_{12} - cm_{11})\partial_\xi + (m_{22} - cm_{21})\partial_\eta \end{aligned}$$

implies

$$\begin{cases} 0 = m_{12} - cm_{11} \\ B = m_{22} - cm_{21}. \end{cases} \tag{4}$$

Solving (3) and (4), we obtain

$$\begin{cases} m_{11} = \dfrac{A}{2c}, & m_{12} = \dfrac{A}{2}, \\ m_{21} = -\dfrac{B}{2c}, & m_{22} = \dfrac{B}{2}. \end{cases}$$

That is,

$$\begin{cases} \xi := m_{11}x + m_{12}t = \dfrac{A}{2c}x + \dfrac{A}{2}t = x + ct \\ \eta := m_{21}x + m_{22}t = -\dfrac{B}{2c}x + \dfrac{B}{2}t = x - ct, \end{cases}$$

if we choose $A := 2c$ and $B := -2c$. In terms of the above ξ and η, the wave equation $\underbrace{(\partial_t + c\partial_x)}_{=A\partial_\xi}\underbrace{(\partial_t - c\partial_x)}_{=B\partial_\eta} u = \partial_{tt}u - c^2\partial_{xx}u = 0$ implies

$$\begin{aligned} \partial_\xi\partial_\eta u &= 0 \\ \partial_\xi u &= h(\xi) \\ u &= f(\xi) + g(\eta), \end{aligned}$$

where $g := g(\eta)$ and $h := h(\xi)$ are arbitrary functions arising from direct integrations with respect to ξ and η respectively. The function $f := f(\xi)$ is an anti-derivative of $g := g(\xi)$, so f is also an arbitrary function. Hence, in terms of t and x,

$$\boxed{u(t,x) = f(x+ct) + g(x-ct).}$$

3.4 Cauchy Problem – D'Alembert's Formula

Cauchy/Initial-Value Problem

Consider the wave equation

$$\partial_{tt} u - c^2 \partial_{xx} u = 0, \quad \text{for } -\infty < x < \infty, \text{ and } t > 0, \qquad \text{(1DWaveEqt)}$$

with the initial conditions

$$u|_{t=0} = \phi(x), \qquad \text{(InitPos)}$$

$$\partial_t u|_{t=0} = \psi(x). \qquad \text{(InitVec)}$$

The general solution to (1DWaveEqt) is

$$u(t,x) = f(x+ct) + g(x-ct). \qquad \text{(GSF)}$$

Substituting the solution formula (GSF) into the initial conditions (InitPos) and (InitVec), we have

$$\begin{cases} f(x) + g(x) = \phi(x), \\ cf'(x) - cg'(x) = \psi(x). \end{cases}$$

To find f and g, we first differentiate $f(x)+g(x)=\phi(x)$ with respect to x, and obtain

$$f'(x)+g'(x)=\phi'(x).$$

Solving

$$\begin{cases} \quad f'(x)+g'(x)=\phi'(x), \\ cf'(x)-cg'(x)=\psi(x), \end{cases}$$

we obtain

$$\begin{cases} f'(x)=\dfrac{1}{2}\phi'(x)+\dfrac{1}{2c}\psi(x), \\ g'(x)=\dfrac{1}{2}\phi'(x)-\dfrac{1}{2c}\psi(x). \end{cases}$$

Direct integrations yield, for any $s\in\mathbb{R}$,

$$\begin{cases} f(s)=\dfrac{1}{2}\phi(s)+\dfrac{1}{2c}\Psi(s)+A, \\ g(s)=\dfrac{1}{2}\phi(s)-\dfrac{1}{2c}\Psi(s)+B, \end{cases}$$

where A and B are constants, and $\Psi'=\psi$.

In order to eliminate A and B, we first add both equations in

$$\begin{cases} f(s) = \dfrac{1}{2}\phi(s) + \dfrac{1}{2c}\Psi(s) + A, \\ g(s) = \dfrac{1}{2}\phi(s) - \dfrac{1}{2c}\Psi(s) + B, \end{cases} \tag{5}$$

and then use the identity $f(x) + g(x) = \phi(x)$ to obtain

$$\phi(s) = f(s) + g(s) = \phi(s) + A + B.$$

Hence,

$$A + B = 0. \tag{6}$$

Recall

The identity $f(x) + g(x) = \phi(x)$ is a direct consequence of the IC (InitPos).

Substituting (5) and (6) into (GSF), we finally obtain

$$
\begin{aligned}
u(t,x) &= f(x+ct) + g(x-ct) \\
&= \left\{ \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\Psi(x+ct) + A \right\} \\
&\quad + \left\{ \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\Psi(x-ct) + B \right\} \\
&= \frac{1}{2}\left\{\phi(x+ct) + \phi(x-ct)\right\} + \frac{1}{2c}\left\{\Psi(x+ct) - \Psi(x-ct)\right\} \\
&= \frac{1}{2}\left\{\phi(x+ct) + \phi(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds,
\end{aligned}
$$

where we used the fundamental theorem of calculus in the last equality.

d'Alembert's formula

$$
u(t,x) = \frac{1}{2}\left\{\phi(x+ct) + \phi(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.
$$

Examples

Example

Question: Solve

$$\begin{cases} \partial_{tt} u = 4\partial_{xx} u, & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u\big|_{t=0} = x, \\ \partial_t u\big|_{t=0} = e^x. \end{cases}$$

Solution: Applying d'Alembert's formula with $c = 2$, $\phi(x) := x$ and $\psi(x) := e^x$, we have

$$\begin{aligned} u(t,x) &= \frac{1}{2}\left\{\phi(x+ct) + \phi(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds \\ &= \frac{1}{2}\left\{(x+2t) + (x-2t)\right\} + \frac{1}{4}\int_{x-2t}^{x+2t} e^s\, ds \\ &= x + \frac{1}{4}\left[e^s\right]_{s=x-2t}^{x+2t} = x + \frac{1}{4}e^{x+2t} - \frac{1}{4}e^{x-2t}. \end{aligned}$$

Example (Creating Solution to the Wave Equation)

It follows from (GSF) (with $f(\xi) := g(\xi) := \sin\beta\xi$) that the wave equation (1DWaveEqt) has the following solution:

$$
\begin{aligned}
u(t,x) &= f(x+ct) + g(x-ct) \\
&= \sin(x\beta + c\beta t) + \sin(x\beta - c\beta t) \\
&= 2\sin\left(\frac{(x\beta + c\beta t) + (x\beta - c\beta t)}{2}\right) \\
&\qquad \cdot \cos\left(\frac{(x\beta + c\beta t) - (x\beta - c\beta t)}{2}\right) \\
&= 2\sin\beta x \cos c\beta t,
\end{aligned}
$$

where we applied the sum-to-production formula $\sin\theta + \sin\varphi = 2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$ in the second last equality.

Indeed, this product form solution is a <u>*building block*</u> *of the method of separation of variables.*

3.5 Principle of Causality and Finite Speed of Propagation

Principle of Causality and Finite Speed of Propagation

Let u satisfy

$$\partial_{tt} u - c^2 \partial_{xx} u = 0,$$

where c is a constant. Then

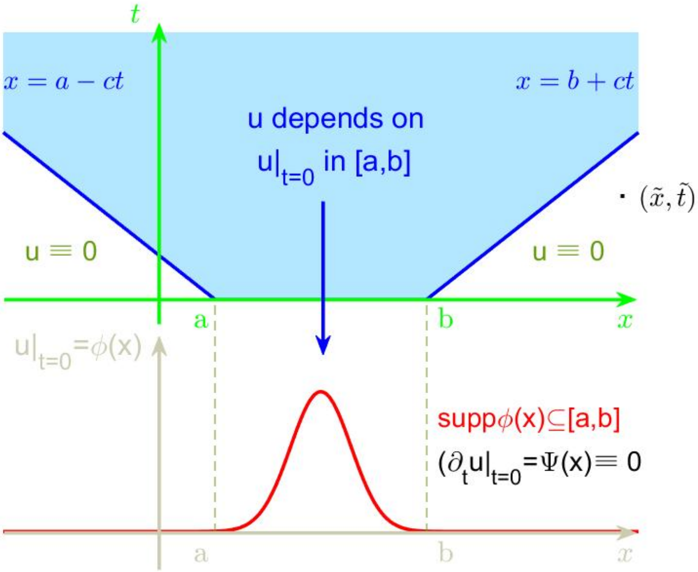
Principle of Causality and Finite Speed of Propagation

The solution u **CANNOT** propagate faster than the given speed c.

In the following, we will discuss

- Finite Speed of Propagation,
- Domain of Dependence (of a Point),
- Domain of Dependence (of an Interval),
- Domain of Influence (of a Point),
- Domain of Influence (of an Interval).

Finite Speed of Propagation



Reasoning – Zoom in at the Point $(\tilde{x}, \tilde{t})$

It follows from d'Alembert's formula that

$$
\begin{aligned}
& u(\tilde{t}, \tilde{x}) \\
= & \frac{1}{2}\{\underbrace{\phi(\tilde{x} + c\tilde{t})}_{=0} + \underbrace{\phi(\tilde{x} - c\tilde{t})}_{=0}\} \\
& + \frac{1}{2c}\int_{\tilde{x}-c\tilde{t}}^{\tilde{x}+c\tilde{t}} \underbrace{\psi(s)}_{\equiv 0}\, ds \\
= & \, 0.
\end{aligned}
$$

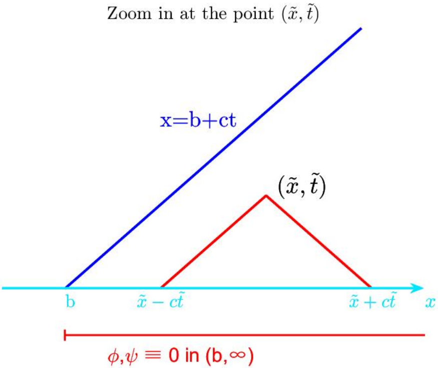


Figure: Zoom in at the Point $(\tilde{x}, \tilde{t})$

Domain of Dependence of a Point

The domain of dependence (also called the "past history" in the Textbook) of $(\tilde{x}, \tilde{t})$ is defined as

$$\Delta := \left\{(x,t) \in \mathbb{R} \times \mathbb{R}^+; \; \tilde{x} - c(\tilde{t} - t) \leq x \leq \tilde{x} + c(\tilde{t} - t)\right\}.$$

Remark

The condition $t \leq \tilde{t}$ is <u>hidden</u> in the *definition* of Δ, since we request $\tilde{x} - c(\tilde{t} - t) \leq \tilde{x} + c(\tilde{t} - t)$.

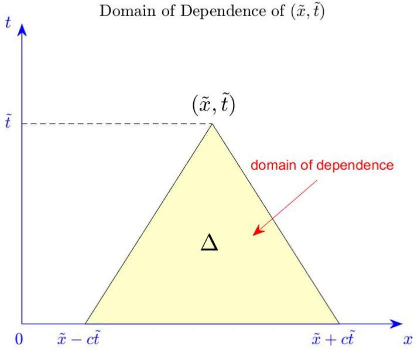


Figure: Domain of Dependence of $(\tilde{x}, \tilde{t})$

Moral

The value $u(\tilde{t}, \tilde{x})$ **ONLY** depends on values of u in the domain of dependence Δ of the point $(\tilde{x}, \tilde{t})$.

Reason

It follows from d'Alembert's formula that

$$
\begin{aligned}
u(\tilde{t}, \tilde{x}) &= \frac{1}{2}\left\{\phi(\tilde{x}+c\tilde{t})+\phi(\tilde{x}-c\tilde{t})\right\}+\frac{1}{2c}\int_{\tilde{x}-c\tilde{t}}^{\tilde{x}+c\tilde{t}}\psi(s)\,ds \\
&= \frac{1}{2}\left\{u|_{t=0}(\tilde{x}+c\tilde{t})+u|_{t=0}(\tilde{x}-c\tilde{t})\right\}+\frac{1}{2c}\int_{\tilde{x}-c\tilde{t}}^{\tilde{x}+c\tilde{t}}\partial_t u|_{t=0}(s)\,ds.
\end{aligned}
$$

Also, for any $t_0 \le \tilde{t}$,

$$
\begin{aligned}
u(\tilde{t}, \tilde{x}) = \frac{1}{2}&\left\{u|_{t=t_0}(\tilde{x}+c(\tilde{t}-t_0))+u|_{t=t_0}(\tilde{x}-c(\tilde{t}-t_0))\right\} \\
&+\frac{1}{2c}\int_{\tilde{x}-c(\tilde{t}-t_0)}^{\tilde{x}+c(\tilde{t}-t_0)}\partial_t u|_{t=t_0}(s)\,ds.
\end{aligned}
$$

Domain of Dependence of an Interval

We can also define the *domain of dependence* for an interval. For example, the domain of dependence of the interval $I := (x_0, x_1)$ at the time $\tilde{t}$ is

$$\Delta := \big\{(x,t) \in \mathbb{R} \times \mathbb{R}^+; \; x_0 - c(\tilde{t} - t) \leq x \leq x_1 + c(\tilde{t} - t)\big\}.$$

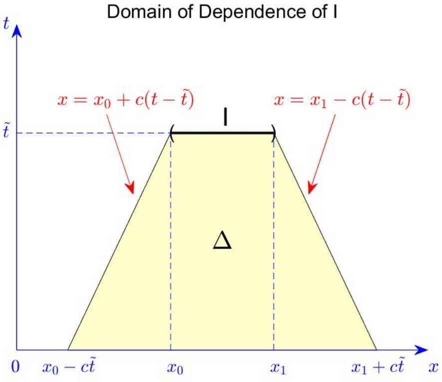
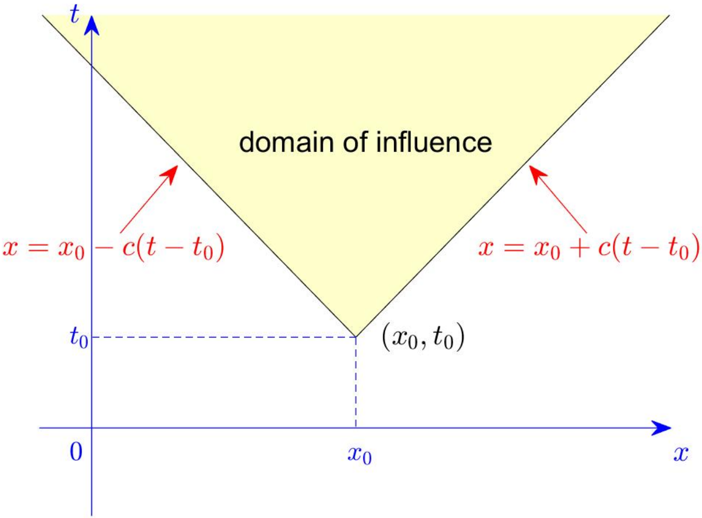
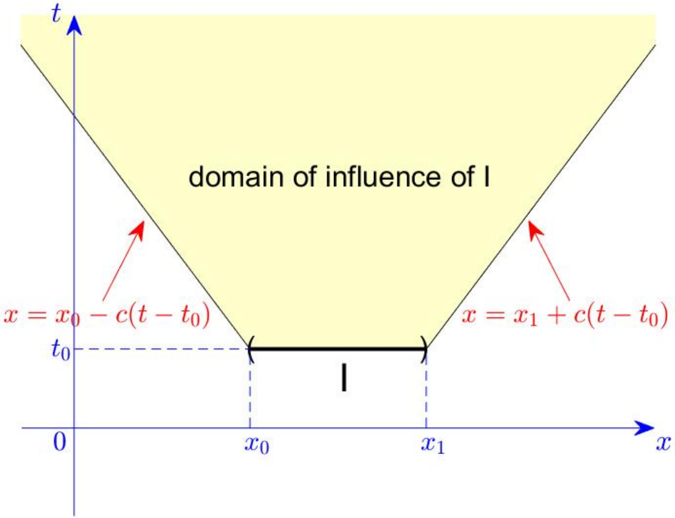


Figure: Domain of Dependence of I

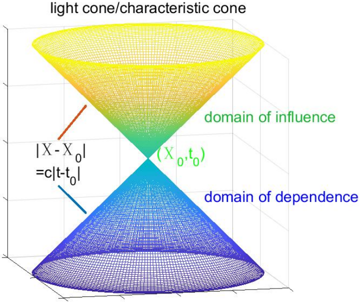
Domain of Influence of a Point



Domain of Influence of a Point



Domains of Dependence and Influence in Higher Spatial Dimensions



One can also define the *domains of dependence and influence* in higher spatial dimensions. The major adjustment is to use the magnitude/length $|X - X_0|$ to replace the numerical difference $x - x_0$ (which we used in one spatial dimension).

Remark

The terminology "light cone"/"characteristic cone" is a terminology borrowed from the theory of relativity.

For more details, see Chapter 9 of the Textbook.