1. (1) *Proof.* By the fundamental theorem of algebra, the irreducible elements in R are polynomials of degree one. So we assume that $f = ax + b$, where $a, b \in \mathbb{C}$ and a is nonzero. Now for any $g \in R$, note that R is an Euclidean domain, we can write $g = f \cdot h + r$, where $h, r \in R$ and $\deg(r) < \deg(f)$. Since $\deg(f) = 1$, r is a constant. Thus we have $R/\langle f \rangle$ is a subfield of $\mathbb{C}$. Consider the set $\{ax + c : c \in C\}$. The projection of this set under f is exactly the set $\mathbb{C}$. Therefore we have $R/\langle f \rangle = \mathbb{C}$. $\square$

   (2) *Proof.* By the fundamental theorem of algebra, f is of degree one or of degree two and has delta $\Delta < 0$.

   Case 1: $f = ax + b$, where $a, b \in \mathbb{R}$ and a is nonzero. Consider the set $\{ax + c : c \in \mathbb{R}\} \subset R$, the projection of this set under f is exactly the set $\mathbb{R}$. So $\mathbb{R}$ is a subfield of a field isomorphic to $R/\langle f \rangle$. Note that R is a Euclidean domain, so every $g \in R$ can be written as $g = f \cdot h + r$, where $h, r \in R$ and $\deg(r) < \deg(f) = 1$. Thus $\deg(r) = 0$, r is a constant. Therefore $R/\langle f \rangle \cong \mathbb{R}$.

   Case 2: $f = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, a is nonzero and $\Delta = b^2 - 4ac < 0$. Since R is a Euclidean domain, any $g \in R$ can be written as $g = f \cdot h + r$, where $h, r \in R$ and $\deg(r) < \deg(f) = 2$. So r has degree one or two. Thus $R/\langle f \rangle$ is isomorphic to a subfield of $\mathbb{C}$. And the set $\{ax^{+}bx + d : d \in \mathbb{R}\}$ and the set $\{ax^2 + ex + c : e \in \mathbb{R}\}$ shows that $R/\langle f \rangle$ is isomorphic to $\mathbb{C}$ itself. Therefore either $R/\langle f \rangle \cong \mathbb{R}$ or $R/\langle f \rangle \cong \mathbb{C}$. $\square$

2. (1) *Proof.* Assume to the contrary that f is reducible. Then f has a linear factor since f is a cubic function. However $f(0) = f(1) = f(2) = 1$, which means f does not have linear factor. Thus f is irreducible. This field has $3^3 = 27$ elements. $\square$

   (2) *Solution.*

	x^2	x^2+1	x^2+2	x^2+x	x^2+x+1	x^2+x+2	x^2+2x	x^2+2x+1	x^2+2x+2
x^2	x^2+2x	$2x^2+2x$	$2x$	x^2+x+2	$2x^2+x+2$	$x+2$	x^2+x+1	$2x^2+x+1$	$x+1$
x^2+1		$2x+1$	x^2+2x+2	$2x^2+x+2$	x	x^2+x+1	$2x^2+1$	2	x^2
x^2+2			$2x^2+2x+1$	$2x+2$	x^2+2x+1	$2x^2+2x$	$2x+1$	x^2+2x	$2x^2+2x+2$
x^2+x				$2x^2+x+1$	$2x+1$	x^2+1	$2x$	x^2	$2x^2+x$
x^2+x+1					x^2+2	$2x^2+x$	x^2+x	$2x^2+2x+1$	2
x^2+x+2						$2x+2$	$2x^2$	$x+2$	x^2+2x+1
x^2+2x							$2x^2+2$	$2x+2$	x^2+x+2
x^2+2x+1								x^2+x	$2x^2+1$
x^2+2x+2									$2x$

3. (1) *Solution.* f is irreducible over $\mathbb{Q}$. By Eisenstein's Criteria, $3|a_0, a_1, \ldots, a_8$, but $3 \nmid a_9$ and $9 \nmid a_0$.

   (2) *Solution.* f is irreducible over $\mathbb{Q}$. Take $p = 2$, note that $\pi_2(f) = x^4 + x + 1$ has no root over $Z_2[x]$, thus it has no proper factorization over $\mathbb{Z}[x]$, and therefore is irreducible over $\mathbb{Q}[x]$.

   (3) *Solution.* f is irreducible over $\mathbb{Q}$. Take $p = 3$, note that $\pi_2(f) = x^4 + x^3 + 2x + 1$ has no root over $Z_3[x]$ ($\pi_2(f)(0) = 1$,$\pi_2(f)(1) = 2$,$\pi_2(f)(2) = 2$), thus it has no proper factorization over $\mathbb{Z}[x]$, and therefore is irreducible over $\mathbb{Q}[x]$.

4. (1) *Proof.* For any $a \in \mathbb{Z}$, $f = x - a \in \mathbb{Z}[x]$ is a monic polynomial such that a is its root, so $\mathbb{Z} \subset \mathbb{Q} \cap \bar{\mathbb{Z}}$. If a rational number $\frac{b}{a}$ is in $\bar{\mathbb{Z}}$ where $\gcd(a, b) = 1$ and $a \neq 1$, assume that $f = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a monic function which has $\frac{b}{a}$ as its root. Then

$$f(\frac{b}{a}) = \frac{b^n}{a^n} + a_{n-1}\frac{b^{n-1}}{a^{n-1}} + \cdots + a_1\frac{b}{a} + a_0 = 0,$$

$$b^n + a_{n-1}b^{n-1}a + \cdots + a_1 b a^{n-1} + a_0 a^n = 0.$$

   So $a|b^n = -a_{n-1}b^{n-1}a + \cdots + a_1 b a^{n-1} + a_0 a^n$, contradicts the assumption that $\gcd(a, b) = 1$.

   Recall that the intersection of two field is again a field, we deduce that $\bar{\mathbb{Z}}$ is not a field since $\mathbb{Z}$ is not a field. □

   (2) (a) *Solution.*

$$R(f, g) = \det \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} = 8.$$

   (b) *Solution.*

$$R(f, g) = \det \begin{bmatrix} b & a & 1 \\ a & 2 & 0 \\ 0 & a & 2 \end{bmatrix} = 4b - a^2.$$

(c) *Solution.*

$$R(f,g) = \det \begin{bmatrix} b & a & 0 & 1 & 0 \\ 0 & b & a & 0 & 1 \\ a & 0 & 3 & 0 & 0 \\ 0 & a & 0 & 3 & 0 \\ 0 & 0 & a & 0 & 3 \end{bmatrix} = 27b^2 + 4a^3.$$

(3) *Proof.* Since $f(x)$ has a root α and $g(z-x)$ has a root $z-\beta$, the resultant has $\alpha-(z-\beta) = \alpha+\beta-z$ as its factor. Therefore $P(z)$ has a root $\alpha+\beta$.

Since $f(x)$ has a root α and $x^n g(\frac{z}{x})$ has a root $\frac{z}{\beta}$, the resultant has $\alpha - \frac{z}{\beta} = \beta(\alpha\beta - z)$ as its factor. Therefore $Q(z)$ has a root $\alpha\beta$. Here we multiply x^n in front of $g(\frac{z}{x})$ in order to make $Q(z)$ monic. $\square$

(a) *Solution.* $f(x) = x^2+2x+2$ has $\alpha = 1+i$ as its root and $g(x) = x^2-2$ has $\beta = \sqrt{2}$ as its root. Therefore

$$P(z) = z^4 - 4z^3 + 4z^2 + 8,$$

$$Q(z) = z^4 + 16.$$

(b) *Solution.* $f(x) = x^2 - 2$ has $\alpha = \sqrt{2}$ as its root and $g(x) = x^2 + 3$ has $\beta = i\sqrt{2}$ as its root. Therefore

$$P(z) = z^4 + 2z^2 + 25,$$

$$Q(z) = z^2 + 6.$$