THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 6 Solution

Problem 1.

(i) Direct computation yields

$$\begin{aligned}\partial_t e &= \partial_t u\,\partial_{tt} u + \partial_x u\,\partial_{tx} u\\ &= \partial_t u\,\partial_{xx} u + \partial_x u\,\partial_{tx} u\\ &= \partial_x(\partial_x u\,\partial_t u) = \partial_x p.\end{aligned}$$

(ii) Direct computation yields

$$\frac{1}{2}(\partial_t u \pm \partial_x u)^2 = \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\partial_x u|^2 \pm \partial_t u\,\partial_x u = e \pm p.$$

(iii) For $t \geq 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}E(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{a+t}^{b-t} e(t,x)\,\mathrm{d}\,x\\ &= \int_{a+t}^{b-t} \partial_t e(t,x)\,\mathrm{d}\,x - e(t,b-t) - e(t,a+t) \quad \text{(by hint)}\\ &= \int_{a+t}^{b-t} \partial_x p(t,x)\,\mathrm{d}\,x - e(t,b-t) - e(t,a+t) \quad \text{(by (i))}\\ &= p(t,b-t) - p(t,a+t) - e(t,b-t) - e(t,a+t).\end{aligned}$$

(iv) Apply parts (ii) and (iii)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}E(t) &= p(t,b-t) - p(t,a+t) - e(t,b-t) - e(t,a+t)\\ &= -[e(t,b-t) - p(t,b-t)] - [e(t,a+t) + p(t,a+t)]\\ &\leq 0. \quad \text{Because from (ii) we have } e \pm p \geq 0.\end{aligned}$$

(v) If $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), then $\partial_x u(0,x) \equiv 0$ on (a,b). Thus,

$$E(0) = \int_a^b \frac{|\partial_t u(0,x)|^2}{2} + \frac{|\partial_x u(0,x)|^2}{2} \,\mathrm{d}\, x = 0.$$

On the other hand, for $0 \le t \le (b-a)/2$,

$$0 \le E(t) \le E(0) = 0 \Rightarrow E(t) = 0.$$

That is,

$$\int_{a+t}^{b-t} \frac{|\partial_t u(t,x)|^2}{2} + \frac{|\partial_x u(t,x)|^2}{2} \,\mathrm{d}\, x = 0.$$

It follows that $\partial_t u(t,x) = \partial_x u(t,x) \equiv 0$. It implies that u is a constant for $a+t \le x \le b-t$. Because $u|_{t=0} \equiv 0$ on (a,b), it follows that $u \equiv 0$ in

$$\Delta := \{(t,x) \in [0,\infty) \times (-\infty,\infty) : \ a+t \le x \le b-t\}.$$

Food for Thought. Consider

$$E(t) := \int_{a+ct}^{b-ct} e(t,x) \, dx,$$

where

$$e(t,x) := \frac{1}{2}|\partial_t u|^2 + \frac{c^2}{2}|\partial_x u|^2,$$

then

(i): the equation becomes $\partial_t e = c^2 \partial_x p$.

(ii): the equation becomes $e \pm cp = \frac{1}{2}(\partial_t u \pm c\partial_x u)^2$.

(iii): the equation becomes

$$\frac{dE}{dt}(t) = c^2 p(t, b-ct) - c^2 p(t, a+ct) - ce(t, b-ct) - ce(t, a+ct).$$

(v): the region becomes $\Delta := \{(x,t) \in (-\infty,\infty) \times [0,\infty);\ a+ct \le x \le b-ct\}$.

Food for Thought. Please read the following problem and answer.

Let u satisfy the following PDE

$$\partial_{tt} u - c^2 \partial_{xx} u = -\alpha u \qquad \text{for } -\infty < x < \infty \text{ and } \alpha, c, t > 0. \tag{1}$$

Given finite interval (a,b), we define the local energy by

$$E(t) := \int_{a+Mt}^{b-Mt} e(t,x)\,dx, \text{ where } e(t,x) := \tfrac{1}{2}|\partial_t u|^2 + \tfrac{c^2}{2}|\partial_x u|^2 + \tfrac{\alpha}{2}|u^2|.$$

(i) Let $p(t,x) := \partial_t u\,\partial_x u$. Prove that $\partial_t e = c^2 \partial_x p$.

(ii) Show that $e \pm cp = \frac{1}{2}(\partial_t u \pm c\partial_x u)^2 + \frac{\alpha}{2}u^2$.

(iii) Using part (i), verify via a direct differentiation that

$$\frac{dE}{dt}(t) = c^2[p(t,b-Mt) - p(t,a+Mt)] - M[e(t,b-Mt) + e(t,a+Mt)].$$

(Hint: Use $\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\,dx = \int_{a(t)}^{b(t)} \partial_t f(t,x)\,dx + f(t,b(t))b'(t) - f(t,a(t))a'(t)$.)

(iv) Suppose $M \ge c$. Using (ii) and (iii), show that $\frac{dE}{dt} \le 0$. Hence if $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), show that $u \equiv 0$ in $\Delta := \{(x,t) \in (-\infty,\infty)\times[0,\infty) : a+Mt \le x \le b-Mt\}$.

Let u satisfy the following PDE

$$\partial_{tt}u - c^2\partial_{xx}u + \alpha u = 0 \qquad \text{for } -\infty < x < \infty \text{ and } c,\alpha,t > 0. \tag{2}$$

Given finite interval (a,b), we define the local energy by

$$E(t) := \int_{a+Mt}^{b-Mt} e(t,x)\,dx, \text{ where } e(t,x) := \tfrac{1}{2}|\partial_t u|^2 + \tfrac{c^2}{2}|\partial_x u|^2 + \tfrac{\alpha}{2}|u^2|.$$

(i)

$$\begin{aligned}
\partial_t e &= \partial_t u \cdot \partial_{tt}u + c^2\partial_x u \cdot \partial_{tx}u + \alpha u \partial_t u \\
&= \partial_t u(\partial_{tt}u + \alpha u) + c^2\partial_x u \cdot \partial_{tx}u \\
&= c^2[\partial_t u \cdot \partial_{xx}u + \partial_x u \cdot \partial_{tx}u] \ \text{ (by (1))} \\
&= c^2\partial_x(\partial_x u \partial_t u) = c^2\partial_x p.
\end{aligned}$$

(ii)

$$\begin{aligned}\frac{1}{2}(\partial_t u \pm c\partial_x u)^2 + \frac{\alpha}{2}u^2 &= \frac{1}{2}|\partial_t u|^2 + \frac{c^2}{2}|\partial_x u|^2 + \frac{\alpha}{2}|u|^2 \pm c\partial_t u \partial_x u \\ &= e \pm cp.\end{aligned}$$

(iii) For $t \geq 0$,

$$\begin{aligned}\frac{dE}{dt}(t) &= \frac{d}{dt}\int_{a+Mt}^{b-Mt} e(t,x)\,dx \\ &= \int_{a+Mt}^{b-Mt} \partial_t e(t,x)\,dx - Me(t,b-Mt) - Me(t,a+Mt)) \quad \text{(by hint)} \\ &= c^2\int_{a+Mt}^{b-Mt} \partial_x p(t,x)\,dx - M[e(t,b-Mt) + e(t,a+Mt))] \quad \text{(by (i))} \\ &= c^2[p(t,b-Mt) - p(t,a+Mt)] - M[e(t,b-Mt) + e(t,a+Mt)].\end{aligned}$$

(iv) For $t \geq 0$,

$$\begin{aligned}&\frac{dE}{dt}(t) = c^2[p(t,b-Mt) - p(t,a+Mt)] - M[e(t,b-Mt) + e(t,a+Mt)] \quad \text{(by (iii))} \\ \leq\ & c[cp(t,b-Mt) - cp(t,a+Mt) - e(t,b-Mt) - e(t,a+Mt)] \\ \leq\ & -c[e(t,b-Mt) - cp(t,b-Mt) + e(t,a+Mt) + cp(t,a+Mt)] \\ \leq\ & 0 \text{ (by (ii))}.\end{aligned}$$

If $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), then $\partial_x u(0,x) \equiv 0$ on (a,b). Thus,

$$E(0) = \int_a^b \frac{|\partial_t u(0,x)|^2}{2} + \frac{c^2|\partial_x u(0,x)|^2}{2} + \frac{\alpha|u(0,x)|^2}{2} dx = 0.$$

On the other hand, for $0 \leq t \leq (b-a)/2M$,

$$0 \leq E(t) \leq E(0) = 0 \Longrightarrow E(t) = 0.$$

So

$$\int_{a+Mt}^{b-Mt} |u(t,x)|^2 dx \leq E(t) = 0$$

and hence $u(t,x) = 0$ for $a + Mt \leq x \leq b - Mt$, that is, $u \equiv 0$ in $\Delta := \{(x,t) \in (-\infty,\infty) \times [0,\infty) :\ a + Mt \leq x \leq b - Mt\}$.

Problem 2.

(a) Let u_1 and u_2 be two solutions to

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = g. \end{cases}$$

Define $\tilde{u} \coloneqq u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \Delta\tilde{u} = \tilde{u} & \text{in } \Omega, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \end{cases}$$

Multiplying $\Delta\tilde{u} = \tilde{u}$ by $\tilde{u}$, and then integrating over Ω, we have

$$\int_\Omega \tilde{u}\Delta\tilde{u}\,dx = \int_\Omega \tilde{u}^2\,dx.$$

Recall the following identity for a smooth vector field F and a smooth scalar-valued function g,

$$\nabla\cdot(Fg) = g\nabla\cdot F + F\cdot\nabla g.$$

Thus, by letting $F = \Delta\tilde{u}$, $g = \tilde{u}$, the left hand side may be rewritten as

$$\begin{aligned} \int_\Omega \tilde{u}\Delta\tilde{u}\,dx &= \int_\Omega \tilde{u}\nabla\cdot(\nabla\tilde{u})\,dx = \int_\Omega \nabla\cdot(\tilde{u}\nabla\tilde{u}) - \nabla\tilde{u}\cdot(\nabla\tilde{u})\,dx \\ &= \int_{\partial\Omega}(\tilde{u}\nabla\tilde{u})\cdot n\,d\sigma - \int_\Omega |\nabla\tilde{u}|^2\,dx \quad \text{(by divergence theorem)} \\ &= \int_{\partial\Omega}\tilde{u}(\nabla\tilde{u}\cdot n)\,d\sigma - \int_\Omega |\nabla\tilde{u}|^2\,dx \\ &= -\int_\Omega |\nabla\tilde{u}|^2\,dx \quad \text{(by the boundary condition } \tilde{u}|_{\partial\Omega} = 0\text{)}. \end{aligned}$$

Therefore, we obtain

$$-\iiint_\Omega |\nabla\tilde{u}|^2\,\,dx = \iiint_\Omega \tilde{u}^2\,dx,$$

or equivalently,

$$\iiint_\Omega |\nabla\tilde{u}|^2 + \tilde{u}^2\,dx = 0.$$

Since $|\nabla \tilde{u}|^2 + \tilde{u}^2 \geq 0$, it follows from the first vanishing theorem that

$$|\nabla \tilde{u}|^2 + \tilde{u}^2 \equiv 0,$$

which implies

$$\tilde{u} \equiv 0.$$

That is, $u_1 \equiv u_2$.

(b) Let u_1 and u_2 be two solutions to

$$\begin{cases} \partial_t u - 5\partial_{xx} u = 11\partial_x u - 8u + f \quad \text{for } 0 < x < L \text{ and } t > 0, \\ u(0,x) = \phi(x), \\ u(t,0) = g(t), \\ u(t,L) = h(t). \end{cases}$$

Define $\tilde{u} := u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} - 5\partial_{xx} \tilde{u} = 11\partial_x \tilde{u} - 8\tilde{u} \quad \text{for } 0 < x < L \text{ and } t > 0, \\ \tilde{u}(0,x) = 0, \\ \tilde{u}(t,0) = 0, \\ \tilde{u}(t,L) = 0. \end{cases}$$

Multiplying $\partial_t \tilde{u} = 5\partial_{xx}\tilde{u} + 11\partial_x \tilde{u} - 8\tilde{u}$ by $\tilde{u}$, and then integrating with respect to x, we have

$$\int_0^L \tilde{u}\partial_t \tilde{u}\, dx = \int_0^L \tilde{u}\left(5\partial_{xx}\tilde{u} + 11\partial_x \tilde{u} - 8\tilde{u}\right)\, dx.$$

Integrating by parts on the right hand side and using the boundary conditions for $\tilde{u}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L |\tilde{u}|^2\, dx &= 5\left[\tilde{u}\partial_x \tilde{u}\right]_{x=0}^L - \int_0^L 5|\partial_x \tilde{u}|^2\, dx + 11\int_0^L \partial_x\left(\frac{\tilde{u}^2}{2}\right) dx - 8\int_0^L |\tilde{u}|^2\, dx \\ &= 11\left[\frac{\tilde{u}^2}{2}\right]_{x=0}^L - \int_0^L 5|\partial_x \tilde{u}|^2 + 8|\tilde{u}|^2\, dx \\ &\leq 0. \end{aligned}$$

Define

$$E(t) := \int_0^L |\tilde{u}(t,x)|^2 \, dx.$$

The above inequality can be written as

$$\frac{d}{dt}E(t) \le 0.$$

A direct integration yields, for any $t \ge 0$,

$$E(t) \le E(0).$$

Using the initial condition $\tilde{u}(0,x) = 0$, we have

$$E(0) = 0.$$

It follows from the definition of E that $E(t) \ge 0$, so

$$0 \le E(t) \le E(0) = 0,$$

which implies $E(t) \equiv 0$, and hence,

$$|\tilde{u}|^2 \equiv 0.$$

This implies

$$\tilde{u} \equiv 0.$$

That is, $u_1 \equiv u_2$.

(c) Let u_1 and u_2 be two solutions to

$$\begin{cases} \partial_{tt}u - 24\partial_{xx}u = -\sinh\left(11t + x^8\right)\partial_t u + f & \text{for } -L < x < L \text{ and } t > 0, \\ u(0,x) = \phi(x), \\ \partial_t u(0,x) = \psi(x), \\ u(t,-L) = g(t), \\ \partial_x u(t,L) = h(t). \end{cases}$$

Define $\tilde{u} := u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \partial_{tt}\tilde{u} - 24\partial_{xx}\tilde{u} = -\sinh\left(11t + x^8\right)\partial_t\tilde{u} \quad \text{for } -L < x < L \text{ and } t > 0, \\ \tilde{u}(0,x) = 0, \\ \partial_t\tilde{u}(0,x) = 0, \\ \tilde{u}(t,-L) = 0, \\ \partial_x\tilde{u}(t,L) = 0. \end{cases}$$

Notice that a differentiation with respect to x gives $\partial_x\tilde{u}(t,-L) = 0$ and $\partial_x\tilde{u}(0,x) = 0$. Multiplying $\partial_{tt}\tilde{u} - 24\partial_{xx}\tilde{u} = -\sinh(11t + x^8)\partial_t\tilde{u}$ by $\partial_t\tilde{u}$, and then integrating with respect to x, we have

$$\int_{-L}^{L} \partial_t\tilde{u}\left(\partial_{tt}\tilde{u} - 24\partial_{xx}\tilde{u}\right)\, dx = \int_{-L}^{L} -\sinh\left(11t + x^8\right)|\partial_t\tilde{u}|^2\, dx.$$

Integrating by parts on the left hand side and using the boundary conditions for $\tilde{u}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-L}^{L} |\partial_t\tilde{u}|^2 + 24|\partial_x\tilde{u}|^2\, dx = -\int_{-L}^{L} \sinh\left(11t + x^8\right)|\partial_t\tilde{u}|^2\, dx \leq 0.$$

Define

$$E(t) := \int_{-L}^{L} |\partial_t\tilde{u}(t,x)|^2 + 24|\partial_x\tilde{u}(t,x)|^2\, dx.$$

The above inequality can be written as

$$\frac{d}{dt}E(t) \leq 0.$$

A direct integration yields, for any $t \geq 0$,

$$E(t) \leq E(0).$$

Using the initial conditions $\partial_t\tilde{u}(0,x) = 0$ and the deduced condition $\partial_x\tilde{u}(0,x) = 0$, we have

$$E(0) = 0.$$

It follows from the definition of E that $E(t) \geq 0$, so

$$0 \leq E(t) \leq E(0) = 0,$$

which implies $E(t) \equiv 0$, and hence,

$$|\partial_t \tilde{u}|^2 + 24\,|\partial_x \tilde{u}|^2 \equiv 0.$$

This implies

$$\tilde{u} \equiv C,$$

where C is a constant. To determine the constant C, we use the initial condition $\tilde{u}(0,x) = 0$, and obtain $C = 0$. Thus,

$$\tilde{u} \equiv 0.$$

That is, $u_1 \equiv u_2$.

Problem 3.

(i) Note that

$$f'(\alpha) = \frac{1}{1+\alpha^2} - \frac{1}{\sqrt{1+\alpha^2}} = \frac{1-\sqrt{1+\alpha^2}}{1+\alpha^2} \leq 0,$$

so f is monotonic decreasing. Then, for any $\alpha, \beta \in \mathbb{R}$,

$$(f(\alpha) - f(\beta))(\alpha - \beta) \leq 0.$$

(ii) Let $\tilde{u} = u_1 - u_2$ and $\tilde{\phi} = \phi_1 - \phi_2$, then $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} - k\partial_{xx}\tilde{u} = f(u_1) - f(u_2) & \text{for } 0 < x < L \text{ and } t > 0 \\ \tilde{u}|_{t=0} = \tilde{\phi} & \text{for } 0 < x < L \\ \left(-\partial_x \tilde{u} + \dfrac{8}{11}\tilde{u}\right)\Big|_{x=0} = 0 & \text{for } t > 0 \\ \tilde{u}|_{x=L} = 0 & \text{for } t > 0. \end{cases}$$

Multiplying $\partial_t \tilde{u} - k\partial_{xx}\tilde{u} = f(u_1) - f(u_2)$ by $\tilde{u}$, and then integrating with respect to x over $[0, L]$, we have

$$\int_0^L \tilde{u}\partial_t \tilde{u}\, dx - k\int_0^L \tilde{u}\partial_{xx}\tilde{u}\, dx = \int_0^L (f(u_1) - f(u_2))(u_1 - u_2)\, dx \leq 0$$

That means

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L |\tilde{u}|^2\,dx &\le k\left[|\tilde{u}\partial_x\tilde{u}|\right]_{x=0}^L - k\int_0^L |\partial_x\tilde{u}|^2\,dx \\ &= -\frac{8}{11}k(\tilde{u}(t,0))^2 - k\int_0^L |\partial_x\tilde{u}|^2\,dx \le 0\end{aligned}$$

Hence,

$$0 \le \int_0^L |\tilde{u}(t,x)|^2\,dx \le \int_0^L |\tilde{u}(0,x)|^2\,dx = \int_0^L |\tilde{\phi}|^2\,dx$$

That is

$$\|u_1 - u_2\|_{L^2([0,L])} \le \|\phi_1 - \phi_2\|_{L^2([0,L])}.$$

Food for Thought. As long as f is monotonic decreasing, we can show the stability by using the same procedure as above.

Problem 4.

1. Using a direct differentiation and the PDE, we have

$$\begin{aligned}E'(t) &= \int_0^2 \partial_t u\,dx = \int_0^2 \partial_{xx}u - u + 6x + 3x^2 - x^3\,dx \\ &= [\partial_x u]_{x=0}^2 - E(t) + \int_0^2 6x + 3x^2 - x^3\,dx \\ &= -E(t) + 16.\end{aligned}$$

On the other hand, using the IC, we can compute

$$E(0) := \int_0^2 u(0,x)\,dx = \int_0^2 1 + \cos\pi x\,dx = 2.$$

Solving

$$\begin{cases} E'(t) = -E(t) + 16 \\ E(0) = 2, \end{cases}$$

we finally obtain

$$E(t) = 16 - 14e^{-t}.$$

2. Let u_1 and u_2 be two solutions to

$$\begin{cases} \partial_t u - \partial_{xx} u = -u + 6x + 3x^2 - x^3, & \text{for } 0 < x < 2 \text{ and } t > 0, \\ \partial_x u|_{x=0} = \partial_x u|_{x=2} = 0, & \text{for } t \geq 0, \\ u|_{t=0} = 1 + \cos \pi x, & \text{for } 0 < x < 2. \end{cases}$$

Define $\tilde{u} \coloneqq u_1 - u_2$. Then $\tilde{u}$ satisfies

$$\begin{cases} \partial_t \tilde{u} - \partial_{xx} \tilde{u} = -\tilde{u}, & \text{for } 0 < x < 2 \text{ and } t > 0, \\ \partial_x \tilde{u}|_{x=0} = \partial_x \tilde{u}|_{x=2} = 0, & \text{for } t \geq 0, \\ \tilde{u}|_{t=0} \equiv 0, & \text{for } 0 < x < 2. \end{cases} \tag{3}$$

Multiplying $(3)_1$ by $\tilde{u}$, and then integrating with respect to x over the interval $[0, 2]$, we have

$$\begin{aligned} \int_0^2 \tilde{u}\partial_t \tilde{u}\, dx - \int_0^2 \tilde{u}\partial_{xx}\tilde{u}\, dx &= -\int_0^2 |\tilde{u}|^2\, dx \\ \frac{1}{2}\frac{d}{dt}\int_0^2 |\tilde{u}|^2\, dx + \int_0^2 |\partial_x \tilde{u}|^2\, dx &= -\int_0^2 |\tilde{u}|^2\, dx \\ \frac{1}{2}\frac{d}{dt}\int_0^2 |\tilde{u}|^2\, dx &\leq -\int_0^2 |\tilde{u}|^2\, dx. \end{aligned}$$

By the Grönwall' lemma and the initial data for $\tilde{u}$, for any $t \geq 0$,

$$0 \leq \int_0^2 |\tilde{u}(t,x)|^2\, dx \leq \left(\int_0^2 |\tilde{u}(0,x)|^2\, dx \right) e^{-2t} \equiv 0,$$

which implies

$$\int_0^2 |\tilde{u}(t,x)|^2 \equiv 0$$

for all $t \geq 0$. By the first vanishing theorem,

$$\tilde{u} \equiv 0,$$

and hence,

$$u_1 \equiv u_2.$$

This shows the uniqueness.

3. Substituting an ansatz $u_E(x) := a_0 + a_1 x + a_2 x^2 + a_3 x^3$ into

$$-u_E'' = -u_E + 6x + 3x^2 - x^3, \tag{4}$$

we obtain

$$-2a_2 - 6a_3 x = -a_0 + (6 - a_1)x + (3 - a_2)x^2 - (1 + a_3)x^3.$$

Comparing coefficients, we have

$$\begin{cases} -2a_2 = -a_0, \\ -6a_3 = 6 - a_1, \\ \quad 0 = 3 - a_2, \\ \quad 0 = -1 - a_3. \end{cases}$$

Solving the above linear system for a_i's, we obtain the unique solution as follows:

$$a_0 = 6, \quad \text{and} \quad a_1 = 0, \quad \text{and} \quad a_2 = 3, \quad \text{and} \quad a_3 = -1.$$

This implies that

$$u_E(x) = 6 + 3x^2 - x^3$$

solves (4). One can also check that $u_E'(x) = 6x - 3x^2$ also satisfies the boundary conditions

$$u_E'(0) = u_E'(2) = 0.$$

Due to the uniqueness to the boundary-value problem

$$\begin{cases} \quad -u_E'' = -u_E + 6x + 3x^2 - x^3, \quad \text{for } 0 < x < 2, \\ u_E'(0) = u_E'(2) = 0, \end{cases}$$

we know that

$$u_E(x) = 6 + 3x^2 - x^3$$

is the unique equilibrium solution.

4. Define $v := u - u_E$. Then v satisfies

$$\begin{cases} \partial_t v - \partial_{xx} v = -v, & \text{for } 0 < x < 2 \text{ and } t > 0, \\ \partial_x v|_{x=0} = \partial_x v|_{x=2} = 0, & \text{for } t \geq 0, \\ v|_{t=0} = 1 + \cos \pi x - u_E, & \text{for } 0 < x < 2. \end{cases} \tag{5}$$

Using a similar computation as in part (b), we will obtain

$$\int_0^2 |u(t,x) - u_E(x)|^2 \, dx =: \int_0^2 |v(t,x)|^2 \, dx \leq \left(\int_0^2 |v(0,x)|^2 \, dx \right) e^{-2t} \to 0^+,$$

as $t \to \infty$.

Problem 5.

(a) Direct computation yields that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,t} M(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t} \int_0^3 u(t,x) \,\mathrm{d}\,x \\ &= \int_0^3 \partial_t u(t,x) \,\mathrm{d}\,x \\ &= \int_0^3 \frac{4}{\pi} \partial_{xx} u(t,x) + 2(t+1) u \,\mathrm{d}\,x \\ &= \frac{4}{\pi} \partial_x u(t,x) \Big|_{x=0}^3 + 2(t+1) \int_0^3 u(t,x) \,\mathrm{d}\,x \\ &= \frac{4}{\pi} (\partial_x u(t,3) - \partial_x u(t,0)) + 2(t+1) M(t) \\ &= \frac{4}{\sqrt{\pi}} + 2(t+1) M(t). \end{aligned}$$

The initial condition is given by

$$M(0) = \int_0^3 u(0,x) \,\mathrm{d}\,x = \frac{2}{81} \int_0^3 x^5 \,\mathrm{d}\,x = 0.$$

Thus, $M(t)$ satisfies the equation

$$\begin{cases} M'(t) - 2(t+1) M(t) = \frac{4}{\sqrt{\pi}}, \\ M(0) = 3. \end{cases}$$

The integrating factor is

$$\mu(t) = e^{\int -2(t+1)\,\mathrm{d}\,t} = e^{-(t+1)^2}.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left[e^{-(t+1)^2}M(t)\right] = \frac{4}{\sqrt{\pi}}e^{-(t+1)^2}.$$

Direct integration gives that

$$M(t) = e^{(t+1)^2}\frac{4}{\sqrt{\pi}}\int_0^t e^{-(s+1)^2}\,\mathrm{d}\,s + e^{(t+1)^2}C_1.$$

Plug in the initial data $M(0) = 3$ we have $C_1 = \frac{3}{e}$. Thus,

$$M(t) = 2e^{(t+1)^2}(\mathrm{erf}(t+1) - \mathrm{erf}(1)) + 3e^{t^2+2t}.$$

(b) (i) Direct computation yields that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}E(t) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\int_0^L u(t,x)\,\mathrm{d}\,x \\
&= \int_0^L \partial_t u(t,x)\,\mathrm{d}\,x \\
&= 5\int_0^L \partial_{xx}u(t,x)\,\mathrm{d}\,x + (e^{-t^2}+1)\int_0^L x\sin\frac{\pi x}{L}\,\mathrm{d}\,x \\
&= 5\partial_x u(t,x)\Big|_{x=0}^L + (e^{-t^2}+1)(-\frac{L}{\pi})\left[x\cos\frac{\pi x}{L}\Big|_{x=0}^L - \int_0^L \cos\frac{\pi x}{L}\,\mathrm{d}\,x\right] \\
&= 5(3b - 2 - te^{-t}) + (e^{-t^2}+1)(\frac{L^2}{\pi})
\end{aligned}$$

(ii) The initial data is given by

$$E(0) = \int_0^L u(0,x)\,\mathrm{d}\,x = \int_0^L \sin^2\frac{2023\pi x}{L}\,\mathrm{d}\,x = \frac{L}{2}.$$

Direct integration gives that

$$\begin{aligned}
E(t) &= E(0) + \int_0^t E'(s)\,\mathrm{d}\,s \\
&= \frac{L}{2} + (15b - 10 + \frac{L^2}{\pi})t - 5\int_0^t se^{-s}\,\mathrm{d}\,s + \frac{L^2}{\pi}\int_0^t e^{-s^2}\,\mathrm{d}\,s \\
&= \frac{L}{2} + (15b - 10 + \frac{L^2}{\pi})t + 5(te^{-t} + e^{-t} - 1) + \frac{L^2}{2\sqrt{\pi}}\mathrm{erf}(t)
\end{aligned}$$

(iii) The limit $\lim_{t\to\infty} E(t)$ exists if

$$15b - 10 + \frac{L^2}{\pi} = 0.$$

That is, $b = \frac{10\pi - L^2}{\pi}$, and the limit is $\frac{L}{2} - 5 + \frac{L^2}{2\sqrt{\pi}}$.

Food for Thought. Suppose $\lim_{t\to\infty} u(t,x) = u_E(x)0$, then

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty} \int_0^L u(t,x)dx = \int_0^L u_E(x)dx.$$

So $\lim_{t\to\infty} E(t)$ exists is necessary for the existence of a steady state solution.

Problem 6. By D' Alembert's formula, we have

$$u(t,x) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

(i) Correct. By assumption, $\phi(x+ct)$, $\phi(x-ct)$ and $\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds$ are non-negative, hence u is also non-negative.

(ii) Correct. For $-\infty < x < \infty$ and $t \geq 0$,

$$|u(x,t)| \leq \frac{|\phi(x+t)| + |\phi(x-t)|}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} |\psi(s)|ds \leq \max_{x\in(-\infty,\infty)} |\phi(x)| + \frac{1}{2c}\int_{-\infty}^{\infty} |\psi(s)|ds.$$

So

$$\max_{\substack{-\infty<x<\infty \\ t\geq 0}} |u(t,x)| \leq \max_{x\in(-\infty,\infty)} |\phi(x)| + \frac{1}{2c}\int_{-\infty}^{\infty} |\psi(s)|ds.$$

(iii) Incorrect. Take $\phi(x) = x^2$, $\psi(x) = 0$ and $c = 1$, then $u(x,t) = x^2 + t^2$. Thus

$$E(t) = 2\int_0^1 t^2 + x^2 dx = 2t^2 + \frac{2}{3}.$$

Remark. Multiplying the PDE $\partial_{tt}u = c^2\partial_{xx}u$ by $\partial_t u$, and then integrating with respect to x over $(0,1)$, we have

$$\int_0^1 \partial_{tt}u \cdot \partial_t u \; dx = c^2 \int_0^1 \partial_t u \partial_{xx} u \; dx$$
$$\Longrightarrow \frac{1}{2}\frac{d}{dt}\int_0^1 |\partial_t u|^2 \; dx = c^2[\partial_t u \partial_x u]_{x=0}^1 - c^2\int_0^1 \partial_{tx}u \cdot \partial_x u dx$$
$$\Longrightarrow E'(t) = c^2[\partial_t u \partial_x u]_{x=0}^1.$$

If $u(x,t) = x^2 + t^2$ with $c = 1$, then $E'(t) = 2t[2x]_{x=0}^1 = 4t \neq 0$.