THE UNIVERSITY OF HONG KONG

MATH Introduction to topology

Midterm 1

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TIME: 90 MINUTES

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This Examination paper consists of 6 pages (including this one). Make sure you have all 6.

INSTRUCTIONS:

No communication devices allowed. Calculator is allowed. One A4 page (one sided) cheating sheet allowed.

MARKING:

Q1	/50
Q2	/50
TOTAL	81 /100

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Q1 [50 marks]

Let $\mathbb{R}^{Zar}$ be $\mathbb{R}$ equipped with the Zariski topology.

(a) Show that $\mathbb{R}^{Zar}$ is not Hausdorff.

(b) Show that $\mathbb{R}^{Zar}$ is compact.

(c) Write down the definition of Zariski topology on $\mathbb{R}^2$ by specifying all closed subsets.

(d) Is the product topology on $\mathbb{R}^{Zar} \times \mathbb{R}^{Zar}$ the same as the Zariski topology on $\mathbb{R}^2$? Justify your answer.
cofinite cofinite.

(e) Let U_1, U_2 be two nonempty (Zariski) open subsets in $\mathbb{R}^2$. Show that $U_1 \cap U_2$ is nonempty.

(f) Let $I = \{(x,0) | 0 < x < 1\} \subset \mathbb{R}^2$. Compute the closure of I in $\mathbb{R}^2$ with the Zariski topology.

(g) Show that the addition map $+ : \mathbb{R}^{Zar} \times \mathbb{R}^{Zar} \to \mathbb{R}^{Zar}$ is not continuous when the domain is equipped with the product topology.

(h) Show that the addition map $+ : \mathbb{R}^2 \to \mathbb{R}^{Zar}$ is continuous if $\mathbb{R}^2$ is equipped with the Zariski topology.

Q1.(a) Proof: Consider the two distinct elements $0, 1$ of $\mathbb{R}^{Zar}$

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Assume to the contrary that $\exists U_0, U_1 \in \mathcal{O}^{Zar}$, $0 \in U_0$ and $1 \in U_1$ and $U_0 \cap U_1 = \phi$

$0 \in U_0 \Rightarrow 0 \notin U_0^c \Rightarrow$ The closed set U_0^c is finite as $\mathcal{O}^{Zar} = \mathcal{O}^{Cofinite}$

$1 \in U_1 \Rightarrow 1 \notin U_1^c \Rightarrow$ The closed set U_1^c is finite as $\mathcal{O}^{Zar} = \mathcal{O}^{Cofinite}$ $\Big\} \Rightarrow \mathbb{R} = U_0^c \cup U_1^c$ is finite

This is a contradiction, so our assumption is false, U_0, U_1 fail to exist, $\mathbb{R}^{Zar}$ is not Hausdorff.

(b) Proof: For all open cover $\mathcal{U}$ of $\mathbb{R}^{Zar}$.

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Step 1: $\mathcal{U}$ cannot be empty, so choose U_0 from $\mathcal{U}$. WLOG, assume that $U_0 \neq \phi$.

Step 2: As $U_0 \neq \phi$, U_0 is cofinite.

If $U_0 = \mathbb{R}$, then $\{U_0\}$ is a finite subcover of $\mathcal{U}$.

If $U_0 \neq \mathbb{R}$, then $U_0 = (\{x_k\}_{k=1}^m)^c$

For each x_k, as $\mathcal{U}$ covers x_k, choose U_k from $\mathcal{U}$ such that $x_k \in U_k$

Now $\{U_k\}_{k=0}^m$ is a finite subcover of $\mathcal{U}$

(c) Solution: For all $C \in \mathcal{P}(\mathbb{R}^{2Zar})$, C is closed in $\mathbb{R}^{2Zar}$ if and only if $C = Z(T)$ for some $T \subseteq \mathbb{R}[x,y]$

Here, $Z(T) = \{(u,v) \in \mathbb{R}^2 : \text{For all polynomial } p(x,y) \in T, p(u,v) = 0\}$.

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continued..

(d) The product topology on $\mathbb{R}^{Zar} \times \mathbb{R}^{Zar}$ is not the same as the Zariski topology $\mathbb{R}^{2,Zar}$.

To prove this, notice that $\{(u,v)\in\mathbb{R}^2 : u=v\} = Z(\{x-y\})$ is closed in the Zariski topology $\mathbb{R}^{2,Zar}$.

I want to prove that $\{(u,v)\in\mathbb{R}^2 : u=v\}$ is not closed in the product topology.

Proof: Assume to the contrary that $\{(u,v)\in\mathbb{R}^2 : u\neq v\}$ is open in the product topology $\mathbb{R}^{Zar}\times\mathbb{R}^{Zar}$

For the element $(1,0)\in\{(u,v)\in\mathbb{R}^2 : u\neq v\}$, we should be able to find two small enough open sets U_1, V_0 in $\mathbb{R}^{Zar}$, such that $(1,0)\in U_1\times V_0$ and $U_1\times V_0 \subseteq \{(u,v)\in\mathbb{R}^2 : u\neq v\}$.

As $1\in U_1$, U_1 is cofinite (not empty), so there exists $A_1\in\mathbb{R}$, such that U_1 contains the whole interval $(A_1,+\infty)$

As $0\in V_0$, V_0 is cofinite (not empty), so there exists $B_0\in\mathbb{R}$, such that V_0 contains the whole interval $(B_0,+\infty)$.

Take $C=\max\{A_1,B_0\}+1$. $C\in(A_1,+\infty)\cap(B_0,+\infty) \Rightarrow C\in U_1\cap V_0 \Rightarrow (C,C)\in U_1\times V_0 \Rightarrow U_1\times V_0 \nsubseteq \{(u,v)\in\mathbb{R}^2 : u\neq v\}$

A contradiction arises, so our assumption is false, and $\{(u,v)\in\mathbb{R}^2 : u=v\}$ is not closed in $\mathbb{R}^{Zar}\times\mathbb{R}^{Zar}$

(e) Proof: Assume that U_1 contains some (u_1,v_1) and U_2 contains some (u_2,v_2).

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Case 1: If $(u_1,v_1)=(u_2,v_2)$, then $U_1\cap U_2$ contains it. $U_1\cap U_2\neq\emptyset$.

Case 2: If $(u_1,v_1)\neq(u_2,v_2)$, then there is a unique line $L=\{(1-t)(u_1,v_1)+t(u_2,v_2)\in\mathbb{R}^{2,Zar} : t\in\mathbb{R}\}$ crossing (u_1,v_1) and (u_2,v_2). Notice that L is homeomorphic to $\mathbb{R}^{Zar}$,

Need to prove it!

so $U_1\cap L$, $U_2\cap L$ are two open subsets in $\mathbb{R}^{Zar}$ which are nonempty.

$\boxed{\text{According to 1(a)}}$, $(U_1\cap L)\cap(U_2\cap L)\neq\emptyset$, so it follows that $U_1\cap U_2\neq\emptyset$.

(f) I claim that $\bar{I}=\mathbb{R}^{Zar}\times\{0\}=Z(\{y\})$.

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Proof: First, $Z(\{y\})$ is a closed set that contains $I=(0,1)\times\{0\}$.

Second, for all closed subset $Z(T)$ of $Z(\{y\})$, $y\in T$.

I want to prove that T cannot contain any $f(x,y)$ such that $f(x,y), y$ are coprime.

If this is the case, then $Z(T)$ will be the emptyset or a singleton $\{(u_0,0)\}$, contradiction with $Z(T)\supseteq I$

No!

Hence, $\bar{I}=Z(T)=Z(\{y\})=\mathbb{R}^{Zar}\times\{0\}$.

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(g) Proof: $\boxed{\text{As mentioned in 1(d)}}$, in $\mathbb{R}^{Zar}\times\mathbb{R}^{Zar}$, one can show that $\{(u,v)\in\mathbb{R}^2 : u=-v\}$ is not closed.

However, $\{(u,v)\in\mathbb{R}^2 : u=v\}$ is the inverse image of a closed set $\{0\}$ via $+$, so $+$ is not continuous.

(h) Proof: For all closed set C in $\mathbb{R}^{Zar}$

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Case 1: If $C=\emptyset$, or $C=\mathbb{R}^{Zar}$, then $+^{-1}(C)=\emptyset$, or $+^{-1}(C)=\mathbb{R}^{2,Zar}$ is closed.

Case 2: If $C\neq\emptyset$, and $C\neq\mathbb{R}^{Zar}$, then $C=\{c_k\}_{k=1}^{m}$ for some c_k in $\mathbb{R}^{Zar}$.

This implies $+^{-1}(C)=Z(\{(x+y-c_1)(x+y-c_2)\cdots(x+y-c_m)\})$ is closed in $\mathbb{R}^{2,Zar}$

Hence, $+$ is continuous.

$$\frac{1}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}+\frac{1}{(x_2-x_3)(x_2-x_1)(x_2-x_4)}+\cdots$$

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Q2 [50 marks]

Let $M_{n,m}$ be the space of $n \times m$ real matrices. We may identify $M_{n,m}$ with $\mathbb{R}^{nm}$ and equip it with the metric topology given by the euclidean norm. Denote by $GL_n(\mathbb{R}) \subset M_{n,n}$ the set of $n \times n$ invertible matrices.

(a) Show that $GL_n(\mathbb{R})$ is a topological group with respect to the subspace topology.

(b) Show that $GL_n(\mathbb{R})$ is not connected.

(c) Show that the subset $V := \{(x,y,z) \in \mathbb{R}^3 | xy - z > 0\}$ is connected. **Hint:** Cover V by two open subsets $V \cap \{x \neq 0\}$ and $V \cap \{z \neq 0\}$!

(d) Let $GL_n^+(\mathbb{R})$ (resp. $GL_n^-(\mathbb{R})$) be the subset of $GL_n(\mathbb{R})$ consisting of matrices with positive (resp. *negative*) determinant. Prove that $GL_n^+(\mathbb{R})$ and $GL_n^-(\mathbb{R})$ are connected components of $GL_n(\mathbb{R})$.

Consider the action $GL_2(\mathbb{R}) \times M_{2,4} \to M_{2,4}$ by left multiplication, i.e.

$$(g, A) \mapsto gA$$

for $g \in GL_2(\mathbb{R})$ and $A \in M_{2,4}$. Let $M_{2,4}^k \subset M_{2,4}$ be the subset consisting of matrices of rank k. Clearly, k takes value in $\{0, 1, 2\}$.

Not open, not closed.

Det is continuous $\Rightarrow A^{-1} = \frac{1}{|A|}A^*$ is cont.

(e) Prove the orbit space $M_{2,4}/GL_2(\mathbb{R})$ is not Hausdorff.

(f) Is the subspace $M_{2,4}^1/GL_2(\mathbb{R}) \subset M_{2,4}/GL_2(\mathbb{R})$ open or closed? Justify your answer. (2)

(g) Given $A \in M_{2,4}^1$, we denote $A = (w_1, w_2)^T$ where w_1, w_2 are the rows of A. By the rank one condition, there exists scalars $t_1, t_2 \in \mathbb{R}$ such that $||t_1 w_1 + t_2 w_2|| = 1$. Show that the orbit space $M_{2,4}^1/GL_2(\mathbb{R})$ is homeomorphic to $\mathbb{RP}^3$, in particular it is Hausdorff.

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Q2.(a) Proof: On $GL_n(\mathbb{R})$, there is a well-defined binary operation $\circ: GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R}), (A,B) \mapsto AB$.
This is well-defined because A has inverse A^{-1} and B has inverse $B^{-1} \Rightarrow AB$ has inverse $B^{-1}A^{-1}$.
As composition of functions is associative, $\circ$ is associative. Identity I and Inverses A^{-1} exist.

-1 is cont.

Notice that dot product $(a_1, \cdots, a_n) \cdot (b_1, \cdots, b_n) = a_1 b_1 + \cdots + a_n b_n$ is continuous, so $\circ$ is continuous.

(b) Proof: Define a continuous function $Det: GL_n(\mathbb{R}) \to \mathbb{R}^\times$, $Det(A) =$ The determinant of A.

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In linear algebra, we proved that Det is well-defined, so this gives a nontrivial open partition $\{Det^{-1}(\text{Negative reals}), Det^{-1}(\text{Positive reals})\}$ of $GL_n(\mathbb{R})$, $GL_n(\mathbb{R})$ is not connected.

(c) Proof: For all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in V$.

0

Consider the circle $(0,0,0), (x_1, y_1, z_1), (x_2, y_2, z_2)$

is P a permutation matrix or generalized permutation matrix?

continued..

(d) Proof: For all matrix $A \in GL_n(\mathbb{R})$, do LPU decomposition $A = L_n L_{n-1} \cdots L_1 P U_1 \cdots U_{m-1} U_m$.

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Notice that $L_k = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & a_k & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ can be rewritten into a path $l_k : [0,1] \to GL_n(\mathbb{R})$, $l_k(t) = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & a_k t & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$, similar for R_k.

So A is path connected to P.

Now every even permutation can be decomposed into a product of 3 cycles like $P = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 1 & 0 & 0 & \\ & & & \ddots \end{pmatrix}$,

and for $P = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 1 & 0 & 0 & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$, it can be rewritten into a path $p(t) = \begin{pmatrix} 1-t & t & 0 & \\ 0 & 1-t & t & \\ t & 0 & 1-t & \\ & & & 1 \end{pmatrix}$ to I so $GL_n(\mathbb{R})$ is path connected, thus a connected comp.

(e) Proof: Assume to the contrary that we can separate $\pi\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\pi\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ by open sets U_0, U_1,

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now $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ are separated by saturated open sets $\pi^{-1}(U_0), \pi^{-1}(U_1)$.

According to Archimedian Property of real numbers, there exists $N \in \mathbb{N}$ such that $\frac{1}{N}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \pi^{-1}(U_0)$

but $\pi^{-1}(U_1)$ is saturated, so $\frac{1}{N}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \pi^{-1}(U_1)$ contradicting to $\pi^{-1}(U_0) \cap \pi^{-1}(U_1) = \varnothing$.

(f) The subspace is open but not closed. vk=1!

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Proof: For all $\pi(x) \in M^1_{2,4}(\mathbb{R})$, x is of full rank, so x has a $2*2$ submatrix with nonzero determinant.

Choose $r > 0$, such that on the open ball $B(x, r)$, the same determinant doesn't go to 0

as π is an open map, $\pi(B(x,r))$ will be an open set such that $\pi(x) \in \pi(B(x,r)) \subseteq M^1_{2,4}(\mathbb{R})$, so the subspace is open.

Take the same argument in (e), we find that $M^1_{2,4}(\mathbb{R})$ has a limit point $\pi\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ not in $M^1_{2,4}(\mathbb{R})$, so it is not closed.

(g) Proof: Notice that $(\vec{w}_1, \vec{w}_2) \mapsto t_1\vec{w}_1 + t_2\vec{w}_2$ where $\|t_1\vec{w}_1 + t_2\vec{w}_2\| = 1$ gives a continuous surjection

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from $M^1_{2,4}(\mathbb{R})$ to $S^3 \subseteq \mathbb{R}^4$ if we choose $t_1 = t_2 = \|\vec{w}_1 + \vec{w}_2\|^{-1} \neq 0$. Identify $\vec{w}$ with $-\vec{w}$,

so we get a continuous surjection σ from $M^1_{2,4}(\mathbb{R})$ to $\mathbb{R}P^3$.

not well defined!

Consider the following diagram:

$$\begin{tikzcd} O^1_{2,4}(\mathbb{R}) \arrow[d, "\pi"'] \arrow[dr, "\sigma"] & \\ O^1_{2,4}(\mathbb{R})/O_2(\mathbb{R}) \arrow[r, "\tilde{\sigma}"] & \mathbb{R}P^3 \end{tikzcd}$$

Notice that $O^1_{2,4}(\mathbb{R})$ is compact, $\mathbb{R}P^3$ is Hausdorff, so $\tilde{\sigma}$ is a homeomorphism.

so $M^1_{2,4}(\mathbb{R}) / GL_2(\mathbb{R}) \cong O^1_{2,4}(\mathbb{R}) / O_2(\mathbb{R}) \cong \mathbb{R}P^3$.