MATH3301 Tutorial 4

1. (a) Let $G = \{f_1, f_2, f_3, f_4\}$ where f_1, f_2, f_3, f_4 are real-valued functions on $\mathbb{R}^\times$ defined as:

$$f_1(x) = x, \quad f_2(x) = \frac{1}{x}, \quad f_3(x) = -x, \quad f_4(x) = -\frac{1}{x} \qquad (x \in \mathbb{R}^\times).$$

Is $(G, \circ)$ a group under the function composite $\circ$? Explain your answer. If it is a group, is it an abelian group, a cyclic group?

(b) Let $GL_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \right\}$.

Show that under matrix multiplication, $GL_2(\mathbb{Z}_2)$ is a *non-abelian* group of order 6.

(c) Compute the order of every element in $GL_2(\mathbb{Z}_2)$.

(d) Is $GL_2(\mathbb{Z}_2) \cong S_3$? Explain why if no and give an isomorphism if yes.

2. A *lattice diagram for (subgroups of) a group* is a diagram so that a line running downward from a subgroup H to a subgroup K means that K is a subgroup of H. e.g.

$$\begin{tikzcd} \mathbb{Z}_4 \arrow[d, dash] \\ \{0,2\} \arrow[d, dash] \\ \{0\} \end{tikzcd}$$

Lattice diagram for $\mathbb{Z}_4$

$$\begin{tikzcd} & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \{(0,0),(0,1)\} \arrow[dr, dash] & \{(0,0),(1,0)\} \arrow[d, dash] & \{(0,0),(1,1)\} \arrow[dl, dash] \\ & \{(0,0)\} & \end{tikzcd}$$

Lattice diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2$

(a) Draw the lattice diagrams for the groups: (i) $\mathbb{Z}_{12}$ and (ii) $GL_2(\mathbb{Z}_2)$.

[Hint: Use the Lagrange theorem to figure out the *possible* order of subgroups.]

(b) Which subgroup(s) in $\mathbb{Z}_{12}$ is isomorphic to $\mathbb{Z}_4$? Answer the same question for $GL_2(\mathbb{Z}_2)$.

3. Consider the group $G = \mathbb{Z}_8^\times \times \mathbb{Z}_{12}$, and the subgroups $H = \langle 3 \rangle \times \langle 4 \rangle$ and $K = \langle 3 \rangle \times \langle 6 \rangle$.

   (a) Find all the elements in G whose orders are 6.
   (b) Is $\langle (3,4) \rangle = H$? Is $\langle (3,6) \rangle = K$? [Hint: Evaluate $\mathrm{ord}((3,4))$ and $\mathrm{ord}((3,6))$.]
   (c) Evaluate all (left-)cosets of H and $[G : H]$. Do the same for K.
   (d) Draw the lattice diagram of G.

4. Let $\phi : G_1 \to G_2$ be a homomorphism between groups. We abuse the notation with e for the identity elements in G_1 and G_2 . Prove the following statements:

   (a) The image $\phi(H)$ is a subgroup of G_2, for any subgroup H of G_1.

   (b) The pre-image $\phi^{-1}(K)$ is a subgroup of G_1, for any subgroup K of G_2.

   (c) If one of G_1 or G_2 is a finite abelian group, then $\phi(G_1)$ is finite abelian.

   (d) If one of G_1 or G_2 is cyclic, then $\phi(G_1)$ is cyclic.

5. Let $\phi : G_1 \to G_2$ be a homomorphism between groups. We abuse the notation with e for the identity elements in G_1 and G_2 . Prove or disprove, with justification, the statements:

   (a) If $\psi : G_2 \to G_1$ is a function such that both $\phi \circ \psi$ and $\psi \circ \phi$ are the identity map, then ψ is a group isomorphism.

   (b) If ϕ is bijective, then $\phi^{-1}(g) = \phi(g^{-1})$ for all $g \in G_1$.

   (c) The pre-image $\phi^{-1}(K)$ is abelian if K is an abelian subgroup of G_2.

   (d) The pre-image $\phi^{-1}(K)$ is cyclic if K is a cyclic subgroup of G_2.

   (e) If $|G_1| = 2024$ and $K := \phi^{-1}(\{e\})$, then $|K|$ is not divisible by primes greater than 24.

6. Evaluate with explanation the number of nontrivial homomorphisms ϕ in each case below:

   (a) $\phi : \mathbb{Z} \to \mathbb{Z}$.

   (b) $\phi : \mathbb{Z} \to \mathbb{Z}$ is surjective.

   (c) $\phi : \mathbb{Z} \to \mathbb{Z}_2$.

   (d) $\phi : \mathbb{Z}_2 \times \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_5$.

   (e) $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_4$.

   (f) $\phi : \mathbb{Z}_4 \to \mathbb{Z}_{12}$.

   (g) $\phi : \mathbb{Z} \times \mathbb{Z} \to 2\mathbb{Z}$.

   (h) $\phi : 2\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$.

End