MATH3301 Tutorial 11

1. Let $0 \neq f$ be an irreducible polynomial in $K[t]$ where K is a field. Suppose E is a field extension of K (i.e. E is a field that contains the field K). Let $\alpha \in E$ such that $f_E(\alpha) = 0$ (here f is viewed as a polynomial in $E[t]$).

   (a) If $0 \neq g \in K[t]$ and $g_E(\alpha) = 0$, show that $\deg g \geq \deg f$.

   (b) Prove that the ideal generated by f in $K[t]$, $\langle f \rangle$ is equal to $\{g \in K[t] : g_E(\alpha) = 0\}$.

2. (a) Let R be a nonzero commutative ring.

      i. If $x^{2015} + x = 0$ for all $x \in R$, show that every *nonzero prime* ideal P of R is maximal.

      ii. If R is an integral domain of 49 elements, must $R[t]/\langle t^2 + t + 1 \rangle$ be a field?

   (b) Let F be a field with $|F| \in (2015, 2200)$. Suppose
   $$\underbrace{1 + \cdots + 1}_{10} \neq 0 \text{ but } \underbrace{1 + \cdots + 1}_{r} = 0$$
   for some $r < 20$. What is the characteristic of F and the cardinality of F?
   Is $t^2 - 3$ an irreducible polynomial ring $F[t]$? Explain.

3. (a) Are $\mathbb{R}[t]/\langle t^2 + t + 1 \rangle$ and $\mathbb{F}_{13}[t]/\langle t^2 + t + 1 \rangle$ fields? Explain.

   (b) Let $\bar{t}$ be the image of t in the field $\mathbb{Z}_2[t]/\langle t^2 + t + 1 \rangle$.
   Evaluate the order of $\bar{t}^2 + 1$ in the field.

4. Consider the polynomial $f(t) = t^2 + 2t + 3$ in $\mathbb{Z}_5[t]$, and the quotient ring $F = \mathbb{Z}_5[t]/\langle f \rangle$.

   (a) Show that F is a finite field and evaluate $|F|$. Let $\bar{t}$ be the image of t under the natural projection. Evaluate the order of $4\bar{t} + 1$ in the group $F^\times$.

   (b) Is it possible to construct a ring homomorphism from F to $\mathbb{Z}_{25}$? Construct a ring homomorphism if yes, or explain if no.

5. (a) Let $\bar{t}$ be the image of t in $F := \mathbb{Z}_5[t]/\langle f \rangle$. Show that $\bar{t}^3 = -1$. Find a generator of $F^\times$.

   (b) Show that all quadratic polynomials in $\mathbb{F}_{p^2}$ whose coefficients and constant lie in $\mathbb{F}_p$ are reducible where p denotes any prime number.

6. Let $R[t]$ be the polynomial ring over the nonzero commutative ring R. Show that $R^\times$ is isomorphic to a cyclic group if (i) $R^\times$ is the group of units in R and $|R^\times| = 24$, and (ii) $|\{\alpha \in R : f_R(\alpha) = 0\}| \leq \deg f$ for every $0 \neq f \in R[t]$.

End