THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 5 Solution

Problem 1.

(i) Compute the derivatives directly

$$\partial_t u_L(t,x) = \partial_t u(t,x) - \frac{2k(M_T - M_0)}{L^2},$$
$$\partial_x u_L(t,x) = \partial_x u(t,x) - \frac{2x(M_T - M_0)}{L^2},$$
$$\partial_{xx} u_L(t,x) = \partial_{xx} u(t,x) - \frac{2(M_T - M_0)}{L^2}.$$

And hence,

$$\begin{aligned}
&\partial_t u_L(t,x) - k\partial_{xx} u_L(t,x) \\
=&\partial_t u(t,x) - \frac{2k(M_T - M_0)}{L^2} - k\left(\partial_{xx} u(t,x) - \frac{2(M_T - M_0)}{L^2}\right) = 0.
\end{aligned}$$

(ii) Apply the maximum principle to u_L over the domain $[0,T]\times[-L,L]$, and it follows that

$$\max_{\substack{0\le t\le T\\ -L\le x\le L}} u_L(t,x) = \max\left\{\max_{-L\le x\le L} u_L(0,x), \max_{0\le t\le T} u(t,-L), \max_{0\le t\le T} u(t,L)\right\}.$$

Notice that the terms on the right-hand side are all less than 0:

$$\begin{aligned}
u_L(0,x) &= u(0,x) - \left(M_0 + \frac{M_T - M_0}{L^2}x^2\right) \le u(0,x) - M_0 \le 0,\\
u_L(t,\pm L) &= u(t,\pm L) - \left(M_0 + \frac{M_T - M_0}{L^2}(L^2 + 2kt)\right)\\
&= u(t,\pm L) - M_T - \frac{M_T - M_0}{L^2}2kt \le 0.
\end{aligned}$$

And hence, we can conclude that

$$\max_{\substack{0 \le t \le T \\ -L \le x \le L}} u_L(t,x) \le 0.$$

(iii) Let $(t_0, x_0) \in [0,T] \times (-\infty, \infty)$ with $|x_0| \le L$. Then by applying part (ii) we have

$$u(t_0, x_0) \le M_0 + \frac{M_T - M_0}{L^2}(x_0^2 + 2kt_0).$$

(iv) Let $L \to \infty$, then by (iii) we have $u(t_0, x_0) \le M_0$. It follows that

$$M_T = \max_{\substack{0 \le t \le T \\ -\infty < x < \infty}} u(t,x) \le M_0 = \max_{-\infty < x < \infty} \phi(x) = \max_{-\infty < x < \infty} u(0,x) \le M_T.$$

Thus, we have

$$\max_{\substack{0 \le t \le T \\ -\infty < x < \infty}} u(t,x) = \max_{-\infty < x < \infty} \phi(x).$$

Problem 2. The function u is continuous on compact set $\bar{\Omega}$, then by the extreme value theorem, the maximum value is attained. Assume to the contrary that the minimum is attained at $(x_0, y_0) \in \Omega$, then it follows that at this point

$$\partial_{xx} u \le 0, \quad \partial_{yy} u \le 0 \quad \text{and} \quad \partial_y u = 0.$$

Hence,

$$\left(10x_0^4 + y_0^{2024}\right) \partial_{xx} u(x_0, y_0) + 5\, \partial_{yy} u(x_0, y_0) + \left(y_0^{330} \ln \frac{e^{x_0}}{1 + e^{x_0}}\right) \partial_y u(x_0, y_0) \le 0.$$

and

$$121x^2 + 22xy^5 + y^{10} + 1 = (11x + y^5)^2 + 1 > 0.$$

Contradiction. Thus, the maximum point can only be attained on $\partial\Omega$. That is to say,

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Problem 3.

(i) Consider $u_\epsilon = u + \epsilon x_1^2$ for some $\epsilon > 0$. It has the properties that

$$\partial_{x_d} u_\epsilon = \partial_{x_d} u, \quad \partial_{x_1}^2 u_\epsilon = \partial_{x_1}^2 u + 2\epsilon, \quad \text{and } \partial_{x_k}^2 u_\epsilon = \partial_{x_k}^2 u \text{ for } k \neq 1.$$

This u_ϵ satisfies that

$$\begin{aligned}\sum_{k=1}^{d} k^k \partial_{x_k}^2 u_\epsilon - (19x_2^{11}\sinh u)\partial_{x_d} u_\epsilon &= \sum_{k=1}^{d} k^k \partial_{x_k}^2 u + 2\epsilon - (19x_2^{11}\sinh u)\partial_{x_d} u \\ &= 2\epsilon > 0.\end{aligned}$$

Now we prove the maximum principle for u_ϵ. Suppose the maximum of u_ϵ is attained at $\tilde{x} \in D$, then at this point it follows

$$\partial_{x_k}^2 u_\epsilon \leq 0, \text{ and } \partial_{x_k} u_\epsilon = 0.$$

Contradiction. Thus, the maximum of u_ϵ is attained on ∂D. That is,

$$\max_{\bar{D}} u_\epsilon = \max_{\partial D} u_\epsilon.$$

Thus, the original u satisfies

$$\max_{\bar{D}} u \leq \max_{\bar{D}} u_\epsilon = \max_{\partial D} u_\epsilon = \max_{\partial D} u + \epsilon \max_{\partial D} x_1^2.$$

Notice that $\max_{\partial D} x_1^2$ is bounded, because

$$x_1^2 \leq 99x_1^2 \leq (10x_1 - 22)^2 + 220^2 - 484 < 2024 + 220^2 - 484 < \infty.$$

So we can let $\epsilon \to 0$ and obtain that

$$\max_{\bar{D}} u = \max_{\partial D} u.$$

Consider $u_\delta = u - \delta x_1^2$ for some $\delta > 0$. It has the properties that

$$\partial_{x_d} u_\delta = \partial_{x_d} u, \quad \partial_{x_1}^2 u_\delta = \partial_{x_1}^2 u - 2\delta, \quad \text{and } \partial_{x_k}^2 u_\delta = \partial_{x_k}^2 u \text{ for } k \neq 1.$$

This u_δ satisfies that

$$\begin{aligned}\sum_{k=1}^{d} k^k \partial_{x_k}^2 u_\delta - (19x_2^{11}\sinh u)\partial_{x_d} u_\delta &= \sum_{k=1}^{d} k^k \partial_{x_k}^2 u - 2\delta - (19x_2^{11}\sinh u)\partial_{x_d} u \\ &= 2\delta < 0.\end{aligned}$$

Now we prove the minimum principle for u_δ. Suppose the maximum of u_ϵ is attained at $\bar{x} \in D$, then at this point it follows

$$\partial_{x_k}^2 u_\epsilon \geq 0, \text{ and } \partial_{x_k} u_\epsilon = 0.$$

Contradiction. Thus, the minimum of u_ϵ is attained on ∂D. That is,

$$\min_{\bar{D}} u_\delta = \min_{\partial D} u_\delta.$$

Thus, the original u satisfies

$$\min_{\bar{D}} u \geq \min_{\bar{D}} u_\delta = \min_{\partial D} u_\delta \geq \min_{\partial D} u - \delta \max_{\partial D} x_1^2.$$

Recall we have proved that $\max_{\partial D} x_1^2$ is bounded, so we can let $\delta \to 0$ and obtain that

$$\min_{\bar{D}} u = \min_{\partial D} u.$$

In conclusion, we have

$$\max_{\bar{D}} |u| = \max_{\partial D} |u|.$$

(ii) (a) Observe that

$$e^{w^2} - w^2 + \frac{1}{2}w^4 = \sum_{n=0}^{\infty} \frac{w^{2n}}{n!} - w^2 + \frac{1}{2}w^4 = 1 + w^4 + \sum_{n=2}^{\infty} \frac{w^{2n}}{n!} \geq 1,$$

so the minimum value is 1, which is obtained when $w = 0$.

(b) Notice that by part (a) the right-hand side of equation is positive because

$$e^{u^2} - u^2 + \frac{1}{2}u^4 - \frac{3}{5} \geq 1 - \frac{3}{5} = \frac{2}{5} > 0.$$

Assume first the minimum is attained by an interior point (t_0, x_0) for $0 < t < T$ and $0 < x < L$. Then at this point,

$$\partial_{xx} u > 0, \quad \partial_x u = 0 \quad \text{and } \partial_t u = 0.$$

Hence,

$$\partial_t u(t_0, x_0) + \left(x_0^2 + t_0\right) \sin\left(\partial_x u(t_0, x_0)\right) - 24\, \partial_{xx} u(t_0, x_0) < 0.$$

Contradiction. The minimum cannot be attained by an interior point. Now suppose that the minimum is attained at (T, x_0) for $0 < x_0 < L$. Then at this point,

$$\partial_{xx} u \geq 0, \quad \partial_x u = 0, \quad \text{and } \partial_t u \leq 0.$$

Hence,

$$\partial_t u(T, x_0) + \left(x_0^2 + T\right) \sin\left(\partial_x u(T, x_0)\right) - 24\, \partial_{xx} u(T, x_0) \leq 0$$

Contradiction. The minimum cannot be attained on this line either. Thus, the minimum can only be attained on the other three boundaries, that is,

$$\min_{\substack{0 \leq t \leq T \\ 0 \leq x \leq L}} u(t, x) = \min\left\{ \min_{0 \leq x \leq L} u(0, x), \min_{0 \leq t \leq L} u(t, 0), \min_{0 \leq t \leq T} u(t, L) \right\}.$$

Problem 4.

(a) Since u is a continuous function defined on a compact set $\bar{\Omega}$, it follows from the extreme value theorem that there exists a point $\tilde{x} \in \bar{\Omega}$ such that

$$u(\tilde{x}) = \max_{\bar{\Omega}} u. \tag{1}$$

Now, we can divide our analysis into two distinct cases, namely $\max_{\bar{\Omega}} u \leq 0$ or $\max_{\bar{\Omega}} u > 0$, as follows.

Case 1: $\max_{\bar{\Omega}} u \leq 0$**.** It follows from the definition of u^+ that for any $x \in \partial\Omega$,

$$u^+(x) := \max\{u(x), 0\} \geq 0,$$

so

$$\max_{\partial\Omega} u^+ \geq 0,$$

and hence, using the hypothesis $\max_{\bar{\Omega}} u \le 0$, we obtain

$$\max_{\bar{\Omega}} u \le 0 \le \max_{\partial\Omega} u^+.$$

Case 2: $\max_{\bar{\Omega}} u > 0$**.** Seeking for a contradiction, we assume

$$\tilde{x} \in \Omega. \tag{2}$$

Then, by the first and second order tests for local/interior maximum in elementary calculus, we know that for any $i = 1, 2, \ldots, d$,

$$\partial_{x_i} u(\tilde{x}) = 0 \quad \text{and} \quad \partial_{x_i}^2 u(\tilde{x}) \le 0,$$

which imply

$$\nabla u(\tilde{x}) = 0 \quad \text{and} \quad \Delta u(\tilde{x}) \le 0. \tag{3}$$

In addition, using (1) and the hypothesis $\max_{\bar{\Omega}} u > 0$, we actually have

$$u(\tilde{x}) = \max_{\bar{\Omega}} u > 0. \tag{4}$$

Evaluating the given inequality

$$0 \ge -\Delta u + b \cdot \nabla u + cu.$$

at $x = \tilde{x}$, and using both (3) and (4) on the right hand side, we finally obtain the following contradiction:

$$0 \ge \underbrace{-\Delta u(\tilde{x})}_{\ge 0} + \underbrace{b(\tilde{x}) \cdot \nabla u(\tilde{x})}_{=0} + \underbrace{c(\tilde{x}) u(\tilde{x})}_{>0} > 0,$$

where we used the hypothesis that $c(x) > 0$ for all $x \in \Omega$. Therefore, the Assumption (2) is WRONG, so

$$\tilde{x} \in \partial\Omega,$$

and hence, by (1),

$$\max_{\bar{\Omega}} u = u(\tilde{x}) = \max_{\partial\Omega} u^+.$$

In conclusion, since

$$\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+,$$

holds in both cases, we complete the proof.

(b) Let u_1 and u_2 be two solutions to the Dirichlet problem

$$\begin{cases} -\Delta u + b \cdot \nabla u + cu = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = g. \end{cases}$$

Define $\tilde{u} := u_1 - u_2$. Then $\tilde{u}$ satisfies the Dirichlet problem

$$\begin{cases} -\Delta U + b \cdot \nabla U + cU = 0, & \text{in } \Omega, \\ U|_{\partial\Omega} = 0. \end{cases} \tag{5}$$

Applying Part (a) and using the boundary condition $\tilde{u}|_{\partial\Omega} \equiv 0$, we obtain

$$\max_{\bar{\Omega}} \tilde{u} \leq \max_{\partial\Omega} \tilde{u}^+ = 0. \tag{6}$$

Indeed, $-\tilde{u}$ also satisfies the same Dirichlet problem (5), and hence, applying Part (a) again and using the boundary condition $-\tilde{u}|_{\partial\Omega} \equiv 0$, we obtain

$$\max_{\bar{\Omega}} (-\tilde{u}) \leq \max_{\partial\Omega} (-\tilde{u})^+ = 0,$$

which implies

$$\min_{\bar{\Omega}} \tilde{u} \geq 0. \tag{7}$$

Combining (6) and (7), we finally obtain

$$\tilde{u} \equiv 0 \quad \text{in } \bar{\Omega},$$

and this proves the uniqueness.

(c) Solving the ODE

$$-\frac{d^2h}{dx_1^2} = h$$

yields

$$h(x_1) = A\cos x_1 + B\sin x_1.$$

Using the boundary condition $h\left(-\frac{\pi}{2}\right) = h\left(\frac{\pi}{2}\right) = 0$, we find that

$$B = 0,$$

and hence,

$$h(x_1) = A\cos x_1.$$

In particular, we can choose $A := 1$, and obtain

$$h(x_1) = \cos x_1.$$

(d) The statement is CORRECT, and its proof is as follows.

In the underlying domain $\bar{\Omega} = \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \times (0, L)$, $|x_1| \le \frac{\pi}{3}$, so $h(x_1) := \cos x_1$ satisfies

$$1 \ge h \ge \frac{1}{2} > 0 \quad \text{in } \bar{\Omega}, \tag{8}$$

and hence, we can consider

$$\boxed{w := \frac{v}{h}}.$$

Then w satisfies the boundary condition $w|_{\partial\Omega} = 0$, the identity

$$\nabla w = \frac{1}{h}\nabla v - \frac{\nabla h}{h^2} v$$

and equality

$$\begin{aligned}
\Delta w = \nabla \cdot \nabla w = \nabla \cdot \left(\frac{1}{h}\nabla v - \frac{\nabla h}{h^2} v\right) \\
&= \frac{1}{h}\Delta v - \left(\frac{2\nabla h}{h^2}\right) \cdot \nabla v + \left(\frac{2|\nabla h|^2}{h^3}\right) v - \left(\frac{\Delta h}{h^2}\right) v \\
&= \frac{1}{h}\Delta v - \left(\frac{2\nabla h}{h}\right) \cdot \left(\frac{1}{h}\nabla v - \frac{\nabla h}{h^2} v\right) - \left(\frac{\Delta h}{h}\right)\left(\frac{v}{h}\right) \\
&= \frac{1}{h}\Delta v - \left(\frac{2\nabla h}{h}\right) \cdot \nabla w - \left(\frac{\Delta h}{h}\right) w.
\end{aligned}$$

Therefore, using the fact that $h > 0$ and

$$-\Delta v + av \le 0,$$

we have

$$-\Delta w - \left(\frac{2\nabla h}{h}\right) \cdot \nabla w - \left(\frac{\Delta h}{h}\right) w = -\frac{1}{h}\Delta v \le -aw,$$

which is equivalent to

$$\boxed{-\Delta w - \left(\frac{2\nabla h}{h}\right) \cdot \nabla w + \left(a - \frac{\Delta h}{h}\right) w \le 0.}$$

On the other hand, a direct computation yields

$$-\Delta h := -\partial_{x_1}^2 h - \partial_{x_2}^2 h = -\partial_{x_1}^2 (\cos x_1) = \cos x_1 = h,$$

and hence, using the assumption that $a(x) > -1$ for all $x \in \bar{\Omega}$, we have

$$a - \frac{\Delta h}{h} = a + 1 > 0.$$

Therefore, we can apply Part (a) to w and the boundary condition $w|_{\partial\Omega} = 0$, and obtain

$$\max_{\bar{\Omega}} w \le \max_{\partial\Omega} w^+ = 0,$$

since $w := \frac{v}{h}$. In particular, for any $x := (x_1, x_2, \dots, x_d) \in \bar{\Omega}$,

$$\frac{v(x)}{h(x_1)} = w(x) \le \max_{\bar{\Omega}} w \le 0,$$

which implies

$$v(x) \le 0$$

since $h > 0$. Taking the maximum over $\bar{\Omega}$, we finally show

$$\max_{\bar{\Omega}} v \le 0.$$

Problem 5. Firstly, we will prove the maximum principle for

$$\partial_t u - a\partial_x u - k\partial_{xx} u = 0 \quad \text{for } 0 < x < L \text{ and } 0 < t < T.$$

Proof. Let $\Omega = (0,T) \times (0,L)$ and $\Gamma := \{(t,x) \in \Omega;\ t = 0 \text{ or } x = 0 \text{ or } L\}$.

Step 1: Show that if $\partial_t v - a\partial_x v - k\partial_{xx} v < 0$, then $\max_{\bar{\Omega}} v = \max_{\Gamma} v$.

As v is continuous over $\overline{\Omega}$, it follows from the extreme value theorem that the maximum exists on $\overline{\Omega}$, say $v(t_0, x_0) = \max_{\bar{\Omega}} v$.

Now we will show that $(t_0, x_0) \notin \overline{\Omega} \smallsetminus \Gamma$. Assume on the contrary that $(t_0, x_0) \in \bar{\Omega} \smallsetminus \Gamma$.

If $\boxed{(t_0, x_0) \in (0, T) \times (0, L)}$, then $\partial_x v(t_0, x_0) = \partial_t v(t_0, x_0) = 0$ and $\partial_{xx} v(t_0, x_0) \le 0$. Thus

$$\partial_t v - a\partial_x v - k\partial_{xx} v = -k\partial_{xx} v \ge 0$$

which give a contradiction.

If $\boxed{t_0 = T}$ and $\boxed{x_0 \in (0, L)}$, then $\partial_x v(t_0, x_0) = 0$, $\partial_t v(t_0, x_0) \ge 0$ and $\partial_{xx} v(t_0, x_0) \le 0$. Thus

$$\partial_t v - a\partial_x v - k\partial_{xx} v = \partial_t v - k\partial_{xx} v \ge 0$$

which also give a contradiction.

Therefore, $(t_0, x_0) \in \Gamma$ and hence $\max_{\bar{\Omega}} v = v(t_0, x_0) = \max_{\Gamma} v$.

<u>Step 2</u>: Show that if $\partial_t u - a\partial_x u - k\partial_{xx} u \le 0$, then $\max_{\bar{\Omega}} u = \max_{\Gamma} u$.

Given any $\epsilon > 0$ and a fixed $\lambda > \max(-\frac{a}{k}, 0)$, we define, for any $(x, y) \in \bar{\Omega}$,

$$v_\epsilon(t, x) := u(t, x) + \epsilon e^{\lambda x}.$$

Then

$$\partial_t v_\epsilon - a\partial_x v_\epsilon - k\partial_{xx} v_\epsilon = \partial_t u - a\partial_x u - k\partial_{xx} u - \epsilon\lambda e^{\lambda x}(a + k\lambda) < 0.$$

By Step 1,

$$\max_{\bar{\Omega}} u \le \max_{\bar{\Omega}} (u + \epsilon e^{\lambda x}) = \max_{\bar{\Omega}} v_\epsilon = \max_{\Gamma} v_\epsilon = \max_{\Gamma} (u + \epsilon e^{\lambda x}) \le \max_{\Gamma} u + \epsilon e^{\lambda L}$$

and hence taking $\epsilon \to 0^+$, we get

$$\max_{\bar{\Omega}} u \le \max_{\Gamma} u.$$

On the other hand, as $\Gamma \subset \bar{\Omega}$, $\max_{\bar{\Omega}} u \ge \max_{\Gamma} u$. Thus, $\max_{\bar{\Omega}} u = \max_{\Gamma} u$. $\square$

Let $v \coloneqq u_1 - u_2$. Then v satisfies the parabolic equation

$$\partial_t v - a\partial_x v - k\partial_{xx} v = 0 \qquad \text{for } 0 < x < L \text{ and } 0 < t < T$$

subject to the initial and boundary conditions:

$$\begin{cases} v|_{t=0} = \phi_1 - \phi_2 \le 0 \\ v|_{x=0} = g_1 - g_2 \le 0 \\ v|_{x=L} = h_1 - h_2 \le 0. \end{cases}$$

Let $\Omega \coloneqq (0,T) \times (0,L)$ and Γ is the parabolic boundary of Ω, that is

$$\Gamma \coloneqq \{(t,\, x) \in \Omega;\; t = 0 \text{ or } x = 0 \text{ or } L\}.$$

Then it follows from the maximum principle that for all $(t,\, x) \in \bar{\Omega}$,

$$u_1(t,\, x) - u_2(t,\, x) = v(t,\, x) \le \max_{\bar{\Omega}} v = \max_{\Gamma} v \le 0,$$

and hence $u_1 \le u_2$ on $\bar{\Omega}$.

Problem 6.

(a) The proof can be divided into two steps, in which the first step will show the conclusion by a stronger assumption, and the second step will weaken the assumption used in the first step.

Step 1. Maximum Principle for Strict Inequality. Let $w \in C^2((0,T) \times (0,L)) \cap C([0,T] \times [0,L])$ satisfy

$$\partial_t w + b\partial_x w - k\partial_{xx} w < 0. \tag{9}$$

Since w is a continuous function on a compact set $[0,T] \times [0,L]$, it then follows from the extreme value theorem that there exists a point $(t_0, x_0) \in [0,T] \times [0,L]$ such that

$$w(t_0, x_0) = \max_{\substack{0 \le x \le L \\ 0 \le t \le T}} w(t, x).$$

Seeking for a contradiction, we assume that $(t_0, x_0) \in (0,T] \times (0,L)$. By the first and second order tests for local/interior maximum in elementary calculus, we know that

$$\partial_t w(t_0,x_0) \geq 0, \quad \partial_x w(t_0,x_0) = 0, \quad \text{and } \partial_{xx} w(t_0,x_0) \leq 0.$$

and hence, at this (t_0, x_0), we actually have

$$\partial_t w(t_0,x_0) + b\partial_x w(t_0,x_0) - k\partial_{xx} w(t_0,x_0) \geq 0,$$

which contradicts with Inequality (9). This means that the assumption "$(t_0,x_0) \in (0,T] \times (0,L)$" is wrong, so $t_0 = 0$ or $x_0 = 0$ or $x_0 = L$, which implies

$$\max_{\substack{0\leq x\leq L\\ 0\leq t\leq T}} w(t,x) = \max\left\{\max_{0\leq x\leq L} w(0,x), \max_{0\leq t\leq T} w(t,0), \max_{0\leq t\leq T} w(t,L)\right\}.$$

Step 2. Maximum Principle for Weak Inequality. Let $u \in C^2((0,T)\times(0,L)) \cap C([0,T]\times[0,L])$ satisfies

$$\partial_t u + b\partial_x u - k\partial_{xx} u \leq 0.$$

Let $\lambda > \dfrac{|b|+1}{k} > 0$ be a positive constant. For any $\epsilon > 0$, we define

$$w_\epsilon(t,x) := u(t,x) + \epsilon e^{\lambda x}.$$

Then

$$\begin{aligned}\partial_t w_\epsilon + b\partial_x w_\epsilon - k\partial_{xx} w_\epsilon &= \partial_t u + b\partial_x u - k\partial_{xx} u + \epsilon\lambda b e^{\lambda x} - \epsilon\lambda^2 k e^{\lambda x}\\ &\leq -\epsilon\lambda\left(\lambda k + b\right) e^{\lambda x} < 0.\end{aligned}$$

Therefore, applying the result in Step 1, we have

$$\max_{\substack{0\leq x\leq L\\ 0\leq t\leq T}} w_\epsilon(t,x) = \max\left\{\max_{0\leq x\leq L} w_\epsilon(0,x), \max_{0\leq t\leq T} w_\epsilon(t,0), \max_{0\leq t\leq T} w_\epsilon(t,L)\right\},$$

which implies

$$\begin{aligned}\max_{\substack{0\le x\le L\\0\le t\le T}} u(t,x) &\le \max_{\substack{0\le x\le L\\0\le t\le T}} w_\epsilon(t,x)\\ &= \max\left\{\max_{0\le x\le L} w_\epsilon(0,x), \max_{0\le t\le T} w_\epsilon(t,0), \max_{0\le t\le T} w_\epsilon(t,L)\right\}\\ &\le \max\left\{\max_{0\le x\le L} u(0,x), \max_{0\le t\le T} u(t,0), \max_{0\le t\le T} u(t,L)\right\} + \epsilon e^{\lambda L}.\end{aligned}$$

Passing to the limit as $\epsilon \to 0^+$ in the above inequality, we obtain

$$\max_{\substack{0\le x\le L\\0\le t\le T}} u(t,x) \le \max\left\{\max_{0\le x\le L} u(0,x), \max_{0\le t\le T} u(t,0), \max_{0\le t\le T} u(t,L)\right\}.$$

Since $\{(t,x) \in [0,T]\times[0,L];\ t=0,\ x=0,\ x=L\} \subset [0,T]\times[0,L]$, it follows from the definition of maximum that

$$\max_{\substack{0\le x\le L\\0\le t\le T}} u(t,x) \ge \max\left\{\max_{0\le x\le L} u(0,x), \max_{0\le t\le T} u(t,0), \max_{0\le t\le T} u(t,L)\right\},$$

and hence,

$$\max_{\substack{0\le x\le L\\0\le t\le T}} u(t,x) = \max\left\{\max_{0\le x\le L} u(0,x), \max_{0\le t\le T} u(t,0), \max_{0\le t\le T} u(t,L)\right\}.$$

(b) Now, let u_1 and u_2 be solutions to the initial and boundary value problem

$$\begin{cases}\partial_t u + b\partial_x u - k\partial_{xx} u = f(t,x) & \text{for } 0<x<L \text{ and } 0<t<T\\ u(t,0)=g(t) & \text{for } 0<t\le T\\ u(t,L)=h(t) & \text{for } 0<t\le T\\ u(0,x)=\phi(x) & \text{for } 0\le x\le L,\end{cases}$$

where the given functions f, g, h and ϕ are the SAME for both u_1 and u_2. Define $\tilde u := u_1 - u_2$. Then $\tilde u$ satisfies

$$\begin{cases}\partial_t \tilde u + b\partial_x \tilde u - k\partial_{xx}\tilde u = 0 & \text{for } 0<x<L \text{ and } 0<t<T\\ \tilde u(t,0)=0 & \text{for } 0<t\le T\\ \tilde u(t,L)=0 & \text{for } 0<t\le T\\ \tilde u(0,x)=0 & \text{for } 0\le x\le L.\end{cases}$$

Applying part (a) to $\tilde{u}$ and using the initial and boundary conditions for $\tilde{u}$, we have

$$\max_{\substack{0\le x\le L\\0\le t\le T}} \tilde{u}(t,x) = \max\left\{\max_{0\le x\le L}\tilde{u}(0,x), \max_{0\le t\le T}\tilde{u}(t,0), \max_{0\le t\le T}\tilde{u}(t,L)\right\} = 0,$$

which implies $u_1 \le u_2$. Repeating the SAME argument but interchanging the roles of u_1 and u_2, we also have $u_2 \le u_1$. Combining both inequalities yields $u_1 \equiv u_2$. This completes the proof of uniqueness.

(c) Let $u := v^2$. Then

$$\partial_t u = 2v\partial_t v,$$
$$\partial_{xx} u = \partial_x (2v\partial_x v) = 2v\partial_{xx} v + 2\left|\partial_x v\right|^2$$

and hence, using the given equation $\partial_t v - k\partial_{xx} v = -v^5$, we have

$$\partial_t u - k\partial_{xx} u = 2v\left(\partial_t v - k\partial_{xx} v\right) - 2k\left|\partial_x v\right|^2 = -2v^6 - 2k\left|\partial_x v\right|^2 \le 0.$$

Applying part (a) to u with $b := 0$, we have

$$\begin{aligned}\max_{\substack{0\le x\le L\\0\le t\le T}} |v(t,x)|^2 &= \max_{\substack{0\le x\le L\\0\le t\le T}} u(t,x)\\ &= \max\left\{\max_{0\le x\le L} u(0,x), \max_{0\le t\le T} u(t,0), \max_{0\le t\le T} u(t,L)\right\}\\ &= \max\left\{\max_{0\le x\le L} |v(0,x)|^2, \max_{0\le t\le T} |v(t,0)|^2, \max_{0\le t\le T} |v(t,L)|^2\right\},\end{aligned}$$

which is equivalent to

$$\max_{\substack{0\le x\le L\\0\le t\le T}} |v(t,x)| = \max\left\{\max_{0\le x\le L} |v(0,x)|, \max_{0\le t\le T} |v(t,0)|, \max_{0\le t\le T} |v(t,L)|\right\}.$$

Problem 7.

(i) Let $\Omega := (0,T)\times(-L,L)$ and Γ is the parabolic boundary of Ω, that is

$$\Gamma := \left\{(t,x)\in\Omega;\ t = 0 \text{ or } x = -L \text{ or } L\right\}.$$

(a) Let $v(t,x) := u(t,-x)$. Then v is a solution to the same initial and boundary value problem

$$\begin{cases} \partial_t v = k\partial_{xx} v & \text{for } -L < x < L \text{ and } t > 0 \\ v|_{t=0} = \phi(-x) = \phi(x) \\ v|_{x=-L} = v|_{x=L} \equiv 0, \end{cases}$$

Then $w := u - v$ satisfies the same heat equation $\partial_t w = k\partial_{xx} w$ on Ω subject to $w|_{t=0} = w|_{x=-L} = w|_{x=L} = 0$. By the maximum principle, we obtain

$$\max_{\bar{\Omega}} |w| = \max_{\Gamma} |w| = 0$$

which implies $w = 0$ on $\bar{\Omega}$ and hence $v = u$ on $\bar{\Omega}$. As $T > 0$ is arbitrary, $u(t,-x) = u(t,x)$ for any $t \geq 0$ and $x \in [-L, L]$.

(b) Let $\tilde{v}(t,x) := -u(t,-x)$. Then $\tilde{v}$ is a solution to the same initial and boundary value problem

$$\begin{cases} \partial_t \tilde{v} = k\partial_{xx} \tilde{v} & \text{for } -L < x < L \text{ and } t > 0 \\ \tilde{v}|_{t=0} = -\phi(-x) = \phi(x) \\ \tilde{v}|_{x=-L} = \tilde{v}|_{x=L} \equiv 0, \end{cases}$$

Then $w = u - \tilde{v}$ satisfies the same heat equation $\partial_t w = k\partial_{xx} w$ on Ω subject to $w|_{t=0} = w|_{x=-L} = w|_{x=L} = 0$. By the maximum principle, we obtain

$$\max_{\bar{\Omega}} |w| = \max_{\Gamma} |w| = 0$$

which implies $w = 0$ on $\bar{\Omega}$ and hence $\tilde{v} = u$ on $\bar{\Omega}$. As $T > 0$ is arbitrary, $u(t,-x) = -u(t,x)$ for any $t \geq 0$ and $x \in [-L, L]$.

(ii) First, we assume that both h and ϕ are bounded on $\mathbb{R}_{\geq 0} := [0, \infty)$. Otherwise, $\max\left\{\max_{x \geq 0} \phi(x), \max_{t \geq 0} h(t)\right\}$ does not exist.

To find the characteristic curves of the PDE,

$$\begin{cases} \dfrac{dt}{ds} = 1, \ t(0) = t_0 \\ \dfrac{dx}{ds} = c, \ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} t = s + t_0 \\ x = cs + x_0 \end{cases}$$

Then $x = c(t - t_0) + x_0 = ct - ct_0 + x_0$, where $c > 0$. For any $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, the characteristic curve can be represented by

$$C_{t_0,x_0} = \{(t, x) : x = ct - ct_0 + x_0\}.$$

If $x_0 \geq ct_0$, then $(0, x_0 - ct_0) \in C_{t_0,x_0} \cap (\{0\} \times \mathbb{R}_{\geq 0})$. On the other hand, if $x_0 < ct_0$, then $(t_0 - x_0/c, 0) \in C_{t_0,x_0} \cap (\mathbb{R}_{\geq 0} \times \{0\})$. As $u(t,x)$ remains unchanged along each characteristic curves C_{t_0,x_0},

$$\begin{aligned} u(t_0, x_0) &= \begin{cases} u(0, x_0 - ct_0) = \phi(x_0 - ct_0) & \text{if } x_0 \geq ct_0 \\ u(t_0 - x_0/c, 0) = h(t_0 - x_0/c) & \text{if } x_0 < ct_0 \end{cases} \\ &\leq \max\left\{\max_{x\geq 0} \phi(x), \max_{t\geq 0} h(t)\right\} \end{aligned}$$

So $\max\limits_{t,x\geq 0} u(t,x) \leq \max\left\{\max\limits_{x\geq 0} \phi(x), \max\limits_{t\geq 0} h(t)\right\}$. On the other hand,

$$\max\left\{\max_{x\geq 0} \phi(x), \max_{t\geq 0} h(t)\right\} = \max_{tx=0} u(t,x) \leq \max_{t,x\geq 0} u(t,x).$$

Thus, $\max\limits_{t,x\geq 0} u(t,x) = \max\left\{\max\limits_{x\geq 0} \phi(x), \max\limits_{t\geq 0} h(t)\right\}$.