Klein bottle

$$X = \mathbb{R}^2 / \mathbb{Z} \times' \mathbb{Z}$$

$$(m,n)\cdot(m',n') = (m+(-1)^n m',\ n+n')$$

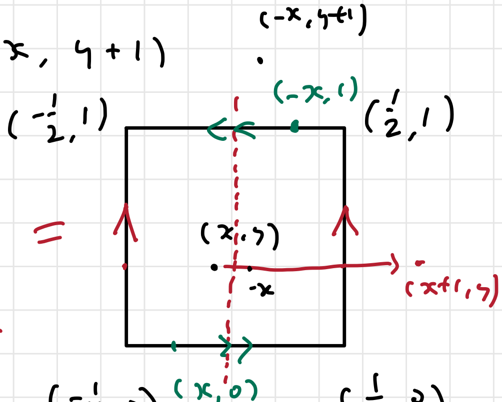
$$(m,n)(x,y) = (m+(-1)^n x,\ n+y)$$

$a=(1,0)$ $\quad$ $b=(0,1)$

$$a(x,y) = (x+1,\ y)$$

$$b(x,y) = (-x,\ y+1)$$

previous definition of X as adjunction space. $=$



5) [lens spaces]

$$\zeta = e^{\frac{2\pi i}{p}}$$

$p, q \in \mathbb{N}$ $\quad$ $\gcd(p,q)=1$

$$\mu_p = \left\{ e^{\frac{2\pi i \cdot k}{p}} \ \middle|\ k = 0, \dots, p-1 \right\}$$

$1, \zeta, \zeta^2, \dots, \zeta^{p-1}$

$$S^3 = \{ v \in \mathbb{R}^4 \mid \|v\|^2 = 1 \} \subset \mathbb{C}^2$$

$$\|v\|^2 = |z_1|^2 + |z_2|^2 = 1 \qquad z_1\ z_2$$

$$\zeta\cdot(z_1, z_2) = (\zeta z_1,\ \zeta^q z_2)$$

$$|\zeta z_1| = |z_1| \qquad |\zeta^q z_2| = |z_2|$$

μ_p acts freely.

$v = (z_1, z_2)$

$$G_v = \{ \zeta^k \mid \zeta^k z_1 = z_1,\ \zeta^{kq} z_2 = z_2 \} = \langle e \rangle$$

if $z_1 \neq 0$ $\quad$ $\zeta^k = 1$ $\quad$ $k/p \in \mathbb{Z}$ $\quad$ $p \mid k$

if $z_2 \neq 0$ $\quad$ $\zeta^{kq} = 1$ $\quad$ $p \mid kq \Rightarrow p \mid k$

$$L(p,q) = S^3 / {}_q\mu_p \xleftarrow{\pi} S^3$$

is compact connected. T2 (cos μ_p is finite)

if $p_1 \neq p_2$ $\quad L(p_1,q_1) \not\cong L(p_2,q_2)$

if $p_1 = p_2 = p$ $\quad L(p,q_1) \cong L(p,q_2) \iff q_1 = \pm q_2^{\pm 1}$ in $\mathbb{Z}/p\mathbb{Z}$

$q_1 = \pm q_2$ in $\mathbb{Z}/p\mathbb{Z}$

means $q_1 \equiv \pm q_2 \mod p$ $\quad$ $\gcd(q_2, p) = 1$

6) Homogeneous space

right action

$H \subset G$ top sub gp $\quad G \times H \to G$

$(g,h) \to gh$

orbit space G/H : homogeneous

if $H \subset G$ is closed subgp

then G/H is T2.

Pf. $\bar g_1, \bar g_2$ $\quad$ $g_1 H$ $\quad$ $g_2 H$

$\bar g_2 \in \bar g_1 H$ $\quad$ $\underline{\bar g_1^{-1} \bar g_2 \notin H}$ $\quad$ $\pi : G \to G/H$

$f : G \times G \to G$ cont.

$(g_1, g_2) \mapsto g_1^{-1} g_2$

$$f^{-1}H = \{(g_1,g_2) \mid g_1^{-1} g_2 \in H\} \subset G \times G$$

closed

$(\bar g_1, \bar g_2) \notin f^{-1}H$ $\quad \exists\ U_1, U_2$ open

$\bar g_1 \in U_1$, $\bar g_2 \in U_2$

$U_1 \times U_2 \cap f^{-1}H = \emptyset$

π is open

$\underbrace{\pi(U_1)}_{\ni\, \bar g_1 H} \cap \underbrace{\pi(U_2)}_{\ni\, \bar g_2 H} = \emptyset$

7) $G \times X \to X$ G is compact gp. X is $T2$

homogeneous space

$\forall x \in X$ $\phi_x : \boxed{G/G_x} \xrightarrow{\cong} Gx$ is a homeo

$g \mapsto gx$

pf. $G_x = \{ g \mid gx = x \}$ G/G_x is compact

$h \in G_x$ $g \cdot h \to g \cdot hx = gx$

ϕ_x is bij. surj ✓ $g_1 x = g_2 x$

$g_2^{-1} g_1 x = x$ $g_2^{-1} g_1 \in G_x$

inj ✓ $g_1 = g_2 \cdot (g_2^{-1} g_1)$

Cont bij from compact space to T_2 space is a homeo.

8) $M = Mat_{n \times n}(\mathbb{C})$ $G = GL_n \mathbb{C}$

$G \times M \to M$

$(g, A) \to gAg^{-1}$

Ex 1) $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \in M$

$G \cdot D = \{ gDg^{-1} \mid g \in G \} \subset M$

is closed.

$$g_t \cdot D g_t^{-1} = \frac{g_t}{\|g_t\|} \cdot D \left(\frac{g_t}{\|g_t\|} \right)^{-1}$$

$$\longrightarrow \bar{g}_\infty \cdot D g_\infty^{-1}$$

2) $M^{ss} = \{\text{diagonalizable } A\} \subset M$

$G \curvearrowright M^{ss}$ $\qquad A \in M^{ss}$, if

$gAg^{-1} = D \qquad hAh^{-1} = h(g^{-1}Dg)h^{-1}$

$A = g^{-1}Dg \qquad = (hg^{-1})D(hg^{-1})^{-1}$

$$\boxed{M^{ss}/G \cong \mathbb{C}^n/S_n}$$

S_n all permutations of $\{1, 2, \dots, n\}$

$\sigma \in S_n \qquad \sigma(i) \in \{1, \dots n\}$

$\sigma(z_1, \dots z_n) = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$

$$M^{ss} \xrightarrow{f} \underbrace{\mathbb{C}^n/S_n}_{} \quad T_2$$

$A \longrightarrow$ ~~unordered~~ ~unordered dupe $\{\lambda_1, \dots \lambda_n\}$

M^{ss}/G

$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \to GD$

two diagonalizable matrices are "similar" iff they have the same spectrum

$\underline{G}D = f^{-1}[D]$ is closed since $[D]$ is closed in M^{ss}

3) $N \subset M$ of nilpotent matrices

$N = \{A \mid A^d = 0\} = \text{Spec}(A) = \{0\}$

Show.

a) N is path conn.

b) G acts on N $\quad \underline{N/G \text{ is finite}}$

c) $N/G \subset M/G$ is the closure of $[0] \leftarrow$ orbit of 0.

$N \subset M$ is closed

9) $G = GL_n(\mathbb{R})$ B = invertible upper Δ matrices

B_- = . . . lower Δ - - -

a) B, B_- are subgp of G

G/B homogeneous space B acts from the right

$G \times B \to G$ $\quad$ $B_- \times G \longrightarrow G$

$(g, b) \mapsto g \cdot b$ $\quad$ $(b_-, g) \mapsto b_- \cdot g$

b) B_- acts on G/B from the left

$b_- gB = b_- g \cdot B$

c) show $B_- \backslash (G/B)$ is finite of cardinality $n!$

d) $B_- \cdot I_n \cdot B$ is an orbit in G/B ✓ is open.