Chapter 6. Heat Equations on the Whole Real Line

MATH4406 Introduction to Partial Differential Equations

The University of Hong Kong

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This chapter is related to the materials in Section 2.4 of the Textbook.

6.1 Symmetries for the Heat Equations

IVP for the Heat Equations on the Whole Real Line

Aim of This Chapter

Solve the following initial-value problem (IVP) for the heat equation:

$$\begin{cases} \partial_t u - k\partial_{xx} u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u\big|_{t=0} = \phi, \end{cases}$$

where $k > 0$ is a given constant, and $\phi := \phi(x)$ is a given initial data.

Question

How can we solve this IVP?

Answers

- Applying the method of Fourier transforms.
- Using symmetries (a.k.a. invariant transformations).

Comments on Two Methods

Comment on the Method of Fourier Transforms

We will NOT explain this method now because it only works for linear PDE, for which you can find a lot of information from classical textbooks or on the internet.

Comment on Symmetries/Invariances

The invariances are very important in understanding the PDE because

- they tell you something about the underlying physics; and
- they give you hints on building/constructing a solution.

Further Comment on Symmetries/Invariances

The invariant transformations may also provide us the information for conserved quantities; see Noether's theorem for instance.

Theorem (Invariances)

Let $u \in C^\infty$ be a solution to

$$\partial_t u - k\partial_{xx} u = 0, \tag{Heat}$$

then so is the v defined in each of the cases below.

i (Translation) *For any x_0, $t_0 \in \mathbb{R}$,*

$$v(t,x) := u(t - t_0, x - x_0).$$

ii (Differentiation) *For any α, $\beta = 0, 1, 2, \cdots$, $v := \partial_t^\alpha \partial_x^\beta u$.*

iii (Convolution) *For any $g : \mathbb{R} \to \mathbb{R}$,*

$$v(t,x) := (u * g)(t,x) := \int_{-\infty}^{\infty} u(t, x - y) g(y) \, dy.$$

iv (Dilation/Scaling) *For any $a > 0$,*

$$v(t,x) := u(at, \sqrt{a}x).$$

Proof of Part i and ii: Students' self-exercise.
Proof of Part iii: A direct differentiation yields

$$(\partial_t - k\partial_{xx})v = \int_{-\infty}^{\infty} (\partial_t - k\partial_{xx})\left\{u(t, x-y)g(y)\right\}\, dy$$
$$= \int_{-\infty}^{\infty} \underbrace{\{\partial_t u - k\partial_{xx} u\}(t, x-y)}_{=0}\, g(y)\, dy,$$

since $\partial_t u - k\partial_{xx} u = 0$.
Proof of Part iv: It follows from the chain rule that

$$\partial_t v(t,x) = \partial_t\left\{u(at, \sqrt{a}x)\right\} = a\partial_t u(at, \sqrt{a}x)$$

and

$$\partial_{xx} v(t,x) = \partial_{xx}\left\{u(at, \sqrt{a}x)\right\} = (\sqrt{a})^2\partial_{xx} u(at, \sqrt{a}x) = a\partial_{xx} u(at, \sqrt{a}x),$$

so

$$(\partial_t - k\partial_{xx})v = a\underbrace{(\partial_t u - k\partial_{xx} u)}_{=0} = 0,$$

since $\partial_t u - k\partial_{xx} u = 0$.

6.2 Self-Similar Solutions to the Heat Equations

Self-Similar Solutions to the Heat Equations

Aim of This Section

Let $k > 0$ be a given constant. We aim at finding a solution u to the heat equation

$$\partial_t u - k\partial_{xx} u = 0,$$

such that

$$u(t,x) = u(at, \sqrt{a}x), \quad \text{for all } a > 0. \tag{1}$$

Remark

The solution u that satisfies (1) is called a *self-similar solution*.

Question

Why should such a self-similar solution exist?

Answer

Part (iv) of the Theorem in Section 6.1 + Uniqueness.

Properties for Self-Similar Solutions

Self-Similar Functions Can be Seen as Functions of a Single Variable.

Let u satisfy the property

$$u(t,x) = u(at, \sqrt{a}x),$$

for all $a > 0$, then for any $t > 0$, we can choose $a := \dfrac{1}{4kt}$, so that

$$u(t,x) = u\left(\left(\frac{1}{4kt}\right)t, \sqrt{\frac{1}{4kt}}x\right) = u\left(\frac{1}{4k}, \frac{x}{\sqrt{4kt}}\right) =: g\left(\frac{x}{\sqrt{4kt}}\right),$$

where $g := g(p)$ is a function of a single variable.

Moral

PDE for self-similar solution $u \longleftrightarrow$ ODE for g.

ODE for g

Let $u(t,x) := g\left(\frac{x}{\sqrt{4kt}}\right)$ and $p := \frac{x}{\sqrt{4kt}}$. It follows from the chain rule that

$$\partial_t u(t,x) = \underbrace{\partial_t \left(\frac{x}{\sqrt{4kt}}\right)}_{=-\frac{1}{2}\frac{x}{\sqrt{4k}}t^{-\frac{3}{2}}} g'(p) = -\frac{1}{2t} p g'(p)$$

$$\partial_{xx} u(t,x) = \cdots (\textit{leave this to students}) \cdots = \frac{1}{4kt} g''(p).$$

Substituting the ansatz $u(t,x) := g\left(\frac{x}{\sqrt{4kt}}\right)$ into the heat equation

$$\partial_t u - k\partial_{xx} u = 0,$$

we have

$$-\frac{1}{4t}\left\{2pg'(p) + g''(p)\right\} = 0$$

$$2pg'(p) + g''(p) = 0.$$

Question

How to solve $g''(p) + 2pg'(p) = 0$?

Answer

Method of integrating factors.

Let $w(p) := g'(p)$. Then

$$
\begin{aligned}
w' + 2pw &= 0 \\
\left(e^{p^2} w\right)' = e^{p^2}\left(w' + 2pw\right) &= 0 \\
e^{p^2} w &= C_1 \\
w &= C_1 e^{-p^2} \\
g(p) &= C_1 \int_0^p e^{-\sigma^2}\, d\sigma + C_2,
\end{aligned}
$$

where C_1 and C_2 are arbitrary constants.

Conclusion

For any constants C_1 and C_2,

$$u(t,x) := g\left(\frac{x}{\sqrt{4kt}}\right) = C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\sigma^2}\, d\sigma + C_2$$

is a special solution to $\partial_t u - k\partial_{xx} u = 0$ for all $t > 0$ and $-\infty < x < \infty$.

Further Observations

- For $x > 0$,

$$\lim_{t\to 0^+} C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\sigma^2}\, d\sigma + C_2 = C_1 \int_0^{\infty} e^{-\sigma^2}\, d\sigma + C_2 = C_1 \frac{\sqrt{\pi}}{2} + C_2.$$

- For $x < 0$,

$$\lim_{t\to 0^+} C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\sigma^2}\, d\sigma + C_2 = C_1 \int_0^{-\infty} e^{-\sigma^2}\, d\sigma + C_2 = -C_1 \frac{\sqrt{\pi}}{2} + C_2.$$

Example

Question: Solve

$$\begin{cases} \partial_t Q - k\partial_{xx} Q = 0 \quad \text{for } -\infty < x < \infty \text{ and } t > 0, \\ Q(0,x) = H(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \end{cases}$$

where H is the Heaviside (step) function.

Solution: Since the IC $Q(0,x)$ is invariant under the scaling

$$x \mapsto \sqrt{a}x,$$

we believe that the solution to this problem should be self-similar, i.e.,

$$Q(t,x) := C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\sigma^2}\, d\sigma + C_2,$$

for some constants C_1 and C_2.

Example (Continued)

In order to determine the constants C_1 and C_2, we use the IC

$$H(x) = \lim_{t \to 0^+} Q(t,x) = \begin{cases} C_1 \dfrac{\sqrt{\pi}}{2} + C_2 & \text{if } x > 0 \\ -C_1 \dfrac{\sqrt{\pi}}{2} + C_2 & \text{if } x < 0 \end{cases}$$

(according to "Further Observations" just before this example), which implies

$$\begin{cases} C_1 \dfrac{\sqrt{\pi}}{2} + C_2 = 1 \\ -C_1 \dfrac{\sqrt{\pi}}{2} + C_2 = 0. \end{cases}$$

Solving the above system, we obtain $C_1 = \dfrac{1}{\sqrt{\pi}}$ and $C_2 = \dfrac{1}{2}$. Hence,

$$Q(t,x) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\sigma^2}\, d\sigma + \frac{1}{2}.$$

Question

Can we construct more solutions?

Example

Define

$$S(t,x) := \partial_x Q(t,x) = \partial_x \left\{ \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-\sigma^2}\, d\sigma + \frac{1}{2} \right\} = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}},$$

for any $t > 0$ and $-\infty < x < \infty$. Then S also satisfies *(because of part (ii) of Theorem (Invariances) in the last section)*

$$\partial_t S - k\partial_{xx} S = 0.$$

Question

What is the IC that S satisfies?

Example (Continued)

$$\lim_{t\to 0^+} S(t,x) = \lim_{t\to 0^+} \partial_x Q(t,x) = \lim_{t\to 0^+} \partial_x H(x) = \delta_0(x),$$

where the first equality follows from the definition of S, the second identity follows from the IC for Q, and the δ_0 is the Dirac delta function.

Question

What is δ_0?

Answer

The Dirac delta function δ_0 is the “generalized” function such that for any smooth function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{-\infty}^{\infty} \delta_0(y) f(y)\, dy := f(0).$$

Remarks on Dirac Delta Function δ_0

1. No (classical) function can fulfill $\int_{-\infty}^{\infty} \delta_0(y) f(y)\, dy := f(0)$, for all smooth function f.
2. You may "think" (but it is NOT true):

$$\delta_0(x) := \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

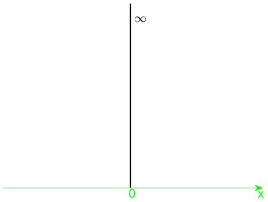
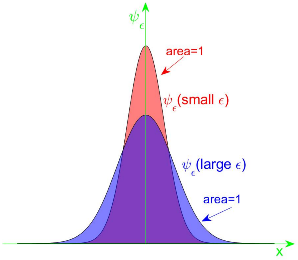


Figure: "Graph" of δ_0, i.e., density function of a unit mass at $x = 0$.

More precisely, passing to the limit as $\epsilon \to 0^+$ of



General Solution

Example

Given any smooth function $\phi := \phi(x)$. For any $t > 0$ and $-\infty < x < \infty$, we define

$$u(t,x) := (S * \phi)(t,x) := \int_{-\infty}^{\infty} S(t, x-y)\phi(y)\, dy$$
$$= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy,$$

since $S(t,x) := \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$. Then according to part (iii) of Theorem (Invariances) in the last section, u is also a solution to

$$\partial_t u - k\partial_{xx} u = 0.$$

Question

What is the IC that u satisfies?

Example (Continued)

$$\lim_{t\to 0^+} u(t,x) := \lim_{t\to 0^+} \int_{-\infty}^{\infty} S(t,x-y)\phi(y)\,dy$$
$$= \int_{-\infty}^{\infty} \lim_{t\to 0^+} S(t,x-y)\phi(y)\,dy$$
$$= \int_{-\infty}^{\infty} \delta_0(x-y)\phi(y)\,dy = \phi(x).$$

General Solution Formula to the Heat Equation on $\mathbb{R}$

The function

$$u(t,x) := \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\,dy$$

is a solution to

$$\begin{cases} \partial_t u - k\partial_{xx} u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u\big|_{t=0} = \phi. \end{cases}$$

In the literature, the heat kernel $S(t, x-y) := \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}}$ has many different names, such as

- source function,
- Gaussian,
- Green's function,
- propagator,
- fundamental solution, etc.

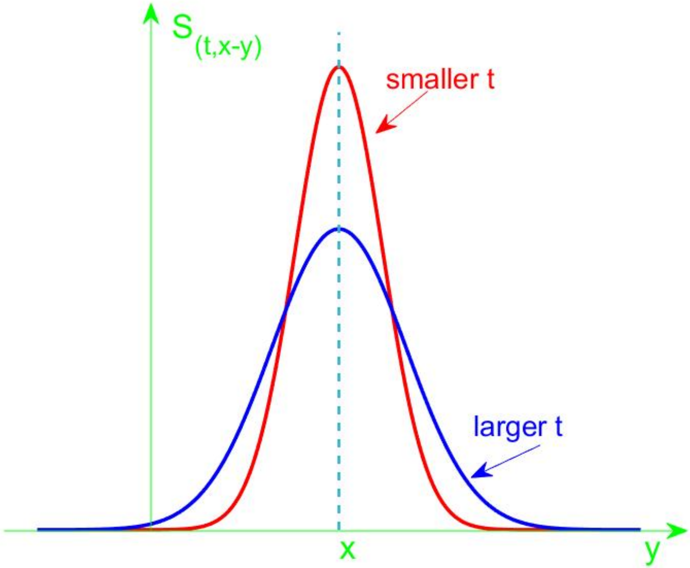
Remark

The heat equation is highly related to the Brownian motion. The Green's function for the heat equation looks like the probability density of the normal distribution is not an accident.

In-Class Discussion

What is the relationship between the heat equation and the Brownian motion?

Heat Kernel For Positive Time t>0



6.3 Properties and Examples for Solutions to the Heat Equations

Solution to the Heat Equation on the Whole Real Line

Consider the Cauchy problem

$$\begin{cases} \partial_t u - k\partial_{xx} u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u\big|_{t=0} = \phi, \end{cases}$$

where $k > 0$ is a given constant, and ϕ is a given initial data. We have already shown that

General Solution Formula to the Heat Equation on $\mathbb{R}$

The function

$$u(t,x) := \int_{-\infty}^{\infty} S(t, x-y)\phi(y)\, dy$$

is a solution to the above Cauchy problem, where the heat kernel

$$S(t, x-y) := \frac{1}{\sqrt{4k\pi t}} e^{-\frac{(x-y)^2}{4kt}}.$$

Properties

For any $-\infty < x < \infty$ and $t > 0$, the solution

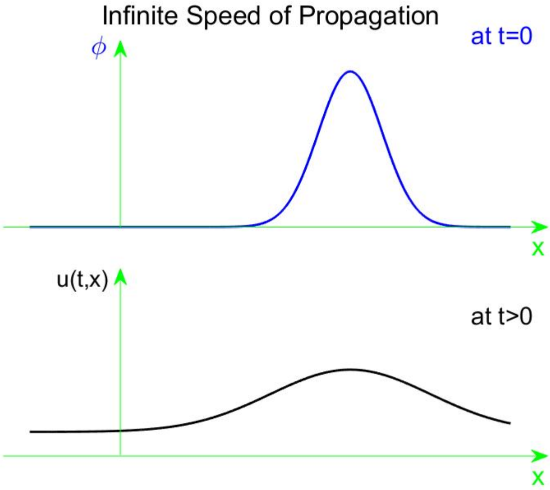
$$u(t,x) := \int_{-\infty}^{\infty} S(t, x-y)\phi(y)\, dy = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy.$$

has the following properties:

1. **Positivity:** If $\phi \geq 0$, then $u \geq 0$ because $S(t, x-y) \geq 0$.
2. **Decay at Infinity**: If $\int_{-\infty}^{\infty} |\phi(y)|\, dy < \infty$, then

$$\begin{aligned} |u(t,x)| &\leq \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} |\phi(y)|\, dy \\ &\leq \frac{1}{\sqrt{4k\pi t}} \underbrace{\left(\max_{-\infty<y<\infty} e^{-\frac{(x-y)^2}{4kt}} \right)}_{=1} \int_{-\infty}^{\infty} |\phi(y)|\, dy \\ &= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} |\phi(y)|\, dy \to 0. \end{aligned}$$

3 **Infinite Speed of Propagation:**



Moral

Immediate Effect: for any $t > 0$, $u(t, x) > 0$.

4 **Smoothing/Regularizing Effect:** For any $t > 0$,

ALL the partial derivatives $\partial_t u$, $\partial_x u$, $\partial_{tt} u$, $\partial_{tx} u$, $\partial_{xx} u$, $\cdots$ exist

because

$$\partial_t^\alpha \partial_x^\beta u(t,x) = \int_{-\infty}^{\infty} \left(\partial_t^\alpha \partial_x^\beta S(t, x-y) \right) \phi(y)\, dy,$$

and the heat kernel $S(t, x-y)$ is smooth.

Moral

We have an explicit solution formula for the heat equation on the whole real line, so we can study the solutions (as well as their properties) easily.

Now, we are going to provide some examples on how we use the explicit solution formula.

Example

Question: Solve $\partial_t u - \frac{1}{4}\partial_{xx} u = 0$ and $u|_{t=0} = e^{2x}$.

Solution: It follows from the explicit solution formula that

$$u(t,x) = \int_{-\infty}^{\infty} S(t, x-y)\phi(y)\, dy = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{t}} e^{2y}\, dy.$$

Since $-\frac{(x-y)^2}{t} + 2y = 2x + t - \frac{(y-t-x)^2}{t}$, we have

$$u(t,x) = \frac{1}{\sqrt{\pi t}} e^{2x+t} \int_{-\infty}^{\infty} e^{-\frac{(y-t-x)^2}{t}}\, dy = e^{2x+t}$$

because $(p := (y-t-x)/\sqrt{t})$

$$\int_{-\infty}^{\infty} e^{-\frac{(y-t-x)^2}{t}}\, dy = \sqrt{t} \int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi t}.$$

Further Discussions

In-Class Discussion

Do you know why

$$\frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \, dy \equiv 1?$$

Exercise

Solve

$$\begin{cases} \partial_t u - k\partial_{xx} u = 0 & \text{for } x \in \mathbb{R} \text{ and } t > 0 \\ u|_{t=0} = x^4, \end{cases}$$

where $k > 0$ is a given constant.
(Answer: $u(t,x) = x^4 + 12kx^2 t + 12k^2 t^2$.)