THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 5 Solution

Problem 1.

(i) By assumption,

$$a(t,x)\partial_t u + b(t,x)\partial_x u - k(t,x)\partial_{xx} u < 0 \text{ on } \bar{\Omega} \setminus \Gamma \text{ with } a,k \geq 0.$$

As u is continuous over $\overline{\Omega}$, it follows from the extreme value theorem that the maximum exists on $\overline{\Omega}$, say $u(t_0,x_0) = \max_{\bar{\Omega}} u$. Assume on the contrary that $(t_0,x_0) \in \bar{\Omega} \setminus \Gamma$.

If $\boxed{(t_0,x_0) \in (0,T) \times (0,L)}$, then $\partial_x u(t_0,x_0) = \partial_t u(t_0,x_0) = 0$ and $\partial_{xx} u(t_0,x_0) \leq 0$. Thus

$$a(t_0,x_0)\partial_t u + b(t_0,x_0)\partial_x u - k(t_0,x_0)\partial_{xx} u = -k(t_0,x_0)\partial_{xx} u \geq 0,$$

which give a contradiction.

If $\boxed{t_0 = T}$ and $\boxed{x_0 \in (0,L)}$, then $\partial_x u(t_0,x_0) = 0$, $\partial_t u(t_0,x_0) \geq 0$ and $\partial_{xx} u(t_0,x_0) \leq 0$. Thus

$$a(t_0,x_0)\partial_t u + b(t_0,x_0)\partial_x u - k(t_0,x_0)\partial_{xx} u = a(t_0,x_0)\partial_t u - k(t_0,x_0)\partial_{xx} u \geq 0,$$

which also give a contradiction.

Therefore, $(t_0,x_0) \in \Gamma$ and hence $\max_{\bar{\Omega}} u = u(t_0,x_0) = \max_{\Gamma} u$.

(ii) Let $w := u - v$. Then

$$a(t,x)\partial_t w + b(t,x)\partial_x w - k(t,x)\partial_{xx} w \leq f < 0 \text{ on } \bar{\Omega} \setminus \Gamma \text{ with } a,k \geq 0,$$

and $w \leq 0$ on Γ. By (i), for any $(t, x) \in \bar{\Omega}$,

$$u(t,x) - v(t,x) = w(t,x) \leq \max_{\bar{\Omega}} w = \max_{\Gamma} w \leq 0$$

and hence $v \geq u$ in $\bar{\Omega}$.

(iii) By assumption,

$$a(t,x)\partial_t u - k(t,x)\partial_{xx} u \leq 0 \text{ on } \bar{\Omega} \setminus \Gamma \text{ with } a, k \geq 0.$$

Given any $\epsilon > 0$, we define, for any $(t, x) \in \bar{\Omega}$,

$$v_\epsilon(t,\, x) := u(t,\, x) + \epsilon x^2.$$

Then

$$a(t,x)\partial_t v_\epsilon - k(t,x)\partial_{xx} v_\epsilon = (a(t,x)\partial_t u - k(t,x)\partial_{xx} u) - 2\epsilon k(t,x) < 0.$$

By (i),

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}}(u + \epsilon x^2) = \max_{\bar{\Omega}} v_\epsilon = \max_{\Gamma} v_\epsilon = \max_{\Gamma}(u + \epsilon x^2) \leq (\max_{\Gamma} u) + \epsilon L^2$$

and hence taking $\epsilon \to 0^+$, we get

$$\max_{\bar{\Omega}} u \leq \max_{\Gamma} u.$$

On the other hand, as $\Gamma \subset \bar{\Omega}$, $\max_{\Gamma} u \leq \max_{\bar{\Omega}} u$. Thus, $\max_{\bar{\Omega}} u = \max_{\Gamma} u$.

(iv) By assumption, we have

$$\partial_t u + 4\partial_x u - (t^2 + 1)\partial_{xx} u = 1 - e^{2x} < 0 \text{ on } \bar{\Omega} \setminus \Gamma$$

and hence by (i),

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u.$$

Problem 2.

(i) By assumption,

$$-\sum_{i=1}^{d} a_i(x)\partial_{x_i}^2 u + \sum_{i=2}^{d} b_i(x)\partial_{x_i} u < 0 \text{ on } \Omega \text{ with } a_i \geq 0.$$

As u is continuous over $\overline{\Omega}$, it follows from the extreme value theorem that the maximum exists on $\overline{\Omega}$, say $u(\tilde{x}) = \max_{\bar{\Omega}} u$. Now we will show that $\tilde{x} \notin \Omega$. Assume on the contrary that $\tilde{x} \in \Omega$. Then then $\partial_{x_i} u(\tilde{x}) = 0$ and $\partial_{x_i}^2 u(\tilde{x}) \leq 0$. Thus

$$-\sum_{i=1}^{d} a_i(x)\partial_{x_i}^2 u + \sum_{i=2}^{d} b_i(x)\partial_{x_i} u = -\sum_{i=1}^{d} a_i(x)\partial_{x_i}^2 u \geq 0,$$

which gives a contradiction.

Therefore, $\tilde{x} \in \partial\Omega$ and hence $\max_{\bar{\Omega}} u = u(\tilde{x}) = \max_{\partial\Omega} u$.

(ii) By assumption,

$$-a_1(x)\partial_{x_1}^2 u - \sum_{i=2}^{d} a_i(x)\partial_{x_i}^2 u + \sum_{i=2}^{d} b_i(x)\partial_{x_i} u \leq 0 \text{ on } \Omega \text{ with } a_i \geq 0\ (2 \leq i \leq d), a_1 > 0.$$

Given any $\epsilon > 0$, we define, for any $x \in \bar{\Omega}$,

$$v_\epsilon(x) := u(x) + \epsilon x_1^2.$$

Then

$$\begin{aligned} & -a_1(x)\partial_{x_1}^2 v_\epsilon - \sum_{i=2}^{d} a_i(x)\partial_{x_i}^2 v_\epsilon + \sum_{i=2}^{d} b_i(x)\partial_{x_i} v_\epsilon \\ = \quad & -a_1(x)\partial_{x_1}^2 u - \sum_{i=2}^{d} a_i(x)\partial_{x_i}^2 u + \sum_{i=2}^{d} b_i(x)\partial_{x_i} u - 2\epsilon\, a_1(x) < 0 \end{aligned}$$

By (i),

$$\max_{\bar{\Omega}} u \leq \max_{\bar{\Omega}} (u + \epsilon x_1^2) = \max_{\bar{\Omega}} v_\epsilon = \max_{\partial\Omega} v_\epsilon = \max_{\partial\Omega} (u + \epsilon x_1^2) \leq (\max_{\partial\Omega} u) + \epsilon L^2,$$

where the positive constant $L := \max\{|x_1| : x = (x_1, \cdots, x_d) \in \Omega\} < \infty$. Hence taking $\epsilon \to 0^+$, we get

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u.$$

On the other hand, as $\partial\Omega \subset \bar{\Omega}$, $\max_{\bar{\Omega}} u \geq \max_{\partial\Omega} u$. Thus, $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

(iii) By assumption,

$$-a_1(x)\partial_{x_1}^2 u - \sum_{i=2}^{d} a_i(x)\partial_{x_i}^2 u + \sum_{i=2}^{d} b_i(x)\partial_{x_i} u \equiv 0 \text{ on } \Omega \text{ with } a_i \geq 0 \ (2 \leq i \leq d), a_1 > 0.$$

By (ii), we obtain $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. On the other hand, as

$$-a_1(x)\partial_{x_1}^2(-u) - \sum_{i=2}^{d} a_i(x)\partial_{x_i}^2(-u) + \sum_{i=2}^{d} b_i(x)\partial_{x_i}(-u) \equiv 0 \text{ on } \Omega,$$

it follows from (ii) that

$$\min_{\bar{\Omega}} u = -\max_{\bar{\Omega}}(-u) = -\max_{\partial\Omega}(-u) = \min_{\partial\Omega} u.$$

Thus,

$$\max_{\bar{\Omega}} |u| = \max(|\max_{\bar{\Omega}} u|, |\min_{\bar{\Omega}} u|) = \max(|\max_{\partial\Omega} u|, |\min_{\partial\Omega} u|) = \max_{\partial\Omega} |u|.$$

Problem 3.

(i) Let $\Omega = (0, T) \times (0, L)$ and $\Gamma := \{(t, x) \in \Omega;\ t = 0 \text{ or } x = 0 \text{ or } L\}$. As u is continuous over $\overline{\Omega}$, it follows from the extreme value theorem that the minimum exists on $\overline{\Omega}$, say $u(t_0, x_0) = \min_{\bar{\Omega}} u$. Now we will show that $(t_0, x_0) \notin \overline{\Omega} \setminus \Gamma$. Assume on the contrary that $(t_0, x_0) \in \bar{\Omega} \setminus \Gamma$.

If $\boxed{(t_0, x_0) \in (0, T) \times (0, L)}$, then $\partial_x u(t_0, x_0) = \partial_t u(t_0, x_0) = 0$ and $\partial_{xx} u(t_0, x_0) \geq 0$. Thus

$$LHS = -\partial_{xx} u(t_0, x_0) \leq 0 < 1 = RHS$$

which give a contradiction.

If $\boxed{t_0 = T}$ and $\boxed{x_0 \in (0, L)}$, then $\partial_x u(t_0, x_0) = 0$, $\partial_t u(t_0, x_0) \le 0$ and $\partial_{xx} u(t_0, x_0) \ge 0$. Thus

$$LHS = \partial_t u(t_0, x_0) - \partial_{xx} u(t_0, x_0) \le 0 < 1 = RHS$$

which also give a contradiction.

Therefore, $(t_0, x_0) \in \Gamma$ and hence $\min_{\bar{\Omega}} u = u(t_0, x_0) = \min_{\Gamma} u$.

(ii) Step 1: Show that if $\partial_{xx} v + 2\partial_{yy} v - v^4 \partial_y v > 0$, then $\max_{\bar{D}} v = \max_{\partial D} v$.

As v is continuous over $\overline{D}$, it follows from the extreme value theorem that the maximum exists on $\overline{D}$, say

$$v(x_0, y_0) = \max_{\bar{D}} v.$$

Assume on the contrary that $(x_0, y_0) \in D$. Then $\partial_y v(x_0, y_0) = 0$ and $\partial_{xx} v(x_0, y_0)$, $\partial_{yy} v(x_0, y_0) \le 0$. Thus

$$\partial_{xx} v(x_0, y_0) + 2\partial_{yy} v(x_0, y_0) - u^4 \partial_y v(x_0, y_0) = \partial_{xx} v(x_0, y_0) + 2\partial_{yy} v(x_0, y_0) \le 0,$$

which give a contradiction.

Step 2: Show that if $\partial_{xx} u + 2\partial_{yy} u - u^4 \partial_y u \ge 0$, then $\max_{\bar{D}} u = \max_{\partial D} u$.

Given any $\epsilon > 0$, we define, for any $(x, y) \in \bar{D}$,

$$v_\epsilon(x,\, y) := u(x,\, y) + \epsilon x^2.$$

Then

$$\partial_{xx} v_\epsilon + 2\partial_{yy} v_\epsilon - u^4 \partial_y v_\epsilon = \partial_{xx} u + 2\partial_{yy} u + 2\epsilon - u^4 \partial_y u > 0.$$

By Step 1,

$$\max_{\bar{D}} u \le \max_{\bar{D}} (u + \epsilon x^2) = \max_{\bar{D}} v_\epsilon = \max_{\partial D} v_\epsilon = \max_{\partial D} (u + \epsilon x^2) \le \max_{\partial D} u + \epsilon$$

and hence taking $\epsilon \to 0^+$, we get

$$\max_{\bar{D}} u \le \max_{\partial D} u.$$

On the other hand, as $\partial D \subset \bar{D}$, $\max_{\bar{D}} u \ge \max_{\partial D} u$. Thus, $\max_{\bar{D}} u = \max_{\partial D} u$.

Step 3: Show that if $\partial_{xx} u + 2\partial_{yy} u - u^4 \partial_y u \equiv 0$, then $\max_{\bar{D}} |u| = \max_{\partial D} |u|$.

By Step 2, we obtain $\max_{\bar{D}} u = \max_{\partial D} u$. On the other hand, as

$$\partial_{xx}(-u) + 2\partial_{yy}(-u) - (-u)^4 \partial_y(-u) \equiv 0 \text{ on } D,$$

it follows from Step 2 that

$$\min_{\bar{D}} u = -\max_{\bar{D}}(-u) = -\max_{\partial D}(-u) = \min_{\partial D} u.$$

Thus,

$$\max_{\bar{D}} |u| = \max(|\max_{\bar{D}} u|, |\min_{\bar{D}} u|) = \max(|\max_{\partial D} u|, |\min_{\partial D} u|) = \max_{\partial D} |u|.$$

Problem 4. Let $v := u_1 - u_2$. Then v satisfies the Laplace equation

$$\Delta u := \partial_{xx} u + \partial_{yy} u = 0 \qquad \text{on } \Omega = [-1, 1] \times [-1, 1]$$

subject to the boundary conditions:

$$\begin{cases} v|_{x=-1} = g_1 - g_2 \le 0 \\ v|_{x=1} = h_1 - h_2 \le 0 \\ v|_{y=-1} = \phi_1 - \phi_2 \le 0 \\ v|_{y=1} = \psi_1 - \psi_2 \le 0. \end{cases}$$

Then it follows from the maximum principle that for all $(x, y) \in \bar{\Omega}$,

$$u_1(x, y) - u_2(x, y) = v(x, y) \le \max_{\bar{\Omega}} v = \max_{\partial \Omega} v \le 0,$$

and hence $u_1 \le u_2$ on $\bar{\Omega}$.