Chapter 2. First-Order Equations and Method of Characteristics

MATH4406 Introduction to Partial Differential Equations

The University of Hong Kong

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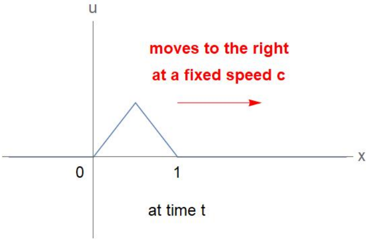
This chapter is related to the materials in Section 1.2, 14.1 of the Textbook.

2.1 Transport Equation

Transport Equation

In Chapter 1 we have already seen that the physical phenomenon



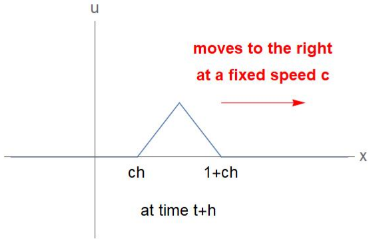


Figure: Transportation to the Right at a Fixed Speed c.

can be described by the *transport equation*

$$\partial_t u + c\partial_x u = 0, \tag{Trans}$$

where the propagation speed c is a given constant.

How to Solve the Transport Equation?

Question

How to solve the transport equation?

Two Main Methods

1. Coordinate Method,
   - Main Idea – Under some coordinate frame(s), the underlying equation becomes simpler, so that we are able to solve via elementary methods, such as direct integrations, etc.
   - Advantage – The computation in this method is simple.
   - Disadvantage – This method is hard to be extended to the variable coefficient case.
2. Method of Characteristics (also called "Geometric Method" in the Textbook).
   - Main Idea – On certain curves (that we have to find out), the PDE behaves like an ODE.
   - Advantage – This (geometric) method can be extended to the variable coefficient case.

Coordinate Method (Special Case: $c = 0$)

Example (Special Case: $c = 0$)

In this case the equation (Trans) becomes

$$\partial_t u = 0.$$

A direct integration (with respect to t) yields

$$u(t, x) = f(x)$$

for some arbitrary function f. In addition, if the IC $u|_{t=0} = g(x)$ is given, then $f \equiv g$, and hence,

$$u(t, x) = g(x).$$

Moral

If there is only one partial derivative (instead of two different partial derivatives), then a direct integration will solve the problem.

Coordinate Method (WLOG, we assume $c \neq 0$)

Coordinate Method

Let $\xi := x + ct$ and $\eta := x - ct$. Then

$$x = \frac{1}{2}\xi + \frac{1}{2}\eta \quad \text{and} \quad t = \frac{1}{2c}\xi - \frac{1}{2c}\eta.$$

Then viewing u as a function of (ξ, η), namely $u := u(\xi, \eta)$, we have

$$\partial_\xi u = \underbrace{\partial_\xi t}_{=\frac{1}{2c}} \partial_t u + \underbrace{\partial_\xi x}_{=\frac{1}{2}} \partial_x u = \frac{1}{2c} \underbrace{(\partial_t u + c\partial_x u)}_{=0} = 0,$$

where we applied the chain rule in the first equality, and the transport equation $\partial_t u + c\partial_x u = 0$ in the last equality. A direct integration yields

$$u = f(\eta) = f(x - ct),$$

for some arbitrary function f.

Density along a Given Curve

For any given function $\gamma := \gamma(s)$, we define

$$v(s) := u(s, \gamma(s)).$$

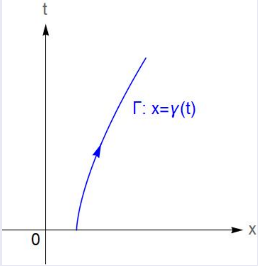


Figure: The Curve $x = \gamma(t)$

Physical Meaning

This $v(s)$ represents the density of the underlying quantity at the position $\gamma(s)$ at a particular time s.

It follows from the chain rule that

$$\begin{aligned}\frac{d}{ds}v = \frac{d}{ds}u(s, \gamma(s)) &= (\partial_s s)\partial_t u + (\partial_s \gamma(s))\partial_x u \\ &= \partial_t u + \gamma'(s)\partial_x u.\end{aligned}$$

Main Idea in the Method of Characteristics

Observation

Comparing the chain rule

$$\partial_t u(s,\gamma(s)) + \gamma'(s)\partial_x u(s,\gamma(s)) = \frac{d}{ds} v$$

and the transport equation at $(t,x) = (s,\gamma(s))$:

$$\partial_t u(s,\gamma(s)) + c\partial_x u(s,\gamma(s)) = 0,$$

we find that if $\gamma'(s) = c$, then the LHS of both equations are the SAME, and hence,

$$\begin{aligned}\frac{d}{ds}(v(s)) &= \partial_t u(s,\gamma(s)) + \gamma'(s)\partial_x u(s,\gamma(s)) \\ &= \partial_t u(s,\gamma(s)) + c\partial_x u(s,\gamma(s)) = 0.\end{aligned}$$

System of ODE

In other words, we can solve the initial-value problem

$$\begin{cases} \partial_t u + c\partial_x u = 0, & \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ u(0,x) = g(x), & \text{for all } x \in \mathbb{R}, \end{cases}$$

by solving the following initial value problem of ODE:

Equations for Characteristics

For any $x_0 \in \mathbb{R}$, we consider

$$\begin{cases} \frac{d}{ds}\gamma = c, & \gamma(0) = x_0, \\ \frac{d}{ds}v = 0, & v(0) = g(x_0). \end{cases} \tag{CharTE}$$

Here, $v(t) := u(t, \gamma(t))$.

Solving (CharTE), we obtain

$$\boxed{\gamma(s) = x_0 + cs, \quad \text{and} \quad v(s) = g(x_0).}$$

How to find $u(t,x)$?

Solving $(t,x) = (s,\gamma(s)) = (s, x_0 + cs)$, we obtain

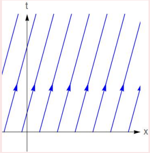
$$s = t \quad \text{and} \quad x_0 = x - ct,$$

and hence,

$$u(t,x) = v(t) = g\left(x - ct\right).$$

Moral

The unknown u is constant along the characteristic curve $x - ct =$ constant.



2.2 Coordinate Method

Coordinate Method

Example (Constant Coefficients)

Question: Solve
$$a\partial_x u + b\partial_y u = u,$$
where a and b are real numbers and $a^2 + b^2 \neq 0$.
Solution: Let
$$\boxed{\begin{aligned} \xi &:= ax + by \\ \eta &:= bx - ay. \end{aligned}}$$
Then
$$\partial_x u = a\partial_\xi u + b\partial_\eta u \quad \text{and} \quad \partial_y u = b\partial_\xi u - a\partial_\eta u,$$
and hence,
$$\begin{aligned} u = a\partial_x u + b\partial_y u &= a(a\partial_\xi u + b\partial_\eta u) + b(b\partial_\xi u - a\partial_\eta u) \\ &= (a^2 + b^2)\partial_\xi u. \end{aligned}$$

Example (Continued)

Applying the method of integrating factors to $(a^2 + b^2)\partial_\xi u = u$, we obtain (via integrating with respect to ξ)

$$u = f(\eta)e^{\frac{1}{a^2+b^2}\xi},$$

for any arbitrary function f. Equivalently, in terms of x and y, we have

$$u = f(bx - ay)e^{\left(\frac{1}{a^2+b^2}\right)(ax+by)}.$$

Comments

1. For beginners, the change of variables is <u>NOT</u> easy to guess.
2. This method CANNOT be extended to equations with non-constant/variable coefficients <u>**EASILY**</u>.
3. This method is purely algebraic.

2.3 Method of Characteristics

Geometric Method

Consider

$$a\partial_x u + b\partial_y u = 0, \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{1}$$

where a and b are two given real numbers.

Geometric Observation

Let $v := (a, b)$. Then using $\nabla u := (\partial_x u, \partial_y u)$, we have

$$\frac{\partial u}{\partial v} = v \cdot \nabla u = (a, b) \cdot (\partial_x u, \partial_y u) = a\partial_x u + b\partial_y u = 0,$$

where the last equality follows from (1). This means that

The value of u is constant along straight lines that are parallel to the direction v.

Remark: The equation for straight lines that are parallel to v is

$$bx - ay = \text{constant}.$$

By the *Geometric Observation* that mentioned on the last slide,

$$u(x, y) = f(bx - ay),$$

for some arbitrary function f.

In particular, for the points (x_1, y_1) and (x_2, y_2) in the figure on the RHS, we have

$$u(x_1, y_1) = u(x_2, y_2).$$

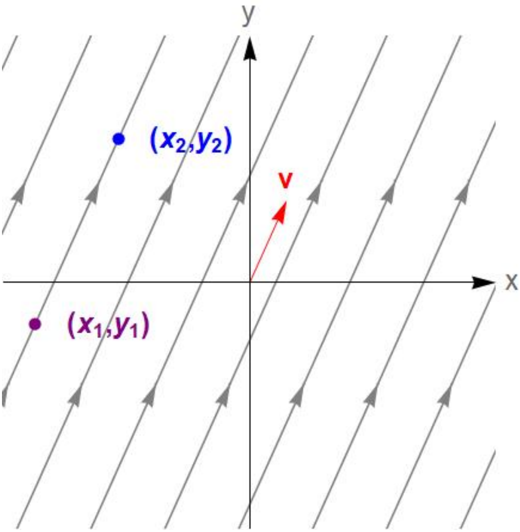


Figure: Characteristics for $a\partial_x u + b\partial_y u = 0$.

Further Discussion on the Geometric Method

Main Idea of Geometric Method

On certain curves (that we have to choose appropriately), the PDE behaves like an ODE.

Moral

The geometric method can be extended to equations with variable coefficients.

Remark

In the case of variable coefficients, we will see that the characteristic curves may no longer be straight lines.

Moral

In general, people usually use the geometric method (i.e., method of characteristics) to solve first-order equations.

First-Order Equations with Variable Coefficients

Consider

$$a(x,y)\partial_x u + b(x,y)\partial_y u = f(u,x,y), \qquad \text{(1stEqt)}$$

where a, b and f are given real-valued functions.

Main Idea of Method of Characteristics

Find a curve $\Gamma(s) := (X(s), Y(s))$ such that

$$\begin{cases} \dfrac{dX}{ds} = a(X,Y) \\ \dfrac{dY}{ds} = b(X,Y). \end{cases}$$

Then

$$\begin{aligned} \frac{d}{ds}u(X(s),Y(s)) &= \frac{dX}{ds}\partial_x u(X,Y) + \frac{dY}{ds}\partial_y u(X,Y) \\ &= a(X,Y)\partial_x u(X,Y) + b(X,Y)\partial_y u(X,Y) = f(u(X,Y),X,Y). \end{aligned}$$

Algorithm for Solving $a(x,y)\partial_x u + b(x,y)\partial_y u = f(u,x,y)$

1. Solve for $X(s)$ and $Y(s)$ in

$$\begin{cases} \dfrac{dX}{ds} = a(X,Y) \\ \dfrac{dY}{ds} = b(X,Y). \end{cases}$$

2. Solve for $W(s)$ in

$$\frac{dW}{ds} = f(W,X,Y).$$

3. Try to find u from the relationship

$$W(s) = u(X(s),Y(s)).$$

Let us see some examples as follows.

Examples of Using Method of Characteristics

Example

Question: Solve

$$x\partial_x u + y\partial_y u = 0, \quad \text{for } x > 0.$$

Solution: The underlying system is of the form (1stEqt) with $a(x, y) := x$, $b(x, y) := y$ and $f(u, x, y) \equiv 0$. Thus, we consider the characteristic curve $\Gamma(s) := (X(s), Y(s))$ that satisfies

$$\begin{cases} \dfrac{dX}{ds} = X, & X(0) = x_0, \\ \dfrac{dY}{ds} = Y, & Y(0) = y_0. \end{cases}$$

Solving the above system, we obtain

$$\boxed{X(s) = x_0 e^s \quad \text{and} \quad Y(s) = y_0 e^s.}$$

Example (Continued)

Furthermore, it follows from the underlying equation that $W(s) := u(X(s), Y(s))$ satisfies

$$\frac{dW}{ds} \equiv 0 \quad \text{and} \quad W(0) = u(x_0, y_0),$$

so for any $s \in \mathbb{R}$,

$$\boxed{u(X(s), Y(s)) =: W(s) \equiv W(0) = u(x_0, y_0).}$$

Combining the above boxed results, we obtain

$$\boxed{u\left(x_0 e^s, y_0 e^s\right) = u(x_0, y_0).}$$

Technical Question

How can we express $u(x, y)$ in terms of x and y?

Example (Continued)

Now, set

$$x := x_0 e^s \quad \text{and} \quad y := y_0 e^s.$$

Then in order to eliminate s, we consider their quotient:

$$\frac{y}{x} = \frac{y_0}{x_0},$$

or equivalently,

$$y_0 = \frac{x_0 y}{x}.$$

Finally, we have

$$u(x, y) = u(x_0, y_0) = u\left(x_0, \frac{x_0 y}{x}\right).$$

In-Class Discussion

Are we able to eliminate x_0 as well?

Example (Continued)

Indeed, we have the freedom to choose x_0, so without loss of generality, let us choose $x_0 := 1$. Then

$$u(x, y) = u\left(x_0, \frac{x_0 y}{x}\right) = u\left(1, \frac{y}{x}\right) =: f\left(\frac{y}{x}\right).$$

Here, f is any arbitrary function that can be determined by additional given condition(s).

Remark

For example, a condition like

$$u|_{x=1} = h(y)$$

for some given h will be sufficient to completely determine the arbitrary function f, since the straight line $x = 1$ cuts all characteristics. The situation is depicted on the next slide.

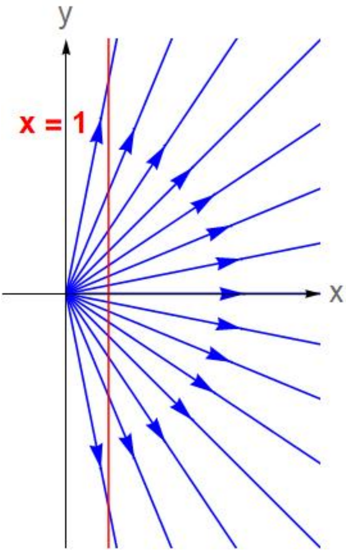


Figure: Characteristics for $x\partial_x u + y\partial_y u = 0$.

Remark

The last example can also be solved by using the coordinate method.

Exercise

Solve the last example by using the following change of variables:

$$r := \sqrt{x^2 + y^2},$$
$$\theta := \tan^{-1}\left(\frac{y}{x}\right).$$

Example

Question: Solve

$$\begin{cases} 2\partial_x u + xy\partial_y u = u, & \text{for } (x,y) \in \mathbb{R}^2 \\ u|_{x=0} = h(y), \end{cases}$$

where h is a given function.

Solution: The underlying system is of the form (1stEqt) with $a(x,y) \equiv 2$, $b(x,y) := xy$ and $f(u,x,y) := u$. Thus, we consider the characteristic curve $\Gamma(s) := (X(s), Y(s))$ that satisfies

$$\begin{cases} \dfrac{dX}{ds} = a(X,Y) = 2, & X(0) = x_0, \\ \dfrac{dY}{ds} = b(X,Y) = XY, & Y(0) = y_0. \end{cases}$$

Solving the above system (that is partially decoupled), we obtain

$$\boxed{X(s) = x_0 + 2s \quad \text{and} \quad Y(s) = y_0 e^{x_0 s + s^2}.}$$

Example (Continued)

Furthermore, it follows from the underlying equation that $W(s) := u(X(s), Y(s))$ satisfies

$$\frac{dW}{ds} = f(W(s), X(s), Y(s)) = W(s) \quad \text{and} \quad W(0) = u(x_0, y_0),$$

so for any $s \in \mathbb{R}$,

$$\boxed{u(X(s), Y(s)) =: W(s) = W(0)e^s = u(x_0, y_0)e^s.}$$

Combining the above boxed results, we obtain

$$\boxed{u\left(x_0 + 2s, y_0 e^{x_0 s + s^2}\right) = u(x_0, y_0)e^s.}$$

Moral

Before expressing $u(x, y)$ in terms of x and y, we should use the data $u|_{x=0} = h(y)$ first.

Example (Continued)

Evaluating $u\left(x_0 + 2s, y_0 e^{x_0 s + s^2}\right) = u(x_0, y_0)e^s$ at $x_0 = 0$, and using $u|_{x=0} = h(y)$, we obtain

$$u\left(2s, y_0 e^{s^2}\right) = u(0, y_0)e^s = h(y_0)e^s.$$

Now, we set

$$x := 2s \quad \text{and} \quad y := y_0 e^{s^2}.$$

Then we can express s and y_0 in terms of x and y as follows:

$$s = \frac{1}{2}x \quad \text{and} \quad y_0 = ye^{-\frac{1}{4}x^2}.$$

Hence, the final answer is

$$\boxed{u(x, y) = h\left(ye^{-\frac{1}{4}x^2}\right) e^{\frac{1}{2}x}.}$$

Moral

The Boundary Condition (BC) CANNOT be imposed arbitrary; otherwise, the Boundary Value Problem (BVP) may NOT have a solution. The BC should be imposed according to the structure of characteristic curves for the Partial Differential Equation (PDE).

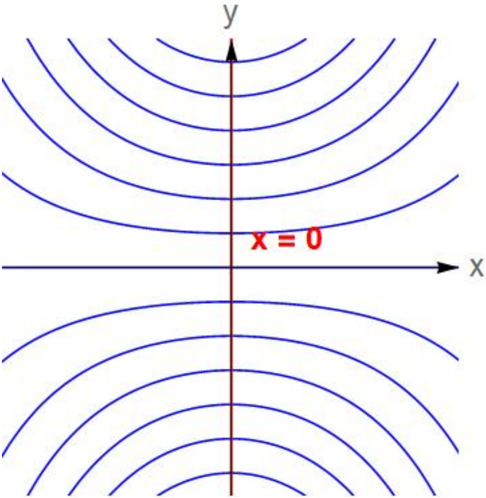


Figure: Characteristics for $2\partial_x u + xy\partial_y u = u$.