Chapter 5. Energy Methods

MATH4406 Introduction to Partial Differential Equations

The University of Hong Kong

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This chapter is related to the materials in Section 2.2-2.3 of the Textbook.

5.1 What Are Energy Methods?

Energy Methods

Question

If the underlying PDE does not have any explicit solution formula, then how should we study the behavior of its solutions?

Possible Answer

One may manipulate the underlying PDE by

- multiplying the equation by a function (e.g., the solution itself),
- integrating the equation, or
- differentiating the equation, etc.,

and hope to extract some useful information.

Main Idea (A Priori Estimate)

Assuming the existence of smooth solution to the underlying PDE, one is allowed to use the above manipulations to study the solution behavior, such as uniqueness, stability, etc.

What Are Energy Methods?

Energy Methods

- arise from physical observations for the underlying equation/system;
- are manipulating the equation by multiplication, integration, differentiation, etc.;
- are highly related to the conserved quantities;
- can study the solution to the underlying PDE **WITHOUT** using ANY explicit solution formula,
- applies to different types of PDE.

Philosophy

While applying the energy methods, we are assuming the existence of sufficiently smooth solutions, or have already showed it via other means. Our aim is to find out some useful information (e.g., estimates) for the solutions that satisfy the underlying PDE.

5.2 Application to the Wave Equations: Conservation of Mechanical Energy

Vibrating String

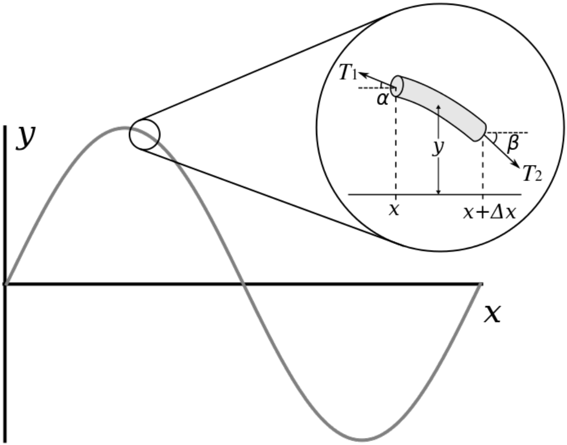


Figure: Illustration for a vibrating string, by Nicoguaro; from `https://commons.wikimedia.org/wiki/File:StringParameters.svg`

Mechanical Energy for the Wave Equations

Consider the wave equation that describes the vibrating string:

$$\rho \partial_{tt} u = T \partial_{xx} u, \tag{1DWave}$$

for $-\infty < x < \infty$ and $t > 0$, where the density ρ and tension T are two given positive constants. Define

Mechanical Energy

$$E(t) := \underbrace{\frac{1}{2} \int_{-\infty}^{\infty} \rho \left| \partial_t u(t,x) \right|^2 \, dx}_{\text{Kinetic Energy (K.E.)}} + \underbrace{\frac{1}{2} \int_{-\infty}^{\infty} T \left| \partial_x u(t,x) \right|^2 \, dx}_{\text{Potential Energy (P.E.)}}.$$

Remark

It follows from the definition that for any t,

$$E(t) \geq 0.$$

Lesson from Physics

Moral

It follows from the **physical observation** that the mechanical energy $E(t)$ (= K.E. + P.E.) is conserved (namely time-independent). Therefore, one may guess that if Equation (1DWave) is a good model/approximation for describing the vibrating string, then Equation (1DWave) should also capture the physical laws, and hence, to keep $E(t)$ remain unchanged.

Proposition (Conservation of Mechanical Energy)

Let $u := u(t,x) \in C^2([0,\infty) \times \mathbb{R})$ satisfy the 1D wave equation (1DWave), and

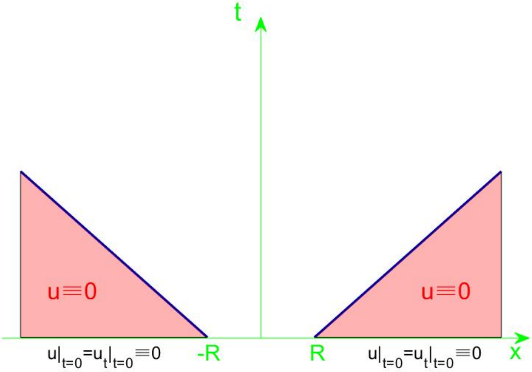
$$\lim_{x \to \pm\infty} \partial_t u \partial_x u = 0. \tag{Lim∞}$$

Then E is conserved, namely for any $t \geq 0$,

$$E(t) \equiv E(0).$$

Remarks

- We can see (Lim∞) as a Boundary Condition (BC) at $x = \pm\infty$.
- The condition (Lim∞) can be fulfilled easily; for example, if both $u|_{t=0}$ and $\partial_t u|_{t=0}$ vanish for $|x| > R$ for some constant $R \geq 0$, then it follows from the *finite speed of propagation* that (Lim∞) is satisfied.



Further Remarks

- The vanishing condition (Lim∞) is a very mild assumption (in the sense of applications), since we generally believe that the event happens in the far field (say, $x = \pm\infty$)/deep universe should not affect our life on the Earth.
- Given that $u|_{t=0} = \phi$ and $\partial_t u|_{t=0} = \psi$. Then

$$
\begin{aligned}
E(0) &:= \frac{1}{2}\int_{-\infty}^{\infty} \rho \left|\partial_t u(0,x)\right|^2 \, dx + \frac{1}{2}\int_{-\infty}^{\infty} T \left|\partial_x u(0,x)\right|^2 \, dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \rho \left|\psi(x)\right|^2 \, dx + \frac{1}{2}\int_{-\infty}^{\infty} T \left|\phi'(x)\right|^2 \, dx.
\end{aligned}
$$

- In the energy E here, we integrate over the whole spatial domain, namely over $\mathbb{R}$. Physically, this includes everything in the underlying problem, so people sometimes call this as a global(-in-space) energy. Later on, while discussing the *Causality*, we will consider a local(-in-space) energy that only integrates over a finite interval.

The First Proof of Proposition

Method 1 (Direct Computation)

$$\begin{aligned}\frac{d}{dt}E &= \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty}\rho\left|\partial_t u\right|^2\ dx + \frac{1}{2}\int_{-\infty}^{\infty}T\left|\partial_x u\right|^2\ dx\right) \\ &= \int_{-\infty}^{\infty}\rho\partial_t u\partial_{tt}u\ dx + \int_{-\infty}^{\infty}T\partial_x u\partial_{tx}u\ dx.\end{aligned} \tag{1}$$

Using (1DWave) and integration by parts, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\rho\partial_t u\partial_{tt}u\ dx &= \int_{-\infty}^{\infty}T\partial_t u\partial_{xx}u\ dx \\ &= \underbrace{\left[T\partial_t u\partial_x u\right]_{x=-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty}T\partial_{tx}u\partial_x u\ dx,\end{aligned} \tag{2}$$

since the BC (Lim∞). Combining (1) and (2), we have $\dfrac{d}{dt}E \equiv 0$.

Method 1 (Continued).

Integrating $\dfrac{d}{dt}E \equiv 0$, we finally obtain

$$E(t) \equiv E(0).$$

$\square$

Remarks

- The proof in Method 1 is usually found in physics textbooks.
- This proof is somewhat cheating, because it is not the way how we "discover" the energy.

Question

How does a mathematican think?

The Second Proof of Proposition

Method 2 (Manipulating the PDE; Energy Method)

Multiplying (1DWave) by $\partial_t u$, and then integrating with respect to x over $\mathbb{R}$, we have

$$\int_{-\infty}^{\infty} \rho \partial_{tt} u \partial_t u \, dx = \int_{-\infty}^{\infty} T \partial_{xx} u \partial_t u \, dx. \tag{3}$$

Applying integration by parts to the RHS of (3), we have

$$\begin{aligned} \int_{-\infty}^{\infty} T \partial_{xx} u \partial_t u \, dx &= \underbrace{[T \partial_t u \partial_x u]_{x=-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} T \partial_x u \partial_{tx} u \, dx \\ &= -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} T \left|\partial_x u\right|^2 \, dx\right), \end{aligned} \tag{4}$$

since the BC (Lim∞) and $\partial_t \left(\frac{1}{2}\left|\partial_x u\right|^2\right) = \partial_x u \partial_{tx} u$.

Method 2 (Continued).

On the other hand, since $\partial_t \left(\frac{1}{2}|\partial_t u|^2\right) = \partial_{tt} u \partial_t u$, we also know that the LHS of (3) becomes

$$\int_{-\infty}^{\infty} \rho \partial_{tt} u \partial_t u \, dx = \frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho |\partial_t u|^2 \, dx\right). \tag{5}$$

Combining (3)-(5), we finally obatin

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho |\partial_t u|^2 \, dx + \frac{1}{2}\int_{-\infty}^{\infty} T |\partial_x u|^2 \, dx\right)}_{=E(t)} = 0.$$

A direct integration completes the proof. □

Moral

We "define"/choose $E(t)$ appropriately, so that it is conserved.

In-Class Discussion

In-Class Discussion

Setting: Consider

$$\partial_{tt} u - c^2 \partial_{xx} u = -\alpha u,$$

for $-\infty < x < \infty$ and $t > 0$, where the wave speed c and restoring coefficient α are two given positive constants.

Aim: Find an appropriate energy.

After some in-class discussion, we have the following

Conclusions

1. We should define the energy

$$E(t) := \frac{1}{2} \int_{-\infty}^{\infty} |\partial_t u(t,x)|^2 + c^2 |\partial_x u(t,x)|^2 + \alpha |u(t,x)|^2 \, dx \geq 0.$$

2. $E(t) = E(0)$ for all t, provided that $\lim_{x \to \pm\infty} \partial_t u \partial_x u = 0$.

5.3 Application to the Wave Equations: Causality via Energy Method

IVP for the Wave Equation with a Restoring Force

Let $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfy the Cauchy problem for the wave equation with a restoring force:

$$\begin{cases} \partial_{tt} u - c^2 \partial_{xx} u = -\alpha u \\ u\big|_{t=0} = \phi \\ \partial_t u\big|_{t=0} = \psi, \end{cases}$$

where the wave speed $c > 0$, and the restoring coefficient $\alpha > 0$ are two given constants.

Question

Do we still have the *Finite Speed of Propagation*?

Technical Difficulty

We have NOT derived ANY explicit solution formula for u yet.

Finite Speed of Propagation

For any closed interval $[a, b]$, for any constant $M \geq c$,

$$\text{if } \phi|_{[a,b]} \equiv \psi|_{[a,b]} \equiv 0, \text{ then } u|_{\Delta} \equiv 0,$$

where

$$\Delta := \{(t, x) \in [0, \infty) \times \mathbb{R};\ a + Mt \leq x \leq b - Mt\}.$$

Remark: *The condition $a + Mt \leq b + Mt$ implies $t \leq \dfrac{b-a}{2M}$.*

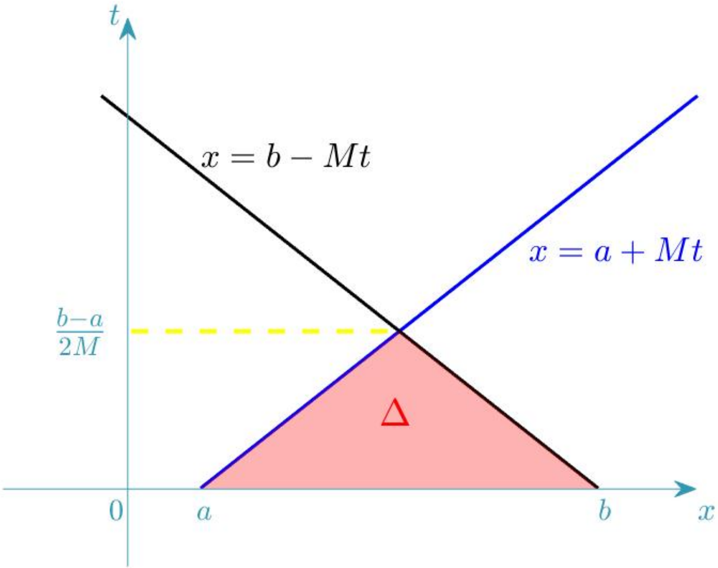
Question

Why should some type(s) of *Finite Speed of Propagation* hold?

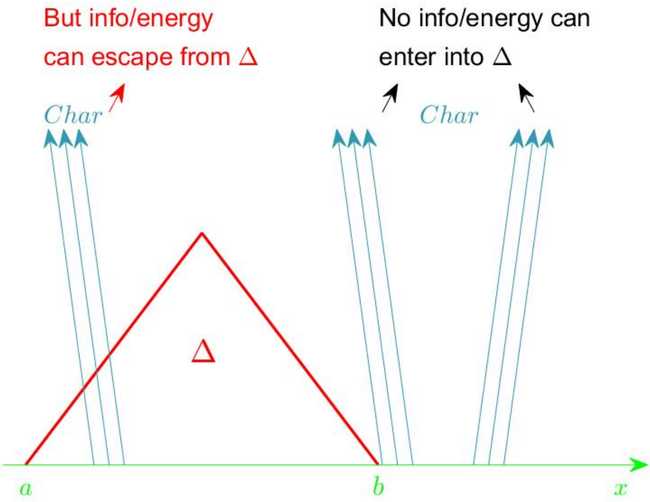
Answer

The PDE $\partial_{tt} u - c^2 \partial_{xx} u = -\alpha u$ is *hyperbolic*.

$$\Delta := \{(t,x) \in [0,\infty) \times \mathbb{R};\ a + Mt \leq x \leq b - Mt\}.$$



Philosophy behind the Causality



Thus, the local(-in-space) energy can only decrease.

How to Prove the Finite Speed of Propagation?

Tool: Local(-in-Space) Energy

For any $0 \leq t \leq \dfrac{b-a}{2M}$, we define

$$E(t) := \int_{a+Mt}^{b-Mt} e(t,x)\, dx,$$

where the energy density

$$e := \frac{1}{2}\left|\partial_t u\right|^2 + \frac{1}{2}c^2\left|\partial_x u\right|^2 + \frac{1}{2}\alpha\left|u\right|^2.$$

Remark

We have already seen this energy density e in our previous discussion on the wave equation with a restoring force; see the in-class discussion in the last section.

Exercise

If $M \geq c$, then

$$\frac{dE}{dt} \leq 0.$$

Remarks

- The checking is just a (simple but tedious) direct computation. We will skip it here, but a simplified case will be assigned in the homework.
- Also, you can talk with us after class or during the office hours. We will be happy to work with you together.
- Please try to verify this fact after finishing the homework. It will be a good exercise.

Consequence

$$E(t) \leq E(0), \quad \text{for all } 0 \leq t \leq \frac{b-a}{2M}.$$

Initial Local(-in-Space) Energy

Since $\phi|_{[a,b]} \equiv \psi|_{[a,b]} \equiv 0$, we have

$$\begin{aligned} E(0) &:= \frac{1}{2}\int_a^b |\partial_t u|^2 + c^2 |\partial_x u|^2 + \alpha |u|^2 \, dx\Big|_{t=0} \\ &:= \frac{1}{2}\int_a^b |\psi|^2 + c^2 |\phi'|^2 + \alpha |\phi|^2 \, dx\Big|_{t=0} \\ &= 0. \end{aligned}$$

Conclusion

Since $e(t,x) \geq 0$, $E(t) \geq 0$. We finally have

$$0 \leq E(t) \leq E(0) = 0,$$

since $\dfrac{dE}{dt} \leq 0$ and $\phi|_{[a,b]} \equiv \psi|_{[a,b]} \equiv 0$. Thus, $E(t) \equiv 0$, which implies $u \equiv 0$ in the region Δ. This proves the *Finite Speed of Propagation*.

Moral

We can prove the *Finite Speed of Propagation* **WITHOUT** knowing any explicit solution formula.

Moral

Energy Methods also apply to problems **WITHOUT** any explicit solution formula.

Example

What else can the Energy Methods show?

- L^2 stability[a] of the heat equation,
- existence to PDE?!
- understanding the quantitative behavior of solutions, e.g., asymptotic limits, long-term behavior, etc.

[a] We will discuss this in the next section.

5.4 Application to the Heat Equations: Uniqueness and Stability to the Initial and Boundary-Value Problem

Energy Method for the Heat Equations

Let the diffusivity parameter $k > 0$, the source term $f := f(t, x)$, the boundary data $g := g(t)$ and $h := h(t)$ be given. For any $i = 1, 2$, let u_i satisfy the following Initial and Boundary-Value Problem (IBVP):

$$\begin{cases} \partial_t u_i - k\partial_{xx} u_i = f & \text{in } (0, \infty) \times (0, L) \\ u_i\big|_{t=0} = \phi_i & \\ u_i\big|_{x=0} = g & \\ u_i\big|_{x=L} = h, & \end{cases}$$

where the given initial data ϕ_i may be different for different i.

Question

Can we estimate the difference between u_1 and u_2 in terms of that between ϕ_1 and ϕ_2? (If yes, in what sense?)

Denote $\tilde{u} := u_1 - u_2$ and $\tilde{\phi} := \phi_1 - \phi_2$. Then $\tilde{u}$ satisfies the following IBVP:

$$\begin{cases} \partial_t \tilde{u} - k\partial_{xx}\tilde{u} = 0 & \text{in } (0,\infty)\times(0,L) \\ \tilde{u}|_{t=0} = \tilde{\phi} & \\ \tilde{u}|_{x=0} \equiv 0 & \\ \tilde{u}|_{x=L} \equiv 0. & \end{cases}$$

Multiplying the PDE $\partial_t \tilde{u} = k\partial_{xx}\tilde{u}$ by $\tilde{u}$, and then integrating with respect to x over the spatial interval $(0, L)$, we have

$$\begin{aligned} \int_0^L \tilde{u}\partial_t \tilde{u}\, dx &= k\int_0^L \tilde{u}\partial_{xx}\tilde{u}\, dx \\ \frac{1}{2}\frac{d}{dt}\left(\int_0^L |\tilde{u}|^2\, dx\right) &= k\underbrace{\left[\tilde{u}\partial_x\tilde{u}\right]_{x=0}^L}_{=0} - k\int_0^L |\partial_x \tilde{u}|^2\, dx \\ &\le 0, \end{aligned}$$

because of the BC $\tilde{u}|_{x=0} \equiv \tilde{u}|_{x=L} \equiv 0$.

A direct integration yields

$$\int_0^L |\tilde{u}(t,x)|^2 \, dx \le \int_0^L |\tilde{u}(0,x)|^2 \, dx = \int_0^L \left|\tilde{\phi}(x)\right|^2 \, dx,$$

because of the BC $\tilde{u}|_{t=0} = \tilde{\phi}$. In other words,

$$\int_0^L |u_1 - u_2|^2 \, dx \le \int_0^L |\phi_1 - \phi_2|^2 \, dx,$$

which is the *stability* in the L^2-norm:

$$\|f\|_{L^2([0,L])} := \left(\int_0^L |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

In particular, if $\phi_1 \equiv \phi_2$, then for any $t \geq 0$,

$$\int_0^L |u_1 - u_2|^2 \, dx \equiv 0,$$

which implies

$$u_1 \equiv u_2,$$

due to the

Theorem (First Vanishing Theorem; see Appendix A.1 of Textbook)

Let $\Omega \subseteq \mathbb{R}^d$ be open, and $f : \Omega \to [0, \infty)$ be a continuous function such that

$$\int_\Omega f(x) \, dx = 0.$$

Then

$$f \equiv 0 \quad \text{on } \Omega.$$

Thus, the *uniqueness* holds as well!!

Remarks

- For the heat equation,

$$\underbrace{\int_0^L |u|^2 \, dx}_{L^2 \text{ Energy}} \neq \underbrace{\int_0^L u \, dx}_{\substack{\text{Thermal} \\ \text{Energy}}} .$$

- The L^2 energy is a mathematical concept (or generalization) only.
- The thermal energy is a *physical quantity*.

Food for Thought

What can we obtain if we change the Dirichlet boundary conditions (i.e., $u|_{x=0} = g$ and $u|_{x=L} = h$) to other types of boundary conditions, such as Neumann boundary conditions, or Robin boundary conditions?

5.5 Application to the Heat Equations: Asymptotic Behavior (Advanced Topic)

Long-Time Behavior for Solutions to the Heat Equations

Let $u := u(t,x)$ be the temperature that satisfies the following Initial and Boundary-Value Problem (IBVP):

$$\begin{cases} \partial_t u - k\partial_{xx} u = f & \text{in } (0,\infty)\times(0,L) \\ u|_{t=0} = \phi \\ u|_{x=0} = 0 \\ \partial_x u|_{x=L} = 0, \end{cases}$$

where the constant $k > 0$, the initial data $\phi := \phi(x)$ and the time-independent heat source $f := f(x)$ are given.

Question

When the time t is large, can we describe the behavior of the solution u? (If yes, in what sense? And how to show the asymptotic/long-time behavior?)

Equilibrium Solution/Temperature

Moral (Lesson from Physics)

As the time $t \to \infty$, the temperature $u := u(t,x)$ should converge to some dynamical equilibrium temperature $u_E := u_E(x)$, namely

$$u(t,x) \to u_E(x).$$

The equilibrium temperature $u_E := u_E(x)$ should satisfy the mixed Dirichlet-Neumann problem for Poisson's equation:

$$\begin{cases} -k\partial_{xx} u_E = f & \text{in } (0,L) \\ u_E|_{x=0} = 0 \\ \partial_x u_E|_{x=L} = 0. \end{cases}$$

Questions

- How can we rigorously verify this assertion?
- For the convergence $u \to u_E$, in what sense are we able to show?

Denote $\tilde{u}(t,x) := u(t,x) - u_E(x)$ and $\tilde{\phi}(x) := \phi(x) - u_E(x)$. Then $\tilde{u}$ satisfies the following IBVP:

$$\begin{cases} \partial_t \tilde{u} - k\partial_{xx}\tilde{u} = 0 & \text{in } (0,\infty)\times(0,L) \\ \tilde{u}|_{t=0} = \tilde{\phi} \\ \tilde{u}|_{x=0} \equiv 0 \\ \partial_x \tilde{u}|_{x=L} \equiv 0. \end{cases}$$

Multiplying the PDE $\partial_t \tilde{u} = k\partial_{xx}\tilde{u}$ by $\tilde{u}$, and then integrating with respect to x over the spatial interval $(0,L)$, we have

$$\int_0^L \tilde{u}\partial_t \tilde{u}\ dx = k\int_0^L \tilde{u}\partial_{xx}\tilde{u}\ dx$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |\tilde{u}|^2\ dx\right) = k\underbrace{\left[\tilde{u}\partial_x\tilde{u}\right]_{x=0}^{L}}_{=0} - k\int_0^L |\partial_x \tilde{u}|^2\ dx$$

$$= -k\int_0^L |\partial_x \tilde{u}|^2\ dx,$$

because of the BC $\tilde{u}|_{x=0} \equiv \partial_x \tilde{u}|_{x=L} \equiv 0$.

Remark

When we study the uniqueness and stability to the IBVP for the heat equations, we just apply the "cheap"/elementary bound

$$-k \int_0^L |\partial_x \tilde{u}|^2 \, dx \leq 0,$$

which is not sufficient to show the convergence $u \to u_E$, because this inequality **CANNOT** "capture" the negative effect that causes the decay.

Question

How can we do a better job?

Answer

Apply a better inequality that can compare

$$\int_0^L |\partial_x \tilde{u}|^2 \, dx \quad \text{and} \quad \int_0^L |\tilde{u}|^2 \, dx.$$

Poincaré Inequality

From Real/Elementary Analysis, we have the

Proposition (Poincaré Inequality)

Let $f : [0, L] \to \infty$ be a C^1 function such that $f(0) = 0$. Then

$$\int_0^L |f(x)|^2 \, dx \leq 4L^2 \int_0^L \left|f'(x)\right|^2 \, dx.$$

Applying the Poincaré inequality to our previous inequality

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |\tilde{u}|^2 \, dx\right) \leq -k \int_0^L |\partial_x \tilde{u}|^2 \, dx,$$

we have

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |\tilde{u}|^2 \, dx\right) \leq -\frac{k}{4L^2} \int_0^L |\tilde{u}|^2 \, dx.$$

Applying the

Lemma (Grönwall's Inequality in the Differential Form)

Let $I := I(t) \geq 0$ be a C^1 function such that

$$\frac{d}{dt} I(t) \leq \beta(t) I(t),$$

for some function $\beta := \beta(t)$. Then

$$I(t) \leq I(0) e^{\int_0^t \beta(s)\, ds}.$$

to our previous inequality

$$\frac{1}{2} \frac{d}{dt} \left(\int_0^L |\tilde{u}|^2 \, dx \right) \leq -\frac{k}{4L^2} \left(\int_0^L |\tilde{u}|^2 \, dx \right),$$

we obtain

$$\int_0^L |\tilde{u}(t,x)|^2 \, dx \leq \left(\int_0^L \left|\tilde{\phi}(x)\right|^2 \, dx \right) e^{-\frac{k}{2L^2} t}.$$

As a result, we have

$$\int_0^L |u - u_E|^2 \, dx \le \left(\int_0^L |\phi - u_E|^2 \, dx \right) e^{-\frac{k}{2L^2} t} \to 0,$$

as $t \to \infty$. In other words, we have the L^2 convergence (with exponential decay):

$$\lim_{t\to\infty} \|u(t,\cdot) - u_E\|_{L^2([0,L])} = 0.$$

where the L^2-norm is defined as follows:

$$\|f\|_{L^2([0,L])} := \left(\int_0^L |f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Conclusion

Using the Energy Method (together with some inequalities in Analysis), we have rigorously verified the L^2 convergence of $u \to u_E$, as the time $t \to \infty$.

Comments

- The Energy Methods apply to equations/systems without any explicit solution formula, and are able to show qualitative/quantitative results, such as asymptotic limit (namely the solution behavior for $t \to \infty$).
- In order to "compensate" the lack of solution formula, we apply different results from Mathematical/Real Analysis (e.g., inequalities) to provide a better estimate on the solution.
- In particular, in the previous argument, we applied
  - Poincaré Inequality, and
  - Grönwall's Inequality.

Final Question

How to show the Poincaré Inequality, and Grönwall's Inequality?

Proof of the Poincaré Inequality

Let $f : [0, L] \to \infty$ be a C^1 function such that $f(0) = 0$. Then applying the fundamental theorem of calculus to $|f(x)|^2$, we know that for any $x \in [0, L]$,

$$|f(x)|^2 = \underbrace{|f(0)|^2}_{=0} + 2\int_0^x f(y)f'(y)\,dy = 2\int_0^x f(y)f'(y)\,dy.$$

By the Cauchy-Schwarz inequality $\Big($Recall: for any functions f and g, $\displaystyle\int_0^L |f(y)g(y)|\,dy \le \|f\|_{L^2([0,L])}\|g\|_{L^2([0,L])}\Big)$,

$$\begin{aligned}|f(x)|^2 &\le 2\left|\int_0^x f(y)f'(y)\,dy\right| \le 2\int_0^x |f(y)f'(y)|\,dy \\ &\le 2\int_0^L |f(y)f'(y)|\,dy \le 2\|f\|_{L^2([0,L])}\|f'\|_{L^2([0,L])}.\end{aligned}$$

Integrating $|f(x)|^2 \leq 2\|f\|_{L^2([0,L])}\|f'\|_{L^2([0,L])}$ with respect to x over $[0, L]$, and using the fact that $\|f\|_{L^2([0,L])}\|f'\|_{L^2([0,L])}$ is just a number, we obtain

$$\begin{aligned}\int_0^L |f(x)|^2 \, dx &\leq 2\int_0^L \|f\|_{L^2([0,L])}\|f'\|_{L^2([0,L])} \, dx \\ \|f\|^2_{L^2([0,L])} &\leq 2L\|f\|_{L^2([0,L])}\|f'\|_{L^2([0,L])} \\ \|f\|_{L^2([0,L])} &\leq 2L\|f'\|_{L^2([0,L])}.\end{aligned}$$

Taking the square, we obtain the Poincaré Inequality

$$\int_0^L |f(x)|^2 \, dx \leq 4L^2 \int_0^L \left|f'(x)\right|^2 \, dx.$$

Q.E.D.

Proof of Grönwall's Inequality

Let $I := I(t) \geq 0$ be a C^1 function such that

$$\frac{d}{dt}I(t) - \beta(t)I(t) \leq 0,$$

for some function $\beta := \beta(t)$. Then multiplying the "integrating factor" $e^{-\int_0^t \beta(s)\,ds}$, we obatin

$$\frac{d}{dt}\left(I(t)e^{-\int_0^t \beta(s)\,ds}\right) = e^{-\int_0^t \beta(s)\,ds}\left(\frac{d}{dt}I(t) - \beta(t)I(t)\right) \leq 0.$$

A direct integration yields

$$I(t)e^{-\int_0^t \beta(s)\,ds} - I(0) \leq 0,$$

which is equivalent to Grönwall's Inequality

$$I(t) \leq I(0)e^{\int_0^t \beta(s)\,ds}.$$