Algebraic extensions and algebraically losed fields

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1. Algebraic extensions;
2. Algebraically closed fields.

Definition. A field extension $K \to L$ is said to be algebraic if every $a \in L$ is algebraic over K.

Recall that every finite extension is algebraic.

~~Examples:~~

$K(x)/K$

finite ext

Theorem If $K \subset L$ and $L \subset M$ are algebraic field extensions, so is $K \subset M$.

Proof. Let $a \in M$ be arbitrary.

use the fact that a simple ext is finite iff α is algebraic

- As the extension $L \subset M$ is algebraic, there exists $f(x) = \sum_{i=0}^{n} \alpha_i x^i \in L[x]$ such that $f(a) = 0$. Let
$$L' = K(\alpha_0, \alpha_1, \ldots, \alpha_n) \subset L.$$
so $f(x) \in L'[x]$
- As L is an algebraic extension of K, every $\alpha_i \in L$ is algebraic over K. Thus L' is a finite extension of K.
- Since a is algebraic over L', $L'(a)$ is a finite extension over L'.
- By the Tower Theorem, $L'(a)$ is a finite extension of K, so $a \in L'(a)$ is algebraic over K.

$a \in (L/M)^{alg} \Rightarrow f(a)=0,\ f \in K[M]$, so $L'(a)$ is finite, L'/K is finite

Definition.: If L is a field extension of K, set

$$\overline{K}^L = \text{the set of all elements in } L \text{ that are algebraic over } K$$

and call $\overline{K}^L$ the relative algebraic closure of K in L,

Theorem: For any field extension L of K,

1. the ~~subset~~ $\overline{K}^L$ ~~of L~~ is a subfield of L;
2. $\overline{K}^L$ is is an algebraic extension of K.

③ $\overline{K}^L \subset L$ is purely transcendental, i.e. $\forall a \in L \setminus \overline{K}^L$ is transcendental over $\overline{K}^L$

Proof.

- Assume that $a, b \in \overline{K}^L$. ~~and $b \neq 0$.~~
- Then $K(a, b)$ is a finite extension over K;
- So every element in $K(a, b)$ are algebraic over K.
- Thus $a \pm b, ab, a/b \in \overline{K}^L$.

If $b \neq 0$, then $\frac{1}{b} \in \overline{K}^L$

③ Suppose $a \in L \setminus \overline{K}^L$ is algebraic over $\overline{K}^L$. Then $K \subset \overline{K}^L$ & $\overline{K}^L \subset \overline{K}^L(a)$ are algebraic
contradiction.

Q.E.D.

Recall Fundamental Theorem of Algebra:

Every polynomial $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

Definition. A field L is said to be algebraically closed if every $f(x) \in L[x]$ has a root in L.

Thus the field $\mathbb{C}$ is algebraically closed.

Question. Are there algebraically closed fields other than $\mathbb{C}$?

Theorem: If C is an algebraically closed field and $K \subset C$ is any subfield, then the relative algebraic closure $\overline{K}^C$ of K in C is algebraically closed.

Proof. Let $L = \overline{K}^C \subset C$. Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + x^n \in L[x]$$

be arbitrary and non-zero. We need to show that f has a root in L.

- Since C is algebraically closed, f has a root α in C.
- $L(\alpha)$ is an algebraic extension of L.
- Since L is an algebraic extension of K, $L(\alpha)$ is an algebraic extension over K.
- Thus α is algebraic over K. Hence $\alpha \in L$.

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Definition. The relative closure of $\mathbb{Q}$ in $\mathbb{C}$ is denoted by $\overline{\mathbb{Q}}$, and simply called the algebraic closure of $\mathbb{Q}$. , so $\overline{\mathbb{Q}}$ is algebraically closed.

Lemma. The algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ is a countable and $[\overline{\mathbb{Q}} : \mathbb{Q}] = +\infty$.

Proof.

- As $\mathbb{Q}$ is countable, $\mathbb{Q}[x]$ is countable.
- As every element in $\overline{\mathbb{Q}} \subset \mathbb{C}$ is a root of some $f \in \mathbb{Q}[x]$, $\overline{\mathbb{Q}}$ is countable.
- For any integer n, $\mathbb{Q}(2^{1/n})$ has degree n over $\mathbb{Q}$ because the minimal polynomial of $2^{1/n}$ is $x^n - 2$. Thus $[\overline{\mathbb{Q}} : \mathbb{Q}] \geq n$ for every n, so $[\overline{\mathbb{Q}} : \mathbb{Q}] = +\infty$.
  irreducible by Eisenstein

Lemma. If $\alpha \in \overline{\mathbb{Q}}$ and $\alpha > 0$, then for any integer $n \geq 1$, $\alpha^{1/n} \in \overline{\mathbb{Q}}$.

Proof. Let $\beta = \alpha^{1/n}$.

- Since $\beta^n = \alpha$, β is algebraic over $L = \mathbb{Q}(\alpha)$, so $L(\beta)$ is an algebraic extension of L.
- Since L is an algebraic extension of $\mathbb{Q}$, $L(\beta)$ is an algebraic extension of $\cancel{\mathbb{Q}}$ L. Thus β is algebraic over $\mathbb{Q}$.
- Hence $\beta \in \overline{\mathbb{Q}}$.

Example. One has

$$-2i\sqrt[3]{9-\sqrt{2}}+\frac{\sqrt{\sqrt{2}+\sqrt[3]{\sqrt{5}+3}}}{-3+i\sqrt{\sqrt{7}+2\sqrt[3]{\sqrt{5}+3}}}\in\overline{\mathbb{Q}}.$$