ASSIGNMENT I, ALGEBRA II, HKU, SPRING 2025

DUE AT 11:59PM ON FRIDAY MARCH 06, 2025

(1) Determine the Smith Normal Form of the matrix
$$A = \begin{pmatrix} x+1 & x & 1 \\ x & 0 & x \\ x & 1 & x^2 \end{pmatrix} \in M_{3,3}(\mathbb{F}_2[x]).$$

(2) For the following $A \in M_{m,n}(\mathbb{Z})$, find the Smith Normal Form Λ of A and matrices $P \in GL(n, \mathbb{Z})$ and $Q \in GL(m, \mathbb{Z})$ such that $PAQ = \Lambda$.
$$1).\ A = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in M_{1,2}(\mathbb{Z}), \qquad 2).\ A = \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix} \in M_{2,2}(\mathbb{Z}).$$

(3) If R is a non-zero commutative ring with identity and every sub-module of every free R-module is free, show that R is a principal domain.

(4) If R is a PID, show that sub-modules of a cyclic R-module are cyclic.

(5) Let V be a finite dimensional vector space over a field K and let $T \in \mathrm{End}_K(V)$. Equip V with the $K[x]$-module structure defined by
$$\left(\sum_i \alpha_i x^i \right)(v) = \sum_i \alpha_i T^i(v), \qquad v \in V.$$
Show that V is a finitely generated torsion $K[x]$-module.

(6) Let R be a commutative ring and M an R-module. If $N_1 \subset N_2 \subset \cdots$ is an ascending chain of sub-modules of M, show that $\bigcup_{i=1}^{\infty} N_i$ is a sub-module of M.

(7) Let A be a $\mathbb{Z}$-module, $a \in A$, and n a positive integer. Prove that he map
$$\phi_n : \ \mathbb{Z}/n\mathbb{Z} \longrightarrow A, \ \ \phi(\bar{k}) = ka$$
is a well-defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$, where $A_n = \{a \in A : na = 0\}$.

(8) Find all elements of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/30\mathbb{Z}, \mathbb{Z}/21\mathbb{Z})$. Show in general that for any positive integers m and n one has
$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m,n)\mathbb{Z},$$
where (m, n) is the gcd of m and n.

(9) Compute the Smith normal form of $A = \begin{pmatrix} -7 & 0 & -6 \\ 6 & 3 & 0 \\ 6 & 0 & 6 \end{pmatrix}$ as an element in $M_{3,3}(\mathbb{Z})$; Let N be the sub-module of $\mathbb{Z}^3$ generated by the columns of A. What can you say about the quotient $\mathbb{Z}^3/N$ as an abelian group?

(10) Let R be a commutative ring. Let M be an R-module and $N \subset M$ a sub-module. Show that if both N andM/N are finitely generated, so is M.