THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 3

Due 3:30pm[1], September 27th (Friday), **in-class**.

Aim of this Homework: *In this assignment you will study the compatibility conditions for different problems, and apply the change of variables technique to rewrite PDE in better forms, as well as solve PDE by using the coordinate method.*

Reading Assignment: Read the following material(s):

(i) Section 1.2 and 2.1 of the textbook.

Instruction: Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the "Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

Problem 1 (Vibrating String with Gravity)**.** Let $u := u(t,x)$ be the vertical displacement of a vibrating string $(0 < x < L)$ with fixed ends. Under the effect of gravity, u satisfies the one-dimensional wave equation with nonhomogeneous term

$$\partial_{tt}u = c^2\partial_{xx}u - g \tag{1}$$

[1]You are expected to submit your homework before the beginning of Friday lecture **in-class**.

subject to the Dirichlet boundary conditions

$$u|_{x=0} = u|_{x=L} = 0, \tag{2}$$

where the given positive constants c and g are the speed of propagation and the gravity respectively.

(i) Find the time-independent/equilibrium solution $u_E := u_E(x)$ by solving the following boundary value problem of ODE:

$$\begin{cases} c^2 u_E'' = g \\ u_E|_{x=0} = u_E|_{x=L} = 0. \end{cases}$$

(ii) Define $v(t,x) := u(t,x) - u_E(x)$. Verify that v satisfies

$$\partial_{tt} v = c^2 \partial_{xx} v.$$

Problem 2. In the following problems, find the compatibility conditions for the existence. In other words, if the problem has a solution, then what is/are the constraint(s) for the data? You are asked to find the compatibility conditions only, and not required to solve the problems.

(i) Let β be a constant, and $\Omega := \{(x,y) \in \mathbb{R}^2;\ 0 < x < 1,\ 0 < y < 3\}$. Consider the following Neumann problem:

$$\begin{cases} -4\partial_{xx} u - \partial_{yy} u = y^2 \cos^2 \pi x & \text{in } \Omega \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = \beta. \end{cases}$$

Find all the possible value(s) of β for which the above Neumann problem has a solution. (Hint: applying the divergence theorem.)

(ii) Consider the following Neumann problem for the Poisson's equation:

$$\begin{cases} -\Delta u = \left(\displaystyle\sum_{k=1}^{d} x_k^2\right) & \text{in } B \\ \displaystyle\sum_{k=1}^{d} x_k \partial_{x_k} u = C \sum_{k=1}^{d} x_k^2 \sum_{\substack{j=1 \\ j \neq k}}^{d} x_j^2 & \text{on } \partial B, \end{cases}$$

where $u := u(x_1, x_2, \ldots, x_d)$ is the unknown, $d \geq 2$ is an integer, C is a given constant, and the unit ball

$$B := \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d;\ \sum_{k=1}^{d} x_k^2 < 3 \right\}.$$

(Hint: applying the divergence theorem <u>twice</u>.)

Problem 3. In the following problems, find the compatibility conditions for the existence. In other words, if the problem has a solution, then what is/are the constraint(s) for the data? You are asked to find the compatibility conditions only, and not required to solve the problems.

(i) Consider

$$\begin{cases} \partial_y u = \tan\left(\dfrac{\left(1 + e^{-x^2}\right) y}{2} \right) & \text{for } -\infty < x < \infty \text{ and } 0 < y < 1 \\ u(x,0) = e^{-x^2} \\ u(x,1) = g(x), \end{cases}$$

where $u := u(x,y)$ is the unknown, g is a given function.

(ii) Consider

$$\begin{cases} 3\partial_t u - 5\partial_x u = 0 & \text{for } -\infty < t < \infty \text{ and } -2 < x < 3 \\ u(t,-2) = g(t) \\ u(t,3) = h(t) \end{cases}$$

where $u := u(t,x)$ is the unknown, g and h are given functions.
(Hint: what are the characteristics?)

Problem 4. In this problem, you are going to solve the following boundary value problem

$$\begin{cases} -y\partial_x u + x\partial_y u = u & \text{for } x, y > 0 \\ u|_{y=0} = \sin x \end{cases}$$

by using the coordinate method.

(i) Rewrite the PDE in terms of the polar coordinates $r := \sqrt{x^2+y^2}$ and $\theta := \tan^{-1}\left(\frac{y}{x}\right)$.

(ii) Express the boundary condition $u|_{y=0} = \sin x$ in terms of r and θ.

(iii) Solve u in terms of r and θ.

(iv) Express your final answer u in terms of x and y.

The following problem(s) is/are *optional*:

Problem 5. In this problem, you are going to solve the following boundary-value problem

$$\begin{cases} 3y\partial_x u - 3x\partial_y u = 7u & \text{for } -\infty < x < \infty \text{ and } y > 0 \\ u|_{y=0} = x \operatorname{erf} x & \text{for } x > 0 \end{cases}$$

by using the coordinate method.

(i) Rewrite the PDE in terms of the polar coordinates $r := \sqrt{x^2+y^2}$ and $\theta := \tan^{-1}\left(\frac{y}{x}\right)$.

(ii) Express the boundary condition

$$u|_{y=0} = x \operatorname{erf} x \qquad \text{for } x > 0$$

in terms of r and θ.

(iii) Solve for u in terms of r and θ.

(iv) Express your final answer u in terms of x and y.