

# Elliptic Functions, Part 3

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The teaching of maths is to make everything obvious.

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## 1 The Weierstrass Problem on $\mathbb{C}/L$

Let us consider Weierstrass data

$$\{(x_k, n_k)\}_{k=1}^s,$$

where each  $x_k$  is a distinct point on  $X = \mathbb{C}/L$ , and  $n_k \in \mathbb{Z}$  is an integer (possibly positive or negative) associated to  $x_k$ .

To lift these points to the universal covering space, choose representatives

$$a_k \in \mathbb{C} \text{ such that } \pi(a_k) = x_k,$$

where  $\pi : \mathbb{C} \rightarrow X = \mathbb{C}/L$  is the canonical projection.

In the simpler case, we assume the following condition holds:

$$\sum_{k=1}^s n_k a_k = 0.$$

Observation: Recall that the shifted Weierstrass sigma function,

$$\sigma_c(z) := \sigma(z + c),$$

has a *simple zero at*  $z = -c$  and no other zeros.

So a possible candidate for the desired function is

$$f(z) = \prod_{k=1}^s \sigma(z - a_k)^{n_k}.$$

Each factor  $\sigma(z - a_k)$  contributes a zero of order  $n_k$  at  $z = a_k$  (if  $n_k > 0$ ), or a pole of order  $|n_k|$  (if  $n_k < 0$ ).

We have

$$\text{ord}_{a_k}(f) = n_k, \quad k = 1, 2, \dots, s,$$

and for any  $a \not\equiv a_k \pmod{L}$  for all  $k$ ,

$$\text{ord}_a(f) = 0.$$

It remains to check whether  $f$  is an elliptic function with respect to the lattice  $L$ . Recall that the sigma function satisfies

$$\sigma(z + \omega) = \exp(A_\omega z + B_\omega) \sigma(z), \quad \forall \omega \in L.$$

Therefore,

$$f(z + \omega) = \prod_{k=1}^s \sigma(z + \omega - a_k)^{n_k}, \quad f(z) = \prod_{k=1}^s \sigma(z - a_k)^{n_k}.$$

So we have

$$\frac{f(z + \omega)}{f(z)} = \prod_{k=1}^s \left( \frac{\sigma((z - a_k) + \omega)}{\sigma(z - a_k)} \right)^{n_k}.$$

By applying the transformation formula  $\sigma(z + \omega) = \exp(A_\omega z + B_\omega) \sigma(z)$ , we obtain

$$\frac{f(z + \omega)}{f(z)} = \prod_{k=1}^s (\exp(A_\omega(z - a_k) + B_\omega))^{n_k} = \prod_{k=1}^s \exp(n_k(A_\omega(z - a_k) + B_\omega)).$$

Expanding inside the product,

$$\frac{f(z + \omega)}{f(z)} = \prod_{k=1}^s \exp(n_k A_\omega z - n_k A_\omega a_k + n_k B_\omega) = \exp \left( \sum_{k=1}^s (n_k A_\omega z - n_k A_\omega a_k + n_k B_\omega) \right).$$

Combine terms in the exponent:

$$\frac{f(z + \omega)}{f(z)} = \exp \left( A_\omega z \sum_{k=1}^s n_k - A_\omega \sum_{k=1}^s n_k a_k + B_\omega \sum_{k=1}^s n_k \right) = \exp \left( (A_\omega z + B_\omega) \sum_{k=1}^s n_k \right) \exp \left( -A_\omega \sum_{k=1}^s n_k a_k \right)$$

Since  $\sum_{k=1}^s n_k = \sum_{k=1}^s n_k a_k = 0$ , both exponential factors equal 1, hence  $\frac{f(z + \omega)}{f(z)} = 1$ , so  $f$  is elliptic with respect to  $L$ .

If the coefficients  $n_k$  satisfy

$$\sum_{k=1}^s n_k = 0 \quad \text{and} \quad \sum_{k=1}^s n_k a_k = 0,$$

then the exponent equals zero, and therefore

$$\frac{f(z + \omega)}{f(z)} = 1 \quad \text{for all } \omega \in L.$$

Thus  $f$  is an elliptic function with respect to  $L$ .

Next, we show that it suffices to require the weaker condition

$$\sum_{k=1}^s n_k a_k \equiv 0 \pmod{L},$$

instead of the strict equality  $\sum_{k=1}^s n_k a_k = 0$ .

Recall that we have  $\pi : \mathbb{C} \rightarrow \mathbb{C}/L$ , and the points  $a_k$  are chosen so that  $\pi(a_k) = x_k$ . We can replace each  $a_k$  by

$$a'_k = a_k + w_0, \quad w_0 \in L.$$

Difficulty: If we replace  $(x_s, a_s)$  by  $(x_s, a_s + w_0)$  and define

$$h(z) = \prod_{k=1}^{s-1} \sigma(z - a_k)^{n_k} \cdot \sigma(z - (a_s + w_0))^{n_s},$$

then

$$\frac{h(z + \omega)}{h(z)} = \exp\left(-\left(\sum_{k=1}^s n_k a'_k\right) A_\omega\right),$$

where  $a'_k = a_k$  for  $k \neq s$ , and  $a'_s = a_s + w_0$ .

Hence,

$$\frac{h(z + \omega)}{h(z)} = \exp\left(-\left(\sum_{k=1}^s n_k a_k + n_s w_0\right) A_\omega\right).$$

Let

$$\mu = \sum_{k=1}^s n_k a_k.$$

By assumption, we know that  $\mu \in L$ .

The Weierstrass periodicity condition will be solved by setting

$$\mu + n_s w_0 = 0, \quad \text{i.e.} \quad w_0 = -\frac{\mu}{n_s} \in \frac{1}{n_s} L.$$

The difficulty arises because  $n_s$  may not be  $\pm 1$ . A solution in the general case is obtained as follows (we may assume  $n_s \geq 1$ ):

$$h(z) = \prod_{k=1}^{s-1} \sigma(z - a_k)^{n_k} \cdot \sigma(z - a_s)^{n_s-1} \cdot \sigma(z - a'_s),$$

where  $a'_s = a_s - w_0$  for some  $w_0 \in L$  to be determined.

Recall that

$$\sigma(z + \omega) = e^{A_\omega z + B_\omega} \sigma(z).$$

Then

$$\sigma(z - a'_s) = \sigma(z - (a_s - w_0)) = \sigma((z - a_s) + w_0) = e^{A_{w_0}(z-a_s)+B_{w_0}} \sigma(z - a_s),$$

and

$$\frac{\sigma(z - a'_s + \omega)}{\sigma(z - a'_s)} = e^{A_\omega(z-a'_s)+B_\omega}.$$

Note that

$$h(z) = \frac{f(z)}{\sigma(z - a_s)} \sigma(z - a'_s).$$

Hence

$$\frac{h(z + \omega)}{h(z)} = \frac{f(z + \omega)}{f(z)} \cdot \frac{\sigma(z - a_s)}{\sigma(z - a_s + \omega)} \cdot \frac{\sigma(z - a'_s + \omega)}{\sigma(z - a'_s)}.$$

Using the quasi-periodicity of  $\sigma$ ,

$$\frac{\sigma(z - a_s + \omega)}{\sigma(z - a_s)} = e^{A_\omega(z-a_s)+B_\omega}, \quad \frac{\sigma(z - a'_s + \omega)}{\sigma(z - a'_s)} = e^{A_\omega(z-a'_s)+B_\omega},$$

we obtain

$$\frac{h(z + \omega)}{h(z)} = \frac{f(z + \omega)}{f(z)} e^{-A_\omega(z-a_s)-B_\omega} e^{A_\omega(z-a'_s)+B_\omega} = \frac{f(z + \omega)}{f(z)} \exp(A_\omega(a_s - a'_s)).$$

Since

$$\frac{f(z + \omega)}{f(z)} = \exp\left(-A_\omega \sum_{k=1}^s n_k a_k\right),$$

we conclude that

$$\frac{h(z + \omega)}{h(z)} = \exp(-A_\omega \mu + A_\omega(a_s - a'_s)) = \exp(-A_\omega \mu + A_\omega w_0).$$

Thus, periodicity holds ( $\frac{h(z+\omega)}{h(z)} = 1$ ) when  $w_0 = \mu$ .

## 2 The Field of Meromorphic Functions on $\mathbb{C}/L$

We have already established the necessary and sufficient conditions for solving both the Mittag-Leffler (ML) problem and the Weierstrass (WP) problem.

In this section, we describe the function field  $\mathcal{M}(X)$  explicitly.

Recall that for the Eisenstein series  $E_k$  with  $k \geq 3$ , we have

$$\wp'(z) = -2E_3(z).$$

As will be seen, the functions  $\wp$  and  $\wp'$  are the most important ones on  $X$ ; every meromorphic function on  $X$  will turn out to be expressible in terms of them.

We will prove the following:

1.  $\wp$  and  $\wp'$  are algebraically related;
2.  $\wp$  and  $\wp'$  generate  $\mathcal{M}(X)$  as a field.

## 2.1 An Algebraic Relation Between $\wp$ and $\wp'$

What tool can we use to prove that an elliptic function  $f \equiv 0$ ? The basic principle is the *maximum principle*, which implies that any holomorphic elliptic function must be constant.

**Basic technique:** use the Laurent series expansion at the poles. If the Laurent expansion of  $f$  at each possible pole actually reduces to a Taylor series (i.e., all terms of negative degree vanish) and the constant term at one possible pole is zero, then  $f \equiv 0$ .

**Objective:** to find an algebraic relation between  $\wp$  and  $\wp'$ , the simpler the better.

Observation:  $\wp$  has a double pole at each lattice point  $\omega \in L$  and no other poles, while  $\wp'$  has a triple pole at each  $\omega \in L$  and no other poles. Thus,

$$\text{ord}_\omega(\wp^3) = 6, \quad \text{ord}_\omega((\wp')^2) = 6.$$

This suggests a connection between  $\wp^3$  and  $(\wp')^2$ .

Recall that

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L^*} \left( \frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \quad E_3(z) = \sum_{\omega \in L} \frac{1}{(z+\omega)^3}, \quad \wp'(z) = -2E_3(z).$$

Since  $\wp$  is even and  $\wp'$  is odd, we can write their local expansions near  $z = 0$  as

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots, \quad \wp'(z) = -\frac{2}{z^3} + b_1 z + b_3 z^3 + \cdots.$$

Therefore,

$$\wp^3(z) = \frac{1}{z^6} + \cdots, \quad (\wp'(z))^2 = \frac{4}{z^6} + \cdots,$$

which leads us to *guess* that

$$(\wp'(z))^2 = 4\wp^3(z) + \cdots ?$$

We will make this relation precise below.

We begin by expanding  $(\wp'(z))^2$  and  $\wp^3(z)$  using their Laurent series near  $z = 0$ .

Since

$$\wp'(z) = -\frac{2}{z^3} + b_1 z + b_3 z^3 + \cdots,$$

we have

$$\begin{aligned} (\wp'(z))^2 &= \left( -\frac{2}{z^3} + b_1 z + b_3 z^3 + \cdots \right)^2 \\ &= \frac{4}{z^6} + 2 \left( -\frac{2}{z^3} \right) (b_1 z + b_3 z^3 + \cdots) + \underbrace{(b_1 z + b_3 z^3 + \cdots)^2}_{\text{Taylor expansion, constant term 0}} \\ &= \frac{4}{z^6} - \frac{4b_1}{z^2} - 4b_3 + \cdots. \end{aligned}$$

Next, for  $\wp(z)$ , recall

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \dots.$$

Using  $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ , we get

$$\begin{aligned}\wp^3(z) &= \left( \frac{1}{z^2} + (a_2 z^2 + a_4 z^4 + \dots) \right)^3 \\ &= \frac{1}{z^6} + 3 \frac{1}{z^4} (a_2 z^2 + a_4 z^4 + \dots) + \underbrace{\frac{3}{z^2} (a_2 z^2 + a_4 z^4 + \dots)^2}_{\text{Taylor expansion, constant term 0}} \\ &= \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + \dots.\end{aligned}$$

Hence,

$$4\wp^3(z) = \frac{4}{z^6} + \frac{12a_2}{z^2} + 12a_4 + \dots$$

Subtracting gives

$$(\wp'(z))^2 - 4\wp^3(z) = \frac{-4b_1 - 12a_2}{z^2} + (-4b_3 - 12a_4) + \dots.$$

Define

$$g_2 = -4b_1 - 12a_2, \quad g_3 = -4b_3 - 12a_4,$$

and set

$$f(z) = (\wp'(z))^2 - 4\wp^3(z) - g_2\wp(z) - g_3.$$

Then, the principal part of  $f$  at  $z = 0$  vanishes, and the constant term there is zero. Since  $f$  is elliptic, its principal part at every lattice point  $\omega \in L$  also vanishes, and its constant term is zero. Hence, by the basic principle for elliptic functions, we have  $f \equiv 0$ .

**Theorem 2.1.** *Let  $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  be a lattice. Then there exist constants  $g_2, g_3 \in \mathbb{C}$  (depending only on  $L$ ) such that the Weierstrass functions  $\wp$  and  $\wp'$  satisfy the identity*

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3 \quad \text{for all } z \in \mathbb{C}.$$

## 2.2 The Generation of $\mathcal{M}(X)$ by $\wp$ and $\wp'$

We now turn to the explicit description of the field of meromorphic functions on  $X = \mathbb{C}/L$ .

**Theorem 2.2.**  *$\mathcal{M}(X)$  is generated by the Weierstrass functions  $\wp$  and  $\wp'$ .*

Reduction: Let  $f$  be an elliptic function. We decompose  $f$  into its even and odd parts:

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2} = f^{\text{even}}(z) + f^{\text{odd}}(z).$$

Define

$$\mathcal{M}^{\text{even}}(X) = \{f \in \mathcal{M}(X) \mid f(-z) = f(z)\}, \quad \mathcal{M}^{\text{odd}}(X) = \{f \in \mathcal{M}(X) \mid f(-z) = -f(z)\}.$$

Then we have  $\wp \in \mathcal{M}^{\text{even}}(X)$  and  $\wp' \in \mathcal{M}^{\text{odd}}(X)$ . Moreover, if  $h \in \mathcal{M}^{\text{odd}}(X)$ , then  $h\wp' \in \mathcal{M}^{\text{even}}(X)$ , since the product of two odd functions is even.

**Proposition 2.3.** *The field of even elliptic functions on  $X = \mathbb{C}/L$  is generated over  $\mathbb{C}$  by the Weierstrass function  $\wp$ ; that is,*

$$\mathcal{M}^{\text{even}}(X) = \mathbb{C}(\wp) = \left\{ \frac{P(\wp(z))}{Q(\wp(z))} \mid P, Q \in \mathbb{C}[x], Q \not\equiv 0 \right\}.$$

*Proof.* Let  $f \in \mathcal{M}^{\text{even}}(X)$ , so that  $f(z) = f(-z)$ . We proceed in two stages.

**Construction of a meromorphic function holomorphic at the lattice points.** The Laurent expansion of  $f$  near  $z = 0$  involves only even powers:

$$f(z) = \sum_{m=k}^{\infty} a_{2m} z^{2m},$$

since  $f$  is even. Hence  $\text{ord}_0(f) = 2s$  for some  $s \in \mathbb{Z}$ .

If  $f$  has a pole of order  $2n$  at 0 (so  $\text{ord}_0(f) = -2n$ ), define

$$h_0(z) = \frac{f(z)}{\wp(z)^n}.$$

Since  $\text{ord}_0(\wp^n) = -2n$ , we get  $\text{ord}_0(h_0) = 0$ , so  $h_0$  is holomorphic at 0. Because  $h_0$  is also even and elliptic, its periodicity ensures  $h_0(z + \omega) = h_0(z)$  for all  $\omega \in L$ ; hence  $h_0$  is holomorphic at every lattice point.

Thus we have produced a meromorphic function on  $\mathbb{C}$ , elliptic with respect to  $L$ , that is holomorphic at all lattice points.

**Determination of the function via the Weierstrass problem on  $X$ .** According to the Weierstrass problem on the torus  $X$ , once we specify a divisor

$$D = \sum_{a \in X} n_a [a], \quad \text{with} \quad \sum_a n_a = 0,$$

there exists a meromorphic function on  $X$  having zeros and poles exactly as prescribed by  $D$ , unique up to multiplication by a nonzero constant.

Because  $f$  is even, its zeros and poles occur in symmetric patterns. We distinguish two cases:

**Case 1.** When  $a \not\equiv -a \pmod{L}$ :

Consider the elliptic function  $\wp(z) - \wp(a)$ . Since  $\wp$  is even,  $\wp(-a) = \wp(a)$ . Therefore,  $\wp(z) - \wp(a)$  has simple zeros at  $z = a$  and  $z = -a$ , and a double pole at each lattice point  $\omega \in L$ . Specifically,

$$\text{ord}_0(\wp - \wp(a)) = -2, \quad \text{ord}_a(\wp - \wp(a)) = \text{ord}_{-a}(\wp - \wp(a)) = 1,$$

and there are no other zeros or poles modulo  $L$ .

For points  $a_k$  with  $2a_k \not\equiv 0 \pmod{L}$ , we have symmetric pairs  $\{a_k, -a_k\}$  with equal orders:

$$\text{ord}_{a_k}(f) = \text{ord}_{-a_k}(f) = n_k.$$

**Case 2.** When  $a \equiv -a \pmod{L}$ :

This occurs exactly when  $2a \equiv 0 \pmod{L}$ , i.e. when  $a$  is a *half-period*. The half-periods of the lattice  $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$  are

$$0, \quad \frac{\omega_1}{2}, \quad \frac{\omega_2}{2}, \quad \omega_3 := \frac{\omega_1 + \omega_2}{2}.$$

At each half-period  $\omega_i/2$  (where  $i = 1, 2, 3$ ), we have  $\wp'(\omega_i/2) = 0$ , so the function  $\wp(z) - \wp(\omega_i/2)$  has a double zero at  $z = \omega_i/2$ . The orders at half-periods must be even:

$$\text{ord}_{\frac{\omega_i}{2}}(f) = 2t_i, \quad t_i \in \mathbb{Z}.$$

Combining both cases, define

$$h(z) = \prod_{k=1}^s (\wp(z) - \wp(a_k))^{n_k} \prod_{i=1}^3 (\wp(z) - \wp(\frac{\omega_i}{2}))^{t_i}.$$

Each factor is even and elliptic, and the overall divisor of  $h$  coincides with that of  $f$ .

By the Weierstrass problem on  $X$ , the meromorphic function with this divisor is unique up to a multiplicative constant. Therefore the quotient  $f/h$  has no zeros or poles, hence  $f/h \equiv c \in \mathbb{C}^\times$ .

We obtain

$$f(z) = c \prod_{k=1}^s (\wp(z) - \wp(a_k))^{n_k} \prod_{i=1}^3 (\wp(z) - \wp(\frac{\omega_i}{2}))^{t_i},$$

so  $f$  is a rational function of  $\wp(z)$ .

Conversely, every rational function of  $\wp$  is even and elliptic. Hence

$$\mathcal{M}^{\text{even}}(X) = \mathbb{C}(\wp).$$

□

### 2.3 The Projective Embedding of the Complex Torus $\mathbb{C}/L$

The *complex projective space*  $\mathbb{P}^n(\mathbb{C})$  is defined as

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim,$$

where the equivalence relation is

$$u \sim v \iff \exists \lambda \in \mathbb{C}^* \text{ such that } u = \lambda v.$$

Thus, two nonzero vectors in  $\mathbb{C}^{n+1}$  represent the same point in  $\mathbb{P}^n$  if they differ by a nonzero scalar multiple. The equivalence class of  $(z_0, z_1, \dots, z_n)$  is written as

$$[z_0 : z_1 : \dots : z_n],$$

and these are called the *homogeneous coordinates* of the point.

The projective space  $\mathbb{P}^n$  can be covered by the open sets

$$U_k = \{[z_0 : \dots : z_n] \mid z_k \neq 0\}, \quad k = 0, 1, \dots, n.$$

Then

$$\mathbb{P}^n = \bigcup_{k=0}^n U_k.$$

On  $U_k$ , we define the *inhomogeneous coordinates* by dividing by  $z_k$ :

$$[z_0 : \cdots : z_k : \cdots : z_n] \mapsto \left( \frac{z_0}{z_k}, \dots, \frac{z_{k-1}}{z_k}, \widehat{\frac{z_k}{z_k} = 1}, \frac{z_{k+1}}{z_k}, \dots, \frac{z_n}{z_k} \right),$$

where the  $\widehat{\phantom{x}}$  symbol indicates that the corresponding component is *omitted* (since  $z_k/z_k = 1$  is fixed).

Thus, in these coordinates,

$$U_k \cong \mathbb{C}^n,$$

with coordinates

$$(z_0/z_k, \dots, z_{k-1}/z_k, z_{k+1}/z_k, \dots, z_n/z_k).$$

When  $n = 1$ , we obtain the *complex projective line*

$$\mathbb{P}^1 = \{[\xi_0 : \xi_1] \mid (\xi_0, \xi_1) \neq (0, 0)\},$$

covered by

$$U_0 = \{[\xi_0 : \xi_1] \mid \xi_0 \neq 0\}, \quad U_1 = \{[\xi_0 : \xi_1] \mid \xi_1 \neq 0\}.$$

On  $U_0$  we set  $z = \xi_1/\xi_0$ , and on  $U_1$  we set  $w = \xi_0/\xi_1$ . On the overlap  $U_0 \cap U_1$ , these coordinates satisfy  $zw = 1$ . Thus each  $U_i \cong \mathbb{C}$ , their intersection  $U_0 \cap U_1 \cong \mathbb{C}^*$ , and

$$\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\},$$

the Riemann sphere.

For  $n = 2$ , we obtain the *complex projective plane*

$$\mathbb{P}^2 = \bigcup_{k=0}^2 U_k,$$

where, for example, on  $U_0$  (with  $\xi_0 \neq 0$ ) the inhomogeneous coordinates are

$$[\xi_0 : \xi_1 : \xi_2] \mapsto \left( z_1^{(0)} = \frac{\xi_1}{\xi_0}, z_2^{(0)} = \frac{\xi_2}{\xi_0} \right),$$

so  $U_0 \cong \mathbb{C}^2$ . Similarly,  $U_1$  and  $U_2$  are also copies of  $\mathbb{C}^2$ , glued together through the appropriate rational transition functions.

$$\begin{aligned} \mathbb{P}^2 &= U_0 \cup \left\{ [\xi_0 : \xi_1 : \xi_2] \mid (\xi_0, \xi_1, \xi_2) \in \mathbb{C}^3 \setminus \{0\}, \xi_0 = 0 \right\} \\ &= U_0 \cup \left\{ [0 : \xi_1 : \xi_2] \mid (\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{0\} \right\}. \end{aligned}$$

The second set is naturally identified with the projective line:

$$\left\{ [0 : \xi_1 : \xi_2] \mid (\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{0\} \right\} \cong \mathbb{P}^1.$$

Hence,

$$\mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{P}^1 = \mathbb{C}^2 \sqcup \mathbb{C} \sqcup \{\text{pt}\}.$$

More generally, one obtains the recursive cell decomposition

$$\boxed{\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \{\text{pt}\}}.$$

**Theorem 2.4.** Let  $X = \mathbb{C}/L$  be a complex torus, where  $L$  is a lattice in  $\mathbb{C}$ . Then  $X$  is biholomorphic to an algebraic curve  $E \subset \mathbb{P}^2$ ; that is,

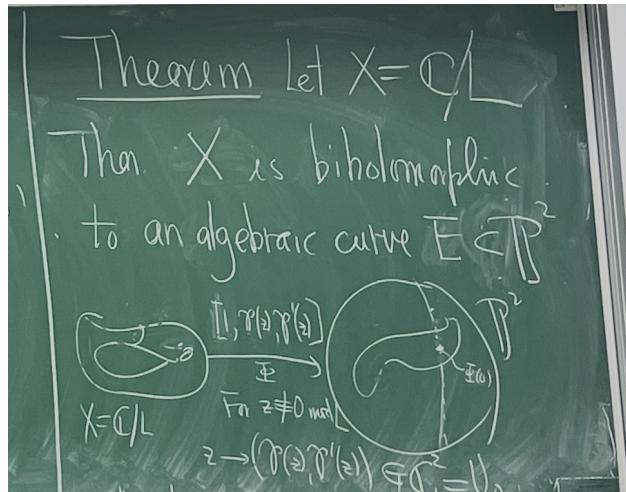
$$X = \mathbb{C}/L \xrightarrow[\Phi]{[1:\wp(z):\wp'(z)]} E \subset \mathbb{P}^2,$$

where  $\wp(z)$  denotes the Weierstrass  $\wp$ -function associated with  $L$ , and  $E$  is given by the Weierstrass equation

$$E : y^2 = 4x^3 + g_2x + g_3.$$

For  $z \not\equiv 0 \pmod{L}$ ,

$$z \mapsto (\wp(z), \wp'(z)) \in \mathbb{C}^2 = U_0.$$



If we write the projective plane as

$$\mathbb{P}^2 = \{[\omega_0, \omega_1, \omega_2] : (\omega_0, \omega_1, \omega_2) \in \mathbb{C}^3 \setminus \{0\}\},$$

then on the affine chart

$$U_0 = \{\omega_0 \neq 0\}, \quad x = \frac{\omega_1}{\omega_0}, \quad y = \frac{\omega_2}{\omega_0},$$

the Weierstrass equation of the elliptic curve takes the form

$$y^2 = 4x^3 + g_2x + g_3.$$

Multiplying by  $\omega_0^4$  yields the homogeneous cubic equation

$$\omega_0^2 \omega_2^2 = 4\omega_1^3 + g_2 \omega_1 \omega_0^2 + g_3 \omega_0^3.$$

Hence, the projective curve  $E$  is given by

$$E = \left\{ [\omega_0, \omega_1, \omega_2] \in \mathbb{P}^2 : P(\omega_0, \omega_1, \omega_2) = 0 \right\},$$

where  $P$  is the homogeneous cubic polynomial

$$P(\omega_0, \omega_1, \omega_2) = \omega_0^2 \omega_2^2 - (4\omega_1^3 + g_2 \omega_1 \omega_0^2 + g_3 \omega_0^3).$$

**Definition 2.5** (Projective embedding of a compact Riemann surface). Let  $X$  be a compact Riemann surface, and

$$\Phi : X \longrightarrow \mathbb{P}^N$$

be a holomorphic mapping. We call  $\Phi$  a *holomorphic embedding* if and only if:

- (a)  $\Phi$  is a holomorphic immersion at every point  $x \in X$ .
- (b)  $\Phi$  separates points, i.e.  $\Phi(x) \neq \Phi(y)$  whenever  $x \neq y$ ,  $x, y \in X$ .

*Remark 2.6.* A *holomorphic mapping* means a continuous map that is holomorphic with respect to local holomorphic coordinate charts. For example, if  $x_0 \in \mathbb{P}^N = \bigcup_{k=0}^N U_k$ , where  $U_k \cong \mathbb{C}^N$ , then near  $x_0$  the map  $\Phi$  can be written in inhomogeneous coordinates as

$$\Phi(z) = (f_1(z), \dots, f_N(z)),$$

where each  $f_k$  is holomorphic.

**Definition 2.7** (Holomorphic immersion). A holomorphic map

$$\Phi : X \longrightarrow \mathbb{P}^N$$

is said to be a *holomorphic immersion* at  $x_0 \in X$  if the differential

$$d\Phi_{x_0} : T_{x_0}(X) \longrightarrow T_{\Phi(x_0)}(\mathbb{P}^N) \cong \mathbb{C}^N$$

is injective.

When  $\dim_{\mathbb{C}} X = 1$  and on a local coordinate chart we write  $\Phi = (f_1, \dots, f_N)$ , this condition simply means that

$$f'_k(x_0) \neq 0 \quad \text{for some } k, 1 \leq k \leq N.$$

**Theorem 2.8.** Let  $L \subset \mathbb{C}$  be a lattice, and define

$$\tilde{\Phi} : \mathbb{C} \longrightarrow \mathbb{P}^2, \quad \tilde{\Phi}(z) = [1 : \wp(z) : \wp'(z)] \in U_0.$$

For every point  $z \in \mathbb{C} \setminus L$ , the map  $\tilde{\Phi}$  is holomorphic. It extends to a holomorphic mapping on all of  $\mathbb{C}$ , still denoted by the same symbol

$$\tilde{\Phi} : \mathbb{C} \longrightarrow \mathbb{P}^2.$$

Furthermore,  $\tilde{\Phi}$  is invariant under translation by any  $\omega \in L$ ; that is,

$$\tilde{\Phi}(z + \omega) = \tilde{\Phi}(z), \quad \forall \omega \in L.$$

Hence,  $\tilde{\Phi}$  descends to a well-defined holomorphic map

$$\Phi : X = \mathbb{C}/L \longrightarrow \mathbb{P}^2.$$

Moreover,  $\Phi$  is a holomorphic embedding, mapping  $X$  biholomorphically onto a smooth projective curve

$$Z = \Phi(X),$$

which is defined by a homogeneous cubic polynomial.

*Proof.* The map

$$\tilde{\Phi} : \mathbb{C} \setminus L \longrightarrow U_0 \subset \mathbb{P}^2, \quad \tilde{\Phi}(z) = [1 : \wp(z) : \wp'(z)],$$

is holomorphic and invariant under translation by any lattice point  $\omega \in L$ , since both  $\wp$  and  $\wp'$  are elliptic with respect to  $L$ . By definition, for every  $z \in \mathbb{C} \setminus L$ ,

$$\tilde{\Phi}(z) = [1 : \wp(z) : \wp'(z)].$$

**Extension to a holomorphic map on all of  $\mathbb{C}$ .** Near  $z = 0$ , the Laurent expansions are

$$\wp(z) = \frac{1}{z^2} + h(z), \quad \wp'(z) = -\frac{2}{z^3} + h'(z),$$

where  $h$  is a holomorphic function (given by a Taylor expansion). Hence

$$\tilde{\Phi}(z) = [1 : \frac{1}{z^2} + h(z) : -\frac{2}{z^3} + h'(z)].$$

Multiplying by  $z^3$  for homogenization, we get

$$\tilde{\Phi}(z) = [z^3 : z^3h(z) + z : -2 + z^3h'(z)] = \left[ \frac{z^3}{-2 + z^3h'(z)} : \frac{z + z^3h(z)}{-2 + z^3h'(z)} : 1 \right] \in U_2.$$

Define  $\tilde{\Phi}(0) = [0 : 0 : 1] \in U_2$ . This gives a holomorphic extension near  $z = 0$ . The same argument works near any  $\omega \in L$ , so we obtain a global holomorphic map

$$\tilde{\Phi} : \mathbb{C} \longrightarrow \mathbb{P}^2,$$

with  $\tilde{\Phi}(\omega) = [0 : 0 : 1]$  for all  $\omega \in L$ . Hence  $\tilde{\Phi}$  descends to a holomorphic map

$$\Phi : X = \mathbb{C}/L \longrightarrow \mathbb{P}^2.$$

Claim 1  $\Phi : X \rightarrow \mathbb{P}^2$  is a holomorphic immersion.

(a) Near  $z = 0$ , using the previous expansion,

$$\tilde{\Phi}(z) = (-\frac{z^3}{2} + \dots, -\frac{z}{2} + \dots) \in U_2 \cong \mathbb{C}^2.$$

Hence  $\tilde{\Phi}'(0) = (0, -\frac{1}{2}) \neq (0, 0)$ , so  $\tilde{\Phi}$  is an immersion at 0, and therefore also at every  $\omega \in L$ .

(b) If  $z_0 \in \mathbb{C} \setminus L$  and  $\tilde{\Phi}$  were not immersive at  $z_0$ , then  $\wp'(z_0) = \wp''(z_0) = 0$ . Consider  $f(z) = \wp(z) - \wp(z_0)$ . Then  $f(z_0) = f'(z_0) = f''(z_0) = 0$ , so  $\text{ord}_{z_0}(f) \geq 3$ . However,  $f$  has only double poles at lattice points and no other poles; by the principle that for elliptic functions the number of zeros equals the number of poles (counting multiplicities), this is impossible. Hence  $\Phi$  is immersive everywhere.

Claim 2  $\Phi : X \rightarrow \mathbb{P}^2$  separates points.

We argue by contradiction. Note first that  $\Phi(0) = [0 : 0 : 1] \notin U_0$ , so  $\Phi(0) \neq \Phi(x)$  for all  $x \in X \setminus \{0\}$ . It remains to consider  $x, y \in X \setminus \{0\}$  with  $\Phi(x) = \Phi(y)$ .

Choose lifts  $a, b \in \mathbb{C}$  such that  $\pi(a) = x$ ,  $\pi(b) = y$ , where  $\pi : \mathbb{C} \rightarrow X = \mathbb{C}/L$  is the projection. In the affine chart  $U_0 \cong \mathbb{C}^2$ ,  $\Phi(z) = (\wp(z), \wp'(z))$ , so  $\Phi(a) = \Phi(b)$  implies

$$\wp(a) = \wp(b), \quad \wp'(a) = \wp'(b).$$

Consider  $f(z) = \wp(z) - \wp(a)$ .

*Case 1.*  $a \not\equiv -a \pmod{L}$ . Since  $\wp$  is even,  $f(z)$  has precisely two simple zeros at  $a$  and  $-a$ . If  $\wp(a) = \wp(b) = \wp(-a)$  and  $a \not\equiv b \pmod{L}$ , then  $b \equiv -a \pmod{L}$ . But then, from  $\wp'(a) = \wp'(b)$  and the oddness of  $\wp'$ , we get  $\wp'(a) = \wp'(-a) = -\wp'(a)$ , so  $\wp'(a) = 0$ . This gives  $\text{ord}_a(f) \geq 2$  and  $\text{ord}_{-a}(f) \geq 2$ , contradicting the zero–pole count for elliptic functions.

*Case 2.*  $a \equiv -a \pmod{L}$ ; that is,  $2a \equiv 0$ , so  $a$  is a half-period:

$$a \equiv \frac{\omega_i}{2} \pmod{L}, \quad i = 1, 2, 3.$$

Suppose  $a, b \in \{\frac{\omega_i}{2} + L\}$  with  $a \not\equiv b \pmod{L}$  and  $\wp(a) = \wp(b)$ . But then  $f(z) = \wp(z) - \wp(a)$  has zeros of order 2 at both  $a$  and  $b$ , which again contradicts the zero–pole count. Therefore no such distinct  $a, b$  exist.

Thus,  $\Phi$  separates points.

Since  $\Phi$  is both a holomorphic immersion and separates points, it is a holomorphic embedding. Its image is the smooth cubic curve

$$Z = \Phi(X) = \{[\omega_0 : \omega_1 : \omega_2] \in \mathbb{P}^2 \mid P(\omega_0, \omega_1, \omega_2) = 0\},$$

where  $P$  is the homogeneous cubic polynomial

$$P(\omega_0, \omega_1, \omega_2) = \omega_0^2 \omega_2^2 - (4\omega_1^3 + g_2 \omega_1 \omega_0^2 + g_3 \omega_0^3).$$

□