$a(ab)^{n-1}b = a^n b^n = (ab)^n = (ab)^{n-1}ab$

$a(ab)^{n-1} = (ab)^{n-1}a$, sim, $(ab)^{n-1}b = b(ab)^{n-1}$

MATH3301 Tutorial 3

1. Let G be a group and $n \in \mathbb{Z}$ (some). Suppose that for all $a, b \in G$,
$$(ab)^{n-1} = a^{n-1}b^{n-1}, \quad (ab)^n = a^n b^n, \quad (ab)^{n+1} = a^{n+1}b^{n+1}.$$
Show that G is abelian.

2. For a group G and $a \in G$, define the maps l_a and $r_a : G \to G$ by
$$l_a(g) = ag, \quad r_a(g) = ga, \quad g \in G,$$
and call them, respectively, the **left translation** and **the right translation of G by a**.
   (a) Show that both maps $l_a, r_a : G \to G$ are bijective.
   (b) Are l_a and r_a in general isomorphisms? Explain your answer.
   (c) Is the map $c_a : G \to G$ defined by $c_a = l_a \circ r_a^{-1}$ an isomorphism? Justify your answer. (The map c_a is called the **conjugation on G by a**.)

3. (a) Construct an isomorphism from $\mathbb{Z}_8^\times$ to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
   (b) Construct an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_{>0}, \times)$.

4. Let H be a subset of the group G. Show that H is a subgroup of G if and only if $H \neq \emptyset$ and $gh^{-1} \in H$ whenever $g, h \in H$.

5. Let G be a finite group and $x, y \in G$. Prove or disprove, with justifications, the following:
   (a) $\text{ord}(x^m) = m \cdot \text{ord}(x)\ \forall\ m \in \mathbb{N}$ (b) $\text{ord}(x^{-1}) = \text{ord}(x)$
   (c) $\text{ord}(xy) = \text{ord}(yx)$ (d) $\text{ord}(xy) = \text{ord}(x) + \text{ord}(y)$ (e) $\text{ord}(x) = \text{ord}(yxy^{-1})$

6. (a) Let $\phi : G \to G'$ be a homomorphism and $a \in G$. Show that $\text{ord}(\phi(a))$ divides $\text{ord}(a)$ if $\text{ord}(a) < \infty$.
   (b) Show that H and K defined below are groups.
   - $H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ under matrix multiplication.
   - $K = \{1, -1, i, -i\}$ under multiplication (of complex numbers).

   (c) Find all homomorphisms from H to K and state which one is an isomorphism.

End