Zariski top on k^n

$A = k[x_1, \cdots, x_n] \supset T$

$Z(T) = \{x \in k^n \mid f(x) = 0 \ \forall f \in T\}$

$\mathcal{C} = \{Z(T) \mid T\} \cup \{\emptyset, k^n\}$

Properties

A comm. ring with unit

$T \subset A$

$I(T) := \{\sum_{i=1}^{N} f_i g_i \mid f_i \in T \ \ g_i \in A\}$

a) $Z(T) = Z(I(T))$

b) I_α ideals of A

$$\bigcap_\alpha Z(I_\alpha) = Z(\sum I_\alpha)$$

$$\sum I_\alpha = \{\sum_{i=1}^{N} f_{\alpha_i} \mid f_{\alpha_i} \in I_{\alpha_i}\}$$

$$\bigcup_{i=1}^{N} Z(I_{\alpha_i}) = Z(\bigcap_{i=1}^{N} I_{\alpha_i})$$

c) $k = \mathbb{R}$ $\quad n = 1 \quad A = \mathbb{R}[x]$

Zariski top on $\mathbb{R}$ = cofinite top on $\mathbb{R}$

closed set of LHS

$Z(I)$ $\quad I \subset \mathbb{R}[x]$ is principal $\exists f \in \mathbb{R}[x]$

$\shortparallel$

$Z(f)$ $\quad (f) = I = \{fg \mid g \in \mathbb{R}[x]\}$

$f = a_0 + a_1 x + \cdots + a_d x^d \quad \mathbb{R}$

$Z(f) = \mathbb{R}$ roots of f

$|Z(f)| \leq d$

$\{\alpha_1, \cdots, \alpha_d\} \subset \mathbb{R}$

$f = \prod_{i=1}^{d} (x - \alpha_i) \quad Z(f) = \{\alpha_1, \cdots, \alpha_d\}$

Abstract Zariski top

A: comm. ring

$\mathfrak{p} \subset A$ ideal is called prime

if $\quad ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$

$\operatorname{Spec} A :=$ set of all prime ideals of A.

$\mathcal{C} = Z(I) \quad I$ ideal

$Z(I) = \{\mathfrak{p} \mid \mathfrak{p} \supset I\}$

this defines a top.

Example $A = \mathbb{Z}$

$T \subset \mathbb{Z}$ $\quad \gcd(T)$ the greatest common divisor

$\operatorname{lcm}(T)$ least common multiple

Any ideal of $\mathbb{Z}$ is principal

$I = (n)$

$I(T) = (\gcd(T))$

chinese remainder theorem.

e.g.

$$(6, 8) = \{ \underbrace{6n + 8m}_{\text{even } \#} \mid n, m \in \mathbb{Z} \}$$

$\rotatebox{90}{=}$

$(6) + (8)$

$\gcd(6,8) = 2$

$T = \{ n_\alpha \}_{\alpha \in \Lambda}$

$$\boxed{I(T) = \sum_{\alpha} (n_\alpha) = (\gcd(T))}$$

$n_1 \cdots n_k$

$$\boxed{\bigcap_{i=1}^{k} (n_i) = (\operatorname{lcm}(n_1, \cdots, n_k))}$$

$$(6) \cap (8) = \{ n \mid 6 \mid n, 8 \mid n \}$$

$$= \{ n \mid 24 \mid n \}$$

$\operatorname{lcm}(6,8) = 24$

7) (n) is prime iff n is prime $\boxed{+\ n=0}$

$ab \in (n) \implies a \in (n)$ $b \in (n)$

$n \mid ab$ $n \mid a$ or $n \mid b$

2) $Z(n) = \{ (p) \supset (n) \} \iff p \mid n$

$= \{ (p) \mid p \text{ prime factor} \}$

$\bigcap Z(n_\alpha) = \{ (p) \mid p \text{ is prime factors of all } n_\alpha \}$

$= \{ (p) \mid p \text{ prime factors of } \gcd(n_\alpha) \}$

$= Z(\gcd_\alpha(n_\alpha))$

Def'n Let $\mathcal{O}, \mathcal{O}'$ topologies on X.

$\mathcal{O} \supset \mathcal{O}'$ $\mathcal{O}$ is *finer* than $\mathcal{O}'$

$\mathcal{O}'$ is *coarser* than $\mathcal{O}$

Ex Zariski top on $\mathbb{R}$ is coarser than metric top

Def'n $(X, \mathcal{O})$ top space

$\mathcal{B} \subset \mathcal{O}$ is called a *basis*

if every element in $\mathcal{O}$ is a union of elements in $\mathcal{B}$.

$\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{O}$ is called *sub basis*

if any element of $\mathcal{B}$ is a finite $\bigcap$ of elements in $\mathcal{B}_0$

Example

1) (X, d) metric space

$\{B(x,r) \mid x \in X,\ r>0\}$ is a basis

$B(x,r) = \{y \mid d(x,y)<r\}$

2) $(\mathbb{R}, |\cdot|)$

$\{(a,+\infty), (-\infty, b)\}$ is a subbasis

3) (k^n, Zar)

$\{Z(f)^c \mid f \in k[x_1, \cdots x_n]\}$ is basis

Q: what a subbasis ?

Rmk if $\mathbb{B} \subset P(X)$ satisfies

X is covered by elements of $\mathbb{B}$

$B_1, B_2 \in \mathbb{B}$ $\quad B_1 \cap B_2 \in \mathbb{B}$ ✓

then arbitrary union of elements in $\mathbb{B}$ is a top.

We call the top is generated by $\mathbb{B}$

$$\left(\bigcup_\alpha B_\alpha\right) \cap \left(\bigcup_\alpha A_\alpha\right)$$

$$= \bigcup_\alpha \left(B_\alpha \cap \left(\bigcup A_\beta\right)\right)$$

$$= \bigcup_\alpha \left(\bigcup_\beta (B_\alpha \cap A_\beta)\right)$$

Def'n $f: X \longrightarrow Y$ is called **continuous**
$\mathcal{O}_X \qquad \mathcal{O}_Y$

if $\forall\, V \in \mathcal{O}_Y \quad f^{-1}(V) \in \mathcal{O}_X$

$f^{-1}(V) := \{ x \in X \mid f(x) \in V \}$

equivalently $\quad (f^{-1}(V))^c = f^{-1}(V^c)$

$A \in \mathcal{C}_Y \quad f^{-1}(A) \in \mathcal{C}_X$

Rmk it suffices to check continuity

of $\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{O}$

$$f^{-1}\Big(\bigcup_\alpha (U_\alpha \cap V_\alpha)\Big) = \bigcup_\alpha f^{-1}(U_\alpha \cap V_\alpha)$$
$$= \bigcup_\alpha \big(f^{-1}(U_\alpha) \cap f^{-1}(V_\alpha)\big)$$

Def'n $(X, \mathcal{O}_X) \supset Y$

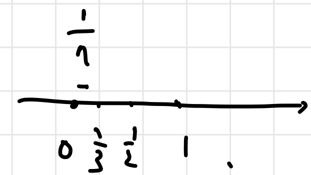
Def'n $\mathcal{O}_Y = \{ U \cap Y \mid U \in \mathcal{O}_X \}$

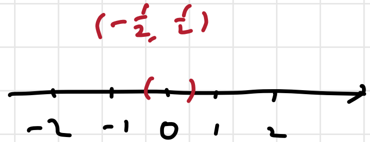
we call $(Y, \mathcal{O}_Y)$ subspace

Example

$\mathbb{Z} \hookrightarrow (\mathbb{R}, |\cdot|)$ discrete

Assign $(\mathrm{Spec}\,\mathbb{Z}, \mathrm{Zar})$ is not discrete





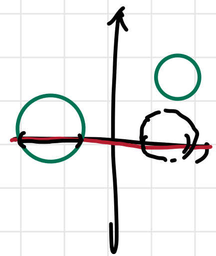
Rmk $Y \overset{i}{\subset} (X, \mathcal{O}_X)$ then

the subspace top on Y is the

coarsest top s.t. i is cont.

Ex $\mathbb{R} \longrightarrow (\mathbb{R}^2, |\cdot|)$

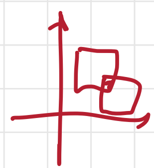
Show that the subspace top = metric top.



Def'n X, Y top spaces

$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ Cartesian product

$\mathcal{O}_\square$ the top generated by the basis ⇐ box top



$$\mathcal{B} = \left\{ U \times V \;\middle|\; \begin{array}{l} U \in \mathcal{O}_X \\ V \in \mathcal{O}_Y \end{array} \right\}$$

$$U_1 \times V_1 \cap U_2 \times V_2 = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

$\{X_\lambda \mid \lambda \in \Lambda\}$ collection of top sp.

$\prod_{\lambda \in \Lambda} X_\lambda$ Cartesian product

we still $\mathcal{O}_\square$

$\mathcal{O}_\Pi$ product top

$$\prod_{\lambda \in \Lambda} X_\lambda \xrightarrow{p_\lambda} X_\lambda \qquad \mathcal{O}_\lambda$$

$\mathcal{O}_\Pi$ = coarsest top s.t. p_λ are continuous

$$\{p_\lambda^{-1}(U_\lambda) \mid U_\lambda \in \mathcal{O}_\lambda\} \text{ generate } \mathcal{O}_\Pi$$

sub basis

$$p_\lambda^{-1}(U_\lambda) = \prod_{\mu \neq \lambda} X_\mu \times U_\lambda$$

Properties

1) $\mathcal{O}_\square = \mathcal{O}_\pi$ if Λ is finite

2)

$$\begin{tikzcd} & \prod_{\lambda\in\Lambda} X_\lambda , \mathcal{O}_\pi \arrow[d, "p_\lambda"] \\ Y \arrow[ur, "f=(f_\lambda)_{\lambda\in\Lambda}"] \arrow[r, "f_\lambda"'] & X_\lambda \end{tikzcd}$$

Then f is continuous $\Leftrightarrow$ f_λ are cont $\forall \lambda \in \Lambda$

3) $\mathcal{O}_\pi$ is coarser than $\mathcal{O}_\square$

$\neq$ if $|\Lambda| = \infty$

$X \xrightarrow{f} Y \xrightarrow{g} Z$ cont.

$g\circ f$ cont.

$(g\circ f)^{-1}(W) = f^{-1}(g^{-1}W)$

Example

$\Lambda = \mathbb{N} = \{1, 2, \dots\}$

$\mathbb{R}, |\cdot|$ $\qquad \mathbb{R}^{\mathbb{N}} = \overset{1}{\mathbb{R}} \times \overset{2}{\mathbb{R}} \times \dots \times \mathbb{R} \times \dots$

$U_n = (-\frac{1}{n}, \frac{1}{n})$ $\qquad \prod_{n\in\mathbb{N}} U_n \notin \mathcal{O}_\pi$

$\Delta : \mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}} \xrightarrow{p_n} \mathbb{R}$

$\Delta(x) \to (x, x, x \dots)$

$\Delta_n = id_{\mathbb{R}}$ is cont. $\qquad x \in (-\frac{1}{n}, \frac{1}{n})$ $\forall n$

by (2) $\Delta^{-1}(\prod_{n\in\mathbb{N}} U_n) = 0$.

$(X, \sim)$ top space with eq. relation.

$\downarrow p$

$[X]$ is set of eq classes

Def'n the quotient top $\mathcal{O}_{[X]}$ is defined to be the finest top. s.t p is continuous.

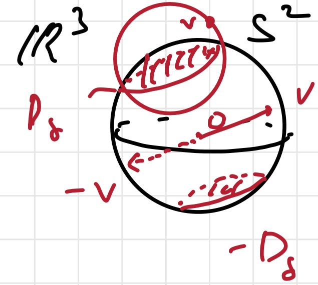
i.e. $\mathcal{O}_{[X]} = \{ [U] \subset [X] \mid p^{-1}([U]) \in \mathcal{O}_X \}$

Example $S^{n-1} = \{ v \in \mathbb{R}^n \mid |v| = 1 \}$

$v \sim -v$

$B(v_0, \delta)$

$= \{ v \in \mathbb{R}^3 \mid |v - v_0| < \delta \}$



$D_{v_0,\delta} = B(v_0, \delta) \cap S^2$

$\delta < \sqrt{2}$ $\quad p^{-1}(p(D_{v_0,\delta})) = D_{v_0,\delta} \sqcup -D_{v_0,\delta}$

$v \to -v$ map one hemi-sphere to the other

$\delta > \sqrt{2}$

$p^{-1}(p(D_{v_0,\delta})) = S^2$

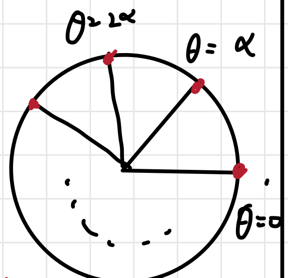
Assign. Show that p maps open set to open sets

Example $S^1 \xrightarrow{p} S^1/\sim$

$\theta \sim \theta + \alpha$

(mod 2π)

α/π irrational e.g. $\alpha = \sqrt{2}\pi$



$S^1 \setminus \{\alpha\}$ open

but $p(S^1 \setminus \{\alpha\})$ is not open

π

$p^{-1}(p(S^1 \setminus \{\alpha\}))$ is not open

Ex dense, infinite.

also not open

Def'n 1) X $(Y_\alpha, \mathcal{O}_\alpha)$ top spaces

$f_\alpha : X \to Y_\alpha$

Define the initial top $\mathcal{O}_X$ on X to be the coarsest top s.t f_α are cont.

$\mathcal{O}_X$ gen. by $\{f_\alpha^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{O}_\alpha\}$

2) $g_\alpha : Y_\alpha \to X$

Define the final top on X to be the finest top s.t g_α are cont.

$\mathcal{O}_X$ gen. by $\{V \mid g_\alpha^{-1}(V) \in \mathcal{O}_\alpha\}$

Property

$$\begin{tikzcd} & Z \arrow[dl, "\phi"'] \arrow[dr, "f_\alpha \circ \phi"] & \\ X \arrow[rr, "f_\alpha"] & & Y_\alpha \end{tikzcd}$$

ϕ is cont

$\iff$ $f_\alpha \circ \phi$ is cont

$\forall \alpha$

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "\psi"] & & Y_\alpha \arrow[ll, "g_\alpha"] \arrow[ul, "\psi \circ g_\alpha"'] \end{tikzcd}$$

ψ is cont

$\iff$ $\psi \circ g_\alpha$ is cont

$\forall \alpha$.