Richard Haberman's "Applied Partial Differential Equations: with Fourier Series and Boundary Value Problems": Chapter 2. Method of Separation of Variables

2.1 Introduction

PDE Problem

In chapter 1, we have derived the heat equations for a 1D rod. For example, the temperature u may satisfy the following IBVP:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho} && \text{for } t > 0 \text{ and } 0 < x < L && (2.1.1)\\
u(x,0) &= f(x) && \text{for } 0 < x < L && (2.1.2)\\
u(0,t) &= T_1(t) && \text{for } t > 0 && (2.1.3)\\
u(L,t) &= T_2(t) && \text{for } t > 0.
\end{aligned}
$$

How can we solve it?

We can apply **Method of Separation Variables** provided that the PDE and the boundary conditions are linear and homogeneous.

2.2 Linearity

Terminology

Operator

- An operator is a function that takes a function as an argument instead of numbers as we are used to dealing with in functions.
- Some examples of operators:

$$L = \frac{d}{dx}, \qquad L = \int \cdot \, dx, \qquad L = \frac{\partial}{\partial x}.$$

- Or equivalently, if we plug in a function, say $u(x)$, we get,

$$L(u) = \frac{du}{dx}, \qquad L(u) = \int u \, dx, \qquad L(u) = \frac{\partial u}{\partial x}.$$

- An important example is the **heat operator**:

$$L(u) = \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2}. \tag{2.2.2}$$

Terminology

Linear operator

An operator L is **linear** if L satisfies the *linearity property*: for any functions u_1 and u_2, and any arbitrary constants c_1 and c_2,

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2). \tag{2.2.1}$$

Example (Heat operator)

Heat operator is a linear operator: let $L := \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$, then

$$\begin{aligned}
L(c_1 u_1 + c_2 u_2) &= \frac{\partial}{\partial t}(c_1 u_1 + c_2 u_2) - k\frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2) \\
&= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} - kc_2\frac{\partial^2 u_2}{\partial x^2} \\
&= c_1\left(\frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2}\right) + c_2\left(\frac{\partial u_2}{\partial t} - k\frac{\partial^2 u_2}{\partial x^2}\right) \\
&= c_1 L(u_1) + c_2 L(u_2).
\end{aligned}$$

Terminology

Linear equation

A **linear equation** for the *unknown* u is of the form

$$L(u) = f, \tag{2.2.3}$$

where L is a linear operator and f is known.

Example (Heat equation)

Heat equation is a linear PDE:

$$L(u) = \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f(x,t). \tag{2.2.4}$$

Or, in a more common form,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + f(x,t). \tag{2.2.5}$$

Terminology

Homogeneous / Nonhomogeneous

Consider a linear equation

$$L(u) = f.$$

- If $f = 0$, then the equation is said to be **homogeneous**.
- Otherwise, if $f \neq 0$, the equation is said to be **nonhomogeneous**.

Example (Homogeneous heat equation)

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0. \tag{2.2.6}$$

Terminology

Trivial solution

- From the linearity property (2.2.1):

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2),$$

  we have $L(0) = 0$ (let $c_1 = c_2 = 0$).
- $u \equiv 0$ is always a solution of a linear homogeneous equation.
- For example, $u \equiv 0$ satisfies the heat equation (2.2.6).
- We call $u \equiv 0$ the trivial solution of a linear homogeneous equation.

Remark

- In fact, there is a nice way of determining if an equation is homogeneous or not:

  If $u \equiv 0$ satisfies a *linear* equation, then it must be that $f = 0$ and hence the linear equation is homogeneous.

Principle of Superposition

If u_1 and u_2 are solutions to a linear homogeneous equation $L(u) = 0$, then so is $c_1 u_1 + c_2 u_2$ where c_1 and c_2 are any constants.

Proof

Let u_1 and u_2 be solutions of a *linear* **homogeneous** equations, i.e.,

$$L(u_1) = L(u_2) = 0.$$

Using the *linearity property*, we have

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2) = 0.$$

Thus, $c_1 u_1 + c_2 u_2$ also satisfies $L(u) = 0$.

Moral:

Principle of superposition is an *important* idea to solve the linear PDE. See Subsection 2.5.1 for instance.

Boundary conditions (BCs)

We can also extend the ideas of linearity and homogeneity to BCs.

Example (Heat equation)

- Examples of linear boundary conditions:

$$u(0, t) = f(t) \tag{2.2.7}$$

$$\frac{\partial u}{\partial x}(L, t) = g(t) \tag{2.2.8}$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \tag{2.2.9}$$

$$-K_0 \frac{\partial u}{\partial x}(L, t) = h[u(L, t) - g(t)]. \tag{2.2.10}$$

- Only (2.2.9) is satisfied by $u \equiv 0$, and hence, is homogeneous.
- It is not necessary that a BC be $u(0, t) = 0$ for $u \equiv 0$ to satisfy it.

2.3 Heat Equation with Zero Temperatures at Finite Ends

2.3.1 Introduction

We begin by studying the heat equation with no sources and constant thermal coefficients:

$$\text{PDE:} \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.3.1}$$

$$\text{BC:} \quad u(0,t) = 0 \text{ and } u(L,t) = 0 \tag{2.3.2}$$

$$\text{IC:} \quad u(x,0) = f(x). \tag{2.3.3}$$

Why do we start with (2.3.1) - (2.3.3)?

- It can be solved by the **method of separation of variables**.
- It is a relevant physical problem:
  - 1D rod with no heat sources; and
  - both ends immersed in a 0° temperature bath.
- Using the solution of (2.3.1) - (2.3.3) as a building block, we can then solve the nonhomogeneous problem (2.1.1) - (2.1.3).

2.3.2 Separation of Variables

Main idea

In order to solve the IBVP (2.3.1) - (2.3.3), we first try to find some special solutions to

$$\text{PDE:} \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.3.1}$$

$$\text{BC:} \qquad u(0,t) = 0 \text{ and } u(L,t) = 0. \tag{2.3.2}$$

Then using these special solutions as building blocks, we will construct the solution to IBVP (2.3.1) - (2.3.3).

Separation of Variables

We attempt to determine solutions in the *product form*

$$\boxed{u(x,t) = \phi(x)G(t).} \tag{2.3.4}$$

- $\phi(x)$ is only a function of x and $G(t)$ is a function of t.
- The *product* (2.3.4) must satisfy the linear homogeneous PDE (2.3.1) and BC (2.3.2), but not the IC (2.3.3).

Why does the product form (2.3.4) work?

The product form (2.3.4) works because it reduces a PDE to ODEs.

According to the product form (2.3.4): $u(x,t) = \phi(x)G(t)$, we have

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t).$$

Substituting these into the heat equation (2.3.1): $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$,

$$\phi(x)\frac{dG}{dt} = k\frac{d^2\phi}{dx^2}G(t). \tag{2.3.5}$$

We can "**separate variables**" by dividing both sides of (2.3.5) by $k\phi(x)G(t)$, and obtain

$$\boxed{\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2}.} \tag{2.3.6}$$

Recall that $G := G(t)$ and $\phi := \phi(x)$, so

$$\underbrace{\frac{1}{kG}\frac{dG}{dt}}_{\text{function of } t \text{ only}} = \underbrace{\frac{1}{\phi}\frac{d^2\phi}{dx^2}}_{\text{function of } x \text{ only}}. \tag{2.3.6}$$

The LHS is only a function of t and the RHS only a function of x.

Separation Constant

What will happen if function of time = function of space?
Both sides of (2.3.6) must equal the same constant:

$$\boxed{\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda,} \tag{2.3.7}$$

where λ is some constant known as the **separation constant**.

Separation Constant

According to

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda, \tag{2.3.7}$$

we obtain two ODEs, one for $\phi(x)$ and one for $G(t)$:

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda\phi;} \tag{2.3.8}$$

$$\boxed{\frac{dG}{dt} = -\lambda kG.} \tag{2.3.9}$$

Remarks

- λ is a constant.
- The minus sign, which was introduced for convenience, will be explained in Subsection 2.3.3.
- The product solution $u(x,t) = \phi(x)G(t)$ must also satisfy the two homogeneous BCs: $u(0,t) = 0$ and $u(L,t) = 0$.

Trivial vs. Nontrivial Solutions

Trivial solution

$u \equiv 0$ is always a solution to the linear homogeneous heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}. \tag{2.3.1}$$

Example (Product form (2.3.4): $u(x,t) = \phi(x)G(t)$)

If $\phi(x) \equiv 0$ or $G(t) \equiv 0$, then (2.3.4) will be a trivial solution:

$$u(x,t) := \phi(x)G(t) \equiv 0.$$

Nontrivial solution

The trivial solution $u \equiv 0$ is not very interesting because it is not a building block of IBVP (2.3.1) - (2.3.3), so while we are finding special solution with the product form (2.3.4), we want to avoid the trivial solution by requiring

$$\phi(x) \not\equiv 0 \quad \text{and} \quad G(t) \not\equiv 0.$$

Boundary Conditions for ϕ

Let us recall that the product solution $u(x,t) := \phi(x)G(t)$ should also satisfy the boundary conditions (BC):

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0. \tag{2.3.2}$$

Substituting the product form $u(x,t) := \phi(x)G(t)$ into the BC (2.3.2), we have

$$\phi(0)G(t) = u(0,t) = 0 \quad \text{and} \quad \phi(L)G(t) = u(L,t) = 0.$$

Since we assume $G(t) \not\equiv 0$, we finally obtain

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

Boundary conditions for ϕ

To obtain the nontrivial solution, ϕ also satisfies the BC:

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

Summary

In order to use the method of separation of variables, we

- Determine solutions in the product form:

$$\boxed{u(x,t) = \phi(x)G(t).} \qquad (2.3.4)$$

- Plug the product solution into the heat equation (2.3.1), and then *separate the variables* so that LHS is only a function of t and the RHS is only a function of x.
- Introduce the separation constant λ. This will produce two ODEs.
- Find the nontrivial solutions. (See Subsections 2.3.3 and 2.3.4.)

Conclusion

The product form (2.3.4) provides a good way to find special solutions because it reduces the heat equation to ODEs.

2.3.3 Time Dependent ODE

In Subsection 2.3.2, we were looking for special solution(s) in the product form

$$u(x,t) = \phi(x)G(t) \tag{2.3.4}$$

which solves the heat equation with the homogeneous BCs:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.3.1}$$

$$u(0,t) = 0 \text{ and } u(L,t) = 0. \tag{2.3.2}$$

Using **separation of variables**, we found that ϕ and G satisfy two ODEs:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{and} \quad \frac{dG}{dt} = -\lambda kG.$$

Aim of this subsection

We will solve the t-dependent ODE:

$$\boxed{\frac{dG}{dt} = -\lambda kG.} \tag{2.3.12}$$

Solving the t-dependent ODE

How to solve $\dfrac{dG}{dt} = -\lambda k G$?

Let us solve it as follows: (denoting $c := e^C$)

$$\begin{aligned} \frac{dG}{G} &= -\lambda k dt \\ \int \frac{dG}{G} &= -\int \lambda k dt \\ \ln G &= C - \lambda k t \\ G &= e^C e^{-\lambda k t} =: c e^{-\lambda k t}. \end{aligned}$$

Final Answer:

$$G(t) = c e^{-\lambda k t} \tag{2.3.13}$$

where $c := G(0)$ is the integration constant.

Long Time Behavior of the General Solution $G(t)$

Long time behavior of G(t)

According to the solution formula

$$G(t) = ce^{-\lambda kt}, \tag{2.3.13}$$

the long time behavior of $G(t)$ depends on the sign of the separation constant λ. As $t \to +\infty$, if

- $\lambda > 0$: $G(t)$ exponentially decays.
- $\lambda < 0$: $G(t)$ exponentially grows.
- $\lambda = 0$: $G(t)$ remains constant in time.

Example (Equilibrium solution in Subsection 1.4.1)

Applying the result of Subsection 1.4.1 to (2.3.1) - (2.3.2), we know that the steady state solution $\equiv 0$. Thus,

$$u(x,t) := \phi(x)G(t) \to 0 \text{ as } t \to +\infty.$$

That is, $G(t)$ should NOT grow in time, so we expect $\lambda \geq 0$.

2.3.4 Boundary Value (Eigenvalue) Problem

Aim of this subsection

Solving the Boundary Value / Eigenvalue Problem (BVP):

$$\boxed{\begin{aligned} \frac{d^2\phi}{dx^2} &= -\lambda\phi \\ \phi(0) = \phi(L) &= 0. \end{aligned}} \tag{2.3.14}$$

Eigenvalues and Eigenfunctions

- For arbitrary λ, there *may NOT exist* nontrivial solutions (i.e. $\phi \not\equiv 0$) to the BVP (2.3.14).
- For those values of λ that give nontrivial solutions, we will call such λ an **eigenvalue** for the BVP (2.3.14).
- The nontrivial solutions $\phi(x)$ will be called **eigenfunctions** for the BVP corresponding to the given eigenvalue λ.

How to determine the eigenvalues λ?

The equation

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

is a linear and homogeneous second-order ODE with **constant coefficients**, so we attempt to find the solutions of the form $\phi = e^{rx}$. Substituting $\phi = e^{rx}$ into the ODE, we have

$$r^2 e^{rx} = -\lambda e^{rx}$$

Canceling e^{rx} both sides, we obtain the characteristic polynomial

$$\boxed{r^2 = -\lambda.}$$

We will study the eigenvalue problem in the following three cases:

Case 1: $\lambda > 0$

Case 2: $\lambda = 0$

Case 3: $\lambda < 0$

Case 1: $\lambda > 0$

Let us consider the BVP with $\lambda > 0$:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$
$$\phi(0) = \phi(L) = 0.$$

General Solution

- If $\phi = e^{rx}$, then $r^2 = -\lambda$.
- Thus, $r = \pm i\sqrt{\lambda}$ and $\phi = e^{\pm i\sqrt{\lambda}x} = \cos\sqrt{\lambda}x \pm i \sin\sqrt{\lambda}x$.
- We want real independent solutions $\Rightarrow$ Choose $\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$ in this case (but $e^{\pm i\sqrt{\lambda}x}$ can be used).

Hence, the general solution: for any arbitrary constants c_1 and c_2,

$$\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, \tag{2.3.18}$$

which is an arbitrary linear combinations of two independent solutions $\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$.

Is λ arbitrary in (2.3.18)?

Let us recall that the general solution

$$\phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x, \tag{2.3.18}$$

should also satisfy the boundary condition (BC):

$$\phi(0) = \phi(L) = 0.$$

Using the BC $\phi(0) = 0$, we have

$$c_1 = 0.$$

Thus, $\phi(x) = c_2 \sin\sqrt{\lambda}x$. Using the BC $\phi(L) = 0$, we obtain

$$c_2 \sin\sqrt{\lambda}L = 0.$$

Hence, the eigenvalues λ must satisfy

$$\sin\sqrt{\lambda}L = 0, \tag{2.3.19}$$

which implies $\sqrt{\lambda}L = n\pi$ for $n = 1, 2, 3, \ldots$. That is,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad \text{for any } n = 1, 2, 3, \ldots. \tag{2.3.20}$$

Eigenvalues and Eigenfunctions when $\lambda > 0$

Eigenvalues and Eigenfunctions when $\lambda > 0$

Thus, the eigenvalues λ are

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots . \tag{2.3.20}$$

The eigenfunction corresponding to the eigenvalue $\lambda = \left(\frac{n\pi}{L}\right)^2$ is

$$\phi(x) = c_2 \sin \sqrt{\lambda} x = c_2 \sin \frac{n\pi x}{L}, \tag{2.3.21}$$

where c_2 is an arbitrary constant.

Remarks

- For convenience sake, we can pick a value for c_2, say $c_2 = 1$.
- Any specific eigenfunction can always be multiplied by an arbitrary constant, since the PDE and BCs are linear and homogeneous.

Case 2: $\lambda = 0$

When $\lambda = 0$, the ODE $\dfrac{d^2\phi}{dx^2} = -\lambda\phi$ becomes

$$\frac{d^2\phi}{dx^2} = 0,$$

which implies

$$\phi(x) = c_1 + c_2 x.$$

Using the BCs $\phi(0) = 0$ and $\phi(L) = 0$, we obtain

$$\phi \equiv 0,$$

which is the trivial solution.

Conclusion

$\lambda = 0$ is **NOT** an eigenvalue for the BVP (2.3.14).

Case 3: $\lambda < 0$

Denote $s := -\lambda > 0$, then the ODE becomes

$$\frac{d^2\phi}{dx^2} = s\phi. \tag{2.3.23}$$

The general solution is

$$\phi = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}, \tag{2.3.24}$$

correspond to $r = \pm\sqrt{s} = \pm\sqrt{-\lambda}$ of the characteristic polynomial. However, we may also rewrite the general solution by using the hyperbolic functions

$$\cosh z \equiv \frac{e^z + e^{-z}}{2} \text{ and } \sinh z \equiv \frac{e^z - e^{-z}}{2},$$

which are the linear combinations of exponentials.

Therefore, the general solution can be written as

$$\phi = c_3 \cosh \sqrt{s}x + c_4 \sinh \sqrt{s}x. \tag{2.3.25}$$

Using the BCs $\phi(0) = \phi(L) = 0$, we have

$$c_3 = 0 \quad \text{and} \quad c_4 \sinh \sqrt{s}L = 0.$$

Since $\sqrt{s}L > 0$, $\sinh \sqrt{s}L > 0$, which implies $c_4 = 0$. Therefore,

$$\phi \equiv 0,$$

which is the trivial solution.

Conclusion

There is NO negative eigenvalues for the BVP (2.3.14).

Summary

In this section, we consider the the Boundary Value / Eigenvalue Problem (BVP):

$$\boxed{\begin{aligned} &\frac{d^2\phi}{dx^2} = -\lambda\phi \\ &\phi(0) = \phi(L) = 0. \end{aligned}} \tag{2.3.14}$$

and we obtain the following information:

The Eigenvalues and Eigenfunctions for this BVP

- The eigenvalues $\lambda > 0$,
- $\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots,$
- The corresponding eigenfunctions: $\phi(x) = \sin \frac{n\pi x}{L}$.

In Subsection 2.3.2, we applied separation of variables to reduce our PDE problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(0,t) = u(L,t) = 0$$

to two ODE problems:

$$\frac{dG}{dt} = -\lambda kG$$
$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{with} \quad \phi(0) = \phi(L) = 0.$$

In Subsection 2.3.3 and 2.3.4, we solved these ODEs and obtained

$$\boxed{G(t) = ce^{-\lambda kt}} \quad \text{and} \quad \boxed{\phi(x) = c_2 \sin\sqrt{\lambda}x}$$

where $\lambda = \left(\frac{n\pi}{L}\right)^2 > 0$ for any $n = 1, 2, 3, \ldots$.

Product Solutions of Heat Equation

The product solutions of the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$u(0,t) = u(L,t) = 0$$

are

$$\boxed{u(x,t) = \phi(x)G(t) = B\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t},} \qquad (2.3.26)$$

where $n = 1, 2, 3, \ldots$ and $B(= cc_2)$ is an arbitrary constant.

Remarks:

- $\lim_{t\to\infty} u(x,t) = 0.$
- $u(x,t)$ satisfies an initial condition $u(x,0) = B\sin\frac{n\pi x}{L}$.

Initial Value Problems (IVP) with Special Initial Data

Example (Solving an IVP of the heat equation)

Consider the following IVP:

$$\begin{aligned}
\text{PDE}: &\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\
\text{BC}: &\quad u(0,t) = u(L,t) = 0 \\
\text{IC}: &\quad u(x,0) = 2\sin\frac{7\pi x}{L}.
\end{aligned}$$

By putting $n = 7$ and $B = 2$ into the product form solution

$$\boxed{u(x,t) = \phi(x)G(t) = B\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t},} \tag{2.3.26}$$

the solution of this example is

$$u(x,t) = 2\sin\frac{7\pi x}{L}e^{-k(7\pi/L)^2 t}.$$

Principle of Superposition

Principle of Superposition

If $u_1, u_2, \cdots, u_M$ are solutions of a linear homogenous problem, then any linear combination of these is also a solution:

$$c_1 u_1 + c_2 u_2 + \cdots + c_M u_M = \sum_{n=1}^{M} c_n u_n,$$

where c_n are arbitrary constants.

Example (Heat equation with Zero BCs)

From the method of separation of variables, we know that $\sin \frac{n\pi x}{L} \cdot e^{-k(n\pi/L)^2 t}$ is a solution for all positive integer n.
By principle of superposition, any linear combination of these solutions

$$u(x,t) = \sum_{n=1}^{M} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \tag{2.3.27}$$

is also a solution for any finite M with IC

$$u(x,0) = f(x) = \sum_{n=1}^{M} B_n \sin \frac{n\pi x}{L}. \tag{2.3.28}$$

What if $M \to \infty$?

Claim (we will prove this in Chapter 3)

Any nice initial condition $f(x)$ can be written as an infinite linear combination of $\sin \frac{n\pi x}{L}$, known as **Fourier sine series**:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}. \tag{2.3.29}$$

Therefore, the solution of our heat problem is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}. \tag{2.3.30}$$

Conclusion

According to the *principle of superposition* and the *above claim*, the IVP (2.3.1) - (2.3.3) of the heat equation with nice initial data $f(x)$ can be solved by the solution formula (2.3.30).

2.3.6 Orthogonality of Sines

In Subsection 2.3.5, we found the solution formula

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \tag{2.3.30}$$

for the IBVP (2.3.1) - (2.3.3) provided that the initial condition

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}. \tag{2.3.31}$$

Question of this subsection

How to determine B_n?

Multiplying both sides of (2.3.31) by $\sin\frac{m\pi x}{L}$ and then integrating from $x=0$ to $x=L$, we have

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx=\sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx. \tag{2.3.34}$$

Let us state without proof the following integral property:

Important Fact (Exercise 2.3.5)

$$\boxed{\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n, \end{cases}} \tag{2.3.32}$$

where m and n are positive integers.

Consider equation (2.3.34) again:

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx. \tag{2.3.34}$$

Using the integral property (2.3.32), we know that only the term $n = m$ remains in the R.H.S.!

Hence, (2.3.34) becomes

$$\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx = B_m \underbrace{\int_0^L \sin^2 \frac{m\pi x}{L}\, dx}_{=\frac{L}{2}}.$$

Finally, we find the

Formula for B_m:

$$\boxed{B_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L}\, dx.} \tag{2.3.35}$$

Orthogonality

Orthogonality

Two functions $A(x)$ and $B(x)$ are **orthogonal** over the interval $0 \leq x \leq L$ if

$$\int_0^L A(x)B(x)\,dx = 0.$$

A set of functions each member of which is orthogonal to every other member is called an **orthogonal set of functions**.

Example ($\sin \frac{n\pi x}{L}$)

Consider the eigenfunctions $\sin \dfrac{n\pi x}{L}$ of the BVP

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \text{ with } \phi(0) = \phi(L) = 0.$$

They are mutually orthogonal because of the integral property (2.3.32).

Summary of Section 2.3

Let us summarize the method of separation of variables for the heat equation:

$$\begin{aligned} \text{PDE :} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ \text{BC :} \quad & u(0,t) = u(L,t) = 0 \\ \text{IC :} \quad & u(x,0) = f(x). \end{aligned}$$

1. Make sure both the PDE and BCs are linear and homogenous.
2. Ignore the nonzero IC for the moment.
3. Separate the variables and introduce a separation constant λ.
4. Determine λ as the eigenvalues of a BVP.
5. Solve other differential equations. Record all product solutions of the PDE obtainable by this method.
6. Apply the principle of superposition.
7. Attempt to satisfy the IC.
8. Determine coefficients using the orthogonality of the eigenfunctions.

Solve the following IBVP:

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad \text{for } t > 0 \text{ and } 0 < x < 1$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = x(1-x).$$

Recall: The general solution of the IBVP

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0, \quad \text{and } u(x,0) = f(x)$$

is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}. \tag{2.3.30}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{2.3.35}$$

2.4 Worked Examples with the Heat Equation

2.4.1 Heat Conduction in a Rod with Insulated Ends

Let us consider the heat conduction with **insulated ends** in a 1D rod with *constant thermal properties* and *no heat sources*:

$$\text{PDE}: \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{2.4.1}$$

$$\text{BC}_1: \quad \frac{\partial u}{\partial x}(0, t) = 0 \tag{2.4.2}$$

$$\text{BC}_2: \quad \frac{\partial u}{\partial x}(L, t) = 0 \tag{2.4.3}$$

$$\text{IC}: \quad u(x, 0) = f(x). \tag{2.4.4}$$

Remarks

- This problem is similar to the problem in Section 2.3. The only difference is the BCs.
- Both the PDE and BCs are linear and homogeneous.

$\Rightarrow$ we can apply the the **method of separation of variables**.

Applying Separation of Variables

Similar to the example in Section 2.3, we first seek for the product solutions:

$$u(x,t) = \phi(x)G(t), \tag{2.4.5}$$

and obtain

$$\frac{dG}{dt} = -\lambda k G \qquad \text{and} \qquad \frac{d^2\phi}{dx^2} = -\lambda\phi,$$

where λ is the separation constant. Solving $G(t)$, we have

$$G(t) = ce^{-\lambda k t}. \tag{2.4.8}$$

Regarding $\phi(x)$, it is **different** here! Indeed, we now have insulated BCs, so we have to consider the following BVP:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{and} \quad \frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0.$$

We will discuss this BVP in three cases: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

When $\lambda > 0$, the general solution of $\phi'' = -\lambda\phi$ is

$$\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x. \qquad (2.4.12)$$

Differentiating (2.4.12) w.r.t. x, we have

$$\frac{d\phi}{dx} = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x\right).$$

Using the insulated BC $\dfrac{d\phi}{dx}(0) = 0$, we have $c_2 = 0$. Thus,

$$\phi = c_1 \cos\sqrt{\lambda}x \quad \text{and} \quad \frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x.$$

Using another BC $\dfrac{d\phi}{dx}(L) = 0$, we have

$$-c_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0.$$

For nontrivial solutions, $c_1 \neq 0$, and hence, $\sin\sqrt{\lambda}L = 0$.

The eigenvalues for $\lambda > 0$ are the same as the problem in Section 2.3,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots, \tag{2.4.14}$$

but the corresponding eigenfunctions are cosines (not sines),

$$\phi(x) = c_1 \cos\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots. \tag{2.4.15}$$

Hence, the product solution is

$$\boxed{u(x,t) = A\cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}, \qquad n = 1, 2, 3, \ldots,} \tag{2.4.16}$$

where A is an arbitrary constant.

Next task:

We have to check whether there is any other eigenvalue!

Case: $\lambda = 0$

When $\lambda = 0$, the general solution of $\phi'' = -\lambda\phi = 0$ is

$$\phi = c_1 + c_2 x,$$

where c_1 and c_2 are arbitrary constants. The derivative of ϕ is

$$\frac{d\phi}{dx} = c_2.$$

Both BCs $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$ give the same condition $c_2 = 0$. Hence there are nontrivial solutions $\phi(x) = c_1$. Note that the time-dependent part is also a constant:

$$e^{-\lambda k t} \equiv 1 \quad \text{when } \lambda = 0.$$

Hence, for $\lambda = 0$, the product solution is

$$\boxed{u(x,t) = A,}$$

where A is an arbitrary constant.

Case: $\lambda < 0$

Question:

Are there any eigenvalues $\lambda < 0$ in this problem?

Answer:

No, We do not expect $\lambda < 0$.

Reasons:

(i) When $\lambda < 0$, the time dependent part

$$G(t) = ce^{-\lambda kt}$$

grows exponentially as $t \to +\infty$.

(ii) We cannot find a nontrivial linear combination of exponentials that would have a zero slope at both $x = 0$ and $x = L$ for $\lambda < 0$. (Exercise 2.4.4).

Applying Principle of Superposition

Using the principle of superposition, we can construct the solution to the IBVP (2.4.1) - (2.4.4) by the linear combination of **ALL** product solutions:

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt},} \tag{2.4.19}$$

which is equivalent to

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \tag{2.4.20}$$

because $\cos 0 = 1$ and $e^0 = 1$. The IC $u(x,0) = f(x)$ is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}, \tag{2.4.21}$$

for $0 \le x \le L$.

Determining A_n

Last question:

How to find A_n?

As before, we multiply

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \qquad (2.4.21)$$

by $\cos m\pi x/L$ and integrate from 0 to L:

$$\int_0^L f(x) \cos \frac{m\pi x}{L}\, dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx$$

for $m = 0,\ 1,\ 2,\ \cdots$.

Orthogonality

Calculus Fact

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases} \qquad (2.4.22)$$

for n and m nonnegative integers.

Using the **orthogonality relation** (2.4.22), we have

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = A_m\int_0^L \cos^2\frac{m\pi x}{L}\,dx = \begin{cases} L\,A_0 & m = 0 \\ \frac{L}{2}\,A_m & m \neq 0. \end{cases}$$

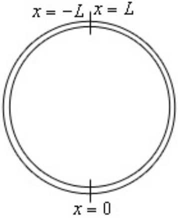
Finally, we obtain

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad (2.4.23)$$

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{for } m \geq 1. \qquad (2.4.24)$$

2.4.2 Heat Conduction in a Thin Insulated Circular Ring

Consider a thin wire with length $2L$ (with lateral sides insulated) is bent into the shape of a circle:



Assumptions

- Assume the temperature u in the wire is constant along cross sections of the bent wire provided the wire is thin enough.
- Assume the wire is very tightly connected to itself at the ends ($x = -L$ to $x = L$), we have $u(-L, t) = u(L, t)$.
- Since the heat flux is continuous, $\dfrac{\partial u}{\partial x}(-L, t) = \dfrac{\partial u}{\partial x}(L, t)$.

Formulating the IBVP

According to the previous description, the temperature $u := u(x, t)$ of the wire should satisfy a 1D heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{2.4.25}$$

$$u(-L, t) = u(L, t) \tag{2.4.26}$$

$$\frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t) \tag{2.4.27}$$

$$u(x, 0) = f(x). \tag{2.4.28}$$

Remark

Both the PDE and BCs are linear and homogenous.

$\Rightarrow$ We can apply the method of separation of variables.

Product Solutions

As before, we have the product solutions

$$u(x, t) = \phi(x)G(t),$$

where $G(t) = ce^{-\lambda kt}$.
The corresponding boundary value / eigenvalue problem (BVP) is

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{2.4.29}$$

$$\phi(-L) = \phi(L) \tag{2.4.30}$$

$$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L). \tag{2.4.31}$$

Remarks

- The BCs each involve both ends (also called the **mixed** type).
- These specific BCs are referred to as **periodic boundary conditions**.

When $\lambda > 0$, the general solution of $\phi'' = -\lambda\phi$ is again

$$\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x,$$

and the derivative of ϕ is

$$\frac{d\phi}{dx} = \sqrt{\lambda}(-c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x).$$

The BCs $\phi(-L) = \phi(L)$ and $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$ implies

$$c_1 \cos\sqrt{\lambda}(-L) + c_2 \sin\sqrt{\lambda}(-L) = c_1 \cos\sqrt{\lambda}L + c_2 \sin\sqrt{\lambda}L$$
$$-c_1 \sin\sqrt{\lambda}(-L) + c_2 \cos\sqrt{\lambda}(-L) = -c_1 \sin\sqrt{\lambda}L + c_2 \cos\sqrt{\lambda}L$$

Using cos is even and sin is odd, we have

$$c_2 \sin\sqrt{\lambda}L = 0, \tag{2.4.32}$$

$$c_1 \sin\sqrt{\lambda}L = 0. \tag{2.4.33}$$

If $\sin\sqrt{\lambda}L \neq 0$, then $c_1 = c_2 = 0$, which is just the trivial solution. Thus, in order to obtain nontrivial solutions, we must have

$$\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, \ldots.$$

Remark

We chose the wire to have length $2L$ so that the eigenvalues have the same formula as before.

Eigenfunctions

Since both c_1 and c_2 are arbitrary, both $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$ are eigenfunctions corresponding to the eigenvalue $\lambda = (n\pi/L)^2$,

$$\phi(x) = \cos\frac{n\pi x}{L}, \ \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots. \tag{2.4.35}$$

In fact, any linear combination of $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ is an eigenfunction:

$$\phi(x) = c_1\cos\frac{n\pi x}{L} + c_2\sin\frac{n\pi x}{L}.$$

Eigenvalues for $\lambda \leq 0$

When $\lambda = 0$, the general solution of $\phi'' = -\lambda\phi = 0$ is

$$\phi = c_1 + c_2 x.$$

The BC $\phi(-L) = \phi(L)$ implies

$$c_1 - c_2 L = c_1 + c_2 L$$
$$c_2 = 0.$$

Eigenfunction for $\lambda = 0$

The eigenfunction is

$$\phi(x) \equiv c_1$$

where c_1 is any non-zero constant.

Eigenvalues for $\lambda < 0$?

It can be shown that there are no eigenvalues for $\lambda < 0$.

Principle of Superposition and Initial Condition (IC)

According to the Principle of Superposition, the general solution consists of an arbitrary linear combination of **ALL** product solutions:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}.$$

Therefore, the IC $u(x,0) = f(x)$ is satisfied provided that

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \\ &= \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}. \end{aligned} \tag{2.4.39}$$

Orthogonality Conditions

To find the Fourier's coefficients a_n and b_n, we need the following

Orthogonality Conditions

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases} \tag{2.4.40}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases} \tag{2.4.41}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0, \tag{2.4.42}$$

where n and m are arbitrary (nonnegative) integers.

Remark

The eigenfunctions form an orthogonal set.

Multiplying

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{2.4.39}$$

by either $\cos \frac{m\pi x}{L}$ or $\sin \frac{m\pi x}{L}$, and then integrating from $x = -L$ to $x = L$, we have

$$\int_{-L}^{L} f(x) \left\{ \begin{matrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{matrix} \right\} dx = \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \left\{ \begin{matrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{matrix} \right\} dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \left\{ \begin{matrix} \cos \frac{m\pi x}{L} \\ \sin \frac{m\pi x}{L} \end{matrix} \right\} dx.$$

Using the orthogonality conditions, we find that

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx = a_m \int_{-L}^{L} \cos^2 \frac{m\pi x}{L}\, dx = \begin{cases} 2La_0 & m = 0 \\ La_m & m > 0 \end{cases}$$

$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx = b_m \int_{-L}^{L} \sin^2 \frac{m\pi x}{L}\, dx = Lb_m.$$

Finally, we obtain the coefficients

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \\ a_m &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx, \quad m \geq 1 \\ b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx, \quad m \geq 1. \end{aligned}} \tag{2.4.43}$$

2.4.3 Summary of Boundary Value Problems

TABLE 2.4.1: Boundary Value Problems for $\dfrac{d^2\phi}{dx^2} = -\lambda\phi$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\dfrac{d\phi}{dx}(0) = 0$ $\dfrac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin \dfrac{n\pi x}{L}$	$\cos \dfrac{n\pi x}{L}$	$\sin \dfrac{n\pi x}{L}$ and $\cos \dfrac{n\pi x}{L}$
Series	$f(x) = \displaystyle\sum_{n=1}^{\infty} B_n \sin \dfrac{n\pi x}{L}$	$f(x) = \displaystyle\sum_{n=0}^{\infty} A_n \cos \dfrac{n\pi x}{L}$	$f(x) = \displaystyle\sum_{n=0}^{\infty} a_n \cos \dfrac{n\pi x}{L}$ $+ \displaystyle\sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{L}$
Coefficients	$B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x) \sin \dfrac{n\pi x}{L}\, dx$	$A_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)\, dx$ $A_n = \dfrac{2}{L}\displaystyle\int_0^L f(x) \cos \dfrac{n\pi x}{L}\, dx$	$a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^L f(x)\, dx$ $a_n = \dfrac{1}{L}\displaystyle\int_{-L}^L f(x) \cos \dfrac{n\pi x}{L}\, dx$ $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^L f(x) \sin \dfrac{n\pi x}{L}\, dx$

Homogeneous Dirichlet BCs ($u(0,t) = u(L,t) = 0$)

Gives rise to a Fourier sine series:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin \frac{n\pi x}{L} \qquad B_n = \frac{2}{L}\int_0^L u(x,0) \sin \frac{n\pi x}{L} dx$$

Homogeneous Neumann BCs ($u_x(0,t) = u_x(L,t) = 0$)

Gives rise to a Fourier cosine series: $\quad A_0 = \frac{1}{L}\int_0^L u(x,0)dx$

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \cos \frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L u(x,0) \cos \frac{n\pi x}{L} dx$$

Fourier Series Solutions of 1D Heat Equation

Periodic BCs ($u(-L,t) = u(L,t)$ and $u_x(-L,t) = u_x(L,t)$)

Gives rise to a mixed Fourier sine/cosine series:

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\frac{n\pi x}{L}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} u(x,0)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} u(x,0)\cos\frac{n\pi x}{L}\,dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} u(x,0)\sin\frac{n\pi x}{L}\,dx$$

Moral:

One can understand Dirichlet and Neumann problems as special cases of the periodic problem.

Solve the following IBVP:

$$\begin{aligned} \frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial x^2} \qquad \text{for } t > 0 \text{ and } -2 < x < 2 \\ u(-2,t) &= u(2,t) \\ \frac{\partial u}{\partial x}(-2,t) &= \frac{\partial u}{\partial x}(2,t) \\ u(x,0) &= 4\sin\frac{5\pi x}{2} + 6\cos\frac{7\pi x}{2}. \end{aligned}$$

Hint:

Comparing coefficients of the trigonometric polynomials!!

2.5 Laplace's Equation: Solutions and Qualitative Properties

Aim of this section

Study the Laplace's equation $\nabla^2 u = 0$ by the method of separation variables.

Consider the equilibrium temperature inside a rectangle $(0 \le x \le L,\ 0 \le y \le H)$ when the temperature is prescribed function of position (independent of time t) on the boundary. Then the equilibrium temperature $u := u(x, y)$ satisfies Laplace's equation with the following BCs:

$$\text{PDE}: \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.5.1}$$

$$\text{BC1}: \quad u(0, y) = g_1(y) \tag{2.5.2}$$

$$\text{BC2}: \quad u(L, y) = g_2(y) \tag{2.5.3}$$

$$\text{BC3}: \quad u(x, 0) = f_1(x) \tag{2.5.4}$$

$$\text{BC4}: \quad u(x, H) = f_2(x), \tag{2.5.5}$$

where $f_1(x)$, $f_2(x)$, $g_1(y)$, $g_2(y)$ are given functions of x and y.

Principle of Superposition

Are the PDE and BCs linear and homogeneous?

- The PDE is linear and homogeneous.
- The BCs are linear, but not homogeneous.

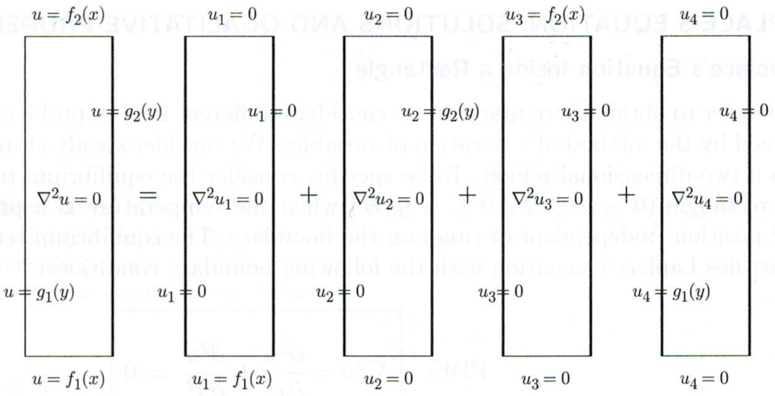
$\Rightarrow$ We cannot apply the method of separation of variable in its present form.

Moral

Hence, we apply the **principle of superposition** so that we can break our problem into four problems, each having one nonhomogeneous condition. Let

$$\boxed{u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),} \qquad (2.5.6)$$

where each $u_i(x,y)$ satisfies Laplace's equation with one nonhomogeneous BC and the related three homogeneous BCs, see the following figure.


$u = f_2(x)$
$u = g_2(y)$
$\nabla^2 u = 0$
$u = g_1(y)$
$u = f_1(x)$
$=$
$u_1 = 0$
$u_1 = 0$
$\nabla^2 u_1 = 0$
$u_1 = 0$
$u_1 = f_1(x)$
$+$
$u_2 = 0$
$u_2 = g_2(y)$
$\nabla^2 u_2 = 0$
$u_2 = 0$
$u_2 = 0$
$+$
$u_3 = f_2(x)$
$u_3 = 0$
$\nabla^2 u_3 = 0$
$u_3 = 0$
$u_3 = 0$
$+$
$u_4 = 0$
$u_4 = 0$
$\nabla^2 u_4 = 0$
$u_4 = g_1(y)$
$u_4 = 0$

Why dose the sum $u = u_1 + u_2 + u_3 + u_4$ satisfy satisfy our problem?

Reasons

- u_1, u_2, u_3 and u_4 satisfy the *linear* and *homogeneous* Laplace's equation and so does u by the principle of superposition.
- At $x = 0$: $u_1 = u_2 = u_3 = 0$ and $u_4 = g_1(y)$. Therefore $u = u_1 + u_2 + u_3 + u_4 = g_1(y)$, the desired nonhomogeneous condition.
- We can check that all four nonhomogeneous conditions have been satisfied similarly.

Solving the four problems are similar, so we will solve only for $u_4(x, y)$ by the method of separation of variables.

Solving for u_4

Consider the problem for u_4:

$$\begin{aligned}
\text{PDE}: &\quad \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 && (2.5.7)\\
\text{BC1}: &\quad u_4(0, y) = g_1(y) && (2.5.8)\\
\text{BC2}: &\quad u_4(L, y) = 0 && (2.5.9)\\
\text{BC3}: &\quad u_4(x, 0) = 0 && (2.5.10)\\
\text{BC4}: &\quad u_4(x, H) = 0. && (2.5.11)
\end{aligned}$$

Let us ignore the nonhomogeneous condition $u_4(0, y) = g_1(y)$ for the moment, and consider the product solution

$$u_4(x, y) = h(x)\phi(y) \tag{2.5.12}$$

where h and ϕ are nontrival (i.e. $h \not\equiv 0$ and $\phi \not\equiv 0$).

Boundary Conditions

Substituting the product form $u_4(x, y) = h(x)\phi(y)$ into the BCs (2.5.9) - (2.5.11), we have

$$
\begin{aligned}
h(L)\phi(y) &= u_4(L, y) = 0 \\
h(x)\phi(0) &= u_4(x, 0) = 0 \\
h(x)\phi(H) &= u_4(x, H) = 0
\end{aligned}
$$

which imply

$$h(L) = 0,\ \phi(0) = 0 \text{ and } \phi(H) = 0.$$

Remarks

- The y-dependent solution $\phi(y)$ has two homogeneous BCs and will become an eigenvalue problem in y.
- The x-dependent solution $h(x)$ has only one homogeneous BC.
- Hence, we solve for ϕ first.

Substituting the product form $u_4(x, y) = h(x)\phi(y)$ into Laplace's equation $\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$, we have

$$\phi(y)\frac{d^2 h}{dx^2} + h(x)\frac{d^2 \phi}{dy^2} = 0.$$

We can "separate the variables" by dividing by $h(x)\phi(y)$, and obtain

$$\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\phi}\frac{d^2 \phi}{dy^2}. \tag{2.5.16}$$

Since L.H.S. of (2.5.16) is a function only of x and R.H.S. of (2.5.16) is a function only of y, both must equal a separation constant λ. That is,

$$\boxed{\frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\phi}\frac{d^2 \phi}{dy^2} = \lambda.} \tag{2.5.17}$$

Eigenvalue Problem of ϕ

Let us first consider the boundary value/eigenvalue problem (BVP) of $\phi := \phi(y)$:

$$\frac{d^2\phi}{dy^2} = -\lambda\phi$$
$$\phi(0) = \phi(H) = 0.$$

Remark:

We have solved this BVP in Subsection 2.3.4, but here the length of the interval is H.

Based on our previous computations in Subsection 2.3.4, we have

Eigenvalues and eigenfunction

This BVP has the following eigenvalues and eigenfunctions:

$$\boxed{\begin{aligned} \lambda &= \left(\frac{n\pi}{H}\right)^2 \\ \phi(y) &= \sin\frac{n\pi y}{H} \end{aligned}} \qquad n = 1, 2, 3, \ldots. \tag{2.5.23}$$

General Solution for h

Regarding $h := h(x)$, we have the following ODE and one BC:

$$\frac{d^2 h}{dx^2} = \lambda h = \left(\frac{n\pi}{H}\right)^2 h \qquad \text{with} \;\; h(L) = 0,$$

but it is NOT a BVP because it does not have two BCs. Noting that $\lambda = \left(\frac{n\pi}{H}\right)^2 > 0$, the general solution can be written as

$$h(x) = a_1 \cosh \frac{n\pi}{H}(x - L) + a_2 \sinh \frac{n\pi}{H}(x - L). \qquad (2.5.25)$$

Using $h(L) = 0$, we have $a_1 = 0$, and hence

$$\boxed{h(x) = a_2 \sinh \frac{n\pi}{H}(x - L).} \qquad (2.5.26)$$

Product Solutions

The product solutions are

$$u_4(x,y) = \phi(y)h(x) = A \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x-L), \qquad (2.5.27)$$

where $n = 1, 2, 3, \ldots$ and A is any non-zero arbitrary constant.

Remarks

The product solution u_4 in (2.5.27) satisfies the Laplace's equation

$$\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \qquad (2.5.7)$$

as well as the three require homogeneous boundary conditions:

$$u_4(L,y) = 0 \qquad (2.5.9)$$

$$u_4(x,0) = 0 \qquad (2.5.10)$$

$$u_4(x,H) = 0. \qquad (2.5.11)$$

Principle of Superposition and Determine the Coefficients

Applying the principle of superposition to the production solutions (2.5.27), we obtain the general solution

$$\boxed{u_4(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x - L).} \tag{2.5.28}$$

To satisfy the nonhomogeneous BC $u_4(0, y) = g_1(y)$, we have

$$g_1(y) = -\sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H} L.$$

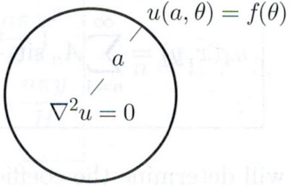
Finally, orthogonality condition implies that

$$-A_n \sinh \frac{n\pi}{H} L = \frac{2}{H} \int_0^H g_1(y) \sin \frac{n\pi y}{H}\, dy,$$

and hence,

$$\boxed{A_n = -\frac{2}{H \sinh(n\pi L/H)} \int_0^H g_1(y) \sin \frac{n\pi y}{H}\, dy.} \tag{2.5.29}$$

Consider a thin circular disk of radius a (with constant thermal properties and no heat sources) with the temperature prescribed on the boundary:



The temperature u satisfies Laplace's equation: $\nabla^2 u = 0$. Using **polar coordinates** (r, θ), we can rewrite our problem as:

$$\text{PDE}: \qquad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (2.5.30)$$

$$\text{BC}: \qquad u(a, \theta) = f(\theta) \qquad (2.5.31)$$

where $0 \le r \le a$ and $-\pi \le \theta \le \pi$.

Conditions in Polar Coordinates

Mathematically, we need conditions at the endpoints of the coordinate system, i.e. $r = 0$, a and $\theta = \pm\pi$.
First, we will prescribe that the temperature is finite at $r = 0$:

$$|u(0, \theta)| < \infty. \tag{2.5.32}$$

Next, the temperature $u(r, \theta)$ and the heat flow in the θ-direction are continuous, which imply **periodicity conditions**:

$$\begin{aligned} u(r, -\pi) &= u(r, \pi) \\ \frac{\partial u}{\partial \theta}(r, -\pi) &= \frac{\partial u}{\partial \theta}(r, \pi). \end{aligned} \tag{2.5.33}$$

Moral:

These three conditions are linear and homogeneous. Our problems now appears similar to Laplace's equation inside a rectangle. Hence, we can apply the method of separation of variables.

Now we consider the product solutions $u(r,\theta) = \phi(\theta)G(r)$ that satisfy the PDE and the three homogeneous conditions. Substituting $u(r,\theta) = \phi(\theta)G(r)$ into the periodicity conditions

$$u(r,-\pi) = u(r,\pi) \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi), \tag{2.5.33}$$

we obtain

$$\phi(-\pi) = \phi(\pi) \qquad \text{and} \qquad \frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi). \tag{2.5.35}$$

Substituting the product form $u(r,\theta) = \phi(\theta)G(r)$ into the Laplace's equation $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, dividing by $(1/r^2)G(r)\phi(\theta)$ and introducing a separation constant λ, we have

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda. \tag{2.5.36}$$

Eigenvalue Problem of ϕ

Then we have to determine λ by the following eigenvalue problem:

$$\begin{gathered} \frac{d^2\phi}{d\theta^2} = -\lambda\phi \\ \phi(-\pi) = \phi(\pi) \quad \text{and} \quad \frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi). \end{gathered} \tag{2.5.37}$$

This problem is identical to the eigenvalue problem in the circular wire (i.e. Subsection 2.4.2). According to Subsection 2.4.2,

Eigenvalues and eigenfunctions in (2.5.37)

The eigenvalues are

$$\lambda = \left(\frac{n\pi}{L}\right)^2 = n^2 \qquad (L = \pi \text{ in this case}), \qquad n = 0, 1, 2, \cdots ,$$

with the corresponding eigenfunctions $\sin n\theta$ and $\cos n\theta$.

Solving G

The r-dependent problem is

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda = n^2,$$

which is equivalent to

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0. \tag{2.5.41}$$

(2.5.41) is a second-order linear and homogenous with nonconstant coefficients ODE. The simplest way to solve it: Substituting the special form $G(r) := r^p$ into (2.5.41), we determine that

$$\begin{aligned} [p(p-1) + p - n^2]r^p &= 0 \\ p^2 - n^2 &= 0. \end{aligned}$$

Thus, $p = \pm n$. According to the principle of superposition, the general solution is

$$G(r) = \begin{cases} c_1 r^n + c_2 r^{-n} & \text{if } n = 1,\ 2,\ 3, \cdots \\ c_1 + c_2 \ln r & \text{if } n = 0. \end{cases}$$

The general solution of $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = n^2$ is

$$G(r) = \begin{cases} c_1 r^n + c_2 r^{-n} & \text{if } n = 1,\ 2,\ 3, \cdots \\ c_1 + c_2 \ln r & \text{if } n = 0. \end{cases}$$

We have prescribed $|u(0, \theta)| < \infty$, hence

$$|G(0)| < \infty.$$

The boundedness condition $|G(0)| < \infty$ implies that

$$c_2 = 0.$$

Hence,

$$G(r) = c_1 r^n, \qquad n \geq 0.$$

Product solutions by the method of separation of variables are

$$r^n \cos n\theta \text{ (for } n \geq 0\text{) and } r^n \sin n\theta \text{ (for } n \geq 1\text{)}.$$

Principle of Superposition and Fourier's Coefficients

By the principle of superposition, for $0 \leq r < a$ and $-\pi < \theta \leq \pi$,

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta,} \tag{2.5.45}$$

According to the nonhomogeneous condition $u(a,\theta) = f(\theta)$,

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta, \tag{2.5.46}$$

Using the orthogonality formulae, it follows that for $n \geq 1$,

$$\boxed{\begin{aligned} A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta \\ A_n &= \frac{1}{a^n \pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \\ B_n &= \frac{1}{a^n \pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta. \end{aligned}} \tag{2.5.47}$$

2.5.4 Qualitative Properties of Laplace's Equation

Functions satisfying $\nabla^2 u = 0$ are called **harmonic**.

Mean Value Property

If u is harmonic on a ball centered at x_0, then $u(x_0)$ equals the average value of $u(x)$ on that ball (i.e., the integral of u on the ball divided by the area or volume of that ball in either 2 or 3 dimensions).

Maximum Principle and Minimum Principle

For a harmonic function u on a domain D, the maximum value of u cannot occur in the interior, only at the boundary. Same with minimum.

Uniqueness of Solutions

A harmonic function on D is completely determined by its boundary values (i.e., there cannot be two distinct harmonic functions which agree on the boundary).

Existence of the Laplace's Equation

The only remaining issue is the **existence** of solutions (which is closely related to but distinct from the question of uniqueness). It can be shown that solutions will exist under fairly general hypotheses and on fairly general sorts of domains (anything piecewise smooth with no crazy cusps is okay).

If you impose Neumann (flux) data, there will also be a solvability criterion: the net flux at the boundary should be zero.

Question(s) for Further Discussion (Section 2.5)

Consider the Laplace equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

inside the quarter-circle of radius 1 $\left(0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\right)$ subject to the following Neumann boundary conditions

$$\frac{\partial u}{\partial \theta}(r,0) = 0, \qquad \frac{\partial u}{\partial \theta}(r,\frac{\pi}{2}) = 0,$$
$$\frac{\partial u}{\partial r}(1,\theta) = \cos 2\theta.$$

(i) Is there any solution to the above problem? Explain briefly.

(ii) Solve the above problem explicitly.