Sheaf Cohomology

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> You have to be able to pose questions otherwise there will be no mathematics.
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Contents

In topology, let X be a topological space. We have the singular (co)homology groups

$$H_p(X, \mathbb{Z}), \qquad H^p(X, \mathbb{Z}).$$

For example, for a compact Riemann surface X of genus g,

$$H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}, \qquad H^1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

In this case, $H_p(X, \mathbb{Z}) = H_p^{\mathrm{sing}}(X, \mathbb{Z})$.

Cellular (co)homology: Cells of dimension n are homeomorphic to $\mathbb{R}^n$.

Example:

$$\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{P}^{n-1} = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \{0\}.$$

Then

$$H^k(\mathbb{P}^m, \mathbb{Z}) \text{ is the same, and } H_{2k}(\mathbb{P}^m, \mathbb{Z}) \cong \mathbb{Z} \quad \text{for } k = 0, 1, 2, \ldots, m.$$

1 Čech Cohomology for $\mathbb{Z}$

Let X be a topological manifold, and let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X such that every finite intersection

$$U_{\alpha_0\alpha_1\cdots\alpha_m} := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_m}$$

is either empty or connected.

Let $p > 0$ be an integer. Define the abelian group of p–cochains

$$C^p(\mathcal{U}, \mathbb{Z})$$

as follows:

A p–cochain is a collection of integers

$$\{\, c_{\alpha_0\alpha_1\cdots\alpha_p} \,\}$$

assigned to each nonempty $(p+1)$–fold intersection $U_{\alpha_0\alpha_1\cdots\alpha_p}$.

The cochain is *alternating*, meaning that for every permutation $\sigma \in \mathrm{Perm}\{0, 1, \ldots, p\}$,

$$c_{\alpha_{\sigma(0)}\alpha_{\sigma(1)}\cdots\alpha_{\sigma(p)}} = \mathrm{sign}(\sigma)\, c_{\alpha_0\alpha_1\cdots\alpha_p},$$

where if σ is a composition of q transpositions, then

$$\mathrm{sign}(\sigma) = (-1)^q.$$

Define the coboundary operator

$$\delta_p : C^p(\mathcal{U}, \mathbb{Z}) \longrightarrow C^{p+1}(\mathcal{U}, \mathbb{Z})$$

as follows.

For a cochain $c \in C^p(\mathcal{U}, \mathbb{Z})$, the cochain $e = \delta_p c \in C^{p+1}(\mathcal{U}, \mathbb{Z})$ is defined by

$$(\delta_p c)_{\alpha_0\alpha_1\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i\, c_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_{p+1}},$$

where the hat $\widehat{\alpha_i}$ means that the index α_i is omitted.

Lemma 1.1. *For all $p \geq 0$, one has*

$$\delta_{p+1} \circ \delta_p = 0.$$

Example 1.2. We check the case $p = 1$, that is, $\delta_2 \circ \delta_1 = 0$.

Let $c = \{c_{\alpha_0\alpha_1}\}$ be a 1–cochain. Then $\delta_1 c = e = \{e_{\alpha_0\alpha_1\alpha_2}\}$ is given by

$$e_{\alpha_0\alpha_1\alpha_2} = c_{\alpha_1\alpha_2} - c_{\alpha_0\alpha_2} + c_{\alpha_0\alpha_1}.$$

Now apply δ_2 to e:

$$(\delta_2 e)_{\alpha_0\alpha_1\alpha_2\alpha_3} = e_{\alpha_1\alpha_2\alpha_3} - e_{\alpha_0\alpha_2\alpha_3} + e_{\alpha_0\alpha_1\alpha_3} - e_{\alpha_0\alpha_1\alpha_2}.$$

Expanding each $e_{\alpha_i\alpha_j\alpha_k}$ in terms of the $c_{\bullet\bullet}$ shows that all terms cancel, hence

$$(\delta_2 e)_{\alpha_0\alpha_1\alpha_2\alpha_3} = 0.$$

Definition 1.3.

(a) A cochain $c \in C^p(\mathcal{U}, \mathbb{Z})$ is called a *p–cocycle* if and only if $\delta_p c = 0$. The set of all p–cocycles is denoted

$$Z^p(\mathcal{U}, \mathbb{Z}) := \ker(\delta_p).$$

(b) A cochain $c \in C^p(\mathcal{U}, \mathbb{Z})$ is called a *p–coboundary* if there exists $b \in C^{p-1}(\mathcal{U}, \mathbb{Z})$ such that $c = \delta_{p-1} b$. The set of all p–coboundaries is denoted

$$B^p(\mathcal{U}, \mathbb{Z}) := \mathrm{im}(\delta_{p-1}).$$

Lemma 1.4.

$$B^p(\mathcal{U}, \mathbb{Z}) \subset Z^p(\mathcal{U}, \mathbb{Z}) \quad \textit{as a subgroup of } C^p(\mathcal{U}, \mathbb{Z}).$$

Proof. Let $c = \delta_{p-1} b \in B^p(\mathcal{U}, \mathbb{Z})$. Then

$$\delta_p c = \delta_p(\delta_{p-1} b) = (\delta_p \circ \delta_{p-1})(b) = 0,$$

since $\delta_p \circ \delta_{p-1} = 0$.

□

Definition 1.5.

(a) For a given open cover $\mathcal{U}$ of X, the p–th Čech cohomology group of the cover is defined as

$$H^p(\mathcal{U}, \mathbb{Z}) := Z^p(\mathcal{U}, \mathbb{Z}) \big/ B^p(\mathcal{U}, \mathbb{Z}).$$

(b) The Čech cohomology of X with coefficients in $\mathbb{Z}$ is defined as the direct limit over all open covers:

$$\check{H}^p(X, \mathbb{Z}) := \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathbb{Z}).$$

If $\mathcal{U} = \{U_\alpha\}$ and $\mathcal{V} = \{V_i\}$ are two open covers such that, for every i, there exists an $\alpha(i)$ with

$$V_i \subset U_{\alpha(i)},$$

then $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$. In this case there is a natural *restriction homomorphism*

$$H^p(\mathcal{U}, \mathbb{Z}) \xrightarrow{\mathrm{Res}} H^p(\mathcal{V}, \mathbb{Z}),$$

which is compatible with the direct limit system that defines $\check{H}^p(X, \mathbb{Z})$.

Theorem 1.6. *Let $\mathcal{U}$ be an* acyclic cover *for the constant sheaf $\mathbb{Z}$ over X —for example, if every finite intersection $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is contractible (hence acyclic) for all $p, \alpha_0, \dots, \alpha_p$. Then we have a natural isomorphism*

$$\check{H}^p(X, \mathbb{Z}) \cong H^p(\mathcal{U}, \mathbb{Z}).$$

Theorem 1.7. *For a reasonable topological space (e.g., paracompact, such as a manifold), the Čech cohomology with constant coefficients agrees with singular cohomology:*

$$\check{H}^p(X, \mathbb{Z}) \cong H^p_{\text{sing}}(X, \mathbb{Z}).$$

Observation:
Let $c \in C^0(\mathcal{U}, \mathbb{Z})$, given by local sections $c = (c_\alpha)$. Then $\delta c \in C^1(\mathcal{U}, \mathbb{Z})$ is defined by

$$(\delta c)_{\alpha_0 \alpha_1} = c_{\alpha_1} - c_{\alpha_0} \quad \text{on } U_{\alpha_0} \cap U_{\alpha_1}.$$

Hence, $\delta c = 0$ if and only if

$$c_{\alpha_1} = c_{\alpha_0} \quad \text{on } U_{\alpha_0} \cap U_{\alpha_1},$$

i.e. the local data $\{c_\alpha\}$ glue together to define a global element of $C^0(X, \mathbb{Z})$.

2 Čech Cohomology of Sheaves

Definition 2.1 (Sheaf of Abelian groups over topological manifolds X)**.** A continuous map

$$\pi : \mathcal{F} \to X$$

is called a *sheaf of abelian groups over* X if the following conditions hold:

1. $\mathcal{F}$ is a topological space and π is a continuous map which is a *local homeomorphism.*
2. For each $x \in X$, the fibre
$$\mathcal{F}_x := \pi^{-1}(x)$$
is an abelian group.
3. Let $e_x \in \mathcal{F}_x$ denote the identity element of the group $\mathcal{F}_x$. Then the map
$$e : X \longrightarrow \mathcal{F}, \qquad x \longmapsto e_x,$$
is continuous.
4. Denote by
$$\mathcal{F} \times_\pi \mathcal{F} = \{(a, b) \in \mathcal{F} \times \mathcal{F} \mid \pi(a) = \pi(b)\}$$
the fibre product of $\mathcal{F}$ with itself over X. Then the group operations (pointwise addition and inverse)
$$(a, b) \longmapsto a \cdot b, \qquad a \longmapsto a^{-1},$$
defined fibrewise on $\mathcal{F} \times_\pi \mathcal{F}$, are continuous maps.

Remark 2.2. The condition that "π is a local homeomorphism" means that for every $a \in \mathcal{F}$ with $\pi(a) = x$, there exist open neighbourhoods $\mathcal{W}$ of a in $\mathcal{F}$ and U of x in X such that the restriction

$$\pi|_{\mathcal{W}} : \mathcal{W} \to U$$

is a homeomorphism. Intuitively, each small open set $U \subset X$ is locally identified with its "slice" $\mathcal{W} \subset \mathcal{F}$ lying above it.

Example 2.3 (Constant Sheaf). Let

$$\mathcal{F} = X \times \mathbb{Z},$$

with the product topology, where $\mathbb{Z}$ carries the discrete topology.

Then the projection

$$\pi : \mathcal{F} \longrightarrow X, \qquad (x, n) \longmapsto x,$$

is continuous and a local homeomorphism. For each $x \in X$, the fibre

$$\mathcal{F}_x = \pi^{-1}(x) = \{(x, n) \mid n \in \mathbb{Z}\}$$

is naturally identified with the group $\mathbb{Z}$ by $(x, n) \mapsto n$. Each fibre $\mathcal{F}_x$ therefore carries the abelian group structure of $\mathbb{Z}$, and the operations of addition and inverse

$$(x, n_1) + (x, n_2) = (x, n_1 + n_2), \qquad -(x, n) = (x, -n),$$

are continuous because $\mathbb{Z}$ is discrete. The identity section $e(x) = (x, 0)$ is also continuous.

Geometric interpretation. Since $\mathbb{Z}$ is discrete, the total space decomposes as a disjoint union

$$\mathcal{F} = \bigsqcup_{n \in \mathbb{Z}} (X \times \{n\}),$$

where each piece $X \times \{n\}$ is homeomorphic to X. Hence $\mathcal{F}$ can be viewed as a countable family of "layers," each one a copy of X, indexed by the integers $n \in \mathbb{Z}$. Over each point $x \in X$, the fibre $\mathcal{F}_x$ consists of countably many points (x, n) stacked vertically, one for each n, forming a discrete abelian group isomorphic to $\mathbb{Z}$.

Now let X be a topological manifold. We construct another fundamental example: the sheaf of germs of continuous functions on X.

Consider all triples (U, f, x) where U is a connected open neighbourhood of $x \in X$, and $f : U \to \mathbb{C}$ is a continuous function.

Define an equivalence relation:

$$(U, f, x) \sim (V, h, y) \quad \Longleftrightarrow \quad \begin{cases} x = y, \\ \text{and there exists an open neighbourhood } W \subset U \cap V \text{ of } x \\ \text{such that } f|_W = h|_W. \end{cases}$$

Lemma 2.4. *Let*

$$\mathcal{F} := \{(U, f, x)\} / \sim$$

be the set of equivalence classes of such triples, and define

$$\pi : \mathcal{F} \to X, \qquad \pi([U, f, x]) = x.$$

Then $\mathcal{F}$, with the natural topology induced by neighbourhoods of the form

$$\{[V, f|_V, x] \mid x \in V \subset U\},$$

is a sheaf of abelian groups over X. Each fibre $\mathcal{F}_x$ consists of germs of continuous functions at x. This is called the sheaf of germs of continuous functions *on X.*

Example 2.5 (Sheaf of Holomorphic Functions). In this example we denote by

$$\mathcal{F} = \mathcal{O}_X$$

the *sheaf of germs of holomorphic functions* on a complex manifold X.

Given a germ $f_x \in \mathcal{O}_x$ (the equivalence class of a holomorphic function f defined near x), the connected component of the domain of definition containing x determines the *maximal domain of existence* of that germ. Geometrically, one can think of this as the *Riemann surface* spread over $\mathbb{C}$ that describes a locally defined holomorphic function.

Example 2.6 (Sheaf of Holomorphic Sections of a Vector Bundle). Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. Then the sheaf

$$\mathcal{O}(E) \quad \text{(often written simply as } \mathcal{E}\text{)}$$

is the *sheaf of germs of holomorphic sections of* E. That is, the fibre of the projection

$$\pi : \mathcal{O}(E) \to X$$

at a point $x \in X$ consists of all germs of local holomorphic sections $s : U \to E$ defined in a neighbourhood U of x.

Example 2.7 (Sheaf of Nowhere–Vanishing Holomorphic Functions). Let

$$\mathcal{O}_X^* \subset \mathcal{O}_X$$

denote the subset consisting of germs of *nowhere-vanishing* holomorphic functions. The projection

$$\pi : \mathcal{O}_X^* \to X$$

is again a local homeomorphism. Each fibre $\mathcal{O}_x^*$ is the multiplicative group of nonzero complex numbers under the operation

$$(f_x, h_x) \longmapsto (fh)_x, \qquad \text{with identity } e_x = 1.$$

Thus $\mathcal{O}_X^*$ is a sheaf of *abelian groups under multiplication.*

3 Mittag–Leffler Problem

Definition 3.1. With the terminology introduced in Definition 1.5, an open cover $\mathcal{U} = \{U_\alpha\}$ of a topological space X is called an *acyclic cover for a sheaf* $\mathcal{F}$ if for all finite intersections $U_{\alpha_0\ldots\alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ we have

$$H^k(U_{\alpha_0\ldots\alpha_p}, \mathcal{F}) = 0 \quad \text{for all } k > 0.$$

Example 3.2 (Acyclic Cover on a Riemann Surface). Let X be a Riemann surface and let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. If $\mathcal{U} = \{U_\alpha\}$ is a covering of X by open sets each biholomorphic to a domain in $\mathbb{C}$, then $\mathcal{U}$ is an *acyclic cover* for $\mathcal{O}_X$.

This follows from the fact that each U_α is a Stein domain, and by the Cauchy–Riemann equations we have

$$H^k(U_\alpha, \mathcal{O}_X) = 0 \quad \text{for } k > 0.$$

Fact: For any Riemann surface X, we have

$$H^2(X, \mathcal{O}_X) = 0.$$

Example 3.3 (Computation for the Projective Line). Let $X = \mathbb{P}^1$ and consider the standard covering

$$\mathcal{U} = \{U_0, U_1\}, \qquad U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\} \cong \mathbb{C}, \quad U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\} \cong \mathbb{C}.$$

Since $\mathcal{U}$ consists of Stein open sets, it is acyclic for $\mathcal{O}_X$, and hence

$$C^2(\mathcal{U}, \mathcal{O}) = 0 \;\Rightarrow\; H^2(\mathbb{P}^1, \mathcal{O}) = 0.$$

Remark 3.4 (Dolbeault Cohomology Perspective). To compute cohomology more analytically, one introduces the $\bar{\partial}$–complex (Dolbeault complex)

$$0 \longrightarrow \mathcal{A}^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2} \longrightarrow \cdots$$

where $\mathcal{A}^{0,q}$ denotes the sheaf of smooth $(0,q)$-forms. The Dolbeault theorem identifies

$$H^q(X, \mathcal{O}_X) \;\cong\; H^{0,q}_{\bar{\partial}}(X).$$

On a Riemann surface ($\dim_{\mathbb{C}} X = 1$), there are no nonzero $(0,2)$-forms, so $H^2(X, \mathcal{O}_X) = 0$.

Example 3.5 (Čech–Dolbeault Description of Cocycles). Let $\mathcal{U} = \{U_0, U_1\}$ be the standard covering of $\mathbb{P}^1$, and let $\mathcal{O} = \mathcal{O}_X$ denote the sheaf of holomorphic functions. Suppose $s_0 \in \mathcal{O}(U_0)$ and $s_1 \in \mathcal{O}(U_1)$ are local holomorphic sections.

The *Čech coboundary operator* acts on a 0–cochain $s = (s_0, s_1) \in C^0(\mathcal{U}, \mathcal{O})$ according to the general formula

$$(\delta^0 s)_{\alpha\beta} = s_\beta|_{U_\alpha \cap U_\beta} - s_\alpha|_{U_\alpha \cap U_\beta}.$$

For our two–set cover, this gives

$$(\delta^0 s)_{01} = s_1|_{U_0 \cap U_1} - s_0|_{U_0 \cap U_1}.$$

Up to a sign convention, we may write the same difference as

$$\phi_{01} = s_0 - s_1,$$

which is a holomorphic function on the overlap $U_0 \cap U_1$. Hence

$$\phi = \{\phi_{01}\} \in C^1(\mathcal{U}, \mathcal{O}),$$

and since there is no triple intersection, this automatically satisfies $\delta^1 \phi = 0$. Thus ϕ defines a 1–*cocycle* in $C^1(\mathcal{U}, \mathcal{O})$.

If one can find local sections s_0, s_1 such that $\phi_{01} = s_0 - s_1 = (\delta^0 s)_{01}$, then the cocycle ϕ is a coboundary, implying

$$H^1(\mathcal{U}, \mathcal{O}) = 0.$$

Because the covering $\mathcal{U}$ is acyclic for $\mathcal{O}$, we conclude

$$H^1(\mathbb{P}^1, \mathcal{O}) = 0.$$

Proposition 3.6 (Mittag–Leffler Problem)**.** *On a complex manifold X, if $H^1(X, \mathcal{O}_X) = 0$, then the* Mittag–Leffler problem *for meromorphic functions on X has a solution.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an acyclic open cover of X for the sheaf $\mathcal{O}_X$. Suppose that on each U_α we are given a meromorphic function $f_\alpha \in \mathcal{M}(U_\alpha)$, and that on overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$ the differences

$$s_{\alpha\beta} := f_\alpha - f_\beta \in \Gamma(U_{\alpha\beta}, \mathcal{O}_X)$$

are holomorphic.

First, we check that $\{s_{\alpha\beta}\}$ defines a Čech 1–cocycle in $Z^1(\mathcal{U}, \mathcal{O}_X)$. Recall that the Čech coboundary for a 1–cochain $\{s_{\alpha\beta}\}$ is given by

$$(\delta s)_{\alpha\beta\gamma} = s_{\beta\gamma} - s_{\alpha\gamma} + s_{\alpha\beta}.$$

Substituting $s_{\mu\nu} = f_\mu - f_\nu$, we get

$$(\delta s)_{\alpha\beta\gamma} = (f_\beta - f_\gamma) - (f_\alpha - f_\gamma) + (f_\alpha - f_\beta) = 0.$$

Hence $(\delta s)_{\alpha\beta\gamma} = 0$ on all triple intersections, and therefore $\{s_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}_X)$.

Since $H^1(X, \mathcal{O}_X) = 0$ and $\mathcal{U}$ is acyclic, we have $H^1(\mathcal{U}, \mathcal{O}_X) = 0$. This means that every 1–cocycle is a 1–coboundary; that is,

$$\{s_{\alpha\beta}\} \in B^1(\mathcal{U}, \mathcal{O}_X).$$

Hence there exists a 0–cochain $\{c_\alpha\}$ with $c_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X)$ such that

$$(\delta c)_{\alpha\beta} = c_\beta - c_\alpha = s_{\alpha\beta}.$$

By the definition of $s_{\alpha\beta}$, this means

$$f_\alpha - f_\beta = c_\beta - c_\alpha \quad \text{on } U_{\alpha\beta}.$$

Now define holomorphic functions

$$h_\alpha := f_\alpha + c_\alpha \quad \text{on } U_\alpha.$$

On intersections $U_{\alpha\beta}$ we have

$$h_\alpha - h_\beta = (f_\alpha - f_\beta) + (c_\alpha - c_\beta) = s_{\alpha\beta} - s_{\alpha\beta} = 0.$$

Thus the functions h_α agree on overlaps and therefore glue together to give a global meromorphic function

$$h \in \Gamma(X, \mathcal{M}_X), \quad \text{with } h|_{U_\alpha} = h_\alpha = f_\alpha + c_\alpha.$$

On each U_α, the difference between h and f_α is

$$h|_{U_\alpha} - f_\alpha = c_\alpha,$$

where c_α is holomorphic. Since adding a holomorphic function does not change the polar part of a meromorphic function, the poles and their coefficients of h and f_α coincide. In the quotient sheaf $\mathcal{M}_X/\mathcal{O}_X$, this means

$$[h|_{U_\alpha}] = [f_\alpha],$$

so h and f_α have the same principal part on U_α.

Consequently, h is a global meromorphic function on X having on each U_α the prescribed principal part of f_α. This proves that the Mittag–Leffler problem has a global solution. □

For the multiplicative sheaf $\mathcal{O}^*$ and an acyclic cover $\mathcal{U}$ for $\mathcal{O}^*$, we have

$$H^1(X, \mathcal{O}^*) = H^1(\mathcal{U}, \mathcal{O}^*) = \{1\}$$

if and only if every 1–cocycle in $\mathcal{O}^*$ is a 1–coboundary.

A 1–*cocycle* in the multiplicative sheaf has components

$$\{\phi_{\alpha\beta}\}_{\alpha,\beta} \quad \text{such that} \quad \phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1 \quad \text{on } U_{\alpha\beta\gamma}.$$

It is a 1–*coboundary* if there exist $\{\psi_\alpha\}$ (a 0–cochain with values in $\mathcal{O}^*$) such that

$$\phi_{\alpha\beta} = (\delta^0\psi)_{\alpha\beta} = \frac{\psi_\beta}{\psi_\alpha}.$$

Hence

$$H^1(X, \mathcal{O}^*) = \{1\} \quad \iff \quad \forall \{\phi_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*),\ \exists \{\psi_\alpha\} \text{ with } \phi_{\alpha\beta} = \frac{\psi_\beta}{\psi_\alpha}.$$

Then $\mathrm{Pic}(X)$ is trivial if and only if $H^1(X, \mathcal{O}^*) = 0$.

Since holomorphic line bundles on X are described by their transition functions $\{\phi_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$, two line bundles E and E' defined on the same cover by $\{\phi_{\alpha\beta}\}$ and $\{\phi'_{\alpha\beta}\}$ are *isomorphic* if there exist nowhere–vanishing holomorphic functions $\{\psi_\alpha\}$ on $\{U_\alpha\}$ such that

$$\frac{\phi'_{\alpha\beta}}{\phi_{\alpha\beta}} = \frac{\psi_\beta}{\psi_\alpha}.$$

That is, the ratio of their 1–cocycles is a 1–coboundary.

Equation above means the ratio $\{\phi'_{\alpha\beta}/\phi_{\alpha\beta}\}$ is of the special form

$$\frac{\phi'_{\alpha\beta}}{\phi_{\alpha\beta}} = (\delta^0\psi)_{\alpha\beta} = \frac{\psi_\beta}{\psi_\alpha},$$

that is, $\phi'/\phi \in B^1(\mathcal{U}, \mathcal{O}^*)$, the group of 1–coboundaries.

Thus two cocycles $\phi, \phi' \in Z^1(\mathcal{U}, \mathcal{O}^*)$ represent *isomorphic line bundles* if and only if they differ by a coboundary. This defines an equivalence relation on $Z^1(\mathcal{U}, \mathcal{O}^*)$:

$$\phi' \sim \phi \quad \iff \quad \exists\, \psi \text{ such that } \phi'_{\alpha\beta} = (\delta^0\psi)_{\alpha\beta}\phi_{\alpha\beta}.$$

Hence the set of isomorphism classes of holomorphic line bundles is precisely the group of 1–cocycles modulo 1–coboundaries:

$$\mathrm{Pic}(X) = Z^1(\mathcal{U}, \mathcal{O}^*)/B^1(\mathcal{U}, \mathcal{O}^*) \cong H^1(X, \mathcal{O}^*).$$

In particular, $\mathrm{Pic}(X)$ is *trivial* (i.e. every holomorphic line bundle over X is isomorphic to the trivial one) if and only if

$$H^1(X, \mathcal{O}^*) = 0,$$

since this condition means every 1–cocycle $\{\phi_{\alpha\beta}\}$ is a 1–coboundary of the form $\phi_{\alpha\beta} = \psi_\beta/\psi_\alpha$, so all transition functions can be trivialized by changing local frames.

4 Sequences in Sheaf Cohomology

Definition 4.1 (Short exact sequence of sheaves)**.** A sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{\sigma} \mathcal{F}'' \longrightarrow 0$$

is called a short exact sequence of sheaves if and only if:

1. i is injective;
2. σ is surjective;
3. $\mathrm{im}(i) = \ker(\sigma)$.

Theorem 4.2 (Long exact sequence in cohomology)**.** *Given a short exact sequence of sheaves as in Definition 4.1,*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{\sigma} \mathcal{F}'' \longrightarrow 0,$$

there exists a natural long exact sequence in sheaf cohomology:

$$\begin{aligned} 0 \longrightarrow & H^0(X, \mathcal{F}') \xrightarrow{i_*} H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) \xrightarrow{\sigma_*} H^0(X, \mathcal{F}'') \xrightarrow{\delta} H^1(X, \mathcal{F}') \\ & \xrightarrow{i_*} H^1(X, \mathcal{F}) \xrightarrow{\sigma_*} H^1(X, \mathcal{F}'') \xrightarrow{\delta} H^2(X, \mathcal{F}') \longrightarrow \cdots \end{aligned}$$

where δ denotes the connecting homomorphisms in cohomology.

Example 4.3 (Connecting homomorphism in the long exact sequence). Let $s \in H^0(X, \mathcal{F}'') = \Gamma(X, \mathcal{F}'')$ be a global section. Choose an open cover $\mathcal{U} = \{U_\alpha\}$ of X such that, by Definition 4.1, each restriction $s|_{U_\alpha}$ can be locally lifted to a section $t_\alpha \in \Gamma(U_\alpha, \mathcal{F})$ satisfying

$$\sigma(t_\alpha) = s|_{U_\alpha}.$$

On the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have

$$\sigma_*(t_\alpha - t_\beta) = \sigma(t_\alpha) - \sigma(t_\beta) = s|_{U_\alpha} - s|_{U_\beta} = 0.$$

Hence

$$u_{\alpha\beta} := t_\alpha - t_\beta \in \Gamma(U_{\alpha\beta}, \mathcal{F}').$$

Moreover, on triple overlaps $U_{\alpha\beta\gamma}$,

$$u_{\alpha\beta} + u_{\beta\gamma} + u_{\gamma\alpha} = 0,$$

so that $\{u_{\alpha\beta}\}$ is a 1–cocycle:

$$\{u_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{F}').$$

The cohomology class $[\{u_{\alpha\beta}\}] \in H^1(X, \mathcal{F}')$ is by definition the image $\delta(s)$ of s under the connecting homomorphism

$$\delta : H^0(X, \mathcal{F}'') \longrightarrow H^1(X, \mathcal{F}').$$

Claim:

$$\varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{O}^*) \cong \operatorname{Pic}(X).$$

Remark 4.4. $\operatorname{Pic}(X)$ is trivial $\iff H^1(X, \mathcal{O}^*) = \{1\}$.

Sketch of the idea. Let L and L' be holomorphic line bundles on X, defined respectively by transition functions

$$\{\phi_{\alpha\beta}\} \subset \mathcal{O}^*(U_{\alpha\beta}) \quad \text{and} \quad \{\phi'_{\alpha\beta}\} \subset \mathcal{O}^*(U_{\alpha\beta})$$

on an open cover $\mathcal{U} = \{U_\alpha\}$ of X. These satisfy

$$\phi_{\alpha\beta}\phi_{\beta\alpha} = 1, \qquad \phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1,$$

for all indices α, β, γ, which means that $\{\phi_{\alpha\beta}\}$ defines a Čech 1–cocycle:

$$\{\phi_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*).$$

Two bundles L and L' are isomorphic if and only if their transition functions differ by a Čech 1–coboundary; that is, there exist local nowhere-vanishing holomorphic functions $\{\psi_\alpha\} \subset \mathcal{O}^*(U_\alpha)$ such that

$$\frac{\phi'_{\alpha\beta}}{\phi_{\alpha\beta}} = \frac{\psi_\alpha}{\psi_\beta}, \qquad \forall\, \alpha, \beta.$$

Hence $\{\phi_{\alpha\beta}\}$ and $\{\phi'_{\alpha\beta}\}$ represent the same cohomology class in $H^1(\mathcal{U}, \mathcal{O}^*)$.

The correspondence

$$[\{\phi_{\alpha\beta}\}] \longmapsto [L]$$

defines a bijection between elements of $H^1(\mathcal{U}, \mathcal{O}^*)$ and isomorphism classes of line bundles defined with respect to the cover $\mathcal{U}$. Passing to the direct limit over refinements of covers gives the natural identification

$$\varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{O}^*) \cong \operatorname{Pic}(X).$$

Theorem 4.5 (Exponential sequence on a Riemann surface X)**.** *There is a short exact sequence of sheaves*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & \mathcal{O} & \xrightarrow{\exp(2\pi i\cdot)} & \mathcal{O}^* & \longrightarrow & 1, \\ & & & & f_x & \longmapsto & e^{2\pi i f_x} & & \end{array}$$

called the exponential sequence *on X.*

The associated long exact sequence in cohomology is

$$H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O}^*) \cong \operatorname{Pic}(X) \xrightarrow{c_1} H^2(X, \underline{\mathbb{Z}}) \longrightarrow H^2(X, \mathcal{O}),$$

where c_1 denotes the first Chern class map.

Example 4.6 ($X = \mathbb{P}^1$). For the Riemann sphere $\mathbb{P}^1$, the exponential sequence yields

$$H^1(\mathbb{P}^1, \mathcal{O}) \longrightarrow \mathrm{Pic}(\mathbb{P}^1) \longrightarrow H^2(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z} \longrightarrow H^2(\mathbb{P}^1, \mathcal{O}) = 0.$$

Hence

$$\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}.$$

Example 4.7 ($X = \mathbb{C}/L$ (complex torus)). On a complex torus $X = \mathbb{C}/L$ with lattice $L \cong \mathbb{Z}^2$, the corresponding portion of the long exact sequence is

$$\begin{array}{ccccccccc}
H^1(X, \underline{\mathbb{Z}}) & \hookrightarrow & H^1(X, \mathcal{O}) & \longrightarrow & H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X) & \xrightarrow{c_1} & H^2(X, \underline{\mathbb{Z}}) & \longrightarrow & 0 \\
\parallel & & & & \parallel & & & & \\
\mathbb{Z}^2 & \longrightarrow & \mathbb{C} & \twoheadrightarrow & \mathbb{C}/\mathbb{Z}^2 = \mathrm{Jac}(X) = \mathrm{Pic}_0(X). & & & &
\end{array}$$

Thus, $\mathrm{Pic}_0(X)$ is the group of degree-zero line bundles on X, and one has

$$\mathrm{Pic}(X) \;\cong\; \mathrm{Pic}_0(X) \times \mathbb{Z},$$

where the integer factor corresponds to the image of the first Chern class $c_1 : H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z})$.