Modules of rings: basic definitions and examples

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In this file: Basic definitions and examples of modules of rings (§2.2.1-2.2.4 of Lecture notes):

- Modules, submodules, and module homomorphisms;
- Annihilators and torsion modules;
- Cyclic modules over PIDs;
- Free modules.

Definitions.

1. A left module over a ring R is an abelian group M together with a map $R \times M \to M, (r, m) \mapsto rm$, satisfying
$$\begin{aligned} 1m &= m, \\ (r_1 r_2)m &= r_1(r_2 m), \\ (r_1 + r_2)m &= r_1 m + r_2 m, \\ r(m_1 + m_2) &= rm_1 + rm_2, \end{aligned}$$
where $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$.
2. Given a left R-module M, a sub-module of M is an abelian subgroup N of M such that $rn \in N$ for all $r \in R$ and $n \in N$.
3. Right R-modules are defined similarly.
4. We will assume that R is commutative.
5. When R is a field, R-modules are precisely vector spaces over R.

Examples. $R = \mathbb{C}[x_1, \cdots, x_n]$

1. A module over a field K is the same as a K-vector space.
2. R is an R-module by multiplication; sub-modules are the ideals of R.
3. For any ideal $I \subset R$ an ideal, then R/I is an R-module:
$$r(r_1 + I) = rr_1 + I, \quad r, r_1 \in R.$$
4. An abelian group is a $\mathbb{Z}$-module. $n \cdot a = \begin{cases} \overbrace{a + \cdots + a}^{n} & n \geq 0 \\ -(\overbrace{a + \cdots + a}^{-n}) & n \leq 0 \end{cases}$
5. If M is an R-module and $S \subset M$, then
$$RS := \{r_1 s_1 + \cdots + r_n s_n : n \in \mathbb{N}, r_i \in R, s_i \in S\}$$
is a sub-module of M called the sub-module of M generated by S.
6. An R-module M is said to be finitely generated if $\exists$ a finite subset S of M s.t. $M = RS$.

$R = K[x]$

One of our main examples: $K[x]$-modules for a field K are in bijection with

Let M be an R-module. Then, by def, $(M,+)$ is an abelian group, and have a map

$$K[x] \times M \longrightarrow M, \quad (f, m) \longmapsto fm$$

$$\Rightarrow \quad \cup$$

1) $K \times M \longrightarrow M \quad (k, m) \longmapsto km$, $k \in K$

2) $x \in K[x] = R$, $T: M \longrightarrow M$, $m \overset{T}{\longmapsto} xm$

$$\Rightarrow T \in \mathrm{End}_K(M) \begin{cases} T(m_1+m_2) = Tm_1 + Tm_2 \\ T(km) = k\,T(m) \end{cases}$$

Lemma: The assignment

$$\{K[x]\text{-modules}\} \longrightarrow \{\text{Pairs } (M,T), \text{ where } M \text{ is a } K\text{-vector space and } T \in \mathrm{End}_K(M)\}$$
$$\overset{?}{\longleftarrow}$$

is a bijection.

Given (M,T), can make $(M,+)$ into a $K[x]$ module

by $\left(f(x) = \sum_{a_i \in K} a_i x^i,\ m\right) \longmapsto \sum a_i T^i m$

More generally, if K is a field, have bijection

$$\{K[x_1,\dots,x_n]\text{-modules}\} \longleftrightarrow \left\{(M, T_1,\dots,T_n) : \begin{array}{l} \text{where} \\ M \text{ is a } K\text{-vector space} \\ T_1,\dots,T_n \in \mathrm{End}_K(M) \\ T_iT_j = T_jT_i \quad \forall i,j=1,\dots,n \end{array}\right\}$$

Definitions:

- The direct sum of R-modules $M_1, \ldots, M_n$, denoted as $M_1 \oplus \cdots \oplus M_n$, is the direct product abelian group $M_1 \times \cdots \times M_n$ equipped with the R-module structure given by

$$r(m_1, \ldots, m_n) = (rm_1, \ldots, rm_n), \qquad r \in R,\ m_j \in M_j.$$

- $M^n = M \oplus \cdots \oplus M$ (n copies).
- Have R-module R^n for each integer $n \geq 1$.
- An R-module M is said to be finitely generated if there exists a finite subset S of M such that $RS = M$.

Goal of this chapter:

Classify finitely generated modules of PIDs, up to isomorphisms.

Let K be a field, eg $K = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Q}$ . . .

Let V be a finite dim. vector space over K.

Let $T \in \mathrm{End}_K(V)$. Regard V as a $K[x]$-module

via

$$k \ni k \cdot m = km \qquad m \in M$$
$$xm = Tm$$
$$x^2 m = T^2 m = T(Tm)$$
$$(1+2x^2)m = m + 2T^2 m \quad \ldots$$

If $\{v_1, \ldots, v_n\}$ is basis of V as a K-vector space,

then $V = Kv_1 + Kv_2 + \cdots + Kv_n$, so V is

a finitely generated $K[x]$-module.

On the other hand, here are some examples of $K[x]$-module

$K[x]/\langle x^2+1 \rangle$, $K[x]/\langle x-3 \rangle$

$\bar{1}$, $\bar{x}$, $\overline{x^2} = \overline{-1} = -\bar{1}$

Continue on Monday Feb 24, 2025