20241121 MATH3541 NOTE 10[1]

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Contents

1 Introduction

We've proven that the fundamental functor π_1 is a functor.
However, beyond the trivial case where X is a convex subset of a normed vector space, we still don't know how to compute the fundamental group $\pi_1(X)$ of X.
Hence, we shall introduce covering space to compute the fundamental group.

2 Covering Map

2.1 Homeomorphism, Local Homeomorphism and Covering Map

Homeomorphism is an attempt to identify topological spaces.

Definition 2.1. (**Homeomorphism**)
Let X, Y be two topological spaces, and $f : X \to Y$ be a function. If f has an inverse g, and both f and g are continuous, then f is a homeomorphism.

Sometimes homeomorphism is too strong, so we introduce local homeomorphism.

Definition 2.2. (**Local Homeomorphism**)
Let X, Y be two topological spaces, and $f : X \to Y$ be a function. If every $x \in X$ has an open neighbour U, such that $f(U)$ is open in Y, and the restricted function $f|_U : U \to f(U)$ is a homeomorphism, then f is a local homeomorphism.

Proposition 2.3. If $f : X \to Y$ is a homeomorphism,
then f is a local homeomorphism.

Proof. For all $x \in X$, there exists an open neighbour X of x, such that $f(X) = Y$ is open in Y, and the restricted function $f|_X = f$ is a homeomorphism.
Hence, f is a local homeomorphism. Quod. Erat. Demonstrandum. □

Proposition 2.4. If $f : X \to Y$ is a local homeomorphism,
then f is continuous and open.

Proof. We may divide our proof into two parts.
(1) We prove that f is open.
For all open subset $\mathfrak{U}$ of X, we wish to show that $f(\mathfrak{U})$ is open in Y.
That is, for all $y \in f(\mathfrak{U})$, we wish to find an open neighbour $\mathfrak{V}$ of y with $\mathfrak{V} \subseteq f(\mathfrak{U})$.
As $y \in f(\mathfrak{U})$, there exists $x \in \mathfrak{U}$, such that $y = f(x)$.
As f is a local homeomorphism, x has an open neighbour U, such that $f(U)$ is open in

Y, and the restricted function $f|_U : U \to f(U)$ is a homeomorphism. Now:

$$\begin{aligned}
\mathfrak{U} \text{ is open in } X &\implies U \cap \mathfrak{U} \text{ is open in } U, \text{ apply the homeomorphism } f|_U \\
&\implies \mathfrak{V} = f|_U (U \cap \mathfrak{U}) \text{ is open in } f(U), \text{ where } f(U) \text{ is open in } Y \\
&\implies \mathfrak{V} \text{ is an open neighbour of } y \text{ with } \mathfrak{V} \subseteq f(U)
\end{aligned}$$

(2) We prove that f is continuous.
For all open subset $\mathfrak{V}$ of Y, we wish to show that $f^{-1}(\mathfrak{V})$ is open in X.
That is, for all $x \in f^{-1}(\mathfrak{V})$, we wish to find an open neighbour $\mathfrak{U}$ of x with $\mathfrak{U} \subseteq f^{-1}(\mathfrak{V})$.
As f is a local homeomorphism, x has an open neighbour U, such that $f(U)$ is open in Y, and the restricted function $f|_U : U \to f(U)$ is a homeomorphism. Now:

$$\begin{aligned}
\mathfrak{V} \text{ is open in } Y &\implies f(U) \cap \mathfrak{V} \text{ is open in } f(U), \text{ apply the homeomorphism } f|_U \\
&\implies \mathfrak{U} = f|_U^{-1} (f(U) \cap \mathfrak{V}) \text{ is open in } U, \text{ where } U \text{ is open in } X \\
&\implies \mathfrak{U} \text{ is an open neighbour of } x \text{ with } \mathfrak{U} \subseteq U
\end{aligned}$$

Quod. Erat. Demonstrandum. □

However, local homeomorphism is too weak to "record different decks".

Definition 2.5. (**Covering Map**)
Let X, Y be two topological spaces, and $f : X \to Y$ be a continuous surjection.
If for all $y \in Y$, there exists an open neighbour V of y, such that:
(1) $f^{-1}(V)$ is homeomorphic to $\coprod_{\lambda \in I} U_\lambda$, where each U_λ is open in X.
(2) Each restricted map $f|_{U_\lambda} : U_\lambda \to V$ is a homeomorphism.
Then, f is a covering map.

Proposition 2.6. Let $(X_\lambda)_{\lambda \in I}$ be an indexed family of topological spaces,
X be the coproduct space of $(X_\lambda)_{\lambda \in I}$, Y be a topological space,
and $(f_\lambda : X_\lambda \to Y)_{\lambda \in I}$ be an indexed family of homeomorphisms.
The coproduct f of $(f_\lambda)_{\lambda \in I}$ is a covering map.

Proof. For all $y \in Y$, there exists an open neighbour Y of y, such that:
(1) $f^{-1}(Y) = X \cong \coprod_{\lambda \in I} X_\lambda \times \{\lambda\}$, where each $X_\lambda \times \{\lambda\}$ is open in X.
(2) Each restricted map $f|_{X_\lambda} (x, \lambda) = f_\lambda(x)$ is a homeomorphism.
Hence, f is a covering map. Quod. Erat. Demonstrandum. □

Remark: *Coproduct is not an interesting covering space.*

Proposition 2.7. If $f : X \to Y$ is a covering map,
then f is a local homeomorphism.

Proof. For all $x \in X$, we wish to find an open neighbour U of x, such that $f(U)$ is open in Y, and the restricted function $f|_U : U \to f(U)$ is a homeomorphism.
As f is a covering map, there exists an open neighbour V of $f(x)$, such that:
(1) $f^{-1}(V)$ is homeomorphic to $\coprod_{\lambda \in I} U_\lambda$, where each U_λ is open in X.
(2) Each restricted map $f|_{U_\lambda} : U_\lambda \to V$ is a homeomorphism.
As $f(x) \in V$, $x \in f^{-1}(V) = \bigcup_{\lambda \in I} U_\lambda$, so x is in some U_μ.
Hence, there exists an open neighbour U_μ of x, such that $f(U_\mu) = V$ is open in Y, and the restricted function $f|_{U_\mu} : U_\mu \to V$ is a homeomorphism.
Quod. Erat. Demonstrandum. □

Remark: *As a consequence, every covering map f is open.*

2.2 Lifting Properties

Covering map is introduced to "record the information of a loop in vertical displacement".

Theorem 2.8. (**Special Lifting Property**)
Let $p : \widetilde{X} \to X$ be a covering map.
For all path $\gamma : [0,1]_t \to X$ downstairs with initial point $\gamma(0) \in X$,
for all initial point $\widetilde{\gamma}(0) \in \widetilde{X}$ upstairs with projection $p(\widetilde{\gamma}(0)) = \gamma(0)$, there exists a unique path $\widetilde{\gamma} : [0,1]_t \to \widetilde{X}$ upstairs with projection $p \circ \widetilde{\gamma} = \gamma$.

Proof. We may divide our proof into seven steps.
(1) For each point $\xi \in X$ downstairs, choose an open neighbour U_ξ of ξ, such that $p^{-1}(U_\xi) \cong \coprod_{\lambda_\xi \in I_\xi} \widetilde{U}_\xi^{\lambda_\xi}$ and each $p|_{\widetilde{U}_\xi^{\lambda_\xi}} : \widetilde{U}_\xi^{\lambda_\xi} \to U_\xi$ is a homeomorphism.
(2) γ pulls the open cover $(U_\xi)_{\xi \in X}$ of X back to that $(\gamma^{-1}(U_\xi))_{\xi \in X}$ of $[0,1]_t$.
(3) There exists a partition $\pi_t : 0 = t^0 < t^1 < \cdots < t^{n-1} < t^n = 1$ of the compact set $[0,1]_t$, such that each $T_k = [t^k, t^{k+1}]$ is entirely contained in some $\gamma^{-1}(U_{\xi_k})$.
(4) There exists a unique $\mu_{\xi_0} \in I_{\xi_0}$, such that $\widetilde{\gamma}(t^0) = \widetilde{\gamma}(0) \in \widetilde{U}_{\xi_0}^{\mu_{\xi_0}}$ upstairs.
(5) Construct a path $\widetilde{\gamma}|_{T_k} = p|_{\widetilde{U}_{\xi_0}^{\mu_{\xi_0}}}^{-1} \circ \gamma|_{T_k}$.
(6) Now we've found a new point $\widetilde{x}_1 = \widetilde{\gamma}(t^1)$ upstairs with projection $x_1 = \gamma(t^1)$. Repeat the procedure inductively, and the concatenation generates a path $\widetilde{\gamma}$.
(7) For all open covers $(U_\xi)_{\xi \in X}, (U'_\xi)_{\xi \in X}$ chosen in (1), on each block of $\pi_t \cup \pi'_t$, the paths $\widetilde{\gamma}, \widetilde{\gamma}'$ are identified, so $\widetilde{\gamma}, \widetilde{\gamma}'$ are globally identified.
Quod. Erat. Demonstrandum. □

Remark: *The main purpose of this theorem is to lift paths.*

Theorem 2.9. (**General Lifting Property**)
Let $p : \widetilde{X} \to X$ be a covering map.
For all continuous function $H : [0,1]^n \to X$ with initial point $H(\mathbf{0}) \in X$,
for all initial point $\widetilde{H}(\mathbf{0})$ upstairs with projection $p(\widetilde{H}(\mathbf{0})) = H(\mathbf{0})$, there exists a unique continuous function $\widetilde{H} : [0,1]^n \to \widetilde{X}$ upstairs with projection $p \circ \widetilde{H} = H$.

Proof. We may divide our proof into seven steps.

(1) For each point $\xi \in X$ downstairs, choose an open neighbour U_ξ of ξ,
such that $p^{-1}(U_\xi) \cong \coprod_{\lambda_\xi \in I_\xi} \widetilde{U}_\xi^{\lambda_\xi}$ and each $p|_{\widetilde{U}_\xi^{\lambda_\xi}} : \widetilde{U}_\xi^{\lambda_\xi} \to U_\xi$ is a homeomorphism.

(2) H pulls the open cover $(U_\xi)_{\xi \in X}$ of X back to that $(H^{-1}(U_\xi))_{\xi \in X}$ of $[0,1]^n$.

(3) There exists a partition $\pi^n = \prod_{l=1}^n \pi_{t_l}$ of the compact set $[0,1]^n = \prod_{l=1}^n [0,1]_{t_l}$,
such that each $T_{\mathbf{k}}^n = \prod_{l=1}^n [t_l^{k_l}, t_l^{k_l+1}]$ is entirely contained in some $H^{-1}(U_{\xi_{\mathbf{k}}})$.

(4) There exists a unique $\mu_{\xi_{\mathbf{0}}} \in I_{\xi_{\mathbf{0}}}$, such that $\widetilde{H}(t_l^0)_{l=1}^n = \widetilde{H}(\mathbf{0}) \in \widetilde{U}_{\xi_{\mathbf{0}}}^{\mu_{\xi_{\mathbf{0}}}}$ upstairs.

(5) Construct a continuous function $\widetilde{H}\Big|_{T_{\mathbf{k}}^n} = p|_{\widetilde{U}_{\xi_{\mathbf{0}}}^{\mu_{\xi_{\mathbf{0}}}}}^{-1} \circ H|_{T_{\mathbf{k}}^n}$.

(6) Repeat the procedure inductively in n directions,
and the concatenation generates a continuous function $\widetilde{H}$.

(7) For all open covers $(U_\xi)_{\xi \in X}, (U'_\xi)_{\xi \in X}$ chosen in (1), on each block of $\pi^n \cup \pi'^n$,
the two continuous functions $\widetilde{H}, \widetilde{H}'$ are identified, so $\widetilde{H}, \widetilde{H}'$ are globally identified.

Quod. Erat. Demonstrandum. □

Remark: *The main purpose of this theorem is to lift path homotopies.*

Proposition 2.10. Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map.
The map p' obtained by the fundamental functor π_1 is a group embedding.

Proof. For all $[\![\widetilde{\gamma}]\!] \in \pi_1(X, x_0)$:

$$p'([\![\widetilde{\gamma}]\!]) = [\![e_{x_0}]\!] = p'([\![\widetilde{e}_{\widetilde{x}_0}]\!]) \implies [\![p \circ \widetilde{\gamma}]\!] = [\![p \circ \widetilde{e}_{\widetilde{x}_0}]\!] \implies p \circ \widetilde{\gamma} \approx p \circ \widetilde{e}_{\widetilde{x}_0} \implies \widetilde{\gamma} \approx \widetilde{e}_{\widetilde{x}_0}$$

Quod. Erat. Demonstrandum. □

Remark: *This is a direct consequence of homotopy lifting property.*

Theorem 2.11. (**Universal Property of Covering Space**[2])
Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map.
For all continuous function $f : (Y, y_0) \to (X, x_0)$,
where Y is path connected and locally path connected:

$$f'(\pi_1(Y, y_0)) \leq p'(\pi_1(\widetilde{X}, \widetilde{x}_0)) \iff \exists! \text{ lift } \widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0) \text{ of } f$$

Proof. We may divide our proof into two parts.

"if" direction: Assume that $\exists!$ lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f.
For all $[\![\gamma]\!] \in f'(\pi_1(Y, y_0))$, there exists $[\![\sigma]\!] \in \pi_1(Y, y_0)$, such that:

$$[\![\gamma]\!] = f'([\![\sigma]\!]) = [\![f \circ \sigma]\!]$$

Hence, there exists $[\![\widetilde{f} \circ \sigma]\!] \in \pi_1(Y, y_0)$, such that:

$$[\![\gamma]\!] = [\![p \circ \widetilde{f} \circ \sigma]\!] = p'([\![\widetilde{f} \circ \sigma]\!])$$

This implies $[\![\gamma]\!] \in p'(\pi_1(\widetilde{X}, \widetilde{x}_0))$.
"only if" direction: Assume that $f'(\pi_1(Y, y_0)) \leq p'(\pi_1(\widetilde{Y}, \widetilde{y}_0))$.
For all $y \in Y$, define $\widetilde{f}(y) = \widetilde{x}$ if there exists a path σ from y_0 to y,
such that $\gamma = f \circ \sigma$ is from x_0 to $x = f(y)$, and γ has a lift $\widetilde{\gamma}$ from $\widetilde{x}_0$ to $\widetilde{x}$ with $p(\widetilde{x}) = x$.
(1) We show that $\widetilde{f}$ has at least one image $\widetilde{x}$ under an arbitrary $y \in Y$.
As Y is path connected, there is at least one path σ from y_0 to y.
As $[\![\gamma]\!] = [\![f \circ \sigma]\!] = f'([\![\sigma]\!])$ is in $p'(\pi_1(\widetilde{X}, \widetilde{x}_0))$, γ has a lift $\widetilde{\gamma}$ from $\widetilde{x}_0$ to $\widetilde{x}$ with $p(\widetilde{x}) = x$.
(2) We show that $\widetilde{f}$ has at most one image $\widetilde{x}$ under an arbitrary $y \in Y$.
For all paths σ, σ' from y_0 to y, we may construct the following loop:

$$[\![\gamma]\!] \star [\![\gamma']\!]^{-1} = f'([\![\sigma]\!]) \star f'([\![\sigma']\!])^{-1} = f'([\![\sigma]\!] \star [\![\sigma']\!]^{-1}) \text{ is in } p'(\pi_1(\widetilde{X}, \widetilde{x}_0))$$

As this loop $[\![\gamma]\!] \star [\![\gamma']\!]^{-1}$ pulls back to a loop $[\![\widetilde{\alpha}]\!]$ in $\widetilde{X}$, it must be true that $\widetilde{x} = \widetilde{x}'$.
(3) We show that $\widetilde{f}$ is continuous. Our strategy is to prove the following statement:

$$\forall y \in Y, \forall \text{ open neighbour } \widetilde{U} \text{ of } \widetilde{f}(y), \exists \text{ open neighbour } V \text{ of } x, \widetilde{f}(V) \subseteq \widetilde{U}$$

For all open neighbour $\widetilde{U}$ of $\widetilde{f}(y)$, we wish to separate it with other unrelated points.
Choose an evenly covered open neighbour U_y of $f(y)$, such that U_y has a unique sheet $\widetilde{U}_y$ above containing $\widetilde{f}(y)$, where $\widetilde{f}(y)$ is a preimage of $f(y)$ under p.
Notice that $f^{-1}(p(\widetilde{U} \cap \widetilde{U}_y))$ is an open neighbour of y. As Y is locally path connected, $f^{-1}(p(\widetilde{U} \cap \widetilde{U}_y))$ contains a path connected neighbour V of y.
We want to show $\widetilde{f}(V) \subseteq \widetilde{U}$, i.e., for all $z \in V$, $\widetilde{f}(z) \in \widetilde{U}$.
As V is path connected, we may connect y_0 to y in Y, and then connect y to z in V.
Now consider the path that connects $f(y)$ to $f(z)$ in $f(V) \subseteq U_y$.
This path can be lifted within one sheet.
As the initial point $\widetilde{f}(y) \in \widetilde{U}_y$, the whole path is restricted to stay in $\widetilde{U}_y$. Now apply the local inverse $p|_{\widetilde{U}_y}^{-1}$ to $f(z) \in p(\widetilde{U} \cap \widetilde{U}_y)$, we get $\widetilde{f}(z) \in \widetilde{U} \cap \widetilde{U}_y \subseteq \widetilde{U}$, and we are done.
(4) It follows from the unique path lifting property that such lifting is unique.
Quod. Erat. Demonstrandum. □

2.3 Constructions of Covering Map

Proposition 2.12. If $f : X \to Y$ is a homeomorphism, and $\mathfrak{U}$ is a subset of X, then the restricted map $f|_{\mathfrak{U}} : \mathfrak{U} \to f(\mathfrak{U})$ is a homeomorphism.

Proof. As $f : X \to Y$ is a homeomorphism, f has an inverse g, and both f and g are continuous. The inverse relationship and continuity are preserved if we restrict f to $f|_{\mathfrak{U}}$ and restrict g to $g|_{f(\mathfrak{U})}$. Hence, the restricted map $f|_{\mathfrak{U}}$ is a homeomorphism.
Quod. Erat. Demonstrandum. □

Proposition 2.13. If $f : X \to Y$ is a local homeomorphism, and $\mathfrak{U}$ is a subset of X, then the restricted map $f|_{\mathfrak{U}} : \mathfrak{U} \to f(\mathfrak{U})$ is a local homeomorphism.

Proof. For all $x \in \mathfrak{U}$, as f is a local homeomorphism, there exists an open neighbour U of x, such that $f(U)$ is open in Y and the restricted function $f|_U : U \to f(U)$ is a homeomorphism. As there exists an open neighbour $U \cap \mathfrak{U}$ of x, such that $f|_{U\cap\mathfrak{U}} = (f|_U)|_{\mathfrak{U}}$ is a homeomorphism, we may conclude that $f|_{\mathfrak{U}}$ is local homeomorphism.
Quod. Erat. Demonstrandum. □

Proposition 2.14. If $f : X \to Y$ is a covering map, and $\mathfrak{U}$ is a subset of X, then the restricted map $f|_{\mathfrak{U}} : \mathfrak{U} \to f(\mathfrak{U})$ is a covering map.

Proof. For all $y \in f(\mathfrak{U})$, as f is a covering map, there exists an open neighbour V of y, such that $f^{-1}(V)$ is homeomorphic to a disjoint union $\coprod_{\lambda\in I} U_\lambda$ of open subsets of X, and each restricted map $f|_{U_\lambda} : U_\lambda \to V$ is a homeomorphism. Hence, there exists an open neighbour $f(\mathfrak{U}) \cap V$ of y, such that $f|_{\mathfrak{U}}^{-1}(f(\mathfrak{U}) \cap V)$ is homeomorphic to a disjoint union $\coprod_{\lambda\in I} \mathfrak{U} \cap U_\lambda$ of open subsets of $\mathfrak{U}$, and each restricted map $f|_{\mathfrak{U}\cap U_\lambda} = (f|_U)|_{\mathfrak{U}}$ is a homeomorphism. This implies the restricted map $f|_{\mathfrak{U}}$ is a covering map.
Quod. Erat. Demonstrandum. □

Proposition 2.15. Define Obj = [All topological space]. $X \sim Y$ if there exists a homeomorphism $f : X \to Y$ is an equivalence relation on Obj.

Proof. We may divide our proof into three parts.
(1) For all $X \in$ Obj:
There exists a homeomorphism $e_X : x \mapsto x$ from X to X.
(2) For all $X, Y \in$ Obj:
If there exists a homeomorphism f from X to Y,
then f has an inverse g and both f and g are continuous,
so g has an inverse f and both g and f are continuous.
Hence, there exists a homeomorphism g from Y to X.
(3) For all $X, Y, Z \in$ Obj:
If there exist a homeomorphism f from X to Y and a homeomorphism u from Y to Z,
then f has an inverse g and u has an inverse v and all f, g, u, v are continuous,
so $u \circ f$ has an inverse $g \circ v$ and both $u \circ f$ and $g \circ v$ are continuous.
Hence, there exists a homeomorphism $u \circ f$ from X to Z.
To conclude, $\sim$ is an equivalence relation on Obj. Quod. Erat. Demonstrandum. □

Proposition 2.16. If $f : X \to Y, g : Y \to Z$ are two local homeomorphisms, then $g \circ f : X \to Z$ is a local homeomorphism.

Proof. We may divide our proof into three steps.
(1) For all $x \in X$, for some open neighbour U_x of x, $f(U_x)$ is open in Y,
and the restricted map $f|_{U_x} : U_x \to f(U_x)$ is a homeomorphism.
(2) For all $y \in Y$, for some open neighbour V_y of y, $g(V_y)$ is open in Z,
and the restricted map $g|_{V_y} : V_y \to g(V_y)$ is a homeomorphism.

(3) For all $x \in X$, for some open neighbour $W_x = U_x \cap f^{-1}(V_{f(x)})$ of x, $g \circ f(W_x)$ is open in Z, and the restricted map $g \circ f|_{W_x} = g|_{V_{f(x)}} \circ f|_{W_x}$ is a homeomorphism.
Hence, $g \circ f$ is a local homeomorphism. Quod. Erat. Demonstrandum. $\square$

Remark: *However, when it comes to covering map, things becomes more complicated.*

Theorem 2.17. Let X, Y, Z be three topological spaces,
and $f : X \to Y, g : Y \to Z$ be two continuous functions.
If f, g are covering maps, and each $g^{-1}(\{z\})$ is finite,
then $h = g \circ f$ is a covering map.

Proof. We may divide our proof into four steps.
Step 1: For all $z \in Z$, for some open neighbour W_z of z, $g^{-1}(W_z) \cong \coprod_{k=1}^{m} \mathfrak{V}_k$,
and each restricted map $g|_{\mathfrak{V}_k} : \mathfrak{V}_k \to W_z$ is a homeomorphism.
Step 2: For some open neighbour $V_{y_k} \subseteq \mathfrak{V}_k$ of $y_k = g|_{\mathfrak{V}_k}^{-1}(z)$, $f^{-1}(V_{y_k}) \cong \coprod_{\lambda_k \in I_k} \mathfrak{U}_{\lambda_k}$,
and each restricted map $f|_{\mathfrak{U}_{\lambda_k}} : \mathfrak{U}_{\lambda_k} \to V_{y_k}$ is a homeomorphism.
Step 3: As g, f are covering maps, g, f are surjective and continuous,
which implies $h = g \circ f$ is surjective and continuous.
Step 4: Define an open neighbour $U_z = \bigcap_{k=1}^{m} g|_{\mathfrak{V}_k}(V_{y_k})$ of z.
The following set-theoretic result holds:

$$
\begin{aligned}
h^{-1}(U_z) = f^{-1}(g^{-1}(U_z)) = f^{-1}\left(\coprod_{k=1}^{m} g|_{\mathfrak{V}_k}^{-1}(U_z)\right) = \coprod_{k=1}^{m} f^{-1}(g|_{\mathfrak{V}_k}^{-1}(U_z)) \\
= \coprod_{k=1}^{m} \coprod_{\lambda_k \in I_k} f|_{\mathfrak{U}_{\lambda_k}}^{-1}(g|_{\mathfrak{V}_k}^{-1}(U_z)) = \coprod_{k=1}^{m} \coprod_{\lambda_k \in I_k} h|_{\mathfrak{U}_{\lambda_k}}^{-1}(U_z)
\end{aligned}
$$

Hence, $h^{-1}(U_z)$ is homeomorphic to the above coproduct of open subsets of X,
and each restricted map $h|_{\mathfrak{U}_{\lambda_k}} : \mathfrak{U}_{\lambda_k} \to U_z$ is a homeomorphism.
Hence, h is a covering map. Quod. Erat. Demonstrandum. $\square$

Theorem 2.18. Let X, Y, Z be three topological spaces,
and $f : X \to Y, g : Y \to Z$ be two continuous functions.
If Z is locally path connected, and $f, h = g \circ f$ are two covering maps,
then $g : Y \to Z$ is a covering map.

Proof. We may divide our proof into four steps.
Step 1: As $h : X \to Z$ is a covering map, for all $z \in Z$, for some open neighbour W_z of z,
$h^{-1}(W_z) \cong \coprod_{\nu \in K} \mathfrak{W}_\nu$, and each restricted map $h|_{\mathfrak{W}_\nu} : \mathfrak{W}_\nu \to W_z$ is a homeomorphism.
As Z is locally path connected, replace W_z with a path connected open neighbour of z.
Step 2: As the covering map $h = g \circ f$ is surjective, g is surjective.
As $h = g \circ f$ is continuous and the covering map f is surjective and open, g is continuous.

$$\forall \text{ open subset } W \text{ of } Z, g^{-1}(W) = f(f^{-1}(g^{-1}(W))) = f(h^{-1}(W)) \text{ is open in } Y$$

Step 3: Partition $g^{-1}(W_z)$ by all its path connected components.

$$g^{-1}(W_z) = \coprod_{\mu \in J} \mathfrak{V}_\mu$$

(1) As the covering map f is surjective and open, each $\mathfrak{V}_\mu$ is open:

$$\begin{aligned}\mathfrak{V}_\mu &= f(f^{-1}(\mathfrak{V}_\mu)) = f(\{x \in X : f(x) \in \mathfrak{V}_\mu\}) = f(\{x \in h^{-1}(W_z) : f(x) \in \mathfrak{V}_\mu\}) \\ &= \{f(x) \in h^{-1}(W_z) : f(x) \in \mathfrak{V}_\mu\} = \bigcup_{f(\mathfrak{W}_\nu) \subseteq \mathfrak{V}_\mu} f(\mathfrak{W}_\nu)\end{aligned}$$

(2) We wish to show that each $f(\mathfrak{W}_\nu)$ is equal to the path connected component $\mathfrak{V}_\mu$ it lives in, so which representative $f(\mathfrak{W}_\nu)$ we choose for $\mathfrak{V}_\mu$ is not important.
Assume to the contrary that some $f(\mathfrak{W}_\nu) \subsetneq \mathfrak{V}_\mu$.
Fix $x_\nu \in \mathfrak{W}_\nu$, $y_\nu = f(x_\nu) \in f(\mathfrak{W}_\nu)$ and $y \in \mathfrak{V}_\mu \backslash f(\mathfrak{W}_\nu)$.
As $\mathfrak{V}_\mu$ is path connected, there exists a path $\gamma : [0, 1] \to \mathfrak{V}_\mu$ from y_ν to y.
As f is a covering map, for certain initial point $x_\nu \in X$,
the path γ downstairs has a unique lift $\widetilde{\gamma} : [0, 1] \to X$ upstairs.
On one hand, $f \circ \widetilde{\gamma}([0, 1]) = \gamma([0, 1]) \subseteq \mathfrak{V}_\mu$, $\widetilde{\gamma}([0, 1]) \subseteq f^{-1}(\mathfrak{V}_\mu) = \coprod_{f(\mathfrak{W}_\nu) \subseteq \mathfrak{V}_\mu} \mathfrak{W}_\nu$,
the path $\widetilde{\gamma}$ should stay in the sheet $\mathfrak{W}_\nu$ where its initial point x_ν lies in.
On the other hand, $y = \gamma(1) = f \circ \widetilde{\gamma}(1)$ goes out of the range $f(\mathfrak{W}_\nu)$, a contradiction.
Hence, our assumption is wrong, and we've proven that $f(\mathfrak{W}_\nu) = \mathfrak{V}_\mu$.
Step 4: For each $\mu \in J$, choose $\nu_\mu \in K$, such that $f(\mathfrak{W}_{\nu_\mu}) = \mathfrak{V}_\mu$.
We've already set up a homeomorphism $h|_{\mathfrak{W}_{\nu_\mu}} : \mathfrak{W}_{\nu_\mu} \to W_z$.
If we restrict the covering map f to $f|_{\mathfrak{W}_{\nu_\mu}} : \mathfrak{W}_{\nu_\mu} \to \mathfrak{V}_\mu$,
then $f|_{\mathfrak{W}_{\nu_\mu}}$ is surjective, open and continuous.
As $h|_{\mathfrak{W}_{\nu_\mu}} = g|_{\mathfrak{V}_\mu} \circ f|_{\mathfrak{W}_{\nu_\mu}}$ is bijective, $f|_{\mathfrak{W}_{\nu_\mu}}$ must be injective.
As $f|_{\mathfrak{W}_{\nu_\mu}}$ is a homeomorphism, $g|_{\mathfrak{V}_\mu} = h|_{\mathfrak{W}_{\nu_\mu}} \circ f|^{-1}_{\mathfrak{W}_{\nu_\mu}}$ is a homeomorphism.
Hence, g is a covering map. Quod. Erat. Demonstrandum. □

> **Theorem 2.19.** Let X, Y, Z be three topological spaces,
> and $f : X \to Y, g : Y \to Z$ be two continuous functions.
> If Y is path connected, and $g, h = g \circ f$ are covering maps,
> then $f : X \to Y$ is a covering map.

Proof. We may divide our proof into four steps.
Step 1: As $g : Y \to Z$ is a covering map, for all $z \in Z$, for some open neighbour V_z of z, $g^{-1}(V_z) \cong \coprod_{\mu \in J} \mathfrak{V}_\mu$, and each restricted map $g|_{\mathfrak{V}_\mu} : \mathfrak{V}_\mu \to V_z$ is a homeomorphism.
Step 2: As $h : X \to Z$ is a covering map, for all $z \in Z$, for some open neighbour W_z of z, $h^{-1}(W_z) \cong \coprod_{\nu \in K} \mathfrak{W}_\nu$, and each restricted map $h|_{\mathfrak{W}_\nu} : \mathfrak{W}_\nu \to W_z$ is a homeomorphism.
Step 3: We wish to show that the continuous function $f : X \to Y$ is surjective.
(1) For all $y \in Y$, we wish to find a preimage x of y under f.
(2) Fix a point $x_0 \in X$, and define $y_0 = f(x_0) \in Y$,$z_0 = h(x_0) = g(y_0) \in Z$.
(3) As Y is path connected, there exist a path $\nu : [0, 1] \to Y$ from y_0 to y.

(4) Project the path ν in Y to the path $\sigma = g \circ \nu$ in Z via g.
(5) As h is a covering map, there exists a unique lift μ of σ with initial point $x_0 \in X$.
(6) Project the path μ in X to the path $\nu' = f \circ \mu$ in Y via f.
(7) As g is a covering map and the initial points $\nu(0) = y_0, \nu'(0) = f \circ \mu(0) = f(\mu(0)) = f(x_0) = y_0$ are identical, $y = \nu(1) = \nu'(1) = f \circ \mu(1) = f(\mu(1)) \in f(X)$, f is surjective.
Step 4: For all $y \in Y$, choose the open neighbour $U_y = g^{-1}(V_{g(y)} \cap W_{g(y)})$ of y.
The following set-theoretic result holds:

$$f^{-1}(U_y) = f^{-1}(g^{-1}(V_{g(y)} \cap W_{g(y)})) = \coprod_{\nu \in K} f|_{\mathfrak{W}_\nu}^{-1}\left(\coprod_{\mu \in J} g|_{\mathfrak{V}_\mu}^{-1}(V_{g(y)} \cap W_{g(y)})\right)$$
$$= \coprod_{\nu \in K} \coprod_{\mu \in J} f|_{\mathfrak{W}_\nu}^{-1}(g|_{\mathfrak{V}_\mu}^{-1}(V_{g(y)} \cap W_{g(y)})) = \coprod_{\nu \in K} \coprod_{\mu \in J} h|_{f^{-1}(\mathfrak{V}_\mu) \cap \mathfrak{W}_\nu}^{-1}(V_{g(y)} \cap W_{g(y)})$$

Hence, $f^{-1}(U_y)$ is homeomorphic to the above coproduct of open subsets of X,
and each restricted map $g|_{\mathfrak{V}_\mu \cap f(\mathfrak{W}_\nu)}^{-1} \circ h|_{f^{-1}(\mathfrak{V}_\mu) \cap \mathfrak{W}_\nu}$ is a homeomorphism.
To conclude, f is a covering map. Quod. Erat. Demonstrandum. □

Remark: *In order to ensure that the composition is a covering map, one should assume that the target space has a universal cover.*

Definition 2.20. (**Simple Connectedness**)
Let X be a topological space. If X is path connected, and the fundamental group $\pi_1(X, x_0)$ at every base point $x_0 \in X$ is isomorphic to the trivial group $\{e\}$, then X is simply connected.

Definition 2.21. (**Universal Covering Map**)
Let X, Y be two topological spaces, and $f : X \to Y$ be a covering map.
If X is simply connected and locally connected, then f is universal.

Theorem 2.22. Let X, Y, Z, W be four topological spaces,
and $f : X \to Y, g : Y \to Z, \zeta : W \to Z$ be three continuous functions.
If Z is locally path connected, f, g are covering maps,
and ζ is a universal covering map, then $h = g \circ f$ is a covering map.

Proof. We may divide our proof into three steps.
Step 1: Treat $g : (Y, y_0) \to (Z, z_0)$ as a cover, and $\zeta : (W, w_0) \to (Z, z_0)$ as a map.
As ζ is universal, $\zeta'(\pi_1(W, w_0)) = \{\zeta'([\![e_{w_0}]\!])\} = \{[\![\zeta \circ e_{w_0}]\!]\} = \{[\![e_{z_0}]\!]\} \leq g'(\pi_1(Y, y_0))$.
As W is path connected and locally path connected, the universal property of covering space suggests that $\zeta : (W, w_0) \to (Z, z_0)$ has a lift $\eta : (W, w_0) \to (Y, y_0)$.
Step 2: Treat $f : (X, x_0) \to (Y, y_0)$ as a cover, and $\eta : (W, w_0) \to (Y, y_0)$ as a map.
As η is universal, $\eta'(\pi_1(W, w_0)) = \{\eta'([\![e_{w_0}]\!])\} = \{[\![\eta \circ e_{w_0}]\!]\} = \{[\![e_{y_0}]\!]\} \leq f'(\pi_1(X, x_0))$.
As W is path connected and locally path connected, the universal property of covering space suggests that $\eta : (W, w_0) \to (Y, y_0)$ has a lift $\xi : (W, w_0) \to (X, x_0)$.

Step 3: As Z is locally path connected, and $\zeta = g \circ \eta = g \circ f \circ \xi = h \circ \xi, \xi$ are covers, it follows from **Theorem 2.18.** that $h : X \to Z$ is a covering map.
Quod. Erat. Demonstrandum. $\square$

2.4 Examples and Non-examples of Covering Map

Proposition 2.23. $f : \mathbb{C}^\times \to \mathbb{C}^\times, z \mapsto z^n (n \neq 0)$ is a covering map.

Proof. Define $X = \mathbb{C}^\times, Y = \mathbb{C}^\times, \alpha = \pi/|n|$. As Y is a topological group,
it suffices to show that $y = 1$ has an open neighbour $V = \mathrm{e}^{\mathbb{R}+\mathrm{i}(-\pi,+\pi)}$ with:
(1) $f^{-1}(V) \cong \coprod_{k=0}^{n-1} U_k$, where each $U_k = \mathrm{e}^{\mathbb{R}+2k\mathrm{i}\alpha+\mathrm{i}(-\alpha,+\alpha)}$ is open in X.
(2) Each restricted map $f|_{U_l} : U_l \to V$ is a homeomorphism.
Hence, the continuous surjection f is a covering map. Quod. Erat. Demonstrandum. $\square$

Proposition 2.24. $f : \mathbb{C} \to \mathbb{C}, z \mapsto z^n (n \geq 2)$ is not a local homeomorphism.

Proof. Define $\alpha = \pi/n$, $U_r = B(0, r)$.
For all $r > 0$, the restricted function $f|_{U_r} : U_r \to f(U_r)$ is not injective:

$$\forall 0 < s < r \text{ and } 0 \leq k < n,\ f|_{U_r}(s\mathrm{e}^{2k\mathrm{i}\alpha}) = s^n$$

Hence, f is not a local homeomorphism. Quod. Erat. Demonstrandum. $\square$

Remark: *As a corollary, f is not a covering map.*
As proposed by Prof. Hua, this example provides us with a general technique to prove that "the power map" on a multiplicative topological group is not a covering map.

Proposition 2.25. $f : \mathbb{C} \to \mathbb{C}^\times, z \mapsto \mathrm{e}^z$ is a covering map.

Proof. Define $X = \mathbb{C}, Y = \mathbb{C}^\times$. As Y is a topological group,
it suffices to show that $y = 1$ has an open neighbour $V = \mathrm{e}^{\mathbb{R}+\mathrm{i}(-\pi,+\pi)}$ with:
(1) $f^{-1}(V) \cong \coprod_{k\in\mathbb{Z}} U_k$, where each $U_k = \mathbb{R} + 2k\mathrm{i}\pi + \mathrm{i}(-\pi, +\pi)$ is open in X.
(2) Each restricted map $f|_{U_l} : U_l \to V$ is a homeomorphism.
Hence, the continuous surjection f is a covering map. Quod. Erat. Demonstrandum. $\square$

Proposition 2.26. $f : \mathbf{GL}_2(\mathbb{C}) \to \mathbf{GL}_2(\mathbb{C}), A \mapsto A^n (n \geq 2)$ and $g : \mathbb{H}^\times \to \mathbb{H}^\times, \mathbf{q} \mapsto \mathbf{q}^n (n \geq 2)$ are not local homeomorphisms.

Proof. We may divide our proof into two parts.
(1) Define $\alpha = \pi/n, R = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\alpha} \end{pmatrix}, U_r = B_{\text{Frobenius}}(R, r)$.

For all $r > 0$, the restricted function $f|_{U_r} : U_r \to f(U_r)$ is not injective:

$$\forall 0 < |t| < r,\, f|_{U_r} \begin{pmatrix} 1 & t \\ 0 & \mathrm{e}^{2\mathrm{i}\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, f is not a local homeomorphism.
(2) Fix an imaginary unit $\mathbf{s}$, and define $\alpha = \pi/n, V_r = B(\mathrm{e}^{\mathbf{s}\alpha}, r)$.
For all $r > 0$, the restricted function $g|_{V_r} : V_r \to g(V_r)$ is not injective:

$$\forall \text{ imaginary unit } \mathbf{t} \text{ with } |\mathbf{t} - \mathbf{s}| < r/\sin\alpha, g(\mathrm{e}^{\mathbf{t}\alpha}) = -1$$

Hence, g is not a local homeomorphism.
Quod. Erat. Demonstrandum. □

Remark: *To prove that f, g are not local homeomorphisms, we implicitly make use of the fact that U_r, V_r contain infinitely many elements in the forms* $\begin{pmatrix} 1 & t \\ 0 & \mathrm{e}^{2\mathrm{i}\alpha} \end{pmatrix}, \mathrm{e}^{\mathbf{t}\alpha}$.

Proposition 2.27. $f : \mathbf{M}_2(\mathbb{C}) \to \mathbf{GL}_2(\mathbb{C}), A \mapsto \mathrm{e}^A$ and $g : \mathbb{H} \to \mathbb{H}^\times, \mathbf{q} \mapsto \mathrm{e}^{\mathbf{q}}$ are not local homeomorphisms.

Proof. We may divide our proof into two parts.
(1) Define $R = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi\mathrm{i} \end{pmatrix}, U_r = B_{\text{Frobenius}}(R, r)$.
For all $r > 0$, the restricted function $f|_{U_r} : U_r \to f(U_r)$ is not injective:

$$\forall 0 < |t| < r, f \begin{pmatrix} 0 & t \\ 0 & 2\pi\mathrm{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, f is not a local homeomorphism.
(2) Fix a pure imaginary unit $\mathbf{s}$, and define $V_r = B(\mathbf{s}\pi, r)$.
For all $r > 0$, the restricted function $g|_{V_r} : V_r \to g(V_r)$ is not injective:

$$\forall \text{ imaginary unit } \mathbf{t} \text{ with } |\mathbf{t} - \mathbf{s}| < r/\pi, g(\mathbf{t}\pi) = -1$$

Hence, g is not a local homeomorphism.
Quod. Erat. Demonstrandum. □

Remark: *To prove that f, g are not local homeomorphisms, we implicitly make use of the fact that U_r, V_r contain infinitely many elements in the forms* $\begin{pmatrix} 0 & t \\ 0 & 2\pi\mathrm{i} \end{pmatrix}, \mathbf{t}\pi$.

Definition 2.28. (**Hawaii Earring $\mathbb{H}_N$**)
Define $\mathbb{S}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$.
Define the subspace $\bigcup_{n=N}^{+\infty} \mathbb{S}\left(\frac{\mathrm{i}}{3^n}, \frac{1}{3^n}\right)$ of $\mathbb{C}$ as the Hawaii earring $\mathbb{H}_N$.

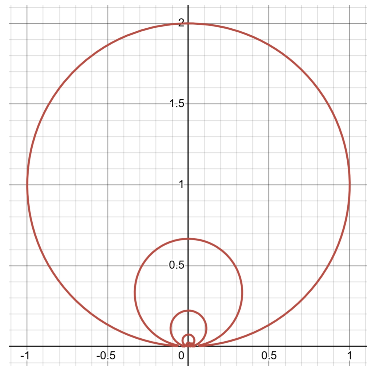

Figure 1: Hawaii Earring $\mathbb{H}_0$

Proposition 2.29. The Hawaii earring $\mathbb{H}_N$ is not a wedge sum.

Proof. Consider the point $0 \in \mathbb{H}_N$.
For all $\epsilon > 0$, there exists $M \geq N$, such that for all $n \geq M$, the diameter $\frac{2}{3^n} < \epsilon$.
This implies the partial union $\bigcup_{n=M}^{+\infty} \mathbb{S}\left(\frac{\mathrm{i}}{3^n}, \frac{1}{3^n}\right)$ is contained in $\mathbb{D}(0, \epsilon)$.
That is, every open neighbour of $0 \in \mathbb{H}_N$ contains infinitely many circles,
which is not true in the wedge sum case. Quod. Erat. Demonstrandum. $\square$

Definition 2.30. (**Hawaii Necklace** $\mathbb{Y}$)
Define the following subspace of $\mathbb{C}$ as the Hawaii Necklace $\mathbb{Y}$:

$$(\mathbb{R} \times \{0\}) \cup \left(\bigcup_{m \in \mathbb{Z}} m + \mathbb{H}_1 \right)$$

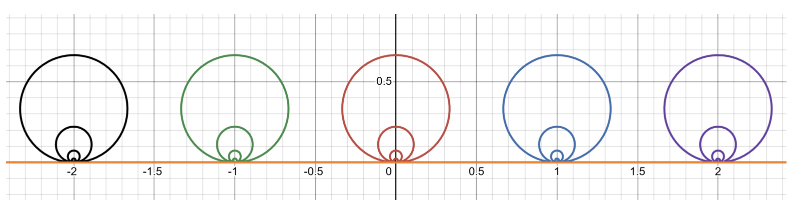

Figure 2: Hawaii Necklace $\mathbb{Y}$

Proposition 2.31. Define a function $p : \mathbb{Y} \to \mathbb{H}_0$ by:
(1) If $z = m + \frac{\theta}{2\pi}(0 \leq \theta < 2\pi)$ is on the line segment $[m, m+1) \times \{0\}$,
then $p(z) = \mathrm{i} - \mathrm{i}e^{\mathrm{i}\theta}$ is on the 1^{st} circle $\mathbb{S}(\mathrm{i}, 1)$.
(2) If $z = m + \frac{\mathrm{i} - \mathrm{i}e^{\mathrm{i}\theta}}{3^n}(0 \leq \theta < 2\pi, n \geq 0)$ is on the n^{th} circle $\mathbb{S}\left(\frac{\mathrm{i}}{3^n}, \frac{1}{3^n}\right)$,
then $p(z) = \frac{\mathrm{i} - \mathrm{i}e^{\mathrm{i}\theta}}{3^{n+1}}$ is on the $(n+1)^{\text{th}}$ circle $\mathbb{S}\left(\frac{\mathrm{i}}{3^{n+1}}, \frac{1}{3^{n+1}}\right)$.
p is a covering map.

Proof. For all $z_0 \in \mathbb{H}_0$, we wish to find an open neighbour U_{z_0} of z_0, such that $p^{-1}(U_{z_0}) \cong \coprod_{\lambda \in I} \mathfrak{U}_\lambda$, and each restricted map $p|_{\mathfrak{U}_\lambda} : \mathfrak{U}_\lambda \to U_{z_0}$ is homeomorphism.
Case 1: If $z_0 = 0$, then choose:

$$U_{z_0} = \mathbb{H}_0 \backslash \{2\mathrm{i}\}$$

Case 2: If $z_0 = \frac{\mathrm{i} - \mathrm{i}e^{\mathrm{i}\theta_0}}{3^n} (0 < \theta_0 < 2\pi, n \geq 0)$, then choose:

$$U_{z_0} = \mathbb{S}\left(\frac{\mathrm{i}}{3^n}, \frac{1}{3^n}\right) \backslash \{0\}$$

Hence, p is a covering map. Quod. Erat. Demonstrandum. $\square$

Proposition 2.32. Define $\partial\mathbb{E}_n = \{x + \mathrm{i}y \in \mathbb{C} : 3^n x^2 + y^2 = 1\}$.
(1) For all $n_1, n_2 \geq 1$:

$$n_1 \leq n_2 \implies \partial\mathbb{E}_{n_1} \cap \mathbb{S}\left(+\mathrm{i} - \frac{\mathrm{i}}{3^{n_2}}, \frac{1}{3^{n_2}}\right) = \{+\mathrm{i}\}$$

(2) For all $n_1, n_2 \geq 1$:

$$n_1 \leq n_2 \implies \partial\mathbb{E}_{n_1} \cap \mathbb{S}\left(-\mathrm{i} + \frac{\mathrm{i}}{3^{n_2}}, \frac{1}{3^{n_2}}\right) = \{-\mathrm{i}\}$$

Proof. It suffices to prove (1) by solving the following system:

$$\begin{cases} 3^{n_1}x^2 + y^2 & = & 1 \\ 3^{2n_2}x^2 + (3^{n_2}y - 3^{n_2} + 1)^2 & = & 1 \end{cases}$$

Step 1: Note that $y \neq -1$:

$$\begin{aligned} y = -1 &\implies 1 = 3^{2n_2}x^2 + (-2 \cdot 3^{n_2} + 1)^2 \geq 25 \\ &\implies \text{Contradiction} \end{aligned}$$

Hence, the following calculation is valid as $y + 1 \neq 0$:

$$\begin{aligned} 3^{n_1}x^2 + y^2 = 1 &\implies 3^{n_1}x^2 = (1+y)(1-y) \\ &\implies y = 1 - \frac{3^{n_1}x^2}{1+y} \end{aligned}$$

Step 2: Note that $y \neq 1 - \frac{2}{3^{n_2}}$:

$$\begin{aligned} y = 1 - \frac{2}{3^{n_2}} &\implies x = \pm\frac{1}{3^{n_2}}\sqrt{1 - (3^{n_2}y - 3^{n_2} + 1)^2} = 0 \\ &\implies 3^{n_1}x^2 + y^2 = \left(1 - \frac{2}{3^{n_2}}\right)^2 < 1 \\ &\implies \text{Contradiction} \end{aligned}$$

Hence, the following calculation is valid as $3^{n_2}y - 3^{n_2} + 2 \neq 0$:

$$3^{2n_n}x^2 + (3^{n_2}y - 3^{n_2} + 1)^2 = 1 \implies 3^{2n_n}x^2 = 3^{n_2}(1-y)(3^{n_2}y - 3^{n_2} + 2)$$
$$\implies y = 1 - \frac{3^{n_2}x^2}{3^{n_2}y - 3^{n_2} + 2}$$

Step 3: When $x = 0$, we get a solution $x + \mathrm{i}y = +\mathrm{i}$.
Step 4: When $x \neq 0$, the following calculation leads to a contradiction:

$$1 - \frac{3^{n_1}x^2}{1+y} = y = 1 - \frac{3^{n_2}x^2}{3^{n_2}y - 3^{n_2} + 2} \implies \frac{3^{n_1}}{1+y} = \frac{3^{n_2}}{3^{n_2}y - 3^{n_2} + 2}$$
$$\implies y = 1 + \frac{\frac{1}{3^{n_1}} - \frac{1}{3^{n_2}}}{1 - \frac{1}{3^{n_1}}} \geq 1$$

To conclude, the original system has a unique solution $x + y\mathrm{i} = +\mathrm{i}$.
Quod. Erat. Demonstrandum. $\square$

Definition 2.33. (**Hawaii Necklace** $\mathbb{X}$)
For each $m \in \mathbb{Z}$, define the following subspace $\mathbb{P}_m$ of $\mathbb{C}$:

$$\mathbb{P}_m = \overbrace{(\mathbb{H}_{|m|+2} - \mathrm{i})}^{\text{Lower Ring}} \cup \overbrace{\left(\bigcup_{n=2}^{|m|+2} \partial\mathbb{E}_n\right)}^{(|m|+2) \text{ Ellipses}} \cup \overbrace{(\mathrm{i} - \mathbb{H}_{|m|+2})}^{\text{Upper Ring}}$$

Define the following subspace of $\mathbb{C}$ as the Hawaii Necklace $\mathbb{X}$:

$$(\mathbb{R} \times \{\pm 1\}) \cup \left(\bigcup_{m \in \mathbb{Z}} m + \mathbb{P}_m\right)$$

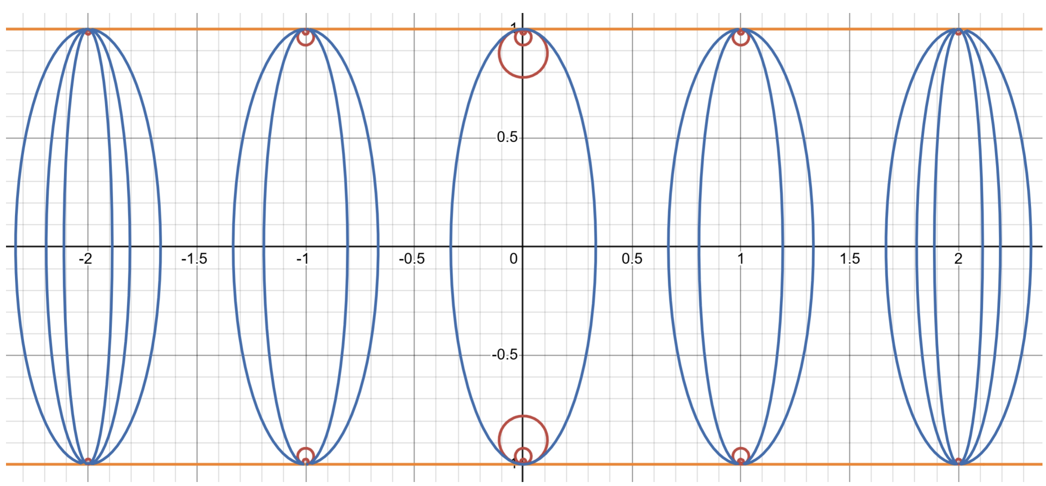

Figure 3: Hawaii Necklace $\mathbb{X}$

Example 2.34. The Hawaii Necklace $\mathbb{X}$ has a two sheet cover q on the Hawaii Necklace $\mathbb{Y}$. However, as every open neighbour U_z of the branch point $z = 0 \in \mathbb{H}_0$ cuts through finitely many boundary circles, $r = q \circ p$ is not a cover.

3 Monodromy Action and Deck Transformation

3.1 Monodromy Action

Example 3.1. For all $0 < \theta < 2\pi$, the following initial value problem:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{A(t)}{t}, A(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Has a unique solution on $\mathrm{e}^{\mathbb{R}+\mathrm{i}[0,\theta]}$:

$$A(t) = \frac{t^{+\sqrt{2}} + t^{-\sqrt{2}}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{t^{+\sqrt{2}} - t^{-\sqrt{2}}}{2\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

However, if we evaluate $A(t)$ at $t = \mathrm{e}^{\mathrm{i}\theta}$ and take limit $\theta \to 2\pi$:

$$\lim_{\theta \to 2\pi} A(\mathrm{e}^{\mathrm{i}\theta}) = \frac{\cos 2\sqrt{2}\pi}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\mathrm{i}\sin 2\sqrt{2}\pi}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \neq A(1)$$

Hence, the solution cannot be extended to $\mathbb{C}^\times$ in a continuous way.

Remark: *In this case, $\sqrt{2} \notin \mathbb{Q}$, so there is no chance for $A(t)$ to return to $A(1)$ after finitely many turns. However, if $A(t)$ does return to $A(1)$, then may "count the order of the singularity", which is the idea behind monodromy action.*

Theorem 3.2. (**Monodromy Action**)
Let Y, Z be two topological spaces, and $p : Y \to Z$ be a covering map. For all $z \in Z$, define $G = \pi_1(Z, z)$ and $X = p^{-1}(\{z\})$. $* : G \times X \to X, [\![\zeta]\!] * x = \eta(1)$ is a contravariant left action, where η is a lift of ζ with initial point x upstairs.

Proof. We may divide our proof into three parts.
Part 1: We prove that $*$ is well-defined.
(1) For all path ζ downstairs and initial point x upstairs,
there exists a unique lift η. Hence, $[\![\zeta]\!] * x$ has at least one image $\eta(1)$.
(2) For all path homotopy J downstairs and initial point x upstairs,
there exists a unique lift I. Hence, $[\![\zeta]\!] * x$ has at most one image $\eta(1)$.
To conclude, $*$ is well-defined.

Part 2: For all $x \in X$, e_z has a lift e_x with initial point x upstairs, so:

$$[\![e_z]\!] * x = e_x(1) = x$$

Part 3: For all $[\![\zeta]\!], [\![\zeta']\!] \in G$ and $x \in X$,
assume that η' is a lift of ζ' with initial point x upstairs,
and η is a lift of ζ with initial point $\eta'(1)$ upstairs.
Now $\eta' \star_c \eta$ is a lift of $\zeta' \star_c \zeta$ with initial point x upstairs, and:

$$[\![\zeta]\!] * ([\![\zeta']\!] * x) = [\![\zeta]\!] * \eta'(1) = \eta(1) = \eta' \star_c \eta(1) = [\![\zeta' \star_c \zeta]\!] * x = ([\![\zeta']\!] \star [\![\zeta]\!]) * x$$

Hence, $*$ is a contravariant left action. Quod. Erat. Demonstrandum. □

Remark: *If Y is path connected, then the monodromy action $*$ is transitive.*

Example 3.3. Define $Y = Z = \mathbb{C}^\times$, $p(y) = y^n (n \neq 0)$ and $z = 1$.
The monodromy action of (Z, z) on Y is the rotation of $\mathrm{e}^{2\mathrm{i}\alpha\mathbb{Z}}$, where $\alpha = \pi/|n|$.

Example 3.4. Define $Y = \mathbb{C}$, $Z = \mathbb{C}^\times$, $p(y) = \mathrm{e}^y$ and $z = 1$.
The monodromy action of (Z, z) on Y is the translation of $\mathbb{Z}$.

Example 3.5. First, define $Y = \mathbb{S}^n$ and $\sim : Y \to Y, \mathbf{y} \sim \mathbf{y}'$ if $\mathbf{y} = \pm\mathbf{y}'$.
Then, define $Z = \widetilde{Y}$ and $p : Y \to Z = \widetilde{Y}, \mathbf{y} \mapsto z = \widetilde{\mathbf{y}}$.
For all $z \in Z$, the monodromy action of (Z, z) on Y is the reflection of ± 1.

The rest of this subsection devotes to an interesting multivariable calculus problem. The technique of solving this problem is related to the idea of monodromy action.

Definition 3.6. (**Regular Surface**)
Let Σ be a subset of $\mathbb{R}^n$. If for all $\mathbf{x} \in \Sigma$, there exists an open neighbour U of $\mathbf{x}$ and a bijection $\mathbf{f}$ from the unit disk to U, such that $\mathbf{f}$ is continuously differentiable, $\mathbf{f}^{-1}$ is continuous, and $\mathrm{Rank}(D\mathbf{f}) = 2$, then Σ is a regular surface.

Theorem 3.7. There is no regular nonplanar surface $\Sigma \subseteq \mathbb{R}^3 \backslash \{\mathbf{0}\}$,
such that for some nonsingular, nonsymmetric matrix $Q = (q_{i,j})$,
for all position vector $\mathbf{x} \in \Sigma$ and differential vector $\mathrm{d}\mathbf{x} \in \mathrm{Tan}(\Sigma, \mathbf{x})$, $\mathrm{d}\mathbf{x}^T Q \mathbf{x} = 0$.

Proof. Assume to the contrary that such regular surface Σ exists.
Step 1: By rotation and scaling, we may assume that Q is in the following form:

$$Q = M + N, \text{ where } M = (m_{i,j}) \text{ is symmetric and } N = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2: We wish to find a point $\mathbf{a} = (a_1, a_2, a_3)^T \in \Sigma$, such that the 3^{rd} component $m_{3,1}a_1 + m_{3,2}a_2 + m_{3,3}a_3$ of the normal vector $Q\mathbf{a}$ at $\mathbf{a}$ is nonzero.
Assume to the contrary that such point doesn't exist.
Case 2.1: $(m_{3,1}, m_{3,2}, m_{3,3})^T = \mathbf{0}$, contradicting to Q is nonsingular.
Case 2.2: $(m_{3,1}, m_{3,2}, m_{3,3})^T \neq \mathbf{0}$, contradicting to Σ is nonplanar.
This implies such point $\mathbf{a}$ should exist.
Step 3: As Σ is regular, there exists an open neighbour U of $\mathbf{a}$ and a bijection $\mathbf{f}$ from the unit disk to U, such that $\mathbf{f}$ is continuously differentiable, and $\text{Rank}(D\mathbf{f}) = 2$ over the unit disk. As the 3^{rd} component of $\frac{\partial \mathbf{f}}{\partial u_1} \times \frac{\partial \mathbf{f}}{\partial u_2}$ is nonzero over the unit disk, i.e.:

$$\text{Det}\begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix} \neq 0$$

We may further shrink U, such that the induced map $(u_1, u_2)^T \mapsto (f_1, f_2)^T$ from the unit disk to the projection of U on the plane $x_3 = 0$ is a diffeomorphism.
Step 4: Choose $r > 0$, such that the continuously differentiable loop $\gamma(\theta) = (a_1 + r\cos\theta, a_2 + r\sin\theta)^T$ is contained in the projection of U. Use the diffeomorphisms $\mathbf{f}, (f_1, f_2)^T$ to lift γ upstairs to a continuously differentiable loop Γ in Σ. Along this Γ:

$$0 = \frac{1}{2}\oint_\Gamma \mathrm{d}(\mathbf{x}^T M\mathbf{x}) = \oint_\Gamma \mathrm{d}\mathbf{x}^T M\mathbf{x} = -\oint_\Gamma \mathrm{d}\mathbf{x}^T N\mathbf{x} = \oint_\Gamma x\mathrm{d}y - y\mathrm{d}x = 2\pi r^2$$

We get a contradiction. Hence, our assumption is wrong, and we've proven that such surface Σ doesn't exist. Quod. Erat. Demonstrandum. $\square$

3.2 Deck Transformation

To "globalize" a monodromy action, we define deck transformation.

Theorem 3.8. (**Deck Transformation**)
Let X, Y be two topological spaces, $p : X \to Y$ be a covering map, and $\text{Aut}(X)$ be the homeomorphism group of X. $\text{Dec}(p) = \{\sigma \in \text{Aut}(X) : p \circ \sigma = p\} \leq \text{Aut}(X)$.

Proof. We may divide our proof into three parts.
Part 1: $p \circ e = p$, so $\text{Dec}(p)$ contains e.
Part 2: For all $f_1, f_2 \in \text{Aut}(X)$:

$$\begin{aligned} f_1 \in \text{Dec}(p) \text{ and } f_2 \in \text{Dec}(p) &\implies p \circ f_1 = p \text{ and } p \circ f_2 = p \\ &\implies p \circ f_1 \circ f_2 = p \circ f_2 = p \implies f_1 \circ f_2 \in \text{Dec}(p) \end{aligned}$$

Part 3: For all $f \in \text{Aut}(X)$:

$$f \in \text{Dec}(p) \implies p \circ f = p \implies p \circ f^{-1} = p \implies f^{-1} \in \text{Dec}(p)$$

Hence, $\text{Dec}(p) \leq \text{Aut}(X)$. Quod. Erat. Demonstrandum. $\square$

Definition 3.9. (**Normal Covering Map**)
Let X, Y be two topological spaces, where X is path connected and locally path connected, and $p : X \to Y$ be a covering map. If for all $y \in Y$, $\mathrm{Dec}(p) \to X = p^{-1}(\{y\}), f * x = f(x)$ is transitive, then p is normal.

Proposition 3.10. Let X, Y be two topological spaces, where X is path connected and locally path connected, and $p : X \to Y$ be a covering map.

$$p \text{ is normal} \iff \forall x_0 \in X, p'(\pi_1(X, x_0)) \text{ is normal in } \pi_1(Y, p(x_0))$$

Proof. We may divide our proof into two parts.
"if" direction: Assume that $\forall x_0 \in X$, $p'(\pi_1(X, x_0))$ is normal in $\pi_1(Y, p(x_0))$.
For all $y \in Y$, for all $x_1, x_2 \in p^{-1}(\{y\})$, we wish to find $d \in \mathrm{Dec}(p)$, such that $d(x_1) = x_2$.
We have the following three conditions:
(1) $(X, x_1), (X, x_2)$ are path connected and locally path connected.
(2) $p_1 = p : (X, x_1) \to (Y, y)$, $p_2 = p : (X, x_2) \to (Y, y)$ are covering maps.
(3) $\pi_1(X, x_1), \pi_1(X, x_2)$ are conjugate, $p_1'(\pi_1(X, x_1)), p_2'(\pi_1(X, x_2))$ are conjugate,
so the normal subgroups $p_1'(\pi_1(X, x_1)), p_2'(\pi_1(X, x_2))$ of $\pi_1(Y, p(x_0))$ are equal.
Combine the three conditions above,
p_1, p_2 have unique lifts $d_{1,2} : (X, x_1) \to (X, x_2), d_{2,1} : (X, x_2) \to (X, x_1)$ upstairs.
Similarly, $p_2 \circ p_1 : (X, x_1) \to (X, x_1)$, $p_1 \circ p_2 : (X, x_2) \to (X, x_2)$ are identities.
This implies $d = d_{1,2} \in \mathrm{Dec}(p)$ is the desired deck transformation, such that $d(x_1) = x_2$.
"only if" direction: Assume that p is normal.
For all $[\![n]\!] \in p'(\pi_1(X, x_0))$ and $[\![g]\!] \in \pi_1(Y, p(x_0))$,
we wish to prove $[\![g]\!]^{-1} \star [\![n]\!] \star [\![g]\!] \in p'(\pi_1(X, x_0))$.
As $[\![n]\!] \in p'(\pi_1(X, x_0))$, there exists $[\![m]\!] \in \pi_1(X, x_0)$, such that $[\![n]\!] = p'([\![m]\!]) = [\![p \circ m]\!]$.
As p is a covering map, the path homotopy class $[\![g]\!]$ downstairs has a unique lift $[\![f]\!]$ upstairs with initial point $f(0) = x_0$. Although $[\![f]\!]$ may not be a loop, we do have:

$$\begin{aligned} [\![g]\!]^{-1} \star [\![n]\!] \star [\![g]\!] &= p'([\![f]\!])^{-1} \star p'([\![m]\!]) \star p'([\![f]\!]) \\ &= p'([\![f]\!]^{-1} \star [\![m]\!] \star [\![f]\!]) \end{aligned}$$

It suffices to find $d \in \mathrm{Dec}(p)$, such that the base point $f(1)$ becomes $f(0) = x_0$:

$$[\![f]\!]^{-1} \star [\![m]\!] \star [\![f]\!] \mapsto [\![d \circ f]\!]^{-1} \star [\![d \circ m]\!] \star [\![d \circ f]\!] \in \pi_1(X, x_0)$$

Hence, we've proven the logical equivalency. Quod. Erat. Demonstrandum. $\square$

Proposition 3.11. Let X, Y be two topological spaces, where X is path connected and locally path connected, and $p : X \to Y$ be a covering map.
For all $x_0 \in X$, the following map from the normalizer subgroup of $p'(\pi_1(X, x_0))$ in $\pi_1(Y, p(x_0))$ to $\mathrm{Dec}(p)$ is a surjective contravariant group homomorphism:

$$\sigma : [\![\eta]\!] \mapsto [\text{A deck transformation from } (X, x_0) \text{ to } (X, \xi(1))]$$

Here, $[\![\xi]\!]$ is the lift of $[\![\eta]\!]$ with initial point x_0 upstairs.
In addition, $\mathrm{Ker}(\sigma) = p'(\pi_1(X, x_0))$.

Proof. We may divide our proof into four parts.
Part 1: We prove that σ is well-defined.
As $[\![\eta]\!]$ is in the normalizer subgroup of $p'(\pi_1(X, x_0))$ in $\pi_1(Y, p(x_0))$, **Proposition 3.10., "if" direction** suggests the existence of $d \in \mathrm{Dec}(p)$ from x_0 to $\xi(1)$. It follows from the unique path lifting property that such deck transformation d is unique.
Part 2: We prove that σ is surjective.
For all $d \in \mathrm{Dec}(p)$, choose a path homotopy class $[\![\xi]\!]$ from x_0 to $d(x_0)$ upstairs.
For some path homotopy class $[\![\eta]\!] = p'([\![\xi]\!])$ downstairs, $d = \sigma([\![\eta]\!])$, so σ is surjective.
Part 3: We prove that σ is a contravariant group homomorphism.
For all $[\![\eta_1]\!], [\![\eta_2]\!]$ downstairs, assume that $[\![\xi_1]\!], [\![\xi_2]\!]$ are the lifts of $[\![\eta_1]\!], [\![\eta_2]\!]$ with common initial point x_0 upstairs. Define $d_1 = \sigma([\![\eta_1]\!]), d_2 = \sigma([\![\eta_2]\!]) \in \mathrm{Dec}(p)$, note that:

$$\begin{aligned}
p'([\![\xi_1]\!] \star [\![d_1 \circ \xi_2]\!]) &= p'([\![\xi_1]\!]) \star p'([\![d_1 \circ \xi_2]\!]) = p'([\![\xi_1]\!]) \star [\![p \circ d_1 \circ \xi_2]\!] \\
&= p'([\![\xi_1]\!]) \star [\![p \circ \xi_2]\!] = p'([\![\xi_1]\!]) \star p'([\![\xi_2]\!]) = [\![\eta_1]\!] \star [\![\eta_2]\!] \\
\sigma([\![\eta_1]\!] \star [\![\eta_2]\!]) &= [\text{A deck transformation from } (X, x_0) \text{ to } (X, \xi_1 \star_c d_1 \circ \xi_2(1))] \\
&= [\text{A deck transformation from } (X, x_0) \text{ to } (X, d_1 \circ \xi_2(1))] \\
&= [\text{A deck transformation from } (X, d_1(x_0)) \text{ to } (X, d_1 \circ \xi_2(1))] \\
&\circ [\text{A deck transformation from } (X, x_0) \text{ to } (X, \xi_1(1))] \\
&= [\text{A deck transformation from } (X, x_0) \text{ to } (X, \xi_2(1))] \\
&\circ [\text{A deck transformation from } (X, x_0) \text{ to } (X, \xi_1(1))] \\
&= d_2 \circ d_1 = \sigma([\![\eta_2]\!]) \circ \sigma([\![\eta_1]\!])
\end{aligned}$$

Hence, σ is a contravariant group homomorphism.
Part 4: We prove that $\mathrm{Ker}(\sigma) = p'(\pi_1(X, x_0))$.

$$\begin{aligned}
\mathrm{Ker}(\sigma) &= \{[\![\eta]\!] \text{ downstairs} : \sigma([\![\eta]\!]) = [\text{The identity deck transformation}]\} \\
&= \{[\![\eta]\!] \text{ downstairs} : \text{The lift } [\![\xi]\!] \text{ of } [\![\eta]\!] \text{ upstairs is a loop}\} = p'(\pi_1(X, x_0))
\end{aligned}$$

Quod. Erat. Demonstrandum. □

Remark: *Apply the first group isomorphism theorem, then $\widetilde{\sigma}$ is a group isomorphism. When p is normal, the normalizer is equal to $\pi_1(Y, y_0)$, so this is the monodromy action.*

Example 3.12. Let X, Y be two topological spaces, and $p : X \to Y$ be a covering map. If p is universal, then $\forall x_0 \in X, \pi_1(Y, p(x_0)) \cong \mathrm{Dec}(p)$

Remark: *Afterwards, we further convert* $\mathrm{Dec}(p)$ *to another group* G *by properly discontinuous left action. Eventually,* $\pi_1(Y, p(y_0)) \cong \mathrm{Dec}(p) \cong G$*, where* G *is computable.*

Definition 3.13. (**Properly Discontinuous Left Action**)
Let $\widetilde{X}$ be a topological space, G be a subgroup $\mathrm{Aut}(\widetilde{X})$, and $* : G \times \widetilde{X} \to \widetilde{X}$ be the induced left action. If every $\widetilde{x} \in \widetilde{X}$ has an open neighbour $\widetilde{U}$, such that for all $g \in G$, $\widetilde{U} \cap g * \widetilde{U} \neq \emptyset$ implies $g = e$, then $*$ is properly discontinuous.

Lemma 3.14. Let $\widetilde{X}$ be a topological space, G be a subgroup of $\mathrm{Aut}(\widetilde{X})$, $*$ be the induced left action, and $p : \widetilde{X} \to X = \widetilde{X}/G, \widetilde{x} \mapsto G * \widetilde{x}$ be the quotient map. If $*$ is properly discontinuous, then each $\ell_g : \widetilde{X} \to \widetilde{X}, \widetilde{x} \mapsto g * \widetilde{x}$ is continuous.

Theorem 3.15. Let $\widetilde{X}$ be a topological space, G be a subgroup of $\mathrm{Aut}(\widetilde{X})$, $* : G \times \widetilde{X} \to \widetilde{X}$ be the induced left action, and $p : \widetilde{X} \to X = \widetilde{X}/G, \widetilde{x} \mapsto G * \widetilde{x}$ be the quotient map.

$$* \text{ is properly discontinuous} \iff p \text{ is a covering map}$$

Proof. As every quotient map is a continuous surjection, p is a covering map iff for each x downstairs, there exists an open neighbour U_x of x, such that:
(1) $p^{-1}(U_x)$ is homeomorphic to a coproduct $\coprod_{\widetilde{x} \in p^{-1}(\{x\})} \widetilde{\mathfrak{U}}_{\widetilde{x}}$ of open subsets of $\widetilde{X}$.
(2) Each restricted map $p|_{\widetilde{\mathfrak{U}}_{\widetilde{x}}} : \widetilde{\mathfrak{U}}_{\widetilde{x}} \to U_x$ is a homeomorphism.
The first criterion is equivalent to $*$ is properly discontinuous.
The second criterion holds because every quotient map maps saturated opens to opens.
Quod. Erat. Demonstrandum. $\square$

Theorem 3.16. Let $\widetilde{X}$ be a path connected and locally path connected topological space, G be a subgroup of $\mathrm{Aut}(\widetilde{X})$, $* : G \times \widetilde{X} \to \widetilde{X}$ be the induced left action, and $p : \widetilde{X} \to X = \widetilde{X}/G, \widetilde{x} \mapsto G * \widetilde{x}$ be the quotient map.

$$* \text{ is properly discontinuous} \implies G = \mathrm{Dec}(p)$$

Proof. We may divide our proof into two parts.
"$\subseteq$" inclusion: For all $g \in G$:

$$\widetilde{x} \in \widetilde{X}, p \circ g(\widetilde{x}) = G * (g * \widetilde{x}) = (Gg) * \widetilde{x} = G * \widetilde{x} = p(\widetilde{x})$$

Hence, $p \circ g = p$, $g \in \mathrm{Dec}(p)$, so $G \subseteq \mathrm{Dec}(p)$.

"$\supseteq$" inclusion: For all $d \in \mathrm{Dec}(p)$:
(1) Choose $\widetilde{x}_0 \in \widetilde{X}$ upstairs, and define $x_0 = p(\widetilde{x}_0)$ downstairs.
(2) As d is a deck transformation, $\widetilde{x}_0, d(\widetilde{x}_0)$ are in the same fibre $p^{-1}(\{x_0\})$.
(3) As $p^{-1}(\{x_0\}) = \{\widetilde{x} \in \widetilde{X} : p(\widetilde{x}) = x_0\} = \{\widetilde{x} \in \widetilde{X} : G * \widetilde{x} = G * \widetilde{x}_0\} = G * \widetilde{x}_0$,
there exists $g \in G$, such that $d(\widetilde{x}_0) = g * \widetilde{x}_0 = g(\widetilde{x}_0)$.
(4) As $(\widetilde{X}, \widetilde{x}_0)$ is path connected and locally path connected, $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, and $p'(\pi_1(\widetilde{X}, \widetilde{x}_0)) = p'(\pi_1(\widetilde{X}, \widetilde{x}_0))$, p has a unique lift $g^{-1} \circ d = e$ upstairs, so we've proven that $d = g \in G$, which implies $G \supseteq \mathrm{Dec}(p)$.
To conclude, $G = \mathrm{Dec}(p)$. Quod. Erat. Demonstrandum. $\square$

Example 3.17. Let $\widetilde{X}$ be a simply connected and locally path connected topological space, G be a subgroup of $\mathrm{Aut}(\widetilde{X})$, $* : G \times \widetilde{X} \to \widetilde{X}$ be the induced left action, and $p : \widetilde{X} \to X = \widetilde{X}/G, \widetilde{x} \mapsto G * \widetilde{x}$ be the quotient map.

$$* \text{ is properly discontinuous} \implies \forall x_0 \in X, \pi_1(X, x_0) \cong \mathrm{Dec}(p) = G$$

3.3 Applications

Theorem 3.18. Let X be a normed vector space over field $\mathbb{R}$.
Every nonempty convex subset U of X is simply connected.

Proof. For all $\mathbf{x}_0 \in U$, for all loop $\boldsymbol{\gamma} : [0,1]_t \to U$ with base point $\mathbf{x}_0$,
there exists a path homotopy $\mathbf{H} : [0,1]_s \times [0,1]_t \to U, \mathbf{H}(s,t) = s\boldsymbol{\gamma}(t)$ from $e_{\mathbf{x}_0}$ to $\boldsymbol{\gamma}$.
Hence, $\pi_1(U, \mathbf{x}_0) = \{[\![e_{\mathbf{x}_0}]\!]\}$, U is simply connected. Quod. Erat. Demonstrandum. $\square$

Theorem 3.19. Let X be a normed vector space over field $\mathbb{R}$.
Every nonempty convex subset U of X is locally path connected.

Proof. For all $\mathbf{x}_0 \in U$, for all open neighbour V of $\mathbf{x}_0$ in U,
we wish to find a path connected open neighbour $\mathfrak{V}$ of $\mathbf{x}_0$ in U with $\mathfrak{V} \subseteq V$.
(1) As V is an open neighbour of $\mathbf{x}_0$ in the subset U of X,
there exists an open neighbour W of $\mathbf{x}_0$ in X, such that $V = U \cap W$.
(2) As W is an open neighbour of $\mathbf{x}_0$ in X,
there exists an open ball $\mathfrak{W}$ centered at $\mathbf{x}_0$ with $\mathfrak{W} \subseteq W$.
(3) As $U, \mathfrak{W}$ are convex in X, $U \cap \mathfrak{W}$ is convex in X, so $U \cap \mathfrak{W}$ is path connected.
(4) As $\mathbf{x}_0$ has a path connected open neighbour $\mathfrak{V} = U \cap \mathfrak{W}$ of $\mathbf{x}_0$ in U with $\mathfrak{V} \subseteq V$,
U is locally path connected. Quod. Erat. Demonstrandum. $\square$

Proposition 3.20. In $\mathbb{C}$, define $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$.

$$\forall \mathrm{e}^{\mathrm{i}\theta_0} \in \mathbb{S}, \pi_1(\mathbb{S}, \mathrm{e}^{\mathrm{i}\theta_0}) \cong \mathbb{Z}$$

Proof. As $\widetilde{X} = \mathbb{R}$ is convex in $\mathbb{R}$, $\widetilde{X}$ is simply connected and locally path connected.
Consider the subgroup $G = [\text{All translation } t_n : \theta \mapsto 2n\pi + \theta]$ of $\mathrm{Aut}(\widetilde{X})$.
Step 1: We prove that the induced left action $*$ is properly discontinuous.
For the identity element 0 in the topological group $\widetilde{X} = \mathbb{R}$, there exists an open neighbour $\widetilde{U} = (-\pi, +\pi)$ of 0, such that for all t_n in the group G:

$$\widetilde{U} \cap t_n * \widetilde{U} \neq \emptyset \implies (-\pi, +\pi) \cap (2n\pi - \pi, 2n\pi + \pi) \neq \emptyset \implies n = 0$$

Hence, $*$ is properly discontinuous, so for all base point x_0 in the quotient space $X = \widetilde{X}/G$, the fundamental group $\pi_1(X, x_0)$ is isomorphic to G, and G is isomorphic to $\mathbb{Z}$.
Step 2: We prove that the quotient space X is homeomorphic to $\mathbb{S}$.
Follow the argument in **Proposition 2.25.**, $\sigma : \mathbb{R} \to \mathbb{S}, \theta \mapsto \mathrm{e}^{\mathrm{i}\theta}$ is a covering map.
As σ is a surjective local homeomorphism, the quotient map $\widetilde{\sigma}$ is a homeomorphism, whose domain is $\mathbb{R}/[\text{The equivalence relation induced by } \sigma] = \widetilde{X}/G$. To conclude:

$$\forall \mathrm{e}^{\mathrm{i}\theta_0} \in \mathbb{S}, \pi_1(\mathbb{S}, \mathrm{e}^{\mathrm{i}\theta_0}) \cong \pi_1(\widetilde{X}/G, G * \theta_0) \cong G \cong \mathbb{Z}$$

Quod. Erat. Demonstrandum. □

Theorem 3.21. Let X be a topological space.
If X has two open subsets X_1, X_2 with:
(1) X_1, X_2 are simply connected.
(2) $X_1 \cap X_2$ is nonempty and path connected.
(3) $X_1 \cup X_2 = X$.
Then X is simply connected.

Proof. Choose $x_0 \in X_1 \cap X_2$. It suffices to prove that every loop γ with base point x_0 is homotopic to the identity loop e_{x_0} at x_0. Consider the open preimages $\gamma^{-1}(X_1), \gamma^{-1}(X_2)$:
(1) As $[0,1]$ is a subset of $\mathbb{R}$, the Lindelöf property suggests:

$$\gamma^{-1}(X_1) \cong \coprod_{\lambda \in I_1} \mathfrak{U}_1^\lambda, \gamma^{-1}(X_2) \cong \coprod_{\lambda \in I_2} \mathfrak{U}_2^\lambda$$

Here, the index sets I_1, I_2 are countable, and the families $\mathcal{U}_1 = \{\mathfrak{U}_1^\lambda\}_{\lambda \in I_1}, \mathcal{U}_2 = \{\mathfrak{U}_2^\lambda\}_{\lambda \in I_2}$ consist of disjoint open connected subsets of $[0,1]$.
(2) As $[0,1]$ is compact, the open cover $\mathcal{U}_1 \cup \mathcal{U}_2$ of it has a finite subcover.
(3) As the open subsets in $\mathcal{U}_1, \mathcal{U}_2$ are disjoint, WLOG,
assume that the finite subcover mentioned above is of the following form:

$$\begin{array}{llll}
0 \in \mathfrak{U}_1^1 = [0, b_1^1), \mathfrak{U}_2^2 = (a_2^2, b_2^2), \cdots, & \mathfrak{U}_1^{k-1} = (a_1^{k-1}, b_1^{k-1}), & \mathfrak{U}_2^k = (a_1^k, 1] \ni 1 \\
0 \in \mathfrak{U}_1^1 = [0, b_1^1) < \cdots < & \mathfrak{U}_1^{k-1} = (a_1^{k-1}, b_1^{k-1}) & \not\ni 1 \\
0 \notin \quad \mathfrak{U}_2^2 = (a_2^2, b_2^2) < \cdots < & & \mathfrak{U}_2^k = (a_1^k, 1] \ni 1
\end{array}$$

(4) Decompose γ into the concatenation $\gamma_1 \star_{c_1} \gamma_2 \star_{c_2} \cdots \star_{c_{k-2}} \gamma_{k-1} \star_{c_{k-1}} \gamma_k$.

For each γ_l, it is contained in X_1 or X_2. Note that the initial point and end point of γ_l lie in the path connected intersection, γ_l is homotopic to some path σ_l in $X_1 \cap X_2$.
(5) Repeat this process inductively, then loop γ is homotopic to some loop σ in $X_1 \cap X_2$. It suffices to apply the simple connectedness of X_1 and prove that σ is homotopic to the identity loop e_{x_0} at x_0. Hence, X is simply connected. Quod. Erat. Demonstrandum. $\square$

Proposition 3.22. In $\mathbb{R}^3$, define $\mathbb{S}^2 = \{(x, y, z)^T \in \mathbb{R}^3 : \|(x, y, z)^T\| = 1\}$.

$$\mathbb{S}^2 \text{ is simply connected and locally path connected}$$

Proof. Define $X = \mathbb{S}^3, X_1 = \mathbb{S}^3 \backslash \{(0, 0, -1)^T\}, X_2 = \mathbb{S}^3 \backslash \{(0, 0, +1)^T\}$.
Step 1: By stereographic projection, $X_1, X_2 \cong \mathbb{R}^2$, where $\mathbb{R}^2$ is convex in $\mathbb{R}^2$. Hence, X_1, X_2 are simply connected.
Step 2: There exists a deformation retraction from $\mathbb{S}$ to $X_1 \cap X_2$:

$$\mathbf{H} : (X_1 \cap X_2) \times [0, 1]_t \to X_1 \cap X_2$$

$$\mathbf{H}((\cos \phi \cos \theta, \sin \phi \cos \theta, \sin \theta)^T, t) = (\cos \phi \cos t\theta, \sin \phi \cos t\theta, \sin t\theta)^T$$

Hence, $X_1 \cap X_2 \sim \mathbb{S}$, $\mathbb{S}$ is path connected implies X_1 is path connected.
Step 3: All conditions in **Theorem 3.21.** hold, so $X = \mathbb{S}^2$ is simply connected.
Step 4: As every spherical crown is path connected, $\mathbb{S}^2$ is locally path connected.
Quod. Erat. Demonstrandum. $\square$

Remark: *The same proof works for $\mathbb{S}^n$, where $n \geq 2$.*

Theorem 3.23. Let $(X, x_0), (Y, y_0)$ be two topological spaces. The following map from $\pi_1(X \times Y, (x_0, y_0))$ to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ is a group isomorphism:

$$\sigma : [\![(\xi, \eta)]\!] \mapsto ([\![\xi]\!], [\![\eta]\!])$$

Proof. We may divide our proof into three parts.
Part 1: We prove that σ is well-defined.
For all $[\![(\xi_1, \eta_1)]\!], [\![(\xi_2, \eta_2)]\!] \in \pi_1(X \times Y, (x_0, y_0))$:

$$\begin{aligned} [\![(\xi_1, \eta_1)]\!] = [\![(\xi_2, \eta_2)]\!] &\implies (\xi_1, \eta_1) \approx (\xi_2, \eta_2) \implies \xi_1 \approx \xi_2 \text{ and } \eta_1 \approx \eta_2 \\ &\implies [\![\xi_1]\!] = [\![\xi_2]\!] \text{ and } [\![\eta_1]\!] = [\![\eta_2]\!] \implies ([\![\xi_1]\!], [\![\eta_1]\!]) = ([\![\xi_2]\!], [\![\eta_2]\!]) \end{aligned}$$

Part 2: We prove that the surjective map σ is injective.
For all $([\![\xi_1]\!], [\![\eta_1]\!]), ([\![\xi_2]\!], [\![\eta_2]\!]) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$:

$$\begin{aligned} ([\![\xi_1]\!], [\![\eta_1]\!]) = ([\![\xi_2]\!], [\![\eta_2]\!]) &\implies [\![\xi_1]\!] = [\![\xi_2]\!] \text{ and } [\![\eta_1]\!] = [\![\eta_2]\!] \implies \xi_1 \approx \xi_2 \text{ and } \eta_1 \approx \eta_2 \\ &\implies (\xi_1, \eta_1) \approx (\xi_2, \eta_2) \implies [\![(\xi_1, \eta_1)]\!] = [\![(\xi_2, \eta_2)]\!] \end{aligned}$$

Part 3: We prove that σ is a group homomorphism.
For all $[\![(\xi_1, \eta_1)]\!], [\![(\xi_2, \eta_2)]\!] \in \pi_1(X \times Y, (x_0, y_0))$:

$$\begin{aligned}\sigma([\![(\xi_1, \eta_1)]\!] \star [\![(\xi_2, \eta_2)]\!]) &= \sigma([\![(\xi_1, \eta_1) \star_c (\xi_2, \eta_2)]\!]) = \sigma([\![(\xi_1 \star_c \xi_2, \eta_1 \star_c \eta_2)]\!]) \\ &= ([\![\xi_1 \star_c \xi_2]\!], [\![\eta_1 \star_c \eta_2]\!]) = ([\![\xi_1]\!] \star [\![\xi_2]\!], [\![\eta_1]\!] \star [\![\eta_2]\!]) \\ &= ([\![\xi_1]\!], [\![\eta_1]\!]) \star ([\![\xi_2]\!], [\![\eta_2]\!]) = \sigma([\![(\xi_1, \eta_1)]\!]) \star \sigma([\![(\xi_2, \eta_2)]\!])\end{aligned}$$

Hence, the bijective group homomorphism σ is a group isomorphism.
Quod. Erat. Demonstrandum. □

Remark: *The same proof works for arbitrary Cartesian product.*

Proposition 3.24. Define $\mathbb{T}^2 = \mathbb{S} \times \mathbb{S}$.

$$\forall (z_0, w_0) \in \mathbb{T}^2, \pi_1(\mathbb{T}^2, (z_0, w_0)) \cong \mathbb{Z} \times \mathbb{Z}$$

Proof. For all $(z_0, w_0) \in \mathbb{T}^2$:

$$\pi_1(\mathbb{T}^2, (z_0, w_0)) = \pi_1(\mathbb{S} \times \mathbb{S}, (z_0, w_0)) \cong \pi_1(\mathbb{S}, z_0) \times \pi_1(\mathbb{S}, w_0) \cong \mathbb{Z} \times \mathbb{Z}$$

Quod. Erat. Demonstrandum. □

Proposition 3.25. In $\mathbb{S}^2$, define $\mathbb{RP}^2 = \mathbb{S}^2/G$.

$$\forall [(x_0, y_0, z_0)^T] \in \mathbb{RP}^2, \pi_1(\mathbb{RP}^2, [(x_0, y_0, z_0)^T]) \cong \{\pm 1\}$$

Here:

$$G = \left\{ \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

Proof. Consider the simply connected and locally path connected space $\widetilde{X} = \mathbb{S}^2$.
We prove that the induced left action $*$ is properly discontinuous.
For all element $(x, y, z)^T$ of the topological space $\widetilde{X}$, there exists an open neighbour $\widetilde{U} = \{(\xi, \eta, \zeta)^T \in \mathbb{S}^2 : x\xi + y\eta + z\zeta > 0\}$ of $(x, y, z)^T$, such that:

$$\widetilde{U} \cap \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} * \widetilde{U} = \emptyset$$

Hence, $*$ is properly discontinuous, and:

$$\forall [(x_0, y_0, z_0)^T] \in \mathbb{RP}^2, \pi_1(\mathbb{RP}^2, [(x_0, y_0, z_0)^T]) = \pi_1(\widetilde{X}/G, [(x_0, y_0, z_0)^T]) \cong G \cong \{\pm 1\}$$

Quod. Erat. Demonstrandum. $\square$

Remark: *The same proof works for* $\mathbb{RP}^n = \mathbb{S}^n/G$.

Proposition 3.26. In $\mathbb{R}^2$, define $\mathbb{K}^2 = \mathbb{R}^2/G$.

$$\forall [(x_0, y_0)^T] \in \mathbb{K}^2, \pi_1(\mathbb{K}^2, [(x_0, y_0)^T]) \cong G$$

Here:

$$G = \left\{ \left(\begin{array}{cc|c} (-1)^n & 0 & m \\ 0 & 1 & n \\ \hline 0 & 0 & 1 \end{array} \right) : m, n \in \mathbb{Z} \right\}$$

Proof. As $\widetilde{X} = \mathbb{R}^2$ is convex in $\mathbb{R}^2$, $\widetilde{X}$ is simply connected and locally path connected.
We prove that the induced left action $*$ is properly discontinuous.
For all element $(x, y)^T$ in the topological space $\widetilde{X} = \mathbb{R}^2$, there exists an open neighbour $\widetilde{U} = B\left((x, y)^T, \frac{1}{2}\right)$ of $(x, y)^T$, such that for all $\left(\begin{array}{cc|c} (-1)^n & 0 & m \\ 0 & 1 & n \\ \hline 0 & 0 & 1 \end{array} \right)$ in the group G:

$$\begin{aligned} \widetilde{U} \cap \left(\begin{array}{cc|c} (-1)^n & 0 & m \\ 0 & 1 & n \\ \hline 0 & 0 & 1 \end{array} \right) * \widetilde{U} \neq \emptyset \implies & B\left(x, \frac{1}{2}\right) \cap B\left(m + (-1)^n x, \frac{1}{2}\right) \neq \emptyset \\ & \text{and } B\left(y, \frac{1}{2}\right) \cap B\left(n + y, \frac{1}{2}\right) \neq \emptyset \\ \implies & \left(\begin{array}{cc|c} (-1)^n & 0 & m \\ 0 & 1 & n \\ \hline 0 & 0 & 1 \end{array} \right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \end{aligned}$$

Hence, $*$ is properly discontinuous, and:

$$\forall [(x_0, y_0)^T] \in \mathbb{K}^2, \pi_1(\mathbb{K}^2, [(x_0, y_0)^T]) = \pi_1(\mathbb{R}^2/G, [(x_0, y_0)^T]) \cong G$$

Quod. Erat. Demonstrandum. $\square$

Remark: *We need some technique to reconstruct the group* G *in a simpler way.*

Proposition 3.27. In $\mathbb{S}^3$, define $\mathcal{L}(p,q) = \mathbb{S}^3/G$.

$$\forall [(z_0,w_0)^T] \in \mathcal{L}(p,q), \pi_1(\mathcal{L}(p,q), [(z_0,w_0)^T]) \cong \mathbb{Z}_p$$

Here, $p, q \geq 1$ are coprime, $\zeta = \mathrm{e}^{2\pi\mathrm{i}/p}$, and:

$$G = \left\{ \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{nq} \end{pmatrix} : n \in \mathbb{Z} \right\}$$

Proof. Consider the simply connected and locally path connected space $\widetilde{X} = \mathbb{S}^2$.
We prove that the induced left action $*$ is properly discontinuous.
Step 1: For all $(z,w)^T \in \widetilde{X} = \mathbb{S}^3$ and $n \in (p\mathbb{Z})^c$:

$$\begin{aligned} \left\| \begin{pmatrix} z \\ w \end{pmatrix} - \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{nq} \end{pmatrix} * \begin{pmatrix} z \\ w \end{pmatrix} \right\|^2 &= \|((1-\zeta^n)z, (1-\zeta^{nq})w)\|^2 \\ &= |1-\zeta^n|^2|z|^2 + |1-\zeta^{nq}|^2|w|^2 \\ &\geq \left|2\sin\frac{\pi}{p}\right|^2 |z|^2 + \left|2\sin\frac{\pi}{p}\right|^2 |w|^2 = \left|2\sin\frac{\pi}{p}\right|^2 \end{aligned}$$

Hence, all distinct points in the same orbit are separated by Euclidean distance $|2\sin\frac{\pi}{p}|$.
Step 2: For all element $(z,w)^T$ in the topological space $\widetilde{X}$, there is an open neighbour $\widetilde{U} = B((z,w)^T, |\sin\frac{\pi}{p}|)$ of $(z,w)^T$, such that for all $\begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{nq} \end{pmatrix}$ in the group G:

$$\begin{aligned} \widetilde{U} \cap \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{nq} \end{pmatrix} * \widetilde{U} \neq \emptyset &\implies \left\| \begin{pmatrix} z \\ w \end{pmatrix} - \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{nq} \end{pmatrix} * \begin{pmatrix} z \\ w \end{pmatrix} \right\| < \left|2\sin\frac{\pi}{p}\right| \\ &\implies \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{nq} \end{pmatrix} * \begin{pmatrix} z \\ w \end{pmatrix} \implies \begin{pmatrix} \zeta^n & 0 \\ 0 & \zeta^{nq} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence, $*$ is properly discontinuous, and:

$$\forall [(z_0,w_0)^T] \in \mathcal{L}(p,q), \pi_1(\mathcal{L}(p,q), [(z_0,w_0)^T]) = \pi_1(\widetilde{X}/G, [(z_0,w_0)^T]) \cong G \cong \mathbb{Z}_p$$

Quod. Erat. Demonstrandum. $\square$

4 Seifert–Van Kampen Theorem

4.1 Free Group and Free Product

Definition 4.1. (**Word**)
Let S be a nonempty set.
(1) Define the empty word e in S as the unique word that contains no letter.
(2) For all finite nonempty list $s_1, s_2, \cdots, s_{m-1}, s_m$ in S and finite nonempty list $\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}, \epsilon_m$ in $\{\pm 1\}$, define a nonempty word $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m}$ in S.
Two words are equal iff they arrange letters in completely the same way.

Definition 4.2. (**Free Group**)
Let S be a nonempty set.
Define $\text{Word}(S)$ as the set of all words in S.
Define concatenate word and inverse word as follows:

$$(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m})(t_1^{\sigma_1} t_2^{\sigma_2} \cdots t_{n-1}^{\sigma_{n-1}} t_n^{\sigma_n}) = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m} t_1^{\sigma_1} t_2^{\sigma_2} \cdots t_{n-1}^{\sigma_{n-1}} t_n^{\sigma_n}$$

$$(s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m})^{-1} = s_m^{-\epsilon_m} s_{m-1}^{-\epsilon_{m-1}} \cdots s_2^{-\epsilon_2} s_1^{-\epsilon_1}$$

Define an equivalence relation $\sim$ on $\text{Word}(S)$ by:

$$w \sim w' \iff w = w' \text{ after inserting or deleting } s^{-1}s^{+1}, s^{+1}s^{-1}$$

Define $\langle S \rangle = \text{Word}(S)/\sim$ as the free group generated by S.

Remark: *In practice, we use $\langle a_1, a_2, \cdots, a_n | R_1, R_2, \cdots, R_m \rangle$ to stand for the free group generated by $\{a_1, a_2, \cdots, a_n\}$ under extra identification relations $R_1, R_2, \cdots, R_m$.*

Example 4.3.

$$\mathbb{Z} = \langle 1 \rangle$$

Lemma 4.4. For all positive integer n:

$$\langle g | g^n = e \rangle = \{e, g, g^2, \cdots, g^{n-1}\}$$

Proof. We may divide our proof into two parts.
"$\subseteq$" inclusion: For all $h \in \langle g | g^n = e \rangle$, there exists $a \in \mathbb{Z}$, such that:

$$h = g^a$$

According to the division algorithm, there exist $q, r \in \mathbb{Z}$, such that:

$$a = qn + r \text{ and } 0 \leq r < n$$

Apply the identification rule $g^n = e$:

$$h = g^{qn+r} = (g^n)^q g^r = e^q g^r = g^r \in \{e, g, g^2, \cdots, g^{n-1}\}$$

"$\supseteq$" inclusion: All of $e, g, g^2, \cdots, g^{n-1}$ are words of g.
Hence, $\langle g | g^n = e \rangle = \{e, g, g^2, \cdots, g^{n-1}\}$.
Quod. Erat. Demonstrandum. $\square$

Example 4.5. Define the following function from Word(g) to $\mathbb{Z}$:

$$\#_g : g^{\epsilon_1} g^{\epsilon_2} \cdots g^{\epsilon_{m-1}} g^{\epsilon_m} \mapsto \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{m-1} + \epsilon_m$$

As two words are equal iff they arrange letters in completely the same way, $\#_g$ is well-defined. In addition, for all positive integer n, if we insert or delete:

$$g^{-1} g^{+1}, g^{+1} g^{-1}, g^n$$

Then the remainder $\#_g^n$ of $\#_g$ modulo n is invariant:

Insert/Delete	$\Delta(\#_g)$	$\Delta(\#_g^n)$
$g^{-1}g^{+1}$	0	0
$g^{+1}g^{-1}$	0	0
g^n	$\pm n$	0

Hence, for all $\alpha, \beta \in$ Word(g):

$$\alpha = \beta \text{ in } \langle g | g^n = e \rangle \implies \#_g^n(\alpha) = \#_g^n(\beta)$$

Lemma 4.6. For all positive integer n:

$$|\langle g | g^n = e \rangle| = n$$

Proof. For the words $e, g, g^2, \cdots, g^{n-1} \in \langle g | g^n = e \rangle$,
we compute $\#_g$ and $\#_g^n$:

Word	$\#_g$	$\#_g^n$
e	0	0
g	1	1
g^2	2	2
$\vdots$	$\vdots$	$\vdots$
g^{n-1}	$n-1$	$n-1$

As the evaluations are pairwisely distinct, it follows that:

$$e, \quad g, \quad g^2, \quad \cdots, \quad g^{n-1} \text{ are pairwisely distinct in } \langle g | g^n = e \rangle$$

Apply **Lemma 4.4**, and we get:

$$|\langle g|g^n = e\rangle| = |\{e, g, g^2, \cdots, g^{n-1}\}| = n$$

Quod. Erat. Demonstrandum. $\square$

Theorem 4.7. For all positive integer n,
the following map from $\mathbb{Z}_n$ to $\langle g|g^n = e\rangle$ is a group isomorphism:

$$\widetilde{\phi} : [m]_n \mapsto g^m$$

Proof. We prove that the following map from $\mathbb{Z}$ to $\langle g|g^n = e\rangle$ is a surjective group homomorphism with kernel $n\mathbb{Z}$:

$$\phi : m \mapsto g^m$$

Part 1: For all $h \in \langle g|g^n = e\rangle$, there exists $a \in \mathbb{Z}$, such that:

$$h = g^a = \phi(a)$$

Hence, ϕ is surjective.
Part 2: For all $m_1, m_2 \in \mathbb{Z}$:

$$\phi(m_1 + m_2) = g^{m_1+m_2} = g^{m_1}g^{m_2} = \phi(m_1)\phi(m_2)$$

Hence, ϕ is a group homomorphism.
Part 3: We prove that $\mathrm{Ker}(\phi) = n\mathbb{Z}$.

$$\mathrm{Ker}(\phi) = \{m \in \mathbb{Z} : \phi(m) = e\} = \{m \in \mathbb{Z} : n|m\} = n\mathbb{Z}$$

Hence, the quotient map $\widetilde{\phi} : \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \to \langle g|g^n = e\rangle, [m]_n \mapsto g^m$ is a group isomorphism. Quod. Erat. Demonstrandum. $\square$

Example 4.8. For all positive integer n, in $\langle r, \sigma : r^n = \sigma^2 = (r\sigma)^2 = e\rangle$,
we have the following Cayley table:

Row∗Col	r^l	$r^l\sigma$
r^k	r^{k+l}	$r^{k+l}\sigma$
$r^k\sigma$	$r^{k-l}\sigma$	r^{k-l}

Lemma 4.9. For all positive integer n:

$$\langle r, \sigma : r^n = \sigma^2 = (r\sigma)^2 = e\rangle = \{e, r, r^2, \cdots, r^{n-1}, \sigma, r\sigma, r^2\sigma, \cdots, r^{n-1}\sigma\}$$

Proof. We may divide our proof into two parts.

"$\subseteq$ inclusion:" According to **Example 4.8.**,
for all $h \in \langle r, \sigma : r^n = \sigma^2 = (r\sigma)^2 = e \rangle$, there exist $a, b \in \mathbb{Z}$, such that:

$$h = r^a \sigma^b$$

According to the division algorithm, there exist $\mu, \nu, \alpha, \beta \in \mathbb{Z}$, such that:

$$a = \mu n + \nu \text{ and } 0 \leq \nu < n \text{ and } b = \alpha 2 + \beta \text{ and } 0 \leq \beta < 2$$

Apply the identification rule $r^n = \sigma^2 = e$:

$$\begin{aligned} h &= r^{\mu n + \nu} \sigma^{\alpha 2 + \beta} = (r^n)^\mu r^\nu (\sigma^2)^\alpha \sigma^\beta = e^\mu r^\nu e^\alpha \sigma^\beta = r^\nu \sigma^\beta \\ &\in \{e, r, r^2, \cdots, r^{n-1}, \sigma, r\sigma, r^2\sigma, \cdots, r^{n-1}\sigma\} \end{aligned}$$

"$\supseteq$ inclusion:" All of $e, r, r^2, \cdots, r^{n-1}, \sigma, r\sigma, r^2\sigma, \cdots, r^{n-1}\sigma$ are words of r, σ.
Hence, $\langle r, \sigma : r^n = \sigma^2 = (r\sigma)^2 = e \rangle = \{e, r, r^2, \cdots, r^{n-1}, \sigma, r\sigma, r^2\sigma, \cdots, r^{n-1}\sigma\}$.
Quod. Erat. Demonstrandum. □

Example 4.10. Define the following function from $\mathrm{Word}(r, \sigma)$ to $\mathbb{Z}$:

$$\#_{r,\sigma} : g_1^{\epsilon_1} g_2^{\epsilon_2} \cdots g_{m-1}^{\epsilon_{m-1}} g_m^{\epsilon_m} \mapsto \delta_{g_1,\sigma}\epsilon_1 + \delta_{g_2,\sigma}\epsilon_2 + \cdots + \delta_{g_{m-1},\sigma}\epsilon_{m-1} + \delta_{g_m,\sigma}\epsilon_m$$

As two words are equal iff they arrange letters in completely the same way, $\#_{r,\sigma}$ is well-defined. In addition, for all positive integer n, if we insert or delete:

$$r^{-1}r^{+1}, r^{+1}r^{-1}, \sigma^{-1}\sigma^{+1}, \sigma^{+1}\sigma^{-1}, r^n, \sigma^2, (r\sigma)^2$$

Then the remainder $\#^2_{r,\sigma}$ of $\#_{r,\sigma}$ modulo 2 is invariant:

Insert/Delete	$\Delta(\#_{r,\sigma})$	$\Delta(\#^2_{r,\sigma})$
$r^{-1}r^{+1}$	0	0
$r^{+1}r^{-1}$	0	0
$\sigma^{-1}\sigma^{+1}$	0	0
$\sigma^{+1}\sigma^{-1}$	0	0
r^n	0	0
σ^2	± 2	0
$(r\sigma)^2$	± 2	0

Hence, for all $\alpha, \beta \in \mathrm{Word}(r, \sigma)$:

$$\alpha = \beta \text{ in } \langle r, \sigma | r^n = \sigma^2 = (r\sigma)^2 = e \rangle \implies \#^2_{r,\sigma}(\alpha) = \#^2_{r,\sigma}(\beta)$$

Example 4.11. Define the following function from $\mathrm{Word}(r, \sigma)$ to $\mathbb{Z}$:

$$\begin{aligned}
\%_{r,\sigma}(e) &= 0 \\
\%_{r,\sigma}([\text{Prefix}]r^{+1}) &= +\%_{r,\sigma}([\text{Prefix}]) + 1 \\
\%_{r,\sigma}([\text{Prefix}]r^{-1}) &= +\%_{r,\sigma}([\text{Prefix}]) - 1 \\
\%_{r,\sigma}([\text{Prefix}]\sigma^{+1}) &= -\%_{r,\sigma}([\text{Prefix}]) \\
\%_{r,\sigma}([\text{Prefix}]\sigma^{-1}) &= -\%_{r,\sigma}([\text{Prefix}])
\end{aligned}$$

As two words are equal iff they arrange letters in completely the same way, $\%_{r,\sigma}$ is well-defined. In addition, for all positive integer n, if we insert or delete:

$$r^{-1}r^{+1}, r^{+1}r^{-1}, \sigma^{-1}\sigma^{+1}, \sigma^{+1}\sigma^{-1}, r^n, \sigma^2, (r\sigma)^2$$

Then the remainder $\%^n_{r,\sigma}$ of $\%_{r,\sigma}$ modulo n is invariant:

Insert/Delete	$\Delta(\%_{r,\sigma})$	$\Delta(\%^n_{r,\sigma})$
$r^{-1}r^{+1}$	0	0
$r^{+1}r^{-1}$	0	0
$\sigma^{-1}\sigma^{+1}$	0	0
$\sigma^{+1}\sigma^{-1}$	0	0
r^n	$\pm n$	0
σ^2	0	0
$(r\sigma)^2$	0	0

Hence, for all $\alpha, \beta \in \mathrm{Word}(r, \sigma)$:

$$\alpha = \beta \text{ in } \langle r, \sigma | r^n = \sigma^2 = (r\sigma)^2 = e\rangle \implies \%^n_{r,\sigma}(\alpha) = \%^n_{r,\sigma}(\beta)$$

Lemma 4.12. For all positive integer n:

$$|\langle r, \sigma | r^n = \sigma^2 = (r\sigma)^2\rangle| = 2n$$

Proof. For the words $e, r, r^2, \cdots, r^{n-1}, \sigma, r\sigma, r^2\sigma, \cdots, r^{n-1}\sigma$, we compute $(\#_{r,\sigma}, \%_{r,\sigma})$ and $(\#^2_{r,\sigma}, \%^n_{r,\sigma})$:

Word	$(\#_{r,\sigma}, \%_{r,\sigma})$	$(\#^2_{r,\sigma}, \%^n_{r,\sigma})$	Word	$(\#_{r,\sigma}, \%_{r,\sigma})$	$(\#^2_{r,\sigma}, \%^n_{r,\sigma})$
e	$(0, 0)$	$(0, 0)$	σ	$(1, 0)$	$(1, 0)$
r	$(0, 1)$	$(0, 1)$	$r\sigma$	$(1, -1)$	$(1, n-1)$
r^2	$(0, 2)$	$(0, 2)$	$r^2\sigma$	$(1, -2)$	$(1, n-2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
r^{n-1}	$(0, n-1)$	$(0, n-1)$	$r^{n-1}\sigma$	$(1, -n+1)$	$(1, 1)$

As the evaluations are pairwisely distinct, it follows that:

$$\begin{matrix} e, & r, & r^2, & \cdots, & r^{n-1} \\ \sigma, & r\sigma, & r^2\sigma, & \cdots, & r^{n-1}\sigma \end{matrix} \text{ are pairwisely distinct in } \langle r,\sigma|r^n=\sigma^2=(r\sigma)^2=e\rangle$$

Apply **Lemma 4.9.**, and we get:

$$|\langle r,\sigma|r^n=\sigma^2=(r\sigma)^2=e\rangle| = |\{e,r,r^2,\cdots,\sigma,r\sigma,r^2\sigma,\cdots,r^{n-1}\sigma\}| = 2n$$

Quod. Erat. Demonstrandum. □

Theorem 4.13. For all positive integer n, if we define $\zeta = \mathrm{e}^{2\pi\mathrm{i}/n}$, then:

$$\langle r,\sigma : r^n=\sigma^2=(r\sigma)^2=e\rangle \cong \langle z\mapsto\zeta z, z\mapsto\zeta\overline{z}\rangle$$

Proof. The two groups have the same Cayley table after relabeling:

Row∗Col	$z\mapsto\zeta^l z$	$z\mapsto\zeta^{k+l}\overline{z}$
$z\mapsto\zeta^k z$	$z\mapsto\zeta^{k+l}z$	$z\mapsto\zeta^{k+l}\overline{z}$
$z\mapsto\zeta^k\overline{z}$	$z\mapsto\zeta^{k-l}\overline{z}$	$z\mapsto\zeta^{k-l}z$

Hence, the two groups are isomorphic. Quod. Erat. Demonstrandum. □

Example 4.14. In $\langle x,y|x^4=yxyx^3=y^3xyx=w\rangle$,
we have the following Cayley table:

Row∗Col	w	x	y	xy	x^2	x^3	x^2y	x^3y
w	w	x	y	xy	x^2	x^3	x^2y	x^3y
x	x	x^2	xy	x^2y	x^3	w	x^3y	y
y	y	x^3y	x^2	x	x^2y	xy	w	x^3
xy	xy	y	x^3	x^2	x^3y	x^2y	x	w
x^2	x^2	x^3	x^2y	x^3y	w	x	y	xy
x^3	x^3	w	x^3y	y	x	x^2	xy	x^2y
x^2y	x^2y	xy	w	x^3	y	x^3y	x^2	x
x^3y	x^3y	x^2y	x	w	xy	y	x^3	x^2

Lemma 4.15.

$$\langle x,y|x^4=yxyx^3=y^3xyx=w\rangle = \{w,x,y,xy,x^2,x^3,x^2y,x^3y\}$$

Proof. We may divide our proof into two parts.
"⊆" inclusion: According to **Example 4.14.**,
for all $h\in\langle x,y|x^4=yxyx^3=y^3xyx=e\rangle$, there exist $0\le a<4, 0\le b<1$, such that:

$$h = x^a y^b \in \{w,x,y,xy,x^2,x^3,x^2y,x^3y\}$$

"$\subseteq$" inclusion: All of $w, x, y, xy, x^2, x^3, x^2y, x^3y$ are words of x, y.
Hence, $\langle x, y | x^4 = yxyx^3 = y^3xyx = w\rangle = \{w, x, y, xy, x^2, x^3, x^2y, x^3y\}$.
Quod. Erat. Demonstrandum. $\square$

Example 4.16. Define the following function from $\text{Word}(x, y)$ to $\mathbb{Z}$:

$$\#_{x,y}(e) = 1$$
$$\#_{x,y}(x) = \#_{x,y}(y) = \#_{x,y}(xy) = \#_{x,y}(yx) = 0$$
$$\#_{x,y}([\text{Word with } x^{-1}, y^{-1}]) = \#_{x,y}([\text{Replace } x^{-1}, y^{-1} \text{ with } x^3, y^3])$$
$$\#_{x,y}([\text{Prefix}]x^2) = \#_{x,y}([\text{Prefix}]y^2) = -\#_{x,y}([\text{Prefix}])$$
$$\#_{x,y}([\text{Prefix}]x^2y) = -\#_{x,y}([\text{Prefix}]y)$$
$$\#_{x,y}([\text{Prefix}]y^2x) = -\#_{x,y}([\text{Prefix}]x)$$
$$\#_{x,y}([\text{Prefix}]xyx) = \#_{x,y}([\text{Prefix}]y)$$
$$\#_{x,y}([\text{Prefix}]yxy) = \#_{x,y}([\text{Prefix}]x)$$

As two words are equal iff they arrange letters in completely the same way, $\#_{x,y}$ is well-defined. In addition, if we insert or delete:

$$x^{-1}x^{+1}, x^{+1}x^{-1}, y^{-1}y^{+1}, y^{+1}y^{-1}, x^4, yxyx^3, y^3xyx$$

Then the number $\#_{x,y}$ itself is invariant:

Insert/Delete	$\Delta(\#_{x,y})$
$x^{-1}x^{+1}$	0
$x^{+1}x^{-1}$	0
$y^{-1}y^{+1}$	0
$y^{+1}y^{-1}$	0
x^4	0
$yxyx^3$	0
y^3yxy	0

Hence, for all $\alpha, \beta \in \text{Word}(x, y)$:

$$\alpha = \beta \text{ in } \langle x, y | x^4 = yxyx^3 = y^3xyx = w\rangle \implies \$_{x,y}(\alpha) = \$_{x,y}(\beta)$$

Lemma 4.17.

$$|\langle x, y | x^4 = yxyx^3 = y^3xyx = w\rangle| = 8$$

Proof. For the words $w, x, y, xy, x^2, x^3, x^2y, x^3y$,
we compute $\#_{x,y}$ in four different directions:

Word α	$\#_{x,y}(\alpha)$	$\#_{x,y}(\alpha x^{-1})$	$\#_{x,y}(\alpha y^{-1})$	$\#_{x,y}(\alpha y^{-1}x^{-1})$
w	1	0	0	0
x	0	1	0	0
y	0	0	1	0
xy	0	0	0	1
x^2	-1	0	0	0
x^3	0	-1	0	0
x^2y	0	0	-1	0
x^3y	0	0	0	-1

As the evaluations are pairwisely distinct, it follows that:

$$\begin{matrix} w, & x, & y, & xy \\ x^2, & x^3, & x^2y, & x^3y \end{matrix} \text{ are pairwisely distinct in } \langle x, y | x^4 = yxyx^3 = y^3xyx = w \rangle$$

Apply **Lemma 4.15.**, and we get:

$$|\langle x, y | x^4 = yxyx^3 = y^3xyx = w \rangle| = |\{w, x, y, xy, x^2, x^3, x^2y, x^3y\}| = 8$$

Quod. Erat. Demonstrandum. □

Theorem 4.18. If we define:

$$\mathbf{i} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then:

$$\langle x, y | x^4 = yxyx^3 = y^3xyx = w \rangle \cong \langle \mathbf{i}, \mathbf{j} \rangle$$

Proof. The two groups have the same Cayley table after relabeling:

Row∗Col	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{ij}$	$\mathbf{i}^2$	$\mathbf{i}^3$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{i}^3\mathbf{j}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{ij}$	$\mathbf{i}^2$	$\mathbf{i}^3$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{i}^3\mathbf{j}$
$\mathbf{i}$	$\mathbf{i}$	$\mathbf{i}^2$	$\mathbf{ij}$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{i}^3$	1	$\mathbf{i}^3\mathbf{j}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$\mathbf{i}^3\mathbf{j}$	$\mathbf{i}^2$	$\mathbf{i}$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{ij}$	1	$\mathbf{i}^3$
$\mathbf{ij}$	$\mathbf{ij}$	$\mathbf{j}$	$\mathbf{i}^3$	$\mathbf{i}^2$	$\mathbf{i}^3\mathbf{j}$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{i}$	1
$\mathbf{i}^2$	$\mathbf{i}^2$	$\mathbf{i}^3$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{i}^3\mathbf{j}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{ij}$
$\mathbf{i}^3$	$\mathbf{i}^3$	1	$\mathbf{i}^3\mathbf{j}$	$\mathbf{j}$	$\mathbf{i}$	$\mathbf{i}^2$	$\mathbf{ij}$	$\mathbf{i}^2\mathbf{j}$
$\mathbf{i}^2\mathbf{j}$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{ij}$	1	$\mathbf{i}^3$	$\mathbf{j}$	$\mathbf{i}^3\mathbf{j}$	$\mathbf{i}^2$	$\mathbf{i}$
$\mathbf{i}^3\mathbf{j}$	$\mathbf{i}^3\mathbf{j}$	$\mathbf{i}^2\mathbf{j}$	$\mathbf{i}$	1	$\mathbf{ij}$	$\mathbf{j}$	$\mathbf{i}^3$	$\mathbf{i}^2$

Hence, the two groups are isomorphic. Quod. Erat. Demonstrandum. □

Definition 4.19. (**Free Product**)
Let $(G_1, \circ_1), (G_2, \circ_2)$ be groups. Define the free product of G_1, G_2 as:

$$G_1 * G_2 = \left\langle G_1 \vee_e G_2 \middle| \begin{array}{lcl} \forall g_1, g_1' \in G, \quad g_1 g_1' & = & g_1 \circ_1 g_1' \\ \forall g_2, g_2' \in G_2, \quad g_2 g_2' & = & g_2 \circ_2 g_2' \end{array} \right\rangle$$

Proposition 4.20. If $G_1 = \langle S_1 \rangle, G_2 = \langle S_2 \rangle$, then $G_1 * G_2 = \langle S_1 \sqcup S_2 \rangle$.

Proof.

$$\begin{aligned} G_1 * G_2 &= \left\langle G_1 \vee_e G_2 \middle| \begin{array}{lcl} \forall g_1, g_1' \in G_1, \quad g_1 g_1' & = & g_1 \circ_1 g_1' \\ \forall g_2, g_2' \in G_2, \quad g_2 g_2' & = & g_2 \circ_2 g_2' \end{array} \right\rangle \\ &= \left\langle \langle S_1 \rangle \vee_e \langle S_2 \rangle \middle| \begin{array}{lcl} \forall s_1, s_1' \in S_1, \quad s_1 s_1' & = & s_1 s_1' \\ \forall s_2, s_2' \in S_2, \quad s_2 s_2' & = & s_2 s_2' \end{array} \right\rangle = \langle S_1 \sqcup S_2 \rangle \end{aligned}$$

Quod. Erat. Demonstrandum. $\square$

Proposition 4.21. If G, H are two nontrivial groups, then for some nontrivial $g \in G$ and $h \in H$, $hgh^{-1} \notin G$ and $ghg^{-1} \notin H$, so G, H are not normal in $G * H$.

Proof. Notice that the number of h before g and the number of h after g are invariant under identification, so $hgh^{-1} \notin G$ and $ghg^{-1} \notin H$. Quod. Erat. Demonstrandum. $\square$

The rest of this section devotes to prove the following result:[3]

$$G = \left\{ \frac{a_{1,1}z + a_{1,2}}{a_{2,1}z + a_{2,2}} : \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in \mathbf{M}_2(\mathbb{Z}) \text{ and Det} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = 1 \right\} \cong \mathbb{Z}_2 * \mathbb{Z}_3$$

Lemma 4.22. For all $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}} \in G$, $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}}$ can be reduced to $\frac{1z+0}{0z+1}$ by a word of sheer transformations $\frac{1z+q}{0z+1}, \frac{1z+0}{qz+1} \in G$ and transposition $\frac{0z-1}{1z+0} \in G$.

Proof. We may divide our proof into two steps.
Step 1: We apply division algorithm to kill one of $a_{1,1}, a_{2,1}$.
Case 1.1: If $|a_{1,1}| + |a_{2,1}| \leq 1$, then **Step 1** terminates.
Otherwise, since $a_{1,1}, a_{2,1}$ are coprime, neither of them is zero.
Case 1.2: If $|a_{1,1}| \geq |a_{2,1}|$, then choose $q \in \mathbb{Z}$, such that $0 \leq a_{1,1} + qa_{2,1} < |a_{2,1}|$.
Replace $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}}$ with the following expression, and return to **Case 1.1.**.

$$\frac{r_{1,1}z + r_{1,2}}{a_{2,1}z + a_{2,2}} = \frac{1z + q}{0z + 1} \frac{a_{1,1}z + a_{1,2}}{a_{2,1}z + a_{2,2}}$$

Case 1.3: If $|a_{1,1}| \le |a_{2,1}|$, then choose $q \in \mathbb{Z}$, such that $0 \le a_{2,1} + qa_{1,1} < |a_{1,1}|$. Replace $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}}$ with the following expression, and return to **Case 1.1.**.

$$\frac{a_{1,1}z + a_{1,2}}{r_{2,1}z + r_{2,2}} = \frac{1z+0}{qz+1}\frac{a_{1,1}z + a_{1,2}}{a_{2,1}z + a_{2,2}}$$

As $|a_{1,1}| + |a_{2,1}| < +\infty$, **Step 1** terminates after finitely many steps.

Step 2: As one of $a_{1,1}, a_{2,1}$ is killed, and Det $\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = 1$, $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}}$ must be equal to $\frac{1z+q}{0z+1}$ or $\frac{0z-1}{1z+q}$ after adjusting signs properly.

Case 2.1: If $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}} = \frac{1z+q}{0z+1}$, then we are done.

Case 2.2: If $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}} = \frac{0z-1}{1z+q}$, then do the following multiplication:

$$\frac{0z-1}{1z+0}\frac{0z-1}{1z+q} = \frac{-1z-q}{-0z-1} = \frac{1z+q}{0z+1}$$

Quod. Erat. Demonstrandum. □

Lemma 4.23. For all $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}} \in G$, $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}}$ is a word of:

$$\alpha = \frac{1z+1}{0z+1} \text{ and } \beta = \frac{0z-1}{1z+0}$$

Proof. It suffices to notice the followings:

$$\left(\frac{1z+q}{0z+1}\right)^{-1} = \frac{1z-q}{0z+1} = \left(\frac{1z+1}{0z+1}\right)^{-q} = \alpha^{-q}$$
$$\left(\frac{1z+0}{qz+1}\right)^{-1} = \left(\frac{1z+0}{-qz+1}\right) = \left(\frac{0z-1}{1z+0}\right)\left(\frac{1z+1}{0z+1}\right)^{q}\left(\frac{0z-1}{1z+0}\right) = \beta\alpha^q\beta$$
$$\left(\frac{0z-1}{1z+0}\right)^{-1} = \left(\frac{0z-1}{1z+0}\right) = \beta$$

Quod. Erat. Demonstrandum. □

Lemma 4.24. $\frac{1z+0}{0+1z}$ is not equal to any nontrivial word of:

$$\alpha = \frac{1z+1}{0z+1} \text{ and } \beta = \frac{0z-1}{1z+0}$$

Proof. Consider the left action $*$ of G on the set of all irrational real numbers $\mathbb{I}$. If we define $\gamma = \alpha\beta = \frac{1z-1}{1z+0}, \mathbb{I}_{>0} = \mathbb{I} \cap \mathbb{R}_{>0}, \mathbb{I}_{<0} \cap \mathbb{R}_{<0}$, then:

$$\beta * \mathbb{I}_{>0} \subseteq \mathbb{I}_{<0}$$
$$\gamma * \mathbb{I}_{<0} \subseteq \mathbb{I}_{>0}$$
$$\gamma^{-1} * \mathbb{I}_{<0} \subseteq \mathbb{I}_{>0}$$

For all nontrivial word ω of α, β, it is also a nontrivial word of γ.
Case 1: If ω has odd nontrivial blocks, then ω begins and ends with a β-block, where $\omega * \mathbb{I}_{>0} \subseteq \beta * \mathbb{I}_{>0} \subseteq \mathbb{I}_{<0}$, or ω begins and ends with a γ-block, where $\omega * \mathbb{I}_{<0} \subseteq \gamma^{\pm 1} * \mathbb{I}_{<0} \subseteq \mathbb{I}_{>0}$. In both situations, ω reverses part of the real line, which is not the case of $\frac{1z+0}{0z+1}$.
Case 2: If ω has even nontrivial blocks, then we may do conjugation and assume that ω begin with a γ-block and end with a β-block. There are two situations two consider:
Situation 2.1: If the first block is γ, then $\omega * \mathbb{I}_{>0} \subseteq \gamma * \mathbb{I}_{<0} \subseteq (1, +\infty)$.
Situation 2.2: If the first block is γ^{-1}, then $\omega * \mathbb{I}_{>0} \subseteq \gamma^{-1} * \mathbb{I}_{<0} \subseteq (-\infty, 1)$.
In both situations, ω cuts a whole interval of the real line, which is not the case of $\frac{1z+0}{0z+1}$.
Quod. Erat. Demonstrandum. $\square$

Theorem 4.25.

$$G \cong \mathbb{Z}_2 * \mathbb{Z}_3$$

Proof. Take $\beta = \frac{0z-1}{1z+0}, \gamma = \frac{1z-1}{1z+0}$. From the following calculations:

$$\begin{aligned}
\beta^1 &= \frac{0z-1}{1z+0} \neq \frac{1z+0}{0z+1} \\
\beta^2 &= \frac{1z+0}{0z+1} \\
\gamma^1 &= \frac{1z-1}{1z+0} \neq \frac{1z+0}{0z+1} \\
\gamma^2 &= \frac{0z-1}{1z-1} \neq \frac{1z+0}{0z+1} \\
\gamma^3 &= \frac{1z+0}{0z+1}
\end{aligned}$$

We know that $\mathrm{Ord}(\beta) = 2$ and $\mathrm{Ord}(\gamma) = 3$.
From **Lemma 4.23.**, we know that every $\frac{a_{1,1}z+a_{1,2}}{a_{2,1}z+a_{2,2}}$ is a word of β, γ.
From **Lemma 4.24.**, we know that there is no other nontrivial restrictions on β, γ.
Hence, we may conclude that:

$$G \cong \langle \beta, \gamma | \beta^2 = \gamma^3 = e \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3$$

Quod. Erat. Demonstrandum. $\square$

4.2 Free Product with Amalgamation

In the last section, we see that it is in general hard to count the following number:

$$|\langle x, y | R(x,y) = S(x,y) = T(x,y) = e \rangle|$$

Counting this number is beyond our scope. What we do is to generalize this concept.

Definition 4.26. (**Free Product with Amalgamation**)
Let A, G_1, G_2 be groups, $\phi_1 : A \to G_1, \phi_2 : A \to G_2$ be group homomorphisms. Define the free product of G_1, G_2 with amalgamation as:

$$G_1 *_A G_2 = \left\langle G_1 \vee_e G_2 \;\middle|\; \begin{array}{ccc} \forall g_1, g_1' \in G_1, & g_1 g_1' & = & g_1 \circ_1 g_1' \\ \forall g_2, g_2' \in G_2, & g_2 g_2' & = & g_2 \circ_2 g_2' \\ \forall a \in A, & \phi_1(a) & = & \phi_2(a) \end{array} \right\rangle$$

Remark: *When A is the trivial group, this degenerates to the usual free product.*

Theorem 4.27. (**The Universal Property of Free Product**)
Let A, G_1, G_2, H be groups, and $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \mathbf{f}_1, \mathbf{f}_2$ be group homomorphisms such that the following diagram commutes:

$$\begin{tikzcd} & G_1 \arrow[dr, "\mathbf{f}_1"] & \\ A \arrow[ur, "\boldsymbol{\phi}_1"] \arrow[dr, "\boldsymbol{\phi}_2"'] & & H \\ & G_2 \arrow[ur, "\mathbf{f}_2"'] & \end{tikzcd}$$

Define $\boldsymbol{\iota}_{1,2} : G_{1,2} \to G_1 *_A G_2$ as the natural projection maps. For some unique group homomorphism $\mathbf{g} : G_1 *_A G_2 \to H$, the following diagram commutes:

$$\begin{tikzcd} & G_1 \arrow[dl, "\boldsymbol{\iota}_1"'] \arrow[dr, "\mathbf{f}_1"] & \\ G_1 *_A G_2 \arrow[rr, "\mathbf{g}"] & & H \\ & G_2 \arrow[ul, "\boldsymbol{\iota}_2"] \arrow[ur, "\mathbf{f}_2"'] & \end{tikzcd}$$

Proof. It suffices to check that $\mathbf{g}$ is invariant under three identification relations.

$$\mathbf{g} : G_1 *_A G_2 \mapsto H, [\text{A word of } G_1 \vee_e G_2] \mapsto [\text{A word of } \mathbf{f}_1(G_1) \vee_e \mathbf{f}_2(G_2)]$$

Step 1: For all words in the form $\cdots x_1 x_1' \cdots = \cdots x_1 \circ_1 x_1' \cdots$, where $x_1, x_1' \in G_1$:

$$\begin{aligned} \mathbf{g}(\cdots x_1 x_1' \cdots) &= \cdots \diamond \mathbf{f}_1(x_1) \diamond \mathbf{f}_2(x_1') \diamond \cdots \\ &= \cdots \diamond \mathbf{f}_1(x_1 \circ_1 x_1') \diamond \cdots = \mathbf{g}(\cdots x_1 \circ_1 x_1' \cdots) \end{aligned}$$

Step 2: For all words in the form $\cdots x_2 x_2' \cdots = \cdots x_2 \circ_2 x_2' \cdots$, where $x_2, x_2' \in G_2$:

$$\begin{aligned} \mathbf{g}(\cdots x_2 x_2' \cdots) &= \cdots \diamond \mathbf{f}_2(x_2) \diamond \mathbf{f}_2(x_2') \diamond \cdots \\ &= \cdots \diamond \mathbf{f}_2(x_2 \circ_2 x_2') \diamond \cdots = \mathbf{g}(\cdots x_2 \circ_2 x_2' \cdots) \end{aligned}$$

Step 3: For all words in the form $\cdots \boldsymbol{\phi}_1(a)\cdots = \cdots \boldsymbol{\phi}_2(a)\cdots$, where $a \in A$:

$$\begin{aligned}\mathbf{g}(\cdots \boldsymbol{\phi}_1(a)\cdots) &= \cdots \diamond \mathbf{f}_1 \circ \boldsymbol{\phi}_1(a) \diamond \cdots \\ &= \cdots \diamond \mathbf{f}_2 \circ \boldsymbol{\phi}_2(a) \diamond \cdots = \mathbf{g}(\cdots \boldsymbol{\phi}_2(a)\cdots)\end{aligned}$$

Hence, the function $\mathbf{g}$ is well-defined. The diagram commutes iff we take the definition at the beginning, so this function is unique. As $\mathbf{f}_1, \mathbf{f}_2$ are group homomorphisms, it follows that $\mathbf{g}$ is a group homomorphism. Quod. Erat. Demonstrandum. $\square$

Theorem 4.28. (**Seifert-Van Kampen Theorem**[4])
Let S, X_1, X_2, Y be topological spaces with:

$$\begin{aligned} Y &= X_1 \cup X_2 \\ X_1 &\subseteq Y \text{ is open and path connected} \\ X_2 &\subseteq Y \text{ is open and path connected} \\ S &= X_1 \cap X_2 \text{ is nonempty and path connected}\end{aligned}$$

ϕ_1, ϕ_2, f_1, f_2 be the natural projection maps described in the follow diagram:

$$\begin{tikzcd} & X_1 \arrow[dr, "f_1"] & \\ S \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & & Y \\ & X_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

Define $\iota_{1,2} : X_{1,2} \to Y$ as the natural projection maps, take an arbitrary base point s_0 from S, and we omit this base point in fundamental groups because the context is clear. There exists a unique group isomorphism $\mathbf{g} : \pi_1(X_1) *_{\pi_1(S)} \pi_1(X_2) \to \pi_1(Y)$, such that the following diagram commutes:

$$\begin{tikzcd} & \pi_1(X_1) \arrow[dl, "\boldsymbol{\iota}_1"'] \arrow[dr, "\mathbf{f}_1"] & \\ \pi_1(X_1) *_{\pi_1(S)} \pi_1(X_2) \arrow[rr, "\mathbf{g}"] & & \pi_1(Y) \\ & \pi_1(X_2) \arrow[ul, "\boldsymbol{\iota}_2"] \arrow[ur, "\mathbf{f}_2"'] & \end{tikzcd}$$

Here, $\boldsymbol{\iota}_{1,2} = \pi_1(\iota_{1,2}), \mathbf{f}_{1,2} = \pi_1(f_{1,2})$ are generated by the fundamental functor π_1.

Proof. We may divide our proof into three parts.
Part 1: We apply **Theorem 4.27.** to find a unique group homomorphism $\mathbf{g}$.
Define groups $A = \pi_1(S), G_{1,2} = \pi_1(X_{1,2}), H = \pi_1(Y)$ and group homomorphisms

$\boldsymbol{\phi_{1,2}} = \pi_1(\phi_{1,2}), \mathbf{f}_{1,2} = \pi_1(f_{1,2})$. Now the following diagram commutes:

$$\begin{tikzcd} & G_1 \arrow[dr, "\mathbf{f}_1"] & \\ A \arrow[ur, "\boldsymbol{\phi}_1"] \arrow[dr, "\boldsymbol{\phi}_2"'] & & H \\ & G_2 \arrow[ur, "\mathbf{f}_2"'] & \end{tikzcd}$$

According to **Theorem 4.27.**, for some unique group homomorphism $\mathbf{g} : G_1 *_A G_2 \to H$, the following diagram commutes:

$$\begin{tikzcd} & G_1 \arrow[dl, "\iota_1"'] \arrow[dr, "\mathbf{f}_1"] & \\ G_1 *_A G_2 \arrow[rr, "\mathbf{g}"] & & H \\ & G_2 \arrow[ul, "\iota_2"] \arrow[ur, "\mathbf{f}_2"'] & \end{tikzcd}$$

Part 2: We prove that this group homomorphism $\mathbf{g}$ is surjective.
It suffices to prove that every loop γ in Y with base point s_0 is homotopic to a word of loops in X_1, X_2 at s_0. Consider the open preimages $\gamma^{-1}(X_1), \gamma^{-1}(X_2)$:
(1) As $[0,1]$ is a subset of $\mathbb{R}$, the Lindelöf property suggests:

$$\gamma^{-1}(X_1) \cong \coprod_{\lambda \in I_1} \mathfrak{U}_1^\lambda, \gamma^{-1}(X_2) \cong \coprod_{\lambda \in I_2} \mathfrak{U}_2^\lambda$$

Here, the index sets I_1, I_2 are countable, and the families $\mathcal{U}_1 = \{\mathfrak{U}_1^\lambda\}_{\lambda \in I_1}, \mathcal{U}_2 = \{\mathfrak{U}_2^\lambda\}_{\lambda \in I_2}$ consist of disjoint open connected subsets of $[0,1]$.
(2) As $[0,1]$ is compact, the open cover $\mathcal{U}_1 \cup \mathcal{U}_2$ of it has a finite subcover.
(3) As the open subsets in $\mathcal{U}_1, \mathcal{U}_2$ are disjoint, WLOG,
assume that the finite subcover mentioned above is of the following form:

$$\begin{array}{llllll} 0 \in \mathfrak{U}_1^1 = [0, b_1^1), & \mathfrak{U}_2^2 = (a_2^2, b_2^2), \cdots, & \mathfrak{U}_1^{k-1} = (a_1^{k-1}, b_1^{k-1}), & \mathfrak{U}_2^k = (a_1^k, 1] \ni 1 \\ 0 \in \mathfrak{U}_1^1 = [0, b_1^1) \ < \cdots < & & \mathfrak{U}_1^{k-1} = (a_1^{k-1}, b_1^{k-1}) & \not\ni 1 \\ 0 \notin & \mathfrak{U}_2^2 = (a_2^2, b_2^2) < \cdots < & & \mathfrak{U}_2^k = (a_1^k, 1] \ni 1 \end{array}$$

(4) Decompose γ into the concatenation $\gamma_1 \star_{c_1} \gamma_2 \star_{c_2} \cdots \star_{c_{k-2}} \gamma_{k-1} \star_{c_{k-1}} \gamma_k$. For each γ_l, it is contained in X_1 or X_2. Note that the initial point and end point of γ_l lie in the path connected intersection, so γ_l is homotopic to some loop σ_l in X_1 or X_2.
(5) Repeat this process inductively, then loop γ is homotopic to an alternating word of loops in X_1 and X_2. This means every $[\![\gamma]\!] \in \pi_1(Y)$ in $\pi_1(Y)$ is the image of some word in $\pi_1(X_1) *_A \pi_1(X_2)$ under the group homomorphism $\mathbf{g}$, so $\mathbf{g}$ is surjective.
Part 3: We prove that this group homomorphism $\mathbf{g}$ is injective.
It suffices to prove that every path homotopy $H : [0,1]_t \times [0,1]_s \to Y$ is a word of path

homotopies in X_1, X_2.
(1) As $[0,1]_t \times [0,1]_s$ is a subset of $\mathbb{R}_t \times \mathbb{R}_s$, the Lindelöf property suggests:

$$\gamma^{-1}(X_1) \cong \coprod_{\lambda \in I_1} \mathfrak{U}_1^\lambda, \gamma^{-1}(X_2) \cong \coprod_{\lambda \in I_2} \mathfrak{U}_2^\lambda$$

Here, the index sets I_1, I_2 are countable, and the families $\mathcal{U}_1 = \{\mathfrak{U}_1^\lambda\}_{\lambda \in I_1}, \mathcal{U}_2 = \{\mathfrak{U}_2^\lambda\}_{\lambda \in I_2}$ consist of disjoint open connected subsets of $[0,1]_t \times [0,1]_s$.
(2) As $[0,1]_t \times [0,1]_s$ is compact, the open cover $\mathcal{U}_1 \cup \mathcal{U}_2$ of it has a finite subcover.
(3) As the open subsets in $\mathcal{U}_1, \mathcal{U}_2$ are disjoint, WLOG, assume that the following partition of $[0,1]_t \times [0,1]_s$ satisfies each block is contained in some $\mathfrak{U}_{i,j}$ in that subcover. If this partition is not fine enough, just halven the width of subintervals and test again:

	$t = 0$	$0 < t < \frac{1}{2}$	$t = \frac{1}{2}$	$\frac{1}{2} < t < 1$	$t = 1$
$s = 0$	$(0,0)$	Edge	$(\frac{1}{2}, 0)$	Edge	$(1,0)$
$0 < s < \frac{1}{2}$	Edge	Square $\subseteq \mathfrak{U}_{0,0}$	Edge	Square $\subseteq \mathfrak{U}_{1,0}$	Edge
$s = \frac{1}{2}$	$(0, \frac{1}{2})$	Edge	$(\frac{1}{2}, \frac{1}{2})$	Edge	$(1, \frac{1}{2})$
$\frac{1}{2} < s < 1$	Edge	Square $\subseteq \mathfrak{U}_{0,1}$	Edge	Square $\subseteq \mathfrak{U}_{1,1}$	Edge
$s = 1$	$(0,1)$	Edge	$(\frac{1}{2}, 1)$	Edge	$(1,1)$

(4) Construct the following finite list of loops and path homotopies in X_1 or X_2:

$$\begin{aligned}
\gamma_0 &= H \circ [(0,0) > \text{Edge} > (1/2,0) > \text{Edge} > (1,0)] \\
H_0 &= [\text{The path homotopy from } \gamma_0 \text{ to } \gamma_1 \text{ induced by } H] \\
\gamma_1 &= H \circ [(0,1/2) > \text{Edge} > (1/2,1/2) > \text{Edge} > (1/2,0) > \text{Edge} > (1,0)] \\
H_1 &= [\text{The path homotopy from } \gamma_1 \text{ to } \gamma_2 \text{ induced by } H] \\
\gamma_2 &= H \circ [(0,1/2) > \text{Edge} > (1/2,1/2) > \text{Edge} > (1,1/2)] \\
H_2 &= [\text{The path homotopy from } \gamma_2 \text{ to } \gamma_3 \text{ induced by } H] \\
\gamma_3 &= H \circ [(0,1) > \text{Edge} > (1/2,1) > \text{Edge} > (1/2,1/2) > \text{Edge} > (1,1/2)] \\
H_3 &= [\text{The path homotopy from } \gamma_3 \text{ to } \gamma_4 \text{ induced by } H] \\
\gamma_4 &= H \circ [(0,1) > \text{Edge} > (1/2,1) > \text{Edge} > (1,1)]
\end{aligned}$$

Hence, $H = H_0 \star_{c_1} H_1 \star_{c_2} H_2 \star_{c_3} H_3$, where each H_k is contained in X_1 or X_2.
To conclude, for all loops $[\![\mu]\!], [\![\nu]\!] \in \pi_1(X) *_{\pi_1(S)} \pi_1(X_2)$, $[\![\mu]\!], [\![\nu]\!]$ are identified upstairs in $\pi_1(Y)$ implies they are identified downstairs in $\pi_1(X) *_{\pi_1(S)} \pi_1(X_2)$, so $\mathbf{g}$ is injective. We end up with the desired group isomorphism $\mathbf{g}$. Quod. Erat. Demonstrandum. $\square$

4.3 Applications

Proposition 4.29.

$$\pi_1(\mathbb{S} \vee_1 \mathbb{S}) \cong \mathbb{Z} * \mathbb{Z}$$

Proof. Define the following four topological spaces:

$$
\begin{aligned}
Y &= \mathbb{S} \vee_1 \mathbb{S} \\
X_1 &= \mathbb{S} \vee_1 (\mathbb{S} \backslash \{-1\}) \\
X_2 &= (\mathbb{S} \backslash \{-1\}) \vee_1 \mathbb{S} \\
S &= (\mathbb{S} \backslash \{-1\}) \vee_1 (\mathbb{S} \backslash \{-1\})
\end{aligned}
$$

(1) Set-theoretically, $Y = X_1 \cup X_2$.
(2) The preimages $\mathbb{S}, \mathbb{S} \backslash \{-1\}$ of $X_1 = \mathbb{S} \vee_1 (\mathbb{S} \backslash \{-1\})$ under natural projections are open in $\mathbb{S}, \mathbb{S}$ respectively, so X_1 is open in the final topological space $Y = \mathbb{S} \vee_1 \mathbb{S}$.
(3) The preimages $\mathbb{S} \backslash \{-1\}, \mathbb{S}$ of $X_2 = (\mathbb{S} \backslash \{-1\}) \vee_1 \mathbb{S}$ under natural projections are open in $\mathbb{S}, \mathbb{S}$ respectively, so X_2 is open in the final topological space $Y = \mathbb{S} \vee_1 \mathbb{S}$.
(4) X_1, X_2, S are wedge sums of path connected topological spaces, so X_1, X_2, S are path connected. And, $S \ni 1$ is nonempty.
The four parts above suggests the following:

$$
\pi_1(\mathbb{S} \vee_1 \mathbb{S}) = \pi_1(Y) \cong \pi_1(X_1) *_{\pi_1(S)} \pi_1(X_2) \cong \pi_1(\mathbb{S}) *_{\pi_1(\{1\})} \pi_1(\mathbb{S}) \cong \mathbb{Z} *_{\{e\}} \mathbb{Z} \cong \mathbb{Z} * \mathbb{Z}
$$

Quod. Erat. Demonstrandum. □

Proposition 4.30.

$$
\pi_1(\mathbb{R}^3 \backslash \mathbb{S}) \cong \mathbb{Z}
$$

Proof. We may divide our proof into two steps.
Step 1: We construct a deformation retraction to simplify $\mathbb{R}^3 \backslash \mathbb{S}$.
It suffices to graph the flow lines of the following vector field:

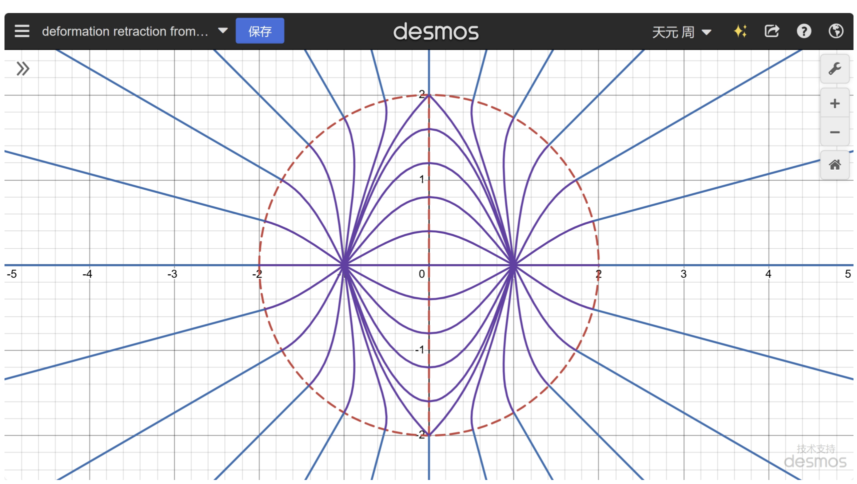

Figure 4: Deformation Retraction From $\mathbb{R}^3 \backslash \mathbb{S}$ To $\mathbb{S}^2 \vee_{\pm 1} [-1, +1]$

Step 2: Retract the line segment inside to a circle attached to the sphere outside, so:

$$\mathbb{R}^3 \backslash \mathbb{S} \cong \mathbb{S}^2 \vee_{\pm 1} [-1, +1] \cong \mathbb{S}^2 \vee_1 \mathbb{S}$$

Define the following four topological spaces:

$$\begin{aligned} Y &= \mathbb{S}^2 \vee_1 \mathbb{S} \\ X_1 &= \mathbb{S}^2 \vee_1 (\mathbb{S}\backslash\{-1\}) \\ X_2 &= (\mathbb{S}^2\backslash\{-1\}) \vee_1 \mathbb{S} \\ S &= (\mathbb{S}^2\backslash\{-1\}) \vee_1 (\mathbb{S}\backslash\{-1\}) \end{aligned}$$

(1) Set-theoretically, $Y = X_1 \cup X_2$.
(2) The preimages $\mathbb{S}^2, \mathbb{S}\backslash\{-1\}$ of $X_1 = \mathbb{S}^2 \vee_1 (\mathbb{S}\backslash\{-1\})$ under natural projections are open in $\mathbb{S}^2, \mathbb{S}$ respectively, so X_1 is open in the final topological space $Y = \mathbb{S}^2 \vee_1 \mathbb{S}$.
(3) The preimages $\mathbb{S}^2\backslash\{-1\}, \mathbb{S}$ of $X_2 = (\mathbb{S}^2\backslash\{-1\}) \vee_1 \mathbb{S}$ under natural projections are open in $\mathbb{S}^2, \mathbb{S}$ respectively, so X_2 is open in the final topological space $Y = \mathbb{S}^2 \vee_1 \mathbb{S}$.
(4) X_1, X_2, S are wedge sums of path connected topological spaces, so X_1, X_2, S are path connected. And, $S \ni 1$ is nonempty.
The four parts above suggests the following:

$$\pi_1(\mathbb{R}^3\backslash\mathbb{S}) \cong \pi_1(Y) \cong \pi_1(X_1) *_{\pi_1(S)} \pi_1(X_2) \cong \pi_1(\mathbb{S}^2) *_{\pi_1(\{1\})} \pi_1(\mathbb{S}) \cong \{e\} *_{\{e\}} \mathbb{Z} \cong \mathbb{Z}$$

Quod. Erat. Demonstrandum. □

Lemma 4.31.

$$\begin{aligned} \mathbb{T}^2\backslash\overline{\mathbb{D}^2} &\sim \mathbb{S} \vee_1 \mathbb{S}, & \pi_1(\mathbb{T}^2\backslash\overline{\mathbb{D}^2}) &\cong \mathbb{Z} * \mathbb{Z} \\ \mathbb{RP}^2\backslash\overline{\mathbb{D}^2} &\sim \mathbb{S}, & \pi_1(\mathbb{RP}^2\backslash\overline{\mathbb{D}^2}) &\cong \mathbb{Z} \\ \mathbb{K}^2\backslash\overline{\mathbb{D}^2} &\sim \mathbb{S} \vee_1 \mathbb{S}, & \pi_1(\mathbb{K}^2\backslash\overline{\mathbb{D}^2}) &\cong \mathbb{Z} * \mathbb{Z} \end{aligned}$$

Proof. If we remove $\overline{\mathbb{D}^2}$ with radius $r < 1$ from the centre, then $|z|^{-t}$ is continuous, so the deformation retraction $H(z, t) = |z|^{-t} z$ is well-defined. Quod. Erat. Demonstrandum. □

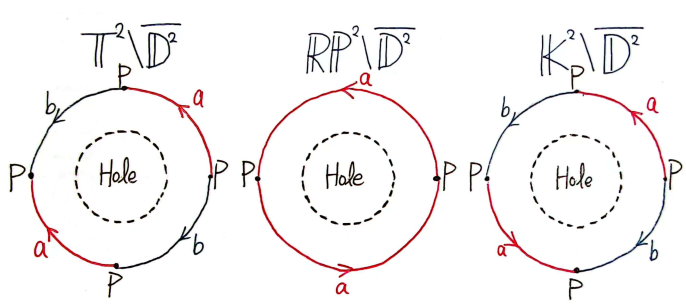

Figure 5: $\mathbb{T}^2\backslash\overline{\mathbb{D}^2}, \mathbb{RP}^2\backslash\overline{\mathbb{D}^2}, \mathbb{K}^2\backslash\overline{\mathbb{D}^2}$

Proposition 4.32.

$$\pi_1(\mathbb{T}^2\#\mathbb{T}^2) \cong \langle s_1, t_1, s_2, t_2 | s_1^{-1}t_1^{-1}s_1t_1 = t_2^{-1}s_2^{-1}t_2s_2\rangle$$

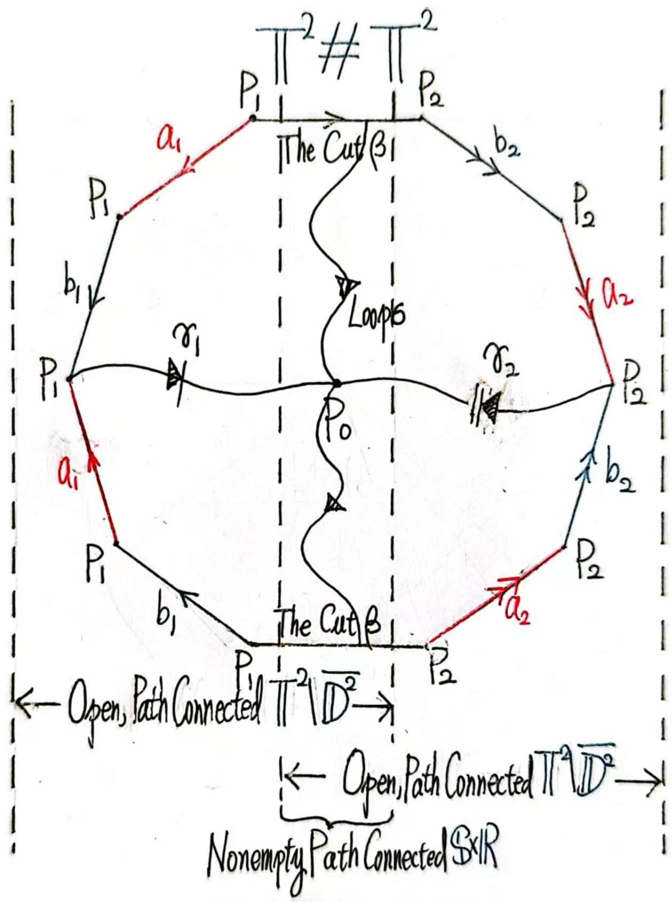

Figure 6: $\mathbb{T}^2\#\mathbb{T}^2$

Proof.

$$\begin{aligned}
\pi_1(\mathbb{T}^2\#\mathbb{T}^2) &\cong \pi_1(\mathbb{T}^2\backslash\overline{\mathbb{D}^2}) *_{\pi_1(\mathbb{S}\times\mathbb{R})} \pi_1(\mathbb{T}^2\backslash\overline{\mathbb{D}^2}) \\
&\cong \langle \gamma_1^{-1}a_1\gamma_1, \gamma_1^{-1}b_1\gamma_1\rangle *_{\langle\sigma\rangle} \langle \gamma_2^{-1}a_2\gamma_2, \gamma_2^{-1}b_2\gamma_2\rangle \\
&\cong \left\langle \begin{matrix} \gamma_1^{-1}a_1\gamma_1, \gamma_1^{-1}b_1\gamma_1 \\ \gamma_2^{-1}a_2\gamma_2, \gamma_2^{-1}b_2\gamma_2 \end{matrix} \middle| \gamma_1^{-1}a_1^{-1}b_1^{-1}a_1b_1\gamma_1 = \gamma_2^{-1}b_2^{-1}a_2^{-1}b_2a_2\gamma_2 \right\rangle \\
&\cong \langle s_1, t_1, s_2, t_2 | s_1^{-1}t_1^{-1}s_1t_1 = t_2^{-1}s_2^{-1}t_2s_2\rangle
\end{aligned}$$

Quod. Erat. Demonstrandum. □

Proposition 4.33.

$$\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) \cong \langle s_1, s_2 | s_1 = s_2 s_1^{-1} s_2 \rangle$$

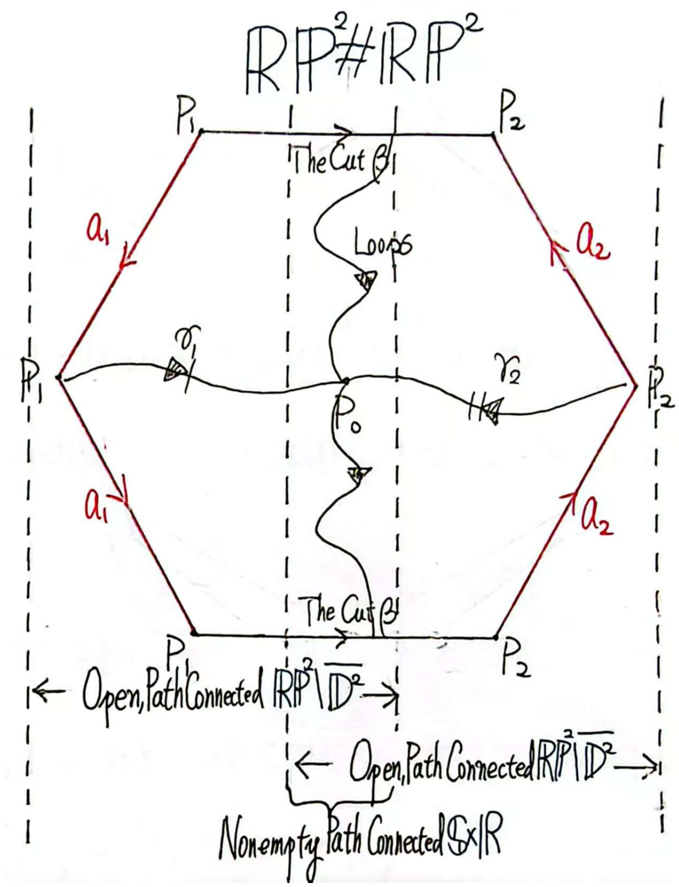

Figure 7: $\mathbb{RP}^2\#\mathbb{RP}^2$

Proof.

$$\begin{aligned}
\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2) &\cong \pi_1(\mathbb{RP}^2\backslash\overline{\mathbb{D}^2}) *_{\pi_1(\mathbb{S}\times\mathbb{R})} \pi_1(\mathbb{RP}^2\backslash\overline{\mathbb{D}^2}) \\
&\cong \langle \gamma_1^{-1} a_1 \gamma_1 \rangle *_{\langle\sigma\rangle} \langle a_2 \rangle \\
&\cong \langle \gamma_1^{-1} a_1 \gamma_1, \gamma_2^{-1} a_2 \gamma_2 | \gamma_1^{-1} a_1^2 \gamma_1 = \gamma_2^{-1} a_2^{-2} \gamma_2 \rangle \\
&\cong \langle a_1, \gamma_1 \gamma_2^{-1} a_2 \gamma_2 \gamma_1^{-1} | a_1^2 = \gamma_1 \gamma_2^{-1} a_2^{-2} \gamma_2 \gamma_1^{-1} \rangle \\
&\cong \langle a_1, \gamma_1 \gamma_2^{-1} \beta^{-1} a_1^{-1} | a_1 = \gamma_1 \gamma_2^{-1} \beta^{-1} a_1^{-1} \gamma_1 \gamma_2^{-1} \beta^{-1} \rangle \\
&\cong \langle s_1, s_2 | s_1 = s_2 s_1^{-1} s_2 \rangle
\end{aligned}$$

Quod. Erat. Demonstrandum. □

Remark: *For 2-dimensional topological manifolds, the orientation of gluing doesn't make any difference, but the proof is not required.*

References

[1] H. Ren, "Template for math notes," 2021.

[2] M. Brittenham, "Lifting properties," 2 2007. [Online]. Available: https://www.math.unl.edu/~mbrittenham2/classwk/872s07/lecnotes/slides.from.class/cls15--lifting.properties.pdf

[3] Chiasme, "An elementary application of ping pong lemma," 2015. [Online]. Available: https://chiasme.wordpress.com/2015/03/08/an-elementary-application-of-ping-pong-lemma/

[4] A. Hatcher, *Algebraic Topology.* Cambridge: Cambridge University Press, 2002.