Algebra II: Tutorial 4

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Problem 1 (Recap on irreducibility)**.** Let $f(x) = x^4 - 2x^2 + 9 \in \mathbb{Q}[x]$.

1. Show that $\pi_p(f(x))$ is reducible for $p = 2, 3, 5, 7$.
2. Show that $f(x)$ is irreducible over $\mathbb{Q}$.

Solution. 1. These are a direct computation. $\pi_2(f) = x^4+1 = (x+1)^4$, $\pi_3(f) = x^4-2x^2 = x^2(x^2+1)$, $\pi_5(f) = x^4+3x^2+4 = (x^2+x+2)(x^2+4x+2)$, and $\pi_7(f) = x^4+5x^2+2 = (x^2+x+3)(x^2+6x+3)$.
2. If f has a proper factorisation containing a linear factor, then f must have a root equal to $\pm 1, \pm 3, \pm 9$. A direct check shows none of these are roots of f. Therefore, f must have a factorisation into two quadratics, say $f = (x^2+ax+b)(x^2+cx+d)$. If $a = 0$, then $c = 0$ and the pair of integers (b, d) must solve the system $b + d = -2\text{m}\ bd = 9$ - this system has no integer solutions. Hence, assume that $a \neq 0$. Then, $d = b = 3$, and $a^2 = 8$, which is again a contradiction. Hence, f has no proper factorisation over $\mathbb{Z}$, and by Gauss's lemma has no proper factorisation over $\mathbb{Q}$.

Problem 2. Suppose that $a, b \in \mathbb{C}$ have same minimal polynomial over $\mathbb{Q}[x]$. Show that $\mathbb{Q}(a) \cong \mathbb{Q}(b)$.

Solution. Denote by $m(x) \in \mathbb{Q}[x]$ the minimal polynomial of a and b. By a theorem in the notes, $\mathbb{Q}(a) \cong \mathbb{Q}[x]/(m(x)) \cong \mathbb{Q}(b)$, proving our claim.

Problem 3. Let p, q be two distinct prime numbers. Calculate $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}]$.

Solution. By the tower theorem, $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}(\sqrt{q})][\mathbb{Q}(\sqrt{q}) : \mathbb{Q}]$. We know that $[\mathbb{Q}(\sqrt{q}) : \mathbb{Q}]$ is equal to the degree of the minimal polynomial of $\sqrt{q}$ over $\mathbb{Q}$. Since $x^2 - q$ is irreducible over $\mathbb{Q}$ with root $\sqrt{q}$, $[\mathbb{Q}(\sqrt{q}) : \mathbb{Q}] = 2$. Similarly, $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}(\sqrt{q})]$ is equal to the degree of the minimal polynomial of $\sqrt{p}$ over $\mathbb{Q}(\sqrt{q})$. It is clear this is at most two. Suppose that the extension has degree one, i.e. $\sqrt{p} \in \mathbb{Q}(\sqrt{q})$. Then, there exist $a, b \in \mathbb{Q}$ such that $\sqrt{p} = a + b\sqrt{q}$, with $b \neq 0$. Squaring both sides and re-arranging, we get that $p - a^2 - b^2 q = 2ab\sqrt{q}$. The left-hand side is rational but $\sqrt{q}$ is irrational; this implies that $a = 0$. In other words, $\sqrt{q} = b\sqrt{q}$; this is a contradiction. Hence, the extension $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}(\sqrt{q})]$ is strictly larger than 1 and smaller than or equal to 2. All in all, $[\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}] = 2 \cdot 2 = 4$.

Problem 4. Suppose that α, β are transcendental over $\mathbb{Q}$. Show that either $\alpha + \beta$ or $\alpha\beta$ is transcendental over $\mathbb{Q}$.

Solution. Suppose not, say $\alpha + \beta$ and $\alpha\beta$ are algebraic. We know that simple algebraic extensions are finite; therefore $[\mathbb{Q}(\alpha + \beta) : \mathbb{Q}] < \infty$ and $[\mathbb{Q}(\alpha\beta) : \mathbb{Q}] < \infty$. By the tower theorem, $[\mathbb{Q}(\alpha\beta, \alpha+\beta) : \mathbb{Q}]$ is equal to the product $[\mathbb{Q}(\alpha\beta, \alpha+\beta) : \mathbb{Q}(\alpha+\beta)][\mathbb{Q}(\alpha+\beta) : \mathbb{Q}]$, which is also finite. Now, notice that $(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$, and so α is algebraic over $\mathbb{Q}(\alpha + \beta, \alpha\beta)$. Again by appealing to the Tower theorem, this implies that $\mathbb{Q}(\alpha)$ is a finite extension of $\mathbb{Q}$, a contradiction.