Normal Extensions

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In this file

1. §3.2.5: Normal extensions.

Normal Extensions.

$L = K[x]/\langle p(x) \rangle$

Definition. An algebraic field extension $K \subset L$ is said to be normal if every irreducible polynomial in $K[x]$ that has a root in L splits over L.
completely

Examples.

- K is a normal extension of itself.
- $\mathbb{Q}[\sqrt[3]{2}]$ is NOT a normal extension of $\mathbb{Q}$.

$f(x) = x^3 - 2$

Lemma. A finite normal extension of K must be a splitting field of some $f(x) \in K[x]$.

Proof. Let $a_1, \ldots, a_n$ be a basis of L over K.

- Each a_i is algebraic over K. Let $p_i \in K[x]$ be the minimal polynomial of a_i over K.
- Let $f = p_1 \cdots p_n$. By assumption, each p_i completely splits over L, so f completely splits over L.
- Let R be the set of all roots of f in L. Then
$$\{a_1, \ldots, a_n\} \subset R.$$
- $L = K(a_1, \ldots, a_n) \subset K(R) \subset L$, so $L = K(R)$.
- By definition, L is a splitting field of f over K.

Q.E.D.

Theorem. Any splitting field over K is $\boxed{\text{a finite normal extension.}}$

Proof. Let L be a splitting field of $f(x) \in K[x]$, and let $p(x) \in K[x]$ be irreducible. Let $\alpha \in L$ be a root of p.

- Let M be a splitting field of $g(x) = f(x)p(x)$ over K. Then both f and p completely split over M. Indeed, as $f(x)p(x)$ is a product of linear factors in $M[x]$, uniqueness of prime factorization implies that both f and p are products of linear factors in $M[x]$.
- By Extension Lemma, can identify $K \subset L \subset M$.
- Want to prove all roots of p in M are in L.
- Let $\beta \in M$ be any root of p. Want to show that $\beta \in L$.

WTS $\text{split}(p(x)) \subseteq \text{split}(f(x))$

Cond.:

$$\begin{tikzcd} & f(x)p(x) & \\ f(x) \arrow[ur] & & p(x) \arrow[ul] \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof cont'd:

- Both $\alpha \in L$ and $\beta \in M$ are roots of p, so have isomorphism
$$\varphi: \ K(\alpha) \longrightarrow K[x]/\langle p(x)\rangle \longrightarrow K(\beta) \subset M$$
with $\varphi|_K = \mathrm{Id}$ and $\varphi(\alpha) = \beta$.

- L is a splitting field of f over $K(\alpha)$. By Extension Lemma again, $\exists\ \tilde{\varphi} : L \to M$ such that
$$\tilde{\varphi}|_{K(\alpha)} = \varphi : \ K(\alpha) \longrightarrow K(\beta).$$
So $\tilde{\varphi}|_K = \mathrm{Id}$ and $\tilde{\varphi}(\alpha) = \varphi(\alpha) = \beta$.

- Now $K \subset M$ is extended to two embeddings $L \subset M$ and $\tilde{\varphi} : L \to M$. Extension Lemma implies that $\tilde{\varphi}(L) = L$, so
$$\beta = \tilde{\varphi}(\alpha) \in L.$$

- Thus all roots of p in M are in L.

Q.E.D.

Conclusion.

Theorem

Splitting fields over $K \Leftrightarrow$ finite and normal extensions of K.

Example. $\mathbb{Q}[\sqrt[3]{2}]$ is not the splitting field of any $f \in \mathbb{Q}[x]$.