20241105 MATH3541 Assignment 5 Part B

Problem 9.

(a) Proof: Recall that the closed upper-half space of $\mathbb{R}^n$ is:

$$\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

A manifold M with boundary is a second countable Hausdorff topological space M locally homeomorphic to $\mathbb{H}^n$.

For all $m \in \operatorname{Int} M$, there exists an interior chart (U, ϕ), where the open subset U of M contains m, $\phi: U \to \mathbb{H}^n$ is a function, $\phi(U)$ is open in $\operatorname{Int} \mathbb{H}^n$ and the restriction $\phi: U \to \phi(U)$ is a homeomorphism. We wish to prove that $U \subseteq \operatorname{Int} M$.

For all $m' \in U$, there exists an interior chart $(U', \phi') = (U, \phi)$, where the open subset $U' = U$ of M contains m', $\phi': U' \to \mathbb{H}^n$ is a function, $\phi'(U') = \phi(U)$ is open in $\operatorname{Int} \mathbb{H}^n$ and the restriction $\phi': U' \to \phi'(U')$ is a homeomorphism. This implies $m' \in \operatorname{Int} M$ and $U \subseteq \operatorname{Int} M$.

Now we've found $U \in \mathcal{O}_M$ with $U \subseteq \operatorname{Int} M$, such that $m \in U$, which implies $m \in (\operatorname{Int} M)^\circ$, i.e., $\operatorname{Int} M \in \mathcal{O}_M$.

The reason why $\operatorname{Int} M$ is n-dimensional is that $\phi(U)$ is open in $\operatorname{Int} \mathbb{H}^n$ and $\operatorname{Int} \mathbb{H}^n$ is open in $\mathbb{R}^n$ imply $\phi(U)$ is open in $\mathbb{R}^n$, and this certainly suggests $\operatorname{Int} M$ has no boundary.

Second countability and Hausdorffness are inherited.

(b) Proof:

In our lecture note, we assume without proof that $\mathrm{Int}M \cap \partial M = \phi$

Hence, $\mathrm{Int}M \in \mathcal{O}_M$ implies $\partial M = (\mathrm{Int}M)^c \in \mathcal{C}_M$

For all $m \in \partial M$, there exists a boundary chart (U, ϕ), where the open subset U of M contains m, $\phi: U \to \mathbb{H}^n$ is a function, $\phi(U)$ is open in $\mathbb{H}^n$, $\phi(U) \cap \partial\mathbb{H}^n \neq \phi$ and the restriction $\phi: U \to \phi(U)$ is a homeomorphism.

Notice that $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}$, so there exists an interior chart (V, ψ), where the open subset $V = U \cap \partial M$ of ∂M contains m, $\psi: V \to \partial\mathbb{H}^n$, $\psi(m') = \phi(m')$ is a function, $\psi(V) = \varphi(U \cap \partial M) = \varphi(U) \cap \varphi(\partial M) = \varphi(U) \cap \partial\mathbb{H}^n$ is open in $\partial\mathbb{H}^n$ and the restriction $\psi: V \to \psi(V)$ is a homeomorphism.

Here $\phi = \varphi$, forgive my bad notation ☺

The reason why ∂M is $(n-1)$-dimensional is that $\psi(U)$ is open in $\partial\mathbb{H}^n$ and $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$, and this certainly suggests ∂M has no boundary.

Second countability and Hausdorffness are inherited.

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Problem 10

(a) Proof: Assume to the contrary that $\forall e^{i\theta} \in \mathbb{S}^1, f(e^{i\theta}) \neq -e^{i\theta}$.

This implies $\forall e^{i\theta} \in \mathbb{S}^1, |f(e^{i\theta}) + e^{i\theta}| > 0$.

We know further that $\forall (e^{i\theta}, t) \in \mathbb{S}^1 \times [0,1], |(1-t)f(e^{i\theta}) + te^{i\theta}| > 0$.

Hence, the following continuous function is well-defined.

$$H: \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1, H(e^{i\theta}, t) = \frac{(1-t)f(e^{i\theta}) + te^{i\theta}}{|(1-t)f(e^{i\theta}) + te^{i\theta}|}$$

To conclude, $f(e^{i\theta}) = H(e^{i\theta}, 0) \sim H(e^{i\theta}, 1) = e(e^{i\theta})$, where e is the identity map.

(b) Proof: Construct the following continuous function:

$$H: \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1, H(e^{i\theta}, t) = e^{i(\theta + \pi t)}$$

To conclude, $e(e^{i\theta}) = H(e^{i\theta}, 0) \sim H(e^{i\theta}, 1) = f(e^{i\theta})$, where e is the identity map.

(c) Proof: I proposed this approach in class.

$$\begin{tikzcd}
f: X \to \mathbb{S}^n \text{ misses some point } \vec{y} \in \mathbb{S}^n \arrow[rr, "\text{Sterographic Projection}"] \arrow[d, "\text{Homotopy } H: X\times[0,1]\to X"'] & & f': X \to \mathbb{R}^n \arrow[d, "\text{Homotopy } H'(x,t)=(1-t)f(x)"] \\
\text{A Constant Map } x \mapsto -\vec{y} & & \text{A Constant Map } x \mapsto \vec{0}
\end{tikzcd}$$

Inverse Sterographic Projection: Homotopy $H'(x,t) = (1-t)f(x)$ $\to$ Homotopy $H: X\times[0,1]\to X$

Problem 11.

(a) Proof: We may divide our proof into two parts.

"if" direction: Assume that every continuous maps $f, f' : [0,1] \to X$ are homotopic to each other.

For all $x, x' \in X$, we wish to find a path $\gamma : [0,1] \to X$ from x to x'.

Construct two continuous functions:

$$f : [0,1] \to X,\ t \mapsto x \ ;\ f' : [0,1] \to X,\ t \mapsto x'$$

According to our assumption, $f \sim f'$, so there exists a continuous function $H : [0,1]^2 \to X$, such that for all $t \in [0,1]$:

$$H(t,0) = f(t) = x \text{ and } H(t,1) = f'(t) = x'$$

Now, $\gamma : [0,1] \to X$, $\gamma(s) = H(0,s)$ will be our desired path, and we've proven that X is path-connected.

"only if" direction: Assume that X is path-connected.

For every continuous maps $f, f' : [0,1] \to X$, we wish to find a continuous function $H : [0,1]^2 \to X$, such that for all $t \in [0,1]$:

$$H(t,0) = f(t) \text{ and } H(t,1) = f'(t)$$

We construct three such functions and glue them together.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
f : [0,1] \to X \arrow[r, "\text{Homotopy}\ H = H_1 * H_2 * H_3"] \arrow[d, "\text{Homotopy}\ H_1(t,s_1) = f(t(1-s_1))"'] & f' : [0,1] \to X \\
\text{A Constant Map } t \mapsto f(0) \arrow[r, "\text{Path } \gamma:[0,1]\to X \text{ From } f(0) \text{ To } f'(0)", "\text{Homotopy}\ H_2(t,s_2) = \gamma(s_2)"'] & \text{A Constant Map } t \mapsto f'(0) \arrow[u, "\text{Homotopy}\ H_3(t,s_3) = f(ts_3)"']
\end{tikzcd}$$

Combine the two parts together, we've proven the biconditional.

(b) Proof: For all topological spaces X_1, X_2, assume that $X_1 \simeq X_2$.

X_1 is path connected $\Leftrightarrow$ Every continuous maps $f_1, f_1' : [0,1] \to X_1$ are homotopic to each other.

$$\sigma^{-1} \Big\uparrow \Big\downarrow \sigma$$

$\Leftrightarrow$ Every continuous maps $f_2, f_2' : [0,1] \to X_2$ are homotopic to each other

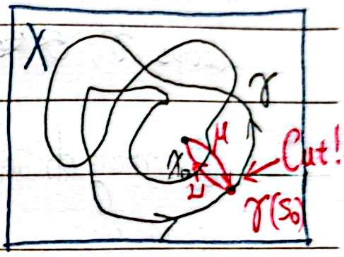
$\Leftrightarrow$ X_2 is path connected.

Hence, path connectedness is invariant under homotopy.

Problem 12:

(a) Proof: For all class $[\gamma] \in [S^1, X]$, we wish to find $[\sigma]_{x_0} \in \pi_1(X, x_0)$, such that $\phi([\sigma]_{x_0}) = [\gamma]$.

Step 1: As X is path-connected, construct a path μ from x_0 to $\gamma(s_0)$ and a path ν from $\gamma(s_0)$ to x_0.



Step 2: Define a loop σ by:

$$\sigma : (S^1, s_0) \to (X, x_0),\ \sigma(s_0 e^{i\theta}) = \begin{cases} \mu(\frac{2}{\pi}\theta), & \text{if } 0 \le \theta \le \frac{\pi}{2}; \\ \gamma(-s_0 e^{2i\theta}), & \text{if } \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}; \\ \nu(\frac{2}{\pi}\theta - 3), & \text{if } \frac{3\pi}{2} \le \theta \le 2\pi; \end{cases}$$

The gluing lemma suggests that σ is well-defined and $[\sigma]_{x_0} \in \pi_1(X, x_0)$.

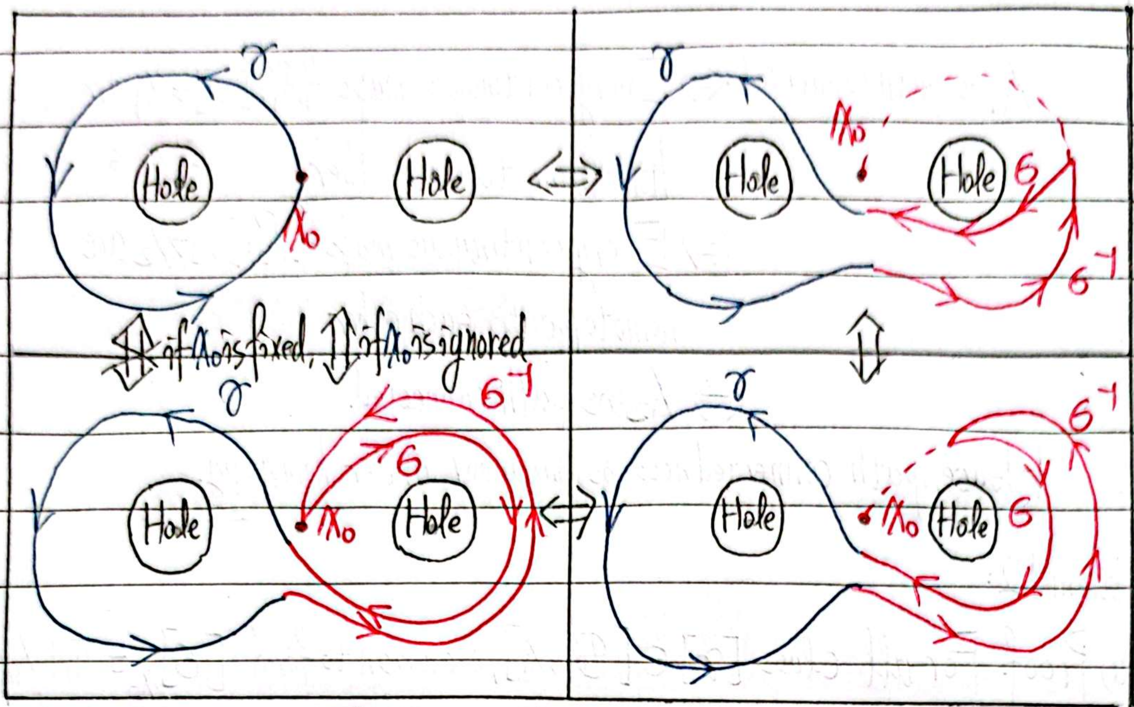
Step 3: Define a homotopy H by:

$$H : S^1 \times [0,1] \to X,\ H(s_0 e^{i\theta}, t) = \begin{cases} \mu(1 - t + \frac{2}{\pi}\theta), & \text{if } 0 \le \theta \le \frac{\pi t}{2}; \\ \gamma(s_0 e^{\frac{2\theta - \pi t}{2-t} i}), & \text{if } \frac{\pi t}{2} \le \theta \le \frac{(4-t)\pi}{2}; \\ \nu(\frac{2}{\pi}\theta - 4 + t), & \text{if } \frac{(4-t)\pi}{2} \le \theta \le 2\pi; \end{cases}$$

The gluing lemma suggests that H is well-defind and continuous, so, $\gamma(s_0 e^{i\theta}) = H(s_0 e^{i\theta}, 0) \sim H(s_0 e^{i\theta}, 1) = \sigma(s_0 e^{i\theta})$.

To conclude, $[\gamma] = [\sigma] = \phi([\sigma]_{x_0})$, ϕ is surjective.

(b) Proof: The fact we want to describe is as follows:



With the basepoint x_0 fixed, the two loops γ, $\sigma\star\gamma\star\sigma^{-1}$ are not homotopic. After ignoring the basepoint x_0, it is possible to "release the band".

We may divide our proof into two parts.

"if" direction: Assume that $[\gamma]_{x_0}$, $[\sigma\star\gamma\star\sigma^{-1}]_{x_0}$ are conjugate in $\pi_1(X,x_0)$.

Define a homotopy H by:

$$H: S^1\times[0,1]\to X,\ H(s_0e^{i\theta},t)=\begin{cases}\sigma\left(s_0e^{2\pi i(1-t+\frac{2}{\pi}\theta)}\right), & \text{if } 0\le\theta\le\frac{\pi t}{2};\\ \gamma\left(s_0e^{\frac{2\theta-\pi t}{2-t}i}\right), & \text{if } \frac{\pi t}{2}\le\theta\le\frac{(4-t)\pi}{2};\\ \sigma^{-1}\left(s_0e^{2\pi i(\frac{2}{\pi}\theta-4+t)}\right), & \text{if } \frac{(4-t)\pi}{2}\le\theta\le 2\pi;\end{cases}$$

The gluing lemma suggests that H is well-defined and continuous,

so $\gamma(s_0e^{i\theta})=H(s_0e^{i\theta},0)\sim H(s_0e^{i\theta},1)=\sigma\star\gamma\star\sigma^{-1}(s_0e^{i\theta})$,

which implies $\phi([\gamma]_{x_0})=[\gamma]=[\sigma\star\gamma\star\sigma^{-1}]=\phi([\sigma\star\gamma\star\sigma^{-1}]_{x_0})$

"only if" direction: Assume that $\phi([\gamma]_{x_0})=\phi([\gamma']_{x_0})$,

i.e., $[\gamma]=[\gamma']$, there exists a continuous function $H:[0,1]^2\to X$, such that for all $s_0e^{i\theta}\in S^1$

$$H(s_0e^{i\theta},0)=\gamma(s_0e^{i\theta}) \quad\text{and}\quad H(s_0e^{i\theta},1)=\gamma'(s_0e^{i\theta})$$

Define a loop σ by:

$$\sigma:(\mathbb{S}^1,s_0)\to(X,x_0),\ \sigma(s_0e^{i\theta})=H(s_0,\tfrac{\theta}{2\pi})$$

As $H(s_0,0)=\gamma(s_0)=x_0=\gamma'(s_0)=H(s_0,1)$, σ is well-defined and $[\sigma]_{x_0}\in\pi_1(X,x_0)$

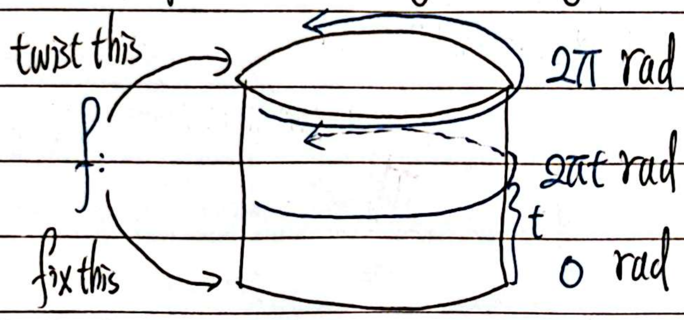
Notice that $\gamma'\sim_{x_0}\sigma\star\gamma\star\sigma^{-1}$, so $[\gamma]_{x_0},[\gamma']_{x_0}$ are conjugate in $\pi_1(X,x_0)$.

The reason why $\gamma'\sim_{x_0}\sigma\star\gamma\star\sigma^{-1}$ is similar to what we've done in "if" direction.

Combine the two parts above, we've proven the biconditional.

Problem 13:

(a) Solution: The map f is obtained by "twisting a cola bottle at one end"



Define a homotopy H by:

$H:\mathbb{S}^1\times[0,1]^2\to\mathbb{S}^1\times[0,1],\ H(e^{i\theta},t,s)=(e^{i(\theta+2\pi st)},t)$

This map is well-defined and continuous,

so $e(e^{i\theta},t)=H(e^{i\theta},t,0)\sim H(e^{i\theta},t,1)=f(e^{i\theta},t)$,

where e is the identity map.

(c) Proof: Assume to the contrary that there exists some homotopy $H:\mathbb{S}^1\times[0,1]^2\to\mathbb{S}^1\times[0,1]$ from e to f, fixing both boundary circles.

Consider the restriction maps $e(1,\cdot)$, $f(1,\cdot)$, $H(1,\cdot,\cdot)$ is also a homotopy from $e(1,\cdot)$ to $f(1,\cdot)$. Do vertical projection homotopy $\pi(e^{i\theta},t,s)=(e^{i\theta},(1-s)t)$, we get a retraction from $\mathbb{S}^1$ to $\{1\}$, which is a contradiction because $\mathbb{S}^1$ is not contractible. Hence, such H fails to exist.