Topological manifold

Def'n A top space X is called 2nd countable if it admit a countable base.

- subspace of 2nd countable space is 2nd countable

- $\mathbb{R}^n$ is 2nd countable

$$B(x, \varepsilon) \qquad x \in \mathbb{Q}^n \quad \varepsilon \in \mathbb{Q}_+$$

Def'n A top. mfd M of dimension $n \in \mathbb{N}$ is a top space that is

1) Hausdorff
2) 2nd countable
3) locally homeomorphic to $\mathbb{R}^n$, i.e

$\forall x \in M$ $\exists$ open nbhd $\mathcal{U}$ of x and a local homeomorphism $\phi : \mathcal{U} \longrightarrow \mathbb{R}^n$.

We call $(\mathcal{U}, \phi)$ a chart of M. An atlas is a collection of charts $(\mathcal{U}_\alpha, \phi_\alpha)$ s.t. $M = \bigcup \mathcal{U}_\alpha$

Rmk M : top mfd of dim n

1) M' another top mfd s.t $M' \cong M$

then $\dim M' = n$

2) $\mathcal{U} \overset{\text{open}}{\subset} M$ then $\mathcal{U}$ is also a top mfd of dim n.

3) to check M is a top mfd it suffices to construct a countable atlas

4) $f : \mathbb{R}^n \to \mathbb{R}$ continuous

$\Gamma_f = \{(x, f(x))\} \subset \mathbb{R}^n \times \mathbb{R}$ is a top mfd of dim n.

5) S^n is a top mfd of dim n. atlas

$\mathbb{R}P^n$ ..... dim n

6) top mfd are locally compact.

$\mathbb{H}^n = \{(x_1 \dots x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$

A top. mfd with boundary of dim n

is a Hausdorff and countable space

that is locally homeomorphic to $\mathbb{H}_n$.

A pt $p \in M$ is called an interior pt

if $\exists$ nbhd U of p and $\phi: U \to \mathbb{H}^n$

a homeomorphism onto the image s.t $\phi(p) \in \mathring{\mathbb{H}}^n$

... boundary pt if ... $\phi(p) \in \partial \mathbb{H}^n$

! boundary pt $\neq$ interior pt

this is not easy to prove

lemma (Ex)

$\operatorname{Int} M$ = set of interior pts

∂M = .. boundary pts

$M = \operatorname{Int} M) \sqcup \partial M$ — Int M is open, ∂M is closed

$\operatorname{Int} M$ is a top mfd of dim n

∂M is a top mfd of dim n-1

$p \in \mathrm{Int}(M)$ $\quad \exists$ nbhd $U \xrightarrow{\ \phi\ } \mathbb{H}$

$$\begin{array}{ccc} U & \xrightarrow{\ \phi\ } & \mathbb{H} \\ \rotatebox{90}{$\in$} & & \\ p & \longrightarrow & \phi(p) \in \mathring{\mathbb{H}} \end{array}$$

$\exists\, p \in V \subset U$ s.t. $\phi(V) \subset \mathring{\mathbb{H}}$

Any pts in V is interior.

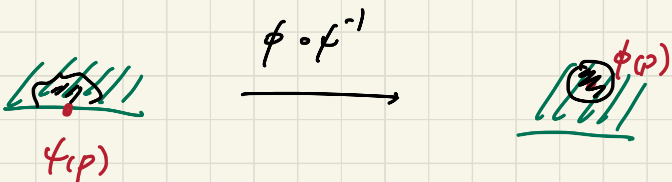
boundary pt $\neq$ interior pt

Suppose p is both a boundary pt & interior pt

$$\begin{tikzcd} \mathbb{H} & U \arrow[l, "\psi"'] \arrow[r, "\phi"] & \mathbb{H} \end{tikzcd}$$

$$\underbrace{\mathbb{H} \xleftarrow{\ \psi\ } U \xrightarrow{\ \phi\ } \mathbb{H}}_{\phi \circ \psi^{-1}}$$

$\psi(p) \in \partial \mathbb{H} \qquad p \qquad \phi(p) \in \mathring{\mathbb{H}}$



Baby example