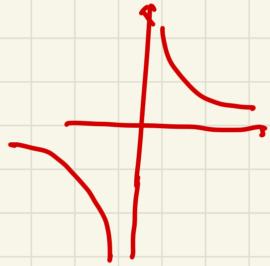
2) $\mathbb{R}^2 \longrightarrow \mathbb{R}$ is open but not closed.

$(x, y) \longmapsto x$



3) $U \overset{\text{open}}{\subset} \mathbb{C}$ $f: U \to \mathbb{C}$

non constant holo. function.

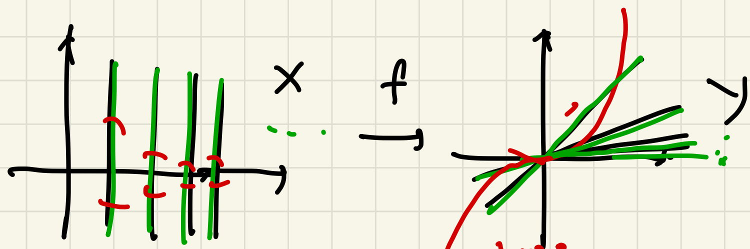
$f(U)$ is open in $\mathbb{C}$ (use the fact that non const. holo. function has discrete zeros.)

Suppose $f: X \twoheadrightarrow Y$ is a cont. surjection

$x_1 \sim x_2$ if $f(x_1) = f(x_2)$ in $Y$

then $X/\sim \overset{\text{bij.}}{\longleftrightarrow} Y$ if is homeo. we call it quotient map!

this is NOT necessarily a homeo. $y = x^3$




$f^{-1}(A)$ is open A is NOT! $A : \{ y > x^3,\ x \geq 0;\ y < x^3,\ x \leq 0 \} \cap Y$

$$X = \bigsqcup_{n \in \mathbb{N}} \{x = n\} \subset \mathbb{R}^2$$

$f : X \longrightarrow \mathbb{R}^2$ $\qquad Y = \operatorname{im} f \subset \mathbb{R}^2$

$(n, y) \longmapsto (ny, y)$ $\qquad f : X \twoheadrightarrow Y$ is cont.

but $f$ is not a quotient map.

if it is so then $A \subset Y$ is open if $f^{-1}A$ is open in $X$

take $A = \bigcup \{ (ny, y) \mid |y| < \frac{1}{n} \}$

$f^{-1}A = \bigsqcup \{ (n, y) \mid |y| < \frac{1}{n} \}$ is open in $X$

$(0, 0) \in A$ $\quad \forall \delta > 0$

$\{x^2 + y^2 < \delta^2\} \cap Y \not\subset A$

**Lemma 4** A cont. surj. $f: X \twoheadrightarrow Y$ is a quotient map $\Leftrightarrow$ $f$ maps saturated open sets to open sets

$f$ is quotient map $\Leftarrow$)
$V \in \mathcal{O}_Y \Leftrightarrow f^{-1}(V) \in \mathcal{O}_X$ ✓
$V = f(f^{-1}(V))$

## Compactness

**Def'n** A top. space $X$ is called **compact** if any open covering of $X$ has a finite subcovering.

i.e suppose $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ $\exists$ a finite $\Lambda' \subset \Lambda$ st

$$X = \bigcup_{\alpha \in \Lambda'} U_\alpha$$

## Examples

1) Every set is compact with trivial top

2) A set with discrete top is compact iff it is finite

3) $(\mathbb{R}, \text{Zariski})$

Every closed set is compact obvious since its finite

Every subset is compact.

$Y \subset \mathbb{R}$ $\quad Y = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha$

fix $\alpha_0$ $\quad \mathcal{U}_{\alpha_0} = \mathbb{R} \setminus \{p_1, \cdots, p_N\}$

let $\mathcal{U}_{\alpha_1} \ni p_1, \cdots \quad \mathcal{U}_{\alpha_N} \ni p_N$ then $Y = \bigcup_{i=0}^{N} \mathcal{U}_{\alpha_i}$

Clearly not every subsets of $\mathbb{R}$ are compact under $|\cdot|$ top.

4) $(\operatorname{Spec} \mathbb{Z}, \text{Zariski})$

Every subset is compact

$$\mathcal{U}_n = (\mathbb{Z}(n))^c = \{ (p) \mid p \nmid n \}$$

is cofinite!

5) $[0,1]$ is compact under $|\cdot|$ top.

**Lemma** $(X, d)$ metric space

any compact subset of $X$ is bounded

pf.

**Thm** A subset $C \subset \mathbb{R}^n$ is compact iff it's closed & bounded. (proved in analysis I)

**Cor** Let $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a cont. function (compact.)

then $f$ has max. min on $\Omega$.

**Rmk** $O(n) \subset Mat_{n \times n}$

$$O(n) = \{A \mid A \cdot A^T = I_n\}$$

$O(n)$ is compact.

$A = (A_1, A_2, \cdots A_n)$

$\|A_i\| = 1$

$\|A\|^2 = \sum \|A_i\|^2 = n \Rightarrow$ bounded needs

$Mat_{n \times n} \longrightarrow Mat_{n \times n}$

$A \longmapsto A \cdot A^T$

# More properties of compact spaces

1) $f: X \longrightarrow Y$ cont. $X$ is compact

$f(X)$ is compact

2) $X, Y$ compact $X \times Y$ is compact w.r.t box (= product) top

[Tychnoff]

3) $X_\alpha$ compact $\prod X_\alpha$ is compact w.r.t product top.

Ex. this fails for box top

$A = \{0,1\}$ $A^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ = all binary sequences

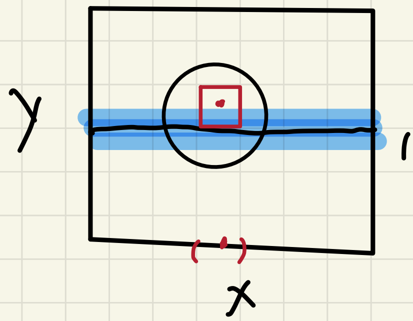
the box top is discrete therefore not compact

but the product top is compact.

4) Closed subsets of compact set are compact

$A \subset X$ $\cup U_\alpha = A$ $\cup U_\alpha \cup (X \setminus A) = X$

## proof of product thm




Let $M_\alpha$ $\alpha \in \Lambda$ be an open covering of $X \times Y$

* $\pi_x$, $\pi_y$ are open map.

fix $y \in Y$ Let $\Lambda_y \subset \Lambda$ finite s.t.

$\{M_\alpha \mid \alpha \in \Lambda_y\}$ is a covering of $\{y\} \times X$

**Lemma** $\exists$ $V_p$ nbhd of $p$ in $Y$ s.t.

$$\bigcup_{\alpha \in \Lambda_y} M_\alpha \supset X \times V_p$$

pf of lemma $\forall x$, $\exists \alpha \in \Lambda_y$ s.t. $(x, y) \in M_\alpha$

choose $(x, y) \in A_\alpha \times B_\alpha \subset M_\alpha$ then $\{A_\alpha \times B_\alpha\}$ covers $y \times X$

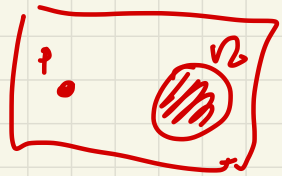
it has finite subcover $\{A_i \times B_i\}_{i \in I}$

then $X \times (\cap B_i) \subset \bigcup_{\alpha \in \Lambda_y} M_\alpha$

by compactness of $Y$ we can choose a finite set of $y$.

# More properties of compact spaces

1) $X$ is Hausdorff. pt & compact subset can be separated.



$\forall x \in \Omega \ \exists \ \mathcal{U}_x \ni x, \ V_x \ni p$

s.t. $\mathcal{U}_x \cap V_x = \emptyset$

by compactness of $\Omega$ $\exists$ $x_1, \dots, x_N$ s.t.

$\Omega \subset \bigcup_{i=1}^{N} \mathcal{U}_{x_i} =: \mathcal{U} \quad V = \bigcap_{i=1}^{N} V_{x_i} \qquad \mathcal{U} \cap V = \emptyset$

2) compact subsets of Hausdorff space is closed.

$X \setminus \Omega \ni x \qquad \mathcal{U} \ni x \quad \mathcal{U} \cap \Omega = \emptyset$

Ex. Zariski top.

3) [Heine-Borel] Compact subsets of $\mathbb{R}^n$ $\iff$ closed & bounded.

it suffices to prove 1D case

Any closed & bounded set is a subset of $[-d, d]^n$