MATH3405
Differential Equations
Appendix A
Existence and Continuous Dependence of Solutions

We shall establish the existence of solutions of of a first order ODE

$$\dot{x} = f(t, x). \tag{1}$$

The method is readily generalized to system of 1st order ODEs. In particular, it covers the existence of solutions of higher order ODEs.

Recall that $\varphi(t)$ is a solution of the IVP

$$\dot{x} = f(t, x), \ \ x(t_0) = x_0 \tag{2}$$

if and only if it is a solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Definition 1 (Picard Iterates)

The *Picard iteration* of the IVP (2) is defined by

$$\varphi_0 \equiv x_0 \quad \text{and} \quad \varphi_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\, ds. \tag{3}$$

Example 2

Consider the IVP $\dot{x} = x$, $x(0) = 1$.Then

$$\begin{aligned}\varphi_1(t) &= 1 + \int_0^t x_0 \, ds = 1 + t \\ \varphi_2(t) &= 1 + \int_0^t 1 + s \, ds = 1 + \left[s + s^2/2\right]_0^t = 1 + t + t^2/2\end{aligned}$$

and in general

$$\varphi_n(t) = 1 + t + t^2/2 + t^3/3! + \cdots + t^n/n!.$$

Note that

$$\varphi(t) := \lim_{n\to\infty} \left(1 + t + t^2/2 + t^3/3! + \cdots + t^n/n!\right) = e^t$$

is a solution of the IVP.

Example 3

Consider the IVP $\dot{x} = x^2$, $x(0) = 1$.Then

$$\begin{aligned}\varphi_1(t) &= 1 + \int_0^t x_0^2 \, ds = 1 + t \\ \varphi_2(t) &= 1 + \int_0^t (1 + s)^2 \, ds = 1 + t + t^2 + \frac{t^3}{3}\end{aligned}$$

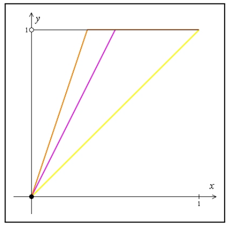
Though every approximation is continuous on $\mathbb{R}$, it will not converge to a solution of the IVP over $\mathbb{R}$!

In fact, the solution of the IVP is $x(t) = \dfrac{1}{1-t}$ which is defined on $(-\infty, 1)$.

Example 4

Consider the sequence

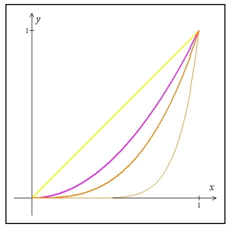
$$\varphi_n(t) = \begin{cases} nt & \text{if } 0 \le t \le 1/n \\ 1 & \text{if } 1/n < t \le 1. \end{cases}$$



Though every $\varphi_n(t)$ is continuous, $\varphi(t) := \lim_{n\to\infty} \varphi_n(t)$ is discontinuous at $t = 0$.

Example 5

Consider the sequence of differentiable functions $\{t^n\}$, $t \in [0, 1]$.



The pointwise limit

$$\lim_{n\to\infty} t^n = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ 1 & \text{if } t = 1, \end{cases}$$

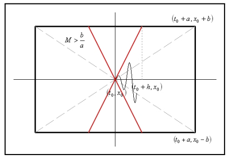
is discontinuous at $t = 1$.

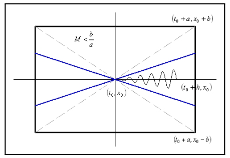
Lemma 6

Let $\{\varphi_n\}$ be the Picard iterates of the IVP (2). If f is continuous on $R := [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$, and if

$$M = \max_{(t,x)\in R} |f(t,x)| (> 0) \text{ and } h = \min\{a, b/M\},$$

the the graphs of $\{\varphi_n\}$ on $[t_0 - h, t_0 + h]$ will remain in R.





Let $\mathcal{C}$ be set of all continuous functions $\varphi(t)$ on $I = [t_0 - h, t_0 + h]$ such that

1. $|\varphi(t) - x_0| \le b$ for all $t \in I$;
2. $|\varphi(t_1) - \varphi(t_2)| \le M|t_1 - t_2|$ for all $t_1, t_2 \in I$.

Define $T : \mathcal{C} \to \mathcal{C}$ by

$$T(\varphi(t)) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds.$$

Clearly, $\varphi(t)$ is a fixed point of T if and only if $\varphi(t)$ is a solution of the IVP (2) on $[t_0 - h, t_0 + h]$.

Clearly, φ is a solution of the above IVP if and only if φ is a fixed point of T:

$$T\varphi = \varphi.$$

An application of the Banach's fixed point theorem leads to

Theorem 7 (Cauchy–Peano Existence Theorem)

If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on a neighbourhood containing (t_0, x_0), then there exists an $h > 0$ and a solution $x = \varphi(t)$ to the IVP (2) on $[t_0 - h, t_0 + h]$.

Let's continue with the IVP $\dot{x} = f(t,x)$, $x(t_0) = x_0$. Suppose that f is continuous in some open set G and $(t_0, x_0) \in G$.
The existence theorem asserts that the IVP has a solution φ_0 over some interval $[t_0 - h_0, t_0 + h_0]$ and the graph of φ_0 remains in G.
We may then consider the IVP

$$\dot{x} = f(t,x),\ x(t_1) = x_1, \quad \text{where } t_1 = t_0 + h_0 \text{ and } x_1 = \varphi_0(t_1).$$

As $(t_1, x_1) \in G$, we may apply the existence theorem again and conclude that it has a solution φ_1 over $[t_1 - h_1, t_1 + h_1]$ and the graph of φ_1 remains in G.
We can continue this process up to some maximal interval $[t_0 + t_R)$.
The same process can be performed on the left hand side of t_0 and finally, the IVP has a solution defined over some maximal interval $(t_0 - t_L, t_0 + t_R)$.

Theorem 8

Let $G = (A, B) \times \mathbb{R}$ where $A, B \in \mathbb{R} \cup \{\pm\infty\}$. If $f(t, x)$ is continuous on G and there is $L > 0$ such that

$$|f(t, x)| \le L|x| \quad \text{for all } x \in \mathbb{R},$$

then a maximal solution of the IVP (2) is defined on (A, B).

Corollary 9

Every solution of a 1st order linear ODE (system) is defined on the whole real line.

Theorem 10 (Gronwall's Inequality)

Let $u : [a, b] \to (0, \infty)$ and $v : [a, b] \to [0, \infty)$ be continuous. If there is $c > 0$ such that

$$v(x) \le c + \int_a^x u(t)v(t)\, dt \quad \text{for all } x \in [a, b],$$

then

$$v(x) \le c \exp\left(\int_a^x u(t)\, dt\right) \quad \text{for all } x \in [a, b].$$

Theorem 11

Suppose that $f(t, x)$ is Lipschitz with respect to x with Lipschitz constant L. Let φ_0 be the solution of the IVP

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

over some interval $[t_0 - h_0, t_0 + h]$. Suppose that $\delta > 0$ is small enough so that there is $0 < h (< h_0)$ such that for any $x_1 \in [x_0 - \delta, x_1 + \delta]$, the IVP

$$\dot{x} = f(t, x), \quad x(t_0) = x_1$$

has a unique solution φ_1 on $[t_0 - h, t_0 + h]$. Then

$$|t - t_0| \le h \quad \Longrightarrow \quad |\varphi_1(t) - \varphi_0(t)| \le |x_1 - x_0|e^{Lh}.$$

Theorem 12

Let G be an open region in $\mathbb{R}^2$ and $I = [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Suppose that

1. *$(t_0, x_0, 0) \in G \times I$;*
2. *$f(t, x, \lambda)$ is Lipschits with respect to (x, λ) on $G \times I$: there is $L > 0$ such that*

   $$|f(t, x_1, \lambda_1) - f(t, x_2, \lambda_2)| \le L\left(|x_1 - x_2| + |\lambda_1 - \lambda_2|\right) \text{ on } G \times I;$$

3. *there is $h > 0$ such that for every $\lambda \in I$, the solution φ_λ of the IVP*

   $$\dot{x} = f(t, x, \lambda), \; x(t_0) = x_0,$$

   is defined on $[t_0 - h, t_0 + h]$.

Then

$$|t - t_0| \le h \quad \Longrightarrow \quad |\varphi_\lambda(t) - \varphi_0(t)| \le \varepsilon h e^{Lh}.$$