Algebra II: Tutorial 9

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Problem 1. Let K be a field. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in K[x]$ and define

$$f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in K[x]$$

be the derivative of $f(x)$.

1. If $\deg(f) \geq 2$, show that f has a repeated root if and only if f and f' share a root.
2. If K has characteristic 0, show that $f'(x) = 0$ if and only if f is a constant polynomial.
3. If K has characteristic $p > 0$, show that $f'(x) = 0$ if and only if there exists $g(x) \in K[x]$ such that $f(x) = g(x^p)$.

Problem 2. Let K be any field, and let f be a non-constant polynomial over K. The following are equivalent:

1. The polynomial f is separable over K,
2. The polynomials f and f' are relatively prime as elements in $K[x]$,
3. The polynomial f has no repeated roots in *any* field extension L of K.

Problem 3. Let K be any field, and let $p \in K[x]$ be irreducible. Then, p is separable over K if and only if $p' \neq 0$. Deduce that if K is a field of characteristic 0, or is a finite field, K is perfect.