20241119 MATH3541 Section B

Problem 11:

(a) Solution:

Define the first copy $\mathbb{S}_1 = \{e^{i\theta_1} \in \mathbb{C} : \theta_1 \in \mathbb{R}\}$ of $\mathbb{S}$.

Define the second copy $\mathbb{S}_2 = \{e^{i\theta_2} \in \mathbb{C} : \theta_2 \in \mathbb{R}\}$ of $\mathbb{S}$

Define an equivalence relation $\sim$ on the disjoint union $\mathbb{S}_1 \sqcup \mathbb{S}_2$ by the partition $[\mathbb{S}_1 \sqcup \mathbb{S}_2]$

(1) If $e^{i\theta_1} \neq 1$, then $[(e^{i\theta_1}, 1)] = \{(e^{i\theta_1}, 1)\} \in [\mathbb{S}_1 \sqcup \mathbb{S}_2]$;

(2) If $e^{i\theta_1} = 1$, then $[(e^{i\theta_1}, 1)] = \{(1, 1), (1, 2)\} \in [\mathbb{S}_1 \sqcup \mathbb{S}_2]$;

(3) If $e^{i\theta_2} \neq 1$, then $[(e^{i\theta_2}, 2)] = \{(e^{i\theta_2}, 2)\} \in [\mathbb{S}_1 \sqcup \mathbb{S}_2]$;

(4) If $e^{i\theta_2} = 1$, then $[(e^{i\theta_2}, 2)] = \{(1, 1), (1, 2)\} \in [\mathbb{S}_1 \sqcup \mathbb{S}_2]$;

Put quotient space topology on $\mathbb{S}_1 \vee \mathbb{S}_2 = [\mathbb{S}_1 \sqcup \mathbb{S}_2]$ and we've constructed a wedge sum.

Define a surjective continuous map $\sigma : X = \mathbb{S}_1 \sqcup \mathbb{S}_2 \to Y = \mathbb{C}$ by:

(1) $\sigma(e^{i\theta_1}, 1) = e^{i\theta_1} - 1$ (2) $\sigma(e^{i\theta_2}, 2) = 1 - e^{-i\theta_2}$

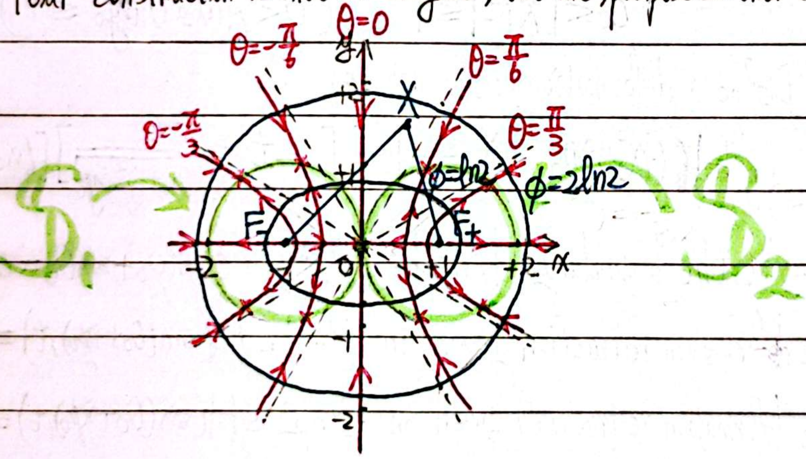
As X is compact and Y is Hausdorff, the quotient map $[\sigma]$ from $[X] = [\mathbb{S}_1 \sqcup \mathbb{S}_2]$ to Y is an embedding, and we are done.

(b) Solution: Your construction is not so elegant, let me propose another one.

(c)

(d)

(e)



Parametrize every $x+iy$ as $\sin(\theta+i\phi) = \sin\theta\cosh\phi + i\cos\theta\sinh\phi$.

Note that $|XF_-| = |1+\sin(\theta+i\phi)| = \sqrt{1+\sin(\theta+i\phi)+\sin(\theta-i\phi)+|\sin(\theta+i\phi)|^2} = \sqrt{\sin^2\theta+2\sin\theta\cosh\phi+\cosh^2\phi} = \cosh\phi+\sin\theta$

$|XF_+| = |1-\sin(\theta+i\phi)| = \sqrt{1-\sin(\theta+i\phi)-\sin(\theta-i\phi)+|\sin(\theta+i\phi)|^2} = \sqrt{\sin^2\theta-2\sin\theta\cosh\phi+\cosh^2\phi} = \cosh\phi-\sin\theta$

$|XF_-|+|XF_+| = 2\cosh\phi$ determines an ellipse, and $|XF_-|-|XF_+| = 2\sin\theta$ determines a branch of hyperbola.

Step1: For any point $x_0+iy_0=\sin(\theta_0+i\phi_0)$ satisfying:

$$|XF_-|\geq|XF_+|\geq 1\quad(\cosh\phi_0+\sin\theta_0\geq\cosh\phi_0-\sin\theta_0\geq 1)$$

Define a deformation retraction by:

$$H(\sin(\theta_0+i\phi_0),t)=\sin[\theta_0+(1-t)i\phi_0+t\,i\,\operatorname{arcosh}(1+\sin\theta_0)]$$

Step2: For any point $x_0+iy_0=\sin(\theta_0+i\phi_0)$ satisfying:

$$1\leq|XF_-|\leq|XF_+|\quad(1\leq\cosh\phi_0+\sin\theta_0\leq\cosh\phi_0-\sin\theta_0)$$

Define a deformation retraction by:

$$H(\sin(\theta_0+i\phi_0),t)=\sin[\theta_0+(1-t)i\phi_0+t\,i\,\operatorname{arcosh}(1-\sin\theta_0)]$$

Step3: For any point $x_0+iy_0=\sin(\theta_0+i\phi_0)$ satisfying:

$$0<|XF_+|\leq 1\quad\left(0<\sqrt{(x_0-1)^2+y_0^2}\leq 1\right)$$

Define a deformation retraction by:

$$H(x_0+iy_0,t)=+1+\left[1-t+\frac{t}{\sqrt{(x_0-1)^2+y_0^2}}\right][(x_0-1)+iy_0]$$

Step4: For any point $x_0+iy_0=\sin(\theta_0+i\phi_0)$ satisfying:

$$0<|XF_-|\leq 1\quad\left(0<\sqrt{(x_0+1)^2+y_0^2}\leq 1\right)$$

Define a deformation retraction by:

$$H(x_0+iy_0,t)=-1+\left[1-t+\frac{t}{\sqrt{(x_0+1)^2+y_0^2}}\right][(x_0+1)+iy_0]$$

Step5: At the boundary $|XF_-|=|XF_+|$ $(\cosh\phi_0+\sin\theta_0=\cosh\phi_0-\sin\theta_0)$:

The deformation retraction given in Step1 is $H(\sin(\theta_0+i\phi_0),t)=i\sinh[(1-t)\phi_0]\cos\theta_0$

The deformation retraction given in Step2 is $H(\sin(\theta_0+i\phi_0),t)=i\sinh[(1-t)\phi_0]\cos\theta_0$

They agrees along the boundary.

Step6: At the boundary $|XF_+|=1$ ($\cosh\phi_0 - \sin\theta_0 = \sqrt{(x_0-1)^2+y_0^2}$):

The deformation retraction given in Step1 is $H(\sin(\theta_0+i\phi_0),t)=\sin(\theta_0+i\phi_0)$

The deformation retraction given in Step3 is $H(x_0+iy_0,t)= x_0+iy_0$

They agrees along the boundary. In addition, they fix the boundary.

Step7: At the boundary $|XF_-|=1$ ($\cosh\phi_0 + \sin\theta_0 = \sqrt{(x_0+1)^2+y_0^2}$):

The deformation retraction given in Step2 is $H(\sin(\theta_0+i\phi_0),t)=\sin(\theta_0+i\phi_0)$

The deformation retraction given in Step4 is $H(x_0+iy_0,t)= x_0+iy_0$

They agrees along the boundary. In addition, they fix the boundary.

Step8: Glue the four deformation retractions,

and we've proven that $\mathbb{C}\setminus\{\pm1\}\sim[S_1\sqcup S_2]$.

Problem 12.

(a) Proof: We may divide our proof into four steps.

Step1: As $p: X\to Z$ is a covering map, for all $z\in Z$, for some open neighbour W_z of z, $p^{-1}(W_z)\cong\coprod_{\nu\in K} W_\nu$, and each restricted map $p|_{W_\nu}: W_\nu \to W_z$ is a homeomorphism. As Z is locally path connected, replace W_z with a path connected open neighbour of z.

Step2: As the covering map $p=r\circ q$ is surjective, r is surjective.

As $p=r\circ q$ is continuous, and the covering map q is surjective and open, we can prove that r is continuous:

$\forall$ open subset W of Z, $r^{-1}(W)=q(q^{-1}(r^{-1}(W)))=q(h^{-1}(W))$ is open in Y

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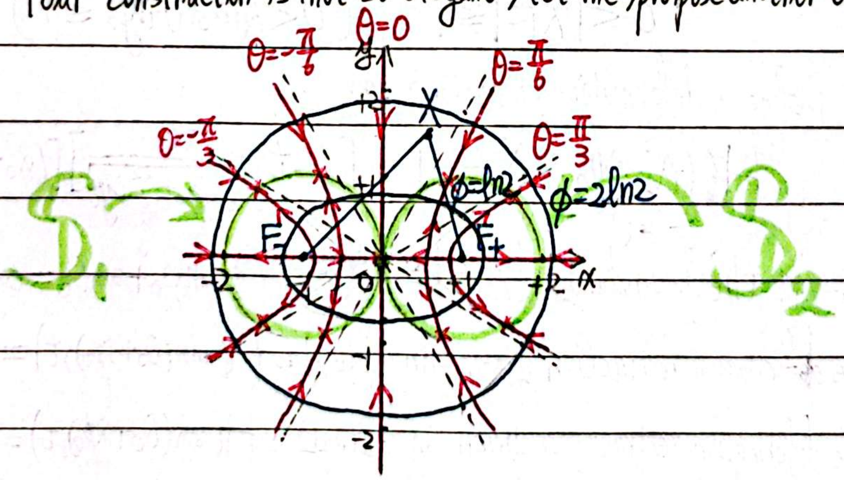

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Problem 12.

(a) Proof: We may divide our proof into four steps.

Step1: As $p: X\to Z$ is a covering map, for all $z\in Z$, for some open neighbour W_z of z, $p^{-1}(W_z)\cong\coprod_{u\in K} W_u$, and each restricted map $p|_{W_u}: W_u\to W_z$ is a homeomorphism. As Z is locally path connected, replace W_z with a path connected open neighbour of z.

Step2: As the covering map $p=r\circ q$ is surjective, r is surjective.

As $p=r\circ q$ is continuous, and the covering map q is surjective and open, we can prove that r is continuous:

$\forall$ open subset W of Z, $r^{-1}(W)=q(q^{-1}(r^{-1}(W)))=q(p^{-1}(W))$ is open in Y

Step 3: Partition $\gamma^{-1}(W_{z})$ by all its path connected components.

$$\gamma^{-1}(W_z) = \coprod_{\mu \in J} V_\mu$$

(1) As the covering map q is surjective and open, each V_μ is open:

$$\begin{aligned} V_\mu &= q(q^{-1}(V_\mu)) = q(\{x \in X : q(x) \in V_\mu\}) \\ &= q(\{x \in p^{-1}(W_z) : q(x) \in V_\mu\}) \ (q(x) \in V_\mu \text{ requires } x \in p^{-1}(W_z)) \\ &= \{q(x) \in p^{-1}(W_z) : q(x) \in V_\mu\} = \bigcup_{q(W_\nu) \subseteq V_\mu} q(W_\nu) \ (\text{open}) \end{aligned}$$

(2) We wish to show that each $q(W_\nu)$ is equal to the path connected component V_μ it lives in, so which representative $q(W_\nu)$ we choose for V_μ is not important.

Assume to the contrary that some $q(W_\nu) \subsetneqq V_\mu$.

Fix $x_\nu \in W_\nu$, $y_\nu = q(x_\nu) \in q(W_\nu)$ and $y \in V_\mu \setminus q(W_\nu)$.

As V_μ is path connected, there exists a path $\gamma : [0,1] \to V_\mu$ from y_ν to y.

As q is a covering map, for certain ~~initial~~ initial point $x_\nu \in X$, the path γ downstairs has a unique lift $\tilde{\gamma} : [0,1] \to X$ upstairs. Now we are ready to deduce the contradiction.

On one hand, $q \circ \tilde{\gamma}([0,1]) = \gamma([0,1]) \subseteq V_\mu$, $\tilde{\gamma}([0,1]) \subseteq q^{-1}(V_\mu)$

$$= \coprod_{f(W_\nu) \subseteq V_\mu} W_\nu \ (\text{because } p^{-1}(W_z) = q^{-1}(\gamma^{-1}(W_z)) = \coprod_{\mu \in J} q^{-1}(V_\mu))$$

the path $\tilde{\gamma}$ should stay in the sheet W_{ν} where its initial point x_{ν} lies in.
On the other hand, $y = \gamma(1) = q \circ \tilde{\gamma}(1)$ goes out of the range $q(W_{\nu})$, a contradiction!
Hence, our assumption is wrong, and we've proven that $q(W_{\nu}) = V_{\mu}$.

Step 4: For each $\mu \in J$, choose $\nu_{\mu} \in K$, such that $q(W_{\nu_{\mu}}) = V_{\mu}$.
We've already set up a homeomorphism $p|_{W_{\nu_{\mu}}} : W_{\nu_{\mu}} \to V_{\mu}$.
If we restrict the covering map q to $q|_{W_{\nu_{\mu}}} : W_{\nu_{\mu}} \to V_{\mu}$,
then $q|_{W_{\nu_{\mu}}}$ is surjective, injective (because $p|_{W_{\nu_{\mu}}} = r|_{V_{\mu}} \circ q|_{W_{\nu_{\mu}}}$ is bijective), open and continuous, so $q|_{W_{\nu_{\mu}}}$ is a homeomorphism.
Hence, $r|_{V_{\mu}} = p|_{W_{\nu_{\mu}}} \circ q|^{-1}_{W_{\nu_{\mu}}}$ is a homeomorphism as well, and we've proven that r is a covering map.

(b) Proof: We may divide our proof into four steps.
Step 1: As $r: Y \to Z$ is a covering map, for all $z \in Z$, for all open neighbour V_z of z, $r^{-1}(V_z) \cong \coprod_{\mu \in J} V_{\mu}$, and each restricted map $r|_{V_{\mu}} : V_{\mu} \to V_z$ is a homeomorphism.
Step 2: As $p: X \to Z$ is a covering map, for all $z \in Z$, for all open neighbour W_z of z, $p^{-1}(W_z) \cong \coprod_{\nu \in K} W_{\nu}$, and each restricted map $p|_{W_{\nu}} : W_{\nu} \to W_z$ is a homeomorphism.

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Step3: We wish to show that the continuous function $q: X \to Y$ is surjective.

(1) For all $y \in Y$, we wish to find a preimage x of y under q.

(2) Fix a point $x_0 \in X$, and define $y_0 = q(x_0) \in Y$, $z_0 = p(x_0) = r(y_0) \in Z$.

(3) As Y is path connected, there exists a path $\upsilon: [0,1] \to Y$ from y_0 to y.

(4) Project the path υ in Y to the path $\sigma = r \circ \upsilon$ in Z via r.

(5) As p is a covering map, there exists a unique lift μ of σ with initial point $x_0 \in X$.

(6) Project the path μ in X to the path $\upsilon' = q \circ \mu$ in Y via q.

(7) As r is a covering map and the initial points $\upsilon(0) = y_0$, $\upsilon'(0) = q \circ \mu(0) = q(\mu(0)) = q(x_0) = y_0$ are identical, $y = \upsilon(1) = \upsilon'(1) = q \circ \mu(1) = q(\mu(1)) \in q(X)$, q is surjective.

Step4: For all $y \in Y$, choose the open neighbour $U_y = r^{-1}(V_{r(y)} \cap W_{r(y)})$ of y. The following set-theoretic result holds.

$$q^{-1}(U_y) = q^{-1}(r^{-1}(V_{r(y)} \cap W_{r(y)})) = \coprod_{\upsilon \in K} q|^{-1}_{W_\upsilon}\Big(\coprod_{\mu \in J} r|^{-1}_{V_\mu}(V_{r(y)} \cap W_{r(y)})\Big) = \coprod_{\upsilon \in K} \coprod_{\mu \in J} q|^{-1}_{W_\upsilon}\big(r|^{-1}_{V_\mu}(V_{r(y)} \cap W_{r(y)})\big)$$

$$= \coprod_{\upsilon \in K} \coprod_{\mu \in J} p|^{-1}_{q^{-1}(V_\mu) \cap W_\upsilon}(V_{r(y)} \cap W_{r(y)})$$

Hence, $q^{-1}(U_y)$ is homeomorphic to the above coproduct of open subsets of X, and each restricted map $r|^{-1}_{V_\mu \cap q(W_\nu)} \circ p|_{q^{-1}(V_\mu)\cap W_\nu}$ is a homeomorphism. To conclude, q is a covering map.

(c) Proof: We may divide our proof into four steps.

Step1: For all $z \in Z$, for some open neighbour W_z of z, $r^{-1}(W_z) \cong \coprod_{k=1}^{m} V_k$, and each restricted map $r|_{V_k}: V_k \to W_z$ is a homeomorphism.

Step2: For some open neighbour $V_{y_k} \subseteq V_k$ of $y_k = r|^{-1}_{V_k}(z)$, $q^{-1}(V_{y_k}) \cong \coprod_{\lambda_k \in I_k} U_{\lambda_k}$, and each restricted map $q|_{U_{\lambda_k}}: U_{\lambda_k} \to V_{y_k}$ is a homeomorphism.

Step3: As q, r are covering maps, q, r are surjective and continuous, which implies $p = r \circ q$ is surjective and continuous.

Step4: Define an open neighbour $U_z = \bigcap_{k=1}^{m} r|_{V_k}(V_{y_k})$ of z.

The following set theoretic result holds:

$$
\begin{aligned}
p^{-1}(U_z) &= q^{-1}(r^{-1}(U_z)) = q^{-1}\Big(\coprod_{k=1}^{m} r|^{-1}_{V_k}(U_z)\Big) \\
&= \coprod_{k=1}^{m} q^{-1}\big(r|^{-1}_{V_k}(U_z)\big) \\
&= \coprod_{k=1}^{m} \coprod_{\lambda_k \in I_k} q|^{-1}_{U_{\lambda_k}}\big(r|^{-1}_{V_k}(U_z)\big) \\
&= \coprod_{k=1}^{m} \coprod_{\lambda_k \in I_k} p|^{-1}_{U_{\lambda_k}}(U_z)
\end{aligned}
$$

Hence, $p^{-1}(U_z)$ is homeomorphic to the above coproduct of open subsets of X, and each restricted map $p|_{U_{\lambda_k}}: U_{\lambda_k} \to U_z$ is a homeomorphism, so p is a covering map.

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(d) Proof: We may divide our proof into three steps.

Step 1: Treat $r:(Y,y_0)\to(Z,z_0)$ as a cover, and $u:(\tilde{Z},\tilde{z}_0)\to(Z,z_0)$ as a map.

As u is universal, $\mathrm{Im}(u')=\{u'([[e_{\tilde{z}_0}]])\}=\{[[u\circ e_{\tilde{z}_0}]]\}$
$=\{[[e_{z_0}]]\}\leq \mathrm{Im}(r')$.

As $\tilde{Z}$ is locally path connected and path connected, the universal property of covering space suggests that $u:(\tilde{Z},\tilde{z}_0)\to(Z,z_0)$ has a lift $v:(\tilde{Z},\tilde{z}_0)\to(Y,y_0)$.

Step 2: Treat $q:(X,x_0)\to(Y,y_0)$ as a cover, and $v:(\tilde{Z},\tilde{z}_0)\to(Y,y_0)$ as a map.

As v is universal, $\mathrm{Im}(v')=\{v'([[e_{\tilde{z}_0}]])\}=\{[[v\circ e_{\tilde{z}_0}]]\}$
$=\{[[e_{y_0}]]\}\leq \mathrm{Im}(q')$.

As $\tilde{Z}$ is locally path connected and path connected, the universal property of covering space suggests that $v:(\tilde{Z},\tilde{z}_0)\to(Y,y_0)$ has a lift $w:(\tilde{Z},\tilde{z}_0)\to(X,x_0)$.

Step 3: As Z is locally path connected, and $u=r\circ v=r\circ q\circ w$
$=p\circ w$, w are covers, it follows from (a) that p is a cover.

Problem 13

(a) Solution: We shall assume that the finite group G has exactly n distinct nontrivial elements $g_1, g_2, \cdots, g_n$. For all $x \in X$, according to the assumption, the orbit $G * x$ of x contains $n+1$ distinct elements $x, g_1 * x, g_2 * x, \cdots, g_n * x$.

As X is Hausdorff, there exist open sets $U_0, U_1, U_2, \cdots, U_n$, such that:

(1) $x \in U_0$ and $g_1 * x \in U_1$ and $g_2 * x \in U_2$ and $\cdots$ and $g_n * x \in U_n$

(2) The sets $U_0, U_1, U_2, \cdots, U_n$ are pairwisely disjoint.

Now, construct $V_0 = U_0 \cap (g_1^{-1} * U_1) \cap (g_2^{-1} * U_2) \cap \cdots \cap (g_n^{-1} * U_n)$

As each g_k is a homeomorphism of X, V_0 is an open neighbour of x, such that:

(1) $x \in V_0$ and $g_1 * x \in V_1 = g_1 * V_0$ and $g_2 * x \in V_2 = g_2 * V_0$ and $\cdots$ and $g_n * x \in V_n = g_n * V_0$;

(2) $V_0 \subseteq U_0$ and $V_1 = g_1 * V_0 \subseteq U_1$ and $V_2 = g_2 * V_0 \subseteq U_2$ and $\cdots$ and $V_n = g_n * V_0 \subseteq U_n$.

The sets $V_0, V_1, V_2, \cdots, V_n$ are pairwisely disjoint.

The existence of this open neighbour V_0 of x ensures that $*$ is properly discontinuous.

(b) Solution: For all compact subset C of X,

for all $x \in \left(\bigcup_{g \in G} g*C\right)^c$,

we wish to find an open neighbour V of x,

such that $V \subseteq \left(\bigcup_{g \in G} g*C\right)^c$

To do so, we first construct an open neighbour U of x,

such that $\overline{U}$ is compact.

As $C \cup \overline{U}$ is compact, our assumption suggests

that there is at most finitely many $g_1, g_2, \cdots, g_n \in G \setminus \{e\}$,

such that
$$g_1*(C \cup \overline{U}) = (g_1*C) \cup (g_1*\overline{U})$$
$$g_2*(C \cup \overline{U}) = (g_2*C) \cup (g_2*\overline{U})$$
$$\vdots \qquad\qquad \vdots$$
$$g_n*(C \cup \overline{U}) = (g_n*C) \cup (g_n*\overline{U}) \text{ intersects } C \cup \overline{U}$$

Hence, $C, g_1*C, g_2*C, \cdots, g_n*C$ are the only possible candidates in $\bigcup_{g \in G} g*C$ that may intersect $\overline{U}$

In the Hausdorff space,

separate the point x and the compact set C by open sets Z_0, W_0

separate the point x and the compact set g_1*C by open sets Z_1, W_1

separate the point x and the compact set g_2*C by open sets Z_2, W_2

$\vdots$

separate the point x and the compact set g_n*C by open sets Z_n, W_n

Now choose $V = U \cap Z_0 \cap Z_1 \cap Z_2 \cap \cdots \cap Z_n$,

and we've proven that $\left(\bigcup_{g \in G} g*C\right)^c$ is open, $\bigcup_{g \in G} g*C$ is closed.

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(c) Solution: For all distinct orbits $G*x_1, G*x_2 \in X/G$:

As x_1 is not in the closed orbit $G*x_2 = \bigcup_{g\in G} g*\underline{\{x_2\}}$ (compact),

we can choose an open neighbour U_1 of x_1, such that $U_1 \cap (G*x_2) = \phi$

Equivalently speaking, $x_2 \notin \bigcup_{g\in G} g*U_1$.

As X is locally compact, choose open neighbours W_1, W_2 of x_1, x_2,

such that $\overline{W_1}, \overline{W_2}$ are compact.

Consider the open neighbours $Z_1 = U_1 \cap W_1$, $Z_2 = W_2$ of x_1, x_2.

(1) $x_2 \notin \bigcup_{g\in G} g*U_1$ implies $x_2 \notin \bigcup_{g\in G} g*Z_1$

(2) $\overline{Z_1}, \overline{Z_2}$ are closed subsets of the compact sets $\overline{W_1}, \overline{W_2}$,

so $\overline{Z_1}, \overline{Z_2}$ are compact and $\overline{Z_1} \cup \overline{Z_2}$ is compact.

According to our assumption, there are finitely many $g_1, g_2, \cdots, g_n \in G\setminus\{e\}$,

such that $g_1*(\overline{Z_1}\cup\overline{Z_2}), g_2*(\overline{Z_1}\cup\overline{Z_2}), \cdots, g_n*(\overline{Z_1}\cup\overline{Z_2})$ intersects $\overline{Z_1}\cup\overline{Z_2}$

That is, $g_1*Z_1, g_2*Z_1, \cdots, g_n*Z_1$ are the only possible candidates in

$\bigcup_{g\in G} g*Z_1$ that intersects Z_2

In the Hausdorff space,

separate x_1, x_2 by open sets V_0, K_0

separate g_1*x_1, x_2 by open sets V_1, K_1

separate g_2*x_1, x_2 by open sets V_2, K_2

$\vdots$ $\vdots$

separate g_n*x_1, x_2 by open sets V_n, K_n

Now there exist open neighbours $M = Z_1 \cap V_0 \cap (g_1^{-1}*V_1) \cap (g_2^{-1}*V_2) \cap \cdots (g_n^{-1}*V_n)$,

$N = Z_2 \cap K_0 \cap K_1 \cap K_2 \cap \cdots \cap K_n$ of x_1, x_2, such that $(\bigcup_{g\in G} g*M) \cap (\bigcup_{g\in G} g*N) = \phi$

Now the saturated neighbours $\bigcup_{g\in G} g*M$, $\bigcup_{g\in G} g*N$ can be mapped to neighbours $\pi(\bigcup_{g\in G} g*M)$, $\pi(\bigcup_{g\in G} g*N)$ to separate $G*x_1, G*x_2$, so X/G is Hausdorff.

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(d) Solution: For all $x \in X$, as X is locally compact, there exists an open neighbour U of x, such that $\overline{U}$ is compact. According to the assumption, there is at most finitely many $g_1, g_2, \cdots, g_n \in G \setminus \{e\}$, such that $g_1 * \overline{U}, g_2 * \overline{U}, \cdots, g_n * \overline{U}$ intersects $\overline{U}$, so $g_1 * \overline{U}, g_2 * \overline{U}, \cdots, g_n * \overline{U}$ are the only possible candidates that may intersect U.

As X is Hausdorff, there exist open sets $V_0, V_1, V_2, \cdots, V_n$, such that:

(1) $x \in V_0$ and $g_1 * x \in V_1$ and $g_2 * x \in V_2$ and $\cdots$ and $g_n * x \in V_n$

(2) The sets $V_0, V_1, V_2, \cdots, V_n$ are pairwisely disjoint.

Now $W = U \cap V_0 \cap (g_1^{-1} * V_1) \cap (g_2^{-1} * V_2) \cap \cdots (g_n^{-1} * V_n)$ will be the desired open neighbour of x such that for all $g \in G \setminus \{e\}$:

(1) If g is equal to some g_k, then $W \cap (g * W) \subseteq V_0 \cap (g_k * (g_k^{-1} * V_k)) = \phi$

(2) If g is not equal to any g_k, then $W \cap (g * W) \subseteq U \cap (g * \overline{U}) = \phi$

The existence of this open neighbour W of x ensures that $*$ is a properly discontinuous.

(e) Solution: For all orbit $G*x \in X/G$;

As X is properly discontinuous,
choose an open neighbour U of x,
such that $\forall g \in G \setminus \{e\}$, $U \cap (g*U) = \phi$
This implies $\pi(\bigcup_{g\in G} g*U) = \pi(U)$,
where $\pi: X \to X/G$, $x \mapsto G*x$ is the natural projection.

As X is locally compact,
choose an open neighbour V of x,
such that $\overline{V}$ is compact in X

Now consider the open neighbour $W = U \cap V$ of x.
As $\pi^{-1}(\pi(W)) = \{x \in X : G*x \in \pi(W) = G*(U\cap V)\}$

$$= \{x \in X : \exists g \in G,\ x \in g*(U\cap V)\}$$
$$= \bigcup_{g\in G} g*W$$
$$\pi^{-1}(\pi(W)) = \pi^{-1}(\pi(\bigcup_{g\in G} g*W)),$$

the open neighbour $\bigcup_{g\in G} g*W$ of x is saturated in X,
so $\pi(W) = \pi(\bigcup_{g\in G} g*W)$ is an open neighbour of $G*x$ in X/G.

Now it remains to show the following:

$$\overline{\pi(W)} = \pi(\overline{W}) \text{ is compact.}$$

It suffices to prove the following general statement:

Statement: Let X, Y be topological spaces with X, Y Hausdorff, and $f: X \to Y$ be a continuous function. If $S \subseteq X$ is compact, then $\overline{f(S)} = f(\overset{S}{\overset{\shortparallel}{\boxed{\overline{S}}}})$

Proof: S is compact $\Rightarrow f(S)$ is compact
$\Rightarrow f(S)$ closed, so $\overline{f(S)} = f(S) = f(\overline{S})$

Hence, $\overline{\pi(W)} = \pi(\overline{W})$ is compact.

That is, we've found an open neighbour $\pi(W)$ of $G * x$, such that $\overline{\pi(W)}$ is compact, so X/G is locally compact.