THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 2

Due 3:30pm[1], September 20th (Friday), **in-class**.

Aim of this Homework: *In this assignment you will study some examples about the well-posedness/ill-posedness and invariant transformations.*

Reading Assignment: Read the following material(s):

(i) Section 1.4-1.6 of the textbook.

Instruction: Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the "Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

Problem 1 (Existence and Uniqueness)**.** In this problem we consider the following boundary-value problem of ODE:

$$\begin{cases} u'' = -\lambda\, u \\ u(-1) = u(1) \\ u'(-1) = u'(1), \end{cases} \tag{1}$$

where $u : [-1, 1] \to \mathbb{R}$ is the unknown, and $\lambda \geq 0$ is a given constant.

[1] You are expected to submit your homework before the beginning of Friday lecture **in-class**.

(i) Prove that the boundary-value problem (1) always has a solution.

(ii) Find all the possible values λ, so that the boundary-value problem (1) has at least two distinct solutions.

Food for Thought. Are you able to analyze the existence and uniqueness of Problem (1) when the constant $\lambda < 0$?

Problem 2. In this problem we are concerned with the following second-order linear PDE: for any $-\infty < x < \infty$ and $0 < y < \frac{\pi}{3}$,

$$\partial_{xy} u - (2 \cot 3y)\, \partial_x u = 0. \tag{2}$$

You will be asked to solve (2) in two different approaches as follows.

(i) Let $v := \partial_x u$. Verify that v satisfies the separable equation

$$\partial_y v - (2 \cot 3y)\, v = 0. \tag{3}$$

Solve for v in (3), and then obtain u by integrating v with respect to x.

(ii) Rewrite the PDE (2) as

$$\partial_x \{\partial_y u - (2 \cot 3y)\, u\} = 0. \tag{4}$$

Integrating (4) with respect to x, you can verify that u satisfies

$$\partial_y u - (2 \cot 3y)\, u = f(y), \tag{5}$$

for some arbitrary function f. Solve for u in (5) by using the method of integrating factors.

(iii) Are the solutions that you obtained in Part (i) and (ii) the SAME? Explain your answer briefly.

Problem 3 (Instability of Backward Heat Equation)**.** The aim of this problem is to illustrate that the initial and boundary value problem of the backward heat equation

$$\begin{cases} \partial_t u + k\partial_{xx} u = 0 & \text{for } t > 0 \text{ and } 0 < x < \pi \\ u|_{x=0} = u|_{x=\pi} = 0 & \\ u|_{t=0} = f & \end{cases} \tag{6}$$

is unstable with respect to the sup norm

$$\|g\|_{\sup} := \sup_{0 \le x \le \pi} |g(x)|.$$

Here, the constant $k > 0$ and the initial data $f : [0, \pi] \to \mathbb{R}$ are given. Complete the following steps:

(i) Verify that for any positive integer n, the function

$$u_n(t, x) := \frac{1}{n} e^{kn^2 t} \sin nx$$

is a solution to the initial and boundary value problem (6) with the initial data

$$f_n(x) := \frac{1}{n} \sin nx.$$

(ii) Prove that

$$\lim_{n \to +\infty} \|f_n\|_{\sup} = 0.$$

(iii) Show that for any $T > 0$,

$$\lim_{n \to +\infty} \|u_n(T, \cdot)\|_{\sup} = +\infty.$$

According to Part (i)-(iii), one may make the following conclusion:

> *no matter how small the initial data is, the solution to* (6) *may be large at any later time* $T > 0$.

In other words,

> *the initial and boundary value problem* (6) *of the backward heat equation is unstable with respect to the sup norm* $\|\cdot\|_{\sup}$.

Physically, the instability is due to the *anti-diffusion*.

Food for Thought. Is the initial and boundary value problem of backward heat equation (6) well-posed?

Problem 4 (Invariant Transformations/Symmetries)**.** Let $d \geq 2$ be an integer, and $u := u(t, x_1, x_2, \ldots, x_d)$ be a C^2 solution to the homogeneous wave equation in d spatial dimensions:

$$\partial_{tt} u - c^2 \Delta u = 0, \tag{7}$$

where the wave speed $c > 0$ is a given constant, and the Laplacian

$$\Delta := \sum_{k=1}^{d} \partial_{x_k x_k}.$$

Show that the v defined in each of the cases below is also a solution to (7).

i (Translation) For any $\tilde{t} \in \mathbb{R}$ and $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_d) \in \mathbb{R}^d$,

$$v(t, x_1, x_2, \ldots, x_d) := u(t - \tilde{t}, x_1 - \tilde{x}_1, x_2 - \tilde{x}_2, \ldots, x_d - \tilde{x}_d).$$

ii (Differentiation) For any positive integers α_1, α_2, $\ldots$, α_d,

$$v := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} u.$$

iii (Convolution) For any continuous function $g : \mathbb{R}^d \to \mathbb{R}$ with compact support,

$$\begin{aligned} &v(t, x_1, x_2, \ldots, x_d) := (v * g)(t, x_1, x_2, \ldots, x_d) \\ := &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(t, x_1 - \tilde{x}_1, x_2 - \tilde{x}_2, \ldots, x_d - \tilde{x}_d) g(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_d)\, d\tilde{x}_1 d\tilde{x}_2 \cdots d\tilde{x}_d. \end{aligned}$$

iv (Dilation/Scaling) For any constant $a > 0$,

$$v(t, x_1, x_2, \dots, x_d) := u(at, ax_1, ax_2, \dots, ax_d).$$

Remark. Symmetries/Invariant transformations play an important role in the study of PDE. For example, they can explain why we have so many conserved quantities, such as energy, momenta, etc., in physical systems; see Noether's theorem

`https://en.wikipedia.org/wiki/Noether%27s_theorem`

for instance. Furthermore, for some PDE, symmetries may also be useful in finding its explicit solution formula; see Chapter 6 of Lecture Slides for more details.

The following problem(s) is/are *optional*:

Problem 5 (Invariant Transformations/Symmetries)**.** Let $u := u(t, x)$ be a C^2 solution to the viscous Burgers' equation: for any t, $x \in \mathbb{R}$,

$$\partial_t u + u \partial_x u = \partial_x^2 u. \tag{8}$$

Show that the v defined in each of the following cases is also a solution to (8).

i (Translation) For any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$,

$$v(t, x) := u(t - t_0, x - x_0).$$

ii (Scaling) For any constant $a > 0$,

$$v(t, x) := a\, u(a^2 t, ax).$$

iii (Galileian Boost) For any constant $\lambda > 0$,

$$v(t, x) := u(t, x - \lambda t) + \lambda.$$

Problem 6. The aim of this problem is to justify the following well-known fact:

`the Laplace's equation` $\Delta u = 0$ `is invariant under rotations.`

Let $u := u(y)$ be a function mapping $\mathbb{R}^r$ to $\mathbb{R}$, and satisfy the Laplace's equation $\Delta_y u := \sum_{i=1}^d \partial_{y_i y_i} u = 0$, and $Q := (q_{ij})_{i,j=1}^d$ be an orthogonal $d \times d$ matrix, that is, $QQ^T = I$. Define

$$v(x) := u(Qx) \qquad \text{for all } x \in \mathbb{R}^d.$$

You are asked to prove that $\Delta_x v := \sum_{i=1}^d \partial_{x_i x_i} v = 0$ by completing the following four steps.

(i) Verify that
$$\sum_{m=1}^d q_{km} q_{lm} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l. \end{cases}$$

(ii) Apply the chain rule to verify that
$$\partial_{x_i} v = \sum_{k=1}^d q_{ki} \partial_{y_k} u.$$

(iii) Apply the chain rule again to verify that
$$\partial_{x_i x_j} v = \sum_{k=1}^d \sum_{l=1}^d q_{ki} q_{lj} \partial_{y_k y_l} u.$$

(iv) Use Part (i) and (iii) to verify that
$$\Delta_x v := \sum_{m=1}^d \partial_{x_m x_m} v = 0.$$