MATH3301 Assignment 1 Part 2

1.

(1) G is an Abelian group.

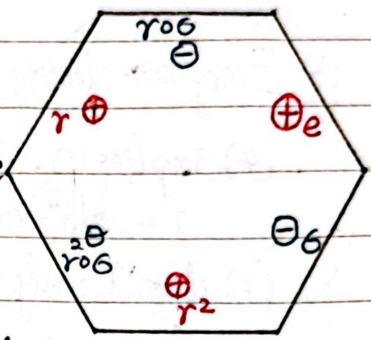
(1) doesn't imply (2), as $\mathbb{Z}$ is an infinite Abelian group under $+$.

(1) doesn't imply (3), as $\mathbb{Z}$ contains no nonidentity element $a \neq 0$ such that $2a = a + a = 0$, and $\mathbb{Z}$ is certainly Abelian.

(1) doesn't imply (4), as $\mathbb{Z}$ contains a nonidentity element $1 \neq 0$ such that $2 \cdot 1 = 1 + 1 \neq 0$, and $\mathbb{Z}$ is certainly Abelian.

(2) G is a finite group of even order.

(2) doesn't imply (1), as $D_6 = \{e, r, r^2, \sigma, r\circ\sigma, r^2\circ\sigma\}$ is a finite group of even order, but $\sigma\circ r = r^2\circ\sigma \neq r\circ\sigma$ for some $r, \sigma \in D_6$, so D_6 is not Abelian.



(2) implies (3). Assume to the contrary that $\forall a \in G - \{e\}$, $a^2 \neq e$, so for each a, $a^{-1} = a$ iff $a = e$. We pair up each a with a^{-1}, which gives the following partition of G:

$$G/\sim = \{\{e\}, \{a_1, a_1^{-1}\}, \{a_2, a_2^{-1}\}, \cdots, \{a_n, a_n^{-1}\}\}$$

This implies
$$|G| = |\{e\}| + |\{a_1, a_1^{-1}\}| + |\{a_2, a_2^{-1}\}| + \cdots + |\{a_n, a_n^{-1}\}|$$
$$= 1 + 2 + 2 + \cdots + 2 = 2n+1,$$
which is odd, so $\neg(3)$ implies $\neg(2)$

(2) doesn't imply (4), as $D_6 = \{e, r, r^2, \sigma, r\circ\sigma, r^2\circ\sigma\}$ is a finite group of even order, some nonidentity element r gives $r^2 \neq e$.

(3) There exists a non-identity element a in G satisfies $a^2=e$

(3) doesn't imply (1), as $D_6=\{e,r,r^2,\sigma,r\circ\sigma,r^2\circ\sigma\}$ contains a non-identity element σ, satisfying $\sigma^2=e$, but $\sigma\circ r=r^2\circ\sigma\neq r\circ\sigma$ for some $\sigma,r\in G$, so D_6 is not Abelian.

(3) doesn't imply (2), as $\mathbb{Q}^\times=\mathbb{Q}\setminus\{0\}$ contains a non-identity element -1, satisfying $(-1)^2=1$, but $\mathbb{Q}^\times$, i.e., the multiplicative group of nonzero rational numbers, is infinite.

(3) doesn't imply (4), as $D_6=\{e,r,r^2,\sigma,r\circ\sigma,r^2\circ\sigma\}$ contains a non-identity element σ, satisfying $\sigma^2=e$, but $r^2\neq e$ for some non-identity element r.

(4) Every non-identity element a in G satisfies $a^2=e$

(4) implies (1). For all $a,b\in G$:

$$a^2=e \text{ and } b^2=e \text{ and } (ab)^2=e \Rightarrow abab=e=aa=aea=abba \Rightarrow ab=ba$$

(4) doesn't imply (2), as mentioned in Tutorial 2, for all set S, if we equip $P(S)$ with a binary operation $\Delta: P(S)\times P(S)\to P(S)$, $A\Delta B=(A\setminus B)\cup(B\setminus A)$, then $P(S)$ is a group satisfying $\forall A\in P(S)$, $A\Delta A=(A\setminus A)\cup(A\setminus A)=\emptyset\cup\emptyset=\emptyset$ (identity). However, if we take $S=\mathbb{N}$ (an infinite set), then $P(S)$ is infinite.

For the implication (4) $\Rightarrow$ (3), it is necessary to assume that G has at least one nonidentity element a, otherwise the discussion here carries no meaning

(4) implies (3). If all $a\in G-\{e\}$ $(\neq\emptyset)$ does the job, then certainly, some $a\in G-\{e\}$ does the job.

2.(a) $H \cup K$ is not always a subgroup of G.

Consider the additive group of real vectors $\mathbb{R}^2$

both $\mathbb{R}\times\{0\}$ and $\{0\}\times\mathbb{R}$ are subgroups of $\mathbb{R}^2$,

but $(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})$ is not a subgroup of $\mathbb{R}^2$ as it is not closed under addition.

$H \cap K$ is always a subgroup of G

(1) $e \in H$ and $e \in K \Rightarrow e \in H \cap K$;

(2) $\forall a_1, a_2 \in G$, $a_1, a_2 \in H \cap K \Rightarrow a_1, a_2 \in H$ and $a_1, a_2 \in K$

$\Rightarrow a_1 a_2 \in H$ and $a_1 a_2 \in K \Rightarrow a_1 a_2 \in H \cap K$

(3) $\forall a \in G$, $a \in H \cap K \Rightarrow a \in H$ and $a \in K \Rightarrow a^{-1} \in H$ and $a^{-1} \in K \Rightarrow a^{-1} \in H \cap K$

Hence, $H \cap K$ is a subgroup of G

(b) $H \cup K$ is a subgroup of G iff $H \subseteq K$ or $K \subseteq H$

"if" direction: If $H \subseteq K$ or $K \subseteq H$, then $H \cup K = K$ or $H \cup K = H$,

so $H \cup K$ is a subgroup of G.

"only if" direction: Assume to the contrary that $H \nsubseteq K$ and $K \nsubseteq H$.

$H \nsubseteq K$ implies the existence of $h \in G$, such that $h \in H$ and $h \notin K$;

$K \nsubseteq H$ implies the existence of $k \in G$, such that $k \in K$ and $k \notin H$;

As $hk \in H \Rightarrow k = h^{-1}(hk) \in H \Rightarrow$ F, it should be true that $hk \notin H$;

As $hk \in K \Rightarrow h = (hk)k^{-1} \in K \Rightarrow$ F, it should be true that $hk \notin K$;

Hence, for some $h, k \in H \cup K$, $hk \notin H \cup K$; $H \cup K$ is not a subgroup of G as it is not closed under group operation in G

3.(a) Proof: First, we prove that $C(H)$ is a subgroup of G.

(1) $e \in G$ and $\forall h \in H$, $eh = he$, so $e \in C(H)$;

(2) For all $g_1, g_2 \in C(H)$:

$\forall h \in H, g_1 h = h g_1$ and $\forall h \in H, g_2 h = h g_2$

$\Rightarrow \forall h \in H, g_1 g_2 h = g_1 h g_2 = h g_1 g_2 \Rightarrow g_1 g_2 \in C(H)$

(3) For all $g \in C(H)$:

$\forall h \in H, gh = hg \Rightarrow \forall h \in H, g^{-1} h = g^{-1} h g g^{-1} = g^{-1} g h g^{-1} = h g^{-1} \Rightarrow g^{-1} \in C(H)$

Combine (1)(2)(3) above, we've proven that $C(H) \leq G$.

Second, we prove that Z is a subgroup of G.

As $Z = \{g \in G : gh = hg \text{ for all } h \in G\} = C(G) \leq G$, it follows that $Z \leq G$.

Third, we prove that $N(H)$ is a subgroup of G.

(1) $e \in G$ and $\forall h \in H$, $ehe^{-1} = e^{-1}he = h \in H$, so $e \in N(H)$;

(2) For all $g_1, g_2 \in N(H)$:

$[\forall h \in H, g_1 h g_1^{-1} \in H \text{ and } g_1^{-1} h g_1 \in H]$

and $[\forall h \in H, g_2 h g_2^{-1} \in H \text{ and } g_2^{-1} h g_2 \in H]$

$\Rightarrow \forall h \in H, [(g_1 g_2) h (g_1 g_2)^{-1} = g_1 (g_2 h g_2^{-1}) g_1^{-1} \in H]$

and $[(g_1 g_2)^{-1} h (g_1 g_2) = g_2^{-1} (g_1^{-1} h g_1) g_2 \in H]$

$\Rightarrow g_1 g_2 \in N(H)$

(3) For all $g \in N(H)$:

$\forall h \in H, g h g^{-1} \in H \text{ and } g^{-1} h g \in H$

$\Rightarrow \forall h \in H, [(g^{-1}) h (g^{-1})^{-1} = g^{-1} h g \in H] \text{ and } [(g^{-1})^{-1} h g^{-1} = g h g^{-1} \in H]$

$\Rightarrow g^{-1} \in N(H)$.

Combine (1)(2)(3) above, we've proven that $N(H) \leq G$.

(b) Z must be abelian. $\forall x_1, x_2 \in Z$,

$$\left.\begin{array}{l} x_1 \in Z \Rightarrow [\forall y \in G,\ x_1 y = y x_1] \\ x_2 \in Z \Rightarrow \qquad x_2 \in G \end{array}\right\} \Rightarrow x_1 x_2 = x_2 x_1.$$

$C(H)$ is not necessarily Abelian.

Choose an arbitrary non Abelian group, say $G = GL_2(\mathbb{R})$.

Choose $H = \{I\}$, then $C(\{I\}) = \{G \in GL_2(\mathbb{R}) : IG = GI\} = GL_2(\mathbb{R})$, which implies $C(\{I\})$ is non Abelian.

$N(H)$ is not necessarily Abelian.

Choose $\overset{\text{Group}}{G} = GL_2(\mathbb{R})$. Choose $H = SL_2(\mathbb{R}) = \{\overset{\text{Matrix}}{G} \in GL_2(\mathbb{R}) : \det(G) = 1\}$.

$$N(SL_2(\mathbb{R})) = \{\overset{\text{Matrix}}{G} \in GL_2(\mathbb{R}) : GHG^{-1} \in SL_2(\mathbb{R}) \text{ and } G^{-1}HG \in SL_2(\mathbb{R}) \text{ for all } H \in SL_2(\mathbb{R})\}$$

$$= \{G \in GL_2(\mathbb{R}) : \underbrace{\det(GHG^{-1})}_{\det(G)\det(H)\det(G)^{-1}} = 1 \text{ and } \underbrace{\det(G^{-1}HG)}_{\det(G)^{-1}\det(H)\det(G)} = 1 \text{ for all } H \in SL_2(\mathbb{R})\}.$$

$= GL_2(\mathbb{R})$, which implies $N(SL_2(\mathbb{R}))$ is non Abelian.