THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 8

Problem 1. The purpose of this problem is to construct a self-similar solution to the following PDE in the upper half-plane:

$$4\partial_{xx}u + \partial_{yy}u = 0 \qquad \text{for } -\infty < x < \infty \text{ and } y > 0. \tag{1}$$

(i) Assume that the solution u to (1) is of the form

$$u(x,y) = g\left(\frac{x}{y}\right)$$

where the function $g : \mathbb{R} \to \mathbb{R}$ will be determined below.

Derive an ODE for g.

(ii) Find the general solution of g and express the general self-similar solution u in terms of x and y only.

Problem 2. Consider the heat equation on the whole line

$$\partial_t u - \partial_{xx}u = 0 \quad \text{for } -\infty < x < \infty \text{ and } t > 0$$

subject to the initial data

$$u(x,0) = \phi(x) \quad \text{for } -\infty < x < \infty$$

where ϕ will be given differently in different parts below.

(i) Solve the initial value problem provided that

$$\phi(x) := e^{3x} \quad \text{for } -\infty < x < \infty.$$

Express your final answer without using any integrals.

(ii) Solve the initial value problem provided that

$$\phi(x) := \begin{cases} 1 & \text{if } |x| < 2 \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

Express your final answer in terms of the Gauss error function

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2} dp.$$

(iii) Solve the initial value problem provided that

$$\phi(x) := \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Express your final answer in terms of the Gauss error function erf as well.

Problem 3. Let $u(t,x)$ be a solution to the following equation

$$\partial_t u - \partial_{xx} u - 2\partial_x u + 2u = 0.$$

And let $w(x,t)$ be a function such that

$$u(x,t) = \exp(\alpha x + \beta t) w(x,t).$$

Find the values of α and β so that the above PDE for u can be transformed into the heat equation w

$$\partial_t w - \partial_{xx} w = 0.$$