THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 3 Solution

Problem 1.

(i) The solution to the given second-order ODE has the general form:

$$u_E(x) = \frac{g}{2c^2}x^2 + C_1 x + C_2$$

for some constants C_1, C_2. Then it follows from the boundary condition $u_E(0) = u_E(L) = 0$ that

$$C_1 = -\frac{gL}{2c^2} \text{ and } C_2 = 0.$$

So we conclude that

$$u_E(x) = \frac{g}{2c^2}x^2 - \frac{gL}{2c^2}x.$$

(ii) It follows that

$$\partial_{tt}v - c^2\partial_{xx}v = \partial_{tt}(u - u_E(x)) - c^2\partial_{xx}(u - u_E(x)) = \partial_{tt}u - c^2\partial_{xx}u + g = 0.$$

Problem 2.

(i) Write $\vec{F} = (4\partial_x u, \partial_y u)$, then

$$\nabla \cdot \vec{F} = 4\partial_{xx}u + \partial_{yy}u = -y^2\cos^2 \pi x.$$

Integrate the equation in Ω,

$$\int_\Omega \nabla \cdot \vec{F} = -\int_\Omega y^2 \cos^2 \pi x = -\left(\int_0^1 \cos^2 \pi x \,\mathrm{d}\,x\right)\left(\int_0^3 y^2 \,\mathrm{d}\,y\right) = -\frac{9}{2}.$$

Apply the divergence theorem,

$$\int_\Omega \nabla\cdot\vec{F} = \int_{\partial\Omega}\vec{F}\cdot n = \int_{\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4}\vec{F}\cdot n,$$

where

$$\begin{aligned}
\gamma_1 &:= \{(x,y)\colon\ 0<x<1,\ y=0\} \quad \text{with } n=(0,-1),\\
\gamma_2 &:= \{(x,y)\colon\ x=1,\ 0<y<3\} \quad \text{with } n=(1,0),\\
\gamma_3 &:= \{(x,y)\colon\ 0<x<1,\ y=3\} \quad \text{with } n=(0,1),\\
\gamma_4 &:= \{(x,y)\colon\ x=0,\ 0<y<3\} \quad \text{with } n=(-1,0).
\end{aligned}$$

Then

$$\begin{aligned}
\int_\Omega \nabla\cdot\vec{F} &= \int_{\gamma_1}\vec{F}\cdot n + \int_{\gamma_2}\vec{F}\cdot n + \int_{\gamma_3}\vec{F}\cdot n + \int_{\gamma_4}\vec{F}\cdot n\\
&= \int_0^1 -\partial_y u(x,0)\,\mathrm{d}\,x + \int_0^3 4\partial_x u(1,y)\,\mathrm{d}\,y + \int_1^0 \partial_y u(x,3)\,\mathrm{d}\,x + \int_3^0 -4\partial_x u(0,y)\,\mathrm{d}\,y\\
&= \int_{\gamma_1}\nabla u\cdot n + 4\int_{\gamma_2}\nabla u\cdot n + \int_{\gamma_3}\nabla u\cdot n + 4\int_{\gamma_4}\nabla u\cdot n = 26\beta.
\end{aligned}$$

Therefore, we can conclude that $\beta = -\frac{9}{52}$.

(ii) Integrate the equation in B,

$$\int_B \Delta u = -\int_B\left(\sum_{k=1}^d x_k^2\right).$$

The outward normal vector at $(x_1,\ldots,x_d)\in\partial B$ is given by $n=\frac{1}{3}(x_1,\ldots,x_d)$, a multiple of its position vector. Then by the divergence theorem, it follows that

$$\begin{aligned}
\int_B \Delta u = \int_B \nabla\cdot\nabla u = \int_{\partial B}\nabla u\cdot n &= \int_{\partial B}(\partial_{x_1}u,\ldots,\partial_{x_d}u)\cdot\frac{1}{\sqrt{3}}(x_1,\ldots,x_d)\\
&= \frac{C}{\sqrt{3}}\int_{\partial B}\sum_{k=1}^d x_k^2\sum_{j=1,j\neq k}^d x_j^2.
\end{aligned}$$

Observe that the integrand can be written as

$$\sum_{k=1}^{d} x_k^2 \sum_{j=1,j\neq k}^{d} x_j^2 = (x_1,\dots,x_d)\cdot\Big(x_1 \sum_{j=1,j\neq 1}^{d} x_j^2,\dots,x_d \sum_{j=1,j\neq d}^{d} x_j^2\Big) = \sqrt{3}n\cdot\vec{F}.$$

Then apply the divergence theorem again,

$$\begin{aligned}\int_B \Delta u = C\int_{\partial B} n\cdot\vec{F} = C\int_B \nabla\cdot\vec{F} &= \int_B \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(x_k \sum_{j=1,j\neq k}^{d} x_j^2\right)\\ &= C\int_B \sum_{k=1}^{d}\sum_{j=1,j\neq k}^{d} x_j^2 = C(d-1)\int_B\left(\sum_{k=1}^{d} x_j^2\right).\end{aligned}$$

So we can conclude that $C = \frac{1}{1-d}$.

Problem 3.

(i) Integrate the equation with respect to y, we arrive at

$$u(x,y) = \int_0^y \tan\left(\frac{\left(1+e^{-x^2}\right)t}{2}\right)\mathrm{d}\,t = -\frac{2}{\left(1+e^{-x^2}\right)}\ln\cos\left(\frac{\left(1+e^{-x^2}\right)y}{2}\right)+f(x).$$

Two boundary conditions imply that

$$\begin{aligned} u(x,0) = e^{-x^2} &\Rightarrow f(x) = e^{-x^2},\\ u(x,1) = g(x) &\Rightarrow g(x) = -\frac{2}{\left(1+e^{-x^2}\right)}\ln\cos\left(\frac{\left(1+e^{-x^2}\right)}{2}\right)+e^{-x^2}.\end{aligned}$$

(ii) The characteristics are governed by

$$\begin{cases}\frac{\mathrm{d}\,t}{\mathrm{d}\,s} = 3, & t(0)=t_0\\ \frac{\mathrm{d}\,x}{\mathrm{d}\,s} = -5, & x(0)=x_0\end{cases} \Rightarrow \begin{cases} t = 3s+t_0\\ x = -5s+x_0\end{cases} \Rightarrow t = -\frac{3}{5}(x-x_0)+t_0.$$

Choose $x_0=-2$, the characteristic curves can be parametrized by t_0,

$$C_{t_0} = \left\{(t,x)\colon t = -\frac{3}{5}(x+2)+t_0\right\}.$$

Note that $u(t,x)$ remains constant along the characteristics, then as $(t_0,-2) \in C_{t_0}$, it follows that

$$g(t_0) = u(t_0,-2) = u(-3+t_0,3) = h(t_0-3).$$

Problem 4.

(i) By chain rule,

$$\frac{\partial u}{\partial x} = \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta},$$
$$\frac{\partial u}{\partial y} = \frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta}.$$

Thus, the PDE in terms of the polar coordinate r and θ is given by

$$\frac{\partial u}{\partial \theta} = u.$$

(ii) Notice that the boundary

$$\{(x,y) \in \mathbb{R}^2 \colon\ x > 0,\ y = 0\} = \{(r,\theta) \in \mathbb{R}^2 \colon\ r > 0,\ \theta = 0\}.$$

Then the boundary condition in the polar coordinate is given by

$$u\Big|_{\theta=0} = \sin r \quad \text{for } r > 0.$$

(iii) From part (i) we have

$$u(r,\theta) = f(r)e^{\theta}$$

From the boundary condition, we can determine that $f(r) = \sin r$. So, the solution in terms of r and θ has the form

$$u(r,\theta) = \sin r e^{\theta}.$$

(iv) Express the answer u in terms of x and y, and we have

$$u(x,y) = \sin\sqrt{x^2+y^2}e^{\arctan\frac{y}{x}}.$$

Problem 5.

(i) By chain rule,

$$\frac{\partial u}{\partial x} = \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta},$$
$$\frac{\partial u}{\partial y} = \frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta}.$$

Thus, the PDE in terms of the polar coordinate r and θ is given by

$$-3\frac{\partial u}{\partial \theta} = 7u.$$

(ii) Notice that the boundary

$$\{(x,y) \in \mathbb{R}^2 \colon\ x > 0,\ y = 0\} = \{(r,\theta) \in \mathbb{R}^2 \colon\ r > 0,\ \theta = 0\}.$$

Then the boundary condition in the polar coordinate is given by

$$u\Big|_{\theta=0} = r \operatorname{erf} r \quad \text{for } r > 0.$$

(iii) From part (i) we have

$$u(r,\theta) = f(r)e^{-\frac{7}{3}\theta}$$

From the boundary condition, we can determine that $f(r) = r \operatorname{erf} r$. So, the solution in terms of r and θ has the form

$$u(r,\theta) = r(\operatorname{erf} r)e^{-\frac{7}{3}\theta}.$$

(iv) Express the answer u in terms of x and y, and we have

$$u(x,y) = \sqrt{x^2+y^2}\left(\operatorname{erf}\sqrt{x^2+y^2}\right)e^{-\frac{7}{3}\arctan\frac{y}{x}}.$$