MATH3541 INTRODUCTION TO TOPOLOGY
ASSIGNMENT V

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG

Due: 12:00 noon, 12th November 2024.

Instructions: Submit solutions to the problems in **Section B** for credit. Problems in Section A should be attempted and may be optionally submitted for feedback.

Guidelines on Writing: *You should write in complete sentences. Do not just give the solution in fragmentary bits and pieces. Clarity of presentation of your argument counts, so explain the meaning of every symbol that you introduce and avoid starting a sentence with a symbol.*

Section A

Problem 1. Let X be a nonempty topological space. Show that the cone CX is always contractible.

Problem 2 (A warm up on the notion of path-homotopy)**.** Let

$$X = \mathbb{R}^3 \backslash \{(0,0,z) \mid z \geq 0\}$$

and consider the two loops α and β in X at $p = (1,0,0)$ given by

$$\alpha(s) = (\cos(\pi s), \sin(\pi s), 0),\ \ \beta(s) = (\cos(\pi s), -\sin(\pi s), 0),\ \ s \in [0,1]$$

Show that the map $F\colon [0,1] \times [0,1] \to X$ given by

$$F(s,t) = (\cos(\pi s), \sin(\pi s)\cos(\pi t), -\sin(\pi s)\sin(\pi t)),$$

is a path-homotopy from α to β.

Problem 3. Let the paths $f, g, h : [0,1] \to X$ be paths in X, use $e_x : [0,1] \to X$ denote the constant path at $x \in X$. Recall that $f \star g$ is the concatenation of the paths f and g. Show via an explicit construction that the following are homotopic:

(a) the three paths f, $e_{f(0)} \star f$, and $f \star e_{f(1)}$;
(b) the two paths $f \star (g \star h)$ and $(f \star g) \star h$.

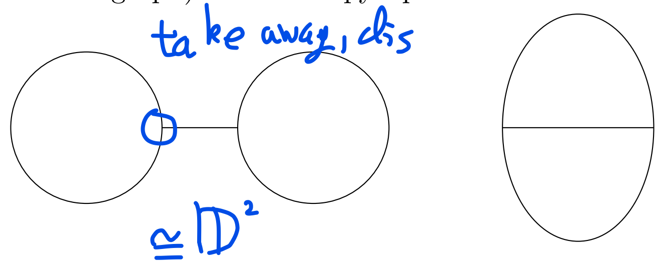
Problem 4 (Building familiarity with homotopy equivalence.)**.** Take the capital letters of the alphabet A, B, C, ... in a sans-serif font and consider them as 26 different graphs in the plane.

(a) Sort them into homeomorphism equivalence classes.
(b) Sort them into homotopy equivalence classes.

Clearly state any unproven assumptions you make.

Problem 5. Give an explicit argument showing that the connected sum $S^1 \vee S^1$ is a deformation retract of the punctured torus $T^2 \setminus \{x_0\}$, where x_0 is some point in the torus.

Problem 6. Show that the two subsets of the plane below (an "eyeglasses" graph and a "theta" graph) are homotopy equivalent but not homeomorphic.



Problem 7. Let $X = [0,1] \times [0,1]$. Prove that any continuous map $f\colon X \to X$ must have a fixed point.

Problem 8. Does the Borsuk—Ulam theorem hold for tori? Specifically, for a continuous map $f : S^1 \times S^1 \to \mathbb{R}^2$, must there exist $(x, y) \in S^1 \times S^1$ such that $f(-x, -y) = f(x, y)$?

Section B

Problem 9 (4 marks)**.** For any n-dimensional manifold M with boundary, prove that

(a) $\mathrm{Int}M$ is an open subset of M and is an n-dimensional manifold without boundary.

(b) ∂M is closed in M and is an $(n-1)$-dimensional manifold without boundary.

Problem 10 (4 marks)**.** Let $f\colon S^1 \to S^1$ be a continuous map.

(a) Suppose that f is not homotopic to the identity. Show that $f(x) = -x$ for some point $x \in S^1$.

(b) Show that the map $f\colon S^1 \to S^1$ defined by $x \mapsto -x$ is homotopic to the identity.

Now let $f\colon X \to S^n$ be a continuous map which is not surjective.

(c) Show that f is homotopic to a constant map from X to S^n.

Problem 11 (4 marks)**.** Let X be a topological space.

(a) Prove that a topological space X is path-connected if and only if all continuous maps $[0,1] \to X$ are homotopic to one another.

(b) Use this to prove that path connectedness is preserved under homotopy equivalence.

Problem 12 (4 marks)**.** Let X be a path-connected space. Regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps

$$(S^1, s_0) \to (X, x_0).$$

Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$, with no conditions on basepoints. We have a natural map $\phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints.

(a) Show that ϕ is surjective.

(b) Prove that $\phi([f]) = \phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

This shows that ϕ induces a bijection between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$.

Problem 13 (4 marks)**.** Consider the map $f : S^1 \times [0, 1] \to S^1 \times [0, 1]$ from the cylinder to itself given by $(\theta, t) \mapsto (\theta + 2\pi t, t)$.

(a) Construct a homotopy from f to the identity map which is constant on one of the boundary circles.

(b) Prove that there is no homotopy from f to the identity map which is constant on both boundary circles. [Hint: consider what f does to some "vertical" path $s \mapsto (\theta_0, s)$.]