**Thm** Let $G \times X \to X$ be a discontinuous action then $\pi_1(X/G) = G$.

pf of this thm & the unproved lemma in the calculation of $\pi_1(S^1)$ use the same technique of covering space, will be provided later.

**Applications**

1) $\pi_1(T^2 = \mathbb{R}^2/\mathbb{Z}^2) \cong \mathbb{Z}^2$

2) $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2 \quad n \geq 2$

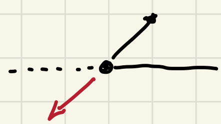
3) $\pi_1(\text{Klein bottle}) = G = \langle a, b \mid aba = b \rangle$

**Non example**

$$\mu_2 \times \mathbb{C} \to \mathbb{C}$$
$$(-1)\cdot z = -z$$

$$\mathbb{C}/\mu_2 \cong \mathbb{C}$$

$$\overline{\mathbb{H}} = \{z \mid \operatorname{Im} z \geq 0\}$$



$$\mathbb{C}/\mu_2 \cong \overline{\mathbb{H}} \big/ (x,0) \sim (-x,0) \longrightarrow \mathbb{C}$$
$$z \longrightarrow z^2$$

clearly $\pi_1(\mathbb{C}/\mu_2) \not\cong \mu_2$

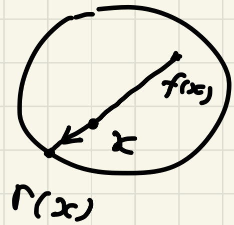
## Thm [Brouwer]

$$D^n = \{ v \in \mathbb{R}^n \mid |v| \le 1 \}$$

Any continuous map $f : D^n \to D^n$ has a fixed pt, i.e. $f(x_0) = x_0$.

## Pf of $n = 2$ case

Suppose $f$ has no fixed point




$$|s(x - f(x)) + x| = 1$$

$$\|$$

$$|(s+1)x - s f(x)|$$

$$\langle (s+1)x - s f(x), (s+1)x - s f(x) \rangle = 1$$

$$(s+1)^2 |x|^2 + s^2 |f(x)|^2 - 2s(s+1)\langle x, f(x) \rangle = 1$$

has two solutions $s_+ > 0$ $s_- < 0$

Define $r(x) = s_+(x - f(x)) + x$

$$F: D^2 \longrightarrow D^2$$

$$F(x,t) = (1-t)\,x + t \cdot r(x)$$

is a deformation retract $D^2 \longrightarrow \partial D^2$

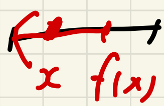
it follows that $D^2 \simeq \partial D^2 \cong S^1$

then $\underset{\{1\}}{\underset{\parallel}{\pi_1(D^2)}} \cong \pi_1(S^1) = \mathbb{Z}$

contradiction!

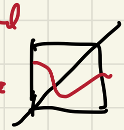
## $n = 1$ case

$f: [0,1] \longrightarrow [0,1]$




if $f$ has no fixed pt

can be proved by mean value thm!



$r(x): [0,1] \longrightarrow \{0,1\}$ is a retraction

contradicts connectedness of $[0,1]$.

# Fundamental thm of algebra

Every poly $f(x) \in \mathbb{C}[x]$ of deg $> 0$ has a root in $\mathbb{C}$.

Pf. Let $f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$

$n > 0$ $\quad a_0 \neq 0$ $\quad$ Assume $f(z)$ has no roots.

$r > 0$

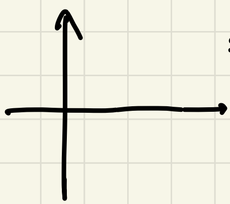
$$g_r : z \longmapsto \frac{f(rz)}{|f(rz)|} \; ; \; S^1 \to S^1$$

$f(z)$ has a root $\iff$ $f(rz)$ has a root

$$g_0 = \frac{a_0}{|a_0|} \qquad g_\infty = z^n$$

$$[0, \infty] \xleftrightarrow{\cong} [0,1] \qquad \frac{t}{1-t}$$

$$g_t := \frac{f\left(\frac{t}{1-t} z\right)}{\left|f\left(\frac{t}{1-t} z\right)\right|}$$



$$\begin{matrix} g_{t=0} & \sim & g_{t=1} \\ \| & & \| \\ g_{r=0} & & g_{r=\infty} \end{matrix}$$

contradiction!

# Covering space

**Def'n** $p : \tilde{X} \to X$ continuous surjection

$p$ is called a covering map if

$\forall\ x \in X\ \exists\ \mathcal{U} \ni x$ s.t

1) $p^{-1}(\mathcal{U}) = \bigsqcup_{\lambda \in \Lambda} \tilde{\mathcal{U}}_\lambda$

2) $p|_{\tilde{\mathcal{U}}_\lambda} : \tilde{\mathcal{U}}_\lambda \xrightarrow{\cong} \mathcal{U}$ is a homeomorphism

**Rmk**

1) $X$ top space $Y$ a discrete space

$\pi_X : X \times Y \to X$ is a covering map

2) $z \longmapsto z^n : \mathbb{C}^* \to \mathbb{C}^*$ is a covering map

$z^n : \mathbb{C} \to \mathbb{C}$ is NOT a covering map.

check fiber at 0

3) $x \to e^{2\pi i x} : \mathbb{R} \to S^1$

is a covering map

3.5) $G \times X \to X$ a top. gp action on $X$ discontinuously

$\pi : X \to X/G$ is a covering map

$x \in \mathcal{U}$ $\quad \bigsqcup_{g \in G} g\mathcal{U} \xrightarrow{\pi} X/G$

since $\pi$ is open map

$\pi(g\mathcal{U})$ is open in $X/G$

$\pi^{-1}(\pi(g\mathcal{U})) = \bigsqcup_{g \in G} g\mathcal{U}$

4) $p : \tilde{X} \to X$, $q : \tilde{Y} \to Y$ are covering map

then so is $(p, q) : \tilde{X} \times \tilde{Y} \to X \times Y$

5) A covering map $p : \tilde{X} \to X$ is called finite

if $p^{-1}(x)$ is finite set $\forall x \in X$

Suppose $p_1 : X \to Y$, $p_2 : Y \to Z$ are

covering map and $p_2$ is $\boxed{\text{finite}}$ then

$p_2 \circ p_1$ is a covering map

cannot be relaxed

see Hawaii ring

6) $p : \tilde{X} \to X$ is covering map

then $\tilde{X}$ is locally a product

of an open subset of $X$ and a discrete space.

$$p^{-1}(\mathcal{U}) = \underbrace{\coprod_{\lambda\in\Lambda} \tilde{\mathcal{U}}_\lambda}_{\substack{\text{subspace top}\\ = \text{disjoint union}}} \xrightarrow{\cong} \underbrace{\mathcal{U}\times\Lambda}_{\text{basis}: \mathcal{B}\times\{\lambda\}}$$

← product top

$\Lambda$ discrete.

$$\tilde{x} \longmapsto (p(\tilde{x}), \lambda)$$

basis $\amalg\lambda$

$$\left(p|_{\tilde{\mathcal{U}}_\lambda}\right)^{-1}(x) \longleftarrow (x, \lambda)$$

Check that both are continuous.

7) Covering map is local homeomorphism

the opposite is not true!

8) if $X$ is connected then

$|p^{-1}(x)|$ is constant

$y \notin W_x$

$\Leftrightarrow W_y \cap W_x = \emptyset$

pf. for $x \neq x' \in X$

$W_x = \{ y \in X \mid |p^{-1}(y)| = |p^{-1}(x)| \}$

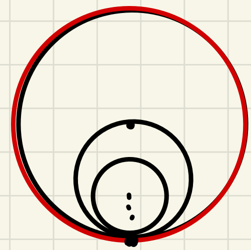
$W_x$ is both open & closed by the covering map property

Since $X$ is conn. $W_x = X$.

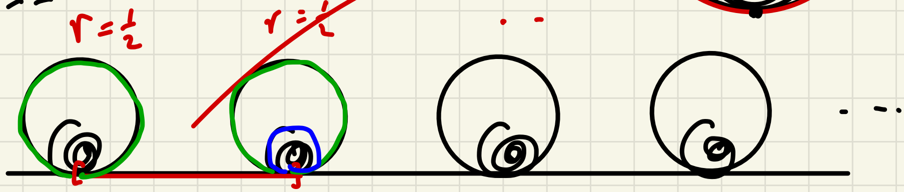
# Hawaii ring

equiped with subspace top.

$$H = \bigcup_{n \geq 1} S\left(\left(0, \tfrac{1}{n}\right), \tfrac{1}{n}\right)$$



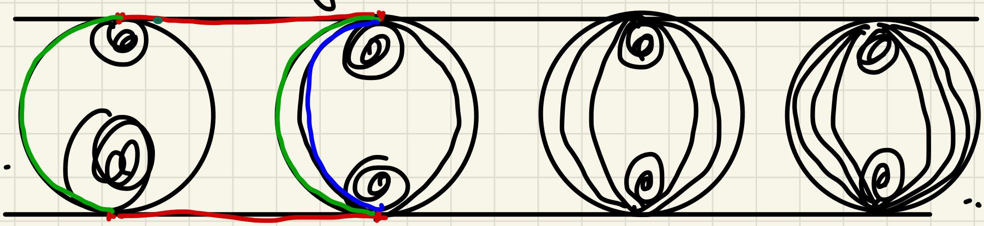
$Y = \cdots$




$p \downarrow$ is a covering map.

$H$

$X = \cdots$



$q : X \to Y$ is 2 : 1 covering map

**Claim**

1) $p \circ q$ is not a covering map

2) $p \circ q$ is a local homeomorphism

3) Hawaii ring $\not\cong$ infinite wedge

not even homotopic!