20240922 MATH3541

Problem 9.

(a) $\mathbb{S}^1 \ncong [0,1]$.

Proof: Assume to the contrary that $\mathbb{S}^1 \cong [0,1]$.

Then ($\mathbb{S}^1$ with a point removed) $\cong$ ($[0,1]$ with $\frac{1}{2}$ removed).

However, ($\mathbb{S}^1$ with a point removed) $\cong \mathbb{R}^1$, which is connected,

while ($[0,1]$ with $\frac{1}{2}$ removed) has a nontrivial open partition $\{[0,\frac{1}{2}),(\frac{1}{2},1]\}$,

so ($[0,1]$ with $\frac{1}{2}$ removed) is not connected, a contradiction.

(b) $\mathbb{S}^1 \ncong \mathbb{S}^2$

Proof: Assume to the contrary that $\mathbb{S}^1 \cong \mathbb{S}^2$.

Then ($\mathbb{S}^1$ with a point removed) $\cong$ ($\mathbb{S}^2$ with a point removed).

As ($\mathbb{S}^1$ with a point removed) $\cong \mathbb{R}^1$ and ($\mathbb{S}^2$ with a point removed) $\cong \mathbb{R}^2$,

we have $\mathbb{R}^1 \cong \mathbb{R}^2$, so ($\mathbb{R}^1$ with 0 removed) $\cong$ ($\mathbb{R}^2$ with $(0,0)$ removed).

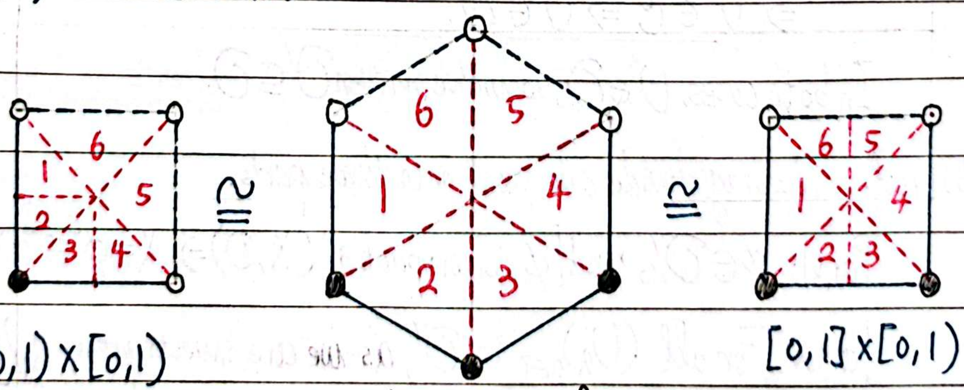
However, ($\mathbb{R}^1$ with 0 removed) has a nontrivial open partition $\{(-\infty,0),(0,+\infty)\}$,

so ($\mathbb{R}^1$ with 0 removed) is not connected, while ($\mathbb{R}^2$ with $(0,0)$ removed)

is path-connected, so ($\mathbb{R}^2$ with $(0,0)$ removed) is connected, a contradiction.

(c) $[0,1)\times[0,1) \cong [0,1]\times[0,1)$

Proof:



To be more specific, we construct a homeomorphism from $[0,1)\times[0,1)$ to $[0,1]\times[0,1)$

by gluing the following 6 disjoint homeomorphisms along their common edges:

$\sigma_1: 1\to 1,\ \sigma_1\left(\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}-1&-1\\-1&1\end{bmatrix}\begin{bmatrix}-1&-1\\0&1\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}$

$\sigma_2: 2\to 2,\ \sigma_2\left(\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}0&-1\\-1&-1\end{bmatrix}\begin{bmatrix}-1&-1\\-1&0\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}$

$\sigma_3: 3\to 3$, $\sigma_3\left(\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}1&0\\-1&-1\end{bmatrix}\begin{bmatrix}0&-1\\-1&-1\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}$

$\sigma_4: 4\to 4$, $\sigma_4\left(\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1&0\\-1&-1\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}$

$\sigma_5: 5\to 5$, $\sigma_5\left(\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}$

$\sigma_6: 6\to 6$, $\sigma_6\left(\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}u\\v\end{bmatrix}\right)=\begin{bmatrix}1/2\\1/2\end{bmatrix}+\begin{bmatrix}-1&0\\1&1\end{bmatrix}\begin{bmatrix}-1&1\\1&1\end{bmatrix}^{-1}\begin{bmatrix}u\\v\end{bmatrix}$

(d) $[0,1)\times[0,1) \ncong [0,1]\times[0,1]$

Proof: $[0,1)\times[0,1)$ has an open cover $\left([0,1-\frac{1}{2n})\times[0,1-\frac{1}{2n})\right)_{n\in\mathbb{N}}$ with no finite subcover, so $[0,1)\times[0,1)$ is not compact. While $[0,1]$ is closed and bounded in $\mathbb{R}$ $\Rightarrow$ $[0,1]$ is compact $\Rightarrow$ $[0,1]\times[0,1]$ is compact. Hence, $[0,1)\times[0,1)\ncong[0,1]\times[0,1]$.

Problem 10.

(a) Proof: For all $U\in\mathcal{O}'$, there are two cases to consider.

Case 1: In this case, $U=\phi$.

By the definition of topological space, $U=\phi\in\mathcal{O}$;

Case 2: In this case, U^c is compact in $(X,\mathcal{O})$.

U^c is compact in a Hausdorff topological space $(X,\mathcal{O})$ $\Rightarrow U^c\in\mathcal{C}\Rightarrow U\in\mathcal{O}$;

In both cases, $U\in\mathcal{O}$, so we've proven $\mathcal{O}'\subseteq\mathcal{O}$

(b) Proof: We may divide our proof into three parts.

Part 1: $\phi\in\mathcal{O}'$, and [ϕ is compact in $(X,\mathcal{O})\Rightarrow X=\phi^c\in\mathcal{O}'$]

Part 2: For all $(U_\lambda)_{\lambda\in I}$ in $\mathcal{O}'$, as we are investigating $\bigcup_{\lambda\in I}U_\lambda$, we may remove ϕ from $(U_\lambda)_{\lambda\in I}$ and assume each U_λ^c is compact in $(X,\mathcal{O})$.
As $(X,\mathcal{O})$ is Hausdorff, each $U_\lambda^c\in\mathcal{C}$, which implies $\left(\bigcup_{\lambda\in I}U_\lambda\right)^c=\bigcap_{\lambda\in I}U_\lambda^c\in\mathcal{C}$.
As $\left(\bigcup_{\lambda\in I}U_\lambda\right)^c$ is a closed subset of a compact topological space U_λ^c, it is compact.
This implies $\bigcup_{\lambda\in I}U_\lambda\in\mathcal{O}'$, i.e., $\mathcal{O}'$ is closed under arbitrary union.

Part 3: For all $(U_k)_{k=1}^{m}$ in $\mathcal{O}'$, as some $U_k=\phi$ implies $\bigcap_{k=1}^{m} U_k=\phi\in\mathcal{O}'$,

we may remove ϕ from $(U_k)_{k=1}^{m}$ and assume that each U_k^c is compact in $(X,\mathcal{O})$.

For all open cover of $\left(\bigcap_{k=1}^{m} U_k\right)^c=\bigcup_{k=1}^{m} U_k^c$, it is an open cover of each U_k^c.

Each U_k^c is compact, so each U_k^c has a finite subcover of the original cover.

Take finite union, and we get a finite subcover of the original cover for $\left(\bigcap_{k=1}^{m} U_k\right)^c$.

This implies $\bigcap_{k=1}^{m} U_k\in\mathcal{O}'$, i.e., $\mathcal{O}'$ is closed under finite intersection.

Combine the three parts above, we've proven that $(X,\mathcal{O}')$ is a topological space.

(c) Proof: It suffices to prove that any nonempty $U_1,U_2\in\mathcal{O}'$ are intersecting.

Assume to the contrary that some nonempty $U_1,U_2\in\mathcal{O}'$ are disjoint.

$U_1,U_2\in\mathcal{O}'\setminus\{\phi\}$ and $U_1\cap U_2=\phi$

$\Rightarrow$ U_1^c, U_2^c are compact in $(X,\mathcal{O})$ and $U_1^c\cup U_2^c=X$

$\Rightarrow$ X is compact in $(X,\mathcal{O})$ $\Rightarrow$ |F

Hence, our assumption is false, and we've proven that $U_1,U_2\in\mathcal{O}'$ are intersecting.

This implies $(X,\mathcal{O}')$ is not Hausdorff as long as $|X|\geq 2$.

(d) Proof: For all $(U_\lambda)_{\lambda\in I}$ in $\mathcal{O}'$, assume that $\bigcup_{\lambda\in I} U_\lambda = X\ (\neq\phi)$

Pick U_{λ_1} from $(U_\lambda)_{\lambda\in I}$. If $U_{\lambda_1}=\phi$, then repeat until it is nonempty.

Notice that $U_{\lambda_1}^c$ is compact in $(X,\mathcal{O})$ and $(U_\lambda)_{\lambda\in I\setminus\{\lambda_1\}}$ covers $U_{\lambda_1}^c$, so we can find $(\lambda_k)_{k=2}^{m}$ in I such that $(U_{\lambda_k})_{k=2}^{m}$ covers $U_{\lambda_1}^c$.

This implies $(U_{\lambda_k})_{k=1}^{m}$ covers X, hence, X is compact in $(X,\mathcal{O}')$.

Problem 11.

(a) Proof: For all $V \in \mathcal{O}_X$, we wish to prove that $f(V) \in \mathcal{O}_Y$.

To prove this, we rewrite $f(V)$ as a union of open subsets of Y.

As $\forall \alpha \in X, \exists U_\alpha \in \mathcal{O}_X$ with $U_\alpha \ni \alpha$, $f|_{U_\alpha}: U_\alpha \to Y$ is open,

$\forall \alpha \in X$, $V \cap U_\alpha$ is open in $U_\alpha \Rightarrow f|_{U_\alpha}(V \cap U_\alpha)$ is open in Y.

Hence, $f(V) = f\left(\bigcup_{\alpha \in X}(V \cap U_\alpha)\right) = \bigcup_{\alpha \in X} f(V \cap U_\alpha) = \bigcup_{\alpha \in X} f|_{U_\alpha}(V \cap U_\alpha) \in \mathcal{O}_Y$

This implies f is open.

(b) Proof: p is a local homeomorphism $\Rightarrow$ p is open.

For all open set U, $p(U)$ is open, so $q \circ p(U) = q(p(U))$ is open.

This implies $q \circ p$ is open.

(c) Proof: Assume that $p(z) = p_1(z) + i p_2(z)$

or $p(z_1, z_2) = p_1(z_1, z_2) + i p_2(z_1, z_2)$ $\quad (z = z_1 + i z_2)$

p is differentiable at $(0,0) \Rightarrow \exists A_1, A_2 \in \mathbb{R}$, $\displaystyle\lim_{(z_1,z_2)\to(0,0)} \frac{|p(z_1,z_2) - p(0,0) - (A_1 + iA_2)(z_1 + iz_2)|}{\sqrt{z_1^2 + z_2^2}} = 0$

$\Rightarrow \displaystyle\lim_{(z_1,z_2)\to(0,0)} \frac{|p_1(z_1,z_2) - p_1(0,0) - (A_1 z_1 - A_2 z_2)|}{\sqrt{z_1^2+z_2^2}} = \lim_{(z_1,z_2)\to(0,0)} \frac{|p_2(z_1,z_2) - p_2(0,0) - (A_2 z_1 + A_1 z_2)|}{\sqrt{z_1^2+z_2^2}} = 0$

$\Rightarrow p_1, p_2$ are differentiable at $(0,0) \Rightarrow \vec{p} = (p_1, p_2)$ is differentiable at $(0,0)$.

As p' is continuous at $(0,0)$ with $p'(0,0) = A_1 + iA_2 \neq 0$

Jacobean Matrix $D\vec{p} = \begin{pmatrix} \frac{\partial p_1}{\partial z_1} & \frac{\partial p_1}{\partial z_2} \\ \frac{\partial p_2}{\partial z_1} & \frac{\partial p_2}{\partial z_2} \end{pmatrix}$ is continuous at $(0,0)$ with

$\det D\vec{p}(0,0) = \det\begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} = A_1^2 + A_2^2 = |A_1 + iA_2|^2 \neq 0$

According to Inverse Function Theorem, $\vec{p}$ is a local homeomorphism, so is p

(d) Proof: As q sends all open disk $B(0,r)$ to open disk $B(0,r^d)$, q is locally open at 0.

(e) Proof: Assume that the non-constant polynomial $p(z)=A^d z^d+\cdots$, where $d\in\mathbb{N}, A\neq 0$.

Notice that if we define $q(z)=\sqrt[d]{p(z)}$ near 0, then:

(1) $\lim\limits_{z\to 0}\dfrac{|q(z)^d-A^dz^d|}{|z|^d}=0$ as $p(z)=q(z)^d$ is a polynomial with leading term A^dz^d;

(2) $\lim\limits_{z\to 0}\dfrac{q(z)^d}{z^d}=A^d$ as $\left|\dfrac{q(z)^d}{z^d}-A^d\right|$ is an infinitesimal;

(3) $\exists$ d^{th} unit root ω, $\lim\limits_{z\to 0}\dfrac{q(z)}{z}=A\omega$ as q is assumed to be continuous;

(4) $\lim\limits_{z\to 0}\dfrac{|q(z)-A\omega z|}{|z|}=0$ as $\left|\dfrac{q(z)}{z}-A\omega\right|$ is an infinitesimal;

(5) $q(z)=\sqrt[d]{p(z)}$ is differentiable at 0, where $q'(0)=A\omega\neq 0$.

So we obtain a differentiable function $q(z)$ near 0 with $q'(z)=\begin{cases}A\omega, & \text{if } z=0;\\ \dfrac{p'(z)}{d\,q(z)^{d-1}}, & \text{if } z\neq 0;\end{cases}$

As $\lim\limits_{z\to 0}q'(z)=\lim\limits_{z\to 0}\dfrac{p'(z)}{d\,q(z)^{d-1}}=\lim\limits_{z\to 0}\dfrac{p'(z)/z^{d-1}}{d[q(z)/z]^{d-1}}=\dfrac{dA^d}{d(A\omega)^{d-1}}=\omega A=q'(0)$,

q is of class C^1 near 0, which implies q is a local homeomorphism.

(f) Proof: It suffices to prove that p is locally open.

For all $z_0\in\mathbb{C}$, there exists $d\in\mathbb{N}$ and $A\in\mathbb{C}\setminus\{0\}$, such that:

$$p(z)=p(z_0)+A^d(z-z_0)^d+\cdots$$

As suggested, there exists a local homeomorphism $q(z)=\sqrt[d]{p(z)-p(z_0)}$ near z_0

Hence, define open map $r(z)=p(z_0)+z^d$, $p=r\circ q$ is open.

Problem 12:

(a) Proof: Assume to the contrary that $\bigcap_{n=1}^{+\infty} A_n = \emptyset$.

According to De Morgan's Law, $\bigcup_{n=1}^{+\infty} A_n^c = X$, so $(A_n)_{n\in\mathbb{N}}$ is an open cover of $(X, \mathcal{O}_X)$.

According to compactness, there exists $N \in \mathbb{N}$, such that $\bigcup_{n=1}^{N} A_n^c = X$.

But $\bigcap_{n=1}^{N} A_n = A_N \neq \emptyset$, a contradiction!

Hence, our assumption is false, and we've proven that $\bigcap_{n=1}^{+\infty} A_n \neq \emptyset$.

(b) Proof: For all nonempty open set U and for all $x \in X$,

as x is not isolated, $\{x\}$ is not open, so $\{x\} \neq U$.

$U \not\subseteq \{x\}$ as $U \neq \emptyset$, so there exists $y \in U$ with $y \neq x$.

As $(X, \mathcal{O}_X)$ is Hausdorff, there exist $V, W \in \mathcal{O}_X$ such that

$y \in V$ and $x \in W$ and $V \cap W = \emptyset$. I can assume $V \subseteq U$ because intersection $V \cap U$ is allowed.

I claim that $x \notin \overline{V}$, because, there exists $W \in \mathcal{O}_X$ with $W \subseteq V^c$,

such that $x \in W$, so this nonempty open set V is our desired set.

(c) Proof: $x_1 \in X$ and X is a nonempty open set

$\Rightarrow$ There exists a nonempty open set $V_1 \subseteq X$, such that $x_1 \notin \overline{V_1}$

$x_2 \in X$ and V_1 is a nonempty open set

$\Rightarrow$ There exists a nonempty open set $V_2 \subseteq V_1$, such that $x_2 \notin \overline{V_2}$

$\vdots$

$x_{n+1} \in X$ and V_n is a nonempty open set

$\Rightarrow$ There exists a nonempty open set $V_{n+1} \subseteq V_n$, such that $x_{n+1} \notin \overline{V_{n+1}}$

$\vdots$

Hence, we've constructed a nested sequence of nonempty open sets $(V_n)_{n\in\mathbb{N}}$,

such that for all $n \in \mathbb{N}$, $x_1, x_2, \cdots, x_n \notin \overline{V_n}$.

This implies $f(\mathbb{N}) = \{x_1, x_2, \cdots, x_n, \cdots\}$ is disjoint with $\bigcap_{n=1}^{+\infty} \overline{V_n}$, which is nonempty.

To conclude, $f(\mathbb{N})$ misses at least one point in X, X is not countable.