MATH6101/MATH7101/EX1/NM/ML/25-26

THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH6101/MATH7101 Intermediate Complex Analysis

Assignment 1

Due Date: October 23, 2025

(Send scanned copies of your solutions to Mimi Lui at mimi@hku.hk.)

1. (a) Let $\{x_1, \dots, x_p\}$ be a set of p distinct points, $p > 0$, on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \amalg \{\infty\}$ and let $n_1, \dots, n_p$ be non-zero integers such that $n_1 + \cdots + n_p = 0$. Show that there exists a nonconstant meromorphic function f on $\mathbb{P}^1$ such that $\mathrm{ord}_{x_k}(f) = n_k$ for $1 \le k \le p$ and $\mathrm{ord}_x(f) = 0$ for every point x on $\mathbb{P}^1$ not belonging to $\{x_1, \dots, x_p\}$. $\Big[$Here $\mathrm{ord}_a(f) = s$ is the zero order at a if $s > 0$, $-\mathrm{ord}_a(f) = -s$ is the pole order at a if $s < 0$, and $\mathrm{ord}_a(f) = 0$ if and only f is holomorphic and non-zero at $a.\Big]$.

   (b) Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ be a lattice and define $X := \mathbb{C}/L$. Let $\{x_1, \dots, x_p\}$, be p distinct points on $X, p > 0$, such that $x_k + x_\ell \neq 0$ on X for $1 \le k, \ell \le p$ (when $X = \mathbb{C}/L$ is regarded as a commutative group). For $1 \le k \le p$ let n_k be non-zero integers such that $n_1 + \cdots + n_p = 0$. Prove using the Weierstrass $\wp$-function that there exists a meromorphic function f on X such that $\mathrm{ord}_{x_k}(f) = \mathrm{ord}_{-x_k}(f) = n_k$, and $\mathrm{ord}_x(f) = 0$ for any $x \in X$ such that $x \notin \{x_1, \dots, x_p; -x_1, \dots, -x_p\}$.

2. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and write $L = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$ for the lattice generated by ω_1 and ω_2. Write $L' := L - \{0\}$.

   (a) Show that for $k \ge 3$ we have $\sum_{\omega \in L'} \frac{1}{|\omega|^k} < \infty$.

   (b) Show that $\sum_{\omega \in L} \frac{z+\omega}{(z+\omega)^6 - 1}$ converges in an appropriate sense to an elliptic function. Describe the nature of the convergence and explain why the limiting function is indeed doubly periodic with respect to L.

   (c) Suppose $\omega_1 = 2$ and $\omega_2 = 2i$ and write $f(z) := \sum_{\omega \in L} \frac{z+\omega}{(z+\omega)^6 - 1}$. Determine the poles of f and the pole order at each of the poles. Regarding f equivalently as a meromorphic function h on $X = \mathbb{C}/L$, counting multiplicities how many zeros of h are there on X? Explain why.

3. Given a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, and writing $L^* = L - \{0\}$ define

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in L^*} \left(\frac{1}{z+\omega} + \frac{z}{\omega^2} - \frac{1}{\omega} \right).$$

Check using 2(a) that the following holds true: For any $R > 0$, on $D(R)$ one can decompose $\zeta(z)$ as a sum of a finite number of meromorphic functions and an infinite sum of holomorphic functions such that the latter sum converges uniformly on $D(R)$ to a holomorphic function.

4. For a given lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, L' := L - \{0\}$, it is assumed known that $\wp(z) := \frac{1}{z^2} + \sum_{\omega \in L'} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right)$ converges in an appropriate sense to a meromorphic function on $\mathbb{C}$.

   (a) Prove that $\wp$ is indeed an elliptic function with respect to L.

   (b) Define

$$\begin{cases} f(z) = (\wp'(z))^2 \\ g(z) = \left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right) \end{cases}.$$

   Show that, counting multiplicities, f and g have the same zeros and the same poles. Hence deduce that

$$(\wp'(z))^2 = 4\left(\wp(z) - \wp\left(\frac{\omega_1}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_2}{2}\right)\right)\left(\wp(z) - \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right).$$

   (c) Assume <u>known</u> that $\wp$ satisfies the equation $(\wp')^2 = 4\wp^3 + a\wp + b$ for some complex numbers a, b (depending on L). Prove that

$$\wp\left(\frac{\omega_1}{2}\right) + \wp\left(\frac{\omega_2}{2}\right) + \wp\left(\frac{\omega_1+\omega_2}{2}\right) = 0.$$