MATH3541 Assignment 1.

SECTION B. Problem 13

(a) Proof: We may divide our proof into two parts.

Part 1: In this part, we prove that $\overline{\prod_{\lambda\in I} A_\lambda} \subseteq \prod_{\lambda\in I}\overline{A_\lambda}$.

For all $x \in \overline{\prod_{\lambda\in I} A_\lambda}$, for all $\lambda\in I$, let's prove that $x(\lambda)\in\overline{A_\lambda}$

That is, for all $V_\lambda \in \mathcal{C}_{X_\lambda}$ with $V_\lambda \supseteq A_\lambda$, let's prove that $x(\lambda)\in V_\lambda$.

As $\pi_\lambda : \prod_{\lambda\in I} X_\lambda \to X_\lambda$, $\pi_\lambda(x) = x(\lambda)$ is continuous,

the preimage of V_λ under π_λ is closed, i.e., $\pi_\lambda^{-1}(V_\lambda)\in\mathcal{C}_X$

As $x\in\overline{\prod_{\lambda\in I} A_\lambda}$ and $\pi_\lambda^{-1}(V_\lambda)\in\mathcal{C}_X$ and $\pi_\lambda^{-1}(V_\lambda)\supseteq\prod_{\lambda\in I}A_\lambda$,

it must be true that $x\in\pi_\lambda^{-1}(V_\lambda)$, so $x(\lambda)\in V_\lambda$, and we are done.

Part 2: In this part, we prove that $\overline{\prod_{\lambda\in I} A_\lambda} \supseteq \prod_{\lambda\in I}\overline{A_\lambda}$

For all $x\in\prod_{\lambda\in I}\overline{A_\lambda}$, let's prove that $x\in\overline{\prod_{\lambda\in I}A_\lambda}$.

That is, for all $V\in\mathcal{C}_X$ with $V\supseteq\prod_{\lambda\in I}A_\lambda$, let's prove that $x\in V$.

Proving this statement is logically equivalent to proving:

$$\forall W\in\mathcal{O}_X \text{ with } x\in W,\ W\cap\prod_{\lambda\in I}A_\lambda\neq\emptyset.$$

For all $W\in\mathcal{O}_X$ with $x\in W$, as X has a subbasis $\mathcal{B}_X=\{\pi_\lambda^{-1}(U_\lambda)\}_{\lambda\in I \text{ and } U_\lambda\in\mathcal{O}_{X_\lambda}}$,

without loss of generality, assume that $W=\bigcap_{k=1}^{n}\pi_{\lambda_k}^{-1}(W_{\lambda_k})$, where each

$W_{\lambda_k}\in\mathcal{O}_{X_{\lambda_k}}$. It is obvious that $x(\lambda_k)\in W_{\lambda_k}$, so $W_{\lambda_k}\cap A_{\lambda_k}\neq\emptyset$,

assume that $z_{\lambda_k}\in W_{\lambda_k}\cap A_{\lambda_k}$, construct the following map:

$$y: I\to\bigcup_{\lambda\in I}X_\lambda,\ y(\mu)=\begin{cases} z_{\lambda_k} & \text{if } \mu=\lambda_k \text{ for some } \lambda_k; \\ \text{arbitrary} & \text{if } \mu\neq\lambda_k \text{ for all } \lambda_k; \end{cases}$$

This $y\in W\cap\prod_{\lambda\in I}A_\lambda$, so $W\cap\prod_{\lambda\in I}A_\lambda\neq\emptyset$, which gives $x\in\overline{\prod_{\lambda\in I}A_\lambda}$

combine the two parts above, we've proven the biconditional.

(b) (c) Proof: We may divide our proof into three parts.

Part 1: In this part, we prove that $(\prod_{\lambda\in I} U_\lambda)^\circ \subseteq \prod_{\lambda\in I} U_\lambda^\circ$ holds for arbitrary I.

For all $x \in (\prod_{\lambda\in I} U_\lambda)^\circ$, $\exists V \in \mathcal{O}_X$ with $V \subseteq \prod_{\lambda\in I} U_\lambda$, $x \in V$.

Hence, as π_λ is an open map, for each $\lambda \in I$, $\exists \pi_\lambda(V) \in \mathcal{O}_{X_\lambda}$ with $\pi_\lambda(V) \subseteq U_\lambda = \pi_\lambda(\prod_{\lambda\in I} U_\lambda)$, $x(\lambda) \in \pi_\lambda(V)$, so $x \in \prod_{\lambda\in I} U_\lambda^\circ$.

For the reason why π_λ is an open map, consider an arbitrary $U \in \mathcal{O}_X$, as X has a subbasis $\mathcal{B}_X = \{\pi_\lambda^{-1}(U_\lambda)\}_{\lambda\in I \text{ and } U_\lambda \in \mathcal{O}_{X_\lambda}}$, without loss of generality, assume that $U = \bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(U_{\lambda_k})$, where each $U_{\lambda_k} \in \mathcal{O}_{X_{\lambda_k}}$.

Case 1: If $\lambda = \lambda_k$ for some λ_k, then $\pi_\lambda(U) = \pi_\lambda(\pi_\lambda^{-1}(U_\lambda)) = U_\lambda \in \mathcal{O}_{X_\lambda}$, as π_λ is surjective;

Case 2: If $\lambda \neq \lambda_k$ for all λ_k, then $\pi_\lambda(U) = X_\lambda \in \mathcal{O}_{X_\lambda}$.

In both cases, $\pi_\lambda(U)$ is open in X_λ, so π_λ is an open mapping.

Part 2: In this part, we prove that $(\prod_{k=1}^{m} U_k)^\circ = \prod_{k=1}^{m} U_k^\circ$, i.e., product commutes with interior if the product is finite.

It suffices to prove $(\prod_{k=1}^{m} U_k)^\circ \supseteq \prod_{k=1}^{m} U_k^\circ$. We prove it directly.

For all $x \in \prod_{k=1}^{m} U_k^\circ$, each $x(k) \in U_k^\circ$, so $\exists V_k \in \mathcal{O}_{X_k}$ with $V_k \subseteq U_k$, $x(k) \in V_k$. This gives the existence of $V = \bigcap_{k=1}^{m} \pi_k^{-1}(V_k) \in \mathcal{O}_X$ with $V \subseteq \prod_{k=1}^{m} U_k$, such that $x \in \bigcap_{k=1}^{m} \pi_k^{-1}(V_k)$. Hence, $x \in (\prod_{k=1}^{m} U_k)^\circ$.

Part 3: In this part, we prove $\prod_{n=1}^{+\infty}(-\frac{1}{n}, \frac{1}{n})$ is not open in $\mathbb{R}^{\mathbb{N}}$, so $(\prod_{n=1}^{+\infty}(-\frac{1}{n}, \frac{1}{n}))^\circ \subset \prod_{n=1}^{+\infty}(-\frac{1}{n}, \frac{1}{n})^\circ$.

Assume to the contrary that $\prod_{n=1}^{+\infty}(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}^{\mathbb{N}}$.

Construct $\Delta: \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, $\Delta(\xi) = (\xi)_{n=1}^{+\infty}$, as $\Delta^{-1}(\prod_{n=1}^{+\infty}(-\frac{1}{n}, \frac{1}{n})) = \{0\}$ is not open in $\mathbb{R}$, Δ is discontinuous. But each $\pi_\lambda \circ \Delta = id_{\mathbb{R}}$ is continuous, which violates "σ is continuous iff each $\pi_\lambda \circ \sigma$ is continuous".

Hence, the equality $(\prod_{\lambda\in I} U_\lambda)^\circ = \prod_{\lambda\in I} U_\lambda^\circ$ doesn't always hold for arbitrary I.

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SECTION B

Problem 14.

(A) The indiscrete topology $\mathcal{O}_A=\{\phi,[0,1]\}$ is the coarest.

(G) $\mathcal{O}_G=\{\phi,(0,1),[0,1]\}\supseteq\mathcal{O}_A=\{\phi,[0,1]\}$, so $\mathcal{O}_A$ is (strictly) coarser than $\mathcal{O}_G$.

(D) The cofinite topology $\mathcal{O}_D=\{\phi, O\in\mathcal{P}([0,1]):[0,1]-O \text{ is finite}\}$ is (strictly) finer than $\mathcal{O}_G$, because:

(i) $\phi\in\mathcal{O}_D$ and $(0,1)\in\mathcal{O}_D$ ($[0,1]-(0,1)=\{0,1\}$ is finite) and $[0,1]\in\mathcal{O}_D$, so $\mathcal{O}_G=\{\phi,(0,1),[0,1]\}\subseteq\mathcal{O}_D$

(ii) There exists $(0,\frac{1}{2})\cup(\frac{1}{2},1)\in\mathcal{P}([0,1])$, such that $(0,\frac{1}{2})\cup(\frac{1}{2},1)\notin\mathcal{O}_G$ and $(0,\frac{1}{2})\cup(\frac{1}{2},1)\in\mathcal{O}_D$ ($[0,1]-(0,\frac{1}{2})\cup(\frac{1}{2},1)=\{0,\frac{1}{2},1\}$ is finite).

(E) The cocountable topology $\mathcal{O}_E=\{\phi, O\in\mathcal{P}([0,1]):[0,1]-O \text{ is countable}\}$ is (strictly) finer than $\mathcal{O}_D$, because:

(i) Forall $O\in\mathcal{P}([0,1])$

$O\in\mathcal{O}_D\Rightarrow[0,1]-O$ is finite or $O=\phi$

$\Rightarrow[0,1]-O$ is countable or $O=\phi\Rightarrow O\in\mathcal{O}_E$,

so $\mathcal{O}_D\subseteq\mathcal{O}_E$

(ii) There exists $\{x\in[0,1]:\forall n\in\mathbb{N}, x\neq\frac{1}{n}\}\in\mathcal{P}([0,1])$, such that $\{x\in[0,1]:\forall n\in\mathbb{N}, x\neq\frac{1}{n}\}\in\mathcal{O}_E$ and $\{x\in[0,1]:\forall n\in\mathbb{N}, x\neq\frac{1}{n}\}\notin\mathcal{O}_D$

(c) The Euclidean (or metric) topology given by basis $\mathcal{B} = \{(a,b)\cap[0,1] \in \mathcal{P}([0,1]) : a,b\in\mathbb{R}, a<b\}$ is (strictly) finer than $\mathcal{O}_D$, because:

(i) For all $C \in \mathcal{P}([0,1])$:

$$C \in \mathcal{C}_D \Rightarrow C \text{ is finite or } C=[0,1]$$

$\Rightarrow$ There exists a strictly increasing finite list $a_1, a_2, \cdots, a_n \in [0,1]$, such that $C=\{a_1, a_2, \cdots, a_n\}$ or $C=\phi$ or $C=[0,1]$

$$\Rightarrow C = \Big(((-1,a_1)\cap[0,1]) \cup ((a_1,a_2)\cap[0,1]) \cup ((a_2,a_3)\cap[0,1]) \cup \cdots \cup ((a_{n-1},a_n)\cap[0,1])\Big)^C$$

$\in \mathcal{C}_C$, so $\mathcal{C}_D \subseteq \mathcal{C}_C$

(ii) There exists $(\frac{1}{3},\frac{2}{3})\cap[0,1] \in \mathcal{P}([0,1])$, such that $(\frac{1}{3},\frac{2}{3})\cap[0,1] \notin \mathcal{O}_D$ and $(\frac{1}{3},\frac{2}{3})\cap[0,1] \in \mathcal{O}_C$

But we cannot compare $\mathcal{O}_E$ and $\mathcal{O}_C$, because:

(i) There exists $(\frac{1}{3},\frac{2}{3})\cap[0,1] \in \mathcal{P}([0,1])$, such that $(\frac{1}{3},\frac{2}{3})\cap[0,1] \notin \mathcal{O}_E$ and $(\frac{1}{3},\frac{2}{3})\cap[0,1] \in \mathcal{O}_C$

(ii) There exists $\{x\in[0,1] ; \forall n\in\mathbb{N}, x\neq\frac{1}{n}\} \in \mathcal{P}([0,1])$, such that $\{x\in[0,1] : \forall n\in\mathbb{N}, x\neq\frac{1}{n}\} \notin \mathcal{O}_C$ (as $x=0$ fails to be interior) and $\{x\in[0,1] : \forall n\in\mathbb{N}, x\neq\frac{1}{n}\} \in \mathcal{O}_E$

(f) $\mathcal{O}_F = \{\underset{[0,1]}{X}\} \cup \mathcal{P}((0,1])$ is (strictly) finer than $\mathcal{O}_G = \{\phi, (0,1), [0,1]\}$

But we cannot compare $\mathcal{O}_F$ and $\mathcal{O}_D$, because:

(i) There exists $[0,1) \in \mathcal{P}([0,1])$ such that:

$$[0,1) \in \mathcal{O}_D \text{ and } [0,1) \notin \mathcal{O}_F$$

(ii) There exists $(\frac{1}{2},1] \in \mathcal{P}([0,1])$ such that:

$$(\tfrac{1}{2},1] \in \mathcal{O}_F \text{ and } (\tfrac{1}{2},1] \notin \mathcal{O}_D$$

so we cannot compare $\mathcal{O}_F$ and $\mathcal{O}_C$, $\mathcal{O}_F$ and $\mathcal{O}_E$ either.

(B) The discrete topology $\mathcal{O}_B = \mathcal{P}([0,1])$ is the finest.

$$\mathcal{O}_A \subset \mathcal{O}_G \subset \mathcal{O}_D \subset \mathcal{O}_E \subset \mathcal{O}_B$$

$$\mathcal{O}_A \subset \mathcal{O}_G \subset \mathcal{O}_D \subset \mathcal{O}_C \subset \mathcal{O}_B$$

$$\mathcal{O}_A \subset \mathcal{O}_G \subset \mathcal{O}_F \subset \mathcal{O}_B$$

$\mathcal{O}_E, \mathcal{O}_C$ cannot compare

$\mathcal{O}_F, \mathcal{O}_D$ cannot compare

$\mathcal{O}_F, \mathcal{O}_C$ cannot compare

$\mathcal{O}_F, \mathcal{O}_E$ cannot compare

(A) The indiscrete topology is not Hausdorff.

$$\exists \underset{(0\neq 1)}{0,1} \in [0,1],\ \forall O_0, O_1 \in \mathcal{O}_A,\ 0 \in O_0 \text{ and } 1 \in O_1 \Rightarrow O_0 = O_1 = [0,1]$$

$$\Rightarrow O_0 \cap O_1 = [0,1] \neq \emptyset$$

(B) The discrete topology is Hausdorff

$$\forall \underset{(x_1\neq x_2)}{x_1, x_2} \in [0,1],\ \exists \{x_1\}, \{x_2\} \in \mathcal{O}_B,\ x_1 \in \{x_1\} \text{ and } x_2 \in \{x_2\} \text{ and } \{x_1\} \cap \{x_2\} = \emptyset$$

(C) The Euclidean (or metric) topology is Hausdorff

$$\forall \underset{(x_1\neq x_2)}{x_1, x_2} \in [0,1],\ \exists B_{\frac{1}{2}d(x_1,x_2)}(x_1), B_{\frac{1}{2}d(x_1,x_2)}(x_2) \in \mathcal{O}_C,\ x_1 \in B_{\frac{1}{2}d(x_1,x_2)}(x_1) \text{ and}$$

$$x_2 \in B_{\frac{1}{2}d(x_1,x_2)}(x_2) \text{ and } B_{\frac{1}{2}d(x_1,x_2)}(x_1) \cap B_{\frac{1}{2}d(x_1,x_2)}(x_2) = \emptyset.$$

(D) The cofinite topology is not Hausdorff:

$\exists \overset{(0\neq 1)}{0,1} \in [0,1], \forall C_0, C_1 \in \mathcal{C}_D, 0 \notin C_0$ and $1 \notin C_1$

$\Rightarrow C_0$ is finite and C_1 is finite $\Rightarrow [0,1] \neq C_0 \cup C_1$, which is finite.

(E) The cocountable topology is not Hausdorff.

$\exists \overset{(0\neq 1)}{0,1} \in [0,1], \forall C_0, C_1 \in \mathcal{C}_E, 0 \notin C_0$ and $1 \notin C_1$

$\Rightarrow C_0$ is countable and C_1 is countable $\Rightarrow [0,1] \neq C_0 \cup C_1$, which is countable.

(F) $\mathcal{O}_F$ is not Hausdorff.

$\exists \overset{(0\neq 1)}{0,1} \in [0,1], \forall O_0, O_1 \in \mathcal{O}_F, 0 \in O_0$ and $1 \in O_1$

$\Rightarrow O_0 = [0,1]$ and $1 \in O_1 \Rightarrow 1 \in O_0$ and $1 \in O_1 \Rightarrow O_0 \cap O_1 \neq \emptyset$

(G) $\mathcal{O}_G$ is not Hausdorff.

$\exists \overset{(\frac{1}{3}\neq\frac{2}{3})}{\frac{1}{3},\frac{2}{3}} \in [0,1], \forall O_{\frac{1}{3}}, O_{\frac{2}{3}} \in \mathcal{O}_F, \frac{1}{3} \in O_{\frac{1}{3}}$ and $\frac{2}{3} \in O_{\frac{2}{3}}$

$\Rightarrow O_{\frac{1}{3}} = O_{\frac{2}{3}} = (0,1) \Rightarrow O_{\frac{1}{3}} \cap O_{\frac{2}{3}} \neq \emptyset$

Problem 15

(a) Proof: Recall that $\mathcal{O}_{X/\sim}$ is the finest topology such that

$$\pi: X \to X/\sim, \ \pi(x) = [x]_\sim \text{ is continuous.}$$

"if" direction: Assume that $f \circ \pi$ is continuous.

For all $W \in \mathcal{O}_Z$, as $f \circ \pi$ is continuous, $(f\circ\pi)^{-1}(W) = \pi^{-1}[f^{-1}(W)] \in \mathcal{O}_X$

Notice that $\forall V \in \mathcal{P}(X/\sim)$, $V \in \mathcal{O}_{X/\sim} \Leftrightarrow \pi^{-1}(V) \in \mathcal{O}_X$

so $f^{-1}(W) \in \mathcal{O}_X$, which implies f is continuous.

"only if" direction: Assume that f is continuous,

As π is continuous, $f \circ \pi$ is continuous as well.

(b) Proof: $X = \mathbb{S}^{n-1} = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\|_2 = 1\}$ is a metric space with Euclidean metric.

Hence, for all $U \in \mathcal{O}_X$, consider the set:

$$-U = \{-\vec{x} \in \mathbb{S}^{n-1} : \vec{x} \in U\}.$$

For all $-\vec{x}_* \in -U$, $\vec{x}_* \in U$, so $\exists r > 0$, $B(\vec{x}_*, r) \subseteq U$,

which implies $\exists r > 0$, $B(-\vec{x}_*, r) \subseteq -U$. Hence $-U \in \mathcal{O}_X$.

For all $U \in \mathcal{O}_X$, $\pi^{-1}(\pi(U)) = U \cup (-U) \in \mathcal{O}_X$,

so $\pi(U) \in \mathcal{O}_{X/\sim}$, which implies π is an open mapping.

(c) Proof: $\mathbb{R}^2$ is a metric space with Euclidean metric.

Notice that:

1. $\forall a \in \mathbb{R}-\{0\}$, $(1,0) = (a0, a^{-1}0) \Rightarrow \Leftarrow$,
   so $(1,0) \not\sim (0,0)$, $[(1,0)]_\sim \neq [(0,0)]_\sim$

2. $\forall V \in \mathcal{O}_{\mathbb{R}^2/\sim}$, $[(0,0)]_\sim \in V \Rightarrow (0,0) \in \pi^{-1}(V)$.
   As $\pi^{-1}(V)$ is open in $\mathbb{R}^2$, $\exists r>0$, $B((0,0),r) \subseteq \pi^{-1}(V)$,
   so $(\frac{1}{2}r,0) \in B((0,0),r) \Rightarrow (\frac{1}{2}r,0) \in \pi^{-1}(V) \Rightarrow [(\frac{1}{2}r,0)]_\sim \in V$
   But $\exists \frac{1}{2}r \in \mathbb{R}-\{0\}$, $(\frac{1}{2}r,0) = (\frac{1}{2}r1, (\frac{1}{2}r)^{-1}0)$,
   so $(\frac{1}{2}r,0) \sim (1,0)$, $[(\frac{1}{2}r,0)]_\sim = [(1,0)]_\sim$, $[(1,0)]_\sim \in V$

Hence, there exist $[(0,0)]_\sim, [(1,0)]_\sim \in \mathbb{R}^2/\sim$, such that forall $V_0, V_1 \in \mathcal{O}_{\mathbb{R}^2/\sim}$, $[(0,0)]_\sim \in V_0$ and $[(1,0)]_\sim \in V_1 \Rightarrow [(1,0)]_\sim \in V_0$ and $[(1,0)]_\sim \in V_1 \Rightarrow V_0 \cap V_1 \neq \emptyset$, so $\mathbb{R}^2/\sim$ is not Hausdorff.

(d) Proof: We identify $D^2 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2+x_2^2 \leq 1\}$ with $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$.

Define $\sigma: \mathbb{C} \to \mathbb{C}$, $\sigma(z) = z^2$. This is obviously a continuously differentiable map.

1. We prove that σ is an open map.
   Forall open subset U of $\mathbb{C}$, pick any $\sigma(z_*) \in \sigma(U)$.
   Case 1: If $z_* \neq 0$, then the Jacobean $\det D\sigma(z_*) = 4|z_*|^2 \neq 0$, by Inverse Function Theorem, there exists an open subset U_{z_*} of U, such that $\sigma(U_{z_*})$ is open in $\mathbb{C}$;
   Case 2: If $z_*=0$, then there exists $r>0$, such that $U_{z_*}=B(0,r) \subseteq U$. Obviously $\sigma(U_{z_*}) \supseteq B(0,r^2)$;
   Forall $a+bi \in B(0,r^2)$, there exists $\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + \mathrm{sgn}(b)\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}i$,
   such that $\sigma\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + \mathrm{sgn}(b)\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}i\right) = a+bi \in \sigma(U)$, so $B(0,r^2) = \sigma(U_{z_*}) \subseteq \sigma(U)$
   In both cases, we've found $U_{z_*} \subseteq U$, such that U_{z_*} is an open neighbour of z_*, so $\sigma(U) = \bigcup_{z \in U} \sigma(U_z)$ is open.
2. By identifying inputs $z, -z$, we make the surjective map σ bijective.
   Hence, $\sigma/\sim : \mathbb{C}/\sim \to \mathbb{C}$, $\sigma/\sim([z]_\sim) = z^2$ is an open continuous bijection, i.e., homeomorphism.
   By restricting our domain to $D^2/\sim$ and codomain to D^2, the restricted σ is also a homeomorphism.

Problem 16. $\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) : p \text{ is prime}\}$

(a) Proof: We may divide our proof into two parts.

Part 1: In this part, we prove that $V((a)) \subseteq \{(p) : p \text{ divides } a\}$

For all $(\alpha) \in V((a)) = \{(p) \in \mathrm{Spec}(\mathbb{Z}) : (a) \subseteq (p)\}$,

Case 1.1. If $(\alpha) = (0)$, then $(a) \subseteq (0)$ implies $a = 0$ implies 0 divides 0.

Hence, $(\alpha) \in \{(p) : p \text{ divides } a\}$

Case 1.2. If $(\alpha) \neq (0)$, then α is prime and $(a) \subseteq (\alpha)$

Assume to the contrary that α doesn't divide a

According to the Division Algorithm, there exists a unique pair $(q, r) \in \mathbb{Z}^2$ with $0 < r < \alpha$ such that $a = q\alpha + r$,

so $r = a + (-q)\alpha \in (\alpha)$, which is a contradiction since $0 < r < \alpha$ implies $r \notin (\alpha)$.

In both cases $(\alpha) \in \{(p) : p \text{ divides } a\}$.

Part 2: In this part, we prove that $\{(p) : p \text{ divides } a\} \subseteq V((a))$

For all $(\alpha) \in \{(p) : p \text{ divides } a\}$, there exists $q \in \mathbb{Z}$, such that $a = q\alpha$.

Hence, $(a) = (q\alpha) = q(\alpha) \subseteq (\alpha)$.

Combine the two parts together, we've proven that $V((a)) = \{(p) : p \text{ divides } a\}$.

(b) Proof: We may divide our proof into three parts.

Part 1: $\emptyset = \{(p) : p \text{ divides } 1\} = V((1)) \in \mathcal{C}_{\mathrm{Spec}(\mathbb{Z})}$

and $\mathrm{Spec}(\mathbb{Z}) = \{(p) : p \text{ divides } 0\} = V((0)) \in \mathcal{C}_{\mathrm{Spec}(\mathbb{Z})}$

Part 2: For all finite collection of ideals $(a_k)_{k=1}^{m}$ of $\mathbb{Z}$,

assume that each $V((a_k)) \in \mathcal{C}_{\mathrm{Spec}(\mathbb{Z})}$

Consider the ideal $(\prod_{k=1}^{m} a_k)$ of $\mathbb{Z}$.

As $p \mid \prod_{k=1}^{m} a_k$ iff $p \mid a_k$ for some a_k, $\bigcup_{k=1}^{m} V((a_k)) = V(\prod_{k=1}^{m}(a_k)) \in \mathcal{C}_{\mathrm{Spec}(\mathbb{Z})}$

Part 3: For all indexed family of ideals $(a_\lambda)_{\lambda \in I}$ of $\mathbb{Z}$,

assume that each $V((a_\lambda)) \in \mathcal{C}_{\mathrm{Spec}(\mathbb{Z})}$

Consider the ideal $(\gcd(a_\lambda)_{\lambda\in I})$ of $\mathbb{Z}$, here, for the case where all $a_\lambda = 0$ we simply define $\gcd(a_\lambda)_{\lambda\in I} = 0$.

As $p \mid \gcd\{a_\lambda\}_{\lambda\in I}$ iff $p \mid a_\lambda$ for all $(a_\lambda)_{\lambda\in I}$, $\bigcap V((a_\lambda)) = V((\gcd\{a_\lambda\}_{\lambda\in I})) \in \mathcal{C}_{Spec(\mathbb{Z})}$

Combine the three parts above, we've proven that $(Spec(\mathbb{Z}), \mathcal{C}_{Spec(\mathbb{Z})})$ is a topological space.

(c) Proof: For all $(p) \in Spec(\mathbb{Z}) = \{(0)\} \cup \{(p) \mid p \text{ is prime}\}$,

we wish to prove that $\forall V \in \mathcal{C}_{Spec(\mathbb{Z})}$ with $V \supseteq \{(0)\}$, $(p) \in V$.

For all $V \in \mathcal{C}_{Spec(\mathbb{Z})}$, $V = V((a))$ for some ideal (a) of $\mathbb{Z}$.

$\{(0)\} \subseteq V \Rightarrow (0) \in V((a)) \Rightarrow (0) \supseteq (a) \Rightarrow a \in (0) \Rightarrow a = 0$
$\Rightarrow (p) \supseteq (a) = (0) \Rightarrow (p) \in V((a)) \Rightarrow \{(p) : p \text{ is prime}\} \cup \{(0)\}$
$\subseteq V$, so $\overline{\{(0)\}} = Spec(\mathbb{Z}) = \{(p) : p \text{ is prime}\} \cup \{(0)\}$.

(d)
In cofinite topology, $\{(0), (2)\}$ is closed,

However, for all $V \in \mathcal{C}_{Spec(\mathbb{Z})}$, $(0) \in V \Rightarrow V = Spec(\mathbb{Z})$
$\Rightarrow V \neq \{(0), (2)\}$, so $\{(0), (2)\}$ is not in Zariski Topology.

Hence, we cannot say that Zariski topology is finer than cofinite topology on $Spec(\mathbb{Z})$.

(e) To show that $\{D(a)\}_{a\in\mathbb{Z}}$ is a basis of Zariski topology, It suffices to prove that any closed set V can be written as $\bigcap_{\lambda\in I} V((a_\lambda)) = V((\gcd\{a_\lambda\}_{\lambda\in I}))$ but that's trivial, because $\mathbb{Z}$ is a principal ideal domain, so every $V \in \mathcal{C}_{Spec(\mathbb{Z})}$ is $V = V((a))$
$= V(\gcd\{a\})$

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To show that $\{D(p)\}_{p \text{ is prime}}$ forms a subbasis,

it suffices to show that $\left\{ \bigcup_{k=1}^{m} V((p_k)) = V(\prod_{k=1}^{m}(p_k)) : (p_k)_{k=1}^{m} \text{ is a list of primes} \right\} = \{V((a))\}_{a \in \mathbb{Z}}$. But that's also trivial, because $\mathbb{Z}$ is a unique factorization domain.

(f) $\overline{X_{diag}} \overset{?}{=} \overline{\{((0),(0))\}} = \overline{\{(0)\} \times \{(0)\}} = \overline{\{(0)\}} \times \overline{\{(0)\}} = X \times X$,

Hence, $\overline{X_{diag}} = X \times X$.

(g) As $\overline{X_{diag}} \neq X_{diag}$, $X_{diag} \notin C_{Spec(\mathbb{Z})}$, so $Spec(\mathbb{Z})$ is not Hausdorff