THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 7

Problem 1. Let u satisfy the following PDE

$$\partial_{tt}u - 4\partial_{xx}u + 2u = 0 \quad \text{for } -\infty < x < \infty \text{ and } t > 0. \tag{1}$$

Given a finite interval (a,b), we define the local energy by

$$E(t) := \int_{a+4t}^{b-4t} e(t,x)\,\mathrm{d}\,x, \quad \text{where } e(t,x) := \frac{1}{2}|\partial_t u|^2 + 2|\partial_x u|^2 + |u|^2.$$

(i) Let $p(t,x) := \partial_t u\,\partial_x u$. Prove that

$$\partial_t e = 4\partial_x p.$$

(ii) Show that

$$e \pm 2p = \frac{1}{2}(\partial_t u \pm 2\partial_x u)^2 + u^2.$$

(iii) Using part (i), verify directly that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}E(t) = 4[p(t,b-4t) - p(t,a+4t) - e(t,b-4t) - e(t,a+4t)].$$

(iv) By (ii) and (iii), show that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}E(t) \le 0.$$

Conclude that if $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), show that $u \equiv 0$ in

$$\Delta := \{(x,t) \in (-\infty,\infty) \times [0,\infty) : \ a + 4t \le x \le b - 4t\}.$$

Problem 2. Apply the **energy method** to show the uniqueness for the following problems:

(i)

$$
\begin{cases}
\partial_t u - 4\partial_{xx} u = -4u \quad \text{for } 0 < x < L,\ t > 0 \\
u|_{t=0} = \phi \\
u|_{x=0} = g \\
\partial_x u|_{x=L} = h
\end{cases}
$$

where ϕ, g and h are given data.

(ii)

$$
\begin{cases}
\partial_{tt} u - 4\partial_{xx} u = -u - \partial_t u \quad \text{for } 0 < x < L,\ t > 0 \\
u|_{t=0} = \phi \\
\partial_t u|_{t=0} = \psi \\
\partial_x u|_{x=0} = g \\
u|_{x=L} = h
\end{cases}
$$

where ϕ, ψ, g and h are given data.

Problem 3. Let the concentration $u \coloneqq u(t,x)$ satisfy

$$
\begin{cases}
\partial_t u - \partial_{xx} u = 2tx \quad \text{for } 0 < x < 1 \text{ and } t > 0 \\
u(0,x) = 3x^2 + 1 \\
\partial_x u(t,0) = 0 \\
\partial_x u(t,1) = 2.
\end{cases}
$$

Compute the total mass

$$
M(t) \coloneqq \int_0^1 u(t,x)\,\mathrm{d}\,x.
$$

Problem 4. Let u be a solution to the following initial value problem

$$
\begin{cases}
\partial_{tt} u - \partial_{xx} u = 0 \quad \text{for } -\infty < x < \infty \text{ and } t > 0 \\
u|_{t=0} = \phi \\
\partial_t u|_{t=0} = 0.
\end{cases}
$$

Decide whether the following statements are correct or not.

(i) If ϕ is bounded, show that

$$\max_{\substack{-\infty<x<\infty \\ t\geq 0}} |u(t,x)| \leq \max_{-\infty<x<\infty} |\phi(x)|.$$

(ii) The local energy defined as

$$E(t) := \frac{1}{2}\int_0^1 |\partial_t u|^2 + |\partial_x u|^2 \,\mathrm{d}\, x$$

is always conserved.