THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 8 Solution

Problem 1.

(a) Integrating $\partial_t u = 3\partial_{xx}u$ with respect to x from 0 to π, and using the boundary conditions $\partial_x u(t,0) = \partial_x u(t,\pi) = 0$, we have

$$M'(t) = \int_0^\pi \partial_t u(t,x)\, dx = 3\int_0^\pi \partial_{xx}u(t,x)\, dx = 3\left[\partial_x u\right]_{x=0}^{\pi} = 0.$$

A direct integration yields

$$\begin{aligned} M(t) \equiv M(0) &= \int_0^\pi u(0,x)\, dx \\ &= \int_0^\pi 3\pi x^2 - 2x^3\, dx = \left[\pi x^3 - \frac{1}{2}x^4\right]_{x=0}^{\pi} = \frac{1}{2}\pi^4, \end{aligned}$$

where we applied the initial condition $u(0,x) = 3\pi x^2 - 2x^3$ in the third equality.

(b) We are going to solve the initial and boundary value problem by the method of separation of variables. Let $u(t,x) := \phi(x)G(t)$. Then G satisfies the ODE

$$G' = -3\lambda G \tag{1}$$

and ϕ satisfies the eigenvalue problem

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi'(0) = 0 \\ \phi'(\pi) = 0. \end{cases} \tag{2}$$

Solving the eigenvalue problem (9), we know that the eigenvalues and eigenfunctions are

$$\lambda = n^2 \quad \text{and} \quad \phi = \cos nx \quad \text{for } n = 0,\ 1,\ 2,\ \cdots.$$

When $\lambda = n^2$, we solve the ODE (8) and find that

$$G(t) = A_n e^{-3n^2 t},$$

where A_n is an arbitrary (integration) constant. Now, we have constructed the product form solutions

$$A_n e^{-3n^2 t} \cos nx \qquad \text{for } n = 0,\ 1,\ 2,\ \cdots.$$

By the principle of superposition, the general solution is

$$u(t,x) = \sum_{n=0}^{+\infty} A_n e^{-3n^2 t} \cos nx.$$

To determine the coefficient A_n, we make use of the initial data:

$$3\pi x^2 - 2x^3 = u(0,x) = \sum_{n=0}^{+\infty} A_n \cos nx.$$

Multiplying both sides by $\cos mx$, and then integrating with respect to x from 0 to π, we have

$$\int_0^\pi (3\pi x^2 - 2x^3) \cos mx\, dx = \sum_{n=0}^{+\infty} A_n \int_0^\pi \cos nx \cos mx\, dx.$$

Using the orthogonality (namely the calculus fact) that

$$\int_0^\pi \cos nx \cos mx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \neq 0 \\ \pi & \text{if } m = n = 0, \end{cases}$$

we obtain

$$A_0 = \frac{1}{\pi} \int_0^\pi (3\pi x^2 - 2x^3)\, dx = \frac{1}{\pi} \left[\pi x^3 - \frac{1}{2} x^4 \right]_{x=0}^{\pi} = \frac{1}{2}\pi^3,$$

and for $m \geq 1$, after applying integration by parts,

$$\begin{aligned}
A_m &= \frac{2}{\pi}\int_0^\pi (3\pi x^2 - 2x^3)\cos mx\,dx = \frac{2}{\pi}\int_0^\pi (3\pi x^2 - 2x^3)\frac{d}{dx}\left(\frac{1}{m}\sin mx\right)dx\\
&= -\frac{2}{\pi m}\int_0^\pi (6\pi x - 6x^2)\sin mx\,dx = \frac{2}{\pi m^2}\int_0^\pi (6\pi x - 6x^2)\frac{d}{dx}\cos mx\,dx\\
&= -\frac{2}{\pi m^2}\int_0^\pi (6\pi - 12x)\cos mx\,dx = -\frac{2}{\pi m^3}\int_0^\pi (6\pi - 12x)\frac{d}{dx}\sin mx\,dx\\
&= -\frac{24}{\pi m^3}\int_0^\pi \sin mx\,dx = -\frac{24}{\pi m^3}\left[-\frac{1}{m}\cos mx\right]_{x=0}^{\pi} = \frac{24}{\pi m^4}\left((-1)^m - 1\right)\\
&= \begin{cases} 0 & \text{if } m \text{ is even}\\ -\frac{48}{\pi m^4} & \text{if } m \text{ is odd}\end{cases}\\
&= \begin{cases} 0 & \text{if } m = 2j \text{ for some integer } j \geq 1\\ -\frac{48}{\pi(2j+1)^4} & \text{if } m = 2j+1 \text{ for some integer } j \geq 0.\end{cases}
\end{aligned}$$

As a result, the solution becomes

$$\begin{aligned}
u(t,x) &= \sum_{n=0}^{+\infty} A_n e^{-3n^2 t}\cos nx = \frac{1}{2}\pi^3 - \sum_{\substack{n=1\\ n \text{ is odd}}}^{+\infty} \frac{48}{\pi n^4} e^{-3n^2 t}\cos nx\\
&= \frac{1}{2}\pi^3 - \sum_{j=0}^{+\infty} \frac{48}{\pi(2j+1)^4} e^{-3(2j+1)^2 t}\cos(2j+1)x.
\end{aligned}$$

(c) No, the assertion is wrong. Indeed,

$$\lim_{t\to\infty} u(t,x) = \lim_{t\to\infty}\left(\frac{1}{2}\pi^3 - \sum_{j=0}^{+\infty}\frac{48}{\pi(2j+1)^4}e^{-3(2j+1)^2 t}\cos(2j+1)x\right) = \frac{1}{2}\pi^3 > 0.$$

Problem 2. We compute the solution by principle of superposition

$$u(x,y) = v(x,y) + w(x,y),$$

where v and w are the solutions of the boundary value problems,

$$\begin{cases} 4\partial_{xx}v + \partial_{yy}v = 0 & \text{for } 0 < x < 2 \text{ and } 0 < y < 5\\ v|_{x=0} = 0 & \text{for } 0 < y < 5\\ v|_{x=2} = \frac{1}{6}\sin 3\pi y & \text{for } 0 < y < 5\\ v|_{y=0} = 0 & \text{for } 0 < x < 2\\ v|_{y=5} = 0 & \text{for } 0 < x < 2,\end{cases} \tag{3}$$

and

$$\begin{cases} 4\partial_{xx}w + \partial_{yy}w = 0 & \text{for } 0 < x < 2 \text{ and } 0 < y < 5 \\ w|_{x=0} = 0 & \text{for } 0 < y < 5 \\ w|_{x=2} = 0 & \text{for } 0 < y < 5 \\ w|_{y=0} = x^2 - 2x & \text{for } 0 < x < 2 \\ w|_{y=5} = 0 & \text{for } 0 < x < 2. \end{cases} \tag{4}$$

For the boundary value problem (3), consider the following product solution

$$v(x,y) = h(x)\phi(y).$$

Step 1 (Derive ODEs):

$$-4h''(x)\phi(y) = h(x)\phi''(y) \Longrightarrow \frac{\phi''(y)}{\phi(y)} = -4\frac{h''(x)}{h(x)} = -\lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2\phi}{dy^2} = -\lambda\phi \quad \text{subject to} \quad \phi(0) = \phi(5) = 0. \tag{5}$$

if $\boxed{\lambda > 0}$, then

$$\begin{aligned} & \phi(y) = c_1 \cos\sqrt{\lambda}y + c_2 \sin\sqrt{\lambda}y \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\ \Longrightarrow \quad & \phi(y) = c_2 \sin\sqrt{\lambda}y \Longrightarrow c_2 \sin 5\sqrt{\lambda} = 0 \ (\because \phi(5) = 0) \\ \Longrightarrow \quad & 5\sqrt{\lambda} = n\pi \ (\because c_1 \neq 0 \text{ for nontrivial solutions}) \\ \Longrightarrow \quad & \lambda_n = \left(\frac{n\pi}{5}\right)^2 \ \text{ for } n = 1, 2, \cdots \end{aligned}$$

with the eigenfunction $\phi_n(y) = c_2 \sin\dfrac{n\pi y}{5}$ for $n = 1, 2, \cdots$.

If $\boxed{\lambda = 0}$, then

$$\begin{aligned} & \phi(y) = c_1 + c_2 y \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \ \Longrightarrow \phi(y) = c_2 y \\ \Longrightarrow \quad & c_2 = 0 \ (\because \phi(5) = 0) \Longrightarrow \phi(y) \equiv 0. \end{aligned}$$

If $\boxed{\lambda < 0}$, then

$$
\begin{aligned}
&\phi(y) = c_1 \cosh \sqrt{-\lambda} y + c_2 \sinh \sqrt{-\lambda} y \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\
\Longrightarrow\ &\phi(y) = c_2 \sinh \sqrt{-\lambda} y \Longrightarrow c_2 \sinh 5\sqrt{-\lambda} = 0 \ (\because \phi(5) = 0) \\
\Longrightarrow\ &c_2 = 0 \ (\because \sinh 5\sqrt{-\lambda} > 0) \Longrightarrow \phi \equiv 0.
\end{aligned}
$$

Step 3 (Solve G): Consider

$$
\frac{d^2 h}{dx^2} = \frac{\lambda}{4} h, \ \ h(0) = 0
$$

the general solution is

$$
h_n(x) = c_1 \cosh \sqrt{\frac{\lambda}{4}} x + c_2 \sinh \sqrt{\frac{\lambda}{4}} x
$$

for $\lambda = \lambda_n$. The condition $h(0) = 0$ further gives $c_1 = 0$, and hence,

$$
h_n(x) = c_2 \sinh \sqrt{\frac{\lambda_n}{4}} x = c_2 \sinh \frac{n\pi x}{10}
$$

Step 4 (Find the solution u): The product solutions $v_n(x,y) = h_n(x)\phi_n(y)$ are

$$
\sinh \frac{n\pi x}{10} \sin \frac{n\pi y}{5} \quad \text{for } n = 1, 2, \cdots.
$$

Superposition yields

$$
v(x,y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi x}{10} \sin \frac{n\pi y}{5}
$$

and

$$
\sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{5} \sin \frac{n\pi y}{5} = v(2,y) = \frac{1}{6} \sin 3\pi y.
$$

So by orthogonality,

$$
\begin{aligned}
&A_n = \left(\sinh \frac{n\pi}{5} \int_0^5 \left(\frac{n\pi y}{5} \right)^2 \ dy \right)^{-1} \int_0^5 \frac{1}{6} \sin 3\pi y \frac{n\pi y}{5} \ dy \\
\Longrightarrow\ &A_n = \frac{1}{15} \left(\sinh \frac{n\pi}{5} \right)^{-1} \int_0^5 \sin 3\pi y \frac{n\pi y}{5} \ dy \\
\Longrightarrow\ &A_n = \begin{cases} \frac{1}{6} \left(\sinh 3\pi \right)^{-1} & \text{if } n = 15 \\ 0 & \text{if } n \neq 15 \end{cases}
\end{aligned}
$$

Thus, $v(x,y) = \frac{1}{6}(\sinh 3\pi)^{-1}\sinh\frac{3\pi x}{2}\sin 3\pi y$.

For the boundary value problem (4), consider the following product solution

$$w(x,y) = g(x)\psi(y).$$

Step 1 (Derive ODEs):

$$-4g''(x)\psi(y) = g(x)\psi''(y) \Longrightarrow \frac{\psi''(y)}{\psi(y)} = -4\frac{g''(x)}{g(x)} = \lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2g}{dx^2} = -\frac{\lambda}{4}g \quad \text{subject to} \quad g(0) = g(2) = 0. \tag{6}$$

if $\boxed{\lambda > 0}$, then

$$\begin{aligned}
& g(x) = c_1\cos\sqrt{\frac{\lambda}{4}}x + c_2\sin\sqrt{\frac{\lambda}{4}}x \Longrightarrow c_1 = 0 \ (\because g(0) = 0) \\
\Longrightarrow\quad & g(x) = c_2\sin\sqrt{\frac{\lambda}{4}}x \Longrightarrow c_2\sin\sqrt{\lambda} = 0 \ (\because g(2) = 0) \\
\Longrightarrow\quad & \sqrt{\lambda} = n\pi \ (\because c_1 \neq 0 \text{ for nontrivial solutions}) \\
\Longrightarrow\quad & \lambda_n = (n\pi)^2 \ \text{ for } n = 1, 2, \cdots
\end{aligned}$$

with the eigenfunction $g_n(x) = c_2\sin\frac{n\pi x}{2}$ for $n = 1, 2, \cdots$.

If $\boxed{\lambda = 0}$, then

$$\begin{aligned}
& g(x) = c_1 + c_2 x \Longrightarrow c_1 = 0 \ (\because g(0) = 0) \ \Longrightarrow g(x) = c_2 x \\
\Longrightarrow\quad & c_2 = 0 \ (\because g(2) = 0) \Longrightarrow \phi(x) \equiv 0.
\end{aligned}$$

If $\boxed{\lambda < 0}$, then

$$g(x) = c_1 \cosh \sqrt{-\frac{\lambda}{4}} x + c_2 \sinh \sqrt{-\frac{\lambda}{4}} x \Longrightarrow c_1 = 0 \ (\because g(0) = 0)$$

$$\Longrightarrow \quad g(x) = c_2 \sinh \sqrt{-\frac{\lambda}{4}} x \Longrightarrow c_2 \sinh \sqrt{-\lambda} = 0 \ (\because g(2) = 0)$$

$$\Longrightarrow \quad c_2 = 0 \ (\because \sinh \sqrt{-\lambda} > 0) \Longrightarrow \phi \equiv 0.$$

Step 3 (Solve G): Consider

$$\frac{d^2\psi}{dx^2} = \lambda\psi, \ \ \psi(5) = 0$$

the general solution is

$$\psi_n(y) = c_1 \cosh \sqrt{\lambda}(y-5) + c_2 \sinh \sqrt{\lambda}(y-5)$$

for $\lambda = \lambda_n$. The condition $\psi(5) = 0$ further gives $c_1 = 0$, and hence,

$$\psi_n(y) = c_2 \sinh \sqrt{\lambda_n}(y-5) = c_2 \sinh n\pi(y-5)$$

Step 4 (Find the solution u): The product solutions $w_n(x,t) = g_n(x)\psi_n(y)$ are

$$\sin \frac{n\pi x}{2} \sinh n\pi(y-5) \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$w(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} \sinh n\pi(y-5)$$

and

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} \sinh(-5n\pi) = w(x,0) = x^2 - 2x.$$

So by orthogonality,

$$
\begin{aligned}
& B_n = \left(\sinh(-5n\pi)\int_0^2 \sin\left(\frac{n\pi x}{2}\right)^2\,dx\right)^{-1}\int_0^2 (x^2-2x)\sin\frac{n\pi x}{2}\,dx \\
\Longrightarrow\quad & B_n = (\sinh(-5n\pi))^{-1}\left(-\frac{2}{n\pi}(x^2-2x)\cos\frac{n\pi x}{2}\Big|_0^2 + \frac{2}{n\pi}\int_0^2 (2x-2)\cos\frac{n\pi x}{2}\,dx\right) \\
\Longrightarrow\quad & B_n = (\sinh(-5n\pi))^{-1}\left(\frac{4}{n^2\pi^2}(2x-2)\sin\frac{n\pi x}{2}\Big|_0^2 - \frac{8}{n^2\pi^2}\int_0^2 \sin\frac{n\pi x}{2}\,dx\right) \\
\Longrightarrow\quad & B_n = (\sinh(-5n\pi))^{-1}\,\frac{16}{n^3\pi^3}\cos\frac{n\pi x}{2}\Big|_0^2 = \frac{16}{n^3\pi^3}(\sinh(-5n\pi))^{-1}((-1)^n-1) \\
\Longrightarrow\quad & B_n = \begin{cases} \frac{-32}{n^3\pi^3}(\sinh(-5n\pi))^{-1} & \text{if } n = 2m-1,\ m\in\mathbb{N} \\ 0 & \text{if } n = 2m,\ m\in\mathbb{N} \end{cases}
\end{aligned}
$$

Thus,

$$
w(x,y) = \sum_{m=1}^{\infty}\frac{-32}{(2m-1)^3\pi^3}(\sinh(-5(2m-1)\pi))^{-1}\sin\frac{(2m-1)\pi x}{2}\sinh(2m-1)(y-5)\pi.
$$

Summing v and w together, we have

$$
u(x,y) = \frac{1}{6}(\sinh 3\pi)^{-1}\sinh\frac{3\pi x}{2}\sin 3\pi y - \sum_{m=1}^{\infty}\frac{32\sin\frac{(2m-1)\pi x}{2}\sinh(2m-1)(y-5)\pi}{(2m-1)^3\pi^3\sinh(-5(2m-1)\pi)}.
$$

Problem 3. We consider the product solution $u(r,\theta) = \phi(\theta)G(r)$.

Step 1 (Derive ODEs):

$$
\Delta u = \frac{\phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{G}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \Longrightarrow \frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.
$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$
\phi''(\theta) = -\lambda\phi(\theta) \quad \text{subject to } \phi(0) = \phi(\pi) = 0,
$$

if $\boxed{\lambda > 0}$, then

$$
\begin{aligned}
& \phi(\theta) = c_1\cos\sqrt{\lambda}\theta + c_2\sin\sqrt{\lambda}\theta \Longrightarrow c_1 = 0\ (\because \phi(0) = 0) \\
\Longrightarrow\quad & \phi(\theta) = c_2\sin\sqrt{\lambda}\theta \Longrightarrow c_2\sin\sqrt{\lambda}\pi = 0\ (\because \phi(\pi) = 0) \\
\Longrightarrow\quad & \sqrt{\lambda}\pi = n\pi\ (\because c_2 \neq 0 \text{ for nontrivial solutions}) \\
\Longrightarrow\quad & \lambda_n = n^2 \text{ for } n = 1, 2, \cdots
\end{aligned}
$$

with the eigenfunction $\phi_n(x) = c_2 \sin(n\theta)$.

If $\boxed{\lambda = 0}$, then

$$\begin{aligned} & \phi(\theta) = c_1 + c_2\theta \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \ \Longrightarrow \phi(\theta) = c_2\theta \\ \Longrightarrow \ & c_2 = 0 \ (\because \phi(\pi) = 0) \ \Longrightarrow \phi \equiv 0. \end{aligned}$$

If $\boxed{\lambda < 0}$, then

$$\begin{aligned} & \phi(\theta) = c_1 \cosh \sqrt{-\lambda}\theta + c_2 \sinh \sqrt{-\lambda}\theta \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\ \Longrightarrow \ & \phi(\theta) = c_2 \sinh \sqrt{-\lambda}\theta \Longrightarrow c_2 \sinh(\sqrt{-\lambda}\pi) = 0 \ (\because \phi(\pi) = 0) \\ \Longrightarrow \ & c_2 = 0 \ (\because \sinh(\sqrt{-\lambda}\pi) > 0) \Longrightarrow \phi \equiv 0. \end{aligned}$$

Step 3 (Solve G): Consider

$$r^2 \frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda_n G = 0,$$

Let $G(r) = r^p$. Then

$$[p(p-1) + p - n^2]r^p = 0 \ \Longrightarrow \ p = \pm n.$$

So the general solution is $G_n(r) = c_1 r^n + c_2 r^{-n}$, for $n = 1, 2, \cdots$.

It follows from the boundedness condition $\lim_{r\to 0} |u(r,\theta)| < \infty$ that

$$c_2 = 0 \Longrightarrow G(r) = c_1 r^n.$$

Step 4 (Find the solution u): The product solutions $u_n(r,\theta) = \phi_n(\theta)G_n(r)$ are

$$r^n \sin(n\theta) \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta).$$

Utilizing the last boundary condition gives,

$$4\sin 5\theta + 6\sin 7\theta = u(3,\theta) = \sum_{n=1}^{\infty} 3^n A_n \sin(n\theta).$$

By linear independence,

$$A_n = \begin{cases} \frac{4}{3^5} = \frac{4}{243} & \text{if } n = 5 \\ \frac{6}{3^7} = \frac{2}{729} & \text{if } n = 7 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $u(r,\theta) = \frac{4}{243} r^5 \sin 5\theta + \frac{2}{729} r^7 \sin 7\theta$.

Problem 4.

We consider the product solution $u(x,t) = \phi(x)G(t)$.

Step 1 (Derive ODEs):

$$\phi(x)G''(t) = 4\phi''(x)G(t) - 5\phi(x)G(t) \Longrightarrow \frac{G''(t)}{G(t)} = \frac{4\phi''(x) - 5\phi(x)}{\phi(x)} = -\lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda - 5}{4}\phi = -\tilde{\lambda}\phi \quad \text{subject to} \quad \phi'(0) = \phi(2) = 0. \tag{7}$$

if $\boxed{\tilde{\lambda} > 0}$, then

$$\begin{aligned} & \phi(x) = c_1 \cos\sqrt{\tilde{\lambda}}x + c_2 \sin\sqrt{\tilde{\lambda}}x \\ \Longrightarrow\ & \phi'(x) = -c_1\sqrt{\tilde{\lambda}}\sin\sqrt{\tilde{\lambda}}x + c_2\sqrt{\tilde{\lambda}}\cos\sqrt{\tilde{\lambda}}x \Longrightarrow c_2 = 0 \ (\because \phi'(0) = 0) \\ \Longrightarrow\ & \phi(x) = c_1 \cos\sqrt{\tilde{\lambda}}x \Longrightarrow c_1\cos(2\sqrt{\tilde{\lambda}}) = 0 \ (\because \phi(2) = 0) \\ \Longrightarrow\ & 2\sqrt{\tilde{\lambda}} = \frac{2n-1}{2}\pi \ (\because c_1 \neq 0 \text{ for nontrivial solutions}) \\ \Longrightarrow\ & \tilde{\lambda}_n = \left(\frac{2n-1}{4}\pi\right)^2 \Longrightarrow \lambda_n = \left(\frac{2n-1}{2}\pi\right)^2 + 5 \text{ for } n = 1, 2, \cdots \end{aligned}$$

with the eigenfunction $\phi_n(x) = c_1 \cos\frac{(2n-1)\pi x}{4}$ for $n = 1, 2, \cdots$.

If $\boxed{\tilde{\lambda} = 0}$, then

$$\begin{aligned} & \phi(x) = c_1 + c_2 x \Longrightarrow c_2 = 0 \ (\because \phi'(0) = 0) \ \Longrightarrow \phi(x) = c_1 \\ \Longrightarrow \ & c_1 = 0 \ (\because \phi(2) = 0) \Longrightarrow \phi(x) \equiv 0. \end{aligned}$$

If $\boxed{\tilde{\lambda} < 0}$, then

$$\begin{aligned} & \phi(x) = c_1 \cosh \sqrt{-\tilde{\lambda}} x + c_2 \sinh \sqrt{-\tilde{\lambda}} x \\ \Longrightarrow \ & \phi'(x) = c_1 \sqrt{-\tilde{\lambda}} \sinh \sqrt{-\tilde{\lambda}} x + c_2 \sqrt{-\tilde{\lambda}} \cosh \sqrt{-\tilde{\lambda}} x \Longrightarrow c_2 = 0 \ (\because \phi'(0) = 0) \\ \Longrightarrow \ & \phi(x) = c_2 \sinh \sqrt{-\tilde{\lambda}} x \Longrightarrow c_2 = 0 \ (\because \phi(2) = 0 \text{ and } \sinh(2\sqrt{-\tilde{\lambda}}) > 0) \\ \Longrightarrow \ & \phi \equiv 0. \end{aligned}$$

Step 3 (Solve G): For $\boxed{\tilde{\lambda} = \tilde{\lambda}_n}$, consider

$$\frac{d^2 G}{dt^2} = -\lambda_n G,$$

the general solution is $G_n(t) = c_1 \cos \sqrt{\lambda_n} t + c_2 \sin \sqrt{\lambda_n} t$. It follows from the condition $\partial_t u(x, 0) = 0$ that

$$G_n'(0) = 0 \Longrightarrow c_2 = 0 \Longrightarrow G_n(t) = c_1 \cos \sqrt{\lambda_n} t = c_1 \cos \left(\sqrt{\left(\frac{2n-1}{2} \pi \right)^2 + 5} t \right).$$

Step 4 (Find the solution u): The product solutions $u_n(x,t) = \phi_n(x) G_n(t)$ are

$$\cos \left(\sqrt{\left(\frac{2n-1}{2} \pi \right)^2 + 5} t \right) \cos \frac{(2n-1)\pi x}{4} \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \left(\sqrt{\left(\frac{2n-1}{2} \pi \right)^2 + 5} t \right) \cos \frac{(2n-1)\pi x}{4}.$$

As $u(x, 0) = x^2 - 4$,

$$x^2 - 4 = \sum_{n=1}^{\infty} A_n \cos \frac{(2n-1)\pi x}{4}.$$

So by orthogonality,

$$A_n = \int_0^2 A_n \cos^2 \frac{(2n-1)\pi x}{4}\ dx = \int_0^2 (x^2-4)\cos\frac{(2n-1)\pi x}{4}\ dx$$

Thus,

$$\begin{aligned}
A_n &= \int_0^2 (x^2-4)\cos\frac{(2n-1)\pi x}{4}\ dx \\
&= \frac{4}{(2n-1)\pi}(x^2-4)\sin\frac{(2n-1)\pi x}{4}\bigg|_0^2 - \frac{8}{(2n-1)\pi}\int_0^2 x\sin\frac{(2n-1)\pi x}{4}\ dx \\
&= \frac{32}{(2n-1)^2\pi^2}x\cos\frac{(2n-1)\pi x}{4}\bigg|_0^2 - \frac{32}{(2n-1)^2\pi^2}\int_0^2 \cos\frac{(2n-1)\pi x}{4}\ dx \\
&= -\frac{128}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{4}\bigg|_0^2 \\
&= -\frac{128}{(2n-1)^3\pi^3}(-1)^{n+1}.
\end{aligned}$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty}\frac{128}{(2n-1)^3\pi^3}(-1)^n\cos\left(\sqrt{\left(\frac{2n-1}{2}\pi\right)^2+5t}\right)\cos\frac{(2n-1)\pi x}{4}.$$

Problem 5.

(a) The right end point (i.e., $x = 2$) is perfectly insulated.

(b) We are going to solve the initial and boundary value problem by the method of separation of variables. Let $u(t,x) := \phi(x)G(t)$. Then G satisfies the ODE

$$\frac{dG}{dt} = -4\lambda G \tag{8}$$

and ϕ satisfies the eigenvalue problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda\phi \\ \phi(0) = 0 \\ \dfrac{d\phi}{dx}(2) = 0. \end{cases} \tag{9}$$

Solving the eigenvalue problem (9), we know that the eigenvalues and eigenfunctions are

$$\lambda = \frac{(2n+1)^2\pi^2}{16} \quad \text{and} \quad \phi = \sin\frac{(2n+1)\pi x}{4} \quad \text{for } n = 0,\ 1,\ 2,\ \cdots.$$

When $\lambda = \frac{(2n+1)^2\pi^2}{16}$, we solve the ODE (8) and find that

$$G(t) = A_n e^{-\frac{(2n+1)^2\pi^2}{4}t}.$$

Now, we have constructed the product form solutions

$$A_n e^{-\frac{(2n+1)^2\pi^2}{4}t} \sin\frac{(2n+1)\pi x}{4} \quad \text{for } n = 0,\ 1,\ 2,\ \cdots.$$

By the principle of superposition, the general solution is

$$u(t,x) = \sum_{n=0}^{+\infty} A_n e^{-\frac{(2n+1)^2\pi^2}{4}t} \sin\frac{(2n+1)\pi x}{4}.$$

To determine the coefficient A_n, we make use of the initial data:

$$6\sin\frac{\pi x}{4} = u(0,x) = \sum_{n=0}^{+\infty} A_n \sin\frac{(2n+1)\pi x}{4}.$$

Comparing the coefficients, we have

$$A_n = \begin{cases} 6 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1. \end{cases}$$

Therefore,

$$u(t,x) = 6e^{-\frac{\pi^2}{4}t} \sin\frac{\pi x}{4}.$$

(c) Yes, the equality is correct. Indeed,

$$\lim_{t\to\infty} u(t,x) = 6\left(\lim_{t\to\infty} e^{-\frac{\pi^2}{4}t}\right) \sin\frac{\pi x}{4} = 0,$$

since $\lim_{t\to\infty} e^{-\frac{\pi^2}{4}t} = 0$.

Problem 6. Following the procedure in chapter 4 of textbook, one may solve the wave equation $\partial_{tt}u = \partial_{xx}u$ with the Dirichlet boundary conditions $u|_{x=0} = u|_{x=2\pi} = 0$ by

$$u(t,x) = \sum_{n=1}^{+\infty} A_n \sin\frac{nx}{2}\cos\frac{nt}{2} + B_n \sin\frac{nx}{2}\sin\frac{nt}{2}. \tag{10}$$

We will skip the derivation of the solution formula (10) *here, but students are expected to provide the skipped details in the exam to show their understanding.*

In order to find A_n and B_n, we make use of the initial conditions $u|_{t=0} = \pi - |\pi - x|$ and $\partial_t u|_{t=0} = 0$:

$$\pi - |\pi - x| = u(0,x) = \sum_{n=1}^{+\infty} A_n \sin\frac{nx}{2} \tag{11}$$

$$0 = \partial_t u(0,x) = \sum_{n=1}^{+\infty} \frac{n}{2} B_n \sin\frac{nx}{2}. \tag{12}$$

Comparing the coefficients of the trigonometric polynomials in (12), we have

$$B_n = 0 \qquad \text{for all } n = 1,\ 2,\ 3,\ \cdots.$$

Using (11) and the L^2-orthogonality of $\sin\frac{nx}{2}$, we have

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_0^{2\pi} (\pi - |\pi - x|)\sin\frac{nx}{2}\,dx \\
&= \frac{8}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n^2\pi} & \text{if } n = 4k+1 \text{ for some non-negative integer } k \\ -\frac{8}{n^2\pi} & \text{if } n = 4k+3 \text{ for some non-negative integer } k. \end{cases}
\end{aligned}$$

Finally, the solution is

$$\begin{aligned}
u(t,x) = &\sum_{k=0}^{+\infty} \frac{8}{(4k+1)^2\pi}\sin\frac{(4k+1)x}{2}\cos\frac{(4k+1)t}{2} \\
&- \sum_{k=0}^{+\infty} \frac{8}{(4k+3)^2\pi}\sin\frac{(4k+3)x}{2}\cos\frac{(4k+3)t}{2}.
\end{aligned}$$

Problem 7.

1. Since the unit normal vector $\vec{n} = -\vec{e_\theta}$ at $\theta = 0$ and $\vec{n} = \vec{e_\theta}$ at $\theta = \pi$, the boundary condition
$$\frac{\partial u}{\partial \theta}\bigg|_{\theta=0} = \frac{\partial u}{\partial \theta}\bigg|_{\theta=\pi} = 0.$$
is equivalent to
$$\frac{\partial u}{\partial \vec{n}} = 0$$
at $\theta = 0$ or π, so the physical meaning of this boundary condition is the boundaries $\theta = 0$ and π at is *perfectly insulated*.

2. **Step 1 (Derive the separated ODEs and BCs):** Ignoring the non-homogeneous boundary condition at $r = 2$ for the moment, let us consider the product form solution $u(r, \theta) = \phi(\theta)G(r)$. Substituting the product form solution $u(r, \theta) = \phi(\theta)G(r)$ into Laplace's equation $\Delta u = 0$, we obtain
$$\Delta u = \frac{\phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{G}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$
A re-arrangement yields
$$\frac{r}{G}\frac{d}{dr}(r\frac{dG}{dr}) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}.$$
Since the left hand side is a function of r only and the right hand side is a function of θ only, we know that both the left and right hand sides are indeed just a constant. As a result, we can introduce the separation constant λ as follows:
$$\frac{r}{G}\frac{d}{dr}(r\frac{dG}{dr}) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} =: \lambda,$$
which implies
$$-\frac{d^2\phi}{d\theta^2} = \lambda\phi \quad \text{and} \quad r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda_n G = 0.$$

Furthermore, using the boundary conditions

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} = \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi} = 0,$$

we also obtain

$$\phi'(0) = \phi'(\pi) = 0.$$

On the other hand, since the origin $r = 0$ is in the underlying physical domain, we also have the following finiteness boundary condition:

$$|u(0, \theta)| < \infty, \quad \text{for any } 0 \leq \theta \leq \pi.$$

This implies

$$|G(0)| < \infty.$$

Step 2 (Finding the product form solutions): Solving the eigenvalue problem

$$-\phi''(\theta) = \lambda\phi(\theta) \quad \text{and} \quad \phi(0) = \phi'(\pi) = 0,$$

we obtain the following eigenvalues λ_n and the eigenfunctions ϕ_n

$$\lambda_n := n^2 \quad \text{and} \quad \phi_n(\theta) := \cos n\theta \quad \text{for } n = 0,\ 1,\ 2,\ \cdots$$

Now, for each $\lambda = \lambda_n$, solving

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda_n G = 0 \quad \text{and} \quad |G(0)| < \infty,$$

we obtain

$$G_n(r) := r^n \quad \text{for } n = 0,\ 1,\ 2,\ \cdots$$

Therefore, all of the product form solutions are as follows: for any $n = 0,\ 1,\ 2,\ \cdots$,

$$\phi_n(\theta)G_n(r) = r^n \cos n.$$

Step 3 (Solving for u): It follows from the principle of superposition that the general solution formula is

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta.$$

Using the non-homogeneous boundary condition

$$u(2,\theta) = f(\theta) := 2\theta^3 - 3\pi\theta^2,$$

we have

$$2\theta^3 - 3\pi\theta^2 =: f(\theta) = u(2,\theta) = \sum_{n=0}^{\infty} 2^n A_n \cos n\theta.$$

Using the orthogonality of $\{\cos n\theta\}_{n=0}^{\infty}$, we have, for any $n = 0, 1, 2, \cdots$,

$$\int_0^{\pi} (2\theta^3 - 3\pi\theta^2) \cos n\theta \, d\theta = 2^n A_n \int_0^{\pi} \cos^2 n\theta \, d\theta = \begin{cases} \pi A_0 & \text{if } n = 0 \\ 2^{n-1}\pi A_n & \text{if } n \geq 1. \end{cases}$$

Computing the integral on the left hand side yields

$$\int_0^{\pi} (2\theta^3 - 3\pi\theta^2) \cos n\theta \, d\theta = \begin{cases} -\frac{1}{2}\pi^4 & \text{if } n = 0 \\ \frac{12}{n^4}\left(1 - (-1)^n\right) & \text{if } n \geq 1, \end{cases}$$

and hence,

$$\begin{aligned} A_n &= \begin{cases} \frac{1}{\pi}\int_0^{\pi} (2\theta^3 - 3\pi\theta^2) \, d\theta & \text{if } n = 0 \\ \frac{1}{2^{n-1}\pi}\int_0^{\pi} (2\theta^3 - 3\pi\theta^2) \cos n\theta \, d\theta & \text{if } n \geq 1 \end{cases} \\ &= \begin{cases} -\frac{1}{2}\pi^3 & \text{if } n = 0 \\ \frac{3}{2^{n-3} n^4 \pi}\left(1 - (-1)^n\right) & \text{if } n \geq 1. \end{cases} \end{aligned}$$

Therefore, the unique solution is

$$u(r,\theta) = -\frac{1}{2}\pi^3 + \sum_{k=0}^{\infty} \frac{3}{2^{2k-3} n^4 \pi} r^{2k+1} \cos(2k+1)\theta.$$

Food for Thought. 1. Part (b) of Problem 4 of Dec 2020 Final Exam:

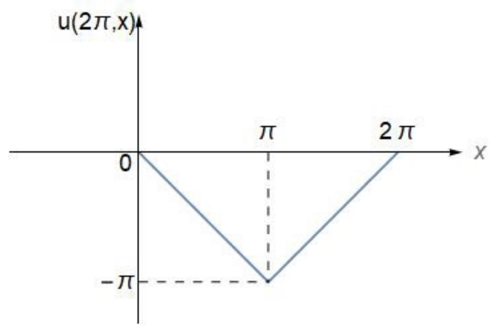
When $t = 2\pi$, by part (a)

$$\begin{aligned}
u(2\pi, x) &= \sum_{k=0}^{+\infty} \frac{8}{(4k+1)^2\pi} \sin\frac{(4k+1)x}{2} \cos(4k+1)\pi \\
&\quad - \sum_{k=0}^{+\infty} \frac{8}{(4k+3)^2\pi} \sin\frac{(4k+3)x}{2} \cos(4k+3)\pi \\
&= -\left(\sum_{k=0}^{+\infty} \frac{8}{(4k+1)^2\pi} \sin\frac{(4k+1)x}{2} - \sum_{k=0}^{+\infty} \frac{8}{(4k+3)^2\pi} \sin\frac{(4k+3)x}{2} \right) \\
&= -(\pi - |\pi - x|),
\end{aligned}$$

because $\cos n\pi = (-1)^n$ and $\pi - |\pi - x|$ has the following Fourier sine series:

$$\pi - |\pi - x| = \sum_{k=0}^{+\infty} \frac{8}{(4k+1)^2\pi} \sin\frac{(4k+1)x}{2} - \sum_{k=0}^{+\infty} \frac{8}{(4k+3)^2\pi} \sin\frac{(4k+3)x}{2}. \tag{13}$$

Thus, the graph of $u(2\pi, x)$ is as follows:



2. Part (c) of Problem 3 Dec 2021 Final Exam:

The statement/inequality

$$\max_{\substack{0\le r\le 2 \\ 0\le\theta\le\pi}} u(r,\theta) \le \max_{0\le\theta\le\pi} f(\theta)$$

is true, and the reasoning is as follows. First of all, it follows from the maximum principle for Laplace's equation that

$$\max_{\substack{0\le r\le 2 \\ 0\le\theta\le\pi}} u(r,\theta) = \max\left\{ \max_{0\le\theta\le\pi} f(\theta), \max_{0\le r\le 2} u(r,0), \max_{0\le r\le 2} u(r,\pi) \right\}.$$

Using the final solution obtained in part (b), we know that

$$u(r,0) = -\frac{1}{2}\pi^3 + \sum_{k=0}^{\infty} \frac{3}{2^{2k-3}n^4\pi} r^{2k+1}$$

and

$$u(r,\pi) = -\frac{1}{2}\pi^3 - \sum_{k=0}^{\infty} \frac{3}{2^{2k-3}n^4\pi} r^{2k+1}.$$

Since all the coefficients $\frac{3}{2^{2k-3}n^4\pi}$ in the infinite sums are positive, one can see that

- $u(r,0)$ is monotonic in r; and
- $u(r,\pi) \le u(r,0)$ for all $r \ge 0$.

As a result, we know that

$$\max_{0\le r\le 2} u(r,\pi) \le \max_{0\le r\le 2} u(r,0) = u(2,0) = f(0).$$

Therefore,

$$\max\left\{\max_{0\le\theta\le\pi} f(\theta), \max_{0\le r\le 2} u(r,0), \max_{0\le r\le 2} u(r,\pi)\right\} = \max_{0\le\theta\le\pi} f(\theta),$$

and the assertion immediately follows.

Problem 8. We consider the product solution $u(r,\theta) = \phi(\theta)G(r)$.

Step 1 (Derive ODEs):

$$\Delta u = \frac{\phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{G}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \Longrightarrow \frac{r}{G}\frac{d}{dr}(r\frac{dG}{dr}) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\phi''(\theta) = -\lambda\phi(\theta) \text{ subject to } \phi(-\pi) = \phi(\pi) \text{ and } \phi'(-\pi) = \phi'(\pi),\ n = 1,2,\ldots$$

if $\boxed{\lambda > 0}$, then

$$
\begin{aligned}
&\phi(\theta) = c_1 \cos\sqrt{\lambda}\theta + c_2 \sin\sqrt{\lambda}\theta \Longrightarrow \phi'(\theta) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\theta + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\theta \\
\Longrightarrow\quad &\begin{cases} c_1\cos\sqrt{\lambda}\pi - c_2\sin\sqrt{\lambda}\pi = \phi(-\pi) = \phi(\pi) = c_1\cos\sqrt{\lambda}\pi + c_2\sin\sqrt{\lambda}\pi \\ c_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi = \phi'(-\pi) = \phi'(\pi) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi \end{cases} \\
\Longrightarrow\quad &\begin{cases} c_2\sin\sqrt{\lambda}\pi = 0 \\ c_1\sin\sqrt{\lambda}\pi = 0 \end{cases} \Longrightarrow \sin\sqrt{\lambda}\pi = 0 \ (\because c_1 \neq 0 \text{ or } c_2 \neq 0 \text{ for nontrivial solutions}) \\
\Longrightarrow\quad &\lambda_n = n^2 \text{ for } n = 1, 2, \cdots
\end{aligned}
$$

with the eigenfunction $\phi_n(x) = c_1\cos n\theta + c_2\sin n\theta$.

If $\boxed{\lambda = 0}$, then

$$\phi(\theta) = c_1 + c_2\theta \Longrightarrow c_2 = 0 \ (\because \phi(-\pi) = \phi(\pi)) \ \Longrightarrow \phi(\theta) = c_1.$$

If $\boxed{\lambda < 0}$, then

$$
\begin{aligned}
&\phi(\theta) = c_1\cosh\sqrt{-\lambda}\theta + c_2\sinh\sqrt{-\lambda}\theta \Longrightarrow \phi'(\theta) = c_1\sqrt{\lambda}\sinh\sqrt{\lambda}\theta + c_2\sqrt{\lambda}\cosh\sqrt{\lambda}\theta \\
\Longrightarrow\quad &\begin{cases} c_1\cosh\sqrt{\lambda}\pi - c_2\sinh\sqrt{\lambda}\pi = \phi(-\pi) = \phi(\pi) = c_1\cosh\sqrt{\lambda}\pi + c_2\sinh\sqrt{\lambda}\pi \\ -c_1\sqrt{\lambda}\sinh\sqrt{\lambda}\pi + c_2\sqrt{\lambda}\cosh\sqrt{\lambda}\pi = \phi'(-\pi) = \phi'(\pi) = c_1\sqrt{\lambda}\sinh\sqrt{\lambda}\pi + c_2\sqrt{\lambda}\cosh\sqrt{\lambda}\pi \end{cases} \\
\Longrightarrow\quad &\begin{cases} c_2\sinh\sqrt{\lambda}\pi = 0 \\ c_1\sinh\sqrt{\lambda}\pi = 0 \end{cases} \Longrightarrow c_1 = c_2 = 0 \ (\because \sinh\sqrt{\lambda}\pi > 0) \\
\Longrightarrow\quad &\phi \equiv 0.
\end{aligned}
$$

Step 3 (Solve G): Consider

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda_n G = 0.$$

For $\boxed{\lambda = 0}$,

$$rG''(r) + G'(r) = 0 \Longrightarrow G'(r) = c_1/r \Longrightarrow G_0(r) = c_1\ln r + c_2.$$

For $\boxed{\lambda = n^2}$ for $n = 1, 2, \cdots$, Let $G(r) = r^p$. Then

$$[p(p-1) + p - n^2]r^p = 0 \ \Longrightarrow \ p = \pm n \Longrightarrow G_n(r) = c_1 r^n + c_2 r^{-n}.$$

Step 4 (Find the solution u): The product solutions $u_n(r,\theta) = \phi_n(\theta)G_n(r)$ are

$$1, \ln r \text{ for } n = 0$$

$$r^n \cos n\theta,\ r^{-n}\cos n\theta,\ r^n \sin n\theta,\ r^{-n}\sin n\theta, \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(r,\theta) = A_0 + \hat{A}_0 \ln r + \sum_{n=1}^{\infty}(A_n r^n \cos n\theta + \hat{A}_n r^{-n}\cos n\theta + B_n r^n \sin n\theta + \hat{B}_n r^{-n}\sin n\theta).$$

Then

$$\begin{cases} 3\sin 3\theta = u(1,\theta) = A_0 + \sum_{n=1}^{\infty}(A_n \cos n\theta + \hat{A}_n \cos n\theta + B_n \sin n\theta + \hat{B}_n \sin n\theta) \\ 15\cos 2\theta - \frac{15}{2}\sin 3\theta = u(2,\theta) = A_0 + B_0 \ln 2 + \sum_{n=1}^{\infty}(2^n A_n \cos n\theta + \frac{\hat{A}_n}{2^n}\cos n\theta + 2^n B_n \sin n\theta + \frac{\hat{B}_n}{2^n}\sin n\theta) \end{cases}$$

By linear independence,

- $A_0 = A_0 + B_0 \ln 2 = 0$ implies $A_0 = B_0 = 0$.
- $A_n + \hat{A}_n = 0$ and $2^n A_n + \dfrac{\hat{A}_n}{2^n} = \begin{cases} 15 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$ imply

$$A_n = \begin{cases} 4 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases} \text{ and } \hat{A}_n = \begin{cases} -4 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

- $B_n + \hat{B}_n = \begin{cases} 3 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$ and $2^n B_n + \dfrac{\hat{B}_n}{2^n} = \begin{cases} -\frac{15}{2} & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$ imply

$$B_n = \begin{cases} -1 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases} \text{ and } \hat{B}_n = \begin{cases} 4 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $u(r,\theta) = 4r^2 \cos 2\theta - 4r^{-2}\cos 2\theta - r^3 \sin 3\theta + 4r^{-3}\sin 3\theta$.

Food for Thought. Yes, for example, the Laplace's equation can be solved on a rectangle. Exercise: solve the Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

in a rectangle $(0 \le x \le L,\ 0 \le y \le H)$ subject to the boundary conditions

$$u(x,0) = u(x,H) = u(L,y) = 0 \quad \text{and} \quad u(0,y) = f(y).$$

Problem 9.

(i) We consider the product solution $u(x,t) = \phi(x)G(t)$.

Step 1 (Derive ODEs):

$$\phi(x)G'(t) = 5\phi''(x)G(t) \Longrightarrow \frac{G'(t)}{G(t)} = \frac{5\phi''(x)}{\phi(x)} = -\lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda}{5}\phi \ \text{ subject to } \ \phi'(0) = \phi'(2) = 0. \tag{14}$$

if $\boxed{\lambda > 0}$, then

$$\begin{aligned}
& \phi(x) = c_1 \cos\sqrt{\frac{\lambda}{5}}x + c_2 \sin\sqrt{\frac{\lambda}{5}}x \\
\Longrightarrow\quad & \phi'(x) = -c_1\sqrt{\frac{\lambda}{5}}\sin\sqrt{\frac{\lambda}{5}}x + c_2\sqrt{\frac{\lambda}{5}}\cos\sqrt{\frac{\lambda}{5}}x \\
\Longrightarrow\quad & c_2\sqrt{\frac{\lambda}{5}} = 0 \ (\because \phi'(0) = 0) \ \Longrightarrow c_2 = 0 \\
\Longrightarrow\quad & \phi'(x) = -c_1\sqrt{\frac{\lambda}{5}}\sin\sqrt{\frac{\lambda}{5}}x \Longrightarrow -c_1\sqrt{\frac{\lambda}{5}}\sin 2\sqrt{\frac{\lambda}{5}} = 0 \ (\because \phi'(2) = 0) \\
\Longrightarrow\quad & 2\sqrt{\frac{\lambda}{5}} = n\pi \ (\because c_1 \neq 0 \text{ for nontrivial solutions}) \\
\Longrightarrow\quad & \lambda_n = 5(\frac{n\pi}{2})^2 \text{ for } n = 1, 2, \cdots
\end{aligned}$$

with the eigenfunction $\phi_n(x) = c_1 \cos\dfrac{n\pi x}{2}$ for $n = 1, 2, \cdots$.

If $\boxed{\lambda = 0}$, then

$$\phi(x) = c_1 + c_2 x \Longrightarrow \phi'(x) = c_2 \Longrightarrow c_2 = 0\ (\because \phi'(0) = \phi'(2) = 0) \Longrightarrow \phi \equiv c_1.$$

If $\boxed{\lambda < 0}$, then

$$\begin{aligned}
& \phi(x) = c_1 \cosh\sqrt{-\frac{\lambda}{5}}x + c_2 \sinh\sqrt{-\frac{\lambda}{5}}x \\
\Longrightarrow\ & \phi'(x) = c_1\sqrt{-\frac{\lambda}{5}} \sinh\sqrt{-\frac{\lambda}{5}}x + c_2\sqrt{-\frac{\lambda}{5}} \cosh\sqrt{-\frac{\lambda}{5}}x \Longrightarrow c_2 = 0\ (\because \phi'(0) = 0) \\
\Longrightarrow\ & \phi'(x) = c_1\sqrt{-\frac{\lambda}{5}} \sinh\sqrt{-\frac{\lambda}{5}}x \Longrightarrow c_1\sqrt{-\frac{\lambda}{5}} \sinh 2\sqrt{-\frac{\lambda}{5}} = 0\ (\because \phi'(2) = 0) \\
\Longrightarrow\ & c_1 = 0\ (\because \sinh 2\sqrt{-\frac{\lambda}{5}} > 0) \Longrightarrow \phi \equiv 0.
\end{aligned}$$

Step 3 (Solve G): Consider

$$\frac{dG}{dt} = -\lambda G,$$

the general solution is $G_n(t) = c$ (for $\lambda = 0$) or $G_n(t) = ce^{-\lambda_n t} = ce^{-\frac{5n^2\pi^2 t}{4}}$ (for $\lambda = \lambda_n$).

Step 4 (Find the solution u): The product solutions $u_n(x,t) = \phi_n(x)G_n(t)$ are

$$\text{a constant } A_0 \text{ and } e^{-\frac{5n^2\pi^2 t}{4}} \cos\frac{n\pi x}{2} \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{5n^2\pi^2 t}{4}} \cos\frac{n\pi x}{2}$$

and

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{2} = u(x,0) = x.$$

So by orthogonality,

$$2A_0 = \int_0^2 A_0 \ dx = \int_0^2 x \ dx = 2 \Longrightarrow A_0 = 1$$

$$A_n = \int_0^2 A_n \cos^2 \frac{n\pi x}{2} \ dx = \int_0^2 x \cos \frac{n\pi x}{2} \ dx = \frac{2}{n\pi} x \sin \frac{n\pi x}{2} \bigg|_0^2 - \frac{2}{n\pi} \int_0^2 \sin \frac{n\pi x}{2} \ dx$$

$$\Longrightarrow \quad A_n = \frac{4}{n^2\pi^2} \cos \frac{n\pi x}{2} \bigg|_0^2 = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

Thus, $u(x,t) = 1 - \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2} e^{-\frac{5(2k-1)^2\pi^2 t}{4}} \cos \frac{(2k-1)\pi x}{2}$.

(ii) Note that

$$\lim_{t\to\infty} \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2} e^{-\frac{5(2k-1)^2\pi^2 t}{4}} \cos \frac{(2k-1)\pi x}{2} = \lim_{t\to\infty} e^{-\frac{5\pi^2 t}{4}} f(t,x) = 0$$

as

$$\begin{aligned} |f(t,x)| &= \left| \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2} e^{-\frac{5[(2k-1)^2-1]\pi^2 t}{4}} \cos \frac{(2k-1)\pi x}{2} \right| \\ &\leq \sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2} < \infty. \end{aligned}$$

Thus

$$\lim_{t\to\infty} u(x,t) = 1 \neq 0.$$

Problem 10.

(i) The left endpoint $x = 0$ is held at zero temperature.

(ii) The right endpoint $x = L$ is insulated (that is, there is no heat transfer at $x = L$).

(iii) We consider the product solution $u(x,t) = \phi(x)G(t)$.

Step 1 (Derive ODEs):

$$\phi(x)G'(t) = k\phi''(x)G(t) \Longrightarrow \frac{G'(t)}{G(t)} = \frac{k\phi''(x)}{\phi(x)} = -\lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda}{k}\phi \quad \text{subject to} \quad \phi(0) = \phi'(L) = 0. \tag{15}$$

if $\boxed{\lambda > 0}$, then

$$\phi(x) = c_1 \cos\sqrt{\frac{\lambda}{k}}x + c_2 \sin\sqrt{\frac{\lambda}{k}}x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0)$$

$$\Longrightarrow \quad \phi(x) = c_2 \sin\sqrt{\frac{\lambda}{k}}x$$

$$\Longrightarrow \quad \phi'(x) = c_2\sqrt{\frac{\lambda}{k}}\cos\sqrt{\frac{\lambda}{k}}x \Longrightarrow c_2\sqrt{\frac{\lambda}{k}}\cos\sqrt{\frac{\lambda}{k}}L = 0 \ (\because \phi'(L) = 0)$$

$$\Longrightarrow \quad \sqrt{\frac{\lambda}{k}}L = (2n-1)\pi/2 \ (\because c_2 \neq 0 \text{ for nontrivial solutions})$$

$$\Longrightarrow \quad \lambda_n = \frac{k(2n-1)^2\pi^2}{4L^2} \quad \text{for } n = 1, 2, \cdots$$

with the eigenfunction $\phi_n(x) = c_2 \sin\dfrac{(2n-1)\pi x}{2L}$ for $n = 1, 2, \cdots$.

If $\boxed{\lambda = 0}$, then

$$\phi(x) = c_1 + c_2 x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \Longrightarrow \phi(x) = c_2 x \Longrightarrow \phi'(x) = c_2$$

$$\Longrightarrow \quad c_2 = 0 \ (\because \phi'(L) = 0) \ \Longrightarrow \phi \equiv 0.$$

If $\boxed{\lambda < 0}$, then

$$\phi(x) = c_1 \cosh\sqrt{-\frac{\lambda}{k}}x + c_2 \sinh\sqrt{-\frac{\lambda}{k}}x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0)$$

$$\Longrightarrow \quad \phi(x) = c_2 \sinh\sqrt{-\frac{\lambda}{k}}x \Longrightarrow \phi'(x) = c_2\sqrt{-\frac{\lambda}{k}}\cosh\sqrt{-\frac{\lambda}{k}}x$$

$$\Longrightarrow \quad c_2\sqrt{-\frac{\lambda}{k}}\cosh\sqrt{-\frac{\lambda}{k}}L = 0 \ (\because \phi'(L) = 0)$$

$$\Longrightarrow \quad c_2 = 0 \ (\because \cosh\sqrt{-\frac{\lambda}{k}}L > 1) \Longrightarrow \phi \equiv 0.$$

Step 3 (Solve G): Consider

$$\frac{dG}{dt} = -\lambda G,$$

the general solution is $G_n(t) = ce^{-\lambda_n t} = ce^{-\frac{k(2n-1)^2\pi^2 t}{4L^2}}$.

Step 4 (Find the solution u): The product solutions $u_n(x,t) = \phi_n(x)G_n(t)$ are

$$e^{-\frac{k(2n-1)^2\pi^2 t}{4L^2}} \sin\frac{(2n-1)\pi x}{2L} \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{k(2n-1)^2\pi^2 t}{4L^2}} \sin\frac{(2n-1)\pi x}{2L}.$$

As $u(x,0) = x^2 - Lx$,

$$x^2 - 2Lx = \sum_{n=1}^{\infty} A_n \sin\frac{(2n-1)\pi x}{2L}.$$

So by orthogonality,

$$A_n = \frac{2}{L}\int_0^L A_n \sin^2\frac{(2n-1)\pi x}{2L}\, dx = \frac{2}{L}\int_0^L (x^2 - 2Lx)\sin\frac{(2n-1)\pi x}{2L}\, dx$$

$$\begin{aligned}
& \int_0^L x^2 \sin\frac{(2n-1)\pi x}{2L}\, dx \\
= \quad & -\frac{2L}{(2n-1)\pi} x^2 \cos\frac{(2n-1)\pi x}{2L}\Big|_0^L + \frac{4L}{(2n-1)\pi}\int_0^L x\cos\frac{(2n-1)\pi x}{2L}\, dx \\
= \quad & \frac{8L^2}{(2n-1)^2\pi^2} x\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L - \frac{8L^2}{(2n-1)^2\pi^2}\int_0^L \sin\frac{(2n-1)\pi x}{2L}\, dx \\
= \quad & \frac{(-1)^{n+1}8L^3}{(2n-1)^2\pi^2} + \frac{16L^3}{(2n-1)^3\pi^3}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L \\
= \quad & \frac{(-1)^{n+1}8L^3}{(2n-1)^2\pi^2} - \frac{16L^3}{(2n-1)^3\pi^3}.
\end{aligned}$$

$$
\begin{aligned}
& \int_0^L x \sin \frac{(2n-1)\pi x}{2L}\, dx \\
= & -\frac{2L}{(2n-1)\pi} x \cos \frac{(2n-1)\pi x}{2L}\Big|_0^L + \frac{2L}{(2n-1)\pi} \int_0^L \cos \frac{(2n-1)\pi x}{2L}\, dx \\
= & \frac{4L^3}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi x}{2L}\Big|_0^L \\
= & \frac{(-1)^{n+1} 4L^3}{(2n-1)^2\pi^2}.
\end{aligned}
$$

Thus,

$$A_n = -\frac{32L^2}{(2n-1)^3\pi^3}.$$

Hence

$$u(x,t) = -\sum_{n=1}^{\infty} \frac{32L^2}{(2n-1)^3\pi^3} e^{-\frac{k(2n-1)^2\pi^2 t}{4L^2}} \sin \frac{(2n-1)\pi x}{2L}.$$

Problem 11.

We consider the product solution $u(x,t) = \phi(x)G(t)$.

Step 1 (Derive ODEs):

$$\phi(x)G''(t) = c^2\phi''(x)G(t) - \beta\phi(x)G(t) \Longrightarrow \frac{G''(t)}{G(t)} = \frac{c^2\phi''(x) - \beta\phi(x)}{\phi(x)} = -\lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda - \beta}{c^2}\phi = -\tilde{\lambda}\phi \quad \text{subject to} \quad \phi(0) = \phi(L) = 0. \tag{16}$$

if $\boxed{\tilde{\lambda} > 0}$, then

$$
\begin{aligned}
& \phi(x) = c_1 \cos\sqrt{\tilde{\lambda}}x + c_2 \sin\sqrt{\tilde{\lambda}}x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\
\Longrightarrow \ & \phi(x) = c_2 \sin\sqrt{\tilde{\lambda}}x \Longrightarrow c_2 \sin(L\sqrt{\tilde{\lambda}}) = 0 \ (\because \phi(L) = 0) \\
\Longrightarrow \ & L\sqrt{\tilde{\lambda}} = n\pi \ (\because c_2 \neq 0 \text{ for nontrivial solutions}) \\
\Longrightarrow \ & \tilde{\lambda}_n = n^2\pi^2/L^2 \Longrightarrow \lambda_n = n^2c^2\pi^2/L^2 + \beta \text{ for } n = 1, 2, \cdots
\end{aligned}
$$

with the eigenfunction $\phi_n(x) = c_2 \sin \frac{n\pi x}{L}$ for $n = 1, 2, \cdots$.

If $\boxed{\tilde{\lambda} = 0}$, then

$$\begin{aligned} & \phi(x) = c_1 + c_2 x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \ \Longrightarrow \phi(x) = c_2 x \\ \Longrightarrow \ & c_2 = 0 \ (\because \phi(L) = 0) \Longrightarrow \phi(x) \equiv 0. \end{aligned}$$

If $\boxed{\tilde{\lambda} < 0}$, then

$$\begin{aligned} & \phi(x) = c_1 \cosh \sqrt{-\tilde{\lambda}} x + c_2 \sinh \sqrt{-\tilde{\lambda}} x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\ \Longrightarrow \ & \phi(x) = c_2 \sinh \sqrt{-\tilde{\lambda}} x \Longrightarrow c_2 = 0 \ (\because \phi(L) = 0 \text{ and } \sinh(L\sqrt{-\tilde{\lambda}}) > 0) \\ \Longrightarrow \ & \phi \equiv 0. \end{aligned}$$

Step 3 (Solve G): For $\boxed{\tilde{\lambda} = \tilde{\lambda}_n}$, consider

$$\frac{d^2 G}{dt^2} = -\lambda_n G,$$

the general solution is $G_n(t) = c_1 \cos \sqrt{\lambda_n} t + c_2 \sin \sqrt{\lambda_n} t$. It follows from the condition $\partial_t u(x, 0) = 0$ that

$$G_n'(0) = 0 \Longrightarrow c_2 = 0 \Longrightarrow G_n(t) = c_1 \cos \sqrt{\lambda_n} t = c_1 \cos(\sqrt{n^2 c^2 \pi^2 / L^2 + \beta} \ t).$$

Step 4 (Find the solution u): The product solutions $u_n(x,t) = \phi_n(x) G_n(t)$ are

$$\cos(\sqrt{n^2 c^2 \pi^2 / L^2 + \beta} \ t) \sin \frac{n\pi x}{L} \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\sqrt{n^2 c^2 \pi^2 / L^2 + \beta} \ t) \sin \frac{n\pi x}{L}.$$

As $u(x, 0) = x^2 - Lx$,

$$x^2 - Lx = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}.$$

So by orthogonality,

$$A_n = \frac{2}{L}\int_0^L A_n \sin^2\frac{n\pi x}{L}\ dx = \frac{2}{L}\int_0^L (x^2 - Lx)\sin\frac{n\pi x}{L}\ dx$$

$$\begin{aligned}
&\int_0^L x^2 \sin\frac{n\pi x}{L}\ dx \\
= \quad & -\frac{L}{n\pi}x^2\cos\frac{n\pi x}{L}\bigg|_0^L + \frac{2L}{n\pi}\int_0^L x\cos\frac{n\pi x}{L}\ dx \\
= \quad & \frac{(-1)^{n+1}L^3}{n\pi} + \frac{2L^2}{n^2\pi^2}x\sin\frac{n\pi x}{L}\bigg|_0^L - \frac{2L^2}{n^2\pi^2}\int_0^L \sin\frac{n\pi x}{L}\ dx \\
= \quad & \frac{(-1)^{n+1}L^3}{n\pi} + \frac{2L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\bigg|_0^L \\
= \quad & \frac{L^3}{n^3\pi^3}[(-1)^{n+1}(n^2\pi^2 - 2) - 2].
\end{aligned}$$

$$\begin{aligned}
&\int_0^L x\sin\frac{n\pi x}{L}\ dx \\
= \quad & -\frac{L}{n\pi}x\cos\frac{n\pi x}{L}\bigg|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\ dx \\
= \quad & \frac{(-1)^{n+1}L^2}{n\pi} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\bigg|_0^L \\
= \quad & \frac{(-1)^{n+1}L^2}{n\pi}.
\end{aligned}$$

Thus,

$$A_n = \frac{2L^2}{n^3\pi^3}[(-1)^{n+1}(n^2\pi^2 - 2) - 2 - (-1)^{n+1}n^2\pi^2] = \begin{cases} -\dfrac{8L^2}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

Hence

$$u(x,t) = -\sum_{m=1}^{\infty}\frac{8L^2}{(2m-1)^3\pi^3}\cos\left(\sqrt{(2m-1)^2c^2\pi^2/L^2 + \beta}\ t\right)\sin\frac{(2m-1)\pi x}{L}.$$

Problem 12. We consider the product solution $u(r,\theta) = \phi(\theta)G(r)$.

Step 1 (Derive ODEs):

$$\Delta u = \frac{\phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{G}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \Longrightarrow \frac{r}{G}\frac{d}{dr}(r\frac{dG}{dr}) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\phi''(\theta) = -\lambda\phi(\theta) \ \text{ subject to } \phi(0) = \phi\left(\frac{\pi}{2}\right) = 0,$$

if $\boxed{\lambda > 0}$, then

$$\begin{aligned} & \phi(\theta) = c_1\cos\sqrt{\lambda}\theta + c_2\sin\sqrt{\lambda}\theta \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\ \Longrightarrow\ & \phi(\theta) = c_2\sin\sqrt{\lambda}\theta \Longrightarrow c_2\sin\frac{\pi\sqrt{\lambda}}{2} = 0 \ (\because \phi\left(\frac{\pi}{2}\right) = 0) \\ \Longrightarrow\ & \frac{\sqrt{\lambda}\pi}{2} = n\pi \ (\because c_2 \neq 0 \text{ for nontrivial solutions}) \\ \Longrightarrow\ & \lambda_n = 4n^2 \text{ for } n = 1, 2, \cdots \end{aligned}$$

with the eigenfunction $\phi_n(x) = c_2\sin(2n\theta)$.

If $\boxed{\lambda = 0}$, then

$$\begin{aligned} & \phi(\theta) = c_1 + c_2\theta \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \ \Longrightarrow \phi(\theta) = c_2\theta \\ \Longrightarrow\ & c_2 = 0 \ (\because \phi\left(\frac{\pi}{2}\right) = 0) \ \Longrightarrow \phi \equiv 0. \end{aligned}$$

If $\boxed{\lambda < 0}$, then

$$\begin{aligned} & \phi(\theta) = c_1\cosh\sqrt{-\lambda}\theta + c_2\sinh\sqrt{-\lambda}\theta \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\ \Longrightarrow\ & \phi(\theta) = c_2\sinh\sqrt{-\lambda}\theta \Longrightarrow c_2\sinh\left(\frac{\sqrt{-\lambda}\pi}{2}\right) = 0 \ (\because \phi\left(\frac{\pi}{2}\right) = 0) \\ \Longrightarrow\ & c_2 = 0 \ (\because \sinh\left(\frac{\sqrt{-\lambda}\pi}{2}\right) > 0) \Longrightarrow \phi \equiv 0. \end{aligned}$$

Step 3 (Solve G): Consider

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda_n G = 0,$$

Let $G(r) = r^p$. Then

$$[p(p-1) + p - (2n)^2]r^p = 0 \implies p = \pm 2n.$$

So the general solution is $G_n(r) = c_1 r^{2n} + c_2 r^{-2n}$, for $n = 1, 2, \cdots$.

It follows from the boundedness condition $\lim\limits_{r\to 0} |u(r,\theta)| < \infty$ that

$$c_2 = 0 \Longrightarrow G(r) = c_1 r^{2n}.$$

Step 4 (Find the solution u): The product solutions $u_n(r,\theta) = \phi_n(\theta)G_n(r)$ are

$$r^{2n}\sin(2n\theta) \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin(2n\theta).$$

As $\partial_r u(r,\theta) = \sum\limits_{n=1}^{\infty} (2n) A_n r^{2n-1} \sin(2n\theta)$,

$$\frac{7}{2}\sin 4\theta + \frac{15}{16}\sin 6\theta = \partial_r u(\frac{1}{2},\theta) = \sum_{n=1}^{\infty} n(\frac{1}{2})^{2n-2} A_n \sin(2n\theta).$$

By linear independence,

$$A_n = \begin{cases} \frac{7}{2}\cdot\frac{2^2}{2} = 7 & \text{if } n = 2 \\ \frac{15}{16}\cdot\frac{2^4}{3} = 5 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $u(r,\theta) = 7r^4 \sin 4\theta + 5r^6 \sin 6\theta$.

Food for Thought. The boundedness condition $\lim\limits_{r\to 0} |u(r,\theta)| < \infty$ is required.