Recall X is connected

if $X = U \sqcup V$, U, V are open

then $U = \emptyset$ or $V = \emptyset$

More properties

1) X space $\{Y_\alpha \mid \alpha \in \Lambda\}$ family of subspaces. & Y_α connected. $X = \bigcup_\alpha Y_\alpha$

and $Y_\alpha \cap Y_\beta \neq \emptyset$ $\forall \alpha, \beta$

Then X is conn.

P.f. $X = X_1 \sqcup X_2$ X_i open

$\forall \alpha$ $Y_\alpha \subset X_1$ or $Y_\alpha \subset X_2$

$\Lambda = \underset{\alpha}{\Lambda_1} \sqcup \underset{\beta}{\Lambda_2}$ $Y_\alpha \cap Y_\beta \neq \emptyset$ contradiction

$\Rightarrow X = X_1$ or X_2.

2) X_1, X_2 conn. $\Rightarrow X_1 \times X_2$ conn.

Pf. "$\Leftarrow$" $X_1 = \overset{\neq\emptyset}{U_1} \sqcup \overset{\neq\emptyset}{V_1}$ $\quad$ $X_1 \times X_2 \xrightarrow{\pi} X_1$

$\pi^{-1}(U_1) \sqcup \pi^{-1}(U_2) = X_1 \times X_2$

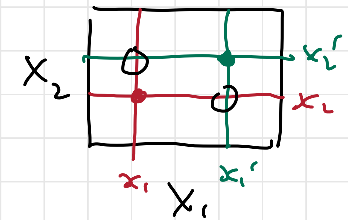
contradiction

"$\Rightarrow$" X_1, X_2 conn.

$X_1 \times X_2 = \bigcup \underbrace{x_1 \times X_2 \cup X_1 \times x_2}_{\text{conn.}}$

$x_1 \in X_1$

$x_2 \in X_2$



Apply property (1)

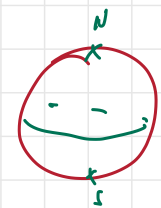
S^n

$\downarrow$

Examples

$\mathbb{R}^n$, $S^n (n \geq 1)$, $S^1 \times S^1 \times \cdots \times S^1$, $\mathbb{RP}^n_{n \geq 1}$

are conn.



$U = S^2 \setminus N$

$V = S^2 \setminus S$

Def'n X space. A max conn. subspace of X is called a **conn. component**

Properties 1) X_1, X_2 are both conn. component of X. then either $X_1 = X_2$ or $X_1 \cap X_2 = \emptyset$

2) $\forall\, x \in X$ $\exists!$ conn. component containing x.

3) All conn. components are closed.

$$X_i \subsetneq \overline{X_i} \subset X$$ violate "max".

4) if X has finitely many conn. components then components are open.

E.g. $\mathbb{Q} \subset (\mathbb{R}, |\cdot|)$

every pt in $\mathbb{Q}$ is conn. component.

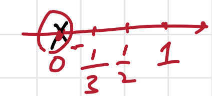
but not discrete.

Def'n X is loc. conn. if $\forall x \in X$ every open nbhd of x contains a conn. open nbhd.

$$\boxed{D^2 \cong \mathbb{R}^2 \quad \text{construct a homeo.!}}$$

Examples

1) $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not loc. conn.



2) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is loc conn.

$$\frac{1}{n} \in \left(\frac{1}{n+1}, \frac{1}{n-1}\right)$$

3) $(\mathbb{R}^n, |\cdot|)$ is loc. conn.

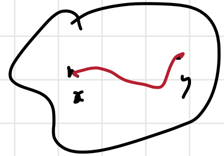
Def'n X is called **path-conn.** if

$\forall\, x, y \in X$ $\exists$ $\gamma : [0,1] \xrightarrow{\text{cont.}} X$

$\gamma(0) = x$ $\gamma(1) = y$.

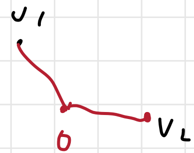
Examples

1) $B(0,\delta) = \{ v \in \mathbb{R}^n \mid |v| < \delta \}$



is path conn. $v_1, \; v_2 \in B(0,\delta)$

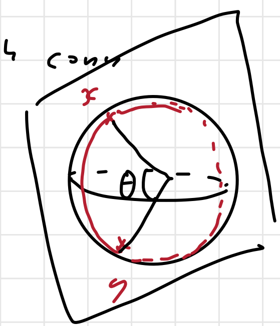
λv_1 $\quad \lambda \in [0,1]$ $\quad \lambda v_2$



$(1-2\lambda) \cdot v_1$ $\quad (2\lambda - 1) v_2$

can also take line connecting $v_1 v_2$
since "ball" is convex.

2) $S^n = \{ |v| = 1 \}$ is path conn



choose hyperplane section
through x & y
$t \cdot \theta$ $\quad [0,1]$

3) Path conn $\Rightarrow$ conn.

Pf. X is p.c. $\quad X = X_1 \sqcup X_2$ $\quad X_i$ open.

$x_1 \in X_1$, $x_2 \in X_2$, γ

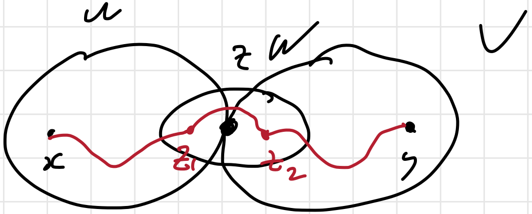
$\gamma^{-1}(X_1) \sqcup \gamma^{-1}(X_2) = [0,1]$, $\gamma^{-1}(X_1) \neq \emptyset$, $\gamma^{-1}(X_2) \neq \emptyset$

Contradicts to connectedness of $[0,1]$.

4) X is called loc. p.c. if
$\forall\, x \in U \subset X$ $\exists$ p.c. nbhd $\subset U$.

loc p.c + conn. $\Rightarrow$ p.c.

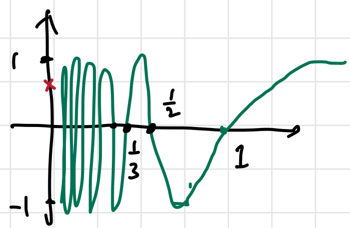
pf. X is loc. p.c. conn.
$x \neq y \in X$ if X is not path. conn.
$\exists\, U \ni x$ $\quad V \ni y$ both path conn.
but $U \cap V = \emptyset$
$\bar{U} \cap \bar{V} \neq \emptyset$ because X conn.
$z \in \bar{U}$ $\quad z \in \bar{V}$ $\quad \exists\, W \ni z$ p.c. $\quad W \cap U \neq \emptyset$, $\quad W \cap V \neq \emptyset$
Contradiction.



5) $C = C_0 \cup \{(0,y) \mid y \in [-1,1]\}$

$C_0 = \{(x, \sin \pi/x) \mid x \in [0,1]\}$

- $C = \overline{C_0}$
- C_0 is path conn
- C is conn but not path conn.



$(0, y_0)$ doesn't have p.c. nbhd.

$y_0 \in [-1,1]$