THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 3 Solution

Problem 1.

(i) Note that $-\nabla \cdot \nabla u = -\Delta u = Cy^2$. Let $\Omega = B_2$.

$$\begin{aligned} & \iint_\Omega \nabla \cdot \nabla u \, dxdy = -C \iint_\Omega y^2 \, dxdy = -C \int_0^{2\pi} \int_0^2 (r^2 \sin^2 \theta)\, r dr d\theta \\ = & \; -4C \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta = -4\pi C. \end{aligned}$$

On the other hand, by the divergence theorem,

$$\begin{aligned} & \iint_\Omega \nabla \cdot \nabla u \, dxdy = \int_{\partial\Omega} \nabla u \cdot \hat{\mathbf{n}} \, ds = \int_0^{2\pi} (\partial_x u, \, \partial_y u) \cdot (\frac{x}{2}, \frac{y}{2}) \, (2d\theta) \\ = & \int_0^{2\pi} (x\partial_x u + y \partial_y u) \, d\theta = \int_0^{2\pi} \sqrt{4-x^2} \, d\theta = \int_0^{2\pi} 2|\sin\theta| \, d\theta \\ = & \; 8 \int_0^{\pi/2} \sin\theta \, d\theta = 8[\cos(0) - \cos(\pi/2)] = 8 \end{aligned}$$

Thus the compatibility condition is $\boxed{C = -2/\pi}$.

(ii)

$$\begin{cases} \dfrac{dx}{ds} = 1, \;\; x(0) = x_0 \\ \dfrac{dy}{ds} = 2, \;\; y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = s + x_0 \\ y = 2s + y_0 \end{cases} \Longrightarrow y = 2(x - x_0) + y_0.$$

Choose $x_0 = 0$, the characteristic curves can be parametrized by y_0:

$$C_{y_0} = \{(x, \, y) : y = 2x + y_0\}.$$

Note that $u(x, y)$ remains unchanged along each characteristic curves and hence for all $(x, \, y) \in C_{y_0}$,

$$u(x \, , y) = u(0, \, y_0) = g(y_0).$$

Take $(2, 4+y_0) \in C_{y_0}$, we get $g(y_0) = u(2, 4+y_0) = h(4+y_0)$.

The compatibility condition is

$$\boxed{g(y_0) = h(4+y_0) \text{ for all } y_0 \in \mathbb{R}.}$$

(iii) Assume $x, y > 0$.

$$\begin{cases} \dfrac{dx}{ds} = x+1, \ x(0) = x_0 \\ \dfrac{dy}{ds} = y-1, \ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} \ln \dfrac{x+1}{x_0+1} = s \\ \ln \dfrac{y-1}{y_0-1} = s \end{cases} \Longrightarrow \frac{x+1}{x_0+1} = e^s = \frac{y-1}{y_0-1}$$

Then $y = (y_0 - 1)\frac{x+1}{x_0+1} + 1$. Choose $x_0 = 0$, the characteristic curves can be parametrized by y_0:

$$C_{y_0} = \{(x,\, y) : y = (y_0-1)(x+1)+1 = (y_0-1)x + y_0\}.$$

If $\boxed{y_0 \geq 1}$, then the PDE always has a solution along the line C_{y_0}.

If $\boxed{0 \leq y_0 < 1}$, then for all $(x,\, y) \in C_{y_0}$, since $u(x, y)$ remains unchanged along each characteristic curves, we have

$$u(x\,, y) = u(0,\, y_0) = g(y_0).$$

Take $(\frac{y_0}{1-y_0}, 0) \in C_{y_0}$, we get $g(y_0) = u(\frac{y_0}{1-y_0}, 0) = h(\frac{y_0}{1-y_0})$.

The compatibility condition is

$$\boxed{g(y_0) = h\left(\frac{y_0}{1-y_0}\right) \text{ for all } y_0 \in [0,1).}$$

Problem 2.

(i)

$$\begin{cases} \dfrac{dx}{ds} = 1, \ x(0) = x_0 \\ \dfrac{dy}{ds} = 2x, \ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0 + s \\ \dfrac{dy}{ds} = 2(x_0+s), \ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0 + s \\ y = 2x_0 s + s^2 + y_0 \end{cases}$$

Then $y = 2x_0(x - x_0) + (x - x_0)^2 + y_0 = x^2 - x_0^2 + y_0$ and hence the characteristic curves can be parametrized by (x_0, y_0):

$$C_{(x_0,y_0)} = \{(x,\, y) : y = x^2 - x_0^2 + y_0\}.$$

Let $W(s) = u(x(s), y(s))$. Then $\frac{dW(s)}{ds} = 2W$ implies that

$$u(x,y) = W(s) = W(0)e^{2s} = u(x_0,\, y_0)e^{2(x-x_0)}.$$

If $\boxed{y_0 > 0 \text{ and } x_0 = 0}$, then for all $(x,\, y) \in C_{(0,\,y_0)} = \{(x,\, y) : y = x^2 + y_0\}$,

$$u(x,y) = u(0,\, y_0)e^{2x} = (e^{y_0} - 1)e^{2x} = (e^{y-x^2} - 1)e^{2x}.$$

If $\boxed{y_0 = x_0 = 0}$, then for all $(x,\, y) \in C_{(0,0)} = \{(x,\, y) : y = x^2\}$,

$$u(x,y) = u(0,0)e^{2x} = 0.$$

If $\boxed{x_0 > 0 \text{ and } y_0 = 0}$, then for all $(x,\, y) \in C_{(x_0,\,0)} = \{(x,\, y) : y = x^2 - x_0^2\}$,

$$u(x,y) = u(x_0,\, 0)e^{2(x-x_0)} = x_0^2 e^{2(x-x_0)} = (x^2 - y)e^{2x - 2\sqrt{x^2-y}}.$$

(ii)

$$\begin{cases} \dfrac{dx}{ds} = 1,\ x(0) = x_0 \\ \dfrac{dy}{ds} = 2x(y+1),\ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0 + s \\ \dfrac{dy}{y+1} = 2(x_0 + s)ds,\ y(0) = y_0 \end{cases} \Longrightarrow \begin{cases} x = x_0 + s \\ \ln(\dfrac{y+1}{y_0+1}) = 2x_0 s + s^2 \end{cases}$$

Then $y = (y_0 + 1)e^{s^2 + 2x_0 s} - 1 = (y_0 + 1)e^{x^2 - x_0^2} - 1$ and hence the characteristic curves can be parametrized by (x_0, y_0):

$$C_{(x_0,y_0)} = \{(x,\, y) : y = (y_0 + 1)e^{x^2 - x_0^2} - 1\}.$$

Let $W(s) = u(x(s), y(s))$. Then $\frac{dW(s)}{ds} = 2$ implies that

$$u(x,y) = W(s) = W(0) + 2s = u(x_0,\, y_0) + 2(x - x_0).$$

If $\boxed{y_0 > 0 \text{ and } x_0 = 0}$, then $C_{(0,y_0)} = \{(x, y) : y = (y_0+1)e^{x^2} - 1\}$ and for all $(x, y) \in C_{(0,y_0)}$, we have

$$u(x,y) = u(0, y_0) + 2x = g(y_0) + 2x = g\big((y+1)e^{-x^2} - 1\big) + 2x.$$

If $\boxed{y_0 = x_0 = 0}$, then for all $(x, y) \in C_{(0,0)} = \{(x, y) : y = e^{x^2} - 1\}$,

$$u(x,y) = u(0,0) + 2x = g(0) + 2x = 2x.$$

If $\boxed{x_0 > 0 \text{ and } y_0 = 0}$, then $C_{(x_0,0)} = \{(x, y) : y = e^{x^2-x_0^2} - 1\}$ and for all $(x, y) \in C_{(x_0,0)}$, we have

$$\begin{aligned} u(x,y) &= u(x_0, 0) + 2(x - x_0) = h(x_0) + 2(x - x_0) \\ &= h\big(\sqrt{x^2 - \ln(y+1)}\big) + 2x - 2\sqrt{x^2 - \ln(y+1)}. \end{aligned}$$

Problem 3.

(i) Note that $\nabla \cdot (\partial_x u, u) = \partial_x^2 u + \partial_y u = 3$. Let $\Omega = [0,1] \times [0,1]$.

$$\iint_\Omega \nabla \cdot (\partial_x u, u)\, dxdy = 3.$$

On the other hand, by the divergence theorem,

$$\begin{aligned} &\iint_\Omega \nabla \cdot (\partial_x u, u)\, dxdy = \int_{\partial\Omega} (\partial_x u, u) \cdot \hat{\mathbf{n}}\, ds \\ =\ & \int_0^1 -u(t, 0)\, dt + \int_0^1 \partial_x u(1, t)\, dt + \int_0^1 u(1-t, 1)\, dt + \int_0^1 -\partial_x u(0, 1-t)\, dt \\ =\ & -1 + 0 + 1 + 0 = 0, \end{aligned}$$

which is impossible. Thus the PDE has no solution.

(ii)

$$\begin{cases} \dfrac{dt}{ds} = t,\ t(0) = t_0 \\ \dfrac{dx}{ds} = 2,\ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} \ln \dfrac{t}{t_0} = s \\ x = 2s + x_0 \end{cases} \Longrightarrow \begin{cases} t = t_0 e^s \\ x = 2s + x_0 \end{cases}$$

Then $t = t_0 e^{(x-x_0)/2}$. Choose $x_0 = 0$, the characteristic curves can be parametrized by $t_0 > 0$:

$$C_{t_0} = \{(t,\, x) : t = t_0 e^{x/2}\}.$$

Now suppose that the problem has a solution $u(t,x)$. Note that $u(t,x)$ is continuous on the closed half plane $\{(t,x) : t \ge 0 \text{ and } x \in \mathbb{R}\}$. Since it remains unchanged along each characteristic curves, for all $x \in \mathbb{R}$ and $t_0 > 0$,

$$u(t_0 e^{x/2}, x) = u(t_0, 0) < \infty.$$

However,

$$\lim_{x\to-\infty} u(t_0 e^{x}, x) = \lim_{x\to-\infty} u(0,x) = \lim_{x\to-\infty} x^4 = +\infty,$$

which is impossible.

(iii)

$$\begin{cases} \dfrac{dt}{ds} = 1,\ t(0) = t_0 \\ \dfrac{dx}{ds} = -(x+t+1),\ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} t = s + t_0 \\ \dfrac{dx}{ds} + x = -s - t_0 - 1,\ x(0) = x_0 \end{cases}$$

$$\Longrightarrow \begin{cases} t = s + t_0 \\ \dfrac{d(xe^s)}{ds} = -se^s - t_0 e^s - e^s,\ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} t = t_0 e^s \\ x = -t + (x_0 + t_0) e^{t_0 - t} \end{cases}$$

Choose $t_0 = 0$, the characteristic curves can be parametrized by x_0:

$$C_{x_0} = \{(t,\, x) : x = x_0 e^{-t} - t\}.$$

Now suppose that the problem has a solution $u(t,x)$. In particular, $W(s) := u(t(s), x(s))$ is differentiable for $s \ge 0$.

Starting from the point on the positive x-axis, say $(t(0), x(0)) = (0, x_0)$ with $x_0 > 0$, and then moving along the characteristic curve C_{x_0} in the first quadrant, it will intersect the positive t-axis at the point $(t(s^*), x(s^*)) = (t^*, 0)$ for some $s^*, t^* > 0$.

As $\frac{dW(s)}{ds} = W^8 \geq 0$, $W(s)$ is an increasing function. Thus

$$W(s^*) \geq W(0) = u(0, x_0) = 6x_0 > 0.$$

On the other hand,

$$W(s^*) = u(t^*, 0) = -4(t^*)4 < 0,$$

which is a contradiction.

Remark. In fact, it suffices to consider a particular charactertistic curve to obtain a contradiction. For example, choose $x_0 = e$, then the characteristic curve C_e will intersect the positive t-axis at $(1, 0)$.