

Rem For our purpose, we may use the weaker estimate:

$$|S(0)|^2 \leq \frac{1}{\pi R^2} \int_{D(R)} |S(z)|^2 dx dy.$$

$$S(z) = \sum_{k \geq 0} a_k z^k, \quad |a_0|^2 \leq \frac{1}{\pi R} \int_{D(R)} |S(z)|^2 dx dy.$$

By a standard covering argument we get that for $K \in \mathbb{N}$ and $\epsilon > 0$

$$\sum_{|\gamma| \geq K} |\gamma'(z)|^2 \leq C_K, \quad \exists C_K > 0.$$

From this, we have $|S(0)|^2 \leq |a_0|^2 \leq \frac{1}{\pi R^2} \int_{D(R)} |S(z)|^2 dx dy$.

For our purpose we may use the weaker estimate $|S(0)|^2 \leq \frac{1}{\pi R} \int_{D(R)} |S(z)|^2 dx dy$.

Because $\int_0^{2\pi} (re^{i\theta})^k (r\bar{e}^{i\theta})^l r dr d\theta = \int_0^R r^{k+l+1} e^{i(k-l)\theta} dr \Big|_0^{2\pi} = 0$.

$$\int_{k,l} \sum a_k \bar{a}_l z^k \bar{z}^l dx dy = \int \sum |a_k|^2 |z|^{2k} dx dy.$$

$$\text{because } \int_0^{2\pi} (re^{i\theta})^k (r\bar{e}^{i\theta})^l r dr d\theta \quad (k \neq l).$$

$$= \int_0^{2\pi} r^{k+l+1} e^{i(k-l)\theta} dr d\theta.$$

$$= \left(\int_0^{2\pi} e^{i(k-l)\theta} d\theta \right) \left(\int_0^R r^{k+l+1} dr \right) = 0.$$

Prop: Let $z_0 \in D$ be arbitrary. There exists $h \in H^\infty(D)$ and $k \geq 2$ such that $f(z) = P_r^k(h)(z_0) \neq 0$.

Fix $z_0 \in D$, write

$$\Gamma = \Gamma' \cup \Gamma'', \text{ where}$$

$$\begin{cases} \Gamma' = \{ \gamma \in \Gamma : |\gamma'(z_0)| \geq 1/2 \}, & e \in \Gamma' \\ \Gamma'' = \{ \gamma \in \Gamma : |\gamma'(z_0)| < 1/2 \}. \end{cases}$$

Clearly Γ' is finite and Γ'' is infinite., as.

$$\sum_{\Gamma} |\gamma'(z_0)|^2 = \sum_{\Gamma'} + \sum_{\Gamma''} < \infty.$$

Say $|\Gamma'| = m$. Pick a polynomial Q (by interpolation) of degree at most $m-1$. st.

$$Q(z_0) = 1 \text{ and } Q(\gamma z_0) = 0 \text{ for all } \gamma \in \Gamma' \setminus \text{id}.$$

Pick k large, and

$$\begin{aligned} f(z) &= P_r^k(Q)(z) = \sum_{\gamma \in \Gamma} Q(\gamma z) (\gamma'(z))^k \\ &= \sum_{\gamma \in \Gamma'} Q(\gamma z) (\gamma'(z))^k + \sum_{\gamma \in \Gamma''} Q(\gamma z) (\gamma'(z))^k \end{aligned}$$

I want to pick k so that $f(z_0) \neq 0$.

$$\text{Note } \sum_{\Gamma'} Q(\gamma z_0) (\gamma'(z_0))^k = 1.$$

$$\text{WTS. } \exists k \gg 0, \left| \sum_{\Gamma''} Q(\gamma z_0) (\gamma'(z_0))^k \right| \leq \frac{1}{2} \text{ and}$$

this suffices to complete the proof.

$$\exists C > 0, \text{ s.t. } |\mathbb{Q}(z)| < C \quad \forall z \in \bar{D}$$

$$\begin{aligned} \Rightarrow \sum_{\gamma \in \Gamma''} \mathbb{Q}(\gamma z_0) (\gamma'(z_0))^k &\leq C \sum_{\gamma \in \Gamma''} |\gamma'(z_0)|^k \\ &= C \sum_{\gamma \in \Gamma''} |\gamma'(z_0)|^2 |\gamma'(z_0)|^{k-2} \\ &< \left(\frac{1}{2} \right)^{k-2} \left(\sum_{\gamma \in \Gamma''} |\gamma'(z_0)|^2 \right) C \end{aligned}$$

\hookrightarrow uniformly bounded

$$\Rightarrow \exists k > 0, \text{ s.t. } \otimes \leq \frac{1}{2}$$

11.