EXERCISES ON THE OUTER AUTOMORPHISMS OF S_6

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1. Introduction

In this class, we investigate the outer automorphism of S_6. Let's recall some definitions, so that we can state what an outer automorphism is.

Definition 1.1. Recall that the **symmetric group on n elements** is the group of all permutations of n elements. So elements of the symmetric group are precisely bijections $\{1, \ldots, n\} \to \{1, \ldots, n\}$ with multiplication in S_n corresponding to composition of maps.

Definition 1.2. An **automorphism** of a group G is an group isomorphism $f : G \to G$.

Exercise 1.3. Let G be a group and $g \in G$ be an element. Check that the map

$$\begin{aligned} \phi_g : G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

is a group automorphism.[1]

Definition 1.4. An automorphism $f : G \to G$ is called **inner** if it is of the form ϕ_g (conjugation by g) as defined in Exercise 1.3. An automorphism $f : G \to G$ is called **outer** if it is not inner.

Of course every group has at most $\#G$ inner automorphisms (though some of these may be the same), at most one for each $g \in G$. A natural question to ask is:

Question 1.5. Which symmetric groups have outer automorphisms? For those that do have outer automorphisms, how many do they have?

The answer is quite simple:

Theorem 1.6.
1. *The symmetric group S_6 is the only symmetric group with an outer automorphism.*
2. *Further, any two outer automorphisms $\phi : S_6 \to S_6$ and $\phi' : S_6 \to S_6$ are related by an inner automorphism. That is, there some $\sigma \in S_6$ with $\phi = \phi_\sigma \circ \phi'$.*

We'll come back to proving this theorem later, but for most of the course, we'll concentrate on investigating various descriptions of this mysterious outer automorphism of S_6.

2. Mystic Pentagons

We now encounter our first incarnation of the outer automorphism of S_6 in terms of mystic pentagons. Recall that a complete graph on n elements is the graph with n points and every pair of points joined by an edge.

Exercise 2.1.
(1) Show there are six ways to color the edges of a complete graph on 5 vertices with two colors so that the edges of one color form a 5-cycle. (See Figure 1.) Here we consider two colorings equivalent if one can move from one to the other by reversing the labels of the colors. We call these six colorings the six **mystic pentagons**.
(2) Verify that if the edges of one color form a 5-cycle, the edges of the other color also form a 5-cycle.

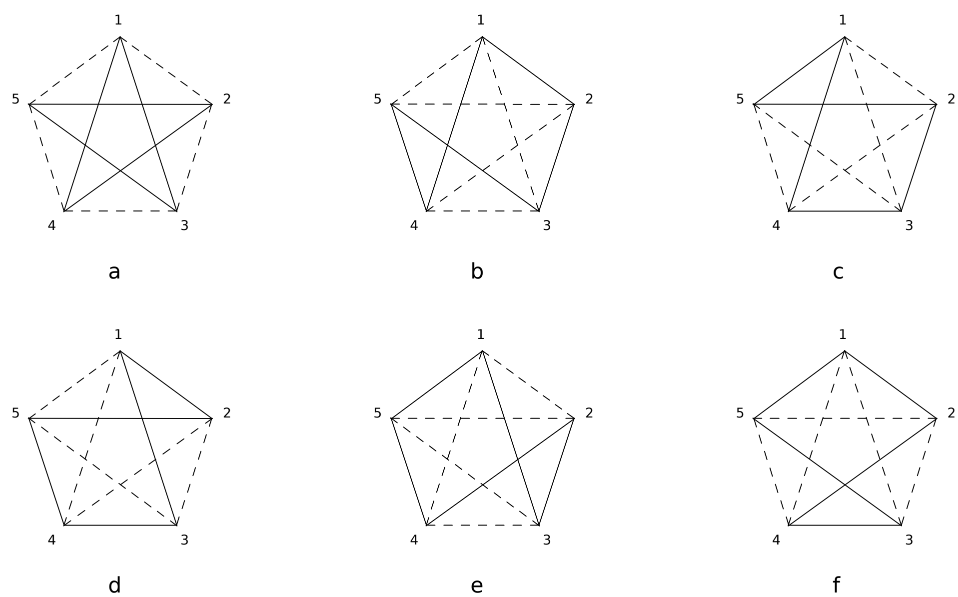

Figure 1. The six mystic pentagons

We label the six mystic pentagons by **a**, **b**, **c**, **d**, **e**, **f** as in Figure 1. For T a set of size n, we use $S_T \simeq S_n$ to denote the symmetric group acting on the set T.

Exercise 2.2.
(1) Use Exercise 2.1 to describe a map $i : S_{\{1,2,3,4,5\}} \to S_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}\}} = S_6$. [2]
(2) Check your understanding of the previous part by computing $i((12))$, where (12) denotes the permuation sending $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 4, 5 \mapsto 5$.
(3) Check that this realizes S_5 as a subgroup of S_6 (i.e., check that the kernel of i is only the identity $e \in S_5$). [3]
(4) Show that the inclusion $i : S_{\{1,2,3,4,5\}} \to S_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}\}}$ is not induced by an inclusion $j : \{1,2,3,4,5\} \to \{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}\}$. More precisely, show there is no inclusion of sets j so that $j(\sigma(k)) = i(\sigma)(j(k))$ for $k \in \{1,2,3,4,5\}, \sigma \in S_{\{1,2,3,4,5\}}$.

Exercise 2.3. Let $\{1,2,3,4,5,6\}$ label the 6 cosets of $S_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}\}}/i(S_5)$. Show the action of $S_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}\}}$ on these 6 cosets induces a map $\phi : S_{\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\mathbf{f}\}} \to S_{\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}\}}$.

Remark 2.4. In the next class, we will relate the mystic pentagons to other problems in order to verify that the map ϕ of Exercise 2.3 is indeed an outer automorphism.

Your homework is to finish up through the previous problem. Also, check out the wobbly nobbly, a dog chew toy exhibiting the outer automorphism! If you have more time, and have not seen the bijection between conjugacy classes and partitions in S_n, we recommend you ask for Appendix C and do the exercises there. Once you have finished that attempt the following.

For the following exercise, it will help to look at a nobbly wobbly. I don't completely understand the answer to the following question myself. It is a bit open ended, but see what you can figure out.

Exercise 2.5 (Loosely phrased, optional, very challenging exercise)**.** Can you figure out how the nobbly wobbly relates to the outer automorphism of S_6? [4]

I don't know a clean way to solve the following exercise directly. See if you can find one. We'll see an indirect way to check the next exercise tomorrow.

Exercise 2.6 (Optional very challenging exercise)**.** Show directly that the map ϕ of Exercise 2.3 is actually an outer automorphism. [5]

3. LABELED TRIANGLES

The action on mystical pentagons, although quite beautiful is not symmetric, since it preferences a certain subset of 5 of the 6 elements, corresponding to the 5 vertices of the pentagon. We now discuss a more symmetric construction.

Definition 3.1. Note that are $20 = \binom{6}{3}$ triangles on 6 vertices labeled $\{1,2,3,4,5,6\}$ A configuration of **labeled triangles** is a way of dividing these 20 triangles into two sets of 10 (which we think of as colored white and black, up to swapping of the colors) so that

(1) two disjoint triangles are are colored different colors
(2) every tetrahedron (subset of 4 vertices) has two triangles of each color

Fact 3.2. There are six configurations of labeled triangles. This can be checked via a tedious combinatorial exercise, which we recommend skipping.

Exercise 3.3. Construct a bijection between mystic pentagons **a,b,c,d,e,f** and the 6 groupings of labeled triangles as follows:

(1) Given a mystic pentagon, construct a 2-coloring of the 20 triangles on 6 vertices as follows: If edge AB is colored black for $A, B \in \{1,2,3,4,5\}$, coloring the triangle $6AB$ black, and the triangle CDE white (where $CDE = \{1,2,3,4,5\} - \{A, B\}$). If edge AB is colored white for $A, B \in \{1,2,3,4,5\}$, coloring the triangle $6AB$ white, and the triangle CDE black (where $CDE = \{1,2,3,4,5\} - \{A, B\}$).
(2) Verify that under the coloring of the previous part, any tetrahedron has precisely 2 triangles of each color.
(3) Conclude that the above construction determines a map from mystic pentagons to groupings of labeled triangles.
(4) Verify the the map from mystic pentagons to groupings of labeled triangles is a bijection.

Exercise 3.4. There is a natural action of S_6 on the 6 groupings of labeled triangles (which we denote **a, b, c, d, e, f** as they are in bijection with mystic pentagons after all) given by permuting the 6 labels, and seeing how that permutes the six groupings. This yields a map $S_6 \to S_{\{\mathbf{a,b,c,d,e,f}\}} \simeq S_6$.

(1) Check that the map $S_6 \to S_{\{\mathbf{a,b,c,d,e,f}\}} \simeq S_6$ is an automorphism.
(2) Check that under this map (12) is sent to $(\mathbf{ad})(\mathbf{bc})(\mathbf{ef})$. (This uses cycle notation. If you have not seen it, as for Appendix C.)
(3) Verify that this automorphism $S_6 \to S_6$ is not inner. [6]

Your homework is to finish the above exercises. If you have finished them, ask for the next handout on labeled icosahedra.

Remark 3.5. It turns out that the outer automorphism constructed in Exercise 3.4 is inverse to that constructed for mystic pentagons in Exercise 2.3. The verification of this seems quite painstaking, so we suggest you skip it.

4. LABELED ICOSAHEDRA

Definition 4.1. A **labeled icosahedron** is an icosahedron with each of its 12 vertices labeled with a number from 1 to 6 so that opposite vertices are labeled by the same number. Two labeled icosahedra are considered equivalent if one can be taken the other by rotations and reflections.

Exercise 4.2. Show that, up to rotations and reflections, there are 12 labeled icosahedra. [7]

Exercise 4.3. From a labeled icosahedra, show that one obtains ten distinct size 3 subsets of $\{1, 2, 3, 4, 5, 6\}$ with each subset given by taking the three vertices of any given face. Show that there is a unique way to pair up the 12 labeled icosahedra into 6 pairs of 2 so that two labeled icosahedra are in the same pair if and only if they have no size 3 subsets in common. [8]

Exercise 4.4. Construct a bijection between the 6 pairs of labeled icosahedra from Exercise 4.3 and groupings of labeled triangles.

Exercise 4.5. Construct a direct bijection between the six pairs of labeled icosahedra from Exercise 4.3 and mystic pentagons. [9]

Exercise 4.6. Check that S_6 acts on the six pairs of labeled icosahedra by permuting the labelings, and that this action defines a map $S_6 \to S_6$ which is an outer automorphism.

5. Synthemes and Pentads

Definition 5.1. A **duad** is a pair of two distinct elements of $\{1,2,3,4,5,6\}$. A **syntheme** is a partition of $\{1,2,3,4,5,6\}$ into three duads. A **pentad** is a set of five synthemes so that each of the $15 = \binom{6}{2}$ duads appears once.

Example 5.2. The triple $12/35/46$ is an example of a syntheme. An example of a pentad is given by

$$\{12/35/46, 23/14/56, 34/25/16, 45/13/26, 15/24/36\}$$

Fact 5.3. There are exactly 6 pentads.

Exercise 5.4. Define an action of S_6 on the 6 pentads and use this to construct a map $\phi : S_6 \to S_6$.

Exercise 5.5. In this exercise, we directly construct a bijection between pentads and mystic pentagons, respecting the S_6 actions.

(1) Given a mystic pentagon, show that for each white edge AB there is a unique black edge CD sharing no vertex in common with AB.
(2) Given a white edge AB and the corresponding black edge CD from the previous exercise, let E denote the remaining vertex. From this data, construct the syntheme $AB/CD/E6$.
(3) Show that from a mystic pentagon one obtains precisely 5 synthemes, each constructed as in the previous part.
(4) Show these 5 synthemes from the previous part form a pentad.
(5) Check that under the above map from mystic pentagons to pentads mystic pentagon **a** from Figure 1 corresponds to the pentad given in Example 5.2.
(6) Show the map constructed above from mystic pentagons to pentads is a bijection.

Exercise 5.6. Construct a direct bijection between pentads and labeled triangles. This will be the composition of your bijections from pentads to mystic pentagons and mystic pentagons to labeled triangles, but try to describe it directly.

Remark 5.7. Using the bijection between pentads and labeled triangles, S_6 acts on both labeled triangles (by permuting 6 labeled vertices of each pair of disjoint triangles) and on pentads (by permuting the 6 numbers appearing in each syntheme). It turns out that the two S_6 actions are identified under this bijection.

Remark 5.8. So, assuming Remark 5.7, the bijection between pentads and labeled triangles implies that the map $\phi : S_6 \to S_6$ of Exercise 5.4 is an outer automorphism, since it is identified with the outer automorphism of S_6 constructed via labeled triangles.

Your homework is to finish all exercises in this section. If you finish them, ask for the handout describing the outer automorphism via Sylow subgroups.

6. Action on the Sylow Subgroups

There is also a description of the outer automorphism via an action of S_5 on the 6 5-Sylow subgroups of S_5.

Exercise 6.1. Let S_5 act on the six mystic pentagons (as in Exercise 2.2). Let $G \subset S_5$ be the stabilizer of pentagon **a** in Figure 1.

(1) Show this stabilizer has order 20.
(2) Show this yields an action of S_5 on S_5/G. Since $\#S_5/G = 6$ this yields a map $\tilde{i} : S_5 \to S_6$.
(3) Show this map is the same as the map $i : S_5 \to S_6$ constructed by sending a permutation of the five vertices of a pentagon to a permutation of the 6 mystic pentagons (see Exercise 2.2). In other words, realize the action of S_5 on the 6 mystic pentagons as the action of S_5 on the cosets S_5/G.

Exercise 6.2. Let S_5 act on itself via conjugation. Explicitly, the action map is $\rho : S_5 \times S_5 \to S_5$ given by $(g, x) \mapsto gxg^{-1}$.

(1) Show that S_5 has 6 subgroups of order 5. These are called 5-Sylow subgroups of S_5. [10]
(2) Under the action ρ, show that these 6 subgroups are permuted. Conclude that S_5 acts on its six subgroups of order 5 via conjugation. (Specifically $g \in S_5$ it sends a subgroup $H \subset S_5$ of order 5 to gHg^{-1}.)
(3) Show that the group G defined in Exercise 6.1 is precisely the stabilizer of the order 5 subgroup generated by (12345).
(4) Conclude that the action of S_5 on its 6 order 5 subgroups by conjugation yields a map $i : S_5 \to S_6$ which is the same as the map i constructed in our investigation of mystic pentagons from Exercise 2.2.

Exercise 6.3. As in Exercise 2.3, use the inclusion $i : S_5 \to S_6$ to construct a a map $\phi : S_6 \to S_6$ which is an outer automorphism of S_6.

7. Proving S_6 is the only symmetric group with an outer automorphism

In this section, we prove the following theorem in a series of exercises.

Theorem 7.1. *(1) The symmetric group S_6 is the only symmetric group with an outer automorphism.*

(2) Further, S_6 has a unique outer automorphism in the sense that any two outer automorphisms $f : S_6 \to S_6$ and $g : S_6 \to S_6$ are related by an inner automorphism. That is, there some $\sigma \in S_6$ with $f = \phi_\sigma \circ g$.

7.1. **Criterion for being an inner automorphism.** Recall that $\sigma \in S_n$ is a **transposition** if it is of the form (ab) (here, we omit all length 1 cycles in notation, so $(12)(3)(4)(5)$ would be notated simply as (12)). The main result we will prove in this subsection, on the way to proving Theorem 7.1 is the following:

Proposition 7.2. *Suppose $f : S_n \to S_n$ is an automorphism. Then it is an inner automorphism if and only if it sends the set of all transpositions to itself.*

Exercise 7.3. Show that for $\sigma \in S_n$, $\sigma^{-1}(12)\sigma = (\sigma(1)\sigma(2))$. Conclude that an inner automorphism sends the set of transpositions to itself, proving one direction of Proposition 7.2.

It only remains to show that if an automorphism sends all transpositions to transpositions then it is inner (i.e., conjugation by some element of G).

Recall that two elements $h_1, h_2 \in G$ are **conjugate** if there is some $g \in G$ with $gh_1g^{-1} = h_2$. A **conjugacy class** is a set of all elements conjugate to any given element. If you would like to see more details, or are feeling lost about conjugacy classes in S_n, ask for Appendix C.

Exercise 7.4. For G any group, show that any automorphism $f : G \to G$ sends conjugacy classes to conjugacy classes. That is, if $c = \{x : x \in c\} \subset S_n$ is a conjugacy class then $f(c) = \{f(x) : x \in c\}$ is also a conjugacy class of G. Show further that $\#c = \#f(c)$.

Exercise 7.5. Recall that an element $g \in G$ has **order** n if $g^n = \mathrm{id}$ but $g^k \neq \mathrm{id}$ for $0 < k < n$. If $f : G \to G$ is an automorphism, show that $\mathrm{ord}(f(g)) = \mathrm{ord}(g)$ for any $g \in G$.

Exercise 7.6. Check explicitly that all automorphisms of S_1, S_2, and S_3 are inner, so we may assume $n > 3$ in Proposition 7.2. [11]

Exercise 7.7. Suppose $n > 3$. Let $f : S_n \to S_n$ be an automorphism sending transpositions to transpositions. In this exercise, we show it is possible to write $f(12) = (ab_2), f(13) = (ab_3), \ldots, f(1n) = (ab_n)$ for some distinct elements $a, b_2, \ldots, b_n \in \{1, \ldots, n\}$ in several steps:

(1) Prove that a product of transpositions $(xy) \cdot (zw)$ has cycle type
   (a) 2^2 if and only if $\# \{x, y\} \cap \{z, w\} = 0$
   (b) 3 if and only if $\# \{x, y\} \cap \{z, w\} = 1$.
   If $\# \{x, y\} \cap \{x, w\} = 2$ show $(ab) \cdot (cd) = \mathrm{id}$.
(2) Suppose $f(12) = (xy)$ and $f(13) = (zw)$. Use that $(12)(13)$ has order 3 to conclude $(xy)(zw)$ has cycle type 3.
(3) Deduce from the previous two parts that if $f(12) = (xy)$ and $f(13) = (zw)$ then $\# \{x, y\} \cap \{z, w\} = 1$. Let a denote the common element, and let b_2 and b_3 denote the remaining distinct elements so that we may assume $f(12) = (ab_2)$ and $f(13) = (ab_3)$.
(4) Show $f(23) = (b_2 b_3)$.
(5) If a denotes the common element from part (3), show that for all k, $f(1k) = (ab_k)$ for some $b_k \in \{1, \ldots, n\}$. [12]
(6) Conclude that for all $2 \leq k \leq n$, with a as in part (3), we can write $f(1k) = (ab_k)$.
(7) Show that the $b_2, \ldots, b_n$ defined in the previous part are pairwise distinct.
(8) Conclude that it is possible to write

$$f(12) = (ab_2), f(13) = (ab_3), \ldots, f(1n) = (ab_n)$$

   for some distinct elements $a, b_2, \ldots, b_n \in \{1, \ldots, n\}$.

Exercise 7.8. Show that if $f : S_n \to S_n$ is an automorphism with $f(12) = (ab_2), f(13) = (ab_3), \ldots, f(1n) = (ab_n)$ for some distinct elements $a, b_2, \ldots, b_n \in \{1, \ldots, n\}$, then we can compose with some inner automorphism to assume that $f(1k) = (1k)$.

Exercise 7.9. (1) Show that the transpositions $(12), (13), \ldots, (1k)$ generate S_n (i.e., any $\sigma \in S_n$ can be written as a product of these transpositions).
(2) Conclude from the previous part that any automorphism of S_n preserving $(12), (13), \ldots, (1k)$ must be the identity.

Exercise 7.10. Conclude that Proposition 7.2 holds. [13]

Your homework is to finish up through Exercise 7.10 But feel free to work further if you would like.

7.2. **Showing that transpositions are preserved.** The main result we will prove in this subsection is the following. The first part is proven in Exercise 7.15 and the second in Exercise 7.16.

Proposition 7.11. *(1) If $f : S_n \to S_n$ is an automorphism and $n \neq 6$, for any transposition $\sigma \in S_n$, $f(\sigma)$ is also a transposition.*

(2) *If $f : S_6 \to S_6$ is an automorphism then the conjugacy class of transpositions is either sent to itself under f or it is sent to the conjugacy class of cycle type 2^3. In the latter case, the conjugacy class of cycle type 2^3 is sent to that of transpositions.*

Exercise 7.12. Using the bijection between conjugacy classes and partitions of Exercise C.10 show that the only cycle types of order 2 are those corresponding to partitions 2^k for some $1 \leq k \leq \lfloor n/2 \rfloor$. That is, such elements are a product of k disjoint transpositions, and representatives for these conjugacy classes are of the form $(12)(34)\cdots((2k-1)(2k))$.

Exercise 7.13. Compute the number of elements in S_n in the conjugacy class of cycle type 2^k. [14]

Exercise 7.14 (Computational but elementary exercise)**.** (1) Show that when $n \neq 6$ the number of transpositions (i.e., the size of the conjugacy class of cycle type 2) is distinct from the number of elements in any conjugacy class of the form 2^k for $k \neq 1$. [15]

(2) If $n = 6$, show that the number of transpositions is equal to the number of elements in the conjugacy class of type 2^3, but different from the number of elements in 2^2.

Exercise 7.15. Show that when $n \neq 6$, f must send the set of transpositions must to itself. This proves Proposition 7.11(1). [16]

Exercise 7.16. Show that when $n = 6$, the set of transpositions must be sent to the set of transpositions or to the conjugacy class of type 2^3. Similarly, show the conjugacy class of type 2^3 is also either sent to that of transpositions or itself. Conclude that Proposition 7.11(2) holds. [17]

7.3. **Combining the above to prove Theorem 7.1.** We are now ready to prove Theorem 7.1.

Exercise 7.17. Prove Theorem 7.1(1) by combining Proposition 7.2 and Proposition 7.11(1).

Exercise 7.18. Show that if $f : S_6 \to S_6$ and $g : S_6 \to S_6$ are two automorphisms which do not send transpositions to transpositions, then $f \circ g^{-1} : S_6 \to S_6$ does send transpositions to transpositions. [18]

Exercise 7.19. Show that if $f : S_6 \to S_6$ and $g : S_6 \to S_6$ are two automorphisms which do not send transpositions to transpositions, then f and g are related by an inner automorphism. That is, show there is some $\sigma \in S_6$ so that $f = \phi_\sigma \circ g$. [19]

Exercise 7.20. Prove Theorem 7.1(2). [20]

If you did not get to this point by the end of Saturday, your homemwork is to finish the above exercises. If you finished this, ask for the next handout on automorphisms of symmetric groups.

8. AUTOMORPHISMS OF SYMMETRIC GROUPS

In this handout, we classify automorphisms of symmetric groups.

Exercise 8.1. Recall that the center of a group G is the set of elements $g \in G$ so that $gh = hg$ for all $h \in G$. For each n, compute the center of S_n.

Exercise 8.2. Determine the group of inner automorphisms of S_n for each n.

Exercise 8.3. Determine the group of automorphisms of S_n, $\mathrm{Aut}(S_n)$, for each $n \geq 1$. [21]

If you have finished the above exercises, ask for the next handout on the outer automorphism in terms of $\mathrm{PGL}_2(\mathbb{Z}/5\mathbb{Z})$.

9. Bonus description: The outer automorphism and $\mathrm{PGL}_2(\mathbb{Z}/5\mathbb{Z})$

Having proven that S_6 is the only symmetric group with an outer automorphism, let's investigate another description of this outer automorphism. This sections assumes you are comfortable with introductory linear algebra.

9.1. Automorphisms of $\mathbb{P}^1(\mathbb{F}_5)$.

Definition 9.1. Let $\mathbb{F}_5$ denote the field on 5 elements, which can be viewed as $\mathbb{Z}/5\mathbb{Z}$ with multiplication and addition taken $\bmod 5$. Let $V = \mathbb{F}_5 \oplus \mathbb{F}_5$ denote a 2 dimensional vector space over $\mathbb{F}_5$.

Exercise 9.2. Show that V has precisely 6 1-dimensional subspaces (which we call lines).

Definition 9.3. Let $\mathbb{P}^1(\mathbb{F}_5)$ denote the set of 6 lines in a 2-dimensional vector space V over $\mathbb{F}_5$.

We make the following ad hoc definition of automorphisms of $\mathbb{P}^1(\mathbb{F}_5)$.

Definition 9.4. An automorphism of $\mathbb{P}^1(\mathbb{F}_5)$ is any map $\mathbb{P}^1(\mathbb{F}_5) \to \mathbb{P}^1(\mathbb{F}_5)$ induced by a 2×2 matrix acting on the 6 lines of $\mathbb{F}_5 \oplus \mathbb{F}_5$.

Exercise 9.5. Show that the automorphisms of $\mathbb{P}^1(\mathbb{F}_5)$ act triply transitively on the 6 points of $\mathbb{P}^1(\mathbb{F}_5)$. That is, show that for any three distinct points x, y, z and any other set of three distinct points a, b, c, (which may overlap with x, y, z) there is a unique automorphism of $\mathbb{P}^1(\mathbb{F}_5)$ sending $x \mapsto a, y \mapsto b, z \mapsto c$. [22]

9.2. The outer automorphism in terms of labelings of $\mathbb{P}^1(\mathbb{F}_5)$.

Exercise 9.6. Label the 6 points of $\mathbb{P}^1(\mathbb{F}_5)$ by the numbers $\{a, b, c, d, e, f\}$. Consider two labelings of $\mathbb{P}^1(\mathbb{F}_5)$ equivalent if there is an automorphism of $\mathbb{P}^1(\mathbb{F}_5)$ taking one labeling to another. Show that there are 6 equivalence classes of labelings of $\mathbb{P}^1(\mathbb{F}_5)$.

Exercise 9.7. Fix names $0, 1, 2, 3, 4, 5, \infty$ for the 6 points of $\mathbb{P}^1(\mathbb{F}_5)$. Here ∞ corresponds to the line spanned by $(1, 0)$ in the 2-dimensional vector space V, and the other lines i for $0 \leq i \leq 5$ correspond to the line spanned by the vector $(i, 1)$ in V. Consider the following representatives for labelings of the 6 points of $\mathbb{P}^1(\mathbb{F}_5)$ by $\{a, b, c, d, e, f\}$:

(1) $L_1 : a = 0, b = 1, c = \infty, d = 2, e = 3, f = 4$
(2) $L_2 : a = 0, b = 1, c = \infty, d = 2, e = 4, f = 3$
(3) $L_3 : a = 0, b = 1, c = \infty, d = 3, e = 2, f = 4$
(4) $L_4 : a = 0, b = 1, c = \infty, d = 3, e = 4, f = 2$
(5) $L_5 : a = 0, b = 1, c = \infty, d = 4, e = 2, f = 3$

(6) $L_6 : a = 0, b = 1, c = \infty, d = 4, e = 3, f = 2$

Show that the transposition (ab) sends L_1 to the labeling $a = 1, b = 0, c = \infty, d = 2, e = 3, f = 4$. Show that this line is sent to L_6 by applying the automorphism corresponding to the matrix

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Show that the transposition $(ab) \in S_6$ acts on the lines by $(L_1L_6)(L_2L_5)(L_3L_4)$.

Exercise 9.8. Verify that the action of S_6 on the six lines $L_1, \ldots, L_6$ yields a map $S_6 \to S_6$ which is an outer automorphism of S_6.

If you have finish this, ask for the next handout on the relation to automorphisms of the symplectic group.

10. Bonus Description: The outer automorphism of S_6 and symplectic groups

We'll now investigate the relation between the other automorphism of S_6 and the symplectic group. This is the last description of the outer automorphism of S_6 we'll do in this class, but check out Wikipedia for a few more descriptions!

Definition 10.1. Let W be a 6-dimensional vector space over $\mathbb{F}_2$, which we use as notation for the ring $\mathbb{Z}/2\mathbb{Z}$. Choose a basis $e_1, \ldots, e_6$ so that every element of W can be written as $\sum_{i=1}^6 x_i e_i =: (x_1, x_2, x_3, x_4, x_5, x_6)$ with $x_i \in \{0, 1\}$. Note that W has an inner product given by

$$(x_1, x_2, x_3, x_4, x_5, x_6) \cdot (y_1, y_2, y_3, y_4, y_5, y_6) = \sum_{i=1}^{6} x_i y_i.$$

Define the vector

$$u := (1, 1, 1, 1, 1, 1).$$

Let

$$U := \{v \in W : v \cdot u = 0\}.$$

Note that $u \in U$ because $u \cdot u = 0$ and so we may define

$$V := U / \operatorname{Span}(u).$$

Exercise 10.2. Show that $|V| = 16$ and each nonzero vector can be uniquely represented as $e_i + e_j$ for $1 \leq i < j \leq 6$. Use this to define a bijection between nonzero vectors in V and duads (pairs of two distinct elements in $\{1, 2, 3, 4, 5, 6\}$).

10.1. The outer automorphism via isotropic subspaces.

Exercise 10.3. Show that S_6 acts on W by permuting the 6 basis vectors $e_1, \ldots, e_6$. Show that this restricts to an action on U and induces an action on V.

Definition 10.4. A subspace of V is **isotropic** if for all $v, w \in V$, $v \cdot w = 0$.

Exercise 10.5.
(1) Show that 2-dimensional isotropic subspaces of V are in bijection with synthemes.
(2) Conclude there are 15 2-dimensional isotropic subspaces of V.
(3) Show that S_6 acts on the collection of 2-dimensional isotropic subspaces via its action on the basis vectors of W.
(4) Show that this action of S_6 on 2-dimensional isotropic subspaces agrees with the action of S_6 on synthemes.

Exercise 10.6. (1) Show that collections of 5 pairwise disjoint 2-dimensional isotropic subspaces are in bijection with pentads.
(2) Show that the action of S_6 on pentads agrees with that on collections of 5 pairwise disjoint isotropic subspaces (which you should show exists).
(3) Conclude there are 6 collections of 5 pairwise disjoint 2-dimensional isotropic subspaces.
(4) Show that this determines a map $S_6 \to S_6$.
(5) Show that this map $S_6 \to S_6$ is the outer automorphism of S_6.

10.2. **Showing $S_6 \simeq \mathrm{Sp}\left(\mathbb{F}_2^4\right)$.**

Definition 10.7. Let $\mathbb{F}_2^{2n}$ be a $2n$ dimensional vector space over with basis vectors $e_1, \ldots, e_n, f_1, \ldots, f_n$. Define an inner product on $\mathbb{F}_2^{2n}$ given by

$$\begin{cases} e_i \cdot e_i = 1 \\ e_i \cdot e_j = 0 & \text{if } i \neq j \\ e_i \cdot e_j = 0 \\ f_i \cdot f_j = 0 \end{cases}$$

Let $\mathrm{GL}(\mathbb{F}_2^n)$ denote the set of invertible matrices from $\mathbb{F}_2^n$ to $\mathbb{F}_2^n$. Define the **symplectic group** with respect to the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ to be the set of $2n \times 2n$ matrices preserving the inner product

$$\mathrm{Sp}(\mathbb{F}_2^{2n}) = \left\{ M \in \mathrm{GL}(\mathbb{F}_2^{2n}) : v \cdot w = Mv \cdot Mw \text{ for all } v, w \in \mathbb{F}_2^{2n} \right\}.$$

Exercise 10.8. Show that the action of S_6 on $V \simeq \mathbb{F}_2^4$ from Exercise 10.3 defines a map $S_6 \to \mathrm{Sp}\left(\mathbb{F}_2^4\right)$ where the symplectic group above is defined with respect to the basis

$$e_1 + e_2, e_2 + e_3, e_4 + e_5, e_5 + e_6.$$

Exercise 10.9. Show that the map $S_6 \to \mathrm{Sp}\left(\mathbb{F}_2^4\right)$ is in fact an isomorphism. For this, you may assume $\#\mathrm{Sp}\left(\mathbb{F}_2^4\right) = 720$. If you are feeling so inclined, prove $\#\mathrm{Sp}\left(\mathbb{F}_2^4\right) = 720$.

APPENDIX A. GROUP THEORY

In this appendix, we review some basic group theory. Technically speaking, this is a prerequisite for the course, but if you haven't seen it before, you may find this appendix helpful.

Definition A.1. A group G is a set G together with a multiplication operation $\cdot : G \times G \to G$ and an **identity** $e \in G$ satisfying the following properties

Identity For every $g \in G$, we have $e \cdot g = g \cdot e = g$.
Associativity For $g, h, k \in G$, we have $(g \cdot h) \cdot k = g \cdot (h \cdot k)$.
Inverses For every $g \in G$, there is an **inverse** denoted $g^{-1} \in G$ so that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

Remark A.2. We will often omit the multiplication operation $\cdot$ from the notation of a group when it is understood from context.

Exercise A.3. Verify, directly from the definition that every group has a unique identity element. Show that for any $g \in G$, g has a unique inverse, and so the name g^{-1} is justified.

Definition A.4. A **homomorphism** of groups $f : G \to H$ is a map of sets such that $f(e_G) = e_H$ and $f(g \cdot_G g') = f(g) \cdot_H f(g')$, where the subscripts denote the identity and multiplication in the corresponding group.

Definition A.5. A group homomorphism $f : G \to H$ is **injective** if $f(g) = f(g') \implies g = g'$. It is **surjective** if for every $h \in H$ there is some $g \in G$ with $f(g) = h$. It is **bijective** (also known as an **isomorphism**) if it is both injective and surjective. If $f : G \to H$ is bijective, we write $G \simeq H$.

Exercise A.6. Show that a group homomorphism $f : G \to H$ is injective if and only if $f^{-1}(e_H) = e_G$.

Exercise A.7. Show that a group homomorphism $f : G \to H$ is bijective if and only if there is a group homomorphism $f^{-1} : H \to G$ so that $f^{-1} \circ f = \mathrm{id}_G, f \circ f^{-1} = \mathrm{id}_H$. [23]

Definition A.8. The **kernel** of a group homomorphism $f : G \to H$ is the set of elements $g \in G$ with $f(g) = e_H$.

Definition A.9. A **subgroup** H of G is a subset $H \subset G$ so that

(1) $e_G \in H$.
(2) For any $g \in H$ we also have $g^{-1} \in H$.
(3) If $g, g' \in H$ then $g \cdot_G g'$ is also in H.

Exercise A.10. Show that the kernel of a group homomorphism is a subgroup.

Definition A.11. A group is **abelian** if for all $a, b \in G$, we have $a \cdot b = b \cdot a$.

Example A.12 (Non-example)**.** The group of permutations of three elements is not abelian because if you first switch elements 1 and 2, and then switch elements 2 and 3, this is not the same as first switching 2 and 3 and then switching 1 and 2. In the first case, you end up sending $1 \mapsto 3 \mapsto 2 \mapsto 1$ while in the second case you end up sending $1 \mapsto 2 \mapsto 3 \mapsto 1$.

A.1. **Normal subgroups and quotients.**

Definition A.13. A subgroup $H \subset G$ is **normal** if for all $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$.

Definition A.14. Let $H \subset G$ be a subgroup. Construct the **quotient** G/H as the set of all elements $g \in G$ modulo the equivalence relation $g_1 \sim g_2$ if there is some $h \in H$ with $g_1 = g_2 h$. The equivalence class of g is called the **coset of** g and the coset is notated gH.

Exercise A.15. Verify that the relation $\sim$ as defined in Definition A.14 is indeed an equivalence relation.

Exercise A.16. Show that if $H \subset G$ is normal then G/H can be given the structure of a group by $gH \cdot g'H = (gg')H$.

Exercise A.17 (Universal property of quotients)**.** For $H \subset G$ a subgroup, show that for any group homomorphism $f : G \to G'$ with $f(H) = e_{G'}$ there is a unique map of sets $\overline{f} : G/H \to G'$ so that the composition of $G \to G/H$ (given by $g \mapsto gH$) with $\overline{f}$ is equal to f. Show that $\overline{f}$ is a group homomorphism when H is normal (the normality hypothesis is only so that G/H can be given the structure of a group, using Exercise A.16).

Definition A.18. Let $S \subset G$ be a subset (which is not necessarily a subgroup). The **subgroup generated by** S is the intersection of all subgroups of G containing S.

Remark A.19. Intuitively, you can obtain the subgroup generated by a set by just throwing in all inverses and then throwing in all repeated products of such elements.

Exercise A.20. Show that the intersection of any collection of subgroups of a group is again a subgroup. Conclude that the subgroup generated by a set is indeed a subgroup.

Appendix B. Group Actions

In case you are not familiar with group actions, we review it here. A group action on a set is an assignment of a permutation of that set for each element of the group. This should be compatible with the multiplication structure on the group. Here is an unconventional definition:

Definition B.1. A **group action** of a group G on a set X is a group homomorphism $G \to S_X$, where S_X denotes the symmetric group on the set X.

Again, intuitively, for each element of G, we are assigning a permutation of the set X.

Example B.2. Let S_n denote the symmetric group on n elements. This acts on the set $\{1, \ldots, n\}$ by sending a permutation $\sigma \in S_n$ to the "same permutation" of the set $\{1, \ldots, n\}$. Under identifying $S_{\{1,\ldots,n\}} = S_n$, the resulting action map $S_n \to S_{\{1,\ldots,n\}} = S_n$ is just the identity map.

Example B.3. In this example, we use cycle notation for S_n. If you are not familiar with this, see Appendix C. Let D_8 denote the symmetries (reflections and rotations) of a square. Then D_8 acts on the four vertices of a square by permuting them. Suppose the square has vertices $(0,0), (1,0), (1,1), (0,1)$. Label those 4 vertices $1, 2, 3$, and 4 in order. Then, the 90 degree counterclockwise rotation maps to the permutation of vertices (1234) because it sends vertex 1 to 2, 2 to 3, 3 to 4, and 4 to 1. Similarly, the vertical reflection about the vertical line $x = 1/2$ maps to the permutation of vertices $(12)(34)$.

Exercise B.4. Compute how the other group elements act, and check this defines a group action.

Here is an alternate definition, which is more commonly given.

Definition B.5 (Alternate definition of group action)**.** A **group action** of G on a set S is a map

$$\begin{aligned} \rho \colon G \times S &\to S \\ (g, s) &\mapsto g \cdot s, \end{aligned}$$

so that

(1) $\rho(e, s) = s$ for $e \in G$ the identity and $s \in S$ arbitrary.
(2) For any $g, h \in G$ and $s \in S$,

$$\rho\left(g, \rho\left(h, s\right)\right) = \rho\left(gh, s\right).$$

Example B.6. Let S_n denote the symmetric group on n elements. This acts on the set $\{1, \ldots, n\}$ via definition Definition B.5 by $\rho(\sigma, s) = \sigma(s)$, where σ is thought of as a bijection $\{1, \ldots, n\} \to \{1, \ldots, n\}$ and we are evaluating this bijection on $s \in \{1, \ldots, n\}$.

Exercise B.7. Check that this is a group action.

Exercise B.8. Explain why Definition B.1 is equivalent to Definition B.5.

Exercise B.9. Check that under the equivalence you produced in Exercise B.8, the action in Example B.2 corresponds to the action in Example B.6.

APPENDIX C. CONJUGACY CLASSES OF THE SYMMETRIC GROUP

In this appendix, we recall the definition of conjugacy classes, cycle notation, and discuss properties of conjugacy classes in the symmetric group.

C.1. **Conjugacy in general.**

First, we recall the general definition of conjugacy.

Definition C.1. For G a group and $g \in G$, the map

$$\begin{aligned} \phi_g \colon G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}$$

is called **conjugation by** g. Two elements h_1 and h_2 are **conjugate** if there is some g so that $\phi_g(h_1) = h_2$. In other words, h_1 is conjugate to h_2 if $gh_1g^{-1} = h_2$. For $h \in G$, the **conjugacy class of** h is the set of elements which are conjugate to h. That is, the conjugacy class is the set of all elements of the form ghg^{-1} for $g \in G$.

Exercise C.2. Define the relation $h_1 \sim h_2$ if there is some $g \in G$ for which $gh_1g^{-1} = h_2$. Check this is an equivalence relation. In other words, conjugacy defines an equivalence relation.

C.2. **Cycle notation in the symmetric group.**

Here, we review cycle notation in the symmetric group. The following definition of cycle type is quite opaque. We advise you skip the following definition and jump to the examples following it.

Definition C.3. For $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with $\sigma \in S_n$, in the symmetric group S_n, we can express σ in **cycle notation** as follows: Let $x_1, \ldots, x_k$ be a maximal subset of $\{1, \ldots, n\}$ with $\sigma^s(x_i) \neq x_j$ for any i, j, s. For $1 \leq i \leq k$ let t_i be the minimal positive integer so that $\sigma^{t_i}(x_i) = x_i$. Then, we write σ in cycle notation as $\prod_{i=1}^k \left(x_i\sigma(x_i)\sigma^2(x_i) \cdots \sigma^{t_i-1}(x_i)\right)$.

Example C.4. See Figure 2 for a picture depicting cycle notation.

Remark C.5. If $s \in S_n$ is an element with certain cycle notation, we often omit all singletons from the cycle notation. So, for example, we denote the element of S_8 with cycle notation $(146837)(2)(5)$ simply as (146837).

Example C.6. Let $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ be the permutation sending $1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 3$. Then, the cycle notation for σ is $(12)(345)$. One can read the first (12) as saying 1 goes to 2, and then 2 goes back to 1 because you reach the parentheses. The second parenthetical reads as sending $3 \mapsto 4 \mapsto 5$, and then 5 is sent back to 3, again looping around from the ")" to the previous "(".

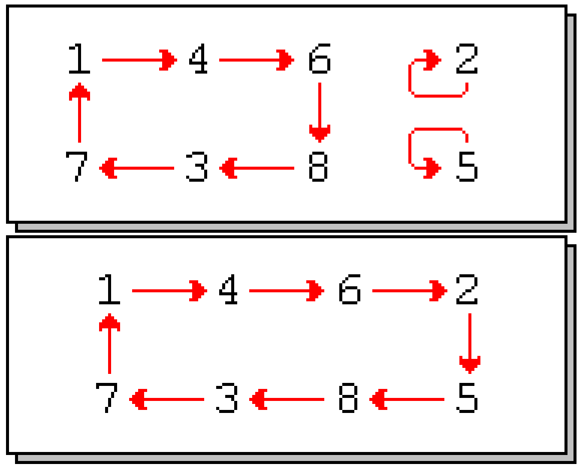

FIGURE 2. This is a depiction of two elements thought of as elements of S_8. The first corresponds the the permutation fixing 2 and 5, and sending $1 \mapsto 4, 4 \mapsto 6, 6 \mapsto 8, 8 \mapsto 3, 3 \mapsto 7, 7 \mapsto 1$. The corresponding cycle notation for the first picture is $(146837)(2)(5)$ (where each parenthesized group of numbers corresponds to a cycle in the above diagram. Similarly, the second permutation has cycle notation (14625837).

Definition C.7. For $\sigma \in S_n$, the **cycle type** of σ is the collection of the lengths of the cycles in the cycle notation for σ. So, for example $(12)(345)$ would have cycle type $2, 3$ (since it has one cycle of length 2 and one cycle of length 3) while $(12)(34)(5)$ would have cycle type $1, 2^2$ (since it has one cycle of length 1 and 2 cycles of length 2). Furthermore, when an element contains cycles of length 1, we often omit them from the notation. So, for example, $(12)(34)(5)$ would be alternatively notated as $(12)(34) \in S_5$ and we would say this has cycle type 2^2. Since $(12)(34)$ is viewed as an element of S_5, it is understood that (5) is a cycle of length 1.

C.3. **Conjugacy in S_n.** We now show that two elements are conjugate if and only if they have the same cycle type and classify the conjugacy classes of S_n.

Exercise C.8. Show that $\sigma, \tau \in S_n$ of the same cycle type are conjugate. [24]

Exercise C.9. Show the converse to the previous exercise: In fact, σ and τ are conjugate if and only if σ and τ have the same cycle type.

Exercise C.10. (1) Conclude from the previous two exercises that cycle types are in bijection with conjugacy classes of S_n.
(2) Show that cycle types are also in bijection with partitions of n.

NOTES

[1] *Hint:* To show it is a bijection, construct an inverse. You can take the inverse to ϕ_g to be ϕ_k for an appropriate $k \in G$.

[2] *Hint:* In general, show that an action of a group G on an n element set yields a map $G \to S_n$.

[3] *Hint:* You can check directly it is an inclusion. For a slicker proof if you know the normal subgroups of S_5, you can also use that the kernel is always normal.

[4] *Possible Hint:* There are six ways to color the nobbly wobbly, up to rotating the shape, see `http://www.celebrationofmind.org/wp-content/uploads/2016/09/NobblyWobbly-ColorVariants.pdf` For another possible relation, the rotations of the dodecahedron is isomorphic to the group A_5, with A_5 the alternating group on 5 elements.

[5] *Hint:* The main difficulty is understanding what the map ϕ is. If you can figure out how ϕ is defined, show $\phi((\mathbf{ad})(\mathbf{bc})(\mathbf{ef}))$ is a transposition.

[6] *Hint:* If it were inner, show the image of the transposition (12) is again a transposition.

[7] *Hint:* Note that the 5 vertices adjacent to any vertex labeled 1 have labels $2, 3, 4, 5$, and 6. Use rotations to fix the locations of two of the vertices.

[8] *Hint:* If vertex 1 is at the top, and the 5 adjacent vertices are $2, 3, 4, 5, 6$ in cyclic order, consider the icosahedra with 1 at the top and vertices $2, 4, 6, 3, 5$ in cyclic order.

[9] *Hint:* Look at the cyclic order of vertices around the vertex 6.

[10] *Hint:* Every such subgroup is generated by a 5 cycle and there are 4 choices of generators for each such subgroup.

[11] *Hint:* To deal with S_3, after conjugating (i.e., composing with an inner automorphism), show you can assume $(12) \mapsto (12)$ and $(13) \mapsto (13)$ Show that this forces the map to be the identity.

[12] *Hint:* Suppose for the sake of contradiction this is not the case. Arguing as in previous parts, but working with the transposition (12) and $(1k)$, show that if $f(1k) = (uv)$ and $u \neq a, v \neq a$, then $u = b_2$ or $v = b_2$. Similarly working with (13) and $(1k)$, if $u \neq a, v \neq a$ show $u = b_3$ or $v = b_3$. Obtain a contradiction to the fact that f is a bijection by using that $f(1k) = b_2 b_3 = f(23)$ (using the previous part).

[13] *Hint:* Combine Exercise 7.7, Exercise 7.8, and Exercise 7.9(2).

[14] *Hint:* You should get $\frac{n!}{k! 2^k (n-2k)!}$.

[15] *Hint:* Use your formula from Exercise 7.13.

[16] *Hint:* Combine Exercise 7.4, Exercise 7.5, Exercise 7.12, and Exercise 7.14(1).

[17] *Hint:* As in Exercise 7.15, combine Exercise 7.4, Exercise 7.5, Exercise 7.12, and Exercise 7.14(2).

[18] *Hint:* Use Proposition 7.11.

[19] *Hint:* Use Exercise 7.18 and Proposition 7.2.

[20] *Hint:* We have previously seen outer automorphisms exist, so it suffices to show that any two outer automorphisms of S_6 are related by an inner automorphism. For this, use Exercise 7.19 and Proposition 7.11.

[21] *Hint:* Use Theorem 7.1 and Exercise 8.1. What does the center of S_n have to do with the group of inner automorphisms? Be careful for low n.

[22] *Hint:* First show that it acts transitively, and then show that the stabilizer of any single point is given by the set of upper triangular matrices. Use this description of the stabilizer to show it acts double transitively and identify the stabilizer of two points.

[23]*Hint:* Show that a map is bijective if and only if there is a unique element of G mapping to any given element of H. Use this to define an inverse map.

[24]*Hint:* Show that if τ is expressed in cycle notation as $(a_1^1 \cdots a_{n_1}^1) \cdots (a_1^k \cdots a_{n_k}^k)$ then

$$\sigma\tau\sigma^{-1} = (\sigma(a_1^1) \cdots \sigma(a_{n_1}^1)) \cdots (\sigma(a_1^k) \cdots \sigma(a_{n_k}^k)).$$