THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 1

Due 3:30pm[1], September 13th (Friday), **in-class**.

Aim of this Homework: *In this assignment you will practice some (but* NOT ALL*) standard techniques that you are expected to have already learned in calculus, linear algebra, mathematical analysis, and ordinary differential equations (ODE). You will find these techniques very useful in this course.*

Reading Assignment: Read the following material(s):

(i) "*Quick Review of Linear Algebra for MATH4406 Introduction to Partial Differential Equations" under the module"Lecture Notes*";

(ii) If you forget how to compute the Jordan normal form for 3 by 3 matrices, then you may have a quick look at Example 3 in the following pdf file:

`https://iuuk.mff.cuni.cz/~rakdver/linalg/lesson15-8.pdf`

which was created by Prof. Zdeněk Dvořák; and

(iii) Section 1.1 and 1.3 of the textbook.

Instruction: Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the

[1] You are expected to submit your homework before the beginning of Friday lecture **in-class**.

"Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

Problem 1. Let $L > 0$ be a constant, and m and n be two positive integers. Compute the following integrals:

(i) $\int_{-L}^{L} \sin \frac{m\pi x}{L} \, \sin \frac{n\pi x}{L} \, dx$
(Hint: product-to-sum formula for trigonometric functions.)

(ii) $\int_0^L \left(\frac{L}{2} - \left| \frac{L}{2} - x \right| \right) \cos \frac{m\pi x}{L} \, dx$

(iii) $\int_{-\pi}^{\pi} e^{imx} \sin mx \, dx$
(Hint: Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.)

Problem 2. Let $f := f(x,y) \in C^1([0,1] \times [0,1])$ satisfy an integral relation

$$\int_0^1 \int_0^1 |\nabla f|^2 \, dxdy = 0$$

and a pointwise condition

$$f(0,0) = 0.$$

Prove or disprove

$$f \equiv 0.$$

Problem 3. Let $\epsilon > 0$ be a constant and $u := u(t)$ be a solution to the initial-value problem of the ordinary differential equation (ODE)

$$\begin{cases} \dfrac{du}{dt} = u^{1+\epsilon} \quad \text{for } t > 0, \\ u(0) = 1. \end{cases}$$

Show that there exists a constant $T > 0$ such that

$$\lim_{t \to T^-} u(t) = \infty.$$

Remark. Typically, a solution (e.g., the u in Problem 3) to the initial-value problem of ordinary/partial differential equation(s) may cease to exist after a finite time (e.g., the T in Problem 3), because a quantity associated with the solution (or actually the solution itself) grows to infinity as the time approaches this specific time. In the literature, this is commonly known as the blowup phenomenon, and one may say that the solution blows up within a finite time.

Problem 4. Let

$$A := \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix},$$

and complete the following parts.

(i) Find all eigenvalues and eigenvectors of A.

(ii) Find an invertible 3×3 real matrix P such that

$$A = PJP^{-1},$$

where the Jordon normal form

$$J := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) Solve the following system of linear ordinary differential equations (ODE):

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x},$$

where the vector

$$\mathbf{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

is the unknown function of t.

Food for Thought. If the matrix A and J are changed to be

$$A := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 4 & -3 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

will you still be able to solve Problem 4?

The following problem(s) is/are *optional*:

Problem 5.

(i) Let

$$u(x,y) := \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}.$$

Verify that u satisfies the Laplace's equation

$$\Delta u = 0,$$

subject to the Neumann boundary condition

$$\partial_y u\big|_{y=0} = 0.$$

(Recall: by definition, $\Delta u := \partial_{xx} u + \partial_{yy} u$.)

(ii) Let $w(t,x) := -2\operatorname{sech}^2(x-4t)$. Verify that w satisfies the Korteweg-De Vries (KdV) equation: for any t, $x \in \mathbb{R}$,

$$\partial_t w = 6w\partial_x w - \partial_{xxx} w.$$

(iii) For any $t > 0$ and $(x,y) \in \mathbb{R}^2$, we define

$$v(t,x,y) := \frac{1}{4\pi t} \int_{-\infty}^{\infty} \operatorname{erf} \tilde{y} \int_{-\infty}^{\infty} \frac{1}{1+\tilde{x}^4} e^{-\frac{(x-\tilde{x})^2+(y-\tilde{y})^2}{4t}} \, d\tilde{x} d\tilde{y},$$

where the function $\operatorname{erf}(\cdot)$ is the Gauss error function. Verify that v satisfies the heat equation in two spatial dimensions:

$$\partial_t v - \Delta v = 0,$$

where the $\Delta v := \partial_x^2 v + \partial_y^2 v$. (Hint: differentiation under an integral sign.)

Problem 6. Let $p > 0$ be a real number, d be a positive integer, $\Omega \subseteq \mathbb{R}^d$ be an open set, and $f : \Omega \to \mathbb{R}$ be a continuous function such that

$$\int_\Omega |f(x)|^p \, dx = 0.$$

Prove that

$$f \equiv 0 \quad \text{on } \Omega.$$

Food for Thought. Will the assertion in Problem 6 still remain true, if f is only assumed to be integrable (instead of continuous)?

Problem 7. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field and $g : \mathbb{R}^2 \to \mathbb{R}$ be a smooth scalar-valued function.

(i) Verify that

$$\nabla \cdot (Fg) = g\nabla \cdot F + F \cdot \nabla g.$$

(ii) Apply the divergence theorem to prove that for any smooth and bounded region $\Omega \subseteq \mathbb{R}^2$,

$$\iint_\Omega g\nabla \cdot F \, dxdy = \oint_{\partial\Omega} gF \cdot n \, d\sigma - \iint_\Omega F \cdot \nabla g \, dxdy$$

where $\partial\Omega$ is the boundary of Ω, $n := \binom{n_1}{n_2}$ is the unit outer normal vector on $\partial\Omega$.

(iii) Choosing $F := \binom{f}{0}$ in part (ii), prove the following integration by parts formula:

$$\iint_\Omega g\partial_x f \, dxdy = \oint_{\partial\Omega} gfn_1 \, d\sigma - \iint_\Omega f\partial_x g \, dxdy.$$

Food for Thought. Can you generalize the result in Part (iii) of Problem 7 to higher dimensional cases?