Department of Mathematics, The University of Hong Kong

MATH4302 (Algebra II), May, 2021

Answer ALL EIGHT questions. Put your University ID on your answer sheets.

This is a **closed-book exam.** Except for Problem 1, you should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your arguments counts. Think carefully before you write.

Problem 1 (16 points). Answer "True" or "False" to each of the following six statements. For this problem only, you do not need to explain your answers.

1) $\mathbb{Z}[x_1, x_2, x_3]$ is a UFD;

2) Every ideal I of a PID is a finitely generated module of the PID;

3) If K is a field, any $K[x]$-module M is naturally a K-vector space;

4) The two polynomials $f(x) = (x^3 - 2)^5(x^8 - 9)^2$ and $g(x) = (x^3 - 2)(x^8 - 9)$ have the same splitting field in $\mathbb{C}$ over $\mathbb{Q}$.

5) There exists a field with 75 elements;

6) There exists a field with 125 elements;

7) There exists an irreducible polynomial of degree 123 in $\mathbb{F}_5[x]$.

8) There are polynomials in $\mathbb{Q}[x]$ of order 6 that are not solvable by radicals.

Problem 2 (14 points). 1) Compute the Smith normal form of $A = \begin{pmatrix} -7 & 0 & -6 \\ 6 & 3 & 0 \\ 6 & 0 & 6 \end{pmatrix}$ as an element in $M_{3,3}(\mathbb{Z})$;

2) Let N be the sub-module of $\mathbb{Z}^3$ generated by the columns of A. What can you say about the quotient $\mathbb{Z}^3/N$ as an abelian group?

Problem 3 (10 points). Classify, up to isomorphisms, all abelian groups of order 24. Identify which one in your list is isomorphic to $\mathbb{F}_{25}\backslash\{0\}$ regarded as an abelian group of order 24 under multiplication in $\mathbb{F}_{25}$, where $\mathbb{F}_{25}$ is a field with 25 elements.

Problem 4 (10 points). Consider the $\mathbb{R}[x]$-module

$$M = \mathbb{R}[x]/\langle x - 2\rangle \oplus \mathbb{R}[x]/\langle (x-2)^2(x+2)\rangle.$$

1) If we regard M as a real vector space, what is the dimension of M?

2) Let $T \in \mathrm{End}_{\mathbb{R}}(M)$ be defined by $Tm = x \cdot m$ for $m \in M$, where $x\cdot$ denotes the action of $x \in \mathbb{R}[x]$ on M. Does T have a Jordan canonical form? If yes, find the Jordan canonical form of T.

Problem 5 (10 points). Let a be any positive real number and let α be the unique real root of the polynomial $p(x) = x^3 + ax + 1$. Can α be constructed by a ruler and compass? Explain your answer.

Problem 6 (15 points). Recall that $\overline{\mathbb{Q}}$ is the set of $\alpha \in \mathbb{C}$ that are algebraic over $\mathbb{Q}$. Outline a proof that $\overline{\mathbb{Q}}$ is a subfield of $\mathbb{C}$ and explain some of the main concepts involved in the proof.

Problem 7 (15 points). 1) Explain the Galois Correspondence for finite Galois extensions;

2) Let p be any prime number and let n be any positive integer. Explain the Galois Correspondence in the example of $\mathbb{F}_{p^n}$ as a Galois extension of $\mathbb{F}_p$.

Problem 8 (10 points). Compute the Galois closure of $\mathbb{Q}(\sqrt[4]{2})$ over $\mathbb{Q}$.

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