THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 2 Solution

Problem 1.

(i) It is clear that $u \equiv 0$ is a solution.

(ii) For $\lambda = 0$, the general solution to ODE $u'' = -\lambda u = 0$ is of the form

$$u(x) = C_1 x + C_2 \text{ for } C_1, C_2 \in \mathbb{R}.$$

By the boundary conditions, the constants C_1, C_2 satisfy $C_1 = 0$, i.e., $u(x) = C_2$. For nontrivial solutions, we can take $C_2 \neq 0$.

For $\lambda > 0$, the general solution to ODE $u'' = -\lambda u$ is of the form

$$u(x) = C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda}x \text{ for } C_1, C_2 \in \mathbb{R}.$$

By the boundary conditions, the constants C_1, C_2 satisfy

$$\begin{aligned} u(-1) = u(1) &\Rightarrow -C_1 \sin\sqrt{\lambda} + C_2 \cos\sqrt{\lambda} = C_1 \sin\sqrt{\lambda} + C_2 \cos\sqrt{\lambda}, \\ u'(-1) = u'(1) &\Rightarrow C_1\sqrt{\lambda}\cos\sqrt{\lambda} + C_2\sqrt{\lambda}\sin\sqrt{\lambda} = C_1\sqrt{\lambda}\cos\sqrt{\lambda} - C_2\sqrt{\lambda}\sin\sqrt{\lambda}. \end{aligned}$$

We wish for nontrivial solutions. So

$$\det\begin{pmatrix} \sin\sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}\sin\sqrt{\lambda} \end{pmatrix} = \sqrt{\lambda}\sin^2\sqrt{\lambda} = 0.$$

It follows that

$$\lambda = (k\pi)^2 \text{ for } k \in \mathbb{Z}.$$

Food for Thought. For $\lambda < 0$, $u \equiv 0$ is still a solution. The general solution instead has the form

$$u(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \text{ for } C_1, C_2 \in \mathbb{R}.$$

By the boundary conditions, the constants C_1, C_2 satisfy

$$\begin{aligned} u(-1) = u(1) &\Rightarrow C_1 e^{-\sqrt{-\lambda}} + C_2 e^{\sqrt{-\lambda}} = C_1 e^{\sqrt{-\lambda}} + C_2 e^{-\sqrt{-\lambda}}, \\ u'(-1) = u'(1) &\Rightarrow C_1\sqrt{-\lambda} e^{-\sqrt{-\lambda}} - C_2\sqrt{-\lambda} e^{\sqrt{-\lambda}} = C_1\sqrt{-\lambda} e^{\sqrt{-\lambda}} - C_2\sqrt{-\lambda} e^{-\sqrt{-\lambda}}. \end{aligned}$$

This implies $C_1 e^{-\sqrt{-\lambda}} = C_1 e^{\sqrt{-\lambda}}$, i.e. $C_1 = C_2 = 0$. Thus $u \equiv 0$ is actually the only solution when $\lambda < 0$.

Problem 2.

(i) Let $v := \partial_x u$. Then

$$\partial_y v - (2\cot 3y)v = \partial_{xy} u - (2\cot 3y)\partial_x u = 0.$$

The integrating factor is given by

$$\exp\left\{\int -2\cot 3y \,\mathrm{d}\, y\right\} = \exp\left\{-\frac{2}{3}\ln(\sin 3y)\right\} = (\sin 3y)^{-\frac{2}{3}}.$$

Then it follows that

$$\frac{\partial}{\partial y}\left[\frac{v(x,y)}{(\sin 3y)^{\frac{2}{3}}}\right] = 0.$$

Thus, we have $v(x,y) = F(x)(\sin 3y)^{\frac{2}{3}}$ for certain function $F(x)$. Since

$$\partial_x u(x,y) = v(x,y) = F(x)(\sin 3y)^{\frac{2}{3}},$$

Then

$$u(x,y) = (\sin 3y)^{\frac{2}{3}} \int F(x)\,\mathrm{d}\,x + G(y)$$

for some function $G(y)$.

(ii) Rewrite the PDE as

$$\frac{\partial}{\partial x}\{\partial_y u - (2\cot 3y)u\} = 0.$$

Integrating with respect to x, it follows that

$$\partial_y u - (2\cot 3y)u = f(y),$$

for some function f. The integrating factor is again given by

$$(\sin 3y)^{-\frac{2}{3}}.$$

That is to say,

$$\frac{\partial}{\partial y}\left[\frac{u(x,y)}{(\sin 3y)^{\frac{2}{3}}}\right] = f(y)(\sin 3y)^{-\frac{2}{3}}.$$

Integrating with respect to y gives that

$$u(x,y) = (\sin 3y)^{\frac{2}{3}} \int \frac{f(y)}{(\sin 3y)^{\frac{2}{3}}}\,\mathrm{d}\,y + g(x)(\sin 3y)^{\frac{2}{3}}.$$

(iii) They are essentially the same. We can take

$$g(x) = \int F(x)\,\mathrm{d}\,x \text{ and } G(y) = (\sin 3y)^{\frac{2}{3}} \int \frac{f(y)}{(\sin 3y)^{\frac{2}{3}}}\,\mathrm{d}\,y.$$

Problem 3.

(i) Let $u_n(t,x) = \frac{1}{n}e^{kn^2t}\sin nx$ be proposed function. Then

$$\partial_t u_n(t,x) + k\partial_{xx}u_n(t,x) = \frac{1}{n}kn^2e^{kn^2t}\sin nx - \frac{k}{n}e^{kn^2t}n^2\sin nx = 0.$$

Moreover, $u_n(t,x)$ satisfies the boundary conditions

$$u_n(t,0) = \frac{1}{n}e^{kn^2t}\sin 0 = 0,$$
$$u_n(t,\pi) = \frac{1}{n}e^{kn^2t}\sin n\pi = 0.$$

and the initial condition

$$u_n(0,x) = \frac{1}{n}e^{kn^2(0)}\sin nx = \frac{1}{n}\sin nx = f(x).$$

(ii) It follows that

$$\|f_n\|_{\sup} \le \frac{1}{n}\|\sin nx\|_{\sup} = \frac{1}{n} \to 0$$

as $n \to \infty$.

(iii) For any $T > 0$,

$$u_n(T,x) = \frac{e^{kn^2T}}{n}\sin nx$$

Then by the L'hôspital's rule, it follws that

$$\lim_{n\to\infty}\|u_n(T,\cdot)\|_{\sup} = \lim_{n\to\infty}\frac{e^{kn^2T}}{n} = \lim_{n\to\infty} 2nkTe^{kn^2T} = +\infty.$$

Food for Thought. Recall the definition for a PDE to be well posed: it must have a solution (existence), the solution must be unique (uniqueness), and small perturbation of data should yield small perturbation in solution with respect to some norm.

This problem shows that the backward heat equation is in general not stable: when we fixed the boundary data $u(t,0) = u(t,\pi) = 0$ and let the initial data $\|f_n\|_{\sup} \to 0$ by $n \to \infty$, we instead have $\|u_n(T,\cdot)\|_{\sup} \to \infty$. In other words, the perturbation of initial data towards the zero function results in large changes of the solution with respect to the sup norm.

Problem 4.

(i) For any $k = 1, \ldots, n$, $\tilde{t} \in \mathbb{R}$ and $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_d) \in \mathbb{R}^d$, we have

$$\begin{aligned}
\partial_t v(t,x_1,x_2,\ldots,x_d) &= \partial_t u(t-\tilde{t}, x_1-\tilde{x}_1, x_2-\tilde{x}_2,\ldots,x_d-\tilde{x}_d),\\
\partial_{tt} v(t,x_1,x_2,\ldots,x_d) &= \partial_{tt} u(t-\tilde{t}, x_1-\tilde{x}_1, x_2-\tilde{x}_2,\ldots,x_d-\tilde{x}_d),\\
\partial_{x_k} v(t,x_1,x_2,\ldots,x_d) &= \partial_{x_k} u(t-\tilde{t}, x_1-\tilde{x}_1, x_2-\tilde{x}_2,\ldots,x_d-\tilde{x}_d),\\
\partial_{x_kx_k} v(t,x_1,x_2,\ldots,x_d) &= \partial_{x_kx_k} u(t-\tilde{t}, x_1-\tilde{x}_1, x_2-\tilde{x}_2,\ldots,x_d-\tilde{x}_d).
\end{aligned}$$

Then the result follows.

(ii) For any $k = 1, \ldots, n$, and positive integers $\alpha_1, \alpha_2, \ldots, \alpha_d$, we have

$$\begin{aligned}
\partial_t v(t, x_1, x_2, \ldots, x_d) &= \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} \partial_t u(t, x_1, x_2, \ldots, x_d), \\
\partial_{tt} v(t, x_1, x_2, \ldots, x_d) &= \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} \partial_{tt} u(t, x_1, x_2, \ldots, x_d), \\
\partial_{x_k} v(t, x_1, x_2, \ldots, x_d) &= \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} \partial_{x_k} u(t, x_1, x_2, \ldots, x_d), \\
\partial_{x_k x_k} v(t, x_1, x_2, \ldots, x_d) &= \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d} \partial_{x_k x_k} u(t, x_1, x_2, \ldots, x_d).
\end{aligned}$$

The the result follows.

(iii) For any $k = 1, \ldots, n$, and any continuous function $g : \mathbb{R}^d \to \mathbb{R}$ with compact support,, we have

$$\begin{aligned}
&\partial_t v(t, x_1, x_2, \ldots, x_d) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\partial_t u(t, x_1 - \tilde{x}_1, \ldots, x_d - \tilde{x}_d)\right] g(\tilde{x}_1, \ldots, \tilde{x}_d)\, d\tilde{x}_1 \cdots d\tilde{x}_d \\
&\partial_{tt} v(t, x_1, x_2, \ldots, x_d) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\partial_{tt} u(t, x_1 - \tilde{x}_1, \ldots, x_d - \tilde{x}_d)\right] g(\tilde{x}_1, \ldots, \tilde{x}_d)\, d\tilde{x}_1 \cdots d\tilde{x}_d \\
&\partial_{x_k} v(t, x_1, x_2, \ldots, x_d) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\partial_{x_k} u(t, x_1 - \tilde{x}_1, \ldots, x_d - \tilde{x}_d)\right] g(\tilde{x}_1, \ldots, \tilde{x}_d)\, d\tilde{x}_1 \cdots d\tilde{x}_d \\
&\partial_{x_k x_k} v(t, x_1, x_2, \ldots, x_d) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\partial_{x_k x_k} u(t, x_1 - \tilde{x}_1, \ldots, x_d - \tilde{x}_d)\right] g(\tilde{x}_1, \ldots, \tilde{x}_d)\, d\tilde{x}_1 \cdots d\tilde{x}_d.
\end{aligned}$$

Then the result follows.

(iv) For any constant $a > 0$, we have

$$\begin{aligned}
\partial_t v(t, x_1, x_2, \ldots, x_d) &= a \partial_t u(at, ax_1, ax_2, \ldots, ax_d), \\
\partial_{tt} v(t, x_1, x_2, \ldots, x_d) &= a^2 \partial_{tt} u(at, ax_1, ax_2, \ldots, ax_d), \\
\partial_{x_k} v(t, x_1, x_2, \ldots, x_d) &= a \partial_{x_k} u(at, ax_1, ax_2, \ldots, ax_d), \\
\partial_{x_k x_k} v(t, x_1, x_2, \ldots, x_d) &= a^2 \partial_{x_k x_k} u(at, ax_1, ax_2, \ldots, ax_d).
\end{aligned}$$

Then the result follows.

Problem 5.

(i) For any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$, we have

$$\begin{aligned} \partial_t v(t,x) &= \partial_t u(t-t_0, x-x_0), \\ \partial_x v(t,x) &= \partial_x u(t-t_0, x-x_0), \\ \partial_{xx} v(t,x) &= \partial_{xx} u(t-t_0, x-x_0). \end{aligned}$$

Then the result follows.

(ii) For any constant $a > 0$, we have

$$\begin{aligned} \partial_t v(t,x) &= a^3 \partial_t u(a^2 t, ax), \\ \partial_x v(t,x) &= a^2 \partial_x u(a^2 t, ax), \\ \partial_{xx} v(t,x) &= a^3 \partial_{xx} u(a^2 t, ax). \end{aligned}$$

Then the result follows.

(iii) For any constant $\lambda > 0$, we have

$$\begin{aligned} \partial_t v(t,x) &= \partial_t u(t, x-\lambda t) - \lambda \partial_x u(t, x-\lambda t), \\ \partial_x v(t,x) &= \partial_x u(t, x-\lambda t), \\ \partial_{xx} v(t,x) &= \partial_{xx} u(t, x-\lambda t). \end{aligned}$$

The the result follows.

Problem 6.

(i)

$$\sum_{m=1}^{d} q_{km} q_{lm} = (k,l) \text{ entry of } QQ^T = (k,l) \text{ entry of } I = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l. \end{cases}$$

(ii) By the chain rule,

$$\partial_{x_i} v = \sum_{k=1}^{d} \partial_{y_k} u \cdot \partial_{x_i} y_k = \sum_{k=1}^{d} \partial_{y_k} u \cdot \partial_{x_i}(q_{k1}x_1 + \cdots + q_{kd}x_d) = \sum_{k=1}^{d} q_{ki}\partial_{y_k} u.$$

(iii) By the chain rule again,

$$\begin{aligned} \partial_{x_i x_j} v &= \sum_{k=1}^{d} q_{ki}\partial_{x_j}(\partial_{y_k} u) = \sum_{k=1}^{d} q_{ki} \sum_{l=1}^{d} \partial_{y_k y_l} u \cdot \partial_{x_j} y_l \\ &= \sum_{k=1}^{d}\sum_{l=1}^{d} q_{ki}\partial_{y_k y_l} u \cdot \partial_{x_j}(q_{l1}x_1 + \cdots + q_{ld}x_d) = \sum_{k=1}^{d}\sum_{l=1}^{d} q_{ki}q_{lj}\partial_{y_k y_l} u. \end{aligned}$$

(iv)

$$\begin{aligned} \Delta_x v &:= \sum_{m=1}^{d} \partial_{x_m x_m} v = \sum_{m=1}^{d} (\sum_{k=1}^{d}\sum_{l=1}^{d} q_{km}q_{lm}\partial_{y_k y_l} u) \quad \text{(by (iii))} \\ &= \sum_{k=1}^{d}\sum_{l=1}^{d} (\sum_{m=1}^{d} q_{km}q_{lm})\partial_{y_k y_l} u \quad \text{(by (i))} \\ &= \sum_{k=1}^{d} \partial_{y_k y_k} u = 0. \end{aligned}$$