THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH3541
Introduction to Topology

13 Dec, 2023 2:30 pm – 5:00 pm

Calculator is allowed. Only approved calculators as announced by the Examinations Secretary can be used in this examination. It is candidates' responsibility to ensure that their calculator operates satisfactorily, and candidates must record the name and type of the calculator used on the front page of the examination script.

Answer ALL **Five** questions

Note: You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

1. (30 points) Answer "True" or "False" for the following ten statements. For this problem only, you do not need to explain your answers.

(1) In any non-empty topological space, every singleton subset is compact;

(2) A subspace of a Hausdorff space is Hausdorff;

(3) In any non-empty topological space, every compact subset is closed;

(4) Every loop in the $\mathbb{RP}^2$ is homotopic to a constant loop;

(5) A path connected topological space is connected;

(6) Composition of two covering maps is a covering map;

(7) $\mathbb{R}^3 \setminus \{(0,0,0)\}$ is homotopic to S^1.

(8) Every compact topological space has finitely many connected components;

(9) The map $z \mapsto z^2 : \mathbb{C} \to \mathbb{C}$ is a covering map.

(10) A topological space X is covered by two simply connected open subsets U and V such that $U \cap V$ is nonempty. Then X is simply connected.

2. (10 points) Let S^2 be the unit sphere in $\mathbb{R}^3$ (with the subspace topology) and let $p_1 = (0,0,1)$.

(1) Construct an explicit homeomorphism from $S^2 \setminus \{p_1\}$ to $\mathbb{R}^2$. You need to justify why your construction is a homeomorphism!

(2) For $p_2 = (1,0,0)$, construct a deformation retract from $S^2 \setminus \{p_1, p_2\}$ to S^1.

3. (25 points) Let $p : \widetilde{X} \to X$ be a covering map, where $\widetilde{X}$ is path connected.

(1) State the definition of $p : \widetilde{X} \to X$ being a universal covering space.

(2) Let G be a topological group acts on a topological space M. State the condition that the action being free and properly discontinuous.

(3) Suppose $\widetilde{X}$ is locally path connected, G acts on $\widetilde{X}$ free and properly discontinuously. Denote $X := \widetilde{X}/G$. Construct a group homomorphism ϕ from $\pi_1(X, x_0)$ to G for a base point $x_0 \in X$.

(4) Use path lifting property to show that ϕ (defined in part (3)) is an isomorphism if $\widetilde{X}$ is simply connected.

4. (15 points) Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

be a monic polynomial with complex coefficients of degree $n > 0$. We may view f as a continuous self map of the complex plane $\mathbb{C}$.

(1) For $r \in [0,1)$ and $f(z) = z^2+z+1$, plot the roots of $f(\frac{rz}{1-r})$ on the complex plane (as a function of r).

(2) We define a function $F(r,z)$ by

$$F(r,z) = \begin{cases} f(\frac{rz}{1-r})/|f(\frac{rz}{1-r})| & r \in [0,1) \\ z^n/|z|^n & r = 1 \end{cases}$$

Show that if $f(z)$ has no complex roots then $F(r,z)$ is a continuous function on $[0,1] \times \mathbb{C}$.

(3) Use part (2) to prove the fundamental theorem of algebra, i.e. any univariate complex polynomial of positive degree has a root.

5. (20 points) Let $\sigma : G \times M \to M$ be a topological group G action on a topological space M. Denote by M/G the orbit space. For $A \in M$ denote by G_A the orbit of A and by $[A]$ the corresponding point in M/G. Let $G = GL_n(\mathbb{C})$ be the topological group of invertible $n \times n$ complex matrices with metric topology. Set M to be the space of *all* $n \times n$ complex matrices with metric topology.

(1) Prove that the formula

$$\sigma(g, A) = gAg^{-1}$$

defines a topological group action of G on M.

(2) Show that M/G is not Hausdorff.

From now on we take $n = 2$. Let $\mathrm{ch} : M \to \mathbb{C}^2$ be the continuous map defined by

$$\mathrm{ch}(A) = (\mathrm{tr}(A), \det(A))$$

where $\mathrm{tr}(A), \det(A)$ are the trace and determinant of the matrix A.

(3) Let $M^\circ \subset M$ be the subset consisting of 2×2 matrices that are diagonalizable. Show that G acts on M° and the orbit space M°/G is Hausdorff.

(4) Prove that ch factors through a continuous bijection ϕ

$$\mathrm{ch} : M^\circ \longrightarrow M^\circ/G \xrightarrow{\phi} \mathbb{C}^2.$$

***** END OF PAPER *****