Claim 1: $\forall$ distinct primes $p, q, r_1, \cdots, r_k$,

$\exists\ A(x), B(x) \in \mathbb{Z}[x]$ s.t.

$$A(x)\,\Phi_{pr_1\cdots r_k}(x) + B(x)\,\Phi_{qr_1\cdots r_k}(x) = 1.$$

Proof: Induction.

Base case $k=0$: Choose $m \in \mathbb{N}$ s.t. $\begin{cases} m \equiv 0 \pmod p \\ m \equiv 1 \pmod q \end{cases}$.

Then $(x^{m-p} + x^{m-2p} + \cdots + 1)\,\Phi_p(x) - (x^{m-q} + x^{m-2q} + \cdots + x)\,\Phi_q(x) = 1$

Inductive step: Assume $\exists\ C(x), D(x) \in \mathbb{Z}[x]$ s.t.

$$C(x)\,\Phi_{pr_1\cdots r_{k-1}}(x) + D(x)\,\Phi_{qr_1\cdots r_{k-1}}(x) = 1.$$

Then $\left[C(x^{r_k})\,\Phi_{pr_1\cdots r_{k-1}}(x)\right]\Phi_{pr_1\cdots r_k}(x) + \left[D(x^{r_k})\,\Phi_{qr_1\cdots r_{k-1}}(x)\right]\Phi_{qr_1\cdots r_k}(x)$

$$= C(x^{r_k})\,\Phi_{pr_1\cdots r_{k-1}}(x^{r_k}) + D(x^{r_k})\,\Phi_{qr_1\cdots r_{k-1}}(x^{r_k}) = 1.$$

Claim 2: $\forall$ distinct primes $p_1, \cdots, p_k$, $\exists\ A_i(x) \in \mathbb{Z}[x]$ s.t.

$$\sum_{i=1}^{k} A_i(x) \cdot \frac{x^n - 1}{x^{\frac{n}{p_i}} - 1} = \Phi_n(x) \quad \text{where } n = p_1 \cdots p_k.$$

Proof: Induction.

Base case $k=1$: $1 \cdot \dfrac{x^{p_1} - 1}{x - 1} = \Phi_{p_1}(x)$.

Base case $k=2$:

$$A_1(x) \cdot \frac{x^{p_1 p_2} - 1}{x^{p_2} - 1} + A_2(x) \cdot \frac{x^{p_1 p_2} - 1}{x^{p_1} - 1} = \Phi_{p_1 p_2}(x)$$

$$\Longleftrightarrow \quad A_1(x)\,\Phi_{p_1}(x) + A_2(x)\,\Phi_{p_2}(x) = 1 \quad \text{(Claim 1)}$$

Inductive step: Assume $\exists\, A_i(x), B_i(x) \in \mathbb{Z}[x]$ s.t.

$$\sum_{i=1}^{k-1} A_i(x) \cdot \frac{x^{p_1 \cdots p_{k-1}} - 1}{x^{\frac{p_1 \cdots p_{k-1}}{p_i}} - 1} = \Phi_{p_1 \cdots p_{k-1}}(x),$$

$$\sum_{i=2}^{k} B_i(x) \cdot \frac{x^{p_2 \cdots p_k} - 1}{x^{\frac{p_2 \cdots p_k}{p_i}} - 1} = \Phi_{p_2 \cdots p_k}(x).$$

Also, $\exists\, C(x), D(x) \in \mathbb{Z}[x]$ s.t.

$$C(x)\,\Phi_{p_1 \cdots p_{k-1}}(x) + D(x)\,\Phi_{p_2 \cdots p_k}(x) = 1. \quad \text{(Claim 1)}$$

Then $\left[C(x)\,A_1(x^{p_k})\right] \cdot \dfrac{x^n - 1}{x^{\frac{n}{p_1}} - 1} + \left[D(x)\,B_k(x^{p_1})\right] \cdot \dfrac{x^n - 1}{x^{\frac{n}{p_k}} - 1}$

$$+ \sum_{i=2}^{k-1} \left[C(x)\,A_i(x^{p_k}) + D(x)\,B_i(x^{p_1})\right] \cdot \frac{x^n - 1}{x^{\frac{n}{p_i}} - 1}$$

$$= C(x) \sum_{i=1}^{k-1} A_i(x^{p_k}) \cdot \frac{x^n - 1}{x^{\frac{n}{p_i}} - 1} + D(x) \sum_{i=2}^{k} B_i(x^{p_1}) \cdot \frac{x^n - 1}{x^{\frac{n}{p_i}} - 1}$$

$$= C(x)\,\Phi_{p_1 \cdots p_{k-1}}(x^{p_k}) + D(x)\,\Phi_{p_2 \cdots p_k}(x^{p_1})$$

$$= C(x)\,\Phi_{p_1 \cdots p_{k-1}}(x)\,\Phi_{p_1 \cdots p_k}(x) + D(x)\,\Phi_{p_2 \cdots p_k}(x)\,\Phi_{p_1 \cdots p_k}(x)$$

$$= \Phi_n(x)\left[C(x)\,\Phi_{p_1 \cdots p_{k-1}}(x) + D(x)\,\Phi_{p_2 \cdots p_k}(x)\right]$$

$$= \Phi_n(x).$$

Claim 3: $\forall\, n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $\exists\, A_i(x) \in \mathbb{Z}[x]$ s.t.

$$\sum_{i=1}^{k} A_i(x) \cdot \frac{x^n - 1}{x^{\frac{n}{p_i}} - 1} = \Phi_n(x).$$

Proof: Induction.

Base case $n=2$: Claim 2.

Inductive step: If n is squarefree, use Claim 2.

If $\exists$ prime p s.t. $p^2 \mid n$, assume $\exists A_i(x) \in \mathbb{Z}[x]$ s.t.

$$\sum_{i=1}^{k} A_i(x) \cdot \frac{x^{\frac{n}{p}}-1}{x^{\frac{n}{pp_i}}-1} = \Phi_{\frac{n}{p}}(x).$$

Then
$$\sum_{i=1}^{k} A_i(x^p) \cdot \frac{x^n-1}{x^{\frac{n}{p_i}}-1}$$

$$= \sum_{i=1}^{k} A_i(x^p) \cdot \frac{(x^p)^{\frac{n}{p}}-1}{(x^p)^{\frac{n}{pp_i}}-1}$$

$$= \Phi_{\frac{n}{p}}(x^p)$$

$$= \Phi_n(x).$$