# MATH3541 INTRODUCTION TO TOPOLOGY ASSIGNMENT III

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HONG KONG

**Due**: 12:00 noon, Tuesday 15th October 2024.

**Instructions**: Submit solutions to the problems in **Section B** for credit. Problems in Section A should be attempted and may be optionally submitted for feedback.

**Guidelines on Writing:** *You should write in complete sentences. Do not just give the solution in fragmentary bits and pieces. Clarity of presentation of your argument counts, so explain the meaning of every symbol that you introduce and avoid starting a sentence with a symbol.*

## Section A

**Problem 1.** Let $X$ be any topological space. Recall that the set $CX$ is called the cone of $X$. Suppose now that $X$ is a compact subset of $\mathbb{R}^n$, and regard $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$ consisting of all elements in $\mathbb{R}^{n+1}$ with the last coordinate being 0. Let $v = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$, and define the *geometric cone* on $X$ to be

$$C_{\text{geom}}X = \{tv + (1-t)x \mid t \in [0,1],\, x \in X\}.$$

Prove that $C_{\text{geom}}X$ is homeomorphic to $CX$.

**Problem 2.** Consider $f\colon [0, 2\pi] \times [-1, 1] \to \mathbb{R}^3$ given by

$$f(s,t) = \left( \left(1 + \frac{t}{2}\cos\left(\frac{s}{2}\right)\right)\cos s,\ \left(1 + \frac{t}{2}\cos\left(\frac{s}{2}\right)\right)\sin s,\ \frac{t}{2}\sin\left(\frac{s}{2}\right) \right).$$

Show that $f$ defines an embedding of the Mobius strip to $\mathbb{R}^3$. Here the Mobius strip is defined as the quotient space of $[0, 2\pi] \times [-1, 1]$ by identifying the point $(0, y)$ with $(2\pi, -y)$ for every $y \in [-1, 1]$.

**Problem 3.** Show that a singleton set $\{x\}$ of any topological space $X$ is compact.

**Problem 4.** Find a topological space and a compact subset whose closure is not compact.

**Problem 5.** Give an example of a compact topological space $X$ and a compact subset $A$ of $X$ such that $A$ is not closed in $X$.

**Problem 6.** Let $C_1, \ldots, C_n$ be a collection of compact subsets of a topological space $X$. Prove that $\cup_{i=1}^n C_i$ is a compact subset of $X$.

**Problem 7.** Let $\{C_\alpha \mid \alpha \in \mathcal{I}\}$ be a collection of compact subsets of a Hausdorff topological space $X$. Show that $\cap_{\alpha\in\mathcal{I}} C_\alpha$ is a compact subset of $X$.

**Problem 8.** Let $f\colon X \to Y$ be a continuous map between topological spaces such that $X$ is compact and $Y$ is Hausdorff. Show that $f$ is a proper map.

**Problem 9.** An $n \times n$ square matrix $A = (a_{ij})$ with complex entries is *unitary* if $A^*A = I_n$, where $A^* = (\overline{a_{ji}})$ denotes the conjugate transpose of $A$. Prove that the set of all $n \times n$ unitary matrices, regarded as a subspace of $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$ and with the subspace topology, is compact.

## Section B

**Problem 10** (4 marks)**.** Let $M_n(k)$ be the set of $n \times n$ matrices with values in a field $k = \mathbb{R}$ or $\mathbb{C}$. Identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ and equip it with the metric topology given by the Euclidean norm,

$$\|A\| = \left( \sum_{i,j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}.$$

Recall that the product of two matrices $A = (a_{ij}), B = (b_{ij}) \in M_n$ is the matrix $C = (c_{ij})$ where $c_{ij} = \sum_{i=1}^{n} a_{ik}b_{kj}$.

(a) Briefly explain why the product of matrices is a continuous map

$$m : M_n(k) \times M_n(k) \to M_n(k)$$

and why the determinant $\det(\cdot) : M_n(k) \to k$ is continuous.

(b) Prove that the set of invertible matrices in $M_n(k)$ is an open subset.

(c) Prove that the set of invertible matrices in $M_n(\mathbb{R})$ is not connected.

(d) Prove that the set of invertible matrices in $M_n(\mathbb{C})$ **is** connected.

**Problem 11** (4 marks)**.** This question is about orbit spaces

(a) Give an action of $\mathbb{Z}$ on $\mathbb{R} \times [0, 1]$ which has the Möbius strip as the orbit space.

(b) Give an action of $\mathbb{Z}/2\mathbb{Z}$ on the torus $T^2$ with the cylinder as the orbit space.

**Problem 12** (5 marks)**.** Let $X = Y = \mathbb{R}^n$ with the Euclidean topology, and $A \subset Y$ be the set $\mathbb{R}^n - \{0\}$. Take the map $f : A \to X$ given by $x \mapsto \frac{x}{\|x\|^2}$, where $\|x\|$ denotes the Euclidean norm. Define the adjunction space $Z = X \cup_f Y$. Denote by $0_X$ and $0_Y$ the points corresponding to the origin in $X$ and $Y$ respectively.

Let $S^n = \{(x_0, \ldots, x_n) \mid \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$ be the $n$-sphere. Construct a homeomorphism $\phi : Z \to S^n$ such that

$$\phi(0_X) = -\phi(0_Y).$$

(HINT: Construct two continuous maps $\phi_X : X \to S^n$, $\phi_Y : Y \to S^n$ which can glue via $f$ on $A$ and $f(A)$. Verify that it is a homeomorphism. It may help to first attempt this in the cases $n = 1$ and $n = 2$ before extending to general $n$.)

**Problem 13** (7 marks)**.** Consider the topological space $X$ whose underlying set is $\mathbb{R}$ and whose topology $\mathcal{O}_l$ of open sets is generated by the basis of half open intervals $[a, b)$ for real numbers $a < b$.

(a) Prove that $\mathcal{O}_l$ is finer than the Euclidean topology on $\mathbb{R}$.
(b) Prove that $X$ is not connected.
(c) Prove that $X$ is not locally connected.

A topological space is $T_4$ if disjoint closed sets can be separated by disjoint open sets. We will show that a product of $T_4$ spaces need not be $T_4$.

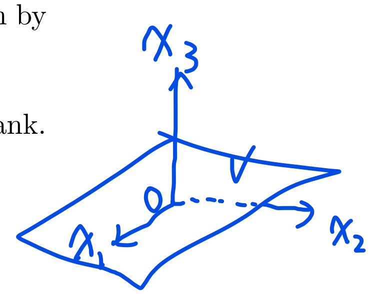
(d) Show that $X$ is $T_4$.
(e) Show that the *anti-diagonal* $\Delta = \{(x, -x) \mid x \in \mathbb{R}\}$ is a discrete subset of the product space $X \times X$.
(f) Show that $\Delta_{\mathbb{Q}} = \{(x, -x) \mid x \in \mathbb{Q}\}$ and $\Delta \setminus \Delta_{\mathbb{Q}}$ are closed sets in $X \times X$ which cannot be separated by open sets.

## Section C

The question(s) in this section are not assessed. But you are strongly recommended to attempt them as preparation for the midterm/exam.

**Problem 14.** Let $G = \mathrm{GL}_k(\mathbb{R})$ denote the topological group of $k \times k$ invertible matrices with entries in $\mathbb{R}$ and let $M = M_{k \times n}$ be the topological space of $k \times n$ matrices entries in $\mathbb{R}$. There is an action of $G$ on $M$ given by left multiplication

$$G \times M \to M, \quad (g, A) \mapsto gA.$$




Let $X = \{A \in M \mid \mathrm{rank}(A) = k\}$ be the subspace of matrices with full rank.

(a) Check that $X$ is an open subset of $M$.
(b) Show that the action of $G$ on $M$ restricts to an action on $X$.
(c) Prove that $X/G$ is Hausdorff.

In the following parts, we work towards the fact that $X/G$ is compact.

(d) Construct a bijection $\phi : X/G \to \{k\text{-dim subspaces of } \mathbb{R}^n\}$.
(e) Show that $\phi$ is induces a bijection from $X/G$ to the set of embeddings

$H$

$S^{n-1} \cap H \cong S^{k-1} \xrightarrow{i} S_{n-1}$

$$\{i : S^{k-1} \hookrightarrow S^{n-1} \mid i(S^{k-1}) = S^{n-1} \cap H\}$$

where $H$ is a $k$-dimensional subspace of $\mathbb{R}^n$.

(f) Let $Y$ be the subset of matrices $\begin{bmatrix} — & w_1 & — \\ & \vdots & \\ — & w_k & — \end{bmatrix} \in M$ whose row vectors $w_1, \ldots, w_k$ are orthonormal. Show that the orthogonal group $\mathrm{O}(k)$ has an action on $Y$ by left multiplication.

find orthonormal basis

(g) Construct bijection between the sets $X/G$ and $Y/\mathrm{O}(k)$.
(h) (Hard!) Show that this gives a homeomorphism $X/G \cong Y/\mathrm{O}(k)$.
(i) Use part (h) to show that $X/G$ is compact.



$Y^T Y = I$

(i) Left is well-defined
(ii) Left$(g, \text{Left}(g', x))$ = Left$(gg', x)$
(iii) Left$(e, x) = x$