THE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH3301: Algebra I

December 18, 2023 | 9:30 am - 12:00 noon

No calculator is allowed in this examination.

Answer ALL FOUR questions

Note: You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts. So **think carefully before you write.**

1. (25%) For each of the following, determine with explanation whether it is true or false.

   (a) The symmetric group S_4 is solvable.

   (b) Suppose H and K are subgroups of a non-abelian group G and
   $$HK = KH.$$
   If H and K are simple groups, then HK has to be a simple group.

   (c) If G is a group of order p^2 where p is a prime, then G is an abelian group.

   (d) Suppose R is a commutative ring with unity and I is a proper ideal of R. If the characteristic $\text{char}(R/I)$ of the quotient ring R/I is p^2 where p is a prime, then I is not a maximal ideal.

2. (25%) Let G be finite group and denote its order by $|G|$. We say that

   - G has Property (*) if for any prime p dividing $|G|$, the group G has a *normal* subgroup of order p, and
   - G has Property (†) if for every divisor d of $|G|$, the group G contains a subgroup of order d.

   [Remark. The trivial group has both Properties (*) and (†).]

   (a) With verification, give an example of G that does not have Property (*).

   (b) Let Q be the family of all quotient groups descended from G, i.e.
   $$Q = \{G/N : N \triangleleft G\}.$$
   If all groups in Q have Property (*), show that G has Property (†).
   [Suggestion: Argue by induction on the order of G.]

   (c) Hence or otherwise, show that all p-groups have Property (†).

3. (25%) Let R be a non-zero commutative ring with unity, and define

$$A = \{x \in R : x \text{ is a zero divisor}\} \cup \{0\}.$$

(a) If R is finite and $y \in R \setminus A$, show that y is a unit of R.

(b) With verification, give an example of R for which A is not an ideal of R.

(c) If A is an ideal of R, show that A is a prime ideal.

(d) If R is finite and A is an ideal of R, show that A is a maximal ideal.

4. (25%) Consider the polynomial ring $\mathbb{F}_3[t]$ where $\mathbb{F}_3 = \mathbb{Z}_3$ is the finite field with 3 elements. Let $f(t) = t^2 + t + 2 \in \mathbb{F}_3[t]$, and define $K = \mathbb{F}_3[t]/\langle f \rangle$.

(a) Show that K is a field extension of $\mathbb{F}_3$.

(b) Let $u = \pi(t)$ where π is the natural projection from $\mathbb{F}_3[t]$ to K.

(i) List all the elements of K in terms of the elements in $\mathbb{F}_3$ and u, and

(ii) evaluate the degree $[K : \mathbb{F}_3]$.

(c) Find the two roots of the polynomial $f(x) = x^2 + x + 2$ in the field K.

(d) Show that the group of units of $K[x]/\langle f \rangle$ is not a cyclic group, where $f(x) = x^2 + x + 2$.

[Hint: Consider the map $\varphi : K[x] \to K \times K$ defined by

$$g(x) = (g(\alpha), g(\beta)) \qquad \text{for any } g \in K[x],$$

for some *suitable* $\alpha, \beta \in K$, and use the Isomorphism Theorem.]

******* END OF PAPER *******