ASSIGNMENT IV, ALGEBRA II, HKU, SPRING 2025

DUE AT 11:59PM ON FRIDAY MARCH 28, 2025

(1) Classify all finite abelian groups of order 72.

(2) Find the invariant factor form and the elementary divisor form of the group
$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_9.$$

(3) Consider $\mathbb{R}[x]/\langle x(x-2)^3 \rangle$ as an $\mathbb{R}[x]$-module. Let
$$V = \mathbb{R}[x]/\langle x(x-2)^3 \rangle$$
as an $\mathbb{R}$-vector space and and let $T : V \to V$ be the $\mathbb{R}$-linear map defined by the action of $x \in \mathbb{R}[x]$. Show that T has a Jordan canonical form, and find its Jordan canonical form.

(4) Find the rational canonical form and the Jordan canonical form of the following two matrices
$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{pmatrix}.$$

(5) State the definition of a simple field extension; Is every simple field extension finite? Explain your answer.

(6) State the definition of an algebraic field extension. Is every algebraic extension finite? Explain your answer.

\mathbb{Q}[\sqrt[n]{2}]

(7) Let K be a field. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in K[x]$ and define
$$f'(x) = a_1 + 2a_2 x + \cdots + n x^{n-1} \in K[x]$$
be the derivative of $f(x)$. (a) If K has characteristic 0, show that $f'(x) = 0$ if and only if f is a constant polynomial. (b) If K has characteristic $p > 0$, show that $f'(x) = 0$ if and only if there exists $g(x) \in K[x]$ such that $f(x) = g(x^p)$.

(8) Show that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic.

(9) Let $K \subset L$ be a field extension and assume that $a \in L$ is algebraic over K. Let $K(a)$ be the sub-field of L generated by K and a. Prove that
$$K(a) = \{f(a) : f \in K[x]\}.$$
Prove that $K(a)$ is a finite extension of K. What is the degree of $K(a)$ over K? (We have done this in class, but please try to write the proofs in your own words and review the definitions if necessary).

(10) For the following $\alpha \in \mathbb{R}$, find the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$:
$$1)\ \alpha = \sqrt{1+\sqrt{3}}; \quad 2)\ \alpha = \sqrt{3-\sqrt{6}}; \quad 3)\ \gamma = \sqrt{3+2\sqrt{2}}.$$