THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 4 Solution

Problem 1.

(i) The equation is elliptic, because

$$\mathcal{D} = (-6)^2 - (9)(16) = -108.$$

(ii) The corresponding matrix with its trace non-negative is given by

$$A = \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix}.$$

Its characteristic equation is given by

$$0 = \det(A - \lambda I) = -\lambda(\sqrt{2} - \lambda)(2\sqrt{2} - \lambda).$$

Since all eigenvalues are non-negative, with one of them being 0, the equation is parabolic.

(iii) The equation is hyperbolic, because

$$\mathcal{D} = (\sqrt{1 + x^2 + y^2})^2 - (x)(y) = 1 + x^2 + y^2 - xy,$$

and for any $x, y \in \mathbb{R}$, we must have

$$xy \le \max\{x^2, y^2\} < 1 + x^2 + y^2.$$

(iv) The corresponding matrix with its trace non-negative is given by

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 3 & 5 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}.$$

Then its characteristic polynomial is given by

$$0 = \det(B - \lambda I) = (-1-\lambda)(5-\lambda)(5-3\sqrt{2}-\lambda)(5+3\sqrt{2}-\lambda).$$

We can conclude that we have one negative eigenvalue. And thus, the equation is hyperbolic.

Problem 2.

(i) First, the characteristic is given by

$$\begin{cases} \frac{\mathrm{d}\,t}{\mathrm{d}\,s} = 1,\ t(0) = 0 \\ \frac{\mathrm{d}\,x}{\mathrm{d}\,s} = 2x - 7,\ x(0) = x_0 \\ \frac{\mathrm{d}\,u}{\mathrm{d}\,s} = 1,\ u(0, x_0) = u_0. \end{cases} \Rightarrow \begin{cases} t = s, \\ x = (x_0 - \frac{7}{2})e^{2s} + \frac{7}{2} \\ u = u_0 + t. \end{cases}$$

First, if $x_0 \geq \frac{7}{2}$, the characteristic curves do not intersect with $x = 0$. So no compatibility condition is required in this case. If $x_0 < \frac{7}{2}$, the characteristic curves intersect both $x = 0$ and $t = 0$. Then for all (t, x) on the characteristics, we have

$$u(t, x) = u(0, x_0) + t = \phi(x_0) + t.$$

Take $x = 0$ we have

$$t = \frac{1}{2}\ln\left(\frac{-7}{2x_0 - 7}\right).$$

Thus, the compatibility condition says that

$$g\left(\frac{1}{2}\ln\left(\frac{-7}{2x_0-7}\right)\right) = \phi(x_0) + \frac{1}{2}\ln\left(\frac{-7}{2x_0-7}\right) \quad \text{for } x_0 \in \left(0, \frac{7}{2}\right).$$

(ii) By the method of characteristics, we have

$$\begin{cases} \frac{\mathrm{d}\,t}{\mathrm{d}\,s} = 1,\ t(0) = t_0, \\ \frac{\mathrm{d}\,x}{\mathrm{d}\,s} = t^2 + x,\ x(0) = x_0, \\ \frac{\mathrm{d}\,u}{\mathrm{d}\,s} = \sqrt{1+u^2},\ u(t_0, x_0) = u_0, \end{cases}$$

$$\Rightarrow \begin{cases} t = s + t_0, \\ x = -(s+t_0)^2 - 2(s+t_0) - 2 + (x_0 + t_0^2 + 2t_0 + 2)e^s, \\ u = \sinh(t - t_0 + \sinh^{-1} u_0). \end{cases}$$

The characteristic curves can be parametrized by (t_0, x_0):

$$C_{(t_0,x_0)} = \{(t,x) : x = -t^2 - 2t - 2 + (x_0 + t_0^2 + 2t_0 + 2)e^{t-t_0}\}.$$

Let $\boxed{t_0 = 0 \text{ and } x_0 > 0}$, then for all $(t,x) \in C_{(0,x_0)} = \{(t,x) : x = -t^2 - 2t - 2 + (x_0 + 2)e^t\}$,

$$u(t,x) = \sinh(t + \sinh^{-1}(\phi(x_0))) = \sinh(t + \sinh^{-1}(\phi(e^{-t}(x + t^2 + 2t + 2) - 2))).$$

Let $\boxed{t_0 > 0 \text{ and } x_0 = 0}$, then for all $(t,x) \in C_{(t_0,0)} = \{(t,x) : x = -t^2 - 2t - 2 + (t_0^2 + 2t_0 + 2)e^{t-t_0}\}$,

$$u(t,x) = \sinh(t - t_0 + \sinh^{-1}(g(t_0))),$$

where t_0 represents the unique solution of the equation

$$x = -t^2 - 2t - 2 + (t_0^2 + 2t_0 + 2)e^{t-t_0}.$$

If $\boxed{t_0 = 0 \text{ and } x_0 = 0}$, then for all $(t,\, x) \in C_{(0,\,0)} = \{(t,\, x) : x = -t^2 - 2t - 2 + 2e^t\}$, by utilizing the compatibility condition $g(0) = \phi(0) = 0$,

$$u(t,x) = \sinh t,$$

(iii) The characteristic is given by

$$\begin{cases} \frac{\mathrm{d}\,t}{\mathrm{d}\,s} = 1,\ t(0) = 0, \\ \frac{\mathrm{d}\,x}{\mathrm{d}\,s} = -27x - 9t - 2024,\ x(0) = x_0, \\ \frac{\mathrm{d}\,u}{\mathrm{d}\,s} = 3\sin u + 4\cos u + 5. \end{cases} \quad \Rightarrow \begin{cases} t = s, \\ x = (x_0 + \frac{6071}{81})e^{-27s} - \frac{s}{3} - \frac{6071}{81}. \end{cases}$$

Integrate along the characteristics, we have

$$u(t,x) = u(0,x_0) + \int_0^s \frac{\mathrm{d}\,u}{\mathrm{d}\,s}\,\mathrm{d}\,s = \int_0^s 5\sin(u+\theta) + 5\,\mathrm{d}\,s \geq 0,$$

where $\theta = \arccos(\frac{3}{5})$.

However, notice that the characteristic also intersects with $x = 0$ and

$$u(t, 0) = -20 \sinh 24t < 0,$$

where the strict inequality applies as this intersection point can't be $(0, 0)$.

The problem is thus not solvable.

Problem 3.

(i) The equation is hyperbolic because

$$\mathcal{D} = \left(-\frac{1}{2}\right)^2 - (1)(-12) = \frac{49}{4} > 0.$$

(ii) The equation (2) can be rewritten as

$$(\partial_t - 4\partial_x)(\partial_t + 3\partial_x)u = 0.$$

To find the general solution to (2), we need to solve

$$\begin{cases} \partial_t v - 4\partial_x v = 0 \\ \partial_t u + 3\partial_x u = v \end{cases}.$$

Note that the general solution of $\partial_t v - 4\partial_x v = 0$ is

$$v = f(x + 4t),$$

for arbitrary function f because v is a constant along the characteristic curve

$$C_{x_0} = \{(t, x) : x = -4t + x_0\}.$$

To solve $\partial_t u + 3\partial_x u = v = f(x + 4t)$, find the characteristic curves,

$$\begin{cases} \dfrac{dt}{ds} = 1, \ t(0) = 0 \\ \dfrac{dx}{ds} = 3, \ x(0) = x_0 \end{cases} \Longrightarrow \begin{cases} t = s \\ x = 3s + x_0 \end{cases}$$

Then $x = 3t + x_0$, and the characteristic curves can be parametrized by x_0:

$$\tilde{C}_{x_0} = \{(t,\, x) : x = 3t + x_0\}.$$

Let $W(s) := u(t(s), x(s))$. Then

$$\frac{dW}{ds} = v(t(s), x(s)) = f(x(s) + 4t(s)) = f(3s + x_0 + 4s) = f(7s + x_0).$$

Integrate $\frac{dW}{ds}$ along the characteristic curve $\tilde{C}_{x_0}$ from $(0, x_0)$ to (t, x), we have

$$\begin{aligned} u(t,x) - u(0,x_0) &= \int_0^t f(7s + x_0)\ ds \\ &= \frac{1}{7}\int_{x_0}^{7t+x_0} f(\tilde{s})\ d\tilde{s} \quad (\tilde{s} = 7s + x_0) \\ &= \frac{1}{7}\int_{x_0}^{x+4t} f(\tilde{s})\ d\tilde{s} = F(x+4t), \end{aligned}$$

where $F' = \dfrac{f}{7}$ with $F(x_0) = 0$. Hence

$$u(t,x) = u(0,x_0) + F(x+4t) = u(0, x-3t) + F(x+4t) = g(x-3t) + F(x+4t),$$

for arbitrary functions g, F.

(iii) Note that

$$g(x) + F(x) = u(0,x) = 6x^3 \tag{1}$$

and

$$-3g'(x) + 4F'(x) = \Big[-3g'(x-3t) + 4F'(x+4t)\Big]_{t=0} = u_t(0,x) = 59x^2.$$

Differentiating (1), we have

$$g'(x) + F'(x) = 18x^2.$$

Hence $g'(x) = \frac{13}{7}x^2$ and $F'(x) = \frac{113}{7}x^2$ implies

$$g(x) = \frac{13}{21}x^3 + C_1 \text{ and } F(x) = \frac{113}{21}x^3 + C_2 \text{ for some } C_1, C_2 \in \mathbb{R}.$$

It follows from (1) that $C_1 + C_2 = 0$. Therefore

$$\begin{aligned} u(t,x) &= g(x-3t) + F(x+4t) = \frac{13}{21}(x-3t)^3 + C_1 + \frac{113}{21}(x+4t)^3 + C_2 \\ &= \frac{13}{21}(x-3t)^3 + \frac{113}{21}(x+4t)^3. \end{aligned}$$

Problem 4.

(a) A direct computation yields that the discriminant

$$\mathcal{D} := \left(-\frac{5}{2}\right)^2 - (1)(6) = \frac{1}{4} > 0,$$

so the equation is hyperbolic.

(b) Since

$$\begin{cases} \xi := m_{11}x + m_{12}t, \\ \eta := m_{21}x + m_{22}t, \end{cases}$$

it follows from the chain rule that

$$\begin{cases} \partial_x = (\partial_x \xi)\partial_\xi + (\partial_x \eta)\partial_\eta = m_{11}\partial_\xi + m_{21}\partial_\eta, \\ \partial_t = (\partial_t \xi)\partial_\xi + (\partial_t \eta)\partial_\eta = m_{12}\partial_\xi + m_{22}\partial_\eta. \end{cases}$$

Thus, we have

$$\begin{cases} \partial_\xi = \partial_t - 2\partial_x = (m_{12} - 2m_{11})\partial_\xi + (m_{22} - 2m_{21})\partial_\eta, \\ -\partial_\eta = \partial_t - 3\partial_x = (m_{12} - 3m_{11})\partial_\xi + (m_{22} - 3m_{21})\partial_\eta. \end{cases}$$

Comparing the coefficients, we obtain the following system of linear equations

$$\begin{cases} 1 = m_{12} - 2m_{11}, \quad 0 = m_{22} - 2m_{21}, \\ 0 = m_{12} - 3m_{11}, \quad -1 = m_{22} - 3m_{21}. \end{cases}$$

Solving the above system, we finally obtain

$$m_{11} = 1, \quad m_{12} = 3, \quad m_{21} = 1, \quad \text{and} \quad m_{22} = 2.$$

(c) Using the factorization of differential operator

$$\partial_{tt} - 5\partial_{tx} + 6\partial_{xx} = (\partial_t - 2\partial_x)(\partial_t - 3\partial_x) = -\partial_\xi \partial_\eta,$$

we can rewrite the equation as

$$\partial_\xi \partial_\eta u = 1.$$

(d) It follows from part (c) that

$$\partial_\xi \partial_\eta u = 1,$$

so a direct integration yields

$$u = \xi\eta + f(\xi) + g(\eta),$$

where both f and g are arbitrary functions. If follows from part (b) that

$$\xi := x + 3t \quad \text{and} \quad \eta := x + 2t,$$

so we finally obtain the general solution formula

$$\begin{aligned} u(t,x) &= (x+3t)(x+2t) + f(x+3t) + g(x+2t) \\ &= x^2 + 5tx + 6t^2 + f(x+3t) + g(x+2t). \end{aligned}$$

(e) Substituting the initial conditions to the general solution formula, we have

$$\begin{cases} x^2 = x^2 + f(x) + g(x) \\ 5x + 2xe^{-x^2} = 5x + 3f'(x) + 2g'(x), \end{cases}$$

or equivalently,

$$\begin{cases} 0 = f(x) + g(x) \\ 2xe^{-x^2} = 3f'(x) + 2g'(x). \end{cases}$$

Solving the above system, we obtain

$$f(x) = -e^{-x^2} + C \quad \text{and} \quad g(x) = e^{-x^2} - C,$$

for any arbitrary constant C. Hence, the final answer is

$$\begin{aligned} u(t,x) &= x^2 + 5tx + 6t^2 + f(x+3t) + g(x+2t) \\ &= x^2 + 5tx + 6t^2 + \left(-e^{-(x+3t)^2} + C\right) + \left(e^{-(x+2t)^2} - C\right) \\ &= x^2 + 5tx + 6t^2 - e^{-(x+3t)^2} + e^{-(x+2t)^2}. \end{aligned}$$

Problem 5.

(i) The discriminant is given by

$$\mathcal{D} = (-6)^2 - (9)(4) = 0.$$

The equation is parabolic.

(ii) The direct computation yields that

$$(3\partial_t - 2\partial_x)^2 = 9\partial_{tt} - 12\partial_{tx} + 4\partial_{xx}.$$

(iii) Denote $v := (3\partial_t - 2\partial_x)u$, it follows that

$$\begin{cases} 3\partial_t v - 2\partial_x v = 0, \\ 3\partial_t u - 2\partial_x u = v. \end{cases}$$

(iv) By the method of characteristic,

$$\begin{cases} \frac{\mathrm{d}\,t}{\mathrm{d}\,s} = 3, \\ \frac{\mathrm{d}\,x}{\mathrm{d}\,s} = -2, \end{cases} \quad \Rightarrow \begin{cases} t = 3s, \\ x = -2s + x_0. \end{cases}$$

Then the general solution to $3\partial_t v - 2\partial_x v = 0$ is given by

$$v(t,x) = g(2t + 3x).$$

(v) To solve $3\partial_t u - 2\partial_x u = g(2t+3x)$, we again apply the method of characteristic,

$$\frac{\mathrm{d}}{\mathrm{d}\,s}u(t(s),x(s)) = g(2t(s)+3x(s)) = g(x_0).$$

Integrate from $(0,x_0)$ to (t,x) along the characteristic

$$u(t,x) - u(0,x_0) = \int_0^s g(x_0)\,\mathrm{d}\,s = sg(x_0).$$

So the general solution is given by

$$u(t,x) = u(0,x_0)+sg(x_0) = f(x+\frac{2}{3}t)+\frac{t}{3}g(x+\frac{2}{3}t) = \tilde{f}(3x+2t)+t\tilde{g}(3x+2t)$$

for some $\tilde{f},\tilde{g}$ arbitrary.

Food for Thought. An initial-value problem is given in the following form:

$$\begin{cases} u|_{t=0} = \phi(x) \\ \partial_t u|_{t=0} = \psi(x). \end{cases}$$

Then from the general solution deduced above, we have

$$u(0,x) = \tilde{f}(3x) = \phi(x) \implies \tilde{f}(x) = \phi\left(\frac{x}{3}\right),$$

and

$$\partial_t u(0,x) = 2\tilde{f}'(3x)+\tilde{g}(3x) = \psi(x) \implies \tilde{g}(x) = \psi\left(\frac{x}{3}\right)-2\tilde{f}'(x) = \psi\left(\frac{x}{3}\right)-\frac{2}{3}\phi'\left(\frac{x}{3}\right).$$

Hence the general solution can be written as

$$u(t,x) = \tilde{f}(3x+2t)+t\tilde{g}(3x+2t) = \phi\left(\frac{3x+2t}{3}\right)+t\left[\psi\left(\frac{3x+2t}{3}\right)-\frac{2}{3}\phi'\left(\frac{3x+2t}{3}\right)\right].$$

Problem 6. By the d'Alembert formula,

$$u(t,x) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds.$$

(i) Then

$$
\begin{aligned}
u(t,0) &= \frac{1}{2}[\phi(ct)+\phi(-ct)] + \frac{1}{2c}\int_{-ct}^{-ct}\psi(s)\,ds \\
&= \frac{1}{2}[\phi(ct)+\phi(-ct)] + \frac{1}{2c}\int_{ct}^{-ct}\psi(s)\,ds \\
&= \frac{1}{2}[\phi(ct)-\phi(ct)] + \frac{1}{2c}\cdot 0 \ (\psi \text{ is odd}) \\
&= 0.
\end{aligned}
$$

(ii) Then there exists a constant $M > 0$ such that if $x+ct < -M$ or $x-ct > M$ then $u = 0$. Because when $x + ct < -M$ or $x - ct > M$ then

$$
\begin{aligned}
u(t,x) &= \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds \\
&= \frac{1}{2}[0+0] + \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,ds \\
&= 0.
\end{aligned}
$$

(iii) One of the weakest sufficient conditions can be $T_\phi = T_\psi$, which we denote as T, since

$$
\begin{aligned}
u(t,x+T) &= \frac{1}{2}[\phi(x+T+ct)+\phi(x+T-ct)] + \frac{1}{2c}\int_{x+T-ct}^{x+T+ct}\psi(s)\,ds \\
&= \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\tilde{s})\,d\tilde{s} \ (\tilde{s} = s-T) \\
&= u(t,x).
\end{aligned}
$$