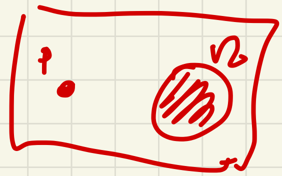
More properties of compact spaces

1) X is Hausdorff. pt & compact subset can be separated.



$\forall x \in \Omega \ \exists \ \mathcal{U}_x^{\ni x}, V_x \ni p$

s.t. $\mathcal{U}_x \cap V_x = \emptyset$

by compactness of Ω $\exists$ $x_1, \dots, x_N$ s.t.

$\Omega \subset \bigcup_{i=1}^{N} \mathcal{U}_{x_i} =: \mathcal{U}$ $\quad V = \bigcap_{i=1}^{N} V_{x_i}$ $\quad \mathcal{U} \cap V = \emptyset$

2) compact subsets of Hausdorff space is closed.

$X \setminus \Omega \ni x$ $\quad \mathcal{U} \ni x$ $\quad \mathcal{U} \cap \Omega = \emptyset$

Ex. Zariski top.

3) [Heine-Borel] Compact subsets of $\mathbb{R}^n$ $\iff$ closed & bounded.

it suffices to prove 1D case

Any closed & bounded set is a subset of $[-d, d]^n$

prove $[0,1]$ is compact using ∞-induction

sup.

* 4) $f: X \to Y$ cont. X compact. Y Hausdorff.

Then a) f is a closed map i.e. $X \overset{\text{sub.t}}{\cong} f(X) \subset Y$

b) if f is inj. the it's a top. embedding

c) if f is bij. then it's a homeo

5) $f: X \to Y$ cont. X compact, Y Hausdorff

a) $[f]: X/\sim_f \to Y$ embedding

b) if f is surj. then f is a quotient map.

Example Real projective plane can be embedded in $\mathbb{R}^4$

$\mathbb{R}P^2 = S^2 / v \sim -v$ is T_2 & compact

The cont. map $S^2 \longrightarrow \mathbb{R}^4$

$$(x,y,z) \longmapsto (x^2-y^2, xy, xz, yz)$$

$x^2+y^2+z^2=1$

induces a top. embedding $\mathbb{R}P^2 \hookrightarrow \mathbb{R}^4$

Def'n A top space X is called **limit pt compact** if every infinite set of X has a limit pt.

Def'n A seq. of pts in X is a map

$$\mathbb{N} \longrightarrow X \qquad \lim_{i\to\infty} x_i = x_\infty \text{ if}$$
$$i \longmapsto x_i \qquad \forall \text{ nbhd } U \text{ of } x_\infty \ \exists N \text{ s.t. } i>N \Rightarrow x_i \in U.$$

($x_\infty \in X$)

X is called **sequentially compact** if any seq in X has a convergent sub seq.

Prop (X, d) metric space TFAE

1) X is compact
2) X is limit pt. compact
3) X is sequentially compact

$$\begin{array}{ccc} & 1) & \\ \swarrow & & \nwarrow \\ 2) & \Longrightarrow & 3) \end{array}$$

proved in analysis I.

1) $\Rightarrow$ 2) $A \subset X$ compact $|A| = \infty$

suppose A has no limit pt then $\forall x \in A$

$\exists\, \mathcal{U} \ni x$ $\mathcal{U} \cap A \setminus \{x\} = \phi$

the we get an open covering of A w/o finite sub covering

2) $\Rightarrow$ 3) WLOG Assume $\overset{\text{image}}{\mathbb{N} \longrightarrow X}$ is an infinite set.

$x_1, x_2 \ldots \longrightarrow a$

$x_{i_1} \in B(a, 1)$ $\frac{1}{2^{n_1}} < d(a, x_{i_1})$

$x_{i_2} \in B(a, \frac{1}{2^{n_1}})$ $\frac{1}{2^{n_2}} < d(a, x_{i_2})$

$\vdots$

we construct a convergent sub seq.

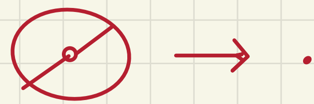
Def'n X is called locally compact

if $\forall\, x \in X$ $\exists$ compact $\Omega \subset X$

s.t $\Omega^\circ \ni x$. is an open nbhd.

Def: $f: X \to Y$ cont. is called **proper** if $\forall\, K \subset Y$ compact $f^{-1}(K)$ is compact

properties of proper map $\bar{B}(0,\delta) \subset \mathbb{R}^2 \to \mathbb{R}^2/_{v \sim \lambda v}$

1) X compact Y Hausdorff 

$f: X \to Y$ cont. then f is proper

pf. $K \subset Y$ compact $\Rightarrow$ K is closed

$+\ Y\ T_2$

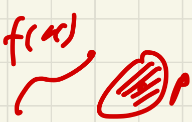
$\Rightarrow f^{-1}(K)$ is closed

$+\ X$ is compact $\Rightarrow f^{-1}(K)$ is compact.

2) Y locally compact Hausdorff

$f: X \to Y$ cont. proper $\Rightarrow$ f is a closed map

$K \subset X$ closed $\quad p \in f(K)^c$ $\exists\ \mathcal{U} \ni p$ s.t.

$\bar{\mathcal{U}}$ is compact $\quad f^{-1}(\bar{\mathcal{U}})$ is compact 

$f(f^{-1}(\bar{\mathcal{U}}) \cap K)$ is compact $\quad \mathcal{U} \setminus f(f^{-1}(\bar{\mathcal{U}}) \cap K) \ni p$

3) $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ $\qquad f(x,y) = y^d + g(x)$

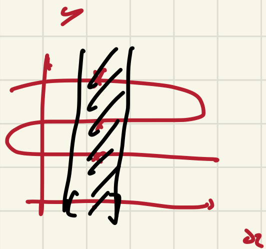
$(x,y) \longrightarrow x$ $\qquad g(x) \in \mathbb{C}[x]$

$\pi : \Gamma \to \mathbb{C}$ is proper $\qquad \Gamma = \{f(x,y) = 0\} \subset \mathbb{C}^2$

a) $X \xrightarrow{f} Y$ cont $\quad Y$ loc. compact

to check properness it suffices to check $f^{-1}(\bar{U})$

b) $\pi^{-1}(\bar{D}(x_0, \delta)) = \left\{ (x,y) \middle| \begin{array}{l} f(x,y) = 0 \\ |x| \leq \delta \end{array} \right\}$



say $g(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$

$|y|^d = |g(x)| \leq \sum |a_i| \delta^m$

$\pi^{-1}(\bar{D}(x_0, \delta))$ is bounded & closed.

4) let $K \subset M_n(\mathbb{R})$ be a compact set

$\{\lambda \in \mathbb{C} \mid \lambda \in \mathrm{Spec}(A), A \in K\}$ is compact

$$\begin{array}{ccc} K & & \mathbb{C} \\ \uparrow \pi_1 & \nearrow \pi_2 & \\ \phi : K \times \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

$(A, t) \longmapsto ch_A(t)$ characteristic poly

ϕ is proper. $\quad \phi^{-1}(0)$ is compact

$\pi_2 \phi^{-1}(0)$ is compact in $\mathbb{C}$.

Connectedness

Def'n $(X, \mathcal{O}_X)$ is called connected if

$X = X_1 \sqcup X_2$ for X_1, X_2 open

$\Rightarrow$ either X_1 or $X_2 = \emptyset$.

TFAE

1) X is conn.

2) if $U \subset X$ is both open & closed

then $U = \emptyset$ or $U = X$

3) $A \cup B = X$ $\quad A, B$ nonempty

$$\Rightarrow \bar{A} \cap B \neq \emptyset \text{ or } A \cap \bar{B} \neq \emptyset$$

Rmk 1) $[0, 1) \cup (1, 2]$ is not connected

2) $\mathbb{Q} \subset \mathbb{R}$ is not connected.

3) X top space $Y \subset X$ conn.

$X = U \sqcup V$ then $x \subset U$

or $x \subset V$.

4) $f : X \to Y$ cont

$U \subset X$ conn. $\Rightarrow f(U)$ is conn

p.f. $f(U) = (f(U) \cap V_1) \sqcup (f(U) \cap V_2)$

V_1, V_2 open in Y

$U = (U \cap f^{-1}V_1) \sqcup (U \cap f^{-1}V_2)$

U conn. $\Rightarrow$ $U \subset f^{-1}V_1$ or $U \subset f^{-1}V_2$

$\Rightarrow$ $f(U) \subset V_1$ or $f(U) \subset V_2$.

5) $Z \subset X$ conn. dense. $\Rightarrow$ X is conn.

p.f. suppose $X = X_1 \sqcup X_2$ open & closed $X_1, X_2 \neq \emptyset$

$Z = (X_1 \cap Z) \sqcup (X_2 \cap Z)$ since Z is

dense $X_1 \cap Z \neq \emptyset$ $X_2 \cap Z \neq \emptyset$

6) $Z \subset X$ conn. $Z \subset Y \subset \bar{Z}$

then Y is conn.

Thm A subset of $(\mathbb{R}, |\cdot|)$ is conn. $\iff$ it's an interval.

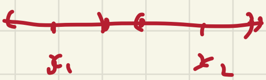
Pf. by def'n. a subset $\underbrace{\text{of } \mathbb{R}}_{I}$ is an interval if it satisfies the following property

$x < y \in I \Rightarrow [x, y] \subset I$ (prove it!)

Suppose I is an interval but not connected

then $I = I_1 \sqcup I_2$ $\quad I_1, I_2$ are open subset of I

Suppose $x_1 < x_2$ for $x_i \in I_i$



Let $y_1 = \max\{x \mid (x_1, x) \subset I_1\}$

$y_2 = \max\{x \mid (x, x_2] \subset I_2\}$

$y_1 \le y_2$ but $y_1 = y_2$ since otherwise

a pt $y_1 < z < y_2$ is not in $I_1 \cup I_2$

Suppose I is conn. $x_1 < x_2 \in I$

if $\exists\ x_1 < x < x_2$ s.t $x \notin I$

then $I = \left((-\infty, x) \cap I \right) \cup \left((x, +\infty) \cap I \right)$

Contradicts connectedness.

Cor [Intermediate value thm]

X connected $f : X \to \mathbb{R}$ cont.

if $x, y \in X$ $f(x) = a < f(y) = b$

$\forall\ a < c < b$ $\exists$ $z \in X$ s.t $f(z) = c$.

Another example

$\mathbb{R} \setminus 0$ is connected w.r.t Zariski top

but not connected w.r.t $|\cdot|$ top.