Topological space

Def X set $\mathcal{O}$: a collection of subsets of X.
is called a topology on X if

1) $\phi, X \in \mathcal{O}$
2) X is closed under arbitrary union
3) X is closed under finite intersection.

- $(X, \mathcal{O})$ is called a top space.
- elements of $\mathcal{O}$ are called open sets
- complement of open sets are closed sets.

Rmk $(X, \mathcal{O})$ top. space $\mathcal{C} = \{\text{closed sets}\}$

1) $\phi, X \in \mathcal{C}$ $\quad \left(\bigcap_{\lambda \in \Lambda} C_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} C_\lambda^c$
2) $\mathcal{C}$ is closed under arbitrary intersection
3) . ... finite union

Examples of top

1) Trivial top.

$$\mathcal{O} = \{ \emptyset, X \}$$

2) Discrete top

$\mathcal{O} = P(X)$: power set of X

3) Cofinite top.

$\mathcal{C} = \mathcal{O}^c = \{\text{finite subsets of } X\}$

4) Metric top.

(X, d) metric space i.e.

- $d(x, y) = d(y, x) \geq 0$
- $d(x, y) + d(y, z) \geq d(x, z)$
- $d(x, x) = 0$

$B(x, r) = \{ y \in X \mid d(x, y) < r \}$

$\mathcal{U} \in \mathcal{O} \iff \forall\, x \in \mathcal{U} \ \exists\, r > 0 \quad B(x, r) \subset \mathcal{U}.$

Check it !

E.g. $X = \mathbb{R}$ $\quad d(x,y) = |x-y|$

5) Zariski top. on k^n

k: field. $\quad f \in k[x_1, \dots x_n] = A$

$T \subset A \quad Z(T) := \{ \vec{x} \in k^n \mid f(\vec{x}) = 0 \ \forall f \in T \}$

$T_1, T_2 \subset A \quad T_1 \cdot T_2 = \{ f_1 f_2 \mid f_i \in T_i \}$

$\bigcap_{\lambda \in \Lambda} Z(T_\lambda) = Z(\bigcup_{\lambda \in \Lambda} T_\lambda) \qquad Z(T_1) \cup Z(T_2)$

$= Z(T_1 T_2)$

$\mathcal{C} := \{ Z(T) \mid T \subset A \}$

p.f. of $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$

$x \in Z(T_1) \iff f(x) = 0 \ \forall f \in T_1 \implies g(x) = 0 \ \forall g \in T_1 T_2$

$\Updownarrow$

$x \in Z(T_1 T_2)$

similarly $Z(T_2) \subset Z(T_1 T_2)$

"$\supset$" equivalently we show $(Z(T_1) \cup Z(T_2))^c \subset Z(T_1 T_2)^c$

$x \in (Z(T_1) \cup Z(T_2))^c \iff \exists f_1 \in T_1, f_2 \in T_2$ s.t.

$f_1(x) \neq 0 \ f_2(x) \neq 0$ then $f_1 f_2(x) \neq 0$

properties of Zariski top

$A = k[x_1, \cdots, x_n]$

a) $Z(T) = Z(I(T))$ $\quad I(T) = \left\{ \sum_{i=1}^{m} f_i g_i \;\middle|\; \begin{matrix} f_i \in T \\ g_i \in A \end{matrix} \right\}$

I, ideal generated by T,

b) I_α ideal of A

$$\bigcap_\alpha Z(I_\alpha) = Z(\sum I_\alpha)$$

$\sum I_\alpha$: $\underbrace{\text{sum of ideal}}_{\text{an ideal}}$

$\| $

$\left\{ \sum_{\alpha=1}^{m} f_\alpha \;\middle|\; f_\alpha \in I_\alpha \right\}$

$$\bigcup_{\alpha=1}^{m} Z(I_\alpha) = Z\left(\bigcap_{\alpha=1}^{m} I_\alpha\right)$$

c) $k = \mathbb{R}$ $\quad n = 1$ $\qquad A = \mathbb{R}[x]$

Zariski top on $\mathbb{R}$ $=$ cofinite top

$I \subset \mathbb{R}[x]$ ideal $\exists\, g \in \mathbb{R}[x]$ s.t

$I = (g) = \{ fg \mid f \in \mathbb{R}[x] \}$

$Z(g) = Z(I) =$ roots of g in $\mathbb{R}$

fundamental thm of alg $\Rightarrow$ $Z(g)$ is finite

$a_1, \cdots, a_N \in \mathbb{R}$ $\quad f = \prod_{i=1}^{N} (x - a_i) \in \mathbb{R}[x]$

$Z(f) = \{a_1, \cdots, a_N\}$