# MATH3301 Tutorial 9

1. Let $R$ be a commutative ring with unity, and $R[t]$ be the polynomial ring over $R$.
   (a) Show that $R[t]$ a commutative ring with unity and $R[t]^{\times} = R^{\times}$.
   (b) Give an example to illustrate that $\deg(fg) \neq \deg f + \deg g$ can happen with $f, g \in R[t]$.
   (c) Suppose $R$ is an integral domain. Show that
   (i) $R[t]$ an integral domain, and (ii) $\deg(fg) = \deg f + \deg g$, $\forall\ f, g \in R[t]$.
   (d) Give an example of an integral domain $R[t]$ where the division algorithm does not hold.

2. (a) Let $S$ be a commutative ring and $R$ be a subring of $S$. For any $a \in S$, define
   $R[a]$ to be the smallest subring of $S$ that contains $a$ and all elements in $R$.
   Show that every element of $R[a]$ is of the form $r_0 + r_1 a + \cdots + r_n a^n$ for some $n \in \mathbb{N}$ and some $r_0, \cdots, r_n \in R$.
   [For example, letting $i = \sqrt{-1}$, the smallest subring of $\mathbb{C}$ containing $i$ and $\mathbb{Z}$ is
   $$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}.$$
   More generally, we define: for $x_1, \cdots, x_m \in S$, $R[x_1, \cdots, x_n] := R[x_1, \cdots, x_{n-1}][x_n]$. ]
   (b) Consider $S = \mathbb{C}$ and $R = \mathbb{Z}$. Describe the elements in the (sub-)rings below:
   i. $\mathbb{Z}[\sqrt{2}]$,
   ii. $\mathbb{Z}[w]$ ($w = e^{2\pi i/3}$),
   iii. $\mathbb{Z}[2^{1/3}, 3^{1/2}]$,
   iv. $\mathbb{Z}[\pi]$ ($\pi$ = area of the unit circle)
   [Take for granted that *there is no nonzero polynomial $P \in \mathbb{Z}[t]$ such that $P(\pi) = 0$.*]
   (c) Let $\mathbb{Z}[t]$ be the polynomial ring over $\mathbb{Z}$. Find elements $a, b$ in $\mathbb{C}$ such that $\mathbb{Z}[a] \cong \mathbb{Z}[t]$ (i.e. $\mathbb{Z}[a]$ is isomorphic to $\mathbb{Z}[t]$ as rings) but $\mathbb{Z}[b] \ncong \mathbb{Z}[t]$. Justify your answer.

3. Let $R$ be a commutative ring. Suppose $L, M, J_1, J_2$ are ideals of $R$.
   (a) Show that $LM \subset L \cap M$. Give an example that $LM \neq L \cap M$.
   (b) Suppose $J_1 + J_2 = R$. Show
   (i) $J_1 \cap J_2 = J_1 J_2$, and
   (ii) $\forall\ a, b \in R$, $(a + J_1) \cap (b + J_2) \neq \varnothing$.

4. (**Chinese Remainder Theorem**)
Let $R$ be a commutative ring, and $J_1, J_2$ be its ideals such that $J_1 + J_2 = R$. Show that

$$R/(J_1 \cap J_2) \cong R/J_1 \times R/J_2 \qquad \text{as rings.}$$

[Hint: Consider the ring homomorphism $f : R \longrightarrow R/J_1 \times R/J_2$, $x \mapsto (\overline{x}, \overline{x})$ where $\overline{x}$ and $\overline{x}$ are the images of $x$ under the natural projections onto $R/J_1$ and $R/J_2$ respectively. ]

5. Figure out the connection between the question below and the Chinese Remainder Theorem.

在《孫子算經》下卷裡有以下問題：今有物不知其數，三三數之剩二，五五數之剩三，七七數之剩二，問物幾何？這個問題俗稱為「韓信點兵」，表述如下:『韓信點兵，三個三個一數餘二人，五個五個數餘三人，七個七個數餘二人，問兵有幾人？』

In the chinese mathematical treatise *The Mathematical Classic of Sunzi*, there is a problem equivalent to a folklore episode of the ancient Chinese General Han Xin's[†] Soldier Roll Call: When the soldiers stood 3 in a row, there were 2 soldiers left over. When they lined up 5 in a row, there were 3 soldiers left over. When they lined up 7 in a row, there were 2 soldiers left over. How many soldiers were there?

$J_1 = 3\mathbb{Z}$ $\qquad$ $J_2 = 5\mathbb{Z}$

*End*

[†] Han Xin (230-196 BC) was a military general and contributed greatly to the founding of the Han dynasty.