20240903 MATH3541 NOTE 1[1]

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Contents

1 Introduction

Today, Prof. Hua introduced the definitions and examples of topological space. [2] Here, the Zariski topology is new to me. I found that whether it is Hausdorff depends on whether the corresponding field is finite. [3] Hence, I take this note to record them.

2 Definitions of Topological Space

Definition 2.1. (**Open Definition of Topological Space**)
Let X be a set, $\mathcal{O}$ be a subset of $\mathcal{P}(X)$. If:
1. $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$;
2. For all $(O_k)_{k=1}^m$ in $\mathcal{O}$, $\cap_{k=1}^m O_k \in \mathcal{O}$;
3. For all $(O_\lambda)_{\lambda \in I}$ in $\mathcal{O}$, $\cup_{\lambda \in I} O_\lambda \in \mathcal{O}$,

then $\mathcal{O}$ is a topology on X. For all $O \in \mathcal{O}$, U is open and O^c is closed.

Definition 2.2. (**Closed Definition of Topological Space**)
Let X be a set, $\mathcal{C}$ be a subset of $\mathcal{P}(X)$. If:
1. $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$;
2. For all $(C_k)_{k=1}^m$ in $\mathcal{C}$, $\cup_{k=1}^m C_k \in \mathcal{C}$;
3. For all $(C_\lambda)_{\lambda \in I}$ in $\mathcal{C}$, $\cap_{\lambda \in I} C_\lambda \in \mathcal{C}$,

then $\mathcal{C}$ is a topology on X. For all $C \in \mathcal{C}$, C is closed and C^c is open.

Proposition 2.3. The two definitions introduced above are logically equivalent.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset^c = X$ and $X^c = \emptyset$,
so the first statements are equivalent;
Part 2: $(\cap_{k=1}^m O_k)^c = \cup_{k=1}^m O_k^c$ and $(\cup_{k=1}^m C_k)^c = \cap_{k=1}^m C_k^c$,
so the second statements are equivalent;
Part 3: $(\cup_{\lambda \in I} O_\lambda)^c = \cap_{\lambda \in I} O_\lambda^c$ and $(\cap_{\lambda \in I} C_\lambda)^c = \cup_{\lambda \in I} C_\lambda^c$,
so the third statements are equivalent,
Combine the three parts above, we've proven that the two definitions introduced above are logically equivalent. Quod. Erat. Demonstrandum. □

3 Examples of Topological Space

Definition 3.1. (**Definition of Indiscrete Topology**)
Let X be a set. We define $\mathcal{O} = \{\emptyset, X\}$ to be the indiscrete topology on X.

Proposition 3.2. The indiscrete topology induces a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset \in \{\emptyset, X\} = \mathcal{O}$ and $X \in \{\emptyset, X\} = \mathcal{O}$;
Part 2: $\emptyset \cap \emptyset = \emptyset$ and $\emptyset \cap X = \emptyset$ and $X \cap \emptyset = \emptyset$ and $X \cap X = X$;
Part 3: $\emptyset \cup \emptyset = \emptyset$ and $\emptyset \cup X = X$ and $X \cup \emptyset = X$ and $X \cup X = X$,
Combine the three parts above, we've proven that the indiscrete topology induces a topological space. Quod. Erat. Demonstrandum. ☐

Definition 3.3. (**Definition of Discrete Topology**)
Let X be a set. We define $\mathcal{O} = \mathcal{P}(X)$ be the discrete topology on X.

Proposition 3.4. The discrete topology induces a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset \in \mathcal{P}(X) = \mathcal{O}$ and $X \in \mathcal{P}(X) = \mathcal{O}$;
Part 2: For all $(O_k)_{k=1}^m$ in $\mathcal{O} = \mathcal{P}(X)$, $\cap_{k=1}^m O_k \in \mathcal{P}(X) = \mathcal{O}$;
Part 3: For all $(O_\lambda)_{\lambda \in I}$ in $\mathcal{O} = \mathcal{P}(X)$, $\cup_{\lambda \in I} O_\lambda \in \mathcal{P}(X) = \mathcal{O}$,
Combine the three parts above, we've proven that the discrete topology induces a topological space. Quod. Erat. Demonstrandum. ☐

Definition 3.5. (**Definition of Finite Complement Topology**)
Let X be a set. We define $\mathcal{C} = \{C, X \in \mathcal{P}(X) : C \text{ is a finite set}\}$ as the finite complement topology on X.

Proposition 3.6. The finite complement topology induces a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset \in \mathcal{P}(X)$ is finite, so $\emptyset \in \mathcal{C}$, and by definition, $X \in \mathcal{C}$;
Part 2: For all $(C_k)_{k=1}^m$ in $\mathcal{C}$:
Case 2.1: If some $C_k = X$, then $\cup_{k=1}^m C_k = X \in \mathcal{C}$;
Case 2.2: If all $C_k \neq X$, then $\cup_{k=1}^m C_k \in \mathcal{P}(X)$ is finite, so $\cup_{k=1}^m C_k \in \mathcal{C}$,
this implies $\mathcal{C}$ is closed under finite union;
Part 3: For all $(C_\lambda)_{\lambda \in I}$ in $\mathcal{C}$:
Case 3.1: If all $C_\lambda = X$, then $\cap_{\lambda \in I} C_\lambda = X \in \mathcal{C}$;
Case 3.2: If some $C_\lambda \neq X$, then $\cap_{\lambda \in I} C_\lambda \in \mathcal{P}(X)$ is finite, so $\cap_{\lambda \in I} C_\lambda \in \mathcal{C}$,
this implies $\mathcal{C}$ is closed under arbitrary intersection,
Combine the three parts above, we've proven that the finite complement topology induces a topological space. Quod. Erat. Demonstrandum. ☐

Definition 3.7. (**Definition of Metric Space Topology**)
Let X be a metric space equipped with metric d. We define $\mathcal{O} = \{O \in \mathcal{P}(X) : \forall x \in O, \exists r > 0, B(x, r) \subseteq O\}$ as the metric space topology on X.

Proposition 3.8. The metric space topology induces a topological space.

Proof. We may divide our proof into three parts.
Part 1: For $\emptyset$, the hypothesis $\forall x \in \emptyset$ is a contradiction, so the conditional is true, which implies $\emptyset \in \mathcal{O}$, and for X, $\forall x \in X, \exists 1 > 0, B(x, 1) \subseteq X$, which implies $X \in \mathcal{O}$;
Part 2: For all $(O_k)_{k=1}^m$ in $\mathcal{O}$, for all $x \in \cap_{k=1}^m O_k$, each O_k gives $r_k > 0$, such that $B(x, r_k) \subseteq O_k$. Hence, there exists $r = \min(r_k)_{k=1}^n > 0$, such that $B(x, r) \subseteq \cap_{k=1}^m O_k \in \mathcal{O}$, this implies $\mathcal{O}$ is closed under finite intersection;
Part 3: For all $(O_\lambda)_{\lambda \in I}$ in $\mathcal{O}$, for all $x \in \cup_{k=1}^m O_k$, x is in some O_k, so there exists $r_k > 0$, such that $B(x, r_k) \subseteq O_k \subseteq \cup_{k=1}^m O_k \in \mathcal{O}$, this implies $\mathcal{O}$ is closed under arbitrary union,
Combine the three parts above, we've proven that the metric space topology induces a topological space. Quod. Erat. Demonstrandum. □

Definition 3.9. (**Definition of Zariski Topology**)
Let $\mathbb{F}$ be a field, $(x_l)_{l=1}^n$ be n independent indeterminates, and $\mathbb{F}[x_l]_{l=1}^n$ be the corresponding polynomial ring. We define $\mathcal{C} = \{C \in \mathcal{P}(\mathbb{F}^n) : \exists T \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n), C \text{ is the solution set of } T\}$ as the Zariski topology on $\mathbb{F}^n$.

Proposition 3.10. The Zariski topology induces a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset \in \mathcal{P}(\mathbb{F}^n)$ is the solution set of $\{1\}$, so $\emptyset \in \mathcal{C}$,
and $\mathbb{F}^n \in \mathcal{P}(\mathbb{F}^n)$ is the solution set of $\{0\}$, so $\mathbb{F}^n \in \mathcal{C}$;
Part 2: For all $(C_k)_{k=1}^m$ in $\mathcal{C}$, assume that each C_k is the solution set of $T_k \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n)$:
Step 2.1: In this step, we prove that $\cup_{k=1}^m C_k$ is contained in the solution set of $\Pi_{k=1}^m T_k$.
Let's prove this directly.
For all $(\xi_l)_{l=1}^n$ in $\cup_{k=1}^m C_k$, $(\xi_l)_{l=1}^n$ is in some C_k, so all polynomial in T_k vanishes at $(\xi_l)_{l=1}^n$. Hence, all polynomial in $\Pi_{k=1}^m C_k$ also vanishes at $(\xi_l)_l^n$, this implies $\cup_{k=1}^m C_k$ is contained in the solution set of $\Pi_{k=1}^m T_k$.
Step 2.2: In this step, we prove that the solution set of $\Pi_{k=1}^m T_k$ is contained in $\cup_{k=1}^m C_k$.
Let's prove the contrapositive.
For all $(\xi_l)_{l=1}^n$ not in $\cup_{k=1}^m C_k$, it is not in each C_k, so each C_k gives $f_k(x_l)_{l=1}^n$ which doesn't vanish at $(\xi_l)_{l=1}^n$. Hence, some polynomial $\Pi_{k=1}^m f_k(x_l)_{l=1}^n$ in $\Pi_{k=1}^m T_k$ doesn't vanish at $(\xi_l)_{l=1}^n$, this implies $(\xi_l)_{l=1}^n$ is not in the solution set of $\Pi_{k=1}^m T_k$.
The two steps above shows $\cup_{k=1}^m C_k$ is indeed a solution set of some $\Pi_{k=1}^m T_k \in \mathcal{P}(\mathbb{F}[x_l]_l^n)$, which implies $\mathcal{C}$ is closed under finite union.
Part 3: For all $(C_\lambda)_{\lambda \in I}$ in $\mathcal{C}$, assume that each C_λ is the solution set of $T_\lambda \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n)$:
Step 3.1: In this step, we prove that $\cap_{\lambda \in I} C_\lambda$ is contained in the solution set of $\cup_{\lambda \in I} T_\lambda$.
Let's prove this directly.
For all $(\xi_l)_{l=1}^n$ in $\cap_{\lambda \in I} C_\lambda$, $(\xi_l)_{l=1}^n$ is in each C_λ, so all polynomial in T_λ vanishes at $(\xi_l)_{l=1}^n$. Hence, all polynomial in $\cup_{\lambda \in I} T_\lambda$ vanishes at $(\xi_l)_{l=1}^n$, this implies $\cap_{\lambda \in I} C_\lambda$ is contained in the solution set of $\cup_{\lambda \in I} T_\lambda$.

Step 3.2: In this step, we prove that the solution set of $\cup_{\lambda\in I}T_\lambda$ is contained in $\cap_{\lambda\in I}C_\lambda$. Let's prove the contrapositive.
For all $(\xi_l)_{l=1}^n$ not in $\cap_{\lambda\in I}C_\lambda$, $(\xi_l)_{l=1}^n$ is not in some C_λ, so some polynomial $f_\lambda(x_l)_{l=1}^n$ in T_λ doesn't vanish at $(\xi_l)_{l=1}^n$. Hence, some polynomial $f_\lambda(x_l)_{l=1}^n$ in $\cup_{\lambda\in I}T_\lambda$ doesn't vanish at $(\xi_l)_{l=1}^n$, this implies $(\xi_l)_{l=1}^n$ is not in the solution set of $\cup_{\lambda\in I}T_\lambda$.
The two steps above shows $\cap_{\lambda\in I}C_\lambda$ is indeed a solution set of some $\cup_{\lambda\in I}T_\lambda \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n)$, which implies $\mathcal{C}$ is closed under arbitrary intersection.
Combine the three parts above, we've proven that the Zariski topology induces a topological space. Quod. Erat. Demonstrandum. □

By the way, Prof. Hua also mentioned in class that for any $T \in \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n)$, the solution set of T is the same thing as the solution set of gen T. As ideal has better structure than an arbitrary set, without loss of generality, we may assume that T is an ideal when dealing with the Zariski topological space $(\mathbb{F}^n, \mathcal{C})$.

4 Hausdorff Space

Definition 4.1. (**Definition of Hausdorff Space**)
Let $(X, \mathcal{O})$ be a topological space. If for all $x_1, x_2 \in X$, there exist $O_1, O_2 \in \mathcal{O}$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$, then $(X, \mathcal{O})$ is Hausdorff.

Proposition 4.2. If a topological space $(X, \mathcal{O})$ is induced by a metric d, then it is Hausdorff.

Proof. For all $x_1, x_2 \in X$, there exist $B(x_1, d(x_1, x_2)/2), B(x_2, d(x_1, x_2)/2) \in \mathcal{O}$, such that $x_1 \in B(x_1, d(x_1, x_2)/2)$ and $x_2 \in B(x_2, d(x_1, x_2)/2)$ and $B(x_1, d(x_1, x_2)/2) \cap B(x_2, d(x_1, x_2)/2) = \emptyset$, so $(X, \mathcal{O})$ is Hausdorff. Quod. Erat. Demonstrandum. □

We will encounter "manifold" afterwards, which "locally resembles $\mathbb{R}^n$".
One criterion for a topological space to be a manifold is that it is Hausdorff.

Proposition 4.3. If X has at least two elements x_1, x_2, then the indiscrete topological space $(X, \mathcal{O})$ is not Hausdorff.

Proof. There exist $x_1, x_2 \in X$, such that for all $O_1, O_2 \in \mathcal{O}$:

$$x_1 \in O_1 \text{ and } x_2 \in O_2 \implies O_1 = O_2 = X \implies O_1 \cap O_2 \neq \emptyset$$

So the three conditions fail to hold simultaneously, which implies $(X, \mathcal{O})$ is not Hausdorff. Quod. Erat. Demonstrandum. □

Proposition 4.4. The discrete topological space $(X, \mathcal{O})$ is induced by the following metric $d : X \times X \to \mathbb{R}$:

$$d(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_1 = x_2; \\ 1 & \text{if} \quad x_1 \neq x_2; \end{cases}$$

So $(X, \mathcal{O})$ is Hausdorff.

Proof. It suffices to show that the metric topology $\mathcal{O}$ contains $\mathcal{P}(X)$.
For all $O \in \mathcal{P}(X)$, for all $x \in O$, there exists $1 > 0$, such that $B(x, 1) = \{x\} \subseteq O$, so $O \in \mathcal{O}$, which implies $\mathcal{P}(X) \subseteq \mathcal{O}$. Quod. Erat. Demonstrandum. □

Proposition 4.5. If X is finite, then the finite complement topological space $(X, \mathcal{O})$ and the discrete topological space $(X, \mathcal{O})$ are identical, thus Hausdorff.

Proposition 4.6. If X is infinite, then the finite complement topological space $(X, \mathcal{C})$ is not Hausdorff.

Proof. There exist $x_1, x_2 \in X$, such that for all $C_1, C_2 \in \mathcal{C}$:

$$x_1 \notin C_1 \text{ and } x_2 \notin C_2 \implies C_1, C_2 \text{ are finite} \implies C_1 \cup C_2 \text{ is finite} \implies C_1 \cup C_2 \neq X$$

So the three conditions fail to hold simultaneously, which implies $(X, \mathcal{C})$ is not Hausdorff.
Quod. Erat. Demonstrandum. □

Proposition 4.7. If a field $\mathbb{F}$ is finite, then the Zariski topological space $(\mathbb{F}^n, \mathcal{C})$ and the discrete topological space $(\mathbb{F}^n, \mathcal{O})$ are identical, thus Hausdorff.

Proof. It suffices to show that the Zariski topology $\mathcal{C}$ contains all singleton $(\xi_l)_{l=1}^n$.
Define $T = \{x_l - \xi_l\}_{l=1}^n \subseteq \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n)$. The solution set of T is $\{(\xi_l)_{l=1}^n\} \in \mathcal{C}$.
Quod. Erat. Demonstrandum. □

To prove that the Zariski topological space $(\mathbb{F}^n, \mathcal{C})$ is not Hausdorff when $\mathbb{F}$ is infinite, we shall prove several lemmas.

Lemma 4.8. If a field $\mathbb{F}$ is infinite, then the Zariski topological space $(\mathbb{F}, \mathcal{C})$ and the finite complement topological space $(\mathbb{F}, \mathcal{C})$ are identical, thus Hausdorff.

Proof. For all $C \in \mathcal{P}(X) \backslash \{X\}$:

$$\begin{aligned} C \text{ is in the Zariski topology} \quad &\iff \quad \exists f(x_l)_{l=1}^n \in \mathbb{F}[x_l]_{l=1}^n \text{ with } \deg f(x_l)_{l=1}^n > -\infty \\ &\qquad\qquad C \text{ is the solution set of } f(x_l)_{l=1}^n \\ &\iff \quad C \text{ is finite} \\ &\iff \quad C \text{ is in the finite complement topology} \end{aligned}$$

Quod. Erat. Demonstrandum. □

Lemma 4.9. If a topological space $(X_1, \mathcal{O}_1)$ is Hausdorff, then its subspace $(X_2, \mathcal{O}_2)$ is also Hausdorff.

Proof. For all $x_1, x_2 \in X_2$, they can be regarded as points in X_1, so there exist $O_1, O_2 \in \mathcal{O}_1$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. Hence, there exists $O_1 \cap X_2, O_2 \cap X_2 \in \mathcal{O}_2$, such that $x_1 \in O_1 \cap X_2$ and $x_2 \in O_2 \cap X_2$ and $(O_1 \cap X_2) \cap (O_2 \cap X_2) = \emptyset$.
Quod. Erat. Demonstrandum. □

Proposition 4.10. If a field $\mathbb{F}$ is infinite, then the Zariski topological space $(\mathbb{F}^n, \mathcal{C})$ is not Hausdorff, where $n \geq 2$.

Proof. Assume to the contrary that the Zariski topological space $(\mathbb{F}^n, \mathcal{C})$ is Hausdorff, then its subspace $(\mathbb{F} \times \{0\}^{n-1}, \mathcal{C}')$ is Hausdorff. But this subspace is homeomorphic to the Zariski topological space $(\mathbb{F}, \mathcal{C}'')$, which is already proven to be not Hausdorff, so we've arrived at a contradiction.
Hence, the assumption is false, and we've proven that the Zariski topological space $(\mathbb{F}^n, \mathcal{C})$ is not Hausdorff. Quod. Erat. Demonstrandum. □

References

[1] H. Ren, "Template for math notes," 2021.

[2] P. J.-H. L. Dr. Tak Wing Ching and P. Z. Hua, "Introduction to topology," 2024, lecture notes, HKU, Spring 2024.

[3] E. Wofsey, "The zariski topology on $\mathbb{F}^n$ is never hausdorff if $n > 0$ and $\mathbb{F}$ is infinite." https://math.stackexchange.com/questions/1918669/when-is-the-zariski-topology-t-2, Sep 2016.