THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 3

Problem 1. In the following problems, find the compatibility conditions for the existence. In other words, if the problem has a solution, then what is/are the constraint(s) for the data? You are asked to find the compatibility conditions only, and not required to solve the problems.

(i) Consider the following Neumann problem for the Poisson's equation:

$$\begin{cases} -\Delta u = Cy^2 & \text{in } B_2 \\ x\partial_x u + y\partial_y u = \sqrt{4 - x^2} & \text{on } \partial B_2 \end{cases}$$

where $u := u(x, y)$ is the unknown, C is a given constant, and the ball

$$B_2 := \left\{(x, y) \in \mathbb{R}^2;\ \sqrt{x^2 + y^2} < 2\right\}.$$

(ii)

$$\begin{cases} \partial_x u + 2\partial_y u = 0 & \text{for } 0 < x < 2 \text{ and } -\infty < y < \infty \\ u(0, y) = g(y) \\ u(2, y) = h(y), \end{cases}$$

where $u := u(x, y)$ is the unknown, g and h are given boundary conditions.

(iii)

$$\begin{cases} (x + 1)\partial_x u + (y - 1)\partial_y u = 0 & \text{for } x > 0 \text{ and } y > 0 \\ u(0, y) = g(y) \\ u(x, 0) = h(x), \end{cases}$$

where $u := u(x, y)$ is the unknown, g and h are given boundary conditions.

Problem 2. Consider the following boundary value problems:

(i)
$$\begin{cases} \partial_x u + 2x\partial_y u = 2u & \text{for } x > 0 \text{ and } y > 0 \\ u|_{x=0} = e^y - 1 \\ u|_{y=0} = x^2, \end{cases}$$

where $u := u(x, y)$ is the unknown.

Solve the above boundary value problem. Express your final answer u in terms of x and y only.

(ii)
$$\begin{cases} \partial_x u + 2x(y+1)\partial_y u = 2 & \text{for } x > 0 \text{ and } y > 0 \\ u|_{x=0} = g(y) \\ u|_{y=0} = h(x), \end{cases}$$

where $u := u(x, y)$ is the unknown, h, g are given boundary data respectively.

Under the compatibility condition $g(0) = h(0) = 0$, solve the above boundary value problem. Express your final answer u in terms of x, y, g and h only.

Problem 3. Consider the following problems:

(i) Using the divergence theorem, show that the following boundary value problem is NOT solvable:

$$\begin{cases} \partial_x^2 u + \partial_y u = 3 & \text{in } [0,1] \times [0,1] \\ \partial_x u(0, y) = \partial_x u(1, y) \equiv 1 \\ u(x, 0) = u(x, 1) \equiv 0, \end{cases}$$

where $u := u(x, y)$ is the unknown.

(ii) Consider

$$\begin{cases} t\partial_t u + 2\partial_x u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi(x). \end{cases} \tag{1}$$

Prove that if $\phi(x) := x^4$, then (1) has no solution.

(iii) Consider

$$\begin{cases} \partial_t u - (x+t+1)\partial_x u = u^8 & \text{for } x > 0 \text{ and } t > 0 \\ u|_{t=0} = 6x \\ u|_{x=0} = -4t^2. \end{cases} \tag{2}$$

Prove that (2) has no solution.