Sheaf Cohomology.

In topology, let X be a topological manifold, we have the singular (co)homology groups.

e.g. $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$ for X compact Riemann surface of genus g. $H_p = H_p^{sig.}$

Cellular (co)homology. Cells $\underset{\text{homeom}}{\cong} \mathbb{R}^m$.

Eg: $H^\bullet(\mathbb{P}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, i.e.

$H^{2k}(\mathbb{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ for $0 \leq k \leq n$ and 0 otherwise

Čech cohomology for $\mathbb{Z}$., X topological mnfd.

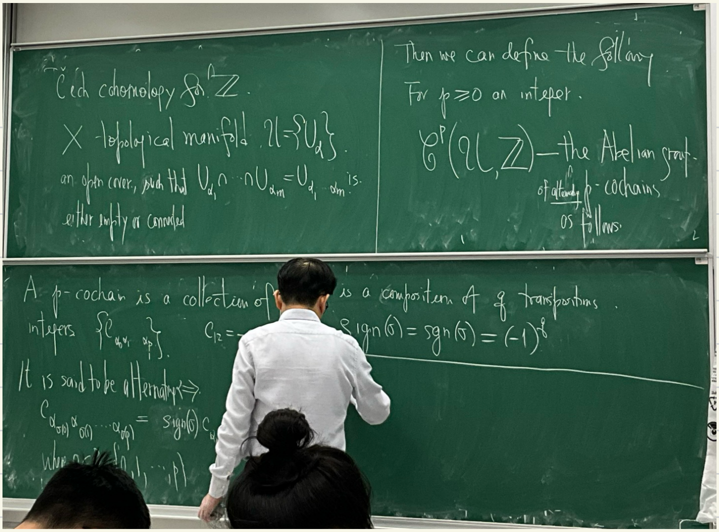
Take $\mathcal{U} = \{U_\alpha\}$ an open cover, such that

$$U_{\alpha_1} \cap \cdots \cap U_{\alpha_m} = U_{\alpha_1,\cdots,\alpha_m} \text{ is either}$$

empty or connected.

Then we can define the following. For $p \geq 0$ an integer

$\mathcal{C}^p(\mathcal{U},\mathbb{Z})$ - the Abelian group of p-cochains as follows.


Čech cohomology for Z.
X topological manifold. 𝒰 = {U_α}.
an open cover, such that U_α1 ∩ ⋯ ∩ U_αm = U_α1⋯αm is
either empty or connected
Then we can define the following
For p ≥ 0 an integer.
C^p(𝒰, Z) — the Abelian group of alternating p-cochains as follows.
A p-cochain is a collection of integers {c_α0α1⋯αp}.
is a composition of q transpositions.
sign(σ) = sgn(σ) = (−1)^q
It is said to be alternating ⇔
c_ασ(0)ασ(1)⋯ασ(p) = sign(σ) c
where σ {0, 1, ⋯, p}

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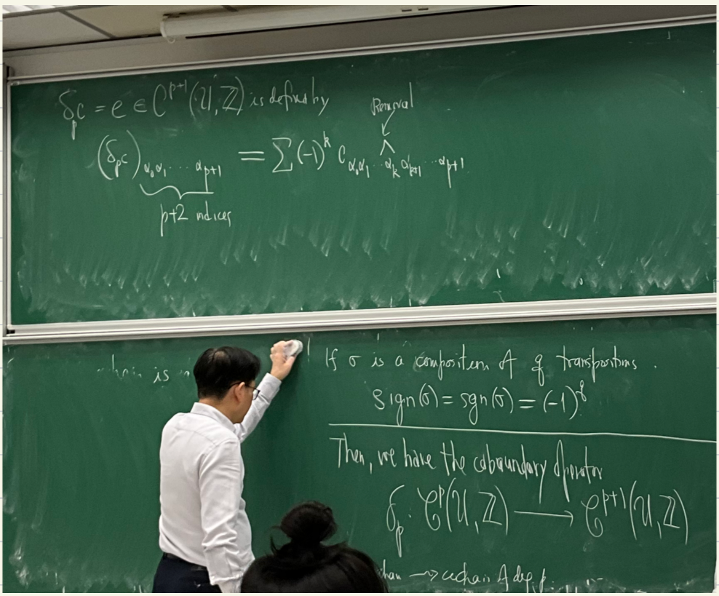
δ_p c = e ∈ C^(p+1)(𝒰, Z) is defined by
Removal
(δ_p c)_α0α1⋯αp+1 = Σ(−1)^k c_α0α1⋯α̂k⋯αp+1
p+2 indices
If σ is a composition of q transpositions.
sign(σ) = sgn(σ) = (−1)^q
Then, we have the coboundary operator
δ_p : C^p(𝒰, Z) ⟶ C^(p+1)(𝒰, Z)

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Definition

(a) $c \in \mathcal{C}^p(\mathcal{U},\mathbb{Z})$ is called a p-cocycle $\iff \delta c = 0$. written $Z^p(\mathcal{U},\mathbb{Z})$

(b) $c \in \mathcal{C}^p(\mathcal{U},\mathbb{Z})$ is called a p-coboundary $\iff$ $\exists\, b \in \mathcal{C}^{p-1}(\mathcal{U},\mathbb{Z})$ such that $c = \delta_{p-1} b$ $(B^p(\mathcal{U},\mathbb{Z}))$

Lemma $B^p(\mathcal{U},\mathbb{Z}) \subset Z^p(\mathcal{U},\mathbb{Z})$ as a subgroup.

Pf. $c = \delta_{p-1} b \Rightarrow \delta_p c = \delta_p(\delta_{p-1}(b)) = (\delta_p \circ \delta_{p-1})(b) = 0$ by lemma

②

Theorem (Assumed).

Let $\mathcal{U}$ be an acyclic cover for $\mathbb{Z}$ over X (e.g. if $U_{\alpha_0 \dots \alpha_p}$ is contractible for all $p, \alpha_0, \dots, \alpha_p$).

Then $\check{H}^p(X,\mathbb{Z}) \cong H^p(\mathcal{U},\mathcal{F})$

Theorem (Assumed)

$$\check{H}^p(X,\mathbb{Z}) \cong H^p_{sing}(X,\mathbb{Z})$$

Observation $c \in \mathcal{C}^0(X,\mathbb{Z})$, $\delta c \in \mathcal{C}^1(X,\mathbb{Z})$

$c = (c_{\alpha_0})$ (1 index) $\quad \delta c = e$ (2 index) $\quad e_{\alpha_0\alpha_1} = c_{\alpha_1} - c_{\alpha_0}$

$\delta c = 0 \iff c_{\alpha_1} = c_{\alpha_0}$ on $U_{\alpha_0\alpha_1}$

$\Rightarrow \check{H}^0(X,\mathbb{Z}) = \mathbb{Z}$, for X connected

①

Definition

(a) $H^p(\mathcal{U},\mathbb{Z}) \overset{\text{def}}{=} Z^p(\mathcal{U},\mathbb{Z}$ … $(\mathcal{U},\mathbb{Z})$

(b) $\check{H}^p(X,\mathbb{Z}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U}$ … $(\mathcal{U},\mathbb{Z}) \xrightarrow{Res} H^p(\mathcal{V},\mathbb{Z})$

… $\mathcal{U} = \{U_\alpha\}$, $\mathcal{V} = \{V_i\}$ $\forall i$, $V_i \subset U_{\alpha(i)}$

$\mathcal{U}, \mathcal{V}$

Sheaf (see handout).

Remember we proved that $H^1(\mathcal{U},\mathcal{O})=0$

for $\mathcal{U}=\{U_0,U_1\}$ on $\mathbb{P}^1$, Indeed.

$$\underset{\varphi_{01}}{\overset{h}{\parallel}} = s_0 - s_1 ,\quad \text{resolving a cocycle into a coboundary}$$

$\mathcal{U}$-acyclic $\Rightarrow H^1(\mathbb{P}^1,\mathcal{O})=0$.

Here $H^p(X,\mathcal{F})$ means $\check{H}^p(X,\mathcal{F})$.

__Proposition__: On a complex manifold X ($\exists$ acyclic covers)

$H^1(X,\mathcal{O})=0$. Mittag-Leffler is solvable.

__Pf__: Let $\mathcal{U}=\{U_\alpha\}_{\alpha\in A}$ be an acyclic cover for $\mathcal{O}_X$.

Suppose on each U_α given $f_\alpha\in M(U_\alpha)$.

$s_{\alpha\beta}=f_\alpha-f_\beta\in\Gamma(U_{\alpha\beta},\mathcal{O})$.

Now $H^1(X,\mathcal{O})=H^1(\mathcal{U},\mathcal{O})$ means.

$s_{\alpha\beta}\in\mathcal{B}^1(\mathcal{U},\mathcal{O})$, i.e. $\exists\ c=\{c_\alpha\}$ 0-cochain.

s.t. $\delta c=\{s_{\alpha\beta}\}$. $\nearrow$ need to show this cocycle.

$(\delta c)_{\alpha\beta}=c_\beta-c_\alpha$, $\quad f_\alpha-f_\beta=s_{\alpha\beta}=c_\beta-c_\alpha$ on $U_{\alpha\beta}$.

$$\Rightarrow\quad f_\alpha+c_\alpha=f_\beta+c_\beta \text{ on } U_{\alpha\beta}.$$

Hence defining $h = f_\alpha + c_\alpha$ on $\mathcal{U}_\alpha$, we have a meromophic function h on X, such that h has the prescribed principal parts f_α on U_α. //

$\mathcal{B}^1(\mathcal{U}, \mathcal{O}) = Z^n(\mathcal{U}, \mathcal{O})$ means

$$s_{\alpha\beta} = c_\beta - c_\alpha.$$

$\mathcal{O}^*$ multiplication $\mathcal{U}$ acyclic cover for $\mathcal{O}^*$

then $H^1(X, \mathcal{O}^*) = H^1(\mathcal{U}, \mathcal{O}^*) = \{1\}$. iff

any $\{P_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ is a 1-coboundary.

i.e. $\exists \{\psi_\alpha\}$ 0-cochain in $\mathcal{O}^*$ s.t. $\dfrac{\varphi_{\alpha\beta}}{1} = \dfrac{\psi_\beta}{\psi_\alpha}$

Claim $\varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{O}^*) \cong \mathrm{Pic}(X)$

<u>Rem</u>: $\mathrm{Pic}(X)$ trivial $\Leftrightarrow H^1(X, \mathcal{O}^*) = 1$

assume for simplicity on a fixed open covering (or refine it otherwise)

Sketch of idea: $L \cong L'$ isom of line bundles over X

where L (resp. L') given by $\{\varphi_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})\}$ (resp. $\varphi'_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$).

Since $\varphi_{\alpha\beta} \cdot \varphi_{\beta\alpha} = 1$, $\forall \alpha, \beta$. $\Rightarrow \{\varphi_{\alpha\beta}\}$ is a Čech-1-cocycle

$$L \cong L' \Leftrightarrow \frac{\varphi_{\alpha\beta}}{\varphi'_{\alpha\beta}} = \frac{\psi_\beta}{\psi_\alpha}, \text{ for some } \{\psi_\alpha \in \mathcal{O}^*(U_\alpha)\}.$$

i.e. $\varphi_{\alpha\beta}$ and $\varphi'_{\alpha\beta}$ differs by a Čech-1-coboundary.

$\Rightarrow [L] \mapsfrom [\{\varphi_{\alpha\beta}\}]$ gives a map $\rightsquigarrow$ pass to direct limit //

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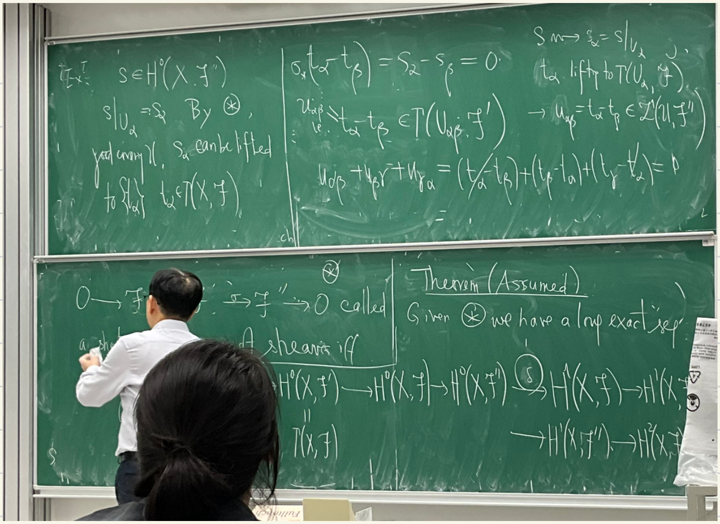

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Snack lemma

Exponential sequence on X. (Riemann Surface)

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to \{1\}$$

$$f_x^{\phi} \longmapsto e^{2\pi i f_x}.$$

To determine $H^1(X, \mathcal{O}^*) \cong \mathrm{Pic}(X)$.

General fact : $\forall$ Riem. surface X, $H^2(X, \mathcal{O}_X)$.

LES

$$H^1(X,\mathcal{O}) \to \underbrace{H^1(X,\mathcal{O}^*)}_{\mathrm{Pic}(X)} \xrightarrow{\sim} H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O})$$

Ex : $X = \mathbb{P}^1$

$$\cancel{H^1(\mathbb{P}^1,\mathcal{O})} \to \mathrm{Pic}(\mathbb{P}^1) \to \cancel{H^2(\mathbb{P}^1,\mathbb{Z})} \to \cancel{H^2(X,\mathcal{O})}$$

(previous result)

$= \mathbb{Z}$ by alg top (singular cohomology)

0

$\Rightarrow \mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}.$

Ex $X = \mathbb{C}/L$

first chern class.

$$\begin{tikzcd}
 & & \mathrm{Pic}(X). & & \\
H^1(X,\mathbb{Z}) \arrow[r, hook] \arrow[d, phantom, "\wr"] & H^1(X,\mathcal{O}) \arrow[r] & H^1(X,\mathcal{O}^*) \arrow[r, "c_1"] \arrow[u, phantom, "\parallel"] \arrow[d, phantom, "\cup"] & H^2(X,\mathbb{Z}) \arrow[r] & 0 \\
\mathbb{Z}^2 \arrow[r] & \mathbb{C} \arrow[r, two heads, "\text{image}"] & \mathbb{C}/\mathbb{Z}^2 = \mathrm{Jac}(X) & & \\
 & & \mathrm{Pic}_0(X). \arrow[u, phantom, "\parallel"] & &
\end{tikzcd}$$

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