THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 6

Due 3:30pm[1], November 8th (Friday), **in-class**.

Aim of this Homework: *In this assignment you will apply the energy methods to study the uniqueness, stability, and qualitative behavior of various types of PDE, including elliptic, parabolic and hyperbolic PDE. Additionally, you will also gain knowledge on computing several fundamental physical quantities, such as the total mass for chemical diffusion problems and the total thermal energy for heat transfer problems.*

Reading Assignment: Read the following material(s):

(i) Section 2.4-2.5 of the textbook.

Instruction: Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the "Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

Problem 1. In this problem you will be asked to justify the **finite speed of propagation** for the wave equation via using the **energy method**.

[1] You are expected to submit your homework before the beginning of Friday lecture **in-class**.

Let u satisfy the one-dimensional wave equation

$$\partial_{tt} u = \partial_{xx} u \qquad \text{for } -\infty < x < \infty \text{ and } t > 0. \tag{1}$$

Here, the speed of propagation c is normalized to be 1 for simplicity. The finite speed of propagation (with speed $c = 1$) for the wave equation (1) can be stated as follows:

> *For any finite spatial interval (a, b), if both initial data $u|_{t=0}$ and $\partial_t u|_{t=0}$ vanish on (a, b), then u must be identically zero in the region $\Delta := \{(x, t) \in (-\infty, \infty) \times [0, \infty);\ a + t \le x \le b - t\}$.*

You are asked to show the above assertion by completing Step (i)-(v) below. Let us begin by introducing a few quantities as follows: for any given finite interval (a, b), we define the local energy by

$$E(t) := \int_{a+t}^{b-t} e(t, x)\, dx,$$

where the energy density e is given by

$$e(t, x) := \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\partial_x u|^2.$$

Let us also denote the momentum density p by

$$p(t, x) := \partial_t u\, \partial_x u.$$

(i) Using the wave equation (1), prove that

$$\partial_t e = \partial_x p.$$

(ii) Show that

$$e \pm p = \frac{1}{2}(\partial_t u \pm \partial_x u)^2.$$

(iii) Using part (i), verify via a direct differentiation that

$$\frac{dE}{dt}(t) = p(t, b-t) - p(t, a+t) - e(t, b-t) - e(t, a+t).$$

(Hint: the following identity may be useful in the computation:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)\,dx = \int_{a(t)}^{b(t)} \partial_t f(t,x)\,dx + f(t,b(t))b'(t) - f(t,a(t))a'(t).$$

For the proof of this identity, see Theorem 3 in Appendix A.3 of the textbook for instance.)

(iv) Apply parts (ii) and (iii) to show that

$$\frac{dE}{dt} \le 0.$$

(v) Apply parts (iv) to prove that if $u|_{t=0} = \partial_t u|_{t=0} \equiv 0$ on (a,b), then

$$u \equiv 0 \qquad \text{in } \Delta$$

where the region $\Delta := \{(x,t) \in (-\infty,\infty) \times [0,\infty);\ a+t \le x \le b-t\}$.

Food for Thought. Which parts of the above computations will have to been adjusted, if the wave speed c is NOT normalized to be 1?

Food for Thought. Are you able to fill in the missing details in Section 5.3 of Lecture Slides?

Problem 2. Apply the **energy method** to prove the uniqueness for each of the following problems.

(i) Let $d \ge 2$ be an integer, and $\Omega \subset \mathbb{R}^d$ be an open, smooth and bounded set. Consider

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = g, \end{cases}$$

where f and g are given functions.

(ii) Consider

$$\begin{cases} \partial_t u - 5\partial_{xx} u = 11\partial_x u - 8u + f & \text{for } 0 < x < L,\ t > 0 \\ u|_{t=0} = \phi \\ u|_{x=0} = g \\ u|_{x=L} = h, \end{cases}$$

where f is a given source term, and ϕ, g and h are given data.

(iii) Consider

$$\begin{cases} \partial_{tt} u - 24\partial_{xx} u = -\sinh\left(11t + x^8\right)\partial_t u + f & \text{for } -L < x < L,\ t > 0 \\ u|_{t=0} = \phi \\ \partial_t u|_{t=0} = \psi \\ u|_{x=-L} = g \\ \partial_x u|_{x=L} = h, \end{cases}$$

where f is a given forcing term, and ϕ, ψ, g and h are given data.

Problem 3. The energy method can also apply to nonlinear problems. For example, in this problem, you will be asked to study the stability for the reaction-diffusion equation via the energy method. Consider the initial and boundary value problem (IVBP)

$$\begin{cases} \partial_t u - k\partial_{xx} u = f(u) & \text{for } 0 < x < L \text{ and } t > 0 \\ u|_{t=0} = \phi \\ \left.\left(-\partial_x u + \dfrac{8}{11}u\right)\right|_{x=0} = \dfrac{1}{2}t\left(1 + \operatorname{erf}\left(\dfrac{t}{\sqrt{2}}\right)\right) \\ u|_{x=L} = 0, \end{cases} \tag{2}$$

where the diffusivity $k > 0$ is a given constant, the initial data $\phi := \phi(x)$ is a given function, and the source term $f : \mathbb{R} \to \mathbb{R}$ accounts for all local chemical reactions and is given by

$$f(\alpha) := \tan^{-1}\alpha - \sinh^{-1}\alpha,$$

for all $\alpha \in \mathbb{R}$. Complete the following parts.

(i) Show that for any two real numbers α and β,

$$(f(\alpha) - f(\beta))(\alpha - \beta) \le 0.$$

(ii) For any $i = 1$ and 2, let the concentration u_i be the C^2 solution to the IVBP (2) with the initial data $\phi := \phi_i$. Apply the energy method to show that for any $t \ge 0$,

$$\|u_1 - u_2\|_{L^2([0,L])} \le \|\phi_1 - \phi_2\|_{L^2([0,L])},$$

where the L^2-norm $\|\cdot\|_{L^2[0,L]}$ is given by

$$\|g\|_{L^2[0,L]} := \left(\int_0^L |g(x)|^2 \; dx\right)^{\frac{1}{2}}.$$

Food for Thought. Are you able to show the stability for the IBVP (2) of reaction-diffusion equation for other non-homogeneous term f?

Problem 4. Do Problem 3 of Dec 2020 Final Exam.

The following problem(s) is/are *optional*:

Problem 5. Complete the following parts.

(a) Let the concentration $u := u(t,x)$ of the underlying chemical satisfy

$$\begin{cases} \partial_t u = \dfrac{4}{\pi}\partial_{xx} u + 2(t+1)u \quad \text{for } 0 < x < 3 \text{ and } t > 0 \\ u(0,x) = \dfrac{2x^5}{81} \\ \partial_x u(t,0) = 6\sqrt{\pi} \\ \partial_x u(t,3) = 7\sqrt{\pi}. \end{cases}$$

Compute the total mass

$$M(t) := \int_0^3 u(t,x)\, dx.$$

Please simplify your final answer by using the error function erf.
(Hint: *It will be useful, if you are able to find out the ordinary differential equation (in t) that the total mass M satisfies.*)

(b) Let L be a positive constant. Consider the heat equation with source

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} + x\left(e^{-t^2} + 1\right)\sin\frac{\pi x}{L} \tag{3}$$

in a rod $0 < x < L$, for $t > 0$. The initial condition is

$$u(0,x) = \sin^2\left(\frac{2023\pi x}{L}\right),$$

and the boundary conditions are

$$\frac{\partial u}{\partial x}(t,0) = te^{-t}, \tag{4}$$

and

$$\frac{\partial u}{\partial x}(t,L) = 3b - 2, \tag{5}$$

where b is a given constant. Denote the total thermal energy in the rod by

$$E(t) := \int_0^L u(t,x)\,dx.$$

(i) Compute $\frac{dE}{dt}$. (Hint: you do <u>not</u> have to solve for $u(t,x)$ first.)

(ii) Using part (i), find E.

(iii) For which value of b does the limit $\lim_{t\to+\infty} E(t)$ exist, and what is this limit?

Food for Thought. In order to have a steady-state solution to Problem (3)-(5), the existence of $\lim_{t\to+\infty} E(t)$ is a necessary condition. Do you know why?

Problem 6. Let u be a solution to the wave equation in a whole real line:

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi & \\ \partial_t u|_{t=0} = \psi, & \end{cases}$$

where the wave speed c is a positive constant. Decide whether the following statements are correct or not. To obtain full credit, you must explain your answers.

(i) If both ϕ and ψ are non-negative functions, then the solution u is also non-negative.

(ii) The following inequality always holds:

$$\max_{\substack{-\infty<x<\infty \\ t\geq 0}} |u(t,x)| \leq \max_{-\infty<x<\infty} |\phi(x)| + \frac{1}{2c}\int_{-\infty}^{\infty} |\psi(x)|\, dx.$$

(iii) The local energy

$$E(t) := \frac{1}{2}\int_0^1 |\partial_t u|^2 + c^2|\partial_x u|^2 \, dx$$

is always conserved.