THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 5

Problem 1. Let $\Omega := (0,T) \times (0,L)$ and $u \in C^2(\bar{\Omega})$ be a solution to

$$a(t,x)\partial_t u + b(t,x)\partial_x u - k(t,x)\partial_{xx} u = f(t,x)$$

where the coefficients a, b and k as well as the non-homogeneous term f are continuous functions in $\bar{\Omega}$, and will be given differently in different parts below. Answer the following questions.

(i) Prove that if $a,\ k \geq 0$ and $f < 0$, then

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u,$$

where Γ is the parabolic boundary of Ω, and given by

$$\Gamma := \{(t,x) \in \Omega;\ t = 0 \text{ or } x = 0 \text{ or } L\}. \tag{1}$$

(ii) Let $v \in C^2(\bar{\Omega})$ satisfy the following inequalities:

$$\begin{cases} a(t,x)\partial_t v + b(t,x)\partial_x v - k(t,x)\partial_{xx} v \geq 0 & \text{in } \bar{\Omega} \\ v \geq u & \text{on } \Gamma. \end{cases}$$

Prove that if $a,\ k \geq 0$ and $f < 0$, then

$$v \geq u \qquad \text{in } \bar{\Omega}.$$

(iii) Prove that if $a \geq 0$, $b \equiv 0$, $k > 0$ and $f \leq 0$, then

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u,$$

where the parabolic boundary Γ is defined by (1).

(iv) Let $a(t,x) \equiv 1$, $b(t,x) \equiv 4$, $k(t,x) := t^2 + 1$, and $f(t,x) := 1 - e^{2x}$. Prove or disprove

$$\max_{\bar{\Omega}} u = \max_{\Gamma} u,$$

where the parabolic boundary Γ is defined by (1).

Problem 2. Let $d \geq 3$ be an integer, and $\Omega \subset \mathbb{R}^d$ be a bounded domain. Assume that $u := u(x) := u(x_1, x_2, \cdots, x_d) \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$-\sum_{i=1}^{d} a_i(x)\partial_{x_i}^2 u + \sum_{i=2}^{d} b_i(x)\partial_{x_i} u = f(x),$$

where the coefficients a_i and b_i as well as the source term f are continuous functions mapping from $\bar{\Omega}$ to $\mathbb{R}$. Try to answer the following questions.

(i) Prove that if $a_i \geq 0$ for all $i = 1, 2, \cdots, d$, and $f < 0$, then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

(ii) Prove that if $a_1 > 0$, $a_i \geq 0$ for all $i = 2, 3, \cdots, d$, and $f \leq 0$, then

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

(iii) Prove that if $a_1 > 0$, $a_i \geq 0$ for all $i = 2, 3, \cdots, d$, and $f \equiv 0$, then

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Problem 3. (i) Let $u := u(t,x) \in C^2([0,T] \times [0,L])$ be a solution to

$$\partial_t u - \partial_{xx} u = |\partial_x u|^2 + 1,$$

for $0 < x < L$ and $0 < t < T$. Show that

$$\min_{\substack{0 \leq x \leq L \\ 0 \leq t \leq T}} u(t,x) \geq \min\left\{ \min_{0 \leq x \leq L} u(0,x), \min_{0 \leq t \leq T} u(t,0), \min_{0 \leq t \leq T} u(t,L) \right\}.$$

(ii) Let

$$D := \left\{ (x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1 \right\},$$

and $u \in C(\bar{D}) \cap C^2(D)$ be a solution to

$$\partial_{xx} u + 2\, \partial_{yy} u = u^4\, \partial_y u.$$

Show that

$$\max_{\bar{D}} |u| = \max_{\partial D} |u|.$$

Problem 4. Let u_1 and u_2 be two solutions to the same Laplace equation

$$\Delta u := \partial_{xx} u + \partial_{yy} u = 0 \qquad \text{in } \Omega = [-1, 1] \times [-1, 1].$$

But u_1 and u_2 satisfy different boundary conditions: for $i = 1, 2$,

$$\begin{cases} u_i|_{x=-1} = g_i \\ u_i|_{x=1} = h_i \\ u_i|_{y=-1} = \phi_i \\ u_i|_{y=1} = \psi_i \end{cases}$$

where g_i, h_i, ϕ_i and ψ_i, are given data. Prove that if

$$\begin{cases} g_1 \le g_2, \\ h_1 \le h_2, \\ \phi_1 \le \phi_2, \\ \psi_1 \le \psi_2, \end{cases}$$

then

$$u_1 \le u_2.$$