20241009 MATH3541 Assignment 3 Section B

Problem 10:

(a) Proof: We may divide our proof into two parts.

Part 1: In this part, we prove that $\circ: M_n(\mathbb{K})\times M_n(\mathbb{K})\to M_n(\mathbb{K})$, $(A,B)\mapsto AB$ is continuous.

To prove this, first notice that $\mathcal{A}:\mathbb{K}^n\to\mathbb{K}^n$, $\mathcal{A}(\vec{x})=A\vec{x}$ is a finite linear combination of projection maps $\vec{x}\mapsto x_i\vec{e_j}$.

These projection maps are continuous according to the definition of product topology, so $\mathcal{A}$ is continuous.

As $\mathcal{A}$ is continuous, $\mathbb{K}^n\to\mathbb{R}$, $\vec{x}\mapsto\|A\vec{x}\|_{Euclid}$ is continuous by composition, so $\|A\vec{x}\|_{Euclid}$ has a unique maximum on the compact domain $\overline{B}(\vec{0},1)$

Define $\|A\|_{spec}=\max\limits_{\|\vec{x}\|_{Euclid}=1}\|A\vec{x}\|_{Euclid}=\max\limits_{\vec{y}\in\mathbb{K}^n\setminus\{\vec{0}\}}\dfrac{\|A\vec{y}\|_{Euclid}}{\|\vec{y}\|_{Euclid}}$

Notice that $\|AB\|_{spec}=\max\limits_{\vec{y}\in\mathbb{K}^n\setminus\{\vec{0}\}}\dfrac{\|AB\vec{y}\|_{Euclid}}{\|\vec{y}\|_{Euclid}}\le\max\limits_{\vec{y}\in\mathbb{K}^n\setminus\{\vec{0}\}}\dfrac{\|A\|_{spec}\|B\vec{y}\|_{Euclid}}{\|\vec{y}\|_{Euclid}}$

$=\|A\|_{spec}\|B\|_{spec}$ and $AB-A_*B_*=(A-A_*)B+A_*(B-B_*)$, so:

Similarly, $\|A\|_{spec}\ge 0$, $\|A+B\|_{spec}\le\|A\|_{spec}+\|B\|_{spec}$.

$\forall\varepsilon>0$:

(i) $\exists\delta_1>0$, $\forall B\in M_n(\mathbb{K})$ with $\|B-B_*\|_{spec}<\delta_1$, $\|B\|_{spec}\le\|B_*\|_{spec}+1$;

(ii) $\exists\delta_2>0$, $\forall A\in M_n(\mathbb{K})$ with $\|A-A_*\|_{spec}<\delta_2$, $\|A-A_*\|_{spec}<\dfrac{\varepsilon/2}{\|B_*\|_{spec}+1}$

(iii) $\exists\delta_3>0$, $\forall B\in M_n(\mathbb{K})$ with $\|B-B_*\|_{spec}<\delta_3$, $\|B-B_*\|_{spec}<\dfrac{\varepsilon/2}{\|A_*\|_{spec}+1}$.

Hence, $\exists\delta=\min\{\delta_1,\delta_2,\delta_3\}$, $\forall A,B\in M_n(\mathbb{K})$ with $\max\{\|A-A_*\|_{spec},\|B-B_*\|_{spec}\}<\delta$,

$\|AB-A_*B_*\|_{spec}=\|(A-A_*)B+A_*(B-B_*)\|_{spec}\le\|A-A_*\|_{spec}\|B\|_{spec}+\|A_*\|_{spec}\|B-B_*\|_{spec}$

$<\dfrac{\varepsilon/2}{\|B_*\|_{spec}+1}(\|B_*\|_{spec}+1)+\dfrac{\varepsilon/2}{\|A_*\|_{spec}+1}(\|A_*\|_{spec}+1)=\varepsilon$

As $\forall C\in M_n(\mathbb{K})$, $\frac{1}{n}\|C\|\le\|C\|_{spec}\le\|C\|$, we've given an ε-δ proof to the continuity of $\circ$.

Part 2: det is a polynomial defined by:

$$\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where $\varepsilon(\sigma) = \begin{cases} +1 & \text{if } \sigma \in A_n; \\ -1 & \text{if } \sigma \in A_n^c; \end{cases}$

Notice that:

(i) The projection map $A \mapsto a_{ij}$ is continuous;

Finite Product ⟹ (ii) The product map $A \mapsto a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$ is continuous;

Finite Linear Combination ⟹ (iii) The determinant $A \mapsto \det(A)$ is continuous;

(b) Proof: $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$) is Hausdorff.

$\Rightarrow$ The singleton $\{0\}$ is closed in $\mathbb{K}$

$\Rightarrow$ The complement $\{0\}^c$ is open in $\mathbb{K}$

$\Rightarrow$ $GL_n(\mathbb{K}) = \det^{-1}(\{0\}^c)$ is open in $M_n(\mathbb{K})$

(c) Proof: As the continuous image of $GL_n(\mathbb{R})$ under det is a discontinuous set $\mathbb{R}\setminus\{0\}$, $GL_2(\mathbb{R})$ must be disconnected

(d) Proof: Notice that every elementary matrix E generates a path in $GL_n(\mathbb{C})$ from A to EA:

$E_1 = \begin{pmatrix} ke^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} (k>0) \Rightarrow \gamma_1 : [0,1] \to GL_2(\mathbb{C}),\ \gamma_1(t) = \begin{pmatrix} k^t e^{it\theta} & 0 \\ 0 & 1 \end{pmatrix} A$

$E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \gamma_2 : [0,1] \to GL_2(\mathbb{C}),\ \gamma_2(t) = \begin{pmatrix} e^{-i\theta}\cos\theta & e^{-i\theta} i\sin\theta \\ e^{-i\theta} i\sin\theta & e^{-i\theta}\cos\theta \end{pmatrix} A$

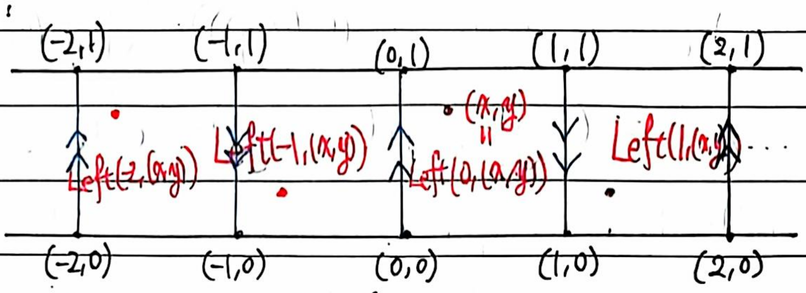
$E_3 = \begin{pmatrix} 1 & 0 \\ ke^{i\theta} & 1 \end{pmatrix} (k>0) \Rightarrow \gamma_3 : [0,1] \to GL_2(\mathbb{C}),\ \gamma_3(t) = \begin{pmatrix} 1 & 0 \\ tke^{i\theta} & 1 \end{pmatrix} A$

This means $\forall A \in GL_2(\mathbb{C})$, $A = E_{i_\mu}^{\mu} E_{i_{\mu-1}}^{\mu-1} \cdots E_{i_1}^{1}$, I is path connected to I

So $GL_n(\mathbb{C})$ is path connected, thus connected.

Problem 11.

(a) Solution:



Define $Left: \mathbb{Z}\times(\mathbb{R}\times[0,1])\to(\mathbb{R}\times[0,1])$, $Left(n,(x,y)) = \left(x+n, \frac{1}{2}+(-1)^{n-1}\frac{1-2y}{2}\right)$.

(i) For all $n, n' \in \mathbb{Z}$ and for all $(x,y)\in\mathbb{R}\times[0,1]$:

$$Left(n, Left(n',(x,y))) = Left\left(n, \left(x+n', \frac{1}{2}+(-1)^{n'-1}\frac{1-2y}{2}\right)\right)$$

$$= \left(x+n'+n, \frac{1}{2}+(-1)^{n-1}\frac{1-2\left(\frac{1}{2}+(-1)^{n'-1}\frac{1-2y}{2}\right)}{2}\right)$$

$$= \left(x+n+n', \frac{1}{2}+(-1)^{n+n'-1}\frac{1-2y}{2}\right) = Left(n+n',(x,y))$$

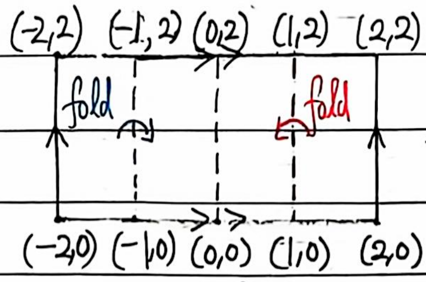
(ii) For all $(x,y)\in\mathbb{R}\times[0,1]$:

$$Left(0,(x,y)) = \left(x+0, \frac{1}{2}+(-1)^{0-1}\frac{1-2y}{2}\right) = (x,y)$$

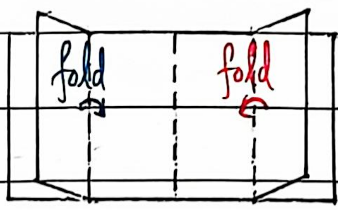
Hence, Left is a well-defined left action of $\mathbb{Z}$ on $\mathbb{R}\times[0,1]$.

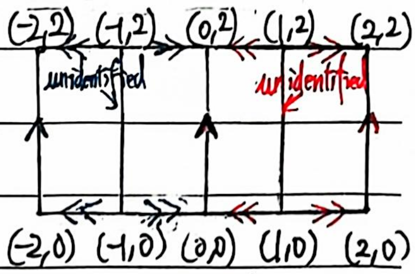
Notice that the edge $\{0\}\times[0,1]$ is glued to the edge $\{1\}\times[0,1]$ with a half twist, and the edges $[0,1]\times\{0\}$, $[0,1]\times\{1\}$ are identified only at endpoints, $(\mathbb{R}\times[0,1])/\mathbb{Z} \cong M^2$ (The Möbius Strip)

(b) Solution:



T^2 (Torus)





(Cylinder) CS^1

Define $Left: (\mathbb{Z}/2\mathbb{Z})\times T^2 \to T^2$ by:

$Left(0,(x,y)) = (x,y)$ $\qquad Left(1,(x,y)) = \begin{cases}(2-x, y), & \text{if } 0\le x\le 2\\ (-2-x, y), & \text{if } -2\le x < 0\end{cases}$

By the definition of torus, this is a WELL-DEFINED FUNCTION.

(i) For all $n, n' \in \mathbb{Z}/2\mathbb{Z}$ and $(x,y) \in \mathbb{T}^2$:

Case 1: $(n,n')=(0,0)$. $Left(0, Left(0,(x,y))) = Left(0,(x,y))$

$$= (x,y) = Left(0,(x,y)) = Left(0+0,(x,y))$$

Case 2: $(n,n')=(1,0)$. $Left(1, Left(0,(x,y))) = Left(1,(x,y))$

$$= \begin{cases} (2-x,y) & \text{if } 0 \le x \le 2 \\ (-2-x,y) & \text{if } -2 \le x \le 0 \end{cases} = Left(1,(x,y)) = Left(0+1,(x,y))$$

Case 3: $(n,n')=(0,1)$. $Left(0, Left(1,(x,y))) = \begin{cases} Left(0,(2-x,y)) & \text{if } 0 \le x \le 2 \\ Left(0,(-2-x,y)) & \text{if } -2 \le x \le 0 \end{cases}$

$$= \begin{cases} (2-x,y) & \text{if } 0 \le x \le 2 \\ (-2-x,y) & \text{if } -2 \le x \le 0 \end{cases} = Left(1,(x,y)) = Left(1+0,(x,y))$$

Case 4: $(n,n')=(1,1)$. $Left(1, Left(1,(x,y))) = \begin{cases} Left(1,(2-x,y)) & \text{if } 0 \le x \le 2 \\ Left(1,(-2-x,y)) & \text{if } -2 \le x \le 0 \end{cases}$

$$= (x,y) = Left(0,(x,y)) = Left(1+1,(x,y))$$

(ii) For all $(x,y) \in \mathbb{T}^2$: $Left(0,(x,y)) = (x,y)$

Hence, $Left$ is a well-defined left action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{T}^2$.

Notice that the edge $[-1,1] \times \{0\}$ is glued to the edge $[-1,1] \times \{2\}$ without twisting, and the edges $\{-1\} \times [0,2]$, $\{1\} \times [0,2]$ are not identified,

$\mathbb{T}^2/(\mathbb{Z}/2\mathbb{Z}) \cong CS^1$.

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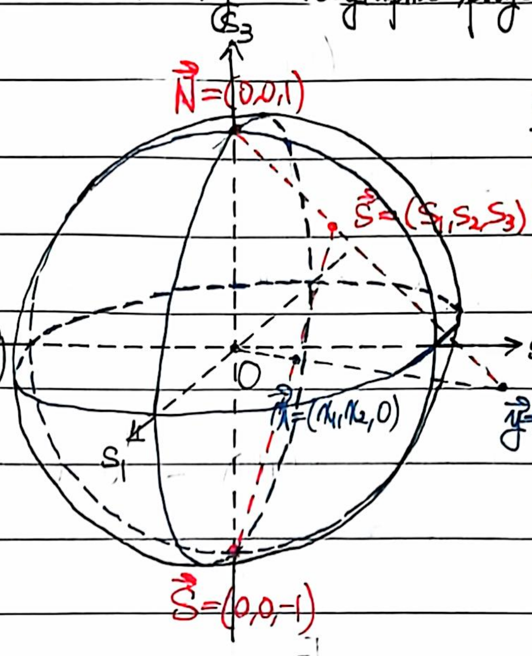
Problem 12. This problem can be solved by stereographic projection(s)

Remark: This graph must be "oblique" because the projection map is oblique (not orthogonal)



$$\begin{cases}\vec{s}=\vec{S}+\lambda(\vec{x}-\vec{S})\\ \vec{s}\cdot\vec{s}=1\end{cases}\Rightarrow \vec{s}=\vec{S}+\frac{2}{1+\|\vec{x}\|^2}(\vec{x}-\vec{S})$$

$$\begin{cases}\vec{s}=\vec{N}+\mu(\vec{x}-\vec{N})\\ \vec{s}\cdot\vec{s}=1\end{cases}$$

$\vec{y}=(y_1,y_2,0)\Rightarrow \vec{s}=\vec{N}+\frac{2}{1+\|\vec{y}\|^2}(\vec{y}-\vec{N})$

Solution: Identify $\vec{x}=(x_1,\cdots,x_n)=(x_1,\cdots,x_n,0)$

$\vec{y}=(y_1,\cdots,y_n)=(y_1,\cdots,y_n,0)$.

Construct $\phi_X: X\to S^n,\ \phi_X(\vec{x})=\vec{S}+\frac{2}{1+\|\vec{x}\|^2}(\vec{x}-\vec{S})\ (\vec{S}=-\vec{e}_{n+1})$

$\phi_Y: Y\to S^n,\ \phi_Y(\vec{y})=\vec{N}+\frac{2}{1+\|\vec{y}\|^2}(\vec{y}-\vec{N})\ (\vec{N}=+\vec{e}_{n+1})$

According to the Arithmetic Properties of Continuity, ϕ_X, ϕ_Y are continuous.

In addition,

the restricted function $\phi_X: X\to S^n\setminus\{\vec{S}\}$ has a continuous inverse $\vec{s}\mapsto\vec{S}+\frac{1}{s_{n+1}+1}(\vec{s}-\vec{S})$;

the restricted function $\phi_Y: Y\to S^n\setminus\{\vec{N}\}$ has a continuous inverse $\vec{s}\mapsto\vec{N}+\frac{1}{s_{n+1}+1}(\vec{s}-\vec{N})$;

so both ϕ_X and ϕ_Y are embeddings.

Notice that $\forall\vec{y}\in A,\ \phi_Y(\vec{y})=\vec{N}+\frac{2}{1+\|\vec{y}\|^2}(\vec{y}-\vec{N})=\frac{\|\vec{y}\|^2-1}{\|\vec{y}\|^2+1}\vec{N}+\frac{2\vec{y}}{1+\|\vec{y}\|^2}$

$$=\frac{\left\|\frac{\vec{y}}{\|\vec{y}\|^2}\right\|^2-1}{\left\|\frac{\vec{y}}{\|\vec{y}\|^2}\right\|^2+1}\vec{S}+\frac{2\left(\frac{\vec{y}}{\|\vec{y}\|^2}\right)}{1+\left(\frac{\vec{y}}{\|\vec{y}\|^2}\right)^2}=\vec{S}+\frac{2}{1+\left\|\frac{\vec{y}}{\|\vec{y}\|^2}\right\|^2}\left(\frac{\vec{y}}{\|\vec{y}\|^2}-\vec{S}\right)=\phi_X(f(\vec{y})),$$

So $\phi=\phi_X\cup_f\phi_Y$ is well defined and $\phi(\vec{0},1)=\vec{N}=-\vec{S}=\phi(\vec{0},2)$

(i) For all $U\in\mathcal{O}_{X\cup_f Y}$, $\phi(U)=\underbrace{\phi_X}_{\text{open mapping}}(\pi_1^{-1}(\pi^{-1}(U)))\cup\underbrace{\phi_Y}_{\text{open mapping}}(\pi_2^{-1}(\pi^{-1}(U)))\in\mathcal{O}_{S^n}$

(ii) For all $V\in\mathcal{O}_{S^n}$, $\phi^{-1}(V)=\underbrace{\pi}_{\text{saturated open mapping}}\Big(\underbrace{\pi_1(\phi_X^{-1}(V))\cup\pi_2(\phi_Y^{-1}(V))}_{\text{saturated open set in } X\sqcup Y}\Big)\in\mathcal{O}_{X\cup_f Y}$

Hence, $\phi: X\cup_f Y\to S^n$ is a homeomorphism, where $\pi: X\sqcup Y\to X\cup_f Y$, $\pi_1: X\to X\sqcup Y$, $\pi_2: Y\to X\sqcup Y$.

Problem 13:

(a) Proof: For every basis element (a,b) of the Euclidean topology on $\mathbb{R}$,

$(a,b) = \bigcup_{n=1}^{+\infty} [a+\frac{b-a}{2^n}, b)$ is a union of basis elements $\left([a+\frac{b-a}{2^n}, b)\right)_{n=1}^{+\infty}$ of the new topology on $\mathbb{R}$.

This implies:

The Euclidean topology on $\mathbb{R}$ $\subseteq$ The new topology on $\mathbb{R}$

(b) Proof: Note that $[0,+\infty) = \bigcup_{n=1}^{+\infty} [0,n)$ is open in the new topological space $\mathbb{R}$

and $(-\infty,0) = \bigcup_{n=1}^{+\infty} [-n,0)$ is open in the new topological space $\mathbb{R}$,

so $\mathbb{R}$ has a nontrivial open partition $\{(-\infty,0), [0,+\infty)\}$, $\mathbb{R}$ is not connected.

(c) Proof: For all $x \in \mathbb{R}$, we wish to find an open neighbour U of x, such that U contains no connected open neighbour V of x.

Take $U = \mathbb{R}$, and assume to the contrary that $\mathbb{R}$ contains a connected open neighbour V of x.

As V is open, $x \in V \Rightarrow x \in V^\circ \Rightarrow \exists [a,b)$ in basis with $[a,b) \subseteq V$, $x \in [a,b)$

Now $[a,b)$ has a nontrivial open partition $\{[a,\frac{a+b}{2}), [\frac{a+b}{2}, b)\}$,

so V has a nontrivial open partition $\{V \cap (-\infty, \frac{a+b}{2}), V \cap [\frac{a+b}{2}, +\infty)\}$, contradicting to V is connected.

Hence, $\mathbb{R}$ contains no connected open neighbour V of x, $\mathbb{R}$ is not locally connected at x.

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(d) Proof: For all disjoint closed subsets V_1, V_2 of $\mathbb{R}$ under new topology, we wish to find disjoint open subsets U_1, U_2 of $\mathbb{R}$ under new topology, such that $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$.

Step 1: For each $x_1 \in V_1$, define $E_{x_1} = \{\varepsilon > 0 : [x_1, x_1+\varepsilon) \subseteq V_2^c\}$.

Recall that $\lim_{n\to+\infty} a_n = a_*$ iff $\forall [a_*, a_*+\varepsilon)\ (\varepsilon>0), \exists N \in \mathbb{N}, \forall n \geq N, a_n \in [a_*, a_*+\varepsilon)$

Assume to the contrary that $E_{x_1} = \emptyset$, so there exists $(a_n)_{n\in\mathbb{N}}$ in V_2 such that $\lim_{n\to+\infty} a_n = x_1$.

This implies $x_1 \notin (V_2^c)^\circ$ and $x_1 \in V_2^c$, contradicting to V_2 is closed.

Hence, our assumption is false, $E_{x_1} \neq \emptyset$, we may take ε_{x_1} and form an open cover $\mathcal{U}_1 = \{[x_1, x_1+\varepsilon_{x_1})\}_{x_1\in V_1}$ of V_1.

Step 2: For each $x_2 \in V_2$, we do the same, so there exists an open cover $\mathcal{U}_2 = \{[x_2, x_2+\varepsilon_{x_2})\}_{x_2\in V_2}$ of V_2

Step 3: For each pair $[x_1, x_1+\varepsilon_{x_1})$, $[x_2, x_2+\varepsilon_{x_2})$, we want to show they are disjoint.

Assume to the contrary that $[x_1, x_1+\varepsilon_{x_1}) \cap [x_2, x_2+\varepsilon_{x_2}) \neq \emptyset$.

This implies $x_2 \in [x_1, x_1+\varepsilon_{x_1})$, $[x_1, x_1+\varepsilon_{x_1}) \not\subseteq V_2^c$

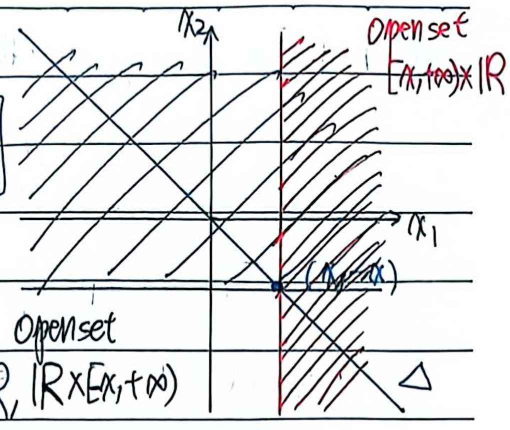
or $x_1 \in [x_2, x_2+\varepsilon_{x_2})$, $[x_2, x_2+\varepsilon_{x_2}) \not\subseteq V_1^c$, a contradiction.

Hence, our assumption is false, and we've proven that $[x_1, x_1+\varepsilon_{x_1}) \cap [x_2, x_2+\varepsilon_{x_2}) = \emptyset$

To conclude, there exist disjoint open subsets $U_1 = \bigcup_{x_1\in V_1} [x_1, x_1+\varepsilon_{x_1})$, $U_2 = \bigcup_{x_2\in V_2} [x_2, x_2+\varepsilon_{x_2})$ of $\mathbb{R}$ under new topology, such that $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$

(e) Proof: For all $(x,-x)\in\Delta$,

$\{(x,-x)\} = \Delta \cap [([x,+\infty)\times\mathbb{R}) \cap (\mathbb{R}\times[-x,+\infty))]$



As $[x,+\infty)$, $\mathbb{R}$, $[-x,+\infty)$, $\mathbb{R}$ are open in $\mathbb{R}$,

$[x,+\infty)\times\mathbb{R}$, $\mathbb{R}\times[-x,+\infty)$ are open in $\mathbb{R}\times\mathbb{R}$,

so $([x,+\infty)\times\mathbb{R})\cap(\mathbb{R}\times[-x,+\infty))$ is open in $\mathbb{R}\times\mathbb{R}$,

which implies $\{(x,-x)\} = \Delta\cap[([x,+\infty)\times\mathbb{R})\cap(\mathbb{R}\times[-x,+\infty))]$ is open in Δ.

(f) Proof:

Step 1: For all $(x_1,x_2)\in\Delta_{\mathbb{Q}}^c$:

If $x_1+x_2\geq 0$, then $\exists$ open $[x_1,+\infty)\times[x_2,+\infty)$ with $[x_1,+\infty)\times[x_2,+\infty)\subseteq\Delta_{\mathbb{Q}}^c$, $(x_1,x_2)\in[x_1,+\infty)\times[x_2,+\infty)$

If $x_1+x_2<0$, then $\exists$ open $[x_1,\frac{x_1-x_2}{2})\times[x_2,\frac{x_2-x_1}{2})$ with $[x_1,\frac{x_1-x_2}{2})\times[x_2,\frac{x_2-x_1}{2})\subseteq\Delta_{\mathbb{Q}}^c$, $(x_1,x_2)\in[x_1,\frac{x_1-x_2}{2})\times[x_2,\frac{x_2-x_1}{2})$

Hence, $(x_1,x_2)\in(\Delta_{\mathbb{Q}}^c)^\circ$. This implies $\Delta_{\mathbb{Q}}$ is closed in $X\times X$.

Similarly, $\Delta_{\mathbb{Q}^c}$ is closed in $X\times X$.

Step 2: Assume to the contrary that for some open subsets R,S of $X\times X$, $\Delta_{\mathbb{Q}}\subseteq R$ and $\Delta_{\mathbb{Q}^c}\subseteq S$ and $R\cap S=\emptyset$.

Start with $r_1=0\in\mathbb{Q}$. $(r_1,-r_1)\in R\Rightarrow\exists\varepsilon_1>0$, $(r_1,-r_1)\in[r_1,r_1+\varepsilon_1)\times[-r_1,-r_1+\varepsilon_1)\subseteq R$.

Take $s_1\in(r_1,r_1+\frac{\varepsilon_1}{2})\cap\mathbb{Q}^c$. $(s_1,-s_1)\in S\Rightarrow\exists\mu_1>0$, $(s_1,-s_1)\in[s_1,s_1+\mu_1)\times[-s_1,-s_1+\mu_1)\subseteq S$

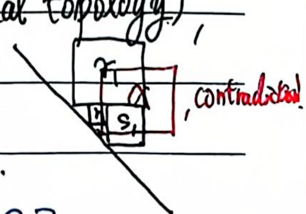
Take $r_2\in(s_1-\frac{\mu_1}{2},s_1)\cap\mathbb{Q}$. $(r_2,-r_2)\in R\Rightarrow\exists\varepsilon_2>0$, $(r_2,-r_2)\in[r_2,r_2+\varepsilon_2)\times[-r_2,-r_2+\varepsilon_2)\subseteq R$

Take $s_2\in(r_2,r_2+\frac{\varepsilon_2}{2})\cap\mathbb{Q}^c$. $(s_2,-s_2)\in S\Rightarrow\exists\mu_2>0$, $(s_2,-s_2)\in[s_2,s_2+\mu_2)\times[-s_2,-s_2+\mu_2)\subseteq S$

Repeat this iteration, we get a strictly increasing sequence $(r_n)_{n=1}^{+\infty}$ and a strictly decreasing sequence $(s_n)_{n=1}^{+\infty}$ with the same limit x (under usual topology).

WLOG, assume that $x\in\mathbb{Q}$. $(x,-x)\in R\Rightarrow\exists\eta>0$, $(x,-x)\in[x,x+\eta)\times[-x,-x+\eta)\subseteq R$.

As $(s_n)_{n=1}^{+\infty}$ is a decreasing sequence with limit x (under usual topology), there exists $N\in\mathbb{N}$, such that $s_N\in[x,x+\frac{\eta}{3})$.



For all $n\geq N$, by construction, $s_n\in[s_N-\frac{\mu_N}{2},s_N]$.

Take limit with respect to usual topology, $x\in[s_N-\frac{\mu_N}{2},s_N]$,

so $([x,x+\eta)\times[-x,-x+\eta))\cap([s_N,s_N+\mu_N)\times[-s_N,-s_N+\mu_N))\neq\emptyset$

contradicting to $R\cap S=\emptyset$. Hence, our assumption is false, R,S don't exist.