If R is a UFD, so is $R[x]$

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1. §1.3.4: Theorem: If R is a UFD, so is $R[x]$

$\Rightarrow$ $R[x_1, x_2, \cdots, x_n]$ is a UFD

$\Rightarrow$ $F[x_1, x_2, \cdots, x_n]$ is a UFD $\forall$ field F.

Let R be a UFD and $F = \mathrm{Frac}(R)$ the fraction field of R. Recall

Recall that $f(x) \in R[x]$ is said to be primitive if f is not constant and

$$\gcd(\text{coefficients of } f) = 1.$$

We have proved the following:

Theorem. Irreducible elements in $R[x]$ are precisely of the two types:

1. Type I: constant polynomials defined by irreducible elements of R;
2. Type II: primitive polynomials $f(x) \in R[x]$ that are irreducible in $F[x]$.

Existence of factorization into irreducibles

$\cup F[x]$

Proposition.: If R is a UFD, every non-zero non-unit $f(x) \in R[x]$ is can be written as a product of irreducible elements in $R[x]$.

Proof. Induction on $\deg(f)$.

- Case 1. $\deg(f) = 0$. Then $f \in R$, non-zero and non-unit.
- Since R is a UFD,
$$f = a_1 a_2 \cdots a_n,$$
where $a_1, \ldots, a_n \in R$ are irreducible in R and thus also in $R[x]$.
- Case 2. $\deg(f) > 0$. Write $f(x) = \alpha g(x)$, where
$$\alpha = \mathrm{cont}(f) \in R$$
and $g(x) \in R[x]$ is primitive.

Proof of Proposition, cont'd:

- Write $\alpha = \alpha_1 \cdots \alpha_m$, where each $\alpha_j \in R$ is irreducible.
- If $g(x) \in R[x]$ is irreducible, we are done.
- Assume that $g \in R[x]$ is not irreducible, then

$$g(x) = h(x)k(x)$$

  with $h(x), k(x) \in R[x]$ both having positive degrees.
- By induction, both $h(x)$ and $k(x)$ are products of irreducible elements in $R[x]$.
- Thus $f(x) = \alpha g(x) = \alpha_1 \cdots \alpha_m h(x)k(x)$ is a product of irreducible elements in $R[x]$.

Q.E.D.

Main Theorem. If R is a UFD, so is $R[x]$.

Proof. Let $f \in R[x]$ be non-zero and non-unit.

- Already know f is a product of irreducible elements in $R[x]$.
- Suppose that

$$f = \alpha_1 \cdots \alpha_m f_1 \cdots f_n = \alpha_1' \cdots \alpha_l' f_1' \cdots f_k',$$

  where

  - $\alpha_1, \ldots, \alpha_m, \alpha_1', \ldots, \alpha_l'$ are irreducible elements in R,
  - $f_1, \ldots, f_n, f_1', \ldots, f_k'$ are non-constant polynomials in $R[x]$ which are primitive and irreducible.

Proof of Main Theorem, cont'd:

- By Gauss' Lemma on products of primitive elements in $R[x]$,

$$f_1 \cdots f_n \in R[x] \quad \text{and} \quad f_1' \cdots f_k' \in R[x]$$

  are primitive.

- By uniqueness of contents,

$$\alpha := \alpha_1 \cdots \alpha_m \quad \text{and} \quad \alpha' := \alpha_1' \cdots \alpha_l'$$

  are both contents of f, so they are associates in R.

- Since R is a UFD, $l = m$ and after a permutation, $\alpha_i = u_i \alpha_i'$ for some unit u_i of R for every $1 \leq i \leq l = m$.

$$\alpha_1 \cdots \alpha_m = u_1 \cdots u_m \; \alpha_1' \cdots \alpha_m'$$

Proof of Main Theorem, cont'd:

- Let $u = u_1 u_2 \cdots u_n$ so that

$$f = f_1 \ldots f_n = u f_1' \cdots f_k'. \in R[x] \subset F[x]$$

- By Gauss' Lemma relating irreducible elements in $F[x]$ and $R[x]$, $f_1, \cdots, f_n, f_1', \ldots, f_k'$ are irreducible in $F[x]$.

$v_j = \frac{s_j}{t_j}$

$s_j, t_j \in R$

- As $F[x]$ is a UFD, we know that $n = k$, and after a permutation $f_j = v_j f_j'$ for some $v_j \in F \setminus \{0\}$ for all $1 \le j \le n = k$.

$\Rightarrow t_j f_j = s_j f_j' \in R[x]$

- By uniqueness of contents again, v_j is a unit in R for each $1 \le j \le n$. $\Rightarrow$ t_j and s_j are associates in R $\Rightarrow v_j \in R$ and is a unit of R.

- We have thus proved uniqueness of factorizations of f into irreducibles.

Q.E.D.

Corollary: If R is a UFD, so is $R[x_1, \ldots, x_n]$ for any $n \geq 1$.

Examples: For any integers $n \geq 1$,

- $\mathbb{Z}[x_1, \ldots, x_n]$ is a UFD;
- For any field F, $F[x_1, \ldots, x_n]$ is a UFD.

Thm: Localizations of UFDs are UFDs.

$\mathbb{C}[x, y] / (x^2 - y^3) \ni \bar{x}, \bar{y}$ $\quad \bar{x}^2 = \bar{y}^3$

is not a UFD.