Algebra II: Tutorial 11

April 27, 2022

Problem 1 (Galois extensions)**.** Determine whether the following extensions $L : K$ are Galois:

1. $K = \mathbb{Q}, L = \mathbb{Q}(\sqrt[3]{2})$.
2. $K = \mathbb{Q}, L = \mathbb{Q}(\sqrt[4]{2})$.
3. $K = \mathbb{Q}(\sqrt{2}), L = \mathbb{Q}(\sqrt[4]{2})$.
4. $K = \mathbb{Q}(i), L = \mathbb{Q}(i, \sqrt[4]{2})$.
5. $K = \mathbb{Q}(t^2), L = \mathbb{Q}(t)$.
6. $K = \mathbb{F}_2(t^2 + t), L = \mathbb{F}_2(t)$.

Solution. The first two are not Galois extensions, the last four are. ■

Problem 2 (Computing Galois groups)**.** Compute $G(L) = \mathrm{Aut}_K(L)$, list all subgroups H of $G(L)$ and determine the corresponding intermediate field L^H for each of the following field extensions L over K:

1. $K = \mathbb{Q}$ and $L = \mathbb{Q}(i + \sqrt{2})$.
2. $K = \mathbb{Q}(i)$ and $L = K(\sqrt[4]{2})$.

Solution.

1. Note that $\sqrt{2}+i$ is a root of the irreducible polynomial $f(x) = x^4-2x^2+9$, so $[L : K] = 4$. Furthermore, f splits completely in L so L is a splitting field for f over $\mathbb{Q}$. Since $\mathbb{Q}$ is a perfect field, this implies that $L : K$ is a Galois extension, and so $G = \mathrm{Aut}_{\mathbb{Q}}(L)$ has order 4. Explicitly, $G = \{Id, \sigma_1, \sigma_2, \sigma_3\}$ where

$$\begin{array}{ccc} \sigma_1(i) = i, & \sigma_2(i) = -i, & \sigma_3(i) = -i, \\ \sigma_1(\sqrt{2}) = -\sqrt{2}, & \sigma_2(\sqrt{2}) = \sqrt{2}, & \sigma_3(\sqrt{2}) = -\sqrt{2}. \end{array}$$

   In particular, each of the automorphisms σ_i has order 2, so $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The proper subgroups of G are isomorphic to $\mathbb{Z}_2$; explicitly they are $G_1 = \{Id, \sigma_1\}, G_2 =$

$\{Id, \sigma_2\}, G_3 = \{Id, \sigma_3\}$. In order to compute L^{G_i}, we choose a convenient basis for elements in L. It is easy to see that the set $\{1, \sqrt{2}, i, i\sqrt{2}\}$ is a $\mathbb{Q}$-basis of L. Then:

$$L^{G_1} = \mathbb{Q}(i), \quad L^{G_2} = \mathbb{Q}(\sqrt{2}), \quad \text{and} \quad L^{G_3} = \mathbb{Q}(i\sqrt{2}).$$

2. L is the splitting field of $f(x) = x^4 - 2$ over $\mathbb{Q}(i)$; the roots of $f(x)$ are $\pm\sqrt[4]{2}$ and $\pm i\sqrt[4]{2}$. In particular, L is a Galois extension of K, and so $|G| = [L : K]$. Note that $f(x)$ is irreducible over K, so $|G| = 4$. Denote by id, σ_1, σ_2 and σ_3 the four K-automorphisms of L. Since L is a simple extension, each automorphism is completely determined by its value on $\alpha = \sqrt[4]{2}$:

$$\sigma_1(\alpha) = -\sqrt[4]{2}, \quad \sigma_2(\alpha) = i\sqrt[4]{2} \quad \text{and} \quad \sigma_3(\alpha) = -i\sqrt[4]{2}.$$

A direct computation shows that σ_1 has order 2 and σ_2, σ_3 have order 4. Hence, $G \cong \mathbb{Z}_4$. There is only one proper subgroup of G, namely $G_1 = \{id, \sigma_1\}$. Using the K-basis $\{1, \alpha, \alpha^2, \alpha^3\}$ of L, it is easy to see that $L^{G_1} = \mathbb{Q}(i, \sqrt{2})$. ■