THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 1 Solution

Problem 1.

(i) By the trignometric identity

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right],$$

we have

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(x)\cos(4x)\,dx &= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(5x) + \cos(-3x)\right]\,dx \\ &= \frac{1}{2}\left[\frac{\sin(5x)}{5} + \frac{\sin(-3x)}{-3}\right]_{-\pi}^{\pi} = 0.\end{aligned}$$

(ii) By applying integration by parts twice,

$$\begin{aligned}\int_{0}^{2\pi} x^2 \sin(mx)\,dx &= \left[-\frac{x^2\cos(mx)}{m}\right]_0^{2\pi} + \int_0^{2\pi} \frac{2x\cos(mx)}{m}\,dx \\ &= -\frac{4\pi^2}{m} + \left[\frac{2x\sin(mx)}{m^2}\right]_0^{2\pi} - \int_0^{2\pi} \frac{2\sin(mx)}{m^2}\,dx \\ &= -\frac{4\pi^2}{m} + \left[\frac{2\cos(mx)}{m^3}\right]_0^{2\pi} = -\frac{4\pi^2}{m}.\end{aligned}$$

Problem 2.

$$\partial_t v = \partial_t\left(t^{-\frac{1}{2}}\exp\left(\frac{2(x-2)^2}{t}\right)\right) = \exp\left(\frac{2(x-2)^2}{t}\right)\left[-\frac{t^{-\frac{3}{2}}}{2} - 2(x-2)^2 t^{-\frac{5}{2}}\right].$$

On the other hand,

$$\partial_x v = \partial_x\left(t^{-\frac{1}{2}}\exp\left(\frac{2(x-2)^2}{t}\right)\right) = t^{-\frac{1}{2}}\exp\left(\frac{2(x-2)^2}{t}\right)\left[\frac{4(x-2)}{t}\right]$$

and hence

$$\partial_{xx} v = t^{-\frac{1}{2}} \exp\left(\frac{2(x-2)^2}{t}\right)\left[\frac{16(x-2)^2}{t^2} + \frac{4}{t}\right].$$

Thus,

$$-\frac{1}{8}\partial_{xx} v = t^{-\frac{1}{2}} \exp\left(\frac{2(x-2)^2}{t}\right)\left[\frac{16(x-2)^2}{t^2} + \frac{4}{t}\right] = \partial_t v.$$

Problem 3. As f is continuous at 0, there exists $\delta > 0$ such that

$$|f(x) - f(0)| < \frac{f(0)}{2}$$

for all $x \in (-\delta, \delta)$. In particular, for all $x \in (-\delta, \delta)$, we have

$$f(x) > \frac{f(0)}{2} > 0$$

and hence

$$\int_{-\delta}^{\delta} f(x)\, dx \geq \frac{f(0)}{2} \int_{-\delta}^{\delta} dx = \delta f(0) > 0.$$

Problem 4. By separation of variables,

$$\frac{du}{u^3} = 4\, dt,$$

thus integration on both sides gives

$$-\frac{u^{-2}}{2} = 4t + C.$$

Substituting the initial condition $u(0) = 2$ deduces $C = -\frac{1}{8}$. Hence,

$$u(t) = \frac{1}{\left(\frac{1}{4} - 8t\right)^{\frac{1}{2}}} = \frac{2}{(1-32t)^{\frac{1}{2}}}.$$

By setting $T = \frac{1}{32}$, we have $\lim_{t\to T^-} u(t) = \infty$.

Problem 5.

(i)

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 4 & -4-\lambda \end{pmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2.$$

So -2 is the only eigenvalue.

To get $E_{-2} = \mathrm{Nul}(A+2I)$, consider $(A+2I)\mathbf{x} = \mathbf{0}$ with the augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = \frac{t}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ where } t \text{ is free.}$$

So we get

$$E_2 = \mathrm{Nul}(A-2I) = \mathrm{Span}\{\begin{bmatrix} 1 & 2 \end{bmatrix}^T\}.$$

Hence an eigenvector in the basis for E_2 are given by $\hat{\mathbf{p}}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$.

(ii) Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{bmatrix}$. Note that $P^{-1}AP = J_2(2)$ is equivalent to

$$\begin{cases} A\mathbf{p}_1 & = -2\mathbf{p}_1 \\ A\mathbf{p}_2 & = \mathbf{p}_1 - 2\mathbf{p}_2 \end{cases}, \text{ i.e., } \begin{cases} (A+2I)\mathbf{p}_1 & = 0 \\ (A+2I)\mathbf{p}_2 & = \mathbf{p}_1 \end{cases}.$$

By (i), we can choose $\mathbf{p}_1 = \hat{\mathbf{p}}_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$. To find $\mathbf{p}_2$, consider the augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 4 & -2 & 2 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right].$$

So the general solution is

$$\mathbf{p}_2 = \begin{bmatrix} \frac{1}{2} + \frac{t}{2} \\ t \end{bmatrix}, \text{ where } t \text{ is free.}$$

Now we can take $t = -1$ to get a particular solution $\mathbf{p}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Thus, $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

(iii) Note that

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0,$$

where $\mathbf{x}_0 = \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It follows from (ii) that

$$e^{tA} = \exp\left(tP\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}P^{-1}\right) = P\exp\left(t\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\right)P^{-1}.$$

As

$$\begin{aligned} \exp\left(t\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\right) &= \exp\left(-2tI + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e^{-2t}\exp\left(t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \\ &= e^{-2t}\left(I + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\right) = \begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}, \end{aligned}$$

the solution is

$$\begin{aligned} e^{tA}\mathbf{x}_0 &= P\begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}P^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t} & te^{-2t} \\ 2e^{2t} & 2te^{-2t} - e^{-2t} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} te^{-2t} + e^{-2t} \\ 2te^{-2t} + e^{-2t} \end{pmatrix} = e^{-2t}\begin{pmatrix} t+1 \\ 2t+1 \end{pmatrix}. \end{aligned}$$