20240910 MATH3541 NOTE 3[1]

Author: Be $\sqrt{-1}$maginative, and nothing will be $\frac{\mathrm{d}}{\mathrm{d}x}$ifficult!
Email: u3612704@connect.hku.hk;
Phone: +852 5693 2134; +86 19921823546;

Contents

1 Introduction

Today, Prof. Hua mentioned interior, closure and derived set. Although the three concepts are already introduced in metric space, certain counter-intuitive examples exist in general topological space. Hence, it is necessary to review these concepts.

Prof. Hua also mentioned Hausdorff space, or T_2 space, which is a member in a sequence of restrictions that "separate a point from another to certain extents". There is an equivalent criterion to characterize Hausdorff space.

2 Review

2.1 Initial Topology and Final Topology

Definition 2.1. (**Initial Topology**)
Let X be a set, $(Y, \mathcal{O}_Y)$ be a topological space, and $\sigma : X \to Y$ be a function. We define $\mathcal{O}_X = \{\sigma^{-1}(V) \in \mathcal{P}(X) : V \in \mathcal{O}_Y\}$ as the initial topology of $(Y, \mathcal{O}_Y)$ on X via σ.

Proposition 2.2. The initial topology inherits a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\emptyset, Y \in \mathcal{O}_Y \implies \emptyset = \sigma^{-1}(\emptyset), X = \sigma^{-1}(Y) \in \mathcal{O}_X$.
Part 2: $\forall(\sigma^{-1}(V_k))_{k=1}^m$ in $\mathcal{O}_X, \cap_{k=1}^m \sigma^{-1}(V_k) = \sigma^{-1}(\cap_{k=1}^m V_k) \in \mathcal{O}_X$.
Part 3: $\forall(\sigma^{-1}(V_\lambda))_{\lambda \in I}$ in $\mathcal{O}_X, \cup_{\lambda \in I} \sigma^{-1}(V_\lambda) = \sigma^{-1}(\cup_{\lambda \in I} V_\lambda) \in \mathcal{O}_X$.
Combine the three parts above, we've proven that initial topology inherits a topological space. Quod. Erat. Demonstrandum. □

Definition 2.3. (**Final Topology**)
Let $(X, \mathcal{O}_X)$ be a topological space, Y be a set, and $\sigma : X \to Y$ be a function. We define $\mathcal{O}_Y = \{V \in \mathcal{P}(Y) : \sigma^{-1}(V) \in \mathcal{O}_X\}$ as the final topology of $(X, \mathcal{O}_X)$ on Y via σ.

Proposition 2.4. The final topology inherits a topological space.

Proof. We may divide our proof into three parts.
Part 1: $\sigma^{-1}(\emptyset) = \emptyset, \sigma^{-1}(Y) = X \in \mathcal{O}_X \implies \emptyset, Y \in \mathcal{O}_Y$.
Part 2: $\forall(V_k)_{k=1}^m$ in $\mathcal{O}_Y, \sigma^{-1}(\cap_{k=1}^m V_k) = \cap_{k=1}^m \sigma^{-1}(V_k) \in \mathcal{O}_X \implies \cap_{k=1}^m V_k \in \mathcal{O}_Y$.
Part 3: $\forall(V_\lambda)_{\lambda \in I}$ in $\mathcal{O}_Y, \sigma^{-1}(\cup_{\lambda \in I} V_\lambda) = \cup_{\lambda \in I} \sigma^{-1}(V_\lambda) \in \mathcal{O}_X \implies \cup_{\lambda \in I} V_\lambda \in \mathcal{O}_Y$.
Combine the three parts above, we've proven that final topology inherits a topological space. Quod. Erat. Demonstrandum. □

2.2 Disjoint Union Topology

Definition 2.5. (**Disjoint Union Topology**)
Let $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ be an indexed family of pairwisely disjoint topological spaces, and $X = \bigsqcup_{\lambda\in I} X_\lambda$ be the disjoint union set.
We define the disjoint union topology $\mathcal{O}_X$ of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ on X as the intersection of each final topology of $(X_\lambda, \mathcal{O}_{X_\lambda})_{\lambda\in I}$ on X via $\pi_\lambda : X_\lambda \to X, \pi_\lambda(x) = x$.

Proposition 2.6. Let $((X_\lambda), \mathcal{O}_{X_\lambda})_{\lambda\in I}$ be an indexed family of pairwisely disjoint topological spaces, and $X = \bigsqcup_{\lambda\in I} X_\lambda$ be the disjoint union set.
If $\mathcal{O}_X$ is the disjoint union topology of $((X_\lambda, \mathcal{O}_{X_\lambda}))_{\lambda\in I}$ on X, then $(X, \mathcal{O}_X)$ has a basis $\mathcal{B}_X = \cup_{\lambda\in I}\mathcal{O}_{X_\lambda}$.

Proof. We may divide our proof into three steps.
Step 1: For all $\lambda \in I$, for all $O_\lambda \in \mathcal{O}(X_\lambda)$, for all $\mu \in I$:

$$\pi_\mu^{-1}(O_\lambda) = \begin{cases} O_\lambda \in \mathcal{O}_{X_\lambda} & \text{if} \quad \mu = \lambda; \\ \emptyset \in \mathcal{O}_{X_\mu} & \text{if} \quad \mu \neq \lambda; \end{cases}$$

This gives $O_\lambda \in \mathcal{O}_X$, so $\mathcal{B}_X \subseteq \mathcal{O}_X$.
Step 2: For all $O \in \mathcal{O}_X$, for all $\lambda \in I$, $O \cap X_\lambda = \pi_\lambda^{-1}(O) \in \mathcal{O}_{X_\lambda}$, so $O \cap X_\lambda \in \mathcal{B}_X$.
Step 3: For all $O \in \mathcal{O}_X$, $O = \cup_{\lambda\in I}(O \cap X_\lambda)$ is a union of elements in $\mathcal{B}_X$.
Combine the three parts above, we've proven that $\mathcal{B}_X$ is a basis of $(X, \mathcal{O}_X)$.
Quod. Erat. Demonstrandum. □

Proposition 2.7. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, and $(-\infty, 0), [0, +\infty)$ as subspaces of $\mathbb{R}$, then the disjoint union topology of $(-\infty, 0), [0, +\infty)$ on $\mathbb{R}$ is different from the original topology.

Proof. As $[0, +\infty)$ belongs to the subspace topology on $[0, +\infty)$, it must lie in the disjoint union topology on $\mathbb{R}$. However, $[0, +\infty)$ doesn't belong to the metric topology on $\mathbb{R}$, so the two topologies on $\mathbb{R}$ are different. Quod. Erat. Demonstrandum. □

3 Interior, Closure and Derived Set

3.1 Definition of Interior, Closure and Derived Set

Definition 3.1. (**Interior**)
Let $(X, \mathcal{O}_X)$ be a topological space, and U be a subset of X.
Define $U^\circ = \{x \in X : \exists V \in \mathcal{O}_X \text{ with } V \subseteq U, x \in V\}$ as the interior of U.
Each element $x \in U^\circ$ is called an interior point of U.

Definition 3.2. (**Closure**)
Let $(X, \mathcal{C}_X)$ be a topological space, and U be a subset of X.
Define $\overline{U} = \{x \in X : \forall V \in \mathcal{C}_X \text{ with } V \supseteq U, x \in V\}$ as the closure of U.
Each element $x \in \overline{U}$ is called an accumulation point of U.

Definition 3.3. (**Derived Set**)
Let $(X, \mathcal{C}_X)$ be a topological space, and U be a subset of X.
Define $\overline{U} = \{x \in X : \forall V \in \mathcal{C}_X \text{ with } V \supseteq U \backslash \{x\}, x \in V\}$ as the derived set of U.
Each element $x \in \overline{U}$ is called a limit point of U.

3.2 Related Properties in General Space

Proposition 3.4. (**Interior/Closure as Members of Topology**)

$$\begin{array}{cccc} (1) & U^\circ & \in & \mathcal{O}_X; \\ (2) & \overline{U} & \in & \mathcal{C}_X; \end{array}$$

Proof.
(1) Define I as the set of all open sets that U contains, so $U^\circ = \cup_{V \in I} V \in \mathcal{O}_X$;
(2) Define I as the set of all closed sets that contains U, so $\overline{U} = \cap_{V \in I} V \in \mathcal{C}_X$;
Quod. Erat. Demonstrandum. $\square$

Proposition 3.5. If $\mathcal{O}_X$ is finer than the cofinite topology, then $U' \in \mathcal{C}_X$.

Proof. We prove $(U')^c \in \mathcal{O}_X$ instead. Notice that:

$$\forall x \in X, [x \in (U')^c \iff \exists W \in \mathcal{O}_X \text{ with } (W \backslash \{x\}) \cap U = \emptyset, x \in W]$$

For all $x \in (U')^c$, there exists $W_x \in \mathcal{O}_X$ with $(W_x \backslash \{x\}) \cap U = \emptyset$, such that $x \in W_x$.
If we could prove $W_x \subseteq (U')^c$, then $(U')^c = \bigcup_{x \in (U')^c} W_x \in \mathcal{O}_X$ and we are done.
For all $y \in W_x$, without loss of generality, assume that $y \neq x$. As $\mathcal{O}_X$ is finer than the cofinite topology, $\{x\}^c \in \mathcal{O}_X$, so there exists $W_x \backslash \{x\} = W_x \cap \{x\}^c \in \mathcal{O}_X$ with $[(W_x \backslash \{x\}) \backslash \{y\}] \cap U = \emptyset$, such that $y \in W_x \backslash \{x\}$.
Hence, $y \in (U')^c$, so $W_x \subseteq (U')^c$ and we are done. Quod. Erat. Demonstrandum. $\square$

Proposition 3.6. If X contains at least two elements x_1, x_2, and $\mathcal{C}_X$ is the indiscrete topology on X, then $\{x_1\}' = \{x_1\}^c \notin \mathcal{C}_X$.

Proof. We may divide our proof into two parts.
Part 1: There exists $\emptyset \in \mathcal{C}_X$ with $\emptyset \supseteq \emptyset = \{x_1\} \backslash \{x_1\}$, such that $x_1 \notin \emptyset$.
Hence, $x_1 \notin \{x_1\}', \{x_1\}' \subseteq \{x_1\}^c$;
Part 2: For all $y \in \{x_1\}^c$, for all $V \in \mathcal{C}_X$ with $V \supseteq \{x_1\} = \{x_1\} \backslash \{y\}$, as $\mathcal{C}_X$ is the

indiscrete topology, $V = X$, so $y \in V$. Hence $y \in \{x_1\}'$, $\{x_1\}^c \subseteq \{x_1\}'$.
As $\{x_1\}' = \{x_1\}^c$ is neither $\emptyset$ nor X, $\{x_1\}' \notin \mathcal{C}_X$. Quod. Erat. Demonstrandum. $\square$

Proposition 3.7. (**Extremal Property**)

$$\begin{array}{lllll} (1) & \forall V \in \mathcal{O}_X, \quad V \subseteq U & \Longrightarrow & V \subseteq U^\circ, & \text{as a consequence,} \quad (U^\circ)^\circ = U^\circ \\ (2) & \forall V \in \mathcal{C}_X, \quad V \supseteq U & \Longrightarrow & V \supseteq \overline{U}, & \text{as a consequence,} \quad \overline{(\overline{U})} = \overline{U} \end{array}$$

Proof.
(1) Define I as the set of all open sets that U contains, so $V \subseteq \cup_{V \in I} V = U^\circ$;
(2) Define I as the set of all closed sets that contains U, so $V \supseteq \cap_{V \in I} V = \overline{U}$;
Quod. Erat. Demonstrandum. $\square$

Proposition 3.8. (**Arbitrary Intersection/Union Property**)

$$\begin{array}{lccc} (1) & \left(\bigcap_{\lambda \in I} U_\lambda\right)^\circ & \subseteq & \bigcap_{\lambda \in I} U_\lambda^\circ; \\ (2) & \left(\bigcup_{\lambda \in I} U_\lambda\right)^\circ & \supseteq & \bigcup_{\lambda \in I} U_\lambda^\circ; \\ (3) & \overline{\left(\bigcap_{\lambda \in I} U_\lambda\right)} & \subseteq & \bigcap_{\lambda \in I} \overline{U_\lambda}. \\ (4) & \overline{\left(\bigcup_{\lambda \in I} U_\lambda\right)} & \supseteq & \bigcup_{\lambda \in I} \overline{U_\lambda}. \end{array}$$

Proof.
(1) For all $x \in \left(\bigcap_{\lambda \in I} U_\lambda\right)^\circ$, $\exists V \in \mathcal{O}_X$ with $V \subseteq \bigcap_{\lambda \in I} U_\lambda, x \in V$.
Hence, for all $\lambda \in I$, $\exists V \in \mathcal{O}_X$ with $V \subseteq U_\lambda, x \in V$, so $x \in \bigcap_{\lambda \in I} U_\lambda^\circ$.
(2) For all $x \in \bigcup_{\lambda \in I} U_\lambda^\circ$, for some $\lambda \in I$, $x \in U_\lambda^\circ$, so $\exists V \in \mathcal{O}_X$ with $V \subseteq U_\lambda, x \in V$.
Hence, $\exists V \in \mathcal{O}_X$ with $V \subseteq \bigcup_{\lambda \in I} U_\lambda, x \in V$, so $x \in \left(\bigcup_{\lambda \in I} U_\lambda\right)^\circ$.
(3) For all $x \in \overline{\left(\bigcap_{\lambda \in I} U_\lambda\right)}$, $\forall V \in \mathcal{C}_X$ with $V \supseteq \bigcap_{\lambda \in I} V_\lambda, x \in V$.
Hence, for all $\lambda \in I$, $\forall V \in \mathcal{C}_X$ with $V \supseteq U_\lambda, x \in V$, so $x \in \bigcap_{\lambda \in I} \overline{U_\lambda}$.
(4) For all $x \in \bigcup_{\lambda \in I} \overline{U_\lambda}$, for some $\lambda \in I$, $x \in \overline{U_\lambda}$, so $\forall V \in \mathcal{C}_X$ with $V \supseteq U_\lambda, x \in V$.
Hence, $\forall V \in \mathcal{C}_X$ with $V \supseteq \bigcup_{\lambda \in I} U_\lambda, x \in V$, so $x \in \overline{\left(\bigcup_{\lambda \in I} U_\lambda\right)}$.
Quod. Erat. Demonstrandum. $\square$

Proposition 3.9. (**Finite Intersection/Union Property**)

$$\begin{array}{lccc} (1) & \left(\bigcap_{k=1}^m U_k\right)^\circ & = & \bigcap_{k=1}^m U_k^\circ; \\ (2) & \overline{\left(\bigcup_{k=1}^m U_k\right)} & = & \bigcup_{k=1}^m \overline{U_k}. \end{array}$$

Proof.
(1) It suffices to prove $\left(\bigcap_{k=1}^m U_k\right)^\circ \supseteq \bigcap_{k=1}^m U_k^\circ$. We prove it directly.
For all $x \in \bigcap_{k=1}^m U_k^\circ$, x is in each U_k°, so $\exists V_k \in \mathcal{O}_X$ with $V_k \subseteq U_k, x \in V_k$.
Hence, $\exists \bigcap_{k=1}^m V_k \in \mathcal{O}_X$ with $\bigcap_{k=1}^m V_k \subseteq \bigcap_{k=1}^m U_k, x \in \bigcap_{k=1}^m V_k$, so $x \in \left(\bigcap_{k=1}^m U_k\right)^\circ$.
(2) It suffices to prove $\overline{\left(\bigcup_{k=1}^m U_k\right)} \subseteq \bigcup_{k=1}^m \overline{U_k}$. We prove its contrapositive.
For all $x \in \bigcap_{k=1}^m \overline{U_k}^c$, x is in each $\overline{U_k}^c$, so $\exists W_k \in \mathcal{O}_X$ with $W_k \cap U_k = \emptyset, x \in W_k$.

Hence, $\exists \bigcap_{k=1}^m W_k \in \mathcal{O}_X$ with $\bigcap_{k=1}^m W_k \cap \bigcup_{k=1}^m U_k = \emptyset, x \in \bigcap_{k=1}^m W_k$, so $x \in \overline{(\bigcup_{k=1}^m U_k)}$. Quod. Erat. Demonstrandum. $\square$

Proposition 3.10. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then:

$$\begin{array}{lccccccc} (1) & \left(\bigcap_{n=1}^{+\infty}\left[0,\frac{1}{n}\right]\right)^\circ & = & \emptyset & \subset & \{0\} & = & \bigcap_{n=1}^{+\infty}\left[0,\frac{1}{n}\right]^\circ; \\ (2) & \overline{\left(\bigcup_{n=1}^{+\infty}\left(\frac{1}{n},1\right)\right)} & = & [0,1] & \supset & (0,1] & = & \bigcup_{n=1}^{+\infty}\overline{\left[\frac{1}{n},1\right]}. \end{array}$$

Proposition 3.11. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then:

$$\begin{array}{lccccccc} (1) & ((-1,0]\cup[0,1))^\circ & = & (-1,1) & \supset & (-1,0)\cup(0,1) & = & (-1,0]^\circ \cup [0,1)^\circ; \\ (2) & \overline{([-1,0)\cap(0,1])} & = & \emptyset & \subset & \{0\} & = & \overline{[-1,0)} \cap \overline{(0,1]}. \end{array}$$

The above examples are trivial, let's investigate some elegant examples.

Definition 3.12. (**Cantor Set**)
Cantor set is a subset of $\mathbb{R}$ collecting all $a \in \mathbb{R}$ that can be written as $\sum_{n=1}^{+\infty} a_n 3^{-n}$, where each $a_n \in \{0,2\}$.

Proposition 3.13. Cantor set is not countable.

Proof. Assume to the contrary that Cantor set is countable, so it is the range of some sequence $\left(\sum_{n=1}^{+\infty} a_{n,m} 3^{-n}\right)_{m=1}^{+\infty}$. There exists $\sum_{n=1}^{+\infty} (2-a_{n,n})3^{-n}$ in Cantor set, such that for all $\sum_{n=1}^{+\infty} a_{n,m}3^{-n}$ in the range, the digits $2-a_{n,n}, a_{n,n}$ differ by 2, so the two series have different values, which implies $\sum_{n=1}^{+\infty}(2-a_{n,n})3^{-n}$ is not in the range, a contradiction. Hence, our assumption is false, and we've proven that Cantor set is uncountable. Quod. Erat. Demonstrandum. $\square$

Proposition 3.14. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then Cantor set is closed in $\mathbb{R}$.

Proof. For all $m \in \mathbb{N}$, construct the following set:

$$C_m = \left\{ a \in \mathbb{R} : a \text{ can be written as } \sum_{n=1}^{+\infty} a_n 3^{-n} \text{ where } a_n \in \{0,2\} \text{ whenever } n \leq m \right\}$$

C_m is a finite union of closed intervals, so C_m is closed in $\mathbb{R}$, and as a consequence, Cantor set $\cap_{m=1}^{+\infty} C_m$ is closed in $\mathbb{R}$. Quod. Erat. Demonstrandum. $\square$

Proposition 3.15. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then for all $\epsilon > 0$, there exists a sequence of open intervals $(I_n)_{n=1}^{+\infty}$, such that $\sum_{n=1}^{+\infty} \mu(I_n) < \epsilon$ and $\cup_{n=1}^{+\infty} I_n$ covers Cantor set, where $\mu(I)$ is the length of I.

Proof. Let $(C_m)_{m=1}^{+\infty}$ be the list of sets constructed in **Proposition 3.14.**. Notice that:

$$\forall n \in \mathbb{N}, \text{the total length of } C_m \text{ is } \left(\frac{2}{3}\right)^m$$

For all $\epsilon > 0$, on one hand, there exists $m \in \mathbb{N}$, such that $\left(\frac{2}{3}\right)^m < \frac{\epsilon}{2}$; On the other hand, there exists a sequence of open intervals $(I_n)_{n=1}^{+\infty}$, such that $\sum_{n=1}^{+\infty} < \left(\frac{2}{3}\right)^m + \frac{\epsilon}{2} < \epsilon$ and $\cup_{n=1}^{+\infty} I_n$ covers C_m. Hence, this sequence of open intervals also covers $\cup_{m=1}^{+\infty} C_m$, i.e., the Cantor set, with appropriate total length. Quod. Erat. Demonstrandum. $\square$

Lemma 3.16. In product topological space $(X, \mathcal{O}_X)$,
π_λ sends all $U \in \mathcal{O}_X$ to $\pi_\lambda(U) \in \mathcal{O}_{X_\lambda}$.

Proof. For all $U \in \mathcal{O}_X$, it is an arbitrary union of blocks, where each block is a finite intersection of sets in $\mathcal{B}_X$. As image set commutes with arbitrary union, so without loss of generality, we may assume that $U = \bigcap_{k=1}^m \pi_{\lambda_k}^{-1}(U_{\lambda_k})$, where each $U_{\lambda_k} \in \mathcal{O}_{X_{\lambda_k}}$.
Case 1: If λ equals to some λ_k, then $\pi_\lambda(U) = \pi_\lambda(\pi_\lambda^{-1}(U_\lambda)) = U_\lambda \in \mathcal{O}_{X_\lambda}$;
Case 2: If λ equals to no λ_k, then $\pi_\lambda(U) = \pi_\lambda(X) = X_\lambda \in \mathcal{O}_{X_\lambda}$.
In both cases, $\pi_\lambda(U) \in \mathcal{O}_{X_\lambda}$. Quod. Erat. Demonstrandum. $\square$

Proposition 3.17. In product topological space $(X, \mathcal{O}_X)$:

$$\left(\prod_{\lambda \in I} U_\lambda\right)^\circ \subseteq \prod_{\lambda \in I} U_\lambda^\circ$$

Proof. For all $x \in \left(\prod_{\lambda \in I} U_\lambda\right)^\circ$, $\exists V \in \mathcal{O}_X$ with $V \subseteq \prod_{\lambda \in I} U_\lambda, x \in V$.
Hence, for all $\lambda \in I$, $\exists \pi_\lambda(V) \in \mathcal{O}_X$ with $\pi_\lambda(V) \subseteq U_\lambda, x(\lambda) \in \pi_\lambda(V)$, so $x \in \prod_{\lambda \in I} U_\lambda^\circ$.
Quod. Erat. Demonstrandum. $\square$

Proposition 3.18. In product topological space $(X, \mathcal{O}_X)$:

$$\left(\prod_{k=1}^m U_k\right)^\circ = \prod_{k=1}^m U_k^\circ$$

Proof. It suffices to prove $(\prod_{k=1}^m U_k)^\circ \supseteq \prod_{k=1}^m U_k^\circ$. We prove it directly.
For all $x \in \prod_{k=1}^m U_k^\circ$, each $x(k) \in U_k^\circ$, so $\exists V_k \in \mathcal{O}_{X_k}$ with $V_k \subseteq U_k, x(k) \in V_k$.
Hence, $\exists \bigcap_{k=1}^m \pi_k^{-1}(V_k) \in \mathcal{O}_X$ with $\bigcap_{k=1}^m \pi_k^{-1}(V_k) \subseteq \prod_{k=1}^m U_k, x \in \bigcap_{k=1}^m \pi_k^{-1}(V_k)$,
so $x \in (\prod_{k=1}^m U_k)^\circ$. Quod. Erat. Demonstrandum. $\square$

Lemma 3.19. In product topological space $(X, \mathcal{O}_X)$,
for all $\sigma : Y \to X$, σ is continuous if and only if each $\pi_\lambda \circ \sigma$ is continuous.

Proof. We may divide our proof into two parts.
"if" direction: Assume that each $\pi_\lambda \circ \sigma$ is continuous.
For all $U \in \mathcal{O}_X$, it is an arbitrary union of blocks, where each block is a finite intersection of sets in $\mathcal{B}_X$. As image set commutes with arbitrary union, so without loss of generality, we may assume that $U = \bigcap_{k=1}^{m} \pi_{\lambda_k}^{-1}(U_{\lambda_k})$, where each $U_{\lambda_k} \in \mathcal{O}_{\lambda_k}$.
Hence, $\sigma^{-1}(U) = \bigcap_{k=1}^{m} (\pi_{\lambda_k} \circ \sigma)^{-1}(U_{\lambda_k}) \in \mathcal{O}_Y$, so σ is continuous.
"only if" direction: Assume that σ is continuous.
For all $\lambda \in I$, σ, π_λ are continuous, so is their composition $\pi_\lambda \circ \sigma$.
Combine the two parts together, we've proven the biconditional.
Quod. Erat. Demonstrandum. □

Proposition 3.20. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then $\prod_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$ is not open in product topological space $\mathbb{R}^{\mathbb{N}}$.

Proof. Assume to the contrary that $\prod_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$ is open in $\mathbb{R}^{\mathbb{N}}$.
Construct $\Delta : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}, \Delta(\xi) = (\xi)_{n=1}^{+\infty}$.
On one hand, $\Delta^{-1}\left(\prod_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)\right) = \{0\}$ is not open in $\mathbb{R}$, so Δ is discontinuous.
On the other hand, each composition function $\pi_n \circ \Delta = id_{\mathbb{R}}$ is continuous.
This violates **Lemma 3.19.**, so our assumption is false,
and we've proven that $\prod_{n=1}^{+\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$ is not open in $\mathbb{R}^{\mathbb{N}}$. Quod. Erat. Demonstrandum. □

Proposition 3.21. If we regard $\mathbb{R}$ as a metric space with Euclidean metric, then in product topological space, $\left(\prod_{n=1}^{+\infty} \left(0, \frac{1}{n}\right)\right)^\circ \subset \prod_{n=1}^{+\infty} \left(0, \frac{1}{n}\right) = \prod_{n=1}^{+\infty} \left(0, \frac{1}{n}\right)^\circ$.

Proof. As $\prod_{n=1}^{+\infty} \left(0, \frac{1}{n}\right)$ is not open in $\mathbb{R}^N$, its interior is a proper subset of itself.
Quod. Erat. Demonstrandum. □

4 Hausdorff Space

4.1 Definition of Hausdorff Space

Definition 4.1. (**Definition of Hausdorff Space**)
Let $(X, \mathcal{O}_X)$ be a topological space.
If for all distinct $x_1, x_2 \in X$, there exist $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$, then $(X, \mathcal{O}_X)$ is Hausdorff.

Proposition 4.2. Every metric space (X, d_X) is Hausdorff.

Proof. For all distinct $x_1, x_2 \in X$, there exist $B(x_1, r), B(x_2, r) \in \mathcal{O}_X$, where $r = \frac{1}{2}d_X(x_1, x_2)$, such that $x_1 \in B(x_1, r)$ and $x_2 \in B(x_2, r)$ and $B(x_1, r) \cap B(x_2, r) = \emptyset$, so $(X, \mathcal{O}_X)$ is Hausdorff. Quod. Erat. Demonstrandum. ☐

Proposition 4.3. If X has at least two distinct elements x_1, x_2, then the indiscrete topological space $(X, \mathcal{O}_X)$ is not Hausdorff.

Proof. There exist distinct $x_1, x_2 \in X$, such that for all $O_1, O_2 \in \mathcal{O}_X$:

$$x_1 \in O_1 \text{ and } x_2 \in O_2 \implies O_1 = O_2 = X \implies O_1 \cap O_2 \neq \emptyset$$

So the three conditions fail to hold simultaneously, which implies $(X, \mathcal{O}_X)$ is not Hausdorff. Quod. Erat. Demonstrandum. ☐

Proposition 4.4. The discrete topological space $(X, \mathcal{O}_X)$ is induced by the following metric $d_X : X \times X \to \mathbb{R}$:

$$d_X(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_1 = x_2; \\ 1 & \text{if} \quad x_1 \neq x_2; \end{cases}$$

So $(X, \mathcal{O}_X)$ is Hausdorff.

Proof. It suffices to show that the metric topology $\mathcal{O}_X$ contains $\mathcal{P}(X)$.
For all $O \in \mathcal{P}(X)$, for all $x \in O$, there exists $1 > 0$, such that $B(x, 1) = \{x\} \subseteq O$, so $O \in \mathcal{O}_X$, which implies $\mathcal{P}(X) \subseteq \mathcal{O}_X$. Quod. Erat. Demonstrandum. ☐

Proposition 4.5. If X is finite, then the finite complement topological space and the discrete topological space are identical, thus Hausdorff.

Proposition 4.6. If X is infinite, then the finite complement topological space $(X, \mathcal{C}_X)$ is not Hausdorff.

Proof. There exist distinct $x_1, x_2 \in X$, such that for all $C_1, C_2 \in \mathcal{C}_X$:

$$x_1 \notin C_1 \text{ and } x_2 \notin C_2 \implies C_1, C_2 \text{ are finite} \implies C_1 \cup C_2 \text{ is finite} \implies C_1 \cup C_2 \neq X$$

So the three conditions fail to hold simultaneously, which implies $(X, \mathcal{C}_X)$ is not Hausdorff. Quod. Erat. Demonstrandum. ☐

Proposition 4.7. If a field $\mathbb{F}$ is finite, then the Zariski topological space and the discrete topological space are identical, thus Hausdorff.

Proof. It suffices to show that the Zariski topology $\mathcal{C}_X$ contains all singleton $\{(\xi_l)_{l=1}^n\}$. Define $T = \{x_l - \xi_l\}_{l=1}^n \subseteq \mathcal{P}(\mathbb{F}[x_l]_{l=1}^n)$. The solution set of T is $\{(\xi_l)_{l=1}^n\} \in \mathcal{C}_X$.
Quod. Erat. Demonstrandum. □

Lemma 4.8. If a field $\mathbb{F}$ is infinite, then the Zariski topological space $\mathbb{F}$ and the finite complement topological space are identical, thus Hausdorff.

Proof. For all $C \in \mathcal{P}(X)\backslash\{X\}$:

$$\begin{aligned} C \text{ is in the Zariski topology} \quad &\iff \quad \exists f(x_l)_{l=1}^n \in \mathbb{F}[x_l]_{l=1}^n \text{ with } \deg f(x_l)_{l=1}^n > -\infty \\ &\qquad\qquad C \text{ is the solution set of } f(x_l)_{l=1}^n \\ &\iff \quad C \text{ is finite} \\ &\iff \quad C \text{ is in the finite complement topology} \end{aligned}$$

Quod. Erat. Demonstrandum. □

Lemma 4.9. If a topological space $(X, \mathcal{O}_X)$ is Hausdorff, then its subspace $(X', \mathcal{O}_{X'})$ is also Hausdorff.

Proof. For all distinct $x_1, x_2 \in X'$, they can be regarded as points in X, so there exist $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. Hence, there exists $O_1 \cap X', O_2 \cap X' \in \mathcal{O}_{X'}$, such that $x_1 \in O_1 \cap X'$ and $x_2 \in O_2 \cap X'$ and $(O_1 \cap X') \cap (O_2 \cap X') = \emptyset$. Quod. Erat. Demonstrandum. □

Proposition 4.10. If a field $\mathbb{F}$ is infinite, then the Zariski topological space $\mathbb{F}^n$ is not Hausdorff, where $n \geq 2$.

Proof. Assume to the contrary that the Zariski topological space $\mathbb{F}^n$ is Hausdorff, then its subspace $\mathbb{F} \times \{0\}^{n-1}$ is Hausdorff. But this subspace is homeomorphic to the Zariski topological space $\mathbb{F}$, which is not Hausdorff, a contradiction.
Hence, the assumption is false, and we've proven that the Zariski topological space $\mathbb{F}^n$ is not Hausdorff. Quod. Erat. Demonstrandum. □

4.2 Related Properties in Hausdorff Space

Proposition 4.11. $(X, \mathcal{O}_X)$ is Hausdorff if and only if the image of the diagonal map $\Delta : X \to X \times X, \Delta(x) = (x, x)$ is closed in $X \times X$.

Proof. We may divide our proof into two parts.
"if" direction: Assume that Image is closed in $X \times X$.
For all distinct $x_1, x_2 \in X$, as $(x_1, x_2) \in \text{Image}^c$, where Image is closed, $\exists O \in \mathcal{O}_X$ with $O \cap \text{Image} = \emptyset, (x_1, x_2) \in O$. Without loss of generality, assume that $O = \pi_1^{-1}(O_1) \cap \pi_2^{-1}(O_2)$ for some $O_1, O_2 \in \mathcal{O}_X$. For all $x \in X$, $x \in O_1 \cap O_2 \implies (x, x) \in \pi_1^{-1}(O_1) \cap$

$\pi_2^{-1}(O_2) \implies \mathbb{F}$, so $O_1 \cap O_2 = \emptyset$ and X is Hausdorff.
"only if" direction: Assume that $(X, \mathcal{O}_X)$ is Hausdorff.
For all $(x_1, x_2) \in \text{Image}^c$, x_1, x_2 are distinct elements of X. As $(X, \mathcal{O}_X)$ is Hausdorff, there exists $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$, so $\exists O_1 \times O_2$ with $(O_1 \times O_2) \cap \text{Image} = \emptyset, (x_1, x_2) \in O_1 \times O_2$, so Image is closed in $X \times X$.
Quod. Erat. Demonstrandum. $\square$

Proposition 4.12. In finite product topological space $(X, \mathcal{O}_X)$, $(X, \mathcal{O}_X)$ is Hausdorff if and only if each $(X_k, \mathcal{O}_{X_k})$ is Hausdorff.

Proof. We may divide our proof into two parts.
"if" direction: Assume that each $(X_k, \mathcal{O}_{X_k})$ is Hausdorff.
For each k, $\forall$ distinct $x_1(k), x_2(k) \in X_k, \exists O_{1,k}, O_{2,k} \in \mathcal{O}_{X_k}, x_1(k) \in O_{1,k}$ and $x_2(k) \in O_{2,k}$ and $O_{1,k} \cap O_{2,k} = \emptyset$. Hence, $\exists O_1 = \prod_{k=1}^m O_{1,k}, O_2 = \prod_{k=1}^m O_{2,k} \in \mathcal{O}_X, x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$, so $(X, \mathcal{O}_X)$ is Hausdorff.
"only if" direction: Assume that $(X, \mathcal{O}_X)$ is Hausdorff.
For each k, pick one element $\xi_k \in X_k$;
For each k, for all distinct $\xi_{1,k}, \xi_{2,k} \in X_k$ define the following two functions:

$$x_1, x_2 \in \mathbb{R}^m, x_1(s) = \begin{cases} \xi_{1,k} & \text{if} \quad s = k; \\ \xi_s & \text{if} \quad s \neq k; \end{cases}, x_2(s) = \begin{cases} \xi_{2,k} & \text{if} \quad s = k; \\ \xi_s & \text{if} \quad s \neq k; \end{cases}$$

As x_1, x_2 are two distinct elements in the Hausdorff space $(X, \mathcal{O}_X)$, there exists $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.
Without loss of generality, we may assume that $O_1 = \bigcap_{s=1}^m \pi_s^{-1}(O_{1,s})$ and $O_2 = \bigcap_{s=1}^m \pi_s^{-1}(O_{2,s})$, where each $O_{1,s}, O_{2,s} \in \mathcal{O}_s$.
For each $s \neq k$, $\xi_s \in O_{1,s} \cap O_{2,s}$, so $\pi_s^{-1}(O_{1,s}) \cap \pi_s^{-1}(O_{2,s}) = \pi_s^{-1}(O_{1,s} \cap O_{2,s}) \neq \emptyset$.
But $\bigcap_{s=1}^m \pi_s^{-1}(O_{1,s}) \cap \bigcap_{s=1}^m \pi_s^{-1}(O_{2,s}) = \bigcap_{s=1}^m \pi_s^{-1}(O_{1,s} \cap O_{2,s}) = \emptyset$, so the set $O_{1,k} \cap O_{2,k}$ must be empty as it controls the unique output that may lead to trouble.
Hence, $(X_k, \mathcal{O}_{X_k})$ is Hausdorff.
Combine the two parts above, we've proven the biconditional.
Quod. Erat. Demonstrandum. $\square$

Proposition 4.13. If $(X, \mathcal{O}_X)$ is Hausdorff, then every sequence $(x_n)_{n=1}^{+\infty}$ in X has at most one limit.

Proof. Assume to the contrary that $(x_n)_{n=1}^{+\infty}$ has two distinct limits x_*, x^*.
As $(X, \mathcal{O}_X)$ is Hausdorff, there exists $O_1, O_2 \in \mathcal{O}_X$, such that $x_1 \in O_1$ and $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. However, there exists $N_1, N_2 \in \mathbb{N}$, such that $\{x_n\}_{n=N_1}^{+\infty} \subseteq O_1$ and $\{x_n\}_{n=N_2}^{+\infty} \subseteq O_2$, which leaves $x_{\max\{N_1, N_2\}}$ no where to go.
Quod. Erat. Demonstrandum. $\square$

References

[1] H. Ren, “Template for math notes,” 2021.