Elliptic Functions, Part 1

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Contents

1 Compact Riemann Surfaces of Genus 1 and Lattices

Definition 1.1 (Genus)**.** The genus $g(X)$ of a compact Riemann surface X is a topological invariant that counts the number of "holes" in the surface.

- $g = 0$: the Riemann sphere $\mathbb{P}^1$ (no holes),
- $g = 1$: a torus (one hole),
- in general, g holes correspond to a g-torus.

Thus, a genus 1 compact Riemann surface is topologically a torus.

Definition 1.2 (Universal Cover)**.** Let X be a connected topological space. A *universal cover* of X is a simply connected space

$$\pi : \tilde{X} \to X$$

such that every covering of X factors through π. Here "simply connected" means $\pi_1(\tilde{X}) = 0$, i.e. $\tilde{X}$ has no nontrivial loops.

Example 1.3. The universal cover of the circle S^1 is $\mathbb{R}$ with covering map $\pi(t) = e^{2\pi i t}$.

The universal cover of a torus is $\mathbb{R}^2$, with covering map given by projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$.

Theorem 1.4 (Uniformization Theorem)**.** *Let X be a simply connected Riemann surface. Then X is biholomorphically equivalent to exactly one of:*

1. $\mathbb{P}^1$ *(the Riemann sphere),*
2. $\mathbb{C}$ *(the complex plane),*
3. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *(the unit disk).*

Proposition 1.5. *If X is a compact Riemann surface of genus 1, then its universal cover is $\mathbb{C}$.*

Proof. By the Uniformization Theorem, the universal cover $\tilde{X}$ of X is biholomorphic to one of $\mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{D}$.

Suppose first that $\tilde{X} = \mathbb{P}^1$. Then X would arise as a quotient of $\mathbb{P}^1$ by a discrete group of automorphisms. Since $\mathbb{P}^1$ is simply connected of genus 0, no such quotient can produce a compact Riemann surface of genus 1.

Next, suppose $\tilde{X} = \mathbb{D}$. Quotients of the unit disk by discrete groups of automorphisms carry a hyperbolic metric and yield compact Riemann surfaces of genus at least 2. Hence this case is also impossible for X.

The only remaining possibility is that $\tilde{X} \cong \mathbb{C}$, and therefore the universal cover of a genus 1 compact Riemann surface is biholomorphic to $\mathbb{C}$. □

Definition 1.6 (Lattice)**.** A *lattice* in $\mathbb{C}$ is a discrete additive subgroup of the form

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\},$$

where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

Proposition 1.7. *Let $\Gamma \subset \mathrm{Aut}(\mathbb{C})$ be a discrete subgroup acting freely on $\mathbb{C}$. Then exactly two cases can occur:*

1. *Γ is generated by one element. In this case, $\Gamma = \{n\omega : n \in \mathbb{Z}\}$ for some $\omega \neq 0$, and the quotient $\mathbb{C}/\Gamma$ is biholomorphic to $\mathbb{C}^\times$, which is not compact.*
2. *Γ is generated by two $\mathbb{R}$–linearly independent elements $\omega_1, \omega_2 \in \mathbb{C}$. In this case*

$$\Gamma = L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

which is a lattice in $\mathbb{C}$, and the quotient $\mathbb{C}/L$ is a compact Riemann surface of genus 1.

Proof. We first note that

$$\mathrm{Aut}(\mathbb{C}) = \{\phi(z) = az + b : a \in \mathbb{C}^\times,\ b \in \mathbb{C}\}.$$

If $a \neq 1$, then $\phi(z) = az + b$ has a fixed point $z_0 = b/(1-a)$. Since Γ must act freely, such maps cannot occur in Γ. Therefore every nontrivial element of Γ must be a translation of the form $z \mapsto z + \ell$.

If Γ is generated by a single nonzero element ω, then the exponential map

$$\exp : \mathbb{C} \to \mathbb{C}^\times, \qquad z \mapsto e^z$$

has kernel $2\pi i\mathbb{Z}$, showing that $\mathbb{C}/\Gamma \cong \mathbb{C}^\times$. This surface is non-compact and has genus 0.

If Γ has two $\mathbb{R}$–linearly independent generators ω_1, ω_2, then Γ is a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The quotient $\mathbb{C}/L$ is compact, since the fundamental parallelogram

$$\{a\omega_1 + b\omega_2 : 0 \leq a, b < 1\}$$

maps bijectively to the quotient space. Topologically this quotient is a torus, hence a compact genus 1 Riemann surface. No other possibility for Γ exists. $\square$

Corollary 1.8. *Every compact Riemann surface X of genus 1 is isomorphic to a quotient*

$$X \cong \mathbb{C}/L,$$

where $L \subset \mathbb{C}$ is a lattice of rank 2.

Proof. This follows directly from the preceding proposition and the fact that the universal cover of a genus 1 compact Riemann surface is $\mathbb{C}$. $\square$

Theorem 1.9 (Moduli of elliptic curves)**.** *Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$ and define the lattice*

$$L_\tau = \mathbb{Z} + \mathbb{Z}\tau,$$

together with the elliptic curve (torus)

$$X_\tau := \mathbb{C}/L_\tau.$$

Then two elliptic curves X_τ and $X_{\tau'}$ are biholomorphic if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

such that

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

Consequently, the moduli space of compact Riemann surfaces of genus 1 is

$$\mathcal{M}_1 \;\cong\; \mathbb{H}/\,\mathrm{SL}(2, \mathbb{Z}).$$

Proof. Every lattice in $\mathbb{C}$ has the form $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\Im(\omega_2/\omega_1) > 0$. Scaling by ω_1^{-1} yields

$$L = \omega_1(\mathbb{Z} + \mathbb{Z}\tfrac{\omega_2}{\omega_1}) = \omega_1 L_\tau, \quad \tau = \frac{\omega_2}{\omega_1} \in \mathbb{H}.$$

Since multiplication by ω_1 gives a biholomorphism $\mathbb{C}/L_\tau \to \mathbb{C}/L$, any elliptic curve is isomorphic to one of the form X_τ.

Suppose $X_\tau \cong X_{\tau'}$. Then there exists a biholomorphism $f : \mathbb{C}/L_\tau \to \mathbb{C}/L_{\tau'}$. Lifting to the universal cover, we get a biholomorphism $\tilde{f} : \mathbb{C} \to \mathbb{C}$ such that $\tilde{f}(L_\tau) = L_{\tau'}$. Since $\mathrm{Aut}(\mathbb{C})$ consists of affine transformations $z \mapsto az + b$, and $\tilde{f}$ must map lattice to lattice, we have $\tilde{f}(z) = \alpha z$ for some $\alpha \in \mathbb{C}^\times$.

Therefore $L_{\tau'} = \alpha L_\tau$, which means

$$\{\alpha, \alpha\tau\} = \{1, \tau'\}$$

as $\mathbb{Z}$-bases. This gives us two cases:

1. $\alpha = 1$ and $\alpha\tau = \tau'$, so $\tau' = \tau$.

2. $\alpha = \tau'$ and $\alpha\tau = 1$, so $\tau'\tau = 1$, giving $\tau' = -1/\tau$.

More generally, there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$\begin{pmatrix} 1 \\ \tau' \end{pmatrix} = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \alpha \begin{pmatrix} a + b\tau \\ c + d\tau \end{pmatrix}.$$

This gives us $1 = \alpha(a + b\tau)$ and $\tau' = \alpha(c + d\tau)$. Solving for α and substituting:

$$\tau' = \frac{c + d\tau}{a + b\tau} = \frac{d\tau + c}{b\tau + a} = \frac{a\tau + c}{b\tau + d}$$

(after swapping $a \leftrightarrow d$ and $b \leftrightarrow c$).

Since both lattices have the same orientation in $\mathbb{C}$, we need $\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc = 1$, so the matrix is in $\mathrm{SL}(2, \mathbb{Z})$.

Conversely, if $\tau' = \frac{a\tau+b}{c\tau+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, then the map $z \mapsto (c\tau + d)z$ gives an isomorphism between X_τ and $X_{\tau'}$.

Therefore the set of isomorphism classes of compact genus 1 Riemann surfaces is exactly $\mathbb{H}/\,\mathrm{SL}(2, \mathbb{Z})$. $\square$

2 Meromorphic Functions on Elliptic Curves

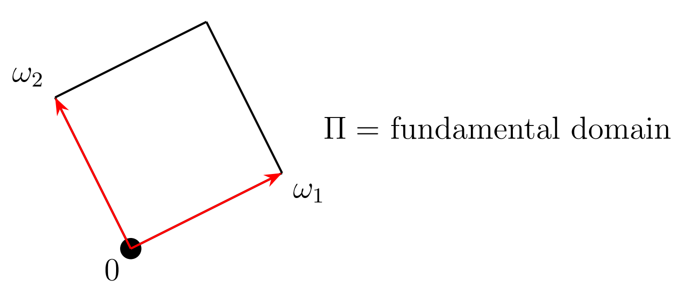
2.1 Elliptic Functions and the Construction Problem

Now fix a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where ω_1 and ω_2 are linearly independent over $\mathbb{R}$. Write $X = \mathbb{C}/L$.

Objective: To study $\mathcal{M}(X)$, the field of meromorphic functions on X, and answer three principal questions.

Problem 2.1 (Question 1)**.** Find an example of $f \in \mathcal{M}(X)$ with $f \not\equiv 0$.

Remark 2.2. Since $X = \mathbb{C}/L$ is compact, any holomorphic function on X must be constant by Liouville's theorem. Therefore, all non-constant meromorphic functions on X must have poles.



Definition 2.3. An *elliptic function* with respect to a lattice $L \subset \mathbb{C}$ is a meromorphic function

$$f : \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$$

such that

$$f(z+w) = f(z) \quad \text{for all } z \in \mathbb{C},\ w \in L.$$

Remark 2.4. Equivalently, an elliptic function is a *doubly periodic* meromorphic function with respect to L. That is, if $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with ω_1, ω_2 linearly independent over $\mathbb{R}$, then

$$f(z+\omega_1) = f(z), \qquad f(z+\omega_2) = f(z) \quad \text{for all } z \in \mathbb{C}.$$

By linearity this implies

$$f\big(z + n_1\omega_1 + n_2\omega_2\big) = f(z) \quad \text{for all } z \in \mathbb{C},\ n_1, n_2 \in \mathbb{Z}.$$

We denote by $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ the Riemann sphere, and recall that the elliptic curve associated to a lattice L is the quotient $X = \mathbb{C}/L$.

Convention. Strictly speaking, an elliptic function is a meromorphic and doubly periodic function

$$f : \mathbb{C} \longrightarrow \mathbb{P}^1$$

with respect to L. However, since $f(z+w) = f(z)$ for all $w \in L$, the function is invariant under translations by elements of L. Therefore f descends to a well-defined meromorphic function on the quotient X:

$$f \in \mathcal{M}(X).$$

In this way, we usually regard elliptic functions as precisely the meromorphic functions on X.

To answer Problem 2.1, we want an explicit example of a nonconstant meromorphic function

$$f \in \mathcal{M}(X), \quad f \not\equiv 0.$$

A general strategy is the following: if Γ is a finite group acting on $\mathbb{C}$ and h is any function on $\mathbb{C}$, then we may form the averaged function

$$h^*(z) := \sum_{\gamma \in \Gamma} h(\gamma z).$$

Lemma 2.5. *The function h^* is invariant under the action of Γ, i.e.*

$$h^*(\gamma_0 z) = h^*(z) \qquad \textit{for all } \gamma_0 \in \Gamma.$$

Proof. Fix $\gamma_0 \in \Gamma$. Then

$$h^*(\gamma_0 z) = \sum_{\mu \in \Gamma} h\big(\mu(\gamma_0 z)\big) = \sum_{\mu \in \Gamma} h\big((\mu\gamma_0)(z)\big).$$

Since $\mu \mapsto \mu\gamma_0$ permutes the group Γ, this sum can be rewritten

$$\sum_{\nu \in \Gamma} h(\nu z) = h^*(z).$$

$\square$

2.2 Eisenstein Series

Theorem 2.6 (Convergence of Lattice Power Sums)**.** *Given an "appropriate function" h on $\mathbb{C}$, we try to define $f(z) = \sum_{w\in L} h(z+w)$, which is well-defined if $\sum_{w\in L} |h(z+w)| < \infty$.*

Take $h(z) = \frac{1}{z^k}$. Then $\sum_{w\in L} \left|\frac{1}{(z+w)^k}\right|$ converges if and only if $k \geq 3$.

Proof. Write $w = n_1\omega_1 + n_2\omega_2$ for lattice points. Define:

$$S_0 = \{0\} \tag{1}$$
$$S_1 = \{n_1\omega_1 + n_2\omega_2 : -1 \leq n_1, n_2 \leq 1\} \tag{2}$$
$$T_1 = S_1 \setminus S_0 = \{n_1\omega_1 + n_2\omega_2 : n_1 = \pm 1 \text{ or } n_2 = \pm 1\} \tag{3}$$
$$S_2 = \{n_1\omega_1 + n_2\omega_2 : -2 \leq n_1, n_2 \leq 2\} \tag{4}$$
$$T_2 = S_2 \setminus S_1 = \{n_1\omega_1 + n_2\omega_2 : n_1 = \pm 2 \text{ or } n_2 = \pm 2\} \tag{5}$$

More generally, $T_n = S_n \setminus S_{n-1}$ for $n \geq 1$.

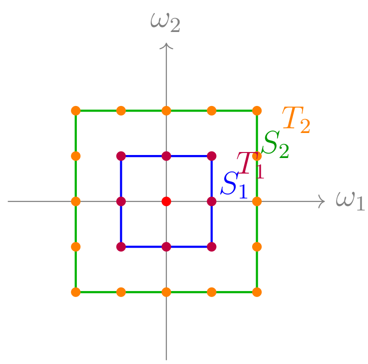


Figure 1: Lattice points for S_2. The red point is the origin. S_1 (blue square) and S_2 (green square) are shown. The purple points form $T_1 = S_1 \setminus S_0$, and the orange points form $T_2 = S_2 \setminus S_1$.

We have $|S_n| = (2n+1)^2$ and $|T_n| = |S_n \setminus S_{n-1}| = 8n$ for $n \geq 1$.

Let $z \notin L$. The question becomes: when is $\sum_{n=0}^{\infty} \sum_{w\in T_n} \left|\frac{1}{(w+z)^k}\right| < \infty$?

Fix $z_0 \notin L$. The question is the same as asking when $\sum_{w\in L^\times} \frac{1}{|w|^k} < \infty$, where $L^\times = L \setminus \{0\}$.

To be more precise, we can write:

$$\sum_{n=0}^{\infty} \sum_{w\in T_n} \left|\frac{1}{(z_0+w)^k}\right| = \sum_{n=0}^{s} \sum_{w\in T_n} \left|\frac{1}{(z_0+w)^k}\right| + \sum_{n=s+1}^{\infty} \sum_{w\in T_n} \left|\frac{1}{(z_0+w)^k}\right|$$

For sufficiently large s, we have $\forall n \geq s+1, \forall w \in T_n$:

$$\frac{|w|}{2} < |z_0 + w| < \frac{3}{2}|w|$$

by applying the triangle inequality. Then there exist constants $c_1, c_2 > 0$ such that:

$$c_1 \sum_{n=s+1}^{\infty} \sum_{w \in T_n} \frac{1}{|w|^k} \leq \sum_{n=s+1}^{\infty} \sum_{w \in T_n} \frac{1}{|z_0 + w|^k} \leq c_2 \sum_{n=s+1}^{\infty} \sum_{w \in T_n} \frac{1}{|w|^k}$$

Hence the absolute convergence of $\sum_{w \in L} \left| \frac{1}{(z_0+w)^k} \right|$ holds if and only if $\sum_{w \in L^\times} \frac{1}{|w|^k} < \infty$.

There exist constants $a, b > 0$ such that $\forall w \in T_n$: $an \leq |w| \leq bn$. To check this for T_1, write Π_1 for the first parallelogram so that $T_1 \subset \partial \Pi_1$ and $T_n \subset \partial \Pi_n$ where $\Pi_n = \{nz : z \in \Pi_1\}$.

If we choose $a, b > 0$ such that:

$$a = \min_{\xi \in \partial \Pi_1} |\xi| \tag{6}$$

$$b = \max_{\xi \in \partial \Pi_1} |\xi| \tag{7}$$

then $a \leq \frac{|w|}{n} \leq b$, so $an \leq |w| \leq bn$ holds for all $w \in T_n$, since $T_n \subset \partial \Pi_n$ and $w \in T_n \Rightarrow \frac{w}{n} \in \partial \Pi_1$.

Hence:

$$\frac{1}{b^k} \left(\sum_{n=1}^{\infty} \frac{8n}{n^k} \right) \leq \sum_{w \in L^\times} \frac{1}{|w|^k} \leq \frac{1}{a^k} \left(\sum_{n=1}^{\infty} \frac{8n}{n^k} \right) \tag{8}$$

$$\frac{8}{b^k} \left(\sum_{n=1}^{\infty} \frac{1}{n^{k-1}} \right) \leq \sum_{w \in L^\times} \frac{1}{|w|^k} \leq \frac{8}{a^k} \left(\sum_{n=1}^{\infty} \frac{1}{n^{k-1}} \right) \tag{9}$$

Conclusion: $\sum_{w \in L^\times} \frac{1}{|w|^k} < \infty \Leftrightarrow k - 1 \geq 2$, i.e., $k \geq 3$. □

Corollary 2.7 (Eisenstein Series Construction)**.** *When $k \geq 3$, we can define the* Eisenstein series*:*

$$E_k(z) = \sum_{w \in L} \frac{1}{(z+w)^k}$$

where E stands for Eisenstein.
The function E_k has a pole of order k at lattice points and no other poles. Moreover, E_k is doubly periodic with respect to L and hence descends to a meromorphic function on the elliptic curve $X = \mathbb{C}/L$.

2.3 From Lattice Sums to Elliptic Functions

Lemma 2.8. *Suppose that the series*

$$\sum_{w \in L} |h(z+w)|$$

is absolutely convergent and uniformly convergent on any given compact set of $\mathbb{C}$, after removing finitely many terms corresponding to poles. Then the function

$$f(z) = \sum_{w \in L} h(z+w)$$

defines a doubly periodic meromorphic function, i.e. an elliptic function with respect to the lattice L.

Proof. By assumption, the series converges absolutely and locally uniformly outside the poles. Hence $f(z)$ defines a meromorphic function on $\mathbb{C}$ with poles only at lattice translates of the poles of h.

It remains to check periodicity. Let $w_0 \in L$. Then

$$f(z+w_0) = \sum_{w \in L} h\big((z+w_0)+w\big).$$

Since $w \mapsto w+w_0$ is a bijection of L, we may reindex the sum. Absolute convergence justifies this rearrangement:

$$f(z+w_0) = \sum_{w \in L} h\big(z+(w+w_0)\big) = \sum_{w' \in L} h(z+w') = f(z).$$

Thus f is periodic with respect to every $w_0 \in L$, hence doubly periodic with respect to the lattice L. □

Remark 2.9. There exists an elliptic function $\wp$, called the *Weierstrass $\wp$-function*, which arises from regularizing the divergent series

$$\sum_{w \in L} \frac{1}{(z+w)^2}.$$

It satisfies the fundamental relation

$$\wp'(z) = -2E_3(z),$$

where E_3 denotes the Eisenstein series of weight 3. Thus $\wp$ may be viewed as the prototype elliptic function associated with the lattice L.

2.4 General Properties of Elliptic Functions

Theorem 2.10 (General Properties of Elliptic Functions)**.** *Let Π be a fundamental domain for $X = \mathbb{C}/L$. Suppose f is an elliptic function such that f has no zeros nor poles on $\partial\Pi$. Then the following holds true:*

(a) $\sum_{a_k \in P(f|_\Pi)} Res(f; a_k) = 0$

(b) $\sum_{a_k \in Z(f|_\Pi)} ord_{a_k}(f) + \sum_{b_l \in P(f|_\Pi)} ord_{b_l}(f) = 0$

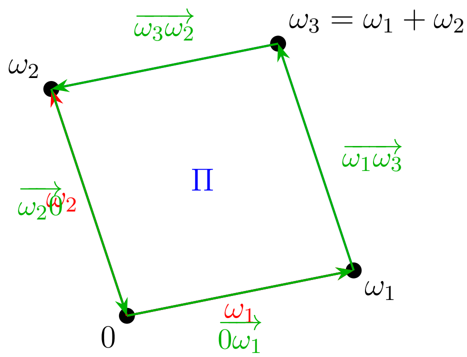
(c) $\sum_{a_k \in Z(f|_\Pi)} ord_{a_k}(f) \cdot a_k \equiv \sum_{b_l \in P(f|_\Pi)} ord_{b_l}(f) \cdot b_l \pmod{L}$

Proof. **Part (a):** By the Residue Theorem, we have

$$\frac{1}{2\pi i}\int_{\partial\Pi} f(\xi)d\xi = \sum_{a_k \in P(f|_\Pi)} \mathrm{Res}(f; a_k) = \sum_{a \in \Pi}{}' \mathrm{Res}(f, a)$$

where $\sum'$ is the sum over a finite number of possibly nonzero terms, and we write $\mathrm{Res}(f, a) = 0$ whenever f is holomorphic at a.

Fundamental Parallelogram with Oriented Boundary



To prove (a), it remains to show that $\int_{\partial\Pi} f(\xi)d\xi = 0$.
Note that

$$\begin{aligned}
\int_{\partial\Pi} f(\xi)d\xi &= \int_{\overrightarrow{0\omega_1}} f(\xi)d\xi + \int_{\overrightarrow{\omega_1,\omega_1+\omega_2}} f(\xi)d\xi && (10)\\
&\quad + \int_{\overrightarrow{\omega_1+\omega_2,\omega_2}} f(\xi)d\xi + \int_{\overrightarrow{\omega_2,0}} f(\xi)d\xi && (11)\\
&= \underbrace{\left(\int_{\overrightarrow{0\omega_1}} - \int_{\overrightarrow{\omega_2,\omega_1+\omega_2}}\right)}_{\mathrm{I}} + \underbrace{\left(\int_{\overrightarrow{\omega_1,\omega_1+\omega_2}} - \int_{\overrightarrow{0\omega_2}}\right)}_{\mathrm{II}} && (12)
\end{aligned}$$

We will show that $\mathrm{I} = 0$ and $\mathrm{II} = 0$ follows easily.
For I, parametrize $\overrightarrow{0\omega_1}$ by $\xi(t) = t\omega_1$, where $\xi : [0,1] \xrightarrow{\cong} \overrightarrow{0\omega_1}$. Then we have

$$\begin{aligned}
\int_{\overrightarrow{0\omega_1}} f(\xi)d\xi &\overset{\text{def}}{=} \int_0^1 f(t\omega_1)\omega_1 dt && (13)\\
&= \int_0^1 f(t\omega_1 + \omega_2)\omega_1 dt \quad \text{(by ellipticity)} && (14)\\
&= \int_{\overrightarrow{\omega_2,\omega_1+\omega_2}} f(\xi)d\xi && (15)
\end{aligned}$$

Part (b): Consider $g(z) = \frac{f'(z)}{f(z)}$, which is elliptic. Apply part (a): $\operatorname{Res}\left(\frac{f'}{f}\right) = \operatorname{ord}_a(f)$, so

$$\sum_{a\in\Pi} \operatorname{ord}_a(f) = \sum_{a_i\in\Pi} \operatorname{Res}(g, a_i) = 0$$

Part (c): The proof is equivalent to showing $\sum'_{a\in\Pi} \operatorname{ord}_a(f)\cdot a \equiv 0 \pmod{L}$.
Proof by Residue Theorem applied to $z\frac{f'(z)}{f(z)}$. First, we analyze the poles of $\frac{f'(z)}{f(z)}$:
If a is a zero of f of order $m > 0$, then near a we have $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$. Thus

$$\frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1}g(z) + (z-a)^m g'(z)}{(z-a)^m g(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$$

Since g is holomorphic and nonzero at a, we see that $\frac{f'}{f}$ has a simple pole at a with residue $m = \mathrm{ord}_a(f)$.

If b is a pole of f of order $n > 0$, then near b we have $f(z) = (z-b)^{-n}h(z)$ where $h(b) \neq 0$. Thus

$$\frac{f'(z)}{f(z)} = \frac{-n(z-b)^{-n-1}h(z) + (z-b)^{-n}h'(z)}{(z-b)^{-n}h(z)} = \frac{-n}{z-b} + \frac{h'(z)}{h(z)}$$

So $\frac{f'}{f}$ has a simple pole at b with residue $-n = \mathrm{ord}_b(f)$.

Therefore, $\frac{f'(z)}{f(z)}$ has only simple poles, and at each point $a \in \Pi$ (zero or pole of f), the residue is exactly $\mathrm{ord}_a(f)$.

Now, using the formula $\mathrm{Res}(gh; a) = g(a)\mathrm{Res}(h; a)$ where h has a simple pole at a and g is holomorphic at a, we get:

$$\frac{1}{2\pi i}\int_{\partial\Pi} \xi \frac{f'(\xi)}{f(\xi)} d\xi = \sum_{a\in\Pi}{}' \mathrm{Res}\left(z\frac{f'(z)}{f(z)}, a\right) = \sum_{a\in\Pi}{}' a \cdot \mathrm{ord}_a(f)$$

We have

$$\int_{\partial\Pi} \xi \frac{f'(\xi)}{f(\xi)} d\xi = \underbrace{\int_{\overrightarrow{0\omega_1}} - \int_{\overrightarrow{\omega_2,\omega_1+\omega_2}}}_{\mathrm{I}'} + \underbrace{\int_{\overrightarrow{\omega_1,\omega_1+\omega_2}} - \int_{\overrightarrow{0\omega_2}}}_{\mathrm{II}'}$$

Using the parametrization in part (a):

$$\int_{\overrightarrow{0\omega_1}} \xi \frac{f'}{f} d\xi = \int_0^1 (t\omega_1)\frac{f'(t\omega_1)}{f(t\omega_1)}\omega_1 dt \tag{16}$$

$$\int_{\overrightarrow{\omega_2,\omega_1+\omega_2}} \xi \frac{f'}{f} d\xi = \int_0^1 (t\omega_1+\omega_2)\frac{f'(t\omega_1+\omega_2)}{f(t\omega_1+\omega_2)}\omega_1 dt \tag{17}$$

Then

$$\begin{aligned}
\mathrm{I}' &= -\omega_2 \int_0^1 \frac{f'(t\omega_1)}{f(t\omega_1)}\,\omega_1\,dt & (18)\\
&= -\omega_2 \int_{\overrightarrow{0\omega_1}} \frac{f'(\xi)}{f(\xi)}\,d\xi & (19)\\
&= -\omega_2 \int_{\overrightarrow{0\omega_1}} d(\log f(\xi)) & (20)\\
&= -\omega_2\big[\log f(\omega_1) - \log f(0)\big] & (21)\\
&= -\omega_2(2\pi i n), \qquad n \in \mathbb{Z}, & (22)
\end{aligned}$$

since $f(\omega_1) = f(0) \Rightarrow e^{\log f(\omega_1) - \log f(0)} = 1 = e^{2\pi i n}$. where $\log f$ can be defined near 0 and analytically continued along a neighborhood of $\overrightarrow{0\omega_1}$.

Thus $\frac{1}{2\pi i}\mathrm{I}' = -n\omega_2$. Similarly, we can show that

$$\frac{1}{2\pi i}\int_{\partial\Pi} \xi \frac{f'(\xi)}{f(\xi)} d\xi = n_1\omega_1 + n_2\omega_2, \quad \exists n_1, n_2 \in \mathbb{Z}$$

so it falls in L. $\square$