QUICK REVIEW OF LINEAR ALGEBRA FOR MATH4406 INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The aim of this article is to review some basic knowledge of linear algebra that is essential for the HKU class MATH4406 Introduction to Partial Differential Equations.

Contents

1. Vector Spaces

In this section we will recall the definition of vector space and provide some examples.

Definition 1.1 (Vector Spaces). *Let X be a non-empty set and K be a field. We call that $(X, +, \cdot)$ is a vector space over the field K if the operations $+ : X \times X \to X$ and $\cdot : K \times X \to X$ satisfy the following properties:*

I. $(X, +)$ forms an Abelian group:

(i) Closure: for all vectors u, $v \in X$,

$$u + v \in X;$$

(ii) Commutative law: for all vectors u, $v \in X$,

$$u + v = v + u;$$

(iii) Associative law: for all vectors u, v, $w \in X$,

$$u + (v + w) = (u + v) + w;$$

(iv) *Additive identity: the set X contains an additive identity element, denoted by 0, such that for any vector $v \in X$,*

$$0 + v = v = v + 0;$$

(v) *Additive inverses: for each vector $v \in X$, there exists an additive inverse of v, denoted by $-v$, such that*

$$v + (-v) = 0 = (-v) + v.$$

II. *The operation $\cdot$ satisfies:*

(i) *Closure: for all scalar $k \in K$ and vector $u \in X$,*

$$k \cdot u \in X;$$

(ii) *Associative law: for all scalars k_1, $k_2 \in K$ and vector $u \in X$,*

$$k_1 \cdot (k_2 \cdot u) = (k_1 k_2) \cdot u;$$

(iii) *Unitary law: for all vector $u \in X$,*

$$1 \cdot u = u,$$

where 1 is the multiplicative identity for the field K.

III. *The operations $+$ and $\cdot$ also satisfy two distributive laws:*

(i) *Distributive law: for all scalar $k \in K$ and vectors u, $v \in X$,*

$$k \cdot (u + v) = k \cdot u + k \cdot v;$$

(ii) *Distributive law: for all scalars k_1, $k_2 \in K$ and vector $u \in X$,*

$$(k_1 + k_2) \cdot u = k_1 \cdot u + k_2 \cdot u.$$

Remark 1.2. *Just in case if you have not learned the concept of a field, then you may also consider K as either $\mathbb{R}$ or $\mathbb{C}$ in this course. When $K := \mathbb{R}$, we call $(X, +, \cdot)$ as a real vector space. Similarly, when $K := \mathbb{C}$, we call $(X, +, \cdot)$ as a complex vector space.*

Remark 1.3 (Terminology/Notation)**.** *In the literature, we usually use the following terminology/notations:*

(i) *Usually, we call the operation $+ : X \times X \to X$ as the vector addition, and $\cdot : K \times X \to X$ as the scalar multiplication.*

(ii) *If the underlying vector addition $+$ and scalar multiplication $\cdot$ are well-understood in the context, then without ambiguity one may call X as a vector space instead of writing $(X, +, \cdot)$.*

(iii) *In the literature, vector spaces are also called linear spaces.*

Example 1.4. *Let $\mathbb{R}$ be the field of real numbers. Then*

$$\mathbb{R}^3 := \{(x, y, z);\ x,\ y,\ z \in \mathbb{R}\}$$

is a vector space over the field $\mathbb{R}$.

Remark 1.5. *The three-dimensional Euclidean space $\mathbb{R}^3$ is very useful in daily life applications, but it is not the only useful physical spaces in applied sciences. In general, one could also consider a d-dimensional Euclidean space*

$$\mathbb{R}^d := \{(x_1, x_2, \cdots, x_d);\ x_i \in \mathbb{R} \text{ for all } i = 1,\ 2,\ \dots,\ d\}$$

over the field $\mathbb{R}$, or the the d-dimensional space

$$\mathbb{C}^d := \{(x_1, x_2, \cdots, x_d);\ x_i \in \mathbb{C} \text{ for all } i = 1,\ 2,\ \dots,\ d\}$$

over the field $\mathbb{C}$, which is the collection of all complex numbers.

Exercise 1.6. *Verify that the set $P_2(\mathbb{R}) := \{a_2x^2 + a_1x + a_0;\ a_i \in \mathbb{R}\}$, which is a collection of all polynomials with degree smaller than or equal to 2, is a vector space over the field $\mathbb{R}$.*

Discussion 1.7. *Do you think that vector spaces in Example 1.4 and Exercise 1.6 are very similar? Indeed, if we make the following identification*

$$\begin{cases} a_2 = x \\ a_1 = y \\ a_0 = z, \end{cases}$$

then you immediately see that both vector spaces have the same structure. More precisely, this relates to the idea of basis, which will be "recalled"[1] in Section 3.

Example 1.8. *In general, for any positive integer k,*

$$P_k(\mathbb{R}) := \left\{ \sum_{i=0}^{k} a_i x^i;\ a_i \in \mathbb{R} \right\}$$

is also a vector space over $\mathbb{R}$.

Example 1.9. *Let $C([0,T]) := \{f : [0,T] \to \mathbb{R};\ f$ is continuous on $[0,T]\}$. Then $C([0,T])$ is a vector space over $\mathbb{R}$.*

Exercise 1.10. *Let k be a positive integer. Define $C^k([0,T]) := \{f : [0,T] \to \mathbb{R};\ f,\ f',\ \cdots,\ f^{(k-1)}$, and $f^{(k)}$ are well-defined and continuous on $[0,T]\}$. Is $C^k([0,T])$ a vector space over $\mathbb{R}$?*

Remark 1.11. *In the literature, the space $C([0,T])$ introduced in Example 1.9 is also denoted by $C^0([0,T])$.*

[1] In general, we use the vocabulary "recalled" in two different contexts: (i) we actually remind you some materials that we have already discussed; or (ii) we just introduce some notion that you should have already learned in other classes/courses. Here, the "recalled" means the later one.

Exercise 1.12. *Define* $C^\infty([0,T]) := \cap_{k=1}^\infty C^k([0,T]) = \{f : [0,T] \to \mathbb{R};$ *f and all of its derivatives are well-defined and continuous on* $[0,T]\}$*. Is* $C^\infty([0,T])$ *a vector space over* $\mathbb{R}$*?*

2. Linear Dependence/Independence

The aim of this section is to recall the concept of linear dependence/independence.

The linear dependence/independence is related to the following vector equation: for any given vector(s) v_1, v_2, $\cdots$, $v_m \in X$, we consider

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0, \tag{1}$$

where 0 on the right hand side of (1) is the zero vector in the vector space X, and α_1, α_2, $\cdots$, α_m are the unknowns in the scalar field K. It is obvious that $(\alpha_1, \alpha_2, \cdots, \alpha_m) = (0, 0, \cdots, 0)$ is a solution to (1). Now, the question is whether $(\alpha_1, \alpha_2, \cdots, \alpha_m) = (0, 0, \cdots, 0)$ is the ONLY solution or not. More precisely, we have the following

Definition 2.1 (Linear Dependence/Independence)**.** *Let X be a vector space over a field K. We say that a set $\{v_1, v_2, \cdots, v_m\} \subset X$ is linearly independent if $(\alpha_1, \alpha_2, \cdots, \alpha_m) = (0, 0, \cdots, 0)$ is the unique solution to the vector equation* (1)*.*

Otherwise[2]*, we say that the set $\{v_1, v_2, \cdots, v_m\}$ is linearly dependent.*

Remark 2.2. *If a set $S := \{v_1, v_2, \cdots, v_m\}$ is linearly independent, then we are NOT able to express any element $v_j \in S$ in terms of a linear combination of other elements in S.*

Example 2.3 (Exponential Functions)**.** *Consider the vector space $X := C^\infty([0,1])$ over the field $K := \mathbb{R}$. For any m distinct real numbers $-\infty < \mu_1 < \mu_2 < \cdots < \mu_m < \infty$, let*

$$S := \{e^{\mu_1 x}, e^{\mu_2 x}, \cdots, e^{\mu_m x}\}.$$

Now, the question is whether S is a linearly independent set or not. Indeed, the answer is affirmative: let us consider the equation

$$\alpha_1 e^{\mu_1 x} + \alpha_2 e^{\mu_2 x} + \cdots + \alpha_m e^{\mu_m x} = 0. \tag{2}$$

[2]That is, there exists a solution to equation (1) with at least one of α_j's being non-zero.

Differentiating (2) with respect to x once, twice, $\cdots$, $(m-1)$-times, we obtain the following system of m equations:

$$\begin{cases} \alpha_1 e^{\mu_1 x} + \alpha_2 e^{\mu_2 x} + \cdots + \alpha_m e^{\mu_m x} = 0 \\ \alpha_1 \mu_1 e^{\mu_1 x} + \alpha_2 \mu_2 e^{\mu_2 x} + \cdots + \alpha_m \mu_m e^{\mu_m x} = 0 \\ \vdots \qquad = \vdots \\ \alpha_1 \mu_1^{m-1} e^{\mu_1 x} + \alpha_2 \mu_2^{m-1} e^{\mu_2 x} + \cdots + \alpha_m \mu_m^{m-1} e^{\mu_m x} = 0, \end{cases}$$

which is equivalent to the following vector equation:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_m \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{m-1} & \mu_2^{m-1} & \cdots & \mu_m^{m-1} \end{pmatrix} \begin{pmatrix} \alpha_1 e^{\mu_1 x} \\ \alpha_2 e^{\mu_2 x} \\ \vdots \\ \alpha_m e^{\mu_m x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It is worth noting that the m by m matrix in the above identity is called the Vandermonde matrix, whose determinant has an explicit formula

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_m \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{m-1} & \mu_2^{m-1} & \cdots & \mu_m^{m-1} \end{vmatrix} = \prod_{1 \le i < j \le m} (\mu_j - \mu_i) \neq 0,$$

since $\mu_1 < \mu_2 < \cdots < \mu_m$. Therefore, its inverse exists, and hence,

$$\begin{pmatrix} \alpha_1 e^{\mu_1 x} \\ \alpha_2 e^{\mu_2 x} \\ \vdots \\ \alpha_m e^{\mu_m x} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_m \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{m-1} & \mu_2^{m-1} & \cdots & \mu_m^{m-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since all $e^{\mu_1 x}$, $e^{\mu_2 x}$, $\cdots$, $e^{\mu_m x}$ are non-zero, we finally have

$$(\alpha_1, \alpha_2, \cdots, \alpha_m) = (0, 0, \cdots, 0).$$

That is, the set S is linearly independent.

Exercise 2.4 (Unique Representation)**.** *Let $\{v_1, v_2, \cdots, v_m\} \subset X$ be a linearly independent set. Suppose that an element $u \in X$ satisfies the following identity:*

$$\sum_{j=1}^{m} \alpha_j v_j = u = \sum_{j=1}^{m} \beta_j v_j,$$

for some scalars α_1, α_2, $\cdots$, α_m, and β_1, β_2, $\cdots$, β_m. Prove that $(\alpha_1, \alpha_2, \cdots, \alpha_m) = (\beta_1, \beta_2, \cdots, \beta_m)$.

3. Basis and Orthonormal Basis

In this section we will recall the ideas of basis and orthonormal basis. Furthermore, we will also recall the definition of dimension.

First of all, let us recall the notion of linear span as follows:

Definition 3.1 (Linear Span)**.** *For any given vectors v_1, v_2, $\cdots$, v_m, we define*

$$\mathrm{span}\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_\mathrm{m}\} := \left\{ \sum_{\mathrm{j}=1}^{\mathrm{m}} \alpha_\mathrm{j} \mathrm{v}_\mathrm{j} \,\middle|\, \alpha_\mathrm{j} \textit{ is a scalar, for all } j = 1,\ 2,\ \cdots,\ m. \right\}$$

Now, we are ready to provide the following

Definition 3.2 (Basis and Dimension)**.** *We say that a set $S := \{v_1, v_2, \cdots, v_m\}$ is a basis of a vector space X if*

(i) $\mathrm{span(S) = X}$*, and*

(ii) S is linearly independent.

Furthermore, we define the dimension of X as

$$\mathrm{dimX := |S|}.$$

In other words, the dimension of X is defined as the number of linearly independent vectors that span the whole vector space X.

We say that the vector space X is finite dimensional if $\mathrm{dimX} < \infty$*. Otherwise, X is infinite dimensional.*

In particular, if $X = \mathbb{R}^d$, then we have a mathematical structure called inner product as follows: for any u, $v \in \mathbb{R}^d$,

$$< u, v >:= \sum_{j=1}^{d} a_j b_j,$$

where $u := (a_1, a_2, \cdots, a_d)$ and $v := (b_1, b_2, \cdots, b_d)$. Using this inner product, we can introduce the idea of orthogonality as follows:

Definition 3.3 (orthogonal)**.** *We say that $S := \{v_1, v_2, \cdots, v_m\}$ is an orthogonal set if*

$$< v_i, v_j >= 0 \quad \textit{for all } i \neq j.$$

Similarly, we say that $S := \{v_1, v_2, \cdots, v_m\}$ is an orthonormal set if for all i, $j = 1$, 2, $\cdots$, m,

$$< v_i, v_j >= \delta_{ij} := \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{if } i = j, \end{cases}$$

where δ_{ij} is the so-called "Kronecker delta".

Finally, we can define

Definition 3.4 (Orthogonal and Orthonormal Basis)**.** *We say that* $S := \{v_1, v_2, \cdots, v_m\}$ *is an orthogonal basis of* X *if*

(i) S *is an orthogonal set, and*
(ii) S *is basis of* X*.*

Similarly, we say that $S := \{v_1, v_2, \cdots, v_m\}$ *is an orthonormal basis of* X *if*

(i) S *is an orthonormal set, and*
(ii) S *is basis of* X*.*

4. Gram-Schmidt Process

The Gram-Schmidt process is a standard procedure/algorithm to construct an orthonormal basis from a given basis that is not orthonormal at the beginning. In applied mathematics, we enjoy performing computations in an orthonormal basis since computations in an orthonormal basis will be highly simplified (compared to using a general basis). However, finding an orthonormal basis directly is not so easy in general. Therefore, the Gram-Schmidt process is a very useful technique in applied mathematics. More precisely, this allows us to find an arbitrary basis first, and then obtain an orthonormal basis via the Gram-Schmidt process.

For the details of Gram-Schmidt Process, please read the following Wikipedia page:
`https://en.wikipedia.org/wiki/Gram%E2%80%93Schmidt_process`
In particular, you only need to read up to the section "Example" in the above Wikipedia page.

5. Eigenvalues and Eigenvectors

In this and the next section, we will discuss several important concepts for linear maps/operators. First of all, we will recall the linearity, eigenvalues and eigenvectors in this section.

Definition 5.1 (Linearity)**.** *Let* X *and* Y *be vector spaces over the same field* K*. A mapping/operator* $T : X \to Y$ *is called linear if for any scalars* k_1*,* $k_2 \in K$*, and any vectors* u*,* $v \in X$*, we have*

$$T(k_1 u + k_2 v) = k_1 Tu + k_2 Tv.$$

Exercise 5.2. *Verify the following facts:*

(i) $T0_X = 0_Y$*, where* 0_X *and* 0_Y *are the zero vectors in* X *and* Y *respectively;*
(ii) If T_1 *and* T_2 *are linear mappings from* X *to* Y*, then so is* $T_1 + T_2$*.*

Exercise 5.3. *Define a map $T : \mathbb{R}^2 \to \mathbb{R}$ by*

$$T(x_1, x_2) := 2x_1 - x_2.$$

Is T a linear map?

Example 5.4 (3×3 Matrix as a Linear Mapping)**.** *Consider the vector spaces $X = Y = \mathbb{R}^3$. Let A be a given 3×3 real-valued matrix. Define the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ as follows: for any $u \in \mathbb{R}^3$,*

$$T(u) := Au.$$

Here, Au is defined by the standard matrix multiplication: more precisely, if

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

then

$$Au := \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + a_{13}u_3 \\ a_{21}u_1 + a_{22}u_2 + a_{23}u_3 \\ a_{31}u_1 + a_{32}u_2 + a_{33}u_3 \end{pmatrix}.$$

One may check that the mapping T is linear.

Discussion 5.5. *Let A be a $m \times n$ complex-valued matrix. Is the mapping $T(u) := Au$, defined from $\mathbb{C}^n$ to $\mathbb{C}^m$, linear?*

Example 5.6 (Evaluation Map)**.** *Let $C([0,1])$ be the collection of continuous and real-valued functions. We define a mapping $T : C([0,1]) \to \mathbb{R}$ by*

$$Tf := f(1).$$

That is, we evaluate the function f at the point 1. One can check that it is a linear map as follows: for any f, $g \in C([0,1])$, and α, $\beta \in \mathbb{R}$,

$$\begin{aligned} T(\alpha f + \beta g) &= (\alpha f + \beta g)(1) \\ &= \alpha f(1) + \beta g(1) \\ &= \alpha T f + \beta T g. \end{aligned}$$

It is also worth noting that the codomain of the linear mapping T is $\mathbb{R}$, which is just the scalar field of the vector space $C([0,1])$. In the literature, we usually call this type of linear mappings as the linear functionals.

Definition 5.7 (Eigenvalue and Eigenvector)**.** *Let X be a vector space over a field K. For any given linear mapping $T : X \to X$, we say that $\lambda \in K$ is an eigenvalue of T with the corresponding eigenvector $u \in X$ if $u \neq 0$ and*

$$Tu = \lambda u.$$

Example 5.8. *Continuing from Example 5.4, we can see that the eigenvalues as well as the eigenvectors of the linear mapping T are the same as that of the matrix A. In particular, if*

$$A := \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

then we have the following eigenvalues and eigenvectors:

$$\begin{cases} \lambda_1 = -2 \qquad u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \\ \lambda_2 = 3 \qquad u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{cases}$$

Remark 5.9 (Characteristic Polynomial)**.** *In the case of matrices, one can also find the eigenvalues by using the characteristic polynomial $p(t) := \det(tI - A)$, where I is the identity matrix. More precisely,*

$$\lambda \text{ is an eigenvalue of } A \text{ if and only if } p(\lambda) = 0.$$

Exercise 5.10. *For the matrix A defined in Example 5.8, one may find that the eigenvalue λ_1 is repeated via using the characteristic polynomial recalled in Remark 5.9. Now, the question is whether you can find another eigenvector corresponding to λ_1 so that the new eigenvector is NOT a multiple of u_1? Explain your answer briefly.*

Example 5.11 (Differential Operator)**.** *Consider the mapping $T : C^\infty([0,T]) \to C^\infty([0,T])$ as follows:*

$$Tf := \frac{d}{dt} f.$$

Due to the linearity of differentiation, one may check that T is a linear operator[3]. For any scalar $\mu \in \mathbb{R}$, the function

$$f_\mu(t) := e^{\mu t}$$

belongs to $C^\infty([0,T])$. A direction computation yields

$$Tf_\mu = \frac{d}{dt} e^{\mu t} = \mu e^{\mu t} = \mu f_\mu,$$

[3]Usually, we will call a linear mapping acting on function spaces (e.g., $C^\infty([0,T])$) as an *operator*.

so μ is an eigenvalue of T and $f_\mu := e^{\mu t}$ is the corresponding eigenfunction[4]*. It is remarkable that the differential operator $T := \frac{d}{dt}$ has infinitely many eigenvalues. This is one of the main difference between the differential operator $T := \frac{d}{dt}$ and matrices, which have finitely many eigenvalues.*

Exercise 5.12. *Define an operator T from $C^\infty([0,T])$ to $C^\infty([0,T])$ by*

$$Tf := \frac{d^2}{dt^2} f.$$

Verify that T is a linear operator. Furthermore, what are the eigenvalues and eigenfunctions?

6. Matrix Representations of Linear Maps/Transformation

In this section we will use an example to illustrate the following

Fact 6.1 (Matrix Representation)**.** *Let X and Y be two* ***FINITE*** *dimensional vector spaces. For any* <u>*linear*</u> *mapping $T : X \to Y$, we can always represent T as a matrix.*

In other words, we can always see all linear mappings between finite dimensional vector spaces as matrices in a certain sense.

Remark 6.2 (Basis Dependence)**.** *It is worth noting that the matrix representation of T depends on the basis that we use in X and Y.*

Let us end this article by providing the following

Example 6.3 (Representation of Differential Operator)**.** *Suppose that*

$$\begin{cases} X := P_2(\mathbb{R}) := \left\{a_2x^2 + a_1x + a_0;\ a_i \in \mathbb{R}\right\} \\ Y := P_1(\mathbb{R}) := \left\{b_1x + b_0;\ b_i \in \mathbb{R}\right\}. \end{cases}$$

Define $T : X \to Y$ by

$$Tf := \frac{d}{dx} f.$$

Can we express the linear mapping T as a matrix?

Yes. To do so, we first choose the basis for X and Y as follows. For the vector space X, let us consider the following basis:

$$B_X := \{u_1, u_2, u_3\} := \{x^2, x, 1\}.$$

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[4]Since f_μ is a function, we usually call it as an *eigenfunction* (instead of an eigenvector).

For the vector space Y, we choose

$$B_Y := \{v_1, v_2\} := \{2x+1, 1\}.$$

Remark 6.4. *In applications, you can choose any basis. The above selection of basis is just an example.*

Using these basis, we can identify the vectors in X and Y as that in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. For example,

$$\begin{pmatrix}4\\3\\5\end{pmatrix}_{B_X} = 4\cdot u_1 + 3\cdot u_2 + 5\cdot u_3 = 4\cdot x^2 + 3\cdot x + 5\cdot 1 = 4x^2+3x+5,$$
$$\begin{pmatrix}7\\13\end{pmatrix}_{B_Y} = 7\cdot v_1 + 13\cdot v_2 = 7\cdot(2x+1) + 13\cdot 1 = 14x+20.$$

Now, let us express the image of each vector u_i in B_X in terms of the basis B_Y as follows:

$$\begin{cases} Tu_1 = \dfrac{d}{dx}x^2 = 2x = (2x+1)-1 = v_1 - v_2 \\ Tu_2 = \dfrac{d}{dx}x = 1 = 0\cdot v_1 + v_2 \\ Tu_3 = \dfrac{d}{dx}1 = 0 = 0\cdot v_1 + 0\cdot v_2. \end{cases}$$

Therefore, we can express T as the following matrix:

$$[T]_{B_X,B_Y} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

Finally, let us end this example by demonstrating how to apply the above matrix representation. Assume that we would like to compute

$$\frac{d}{dx}\left(4404x^2 + 3404x + 2016\right).$$

Of course, we can use the elementary calculus to compute this result quickly, but (in order to illustrate the usefulness of the matrix representation,) we can also compute the result by

$$\begin{aligned}
\frac{d}{dx}\left(4404x^2 + 3404x + 2016\right) &= T\left(4404u_1 + 3404u_2 + 2016u_3\right) \\
&= [T]_{B_X,B_Y}\begin{pmatrix}4404\\3404\\2016\end{pmatrix}_{B_X} \\
&= \begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}4404\\3404\\2016\end{pmatrix}_{B_X} \\
&= \begin{pmatrix}4404\\ -4404+3404\end{pmatrix}_{B_Y} \\
&= \begin{pmatrix}4404\\ -1000\end{pmatrix}_{B_Y} \\
&= 4404v_1 - 1000v_2 \\
&= 4404(2x+1) - 1000(1) \\
&= 8808x + 3404.
\end{aligned}$$