Def'n

$Z = X \cup_f Y \quad \underset{\text{closed}}{A \subset Y} \quad f : A \to X$ $\boxed{\text{closed embedding}}$

$$\begin{tikzcd} X \sqcup Y \arrow[r, "\alpha \sqcup \beta"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowright"] & W \\ Z \arrow[ur, dashed, "\exists \gamma"'] & \end{tikzcd} \qquad \alpha \circ f = \beta|_A$$

continuous

adjuction of α and β.

Embedding of adjunction space

Q1: Given $X \cup_f Y$, how to embed it into a known space (i.e. $\mathbb{R}^n$) ?

Q2: Given a sub space of a known space, How to describe it as an adjunction space ?

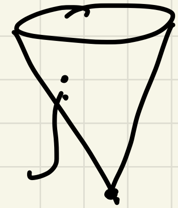
! embed > find an injective continuous map !

Examples

1)

$$\begin{tikzcd} CS^1 \arrow[r, "?"] & \mathbb{R}^3 \\ S^1 \times [0,1] \arrow[u] \arrow[ur] & \end{tikzcd}$$

$\Leftarrow$ compact $\qquad$ $\leftarrow$ hausdorff.



$(\theta, t) \mapsto (t, \theta, t)$ $\qquad$ (r, θ, z)

polar coord.

$\| $

$(t\cos\theta, t\sin\theta, t)$ Cartesian coordin.

2) Ex: construct the map

$$\begin{tikzcd} V_n S^1 \arrow[r] & \mathbb{R}^2 \\ \sqcup S^1 \arrow[u] \arrow[ur] & \end{tikzcd}$$

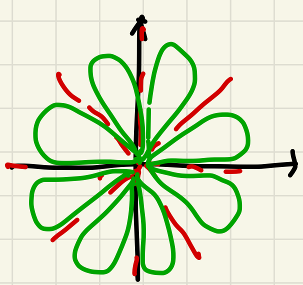
n copies

well defined locally

$$\mathbb{C} \longrightarrow \mathbb{C}$$

$$z \longrightarrow z^{1/n}$$

$$re^{i\theta} \longrightarrow \sqrt[n]{r}\, e^{i\theta/n}$$

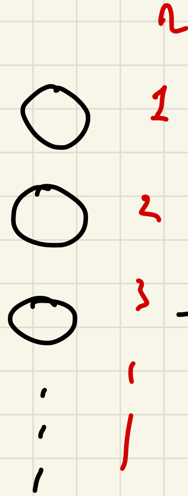


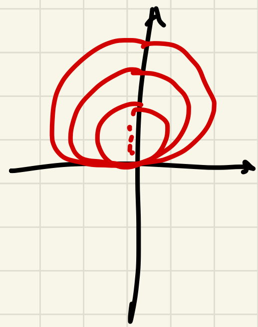
$|z+1| = 1$ $\qquad$ $|w^n+1| = 1$

$\theta \in (0, 2\pi)$

3)

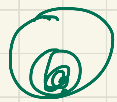
$$\begin{tikzcd} \bigvee_{\mathbb{N}} S^1 \arrow[r, "f"] & \mathbb{R}^2 \\ \bigsqcup_{\mathbb{N}} S^1 \arrow[u] \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$



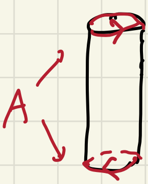


radius $\frac{1}{n}$

f is NOT an embedding!

$\bigvee_{\mathbb{N}} S^1$ is not locally homeomorphic to  at 0.

4) A



$[0,1] \times S^1 = Y$

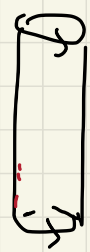
$\uparrow$

$\{0,1\} \times S^2 = A \xrightarrow{f} X \cong S^1$

$(0,\theta) \mapsto \theta$

$(1,\theta) \mapsto -\theta$

Klein bottle



$X \sqcup_f Y \longrightarrow \overset{①}{S^1 \times S^1}$
$\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\quad X$

$X \to S^1 \times \{0\}$

$S^1 \times [0,1] = Y \to S^1 \times [0, 2\pi]$

How about:

$(0,\theta) \mapsto \theta$?

$(1,\theta) \mapsto -\theta$

Topological group

Def'n A group is a nonempty set G with a binary operation $G \times G \xrightarrow{*} G$ s.t with a distinguish element $e \in G$ s.t

1) $(x * y) * z = x * (y * z)$

2) $\forall x \in G \quad \exists \quad x^{-1} \in G$ s.t $x * x^{-1} = e$
$x^{-1} * x = e$

3) $x * e = e * x = x \quad \forall x$

if $x * y = y * x \quad \forall x, y \in G$

then we say G is abelian.

Examples

1) [Additive group]

$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \quad , +.$

2) [Multiplicative group]

$\mathbb{Z}^{\times} = \{\pm 1\} \subset \mathbb{Q}^{\times} = \mathbb{Q} \setminus 0 \subset \mathbb{R}^{\times} = \mathbb{R} \setminus 0 \subset \mathbb{C}^{\times} = \mathbb{C} \setminus 0$

under $\times$.

3) if k is a field, we write $\mathbb{G}_a$ the underlying additive gp. $\mathbb{G}_m = k^{\times}$ the multiplicative gp.

4) [Cyclic group]

a) $\mu_{\xi} = \{ e^{2\pi i \cdot \xi \cdot n} \mid n \in \mathbb{Z} \}$ ξ is irrational

b) $\mu_{\frac{1}{p}}$ p: prime $\mu_{\frac{1}{p}}$ is called the finite cyclic group of order p.

5) Product group

subgroup. normal subgroup

quotient group.

Def'n G, H groups $\phi: G \to H$

is called a **homomorphism** if

$\phi(xy) = \phi(x)\phi(y)$.

isomorphism if ϕ is bijective.

Def'n A **top group** is both a Hausdorff space and a group s.t

$$\left.\begin{array}{l} *: G \times G \to G \\ (\ )^{-1}\ G \to G \end{array}\right. \text{ are cont.}$$

A morphism of top group is a continous group homomorphism.

A top. subgroup = subgroup + subspace top.

Examples

1) $Gl_n \mathbb{R}$

2) $O(n)$ $SO(n)$ $U(n)$

p.f. $GL_n \mathbb{R} = \{ A \text{ invertible } n \times n\ \mathbb{R}\text{-matrices} \}$

clearly it is a gp. $GL_n \mathbb{R} \subset \mathbb{R}^{n^2}$ with sub space top.

$(AB)_{ij} = \sum_k A_{ik} B_{kj}$ is quadratic poly.

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(A) \qquad \operatorname{adj}(A) = \begin{array}{c} \\ \left[\begin{array}{c} C_{ij} \end{array} \right]^{T} \\ \quad j \end{array}$$

$(A^{-1})_{ij}$ is a rational function in A_{ij} $\qquad C_{ij} = \det_i \left[\quad \right]$

(A) $(\mathbb{Q}, +)$ with discrete top

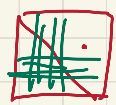
$(\mathbb{R}, +)$ with metric top. $(\mathbb{Q}, +) \hookrightarrow (\mathbb{R}, +)$ is not a sub gp

the sub space top is NOT discrete!

(B) discrete $(\mathbb{Z}, +) \xrightarrow{\phi} \{ e^{2\pi i n \xi} \mid n \in \mathbb{Z} \}$ ξ irr.

$n \longmapsto e^{2\pi i n \xi}$ $\qquad$ $\mathbb{T}$ with sub space top from S^1

ϕ is an isom of gps



but not an isom of top gps.

(C) $(\mathbb{R}, +, \text{Zar})$ is not a top gp even w/o T2 condition

$\mu^{-1}(0) = \{ (x, -x) \mid x \in \mathbb{R} \}$ is not closed under Π top

$\mathbb{R} \times \mathbb{R} \xrightarrow{\mu} \mathbb{R}$ is not cont.

$(x, y) \mapsto x + y$

3) $$\mathbb{R} \longrightarrow S^1 \quad \text{is a group homomorphism}$$
$$\theta \longmapsto e^{i\theta}$$

Action of top gp

A **left action** of G on X is a cont. map

$$\sigma : \overbrace{G \times X}^{\text{product top}} \longrightarrow X \quad \text{s.t.}$$

$$\begin{tikzcd}
G \times G \times X \arrow[r, "(1,\sigma)"] \arrow[d, "(\mu,1)"'] \arrow[dr, phantom, "="] & G \times X \arrow[d, "\sigma"] \\
G \times X \arrow[r, "\sigma"] & X
\end{tikzcd}$$

$$\sigma(gh, x) = \sigma(g, \sigma(h,x))$$

and $\sigma(e, x) = x \quad \forall x \in X$ $\quad$ e unit

right action $\delta : X \times G \longrightarrow X$

$$\begin{tikzcd}
X \times G \times G \arrow[r, "(\delta,1)"] \arrow[d, "(1,\mu)"'] \arrow[dr, phantom, "="] & X \times G \arrow[d, "\delta"] \\
X \times G \arrow[r, "\delta"] & X
\end{tikzcd}$$

$$\delta(x, gh) = \delta(\delta(x,g), h)$$

Left & right action are related as follows

if $\sigma: G \times X \to X$ is a left action

then $\delta: X \times G \to X$ defined by

$$\delta(x, g) := \sigma(g^{-1}, x) \text{ is a right action}$$

$$\delta(\delta(x,g), h) = \sigma(h^{-1}, \delta(x,g)) = \sigma(h^{-1}, \sigma(g^{-1}, x))$$
$$= \sigma(h^{-1}g^{-1}, x) = \sigma((gh)^{-1}, x)$$
$$= \delta(x, gh)$$

Ex. $X = Mat_{n \times n}(\mathbb{R})$

$G = GL_n$ X has both left & right G-action

$(g, A) \mapsto gA$ $\quad (A, g) \mapsto Ag$

$Gx = \{gx \mid g \in G\}$ is called the G-orbit of x. We may define $x_1 \sim x_2$ if $\exists$ $x_1 = g x_2$

note that this is an eq. relation

since $x_1 = g x_2 \iff g^{-1} x_1 = x_2$

Gx = equivalence class of x $\quad X = \bigsqcup Gx$

We call the action **transitive** if $X = Gx$

Fix $x \in X$ the **stabilizer (subgroup) at x**

is $G_x = \{ g \in G / \ gx = x \}$

The action is called **free** if $\forall x$, $G_x = \{e\}$

Orbit space $G \backslash X$ $\left(\text{or } X/G \right) := X/\sim$

$x \sim gx$ with quotient top.

properties

1) fix $g \in G$ $\quad \sigma_g : X \longrightarrow X$

$x \longmapsto gx$

is a homeomorphism