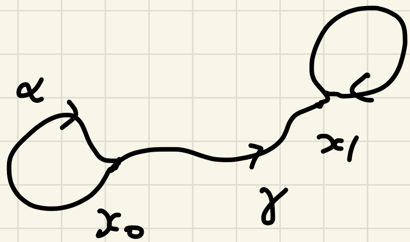
Change of base point

$x_0, x_1 \in X$ Let γ be a path

$\gamma(0) = x_0$ $\gamma(1) = x_1$



$\phi_\gamma : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$[\alpha] \longmapsto [\gamma^{-1} * \alpha * \gamma]$ is a group isomorphism

Ex Show that ϕ_γ only depends on the homotopy class $[\gamma]$.

Fundamental groupoid

path homotopy class (end pts fixed)

$\pi_1(X) = \{ [\gamma] \mid \text{for all } \gamma : I \to X \}$

not necessarily loops

$\gamma_1 * \gamma_2$ defined if $\gamma_1(1) = \gamma_2(0)$ we call it composable

$[\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2]$ if composable

$\forall \gamma, \quad \gamma^{-1}(t) = \gamma(1-t)$

$\forall x_0 \in X \qquad e_{x_0} : I \to x_0 \to X$

$\pi_1(X)$ is not a group (unless $X = \{\bullet\}$)

but a "groupoid".

A groupoid is a <u>category</u> where all morphism are invertible

Define category & Functor!

A space X is called simply connected if

1) X is path conn.

2) $\pi_1(X, x_0) = 1$ for some $x_0 \in X$

by (1) therefore holds for any x_0.

<u>properties of π_1</u>

1) π_1 as a functor

Top^* : category of top spaces with base pts

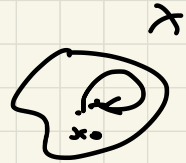
$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

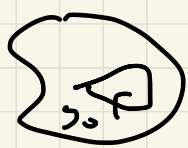
continuous map s.t $f(x_0) = y_0$

Grp : category of groups

π_1 is a functor $Top^* \to Grp$

$$\begin{tikzcd}
(X, x_0) \arrow[r, mapsto] \arrow[d, "f"'] & \pi_1(X, x_0) \arrow[d, "f_*"] \\
(Y, y_0) \arrow[r, mapsto] & \pi_1(Y, y_0)
\end{tikzcd}$$





Define $f_*[\gamma] = [f \circ \gamma]$

$$f \circ \gamma : [0,1] \to X \xrightarrow{f} Y$$

Check

a) f_* is a group homomorphism

b) $(id_X)_* = id_{\pi_1(X, x_0)}$

c) $X \xrightarrow{f} Y \xrightarrow{g} Z$

$x_0 \mapsto y_0 \longmapsto z_0$

$(g \circ f)_* = g_* \circ f_*$

$(g \circ f)_* [\alpha]$

$= [g \circ f \circ \alpha]$

$g_* \circ f_* [\alpha]$

$= [g \circ f \circ \alpha]$

a) $(f \circ \alpha) * (f \circ \beta) = \begin{cases} f \circ \alpha(2t) & t \in [0, \frac{1}{2}] \\ f \circ \beta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$

$\parallel$

$f \circ (\alpha * \beta)$

2) π_1 preserves product

i.e. $\pi_1(X \times Y, (x_0, y_0)) \cong$

$\pi_1(X, x_0) \times \pi_1(Y, y_0)$

pf.

$$\begin{tikzcd} & X \\ {[0,1]} \arrow[ur] \arrow[r, "\gamma"] \arrow[dr] & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} x_0 \\ (x_0, y_0) \arrow[u, mapsto] \arrow[d, mapsto] \\ y_0 \end{tikzcd}$$

$$\begin{tikzcd} & {[\gamma] \in \pi_1(X \times Y, (x_0, y_0))} \arrow[dl] \arrow[dr] & \\ {[\pi_X \circ \gamma] \in \pi_1(X, x_0)} & & {[\pi_Y \circ \gamma] \in \pi_1(Y, y_0)} \end{tikzcd}$$

$\pi_X \circ \gamma \overset{F_X}{\sim} e_{x_0}$ $\qquad F_X : [0,1] \times [0,1] \to X$

$\pi_Y \circ \gamma \overset{F_Y}{\sim} e_{y_0}$ $\qquad F_Y : [0,1] \times [0,1] \to Y$

$\gamma \overset{F_X \times F_Y}{\sim} e_{(x_0, y_0)}$ $\qquad F_X \times F_Y : [0,1]^2 \to X \times Y$

injectivity holds

surjectivity is obvious.

3) π_1 is a homotopy invariant

i.e. if $f:(X, x_0) \to (Y, y_0)$ is a homotopy equivalence

then f_* is a group isomorphism.

Lemma

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] \arrow[from=U, to=D, Rightarrow, "F"] & Y \end{tikzcd}$$

$F(x, 0) = f$

$F(x, 1) = g$

$$x_0 \longrightarrow f(x_0)$$
$$x_0 \longrightarrow g(x_0)$$

$\gamma = F(x_0, t) : [0,1] \to Y$

Then

$$\begin{tikzcd} & \pi_1(Y, f(x_0)) \arrow[dd, "(\ )^\gamma"] \\ \pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] \arrow[r, phantom, "//"] & \\ & \pi_1(Y, g(x_0)) \end{tikzcd}$$

$\gamma(0) = f(x_0)$

$\gamma(1) = g(x_0)$

Apply to

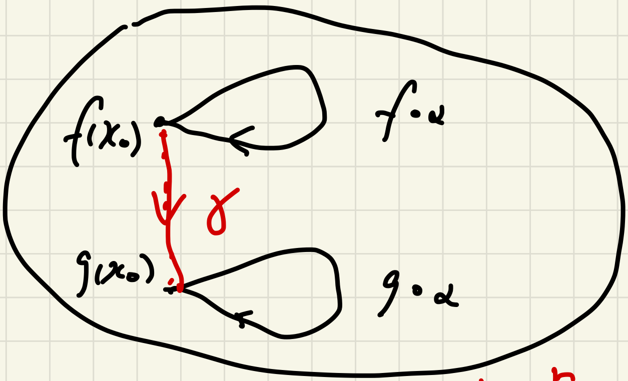
$f \circ f^{-1} \simeq id_Y$

pf. $(f_*[\alpha])^\gamma = [\gamma^{-1} * (f \circ \alpha) * \gamma]$

$\gamma^{-1} * (f \circ \alpha) * \gamma$

$H \wr$

$g \circ \alpha$



$\gamma(1) \to \gamma(-1) = F(x_0, -1)$

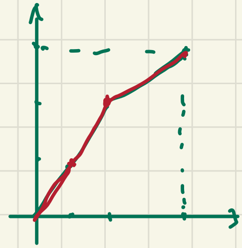
$$H = \begin{cases} \gamma(1-4s) & s \in [0, \frac{1-t}{4}] \\ F\left(\alpha\left(\frac{4s+t-1}{3t+1}\right), t\right) & s \in [\frac{1-t}{4}, \frac{1+t}{2}] \\ \gamma(2s-1) & s \in [\frac{1+t}{2}, 1] \end{cases}$$

$\gamma(t) \to \gamma(1)$

$$H(s,0) = \begin{cases} \gamma(1-4s) & [0, \frac{1}{4}] \\ F(\alpha(4s-1), 0) & [\frac{1}{4}, \frac{1}{2}] \\ \gamma(2s-1) & [\frac{1}{2}, 1] \end{cases}$$

$$H(s,1) = \begin{cases} \gamma(1) & s=0 \\ F(\alpha(s), 1) & [0, 1] \\ \gamma(1) & s=1 \end{cases}$$

Fundamental gp of circle



Thm $\pi_1(S^1) = (\mathbb{Z}, +)$

$$\pi: \mathbb{R} \longrightarrow S^1 \qquad x_0 = 1$$
$$x \longmapsto e^{2\pi i x}$$

$$\alpha: [0,1] \longrightarrow S^1 \qquad \alpha^n = \overbrace{\alpha * \cdots * \alpha}^{n} \qquad n \in \mathbb{Z}$$
$$\alpha(t) = e^{2\pi i t} \qquad \sim e^{2\pi i n t}$$

$$\phi: \mathbb{Z} \longrightarrow \pi_1(S^1, x_0) \quad \text{is a group homomorphism}$$
$$n \longrightarrow [\alpha^n]$$

Need to check

1) ϕ is inj. i.e. $\alpha^n \sim e_{x_0} \iff n = 0$

2) ϕ is surj. i.e. $\forall\, \gamma: [0,1] \to S^1$ s.t. $\gamma(0) = \gamma(1) = 1$ $\exists\, n \in \mathbb{N}$ $\gamma \sim \alpha^n$.