1. (a) State *without proof* the First Isomorphism Theorem.

   (b) Prove or disprove the statement: the product group of two simple groups is simple.

   [Hint. Firstly state the definition of a simple group.]

Ans.

(a) (4 marks) Bookwork.

Common mistakes/inaccuracies: Many students didn't write down how the (injective) homomorphism $\overline{\phi}$ is defined, which is also a key part of the theorem. A few students said that $\overline{\phi}$ is bijective (or isomorphism).

(b) (4 marks) The statement is false. Let G_1 and G_2 be simple groups. Then they are non-trivial groups. Now $\{e\} \times G_2$ is a non-trivial proper subgroup of $G_1 \times G_2$. Direct checking, one sees that $\{e\} \times G_2$ is normal in $G_1 \times G_2$. Hence $G_1 \times G_2$ is not simple.

Common mistakes/inaccuracies: Some student said a simple group is a group without non-trivial proper subgroups. Some student said a simple group is a group without non-trivial normal subgroups.

2. Let $i = \sqrt{-1} \in \mathbb{C}$ and

$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},\ a = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},\ b = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \ \text{ and } \ j = \begin{pmatrix} i & \\ & -i \end{pmatrix},\ k = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Define $D_4 := \{I, a, a^2, a^3, b, ba, ba^2, ba^3\}$ and $Q := \{I, j, j^2, j^3, k, jk, jk^2, jk^3\}$. Under matrix multiplication, we know (and you may take for granted) that both D_4 and Q are groups.

(a) <u>Verify</u> that D_4 is not isomorphic to Q.

(b) In Assignment 2, Part I, Q1 & Q2, we showed that D_4 is isomorphic to *the group G generated by two generators x, y satisfying* **(1)** $x^4 = e$**, (2)** $y^2 = e$ **and (3)** $xy = yx^3$, where e is the identity element of G. Similarly, (we know and you may take for granted that) Q is also isomorphic to *a group generated by two generators x and y satisfying three relations with identity e*. <u>Find</u> the three relations that determine Q.

[Hint. Some relations are the same as the determining relations **(1)**, **(2)**, **(3)** of G.]

(c) Find a subgroup of D_4 which is not normal in D_4. Justify your answer.

Ans.

(a) (4 marks) By direct checking, Q has only one element (j^2) of order 2 , while D_4 has two elements (a^2 and b) of order 4. Thus they are not isomorphic.

Common mistakes/inaccuracies: Some students defined a function from D_4 and Q and showed that this function is not an isomorphism. Then they concluded that D_4 is not isomorphic to Q. This cannot serve as a justification!

(b) (4 marks) $x^4 = e$, $xy = yx^3$ and $x^2 = y^2$.

Common mistakes/inaccuracies: Many students took $x^4 = y^4 = e$. This doesn't work because both D_4 and Q fulfil these relations! Some students took $y^2 = -e$. Note that the group generated by x, y is abstract (i.e. not matrices), $-e$ is not well-defined.

(c) (4 marks) Let $H = \langle b \rangle$. Then $aHa^{-1} = \{I, ba^2\} \not\subset H$. Thus H is a non-normal subgroup of D_4.

Common mistakes/inaccuracies: Some students picked the subgroup $\{I, a, a^2, a^3\}$ $(= \langle a \rangle)$ and tried to show that it is not normal. One should know it's not a right target from the index $[D_4, \langle a \rangle] = 2$!

3. (a) Let G be a p-group of order p^n. Show that for every $1 \le r \le n$, G has a *normal subgroup* of order p^r. [Hint. *Mimic the proof* for Theorem 7.2.4 which yields "Assume G is a p-group of order p^n. Then, for any $1 \le r \le n$, there exists a subgroup of G of order p^r."]

(b) Give an example to illustrate that a p-group may have a non-normal subgroup.

Ans.

(a) (12 marks) We apply induction on n. When $n = 1$, G (of order p) has only two subgroups: the trivial subgroup and the whole group G. Both are normal subgroups.

Let $n > 1$. Assume for all $1 \le m < n$, the statement holds for all groups of order p^m.

Suppose G is a group of order p^n. As G is a p-group, its center Z is non-trivial.

Thus Z is an abelian p-group. By Cauchy's theorem, Z contains an element a of order p.

Then $\langle a \rangle$ is a normal subgroup of G and of order p.

Define $G' := G/\langle a \rangle$. Then G' is a p-group of order p^{n-1}.

By induction assumption, for $1 \le \ell \le n-1$, G' has a normal subgroup H'_ℓ of order p^ℓ.

Set $H_\ell := \pi^{-1}(H'_\ell)$ where $\pi : G \to G'$ is the natural projection.

As π is a homomorphism and H'_ℓ is normal subgroup of G', $H_\ell \lhd G$.

Moreover, $|H_\ell| = p^{\ell+1}$ as $H_\ell/\langle a \rangle = H'_\ell$.

Thus for $2 \le r \le n$, G contains a normal subgroup of order p^r.

Together with $H_1 := \langle a \rangle$. We complete the proof.

Common mistakes/inaccuracies: Some students could not present the induction argument properly even though they are able to apply the necessary ingredients and ideas.

(b) (4 marks) D_4 is a 2-group of order 2^3. By Qn 2 (c), we know it has a non-normal subgroup.

Final remark. Some students commented that there was no bookwork in Test 1. Test 2 has more bookwork but the performance is slightly worse. For your information, the final exam consists of a good number of bookwork type questions. Bookwork is not limited to lecture notes, but also includes assignments and tutorials.

End