

ELLIPTIC FUNCTIONS

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1 Elliptic Functions

§1. THE LIOUVILLE THEOREMS

By a **lattice** in the complex plane \mathbf{C} we shall mean a subgroup which is free of dimension 2 over \mathbf{Z} , and which generates \mathbf{C} over the reals. If ω_1, ω_2 is a basis of a lattice L over \mathbf{Z} , then we also write $L = [\omega_1, \omega_2]$. Such a lattice looks like this:

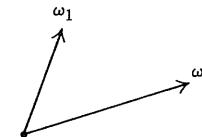


Fig. 1-1

Unless otherwise specified, we also assume that $\text{Im}(\omega_1/\omega_2) > 0$, i.e. that ω_1/ω_2 lies in the upper half plane $\mathfrak{H} = \{x + iy, y > 0\}$. An **elliptic function** f (with respect to L) is a meromorphic function on \mathbf{C} which is L -periodic, i.e.

$$f(z + \omega) = f(z)$$

for all $z \in \mathbf{C}$ and $\omega \in L$. Note that f is periodic if and only if

$$f(z + \omega_1) = f(z) = f(z + \omega_2).$$

An elliptic function which is entire (i.e. without poles) must be constant, because it can be viewed as a continuous function on \mathbf{C}/L , which is compact (homeomorphic to a torus), whence the function is bounded, and therefore constant.

If $L = [\omega_1, \omega_2]$ as above, and $\alpha \in \mathbf{C}$, we call the set consisting of all points

$$\alpha + t_1\omega_1 + t_2\omega_2, \quad 0 \leq t_i \leq 1$$

a **fundamental parallelogram** for the lattice (with respect to the given basis). We could also take the values $0 \leq t_i < 1$ to define a fundamental parallelogram, the advantage then being that in this case we get unique representatives for elements of \mathbf{C}/L in \mathbf{C} .

Theorem 1. Let P be a fundamental parallelogram for L , and assume that the elliptic function f has no poles on its boundary ∂P . Then the sum of the residues of f in P is 0.

Proof. We have

$$2\pi i \sum \text{Res } f = \int_{\partial P} f(z) dz = 0,$$

this last equality being valid because of the periodicity, so the integrals on opposite sides cancel each other.

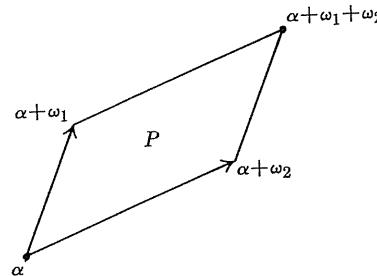


Fig. 1-2

An elliptic function can be viewed as a meromorphic function on the torus \mathbf{C}/L , and the above theorem can be interpreted as saying that the sum of the residues on the torus is equal to 0. Hence:

Corollary. An elliptic function has at least two poles (counting multiplicities) on the torus.

Theorem 2. Let P be a fundamental parallelogram, and assume that the elliptic function f has no zero or pole on its boundary. Let $\{a_i\}$ be the singular points (zeros and poles) of f inside P , and let f have order m_i at a_i . Then

$$\sum m_i = 0.$$

Proof. Observe that f elliptic implies that f' and f'/f are elliptic. We then obtain

$$0 = \int_{\partial P} f'/f(z) dz = 2\pi\sqrt{-1} \sum \text{Residues} = 2\pi\sqrt{-1} \sum m_i,$$

thus proving our assertion.

Again, we can formulate Theorem 2 by saying that the sum of the orders of the singular points of f on the torus is equal to 0.

Theorem 3. Hypotheses being as in Theorem 2, we have

$$\sum m_i a_i \equiv 0 \pmod{L}.$$

Proof. This time, we take the integral

$$\int_{\partial P} z \frac{f'(z)}{f(z)} dz = 2\pi\sqrt{-1} \sum m_i a_i,$$

because

$$\text{res}_{a_i} z \frac{f'(z)}{f(z)} = m_i a_i.$$

On the other hand we compute the integral over the boundary of the parallelogram by taking it for two opposite sides at a time. One pair of such integrals is equal to

$$\int_{\alpha}^{\alpha+\omega_1} z \frac{f'(z)}{f(z)} dz - \int_{\alpha+\omega_2}^{\alpha+\omega_1+\omega_2} z \frac{f'(z)}{f(z)} dz.$$

We change variables in the second integral, letting $u = z - \omega_2$. Both integrals are then taken from α to $\alpha + \omega_1$, and after a cancellation, we get the value

$$-\omega_2 \int_{\alpha}^{\alpha+\omega_1} \frac{f'(u)}{f(u)} du = 2\pi\sqrt{-1} k\omega_2,$$

for some integer k . The integral over the opposite pair of sides is done in the same way, and our theorem is proved.

§2. THE WEIERSTRASS FUNCTION

We now prove the existence of elliptic functions by writing some analytic expression, namely the Weierstrass function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L'} \left[\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right],$$

where the sum is taken over the set of all non-zero periods, denoted by L' . We have to show that this series converges uniformly on compact sets not including the lattice points. For bounded z , staying away from the lattice points, the expression in the brackets has the order of magnitude of $1/|\omega|^3$. Hence it suffices to prove:

Lemma. If $\lambda > 2$, then $\sum_{\omega \in L'} \frac{1}{|\omega|^\lambda}$ converges.

Proof. The partial sum for $|\omega| \leq N$ can be decomposed into a sum for ω in the annulus at n , i.e. $n - 1 \leq |\omega| \leq n$, and then a sum for $1 \leq n \leq N$. In each annulus the number of lattice points has the order of magnitude n . Hence

$$\sum_{|\omega| \leq N} \frac{1}{|\omega|^\lambda} \ll \sum_1^N \frac{n}{n^\lambda} \ll \sum_1^\infty \frac{1}{n^{\lambda-1}}.$$

which converges for $\lambda > 2$.

The series expression for \wp shows that it is meromorphic, with a double pole at each lattice point, and no other pole. It is also clear that \wp is even, i.e.

$$\wp(z) = \wp(-z)$$

(summing over the lattice points is the same as summing over their negatives). We get \wp' by differentiating term by term,

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z - \omega)^3},$$

the sum being taken for all $\omega \in L$. Note that \wp' is clearly periodic, and is odd, i.e.

$$\wp'(-z) = -\wp'(z).$$

From its periodicity, we conclude that there is a constant C such that

$$\wp(z + \omega_1) = \wp(z) + C.$$

Let $z = -\omega_1/2$ (not a pole of \wp). We get

$$\wp\left(\frac{\omega_1}{2}\right) = \wp\left(-\frac{\omega_1}{2}\right) + C,$$

and since \wp is even, it follows that $C = 0$. Hence \wp is itself periodic, something which we could not see immediately from its series expansion.

It is clear that the set of all elliptic functions (with respect to a given lattice L) forms a field, whose constant field is the complex numbers.

Theorem 4. The field of elliptic functions (with respect to L) is generated by \wp and \wp' .

Proof. If f is elliptic, we can write f as a sum of an even and an odd elliptic function as usual, namely

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}.$$

If f is odd, then the product $f\wp'$ is even, so it will suffice to prove that $\mathbf{C}(\wp)$ is the field of even elliptic functions, i.e. if f is even, then f is a rational function of \wp .

Suppose that f is even and has a zero of order m at some point u . Then clearly f also has a zero of the same order at $-u$ because

$$f^{(k)}(u) = (-1)^k f^{(k)}(-u).$$

Similarly for poles.

If $u \equiv -u \pmod{L}$, then the above assertion holds in the strong sense, namely f has a zero (or pole) of even order at u .

Proof. First note that $u \equiv -u \pmod{L}$ is equivalent to

$$2u \equiv 0 \pmod{L}.$$

On the torus, there are exactly four points with this property, represented by

$$0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$$

in a period parallelogram. If f is even, then f' is odd, i.e.

$$f'(u) = -f'(-u).$$

Since $u \equiv -u \pmod{L}$ and f' is periodic, it follows that $f'(u) = 0$, so that f has a zero of order at least 2 at u . If $u \not\equiv 0 \pmod{L}$, then the above argument shows that the function

$$g(z) = \wp(z) - \wp(u)$$

has a zero of order at least 2 (hence exactly 2 by Theorem 2 and the fact that \wp has only one pole of order 2 on the torus). Then f/g is even, elliptic, holomorphic at u . If $f(u)/g(u) \neq 0$ then $\text{ord}_u f = 2$. If $f(u)/g(u) = 0$ then f/g again has a zero of order at least 2 at u and we can repeat the argument. If $u \equiv 0 \pmod{L}$ we use $g = 1/\wp$ and argue similarly, thus proving that f has a zero of even order at u .

Now let u_i ($i = 1, \dots, r$) be a family of points containing one representative from each class $(u, -u) \pmod{L}$ where f has a zero or pole, other than the class of L itself. Let

$$m_i = \text{ord}_{u_i} f \quad \text{if } 2u_i \not\equiv 0 \pmod{L},$$

$$m_i = \frac{1}{2} \text{ord}_{u_i} f \quad \text{if } 2u_i \equiv 0 \pmod{L}.$$

Our previous remarks show that for $a \in \mathbf{C}$, $a \not\equiv 0 \pmod{L}$, the function

$\wp(z) - \wp(a)$ has a zero of order 2 at a if and only if $2a \equiv 0 \pmod{L}$, and has distinct zeros of order 1 at a and $-a$ otherwise. Hence for all $z \not\equiv 0 \pmod{L}$ the function

$$\prod_{i=1}^r [\wp(z) - \wp(u_i)]^{m_i}$$

has the same order at z as f . This is also true at the origin because of Theorem 2 applied to f and the above product. The quotient of the above product by f is then an elliptic function without zero or pole, hence a constant, thereby proving Theorem 4.

Next, we obtain the power series development of \wp and \wp' at the origin, from which we shall get the algebraic relation holding between these two functions. We do this by brute force.

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega \in L'} \left[\frac{1}{\omega^2} \left(1 + \frac{z}{\omega} + \left(\frac{z}{\omega} \right)^2 + \cdots \right)^2 - \frac{1}{\omega^2} \right] \\ &= \frac{1}{z^2} + \sum_{\omega \in L'} \sum_{m=1}^{\infty} (m+1) \left(\frac{z}{\omega} \right)^m \frac{1}{\omega^2} \\ &= \frac{1}{z^2} + \sum_{m=1}^{\infty} c_m z^m\end{aligned}$$

where

$$c_m = \sum_{\omega \neq 0} \frac{m+1}{\omega^{m+2}}.$$

Note that $c_m = 0$ if m is odd.

Using the notation

$$s_m(L) = s_m = \sum_{\omega \neq 0} \frac{1}{\omega^m}$$

we get the expansion

$$\boxed{\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)s_{2n+2}(L)z^{2n},}$$

from which we write down the first few terms explicitly:

$$\boxed{\wp(z) = \frac{1}{z^2} + 3s_4 z^2 + 5s_6 z^4 + \cdots}$$

and differentiating term by term,

$$\boxed{\wp'(z) = \frac{-2}{z^3} + 6s_4 z + 20s_6 z^3 + \cdots}$$

Theorem 5. Let $g_2 = g_2(L) = 60s_4$ and $g_3 = g_3(L) = 140s_6$. Then

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3.$$

Proof. We expand out the function

$$\wp(z) = \wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$$

at the origin, paying attention only to the polar term and the constant term. This is easily done, and one sees that there is enough cancellation so that these terms are 0, in other words, $\wp(z)$ is an elliptic function without poles, and with a zero at the origin. Hence \wp is identically zero, thereby proving our theorem.

The preceding theorem shows that the points $(\wp(z), \wp'(z))$ lie on the curve defined by the equation

$$y^2 = 4x^3 - g_2x - g_3.$$

The cubic polynomial on the right-hand side has a discriminant given by

$$\Delta = g_2^3 - 27g_3^2.$$

We shall see in a moment that this discriminant does not vanish.

Let

$$e_i = \wp\left(\frac{\omega_i}{2}\right), \quad i = 1, 2, 3,$$

where $L = [\omega_1, \omega_2]$ and $\omega_3 = \omega_1 + \omega_2$. Then the function

$$h(z) = \wp(z) - e_i$$

has a zero at $\omega_i/2$, which is of even order so that $\wp'(\omega_i/2) = 0$ for $i = 1, 2, 3$, by previous remarks. Comparing zeros and poles, we conclude that

$$\wp'^2(z) = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Thus e_1, e_2, e_3 are the roots of $4x^3 - g_2x - g_3$. Furthermore, \wp takes on the value e_i with multiplicity 2 and has only one pole of order 2 mod L , so that $e_i \neq e_j$ for $i \neq j$. This means that the three roots of the cubic polynomial are distinct, and therefore

$$\Delta = g_2^3 - 27g_3^2 \neq 0.$$

§3. THE ADDITION THEOREM

Given complex numbers g_2, g_3 such that $g_2^3 - 27g_3^2 \neq 0$, one can ask whether there exists a lattice for which these are the invariants associated to the lattice as in the preceding section. The answer is yes, and we shall prove this in chapter 3. For the moment, we consider the case when g_2, g_3 are given as in the preceding section, i.e. $g_2 = 60s_4$ and $g_3 = 140s_6$.

We have seen that the map

$$z \mapsto (1, \wp(z), \wp'(z))$$

parametrizes points on the cubic curve A defined by the equation

$$y^2 = 4x^3 - g_2x - g_3.$$

This is an affine equation, and we put in the coordinate 1 to indicate that we also view the points as embedded in projective space. Then the mapping is actually defined on the torus \mathbf{C}/L , and the lattice points, i.e. 0 on the torus, are precisely the points going to infinity on the curve. Let A_C denote the complex points on the curve. We in fact get a bijection

$$\mathbf{C}/L - \{0\} \rightarrow A_C - \{\infty\}.$$

This is easily seen: For any complex number α , $\wp(z) - \alpha$ has at most two zeros, and at least one zero, so that already under \wp we cover each complex number α . It is then verified at once that using \wp' separates the points of \mathbf{C}/L lying above α , thus giving us the bijection. If you know the terminology of algebraic geometry, then you know that the curve defined by the above equation is non-singular, and that our mapping is actually a complex analytic isomorphism between \mathbf{C}/L and A_C .

Furthermore, \mathbf{C}/L has a natural group structure, and we now want to see what it looks like when transported to A . We shall see that it is algebraic. In other words, if

$$P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2), \quad P_3 = (x_3, y_3)$$

and

$$P_3 = P_1 + P_2,$$

then we shall express x_3, y_3 as rational functions of (x_1, y_1) and (x_2, y_2) . We shall see that P_3 is obtained by taking the line through P_1, P_2 , intersecting it with the curve, and reflecting the point of intersection through the x -axis, as shown on Fig. 3.

Select $u_1, u_2 \in \mathbf{C}$ and $\notin L$, and assume $u_1 \not\equiv u_2 \pmod{L}$. Let a, b be complex numbers such that

$$\wp'(u_1) = a\wp(u_1) + b$$

$$\wp'(u_2) = a\wp(u_2) + b,$$

in other words $y = ax + b$ is the line through $(\wp(u_1), \wp'(u_1))$ and $(\wp(u_2), \wp'(u_2))$. Then

$$\wp'(z) = (a\wp(z) + b)$$

has a pole of order 3 at 0, whence it has three zeros, counting multiplicities, and two of these are at u_1 and u_2 . If, say, u_1 had multiplicity 2, then by Theorem 3 we would have

$$2u_1 + u_2 \equiv 0 \pmod{L}.$$

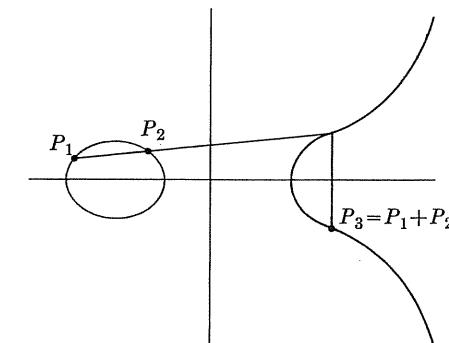


Fig. 1-3

If we fix u_1 , this can hold for only one value of u_2 . Let us assume that we do not deal with this value. Then both u_1, u_2 have multiplicity 1, and the third zero lies at

$$u_3 \equiv -(u_1 + u_2) \pmod{L}$$

again by Theorem 3. So we also get

$$\wp'(u_3) = a\wp(u_3) + b.$$

The equation

$$4x^3 - g_2x - g_3 - (ax + b)^2 = 0$$

has three roots, counting multiplicities. They are $\wp(u_1), \wp(u_2), \wp(u_3)$, and the left-hand side factors as

$$4(x - \wp(u_1))(x - \wp(u_2))(x - \wp(u_3)).$$

Comparing the coefficient of x^2 yields

$$\wp(u_1) + \wp(u_2) + \wp(u_3) = \frac{a^2}{4}.$$

But from our original equations for a and b , we have

$$a(\wp(u_1) - \wp(u_2)) = \wp'(u_1) - \wp'(u_2).$$

Therefore from

$$\wp(u_3) = \wp(-(u_1 + u_2)) = \wp(u_1 + u_2)$$

we get

$$\boxed{\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \frac{1}{4} \left(\frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)} \right)^2}$$

or in algebraic terms,

$$x_3 = -x_1 - x_2 + \frac{1}{4} \left(\frac{y_1 - y_2}{x_1 - x_2} \right)^2.$$

Fixing u_1 , the above formula is true for all but a finite number of $u_2 \not\equiv u_1 \pmod{L}$.

whence for all $u_2 \not\equiv u_1 \pmod{L}$ by analytic continuation.

For $u_1 \equiv u_2 \pmod{L}$ we take the limit as $u_1 \rightarrow u_2$ and get

$$\boxed{\wp(2u) = -2\wp(u) + \frac{1}{4} \left(\frac{\wp''(u)}{\wp'(u)} \right)^2.}$$

These give us the desired algebraic addition formulas. Note that the formulas involve only g_2, g_3 as coefficients in the rational functions.

This is as far as we shall push the study of the \wp -function in general, except for a Fourier expansion formula in Chapter 4. For further information, the reader is referred to Fricke [B2]. For instance one can get formulas for $\wp(nz)$, one can get a continued fraction expansion (done by Frobenius), etc. Classics like Fricke still contain much information which has not yet reappeared in more modern books, nor been made much use of, although history shows that everything that has been discovered along those lines ultimately returns to the center of the stage at some point.