THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 7

Due 3:30pm[1], November 15th (Friday), **in-class**.

Aim of this Homework: *In this assignment you will compute and study solutions to different Cauchy/initial-value problems for various PDE in the whole real line by applying explicit solution formulae or appropriate ansatz. In addition, you will also derive a solution formula for Laplace's equation in the upper half-plane.*

Reading Assignment: Read the following material(s):

(i) Section 3.3-3.4 of the textbook.

Instruction: Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the "Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

Problem 1. Consider the heat equation on the whole line

$$\partial_t u - \frac{1}{6}\partial_{xx} u = 0 \quad \text{for } -\infty < x < \infty \text{ and } t > 0$$

[1] You are expected to submit your homework before the beginning of Friday lecture **in-class**.

subject to the initial data

$$u(x,0) = \phi(x) \quad \text{for } -\infty < x < \infty,$$

where ϕ will be given differently in different parts below.

(i) Solve the initial value problem provided that

$$\phi(x) := e^{-4x} \quad \text{for } -\infty < x < \infty.$$

Express your final answer without using any integrals.

(ii) Solve the initial value problem provided that

$$\phi(x) := \sin\frac{8x}{3} \quad \text{for } -\infty < x < \infty.$$

Express your final answer without using any integrals.

(iii) Solve the initial value problem provided that

$$\phi(x) := \begin{cases} e^{7x} & \text{if } x \geq 5 \\ 0 & \text{if } x < 5. \end{cases}$$

Express your final answer in terms of the Gauss error function

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2}\, dp.$$

(iv) Solve the initial-value problem provided that

$$\phi(x) := \begin{cases} 9 & \text{if } |x| < 2 \\ -1 & \text{if } |x| \geq 2. \end{cases}$$

Express your final answer in terms of the Gauss error function erf as well.

Problem 2. Consider the following initial-value problem:

$$\begin{cases} \partial_t u - k\partial_{xx}u + \alpha\partial_x u + \beta u = 0, & \text{for } -\infty < x < \infty \text{ and } t > 0, \\ u|_{t=0}(x) = \phi(x), & \text{for } -\infty < x < \infty, \end{cases} \tag{1}$$

where k is a given positive constant, α and β are given constants, and ϕ is a given initial data. To find an explicit solution formula for the initial-value problem (1), complete the following parts.

(i) Let

$$\begin{cases} y := x - \alpha t \\ \tau := t. \end{cases}$$

Rewrite (1) in terms of y and τ, and show that

$$\begin{cases} \partial_\tau u - k\partial_{yy}u + \beta u = 0 \\ u|_{\tau=0}(y) = \phi(y). \end{cases}$$

(ii) Define

$$v(y,\tau) := e^{\beta\tau}u(y,\tau).$$

Prove that v satisfies

$$\begin{cases} \partial_\tau v - k\partial_{yy}v = 0 \\ v|_{\tau=0}(y) = \phi(y). \end{cases}$$

(iii) Solve the initial-value problem (1). Express the solution u in terms of x and t only.

Problem 3 (Polynomial Solutions to the Heat Equation)**.** In this problem we aim at solving the following initial-value problem for the heat equation:

$$\begin{cases} \partial_t u - k\partial_{xx}u = 0, & \text{for } -\infty < x < \infty \text{ and } t > 0, \\ u(x,0) = \phi(x) := \sum_{n=0}^{N} a_n x^n, & \text{for } -\infty < x < \infty, \end{cases} \tag{2}$$

where k is a given positive constant, a_n's are some given constants, and $a_N \neq 0$. Since the initial data ϕ is a degree N polynomial, one may guess that the solution u has the ansatz[2]

$$u(t,x) = \sum_{n=0}^{N} A_n(t) x^n, \tag{3}$$

where the coefficient function $A_n : [0,\infty) \to \mathbb{R}$ will be determined below.

(i) Prove that if the solution u is of the form (3), then A_n's satisfy the following system of ODE:

$$\begin{cases} A_n' = k(n+1)(n+2)A_{n+2}, & \text{for } n = 0,1,\cdots,N-2, \\ A_{N-1}' = 0, \\ A_N' = 0, \end{cases} \tag{4}$$

subject to the initial data

$$A_n(0) = a_n, \qquad \text{for } n = 0,1,\cdots,N. \tag{5}$$

(ii) Assuming $N \geq 5$, compute A_N, A_{N-1}, A_{N-2}, A_{N-3}, A_{N-4}, and A_{N-5} via solving (4) and (5). Express your answers in terms of a_n's and t only.

In general, one may compute all A_n's by solving (4) and (5). We leave this for the interested students.

(iii) Solve the initial-value problem (2) provided that

$$\phi(x) := x^6 - 7x^5 + 8.$$

Problem 4. Do Problem 5 of Dec 2020 Final Exam.

The following problem(s) is/are *optional*:

[2] An ansatz is an educated guess; see `https://en.wikipedia.org/wiki/Ansatz` for more details.

Problem 5. The purpose of this problem is to construct a self-similar solution to Laplace's equation in the upper half-plane: for any $-\infty < x < \infty$ and $y > 0$,

$$\partial_{xx}u + \partial_{yy}u = 0. \tag{6}$$

Complete the following parts.

(i) Prove that if u solves Laplace's equation (6), then so does $v(x, y) := u(ax, ay)$ for any positive constant a.

(ii) Assume that the solution u to (6) is of the form

$$u(x, y) = g\left(\frac{x}{y}\right),$$

where the function $g : \mathbb{R} \to \mathbb{R}$ will be determined below. Derive an ODE for g.

(iii) Find the general solution of g.

(iv) Express the general self-similar solution u in terms of x and y only.

(v) For any fixed $x > 0$, compute

$$\lim_{y \to 0^+} u(x, y).$$

(vi) For any fixed $x < 0$, compute

$$\lim_{y \to 0^+} u(x, y).$$

(vii) Solve the following Dirichlet problem:

$$\begin{cases} \partial_{xx}u + \partial_{yy}u = 0, \text{ for } -\infty < x < \infty \text{ and } y > 0, \\ u|_{y=0} = H(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \end{cases}$$

where the function H is the Heaviside (step) function.

(viii) Compute $v := \partial_x u$. Is v also a solution to (6)?

(ix) Find the explicit solution formula for the Dirichlet problem

$$\begin{cases} \partial_{xx} u + \partial_{yy} u = 0, & \text{for } -\infty < x < \infty \text{ and } y > 0, \\ u|_{y=0} = \phi(x), & \text{for } -\infty < x < \infty, \end{cases}$$

where ϕ is an arbitrary given boundary data.

Food for Thought. Are you able to write down the solution formula to the Dirichlet problem

$$\begin{cases} \partial_{xx} u + \partial_{yy} u = 0, & \text{for } x,\ y > 0, \\ u|_{x=0} = 0, & \text{for } 0 < y < \infty, \\ u|_{y=0} = \phi(x), & \text{for } 0 < x < \infty, \end{cases}$$

where ϕ is an arbitrary given boundary data?

Problem 6. Solve

$$\begin{cases} \partial_t u - 4\partial_{xx} u - 9\partial_x u + 5u = 0, & \text{for } -\infty < x < \infty \text{ and } t > 0, \\ u|_{t=0}(x) = e^{6x} + x^3 - 8x^2 + 7, & \text{for } -\infty < x < \infty. \end{cases}$$

Problem 7. Consider the following initial-value problem

$$\begin{cases} \partial_t u - k\partial_{xx} u = f & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi, \end{cases} \tag{7}$$

where $k > 0$ is a given constant, f and ϕ will be given differently in different parts below. Solve the initial-value problem (7) in the following cases:

(i) $f(t,x) := 3$ and $\phi(x) := x^5$;

(ii) $f(t,x) := \cos t$ and $\phi(x) := x^2$;

(iii) $f(t,x) := xe^{-2t}$ and $\phi(x) := \begin{cases} 6 & \text{if } |x| \le 4 \\ 0 & \text{if } |x| > 4. \end{cases}$