THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 2

Problem 1. In this problem we consider the following boundary-value problem of ODE:

$$\begin{cases} u'' - 2u' + 2u = 0 \\ u(0) = u'(L) = 0, \end{cases} \tag{1}$$

where $u : [0, L] \to \mathbb{R}$ is the unknown, and $L > 0$ is a given constant.

(i) Find the general solution of $u'' - 2u' + 2u = 0$.

(ii) Prove that the problem (1) has a unique solution if and only if $L \neq \frac{(4k-1)\pi}{4}$ for any positive integer k.

Problem 2. Given the following second-order linear PDE

$$2\partial_x u + \partial_{xy} u = 0 \tag{2}$$

(i) solve (2) by rewriting it as $\partial_x (2u + \partial_y u) = 0$ and hence considering

$$2u + \partial_y u = g(y), \text{ where } g \text{ is any arbitrary function.}$$

(ii) prove or disprove the v defined in each of the cases below satisfies (2):

(a) $v(x, y) \coloneqq u(x - 1, 2y)$.

(b) $v(x, y) \coloneqq y \left[\partial_x u(x, y)\right]$.

(c) for any $g : \mathbb{R} \to \mathbb{R}$,

$$v(x, y) \coloneqq (u * g)(x, y) \coloneqq \int_{-\infty}^{\infty} u(x - t, y) g(t)\, dt.$$

Problem 3. Solve the following PDE

$$\frac{2}{y+1}\partial_x u + \partial_{xy} u = \frac{e^y}{(y+1)^2}$$

subject to the conditions

$$u\big|_{x=0} = u\big|_{y=0} = 0.$$

(Hint: let $v := \partial_x u$.)

Problem 4. Given the following PDE

$$\partial_{xx} u + \partial_{tt} u = 0 \text{ for } 0 < x < 1 \text{ and } t > 0$$

subject to the initial and boundary value conditions

$$\begin{cases} u\big|_{x=0} = u\big|_{x=1} = 0 \\ u\big|_{t=0} = f \text{ and } \partial_t u\big|_{t=0} = 0. \end{cases}$$

(i) Verify that for any positive integer n, the function

$$u_n(t,x) := \frac{1}{n}\cosh(n\pi t)\sin(n\pi x)$$

is a solution to the initial and boundary value problem with the initial data

$$f_n(x) := \frac{1}{n}\sin(n\pi x).$$

(ii) Let the sup norm be defined by $\|g\|_{\sup} := \sup_{x\in[0,1]} |g(x)|$. Prove that

$$\lim_{n\to+\infty} \|f_n\|_{\sup} = 0 \text{ and } \lim_{n\to+\infty} \|u_n(T,\cdot)\|_{\sup} = +\infty \text{ for any } T > 0.$$

Problem 5. Consider the following heat equation:

$$\partial_t u - \partial_{xx} u = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{3}$$

(i) (Invariant Transformation/Symmetries)

Let $u := u(t,x)$ be a C^2 solution to (3). Show that the $u_k(t,x) := u(k^2 t, kx)$ is also a solution.

(ii) (Linearity and Principle of Superposition)

Let u_1, u_2 be C^2 solutions to (3). Show that any linear combination of u_1, u_2 is also a solution, i.e., $(\alpha u_1 + \beta u_2)$ is also a solution for any constant $\alpha, \beta \in \mathbb{R}$.

Problem 6. Let $u = u(y_1, y_2)$ be a differentiable function of independent variables y_1 and y_2. Suppose that $y_1 = y_1(x_1, x_2)$ and $y_2 = y_2(x_1, x_2)$ are differentiable functions of independent variables x_1 and x_2.

(i) Using the chain rule, for $i = 1, 2$, show that

$$\begin{aligned}\partial_{x_i}^2 u = \partial_{y_1}^2 u \cdot \left(\partial_{x_i} y_1\right)^2 + \partial_{y_1} u \cdot \partial_{x_i}^2 y_1 + 2\partial_{y_1 y_2} u \cdot \partial_{x_i} y_1 \cdot \partial_{x_i} y_2 \\ + \partial_{y_2}^2 u \cdot \left(\partial_{x_i} y_2\right)^2 + \partial_{y_2} u \cdot \partial_{x_i}^2 y_2.\end{aligned}$$

(ii) Given

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

if $\partial_{y_1}^2 u + \partial_{y_2}^2 u = 1$, calculate $\partial_{x_1}^2 u + \partial_{x_2}^2 u$.