# 20250526 MATH3541 NOTE 11[1]

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# Contents

# 1 Abstract

This note assumes the basics of fundamental group. It includes covering map, Galois theory and Seifert-Van Kampen theorem, based on the first chapter book *Algebraic Topology*[2] and the lecture note *Introduction to Topology*[3]. It has two purposes:

(1) Compute the fundamental groups of some topological spaces, including but not limited to $\mathbb{C}^\times, \bigwedge_{\lambda\in I}\mathbb{S}, \mathbb{S}^n, \mathbb{S}^3/\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \frac{\pm 1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k}}{2}\}, \mathbb{T}^2, \mathbb{K}^2, (\mathbb{T}^2)^{\#g}$.

(2) Seek connections between algebraic topology and other subjects. For example, the Lie group structure of $\mathbb{S}^3$, the Galois correspondence for normal covering map, Cayley graph, ping-pong lemma, torus knot, and Nielsen-Schreier theorem.

# 2 Covering Map

## 2.1 Compact-open Topology

**Definition 2.1.** (**Compact-open Topology**)
Let $X, Y$ be topological spaces.

(1) Define $\mathbf{C}(U,V) = \{f \in \mathbf{C}(X,Y) : f(U) \subseteq V\}$,
where $U$ is compact in $X$ and $V$ is open in $Y$.

(2) Define the compact-open topology on $\mathbf{C}(X,Y)$ by the subbasis
$\{\mathbf{C}(U,V) \subseteq \mathbf{C}(X,Y) : U \text{ is compact in } X \text{ and } V \text{ is open in } Y\}$.

**Proposition 2.2.** Let $X, Y, Z$ be topological spaces.
If $g : Y \to Z$ is continuous, then the map below is continuous:

$$\mathbf{C}(X,g) : \mathbf{C}(X,Y) \to \mathbf{C}(X,Z), f \mapsto g\circ f$$

*Proof.*

$$\begin{aligned}\Omega \text{ is open in } \mathbf{C}(X,Z) &\implies \Omega \text{ is generated by } \mathbf{C}(U,W)\\ &\implies \mathbf{C}(X,g)^{-1}(\Omega) \text{ is generated by } \mathbf{C}(U,g^{-1}(W))\\ &\implies \mathbf{C}(X,g)^{-1}(\Omega) \text{ is open in } \mathbf{C}(X,Y)\end{aligned}$$

□

**Proposition 2.3.** Let $X, Y, Z$ be topological spaces.
If $f : X \to Y$ is continuous, then the map below is continuous:

$$\mathbf{C}(f,Z) : \mathbf{C}(Y,Z) \to \mathbf{C}(X,Z), g \mapsto g\circ f$$

*Proof.*

$$\begin{aligned} \Omega \text{ is open in } \mathbf{C}(X,Z) &\implies \Omega \text{ is generated by } \mathbf{C}(U,W) \\ &\implies \mathbf{C}(f,Z)^{-1}(\Omega) \text{ is generated by } \mathbf{C}(f(U),W) \\ &\implies \mathbf{C}(f,Z)^{-1}(\Omega) \text{ is open in } \mathbf{C}(Y,Z) \end{aligned}$$

□

**Proposition 2.4.** Let $X, Y, Z$ be topological spaces.
If $f : X \times Y \to Z$ is continuous, then for all $x \in X$, the map below is continuous:

$$f_x : Y \to Z, y \mapsto f(x,y)$$

*Proof.*

$$\begin{aligned} W \text{ is open in } Z &\implies f^{-1}(W) \text{ is open in } X \times Y \\ &\implies \{x\} \times f_x^{-1}(W) \text{ is open in } \{x\} \times Y \\ &\implies f_x^{-1}(W) \text{ is open in } Y \end{aligned}$$

□

**Proposition 2.5.** Let $X, Y, Z$ be topological spaces.
If $f : X \times Y \to Z$ is continuous, then the map below is continuous:

$$f_X : X \to \mathbf{C}(Y,Z), x \mapsto f_x$$

*Proof.* For all $x_0 \in X$, for all $\mathbf{C}(V,W)$ containing $f_X(x_0) = f_{x_0}$,
we aim to find an open neighbour of $x_0$ contained in $f_X^{-1}(\mathbf{C}(V,W))$.

(1) As $f_{x_0} \in \mathbf{C}(V,W)$, $f(\{x_0\} \times V) = f_{x_0}(V) \subseteq W$,
so $\{x_0\} \times V$ has an open neighbour $\bigcup_{\lambda \in I} U'_\lambda \times V'_\lambda \subseteq f^{-1}(W)$.

(2) As $\{x_0\} \times V$ is compact, we may reduce $I$ to a finite set,
and reduce $\bigcup_{\lambda \in I} U'_\lambda \times V'_\lambda$ to $U' \times V'$, where $U' = \bigcap_{\lambda \in I} U'_\lambda, V' = \bigcup_{\lambda \in I} V'_\lambda$.

(3) It suffices to choose the open neighbour $U'$ of $x_0$.

□

**Proposition 2.6.** Let $X, Y, Z$ be topological spaces.
The map below is continuous:

$$\sigma : \mathbf{C}(X \times Y, Z) \to \mathbf{C}(X, \mathbf{C}(Y,Z)), f \mapsto f_X$$

*Proof.*

$$
\begin{aligned}
\Omega \text{ is open in } \mathbf{C}(X, \mathbf{C}(Y,Z)) &\implies \Omega \text{ is generated by } \mathbf{C}(U, \mathbf{C}(V,W)) \\
&\implies \sigma^{-1}(\Omega) \text{ is generated by } \mathbf{C}(U \times V, W) \\
&\implies \sigma^{-1}(\Omega) \text{ is open in } \mathbf{C}(X \times Y, Z)
\end{aligned}
$$

□

**Proposition 2.7.** Let $X, Y, Z$ be topological spaces with $Y$ regular and locally compact. The map below is continuous:

$$\sigma : \mathbf{C}(X,Y) \times \mathbf{C}(Y,Z) \to \mathbf{C}(X,Z), (f,g) \mapsto g \circ f$$

*Proof.* For all $(f_0, g_0) \in \mathbf{C}(X,Y) \times \mathbf{C}(Y,Z)$, for all $\mathbf{C}(U,W)$ containing $\sigma(f_0, g_0) = g_0 \circ f_0$, we aim to find an open neighbour of $(f_0, g_0)$ contained in $\sigma^{-1}(\mathbf{C}(U,W))$.

(1) As $Y$ is regular and locally compact,
every $y \in Y$ has a precompact open neighbour $V$ with $\bar{V} \subseteq g_0^{-1}(W)$.

(2) As $f_0(U)$ is compact,
$f_0(U)$ has a precompact open neighbour $V$ with $\bar{V} \subseteq g_0^{-1}(W)$.

(3) It suffices to choose the open neighbour $\mathbf{C}(U,V) \times \mathbf{C}(\bar{V}, W)$ of $(f_0, g_0)$.

□

**Proposition 2.8.** Let $X, Y, Z$ be topological spaces with $Y$ regular and locally compact. If $f_X : X \to \mathbf{C}(Y,Z)$ is continuous, then the map below is continuous:

$$f : X \times Y \to Z, (x,y) \mapsto f_X(x)(y)$$

*Proof.* For all $(x_0, y_0) \in X \times Y$, for all open neighbour $W$ of $f(x_0, y_0) = f_X(x_0)(y_0)$, we aim to find an open neighbour of $(x_0, y_0)$ contained in $f^{-1}(W)$.

(1) As $Y$ is regular and locally compact,
$y_0$ has a precompact open neighbour $V$ with $\bar{V} \subseteq f_X(x_0)^{-1}(W)$.

(2) As $f_X$ is continuous, $x_0$ has an open neighbour $U = f_X^{-1}(\mathbf{C}(\bar{V}, W))$.

(3) It suffices to choose the open neighbour $U \times V$ of $(x_0, y_0)$.

□

**Proposition 2.9.** Let $X, Y, Z$ be topological spaces with $X, Y$ regular and locally compact. The map below is continuous:

$$\tau : \mathbf{C}(X, \mathbf{C}(Y,Z)) \to \mathbf{C}(X \times Y, Z), f_X \mapsto f$$

*Proof.* For all $(f_X)_0 \in \mathbf{C}(X, \mathbf{C}(Y, Z))$, for all $\mathbf{C}(K, W)$ containing $\tau((f_X)_0) = f_0$, we aim to find an open neighbour of $(f_X)_0 \in \mathbf{C}(X, \mathbf{C}(Y, Z))$ contained in $\tau^{-1}(\mathbf{C}(K, W))$.

(1) As $f_0$ is continuous, $f_0^{-1}(W)$ is an open neighbour of the compact set $K$.

(2) As $X, Y$ are regular and locally compact, for all $(x, y) \in K$, for some precompact open neighbours $U, V$ of $x, y$, $\bar{U} \times \bar{V} \subseteq f_0^{-1}(W)$, which covers $K$ within $W$.

(3) As $K$ is compact, finitely many $(U_\lambda \times V_\lambda)_{\lambda \in I}$ covers $K$.

(4) It suffices to choose the open neighbour $\bigcap_{\lambda \in I} \mathbf{C}(\bar{U}_\lambda, \mathbf{C}(\bar{V}_\lambda, W))$ of $f_0$.

□

**Remark:** As a consequence, if $Y, Z$ are topological spaces with $Y$ regular and locally compact, then a homotopy from $Y$ to $Z$ is equivalent to a path in $\mathbf{C}(Y, Z)$.

**Proposition 2.10.** Let $X, Y$ be topological spaces, and $X_1, X_2, X_3$ be closed subsets of $X$ with $X = X_1 \cup X_2, X_3 = X_1 \cap X_2$. $\sigma : \mathbf{C}(X_1, Y) \times_{\mathbf{C}(X_3, Y)} \mathbf{C}(X_2, Y) \to \mathbf{C}(X, Y), (f_1, f_2) \mapsto f_1 \cup f_2$ is a homeomorphism.

*Proof.* We may divide our proof into three steps.

(1) According to the gluing lemma, $\sigma$ is a well-defined bijection.

(2) For all subbasis element $\mathbf{C}(U_1, V_1) \times_{\mathbf{C}(X_3, Y)} \mathbf{C}(U_2, V_2)$,
Its image $\mathbf{C}(U_1, V_1) \cap \mathbf{C}(U_2, V_2)$ is open.

(3) For all subbasis element $\mathbf{C}(U, V)$,
Its preimage $\mathbf{C}(U \cap X_1, V) \times_{\mathbf{C}(X_3, Y)} \mathbf{C}(U \cap X_2, V)$ is open.

□

**Proposition 2.11.** Let $X, Y$ be topological spaces, where $X$ is normal, and $X_1, X_2, X_3$ be open subsets of $X$ with $X = X_1 \cup X_2, X_3 = X_1 \cap X_2$. $\sigma$ : $\mathbf{C}(X_1, Y) \times_{\mathbf{C}(X_3, Y)} \mathbf{C}(X_2, Y) \to \mathbf{C}(X, Y), (f_1, f_2) \mapsto f_1 \cup f_2$ is a homeomorphism.

*Proof.* We may divide our proof into three steps.

(1) According to the gluing lemma, $\sigma$ is a well-defined bijection.

(2) For all subbasis element $\mathbf{C}(U_1, V_1) \times_{\mathbf{C}(X_3, Y)} \mathbf{C}(U_2, V_2)$,
Its image $\mathbf{C}(U_1, V_1) \cap \mathbf{C}(U_2, V_2)$ is open.

(3) For all subbasis element $\mathbf{C}(U, V)$,
Its preimage $\mathbf{C}(U \backslash X_4, V) \times_{\mathbf{C}(X_3, Y)} \mathbf{C}(U \backslash X_5, V)$ is open.
Here, $X_4, X_5$ separate $X \backslash X_1, X \backslash X_2$.

□

## 2.2 Definition and Example

**Definition 2.12.** (**Covering Map**)
Let $X, Y$ be topological spaces,
and $f : X \to Y$ be a continuous surjection.
If for all $y \in Y$, for some open neighbour $V$ of $y$:

(1) $f^{-1}(V) \cong \coprod_{\lambda \in I} U_\lambda$, where each $U_\lambda$ is open in $X$.

(2) Each $f_\lambda : U_\lambda \to V, x \mapsto f(x)$ is a homeomorphism.

Then $f$ is a covering map.

**Definition 2.13.** (**Covering Map Homomorphism**)
Let $X, \bar{X}, Y, \bar{Y}$ be topological spaces,
$f : X \to \bar{X}, g : Y \to \bar{Y}$ be continuous,
and $p : X \to Y, \bar{p} : \bar{X} \to \bar{Y}$ be covering maps. If:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & \bar{X} \arrow[d, "\bar{p}"] \\ Y \arrow[r, "g"'] & \bar{Y} \end{tikzcd}$$

Then $(f, g) \in \mathbf{Hom}(p, \bar{p})$ is a homomorphism.

**Remark:** If we consider a fibre bundle $X$ over $Y$, then covering map homomorphism generalizes to bundle map. We will introduce the concept of deck transformation group later. Isomorphic covering maps induce isomorphic deck transformation group. This can be used to compute the deck transformation group, thus computing the fundamental group, because we will have a simple way to compute the deck transformation group of a special type of covering map, namely, free and properly discontinuous group action.

**Proposition 2.14.** Let $X = \mathbb{C}, Y = \mathbb{C}^\times$.
$f : X \to Y, x \mapsto \mathrm{e}^x$ is a covering map.

*Proof.* For all $y = \mathrm{e}^{x_1 + \mathrm{i}x_2} \in Y$, for some open neighbour $V$ of $y$:

$$V = \mathrm{e}^{\mathbb{R} + \mathrm{i}(x_2 - \pi, x_2 + \pi)}$$

It suffices to cover $V$ by:

$$U_k = \mathbb{R} + \mathrm{i}(x_2 + 2k\pi - \pi, x_2 + 2k\pi + \pi)$$

□

**Proposition 2.15.** Let $X = \mathfrak{gl}_2(\mathbb{C}), Y = \mathbf{GL}_2(\mathbb{C})$.
$f : X \to Y, x \mapsto \mathrm{e}^x$ is not a covering map.

*Proof.* It suffices to notice that some $f$-fibre is not discrete:

$$\mathrm{e}^{\begin{pmatrix} 0 & \mathbb{C} \\ 0 & 2\pi\mathrm{i} \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

□

**Proposition 2.16.** Let $X = \mathbb{R}^3, Y = \mathbb{S}^3$.
$f : X \to Y, x \mapsto \mathrm{e}^x$ is not a covering map.

*Proof.* It suffices to notice that some $f$-fibre is not discrete:

$$\mathrm{e}^{2\pi\mathbb{S}^2} = \mathbf{1}$$

□

**Proposition 2.17.** Let $X = \mathbb{C}^\times, Y = \mathbb{C}^\times, n > 0$.
$f : X \to Y, x \mapsto x^n$ is a covering map.

*Proof.* For all $y = \mathrm{e}^{n\alpha_1 + \mathrm{i}n\alpha_2} \in Y$, for some open neighbour $V$ of $y$:

$$V = \mathrm{e}^{\mathbb{R} + \mathrm{i}(n\alpha_2 - \pi, n\alpha_2 + \pi)}$$

It suffices to cover $V$ by:

$$U_k = \mathrm{e}^{\mathbb{R} + \mathrm{i}\left(\alpha_2 + \frac{2k\pi - \pi}{n}, \alpha_2 + \frac{2k\pi + \pi}{n}\right)}$$

□

**Proposition 2.18.** Let $X = \mathbf{GL}_2(\mathbb{C}), Y = \mathbf{GL}_2(\mathbb{C}), n > 0$.
$f : X \to Y, x \mapsto x^n$ is not a covering map.

*Proof.* It suffices to notice that some $f$-fibre is not discrete:

$$\begin{pmatrix} 1 & \mathbb{C} \\ 0 & \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

□

**Proposition 2.19.** Let $X = \mathbb{R}^3, Y = \mathbb{S}^3, n > 0$.
$f : X \to Y, x \mapsto x^n$ is not a covering map.

*Proof.* It suffices to notice that some $f$-fibre is not discrete:

$$\left(\cos\frac{2\pi}{n} + \mathbb{S}^2 \sin\frac{2\pi}{n}\right)^n = 1$$

Quod. Erat. Demonstrandum. □

**Proposition 2.20.** Let $X, Y$ be open in $\mathbb{C}$, and $f : X \to Y$ be a holomorphic surjection. If $\exists x \in X, f'(x) = 0$, then $f$ is not a covering map.

*Proof.* It suffices to notice that $f$ is not injective near $x$:

$$\exists r > 0, \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(x,r)} \frac{f'(a)}{f(a) - f(x)} \mathrm{d}a > 1$$

□

**Proposition 2.21.** Let $n \geq 1$, $X$ be the set of all $n$-complex tuples with distinct entries, and $Y$ be the set of all monic degree $n$ polynomials in $\mathbb{C}[z]$ with distinct roots. $f : X \mapsto Y, (z_1, z_2, z_3, \cdots) \mapsto (z - z_1)(z - z_2)(z - z_3) \cdots$ is a covering map.

*Proof.* According to the fundamental theorem of algebra, $f$ is surjective.
According to the fundamental theorem of algebra, each $f$-fibre has cardinality $n!$.
According to inverse function theorem, $f$ is locally biholomorphic:

$$\begin{aligned}
(z - z_1)(z - z_2)(z - z_3) \cdots &= z^n - \sigma_1 z^{n-1} + \sigma_2 z^{n-2} - \sigma_3 z^{n-3} + \cdots \\
\left(\frac{\partial \sigma_1}{\partial z_k}, \frac{\partial \sigma_2}{\partial z_k}, \frac{\partial \sigma_3}{\partial z_k}, \cdots\right) &= \left(1, \sigma_1 - z_k \frac{\partial \sigma_1}{\partial z_k}, \sigma_2 - z_k \frac{\partial \sigma_2}{\partial z_k}, \cdots\right) \\
&= \left(1, \sigma_1 - z_k, \sigma_2 - z_k \sigma_1 + z_k^2, \cdots\right) \\
\frac{\partial(\sigma_1, \sigma_2, \sigma_3, \cdots)}{\partial(z_1, z_2, z_3, \cdots)} &= \begin{vmatrix} 1 & 1 & 1 & \cdots \\ -z_1 & -z_2 & -z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix} \neq 0
\end{aligned}$$

□

**Definition 2.22.** (**Hawaii Earring $\mathbf{Z}_N(a,b)$**)

(1) Let $a, b \in \mathbb{C}, b \neq a$. Define:

$$\mathbf{S}(a,b) = \left\{ \frac{a+b}{2} + \frac{b-a}{2}\mathrm{e}^{\mathrm{i}t} \in \mathbb{C} : t \in \mathbb{R} \right\}$$

(2) Let $N > 0, a, b \in \mathbb{C}, b \neq a$. Define:

$$\mathbf{Z}_N(a,b) = \bigcup_{n \geq N} \mathbf{S}\left(a, a + \frac{b-a}{n}\right)$$

**Remark:** For all open neighbour $U$ of $a \in \mathbf{Z}_N(a,b)$, for some $K \geq N$, $\mathbf{Z}_K(a,b) \subseteq U$.

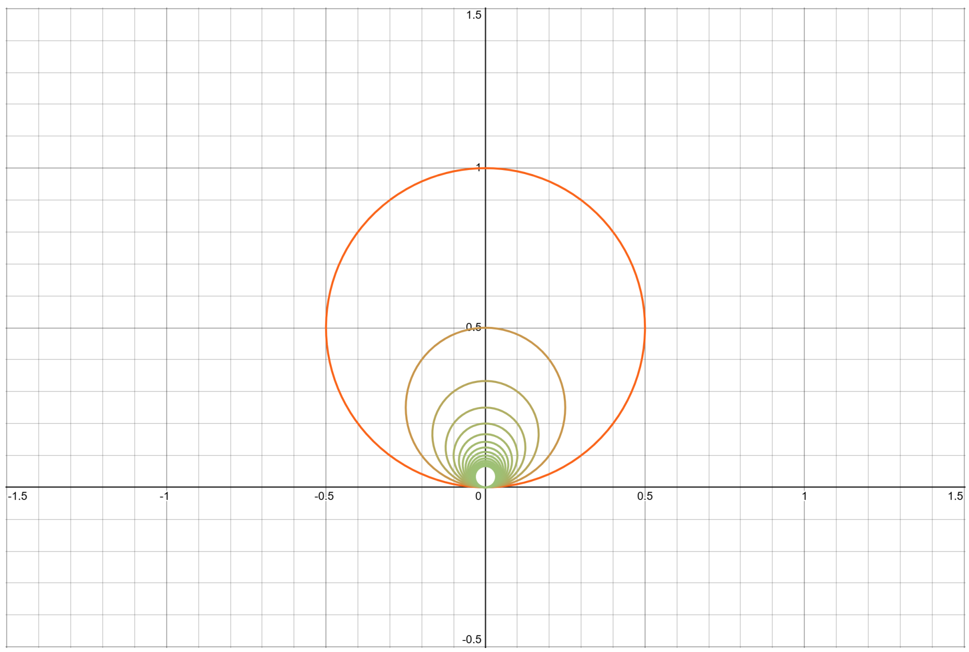



Figure 1: Hawaii Earring $\mathbf{Z}_1(0, \mathrm{i})$

**Definition 2.23.** (**Hawaii Necklace $\mathbf{Y}(a,b)$**)

(1) Let $a, b \in \mathbb{C}, b \neq a$. Define:

$$\mathbf{L}(a,b) = \{a + \mathrm{i}t(b-a) \in \mathbb{C} : t \in \mathbb{R}\}$$

(2) Let $a, b \in \mathbb{C}, b \neq a$. Define:

$$\mathbf{Y}(a,b) = \mathbf{L}(a,b) \cup \bigcup_{n \in \mathbb{Z}} \mathbf{Z}_2(a + \mathrm{i}n(b-a), b + \mathrm{i}n(b-a))$$

**Remark:** There is a covering map $g : \mathbf{Y}(a,b) \to \mathbf{Z}_1(a,b)$.

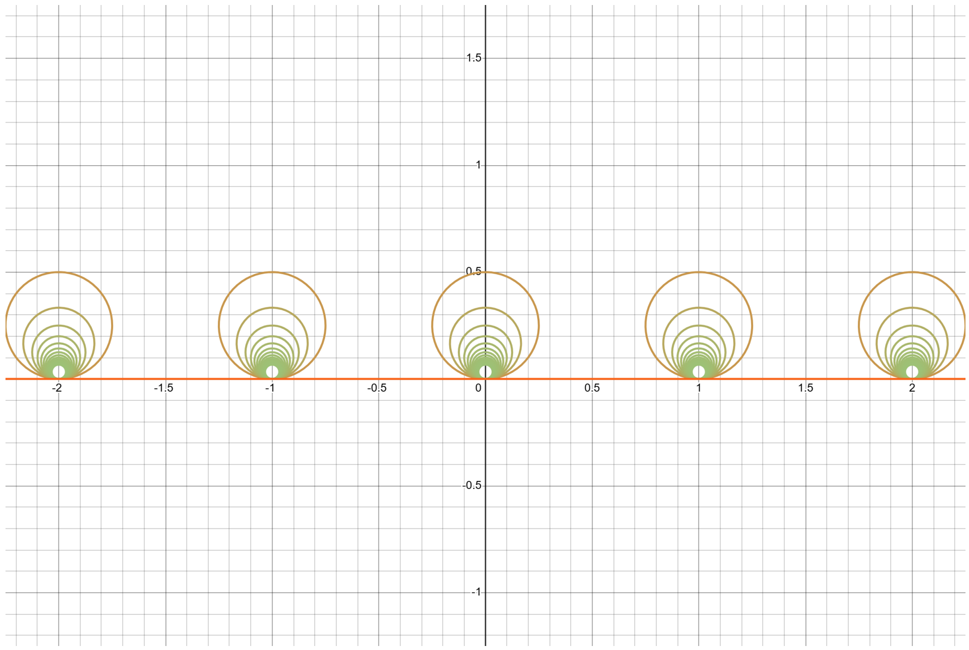



Figure 2: Hawaii Necklace $\mathbf{Y}(0, \mathrm{i})$

**Definition 2.24.** (**Hawaii Necklace** $\mathbf{X}(a, b)$)

(1) Let $M > 0, a, b \in \mathbb{C}, b \neq a$. Define:

$$\mathbf{S}_M(a, b) = \left\{ \frac{a+b}{2} + \frac{b-a}{2} \cos t + \frac{b-a}{2\sqrt{M}} \mathrm{i} \sin t \in \mathbb{C} : t \in \mathbb{R} \right\}$$

(2) Let $M > 0, a, b \in \mathbb{C}, b \neq a$. Define:

$$\mathbf{W}_M(a, b) = \bigcup_{2 \leq m \leq M} \mathbf{S}_m(a, b)$$

(3) Let $a, b \in \mathbb{C}, b \neq a$. Define:

$$\mathbf{X}(a, b) = \mathbf{L}(a, b) \cup \mathbf{L}(b, a) \cup \bigcup_{n \geq 0} \mathbf{W}_{n+2}(a, b) \cup \mathbf{Z}_{n+3}(a, b) \cup \mathbf{Z}_{n+3}(b, a)$$

**Remark:** There is a covering map $f : \mathbf{X}(a, b) \to \mathbf{Y}(a, b)$. However, $h = g \circ f$ is not a covering map, because every open neighbour $W$ of $a \in \mathbf{Z}_1(a, b)$ contains some circle, which is eventually unwinded in $U = h^{-1}(W)$, resulting in the noninjectivity of the restricted $h$. This situation motivates us to study the diagram below:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

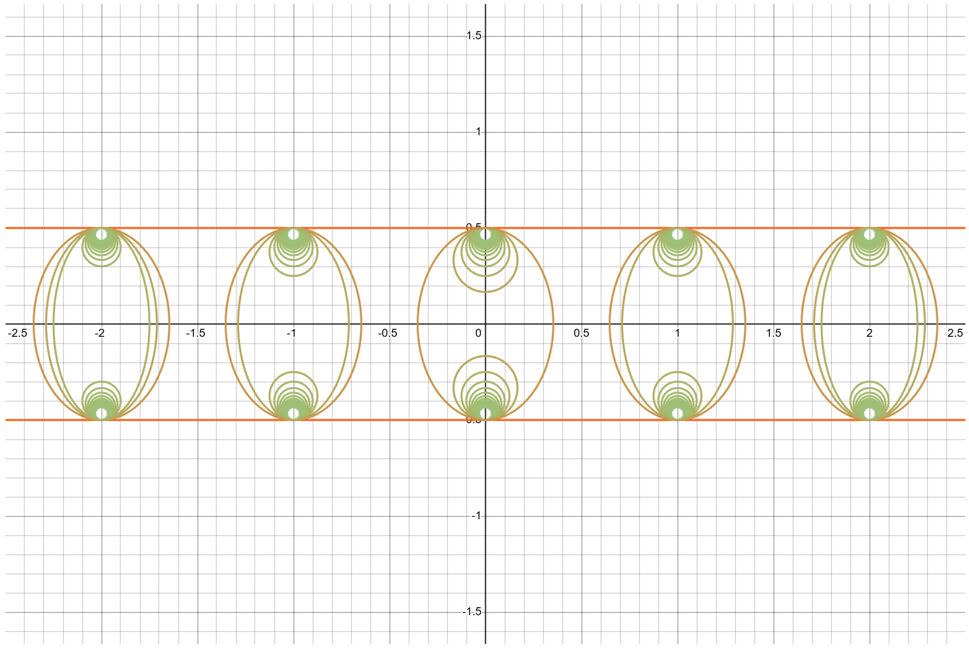



Figure 3: Hawaii Necklace $\mathbf{X}\left(-\frac{\mathrm{i}}{2}, \frac{\mathrm{i}}{2}\right)$

## 2.3 Lifting Property

**Definition 2.25.** (**Lift**)
Let $(\widetilde{X}, \widetilde{x}_0), (X, x_0), (Y, y_0)$ be topological spaces,
$p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map,
and $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0), f : (Y, y_0) \to (X, x_0)$ be continuous.
If $p \circ \widetilde{f} = f$, then $\widetilde{f}$ is a lift of $f$ in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$.

**Proposition 2.26.** (**Path Lifting Property**)
Let $(\widetilde{X}, \widetilde{x}_0), (X, x_0)$ be topological spaces,
$p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map,
and $f : ([0, 1], 0) \to (X, x_0)$ be continuous.
$f$ has a unique lift $\widetilde{f}$ in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$.

*Proof.* For all path $f$ in $X$ from $x_0$:

(1) As $p$ is a covering map,
for all $0 \leq y \leq 1$, $f(y)$ has an evenly covered open neighbour.

(2) As $f$ is continuous,
these open neighbours form a pullback open cover of $[0, 1]$.

(3) As $[0,1]$ is a closed and bounded interval,
WLOG, for some $0 = y_0 < y_1 < y_2 = 1$,
for some evenly covered open subsets $U_1, U_2$ of $X$,
$f([y_0, y_1]) \subseteq U_1, f([y_1, y_2]) \subseteq U_2$.

(4) Define $\widetilde{U}_1, \widetilde{f}_1$ as the lifts of $U_1$, $f|_{[y_0,y_1]}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ under $p$.
Define $\widetilde{U}_2, \widetilde{f}_2$ as the lifts of $U_2$, $f|_{[y_1,y_2]}$ from $\widetilde{x}_1$ to $\widetilde{x}_2$ under $p$.
$\widetilde{f} = \widetilde{f}_1 \star \widetilde{f}_2$ is a lift of $f$ in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$.

(5) For all lifts $\widetilde{f}, \widetilde{f}'$ of $f$ in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$, $\{y \in [0,1] : \widetilde{f}(y) = \widetilde{f}'(y)\}$
is nonempty and clopen in the closed and bounded interval $[0,1]$, so $\widetilde{f} = \widetilde{f}'$.

$\square$

**Proposition 2.27.** (**Homotopy Lifting Property**)
Let $(\widetilde{X}, \widetilde{x}_{00}), (X, x_{00})$ be topological spaces,
$p : (\widetilde{X}, \widetilde{x}_{00}) \to (X, x_{00})$ be a covering map,
and $f : ([0,1]^2, (0,0)) \to (X, x_{00})$ be continuous.
$f$ has a unique lift $\widetilde{f}$ in $\widetilde{X}$ from $\widetilde{x}_{00}$ under $p$.

*Proof.* For all homotopy $f$ in $X$ from $x_{00}$:

(1) As $p$ is a covering map,
for all $0 \leq y, y' \leq 1$, $f(y, y')$ has an evenly covered open neighbour.

(2) As $f$ is continuous,
these open neighbours form a pullback open cover of $[0,1]^2$.

(3) As $[0,1]^2$ is a closed and bounded square,
WLOG, for some $0 = y_0 < y_1 < y_2 = 1$, for some $0 = y'_0 < y'_1 < y'_2 = 1$,
for some evenly covered open subsets $U_{11}, U_{12}, U_{21}, U_{22}$ of $X$,
$f([y_0, y_1] \times [y'_0, y'_1]) \subseteq U_{11}, f([y_0, y_1] \times [y'_1, y'_2]) \subseteq U_{12}$,
$f([y_1, y_2] \times [y'_0, y'_1]) \subseteq U_{21}, f([y_1, y_2] \times [y'_1, y'_2]) \subseteq U_{22}$.

(4) Define $\widetilde{U}_{11}, \widetilde{f}_{11}$ as the lifts of $U_{11}$, $f|_{[y_0,y_1]\times[y'_0,y'_1]}$ from $\widetilde{x}_{00}$ to $\widetilde{x}_{01}, \widetilde{x}_{10}, \widetilde{x}_{11}$ under $p$.
Define $\widetilde{U}_{12}, \widetilde{f}_{12}$ as the lifts of $U_{12}$, $f|_{[y_0,y_1]\times[y'_1,y'_2]}$ from $\widetilde{x}_{01}$ to $\widetilde{x}_{02}, \widetilde{x}_{11}, \widetilde{x}_{12}$ under $p$.
According to path lifting property, the new $\widetilde{x}_{11}$ agrees with the previous one.
Define $\widetilde{U}_{21}, \widetilde{f}_{21}$ as the lifts of $U_{21}$, $f|_{[y_1,y_2]\times[y'_0,y'_1]}$ from $\widetilde{x}_{10}$ to $\widetilde{x}_{11}, \widetilde{x}_{20}, \widetilde{x}_{21}$ under $p$.
According to path lifting property, the new $\widetilde{x}_{11}$ agrees with the previous one.
Define $\widetilde{U}_{22}, \widetilde{f}_{22}$ as the lifts of $U_{22}$, $f|_{[y_1,y_2]\times[y'_1,y'_2]}$ from $\widetilde{x}_{11}$ to $\widetilde{x}_{12}, \widetilde{x}_{21}, \widetilde{x}_{22}$ under $p$.
According to path lifting property, the new $\widetilde{x}_{12}, \widetilde{x}_{21}$ agree with the previous ones.
$\widetilde{f} = \widetilde{f}_{00} \star \widetilde{f}_{01} \star \widetilde{f}_{10} \star \widetilde{f}_{11}$ is a lift of $f$ in $\widetilde{X}$ from $\widetilde{x}_{00}$ under $p$.

(5) For all lifts $\widetilde{f}, \widetilde{f}'$ of $f$ in $\widetilde{X}$ from $\widetilde{x}_{00}$ under $p$, $\{(y, y') \in [0,1]^2 : \widetilde{f}(y, y') = \widetilde{f}'(y, y')\}$
is nonempty and clopen in the closed and bounded square $[0,1]^2$, so $\widetilde{f} = \widetilde{f}'$.

$\square$

**Remark:** This implies every path homotopy class $[\![f]\!]$ in $X$ from $x_0$ has a unique lift $[\![\widetilde{f}]\!]$ in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$, and $p_*: \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is an embedding.

**Proposition 2.28.** (**The Universal Property of Covering Map**)
Let $(\widetilde{X}, \widetilde{x}_0), (X, x_0), (Y, y_0)$ be topological spaces
with $(Y, y_0)$ path connected and locally path connected,
$p: (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering map,
and $f: (Y, y_0) \to (X, x_0)$ be continuous:

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\exists! \widetilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd} \iff \begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, dashed, "\exists! \widetilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd}$$

*Proof.* We may divide our proof into two directions:
**"only if" direction:**
Assume that:

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\exists! \widetilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

We show that:

$$\begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, dashed, "\exists! \widetilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd}$$

(1) As $\pi_1$ is a functor, a lift $\widetilde{f}_*$ of $f_*$ based at $\widetilde{x}_0$ under $p_*$ exists.

(2) As every loop homotopy class $[\![\gamma]\!]$ in $X$ based at $x_0$ has a unique lift $[\![\widetilde{\gamma}]\!]$ in $\widetilde{X}$ based at $\widetilde{x}_0$ under $p_*$, the lift $\widetilde{f}_*$ of $f_*$ based at $\widetilde{x}_0$ under $p_*$ is unique.

**"if" direction:**
Assume that:

$$\begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, dashed, "\exists! \widetilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd}$$

We show that:

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\exists! \widetilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

For all $y \in Y$, choose a path homotopy class $[\![\sigma]\!]$ in $Y$ from $y_0$ to $y$, and define $\widetilde{x} = \widetilde{f}(y)$ as the end point of the lift $[\![\widetilde{\gamma}]\!]$ of $[\![\gamma]\!] = [\![f \circ \sigma]\!]$ in $\widetilde{X}$ from $\widetilde{x}_0$. We show that:

(1) $\widetilde{f}$ is a map.
On one hand, $Y$ is path connected, so $[\![\sigma]\!]$ exists, and $\widetilde{f}(y) = \widetilde{x}$ exists.
On the other hand, $f_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0))) \leq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \cong \pi_1(\widetilde{X}, \widetilde{x}_0)$, so $[\![\sigma_1]\!], [\![\sigma_2]\!]$ with a common endpoint $y$ generates a common $\widetilde{x}$, $\widetilde{f}(y) = \widetilde{x}$ is unique.

(2) $\widetilde{f}$ is continuous.
For all $y \in Y$, for all open neighbour $\widetilde{U}$ of $\widetilde{x} = \widetilde{f}(y)$, we aim to find an open neighbour $V$ of $y$, such that $\widetilde{f}(V) \subseteq \widetilde{U}$. To do so, first shrink $\widetilde{U}$ such that $\widetilde{U}$ evenly covers an open neighbour $U$ of $x = f(y)$, then shrink $f^{-1}(U)$ to a path connected open neighbour $V$ of $y$. Afterwards, $p$ restricts to a homeomorphism $p|_{\widetilde{U}} : \widetilde{U} \to U$, and $\widetilde{f}$ sends the open set $V$ to $\widetilde{f}(V) = p|_{\widetilde{U}}^{-1}(f(V)) \subseteq p|_{\widetilde{U}}^{-1}(U) = \widetilde{U}$.

(3) $\widetilde{f}$ is a lift.
As $\widetilde{f}$ is defined by path lifts in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$,
$\widetilde{f}$ is a lift of $f$ in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$.

(4) $\widetilde{f}$ is unique.
As every path homotopy class $[\![\gamma]\!]$ in $X$ from $x_0$ has a unique lift $[\![\widetilde{\gamma}]\!]$ in $\widetilde{X}$ from $\widetilde{x}_0$ under $p$, the lift $\widetilde{f}$ of $f$ from $\widetilde{x}$ under $p$ is unique.

□

**Remark:** We restate some definitions on path connectedness:

(1) A topological space $X$ is path connected
if for all $x_0, x_1 \in X$, for some $\gamma \in \mathbf{C}([0,1], X)$, $\gamma(0) = x_0, \gamma(1) = x_1$.

(2) A topological space $X$ is locally path connected
if for all $x \in X$, all its open neighbour contains a path connected open neighbour.

(3) A topological space $X$ is simply connected
if it is path connected with trivial fundamental group.

(4) A topological space $X$ is locally simply connected
if for all $x \in X$ all its open neighbour contains a simply connected open neighbour.

(5) A topological space $X$ is semi locally simply connected
if for all $x \in X$, for some path connected open neighbour $U$ of $x$
with inclusion map $\sigma : (U, x) \to (X, x)$, $\sigma_*([\![\gamma]\!]) = [\![e_x]\!]$.

(6) Locally simply connectedness is stronger than semi locally simply connectedness.

**Proposition 2.29.** Let $\widetilde{X}$ be a topological space,
$X$ be a locally path connected topological space,
$Y$ be a compact simply connected manifold with boundary,
and $p : \widetilde{X} \to X$ be a covering map.
$\mathbf{C}(Y, p) : \mathbf{C}(Y, \widetilde{X}) \to \mathbf{C}(Y, X), \widetilde{f} \mapsto p \circ \widetilde{f}$ is a covering map.

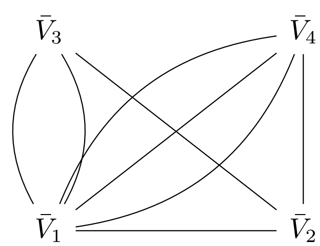
*Proof.* For all $f \in \mathbf{C}(Y, X)$, we aim to find a $\mathbf{C}(Y, p)$-evenly covered open neighbour.

(1) Assume that $\mathcal{V}$ collects all coordinate ball and half ball $V$ of $Y$,
such that $f(\bar{V})$ is contained in some $p$-evenly covered open subset $U$ of $X$.
Note that the closure $\bar{V}$ of $V$ is path connected and compact.

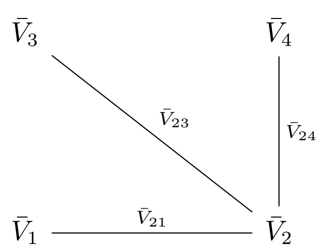
(2) As $Y$ is a manifold with boundary and $p$ is a covering map, $\mathcal{V}$ covers $Y$.

(3) As $Y$ is compact, WLOG, for some $V_1, V_2, V_3, V_4 \in \mathcal{V}$,
for some $p$-evenly covered open subsets $U_1, U_2, U_3, U_4$ of $X$,
$f(\bar{V}_1) \subseteq U_1, f(\bar{V}_2) \subseteq U_2, f(\bar{V}_3) \subseteq U_3, f(\bar{V}_4) \subseteq U_4$.

(4) Define an undirected graph $\mathcal{G}$, whose vertices are $\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4$, and edges are the path connected components of $\bar{V}_1 \cap \bar{V}_2, \bar{V}_1 \cap \bar{V}_3, \bar{V}_1 \cap \bar{V}_4, \bar{V}_2 \cap \bar{V}_3, \bar{V}_2 \cap \bar{V}_4, \bar{V}_3 \cap \bar{V}_4$.
As the union $Y = \bar{V}_1 \cup \bar{V}_2 \cup \bar{V}_3 \cup \bar{V}_4$ is path connected, the graph $\mathcal{G}$ is connected.
WLOG, assume that it is in the form below:




(5) Take a spanning tree $\mathcal{T}$ of the connected graph $\mathcal{G}$:




(6) The spanning tree $\mathcal{T}$ of $\mathcal{G}$ gives us a recipe $\mathcal{R}$ of lifting $f$ under $p$:
Define $y_2 \in \bar{V}_2, x_2 = f(y_2) \in U_2, \widetilde{x}_2 \in p^{-1}(x_2)$, and $U_{21}, U_{23}, U_{24}$ as the path connected components of $f(\bar{V}_{21}), f(\bar{V}_{23}), f(\bar{V}_{24})$ in $U_2 \cap U_1, U_2 \cap U_3, U_2 \cap U_4$.
As $X$ is locally path connected, $U_{21}, U_{23}, U_{24}$ are open.
Define $\widetilde{U}_2, \widetilde{f}_2$ as the lifts of $U_2$, $f|_{\bar{V}_2}$ from $\widetilde{x}_2$ to $\widetilde{U}_{21}, \widetilde{U}_{23}, \widetilde{U}_{24}$ under $p$.
Define $\widetilde{U}_1, \widetilde{f}_1$ as the lifts of $U_1$, $f|_{\bar{V}_1}$ from $\widetilde{U}_{21}$ under $p$.
Define $\widetilde{U}_3, \widetilde{f}_3$ as the lifts of $U_3$, $f|_{\bar{V}_3}$ from $\widetilde{U}_{23}$ under $p$.
Define $\widetilde{U}_4, \widetilde{f}_4$ as the lifts of $U_4$, $f|_{\bar{V}_4}$ from $\widetilde{U}_{24}$ under $p$.
As $Y$ is simply connected and locally path connected, it follows from the universal property that the gluing $\widetilde{f}_1 \star \widetilde{f}_2 \star \widetilde{f}_3 \star \widetilde{f}_4$, which we don't know in advance whether it is well-defined, agrees with the unique lift $\widetilde{f}$ of $f$ in $\widetilde{X}$ from $\widetilde{x}_2$ under $p$, which is well-defined and independent of the choice of $\mathcal{T}$.

(7) $\mathcal{R}$ works over the following open neighbour of $f$:

$$\begin{tikzcd}
\mathbf{C}(\bar{V}_3, U_3) \arrow[dr, dash, "\mathbf{C}(\bar{V}_{23}, U_{23})"] & \mathbf{C}(\bar{V}_4, U_4) \arrow[d, dash, "\mathbf{C}(\bar{V}_{24}, U_{24})"] \\
\mathbf{C}(\bar{V}_1, U_1) \arrow[r, dash, "\mathbf{C}(\bar{V}_{21}, U_{21})"] & \mathbf{C}(\bar{V}_2, U_2)
\end{tikzcd}$$

Hence, it is $\mathbf{C}(Y, p)$-evenly covered.

□

## 2.4 Universal Covering Map

**Definition 2.30.** (**Universal Covering Map**)
Let $\widetilde{X}, X$ be topological spaces, and $p : \widetilde{X} \to X$ be a covering map.
If $\widetilde{X}$ is simply connected and locally path connected, then $p$ is universal.

**Proposition 2.31.** (**The Existence of Universal Covering Map**)
Let $(X, x_0)$ be a path connected and locally path connected topological space. $(X, x_0)$ admits a universal covering map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ iff it is semi locally simply connected.

*Proof.* We may divide our proof into two directions.
**"only if" direction:**
Assume that $(X, x_0)$ admits a unique universal covering map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.

(1) For all $x \in X$, choose a lift $\widetilde{x}$ of $x$ under $p$,
an evenly covered path connected open neighbour $U$ of $x$,
the inclusion map $\sigma : (U, x) \to (X, x)$,
and the lift $\widetilde{U}$ of $U$ based at $\widetilde{x}$ under $p$.

(2) For all loop homotopy class $[\![\gamma]\!]$ in $U$ based at $x$,
it has a unique lift $[\![\widetilde{\gamma}]\!]$ in $\widetilde{U}$ based at $\widetilde{x}$ under $p_*$.

(3) As $[\![\widetilde{\gamma}]\!] = [\![e_{\widetilde{x}}]\!]$ in $\widetilde{X}$ based at $\widetilde{x}$,
$\sigma_*([\![\gamma]\!]) = [\![\gamma]\!] = [\![e_x]\!]$ in $X$ based at $x$.

**"if" direction:**
Assume that $(X, x_0)$ is semi locally simply connected.
We aim to construct a universal covering map $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.

Define a set $\widetilde{X}$ with $\widetilde{x}_0 \in \widetilde{X}$ and a map $p : \widetilde{X} \to X$ with $p(\widetilde{x}_0) = x_0$:

$$\begin{aligned}
\widetilde{X} &= \text{All path homotopy class in } X \text{ from } x_0 \\
\widetilde{x}_0 &= [\![e_{x_0}]\!] \\
p([\![\gamma]\!]) &= \text{The endpoint of } [\![\gamma]\!]
\end{aligned}$$

We show that:

(1) $(\widetilde{X}, \widetilde{x}_0)$ admits an appropriate topology, such that $p$ is a covering map.
As $X$ is a semi locally simply connected, for all $\widetilde{x}_1 \in \widetilde{X}$, its end point $x_1 \in X$ has an open neighbour $U$ with inclusion map $\sigma : (U, x_1) \to (X, x_1)$, such that $\sigma_*$ is trivial. It suffices to generate a topology on $(\widetilde{X}, \widetilde{x}_0)$ by the basis below:

$$\widetilde{U} = \{\widetilde{x} \in \widetilde{X} : \widetilde{x}_1, \widetilde{x} \text{ differ by a path homotopy class in } U \text{ from } x_1 \text{ to } x\}$$

(2) $(\widetilde{X}, \widetilde{x}_0)$ is path connected and locally path connected.
For all $\widetilde{x} \in \widetilde{X}$, there exists a path $\widetilde{\gamma} : [0,1] \to \widetilde{X}, t \mapsto \widetilde{x}|_{[0,t]}$ from $\widetilde{x}_0$ to $\widetilde{x}$.
For all $\widetilde{x} \in \widetilde{X}$ with end point $x$, for all open neighbour $\widetilde{U}$ of $\widetilde{x}$, shrink $\widetilde{U}$ such that $\widetilde{U}$ evenly covers an open neighbour $U$ of $x$, it suffices to lift the path connected open neighbour $V$ of $x$ in $U$ to a path connected open neighbour $\widetilde{V}$ of $\widetilde{x}$ in $\widetilde{U}$.

(3) $\pi_1(\widetilde{X}, \widetilde{x}_0) = \{[\![e_{\widetilde{x}_0}]\!]\}$.
For all loop homotopy class $[\![\widetilde{\gamma}]\!]$ in $\widetilde{X}$ based at $\widetilde{x}_0$, its image $[\![\gamma]\!]$ under $p_*$ is a loop homotopy class in $X$ based at $x_0$. By path lifting property, we may replace $\widetilde{\gamma}$ by the new representative $\widetilde{\gamma} : [0,1] \to \widetilde{X}, t \mapsto [\![\gamma]\!]|_{[0,t]}$ defined before. As $\widetilde{\gamma}$ is a loop, $[\![\gamma]\!] = \widetilde{\gamma}(1) = \widetilde{\gamma}(0) = [\![e_{x_0}]\!]$. By path lifting property, $[\![\widetilde{\gamma}]\!] = [\![e_{\widetilde{x}_0}]\!]$.

□

**Proposition 2.32.** (**The Uniqueness of Universal Covering Map**)
Let $(X, x_0)$ be a path connected and locally path connected topologcial space with universal covering maps $f : (Y, y_0) \to (X, x_0)$, $g : (Z, z_0) \to (X, x_0)$. $f, g$ are isomorphic.

*Proof.* We may divide our proof into two steps.
**Step 1:** We construct a pair of homomorphisms between $f, g$:

$$\begin{tikzcd}
(Y, y_0) \arrow[rr, dashed, shift left, "\exists ! h"] \arrow[dr, "f"'] & & (Z, z_0) \arrow[ll, dashed, shift left, "\exists ! i"] \arrow[dl, "g"] \\
& (X, x_0) &
\end{tikzcd}$$

**Step 2:** We prove that the pair of homomorphisms are inverses to each other.

$$\begin{tikzcd}
(Y, y_0) \arrow[r, dashed, "\exists! h"'] \arrow[rr, dashed, bend left, "\exists! id_Y"] \arrow[dr, "f"'] & (Z, z_0) \arrow[r, dashed, "\exists! i"'] \arrow[d, "g"'] & (Y, y_0) \arrow[dl, "f"] \\
& (X, x_0) &
\end{tikzcd}
\quad
\begin{tikzcd}
(Z, z_0) \arrow[r, dashed, "\exists! i"'] \arrow[rr, dashed, bend left, "\exists! id_Z"] \arrow[dr, "g"'] & (Y, y_0) \arrow[r, dashed, "\exists! h"'] \arrow[d, "f"'] & (Z, z_0) \arrow[dl, "g"] \\
& (X, x_0) &
\end{tikzcd}$$

□

**Remark:** If $X, Y$ are path connected and locally path connected, and $f : Y \to X$ is a covering map, then $X$ admits a universal covering map $g : Z \to X$ iff $Y$ admits a universal covering map $h : Z \to Y$.

## 2.5 Composition

**Proposition 2.33.** Let $X, Y, Z$ be topological spaces, and $f : X \to Y, g : Y \to Z, h : X \to Z$ be continuous maps, where $h = g \circ f$. If $f, g$ are covering maps, and each $g$-fibre is finite, then $h$ is a covering map.

*Proof.* For all $z \in Z$, we aim to find a $h$-evenly covered open neighbour.

(1) As $g$ is a covering map,
$z$ has a $g$-evenly covered open neighbour $W$.

(2) As $f$ is a covering map,
the $g$-fibres $(y_\mu)_{\mu \in J}$ of $z$ have $f$-evenly covered open neighbours $(V_\mu)_{\mu \in J}$.

(3) As $I$ is finite,
$z$ has a $h$-evenly covered open neighbour $W \cap \bigcap_{\mu \in J} g(V_\mu)$.

□

**Proposition 2.34.** Let $X, Y, Z$ be topological spaces, where $Y, Z$ are locally path connected, and $f : X \to Y, g : Y \to Z, h : X \to Z$ be continuous maps, where $h = g \circ f$. If $f, h$ are covering maps, then $g$ is a covering map.

*Proof.* For all $z \in Z$, we aim to find a $g$-evenly covered open neighbour.

(1) As $h$ is a covering map,
$z$ has a $h$-evenly covered open neighbour $W$.

(2) As $Z$ is locally path connected,
the path connected component $\bar{W}$ of $z$ in $W$ is open.

(3) As $Y$ is locally path connected,
the path connected components $(\bar{V}_\mu)_{\mu \in J}$ of $g^{-1}(\bar{W})$ in $Y$ are open.

(4) As the path connected set $\bar{W}$ is contained in $W$,
$\bar{W}$ is $h$-evenly covered by some path connected open subsets $(\bar{U}_\lambda)_{\lambda \in I}$.

(5) As $f$ restricts to a continuous map from $h^{-1}(\bar{W})$ with path connected components $(\bar{U}_\lambda)_{\lambda \in I}$ to $g^{-1}(\bar{W})$ with path connected components $(\bar{V}_\mu)_{\mu \in J}$, for all $\lambda \in I$, for some unique $\mu \in J$, $f(\bar{U}_\lambda) \subseteq \bar{V}_\mu$.

(6) As $f$ is a covering map, the restriction of $f$ from the open set $f^{-1}(\bar{V}_\mu)$ with path connected components $(\bar{U}_{\lambda_\mu})_{\lambda_\mu \in I_\mu}$ to the path connected open set $\bar{V}_\mu$ is a covering map. It follows from path lifting property that for all $\lambda_\mu \in I_\mu$, $f(\bar{U}_{\lambda_\mu}) = \bar{V}_\mu$.

(7) As each $h|_{\bar{U}_\lambda}$ is homeomorphism, each $f|_{\bar{U}_\lambda}, g|_{\bar{V}_\mu} = h|_{\bar{U}_\lambda} \circ f|_{\bar{U}_\lambda}^{-1}$ are homeomorphisms, so $\bar{W}$ is $g$-evenly covered by some open subsets $(\bar{V}_\mu)_{\mu \in J}$.

□

**Proposition 2.35.** Let $X, Y, Z$ be topological spaces, where $Y$ is path connected, and $f : X \to Y, g : Y \to Z, h : X \to Z$ be continuous maps, where $h = g \circ f$. If $g, h$ are covering maps, then $f$ is a covering map.

*Proof.* For all $y \in Y$, we aim to find a $f$-evenly covered open neighbour.

(1) Define $z = g(y)$. Define $(y_\mu)_{\mu \in J}$ as all $g$-lifts of $z$.
For all $\mu \in J$, define $(x_{\lambda_\mu,\mu})_{\lambda_\mu \in I_\mu}$ as all $f$-lifts of $y_\mu$.
As $h = g \circ f$, $(x_{\lambda_\mu,\mu})_{\lambda_\mu \in I_\mu, \mu \in J}$ are all $h$-lifts of $z$.

(2) As $g, h$ are covering maps, for some open neighbours $(U_{\lambda_\mu,\mu})_{\lambda_\mu \in I_\mu, \mu \in J}, (V_\mu)_{\mu \in J}, W$ of $(x_{\lambda_\mu,\mu})_{\lambda_\mu \in I_\mu, \mu \in J}, (y_\mu)_{\mu \in J}, z$, $(U_{\lambda_\mu,\mu})_{\lambda_\mu \in I_\mu, \mu \in J}$ $h$-evenly cover $W$ and $(V_\mu)_{\mu \in J}$ $g$-evenly cover $W$, so every pair of them are homeomorphic.

(3) As $Y$ is path connected, it follows from path lifting property of $h = g \circ f$ that $f$ is surjective, and the trick below shows that $f^{-1}(V_\nu)$ is indeed a coproduct:

$$f^{-1}(V_\nu) = f^{-1}(g|_{V_\nu}^{-1}(W)) = h|_{f^{-1}(V_\nu)}^{-1}(W) \cong \coprod_{\lambda_\mu \in I_\mu, \mu \in J} U_{\lambda_\mu,\mu} \cap f^{-1}(V_\nu)$$

□

**Proposition 2.36.** Let $(X, x_0), (Y, y_0), (Z, z_0)$ be path connected and locally path connected topological spaces, and $f : (X, x_0) \to (Y, y_0), g : (Y, y_0) \to (Z, z_0), h : (X, x_0) \to (Z, z_0)$ be continuous maps, where $h = g \circ f$. If $f, g$ are covering maps, and $Z$ admits a universal covering map $k : (W, w_0) \to (Z, z_0)$, then $h$ is a covering map.

*Proof.* As $f, g$ are covering maps and $k$ is a universal covering map:

$$\begin{tikzcd}
(X, x_0) \arrow[r, "f"] & (Y, y_0) \arrow[d, "g"] \\
(W, w_0) \arrow[u, dashed, "\exists! i"] \arrow[ur, dashed, "\exists! j"] \arrow[r, "k"'] & (Z, z_0)
\end{tikzcd}$$

As $k, i$ are covering maps and $h \circ i = k$, $h$ is a covering map. □

# 3 Galois Theory

## 3.1 Free and Properly Discontinuous Group Action

**Definition 3.1.** (**Free and Properly Discontinuous Group Action**)
Let $\widetilde{X}$ be a topological space, and $\widetilde{G} \leq \mathbf{Aut}(\widetilde{X})$.
If for all $\widetilde{x} \in \widetilde{X}$, for some open neighbour $\widetilde{U}$ of $\widetilde{x}$, for all $\widetilde{g} \in \widetilde{G}$,
$\widetilde{g} = id_{\widetilde{X}}$ or $\widetilde{U} \cap \widetilde{g}(\widetilde{U}) = \emptyset$, then $\widetilde{G}$ is free and properly discontinuous.

**Proposition 3.2.** Let $\widetilde{X}$ be a topological space,
$\widetilde{G} \leq \mathbf{Aut}(\widetilde{X})$, $X = \widetilde{X}/\widetilde{G}$, and $p : \widetilde{X} \to X, \widetilde{x} \mapsto x$.
If $\widetilde{G}$ is free and properly discontinuous, then $p$ is a covering map.

*Proof.* Assume that for all $\widetilde{x} \in \widetilde{X}$, for some open neighbour $\widetilde{U}$ of $\widetilde{x}$, for all $\widetilde{g} \in \widetilde{G}$, $\widetilde{g} = id_{\widetilde{X}}$ or $\widetilde{U} \cap \widetilde{g}(\widetilde{U}) = \emptyset$. For all $x \in X$, choose a representative $\widetilde{x}$ of $x$.
As the quotient map $p$ is open, the quotient $U$ of $\widetilde{U}$ is an open neighbour of $x$, and:

(1) $p^{-1}(U) \cong \coprod_{\widetilde{g} \in \widetilde{G}} \widetilde{g}(\widetilde{U})$, where each $\widetilde{g}(\widetilde{U})$ is open in $\widetilde{X}$.

(2) Each $p|_{\widetilde{g}(\widetilde{U})} : \widetilde{g}(\widetilde{U}) \to U, \widetilde{x} \mapsto p(\widetilde{x})$ is a homeomorphism.

□

**Proposition 3.3.** Let $\widetilde{X}$ be a path connected and locally path connected topological space, $\widetilde{G} \leq \mathbf{Aut}(\widetilde{X})$, $X = \widetilde{X}/\widetilde{G}$, and $p : \widetilde{X} \to X, \widetilde{x} \mapsto x$.
If $p$ is a covering map, then $\widetilde{G}$ is free and properly discontinuous.

*Proof.* Assume that for all $x \in X$, for some open neighbour $U$ of $x$:

(1) $p^{-1}(U) \cong \coprod_{\lambda \in I} \widetilde{U}_\lambda$, where each $\widetilde{U}_\lambda$ is open in $\widetilde{X}$.

(2) Each $p|_{\widetilde{U}_\lambda} : \widetilde{U}_\lambda \to U, \widetilde{x} \mapsto p(\widetilde{x})$ is a homeomorphism.

For all $\widetilde{x} \in \widetilde{X}$, choose a path connected open neighbour $\widetilde{U}$ of $\widetilde{x}$ that covers an evenly covered open neighbour $U$ of $x$, so $\widetilde{U}$ is a path connected component of $p^{-1}(U)$.

For all $\widetilde{g} \in \widetilde{G}$, assume that the path connected components $\widetilde{U}, \widetilde{g}(\widetilde{U})$ of $p^{-1}(U)$ are intersecting, i.e., equal. As $p|_{\widetilde{U}}$ is injective, $\widetilde{g}(\widetilde{x}) = \widetilde{x} = p|_{\widetilde{U}}^{-1}(x)$. As $\widetilde{X}$ is path connected and locally path connected, it follows from the universal property that $\widetilde{g} = id_{\widetilde{X}}$. $\square$

**Proposition 3.4.** Let $\widetilde{X} = \mathbb{C}, \mathrm{i} = \sqrt{-1}$.
$\widetilde{G} = \langle \widetilde{x} \mapsto \widetilde{x} + 1, \widetilde{x} \mapsto \widetilde{x} + \mathrm{i} \rangle$ is free and properly discontinuous.

*Proof.* For all $\widetilde{x} \in \widetilde{X}$, it suffices to choose the open neighbour $\widetilde{U}$ of $\widetilde{x}$:

$$\widetilde{U} = B\left(\widetilde{x}, \frac{1}{2}\right)$$

$\square$

**Proposition 3.5.** Let $\widetilde{X} = \mathbb{C}, \zeta = \frac{1+\sqrt{-3}}{2}$.
$\widetilde{G} = \langle \widetilde{x} \mapsto \widetilde{x} + 1, \widetilde{x} \mapsto \widetilde{x} + \zeta \rangle$ is free and properly discontinuous.

*Proof.* For all $\widetilde{x} \in \widetilde{X}$, it suffices to choose the open neighbour $\widetilde{U}$ of $\widetilde{x}$:

$$\widetilde{U} = B\left(\widetilde{x}, \frac{1}{2}\right)$$

$\square$

**Proposition 3.6.** Let $\widetilde{X} = \mathbb{C} \backslash \mathbb{Z}[\mathrm{i}], \mathrm{i} = \sqrt{-1}$,
$\widetilde{G} = \langle \widetilde{x} \mapsto \mathrm{i}\widetilde{x} \rangle$ is free and properly discontinuous.

*Proof.* For all $\widetilde{x} = \mathrm{e}^{\widetilde{\alpha}_1 + \mathrm{i}\widetilde{\alpha}_2} \in \widetilde{X}$, it suffices to choose the open neighbour $\widetilde{U}$ of $\widetilde{x}$:

$$\widetilde{U} = \mathrm{e}^{\mathbb{R} + \mathrm{i}\left(\widetilde{\alpha}_2 - \frac{\pi}{4}, \widetilde{\alpha}_2 + \frac{\pi}{4}\right)} \backslash \mathbb{Z}[\mathrm{i}]$$

$\square$

**Proposition 3.7.** Let $\widetilde{X} = \mathbb{C} \backslash \mathbb{Z}[\zeta], \zeta = \frac{1+\sqrt{-3}}{2}$,
$\widetilde{G} = \langle \widetilde{x} \mapsto \zeta\widetilde{x} \rangle$ is free and properly discontinuous.

*Proof.* For all $\widetilde{x} = \mathrm{e}^{\widetilde{\alpha}_1 + \mathrm{i}\widetilde{\alpha}_2} \in \widetilde{X}$, it suffices to choose the open neighbour $\widetilde{U}$ of $\widetilde{x}$:

$$\widetilde{U} = \mathrm{e}^{\mathbb{R} + \mathrm{i}\left(\widetilde{\alpha}_2 - \frac{\pi}{6}, \widetilde{\alpha}_2 + \frac{\pi}{6}\right)} \backslash \mathbb{Z}[\zeta]$$

$\square$

**Proposition 3.8.** Let $\widetilde{X} = \mathbb{S}^n$,
$\widetilde{G} = \langle \widetilde{x} \mapsto -\widetilde{x} \rangle$ is free and properly discontinuous.

*Proof.* For all $\widetilde{x} \in \widetilde{X}$, it suffices to choose the open neighbour $\widetilde{U}$ of $\widetilde{x}$:

$$\widetilde{U} = B(\widetilde{x}, \sqrt{2})$$

$\square$

**Example 3.9.** (**Regular Tetrahedron**)
Let $\widetilde{X}$ be a regular tetrahedron. If we take away its vertices, centers of edges, and centers of faces, then the even permutation $A_4$ of the pairs of antipodal vertices and faces is free and properly discontinuous.

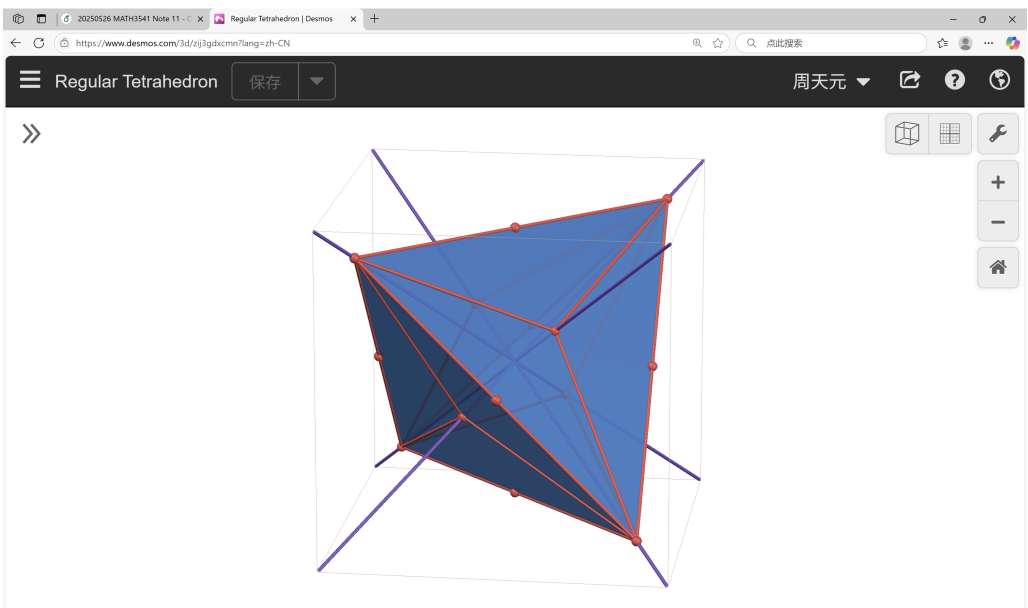



Figure 4: Regular Tetrahedron

**Example 3.10.** (**Regular Hexahedron/Octahedron**)
Let $\widetilde{X}$ be a regular hexahedron/octahedron. If we take away its vertices, centers of edges, and centers of faces, then the permutation $S_4$ of the pairs of antipodal vertices of the regular hexahedron/the pairs of antipodal faces of the regular octahedron is free and properly discontinuous.

**Example 3.11.** (**Regular Dodecahedron/Icosahedron**)
Let $\widetilde{X}$ be a regular dedecahedron/icosahedron. If we take away its vertices, centers of edges, and centers of faces, then the even permutation $A_5$ of the inner regular hexahedrons is free and properly discontinuous.

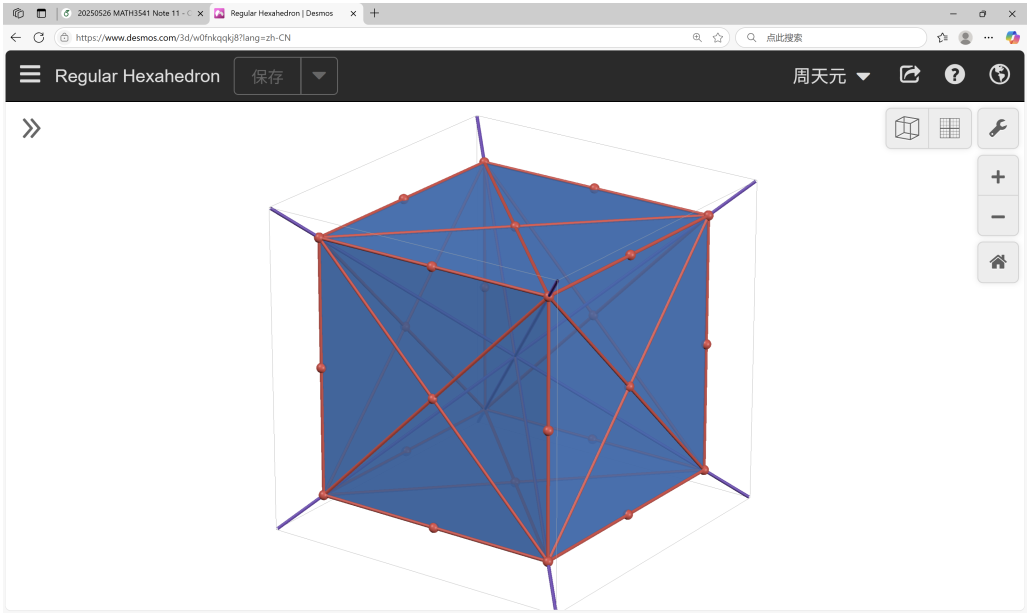



Figure 5: Regular Hexahedron

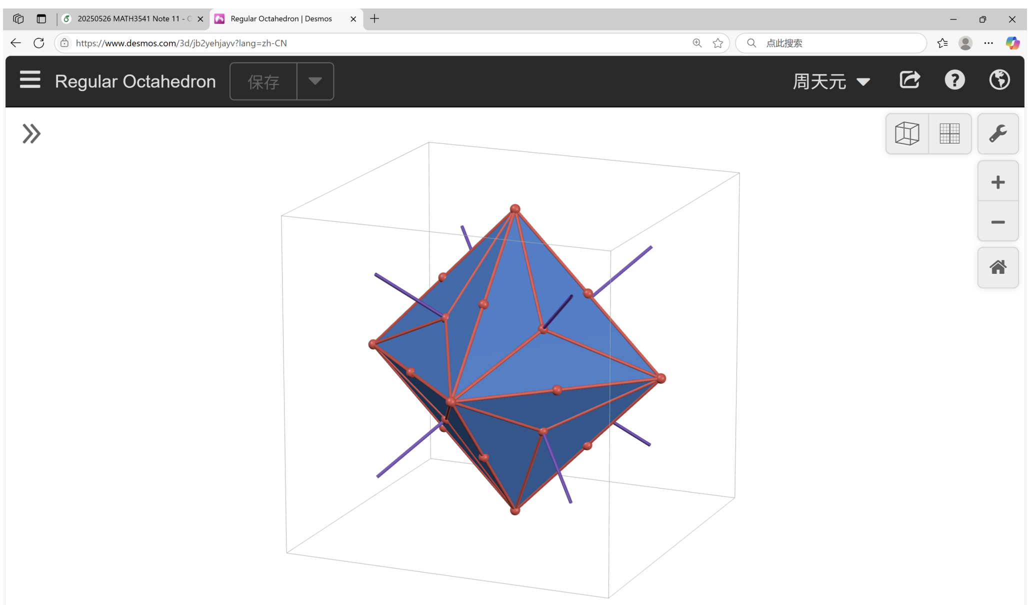



Figure 6: Regular Octahedron

**Proposition 3.12.** Let $p > 0, q \in \mathbb{Z}_p^\times, \zeta = \mathrm{e}^{\frac{2\pi \mathrm{i}}{p}}$, and $\widetilde{X} = \mathbb{S}^3$. $\widetilde{G} = \langle (z_1, z_q) \mapsto (\zeta z_1, \zeta^q z_q) \rangle$ is free and properly discontinuous.

*Proof.* For all $\widetilde{x} \in \widetilde{X}$, it suffices to choose the open neighbour $\widetilde{U}$ of $\widetilde{x}$:

$$\widetilde{U} = B\left(\widetilde{x}, 2\sin\frac{\pi}{2p}\right)$$

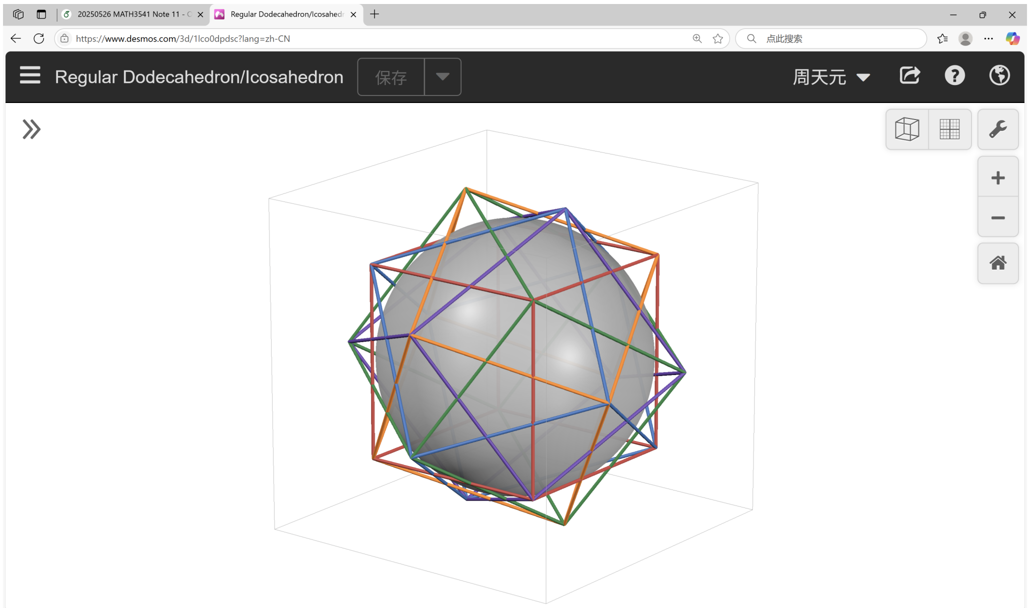



Figure 7: Regular Dodecahedron/Icosahedron

□

**Proposition 3.13.** Let $\widetilde{X} = \mathbb{S}^3$. The group $\widetilde{G}$ is free and properly discontinuous:

$$\widetilde{G} = \left\{ \pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}, \frac{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}}{2} \right\}$$

*Proof.* For all $\widetilde{x} \in \widetilde{X}$, it suffices to choose the open neighbour $\widetilde{U}$ of $\widetilde{x}$:

$$\widetilde{U} = B\left(\widetilde{x}, \sqrt{2 - \sqrt{3}}\right)$$

□

**Remark:** In general, we may take a discrete subgroup $\widetilde{G}$ of a Lie group $\widetilde{X}$.

### 3.2 Deck Transformation

**Definition 3.14.** (**Deck Transformation**)
Let $\widetilde{X}, X$ be topological spaces,
$p : \widetilde{X} \to X$ be a covering map, and $\widetilde{f} \in \mathbf{Aut}(\widetilde{X})$. If:

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\widetilde{f}"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Then $\widetilde{f} \in \mathbf{Deck}(p)$ is a deck transformation of $p$.

**Remark:** If we equip $\mathbf{Aut}(\widetilde{X})$ with compact-open topology, then $\mathbf{Deck}(p)$ is a discrete subgroup of $\mathbf{Aut}(\widetilde{X})$. For convenience, we also define deck transformation $\widetilde{f}_* \in \mathbf{Deck}(p_*)$ if the corresponding group diagram commutes.

**Proposition 3.15.** (**The Universal Property of Deck Transformation**)
Let $\widetilde{X}, X$ be topological spaces with $\widetilde{X}$ path connected and locally path connected, $\widetilde{x}, \widetilde{x}' \in \widetilde{X}, x \in X$, $[\![\widetilde{\gamma}]\!]$ be a path homotopy class in $\widetilde{X}$ from $\widetilde{x}'$ to $\widetilde{x}$, $[\![\gamma]\!]$ be a loop homotopy class in $X$ based at $x$, and $p : \widetilde{X} \to X$ be a covering map with $[\![p \circ \widetilde{\gamma}]\!] = [\![\gamma]\!]$. The conditions below are equivalent:

(1) $[\![\gamma]\!]$ belongs to the normalizer $I$ of $p_*(\pi_1(\widetilde{X}, \widetilde{x}))$ in $\pi_1(X, x)$.

(2) There exists a unique deck transformation $\widetilde{f}_* : \pi_1(\widetilde{X}, \widetilde{x}') \to \pi_1(\widetilde{X}, \widetilde{x})$ of $p_*$.

(3) There exists a unique deck transformation $\widetilde{f} : (\widetilde{X}, \widetilde{x}') \to (\widetilde{X}, \widetilde{x})$ of $p$.

In addition, if we fix $\widetilde{x} \in \widetilde{X}$ and let $\widetilde{x}' \in \widetilde{X}$ vary, then:

(1) $p_*|_I : \pi_1(\widetilde{X}, \widetilde{x}) \to I, [\![\widetilde{\gamma}]\!] \mapsto [\![\gamma]\!]$ is an injective homomorphism.

(2) $q : I \to \mathbf{Deck}(p), [\![\gamma]\!] \mapsto \widetilde{f}$ is a surjective homomorphism.

(3) $\mathbf{Ker}(q) = \mathbf{Im}(p_*|_I)$.

*Proof.* We may divide our proof into two parts.
**Part 1:** We prove the equivalence.

(1) To prove (1) $\iff$ (2), it suffices to notice that:

$$[\![\gamma]\!] \star p_*(\pi_1(\widetilde{X}, \widetilde{x})) \star [\![\gamma]\!]^{-1} = p_*([\![\widetilde{\gamma}]\!] \star \pi_1(\widetilde{X}, \widetilde{x}) \star [\![\widetilde{\gamma}]\!]^{-1}) = p_*(\pi_1(\widetilde{X}, \widetilde{x}'))$$

(2) To prove (2) $\iff$ (3), it suffices to notice that:

$$\begin{tikzcd} \pi_1(\widetilde{X},\widetilde{x}') \arrow[rr, dashed, "\exists! \widetilde{f}_* \in \mathbf{Deck}(p_*)"] \arrow[dr, "p_*"'] & & \pi_1(\widetilde{X},\widetilde{x}) \arrow[dl, "p_*"] \\ & \pi_1(X,x) & \end{tikzcd} \iff \begin{tikzcd} (\widetilde{X},\widetilde{x}') \arrow[rr, dashed, "\exists! \widetilde{f} \in \mathbf{Deck}(p)"] \arrow[dr, "p"'] & & (\widetilde{X},\widetilde{x}) \arrow[dl, "p"] \\ & (X,x) & \end{tikzcd}$$

**Part 2:** We prove the properties of $p_*|_I, q$.

(1) As $p_*(\pi_1(\widetilde{X},\widetilde{x})) \leq I$, $p_*|_I$ is well-defined.

(2) As $p_*$ is an injective homomorphism, $p_*|_I$ is an injective homomorphism.

(3) As $I \leq I$, $q$ is well-defined.

(4) As $\widetilde{X}$ is path connected, $q$ is surjective.

(5) For all loop homotopy classes $[\![\gamma_1]\!], [\![\gamma_2]\!]$ in $X$ based at $x$:
Choose a path lift $[\![\widetilde{\gamma}_2]\!]$ of $[\![\gamma_2]\!]$ in $\widetilde{X}$ from $\widetilde{x}'$ to $\widetilde{x}$ under $p$.
It follows from definition that $q([\![\gamma_2]\!])$ sends $\widetilde{x}'$ to $\widetilde{x}$.
Choose a path lift $[\![\widetilde{\gamma}_1]\!]$ of $[\![\gamma_1]\!]$ in $\widetilde{X}$ from some $\widetilde{x}'' \in \widetilde{X}$ to $\widetilde{x}'$ under $p$.
$[\![q([\![\gamma_2]\!]) \circ \widetilde{\gamma}_1]\!]$ is a path lift of $[\![\gamma_1]\!]$ in $\widetilde{X}$ from $q([\![\gamma_2]\!])(\widetilde{x}'')$ to $q([\![\gamma_2]\!])(\widetilde{x}') = \widetilde{x}$.
It follows from definition that $q([\![\gamma_1]\!])(q([\![\gamma_2]\!])(\widetilde{x}'')) = \widetilde{x}$.
$[\![\widetilde{\gamma}_1]\!] \star [\![\widetilde{\gamma}_2]\!]$ is a path lift of $[\![\gamma_1]\!] \star [\![\gamma_2]\!]$ in $\widetilde{X}$ from $\widetilde{x}''$ to $\widetilde{x}$ under $p$.
It follows from definition that $q([\![\gamma_1]\!] \star [\![\gamma_2]\!])(\widetilde{x}'') = \widetilde{x}$.
It follows from the universal property that $q([\![\gamma_1]\!]) \circ q([\![\gamma_2]\!]) = q([\![\gamma_1]\!] \star [\![\gamma_2]\!])$.

(6) As $\widetilde{f} = id_{\widetilde{X}}$ iff $[\![\widetilde{\gamma}]\!] \in \pi_1(\widetilde{X},\widetilde{x})$, $\mathbf{Ker}(q) = \mathbf{Im}(p_*|_I)$

□

**Remark:** Recall that the normalizer $I$ of a subgroup $H$ of a group $G$ is defined by the collection of all $i \in G$ with $iHi^{-1} = H$. When $p$ is universal and $p$ is isomorphic to some quotient map $r : \widetilde{X} \to \widetilde{X}/\widetilde{G}, \widetilde{x} \mapsto x$, $\pi_1(X,x) \cong \mathbf{Deck}(p) \cong \widetilde{G}$.

**Proposition 3.16.** Let $\widetilde{X}$ be a path connected and locally path connected topological space, $\widetilde{G} \leq \mathbf{Aut}(\widetilde{X}), X = \widetilde{X}/\widetilde{G}$, and $p : \widetilde{X} \to X, \widetilde{x} \mapsto x$.
If $\widetilde{G}$ is free and properly discontinuous, then $\mathbf{Deck}(p) = \widetilde{G}$.

*Proof.* We may divide our proof into two parts.
**"$\leq$" inclusion:** For all $\widetilde{f} \in \mathbf{Deck}(p)$, we aim to prove that $\widetilde{f} \in \widetilde{G}$.
Choose $\widetilde{x}_0 \in \widetilde{X}, x_0 \in X$ with $p(\widetilde{x}_0) = x_0$. As $\widetilde{f} \in \mathbf{Deck}(p)$, $\widetilde{f}(\widetilde{x}_0)$ is another representative of $x_0$, so for some $\widetilde{g} \in \widetilde{G}$, $\widetilde{f}(\widetilde{x}_0) = \widetilde{g}(\widetilde{x}_0)$. As $\widetilde{X}$ is path connected and locally path connected, it follows from the universal property that $\widetilde{f} = \widetilde{g} \in \widetilde{G}$.
**"$\geq$" inclusion:** For all $\widetilde{g} \in \widetilde{G}$, we aim to prove that $\widetilde{g} \in \mathbf{Deck}(p)$.
As $p$ is the quotient map, it is invariant under $\widetilde{g}$, so $\widetilde{g} \in \mathbf{Deck}(p)$. □

**Proposition 3.17.** Let $\widetilde{X}$ be a topological space,
$X$ be a locally path connected topological space,
$Y$ be a compact simply connected manifold with boundary,
and $p : \widetilde{X} \to X$ be a covering map.
$\sigma : \mathbf{Deck}(p) \to \mathbf{Deck}(\mathbf{C}(Y, p)), \widetilde{f} \mapsto \mathbf{C}(Y, \widetilde{f})$ is a group embedding.

*Proof.* We may divide our proof into three parts.

(1) As $\mathbf{C}(Y, p) \circ \mathbf{C}(Y, \widetilde{f}) = \mathbf{C}(Y, p \circ \widetilde{f}) = \mathbf{C}(Y, p)$, $\sigma$ is well-defined.

(2) As $\mathbf{C}(Y, \widetilde{f}_1) \circ \mathbf{C}(Y, \widetilde{f}_2) = \mathbf{C}(Y, \widetilde{f}_1 \circ \widetilde{f}_2)$, $\sigma$ is a homomorphism.

(3) We prove that $\sigma$ has trivial kernel:

$$\begin{aligned}\mathbf{C}(Y, \widetilde{f}) = id_{\mathbf{C}(Y,\widetilde{X})} &\implies \mathbf{C}(Y, \widetilde{f}(\widetilde{x})) = \mathbf{C}(Y, \widetilde{f}) \circ \mathbf{C}(Y, \widetilde{x}) = \mathbf{C}(Y, \widetilde{x}) \\ &\implies \widetilde{f}(\widetilde{x}) = \widetilde{x} \implies \widetilde{f} = id_{\widetilde{X}}\end{aligned}$$

□

**Remark:** Recall that $\mathbf{C}(Y, p) : \mathbf{C}(Y, \widetilde{X}) \to \mathbf{C}(Y, X), \widetilde{f} \mapsto p \circ \widetilde{f}$ is a covering map. $\mathbf{Deck}(\mathbf{C}(Y, p))$ may be larger than $\sigma(\mathbf{Deck}(p))$, as $\mathbf{C}(Y, X)$ may not be path connected.

**Proposition 3.18.** Let $\widetilde{X} = \mathbb{C}, X = \mathbb{C}^\times, p : \widetilde{X} \to X, \widetilde{x} \mapsto \mathrm{e}^{\widetilde{x}}$.

$$\mathbf{Deck}(p) = \langle \widetilde{x} \mapsto \widetilde{x} + 2\pi\mathrm{i} \rangle$$

*Proof.* It suffice to notice that $p$ is isomorphic to the quotient map $q$:

$$q : \widetilde{X} \to \widetilde{X}/\langle \widetilde{x} \mapsto \widetilde{x} + 2\pi\mathrm{i} \rangle, \widetilde{x} \mapsto x$$

□

**Remark:** As $\widetilde{X}$ is simply connected and locally path connected, it follows from the universal property that $\pi_1(X, 1) = \langle [\![\gamma]\!] \rangle$, where $\gamma : [0, 1] \to X, y \mapsto \mathrm{e}^{2\pi\mathrm{i}y}$.

**Proposition 3.19.** Let $\widetilde{X} = \mathbb{C}^\times, X = \mathbb{C}^\times, n > 0, p : \widetilde{X} \to X, \widetilde{x} \mapsto \widetilde{x}^n, \zeta = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$.

$$\mathbf{Deck}(p) = \langle \widetilde{x} \mapsto \zeta\widetilde{x} \rangle$$

*Proof.* It suffices to notice that $p$ is isomorphic to the quotient map $q$:

$$q : \widetilde{X} \to \widetilde{X}/\langle \widetilde{x} \mapsto \zeta\widetilde{x} \rangle, \widetilde{x} \mapsto x$$

□

**Proposition 3.20.** Let $n > 0$, $\widetilde{X}$ be the set of all $n$-complex tuples with distinct entries, $X$ be the set of all monic degree $n$ polgnomials in $\mathbb{C}[z]$ with distinct roots, and $p : \widetilde{X} \mapsto X, (z_1, z_2, z_3, \cdots) \mapsto (z - z_1)(z - z_2)(z - z_3) \cdots$.

$$\mathbf{Deck}(p) = \text{The permutation } S_n \text{ of the } n \text{ roots}$$

*Proof.* It suffices to notice that $p$ is isomorphic to the quotient map $q$:

$$q : \widetilde{X} \to \widetilde{X}/S_n, \widetilde{x} \mapsto x$$

□

### 3.3 The Galois Correspondence

**Definition 3.21.** (**Normal Covering Map**)
Let $Y, X$ be topological spaces with $Y$ path connected and locally path connected, and $f : Y \to X$ be a covering map. If one of the followings holds:

(1) For all $x \in X$, $\mathbf{Deck}(f)$ acts transitively on $f^{-1}(x)$.

(2) For all $x \in X, y \in Y$ with $f(y) = x$, $f_*(\pi_1(Y, y)) \trianglelefteq \pi_1(X, x)$.

Then $f$ is normal.

**Remark:** As $Y$ is path connected and locally path connected, it follows from the universal property that the two conditions are equivalent.

**Example 3.22.** (**The Galois Correspondence**)
Let $Z, X$ be topological spaces with $Z$ path connected and locally path connected, and $h : Z \to X$ be a normal covering map with deck transformation group $G$. The maps below are inverse graph homomorphisms to each other:

$$\begin{tikzcd}
Z \arrow[d, "Z \text{ covers } Y"'] & \{id_Z\} \arrow[d, "\{id_Z\} \leq H"] \\
Y \arrow[r, shift left, "H = \mathbf{Aut}(Z \text{ covers } Y)"] \arrow[d, "Y \text{ covers } X"'] & H \arrow[l, shift left, "Y = Z/H"] \arrow[d, "H \leq G"] \\
X & G
\end{tikzcd}$$

**Example 3.23.** Let $\widetilde{X} = \mathbb{S}, \zeta = \mathrm{e}^{\frac{2\pi\mathrm{i}}{105}}$.
$\widetilde{G} = \langle \zeta \rangle$ gives the Galois correspondence below:

$$\begin{tikzcd}
 & \langle 1 \rangle \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle \zeta^{15} \rangle \arrow[d] \arrow[dr] & \langle \zeta^{21} \rangle \arrow[dl] \arrow[dr] & \langle \zeta^{35} \rangle \arrow[dl] \arrow[d] \\
\langle \zeta^{3} \rangle \arrow[dr] & \langle \zeta^{5} \rangle \arrow[d] & \langle \zeta^{7} \rangle \arrow[dl] \\
 & \langle \zeta \rangle &
\end{tikzcd}$$

**Example 3.24.** Let $\widetilde{X} = \mathbb{S}^3$.
$\widetilde{G} = \left\{ \pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}, \frac{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}}{2} \right\}$ gives the Galois correspondence below:

$$\begin{tikzcd}
 & & & \{1\} \arrow[dll] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
 & \langle -1 \rangle \arrow[dl] \arrow[d] \arrow[dr] & & \left\langle \frac{-1+\mathbf{i}+\mathbf{j}+\mathbf{k}}{2} \right\rangle \arrow[d] & \left\langle \frac{-1+\mathbf{i}-\mathbf{j}+\mathbf{k}}{2} \right\rangle \arrow[d] & \left\langle \frac{-1+\mathbf{i}+\mathbf{j}-\mathbf{k}}{2} \right\rangle \arrow[d] & \left\langle \frac{-1+\mathbf{i}-\mathbf{j}-\mathbf{k}}{2} \right\rangle \arrow[d] \\
\langle \mathbf{i} \rangle \arrow[dr] & \langle \mathbf{j} \rangle \arrow[d] & \langle \mathbf{k} \rangle \arrow[dl] & \left\langle \frac{1+\mathbf{i}+\mathbf{j}+\mathbf{k}}{2} \right\rangle \arrow[dd] & \left\langle \frac{1+\mathbf{i}-\mathbf{j}+\mathbf{k}}{2} \right\rangle \arrow[ddl] & \left\langle \frac{1+\mathbf{i}+\mathbf{j}-\mathbf{k}}{2} \right\rangle \arrow[ddll] & \left\langle \frac{1+\mathbf{i}-\mathbf{j}-\mathbf{k}}{2} \right\rangle \arrow[ddlll] \\
 & \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle \arrow[drr] & & & & & \\
 & & & \widetilde{G} & & &
\end{tikzcd}$$

**Remark:** We will prove $\pi_1(\widetilde{X}, 1) = \{1\}$ by Seifert-Van Kampen theorem. If we assume this, then for all $\widetilde{H} \leq \widetilde{G}$, $\pi_1(\widetilde{X}/\widetilde{H}, 1)$ corresponds to $\widetilde{H}$ under the universal property.

# 4 Seifert-Van Kampen Theorem

## 4.1 Presentation

**Definition 4.1.** (**Presentation**)

(1) Let $S$ be a set. $g \in \mathbf{Word}(S)$ if:

$$\begin{aligned}
&\exists m \geq 0 \\
&\exists s_1, s_2, \cdots, s_{m-1}, s_m \in S \\
&\exists \epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}, \epsilon_m \in \mathbb{Z} \\
&g = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m}
\end{aligned}$$

(2) Let $R \subseteq \mathbf{Word}(S)$. $g, h \in \mathbf{Word}(S)$ are $R$-equivalent if $g = h$ after:

$$\begin{aligned}
&\text{Inserting or deleting } e = s^0, \text{ where } s \in S \\
&\text{Replacing } s^\epsilon s^\delta \text{ by } s^{\epsilon+\delta}, \text{ where } s \in S \text{ and } \epsilon, \delta \in \mathbb{Z} \\
&\text{Inserting or deleting } i \in R
\end{aligned}$$

(3) Define the presentation $\langle S|R\rangle$ as the $R$-quotient of $\mathbf{Word}(S)$ under:

$$\begin{aligned}
g &= s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m} \\
h &= t_1^{\delta_1} t_2^{\delta_2} \cdots t_{n-1}^{\delta_{n-1}} t_n^{\delta_n} \\
gh &= s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m} t_1^{\delta_1} t_2^{\delta_2} \cdots t_{n-1}^{\delta_{n-1}} t_n^{\delta_n}
\end{aligned}$$

**Remark:** When $R = \emptyset$, $\langle S|R\rangle = \langle S\rangle$ is a free group.

**Proposition 4.2.** (**Reduced Form**)
Let $S$ be a set, and $g \in \langle S\rangle$.

$$\begin{aligned}
&\exists! m \geq 0 \\
&\exists! \text{ alternating } s_1, s_2, \cdots, s_{m-1}, s_m \in S \\
&\exists! \text{ nonzero } \epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}, \epsilon_m \in \mathbb{Z} \\
&g = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m}
\end{aligned}$$

*Proof.* The existence of a reduced form is a direct consequence of well-ordering principle. For the uniqueness, we prove by induction on length, where the basis step $e$ is clear. For all $g$ in the free group $\langle S\rangle$, for all similar terms $u, v, w$ of $g$, they are in the form:

$$g = \cdots s_{u-1}^{\epsilon_{u-1}} \overbrace{s^\theta}^{u} s_{u+1}^{\epsilon_{u+1}} \cdots s_{v-1}^{\epsilon_{v-1}} \overbrace{s^\phi}^{v} s_{v+1}^{\epsilon_{v+1}} \cdots s_{w-1}^{\epsilon_{w-1}} \overbrace{s^\psi}^{w} s_{w+1}^{\epsilon_{w+1}} \cdots$$

Assume that there exist reduction strategies $\mathcal{S}, \mathcal{T}$, such that $\mathcal{S}$ combines $u, v$ before simplifying $w$, and $\mathcal{T}$ combines $v, w$ before simplifying $u$. Observe that:

$$s_{u+1}^{\epsilon_{u+1}} \cdots s_{v-1}^{\epsilon_{v-1}} = s_{v+1}^{\epsilon_{v+1}} \cdots s_{w-1}^{\epsilon_{w-1}} = e$$

Hence:

$$g = \cdots s_{u-1}^{\epsilon_{u-1}} s^{\theta+\phi+\psi} s_{w+1}^{\epsilon_{w+1}} \cdots$$

Therefore, the results of $\mathcal{S}, \mathcal{T}$ are equal reduced forms of the same shorter word. $\square$

**Remark:** If $R$ consists of powers of generators, then we may repeat the same proof.

**Example 4.3.** (**The Universal Property of Free Group**)
Let $S$ be a set, and $G$ be a group.
For some injection:

$$\iota : S \to \langle S \rangle, s \mapsto s$$

For all map:

$$f : S \to G$$

For some unique homomorphism:

$$\widetilde{f} : \langle S \rangle \to G, s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m} \mapsto f(s_1)^{\epsilon_1} f(s_2)^{\epsilon_2} \cdots f(s_{m-1})^{\epsilon_{m-1}} f(s_m)^{\epsilon_m}$$

The diagram below commutes:

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "f"'] & \langle S \rangle \arrow[d, "\widetilde{f}"] \\ & G \end{tikzcd}$$

**Definition 4.4.** (**Generate a Subgroup**)
Let $S$ be a set, and $R \subseteq \langle S \rangle$.

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[dr, "f : r \mapsto r"'] & \langle R \rangle \arrow[d, dashed, "\exists ! \widetilde{f}"] \\ & \langle S \rangle \end{tikzcd}$$

Define the subgroup of $\langle S \rangle$ generated by $R$ as $\widetilde{f}(\langle R \rangle)$.

**Remark:** In case that there is no ambiguity, we write it as $\langle R \rangle = \widetilde{f}(\langle R \rangle)$.

**Definition 4.5.** (**Generate a Normal Subgroup**)
Let $S$ be a set, and $R \subseteq \langle S \rangle$.

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "f:s\mapsto s"'] & \langle S \rangle \arrow[d, dashed, "\exists!\widetilde{f}"] \\ & \langle S|R \rangle \end{tikzcd}$$

Define the normal subgroup of $\langle S \rangle$ generated by $R$ as $\mathbf{Ker}(\widetilde{f})$.

**Remark:** In case that there is no ambiguity, we write it as $[R] = \mathbf{Ker}(\widetilde{f})$.

**Proposition 4.6.** Let $G = \langle a, b \rangle$, $|m|, |n| > 1$, $u = a^m b^n$. $\nexists v \in G, \langle u, v \rangle = G$.

*Proof.* Assume to the contrary that $\exists v \in G, \langle u, v \rangle = G$.
**Step 1:** Define $H = [a^m, b^n] \trianglelefteq G = \langle a, b \rangle$.
**Step 2:** Define $N = [u] \trianglelefteq G = \langle u, v \rangle$.
**Step 3:** Since $N = [a^m b^n] \leq H = [a^m, b^n]$, $(G/N)/(H/N) \cong G/H$.
**Step 4:** Since $G/N \cong \mathbb{Z}$ and $H/N \neq \{e\}$, $(G/N)/(H/N)$ is finite.
**Step 5:** Since $\mathbb{Z} \hookrightarrow G/H$, $G/H$ is infinite, contradiction! $\square$

## 4.2 Geometric Characterization

**Definition 4.7.** (**Cayley Graph**)
Let $G$ be a group, and $S \subseteq G$. Define the Cayley graph $\Gamma(G, S)$ of $G$ generated by $S$ as a colored directed graph, where we assign a black vertex for each $g \in G$, and a $s$-colored edge from $g_1$ to $g_2$ for each $g_1, g_2 \in G$ and $s \in S$ with $g_1 s = g_2$.

**Proposition 4.8.** Let $G, \bar{G}$ be groups, and $S \subseteq G$, $\bar{S} \subseteq \bar{G}$.
If $\sigma : G \to \bar{G}$ be a group homomorphism with $\sigma(S) \subseteq \bar{S}$,
then $\sigma : \Gamma(G, S) \to \Gamma(\bar{G}, \bar{S})$ is a colored graph homomorphism.

*Proof.* Assume that there exists a $s$-colored edge from $g_1$ to $g_2$ in $\Gamma(G, S)$, so $g_1 s = g_2$.
As $\sigma : G \to \bar{G}$ is a group homomorphism with $\sigma(S) \subseteq \bar{S}$, $\sigma(g_1)\sigma(s) = \sigma(g_2)$.
Hence, there exists a $\sigma(s)$-colored edge from $\sigma(g_1)$ to $\sigma(g_2)$ in $\Gamma(\bar{G}, \bar{S})$. $\square$

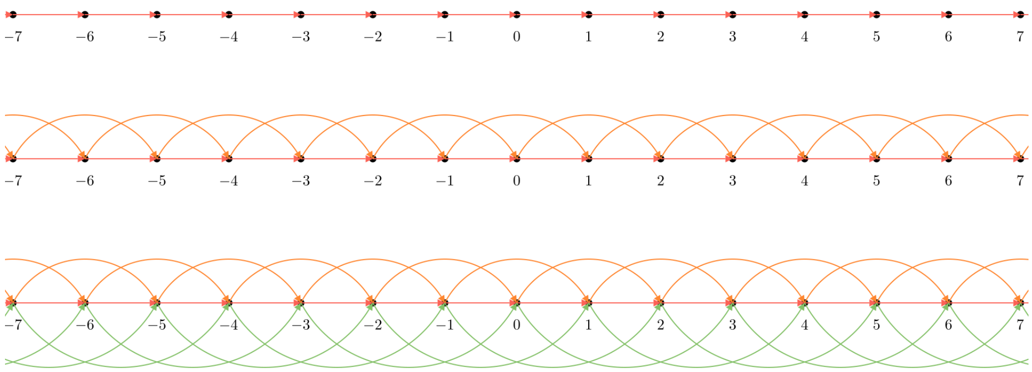



Figure 8: $G = \mathbb{Z}, S_1 = \{1\}, S_2 = \{1, 2\}, S_3 = \{1, 2, 3\}$

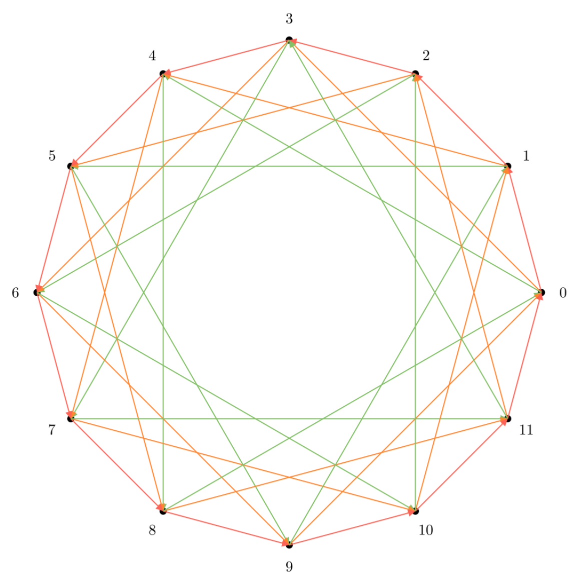



Figure 9: $G = \mathbb{Z}/12\mathbb{Z}, S = \{1, 3, 4\}$

**Proposition 4.9.** (**Ping-pong Lemma**)
Let $X$ be a set, $G \leq \mathbf{Perm}(X)$, $S \subseteq G$ with $|S| \geq 2$, and $f : S \to \mathcal{P}(X)$. If:

(1) For all $s \in S$, $f(s)$ is nonempty.

(2) For all distinct $s_1, s_2 \in S$, $f(s_1), f(s_2)$ are disjoint.

(3) For all distinct $s_1, s_2 \in S$, for all nonzero $\epsilon_1 \in \mathbb{Z}$, $s_1^{\epsilon_1}(f(s_2)) \subseteq f(s_1)$.

Then the subgroup of $G$ generated by $S$ is free.

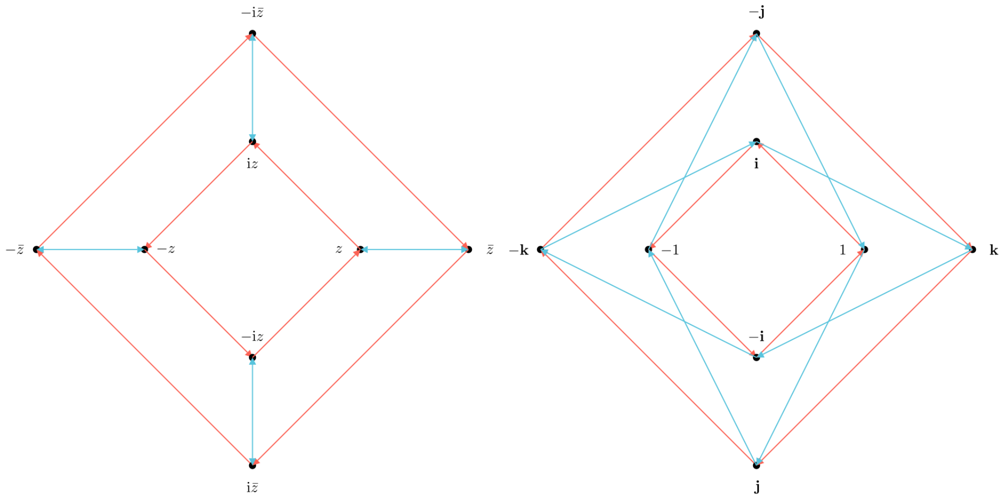



Figure 10: $G_1 = \{\pm z, \pm \mathrm{i}z, \pm \bar{z}, \pm \mathrm{i}\bar{z}\}, S_1 = \{\mathrm{i}z, \bar{z}\}, G_2 = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}, S_2 = \{\mathbf{i}, \mathbf{j}\}$

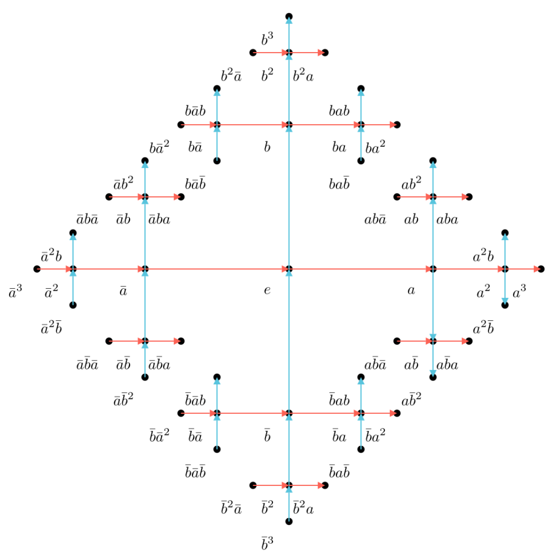



Figure 11: $S = \{a, b\}, G = \langle S \rangle$

*Proof.* It suffices to show that every reduced form is nontrivial.

(1) For all $m \geq 2$,
for all alternating $s_1, s_2, \cdots, s_{m-1}, s_m \in S$,
for all nonzero $\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}, \epsilon_m \in \mathbb{Z}$,
we aim to show that $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m} \neq id_X$.
By conjugation, we may assume WLOG that $s_1 = s_m = s \in S$.
The result follows from $s^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s^{\epsilon_m}(f(s_{m-1})) \subseteq f(s)$.

(2) For the case $m = 1$, as $|S| \geq 2$, choose $s' \in S \backslash \{s\}$.

The result follows from $s(f(s')) \subseteq f(s)$.

$\square$

**Remark:** We may replace nonzero $\epsilon_1 \in \mathbb{Z}$ by nonzero $\epsilon_1 \in \mathbb{Z}/n_1\mathbb{Z}$ for another result.

**Example 4.10.** Let $S$ be a set with $|S| \geq 2$, $G = \langle S \rangle$, $X = G$ with $G$-left translation, and $f : S \to \mathcal{P}(X), s \mapsto$ all nontrivial reduced form $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m}$ in $G$ with $s_1 = s$. It follows that the subgroup of $G$ generated by $S$ is free.

**Example 4.11.** Let $S = \{a^2, ab, b^2\}$, $G$ be the subgroup of $\langle a, b \rangle$ generated by $S$, $X = \langle a, b \rangle$ with $G$-left translation,

(1) $f(a^2)$ be all nontrivial reduced form $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m}$ in $X$ with $s_1 = a, \epsilon_1 \neq 1$.

(2) $f(ab)$ be all nontrivial reduced form $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m}$ in $X$ with $s_1 = a, \epsilon_1 = 1$ or $s_1 = b, \epsilon_1 = -1$.

(3) $f(b^2)$ be all nontrivial reduced form $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{m-1}^{\epsilon_{m-1}} s_m^{\epsilon_m}$ in $X$ with $s_1 = b, \epsilon_1 \neq -1$.

It follows that $G$ is free.

**Remark:** The analogous version of Schröder-Bernstein theorem fails, that is, $G$ is embedded in $\langle a, b \rangle$, $\langle a, b \rangle$ is embedded in $G$, but $G$ is not isomorphic to $\langle a, b \rangle$.

**Example 4.12.** Let $|a|, |b| \geq 2$, $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, $S = \{A, B\}$, $G$ be the subgroup of $\mathbf{SL}_2(\mathbb{R})$ generated by $S$, $X = \mathbb{R}^2$ with $G$-left translation,

(1) $f(A)$ be all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $X$ with $|x_2| < |x_1|$.

(2) $f(B)$ be all $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $X$ with $|x_1| < |x_2|$.

It follows that $G$ is free.

**Remark:** The condition $|a|, |b| \geq 2$ is essential. When $a = -1, b = 1$, the braid relation $ABA = BAB$ and the chain relation $(AB)^6 = I$ hold, giving a presentation of $\mathbf{SL}_2(\mathbb{Z})$.

**Example 4.13.** Let $a(x) = -\frac{1}{x}, b(x) = 1 - \frac{1}{x}, S = \{a, b\}$, $G$ be the subgroup of $\mathbf{PSL}_2(\mathbb{R})$ generated by $S$, $X = \mathbb{R} \backslash \mathbb{Q}$ with $G$-left translation, $f(a) = (-\infty, 0) \backslash \mathbb{Q}$, and $g(b) = (0, +\infty) \backslash \mathbb{Q}$. It follows that $G$ is isomorphic to $\langle a, b | a^2 = b^3 = e \rangle$.

## 4.3 Amalgamation

**Definition 4.14.** (**Amalgamation**)
Let $(P_{\lambda,\mu})_{\lambda,\mu\in I}, (Q_\lambda)_{\lambda\in I}$ be groups, and $(u_{\lambda,\mu} : P_{\lambda,\mu} \to Q_\lambda)_{\lambda,\mu\in I}, (v_{\lambda,\mu} : P_{\lambda,\mu} \to Q_\mu)_{\lambda,\mu\in I}$ be homomorphisms. Define the amalgamation $R$ of $(Q_\lambda)_{\lambda\in I}$ under $(P_{\lambda,\mu})_{\lambda,\mu\in I}$ as the group generated by the disjoint union of $(Q_\lambda)_{\lambda\in I}$, subject to:

(1) For all $\lambda \in I$, the relations within $Q_\lambda$.

(2) For all $\lambda, \mu \in I$, for all $p_{\lambda,\mu} \in P_{\lambda,\mu}$, the relation $u_{\lambda,\mu}(p_{\lambda,\mu}) = v_{\lambda,\mu}(p_{\lambda,\mu})$

**Example 4.15.** (**The Universal Property of Amalgamation**)
Let $(P_{\lambda,\mu})_{\lambda,\mu\in I}, (Q_\lambda)_{\lambda\in I}, R, S$ be groups,
$(u_{\lambda,\mu} : P_{\lambda,\mu} \to Q_\lambda)_{\lambda,\mu\in I}, (v_{\lambda,\mu} : P_{\lambda,\mu} \to Q_\mu)_{\lambda,\mu\in I}$ be homomorphisms,
and $R$ be the amalgamation of $(Q_\lambda)_{\lambda\in I}$ under $(P_{\lambda,\mu})_{\lambda,\mu\in I}$.
There exist homomorphisms $(w_\lambda : Q_\lambda \to R, q_\lambda \mapsto q_\lambda)_{\lambda\in I}$, such that:

$$\begin{tikzcd}
& Q_\lambda \arrow[dr, "w_\lambda"] & \\
P_{\lambda,\mu} \arrow[ur, "u_{\lambda,\mu}"] \arrow[dr, "v_{\lambda,\mu}"'] & & R \\
& Q_\mu \arrow[ur, "w_\mu"'] &
\end{tikzcd}$$

For all homomorphisms $(x_\lambda : Q_\lambda \to S)_{\lambda\in I}$, such that:

$$\begin{tikzcd}
& Q_\lambda \arrow[dr, "x_\lambda"] & \\
P_{\lambda,\mu} \arrow[ur, "u_{\lambda,\mu}"] \arrow[dr, "v_{\lambda,\mu}"'] & & S \\
& Q_\mu \arrow[ur, "x_\mu"'] &
\end{tikzcd}$$

There exists a unique homomorphism:

$$y : R \to S, s_{1,\lambda_1} \cdots s_{m,\lambda_m} \mapsto x_{\lambda_1}(s_{1,\lambda_1}) \cdots x_{\lambda_m}(s_{m,\lambda_m})$$

Such that:

$$\begin{tikzcd}
& Q_\lambda \arrow[dr, "w_\lambda"] \arrow[drr, "x_\lambda"] & & \\
P_{\lambda,\mu} \arrow[ur, "u_{\lambda,\mu}"] \arrow[dr, "v_{\lambda,\mu}"'] & & R \arrow[r, "y"] & S \\
& Q_\mu \arrow[ur, "w_\mu"'] \arrow[urr, "x_\mu"'] & &
\end{tikzcd}$$

**Definition 4.16.** (**The Amalgamation Homomorphism**)
Let $S$ be a topological space with base point $s_0$, $(Q_\lambda)_{\lambda \in I}$ be an open cover of $S$, where each $Q_\lambda \ni s_0$ is path connected, and each $P_{\lambda,\mu} = Q_\lambda \cap Q_\mu$ is path connected. Define $R$ as the amalgamation of $(\pi_1(Q_\lambda, s_0))_{\lambda \in I}$ under $(\pi_1(P_{\lambda,\mu}, s_0))_{\lambda,\mu \in I}$. We construct a homomorphism $y : R \to \pi_1(S, s_0)$:

(1) It follows from definition that the topological diagram commutes:

$$\begin{tikzcd}
 & Q_\lambda \arrow[dr, "x_\lambda : q_\lambda \mapsto q_\lambda"] & \\
P_{\lambda,\mu} \arrow[ur, "u_{\lambda,\mu} : p_{\lambda,\mu} \mapsto p_{\lambda,\mu}"] \arrow[dr, "v_{\lambda,\mu} : p_{\lambda,\mu} \mapsto p_{\lambda,\mu}"'] & & S \\
 & Q_\mu \arrow[ur, "x_\mu : q_\mu \mapsto q_\mu"'] &
\end{tikzcd}$$

(2) It follows from $\pi_1$ is a functor that the group diagram commutes:

$$\begin{tikzcd}
 & \pi_1(Q_\lambda, s_0) \arrow[dr, "(x_\lambda)_*"] & \\
\pi_1(P_{\lambda,\mu}, s_0) \arrow[ur, "(u_{\lambda,\mu})_*"] \arrow[dr, "(v_{\lambda,\mu})_*"'] & & \pi_1(S, s_0) \\
 & \pi_1(Q_\mu, s_0) \arrow[ur, "(x_\mu)_*"'] &
\end{tikzcd}$$

(3) It follows from the universal property that the group diagram commutes:

$$\begin{tikzcd}
 & \pi_1(Q_\lambda, s_0) \arrow[dr, "w_\lambda"] \arrow[drr, "(x_\lambda)_*"] & & \\
\pi_1(P_{\lambda,\mu}, s_0) \arrow[ur, "(u_{\lambda,\mu})_*"] \arrow[dr, "(v_{\lambda,\mu})_*"'] & & R \arrow[r, dashed, "\exists! y"] & \pi_1(S, s_0) \\
 & \pi_1(Q_\mu, s_0) \arrow[ur, "w_\mu"'] \arrow[urr, "(x_\mu)_*"'] & &
\end{tikzcd}$$

We define $y$ as the amalgamation homomorphism.

**Remark:** Under appropriate conditions, we study the bijectivity of $y$:

(1) In **Part 1**, we study the surjectivity. For all loop $\gamma$ in $S$ based at $s_0$, we aim to find loops $\gamma_{1,\lambda_1}, \cdots, \gamma_{1,\lambda_m}$ in $Q_{\lambda_1}, \cdots, Q_{\lambda_m}$ based at $s_0$, such that $\gamma \approx \gamma_{1,\lambda_1} \star \cdots \star \gamma_{m,\lambda_m}$. For convenience, we say that $\gamma$ factorizes into $\gamma_{1,\lambda_1} \star \cdots \star \gamma_{m,\lambda_m}$.

(2) In **Part 2**, we study the injectivity. For all factorizations $\gamma_{1,\lambda_1} \star \cdots \star \gamma_{m,\lambda_m}, \sigma_{1,\mu_1} \star \cdots \star \sigma_{n,\mu_n}$ of the same loop in $S$ based at $s_0$, we aim to apply a sequence of elementary transformations to transform $\gamma_{1,\lambda_1} \star \cdots \star \gamma_{m,\lambda_m}$ to $\sigma_{1,\mu_1} \star \cdots \star \sigma_{m,\mu_m}$.
**Type 1 Elementary Transformation:**

Assume that the factorization $\cdots \star \gamma_{l,\lambda_l} \star \cdots \star \gamma_{r,\lambda_r} \star \cdots$ contains a subfactorization $\gamma = \gamma_{l,\lambda_l} \star \cdots \star \gamma_{r,\lambda_r}$ that belongs to $Q_\lambda = Q_{\lambda_l} = \cdots = Q_{\lambda_r}$. We reduce $\gamma$ as a loop in $Q_\lambda$ based at $s_0$. This corresponds to a relation within $Q_\lambda$.

**Type 2 Elementary Transformation:**

Assume that the factorization $\cdots \star \gamma_{k-1,\lambda_{k-1}} \star \gamma_{k,\lambda_k} \star \gamma_{k+1,\lambda_{k+1}} \star \cdots$ contains a letter $\gamma_{k,\lambda_k}$ that belongs to both $Q_{\lambda_k}$ and $Q_\mu$. We regard the loop $\gamma = \gamma_{k,\lambda_k}$ in $Q_{\lambda_k}$ based at $s_0$ as a loop $\gamma = \gamma_{k,\mu}$ in $Q_\mu$ based at $s_0$. This corresponds to a relation $(u_{\lambda_k,\mu})_*(\llbracket\gamma\rrbracket) = (v_{\lambda_k,\mu})_*(\llbracket\gamma\rrbracket)$.

For convenience, we say that $\gamma_{1,\lambda_1} \star \cdots \star \gamma_{m,\lambda_m}, \sigma_{1,\mu_1} \star \cdots \star \sigma_{n,\mu_n}$ are equivalent if they differ by a sequence of elementary transformations.

**Proposition 4.17.** (**Seifert-Van Kampen Theorem, Part 1**)
Let $S$ be a topological space with base point $s_0$, $(Q_\lambda)_{\lambda \in I}$ be an open cover of $S$, where each $Q_\lambda \ni s_0$ is path connected, and each $P_{\lambda,\mu} = Q_\lambda \cap Q_\mu$ is path connected. The amalgamation homomorphism $y$ is surjective.

*Proof.* For all loop $\gamma$ in $S$ based at $s_0$, we aim to factorize $\gamma$.

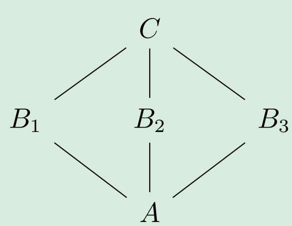
(1) As $\gamma$ is continuous,
the open cover $(Q_\lambda)_{\lambda \in I}$ of $S$ forms a pullback open cover of $[0,1]$.

(2) As $[0,1]$ is a closed and bounded interval,
WLOG, for some $0 = \alpha_0 < \alpha_1 < \alpha_2 = 1$,
for some $Q_1, Q_2$, $\gamma([\alpha_0, \alpha_1]) \subseteq Q_1, \gamma([\alpha_1, \alpha_2]) \subseteq Q_2$.

(3) Define $\gamma_1, \gamma_2$ as the restriction of $\gamma$ to $[\alpha_0, \alpha_1], [\alpha_1, \alpha_2]$.
Choose a path $\gamma_{1,2}$ in $P_{1,2}$ from $s_0$ to $\gamma(\alpha_1)$.
$\gamma$ factorizes into $(\gamma_1 \star \gamma_{1,2}^{-1}) \star (\gamma_{1,2} \star \gamma_2)$.

□

**Example 4.18.** Let $S$ be the graph below with base point $s_0 = A$:




(1) $Q_1 = S \backslash B_1, \pi_1(Q_1, A) = \langle \llbracket AB_2CB_3 \rrbracket \rangle, \pi_1(P_{2,3}, A) = \{\llbracket e_A \rrbracket\}$.

(2) $Q_2 = S \backslash B_2, \pi_1(Q_2, A) = \langle \llbracket AB_3CB_1 \rrbracket \rangle, \pi_1(P_{3,1}, A) = \{\llbracket e_A \rrbracket\}$.

(3) $Q_3 = S \backslash B_3, \pi_1(Q_3, A) = \langle \llbracket AB_1CB_2 \rrbracket \rangle, \pi_1(P_{1,2}, A) = \{\llbracket e_A \rrbracket\}$.

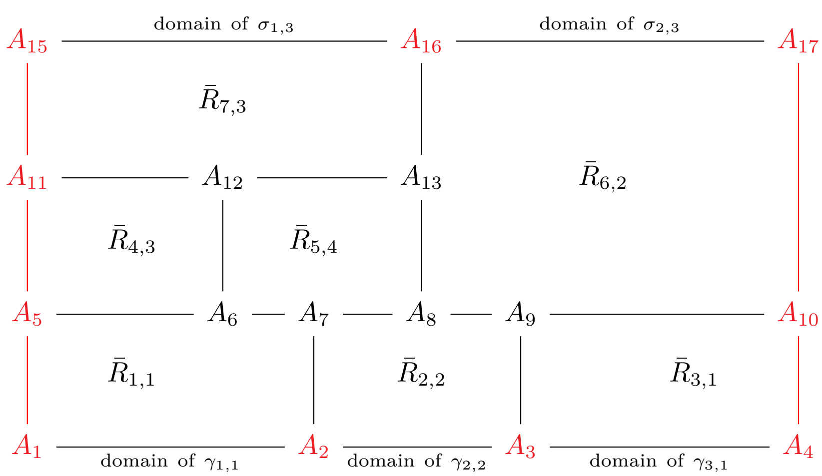
(4) $y(\llbracket AB_2CB_3 \rrbracket \star \llbracket AB_3CB_1 \rrbracket \star \llbracket AB_1CB_2 \rrbracket) = \llbracket e_A \rrbracket$, $y$ is not injective.

**Proposition 4.19.** (**Seifert-Van Kampen Theorem, Part 2**)
Let $S$ be a topological space with base point $s_0$, $(Q_\lambda)_{\lambda\in I}$ be an open cover of $S$, where each $Q_\lambda \ni s_0$ is path connected, each $P_{\lambda,\mu} = Q_\lambda \cap Q_\mu$ is path connected, and each $O_{\lambda,\mu,\nu} = Q_\lambda \cap Q_\mu \cap Q_\nu$ is path connected. The amalgamation homomorphism $y$ is injective.

*Proof.* For all factorizations $\gamma_{1,\lambda_1} \star \cdots \star \gamma_{m,\lambda_m}, \sigma_{1,\mu_1} \star \cdots \star \sigma_{n,\mu_n}$ of the same loop in $S$ based at $s_0$, we aim to show that they are equivalent.

(1) As there is a path homotopy $H$ from $\gamma_{1,\lambda_1} \star \cdots \star \gamma_{m,\lambda_m}$ to $\sigma_{1,\mu_1} \star \cdots \star \sigma_{n,\mu_n}$, the open cover $(Q_\lambda)_{\lambda\in I}$ of $S$ form a pullback open cover of $[0,1]^2$.

(2) As $[0,1]^2$ is a closed and bounded square, WLOG, assume that $m = 3, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 1, n = 2, \mu_1 = 3, \mu_2 = 2$, and partition $[0,1]^2$ into compact subrectangles $\bar{R}_{1,1} = A_1A_2A_7A_5, \bar{R}_{2,2} = A_2A_3A_9A_7, \bar{R}_{3,1} = A_3A_4A_{10}A_9, \bar{R}_{4,3} = A_5A_6A_{12}A_{11}, \bar{R}_{5,4} = A_6A_8A_{13}A_{12}, \bar{R}_{6,2} = A_8A_{10}A_{17}A_{16}, \bar{R}_{7,3} = A_{11}A_{13}A_{16}A_{15}$ with $H(\bar{R}_{1,1}) \subseteq Q_1, H(\bar{R}_{2,2}) \subseteq Q_2, H(\bar{R}_{3,1}) \subseteq Q_1, H(\bar{R}_{4,3}) \subseteq Q_3, H(\bar{R}_{5,4}) \subseteq Q_4, H(\bar{R}_{6,2}) \subseteq Q_2, H(\bar{R}_{7,3}) \subseteq Q_3$. We can apply the path connectedness of $O_{\lambda,\mu,\nu}$, as each of $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}$ has at most 3 edges. Each vertex and edge in red is mapped to $s_0$ under $H$:




(3) As $P_{\lambda,\mu}, O_{\lambda,\mu,\nu}$ are path connected, there exist some auxiliary arcs:

$$\gamma_6 = \text{A path in } O_{1,3,4} \text{ from } s_0 \text{ to } H(A_6)$$
$$\gamma_7 = \text{A path in } O_{1,2,4} \text{ from } s_0 \text{ to } H(A_7)$$
$$\gamma_8 = \text{A path in } P_{2,4} \text{ from } s_0 \text{ to } H(A_8)$$
$$\gamma_9 = \text{A path in } P_{1,2} \text{ from } s_0 \text{ to } H(A_9)$$
$$\gamma_{12} = \text{A path in } P_{3,4} \text{ from } s_0 \text{ to } H(A_{12})$$
$$\gamma_{13} = \text{A path in } O_{2,3,4} \text{ from } s_0 \text{ to } H(A_{13})$$

We perform a sequence of elementary moves to show the equivalence.
For simplicity, we define $H_{i,j} = H|_{A_i,A_j}$, and omit "$\star$":

**Step 1:** $\gamma_{1,1}\gamma_{2,2}\gamma_{3,1} = H_{1,2}H_{2,3}H_{3,4}$

**Step 2:** $H_{1,2}H_{2,3}H_{3,4}H_{4,10}$

**Step 3:** $H_{1,2}H_{2,3}H_{3,9}\gamma_9^{-1}\gamma_9H_{9,10}$

**Step 4:** $H_{1,2}H_{2,7}\gamma_7^{-1}\gamma_7H_{7,8}\gamma_8^{-1}\gamma_8H_{8,9}\gamma_9^{-1}\gamma_9H_{9,10}$

**Step 5:** $H_{1,5}H_{5,6}\gamma_6^{-1}\gamma_6H_{6,7}\gamma_7^{-1}\gamma_7H_{7,8}\gamma_8^{-1}\gamma_8H_{8,9}\gamma_9^{-1}\gamma_9H_{9,10}$

**Step 6:** $H_{5,6}\gamma_6^{-1}\gamma_6H_{6,7}\gamma_7^{-1}\gamma_7H_{7,8}\gamma_8^{-1}\gamma_8H_{8,9}\gamma_9^{-1}\gamma_9H_{9,10}$

**Step 7:** $H_{5,6}\gamma_6^{-1}\gamma_6H_{6,7}\gamma_7^{-1}\gamma_7H_{7,8}\gamma_8^{-1}\gamma_8H_{8,9}\gamma_9^{-1}\gamma_9H_{9,10}H_{10,17}$

**Step 8:** $H_{5,6}\gamma_6^{-1}\gamma_6H_{6,7}\gamma_7^{-1}\gamma_7H_{7,8}\gamma_8^{-1}\gamma_8H_{8,13}\gamma_{13}^{-1}\gamma_{13}H_{13,16}H_{16,17}$

**Step 9:** $H_{5,6}\gamma_6^{-1}\gamma_6H_{6,12}\gamma_{12}^{-1}\gamma_{12}H_{12,13}\gamma_{13}^{-1}\gamma_{13}H_{13,16}H_{16,17}$

**Step 10:** $H_{5,11}H_{11,12}\gamma_{12}^{-1}\gamma_{12}H_{12,13}\gamma_{13}^{-1}\gamma_{13}H_{13,16}H_{16,17}$

**Step 11:** $H_{11,12}\gamma_{12}^{-1}\gamma_{12}H_{12,13}\gamma_{13}^{-1}\gamma_{13}H_{13,16}H_{16,17}$

**Step 12:** $H_{11,15}H_{15,16}H_{16,17}$

**Step 13:** $\sigma_{1,3}\sigma_{2,3} = H_{15,16}H_{16,17}$

$\square$

**Example 4.20.** Let $S = \bigvee_{\lambda\in I}\mathbb{S}$ be the wedge product of $(\mathbb{S})_{\lambda\in I}$ at 1. For each $\lambda \in I$, define a path connected open subset $Q_\lambda = \bigvee_{\mu\in I} T_\mu$ of $S$ containing 1, where $T_\mu = \mathbb{S}$ if $\mu = \lambda$ and $T_\mu = \mathbb{S}\backslash\{-1\}$ if $\mu \neq \lambda$. As each $P_{\lambda,\mu} = Q_\lambda \cap Q_\mu$ is path connected, and each $O_{\lambda,\mu,\nu} = Q_\lambda \cap Q_\mu \cap Q_\nu$ is path connected, $\pi_1(S,1)$ is isomorphic to the free group generated by the generator of each copy of $\mathbb{S}$.

**Example 4.21.** Let $n \geq 2$, $S = \mathbb{S}^n$, $s_0, s_1, s_2$ be distinct points on $S$, and $Q_1 = S\backslash\{s_2\}, Q_2 = S\backslash\{s_1\}$. As $P_{1,2} = Q_1 \cap Q_2$ is path connected, $\pi_1(S, s_0) = \{[\![e_{s_0}]\!]\}$.

**Example 4.22.** $S = \mathbb{T}^2$ is a topological space with base point $s_0$ and open cover $Q_1, Q_2$, where $Q_1 \ni s_0, Q_2 \ni s_0$ are path connected, and $P_{1,2} = Q_1 \cap Q_2$ is path connected. As $\pi_1(Q_1, s_0) = \{e\}, \pi_1(Q_2, s_0) = \langle a, b\rangle$, and the generator of $\pi_1(P_{1,2}, s_0)$ projects to $ab\bar{a}\bar{b} \in \pi_1(Q_2, s_0)$, $\pi_1(S, s_0) = \langle a, b : ab\bar{a}\bar{b} = e\rangle$.

**Example 4.23.** $S = \mathbb{K}^2$ is a topological space with base point $s_0$ and open cover $Q_1, Q_2$, where $Q_1 \ni s_0, Q_2 \ni s_0$ are path connected, and $P_{1,2} = Q_1 \cap Q_2$ is path connected. As $\pi_1(Q_1, s_0) = \{e\}, \pi_1(Q_2, s_0) = \langle a, b\rangle$, and the generator of $\pi_1(P_{1,2}, s_0)$ projects to $ab\bar{a}b \in \pi_1(Q_2, s_0)$, $\pi_1(S, s_0) = \langle a, b : ab\bar{a}b = e\rangle$.

**Remark:** In general, attaching an open polygon to the boundary of a given polygon trivializes the boundary component. Another application of this technique shows that the fundamental group of the $g$-fold connected sum $(\mathbb{T}^2)^{\#g}$ of tori is:

$$\langle a_1, b_1, \cdots, a_g, b_g : a_1b_1\bar{a}_1\bar{b}_1 \cdots a_mb_m\bar{a}_m\bar{b}_m = e\rangle$$

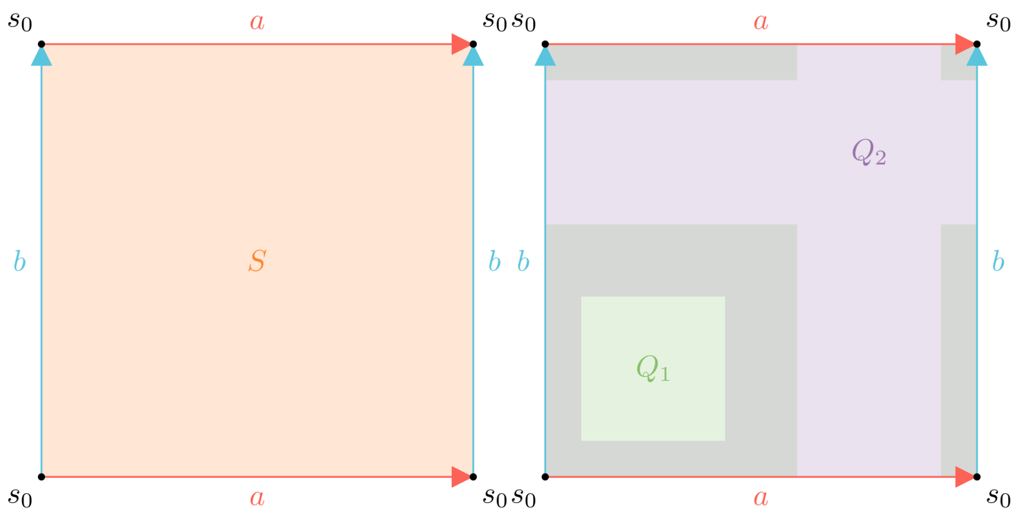



Figure 12: $S = \mathbb{T}^2$

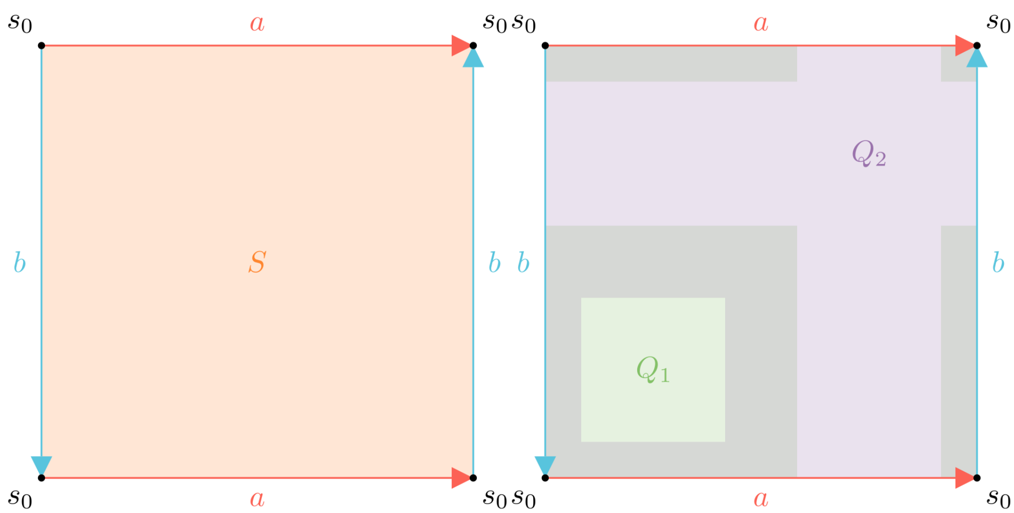



Figure 13: $S = \mathbb{K}^2$

**Definition 4.24.** (**Right Hand Torus Knot**)
Let $m, n$ be nonnegative coprime integers.
Define the $(m, n)$-right hand torus knot as the parametric equations below:

$$
\begin{aligned}
r &= 2 + \cos n\phi \\
x &= r \cos m\phi \\
y &= r \sin m\phi \\
z &= -\sin n\phi
\end{aligned}
$$

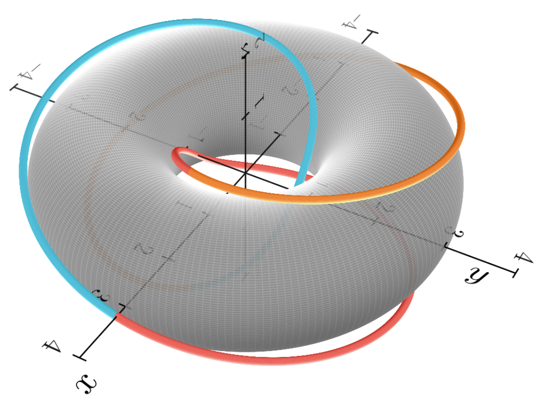



Figure 14: $(3, 2)$-right hand torus knot (view 1)

**Proposition 4.25.** Let $m, n$ be coprime integers, and $S$ be the complement of the image of the $(m, n)$-right hand torus knot with a base point $s_0$.
If we compactify $\mathbb{R}^3$ as $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, then $\pi_1(S, s_0) = \pi_1(S \cup \{\infty\}, s_0)$.

*Proof.* Define $Q_1$ as $S \ni s_0$ and $Q_2$ as a ball neighbour of $\infty$ in $S \cup \{\infty\}$ containing $s_0$. As $\pi_1(P_{1,2}, s_0) = \pi_1(Q_2, s_0) = \{[\![e_{s_0}]\!]\}$, the amalgamation homomorphism is the identity. $\square$

**Proposition 4.26.** Let $m, n$ be coprime integers, and $S$ be the complement of the image of the $(m, n)$-right hand torus knot with a base point $s_0$.
$\pi_1(S \cup \{\infty\}, s_0) = \langle a, b : a^m = b^n \rangle$, where $a$ is freely homotopic to the circle $x^2 + y^2 = 4, z = 0$, $b$ is freely homotopic to the circle $x = y = 0$.

*Proof.* Define $Q_1$ as a torus-shaped neighbour of the torus with central circle $x^2 + y^2 =$

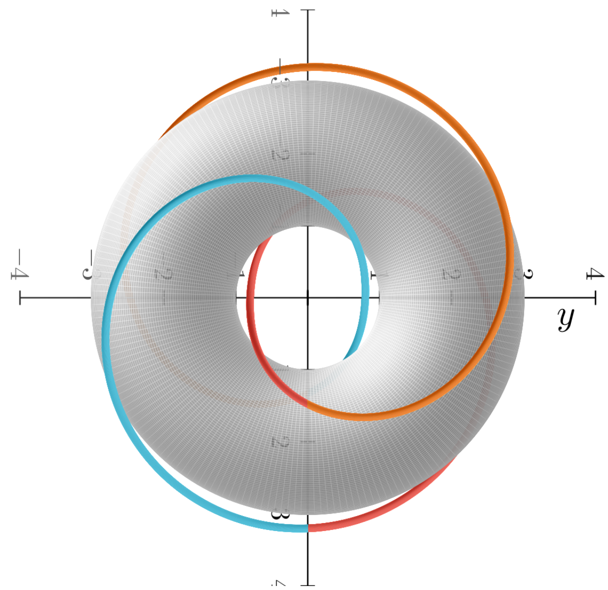



Figure 15: $(3, 2)$-right hand torus knot (view 2)

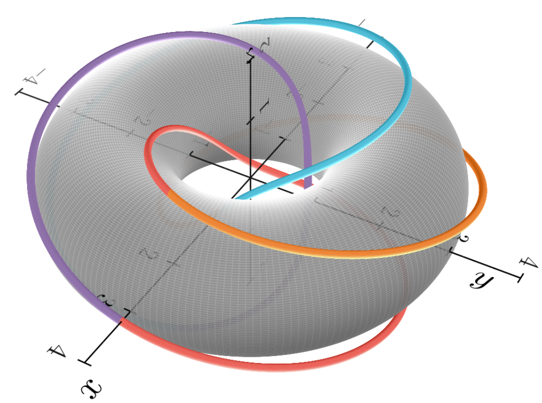



Figure 16: $(4, 3)$-right hand torus knot (view 1)

$4, z = 0$ containing $s_0$ and the knot, and $Q_2$ as a torus-shaped neighbour of the torus with central circle $x = y = 0$ containing $s_0$ and the knot.

(1) Deform $Q_1$ in the way illustrated in the figure, and repeat for $Q_2$.

(2) The deformed version of $Q_1$ is homeomorphic to a special cylinder, whose top circle is the $m^{\text{th}}$ power of $a$. A similar result holds for the deformed version of $Q_2$.

(3) Similar to the cases $\mathbb{T}^2, \mathbb{K}^2, (\mathbb{T}^2)^{\#g}$, we apply Seifert-Van Kampen theorem to show that the deformed version of $S \cup \{\infty\}$ has fundamental group $\langle a, b : a^m = b^n \rangle$.

□

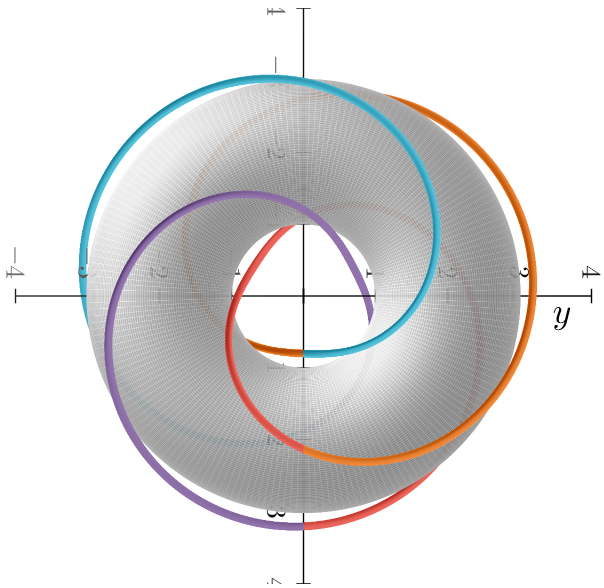



Figure 17: $(4,3)$-right hand torus knot (view 2)

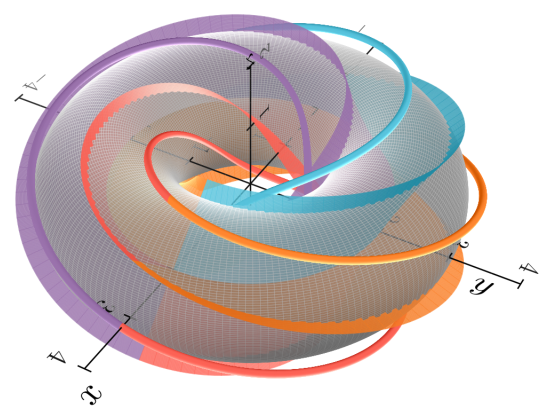



Figure 18: The deformed version of $Q_1$ as colored sheets (view 1)

**Proposition 4.27.** (**Nielsen-Schreier Theorem**)
Let $S$ be a set, $G$ be the free group generated by $S$, and $H \leq G$.
For some $T \subseteq H$, $H$ is the free group generated by $T$.

*Proof.* We may divide our proof into five steps.

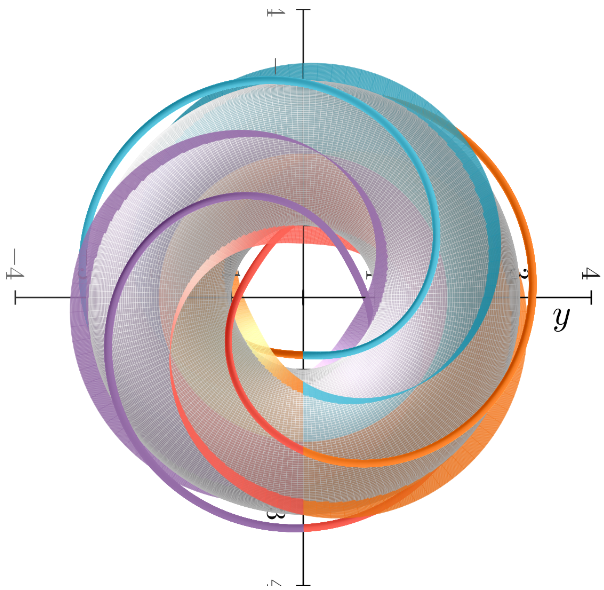



Figure 19: The deformed version of $Q_1$ as colored sheets (view 2)

(1) As $G$ is the free group generated by $S$, $G$ is the fundamental group of some graph $X$, such as the wedge product $\bigvee_{s \in S} \mathbb{S}$ of $(\mathbb{S})_{s \in S}$.

(2) As $X$ is a graph, $X$ is path connected, locally path connected and semi locally simply connected, so $X$ admits a universal covering map $Z$ covers $X$.

(3) As $Z$ covers $X$ is universal, it is normal, so we have the Galois correspondence:

$$\begin{tikzcd}
Z \arrow[d, "Z \text{ covers } Y"'] & \{id_Z\} \arrow[d, "\{id_Z\} \leq H"] \\
Y \arrow[d, "Y \text{ covers } X"'] \arrow[r, shift left, "H = \mathbf{Aut}(Z \text{ covers } Y)"] & H \arrow[d, "H \leq G"] \arrow[l, shift left, "Y = Z/H"] \\
X & G
\end{tikzcd}$$

(4) As $Y$ covers a graph $X$, $Y$ is locally homeomorphic to $X$, so $Y$ is a graph.

(5) As $Z$ covers $Y$ is universal, $H$ is the fundamental group of some graph $Y$, so for some $T \subseteq H$, $H$ is the free group generated by $T$.

$\square$

**Remark:** The analogous theorem fails for $R$-module.

# References


[1] H. Ren, "Template for math notes," 2021.

[2] A. Hatcher, *Algebraic Topology.* Cambridge: Cambridge University Press, 2002.

[3] H. Zheng, "Introduction to topology," 2024, lecture notes.