ASSIGNMENT I, ALGEBRA II, HKU, SPRING 2025

DUE AT 11:59PM ON FRIDAY FEBRUARY 20, 2025

(1) Let R and Q be any commutative rings and let $\phi : R \to Q$ be a ring homomorphism. Recall that for any ideal I of R, the ideal $\phi(I)Q$ of Q is called *the extension of I to Q (by ϕ)* and is denoted as $I^e = \phi(I)Q \subset Q$, and for any ideal J o Q, the ideal $\phi^{-1}(J)$ of R is called *the contraction of J in R (by ϕ)* and is denoted as $J^c = \phi^{-1}(J) \subset R$. Prove that the following statements hold:

1) For any ideal I of R, one has $I \subset (I^e)^c$;

2) For any ideal J of Q, one has $(J^c)^e \subset J$.

(2) Suppose that R is any commutative ring and $D \subset R \backslash \{0\}$ is multiplicatively closed. Let $D^{-1}R$ be the localization of R at D. Prove the following statements:

1) For any ideal J of $D^{-1}R$, one has $J = (J^c)^e$;

2) For any ideal I of R, one has $(I^e)^c = \{r \in R : dr \in I \text{ for some } d \in D\}$. Moreover, $I^e = D^{-1}R$ if and only if $I \cap D \neq \emptyset$.

3) Extension and contraction gives a bijection between prime ideals I of R such that $I \cap D = \emptyset$ and prime ideals of $D^{-1}R$.

(3) Is the ring $\mathbb{C}[x, y, z]/\langle z - 2\rangle$ is a Unique Factorization Domain? Explain your answer.

(4) Show that if R is a UFD, then the intersection of two principal ideals of R is again principal.

(5) Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$.

1) Prove that R is an integral domain and determine its units;

2) Show that the irreducible elements of R are $\pm p$, where $p \in \mathbb{Z}$ is prime, and $f(x) \in \mathbb{Q}[x]$ that are irreducible and have constant term ± 1;

3) Show that R is not a UFD.

(6) Let R be a UFD. Show that any non-zero $f \in R[x]$ can be decomposed as $f(x) = \gamma g(x)$, where $\gamma = \text{cont}(f)$, and $g(x) \in R[x]$ is primitive. Show that any other such product is of the form $f(x) = (\gamma u^{-1})ug(x)$, where $u \in R$ is a unit.

(7) Let R be a UFD and $F = \text{Frac}(R)$. Show that any non-zero $f(x) \in F[x]$ can be decomposed as $f(x) = \alpha g(x)$, where $\alpha \in F$ and $g(x) \in R[x]$ is primitive. Moreover, any other such product is of the form $f(x) = (\alpha u^{-1})ug(x)$, where $u \in R$ is a unit.

(8) Show that $f(x, y) = xy^3 + x^2y^2 - x^5y + x^2 + 1 \in \mathbb{R}[x, y]$ is irreducible.

(9) Show that $x^3 - 6x^2 + 4ix + 1 + 3i$ is irreducible in $R[x]$ where $R = \mathbb{Z}[\sqrt{-1}]$;

(10) Determine whether or not the polynomials are irreducible over $\mathbb{Q}$:

a) $f(x) = 2x^9 + 12x^4 + 36x^3 + 27x + 6$;

b) $f(x) = x^4 + 25x + 7$;

c) $f(x) = (x - 1)(x - 2) \cdots (x - n) - 1$ for each integer $n > 1$.