THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 11 Solution

Problem 1.

(i) We consider the product solution $u(x,t) = \phi(x)G(t)$.

Step 1 (Derive ODEs):

$$\phi(x)G''(t) = 4\phi''(x)G(t) - \phi(x)G(t) \Longrightarrow \frac{G''(t)}{G(t)} = \frac{4\phi''(x) - \phi(x)}{\phi(x)} = -\lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda - 1}{4}\phi = -\tilde{\lambda}\phi \quad \text{subject to} \quad \phi(0) = \phi(\pi) = 0. \tag{1}$$

if $\boxed{\tilde{\lambda} > 0}$, then

$$\begin{aligned} &\phi(x) = c_1 \cos\sqrt{\tilde{\lambda}}x + c_2 \sin\sqrt{\tilde{\lambda}}x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \Longrightarrow \phi(x) = c_2 \sin\sqrt{\tilde{\lambda}}x \\ \Longrightarrow \ &c_2 \sin(\sqrt{\tilde{\lambda}}\pi) = 0 \ (\because \phi(\pi) = 0) \Longrightarrow \sqrt{\tilde{\lambda}}\pi = n\pi \ (\because c_2 \neq 0 \text{ for nontrivial solutions}) \\ \Longrightarrow \ &\tilde{\lambda}_n = n^2 \Longrightarrow \lambda_n = 4n^2 + 1 \text{ for } n = 1, 2, \cdots \end{aligned}$$

with the eigenfunction $\phi_n(x) = c_2 \sin nx$ for $n = 1, 2, \cdots$.

If $\boxed{\tilde{\lambda} = 0}$, then

$$\begin{aligned} &\phi(x) = c_1 + c_2 x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \Longrightarrow \phi(x) = c_2 x \\ \Longrightarrow \ &c_2 = 0 \ (\because \phi(\pi) = 0) \Longrightarrow \phi \equiv 0. \end{aligned}$$

If $\boxed{\tilde{\lambda} < 0}$, then

$$\begin{aligned} &\phi(x) = c_1 \cosh\sqrt{-\tilde{\lambda}}x + c_2 \sinh\sqrt{-\tilde{\lambda}}x \Longrightarrow c_1 = 0 \ (\because \phi(0) = 0) \\ \Longrightarrow \ &\phi(x) = c_2 \sinh\sqrt{-\tilde{\lambda}}x \Longrightarrow c_2 = 0 \ (\because \sinh(\sqrt{-\tilde{\lambda}}\pi) > 0) \\ \Longrightarrow \ &\phi \equiv 0. \end{aligned}$$

Step 3 (Solve G): Consider

$$\frac{d^2G}{dt^2} = -\lambda_n G,$$

the general solution is $G_n(t) = c_1 \cos\sqrt{\lambda_n}t + c_2 \sin\sqrt{\lambda_n}t$. It follows from the condition $u(x,0) = 0$ that

$$G_n(0) = 0 \Longrightarrow c_1 = 0 \Longrightarrow G_n(t) = c_2 \sin\sqrt{\lambda_n}t = c_2 \sin\sqrt{4n^2+1}t.$$

Step 4 (Find the solution u): The product solutions $u_n(x,t) = \phi_n(x)G_n(t)$ are

$$\sin(\sqrt{4n^2+1}t)\sin nx \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{4n^2+1}t)\sin nx.$$

As $\dfrac{\partial u}{\partial t}(x,0) = 3\sin 3x$,

$$3\sin 3x = \sum_{n=1}^{\infty} A_n\sqrt{4n^2+1}\sin nx.$$

By linear independence,

$$A_n = \begin{cases} \frac{3}{\sqrt{4(3)^2+1}} = \frac{3}{\sqrt{37}} & \text{if } n = 3 \\ 0 & \text{if } n \neq 3 \end{cases}.$$

Thus, $u(x,t) = \frac{3}{\sqrt{37}}\sin(3x)\sin(\sqrt{37}t)$.

(ii) As $\lim\limits_{t\to\infty} u(\frac{\pi}{2}, t) = -\dfrac{3}{\sqrt{37}}\lim\limits_{t\to\infty}\sin(\sqrt{37}t)$ does not exist, $\lim\limits_{t\to\infty} u(x,t) \neq 0$.

Problem 2. We consider the product solution $u(r,\theta) = \phi(\theta)G(r)$.

Step 1 (Derive ODEs):

$$\Delta u = \frac{\phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{G}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \Longrightarrow \frac{r}{G}\frac{d}{dr}(r\frac{dG}{dr}) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\phi''(\theta) = -\lambda\phi(\theta) \text{ subject to } \phi'(0) = \phi\Big(\frac{\pi}{2}\Big) = 0,$$

if $\boxed{\lambda > 0}$, then

$$\begin{aligned}
&\phi(\theta) = c_1\cos\sqrt{\lambda}\theta + c_2\sin\sqrt{\lambda}\theta \Longrightarrow \phi'(\theta) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\theta + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\theta \\
\Longrightarrow\ & c_2\sqrt{\lambda} = 0\ (\because\phi'(0)=0)\ \Longrightarrow c_2 = 0 \Longrightarrow \phi(\theta) = c_1\cos\sqrt{\lambda}\theta \\
\Longrightarrow\ & c_1\cos(\frac{\sqrt{\lambda}\pi}{2}) = 0\ (\because\phi\Big(\frac{\pi}{2}\Big)=0)\ \Longrightarrow \frac{\sqrt{\lambda}\pi}{2} = \frac{2n-1}{2}\pi\ (\because c_1 \neq 0 \text{ for nontrivial solutions}) \\
\Longrightarrow\ & \lambda_n = (2n-1)^2 \text{ for } n = 1,2,\cdots
\end{aligned}$$

with the eigenfunction $\phi_n(\theta) = c_1\cos(2n-1)\theta$.

If $\boxed{\lambda = 0}$, then

$$\begin{aligned}
&\phi(\theta) = c_1 + c_2\theta \Longrightarrow \phi'(\theta) = c_2 \Longrightarrow c_2 = 0\ (\because\phi'(0)=0)\ \Longrightarrow \phi(\theta) = c_1 \\
\Longrightarrow\ & c_1 = 0\ (\because\phi\Big(\frac{\pi}{2}\Big)=0)\ \Longrightarrow \phi \equiv 0.
\end{aligned}$$

If $\boxed{\lambda < 0}$, then

$$\begin{aligned}
&\phi(\theta) = c_1\cosh\sqrt{-\lambda}\theta + c_2\sinh\sqrt{-\lambda}\theta \Longrightarrow \phi'(\theta) = c_1\sqrt{-\lambda}\sinh\sqrt{-\lambda}\theta + c_2\sqrt{-\lambda}\cosh\sqrt{-\lambda}\theta \\
\Longrightarrow\ & c_2\sqrt{-\lambda} = 0\ (\because\phi'(0)=0)\ \Longrightarrow c_2 = 0 \Longrightarrow \phi(\theta) = c_1\cosh\sqrt{-\lambda}\theta \\
\Longrightarrow\ & c_1\cosh(\frac{\sqrt{-\lambda}\pi}{2}) = 0\ (\because\phi\Big(\frac{\pi}{2}\Big)=0)\ \Longrightarrow c_1 = 0\ (\because\cosh(\frac{\sqrt{-\lambda}\pi}{2}) > 1) \\
\Longrightarrow\ & \phi \equiv 0.
\end{aligned}$$

Step 3 (Solve G): Consider

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda_n G = 0,$$

Let $G(r) = r^p$. Then

$$[p(p-1) + p - (2n-1)^2]r^p = 0 \implies p = \pm(2n-1).$$

So the general solution is $G_n(r) = c_1 r^{2n-1} + c_2 r^{1-2n}$, for $n = 1, 2, \cdots$.

It follows from the boundedness condition $\lim_{r\to 0} |u(r,\theta)| < \infty$ that

$$c_2 = 0 \Longrightarrow G(r) = c_1 r^{2n-1}.$$

Step 4 (Find the solution u): The product solutions $u_n(r,\theta) = \phi_n(\theta)G(r)$ are

$$r^{2n-1}\cos(2n-1)\theta \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n-1}\cos(2n-1)\theta.$$

As $u(1,\theta) = 4\cos\theta$,

$$4\cos\theta = \sum_{n=1}^{\infty} A_n \cos(2n-1)\theta.$$

By linear independence,

$$A_n = \begin{cases} 4 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

Thus, $u(r,\theta) = 4r\cos\theta$.

Problem 3.

(i) We consider the product solution $u(x,t) = \phi(x)G(t)$.

Step 1 (Derive ODEs):

$$\phi(x)G'(t) = 5\phi''(x)G(t) \Longrightarrow \frac{G'(t)}{G(t)} = \frac{5\phi''(x)}{\phi(x)} = -\lambda.$$

Step 2 (Determine the eigenvalues λ_n and the eigenfunctions ϕ_n):

Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\frac{\lambda}{5}\phi \quad \text{subject to} \quad \phi'(0) = \phi'(2) = 0. \tag{2}$$

if $\boxed{\lambda > 0}$, then

$$\begin{aligned}
& \phi(x) = c_1 \cos\sqrt{\frac{\lambda}{5}}x + c_2 \sin\sqrt{\frac{\lambda}{5}}x \\
\Longrightarrow\ & \phi'(x) = -c_1\sqrt{\frac{\lambda}{5}}\sin\sqrt{\frac{\lambda}{5}}x + c_2\sqrt{\frac{\lambda}{5}}\cos\sqrt{\frac{\lambda}{5}}x \\
\Longrightarrow\ & c_2\sqrt{\frac{\lambda}{5}} = 0 \ (\because \phi'(0) = 0) \ \Longrightarrow c_2 = 0 \\
\Longrightarrow\ & \phi'(x) = -c_1\sqrt{\frac{\lambda}{5}}\sin\sqrt{\frac{\lambda}{5}}x \Longrightarrow -c_1\sqrt{\frac{\lambda}{5}}\sin 2\sqrt{\frac{\lambda}{5}} = 0 \ (\because \phi'(2) = 0) \\
\Longrightarrow\ & 2\sqrt{\frac{\lambda}{5}} = n\pi \ (\because c_1 \neq 0 \text{ for nontrivial solutions}) \\
\Longrightarrow\ & \lambda_n = 5(\frac{n\pi}{2})^2 \text{ for } n = 1, 2, \cdots
\end{aligned}$$

with the eigenfunction $\phi_n(x) = c_1 \cos\frac{n\pi x}{2}$ for $n = 1, 2, \cdots$.

If $\boxed{\lambda = 0}$, then

$$\phi(x) = c_1 + c_2 x \Longrightarrow \phi'(x) = c_2 \Longrightarrow c_2 = 0 \ (\because \phi'(0) = \phi'(2) = 0) \ \Longrightarrow \phi \equiv c_1.$$

If $\boxed{\lambda < 0}$, then

$$\begin{aligned}
&\phi(x) = c_1 \cosh\sqrt{-\frac{\lambda}{5}}x + c_2 \sinh\sqrt{-\frac{\lambda}{5}}x \\
\Longrightarrow\quad &\phi'(x) = c_1\sqrt{-\frac{\lambda}{5}}\sinh\sqrt{-\frac{\lambda}{5}}x + c_2\sqrt{-\frac{\lambda}{5}}\cosh\sqrt{-\frac{\lambda}{5}}x \Longrightarrow c_2 = 0 \ (\because \phi'(0) = 0) \\
\Longrightarrow\quad &\phi'(x) = c_1\sqrt{-\frac{\lambda}{5}}\sinh\sqrt{-\frac{\lambda}{5}}x \Longrightarrow c_1\sqrt{-\frac{\lambda}{5}}\sinh 2\sqrt{-\frac{\lambda}{5}} = 0 \ (\because \phi'(2) = 0) \\
\Longrightarrow\quad &c_1 = 0 \ (\because \sinh 2\sqrt{-\frac{\lambda}{5}} > 0) \Longrightarrow \phi \equiv 0.
\end{aligned}$$

Step 3 (Solve G): Consider

$$\frac{dG}{dt} = -\lambda G,$$

the general solution is $G_n(t) = c$ (for $\lambda = 0$) or $G_n(t) = ce^{-\lambda_n t} = ce^{-\frac{5n^2\pi^2 t}{4}}$ (for $\lambda = \lambda_n$).

Step 4 (Find the solution u): The product solutions $u_n(x,t) = \phi_n(x)G_n(t)$ are

$$\text{a constant } A_0 \text{ and } e^{-\frac{5n^2\pi^2 t}{4}}\cos\frac{n\pi x}{2} \quad \text{for } n = 1, 2, \cdots.$$

Superposition yields

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{5n^2\pi^2 t}{4}}\cos\frac{n\pi x}{2}$$

and

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{2} = u(x,0) = \begin{cases} 0 & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}.$$

So by orthogonality,

$$2A_0 = \int_0^2 A_0\ dx = \int_1^2 1\ dx = 1 \Longrightarrow A_0 = \frac{1}{2}$$

$$A_n = \int_0^2 A_n \cos^2\frac{n\pi x}{2}\ dx = \int_1^2 \cos\frac{n\pi x}{2}\ dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}\bigg|_1^2 \Longrightarrow A_n = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

Thus, $u(x,t) = \dfrac{1}{2} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}e^{-\frac{5n^2\pi^2 t}{4}}\cos\frac{n\pi x}{2}.$

(ii) Note that

$$\lim_{t\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}e^{-\frac{5n^2\pi^2t}{4}}\cos\frac{n\pi x}{2}=\lim_{t\to\infty}e^{-\frac{5\pi^2t}{4}}f(t,x)=0$$

as $f(t,x):=\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}e^{-\frac{5(n^2-1)\pi^2t}{4}}\cos\frac{n\pi x}{2}$ is a bounded function of $t>1$ for a fixed x as for $t>1$,

$$|f(t,x)|\leq\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{5(n^2-1)\pi^2}{4}}\leq\sum_{n=1}^{\infty}e^{-\frac{5(n-1)\pi^2}{4}}<\infty.$$

Thus

$$\lim_{t\to\infty}u(x,t)=\frac{1}{2}\neq 0.$$