Algebra II Assignment 5
Due Friday 22nd April 2022

Please attempt all problems in this assignment and submit your answers (before midnight on Friday 22nd April 2022) by uploading your work to the Moodle page. If you have any questions, feel free to email me at adsg@hku.hk.

Problem 1 (Normal extensions)**.** (We did this in class. The purpose of this problem is for you to review this part of the material. You should try it first to get as far as you can before consulting the lecture/class notes).

1. State the definition of normal field extensions;
2. Show that a finite extension is normal if and only if it is a splitting field.

Problem 2. Factorise $x^9 - x$ and $x^{27} - x$ over $\mathbb{F}_3$ into irreducible factors.

Problem 3 (Splitting fields over finite fields)**.** Construct a splitting field L of $f(x) = x^5 + x + 1 \in \mathbb{F}_2[x]$. What is $|L\colon \mathbb{F}_2|$? How many elements does L have?

Problem 4. Let $\mathbb{F}_{11}$ be the field with 11 elements, and let $\mathbb{F}_{11}^*$ denote the group of units of $\mathbb{F}_{11}$.

1. Find all the generators of $\mathbb{F}_{11}^*$ as a multiplicative group.
2. Compute the product of all elements in $\mathbb{F}_{11}^*$.
3. State and prove a generalisation of 2. in the case where $\mathbb{F}_{11}$ is replaced by any finite field $\mathbb{F}_p$ for p prime.

Problem 5. Prove that in a finite field of even order, every element is a square.

Problem 6. For any field L, show that $\mathrm{Aut}(L) = \mathrm{Aut}_K(L)$, where K is the prime subfield of L.

Problem 7. Let $K \subset L$ be a finite Galois extension and let $G = \mathrm{Aut}_K(L)$. Prove that for any $\alpha \in L$, if $p(x) \in K[x]$ is the minimal polynomial of α over K, then $|G| \geq \deg(p)$.