1. *Sol.* The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is x^3-2. Therefore $\mathbb{Q}(\sqrt[3]{2})$ is a finite extension of $\mathbb{Q}$ of degree 3 and has a basis $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$. In other word, any $\alpha \in \mathbb{Q}(\sqrt[3]{2})$ can be expressed uniquely as a linear combination of $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$.

$$\begin{aligned} 1 =&(a + b\sqrt[3]{2} + c\sqrt[3]{4})(a_1 + b_1\sqrt[3]{2} + c_1\sqrt[3]{4}) \\ =&(aa_1 + 2bc_1 + 2cb_1) + (ab_1 + ba_1 + 2cc_1)\sqrt[3]{2} + (ac_1 + bb_1 + ca_1)\sqrt[3]{4}. \end{aligned}$$

So we have
$$\begin{cases} aa_1 + 2cb_1 + 2bc_1 = & 1, \\ ba_1 + ab_1 + 2cc_1 = & 0, \\ ca_1 + bb_1 + ac_1 = & 0. \end{cases}$$

Therefore
$$\begin{cases} a_1 & = \frac{1}{a^3+2b^3+4c^3}(a^2 - 2bc), \\ b_1 & = \frac{1}{a^3+2b^3+4c^3}(2c^2 - ab), \\ c_1 & = \frac{1}{a^3+2b^3+4c^3}(b^2 - ac). \end{cases}$$

2. (1) *Sol.* $\alpha^2 = 2+\sqrt{2}, \alpha^4 - 4\alpha^2 + 4 = 2$. So $f(\alpha) = 0$ where $f(x) = x^4 - 4x^2 + 2$. To show that it is minimal, we need to show that it has no factors. Since $f(\pm 1) \neq 0$ and $f(\pm 2) \neq 0$, f has no linear factors. Assume that $f(x) = (x^2 + bx + c)(x^2 + ex + f)$, then

$$b + e = 0,\ f + be + c = -4,\ bf + ce = 0,\ cf = 2.$$

Since c cannot equal to f, the first and third equation gives $b = e = 0$. So $f + c = -4$, which is impossible.

(2) *Sol.* Note that $\alpha\beta = \sqrt{2}$, so

$$\alpha = \sqrt{2 + \alpha\beta},\ \alpha^2 = 2 + \alpha\beta.$$

Thus we have $\beta = \frac{\alpha^2-1}{\alpha}$, which shows that $\beta \in \mathbb{Q}(\alpha)$. Note that $2 = -\alpha^4 + 4\alpha^2$, so $1/\alpha = -\alpha^3/2 + 2\alpha$, and

$$\beta = \frac{\alpha^3}{2} - \alpha.$$

3. (1) *Sol.* Let L be a K-vector space of dimension n. Consider the set $\{1, \alpha, ..., \alpha^{n-1}, \alpha^n\}$. Since it has $n+1 > n$ elements, it is linearly dependent, which means there

exists $a_0, a_1, ..., a_n \in K$, which are not all zero, such that

$$a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} + a_n\alpha^n = 0.$$

Let s be the biggest subscript such that $a_s \neq 0$, then

$$f(x) = a_s x^s + a_{s-1}x^s + \cdots + a_1 x + a_0$$

is a polynomial in $K[x]$ such that $f(\alpha) = 0$, which shows that a minimal polynomial exists.

(2) *Proof.* Let $p(x) = a_t x^t + a_{t-1}x^{t-1} + \cdots + a_0$. Then $p(\alpha) = p(\beta) = 0$. Subtract them, we have

$$\begin{aligned} 0 =& a_t(\alpha^t - \beta^t) + a_{t-1}(\alpha^{t-1} - \beta^{t-1}) + \cdots + a_1(\alpha - \beta) \\ =& (\alpha - \beta)\left(a_t \sum_{i=0}^{t-1} \alpha^i \beta^{t-1-i} + a_{t-1} \sum_{i=0}^{t-2} \alpha^i \beta^{t-2-i} + \cdots + a_1\right). \end{aligned}$$

Since $\alpha \neq \beta$, we have the formula in the latter bracket equals zero. Thus $\beta \notin K$, for otherwise by substituting α by x, the latter polynomial has degree $t-1$ and also has α as its root, which contradicts the hypothesis that $p(x)$ is the minimal polynomial.

Now assume to the contrary that $p(x)$ is not the minimal polynomial for β. Let $q(x)$, where $\deg(q) < t$, be the minimal polynomial. Then $q(x)|p(x)$. Write $p(x) = q(x)r(x)$. Since $p(\alpha) = 0$, at least one of $q(\alpha)$ and $r(\alpha)$ is zero. However, it contradicts that $p(x)$ is the minimal polynomial for α since both $q(x)$ and $r(x)$ has degree less then t. □

4. *Sol.* To find the degree of the extension $\mathbb{Q}(\alpha)$ is just to find the degree of the minimal polynomial of α over $\mathbb{Q}$.

(1) $\alpha^2 = 1+\sqrt{3}$, $\alpha^4 - 2\alpha^2 + 1 = 3$. So α is a root of $f(x) = x^4 - 2x^2 - 2$. We claim that $f(x)$ is the minimal polynomial. Since $f(\pm 1)$ and $f(\pm 2)$ are nonzero, $f(x)$ has no linear factors. Assume that $f(x) = (x^2 + bx + c)(x^2 + ex + f)$, then

$$b + e = 0,\ f + be + c = -2,\ bf + ce = 0,\ cf = -2.$$

Since c cannot equal to f, the first and third equation gives $b = e = 0$. So

$f + c = -2$, which is impossible. Therefore the degree is 4.

(2) $\alpha^2 = 3 - \sqrt{6}$, $\alpha^4 - 6\alpha^2 + 9 = 6$. So α is a root of $f(x) = x^4 - 6x^2 + 3$. We claim that $f(x)$ is the minimal polynomial. Since $f(\pm 1)$, $f(\pm 2)$, $f(\pm 6)$ and $f(\pm 2)$ are nonzero, $f(x)$ has no linear factors. Assume that $f(x) = (x^2 + bx + c)(x^2 + ex + f)$, then

$$b + e = 0,\ f + be + c = -6,\ bf + ce = 0,\ cf = 3.$$

Since c cannot equal to f, the first and third equation gives $b = e = 0$. So $f + c = -6$, which is impossible. Therefore the degree is 4.

(c) $\alpha = \sqrt{(1+\sqrt{2})^2} = 1 + \sqrt{2}$, $\alpha^2 - 2\alpha + 1 = 2$. So α is a root of $f(x) = x^2 - 2x - 1$. We claim that $f(x)$ is the minimal polynomial. Since $f(\pm 1)$ are nonzero, $f(x)$ has no linear factors, which means our claim holds. Therefore the degree is 2.

5. *Sol.* $\mathbb{Q}(\sqrt{p})$ is a $\mathbb{Q}$-vector space generated by the basis $\{1, \sqrt{p}\}$. $\mathbb{Q}(\sqrt[3]{q})$ is a $\mathbb{Q}$-vector space generated by the basis $\{1, \sqrt[3]{q}, \sqrt[3]{q^2}\}$. So in order that $\mathbb{Q}(\sqrt{p})$ is a subspace of $\mathbb{Q}(\sqrt[3]{q})$, it suffices to show that $\sqrt{p}$ is a linear combination of $\{1, \sqrt[3]{q}, \sqrt[3]{q^2}\}$. Assume that there are $a, b, c \in \mathbb{Q}$ such that

$$\sqrt{p} = a + b\sqrt[3]{q} + c\sqrt[3]{q^2}.$$

Then

$$p = (a^2 + 2bcq) + (c^2q + 2ab)\sqrt[3]{q} + (b^2 + 2ac)\sqrt[3]{q^2}.$$

So

$$\begin{cases} p &= a^2 + 2bcq, \\ 0 &= c^2q + 2ab, \\ 0 &= b^2 + 2ac. \end{cases}$$

If $a \neq 0$, then $c = -b^2/2a$. Plug it into the second equation, we have $0 = b^4q/4a^2 + 2ab$, $b^3q + 8a^3 = 0$. Thus $b = 0$, for otherwise $\sqrt[3]{q} = -2a/b$ is rational, which is impossible. So $a = 0$, contrary to $a \leq 0$. So $a = 0$, $b = 0$, and $p = 0$. Above all, no such p, q exist.

6. (a) *Sol.* Since all the prime numbers in $R = \mathbb{Z}[i]$ are those p such that $N(p)$ is a prime number or is the square of a $4k + 3$ prime number, $1 + i$ is prime. Since $1 + i$ can divide $6, 4$ and $1 + 3i$ $(1 + 3i = (1 + i)(1 + 2i))$ but $1 + i \nmid 1$

and $(1+i)^2 \nmid 1+3i$, by Eisenstein's criterion, f is irreducible over R.

(b) *Sol.* Write $\mathbb{Q}(i, \alpha_1, \alpha_2, \alpha_3)$ as $\mathbb{Q}(i)(\alpha_1, \alpha_2, \alpha_3)$. We have the relation

$$\mathbb{Q} \subset \mathbb{Q}(i) \subset \mathbb{Q}(i)(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(i, \alpha_1, \alpha_2, \alpha_3).$$

We know that $\mathbb{Q}(i)$ is of degree 2 (i is the root of x^2+1). To show that $\mathbb{Q}(i, \alpha_1, \alpha_2, \alpha_3)$ has degree 6, it suffices to show that $\mathbb{Q}(i)(\alpha_1, \alpha_2, \alpha_3)$ has degree 3 as a $\mathbb{Q}(i)$-vector space, which is equivalent to find a degree 3 minimal polynomial of $\alpha_1, \alpha_2, \alpha_3$ over $\mathbb{Q}[i]$. We claim that $f(x)$ is exactly the desired minimal polynomial. To see that, note that $\mathbb{Z}[i]$ is a unique factorization domain, so Gauss' Lemma tells us that f is irreducible in $\mathbb{Q}[i]$ if and only if the principal part of f, which is exactly f itself, is irreducible in $\mathbb{Z}[i]$, which is proved in 1. Since $\deg(f) = 3$, we conclude that $\mathbb{Q}(i, \alpha_1, \alpha_2, \alpha_3)$ has degree 6.