Algebra II: Tutorial 2

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Problem 1. Let R be an integral domain. Show that if $R[x]$ is a UFD, then R is again a UFD.

Problem 2 (Relation between irreducibility and existence of roots)**.** Let K be any field, and consider the polynomial ring $R = K[x]$.

1. Suppose that $f \in R$ has degree 2 or 3. Show that f is irreducible over K if and only if f has no roots in K.
2. Show that the statement in part 1. no longer holds if we assume that f has degree ≥ 4.
3. Show that the statement in part 1. is no longer true if we replace K by an arbitrary integral domain.

Problem 3 (Irreducibility tests over $\mathbb{Q}$)**.** Determine whether or not the following polynomials are irreducible over $\mathbb{Q}$:

1. $f(x) = x^3 + 5x^2 + 4$,
2. $f(x) = x^4 - 10x^2 + 1$.

Problem 4 (Universal property of polynomial rings)**.** Let K be a commutative ring, and L a ring containing K as a subring. Consider the polynomial ring $K[x]$.

1. For each $\alpha \in L$, show that there exists a unique ring homomorphism

$$ev_\alpha : K[x] \to L,$$

   satisfying the following two conditions:

   (a) $ev_\alpha(k) = k$, for all $k \in K$, and
   (b) $ev_\alpha(x) = \alpha$.

   For each $\alpha \in L$, we call that homomorphism the *evaluation homomorphism at* α.

2. Suppose now that K is an infinite field, and $L = K$. Show that $f \in K[x]$ and $g \in K[x]$ are equal if and only if the evaluations $ev_\alpha(f)$ and $ev_\alpha(g)$ are equal in K for *all* elements $\alpha \in K$.

3. Show that this property does not hold if K is a finite field. (Hint: find two distinct polynomials f and g over $\mathbb{Z}_3$ whose values $f(\alpha)$ and $g(\alpha)$ coincide $\forall \alpha \in \mathbb{Z}_3$.)