Chapter 1. Basics

MATH4406 Introduction to Partial Differential Equations

The University of Hong Kong

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This chapter is related to the materials in Section 1.1, 1.3 - 1.6 of the Textbook.

1.1 What are Partial Differential Equations (PDE)?

What is a Partial Differential Equation (PDE)?

Question

What is a Partial Differential Equation (PDE)?

Answer

An equation consists of partial derivatives.

Definition (Scalar PDE)

An expression (of the form)

$$F(\partial_{x_1}^k u, \partial_{x_1}^{k-1}\partial_{x_2} u, \cdots, \partial_{x_d}^k u, \cdots, \partial_{x_1} u, \partial_{x_2} u, \cdots, \partial_{x_d} u, u, x) = 0, \quad \text{(PDE)}$$

for all $x \in \Omega \subseteq \mathbb{R}^d$, is called a k-th order PDE where

$$F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \cdots \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given, and $u : \Omega \to \mathbb{R}$ is the unknown.

Examples of PDE

Example

1. $u_t + cu_x = 0$ (transport equation)
2. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
3. $u_t + uu_x = 0$ (Burgers' equation)
4. $u_t - u_{xx} = f(u)$ (reaction-diffusion equation)

Remark (Notation)

Throughout this course, we may use different "standard" notations to represent partial derivatives. For example, the partial derivative of u with respect to x can be written as

$$\frac{\partial u}{\partial x}, \quad \partial_x u, \quad \text{or} \quad u_x.$$

They are the **SAME**!!

Linearity

Definition (Linearity)

1. A differential operator[a] $\mathcal{L}$ is *linear* if

$$\mathcal{L}(\alpha u + \beta v) = \alpha \mathcal{L}u + \beta \mathcal{L}v$$

   where α and β are constants.

2. A PDE is *linear* if it is of the form

$$\mathcal{L}u = f$$

   where the differential operator $\mathcal{L}$ is linear, and f is a given function.

3. A linear PDE is *homogeneous* if $f \equiv 0$. That is,

$$\mathcal{L}u = 0.$$

[a] See `https://en.wikipedia.org/wiki/Differential_operator` for instance.

Are These Examples Linear?

Example (Continued from Last Example)

1. $u_t + cu_x = 0$.
   Write $\mathcal{L}u := \partial_t u + c\partial_x u$. Then
   $$\mathcal{L} \text{ is linear and } \mathcal{L}u = 0.$$
   Hence, $u_t + cu_x = 0$ is linear and homogeneous.
2. $\partial_{xx} u + \partial_{yy} u = 0$ is also linear and homogeneous.
3. $u_t + uu_x = 0$ is NOT linear.
4. $u_t - u_{xx} = f(u)$ is linear if and only if
   $$f(u) = a(t, x)u + b(t, x)$$
   for some given functions a and b.

Observations on Linearity

Facts: (Linearity)

- A PDE is linear if and only if it has the form

$$\sum_{\alpha_1+\cdots+\alpha_d\leq k} a_{\alpha_1,\alpha_2,\cdots,\alpha_d}(x)\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}u=f(x)$$

  where f, $a_{\alpha_1,\alpha_2,\cdots,\alpha_d}$'s are given functions.
- The PDE is homogeneous if and only if $f\equiv 0$.
- The PDE is constant-coefficient if $a_{\alpha_1,\cdots,\alpha_d}$'s are constants.

Remark

Using these facts, we can easily see the requirement $f(u)=au+b$ in Example 4.

Examples for Solving PDE

Example (Example 1 on Page 3 of Textbook)

Question: Let $u := u(x, y)$ satisfy $\partial_{xx} u = 0$. What is u?
Solution: Since $\partial_{xx} u = 0$, we know that $\partial_x u$ is independent of x, namely

$$\partial_x u = f(y)$$

for some arbitrary function f. A direct integration yields

$$u = f(y)x + g(y),$$

for some arbitrary function g.

Moral

An anti-derivative of x should have an <u>arbitrary</u> function of y as an "integration constant".

Examples for Solving PDE

Example (Example 2 on Page 3 of Textbook)

Question: Let $u := u(x, y)$ satisfy $\partial_{xx} u = -u$. Find u.
Solution: One may check that

$$u(x, y) = A(y) \cos x + B(y) \sin x,$$

where A and B are arbitrary functions.

Main Idea

If $u := u(x)$, then

$$u(x) = A \cos x + B \sin x,$$

for some constants A and B.
Thus, if $u := u(x, y)$, then

both A and B should depend on y as well.

Examples for Solving PDE

Example (Example 3 on Page 3 of Textbook)

Question: Solve $u_{xy} = 0$.

Solution: Integrating $u_{xy} = 0$ with respect to y, we have

$$u_x = g(x),$$

for any arbitrary g.

Integrating the above equation with respect to x, we obtain

$$u = F(y) + G(x),$$

where F and G are arbitrary functions. Here, $G' = g$.

How to Determine Arbitrary Functions?

Moral

Solutions to PDE may have arbitrary function(s).

Qustion

How to determine these arbitrary function(s)?

Answer

Use Initial Condition(s) (IC) and/or Boundary Condition(s) (BC).

Exercise (Not in the Textbook)

Solve

$$\begin{cases} \partial_x u = y \\ u|_{x=2} = y^2 + y. \end{cases}$$

1.2 Formal Derivations of Partial Differential Equations (PDE)

Transport Equation

Aim

Try to model the following physical phenomenon.

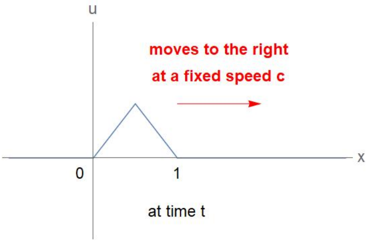


Figure: Transportation to the Right at a Fixed Speed c.

Discussion

Which equation can describe this transport phenomenon?

Vibrating Membrane (in Two Spatial Dimensions)

Discussion

Find a model to describe the vibrations of a drumhead.

Assumptions

- No horizontal movement; and
- the vibration is small.

A useful mathematical tool:

Theorem (Second Vanishing Theorem; see Appendix A.1 of Textbook)

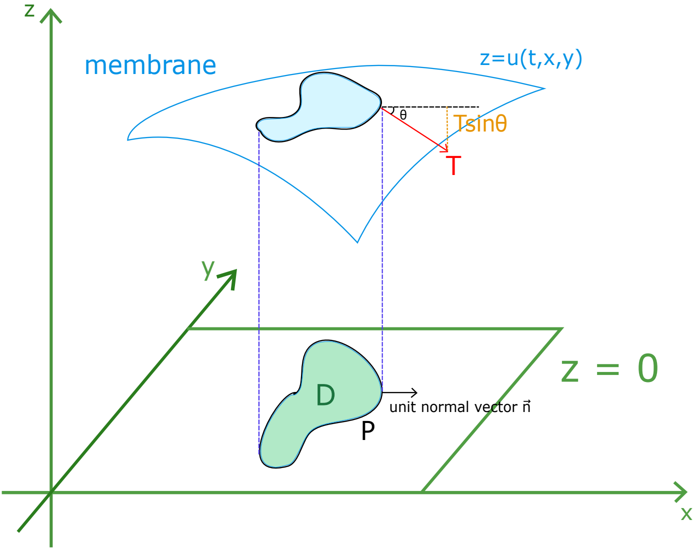
Let $\Omega \subseteq \mathbb{R}^d$ be open, and $f : \Omega \to \mathbb{R}$ be a continuous function such that for any subset $D \subseteq \Omega$,

$$\int_D f(x)\, dx = 0.$$

Then

$$f \equiv 0 \quad \text{on } \Omega.$$

Vibrating Membrane in Two Spatial Dimensions



Formal Derivation

For simplicity, let us consider the situation without gravity or any other external force. In this case,

$$\text{Force acting at } P = T := \text{tension}.$$

Due to the assumption that the horizontal movement is neglectable, we only focus on the vertical forces. In particular,

$$\text{Vertical Force acting at } P = T \sin\theta.$$

Under the hypothesis that $\theta \approx 0$,

$$\begin{aligned}\sin\theta &= \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \cdots \approx \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \cdots = \tan\theta \\ &= \text{Slope at } P \text{ in the direction } n = \frac{\partial u}{\partial n}.\end{aligned}$$

Thus,

$$\text{Vertical Force acting at } P = T\sin\theta \approx T\frac{\partial u}{\partial n}.$$

Hence, using the identity $\dfrac{\partial u}{\partial n} = n \cdot \nabla u$ and the divergence theorem, we have

$$\text{Total External Force acting on } D = \oint_{\partial D} T\frac{\partial u}{\partial n}\, d\sigma$$

$$= \oint_{\partial D} n \cdot (T\nabla u)\ d\sigma = \iint_D \nabla \cdot (T\nabla u)\, dxdy.$$

According to the Newton's Second Law of Motion (in the vertical direction),

$$\iint_D \nabla \cdot (T\nabla u)\, dxdy = \iint_D \rho\partial_{tt} u\, dxdy,$$

where ρ is the density (of the membrane). Since D is arbitrary, it follows from the Second Vanishing Theorem (see Appendix A.1 of Textbook for instance) that

$$\rho\partial_{tt} u = \nabla \cdot (T\nabla u).$$

Membrane with Uniform Material

In addition, if T and ρ are constants, then

$$\rho \partial_{tt} u = \nabla \cdot (T \nabla u)$$

becomes

$$\partial_{tt} u = \frac{T}{\rho} \nabla \cdot \nabla u = c^2 \Delta u,$$

where the wave speed $c := \sqrt{\frac{T}{\rho}}$. This is the *wave equation* in two spatial dimensions.

Remark

It follows from the physical meanings of tension T and density ρ that

$$T \geq 0 \quad \text{and} \quad \rho > 0,$$

so the wave speed $c := \sqrt{\frac{T}{\rho}}$ is always a non-negative number.

Remarks on Wave Equations

Remark 1

If there is any other force, then the wave equation becomes

$$\partial_{tt} u = c^2 \Delta u + f,$$

where f may be:

- gravity $-g$,
- damping $-\alpha \partial_t u$,
- restoring force $-\beta u$, etc.

or their combinations.

Remark on Remark 1

One may also consider a nonlinear forcing term f.

Remark 2

The equation $\partial_{tt} u = c^2 \Delta u$ works for other spatial dimensions:

$$\begin{array}{ll} \text{1-D} & \partial_{tt} u = c^2 \partial_{xx} u \\ \text{2-D} & \partial_{tt} u = c^2 \left(\partial_{xx} u + \partial_{yy} u\right) \\ \text{3-D} & \partial_{tt} u = c^2 \left(\partial_{xx} u + \partial_{yy} u + \partial_{zz} u\right) \\ \vdots & \vdots \\ \text{d-D} & \partial_{tt} u = c^2 \sum_{i=1}^{d} \partial_{x_i x_i} u. \end{array}$$

Notation

In the literature, we also write ∂_t^2 to represent ∂_{tt}. Therefore, one may write the two-dimensional wave equation as follows:

$$\partial_t^2 u = c^2 \left(\partial_x^2 u + \partial_y^2 u\right).$$

Question

How many initial condition(s) (IC) should we impose for the wave equations?

Answer

Since

$$\#(\text{IC we needed}) = \text{order of time-derivative} = 2,$$

we should actually impose 2 IC: e.g.,

$$u|_{t=0} = \text{initial position}$$
$$\partial_t u|_{t=0} = \text{initial velocity}$$

Moral

We also need these two initial conditions for solving Newton's Second Law (of Motion) in the ODE theory.

Diffusion/Heat Equation in Three Spatial Dimensions

Aim

Let u be the density/concentration of an underlying chemical. Derive a PDE that describes the evolution of u.

According to the definition of density/concentration, we have

Total Mass

For any open set $D \subseteq \mathbb{R}^3$,

$$(\text{Total Mass in } D) = \iiint_D u \, dxdydz.$$

Hint

How can we apply the *Local Conservation Law* to derive a PDE for u?

Local Conservation Law

Local Conservation Law

Rate of change of the underlying quantity in time $=$ The underlying quantity flowing across boundaries per unit time $+$ The underlying quantity generated inside the domain per unit time.

Local Conservation Law (in Mathematical Symbols)

Let u be the underlying quantity (i.e., chemical concentration in our case).

$$\frac{d}{dt}\iiint_D u \, dxdydz = -\oiint_{\partial D} F \cdot n \, d\sigma + \iiint_D Q \, dxdydz, \qquad \text{(LCL)}$$

where F is the *flux*, and Q is the *chemical produced per unit time per unit volume*.

Diffusion Flux

Question

What is the relationship between the flux F and the concentration u?

Fick's Law of Diffusion (derived by Adolf Fick in 1855)

The (diffusion) flux F is directly proportional to the concentration gradient (i.e., ∇u), namely

$$F = -k\nabla u, \tag{Fick}$$

for some constant $k > 0$.

Moral

The negative sign in (Fick) represents the fact that the chemical moves from high concentration to low concentration.

How to Obtain an Equation for u?

Applying the divergence theorem

$$\oiint_{\partial D} F \cdot n \, d\sigma = \iiint_D \nabla \cdot F \, dxdydz$$

and Fick's Law (Fick) to the Local Conservation Law (LCL), we have

$$\begin{aligned}
\frac{d}{dt} \iiint_D u \, dxdydz &= -\oiint_{\partial D} F \cdot n \, d\sigma + \iiint_D Q \, dxdydz \\
&= -\iiint_D \nabla \cdot F \, dxdydz + \iiint_D Q \, dxdydz \\
&= -\iiint_D \nabla \cdot (-k\nabla u) \, dxdydz + \iiint_D Q \, dxdydz \\
&= \iiint_D k\Delta u \, dxdydz + \iiint_D Q \, dxdydz,
\end{aligned}$$

since $\nabla \cdot \nabla u = \Delta u$.

On the other hand,

$$\frac{d}{dt}\iiint_D u \; dxdydz = \iiint_D \partial_t u \; dxdydz,$$

and hence, a re-arrangement yields

$$\iiint_D \partial_t u \; dxdydz = \iiint_D k\Delta u + Q \; dxdydz,$$

for any subset $D \subseteq \mathbb{R}^3$. Applying

Theorem (Second Vanishing Theorem; see Appendix A.1 of Textbook)

Let Ω be open, and f be continuous. Then

$$\int_D f(x)\, dx = 0, \; \forall D \subseteq \Omega \quad \Longrightarrow \quad f \equiv 0 \quad on\ \Omega.$$

again, we obtain the *diffusion equation*

$$\boxed{\partial_t u = k\Delta u + Q.}$$

Dynamic Equilibrium

At the dynamic equilibrium, u is independent of t, so $\partial_t u \equiv 0$. Hence, the diffusion/heat equation

$$\partial_t u = k\Delta u + Q$$

becomes

$$0 = k\Delta u + Q.$$

That is,

$$\boxed{-\Delta u = \frac{Q}{k}} \qquad \text{(Poisson equation).}$$

When $Q \equiv 0$, we have

$$\boxed{-\Delta u = 0} \qquad \text{(Laplace's equation).}$$

Remark

The two-dimensional Laplace's equation $\partial_{xx} u + \partial_{yy} u = 0$ plays an important role in the theory of functions of a single complex variable.

1.3 Initial and Boundary Conditions

Initial Condition(s)

Initial Condition(s) (IC):

$$IC = \textit{data given at a particular time } t = t_0.$$

Example

For example,

$$\begin{aligned} u\big|_{t=0} &= g, \\ \partial_t u\big|_{t=0} &= h. \end{aligned}$$

Moral

Usually (but NOT always),

$$(\text{Number of (IC) we needed}) = \left(\underline{\text{highest}} \text{ order of time-derivative}\right).$$

Boundary Condition(s)

Boundary Condition(s) (BC):

BC = data given at a (spatial) location.

Example (Three Typical Types of BC:)

Let Γ be a boundary of a spatial domain Ω, and g be a given function.

BC	Name	Remark
$u\|_{\Gamma} = g$	Dirichlet (or 1st type) BC	$\alpha = 0$ in Robin BC
$\left.\dfrac{\partial u}{\partial n}\right\|_{\Gamma} = g$	Neumann (or 2nd type) BC	$\beta = 0$ in Robin BC
$\left.\alpha\dfrac{\partial u}{\partial n} + \beta u\right\|_{\Gamma} = g$	Robin (or 3rd type) BC	α, β are constants

Some Remarks

Remark

When $g \equiv 0$, the BC is homogeneous; otherwise, the BC is non-homogeneous.

Remark

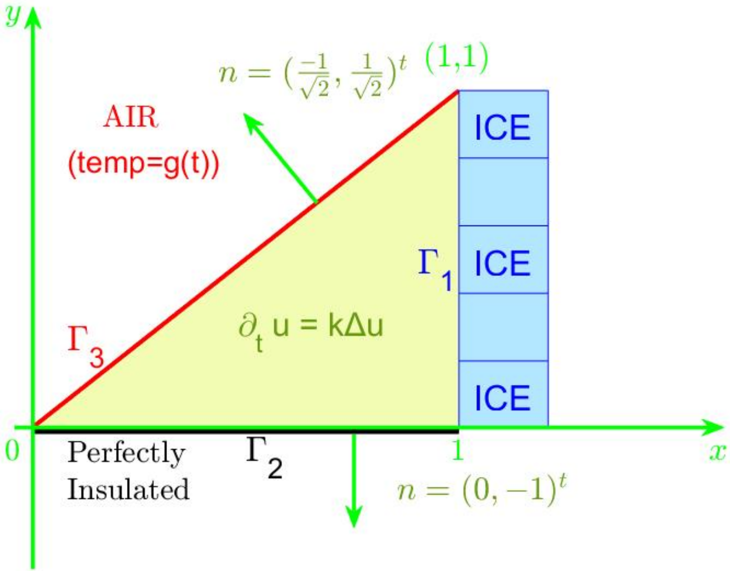
Different types of BC can be imposed on different parts of the boundary.

Example (Heat Transfer with Three Different Types of BC)

The heat equation in two spatial dimensions with three different types of BC; see the next slide.

Remark

The shape of the physical domain is NOT important, one may also impose similar BC in other domains.



y
$n = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^t$
(1,1)
AIR
(temp=g(t))
ICE
Γ_1
ICE
Γ_3
$\partial_t u = k\Delta u$
ICE
0
Perfectly
Insulated
Γ_2
1
x
$n = (0, -1)^t$

Background Information

For the heat equation $\partial_t u = k\Delta u$, the unknown $u(t,x,y)$ stands for the temperature at the location (x,y) at a particular time t.

On the boundary Γ_1, since the ice is *always* 0 degree Celsius, we impose the Dirichlet BC (namely prescribed temperature):

$$u|_{\Gamma_1} \equiv 0,$$

or equivalently,

$$u(t,1,y) \equiv 0.$$

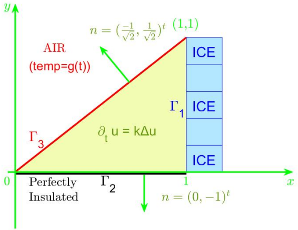


Figure: Heat Transfer with Three Different Types of BC

Heat Transfer with Three Different Types of BC (for Γ_2)

Fourier's Law of Thermal Conduction

$$(\text{Heat Flux}) \propto -\frac{\partial u}{\partial n}.$$

On the boundary Γ_2, since it is perfectly insulated, no heat energy can transfer across Γ_2. Therefore, by Fourier's law, we impose the Neumann BC (namely prescribed flux):

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} \equiv 0,$$

or equivalently,

$$\partial_y u(t,x,0) \equiv 0.$$

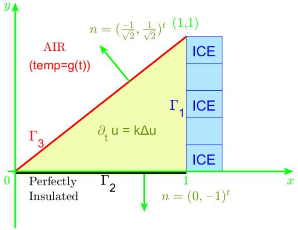


Figure: Heat Transfer with Three Different Types of BC

Heat Transfer with Three Different Types of BC (for Γ_3)

Newton's Law of Cooling

$$(\text{Heat Flux}) \propto (u - g).$$

On the boundary Γ_3, the heat is losing to the surrounding. Due to the convection, we impose the Robin BC (namely Newton's law of cooling):

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_3} \equiv -a(u-g)|_{\Gamma_3},$$

namely for any $t > 0$ and $0 < \eta < 1$,

$$-\frac{1}{\sqrt{2}}\partial_x u(t,\eta,\eta) + \frac{1}{\sqrt{2}}\partial_y u(t,\eta,\eta) = -a\left(u(t,\eta,\eta) - g(t)\right).$$

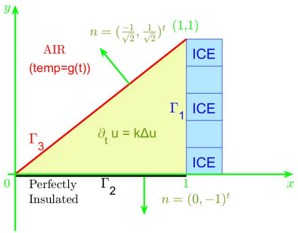


Figure: Heat Transfer with Three Different Types of BC

Periodic BC

Example (Heat Equation in a Circle (NOT in Section 1.4 of the Textbook))

Let $u := u(t,x)$ be the temperature in a huge circular ring. Denote by x the angle of the ring. Then the heat equation (in one spatial dimension) is

$$\partial_t u = k\partial_{xx} u,$$

for $t \geq 0$ and $x \in [0, 2\pi]$.
Physically, the angle $x = 0$ and $x = 2\pi$ represent the SAME point, so we have the following BC:

$$\boxed{\begin{aligned} u(t,0) &= u(t,2\pi) \\ \partial_x u(t,0) &= \partial_x u(t,2\pi). \end{aligned}}$$

The huge circular ring is depicted on the next slide.

Huge Circular Ring

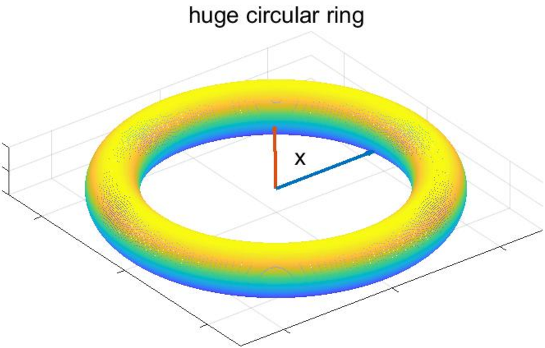


Figure: A Perfectly Insulated and Huge Circular Ring with Uniform Material

1.4 Well-posedness

Well-posedness

Question

What is a well-posed problem?

Answer

Problem = PDE + set of conditions (e.g., IC, BC, etc.)

Well-posedness = 3 properties as follows:

- Existence – has a solution,
- Uniqueness – at most one solution,
- Stability – nearby data $\implies$ nearby solutions.

Question

Why is well-posedness important?

Answer

A good physical model should be a well-posed problem.

Neumann Problem for Poisson's Equation

Consider

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subseteq \mathbb{R}^2 \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

Question

Do we have existence and uniqueness?

Answer (Uniqueness)

No uniqueness, because $u + C$ is also a solution if u is.

Answer (Existence)

The existence requires the following compatibility condition:

$$\boxed{\iint_\Omega f \, dxdy = -\oint_{\partial\Omega} g \, d\sigma.}$$

Compatibility Condition

Let u be a solution to the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subseteq \mathbb{R}^2 \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

Then integrating $-\Delta u = f$ over Ω, applying the divergence theorem, and using BC, we have

$$\begin{aligned} \iint_\Omega f \, dxdy &= -\iint_\Omega \Delta u \, dxdy = -\iint_\Omega \nabla \cdot \nabla u \, dxdy \\ &= -\oint_{\partial\Omega} n \cdot \nabla u \, d\sigma = -\oint_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma = -\oint_{\partial\Omega} g \, d\sigma. \end{aligned}$$

Moral

The compatibility condition $\boxed{\iint_\Omega f \, dxdy = -\oint_{\partial\Omega} g \, d\sigma}$ is a necessary condition for solving the Neumann problem.

1.5 Classification of Second-Order PDE

Classification of Second-Order PDE

Aim of This Section

Classify the **LINEAR** second-order equations.

Consider

$$a_{11}\partial_{xx}u + 2a_{12}\partial_{xy}u + a_{22}\partial_{yy}u + b_1\partial_x u + b_2\partial_y u + cu = f, \quad \text{(2DLPDE)}$$

where a_{11}, a_{12}, a_{22}, b_1, b_2, c, and f are given functions of x and y.

Remark

All of a_{11}, a_{12}, a_{22}, b_1, b_2, c, and f are independent of u.

Remark (Notation in Section 1.6 of Textbook)

Equation (2DLPDE) is the SAME as Equation (1) in Section 1.6 of Textbook, if we identity the coefficients as follows:

$$b_1 := a_1, \quad b_2 := a_2, \quad \text{and} \quad c := a_0.$$

Definition (Discriminant)

Define the discriminant $\mathcal{D} := a_{12}^2 - a_{11}a_{22}$. Then we say that (2DLPDE) is

- elliptic if $\mathcal{D} < 0$,
- parabolic if $\mathcal{D} = 0$, or
- hyperbolic if $\mathcal{D} > 0$.

Example (Laplace's Equation)

Consider Laplace's equation

$$\partial_{xx}u + \partial_{yy}u = 0.$$

In this case, $a_{11} = 1$, $a_{12} = 0$, and $a_{22} = 1$, so the discriminant

$$\mathcal{D} = 0^2 - (1)(1) = -1 < 0.$$

Thus, Laplace's equation $\partial_{xx}u + \partial_{yy}u = 0$ is *elliptic*.

Example (Constant Coefficients)

Consider

$$2\partial_{xx}u + 4\partial_{xy}u - 5\partial_{yy}u + 6\partial_x u = 0.$$

In this case, $a_{11} = 2$, $a_{12} = 4/2 = 2$, and $a_{22} = -5$, so the discriminant

$$\mathcal{D} = a_{12}^2 - a_{11}a_{22} = 2^2 - (2)(-5) = 14 > 0.$$

Thus, the equation $2\partial_{xx}u + 4\partial_{xy}u - 5\partial_{yy}u + 6\partial_x u = 0$ is *hyperbolic*.

Remark

In the last two examples, all of the coefficients (namely a_{11}, a_{12}, a_{22}, b_1, b_2, and c) are constant, so we call them as linear PDE with constant coefficients.

Example (Variable Coefficients)

Consider

$$x^2\partial_{xx}u - 2\partial_{xy}u + y^2\partial_{yy}u = 0.$$

In this case, $a_{11} = x^2$, $a_{12} = -2/2 = -1$, and $a_{22} = y^2$, so the discriminant

$$\mathcal{D} = a_{12}^2 - a_{11}a_{22} = (-1)^2 - (x^2)(y^2) = 1 - x^2y^2.$$

Thus, the equation is

- *elliptic* if $x^2y^2 > 1$.
- *hyperbolic* if $x^2y^2 < 1$.

Remark

$$\mathcal{D} = 0 \text{ if and only if } xy = \pm 1.$$

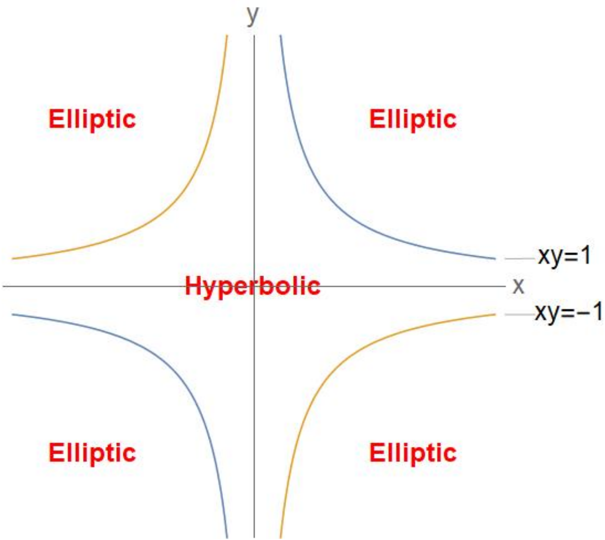


Figure: Regions for $x^2\partial_{xx}u - 2\partial_{xy}u + y^2\partial_{yy}u = 0$.

Classification of 2^{nd}-Order PDE with Constant Coefficients

Theorem (Constant Coefficient Case)

Let a_{11}, a_{12} and a_{22} be constants. Then there exists a 2×2 matrix B such that the linear transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} := B \begin{pmatrix} x \\ y \end{pmatrix}$$

converts (2DLPDE) *to*

$$\begin{cases} \partial_{\xi\xi} u + \partial_{\eta\eta} u + \cdots = 0 & \text{if } \mathcal{D} < 0, \\ \partial_{\xi\xi} u - \partial_{\eta\eta} u + \cdots = 0 & \text{if } \mathcal{D} > 0, \\ \partial_{\xi\xi} u + \cdots = 0 & \text{if } \mathcal{D} = 0, \end{cases}$$

where $\cdots$ represents the lower order terms (that are at most first-order).

For the simple proof, read Page 29 of the Textbook for instance.

Moral

$$\begin{aligned} \text{Elliptic PDE} \quad &\longleftrightarrow \quad \Delta u = 0 \\ \text{Hyperbolic PDE} \quad &\longleftrightarrow \quad \partial_{tt} u - \partial_{xx} u = 0 \\ \text{Parabolic PDE} \quad &\longleftrightarrow \quad \partial_t u - \partial_{xx} u = 0 \end{aligned}$$

Remark

The linear transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} := B \begin{pmatrix} x \\ y \end{pmatrix}$$

means a linear change of coordinates. One may see this as a change of reference frame, and hence, the underlying physics remains unchanged.

Question

Can we classify the linear second-order equations in higher dimensions?

Matrix Representation of Second-Order Terms

Observation on Second-Order Terms

Let $a_{21} := a_{12}$, and $A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then

$$\begin{aligned} A\ D^2 u &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \partial_{xx} u & \partial_{yx} u \\ \partial_{xy} u & \partial_{yy} u \end{pmatrix} \\ &= \begin{pmatrix} a_{11}\partial_{xx} u + a_{12}\partial_{xy} u & a_{11}\partial_{yx} u + a_{12}\partial_{yy} u \\ a_{21}\partial_{xx} u + a_{22}\partial_{xy} u & a_{21}\partial_{yx} u + a_{22}\partial_{yy} u \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{tr}\left(A\ D^2 u\right) &= (a_{11}\partial_{xx} u + a_{12}\partial_{xy} u) + (a_{21}\partial_{yx} u + a_{22}\partial_{yy} u) \\ &= a_{11}\partial_{xx} u + 2a_{12}\partial_{xy} u + a_{22}\partial_{yy} u, \end{aligned}$$

since $a_{21} := a_{12}$. The last line is the second-order terms in (2DLPDE).

Vector Calculus Notations

Observation on First-Order Terms

Let $b := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Then

$$b \cdot \nabla u = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cdot \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = b_1 \partial_x u + b_2 \partial_y u.$$

The right hand side is the first-order terms in (2DLPDE).

Equation (2DLPDE) in Vector Calculus Notations

Using Vector Calculus Notations, we can rewrite Equation (2DLPDE) as

$$\operatorname{tr}\left(A\, D^2 u\right) + b \cdot \nabla u + cu = f. \tag{LPDEVF}$$

Moral

One can easily generalize (LPDEVF) to higher dimensional case.

Generalization via Summation Notations

Using summation notation and the convention that $a_{12} = a_{21}$, we can rewrite

$$a_{11}\partial_{x_1x_1}u + 2a_{12}\partial_{x_1x_2}u + a_{22}\partial_{x_2x_2}u + b_1\partial_{x_1}u + b_2\partial_{x_2}u + cu = f$$

as

$$\sum_{i,j=1}^{2} a_{ij}\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{2} b_i\partial_{x_i}u + cu = f.$$

Now, one can easily to generalize to the case of higher dimensions:

The d-dimensional Second-Order PDE

$$\sum_{i,j=1}^{d} a_{ij}\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{d} b_i\partial_{x_i}u + cu = f, \tag{dDLPDE}$$

where $a_{ij} = a_{ji}$ for all $i,\ j = 1,\ 2,\ \cdots,\ d$.

Revisit Ellipticity, Parabolicity, and Hyperbolicity

Recall

Denote by $A := (a_{ij})_{i,j=1}^{d}$ the $d \times d$ <u>*symmetric*</u> matrix.

Remark

The symmetry condition $a_{ij} = a_{ji}$ (for all i, $j = 1, 2, \cdots, d$) is just a convention instead of an assumption. For example, for any $i < j$, we can always write

$$\alpha \partial_{x_i x_j} u = \frac{\alpha}{2} \partial_{x_i x_j} u + \frac{\alpha}{2} \partial_{x_j x_i} u =: a_{ij} \partial_{x_i x_j} u + a_{ji} \partial_{x_j x_i} u,$$

provided that we define/choose $a_{ij} = a_{ji} = \dfrac{\alpha}{2}$.

Moral

We include all the cases, even if we “assume” the symmetry condition.

Further Convention

In addition, one may also require

$$\boxed{\operatorname{tr} A \geq 0.}$$

Remark

Again, this is also NOT a restriction because if the linear second-order PDE is

$$\operatorname{tr}\left(A\, D^2 u\right) + b \cdot \nabla u + cu = f$$

and $\operatorname{tr} A < 0$, then we can multiply the whole equation by -1 and obtain

$$\operatorname{tr}\left(\tilde{A}\, D^2 u\right) + \tilde{b} \cdot \nabla u + \tilde{c} u = \tilde{f}, \tag{1}$$

where $\tilde{A} := -A$, $\tilde{b} := -b$, $\tilde{c} := -c$ and $\tilde{f} := -f$. Then

$$\operatorname{tr} \tilde{A} > 0.$$

We can just consider (1) instead.

Two Dimensional Case (i.e., $d = 2$)

When $d = 2$ (under the convention $a_{12} = a_{21}$ and $a_{11} + a_{22} =: \operatorname{tr} A \geq 0$), the coefficient matrix

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and its characteristic polynomial is

$$p(\lambda) := \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}^2).$$

One may check that the eigenvalues of A are

$$\lambda_{\pm} = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 + 4\mathcal{D}}}{2},$$

where $\mathcal{D} := a_{12}^2 - a_{11}a_{22}$ is the discriminant for (2DLPDE). Then

- (2DLPDE) is *elliptic* (i.e., $\mathcal{D} < 0$) if and only if $0 < \lambda_- < \lambda_+$;
- (2DLPDE) is *parabolic* (i.e., $\mathcal{D} = 0$) if and only if $0 = \lambda_- < \lambda_+$;
- (2DLPDE) is *hyperbolic* (i.e., $\mathcal{D} > 0$) if and only if $\lambda_- < 0 < \lambda_+$.

Definitions of Ellipticity, Parabolicity, and Hyperbolicity

Definition (Ellipticity, Parabolicity, and Hyperbolicity)

Consider (dDLPDE):

$$\sum_{i,j=1}^{d} a_{ij}\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{d} b_i\partial_{x_i}u + cu = f.$$

Let $A := (a_{ij})_{i,j=1}^{d}$ satisfy the convention that

$$a_{ij} = a_{ji} \quad \text{for all } i \text{ and } j, \quad \text{and} \quad \operatorname{tr} A \geq 0.$$

Then A has d real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$, and we say that (dDLPDE) is

- *elliptic* if $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$;
- *parabolic* if $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$; or
- *hyperbolic* if $\lambda_1 < 0 < \lambda_2 \leq \cdots \leq \lambda_d$.

Question

Why are λ_i's important?

Answer

We can always re-write (dDLPDE) in terms of λ_i's.

Theorem

Let $A \in M_{d\times d}(\mathbb{R})$ be symmetric. Then there exists $\Gamma \in M_{d\times d}(\mathbb{R})$ such that the linear transformation $\xi := \Gamma x$ converts (dDLPDE) *to*

$$\sum_{i=1}^{d} \lambda_i \partial_{\xi_i \xi_i} u + \cdots = 0,$$

where $\cdots$ represents the lower order terms, λ_i's are eigenvalues of A.

Remark

Symbolically,

$$\text{(dDLPDE)} \xrightarrow[\text{becomes}]{\xi := \Gamma x} \sum_{i=1}^{d} \lambda_i \partial_{\xi_i \xi_i} u + \cdots = 0.$$

Main Idea of the Proof

Denote $x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$, $\xi := \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_d \end{pmatrix}$ and $\Gamma := \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1d} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{d1} & \gamma_{d2} & \cdots & \gamma_{dd} \end{pmatrix}$. Then coordinate-wise, $\xi = \Gamma x$ can be written as, for any $k = 1, 2, \cdots, d$,

$$\xi_k = \sum_{m=1}^{d} \gamma_{km} x_m.$$

Main Idea of the Proof (Continued)

Hence,

$$\partial_{x_i}\xi_k = \partial_{x_i}\left(\sum_{m=1}^{d}\gamma_{km}x_m\right) = \sum_{m=1}^{d}\gamma_{km}\underbrace{\partial_{x_i}x_m}_{=\delta_{im}} = \gamma_{ki}.$$

By the chain rule,

$$\partial_{x_i}u = \sum_{k=1}^{d}\partial_{\xi_k}u\partial_{x_i}\xi_k = \sum_{k=1}^{d}\gamma_{ki}\partial_{\xi_k}u.$$

Similarly, we also have

$$\boxed{\partial_{x_i}\partial_{x_j}u = \sum_{k,l=1}^{d}\gamma_{ki}\gamma_{lj}\partial_{\xi_k}\partial_{\xi_l}u.}$$

Main Idea of the Proof (Continued)

Therefore,

$$\begin{aligned}\sum_{i,j=1}^{d} a_{ij}\partial_{x_i}\partial_{x_j}u &= \sum_{i,j=1}^{d} a_{ij}\sum_{k,l=1}^{d}\gamma_{ki}\gamma_{lj}\partial_{\xi_k}\partial_{\xi_l}u \\ &= \sum_{k,l=1}^{d}\left(\sum_{i,j=1}^{d}\gamma_{ki}a_{ij}\gamma_{lj}\right)\partial_{\xi_k}\partial_{\xi_l}u \\ &= \sum_{k,l=1}^{d}\left(\Gamma A\Gamma^T\right)_{kl}\partial_{\xi_k}\partial_{\xi_l}u,\end{aligned}$$

since

$$\sum_{i,j=1}^{d}\gamma_{ki}a_{ij}\gamma_{lj}$$

is just the (k,l)-th entry of the matrix $\Gamma A\Gamma^T$.

Main Idea of the Proof (Continued)

The identity

$$\boxed{\sum_{i,j=1}^{d} a_{ij}\partial_{x_i}\partial_{x_j} u = \sum_{k,l=1}^{d} \left(\Gamma A \Gamma^T\right)_{kl} \partial_{\xi_k}\partial_{\xi_l} u,}$$

holds for any Γ, provided that we define $\xi = \Gamma x$. Since A is symmetric, it follows from the knowledge of Linear Algebra that we can always find a matrix Γ such that

$$\Gamma A \Gamma^T = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \lambda_d \end{pmatrix}.$$

Main Idea of the Proof (Continued)

Using this particular choice of Γ, we finally obtain

$$\boxed{\sum_{i,j=1}^{d} a_{ij}\partial_{x_i}\partial_{x_j}u = \sum_{k=1}^{d}\lambda_k\partial_{\xi_k}\partial_{\xi_k}u.}$$

Since the lower order terms remain lower order under the linear transformation $\xi = \Gamma x$, we complete the proof.

Moral

The physical behavior of a PDE is usually governed by the structure of "highest" order derivatives.

Moral

The physical behavior of a scalar LINEAR second-order PDE is governed by the eigenvalues of its leading order coefficient matrix A.