THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 4

Problem 1. Consider the second order equation

$$\partial_{tt}u + \partial_{tx}u - 2\partial_{xx}u = 0, \tag{1}$$

and complete the following parts.

(i) Is the equation (1) elliptic, parabolic, or hyperbolic? Verify your answer by computing its discriminant $\mathcal{D}$.

(ii) Find the general solution to (1) by the method of characteristics.

Problem 2. Solve the following second order equation

$$\partial_{xx}u - 4xy\partial_{xy}u + 4x^2y^2\partial_{yy}u + 2y(2x^2 - 1)\partial_y u = 0 \tag{2}$$

by the method of characteristics.

(i) Verify that the equation (2) can be written as

$$(\partial_x - 2xy\partial_y)^2 u = 0.$$

(ii) Let $v := (\partial_x - 2xy\partial_y)u = \partial_x u - 2xy\partial_y u$. Verify that u and v satisfy the following system:

$$\begin{cases} \partial_x v - 2xy\partial_y v = 0 \\ \partial_x u - 2xy\partial_y u = v. \end{cases}$$

(iii) Find the general solution to

$$\partial_x v - 2xy\partial_y v = 0.$$

(iv) Apply the method of characteristics to solve

$$\partial_x u - 2xy\partial_y u = v,$$

and prove that the general solution u to (2) has the form

$$u(x,y) = f(ye^{x^2}) + x\, g(ye^{x^2})$$

where f and g are arbitrary functions.

Problem 3. By similar procedures in Problem 2, find the general solution u to

$$y^2\partial_{xx}u - 2xy\partial_{xy}u + x^2\partial_{yy}u - x\partial_x u - y\partial_y u = (-y\partial_x + x\partial_y)^2 u = 0.$$

Problem 4. Consider the following initial value problem for the wave equation:

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = 0 & \text{for } -\infty < x < \infty \text{ and } t > 0 \\ u|_{t=0} = \phi & \\ \partial_t u|_{t=0} = \psi, & \end{cases}$$

where $c > 0$ is a given constant, the functions ϕ and ψ are given initial data, and u is the unknown.

(i) Using the d'Alembert's formula, verify that if both ϕ and ψ are even functions, that is, for any $x \in (-\infty, \infty)$,

$$\begin{cases} \phi(-x) = \phi(x) \\ \psi(-x) = \psi(x), \end{cases}$$

then u is also an even function in x, that is, for any $x \in (-\infty, \infty)$ and $t \geq 0$,

$$u(t,-x) = u(t,x).$$

(ii) Find $u(t,x)$ for $\phi(x) = x^4$ and $\psi(x) = \cos x$.

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Problem 5. Find the solutions to the boundary value problem of the following inhomogeneous PDE

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = -e^x & \text{for } 0 < x < 1 \text{ and } t > 0 \\ u|_{x=0} = u|_{x=1} = 0, \end{cases}$$

(Hint: Find the homogeneous solutions $u_h(x,t)$, and a particular solution $u_p(x)$ which is time-independent.)