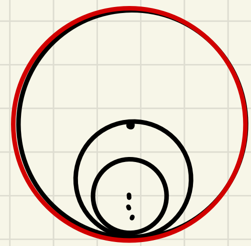
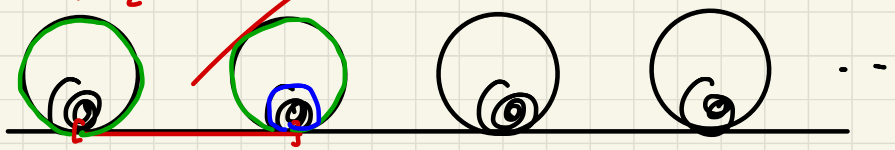
Hawaii ring

$$H = \bigcup_{n \geq 1} S\left(\left(0, \tfrac{1}{n}\right), \tfrac{1}{n}\right)$$

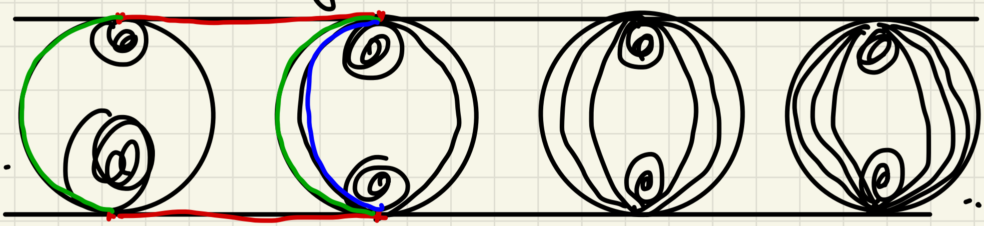


$r = \frac{1}{2}$ $\quad$ $r = \frac{1}{2}$ $\quad$ . .

$Y = \ldots$  $\ldots$

$p \downarrow$ is a covering map.

H

$X = \ldots$  $\ldots$

$q : X \to Y$ is 2:1 covering map

Claim 1) $p \circ q$ is not a covering map

2) $p \circ q$ is a local homeomorphism

3) Hawaii ring $\not\cong$ infinite wedge

not even homotopic!

Thm [Path lifting property]

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ I \arrow[ur, "\exists!\, \tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

$p : \tilde{X} \to X$ a covering map

$\gamma : [0,1] \to X$ a path $x_0 = \gamma(0)$

Fix $\tilde{x}_0 \in p^{-1}(x_0)$ $\exists!$ path $\tilde{\gamma} : [0,1] \to \tilde{X}$

s.t. $\tilde{\gamma}(0) = \tilde{x}_0$ $\quad p \circ \tilde{\gamma} = p$.

Pf. Let $\{U_\alpha\}$ be a covering of X s.t.

$p^{-1}U_\alpha = \bigsqcup_{\lambda \in \Lambda_\alpha} \tilde{U}_\alpha^\lambda \qquad p|_{\tilde{U}_\alpha^\lambda}$ is a homeo.

$\{I \cap U_\alpha\}$ covers I by Lebesgue lemma

$\exists$ a finite partition $t_0 = 0 < t_1 < \cdots < t_n = 1 \quad \forall i$

s.t. $\underbrace{[t_i, t_{i+1}]}_{I_i} \in U_{\alpha_i}$ for some α $\quad \gamma_i = \gamma|_{I_i}$

Let $\lambda_0 \in \Lambda_{\alpha_0}$ s.t. $\tilde{x}_0 \in \tilde{U}_{\alpha_0}^{\lambda_0}$

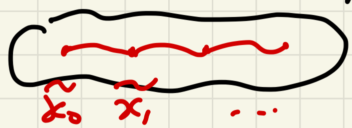
$$\tilde{\gamma}_0 := \left(p|_{\tilde{U}_{\alpha_0}^{\lambda_0}}\right)^{-1} \circ \gamma_0 : I_0 \longrightarrow \tilde{X}$$
$$0 = t_0 \longmapsto \tilde{x}_0$$

set $\tilde{x}_1 = \tilde{\gamma}_0(1)$

Inductively set λ_i s.t. $\tilde{x}_i \in \tilde{U}_{\alpha_i}^{\lambda_i}$

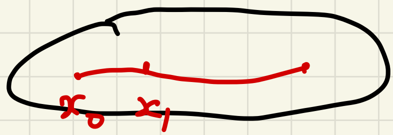
$$\tilde{\gamma}_i := \left(p|_{\tilde{U}_{\alpha_i}^{\lambda_i}}\right)^{-1} \circ \gamma_i : I_i \longrightarrow \tilde{X}$$

then we define the lift $\tilde{\gamma}$. it's clearly unique



in particular

constant path lifts to const. path.



Thm [Homotopy lifting property]

same assumption as above

Let $F(t,s) : I \times I \longrightarrow X$ be a homotopy

$F(t,0) = \gamma(t)$ Then $\exists!\ \tilde{F} : I \times I \to X$

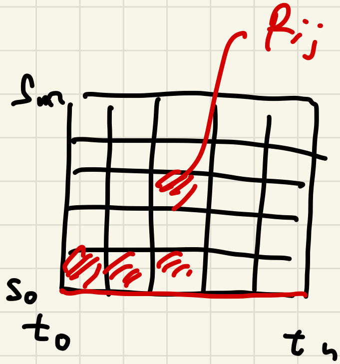
s.t. $\tilde{F}(t,0) = \tilde{\gamma}(t)$ $\quad p \circ \tilde{F} = F$

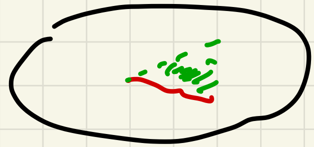
If F is a path homotopy (i.e. $F(0,s) = \gamma(0)$, $F(1,s) = \gamma(1)$)

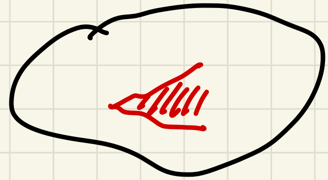
then so is $\tilde{F}$.

p.f. (Sketch)

$I \times I = \bigcup R_{ij}$







by Lebesgue lemma we may assume $R_{ij} \in \mathcal{U}_{\alpha_{ij}}$

$F_{ij} = F|_{R_{ij}}$

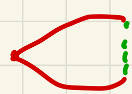
Let $\lambda_{00} \in \Lambda_{\alpha_{00}}$ s.t. $\tilde{x}_0 \in \tilde{\mathcal{U}}_{\alpha_{00}}^{\lambda_{00}}$

$$\tilde{F}_{00} = \left(p|_{\tilde{\mathcal{U}}_{\alpha_{00}}^{\lambda_{00}}}\right)^{-1} \circ F_{00}$$

then we extend to $F_{10} \cdots \quad F_{n-1\,0}$

$F_{11} \cdots \quad F_{n-1,n}$

Suppose F is a path homotopy:



$\tilde{F}(s,1) \in p^{-1}(\gamma(1))$

but $p^{-1}(\gamma(1))$ is discrete $\tilde{F}(s,1)$ must be constant.

Cor α_1, α_2 two loops based at $x_0 \in X$

Fix $\tilde{x}_0$ s.t $p(\tilde{x}_0) = x_0$ $\tilde{\alpha}_1, \tilde{\alpha}_2$ lifts

Then $\tilde{\alpha}_1(1) = \tilde{\alpha}_2(1)$. if $\alpha_1 \sim \alpha_2$.

Thm Let $\tilde{x}_0 \in p^{-1}(x_0)$ $[\alpha] \in \pi_1(X, x_0)$

Set $[\alpha] \bullet \tilde{x}_0 = \tilde{\alpha}(1)$ ← Monodromy action.

This is a group action.

Example $p : \mathbb{R} \longrightarrow S^1$ $\qquad p^{-1}(1) = \mathbb{Z}$

$\theta \longmapsto e^{2\pi i \theta}$ $\qquad x_0 = 1$

1)

$\alpha = e^{2\pi i t}$ $\qquad [\alpha^n] \bullet \mathbb{Z}$ $\qquad m \to m + n$

2) $p : S^2 \to \mathbb{RP}^2 = S^2/_{\{\pm 1\}}$

$\{\pm v\}$ $\qquad [v]$

$-1 \cdot v = -v$