1. (8 points) Let A and B be proper subgroups of a group G.

   (a) Show that $A \cap B$ is also a proper subgroup.

   (b) If $|A| \neq |B|$ and $|G| = pq$ where $p \neq q$ are primes, show that $A \cap B$ is the trivial subgroup.

   Ans.

   (a) (4 marks) $e \in A \cap B$ as both A, B are subgroups. If $x, y \in A \cap B$, then $x, y \in A$ and $x, y \in B$. As A, B are subgroups, $xy^{-1} \in A$ and B respectively. Thus $xy^{-1} \in A \cap B$. So $A \cap B$ is a subgroup.

   Since $A \cap B \subset A \neq G$, $A \cap B \neq G$, i.e $A \cap B$ is proper.

   (b) (4 marks) By Lagrange's theorem, $|A \cap B|$ divides $|A|$ and $|B|$, and both $|A|$ and $|B|$ divide $|G| = pq$. Since both A and B are proper, $|A|, |B| \in \{1, p, q\}$.

   If $|A|$ or $|B| = 1$, then $A \cap B$ is trivial from $|A \cap B| \big| (|A|, |B|) = 1$.

   If $|A|$ and $|B|$ are not equal to 1, then $(|A|, |B|) = 1$. So $|A \cap B| = 1$.

2. (16 points) For each of the following either give an example or say that no such example exists. *Brief explanation is required.*

   (a) A group of order 18 but none of its element is of order 9.

   (b) A group of order 18 which contains two distinct subgroups of order 6.

   (c) An injective homomorphism from S_3 to $\mathbb{Z}_{45}$.

   (d) A group G has a non-trivial proper centre Z. [Recall the centre of a group is always a subgroup.]

   Ans.

   (a) (4 marks) $\mathbb{Z}_3 \times S_3$. Elements of S_3 are of order $1, 2, 3$. Thus elements of $\mathbb{Z}_3 \times S_3$ are of order $1, 2, 3, 6$.

   Note that for an element $(a, b) \in G_1 \times G_2$, $\text{ord}(a, b) = \text{l.c.m.}(\text{ord}(a), \text{ord}(b))$. (Prove it if you don't know it before.)

   (b) (4 marks) $G = \mathbb{Z}_3 \times \mathbb{Z}_6$. Then $A = \mathbb{Z}_3 \times \langle 3 \rangle$ and $B = \{0\} \times \mathbb{Z}_6$ are distinct subgroups of order 6, because the subgroup $\langle 3 \rangle$ of $\mathbb{Z}_6$ has order 2, and $A \neq B$ follows from $(1, 0) \in A$ but not in B.

(c) (4 marks) No such homomorphism, because otherwise the image of S_3 is a non-abelian subgroup of $\mathbb{Z}_{45}$. This is impossible because every subgroup of the abelian group $\mathbb{Z}_{45}$ is abelian.

(d) (4 marks) Let B be any non-abelian group. Then the centre $Z(B)$ of B is a proper subgroup of B since the centre $Z(B)$ is abelian. Consider $\mathbb{Z}_2 \times B$. By direct checking, the centre of $\mathbb{Z}_2 \times B$ is $\mathbb{Z}_2 \times Z(B)$, which is proper for $Z(B) \neq B$ and is non-trivial for $(1, e) \in \mathbb{Z}_2 \times Z(B)$.

[Remark. There are examples (probably easier than above) to answer this question.]

3. (12 points)

(a) Let $x = (2, 4)(2, 3)(1, 3)(1, 2)$ be a product of transpositions and $n \in \mathbb{Z}$.

(i) Express x as a product of disjoint cycles.

(ii) Show that $\alpha(1, 2, 3, 4)(1, 3)\alpha^{-1} = x^n$ for some $\alpha \in S_4$ if and only if n is odd.

(b) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 1 & 3 & 8 & 5 & 2 & 7 \end{pmatrix}$.

(i) Evaluate the order of σ.

(ii) Is $\sigma \in A_8$? Explain your answer.

Ans.

(a) (i) (3 marks) $x = (1, 3)(2, 4)$.

(ii) (3 marks) When n is odd, $x^n = (1, 3)^n(2, 4)^n = (1, 3)(2, 4)$. Note $(1, 2, 3, 4)(1, 3) = (1, 4)(2, 3)$ has the same cycle pattern as x^n. So they are conjugate with each other.

When n is even, $x^n = (1, 3)^n(2, 4)^n$ is the identity element. So it has a different cycle pattern from $(1, 4)(2, 3)$. They are not conjugate with each other.

(b) (i) (3 marks) $\sigma = (1, 4, 3)(2, 6, 5, 8, 7)$. Thus $\text{ord}(\sigma) = 15$.

(ii) (3 marks) Yes, because $(1, 4, 3)$ is a product of two transpositions while $(2, 6, 5, 8, 7)$ is a product of four transpositions. Thus σ is a product 6 transpositions, so it is an even permutation.

[Comment. Many students seems not aware of the result: two elements in S_n are conjugate to each other if and only if they have the same cyclic pattern. Also quite a good number of students mixed up the cycle length and the order of the cycle. Also, an element $x \in S_n$ belongs to A_n based on whether it is an even or odd permutation, rather than its order.

Below are some examples for your understanding:

Let $\sigma = (1,2)$, $\tau = (2,3)$, $\lambda = (1,4)$, $\alpha = (3,4)(1,2)$, $\beta = (2,3,4)$, $\gamma = (1,2,3,4)$. Then,

- σ and τ have the same cycle pattern, so they are conjugate to each other. i.e. $x\sigma x^{-1} = \tau$ for some $x \in S_n$.
- $\mathrm{ord}(\sigma) = \mathrm{ord}(\tau) = 2$. (Well. if two elements h, k of a group G are conjugate to each other, then $\mathrm{ord}(h) = \mathrm{ord}(k)$!)
- α and the product $\tau\lambda$ have the same cyclic pattern, so $y\alpha y^{-1} = \tau\lambda$ for some $y \in S_n$.
- $\mathrm{ord}(\alpha) = \mathrm{ord}((3,4)(1,2)) = 2$ while $\mathrm{ord}(\sigma\tau) = \mathrm{ord}((1,2)(2,3)) = 3$.
- $\sigma\tau$ and β have the same cyclic pattern. (This is less obvious and needs some calculation to check. Check that $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1,2,3)$, which is a 3-cycle and hence of the same pattern as β.)]

End

(Generalized) First Isomorphism Theorem. Let $\phi : G \to G'$ be a homomorphism and $N \lhd G$. If $N \subset \ker \phi$, then there exists a unique homomorphism $\overline{\phi} : G/N \to G'$ such that $\phi = \overline{\phi} \circ \pi$, where $\pi : G \to G/N$ is the natural projection. Moreover, $\ker \overline{\phi} = \ker \phi / N$.

Proof. Define $\overline{\phi} : G/N \to G'$ by $\overline{\phi}(\overline{a}) = \phi(a)$. [Remark: Here $\overline{a} := \pi(a) = aN$, *not* $a \ker \phi$.]
Check

1. $\overline{\phi}$ is a well-defined function.
2. $\overline{\phi}$ is a homomorphism.
3. $\ker \overline{\phi} = \ker \phi / N$

Ex. Show that $f : \mathbb{Z}_{12} \to \mathbb{Z}_{15}$, $f([a]_{12}) = 5[a]_{15}$, is a surjective homomorphism and $\ker f = 3\mathbb{Z}_{12}$. (Here $[a]_n$ denotes the congruence class of a mod n.)

Answer. Consider $\phi : \mathbb{Z} \to \mathbb{Z}_{15}$, $x \mapsto 5[x]_{15}$.
Then ϕ is a well-defined surjective homomorphism (as we know the map $\mathbb{Z} \to \mathbb{Z}_n, a \mapsto [a]_n$ is a homomorphism; by direct checking, the map $\mathbb{Z}_n \to \mathbb{Z}_n, [a]_n \mapsto 5[a]_n$ is a homomorphism, and the composite of two homomorphism is a homomorphism), and $\ker \phi = 3\mathbb{Z}$.
Take $N = 12\mathbb{Z}$.
By the (generalized) First Isomorphism Theorem, we have the homomorphism $\overline{\phi} : \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}_{15}$ such that $\overline{\phi}(\overline{x}) = \phi(x)$ where $\overline{x} = x + 12\mathbb{Z}$, and $\ker \overline{\phi} = 3\mathbb{Z}/12\mathbb{Z}$. Note: $\overline{\phi}$ is onto since ϕ is onto.
Also, we have $\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_{12}$ under the isomorphism $\psi : x + 12\mathbb{Z} \mapsto [x]_{12}$. By direct checking, $\psi(3\mathbb{Z}/12\mathbb{Z}) = \{\psi(3m + 12\mathbb{Z}) : m \in \mathbb{Z}\} = \{[3m]_{12} : m \in \mathbb{Z}\} = \{3[m]_{12} : m \in \mathbb{Z}\} = 3\mathbb{Z}_{12}$.
Thus, $f = \overline{\phi} \circ \psi^{-1}$ (hence is an onto homomorphism) and $\ker f = 3\mathbb{Z}_{12}$.

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathbb{Z}_{15} \\
\mathbb{Z}_{12} & \mathbb{Z}/12\mathbb{Z} \arrow[l, "\psi"'] \arrow[ur, "\overline{\phi}"'] &
\end{tikzcd}$$

$\ker \overline{\phi} = \ker \phi / N = 3\mathbb{Z}/12\mathbb{Z}$