THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Tutorial 8 Solution

Problem 1.

(i) Let $p = \frac{x}{y}$. Then it follows from $u(x,y) = g(p)$ that

$$\begin{aligned}
\partial_x u(x,y) &= \frac{\mathrm{d}\, g}{\mathrm{d}\, p}\frac{\partial p}{\partial x} = g'(p)\frac{1}{y},\\
\partial_y u(x,y) &= \frac{\mathrm{d}\, g}{\mathrm{d}\, p}\frac{\partial p}{\partial y} = g'(p)\frac{-x}{y^2},\\
\partial_{xx} u(x,y) &= \frac{1}{y}\frac{\mathrm{d}\, g'}{\mathrm{d}\, p}\frac{\partial p}{\partial x} = \frac{g''(p)}{y^2},\\
\partial_{yy} u(x,y) &= (-x)\frac{\partial}{\partial y}\frac{g'(p)}{y^2} = \frac{g''(p)x^2 + 2xyg'(p)}{y^4}.
\end{aligned}$$

Then it follows that

$$0 = 4\partial_{xx}u + \partial_{yy}u = 4\frac{g''(p)}{y^2} + \frac{g''(p)x^2 + 2xyg'(p)}{y^4} = (p^2+4)g''(p) + 2pg'(p).$$

(ii) The ODE for $g(p)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}\, p}\Big((p^2+4)g'(p)\Big) = 0.$$

Direct integration yields that

$$g'(p) = \frac{C_1}{p^2+4}$$

for some constant C_1. And then integration again gives that

$$g(p) = \int \frac{C_1}{p^2+4}\,\mathrm{d}\, p + C_2,$$

for some constant C_2. It is also possible to write down the above integral explicitly. Let $p = 2\tan\theta$, then

$$\int \frac{1}{p^2+4}\,\mathrm{d}\,p = \int \frac{1}{4(\tan^2\theta+1)}\,\mathrm{d}(2\tan\theta) = \frac{\theta}{2} + C_3 = \frac{1}{2}\tan^{-1}(\frac{p}{2}) + C_3.$$

Thus, we have

$$u(x,y) = g(\frac{x}{y}) = C_4\tan^{-1}\frac{x}{2y} + C_5,$$

for some constant C_4, C_5.

Problem 2.

(i) Recall the solution formula for the heat equation:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} e^{3y}\,\mathrm{d}\,y.$$

Next we deal with the power terms. Completing the square in y gives that

$$-\frac{(y-x)^2 - 12ty}{4t} = -\frac{(y-x-6t)^2 - 36t^2 - 12tx}{4t}.$$

Then our integral becomes

$$u(x,t) = e^{9t+3x}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(y-x-6t)^2}{4t}}\,\mathrm{d}\,y.$$

We perform the change-of-variables to deal with the power. Let $p = \frac{y-x-6t}{\sqrt{4t}}$ then

$$u(x,t) = e^{9t+3x}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\,\mathrm{d}\,p = e^{9t+3x}.$$

(ii) Recall the solution formula for the heat equation and plug in the initial data,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-2}^{2} e^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}\,y.$$

The goal is to express our integral in terms of the Gauss error function. We do the change-of-variables. Let $p = \frac{y-x}{\sqrt{4t}}$, then we get

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} \int_{\frac{-2-x}{\sqrt{4t}}}^{\frac{2-x}{\sqrt{4t}}} e^{-p^2}\,\mathrm{d}\,p \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\frac{2-x}{\sqrt{4t}}} e^{-p^2}\,\mathrm{d}\,p - \frac{1}{\sqrt{\pi}} \int_0^{\frac{-2-x}{\sqrt{4t}}} e^{-p^2}\,\mathrm{d}\,p \\ &= \frac{1}{2}\operatorname{erf}\left(\frac{2-x}{\sqrt{4t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{-2-x}{\sqrt{4t}}\right). \end{aligned}$$

(iii) Recall the solution formula for the heat equation and plug in the initial data,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4t}} e^{-y} dy.$$

Completing the square in y, the power becomes

$$-\frac{(y-x)^2 + 4ty}{4t} = -\frac{(y-x+2t)^2 - 4t^2 + 4tx}{4t}.$$

Let $p = \frac{y-x+2t}{\sqrt{4t}}$, we get

$$\begin{aligned} u(x,t) &= \frac{e^{t-x}}{\sqrt{\pi}} \int_{\frac{-x+2t}{\sqrt{4t}}}^\infty e^{-p^2}\,\mathrm{d}\,p \\ &= \frac{e^{t-x}}{2} \left(\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-p^2} dp - \frac{2}{\sqrt{\pi}} \int_0^{\frac{-x+2t}{\sqrt{4t}}} e^{-p^2} dp \right) \\ &= \frac{e^{t-x}}{2} \left[1 - \operatorname{erf}\left(\frac{-x+2t}{\sqrt{4t}}\right) \right]. \end{aligned}$$

Problem 3. Let $u(x,t) = \exp(\alpha x + \beta t) w(x,t)$. Then

$$\begin{aligned} \partial_t u &= \exp(\alpha x + \beta t)(\partial_t w + \beta w), \\ \partial_x u &= \exp(\alpha x + \beta t)(\partial_x w + \alpha w), \\ \partial_{xx} u &= \exp(\alpha x + \beta t)(\partial_{xx} w + 2\alpha \partial_x w + \alpha^2 w). \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \partial_t u - \partial_x^2 u - 2\partial_x u + 2u \\ &= \partial_t w + \beta w - (\partial_{xx} w + 2\alpha \partial_x w + \alpha^2 w) - 2(\partial_x w + \alpha w) + 2w. \end{aligned}$$

In order to obtain $\partial_t w - \partial_{xx} w = 0$, we need

$$\begin{cases} -2\alpha - 2 = 0, \\ \beta - \alpha^2 - 2\alpha + 2 = 0, \end{cases} \quad \Rightarrow \begin{cases} \alpha = -1, \\ \beta = -3. \end{cases}$$