Riemann Sphere

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> When you encounter computations, do it, you will see important things from computations.
>
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Contents

This lecture aims to investigate three fundamental problems:

1. **Mittag-Leffler Problem**
2. **Weierstrass Problem**
3. **Structure of $\mathcal{M}(X)$**

where X is a compact Riemann surface and $\mathcal{M}(X)$ is the field of meromorphic functions on X.

1 Riemann Surfaces

Definition 1.1 (Riemann Surface)**.** A **Riemann surface** X is a connected, Hausdorff, second-countable topological space, equipped with a **complex atlas** $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$. This means:

1. Each U_α is an open subset of X, and they form a cover: $\bigcup_{\alpha\in A} U_\alpha = X$.
2. Each **chart** $\phi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism from U_α to an open subset $V_\alpha \subseteq \mathbb{C}$.
3. For any two charts (U_α, ϕ_α) and (U_β, ϕ_β) such that $U_\alpha \cap U_\beta \neq \emptyset$, the **transition map**

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is holomorphic.

2 The Riemann Sphere $\mathbb{P}^1$

Definition 2.1 (Riemann Sphere)**.** The *Riemann sphere* (or complex projective line) is defined as

$$\mathbb{P}^1(\mathbb{C}) = \big(\mathbb{C}\times\{0\}\big) \ \sqcup\ \big(\mathbb{C}\times\{1\}\big) \Big/ \sim$$

where the equivalence relation is

$$(z,0)\sim(w,1) \quad\Longleftrightarrow\quad zw=1.$$

Concretely, $\mathbb{P}^1(\mathbb{C})$ can be viewed as two copies of the complex plane,

$$\mathbb{P}^1 = C_0 \cup \{\infty\} = C_1 \cup \{\infty_1\},$$

glued along the overlap via the coordinate change $z \mapsto \frac{1}{z}$. Thus the Riemann sphere may be interpreted as the complex plane together with a single point at infinity.

Definition 2.2 (Automorphism Group)**.**

$$\mathrm{Aut}(X) = \{\phi : X \to X \mid \phi \text{ is a conformal equivalence}\}$$

where conformal equivalence means ϕ is a bijection that is holomorphic with holomorphic inverse (i.e., biholomorphic).

Remark 2.3. Automorphism = biholomorphism = conformal equivalence.

Definition 2.4 (Fractional Linear Transformation)**.** A *fractional linear transformation* (or *Möbius transformation*) is a map

$$\phi(z) = \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{C}$ and $ad-bc\neq 0$. It is naturally defined on the Riemann sphere $\mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$ by the rules:

1. If $z\in\mathbb{C}$ and $cz+d\neq 0$, then $\phi(z)\in\mathbb{C}$ is given by the usual formula.
2. If $z\in\mathbb{C}$ and $cz+d=0$, then $z=-\frac{d}{c}$, and

$$\lim_{z\to -d/c}\phi(z)=\infty,$$

so we define $\phi(-d/c)=\infty$.

3. At $z = \infty$, we set

$$\phi(\infty) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \begin{cases} \frac{a}{c}, & c \neq 0, \\ \infty, & c = 0. \end{cases}$$

Proposition 2.5. *The set* $\mathrm{FractLin}(\mathbb{P}^1)$ *of fractional linear transformations on* $\mathbb{P}^1$ *forms a group under composition.*

Proof. Every fractional linear transformation

$$\phi(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0,$$

corresponds uniquely to a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{C}) \quad \text{(the group of invertible } 2 \times 2 \text{ matrices).}$$

Now, take two such maps:

$$\phi(w) = \frac{aw+b}{cw+d}, \qquad \varphi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Then their composition is

$$(\phi \circ \varphi)(z) = \phi\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \frac{a(\alpha z + \beta) + b(\gamma z + \delta)}{c(\alpha z + \beta) + d(\gamma z + \delta)}.$$

Expanding,

$$(\phi \circ \varphi)(z) = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}.$$

Thus $(\phi \circ \varphi)(z)$ is again a fractional linear transformation, with matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Since both determinants are nonzero, $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq 0$, so this matrix is invertible. Hence the composition of two fractional linear transformations is again fractional linear.

The identity map corresponds to the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For inverses: If $\phi(z) = \frac{az+b}{cz+d}$, then

$$\phi^{-1}(z) = \frac{dz - b}{-cz + a},$$

which is again of the same form, corresponding to the inverse matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}.$$

Thus: - closure holds via matrix multiplication, - associativity follows from matrix multiplication, - the identity is the identity matrix, - inverses exist from matrix inverses.

Therefore $\mathrm{FractLin}(\mathbb{P}^1)$ is a group under composition. □

Theorem 2.6. $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{FractLin}(\mathbb{P}^1)$.

Proof. Let $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$. We distinguish two cases depending on the image of ∞.

It is a classical result that every bijective entire map $f : \mathbb{C} \to \mathbb{C}$ must be affine linear:

$$f(z) = az + b, \quad a \in \mathbb{C}^\times,\ b \in \mathbb{C}.$$

The proof uses Liouville's theorem and the fact that polynomials of degree ≥ 2 are not injective; thus only degree 1 polynomials yield bijections.

Hence,

$$\mathrm{Aut}(\mathbb{C}) = \{\, z \mapsto az + b \mid a \in \mathbb{C}^\times,\ b \in \mathbb{C} \,\}.$$

Case 1: $\varphi(\infty) = \infty$. Then $\varphi|_{\mathbb{C}}$ is an automorphism of $\mathbb{C}$, so

$$\varphi(z) = az + b, \qquad a \neq 0.$$

This is precisely a fractional linear transformation with $c = 0,\ d = 1$.

Case 2: $\varphi(\infty) \in \mathbb{C}$. Suppose $\varphi(\infty) = s \in \mathbb{C}$. Define

$$\psi(w) = \frac{1}{w - s}, \quad \psi(s) = \infty.$$

Clearly $\psi \in \mathrm{FractLin}(\mathbb{P}^1)$. Then

$$\theta := \psi \circ \varphi \in \mathrm{Aut}(\mathbb{P}^1),$$

and $\theta(\infty) = \infty$. By Case 1, θ is fractional linear, hence so is $\varphi = \psi^{-1} \circ \theta$.

In both cases, φ has the form

$$\varphi(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

Thus every automorphism of $\mathbb{P}^1$ is fractional linear, and we conclude

$$\mathrm{Aut}(\mathbb{P}^1) = \mathrm{FractLin}(\mathbb{P}^1).$$

□

3 The Mittag-Leffler Problem

Definition 3.1 (Meromorphic Function)**.** Let $U \subseteq \mathbb{C}$ be an open set. A function $f : U \to \mathbb{C} \cup \{\infty\}$ is called *meromorphic* on U if

1. f is holomorphic on U except at isolated points, and
2. at each such isolated point $a \in U$, f has a *pole*, i.e. there exists an integer $m \geq 1$ and a holomorphic function g near a with $g(a) \neq 0$ such that

$$f(z) = \frac{g(z)}{(z - a)^m}$$

in a neighborhood of a.

Definition 3.2 (Laurent series)**.** Let f be holomorphic on an annulus

$$A = \{\, z \in \mathbb{C} : r < |z-a| < R \,\},$$

centered at $a \in \mathbb{C}$. Then f admits an expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

which converges absolutely and uniformly on compact subsets of A. This series is called the *Laurent expansion* of f about a.

Definition 3.3 (Pole)**.** Let f be meromorphic near $a \in \mathbb{C}$. The point a is called a *pole* of f of order $m \geq 1$ if the Laurent expansion of f about a takes the form

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n, \qquad c_{-m} \neq 0.$$

Equivalently, a is a pole of order m if $(z-a)^m f(z)$ is holomorphic and nonvanishing at a.

Definition 3.4 (Principal part)**.** Let f be meromorphic near $a \in \mathbb{C}$, and suppose a is a pole of order m. The *principal part* of f at a is the finite sum of the negative power terms in its Laurent expansion,

$$\mathrm{pp}_a(f) \ = \ \frac{c_{-1}}{z-a} + \frac{c_{-2}}{(z-a)^2} + \cdots + \frac{c_{-m}}{(z-a)^m}.$$

Definition 3.5 (Isolated singularity)**.** Let f be a complex function holomorphic on a punctured neighborhood

$$U \setminus \{a\} = \{\, z \in \mathbb{C} : 0 < |z-a| < r \,\}$$

of a point $a \in \mathbb{C}$. Then a is called an *isolated singularity* of f.

Definition 3.6 (Classification of isolated singularities)**.** If a is an isolated singularity of f with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad 0 < |z-a| < r,$$

then:

- If $c_{-n} = 0$ for all $n \geq 1$, the singularity is *removable*.
- If only finitely many negative coefficients are nonzero, i.e. $f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n$ with some $m \geq 1$ and $c_{-m} \neq 0$, then a is a *pole of order m*.
- If infinitely many negative coefficients c_{-n} are nonzero, then a is an *essential singularity*.

Theorem 3.7. *The Mittag–Leffler problem is always solvable on $\mathbb{P}^1$.*

Proof. Let $\{a_1, \ldots, a_m\} \subset \mathbb{P}^1$ be distinct points, and for each a_k let the prescribed principal part be

$$q_k(z) \;=\; \sum_{i=1}^{s_k} \frac{c_{k,i}}{(z-a_k)^i}, \qquad c_{k,i} \in \mathbb{C}.$$

Define

$$f(z) \;=\; \sum_{k=1}^{m} q_k(z) \;=\; \sum_{k=1}^{m}\sum_{i=1}^{s_k} \frac{c_{k,i}}{(z-a_k)^i}.$$

Each q_k has a pole only at a_k, with principal part exactly q_k, and is holomorphic elsewhere. Thus the sum f has, at every a_k, the prescribed principal part q_k, and is holomorphic on $\mathbb{C} \setminus \{a_1, \ldots, a_m\}$. We only need to check the behaviour of f at ∞.

Case 1. $\boldsymbol{\infty \notin \{a_1, \ldots, a_m\}}$. As $z \to \infty$, each term $\frac{c_{k,i}}{(z-a_k)^i} = O(|z|^{-i})$, so $f(z) \to 0$. A function f is holomorphic at ∞ if $\tilde{f}(w) := f(1/w)$ is holomorphic at $w = 0$. Since $\tilde{f}(w) \to 0$ as $w \to 0$, $\tilde{f}$ is holomorphic at $w = 0$. Hence f is holomorphic at ∞.

Case 2. $\boldsymbol{\infty \in \{a_1, \ldots, a_m\}}$. Without loss of generality, assume $a_m = \infty$. Write local coordinate $w = 1/z$ near ∞, so that $w = 0$ corresponds to $z = \infty$.

The prescribed principal part at ∞ is a polynomial in z:

$$q_m(z) \;=\; \sum_{i=1}^{s_m} c_{m,i} z^i.$$

Under the coordinate change $z = 1/w$, this becomes

$$\tilde{q}_m(w) \;=\; q_m(1/w) \;=\; \sum_{i=1}^{s_m} c_{m,i} w^{-i}.$$

This has a pole of order s_m at $w = 0$, as desired.

Now define, in the w-coordinate,

$$\tilde{f}(w) \;=\; \sum_{k=1}^{m-1} q_k(1/w) \;+\; \tilde{q}_m(w).$$

Each $q_k(1/w)$ is holomorphic near $w = 0$ (since $a_k \neq \infty$), while $\tilde{q}_m$ has the the required principal part there. Hence $\tilde{f}$ is meromorphic near $w = 0$ with the correct singularity, so $f(z) = \tilde{f}(1/z)$ is meromorphic near ∞ with the prescribed principal part at ∞.

In both cases, f is meromorphic on $\mathbb{P}^1$ and has exactly the prescribed principal parts at every a_k. Therefore f solves the Mittag–Leffler problem on $\mathbb{P}^1$. □

4 The Weierstrass Problem

Proposition 4.1 (Argument Principle)**.** *Let f be meromorphic in a domain G, and let $a_1, \ldots, a_s$ denote its zeros and poles in G. For each i, let $\mathrm{ord}_{a_i}(f) \in \mathbb{Z}$ denote the order of*

f at a_i (positive for a zero, negative for a pole). If γ is a closed rectifiable curve in G not passing through any of the points $a_1, \ldots, a_s$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{s} n(\gamma; a_i)\ \mathrm{ord}_{a_i}(f),$$

where $n(\gamma; z_0)$ denotes the winding number of γ about z_0, given by

$$n(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z_0}\,d\zeta.$$

Proof. Near a zero a of order $m > 0$, we can factor

$$f(z) = (z-a)^m h(z), \qquad h(a) \neq 0,$$

giving

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)}.$$

Thus $\frac{f'}{f}$ has a simple pole at a with residue $m = \mathrm{ord}_a(f)$. A similar computation at a pole of order m shows that $\mathrm{Res}(\frac{f'}{f}, a) = m = \mathrm{ord}_a(f)$.

By the residue theorem in its general form,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{s} n(\gamma; a_i)\ \mathrm{Res}\Big(\tfrac{f'}{f}, a_i\Big).$$

Since $\mathrm{Res}(\frac{f'}{f}, a_i) = \mathrm{ord}_{a_i}(f)$, this proves the claim. □

Definition 4.2 (Weierstrass Data Set)**.** Let $E = \{a_1, a_2, \ldots, a_s\} \subset \mathbb{P}^1$ be a finite set of distinct points. For each k, $1 \leq k \leq s$, let $n_k \in \mathbb{Z}$. We call $\{(a_1, n_1), \ldots, (a_s, n_s)\}$ a Weierstrass data set, where $n_i \in \mathbb{N}$ for each i.

Theorem 4.3 (Weierstrass Problem)**.** *Given a Weierstrass data set $\{(a_1, n_1), \ldots, (a_s, n_s)\}$ on $\mathbb{P}^1$, there exists a meromorphic function f on $\mathbb{P}^1$ such that f has zeros of order exactly n_i at a_i for each $i = 1, \ldots, s$, and no other zeros.*

Theorem 4.4. *The Weierstrass problem for the data set $\{(a_1, n_1), \ldots, (a_s, n_s)\}$ is solvable if and only if*

$$n_1 + \cdots + n_s = 0.$$

Proof of the Weierstrass Problem. We prove both necessity and sufficiency of the condition $\sum_{k=1}^{s} n_k = 0$.

Necessity (Argument Principle). Let $f \in \mathcal{M}(\mathbb{P}^1)$ with $f \not\equiv 0$. We show that $\sum_{a\in\mathbb{P}^1} \mathrm{ord}_a(f) = 0$.

Without loss of generality, we may assume that the points a_k do not lie on the unit circle. Equivalently, we may choose R so that $\partial D(R) \cap E = \varnothing$.

By the Argument Principle,

$$\frac{1}{2\pi i}\oint_{\partial D(R)} \frac{f'(z)}{f(z)}\,dz = \sum_{a_i \in D(R)} \operatorname{ord}_{a_i}(f),$$

that is, the contour integral counts the number of zeros of f inside $D(R)$ (counted with multiplicity) minus the number of poles inside $D(R)$ (also counted with multiplicity).

Set $z = 1/\xi$, so $dz = -\xi^{-2}\,d\xi$, and define $s(\xi) := f(1/\xi)$. Differentiating gives

$$f'(z) = \frac{d}{dz} s\big(\tfrac{1}{z}\big) = -\frac{1}{z^2}\, s'\big(\tfrac{1}{z}\big) = -\xi^2\, s'(\xi).$$

Hence

$$\frac{f'(z)}{f(z)}\,dz = \frac{-\xi^2\, s'(\xi)}{s(\xi)} \cdot \Big(-\tfrac{1}{\xi^2}\,d\xi\Big) = \frac{s'(\xi)}{s(\xi)}\,d\xi.$$

Therefore

$$\oint_{\partial D(R)} \frac{f'(z)}{f(z)}\,dz = -\oint_{\partial D(\frac{1}{R})} \frac{s'(\xi)}{s(\xi)}\,d\xi.$$

The negative sign happens because of orientation. The left integral computes $\sum_{a_i \in D(R)} \operatorname{ord}_{a_i}(f)$. The right integral computes $\sum_{a_k \in \mathbb{P}^1 \setminus \overline{D(R)}} \operatorname{ord}_{a_k}(f)$, since zeros of $s(\xi) = f(1/\xi)$ inside $|\xi| < 1/R$ correspond to zeros or poles of $f(z)$ outside $|z| > R$.

From the Argument Principle we know that

$$\frac{1}{2\pi i}\oint_{\partial D(R)} \frac{f'(z)}{f(z)}\,dz = \sum_{a_i \in D(R)} \operatorname{ord}_{a_i}(f).$$

Similarly, by the change of variables argument above, we have

$$\frac{1}{2\pi i}\oint_{\partial D(\frac{1}{R})} \frac{s'(\xi)}{s(\xi)}\,d\xi = \sum_{a_k \notin \overline{D(R)}} \operatorname{ord}_{a_k}(f).$$

So we have

$$\sum_{a_i \in D(R)} \operatorname{ord}_{a_i}(f) = -\sum_{a_k \notin \overline{D(R)}} \operatorname{ord}_{a_k}(f).$$

Since every zero or pole a_k of f lies either inside $D(R)$ or outside $\overline{D(R)}$, and since poles contribute negative orders while zeros contribute positive orders, we arrive at

$$\sum_{a_k \in \mathbb{P}^1} \operatorname{ord}_{a_k}(f) = 0.$$

Thus the total number of zeros of f (counted with multiplicity) equals the total number of poles (counted with multiplicity).

Sufficiency (Construction). Suppose we are given distinct points $E = \{a_1, \ldots, a_s\} \subset \mathbb{P}^1$ together with prescribed integers $n_1, \ldots, n_s$ that sum to zero. We will construct a meromorphic function f on $\mathbb{P}^1$ such that $\operatorname{ord}_{a_k}(f) = n_k$ for each k.

Case (a): $\infty \notin E$. Define

$$f(z) := \prod_{k=1}^{s} (z - a_k)^{n_k}.$$

Then clearly

$$\operatorname{ord}_{a_k}(f) = n_k \quad \text{for all } k, \qquad \operatorname{ord}_a(f) = 0 \quad \text{for all } a \in \mathbb{C} \setminus E.$$

Thus f has exactly the prescribed zeros and poles at finite points. It remains to examine the behavior at ∞.

We can write

$$f(z) = \prod_{k=1}^{s} z^{n_k} \left(1 - \tfrac{a_k}{z}\right)^{n_k} = z^{\sum_{k=1}^{s} n_k} h(z),$$

where $h(z) := \prod_{k=1}^{s} \left(1 - \frac{a_k}{z}\right)^{n_k}$ and $\lim_{z \to \infty} h(z) = 1$. Since $\sum_{k=1}^{s} n_k = 0$, we obtain

$$\operatorname{ord}_\infty(f) = \operatorname{ord}_\infty(z^0 \cdot h(z)) = \operatorname{ord}_\infty(h) = 0.$$

Thus f also satisfies the prescribed condition at ∞.

Case (b): $\infty \in E$. Without loss of generality assume $a_s = \infty$. In that case we set

$$f(z) := \prod_{k=1}^{s-1} (z - a_k)^{n_k}.$$

This function has

$$\operatorname{ord}_{a_k}(f) = n_k \quad (k = 1, \ldots, s-1), \qquad \operatorname{ord}_a(f) = 0 \quad \text{for all other finite } a.$$

To determine the order at ∞, note that $\sum_{k=1}^{s-1} n_k = -n_s$, so

$$f(z) = z^{-n_s} \prod_{k=1}^{s-1} \left(1 - \frac{a_k}{z}\right)^{n_k} = z^{-n_s} h(z),$$

where $\lim_{z \to \infty} h(z) = 1$. Therefore

$$\operatorname{ord}_\infty(f) = \operatorname{ord}_\infty(z^{-n_s}) + \operatorname{ord}_\infty(h) = \operatorname{ord}_0(w^{n_s}) + 0 = n_s.$$

In both cases, f realizes exactly the prescribed orders at all points of $\mathbb{P}^1$. This completes the proof. $\square$

5 Structure of $\mathcal{M}(\mathbb{P}^1)$

Definition 5.1 (Rational Functions)**.** Let $\operatorname{Rat}(\mathbb{P}^1)$ denote the field of rational functions on $\mathbb{P}^1$, i.e., functions of the form $f(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials with $Q(z) \not\equiv 0$.

Theorem 5.2 (Structure of $\mathcal{M}(\mathbb{P}^1)$). *We have*

$$\mathcal{M}(\mathbb{P}^1) = \mathrm{Rat}(\mathbb{P}^1),$$

i.e. every meromorphic function on the Riemann sphere is a rational function.

Proof. This follows directly from the solution to the Weierstrass problem.

Let $f \in \mathcal{M}(\mathbb{P}^1)$ with $f \not\equiv 0$. Denote by $E = Z(f) \sqcup P(f)$ the finite set of zeros and poles of f. Write the divisor of f as

$$\{(a_1, n_1), \ldots, (a_s, n_s)\}, \qquad a_k \in E, \quad n_k = \mathrm{ord}_{a_k}(f) \in \mathbb{Z}.$$

By the same argument as in the proof of the Weierstrass problem, the total order of f on the sphere vanishes; that is,

$$\sum_{k=1}^{s} n_k = \sum_{a \in \mathbb{P}^1} \mathrm{ord}_a(f) = 0.$$

By the solution to the Weierstrass problem on $\mathbb{P}^1$, there exists a rational function $h \in \mathrm{Rat}(\mathbb{P}^1)$ whose divisor is exactly $\{(a_k, n_k)\}_{k=1}^{s}$.

Consider $g = f/h$. Then for every $x \in \mathbb{P}^1$,

$$\mathrm{ord}_x(g) = \mathrm{ord}_x(f) - \mathrm{ord}_x(h) = 0,$$

since h was constructed with the same divisor as f.

Hence g has neither zeros nor poles on $\mathbb{P}^1$, i.e. g is a holomorphic function on the compact Riemann surface $\mathbb{P}^1$. By the Maximum Principle (equivalently, by Liouville's theorem), any such function must be constant. Thus $g \equiv c \in \mathbb{C}^\times$.

Therefore $f = ch$ with h rational, so f itself is rational. Since clearly $\mathrm{Rat}(\mathbb{P}^1) \subseteq \mathcal{M}(\mathbb{P}^1)$, we conclude

$$\mathcal{M}(\mathbb{P}^1) = \mathrm{Rat}(\mathbb{P}^1).$$

$\square$