MATH3301 Tutorial 10

1. Let R be a nonzero commutative ring with unity 1_R and characteristic $\text{char}(R) = n > 1$.

   (a) If $m1_R = 0$ where $m \in \mathbb{N}$, show that $n|m$.

   (b) Show that for any ideal $I \neq R$, $\text{char}(R/I)$ divides $\text{char}(R)$.

2. Let $(R, +, \cdot)$ be a nonzero commutative ring, and $R^\times \subset R$ be the group of units. Let $A \subset R$ be the subset of all zero divisors. Consider the following group action of $R^\times$ on R (i.e. $R^\times$ acts on R by):

   $\zeta \lessgtr 0$

   $$R^\times \times R \longrightarrow R : \qquad (g, x) \mapsto g \cdot x \ .$$

   (Remark: $g \cdot x$ is the multiplication of g and x in R, and for simplicity, you may write gx for $g \cdot x$.) For $x \in R$, let G_x and Gx be respectively its stabilizer and orbit for the action of $R^\times$. Also we let R^G be the fixed point set of $R^\times$ in R.

   (a) Show that $A \cap R^\times = \varnothing$.

   (b) If $x \in R$ is **not** a zero divisor, show that G_x is trivial.

   (c) Now suppose R is the ring $\mathbb{Z}_{2p}$ under the above action of $R^\times$, where p is an odd prime.

      i. List the elements in R^G. [Hint: Part (b) may be helpful.]

      ii. Count the number of elements x in R whose stabilizer G_x is a proper subgroup of $R^\times$.

3. Let R be a non-zero commutative ring with unity. Show that

   (a) if R is a field, then R is a PID.

   (b) R is an integral domain if and only if the zero ideal is a prime ideal.

   (c) R is a field if and only if the zero ideal is a maximal ideal.

4. Consider the ring $\mathbb{Z}[t]$.

   (a) Show that $\langle 2 \rangle, \langle t \rangle \subsetneq \langle 2, t \rangle$.

   (b) Show that $\langle 2, t \rangle$ is *not* a principal ideal in $\mathbb{Z}[t]$.

   (c) Show that $\mathbb{Z}[t]/\langle 2, t \rangle$ is a field and $\mathbb{Z}[t]/\langle 2 \rangle$ is *not* a field.

   (d) Show that $\mathbb{Z}[t]/\langle 2 \rangle$ is an integral domain.

   (e) Give an example of a nonzero prime ideal that is not a maximal ideal.

End