THE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH4406
Introduction to Partial Differential Equations
Homework 5

Due 3:30pm[1], November 1st (Friday)[2], **in-class**.

**Aim of this Homework:** *In this assignment you will show and apply the maximum principles to different PDE.*

**Reading Assignment:** Read the following material(s):

(i) Section 2.3 and 6.1 of the textbook.

**Instruction:** Answer Problem 1-4 below and show all your work. In order to obtain full credit, you are NOT required to complete any optional problem(s) or answer the "Food for Thought", but I highly recommend you to think about them. Moreover, if you hand in the optional problem(s), then our TA will also read your solution(s). A correct *answer without supporting work* receives little or NO credit! You should always give precise and adequate explanations to support your conclusions. Clarity of presentation of your argument counts, so **think carefully before you write**.

**Problem 1** (Maximum Principle for the Heat Equation on the Whole Line)**.** The purpose of this problem is to prove the **maximum principle** for the heat equation on the **real line**:

$$\begin{cases} \partial_t u - k\partial_{xx}u = 0 & \text{for } -\infty < x < \infty \text{ and } 0 < t < T \\ u|_{t=0} = \phi, & \end{cases} \tag{1}$$

[1] You are expected to submit your homework before the beginning of Friday lecture **in-class**.

[2] As discussed during the lecture on (Friday) October 4th, the unusually extended deadline for this assignment is due to the absence of lectures on the coming three Fridays.

where $k > 0$ and $\phi$ are given constant and initial data respectively. More precisely, you will be asked to show that if $u$ is a $C^2$ solution to (1) and satisfies the following property:

$$M_T := \max_{\substack{-\infty<x<\infty \\ 0\le t\le T}} u(t,x) < \infty, \tag{2}$$

then

$$\max_{\substack{-\infty<x<\infty \\ 0\le t\le T}} u(t,x) = \max_{-\infty<x<\infty} \phi(x). \tag{3}$$

To prove (3), complete the following steps.

(i) Let $M_0 := \max_{-\infty<x<\infty} \phi(x) < \infty$. For any $L > 0$, we define

$$u_L(t,x) := u(t,x) - \left\{ M_0 + \frac{M_T - M_0}{L^2}(x^2 + 2kt) \right\}.$$

Show that

$$\partial_t u_L - k\partial_{xx} u_L = 0.$$

(ii) Applying the maximum principle to $u_L$ over the domain $[0,T]\times[-L,L]$, prove that

$$\max_{\substack{-L\le x\le L \\ 0\le t\le T}} u_L(x,t) \le 0.$$

(Hint: in order to verify that $u_L \le 0$ on the parabolic boundary of $[0,T]\times[-L,L]$, you have to make use of the definitions of $M_0$ and $M_T$. For example, $M_T \ge M_0$ just because of their definitions.)

(iii) Let $(t_0, x_0)$ be any arbitrary point in $[0,T]\times(-\infty,\infty)$. Show that for any $L \ge |x_0|$, we have

$$u(t_0,x_0) \le \left\{ M_0 + \frac{M_T - M_0}{L^2}(x_0^2 + 2kt_0) \right\}.$$

(iv) Apply part (iii) to prove the assertion (3).

**Remark** (Maximum Principle for Unbounded Solutions)**.** In this problem, you have just shown the maximum principle (3), provided that the $C^2$ solution $u$ of (1) satisfies the boundedness property (2). Indeed, the above result can be generalized by releasing the hypothesis (2). More precisely, the maximum principle (3) still holds if we replace the boundedness condition (2) by the following growth rate condition at infinity: there exist positive constants $a$ and $M$ such that

$$u(t,x) \leq Me^{ax^2} \quad \text{for all } -\infty < x < \infty \text{ and } 0 < t < T.$$

The proof of this generalized result is quite technical, we refer interested students to section 7.1 (b) of [1] for instance.

In homework, tests and final exam, you are allowed to use this generalized version of maximum principle (for the heat equation) if applicable. However, you will NOT be asked to prove this generalized result in the tests and final exam.

**Remark** (Well-posedness for the Heat Equation on the Whole Line)**.** It follows from the maximum principle (as well as the explicit solution formula that we will discuss in class later) that the initial value problem (1) is well-posed in the solution class $\{u \in C^\infty((0,T)\times(-\infty,\infty)) \bigcap C([0,T]\times(-\infty,\infty));\ |u(t,x)| \leq Me^{ax^2}\}$. However, the uniqueness fails if we do not impose the growth rate condition $|u(t,x)| \leq Me^{ax^2}$. In other words, there are many *non-physical* solutions that grow faster than $Me^{ax^2}$ at the infinity. In the literature, we usually call these solutions *non-physical* because they are not observable in experiments. For the construction of *non-physical* solutions, we refer students to section 7.1 (a) of [1] for instance.

**Problem 2.** The method of maximum principles may also apply to linear PDE with variable coefficients and/or source term. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set. Assume that $u := u(x,y) \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\left(10x^4 + y^{2024}\right)\partial_{xx}u + 5\,\partial_{yy}u + \left(y^{330}\ln\frac{e^x}{1+e^x}\right)\partial_y u = 121x^2 + 22xy^5 + y^{10} + 1.$$

Show that

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

**Problem 3.** Maximum principles may also hold for nonlinear problems. Complete the following parts. It is worth noting that *the part (i) and (ii) below are NOT related.* They appear in the same problem just because both of them are related to the theme *maximum principle for nonlinear problems.*

(i) Let $d \geq 2$ be an integer, and

$$D := \left\{ (x_1, x_2, \dots, x_d) \in \mathbb{R}^d;\ \sum_{k=1}^{d} (10x_k - 22k)^2 < 2024 \right\}.$$

Suppose that $u \in C(\bar{D}) \cap C^2(D)$ is a solution to

$$\sum_{k=1}^{d} k^k\ \partial_{x_k}^2 u = \left(19\ x_2^{11}\ \sinh u\right)\ \partial_{x_d} u.$$

Show that

$$\max_{\bar{D}} |u| = \max_{\partial D} |u|.$$

(ii) (a) Compute

$$\min_{-\infty < w < \infty} e^{w^2} - w^2 + \frac{1}{2} w^4.$$

(b) Let $u := u(t,x) \in C^2([0,T] \times [0,L])$ be a solution to

$$\partial_t u + \left(x^2 + t\right) \sin\left(\partial_x u\right) - 24\ \partial_{xx} u = e^{u^2} - u^2 + \frac{1}{2} u^4 - \frac{3}{5},$$

for $0 < x < L$ and $0 < t < T$. Show that

$$\min_{\substack{0 \leq x \leq L \\ 0 \leq t \leq T}} u(t,x) = \min\left\{ \min_{0 \leq x \leq L} u(0,x), \min_{0 \leq t \leq T} u(t,0), \min_{0 \leq t \leq T} u(t,L) \right\}.$$

**Problem 4.** Do Problem 2 of Dec 2023 Final Exam.

The following problem(s) is/are *optional*:

**Problem 5** (Comparison Principle)**.** Let $T$ and $L$ be two positive constants, $u_1$ and $u_2 \in C^2((0,T]\times(-L,L))\cap C([0,T]\times[-L,L])$ be two solutions to the same parabolic equation

$$\partial_t u + a\partial_x u - k\partial_{xx} u = f \qquad \text{for } -L < x < L \text{ and } 0 < t < T,$$

where the parameters $a \in \mathbb{R}$ and $k > 0$ are two given constants, and the source term $f := f(t,x)$ is a given function. However, the solutions $u_1$ and $u_2$ satisfy different initial and boundary conditions: for $i = 1$, $2$,

$$\begin{cases} u_i|_{t=0} = \phi_i \\ u_i|_{x=-L} = g_i \\ u_i|_{x=L} = h_i, \end{cases}$$

where $\phi_i$, $g_i$ and $h_i$ are given data. Prove that if

$$\begin{cases} \phi_1 \le \phi_2 \\ g_1 \le g_2 \\ h_1 \le h_2, \end{cases}$$

then

$$u_1 \le u_2.$$

**Problem 6.** Do Problem 2 of Dec 2022 Final Exam.

**Problem 7.** *The part (i) and (ii) below are NOT related. They appear in the same problem just because both of them are short.*

(i) Let $u$ be a solution to the initial and boundary value problem

$$\begin{cases} \partial_t u = k\partial_{xx} u \qquad \text{for } -L < x < L \text{ and } t > 0 \\ u|_{t=0} = \phi \\ u|_{x=-L} = u|_{x=L} \equiv 0, \end{cases}$$

where $k > 0$ is a given constant. Prove the following statements:

(a) If $\phi$ is even, that is,

$$\phi(-x) = \phi(x),$$

then $u$ is also even in $x$, that is,

$$u(t, -x) = u(t, x).$$

(Hint: let $v(t, x) := u(t, -x)$, then what is the initial and boundary value problem that $v$ satisfies?).

(b) If $\phi$ is odd, that is,

$$\phi(-x) = -\phi(x),$$

then $u$ is also odd in $x$, that is,

$$u(t, -x) = -u(t, x).$$

(ii) Let $u$ be a solution to the following initial and boundary value problem:

$$\begin{cases} \partial_t u + c\,\partial_x u = 0 & \text{for } x \text{ and } t > 0 \\ u|_{x=0} = h(t) & \\ u|_{t=0} = \phi(x), & \end{cases}$$

where $c$ is a given positive constant, $h$ and $\phi$ are given initial and boundary data respectively. Prove or disprove that

$$\max_{t,x\geq 0} u(t, x) = \max\left\{\max_{x\geq 0} \phi(x), \max_{t\geq 0} h(t)\right\}.$$

# References

[1] John, Fritz; Partial differential equations. Reprint of the fourth edition. Applied Mathematical Sciences, 1. Springer-Verlag, New York, 1991.