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Elliptic Functions

With 14 Figures

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Chapter II

General properties of elliptic functions

§ 1. The period parallelogram. Given an elliptic function f, let (ω_1, ω_2) be a pair of *basic* periods for its period-lattice $\{m\omega_1+n\omega_2\}$, where $m, n=0, \pm 1, \pm 2, \ldots$. We may assume, without loss of generality, that $\operatorname{Im}\dfrac{\omega_2}{\omega_1}>0$, by interchanging, if necessary, ω_1 with ω_2. Let P denote the parallelogram, in the complex z-plane, formed by the points $z=x\omega_1+y\omega_2$, where $0\leqq x<1$, and $0\leqq y<1$. The vertices of P are $0, \omega_1, \omega_1+\omega_2$, and ω_2, in that order (noting that $\operatorname{Im}\dfrac{\omega_2}{\omega_1}>0$). In P are included the two sides extending from the origin to ω_1 and to ω_2, but not the other two sides, nor are the two end-points ω_1, ω_2 included. We call P a *fundamental period-parallelogram* associated with the given f (Fig. 4). It is *not* uniquely determined by the period-lattice, since by a proper, unimodular transformation one can pass from the given pair of basic periods (ω_1, ω_2) to another pair of basic periods (ω_1^*, ω_2^*), with $\operatorname{Im}\dfrac{\omega_2^*}{\omega_1^*}>0$.

We take it that the passage from 0 to ω_1, $\omega_1+\omega_2$, ω_2, and back to 0, in that order, defines the positive orientation of the curve ∂P which bounds P; passage in the opposite direction defines the negative orientation.

P:

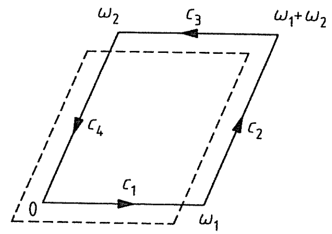

Fig. 4

If $\Omega=m\omega_1+n\omega_2$, and m and n run through all the positive and negative integers as well as zero, Ω runs through all the periods of f, each of them occurring just once.

Let $P_{m,n}$ denote the parallelogram obtained by translating P through the vector Ω. The points of $P_{m,n}$ are given by $x\omega_1+y\omega_2$, with $m\leqq x<m+1$, $n\leqq y<n+1$. Every point of the complex z-plane lies in exactly one such parallelogram, which we call a *period-parallelogram*, and $P_{0,0}=P$, which is a *fundamental period-parallelogram*.

Because of its periodicity, it suffices to study the given elliptic function in the period-parallelogram P defined by $z=x\omega_1+y\omega_2$, $0\leqq x<1$, $0\leqq y<1$. We denote by $\mathscr{L}$ the boundary of P, with the positive orientation, by c_1 the oriented segment extending from 0 to ω_1, by c_2, c_3, c_4 the other three oriented segments. Clearly c_2, c_3 do not belong to P, nor do the end-points of c_1, c_4 except the origin (Fig. 5).

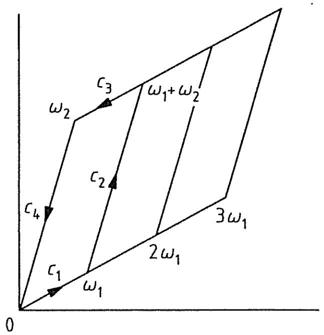

Fig. 5

§ 2. Elementary properties of elliptic functions. The class of elliptic functions has some general properties, which can be obtained by means of Cauchy's theorem of residues.

Theorem 1. *There exists no non-constant elliptic function which is entire.*

Proof. If $E(z)$ is an elliptic function, which is holomorphic on the *closed* parallelogram $\bar{P}$, then $|E(z)|<M<\infty$, for all $z\in P$, and therefore for all complex z, hence by Liouville's theorem, is a constant.

Corollary. *A non-constant elliptic function has at least one pole in any period-parallelogram.*

Theorem 2. *The sum of all the residues of an elliptic function at the poles inside a period-parallelogram is zero.*

Proof. Let $E(z)$ be an elliptic function, and $\mathscr{L}$ the boundary of a period-parallelogram of $E(z)$, positively oriented. If $E(z)$ has no pole on $\mathscr{L}$, then the sum of its residues in the period parallelogram equals

$$R=\frac{1}{2\pi i}\int_{\mathscr{L}} E(z)dz\,.$$

Because of the periodicity of $E(z)$, we have

$$\int_{c_1} E(z)dz+\int_{c_3} E(z)dz=0\,,\qquad \int_{c_4}+\int_{c_2}=0\,,$$

hence $R=0$.

If one or more poles lie on $\mathscr{L}$, then we move the boundary into $\mathscr{L}^*$, such that the sides of $\mathscr{L}^*$ are parallel to those of $\mathscr{L}$, but no poles lie on $\mathscr{L}^*$, and $\mathscr{L}^*$ encloses all the poles in P. This can be done, since poles are isolated points.

Corollary. *A non-constant elliptic function cannot have just one simple pole in a period-parallelogram. It must have therefore at least two simple poles, or at least one pole which is not simple, in any period-parallelogram.*

Theorem 3. *The number of zeros of a non-constant elliptic function in a period-parallelogram P is equal to the number of poles in P, the zeros and poles being counted according to their multiplicity.*

Proof. Let $E(z)$ be a non-constant elliptic function. Then its derivative $E'(z)$ is again an elliptic function, since it is meromorphic, and has as its periods (at least) those of $E(z)$.

The function $\dfrac{E'(z)}{E(z)}$ is likewise elliptic, and the sum of its residues in a period-parallelogram P is zero, by Theorem 2. That is to say,

$$\frac{1}{2\pi i}\int_{\mathscr{L}}\frac{E'(z)}{E(z)}\,dz=(\text{the number of zeros of } E(z) \text{ in } P)-$$
$$(\text{the number of poles of } E(z) \text{ in } P)=0\,,$$

assuming that none of the poles or zeros lie on $\mathscr{L}$. If not, we move the boundary as in the proof Theorem 2.

Definition. The *order* of an elliptic function is the number of its poles in a period-parallelogram, each pole being counted according to its multiplicity.

Theorem 4. *A non-constant elliptic function of order h assumes in a period-parallelogram every complex value exactly h times, taking into account the order of its multiplicity.*

Proof. Let $E(z)$ be a non-constant elliptic function of order h, and let c be a complex number. Set $F(z)=E(z)-c$. The function $F(z)$ has the same poles, and at those poles the same principal parts, as $E(z)$. Therefore $F(z)$ is a non-constant elliptic function of order h (which is independent of c). Its zeros are the roots of the equation $E(z)=c$, and with the same multiplicity in each case. Theorem 3 applied to the function F gives the required result.

Definition. If $a_1, a_2, \ldots, a_h$ denote the zeros of a non-constant elliptic function f in a period-parallelogram P, each of them repeated according to its multiplicity, *the sum of the zeros of f in P* is defined to be $a_1+a_2+\ldots+a_h$. If $b_1, b_2, \ldots, b_h$ denote the poles of f in P, *the sum of the poles of f in P* is similarly defined to be $b_1+b_2+\ldots+b_h$.

Theorem 5. *The sum of the zeros of a non-constant elliptic function in a period-parallelogram differs from the sum of its poles by a period.*

Proof. Let $E(z)$ be the given non-constant elliptic function, and let P be a period-parallelogram of $E(z)$. Let $g(z)$ denote a function which is holomorphic in $\bar{P}$, the *closure* of P. Define

$$G(z)=g(z)\cdot\frac{E'(z)}{E(z)}\,.$$

It is holomorphic in $\bar{P}$ except, possibly, for poles at those points where $E(z)$ has zeros or poles.

Let $a_1, a_2, \ldots, a_h$ be the zeros, and $b_1, b_2, \ldots, b_h$ the poles of $E(z)$ in P. If $a_1 = \ldots = a_k = a$ is a zero of order k, we have the Laurent expansion at a given by

$$\frac{E'(z)}{E(z)} = \frac{k}{z-a} + \ldots + \ldots = \sum_{n=1}^{k} \frac{1}{z-a_n} + \ldots,$$

and correspondingly also

$$\frac{E'(z)}{E(z)} = -\frac{l}{z-b} + \ldots = -\sum_{n=1}^{l} \frac{1}{z-b_n} + \ldots,$$

if $b_1 = \ldots = b_l = b$ is a pole of order l.

If we assume, to begin with, that no poles or zeros of $E(z)$ lie on the boundary $\mathscr{L}$ of P, then by the theorem of residues,

$$\frac{1}{2\pi i} \int_{\mathscr{L}} G(z) dz = \sum_{n=1}^{h} g(a_n) - \sum_{n=1}^{h} g(b_n).$$

If we *choose* $g(z) = z$, then the right-hand side gives the difference between the sum of the zeros of $E(z)$ and the sum of its poles. We wish to show that if

$$\frac{1}{2\pi i} \int_{\mathscr{L}} G(z) dz = \Omega,$$

then Ω is a period of $E(z)$. Because of the periodicity of $E(z)$, we obviously have (see the figure in § 1)

$$\begin{aligned} 2\pi i \cdot \Omega &= \int_{c_1} \{z-(z+\omega_2)\} \frac{E'(z)}{E(z)} dz \\ &\quad + \int_{c_4} \{z-(z+\omega_1)\} \frac{E'(z)}{E(z)} dz \\ &= \omega_1 \cdot \int_0^{\omega_2} \frac{E'(z)}{E(z)} dz - \omega_2 \int_0^{\omega_1} \frac{E'(z)}{E(z)} dz \\ &= \omega_1 [\log E(z)]_0^{\omega_2} - \omega_2 [\log E(z)]_0^{\omega_1}, \end{aligned}$$

where $[\log E(z)]_0^{\omega_1}$ denotes the variation of any fixed branch of the many-valued function $\log E(z)$ as z varies from 0 to ω_1. Since $E(0) = E(\omega_1)$, we have $\log E(0) = \log E(\omega_1) + 2\pi i n_1$, where n_1 is an integer. Similarly $\log E(0) = \log E(\omega_2) + 2\pi i n_2$, where n_2 is an integer. Hence $-\Omega = n_2\omega_1 - n_1\omega_2$, which is a period of $E(z)$.

Should zeros or poles of $E(z)$ lie on $\mathscr{L}$, we move the boundary of P as before, and then apply the above argument.

Remark. The conditions satisfied by the zeros and poles of an elliptic function, as in Theorem 3 or Theorem 5, will in fact also *suffice* for the existence

of an elliptic function with any given set of complex numbers satisfying those conditions as its zeros and poles.

We shall proceed next to *construct* a non-constant elliptic function. In seeking to do so, we note that because of Theorem 2 and its Corollary, the simplest class of elliptic functions, as far as singularities are concerned, are those of order 2. They may be of two types; those which have a double pole, with residue zero, as their only singularity in a period-parallelogram, known as the *Weierstrassian elliptic functions*, and those which have in every period-parallelogram precisely two simple poles, with residues which are equal in absolute value but opposite in sign, known as the *Jacobian elliptic functions*. We shall study the first type in Chapter III, and the second type in Chapter VII.

Notes on Chapter II

For the results of this Chapter, see J. Liouville (Lectures published by C. W. Borchardt), *J. für Math.* 88 (1880), 277–310. Liouville's method of proof is based on the use of Cauchy's theorem of residues, and differs from that of Jacobi's and Weierstrass's. Clearly this presentation is therefore not chronological.

Theorem 1 can be used to prove a number of results on elliptic functions as well as on theta-functions (cf. Chapter V).

That an entire, doubly-periodic function $f(z)$ is a constant, follows also from the Fourier expansion (cf. Notes on Chapter I, § 2)

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z/\omega}, \qquad \omega \neq 0,$$

where ω is a period of f.

Chapter III

Weierstrass's elliptic function $\wp(z)$

§ 1. The convergence of a double series. Let ω_1, ω_2 be two complex numbers, both of them different from zero, and let $\tau=\omega_2/\omega_1$, with $\operatorname{Im}\tau>0$. Let $\omega=m\omega_1+n\omega_2$, where $m, n=0, \pm 1, \pm 2, \ldots$. The points (ω) form a *lattice* $\mathfrak{L}$, say, in the complex plane. We shall consider the series

$$\sum_{\omega}{}' \frac{1}{|\omega|^{\varrho}}=\sum_{\omega\in\mathfrak{L},\,\omega\neq 0}\frac{1}{|\omega|^{\varrho}}=\sum_{\substack{m=-\infty\\(m,n)\neq(0,0)}}^{\infty}\ \sum_{n=-\infty}^{\infty}\frac{1}{|(m\omega_1+n\omega_2)|^{\varrho}},$$

for real values of ϱ.

Theorem 1. *The series $\sum_{\omega}' |\omega|^{-\varrho}$ converges for $\varrho>2$, and diverges for $\varrho\leqq 2$.*

Proof. Define the 'square' partial sums

$$S_k=\sum_{|m|\leqq k,\,|n|\leqq k}{}' |\omega|^{-\varrho}, \qquad k=1,2,3,\ldots,$$

and set $T_k=S_k-S_{k-1}$, with $S_0\equiv 0$. The series $\sum' |\omega|^{-\varrho}$ converges if and only if the series $\sum_{k=1}^{\infty} T_k$ converges, since the terms are all positive. The number of terms in T_k is $8k$, since $(2k+1)^2-1-\{(2k-2+1)^2-1\}=8k$. Each of them is either of the form

$$|k\omega_1+n\omega_2|^{-\varrho}, \qquad |n|\leqq k, \quad \text{or} \quad |m\omega_1+k\omega_2|^{-\varrho}, \qquad |m|\leqq k.$$

The corresponding points (ω) lie on the boundary of the parallelogram with vertices at $k\omega_1+k\omega_2$, $-k\omega_1+k\omega_2$, $-k\omega_1-k\omega_2$, and $k\omega_1-k\omega_2$. Hence there exist two numbers a and b, with $a>0$, $b>0$, and independent of k, such that if ω corresponds to a term in T_k, we have

$$a\cdot k<|\omega|<b\cdot k,$$

and hence

$$8\cdot b^{-\varrho}\cdot k^{1-\varrho}<T_k<8\cdot a^{-\varrho}\cdot k^{1-\varrho},$$

from which it follows that $\sum_{k=1}^{\infty} T_k$ converges if and only if $\sum_{k=1}^{\infty} k^{1-\varrho}$ converges, that is to say, if and only if $\varrho>2$.

Corollary 1. *For any $R>0$, and complex z, the series*

$$\sum_{\omega\in\mathfrak{L},\,|\omega|>2R}|z-\omega|^{-\varrho}, \qquad \varrho>2,$$

converges uniformly in the circle $|z|\leqq R$. Hence the series

$$\sum_{\omega}|z-\omega|^{-\varrho}, \qquad \varrho>2,$$

converges uniformly in every circle of finite radius, if we discard a sufficient number of terms at the beginning.

For, $|z|<\frac{1}{2}|\omega|$, so that $|z|+|\omega|\leqq\frac{3}{2}|\omega|$, and we have

$$\frac{1}{|z-\omega|}\leqq\frac{1}{|\omega|-|z|}\leqq\frac{2}{|\omega|},$$

$$\frac{1}{|z-\omega|}\geqq\frac{1}{|z|+|\omega|}\geqq\frac{2}{3}\cdot\frac{1}{|\omega|},$$

hence

$$\left(\frac{2}{3}\right)^{\varrho}\cdot\frac{1}{|\omega|^{\varrho}}\leqq\frac{1}{|z-\omega|^{\varrho}}\leqq\frac{2^{\varrho}}{|\omega|^{\varrho}}, \quad \text{for} \quad \varrho>0.$$

The Corollary now follows from Theorem 1 and the second inequality; the first inequality shows that the assertion of the Corollary is not true for $\varrho\leqq 2$.

Corollary 2. *The series*

$$\sum_{\omega\in\mathfrak{L},\,\omega\neq 0}\left\{\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right\}$$

converges absolutely for $z\notin\mathfrak{L}$, so that the sum is independent of the order of the terms. For every finite $R>0$, the series converges uniformly in the circle $|z|\leqq R$, after the omission of a sufficient number of initial terms.

Given $R>0$, we have $|\omega|>2R$ except for finitely many points ω; and if $|z|\leqq R$, we have

$$\left|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right|=\left|\frac{z(2\omega-z)}{\omega^2(z-\omega)^2}\right|\leqq\frac{10\cdot|z|}{|\omega|^3}\leqq\frac{10\cdot R}{|\omega|^3},$$

except for finitely many points ω, since

$$\frac{|z|}{|\omega|}<\frac{1}{2}, \quad \text{and} \quad \left|\frac{z(2\omega-z)}{\omega^2(z-\omega)^2}\right|=\left|\frac{\omega z(2-z/\omega)}{\omega^4(1-z/\omega)^2}\right|,$$

where

$$\left|2-\frac{z}{\omega}\right|\leqq 2+\frac{1}{2}, \quad \text{and} \quad \left|1-\frac{z}{\omega}\right|^2\geqq\left(1-\frac{1}{2}\right)^2.$$

Since the series $\sum_{\omega\in\mathfrak{L},\,\omega\neq 0}|\omega|^{-3}$ converges by Theorem 1, the Corollary follows.

§ 2. The elliptic function $\wp(z)$. After Corollary 2 of § 1, we *define*, for complex z,

$$\wp(z)=\frac{1}{z^2}+\sum_{\omega\in\mathfrak{L},\,\omega\neq 0}\left\{\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right\}. \tag{2.1}$$

To indicate the dependence on ω_1, ω_2 (which determine the lattice $\mathfrak{L}$), we sometimes write $\wp(z;\omega_1,\omega_2)$ instead of $\wp(z)$.

The function $\wp(z)$ so defined *is* an elliptic function, which we shall study in considerable detail. Its basic properties are given by the following

Theorem 2. *The function $\wp(z)$ is an elliptic function with the periods ω_1, ω_2. Its poles are given by $z=\omega$. It has the further properties:*
(i) *the principal part of $\wp(z)$ at $z=0$ is $1/z^2$;*
(ii) $\lim_{z\to 0}\left(\wp(z)-\frac{1}{z^2}\right)=0$;
(iii) $\wp(z)=\wp(-z)$;
(iv) $\wp'(-z)=-\wp'(z)$.

Proof. By Corollary 2, combined with Weierstrass's theorem on uniformly convergent series of holomorphic functions, it follows that $\wp(z)$ is a meromorphic function, with *double* poles at $z=\omega=m\omega_1+n\omega_2$; $m,n=0,\pm 1,\pm 2,\ldots$. By considering the difference $\wp(z)-\frac{1}{z^2}$ in a neighbourhood of $z=0$, we obtain (i) and (ii).

To prove (iii) we have only to note that

$$\wp(-z)=\frac{1}{z^2}+\sum_{\omega\in\mathfrak{L},\,\omega\neq 0}\left\{\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right\},$$

and that the set of points $\{-\omega\}$ is the same as the set $\{\omega\}$. Property (iv) follows from (iii).

The uniform convergence, in the sense of Corollary 2, of the (double) series defining $\wp(z)$ permits differentiation term by term, so that the derivative $\wp'(z)$ of $\wp(z)$ is, in fact, given by

$$\wp'(z)=-2\sum_{\omega\in\mathfrak{L}}\frac{1}{(z-\omega)^3},$$

the series converging absolutely for $z\notin\mathfrak{L}$. Since the set of points $\{\omega-\omega_1\}$ is the same as the set $\{\omega\}$, the series for $\wp'(z+\omega_1)$ is just a rearrangement of the series for $\wp'(z)$. Hence $\wp'(z+\omega_1)=\wp'(z)$, likewise $\wp'(z+\omega_2)=\wp'(z)$. Therefore $\wp'(z)$ is an *elliptic function.* By integration, we get the relation: $\wp(z+\omega_1)=\wp(z)+c$, where c is a constant. Setting $z=-\frac{\omega_1}{2}$, we get $\wp\left(\frac{\omega_1}{2}\right)=\wp\left(-\frac{\omega_1}{2}\right)+c$, and since $\wp$ is an even function, it follows that $c=0$, hence $\wp(z+\omega_1)=\wp(z)$, and similarly $\wp(z+\omega_2)=\wp(z)$, so that $\wp(z)$ is an *elliptic function*, with the two periods ω_1, ω_2, with $\operatorname{Im}\frac{\omega_2}{\omega_1}>0$.

Remarks

(i) The higher derivatives of $\wp(z)$ are also elliptic functions (a special case of a general property) given by the corresponding (differentiated) series.

(ii) We may assume that $\left|\frac{\omega_2}{\omega_1}\right|\geqq 1$. In the period-parallelogram defined by $z=\xi\omega_1+\eta\omega_2$, $0\leqq\xi<1$, $0\leqq\eta<1$, $\wp(z)$ has exactly one pole, of the second order, at $z=0$, with residue zero. The points of the lattice $\mathfrak{L}$, which we may call the *period-lattice*, (in as much as ω_1, ω_2 have been shown to be periods of $\wp(z)$), are all poles of $\wp(z)$; no other points are.

(iii) If λ is any complex number, $\lambda\neq 0$, then

$$\wp(\lambda z;\lambda\omega_1,\lambda\omega_2)=\lambda^{-2}\wp(z;\omega_1,\omega_2). \tag{2.2}$$

§ 3. The differential equation associated with $\wp(z)$

Theorem 3. *The elliptic function $\wp(z)$ satisfies the differential equation*

$$\wp'^2(z)=4\wp^3(z)-g_2\wp(z)-g_3,$$

where

$$g_2=60\sum_{\omega\in\mathfrak{L},\,\omega\neq 0}\omega^{-4},\qquad g_3=140\sum_{\omega\in\mathfrak{L},\,\omega\neq 0}\omega^{-6}.$$

Proof. By Corollary 2 of § 1, the series

$$\sum_{|\omega|>2R>0}\left\{\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right\}$$

converges absolutely and uniformly in the circle $|z|\leqq R$. If $|\omega|>2R\geqq|z|$, we have

$$\left\{\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right\}=\frac{2z}{\omega^3}+\frac{3z^2}{\omega^4}+\frac{4z^3}{\omega^5}+\frac{5z^4}{\omega^6}+\ldots.$$

Since $\wp(z)$ is an even function, its Laurent expansion at $z=0$ (i.e. for $0<|z|<\delta$) is given by

$$\wp(z)=\frac{1}{z^2}+b_1z^2+b_2z^4+\ldots+b_nz^{2n}+\ldots,$$

since $\wp(z)-\frac{1}{z^2}=0$ for $z=0$ (Theorem 2). Here

$$b_1=3\sum_{\omega\neq 0}\frac{1}{\omega^4},\qquad b_2=5\sum_{\omega\neq 0}\frac{1}{\omega^6},\qquad b_n=(2n+1)\sum_{\omega\neq 0}\frac{1}{\omega^{2n+2}}.$$

It follows that

$$(\wp(z))^3=z^{-6}(1+b_1z^4+b_2z^6+\ldots)^3=\frac{1}{z^6}+\frac{3b_1}{z^2}+3b_2+\ldots,$$

and

$$(\wp'(z))^2 = z^{-6}(-2+2b_1z^4+4b_2z^6+\dots)^2 = \frac{4}{z^6} - \frac{8b_1}{z^2} - 16b_2 + \dots,$$

so that the difference $(\wp'(z))^2 - 4\wp^3(z)$ has a pole of the second order at $z=0$, with the expansion

$$(\wp'(z))^2 - 4\wp^3(z) = -\frac{20b_1}{z^2} - 28b_2 + z^2(\dots).$$

On setting $g_2 = 20b_1$, $g_3 = 28b_2$, we have

$$(\wp'(z))^2 - 4\wp^3(z) = -\frac{g_2}{z^2} - g_3 + z^2(\dots).$$

It follows that the function

$$(\wp'(z))^2 - 4\wp^3(z) + g_2\wp(z) + g_3,$$

which is obviously *elliptic*, is holomorphic at $z=0$, hence also at all points of the period-lattice. It has therefore no poles at all, so that it reduces to a constant (Theorem 1, Chapter II), which is zero, since the function vanishes for $z=0$, and the theorem follows.

Theorem 4. *The Laurent expansion of $\wp(z)$ at $z=0$ is given by*

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} b_n z^{2n},$$

where

$$b_1 = \frac{g_2}{20}, \qquad b_2 = \frac{g_3}{28}, \qquad b_n = \frac{3}{(2n+3)(n-2)} \cdot \sum_{k=1}^{n-2} b_k b_{n-k-1},$$

for $n=3, 4, \dots$, and g_2, g_3 are defined as in Theorem 3.

Proof. In the course of proof of Theorem 3, we have seen that

$$\wp(z) = \frac{1}{z^2} + b_1z^2 + b_2z^4 + \dots + b_nz^{2n} + \dots.$$

By further differentiation of the equation $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$, we obtain: $2\wp'(z)\wp''(z) = 12(\wp(z))^2\wp'(z) - g_2\wp'(z)$, so that $\wp''(z) = 6(\wp(z))^2 - \frac{1}{2}g_2$, since $\wp'(z)$ is not identically zero. We thus have the identity

$$\frac{6}{z^4} + \sum_{n=1}^{\infty} 2n(2n-1)b_nz^{2n-2} = \frac{6}{z^4}\left(1 + \sum_{n=1}^{\infty} b_nz^{2n+2}\right)^2 - \frac{g_2}{2}.$$

By comparing the coefficients of z^{2n-2} on either side, for $n>2$, we obtain

$$2n(2n-1)b_n = 6(2b_n + b_1b_{n-2} + b_2b_{n-3} + \dots + b_{n-2}b_1).$$

Since $2n(2n-1) - 12 = 2(2n+3)(n-2) \neq 0$, for $n>2$, we have the recurrence formula

$$b_n = \frac{6}{2(2n+3)(n-2)} \cdot \sum_{k=1}^{n-2} b_k b_{n-k-1}, \qquad n=3,4,\dots,$$

and, in particular,

$$b_3 = \frac{1}{3}b_1^2 = \frac{1}{3}\left(\frac{g_2}{20}\right)^2 = \frac{g_2^2}{1200}, \qquad b_4 = \frac{3g_2g_3}{2^4\cdot 5\cdot 7\cdot 11}.$$

Remarks

(i) It follows from the above argument that all the coefficients b_n, $n=1,2,3,\dots$, are polynomials in g_2, g_3, with positive, rational numbers as coefficients.

(ii) The circle of convergence of the Laurent expansion of $\wp(z)$ at $z=0$ passes through the lattice-point $z=\omega_1$, and its radius is, in fact, equal to $|\omega_1|$.

(iii) The coefficients b_n can be expressed in terms of the periods ω. By differentiating the Laurent expansion, we get

$$\wp'(z) + \frac{2}{z^3} = \sum_{n=1}^{\infty} 2nb_nz^{2n-1}.$$

On the other hand, we have, by the definition of $\wp(z)$,

$$\wp'(z) + \frac{2}{z^3} = -\sum_{\omega\neq 0} \frac{2}{(z-\omega)^3},$$

which we can differentiate, term by term, $(2n-1)$ times, and obtain, for $z=0$,

$$(2n)!\,b_n = (2n+1)! \sum_{\omega\neq 0} \frac{1}{\omega^{2n+2}}, \qquad (n=1,2,\dots),$$

so that

$$b_{n-1} = (2n-1)\sigma_n, \qquad n=2,3,\dots,$$

where

$$\sigma_n = \sum_{\omega\neq 0} \frac{1}{\omega^{2n}}, \qquad n=2,3,\dots.$$

It follows that

$$\sum_{\omega\neq 0} \frac{1}{\omega^4} = \sigma_2 = \frac{b_1}{3} = \frac{g_2}{60};$$

$$\sum_{\omega\neq 0} \frac{1}{\omega^6} = \sigma_3 = \frac{b_2}{5} = \frac{g_3}{140};$$

$$\sum_{\omega\neq 0} \frac{1}{\omega^{2n}} = \sigma_n = \frac{b_{n-1}}{2n-1} = P_n(\sigma_2, \sigma_3), \qquad n=2,3,\dots,$$

where $P_n(\sigma_2, \sigma_3)$ is a polynomial in σ_2 and σ_3, with positive, rational numbers as coefficients.

Given a pair of basic periods (ω_1, ω_2) of $\wp(z)$, g_2 and g_3 are uniquely defined by the formulae

$$g_2 = 60 \sum_{m,n}' \frac{1}{(m\omega_1 + n\omega_2)^4}, \qquad g_3 = 140 \sum_{m,n}' \frac{1}{(m\omega_1 + n\omega_2)^6},$$

where m and n run through all the values $0, \pm 1, \pm 2, \ldots$, except $m=n=0$. If

$$\omega_1^* = m\omega_1 + n\omega_2, \qquad \omega_2^* = p\omega_1 + q\omega_2,$$

where m, n, p, q are integers such that $mq - np = +1$, then (ω_1^*, ω_2^*) is *also* a pair of basic periods (see Remark (i), after Theorem 2, Chapter I). Hence $g_2(\omega_1, \omega_2) = g_2(\omega_1^*, \omega_2^*)$, and $g_3(\omega_1, \omega_2) = g_3(\omega_1^*, \omega_2^*)$. For this reason, g_2 and g_3 are commonly referred to as the *invariants* of $\wp(z)$.

We note incidentally that

$$\sum_{\omega \neq 0} \frac{1}{\omega^{2n+1}} = 0, \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

since the terms corresponding to periods ω of opposite signs cancel each other.

If (ω_1, ω_2) is a pair of basic periods of the $\wp$-function, we say that z_1 *is congruent to* z_2 *modulo* (ω_1, ω_2), where z_1, z_2 are complex numbers, if $z_1 - z_2 = m\omega_1 + n\omega_2$, for some integers m and n. We indicate this by writing $z_1 \equiv z_2 \pmod{\omega_1, \omega_2}$.

Theorem 5. *If g_2 and g_3 are the invariants of the function $\wp(z)$, defined by the formulae*

$$g_2 = 60 \sum_{\omega \neq 0} \omega^{-4}, \qquad g_3 = 140 \sum_{\omega \neq 0} \omega^{-6},$$

where $\omega = m\omega_1 + n\omega_2$, (ω_1, ω_2) being a pair of basic periods of the $\wp$-function, and $m, n = 0, \pm 1, \pm 2, \ldots$, then we have

$$g_2^3 - 27g_3^2 \neq 0.$$

Proof. Since the elliptic function $\wp'(z)$ is of order 3, it has only three zeros in the period-parallelogram P consisting of the points z, where $z = x\omega_1 + y\omega_2$, $0 \leq x < 1$, $0 \leq y < 1$. Since $\wp'(z)$ is an *odd* function of z, we have

$$\wp'\left(\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2} + \omega_1\right) = -\wp\left(\frac{\omega_1}{2}\right),$$

hence $\wp'\left(\frac{\omega_1}{2}\right) = 0$. Similarly $\wp'\left(\frac{\omega_2}{2}\right) = 0$, and since

$$\wp'\left(\frac{\omega_1 + \omega_2}{2}\right) = \wp'\left(\frac{-\omega_1 - \omega_2}{2}\right) = -\wp'\left(\frac{\omega_1 + \omega_2}{2}\right),$$

we have also $\wp'\left(\frac{\omega_1 + \omega_2}{2}\right) = 0$. Thus $\frac{\omega_1}{2}, \frac{\omega_2}{2}$, and $\frac{\omega_1 + \omega_2}{2}$ are *the* three zeros of $\wp'(z)$ in P. Since $\{\wp'(z)\}^2 = 4\wp^3(z) - g_2\wp(z) - g_3$, it follows that $\frac{\omega_1}{2}, \frac{\omega_2}{2}$, and $\frac{\omega_1 + \omega_2}{2}$ are *the* three roots of the cubic equation

$$4\wp^3(z) - g_2\wp(z) - g_3 = 0$$

in P. Set

$$e_r = \wp\left(\frac{\omega_r}{2}\right) = \wp\left(\frac{\omega_r}{2}; \omega_1, \omega_2\right), \qquad r = 1, 2, 3; \qquad \omega_3 = \omega_1 + \omega_2. \quad (3.1)$$

Then e_1, e_2, e_3 are *all different*. For consider the function $\wp(z) - e_1$. It has a zero at $z = \frac{\omega_1}{2}$. But $\wp'\left(\frac{\omega_1}{2}\right) = 0$, as we have just seen. Therefore it is a *double* zero. Since $\wp(z)$ has only a *double* pole in P, it follows that the zeros of $\wp(z) - e_1$ in the z-plane are *all* congruent to $\frac{\omega_1}{2}$ modulo (ω_1, ω_2), that is to say, differ from $\frac{\omega_1}{2}$ only by a period.

Similarly the (double) zeros of $\wp(z) - e_2$, and of $\wp(z) - e_3$, are *all* congruent respectively to $\frac{\omega_2}{2}$, and to $\frac{\omega_3}{2}$, modulo (ω_1, ω_2).

It follows that $e_1 \neq e_2 \neq e_3 \neq e_1$. For if $e_1 = e_2$, then $\wp(z) - e_1$ is zero for $z = \frac{\omega_2}{2}$, but $\frac{\omega_2}{2}$ is *not* congruent to $\frac{\omega_1}{2}$ modulo (ω_1, ω_2). Similarly $e_2 \neq e_3$, and $e_3 \neq e_1$.

Since e_1, e_2, e_3 are the roots of the cubic equation

$$4t^3 - g_2 t - g_3 = 0,$$

we have

$$e_1 + e_2 + e_3 = 0; \qquad e_1 \cdot e_2 \cdot e_3 = \frac{g_3}{4};$$

$$e_2e_3 + e_3e_1 + e_1e_2 = -\frac{g_2}{4};$$

and since $e_1 \neq e_2 \neq e_3 \neq e_1$, we have $g_2^3 - 27g_3^2 \neq 0$. (See Appendix I at the end of this chapter.)

Remarks

(i) The *discriminant of the cubic*

$$u \equiv ax^3 + 3bx^2 + 3cx + d,$$

where $a \neq 0$, and a, b, c, d are complex numbers, is *defined* to be

$$D = a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2.$$

If α, β, γ are the three roots of the equation $u=0$, then

$$a^4(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-27D.$$

If follows that a necessary and sufficient condition that the equation $u=0$ have a repeated root is that $D=0$.

(ii) If $a=4$, $b=0$, $3c=-g_2$, $d=-g_3$, then we have $D=16\left(g_3^2-\frac{g_2^3}{27}\right)$. In that case, e_1, e_2, and e_3, are the three roots. Hence

$$4^4(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2=27\cdot\tfrac{16}{27}(g_2^3-27g_3^2),$$

or

$$16(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2=(g_2^3-27g_3^2).$$

(iii) We define

$$\Delta\equiv\Delta(\omega_1,\omega_2)=g_2^3-27g_3^2.$$

The three functions $g_2(\omega_1,\omega_2)$, $g_3(\omega_1,\omega_2)$, and $\Delta(\omega_1,\omega_2)$ are known as the *elementary modular forms*. (cf. Notes on Chapter VI).

(iv) If we set $\tau=\dfrac{\omega_2}{\omega_1}$, we obtain

$$g_2(\omega_1,\omega_2)=\omega_1^{-4}g_2(1,\tau);\qquad g_3(\omega_1,\omega_2)=\omega_1^{-6}g_3(1,\tau);$$

hence $\Delta(\omega_1,\omega_2)=\omega_1^{-12}\Delta(1,\tau)$. (Note that $\operatorname{Im}\tau>0$).

§ 4. The addition-theorem. It is possible to express $\wp(u+v)$ in terms of $\wp(u)$, and $\wp(v)$, and their derivatives, as is shown by the following (which is known as the addition-theorem)

Theorem 6. *If $z_1\not\equiv z_2$ modulo (ω_1,ω_2), then we have*

$$\wp(z_1+z_2)=\frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2-\wp(z_1)-\wp(z_2). \tag{4.1}$$

Proof. Let z_2 be such that $\wp'(z_2)\neq 0$, *and* $\wp(z_2)$ is finite, while $z_1=z$ is variable. Set

$$\varphi(z)=\wp(z+z_2)+\wp(z)+\wp(z_2)-\frac{1}{4}\left(\frac{\wp'(z)-\wp'(z_2)}{\wp(z)-\wp(z_2)}\right)^2.$$

Then $\varphi(z)$ is an elliptic function. The only possible poles of $\varphi(z)$ are at those values of z which differ from $z=0$, or from $z=-z_2$, by a period of $\wp(z)$. (Note that, by assumption, $z\neq z_2$). Near $z=0$, we have the expansions

$$\wp(z)=\frac{1}{z^2}+b_1z^2+\dots,$$

$$\wp'(z)=-\frac{2}{z^3}+2b_1z+\dots,$$

and

$$\frac{1}{4}\left(\frac{\wp'(z)-\wp'(z_2)}{\wp(z)-\wp(z_2)}\right)^2-\wp(z)=\frac{1}{z^2}\left(\frac{1+\frac{1}{2}\wp'(z_2)z^3-b_1z^4+\dots}{1-\wp(z_2)z^2+b_1z^4+\dots}\right)^2-\frac{1}{z^2}-b_1z^2-\dots,$$

$$=2\wp(z_2)+\dots,$$

so that $\varphi(z)$ is holomorphic at $z=0$, with $\varphi(0)=0$.

Near $z=-z_2$, we have the expansions

$$\wp(z+z_2)=\frac{1}{(z+z_2)^2}+\dots,$$

and

$$\frac{1}{4}\left(\frac{\wp'(z)-\wp'(z_2)}{\wp(z)-\wp(z_2)}\right)^2=\left(\frac{\wp'(z_2)-\frac{1}{2}\wp''(z_2)(z+z_2)+\dots}{\wp'(z_2)(z+z_2)-\frac{1}{2}\wp''(z_2)(z+z_2)^2+\dots}\right)^2$$

$$=\frac{1}{(z+z_2)^2}-\dots,$$

so that $\varphi(z)$ is also holomorphic at $z=-z_2$.

As an elliptic function which is entire, $\varphi(z)$ reduces to a constant, which is zero, since $\varphi(0)=0$. This has been proved on the assumption that $z_2\neq 0, \frac{\omega_j}{2}$, $j=1,2,3$. But this assumption can be dropped, since we can interchange z_1 and z_2 in the assertion of the theorem.

Corollary. *If $z_1\not\equiv z_2$ modulo (ω_1,ω_2), we have the relation*

$$\wp(z_1+z_2)+\wp(z_1-z_2)=\frac{\{2\wp(z_1)\wp(z_2)-\frac{1}{2}g_2\}\{\wp(z_1)+\wp(z_2)\}-g_3}{\{\wp(z_1)-\wp(z_2)\}^2},$$

since, by Theorem 6, we have

$$\begin{aligned}\wp(z_1+z_2)+\wp(z_1-z_2)&=\frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2-\wp(z_1)-\wp(z_2)\\&\quad+\frac{1}{4}\left(\frac{\wp'(z_1)+\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2-\wp(z_1)-\wp(z_2)\\&=\frac{1}{2}\left(\frac{\{\wp'(z_1)\}^2+\{\wp'(z_2)\}^2}{\{\wp(z_1)-\wp(z_2)\}^2}\right)-2\{\wp(z_1)+\wp(z_2)\},\end{aligned}$$

while

$$\{\wp'(z_1)\}^2=4\wp^3(z_1)-g_2\wp(z_1)-g_3,\quad \textit{and}\quad \{\wp'(z_2)\}^2=4\wp^3(z_2)-g_2\wp(z_2)-g_3.$$