Complex Numbers (from: Wikipedia – selected & edited slightly & exercises added)

A complex number is a number that can be expressed in the form:

$$a + bi,$$

where a and b are real[1] numbers and i is the *imaginary unit*, satisfying $i^2 = -1$. (i.e., $i = \sqrt{-1}$). For example, $-3.5 + 2i$ is a complex number. It is common to write a for $a + 0i$ and bi for $0 + bi$. Moreover, when the imaginary part is negative, it is common to write $a - bi$ with $b > 0$ instead of $a + (-b)i$, for example $3 - 4i$ instead of $3 + (-4)i$.

The set of all complex numbers is denoted by $\mathbb{C}$.

The real number a in the complex number $z = a + bi$ is called the *real part* of z, and the real number b is often called the *imaginary part*. By this convention the *imaginary part* is a real number – not including the imaginary unit: hence b, not bi, is the imaginary part. The real part a is denoted by $\mathrm{Re}(z)$ or $\Re(z)$, and the imaginary part b is denoted by $\mathrm{Im}(z)$ or $\Im(z)$. For example,

$$\mathrm{Re}(-3.5 + 2i) = -3.5$$
$$\mathrm{Im}(-3.5 + 2i) = 2$$

A real number a can be regarded as a complex number $a + 0i$ with an imaginary part of zero. A pure imaginary number bi is a complex number $0 + bi$ whose real part is zero.

Notation

Some authors write $a + ib$ instead of $a + bi$. In some disciplines, in particular electromagnetism and electrical engineering, j is used instead of i, since i is frequently used for electric current. In these cases complex numbers are written as $a + bj$ or $a + jb$.

Addition and subtraction

Complex numbers are added by adding the real and imaginary parts of the summands. That is to say:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Similarly, subtraction is defined by

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

[1] For simplicity the CSc 115 Assignment 1 will use only integers for the values a and b. It is a straightforward extension to use real numbers.

Some Exercises

1. Re(7 + 4i) = _______________
2. Im(-5 +3i) = _______________
3. Im(6 – 4i) = _______________
4. (7 + 4i) + (1 – 5i) = _______________
5. (-5 + 3i) – (2 – 8i) = _______________
6. (5 + 3i) + 6 = _______________
7. (3 – i) – i = _______________
8. (4 + 6i) + 7i = _______________

Multiplication and division

The multiplication of two complex numbers is defined by the following formula:

$$(a+bi)(c+di) = (ac-bd) + (bc+ad)i.$$

In particular, the square of the imaginary unit is −1:

$$i^2 = i \times i = -1.$$

The preceding definition of multiplication of general complex numbers follows naturally from this fundamental property of the imaginary unit. Indeed, if *i* is treated as a number so that *di* means *d* times *i*, the above multiplication rule is identical to the usual rule for multiplying two sums of two terms.

$(a+bi)(c+di) = ac + bci + adi + bidi$ (distributive law)
$= ac + bidi + bci + adi$ (commutative law of addition—the order of the summands can be changed)
$= ac + bdi^2 + (bc+ad)i$ (commutative law of multiplication—the order of the multiplicands can be changed)
$= (ac-bd) + (bc+ad)i$ (fundamental property of the imaginary unit).

The division of two complex numbers is defined in terms of complex multiplication, which is described above, and real division. Where at least one of *c* and *d* is non-zero:

$$\frac{a+bi}{c+di} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i.$$

Division can be defined in this way because of the following observation:

$$\frac{a+bi}{c+di} = \frac{(a+bi)\cdot(c-di)}{(c+di)\cdot(c-di)} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i.$$

As shown earlier, $c - di$ is the complex conjugate of the denominator $c + di$. The real part c and the imaginary part d of the denominator must not both be zero for division to be defined.

Some Exercises

1. (7 + 4i) (1 – 5i) = _______________
2. (-5 + 3i) / (2 – 8i) = _______________
3. (4 + 6i)7i = _______________
4. (5 + 3i) / 6 = _______________
5. (3 – i) i = ______________