1. [18 marks]

For each of the specified functions in the table below, place an X in <u>all of the appropriate boxes</u> to indicate for which of the sets of values of x the evaluation of the function <u>may have a large relative error</u> using floating-point arithmetic.

Assume that x is an exact real floating-point number within the floating-point system (that is, x has a fixed, finite number of significant digits) and that no overflow or underflow occurs, and that there is no attempt to divide by 0.

Complex arithmetic may be required, but the variable x is real.

No justification for your answers is required. Only the X's in this table will be marked.

NOTE: an X means large relative error.

	x is sufficiently close to 0	x is sufficiently large and positive	x is sufficiently large and negative
$\dfrac{\sqrt{x+1}-1}{\sqrt{x^2+1}+x}$	X		X
$\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$	X		
$\dfrac{x}{x^3-1}+\dfrac{\sqrt{x^2+1}}{x^3+1}$			X

2. (a) [6 marks]
Determine the third order ($n = 3$) Taylor polynomial approximation for $f(x) = \ell n(x+1)$ expanded about $a = 0$. Do not include the remainder term. Show all of your work.

$f(x) = \ln(x+1)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{x+1}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(x+1)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{2}{(x+1)^3}$ $\qquad f'''(0) = 2$

$$\ln(x+1) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$
$$= 0 + (1)(x-0) - \frac{1}{2}(x-0)^2 + \frac{2}{6}(x-0)^3$$
$$= x - \frac{x^2}{2} + \frac{x^3}{6}$$

(b) [4 marks]
Determine the remainder term for the polynomial approximation in (a).

$$f^{(4)}(x) = \frac{-6}{(x+1)^4}$$

$$R = \frac{f^{(4)}(\xi)}{4!}(x-a)^4 = \frac{1}{24} \cdot \frac{-6}{(\xi+1)^4}(x-0)^4$$
$$= \frac{-x^4}{4(\xi+1)^4}$$

(c) [8 marks]
Determine a good upper bound for the truncation error of the Taylor polynomial approximation in (a) when $-0.1 \le x \le 0.2$ by bounding the remainder term.
Note: leave your answer as a product of numeric factors.

$$|R| \le \frac{1}{4} \max_{-.1 \le x \le .2} |x^4| \max_{-.1 \le x \le .2} \left| \frac{1}{(\xi+1)^4} \right|$$

$$= \frac{1}{4}(0.2)^4 \frac{1}{(-.1+1)^4}$$

$$= \frac{1}{4}(0.2)^4 \frac{1}{(0.9)^4} = 0.00061$$

$(x-\pi)^2$

To 6 significant digits, the correct value of $f(3.135)$ is 0.499998 . Is the function $f(x)$ <u>ill-conditioned or well-conditioned</u> when $x = 3.135$? Fully justify your answer, using the notation and definition of condition given in class.

To answer this question, you can simplify the form of $f(x)$ for values of x close to π by using the fourth order Taylor polynomial approximation for $\cos x$ expanded about $a = \pi$, which is

$$\cos x \approx -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}.$$

<u>Note</u>. Do <u>not</u> use the condition number $\frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})}$ to answer this question. You do <u>not</u> need a calculator to answer this question.

given problem, data $x = 3.135$ $\longrightarrow$ exact value $f(x) = 0.49998---$

perturbed problem, data $\hat{x} = 3.135 + \varepsilon$ with $\left|\frac{\varepsilon}{3.135}\right|$ small $\longrightarrow$ exact value $f(3.135+\varepsilon) = \dfrac{1+\cos(3.135+\varepsilon)}{(3.135+\varepsilon-\pi)^2}$

Note that

$$f(x) \approx \frac{1 + \left[-1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24}\right]}{(x-\pi)^2}$$

$$= \frac{1}{2} - \frac{(x-\pi)^2}{24} \quad \text{if } x \text{ is close to } \pi.$$

Thus, $f(3.135+\varepsilon) \approx \dfrac{1}{2} - \dfrac{(3.135+\varepsilon-\pi)^2}{24}$

$$\approx \frac{1}{2} - \frac{(-.006+\varepsilon)^2}{24}$$

$$= \frac{1}{2} - \frac{.000036 - .012\varepsilon + \varepsilon^2}{24}$$

$$\approx 0.4999---- \quad \text{for all } \varepsilon$$

such that $\left|\frac{\varepsilon}{3.135}\right|$ is small.

Thus the problem of computing $f(3.135)$ is well-conditioned.

4. Consider

$$f(x) = \sin(x - \sqrt{3}) - x + \sqrt{3}$$

and its first derivative

$$f'(x) = \cos(x - \sqrt{3}) - 1 .$$

Note that $f(\sqrt{3}) = f'(\sqrt{3}) = 0$ and $\sqrt{3} = 1.73205\cdots$.

(a) [6 marks]
Can the MATLAB built-in function *fzero* be successfully used to compute the zero of $f(x)$ at $x_t = \sqrt{3}$? Answer YES or NO, and justify your answer.

YES : $f''(x) = -\sin(x-\sqrt{3})$, $f''(\sqrt{3}) = 0$

$f'''(x) = -\cos(x-\sqrt{3})$, $f'''(\sqrt{3}) = -1$.

Thus the multiplicity of the zero at $x = \sqrt{3}$ is $m = 3$.

As this is odd, fzero (which uses the Bisection method) can be used to compute this zero.

(b) [4 marks]
Specify one MATLAB statement using the MATLAB built-in function *fzero* that could be used to attempt to compute a zero of $f(x)$ on the interval $[\,1, 3\,]$.

```
fzero('sin(x-sqrt(3))-x+sqrt(3)', [1,3])
```

(c) [6 marks]
If Newton's method were used to compute the zero of $f(x)$ at $x_t = \sqrt{3}$ and it did converge, what would be the order of convergence? Briefly justify your answer.

The order of convergence is $\alpha = 1$ (linear) since the multiplicity of the zero is ≥ 2

$$a = (a_1, a_2, a_3, \dots, a_{n+1})$$

of a polynomial

$$f(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{n+1} x^n$$

and a scalar $x0$, suppose that a MATLAB function with header

function [b, c] = horner (x0 , n , a)

exists to evaluate $f(x0)$ and $f'(x0)$ using Horner's algorithm. Assume that the values are returned from horner.m as

$$b_1 = f(x0) \text{ and } c_2 = f'(x0).$$

Fill in the blanks in the following MATLAB function with header

function root = newton (n , a , x0 , imax , eps)

so that it will compute one zero of a polynomial

$$f(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{n+1} x^n$$

using Newton's method, in which the given MATLAB function horner.m is used to evaluate $f(x)$ and $f'(x)$.

Input parameters for newton.m are

n	the degree of $f(x)$
a	a vector of the coefficients of $f(x)$
x0	the initial approximation to a zero of $f(x)$
imax	maximum number of iterations allowed
eps	a tolerance used to test relative error for convergence

Output parameter for newton.m is

root	the final computed approximation to a zero of $f(x)$

Note. Specify newton.m using MATLAB syntax.
Do not specify the MATLAB function horner.m .
Do not print any approximations to a zero of $f(x)$ within newton.m .
Print 'failed to converge' in newton.m if the function does not converge within imax iterations.

```
function  root = newton ( n , a , x0 , imax , eps )

i = 1 ;

while  i <= imax
    [b,c] = horner(x0,n,a) ;
    root = x0 - b(1)/c(2) ;
    if abs(1-x0/root) < eps

        return

    end
    i = i+1 ;
    x0 = root ;
end

fprintf ( ' failed to converge' )
```

Use Gaussian elimination with partial pivoting to compute the solution of $Ax = b$. Show all of the steps of the forward elimination and the back substitution.

interchange rows 1, 2

$$\left[\begin{array}{cccc|c} -4 & -4 & 0 & 0 & -8 \\ 2 & 1 & 2 & 1.5 & 2 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & -2 & 0 & -2 \end{array}\right]$$

eliminate

$$\left[\begin{array}{cccc|c} -4 & -4 & 0 & 0 & -8 \\ 0 & -1 & 2 & 1.5 & -2 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & -2 & 0 & -2 \end{array}\right]$$

no interchange, eliminate

$$\left[\begin{array}{cccc|c} -4 & -4 & 0 & 0 & -8 \\ 0 & -1 & 2 & 1.5 & -2 \\ 0 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & -2 & 0 & -2 \end{array}\right]$$

interchange rows 3, 4

$$\left[\begin{array}{cccc|c} -4 & -4 & 0 & 0 & -8 \\ 0 & -1 & 2 & 1.5 & -2 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0.5 & 0 \end{array}\right]$$

eliminate

$$\left[\begin{array}{cccc|c} -4 & -4 & 0 & 0 & -8 \\ 0 & -1 & 2 & 1.5 & -2 \\ 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0.5 & -1 \end{array}\right]$$

back-substitution

$$\tfrac{1}{2} x_4 = -1 \implies x_4 = -2$$

$$-2x_3 = -2 \implies x_3 = 1$$

$$x_2 = \frac{-2 - 2(1) - 1.5(-2)}{-1} = \frac{-1}{-1} = 1 \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

$$x_1 = \frac{-8 + 4x_2}{-4} = \frac{-8 + 4}{-4} = 1$$