1. [18 marks]

For each of the specified functions in the table below, place an X in all of the appropriate boxes to indicate for which of the sets of values of x the evaluation of the function may have a large relative error using floating-point arithmetic.

Assume that x is an exact real floating-point number within the floating-point system (that is, x has a fixed, finite number of significant digits) and that no overflow or underflow occurs, and that there is no attempt to divide by 0.

No justification for your answers is required. Only the X's in this table will be marked.

NOTE: an X means large relative error.

	x is close to 0 ($x \neq 0$)	x is sufficiently large and positive	x is sufficiently large in magnitude and negative
$\frac{-x-\sqrt{x^2-0.1}}{2}$			X
$\frac{2}{x-1}+\frac{1}{0.5-x}$	X		
$\frac{e^x-e^{-x}}{x}$	X		
$\frac{x}{x^3+1}-\frac{\sqrt{x^2+1}}{x^3-1}$		X	

2.

2. (a) [6 marks]

Determine the third order ($n = 3$) Taylor polynomial approximation for $f(x) = \ell n(x-1)$ expanded about $a = 2$. Do <u>not</u> include the remainder term.

$f(x) = \ln(x-1)$ $\qquad f(2) = 0$

$f'(x) = \frac{1}{x-1}$ $\qquad f'(2) = 1$

$f''(x) = \frac{-1}{(x-1)^2}$ $\qquad f''(2) = -1$

$f'''(x) = \frac{2}{(x-1)^3}$ $\qquad f'''(2) = 2$

$$f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3$$

$$= 0 + (x-2) - \tfrac{1}{2}(x-2)^2 + \tfrac{2}{6}(x-2)^3$$

(b) [4 marks]

Determine the remainder term for the polynomial approximation in (a).

$$f^{(4)}(x) = \frac{-6}{(x-1)^4}$$

$$R = \frac{f^{(4)}(\xi)}{4!}(x-2)^4 = \frac{-6}{(\xi-1)^4}\frac{1}{24}(x-2)^4 = \frac{-(x-2)^4}{4(\xi-1)^4}$$

(c) [8 marks]

Determine a good upper bound for the truncation error of the Taylor polynomial approximation in (a) when $1.9 \le x \le 2.15$ by bounding the remainder term.

Note: leave your answer as a product of <u>numeric</u> factors (it's not necessary to give your answer as a single number).

$$|R| = \left| \frac{-(x-2)^4}{4(\xi-1)^4} \right|$$ where $1.9 \le x \le 2.15$ and ξ lies between x and 2

$$\le \frac{(2.15-2)^4}{4(1.9-1)^4}$$

$$= \frac{(0.15)^4}{4(0.9)^4}$$

3. Since the exact value of $g(0.00019)$ is $2.000000012\cdots$, this computed approximation is very inaccurate, with a relative error of approximately 47%. Use the Taylor polynomial approximations (expanded about $a = 0$)

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \quad \text{and} \quad e^{-x} \approx 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

which are very accurate when x is close to 0, to show that the above computation of $fl(\, g(0.00019)\,)$ is unstable. Use the notation and definition of stability given in class.

$$g(x) = \frac{e^x - e^{-x}}{x} \xrightarrow{\text{flt-pt.}} \text{computed solution } r = 1.052$$

$x = 0.00019$

perturbed problem with $\hat{x} = 0.00019 + \varepsilon$, $\left|\frac{\varepsilon}{.00019}\right|$ small $\longrightarrow$ exact value is $\hat{r} = \dfrac{e^{\hat{x}} - e^{-\hat{x}}}{\hat{x}}$

For values of x close to 0 (such as $\hat{x} = .00019 + \varepsilon$),

$$\hat{r} \approx \frac{1 + \hat{x} + \frac{\hat{x}^2}{2} + \frac{\hat{x}^3}{6} + \frac{\hat{x}^4}{24} + \frac{\hat{x}^5}{120} - \left[1 - \hat{x} + \frac{\hat{x}^2}{2} - \frac{\hat{x}^3}{6} + \frac{\hat{x}^4}{24} - \frac{\hat{x}^5}{120}\right]}{\hat{x}}$$

$$\approx \frac{2\hat{x} + \frac{2\hat{x}^3}{6} + \frac{2\hat{x}^5}{120}}{\hat{x}}$$

$$= 2 + \frac{\hat{x}^2}{3} + \frac{\hat{x}^4}{60}$$

≈ 2 for all small values of ε

Since $\hat{r}$ is NOT close to $r = 1.052$ for ALL small values of ε, the computation of $fl(g(0.00019))$ is unstable.

4. (a)

$$f(x) = \frac{1}{x} - R$$

in order to determine an iterative formula for computing a zero of $f(x)$. Simplify this iterative formula so that it is in the form

$$x_{i+1} = x_i(c - x_i R)$$

where c is a constant. Note: show all of your work and determine the value of c.

$$f(x) = \frac{1}{x} - R, \quad f'(x) = -\frac{1}{x^2}.$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{\frac{1}{x_i} - R}{-1/x_i^2}$$

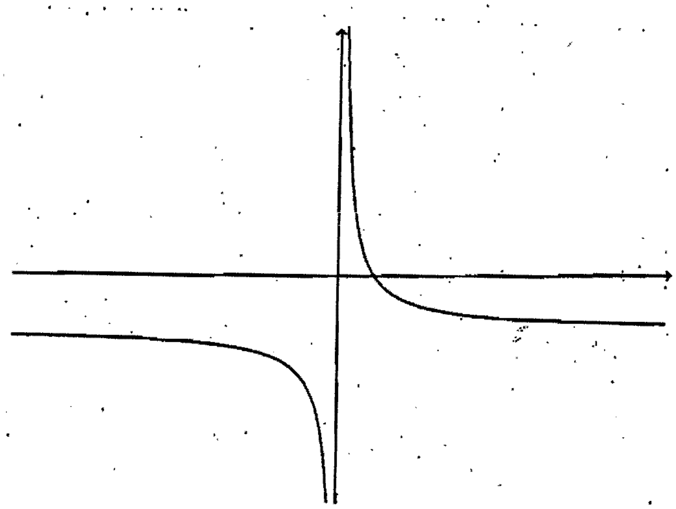
$$= x_i + x_i - x_i^2 R$$

$$= 2x_i - x_i^2 R$$

$$= x_i(2 - x_i R)$$

(b) [8 marks]

This is the graph of $f(x)$. It will help to answer some parts of this question.



If the iterative formula in (a) is used to compute a sequence of values $\{x_1, x_2, x_3, \cdots\}$ that converges to a zero x_t of $f(x)$, what is the value of x_t that this sequence converges to? Answer $1/R$

Let x_0 be the initial approximation for the iterative formula in (a). If this iterative formula is used to compute a sequence of values $\{x_1, x_2, x_3, \cdots\}$, what is the value of $\lim_{i\to\infty} x_i$ if

(i) $x_0 = 0$? Answer 0

(ii) $0 < x_0 < 1/R$? Answer $1/R$

(iii) $x_0 < 0$? Answer $-\infty$

4.

(c) [8 marks]
Fill in the blanks in the following MATLAB code so that the function M-file zero.m could be used to implement the iterative formula from (a), where

x0 is the initial approximation,
imax is the maximum number of iterations allowed, and
eps is used to test relative error.

If the sequence of values $\{x_i\}$ converges, then the function zero.m returns the final value in the variable root, otherwise it prints a message 'failed to converge'.

```
function root = zero( R, x0, imax, eps )
i = 1;
while  i <= imax
    root = x0 * (2 - x0 * R) ;
    if abs(1- x0/root)  < eps
        return
    end
    i = i+1;
    x0 = root ;
end
fprintf( 'failed to converge')
```

5. (a) [8 marks]

Newton's method with the initial approximation $x_0 = 3$ will converge to the zero at $x = 2$ of

$$f(x) = e^{x-2}(14x-12) - 7x^3 + 20x^2 - 26x + 12.$$

What is the order of convergence of this computation? Briefly justify your answer using results discussed in class. Let m = multiplicity of the zero at $x=2$.

$f(2) = 0 \Rightarrow m \geq 1$

$f'(x) = e^{x-2}(14x+2) - 21x^2 + 40x - 26$

$f'(2) = 0 \Rightarrow m \geq 2$

Since $m \geq 2$, Newton's method converges linearly (order of convergence is $\alpha = 1$)

(b) [8 marks]

Can the MATLAB built-in function *fzero* be used to compute the zero at $x = 2$ of $f(x)$? Answer YES or NO, and <u>justify</u> your answer.

Need to determine if the multiplicity m is even or odd.

$f''(x) = e^{x-2}(14x+6) - 42x + 40$

$f''(2) = 0 \Rightarrow m \geq 3$

$f'''(x) = e^{x-2}(14x+30) - 42$

$f'''(2) = 16 \neq 0 \Rightarrow m = 3$

Since the multiplicity is odd, fzero (which uses the Bisection method) can be used. Answer is YES.

(c) [6 marks]

Give 1 MATLAB statement that could be used to compute <u>all</u> of the zeros of the polynomial

$$f(x) = 2x^5 - x^3 + x^2 - 7x - 1$$

using a MATLAB built-in function.

```
roots([2 0 -1 1 -7 -1])
```