Allan Liu
V00806981
CSC 349A
Assignment 5

Q1(a)

$x_0 = 0, x_1 = \frac{2\pi}{3}, x_2 = \frac{4\pi}{3}, x_3 = 2\pi, f(x_0) = 0, f(x_1) = 0.75, f(x_2) = 0.75, f(x_3) = 0$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{\left(x-\frac{2\pi}{3}\right)\left(x-\frac{4\pi}{3}\right)(x-2\pi)}{\left(-\frac{2\pi}{3}\right)\left(-\frac{4\pi}{3}\right)(-2\pi)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)\left(x-\frac{4\pi}{3}\right)(x-2\pi)}{\left(\frac{2\pi}{3}-0\right)\left(\frac{2\pi}{3}-\frac{4\pi}{3}\right)\left(\frac{2\pi}{3}-2\pi\right)}$$

$$= \frac{(x^3 - 2\pi x^2 - \frac{4\pi x^2}{3} + \frac{8\pi x}{3})}{\frac{16\pi}{27}} = \frac{(x^3 - \frac{10\pi x^2}{3} + \frac{8\pi x}{3})}{\frac{16\pi}{27}}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)\left(x-\frac{2\pi}{3}\right)(x-2\pi)}{\left(\frac{4\pi}{3}-0\right)\left(\frac{4\pi}{3}-\frac{2\pi}{3}\right)\left(\frac{4\pi}{3}-2\pi\right)}$$

$$= \frac{(x^3 - 2\pi x^2 - \frac{2\pi x^2}{3} + \frac{4\pi x}{3})}{\frac{-16\pi}{27}} = \frac{(x^3 - \frac{8\pi x^2}{3} + \frac{4\pi x}{3})}{\frac{-16\pi}{27}}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)\left(x-\frac{2\pi}{3}\right)\left(x-\frac{4\pi}{3}\right)}{(2\pi-0)\left(2\pi-\frac{2\pi}{3}\right)\left(2\pi-\frac{4\pi}{3}\right)}$$

Therefore,

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

$$= \frac{(x^3 - \frac{10\pi x^2}{3} + \frac{8\pi x}{3})}{\frac{16\pi}{27}}\ 0.75 + \frac{(x^3 - \frac{8\pi x^2}{3} + \frac{4\pi x}{3})}{\frac{-16\pi}{27}}\ 0.75 = -0.84375x^2 + 1.6875x$$

Q1(b)

```
>> x = [0:0.1:2*pi];
>> y = sin(x).^2;
>> p = -0.84375*x.^2+1.6875*x;
>> plot(x,y,'r', x,p,'b');
```

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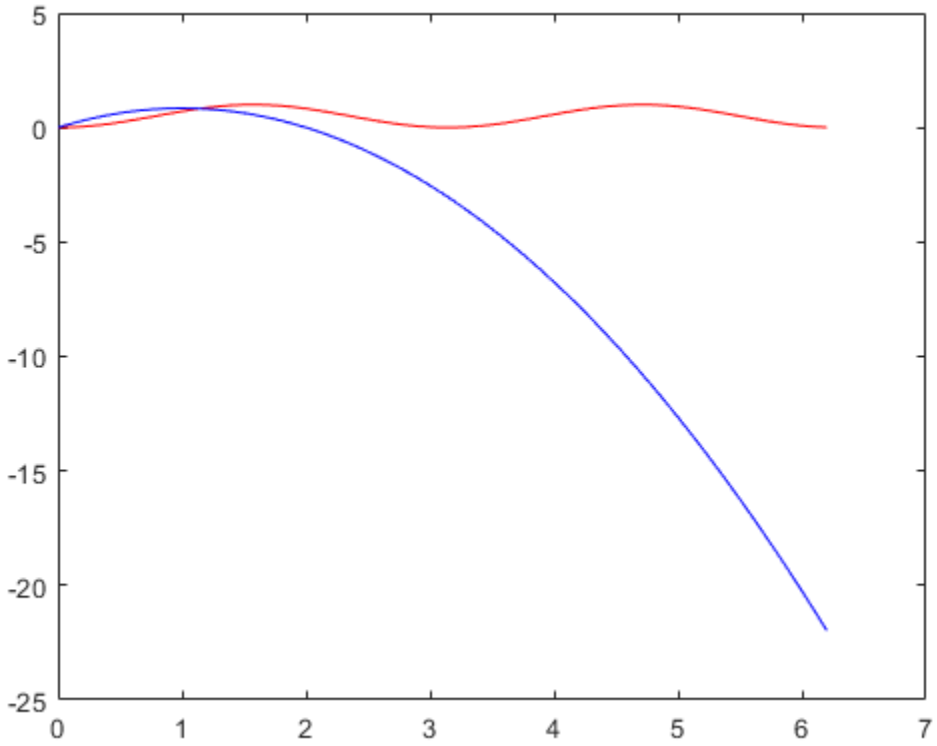

Q2

a)

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0x + c_0x^2 + d_0x^3 & [0 \le x \le \frac{2\pi}{3}] \\ S_1(x) = a_1 + b_1\left(x - \frac{2\pi}{3}\right) + c_1\left(x - \frac{2\pi}{3}\right)^2 + d_1\left(x - \frac{2\pi}{3}\right)^3 & [\frac{2\pi}{3} \le x \le \frac{4\pi}{3}] \\ S_2(x) = a_2 + b_2\left(x - \frac{4\pi}{3}\right) + c_2\left(x - \frac{4\pi}{3}\right)^2 + d_2\left(x - \frac{4\pi}{3}\right)^3 & [\frac{4\pi}{3} \le x \le 2\pi] \end{cases}$$

So we have from question 1:

$x_0 = 0, x_1 = \frac{2\pi}{3}, x_2 = \frac{4\pi}{3}, x_3 = 2\pi, f(x_0) = 0, f(x_1) = 0.75, f(x_2) = 0.75, f(x_3) = 0$

b)

$S_0(x_0) = f(x_0)$

$a_0 + b_0x + c_0x^2 + d_0x^3 = 0 \rightarrow a_0 + b_0(0) + c_0(0)^2 + d_0(0)^3 = 0 \rightarrow a_0 = 0$

$S_1(x_1) = f(x_1)$

$a_1 + b_1\left(x_1 - \frac{2\pi}{3}\right) + c_1\left(x_1 - \frac{2\pi}{3}\right)^2 + d_1\left(x_1 - \frac{2\pi}{3}\right)^3 = 0.75 \rightarrow S_1\left(\frac{2\pi}{3}\right) = 0.75 \rightarrow a_1 = 0.75$

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$S_2(x_2) = f(x_2)$

$$a_2 + b_2\left(x_2 - \frac{4\pi}{3}\right) + c_2\left(x_2 - \frac{4\pi}{3}\right)^2 + d_2\left(x_2 - \frac{4\pi}{3}\right)^3 = 0.75 \rightarrow S_2\left(\frac{4\pi}{3}\right) = 0.75 \rightarrow a_2 = 0.75$$

$S_2(x_3) = f(x_3)$

$$a_2 + b_2\left(x_3 - \frac{4\pi}{3}\right) + c_2\left(x_3 - \frac{4\pi}{3}\right)^2 + d_2\left(x_3 - \frac{4\pi}{3}\right)^3 = 0$$

$$\rightarrow S_2(2\pi) = 0 \rightarrow a_2 + b_2\left(\frac{2\pi}{3}\right) + c_2\left(\frac{2\pi}{3}\right)^2 + d_2\left(\frac{2\pi}{3}\right)^3 = 0 \rightarrow b_2\left(\frac{2\pi}{3}\right) + c_2\left(\frac{4\pi}{9}\right) + d_2\left(\frac{8\pi}{27}\right) = -0.75$$

c)

$S_1(x_1) = S_0(x_1)$

$$a_1 = a_0 + b_0x_1 + c_0x_1^2 + d_0x_1^3 \rightarrow S_1(0) = S_0\left(\frac{2\pi}{3}\right) \rightarrow b_0\left(\frac{2\pi}{3}\right) + c_0\left(\frac{4\pi}{9}\right) + d_0\left(\frac{8\pi}{27}\right) = 0.75$$

$S_2(x_2) = S_1(x_2)$

$$0.75 = a_1 + b_1\left(x_2 - \frac{2\pi}{3}\right) + c_1\left(x_2 - \frac{2\pi}{3}\right)^2 + d_1\left(x_2 - \frac{2\pi}{3}\right)^3$$

$$\rightarrow 0.75 = S_1\left(\frac{4\pi}{3}\right) \rightarrow b_1\left(\frac{2\pi}{3}\right) + c_1\left(\frac{4\pi}{9}\right) + d_1\left(\frac{8\pi}{27}\right) = 0$$

d)

$S_1'(x_1) = S_0'(x_1)$

$$b_1 + 2c_1\left(x_1 - \frac{2\pi}{3}\right) + 3d_1\left(x_1 - \frac{2\pi}{3}\right)^2 = b_0 + 2c_0x_1 + 3d_0x_1^2 \rightarrow S_1'\left(\frac{2\pi}{3}\right) = S_0'\left(\frac{2\pi}{3}\right)$$

$$\rightarrow b_0 - b_1 + c_0\left(\frac{4\pi}{3}\right) + d_0\left(\frac{4\pi}{3}\right) = 0 \rightarrow c_0\left(\frac{4\pi}{3}\right) + d_0\left(\frac{4\pi}{3}\right) - b_1 = -2$$

$S_2'(x_2) = S_1'(x_2)$

$$b_2 + 2c_2\left(x_2 - \frac{4\pi}{3}\right) + 3d_2\left(x_2 - \frac{4\pi}{3}\right)^2 = b_1 + 2c_1\left(x_2 - \frac{2\pi}{3}\right) + 3d_1\left(x_2 - \frac{2\pi}{3}\right)^2 \rightarrow S_2'\left(\frac{4\pi}{3}\right) = S_1'\left(\frac{4\pi}{3}\right)$$

$$\rightarrow b_2 = b_1 + 2c_1\left(\frac{2\pi}{3}\right) + 3d_1\left(\frac{2\pi}{3}\right)^2 \rightarrow b_1 - b_2 + c_1\frac{4\pi}{3} + d_1\frac{4\pi}{3} = 0$$

e)

$S_1''(x_1) = S_0'(x_1)$

$$2c_1 + 6d_1\left(x_1 - \frac{2\pi}{3}\right) = b_0 + 2c_0x_1 + 3d_0x_1^2 \rightarrow S_1''\left(\frac{2\pi}{3}\right) = S_0'\left(\frac{2\pi}{3}\right)$$

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$\rightarrow b_0 + c_0\left(\frac{4\pi}{3}\right) + d_0\left(\frac{4\pi}{3}\right) - 2c_1 = 0$

$S_2''(x_2) = S_1''(x_2)$

$2c_2 + 6d_2\left(x_2 - \frac{4\pi}{3}\right) = 2c_1 + 6d_1\left(\frac{2\pi}{3}\right) \rightarrow 2c_2 = 2c_1 + 6d_1\left(\frac{2\pi}{3}\right) \rightarrow 2c_1 + d_1(4\pi) - 2c_2 = 0$

f) ii)

$S_0'(x_0) = f'(x_0)$

$b_0 + 2c_0x_0 + 3d_0x_0^2 = 2\cos(x_0) \rightarrow S_0'(0) = f'(0) \rightarrow b_0 = 2$

$S_2'(x_0) = f'(x_3)$

$b_2 + 2c_2\left(\frac{2\pi}{3}\right) + 3d_2\left(\frac{2\pi}{3}\right)^2 = 2\cos(x_3) \rightarrow S_2'(0) = f'(0) \rightarrow b_2 + c_2\left(\frac{4\pi}{3}\right) + d_2\left(\frac{4\pi}{3}\right) = 2$

We know $a_0 = 1, a_1 = 0.75, a_2 = 0.75$ *and* $b_0 = 2$ *with unknowns*: $b_1, b_2, c_0, c_1, c_2, d_0, d_1, d_2$.

We have equations:

$b_2\left(\frac{2\pi}{3}\right) + c_2(\frac{4\pi}{9}) + d_2\left(\frac{8\pi}{27}\right) = -0.75$

$b_0\left(\frac{2\pi}{3}\right) + c_0\left(\frac{4\pi}{9}\right) + d_0\left(\frac{8\pi}{27}\right) = 0.75$

$b_1\left(\frac{2\pi}{3}\right) + c_1\left(\frac{4\pi}{9}\right) + d_1\left(\frac{8\pi}{27}\right) = 0$

$c_0\left(\frac{4\pi}{3}\right) + d_0\left(\frac{4\pi}{3}\right) - b_1 = -2$

$b_1 - b_2 + c_1\frac{4\pi}{3} + d_1\frac{4\pi}{3} = 0$

$b_0 + c_0\left(\frac{4\pi}{3}\right) + d_0\left(\frac{4\pi}{3}\right) - 2c_1 = 0$

$2c_1 + d_1(4\pi) - 2c_2 = 0$

$b_2 + c_2\left(\frac{4\pi}{3}\right) + d_2\left(\frac{4\pi}{3}\right) = 2$

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$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\pi/3 & 0 & 0 & 4\pi/9 & 0 & 0 & 8\pi/27 \\
0 & 0 & 0 & 2\pi/3 & 0 & 0 & 4\pi/9 & 0 & 0 & 8\pi/27 & 0 & 0 \\
0 & 0 & 0 & 0 & 2\pi/3 & 0 & 0 & 4\pi/9 & 0 & 0 & 8\pi/27 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 4\pi/3 & 0 & 0 & 4\pi/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 4\pi/3 & 0 & 0 & 4\pi/3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 4\pi/3 & -2 & 0 & 4\pi/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 4\pi & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4\pi/3 & 0 & 0 & 4\pi/3
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \\ c_0 \\ c_1 \\ c_2 \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0.75 \\ 0.75 \\ -0.75 \\ 0.75 \\ 0 \\ -2 \\ 0 \\ 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}$$

Q3(a)

```
>> x = [0 2*pi/3 4*pi/3 2*pi]

x =

         0    2.0944    4.1888    6.2832

>> y = [0 0.75 0.75 0]

y =

         0    0.7500    0.7500         0

>> pp = spline(x, y)

pp =

  struct with fields:

      form: 'pp'
    breaks: [0 2.0944 4.1888 6.2832]
     coefs: [3×4 double]
    pieces: 3
     order: 4
       dim: 1

>> format short
>> [b, c] = unmkpp(pp)

b =

         0    2.0944    4.1888    6.2832
```

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```
c =

    0.0000   -0.0855    0.5371         0
    0.0000   -0.0855    0.1790    0.7500
   -0.0000   -0.0855   -0.1790    0.7500
```

$$S(x) = \begin{cases} S_0(x) = -0.0855x + 0.5371x^2 & [0 \le x \le \frac{2\pi}{3}] \\ S_1(x) = -0.0855\left(x - \frac{2\pi}{3}\right) + 0.1790\left(x - \frac{2\pi}{3}\right)^2 + 0.7500\left(x - \frac{2\pi}{3}\right)^3 & [\frac{2\pi}{3} \le x \le \frac{4\pi}{3}] \\ S_2(x) = -0.0855\left(x - \frac{4\pi}{3}\right) - 0.1790\left(x - \frac{4\pi}{3}\right)^2 + 0.7500\left(x - \frac{4\pi}{3}\right)^3 & [\frac{4\pi}{3} \le x \le 2\pi] \end{cases}$$

Q3(b)

```
>> X1=linspace(0, 2*pi/3,50);
Y1=c(1,1)+c(1,2)*(X1-0)+c(1,3)*(X1-0).^2+c(1,4)*(X1-0).^3;
plot(X1, Y1,'-')
hold on
X2=linspace(2*pi/3, 4*pi/3,50);
Y2=c(2,1)+c(2,2)*(X2-2*pi/3)+c(2,3)*(X2-2*pi/3).^2+c(2,4)*(X2-
2*pi/3).^3;
plot(X2, Y2,'-')
hold on
X3=linspace(4*pi/3, 2*pi,50);
Y3=c(3,1)+c(3,2)*(X3-4*pi/3)+c(3,3)*(X3-4*pi/3).^2+c(3,4)*(X3-
4*pi/3).^3;
plot(X3, Y3,'-')
```