Sturm Liouville Problem

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1 Stum–Liouville Problem

Exercise 3:

If $\lambda = 0$:

$$y = ax + b, y' = a$$

so, $y(0) - hy'(0) = b - ah = 0, \Longrightarrow b = ah$

$y'(1) = a = 0, \Longrightarrow a = 0, b = 0$, which means $y = 0$. So $\lambda = 0$ is not an eigenvalue.

If $\lambda < 0$:

$$y = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$
$$y' = -c_1\sqrt{-\lambda}e^{-\sqrt{-\lambda}x} + c_2\sqrt{-\lambda}e^{\sqrt{-\lambda}x}$$

Impose the boundary conditions and we get:

$$y(0) - hy'(0) = c_1 + c_2 - h(-\sqrt{-\lambda}c_1 + \sqrt{-\lambda}c_2) = (1 + \sqrt{-\lambda}h)c_1 + (1 - \sqrt{\lambda}h)c_2 = 0$$
$$y'(1) = -\sqrt{-\lambda}c_1 e^{-\sqrt{-\lambda}} + \sqrt{-\lambda}c_2 e^{\sqrt{-\lambda}} = 0$$

We set the determinant of the systems to zero and get the following equations to solve:

$$(1+\sqrt{-\lambda}h)\sqrt{-\lambda}e^{\sqrt{-\lambda}}+(1-\sqrt{-\lambda}h)\sqrt{-\lambda}e^{-\sqrt{-\lambda}} = \sqrt{-\lambda}e^{-\sqrt{-\lambda}}[(1+\sqrt{-\lambda}h)e^{2\sqrt{-\lambda}}+1-\sqrt{-\lambda}h] = 0$$

Since $\lambda \neq 0$, we have to solve

$$(1 + \sqrt{-\lambda}h)e^{2\sqrt{-\lambda}} + 1 - \sqrt{-\lambda}h = 0 \Longrightarrow e^{2\sqrt{-\lambda}} = \frac{\sqrt{-\lambda}h - 1}{1 + \sqrt{-\lambda}h}$$

When $h > 1$, there is no solution to this equation. When $0 < h < 1$,there is no solution. When $h = 1$, there is no solution.When $h < 0$, there is 1 solution.

If $\lambda > 0$:

$$y = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$
$$y' = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos\sqrt{\lambda}x$$

Impose the boundary conditions and we get:

$$y(0) - hy'(0) = c_1 - h\sqrt{\lambda}c_2 = 0$$
$$y'(1) = -\sqrt{\lambda}c_1 \sin\sqrt{\lambda} + \sqrt{\lambda}c_2 \cos\sqrt{\lambda} = 0$$

Make some subtitution and we get:

$$c_2\sqrt{\lambda}(-\sqrt{\lambda}h \sin\sqrt{\lambda} + \cos\sqrt{\lambda}) = 0 \Longrightarrow h\sqrt{\lambda} = \cot\sqrt{\lambda}$$

To summarize:

1. If $\lambda = 0$, we get $y = 0$.
2. If $\lambda < 0$, we need to solve the following equation:

$$e^{2\sqrt{-\lambda}} = \frac{\sqrt{-\lambda}h - 1}{1 + \sqrt{-\lambda}h} \tag{1}$$

   which is equivalent to the following with $x > 0$

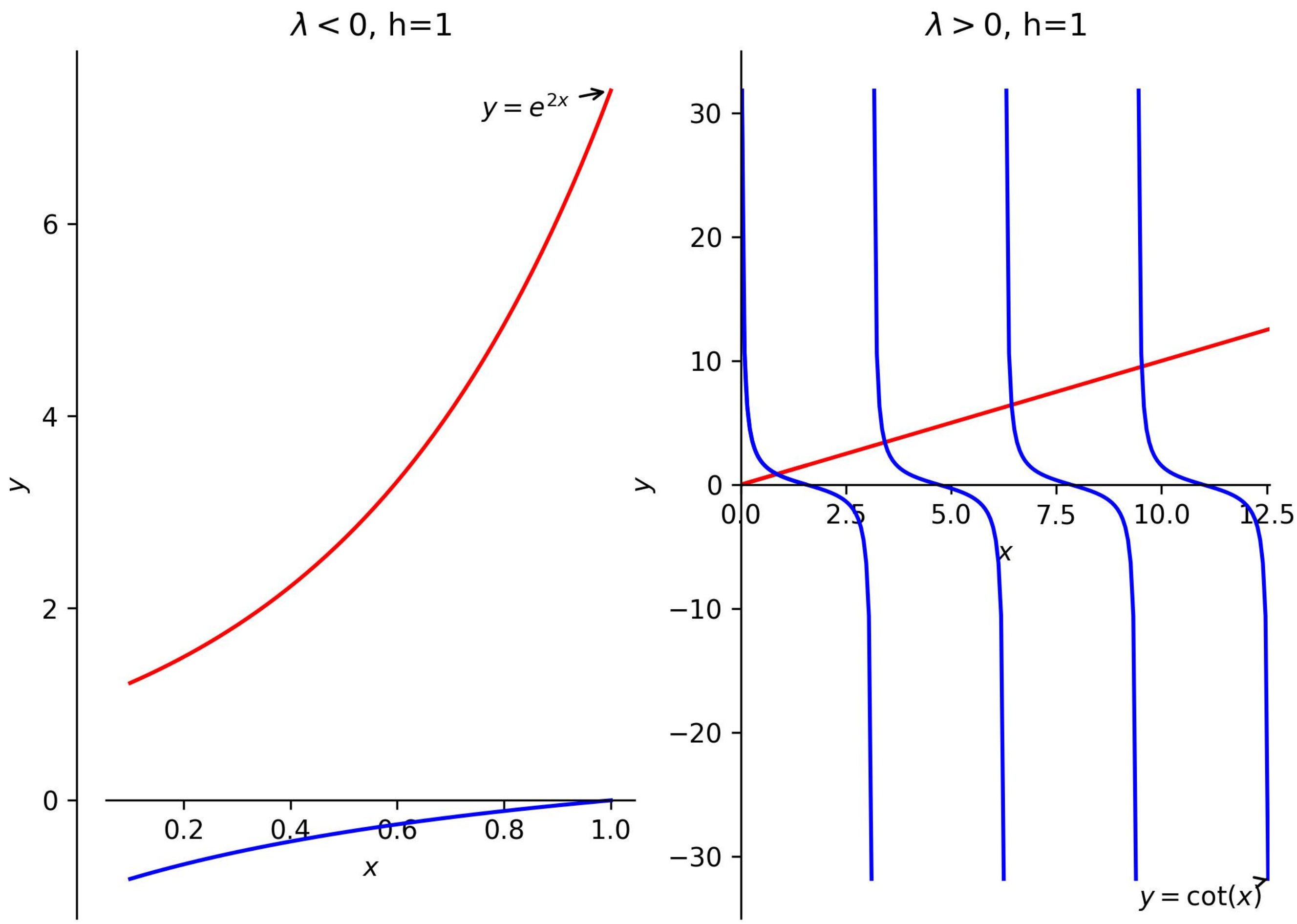
$$e^{2x} = \frac{hx - 1}{1 + hx} \tag{2}$$

3. If $\lambda > 0$, we need to solve the folloing equation:

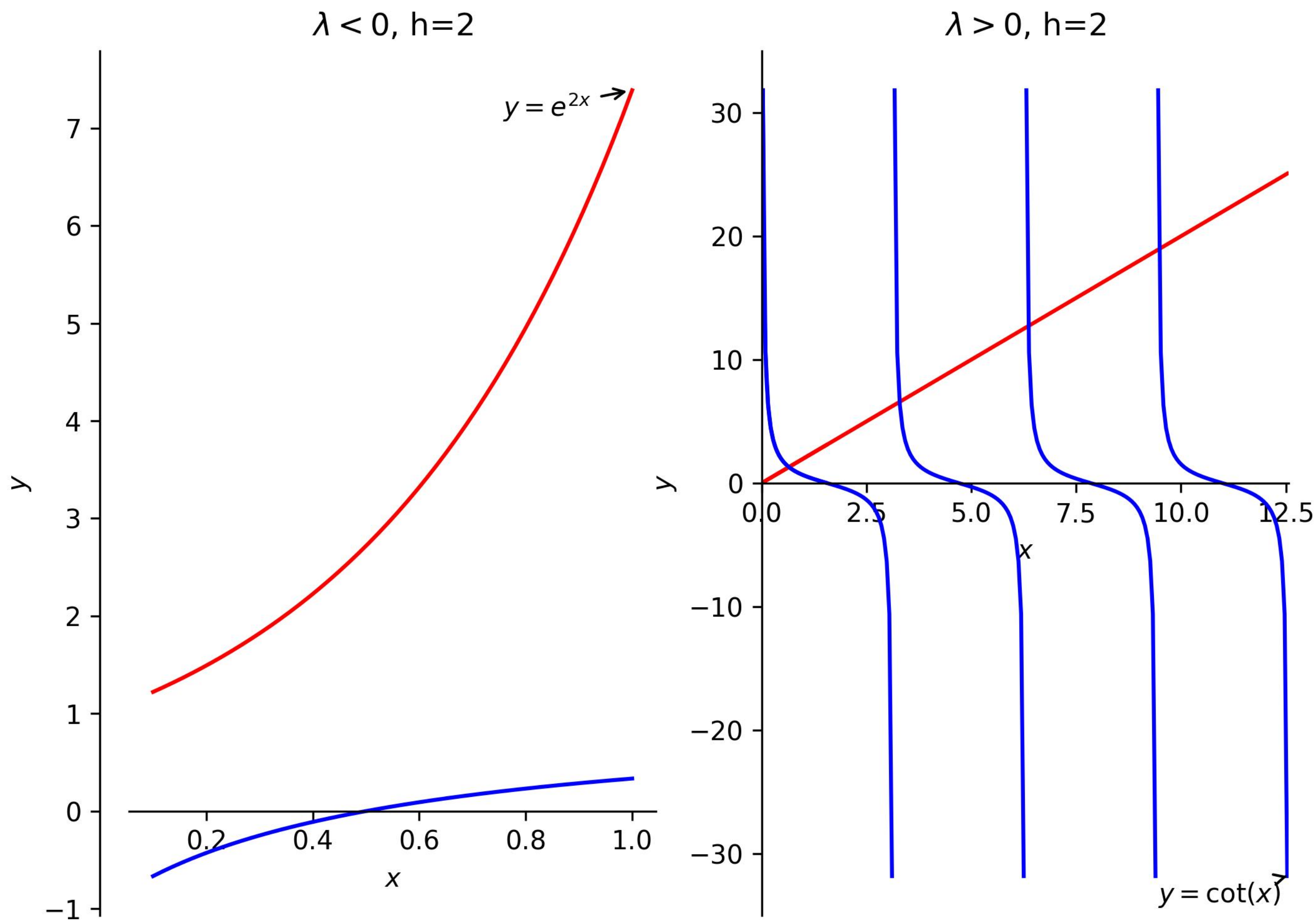
$$h\sqrt{\lambda} = \cot\sqrt{\lambda} \tag{3}$$

   which is equivalent to the following with $x > 0$

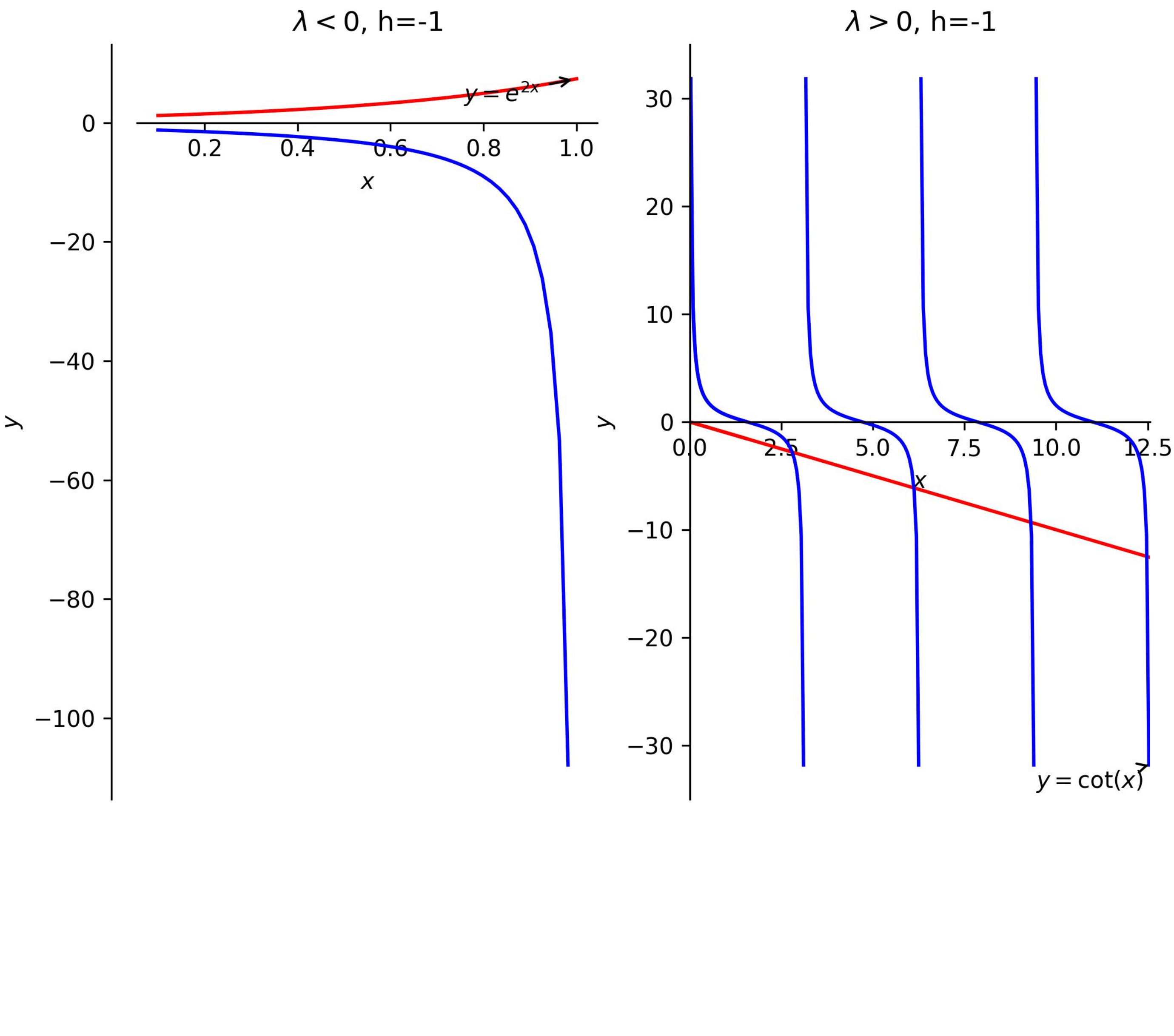
$$hx = \cot x \tag{4}$$



$\lambda < 0$, h=1
$y = e^{2x}$
0
2
4
6
0.2
0.4
0.6
0.8
1.0
x
y
$\lambda > 0$, h=1
30
20
10
0
−10
−20
−30
0.0
2.5
5.0
7.5
10.0
12.5
x
y
$y = \cot(x)$



$\lambda < 0$, h=2
$y = e^{2x}$
7
6
5
4
3
2
1
0
−1
0.2
0.4
0.6
0.8
1.0
x
y
$\lambda > 0$, h=2
30
20
10
0
−10
−20
−30
0.0
2.5
5.0
7.5
10.0
12.5
x
y
$y = \cot(x)$



$\lambda < 0$, h=-1
$y = e^{2x}$
0
−20
−40
−60
−80
−100
0.2
0.4
0.6
0.8
1.0
x
y
$\lambda > 0$, h=-1
30
20
10
0
−10
−20
−30
0.0
2.5
5.0
7.5
10.0
12.5
x
y
$y = \cot(x)$