Lecture 1: Norm and Hilbert Space

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1 Norm

Definition 1 (Vector Space)**.** *A **vector space** V over a field k is a set of vectors which come with addition ($+ : V \times V \to V$) and scalar multiplication ($\cdot : k \times V \to V$) along with some classic axioms: commutativity, associativity, identity, and inverse of addition, identity of multiplication, and distributivity.*

Definition 2 (Norm)**.** *Given a vector space $\boldsymbol{X}$ over a subfield F of the complex numbers $\mathbb{C}$, a norm is a real-values function p: $\boldsymbol{X} \to \mathbb{R}$ with the following properties, where $|s|$ denotes the usual absolute value of a scalr s:*

1. *Positive Definite: $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.*
2. *Homogeneity: $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and $\lambda \in \mathbb{R}$.*
3. *Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.*

Example 1. *Absolute-value norm*
$\|x\| = |x|$.

Example 2. *Euclidean norm*
$\|\boldsymbol{x}\|_2 = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$

Example 3. *Finite-dimensional complex normed spaces*
On an n-dimensional exomplex space $\mathbb{C}^n$, the most common is $\|\boldsymbol{z}\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2} = \sqrt{z_1\overline{z_1} + \cdots + z_n\overline{z_n}}$
In inner product form, this is $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^H \boldsymbol{x}}$

Example 4. *Manhattan norm*
$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|$.

Example 5. *p-norm*

Let $p \geq 1$ be a real number. The p-norm (also called l_p-norm) of a vector $\boldsymbol{x}$ is $\|\boldsymbol{x}\| = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$

As p approaches ∞ the p-norm approaches the infinity norm:

$\|\boldsymbol{x}\| = \max_i |x_i|$

Example 6. *Infinite dimensions*

The generalization of the above norms to an infinite number of components leads to l^p and $\boldsymbol{L}^p$ spaces, with norms

$\|\boldsymbol{x}\| = (\sum_{i\in\mathbb{N}} |x_i|^p)^{1/p}$ *and* $\|f\|_{p,\boldsymbol{X}} = (\int_{\boldsymbol{X}} |f(x)|^p dx)^{1/p}$

2 Hilbert Space

Definition 3. *Inner product*

Inner product is a map

$\langle\cdot,\cdot\rangle : \boldsymbol{V} \times \boldsymbol{V} \rightarrow \boldsymbol{F}$

that satisfies the following three properties for all vectors $x, t, z \in \boldsymbol{V}$ and all scalars $a, b \in \boldsymbol{F}$.

1. *Conjugate symmetry: $\langle x, y\rangle = \overline{\langle y, x\rangle}$*
2. *Linearity in the first argument: $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$.*
3. *Psotive definite: if x is not zero, then $\langle x, x\rangle > 0$*

Definition 4. *Inner product vector space*

*An **inner product vector space** is a vector space $\boldsymbol{V}$ over the field $\boldsymbol{F}$ together with an inner product.*

Definition 5. *Cauchy sequency*

A sequence $x_1, x_2, x_3, \cdots$ in a metric space $(\boldsymbol{X}, d)$ is called Cauchy if for every positive real number $r \geq 0$ there is a positive interger $\boldsymbol{N}$ such that for all positive integers $m, n \geq \boldsymbol{N}$,

$d(x_m, x_n) \leq r$.

Definition 6. *Complete space*

A metric space $(\boldsymbol{X}, d)$ is complete if every Cauchy sequence of points in $\boldsymbol{X}$ has a limit that is also in $\boldsymbol{X}$.

Definition 7. *Hilbert Space*

A Hilbert space H is a real or complex inner product space that is also a complete metrix space with respect to the distance function induced by the inner product.

good examples are needed here.

Example 7. *Lebesgue spaces*

Example 8. *Examples*