MATH 759: PROBLEM SET 2 SOLUTIONS ARE IN CLASS ON MON. SEP. 30.

1. Show that the $sl(n)$ and $so(n)$ are Lie algebras associated to Lie groups $SL(n)$ and $SO(n)$, respectively.

   On a Lie algebra one can define the exponential map (exp : $\mathfrak{g} \to G$ in the following way. Given $X_e \in \mathfrak{g}$, let X be the left-invariant vector field as in the notes. Let $\gamma(t)$ be the solution of
   $$\gamma'(t) = X(\gamma(t)), \qquad \gamma(0) = e.$$
   Define $\exp(X_e) = \gamma(1)$. Compute the exponential map on $sl(n)$.

   Then show that the bracket in the Lie algebra $sl(n)$ can be represented as a commutator of matrices.

2. Let $L_2(V \otimes W)$ be the set of all bilinear functions from $V \times W \to \mathbb{R}$. Show that
   $$(V \otimes W)^* \cong L_2(V, W) \cong V^* \otimes W^*.$$

3. Show that a manifold is orientable if and only if there exists a volume form on the manifold. A volume form on an n-dimensional manifold $\mathcal{M}$ an n-form which is not equal to zero at any point of the manifold.

4. *Pull-back* of a $(0, s)$ tensor. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a differentiable mapping and S a $(0, s)$ tensor on $\mathcal{N}$. The pull back of S is the $(0, s)$ tensor Φ^*S on $\mathcal{M}$ defined as follows:
   $$\Phi^*S|_p(v_1, \ldots, v_s) = S|_{\Phi(p)}(D\Phi v_1, \ldots D\Phi v_s) \quad \text{for all } v_1, \ldots, v_s \in T_p\mathcal{M}.$$
   Pullback of differential forms is defined analogously. Show that

   (i) $\Phi^*(S_1 \otimes S_2) = \Phi^*(S_1) \otimes \Phi^*(S_2)$

   (ii) For any two differential forms ω_1 and ω_2 on $\mathcal{M}$, $\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*(\omega_1) \wedge \Phi^*(\omega_2)$

   (iii) Exterior derivative and pullback commute; that is for any from ω
   $$\Phi^*(d\omega) = d\Phi^*(\omega).$$

5. Let ω be a 1-form on S^2. Think of S^2 as embedded in $\mathbb{R}^3$. Assume that for any $\phi \in SO(3)$, $\phi^*\omega = \omega$. Show that $\omega = 0$.