Department of Mathematical Sciences
Carnegie Mellon University

**21-740** **Functional Analysis II** **Fall 2013**

**Assignment 5**

**(Theoretically) Due on Friday, December 6**

1. (Schrodinger's Equation): Use a semigroup approach to analyze the initial value problem

$$\begin{cases} \frac{1}{i}u_t(t,x) = \Delta u(t,x) - V(x)u(t,x), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \quad x \in \mathbb{R}^n. \end{cases}$$

Here $u_0 : \mathbb{R}^n \to \mathbb{C}$ and $V : \mathbb{R}^n \to \mathbb{R}$ are given functions and $\Delta$ is the Laplacian with respect the spatial variable $x$. Make whatever assumptions and choose whatever spaces you feel are appropriate. Note: The special case $n = 3$,

$$V(x) = -\frac{1}{|x|}, \quad x \in \mathbb{R}^3$$

is of interest. It would be nice (but not essential) if your results apply to this case.

2. Use a semigroup approach to analyze the initial value problem

$$\begin{cases} u_{tt}(t,x) = \Delta u(t,x) + \Delta u_t(t,x), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = v_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Here $u_0, v_0 : \mathbb{R}^n \to \mathbb{C}$ are given functions and $\Delta$ is the Laplacian with respect the spatial variable $x$. Make whatever assumptions and choose whatever spaces you feel are appropriate.

3. Let $X$ be a Banach space and assume that $T : \mathcal{S}(\mathbb{R}^n) \to X$ is linear. Show that $T$ is continuous if and only if there exist $N \in \mathbb{N} \cup \{0\}$ and $K \in \mathbb{R}$ such that

$$\|T\phi\|_X \leq K|||\phi|||_N \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

4. Prove or Disprove: Assume that $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear and continuous and that

$$\tau_h(L\phi) = L(\tau_h\phi) \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n),\ h \in \mathbb{R}^n.$$

Then there is a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$L\phi = u * \phi \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

5. In this problem you may take for granted that if

$$p \in (1,2), \quad q = \frac{p}{p-1},$$

then the Fourier transform maps $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ continuously. Let $s \in \mathbb{R}$ with $0 < s < \frac{n}{2}$ be given. For which value of $r \in [1, \infty]$ can we be sure that

$$H^s(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n) \ ?$$