Department of Mathematical Sciences
Carnegie Mellon University

21-640 **Intro to Functional Analysis** **Spring 2013**

Assignment 2

Due on Monday, February 4.

Solutions to problems marked with an asterisk should be written up and handed in.

1. Let X be a linear space and $\mathcal{C}$ be a collection of linearly independent subsets of X such that

$$\forall S, T \in \mathcal{C}, \quad S \subset T \text{ or } T \subset S,$$

and put $C = \cup \mathcal{C}$. Show that C is linearly independent.

2.* Let X be an infinite-dimensional normed linear space over $\mathbb{K}$. Show that there is a discontinuous linear mapping $L : X \to \mathbb{K}$.

3. Prove or Disprove: Let X and Y be normed linear spaces over $\mathbb{R}$ and $L : X \to Y$ be a continuous mapping such that

$$\forall x, y \in X, \quad L(x+y) = L(x) + L(y).$$

Then L is linear.

4.* Let X be an infinite-dimensional linear space. Show that there exist norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on X that are not equivalent.

5. Let X be a normed linear space. Show that if X has a Schauder basis, then X is separable.

6.* Let X be the set of all real-valued continuous functions on $[0, 1]$ equipped with the norm given by

$$||f||_\infty := \max \{|f(x)| : x \in [0, 1]\},$$

and put

$$K = \{f \in X : f(x) \geq 0 \text{ for all } x \in [0, 1]\}.$$

Given $f \in X$, we write $f \geq 0$ if and only if $f \in K$. Let $L : X \to \mathbb{R}$ be a linear mapping such that

$$Lf \geq 0 \text{ for all } f \in K.$$

Show that L is continuous.

7. Prove or disprove: Let X be a normed linear space and let $x_0 \in X$ and $\delta > 0$ be given. Then
$$\mathrm{cl}(B_\delta(x_0)) = \{x \in X : \|x - x_0\| \leq \delta\}.$$

8. Let l^∞ denote the set of all $\mathbb{K}$-valued sequences $x = (x_k | k \in \mathbb{N})$ that are bounded, equipped with the norm $\| \cdot \|_\infty$ defined by
$$\|x\|_\infty = \sup\{|x_k| : k \in \mathbb{N}\},$$
and c denote the linear manifold in X consisting of all convergent $\mathbb{K}$-valued sequences. Prove or disprove: c is nowhere dense in l^∞.

9.* Let X be a normed linear space. Show that X is complete if and only if every absolutely summable sequence is summable.

10. Prove or Disprove: Let X be an infinite-dimensional Banach space. Then there is a linear manifold $V \subset X$ such that V is not closed.

11.* Give an example of a real Banach space X and a continuous mapping $f : X \to \mathbb{R}$ such that f is unbounded on $\{x \in X : \|x\| \leq 1\}$.

12. Let $(X, \| \cdot \|)$ be a normed linear space over $\mathbb{K}$ and $l : X \to \mathbb{K}$ be a linear functional. Show that l is continuous if and only if $\mathcal{N}(l)$ is closed.

13.* Prove or Disprove: Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed linear spaces and $T : X \to Y$ be a linear mapping. Then T is continuous if and only if $\mathcal{N}(T)$ is closed.

14.* Let $\mathbb{K} = \mathbb{R}$ and let $X = C(\mathbb{R}; \mathbb{R})$, the set of all continuous functions (not necessarily bounded) from $\mathbb{R}$ to $\mathbb{R}$. Does there exist a norm $\| \cdot \|$ on X such that for every $a \in \mathbb{R}$ the mapping
$$f \mapsto f(a)$$
is continuous from $(X, \| \cdot \|)$ to $\mathbb{R}$? Explain.