Homework 1

21-740 Introduction to Functional Analysis II

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Problem 1

I wasn't able to finish this problem.

Problem 2

If M is a finite-dimensional subspace of X, consider a Hamel basis $(x_i : i \in I)$ for X with a subset $(x_i : i \in J)$ $(J \subseteq I)$ that is a Hamel basis for M. Then, $\forall i \in J$, $\exists \alpha_i : M \to \mathbb{K}$ such that, $\forall x \in M$,

$$x = \sum_{i \in J} \alpha_i(x) x_i.$$

Since α_i is linear and M is finite dimensional, α_i is continuous and, by Hahn-Banach, can be extended to some $\beta_i \in X^*$. Define

$$N := \bigcap_{i \in J} \mathcal{N}(\beta_i).$$

Since each β_i is a continuous linear functional, each $\mathcal{N}(\beta_i)$ is closed, and so N is closed. By construction of β_i's, $\forall$ nonzero $x \in M$, $\exists i \in J$ with $\beta_i(x) = \alpha_i(x) \neq 0$, and hence $N \cap M = \{0\}$.

Finally, from the definition of a Hamel basis that, $\forall x \in X$,

$$x \in \mathcal{N}(\beta_i) + \operatorname{span}(x_i : i \in J) = N + M,$$

and so $X = M + N$. Thus, M is complemented. ■

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Problem 3

Let $X = \ell^2$, and let $M := cl(\text{span}\{e^{2i} : i \in \mathbb{N}\})$, where e^i is the unit vector with $e^i_i = 1$, and let $N := cl(\text{span}\{e^{2i} + \frac{1}{i}e^{2i+1} : i \in \mathbb{N}\})$. Note that both M and N are clearly closed subspaces.

Since $x \in N$ has zero in all odd indices (i.e., $x \in M$) only if it has zero in all even indices (i.e., $x = 0$), $N \cap M = \{0\}$. Also, $\forall i \in \mathbb{N}$,

$$e^{2i+1} = i\left(e^{2i} + \frac{1}{i}e^{2i+1}\right) - ie^{2i},$$

and thus $\{e^i : i \in \mathbb{N}\} \subseteq M + N$. Since $(e^i : i \in \mathbb{N})$ is a Schauder basis of ℓ^2, it follows that $M + N$ is dense in X.

Define $x \in \ell^2$ by $x_i = \frac{1}{i}, \forall i \in \mathbb{N}$. If $x = m + n$ for some $m \in M, n \in \mathbb{N}$, then each $n_{2i+1} = \frac{1}{2i+1}$, and so $n_{2i} = 1$. But then $\sum_{i=0}^{\infty} n_i^2 = \infty$, and so $n \notin \ell^2$. Thus, $M + N \neq \ell^2$, as desired. ■

Problem 4

Since A is normal, $\mathcal{N}(A) = \mathcal{N}(A^*) = \mathcal{R}(A)^\perp$. Thus, $\mathcal{R}(A)^\perp = \{0\}$ if and only only if A is injective.

It suffices, now, to show that a linear manifold M is dense in X if and only if $M^\perp = \{0\}$ (we may have shown this at some point, but I didn't see it in the notes). If $M^\perp = \{0\}$, by Proposition 10.7,

$$X =^\perp \{0\} =^\perp (M^\perp) = cl(M),$$

and so M is dense. If M is dense in X and $x \in M^\perp$, then there is a sequence $\{x_n\}_{n=1}^\infty$ with $x_n \to x$ as $n \to \infty$. By Cauchy-Schwarz,

$$\|x\|^2 = (x, x) = (x, x) - (x_n, x) = (x - x_n, x) \leq \|x - x_n\|\|x\| \to 0$$

as $n \to \infty$, and so $x = 0$. ■

Problem 5

($\Rightarrow$) Suppose that A is compact, and let $\{x_n\}_{n=1}^\infty$ be a sequence in X with $x_n \rightharpoonup 0$ (weakly). Since A is compact, $Ax_n \to 0$ (strongly) as $n \to \infty$ (Theorem 11.10). Also, since $\{x_n\}$ is weakly convergent, $\|x_n\|$ is bounded by some $B \in \mathbb{R}$ (Theorem 7.15(i)). Thus, by Cauchy-Schwartz,

$$0 \leq |(Ax_n, x_n)|^2 \leq \|Ax_n\|\|x_n\| \leq \|Ax_n\|B \to 0$$

as $n \to \infty$. Note that this direction does not require $\mathbb{K} = \mathcal{C}$.

($\Leftarrow$) Suppose a sequence $\{x_n\}_{n=1}^\infty$ in X is bounded. By Alaoglu's Theorem (since X is a Hilbert space), $\{x_n\}_{n=1}^\infty$ has a weakly convergent subsequence $\{x_{n_k}\}_{k=1}^\infty$. Suppose $x_{n_k} \rightharpoonup 0$ as $k \to \infty$. Then, $(Ax_{n_k}, x_{n_k}) \to 0$,

Problem 6

Define the map $\exp : \mathcal{L}(X,X) \to \mathcal{L}(X,X)$ by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}.$$

Since

$$\|\exp(A)\| \leq \sum_{n=0}^{\infty} \frac{\|iA\|^n}{n!} = e^{\|iA\|},$$

exp does indeed map bounded operators to bounded operators.

Lemma 1: If $A, B \in \mathcal{L}(X,X)$ commute, then $\exp(A)\exp(B) = \exp(A+B)$.

Proof:

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_{n=0}^{\infty} \frac{(iA)^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{(iB)^m}{m!}\right) \\
&= \sum_{n=0}^{\infty} \frac{n!}{n!} \sum_{m=0}^{n} \left(\frac{(iA)^{n-m}(iB)^m}{m!(n-m)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \sum_{m=0}^{n} \left(\binom{n}{m} A^{n-m}B^m\right) \\
&= \sum_{n=0}^{\infty} \frac{(i(A+B))^n}{n!} = \exp(A+B).
\end{aligned}$$

where the last line follows from the Binomial Theorem, since A and B commute. $\square$

Now observe that, since the adjoint operator is conjugate-linear and $A = A^*$,

$$U^* = \sum_{n=0}^{\infty} \frac{((iA)^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-iA^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i(-A))^n}{n!} = \exp(-A).$$

Thus, by the above lemma (clearly, A and $-A$ commute),

$$UU^* = \exp(A)\exp(-A) = \exp(0) = \exp(-A)\exp(A) = U^*U.$$

It suffices now, by Propositions 1.15 and 1.17, to observe that, since $(i0)^n$ is nonzero only if $n = 0$,

$$\exp(0) = I. \quad \blacksquare$$