Homework 1

21-295 Putnam Seminar
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Due: Friday, September 7, 2012

Putnam 1964/B6

Suppose, for sake of contradiction, that there exist congruent sets A and B with $A \cup B = D$ and $A \cap B = \emptyset$, and let $f : A \to B$ be a geometric transformation between A and B. Since $O := (0, 0) \in D$, O is in exactly one of A and B; without loss of generality, we suppose $O \in A$. Let $Y = f(O) \in B$. Since O is center of the disk and $Y \neq O$ (as $Y \notin A$), there is a unique diameter L of D going through both O and Y. Furthermore, there is a unique diameter L' of D that is perpendicular to L. Let P and Q be the endpoints of L' (i.e., P and Q are the intersection points of L' and the boundary of D).

Since L' is a diameter, $PQ = 2$. Note that any geometric transformation is an isometry (i.e., it preserves the distance between any two points). Since $PO = OQ = 1$, and PY and YQ are hypotenuses of nondegenerate triangles having either PO or OQ as a leg, $PY, YQ > 1$. Thus, every point X on the half-circle with endpoints P and Q (including P and Q) has $XY > 1$, so that $X \notin B$. $PQ = 2$, so that $f(P), f(Q) \in B$ are the endpoints of some diameter of D (as they have distance 2). But this is impossible if B contains no points on the aforementioned half-circle, providing the desired contradiction. ■

Putnam 1962/A6.

Since either $1 \in X$ or $-1 \in X$ and X is closed under multiplication, $1 \cdot 1 = (-1) \cdot (-1) = 1 \in X$. Since X is closed under addition and any postive integer can be constructed by adding 1 a finite number of times, the set of positive integers $\mathbb{Z}^+ \subseteq X$.

Therefore, if, for some $a, b \in \mathbb{Z}^+$, $-\frac{a}{b} \in X$ (i.e., if there were a negative rational number in X), then, since X is closed under multiplication $-a = b \cdot \left(-\frac{a}{b}\right) \in X$. However, this contradicts the fact that $a \in X$, so no negative rational number is in X. Therefore, every postive rational number is in X, so that, since $X \subseteq \mathbb{Q}$, X is the set of positive rationals. ■