Homework 1

21-720 Measure and Integration
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Problem 1

(a) Let $S \subseteq \mathbb{R}^d$ be a countable set, so that there exists a bijection $f : S \to \mathbb{N}$. Let $\epsilon > 0$, and, $\forall i \in \mathbb{N}$, let R_i be the closed hypercube of sidelength $\sqrt[n]{\epsilon 2^{-i}}$ centered at $f^{-1}(i)$, so that R_i has d-dimensional volume $\ell(R_i) = \epsilon 2^{-i}$.

Since, $\forall x \in S$, $x \in R_{f(x)}$, and, since f is a bijection,

$$S \subseteq \bigcup_{x \in S} R_{f(x)} = \bigcup_{i=1}^{\infty} R_i,$$

and, clearly, each R_i is a cell. Thus, by definition of the Lebesgue outer measure,

$$m^*(S) \leq \sum_{i=1}^{\infty} \ell(R_i) = \epsilon \sum_{i=1}^{\infty} 2^{-i} = \epsilon.$$

Since this holds for all $\epsilon > 0$, $\boxed{m^*(S) = 0.}$

(b) $\forall i \in \mathbb{N}$, let C_i be the following recursively defined family of sets:

$$\begin{aligned} C_0 &= [0, 1] \\ C_i &= \frac{C_{i-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{i-1}}{3} \right) \end{aligned}$$

(where addition and multiplication of sets by real numbers denote translation and dilation, respectively). Let C denote the Cantor set. It can be shown inductively $\forall i \in \mathbb{N} \cup \{0\}$, that $C \subseteq C_i$ and that

$$C_i = \bigcup_{k=1}^{2^i} I_k,$$

where each I_k is a cell of length $\ell(I_i) = \left(\frac{1}{3}\right)^i$. Thus, by definition of the Lebesgue outer measure, $\forall i \in \mathbb{N}$.

$$m^*(C) \leq \sum_{i=1}^{2^i} \left(\frac{1}{3}\right)^i = \left(\frac{2}{3}\right)^i.$$

Since this is true for all i, $\boxed{m^*(C) = 0.}$

(c) For notational convenience, let $S = \{x \in [0,1] | x \notin \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, it follows from the result of part (a) that $m^*(\mathbb{Q}) = 0$. Since $[0,1]$ is a cell, $m^* = 1 - 0 = 1$. By sub-additivity of the Lebesgue outer measure,

$$1 = m^*([0,1]) \leq m^*([0,1] \cap \mathbb{Q}) + m^*(S) = m^*(S).$$

By monotonicity, $m^*(S) \leq m^*([0,1]) = 1$. Therefore, $\boxed{m^*(S) = 1.}$

Problem 2

(a) Any subspace V of $\mathbb{R}^d$ is the image of the subspace W whose basis is the set of the first $n := \dim(V)$ canonical basis vectors of $\mathbb{R}^d$, under some rotation. Thus, since all rotations are orthonormal transformations, by the result of part (b) of problem 3, it suffices to show that $\lambda(W) = 0$.

Let $\epsilon > 0$. Since $\mathbb{N}^n$ is countable, let $\mathbf{f} : \mathbb{N} \to \mathbb{N}^n$ be a bijection.

Then, $\forall i \in \mathbb{N}$, let

$$\begin{aligned} I_i =& (f_1(i), f_1(i)+2) \times (f_2(i), f_2(i)+2) \times \cdots \times (f_n(i), f_n(i)+2) \\ & \times \left(-\frac{\epsilon}{2^{n+1+i}}, \frac{\epsilon}{2^{n+1+i}}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq \mathbb{R}^d, \end{aligned}$$

where $f_1(i), \ldots, f_n(i)$ are the components of $\mathbf{f}(i)$.

Since each I_i is a cell, $\lambda(I_i) = \ell(I_i) = \frac{\epsilon}{2^i}$. Thus, by monotonicity and then by subadditivity,

$$\lambda(V) \leq \lambda\left(\bigcup_{i=1}^{\infty} I_i\right) \leq \sum_{i=1}^{\infty} \lambda(I_i) = \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = \epsilon.$$

Since this holds for all $\epsilon > 0$, $\lambda(V) = 0$. $\blacksquare$

(b) Since each segment of the boundary ∂P is a subset of a line, which is the translation of 1-dimensional subspace of $\mathbb{R}^2$, by monotonicity, translational invariance of λ, and the result of part (a), each segment of ∂P has Lebesgue measure 0. Since there finitely many boundary segments, by subadditivity, $\lambda(\partial P) = 0$.

Since any triangle is the image of some triangle having at least 1 side parallel to the x-axis, under a rotation, by the result of part (b) of problem 3, to show the desired result in the case that P is a triangle, we can assume that P has some side parallel to the x-axis. Let I be the smallest open cell containing P, so that I has the same height and base length as P. Let v be the vertex of P that is not on the side parallel to the x-axis, and let I_1 and I_2 be the two cells into which I is split by the vertical line going through v. Then, $P_1 := P \cap I_1$ is a rotation of $R_1 := (P^c \cap I_1) \backslash (\partial P)$, and $P_2 := P \cap I_2$ is a rotation of $R_2 := (P^c \cap I_2) \backslash (\partial P)$. Therefore, by the result of part(b) of problem 3, $\lambda(P_1) = \lambda(R_1)$ and $\lambda(P_2) = \lambda(R_2)$. Since P_1 and P_2 are open and the intersection of their boundaries is closed,

$$\lambda(P_1) + \lambda(R_1) + \lambda(\partial P_1 \cap \partial R_1) = \lambda(I_1),$$
$$\lambda(P_2) + \lambda(R_2) + \lambda(\partial P_2 \cap \partial R_2) = \lambda(I_2).$$

Then, since, as explained above, $\lambda(\partial P_1 \cap \partial R_1) = \lambda(\partial P_2 \cap \partial R_2) = 0$.

$$\lambda(P_1) + \lambda(P_1) = \lambda(P_1) + \lambda(R_1) = \lambda(I_1),$$
$$\lambda(P_2) + \lambda(P_2) = \lambda(P_2) + \lambda(R_2) = \lambda(I_2).$$

Thus, $\lambda(P_1) = \frac{1}{2}\lambda(I_1)$ and $\lambda(P_2) = \frac{1}{2}\lambda(I_2)$, so that

$$\lambda(P) = \lambda(P_1) + \lambda(P_2) = \frac{1}{2}(\lambda(I_1) + \lambda(I_2)) = \frac{1}{2}\lambda(I).$$

Therefore, the Lebesgue measure of a triangle is its area.

Any polygon P can be written as the union of finitely many triangles $T_1, T_2, \ldots, T_k$, which are disjoint except perhaps at their boundaries. Thus,

$$\begin{aligned}
\lambda(P) = \lambda\left(\bigcup_{i=1}^{k} T_i\right) &= \lambda\left(\bigcup_{i=1}^{k} T_i^\circ \cup \partial T_i\right) \\
&= \sum_{i=1}^{k} \lambda(T_i^\circ) + \sum_{i=1}^{k} \lambda(\partial T_i) && \text{(since } T_i^\circ \text{ open, } \partial T_i \text{ closed)} \\
&= \sum_{i=1}^{k} \lambda(T_i^\circ) = \sum_{i=1}^{k} \text{area}(T_i) = \text{area}(P).
\end{aligned}$$

Problem 3

(a) Let $c = \mu(I) \in [0, \infty)$, where $I = [0,1)^d$. Divide I into k^n half-open hypercubes $C_1, \ldots, C_{k^n}$ of sidelength $1/k$. Note that, since each cube is a translation of every other cube, each cube has the same μ measure, so that, since half-open cells are in $\mathcal{L}$,

$$k^n \mu(C_1) = \sum_{i=1}^{k^n} \mu(C_1) = \sum_{i=1}^{k^n} \mu(C_i) = \mu\left(\bigcup_{i=1}^{k^n} C_i\right) = \mu(I) = c.$$

Then, $\mu(C_1) = \frac{c}{k^n} = c\lambda(C_1)$.

By monotonicity, if $S = (a_1, b_1) \times \cdots \times (a_d, b_d)$ is a cell, then there exists a covering family $\mathcal{C}$ of S with cubes of sidelength $1/k$ such that

$$\sum_{i=1}^{d} (1/k)(b_m - a_m - 1/k)^{(d-1)}, \leq \lambda(\cup\mathcal{C}) - \lambda(S) \leq \sum_{i=1}^{d} (1/k)(b_m - a_m)^{(d-1)},$$

where $m = \text{argmax}_m(b_m - a_m)$. Since this holds for all $k \in \mathbb{N}$ and μ agrees with λ on each cube of sidelength $1/k$, by monotonicity, $\mu(S) = c\lambda(S)$, so that we have shown that $\mu = c\lambda$ on half-open cells.

Suppose now that S is any bounded set in $\mathcal{L}$. For any cover of S with countably many disjoint cells $C_1, C_2, \ldots$, there exists a cover of S with countably many disjoint half-open cells $H_1, H_2, \ldots$, such that

$$\sum_{i=1}^{\infty} c\lambda(H_i) \left(\leq \sum_{i=1}^{\infty} c\lambda(C_i) \right) + \epsilon,$$

(we can cover each cell C_i with a half-open cell of sidelength at most $\frac{\sqrt[d]{\epsilon}}{2^i}$ larger than the sidelength of C). Then, by countable additivity,

$$\mu\left(\bigcup_{i=1}^{\infty} H_i\right) = \sum_{i=1}^{\infty} \mu(H_i) \leq \left(\sum_{i=1}^{\infty} c\lambda(H_i)\right) + \epsilon = \lambda\left(\bigcup_{i=1}^{\infty} C_i\right) + \epsilon.$$

Then, taking the infimum over all covers $C_1, C_2, \ldots$ on both sides,

$$\mu(S) \leq c\lambda(S) + \epsilon.$$

Since this is true for all $\epsilon > 0$, $\mu(S) \leq c\lambda(S)$. However, by the same argument, if I is a half-open cell containing S (such an I must exist since S is bounded), then $\mu(I\backslash S) \leq c\lambda(I\backslash S)$ so that $\mu(I) - \mu(S) = c\lambda(I) - \mu(S) \leq c\lambda(I) - c\lambda(S)$, and thus $\mu(S) \geq c\lambda(S)$. Thus, $\mu = c\lambda$ on all bounded sets in $\mathcal{L}$.

Now, letting S be any set in $\mathcal{L}$, S can be written as the union of disjoint bounded sets S_i in $\mathcal{L}$ (since $\mathbb{R}^d$ can be covered by countably many half-open, disjoint cubes $C_1, C_2, \ldots$ of unit sidelength, we can just take $S_i = C_i \cap S$), by countable additivity,

$$\mu(S) = \sum_{i=1}^{\infty} \mu(S_i) = \sum_{i=1}^{\infty} c\lambda(S_i) = c\lambda(S). \quad \blacksquare$$

(b) Since any orthogonal linear transformation is a composition of rotations and reflections, $\forall \delta > 0$, $\forall B(x,r) \in \mathcal{E}_\delta$ (where $\mathcal{E}_\delta$ is as defined in part (B) of problem 4), $T(B(x,r)) = B(T(x), r) \in \mathcal{E}_\delta$, so that, $\rho(T(B(x,r))) = \rho(B(x,r))$. Therefore,

$$\begin{aligned}
H_\alpha(A) &= \lim_{\delta \to 0+} \inf \left\{ \sum_{i=1}^{\infty} \rho(B_i) \,\middle|\, B_i \in \mathcal{E}_\delta, A \subseteq \bigcup_{i=1}^{\infty} B_i \right\} \\
&= \lim_{\delta \to 0+} \inf \left\{ \sum_{i=1}^{\infty} \rho(T(B_i)) \,\middle|\, T(B_i) \in \mathcal{E}_\delta, T(A) \subseteq \bigcup_{i=1}^{\infty} T(B_i) \right\} \\
&= \lim_{\delta \to 0+} \inf \left\{ \sum_{i=1}^{\infty} \rho(B_i) \,\middle|\, B_i \in \mathcal{E}_\delta, T(A) \subseteq \bigcup_{i=1}^{\infty} B_i \right\} = H_\alpha(T(A)).
\end{aligned}$$

Thus, by the result of part (c) of problem 4, $\exists c \in (0, \infty)$ such that

$$\lambda(T(A)) = \frac{1}{c} H_d(T(A)) = \frac{1}{c} H_d(A) = \lambda(A). \quad \blacksquare$$

Problem 4

(a) Clearly, $\mu : [0,1] \to [0,\infty]$, and since $\emptyset \in \mathcal{E}$ and $\rho(\emptyset) = 0$, $\mu^*(\emptyset) = 0$.

Suppose $E \subseteq F \subseteq X$. If $E_1, E_2, \ldots \in \mathcal{E}$ such that $F \subseteq \bigcup_{i=1}^{\infty} E_i$, then $E \subseteq \bigcup_{i=1}^{\infty} E_i$, so that

$$\left\{ \sum_{i=1}^{\infty} \rho(E_i) \,\middle|\, E_i \in \mathcal{E}, F \subseteq \bigcup_{i=1}^{\infty} E_i \right\} \subseteq \left\{ \sum_{i=1}^{\infty} \rho(E_i) \,\middle|\, E_i \in \mathcal{E}, E \subseteq \bigcup_{i=1}^{\infty} E_i \right\}.$$

Then, taking the infimum on both sides gives $\mu^*(E) \le \mu^*(F)$, so that μ^* is monotonic.

Suppose $A_1, A_2, \ldots \subseteq X$. If $\sum_{i=1}^{\infty} \mu^*(A_i) = \infty$, then the inequality

$$m^* \left(\bigcup_{i=1}^{\infty} A_i \right) \le \sum_{i=1}^{\infty} m^*(A_i)$$

trivially holds. Thus, we suppose $\sum_{i=1}^{\infty} \mu^*(A_i) < \infty$. Let $\epsilon > 0$. Since μ^* is an infimum, $\forall i \in \mathbb{N}$, there is a family $\mathcal{G}_i \subseteq \mathcal{E}$ such that $A_i \subseteq \bigcup \mathcal{G}_i$ and

$$\sum_{E \in \mathcal{G}_i} \mu^*(E) \le \mu^*(A_i) + \frac{\epsilon}{2^i}.$$

Taking the sum over all $i \in \mathbb{N}$ gives

$$\sum_{i=1}^{\infty} \sum_{E \in \mathcal{G}_i} \mu^*(E) \le \sum_{i=1}^{\infty} \mu^*(A_i) + \frac{\epsilon}{2^i} = \epsilon + \sum_{i=1}^{\infty} \mu^*(A_i).$$

Since $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{E \in \mathcal{G}_i} E$, by monotonicity of μ^*,

$$\mu^* \left(\bigcup_{i=1}^{\infty} A_i \right) \le \epsilon + \sum_{i=1}^{\infty} \mu^*(A_i).$$

Since this holds for all $\epsilon > 0$,

$$\mu^* \left(\bigcup_{i=1}^{\infty} A_i \right) \le \sum_{i=1}^{\infty} \mu^*(A_i),$$

so that μ^* is countably subadditive. Thus, μ^* is an outer measure, as desired. ■

(b) The result of part (a) implies that $H^*_{\alpha,\delta}$ is an outer measure. Since, $\forall \delta > 0$, $H^*_{\alpha,\delta} : \mathcal{P}(X) \to [0,\infty]$ and $H^*_{\alpha,\delta}(\emptyset) = 0$, taking the limit as $\delta \to 0^+$, $H^*_{\alpha} : \mathcal{P}(X) \to [0,\infty]$ and $H^*_{\alpha}(\emptyset) = 0$.

Since, $\forall \delta > 0$, for $E \subseteq F \in \mathcal{P}(X)$, $H^*_{\alpha,\delta}(E) \le H^*_{\alpha,\delta}(F)$, taking the limit as $\delta \to 0^+$, $H^*_{\alpha}(E) \le H^*_{\alpha}(F)$, so that H^*_{α} is monotonic.

Since $H^*_{\alpha,\delta}$ is nondecreasing in δ (as it is an infimum and $\mathcal{E}_\delta$ becomes smaller as δ decreases), $\forall \delta > 0, H^*_{\alpha,\delta} \leq H^*_\alpha$. Therefore, since, $\forall \delta > 0$, $\forall E_1, E_2, \ldots \in \mathcal{P}(X)$,

$$H^*_{\alpha,\delta}\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i=1}^\infty H^*_{\alpha,\delta}(E_i) \leq \sum_{i=1}^\infty H^*_\alpha(E_i),$$

so that, taking the limit as $\delta \to 0^+$,

$$H^*_\alpha\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i=1}^\infty H^*_\alpha(E_i).$$

Therefore, H^*_α is countably subadditive and thus an outer measure. ■

Since H^*_α is an outer measure, by Caratheodory, it suffices to show that, $\forall B \in \mathcal{B}$, $\forall A \subseteq X$,

$$H^*_\alpha(A) \geq H^*_\alpha(A \cap B) + H^*_\alpha(A \cap B^c),$$

If either $H^*_\alpha(A \cap B) = \infty$ or $H^*_\alpha(A \cap B^c) = \infty$, then, by monotonicity, $H^*_\alpha(A) = \infty$ and the desired result trivially holds. Thus, we assume that each of these measures is finite.

Lemma: If $E, F \in \mathcal{P}$ with $d := \operatorname{dist}(E, F) > 0$, then

$$H^*_\alpha(E \cup F) \geq H^*_\alpha(E \cup F).$$

Proof of Lemma: For $\delta < \frac{d}{2}$, if $B_1, B_2, \ldots \in \mathcal{E}_\delta$, then, each B_i has $B_i \cap E = \emptyset$ or $B_i \cap F = \emptyset$ (for, if $x \in B_i \cap E$ and $y \in B_i \cap F$, then $\operatorname{dist}(E, F) \leq d(x, y) < 2\delta = d$, which is a contradiction). Thus, let $B_{j_1}, B_{j_2}, \ldots \in \mathcal{E}_\delta$ be the subsequence of B_i such that $B_i \cap E \neq \emptyset$, and let $B_{k_1}, B_{k_2}, \ldots \in \mathcal{E}_\delta$ be the subsequence of B_i such that $B_i \cap F \neq \emptyset$ (if either subsequence is finite, we can add countably many empty sets to the sequence). Since we assumed that $H^*_{\alpha,\delta}(E) \leq H^*_\alpha(E) < \infty$ and $H^*_{\alpha,\delta}(F) \leq H^*_\alpha(F) < \infty$, and, in the end, we are concerned only with infima, so that we can ignore infinite sums,

$$\sum_{i=1}^\infty \rho(B_i) = \sum_{i=1}^\infty \rho(B_{j_i}) + \sum_{i=1}^\infty \rho(B_{k_i}).$$

This implies that

$$\begin{aligned}&\left\{\sum_{i=1}^\infty \rho(B_i) \,\middle|\, E \cup F \subseteq \bigcup_{i=1}^\infty B_i, B_i \in \mathcal{E}_\delta\right\}\\ &= \left\{\sum_{i=1}^\infty \rho(B_{j_i}) + \rho(B_{k_i}) \,\middle|\, E \subseteq \bigcup_{i=1}^\infty B_{j_i}, F \subseteq \bigcup_{i=1}^\infty B_{k_i}, B_{j_i}, B_{k_i} \in \mathcal{E}_\delta\right\}.\end{aligned}$$

Rewriting the latter set as the element-wise sum of two sets, by definition of $H^*_{\alpha,\delta}$ and the fact that the infimum is nonincreasing with respect to inclusion,

$$H^*_{\alpha,\delta}(E \cup F) \geq H^*_{\alpha,\delta}(E) + H^*_{\alpha,\delta}(F),$$

so that taking the limit as $\delta \to 0^+$ proves the lemma:

$$H^*_\alpha(E \cup F) \geq H^*_\alpha(E) + H^*_\alpha(F).$$

We now return to showing that, $\forall B \in \mathcal{B}$, $\forall A \subseteq X$,

$$H_\alpha^*(A) \geq H_\alpha^*(A \cap B) + H_\alpha^*(A \cap B^c),$$

Since $\mathcal{B}$ is the σ-algebra generated by all open sets and is closed under the set complement, it is sufficient to this for all closed sets B. Thus, let $B \in \mathcal{B}$, and let $A \subseteq X$. $\forall i \in \mathbb{N}$, define

$$\begin{aligned} B_i &= \{x \in X | \operatorname{dist}(\{x\}, B) < 1/i\}, \\ C_i &= A \cap (B_i \cap B_{i+1}^c) \end{aligned}$$

Using the above lemma and then monotonicity gives

$$H_\alpha^*(A \cap B_i^c) + H_\alpha^*(A \cap B) \leq H_\alpha^*((A \cap B_i^c) \cup (A \cap B)) \leq H_\alpha^*(A \cup B). \quad \blacksquare$$

Thus, it remains only to show that

$$\lim_{i \to \infty} (H_\alpha^*(A \cap B_i^c)) = H_\alpha^*(A \cap B^c).$$

Since B is closed, for all $x \in B^c$, $\operatorname{dist}(\{x\}, B) > 0$, so that, $\forall i \in \mathbb{N}$,

$$A \cap B^c = (A \cap B_i^c) \cup \bigcup_{j=i+1}^{\infty} C_j.$$

Thus, by monotonicity and then subadditivity,

$$H_\alpha^*(A \cap B_i^c) \leq H_\alpha^*(A \cap B^c) \leq H_\alpha^* \left((A \cap B_i^c) \cup \bigcup_{j=i+1}^{\infty} C_j \right) \leq H_\alpha^*(A \cap B_i^c) + \sum_{j=i+1}^{\infty} H_\alpha^*(C_j).$$

Since the above summation is finite and each $H_\alpha^*(F_j) \geq 0$, $\lim_{i \to \infty} \left(\sum_{j=i+1}^{\infty} H_\alpha^*(C_j) \right) = 0$, and taking the limit as $i \to \infty$ in the above inequality:

$$\lim_{i \to \infty} (H_\alpha^*(A \cap B_i^c)) \leq H_\alpha^*(A \cap B^c) \leq \lim_{i \to \infty} (H_\alpha^*(A \cap B_i^c)) \quad \blacksquare$$

(c) $\forall \mathbf{x} \in \mathbb{R}^d$, $\delta > 0$, $\mathcal{E}_\delta = \mathcal{E}_\delta + \mathbf{x}$, since a translated ball is a ball of the same radius, and, $\forall B(\mathbf{y}, r) \in \mathcal{E}_\delta$, $\rho(B(\mathbf{y}, r))) = \rho(B(\mathbf{y} + \mathbf{x}, r)))$ Thus, since, for all Hausdorff measurable sets $E \in \mathcal{P}(\mathbb{R})$

$$H_\alpha(E) = \inf \left\{ \sum_{i=1}^{\infty} \rho(B_i) \middle| B_i \in \mathcal{E}_\delta, E \subseteq \bigcup_{i=1}^{\infty} B_i \right\},$$

$\forall \mathbf{x} \in \mathbb{R}$, $H_\alpha(E) = H_\alpha(E + \mathbf{x})$.

Thus, H_d is translation invariant on $\mathcal{L}$, so that, by the result of problem 3, part (a), $\exists c \geq 0$ such that $H_d = c\lambda$. It remains only to show that $c \in (0, \infty)$. Let $B = B(\mathbf{0}, \sqrt[d]{2})$, let $C_1 = [-1, 1]^d$, and let $C_2 = [-2, 2]^d$.

Since C_1 and C_2 are cells, $\lambda(C_1) = (1 - (-1))^d = 2^d$ and $\lambda(C_2) = (2 - (-2))^d = 4^d$.

Thus, $H_\alpha(B), \lambda(C_1), \lambda(C_2) \in (0, \infty)$. By monotonicity (noting $C_1 \subseteq B \subseteq C_2$),

$$c\lambda(C_1) = H_d(C_1) \leq H_d(B) \leq H_d(C_2) = c\lambda(C_2).$$

The first inequality implies that $c \neq \infty$, and the latter inequality implies that $c \neq 0$. $\quad \blacksquare$

(d) **Lemma:** If, for some $\alpha, \beta \in [0, \infty)$, $\alpha < \beta$ and $H_\alpha(S) < \infty$, then $H_\beta(S) = 0$.

Proof of Lemma: Let $\delta > 0$. Because $H_{\alpha,\delta}$ is an infimum, we can find a sequence of balls $B(x_1, r_1), B(x_2, r_2), \ldots \in \mathcal{E}_\delta$, with $S \subseteq \bigcup_{i=1}^\infty B(x_i, r_i)$, such that

$$\sum_{i=1}^\infty c_\alpha r_i^\alpha = \sum_{i=1}^\infty \rho(B(x_i, r_i)) \leq H_{s,\delta+1}.$$

Furthermore, since $H_{\alpha,\delta}$ is an infimum and $\mathcal{E}_\delta$ becomes smaller as δ decreases, $H_{\alpha,\delta}$ increases as δ decreases. Thus, taking the limit as $\delta \to 0^+$,

$$\sum_{i=1}^\infty c_\alpha r_i^\alpha \leq H_{\alpha,\delta}(S) + 1 \leq H_\alpha(S) + 1.$$

Therefore,

$$\begin{aligned} H_{\beta,\delta}(S) &\leq \sum_{i=1}^\infty c_\beta r_i^\beta \leq \frac{c_\beta}{c_\alpha} \sum_{i=1}^\infty c_\alpha r_i^\alpha r_i^{\beta-\alpha} \\ &\leq \frac{c_\beta}{c_\alpha} \delta^{\beta-\alpha} \sum_{i=1}^\infty c_\alpha r_i^\alpha \qquad \text{(each } r_i \leq \delta\text{)} \\ &\leq \frac{c_\beta}{c_\alpha} \delta^{\beta-\alpha} (H_\alpha(S) + 1). \end{aligned}$$

Since $H_\alpha(S) < \infty$ and $\beta > \alpha$, taking the limit as $\delta \to 0$ proves the lemma.

Let $d = \sup\{\alpha \in [0, \infty] | H_\alpha^*(S) = \infty\}$. Suppose $\alpha \in (d, \infty)$. By choice of d, for $\beta = \frac{\alpha-d}{2} > d$, $H_\beta(S) < \infty$, so that, by the above lemma, $H_\alpha(S) = 0$. On the other hand, suppose $\alpha \in (0, d)$. By the above lemma, if $H_\alpha(S) \neq \infty$, then, $\forall \beta \in (\alpha, d]$, $H_\beta(S) = 0$, contradicting the choice of d as the supremum. Thus, d has the desired properties. Note d is unique, as, if $d' \neq d$ (without loss of generality, $d' > d$), also had the desired properties, then, for $\alpha \in (d, d'$, $\alpha = 0$ and $\alpha = \infty$, which is impossible. ■

(e) **Lemma:** $\forall A \subseteq \mathbb{R}$, $c \in \mathbb{R}$, if cA is the dilation of A by c, then, $H_\alpha(cA) = c^\alpha H_\alpha(A)$.

Proof of Lemma: Note that, if $B_1, B_2, \ldots \in \mathcal{E}_\delta$ with $A \subseteq \bigcup_{i=1}^\infty B_i$, then $cB_1, cB_2, \ldots \in \mathcal{E}_\delta$ with $cA \subseteq \bigcup_{i=1}^\infty cB_i$. Also, for any ball $B(x, r)$, $\rho(cB(x, r)) = \rho(B(cx, cr)) = c^\alpha \rho(B(x, r))$. Thus,

$$H_\alpha(cA) = \lim_{\delta \to 0+} \inf \left\{ \sum_{i=1}^\infty c^\alpha \rho(B_i) \middle| B_i \in \mathcal{E}_\delta, cA \subseteq \bigcup_{i=1}^\infty B_i \right\} = c^\alpha H_\alpha(A),$$

proving the lemma.

The Cantor set C has the property that

$$C = \frac{1}{3} \left(C \cup (C + 2) \right),$$

where addition denotes translation and multiplication denotes dilation.

It was shown in the proof of part (c) that H_d is translation invariant. Note also that, since $C \subseteq [0,1]$, $\operatorname{dist}(C, C+2) > 0$, and thus that, by the Lemma shown in in part (b), $H_\alpha(C \cup (C+2)) = H_\alpha(C) + H_\alpha(C+2)$. Therefore, by the above lemma, $\forall \alpha \in [0, \infty]$,

$$\begin{aligned}
H_\alpha(C) &= H_\alpha\left(\frac{1}{3}(C \cup (C+2))\right) \\
&= \left(\frac{1}{3}\right)^\alpha H_\alpha(C \cup (C+2)) && \text{by above lemma} \\
&= \left(\frac{1}{3}\right)^\alpha H_\alpha(C) + H_\alpha(C+2) && \text{since } \operatorname{dist}(C, C+2) > 0 \\
&= \left(\frac{1}{3}\right)^\alpha 2H_\alpha(C). && \text{(by translation invariance)}
\end{aligned}$$

Suppose, then, that there exists some $\alpha \in (0, \infty)$ such that $H_\alpha(C) \in (0, \infty)$. Then, for that value of α, we can divide both sides of the above equation by $H_\alpha(C)$, so that

$$2 = 3^\alpha.$$

Then,

$$\alpha = \log_3(2) = \boxed{\frac{\ln(2)}{\ln(3)}.}$$