21-484 Notes

JD Nir
jnir@andrew.cmu.edu

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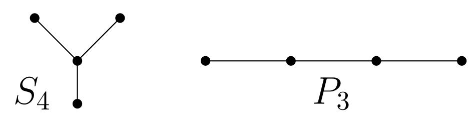
- Recall: There are 4^{4-2} labeled trees with four vertices.

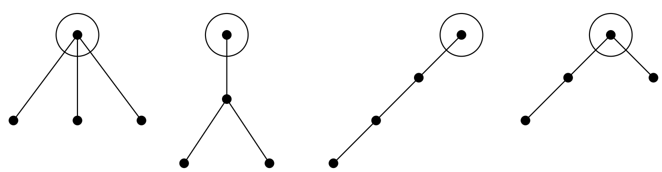
$\rightarrow$ Notice that these are the different unlabeled trees with 4 vertices:



$\rightarrow$ Def: (page 88) - a tree in which one of the vertices is distinguished as the root is called a rooted tree and denoted (T, v).

$\rightarrow$ Remark: People also consider rooted graphs, in which we may have a set of roots: (G, R)

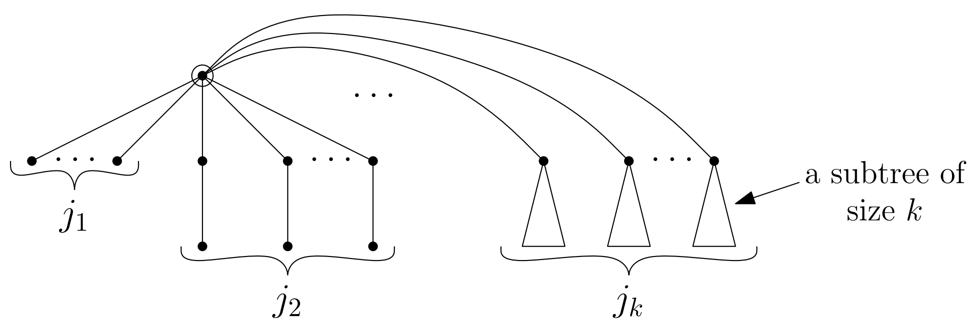
Example: There are 4 rooted trees with 4 vertices.



$= ((\{1,2,3,4\}, \{\{1,2\}, \{2,3\}, \{3,4\}\}), 3)$

- Let a_n be the number of rooted trees with n vertices.

- $a_1 = 1$



$\rightarrow$ Let (T, v) be a rooted tree.

- Let $T_1, \ldots, T_d$ be the subtrees at v.

- let j_i be the number of subtrees of size i.

- In how many ways can we choose the subtrees of size k?

$\binom{a_k + j_k - 1}{j_k}$ – choosing j_k elements out of a_k elements with repetition and without order.

$$\Rightarrow^{n>1} a_n = \sum_{j_1+2j_2+3j_3+\ldots+(n-1)j_{n-1}=n-1} \binom{a_1 + j_1 - 1}{j_1}\binom{a_2 + j_2 - 1}{j_2} \cdots \binom{a_{n-1} + j_{n-1} - 1}{j_{n-1}}$$

$\rightarrow$ got a recursive formula

→ recall: $\dfrac{1}{(1-x)^s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} x^k$ (Newton's generalized Binomial theorem)

→ let $A(z)$ be the generating function for the sequence a_n.

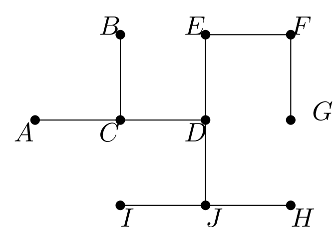
→ $A(z) = \dfrac{z}{(1-z)^{a_1}(1-z^2)^{a_2}(1-z^3)^{a_3}\cdots}$

→ (take log and some simplifying) $A(z) = z \cdot exp(A(z) + \frac{1}{2}A(z^2) + \frac{1}{3}A(z^3) + \ldots)$

→ (not trivial) $a_n = \dfrac{1}{\alpha^{n-1} \cdot n} \cdot \sqrt{\beta/2\pi n} + O(n^{-5/2}\alpha^{-n})$

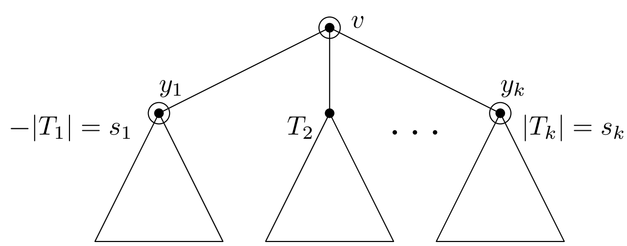
where $1/\alpha \approx 2.955765285652\ldots$ $\alpha\sqrt{\beta/2\pi n} \approx 0.439924012571\ldots$

→ Let T be a tree, v a vertex in T. The weight of v is the size of the maximal subtree at v_1.



weight$(D) = 3$ weight$(E) = \max(2, 7) = 7$

- A vertex of minimal weight is called a centroid.



- $T_1, \ldots, T_k$ are the subtrees at v, their sizes are $s_i = |T_i|$, the root of T_i is the neighbor of v in T_i and it is denoted by y_i.

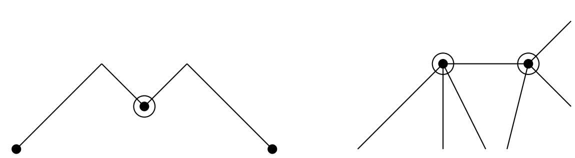
→ weight $(y_i) \geq 1 + s_2 + \ldots + s_k = n - s_1$

→ If there is a centroid of T in T_1, w, then weight$(v) = \max(s_1, \ldots, s_k) \geq$ weight$(w) \geq 1 + s_2 + s_3 + \ldots + s_k$

This is possible only if $s_1 > s_2 + \ldots + s_k$ $*$

→ At most one subtree of a given vertex can contian a centroid of T.

→ There are at most 2 centroids and, if there are two, they are adjacent.



→ $*$ is iff.