Homework 5

21-759 Differential Geometry

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Due: Monday, November 25, 2013

I would be willing to present solutions to Exercises 2, 3, or 7.

Exercise 2

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a) Since p is a critical point of f,

$$\langle A_X(Z), X\rangle = \langle \nabla_Z X, X\rangle = \frac{1}{2}\left(\langle \nabla_Z X, X\rangle + \langle X, \nabla_Z X\rangle\right) = Z\langle X, X\rangle = \frac{1}{2} Zf(p) = 0. \quad \blacksquare$$

b) By the Killing Equation,

$$\begin{aligned}
\frac{1}{2}ZZ\langle X, X\rangle + \langle R(X,Z)X, Z\rangle &= Z\langle \nabla_Z X, X\rangle + \langle R(X,Z)X, Z\rangle \\
&= -Z\langle \nabla_X X, Z\rangle + \langle R(X,Z)X, Z\rangle \\
&= -\langle \nabla_Z \nabla_X X, Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle + \langle \nabla_Z \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X,Z]} X, Z\rangle \\
&= \langle \nabla_{[X,Z]} X, Z\rangle - \langle \nabla_X X, \nabla_Z Z\rangle - \langle \nabla_X \nabla_Z X, Z\rangle \\
&= \langle \nabla_{[X,Z]} X, Z\rangle - \langle \nabla_X \nabla_Z X, Z\rangle \\
&= \langle \nabla_Z X, \nabla_Z X\rangle = \langle A_X(Z), A_X(Z)\rangle.
\end{aligned}$$

Since $\langle \nabla_{[X,Z]} X, Z\rangle = -\langle \nabla_Z X, [X,Z]\rangle$ and $\nabla_X X = 0$ (both also by the Killing Equation). $\blacksquare$

Exercise 3

Let X be a Killing field on M. Define $f : M \to \mathbb{R}$ by $f(p) = \langle X, X\rangle_p$, $\forall p \in \mathcal{M}$, and let $p \in M$ be a minimizer of f on M (since M is compact). Suppose, for sake of contradiction, that $X(p) \neq 0$.

Define $A : T_p\mathcal{M} \to T_p\mathcal{M}$ by $A(y) = \nabla_X Y$, for an extension Y of y, and note that A is linear.

Define $E := \{v \in T_p\mathcal{M} : \langle v, X(p)\rangle = 0\}$, and note that clearly $\dim E = \dim M - 1$, which is odd.

Since p minimizes f on M and M has no boundary, $df_p = 0$, and hence we can apply the results of Exercise 2 at p. Thus, for any $y \in E$,

$$\langle \nabla_Y X, X\rangle(p) = 0,$$

and so $A(E) \subseteq E$. Note that, by the Killing Equation,

$$\langle A(Y), Z\rangle = \langle \nabla_Y X, Z\rangle = -\langle \nabla_Z X, Y\rangle = -\langle A(Z), Y\rangle,$$

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and so $A : E \to E$ is skew-symmetric, and hence

$$\det(A) = \det(-A^T) = (-1)^{\dim E} \det(A^T) = -\det(A),$$

so that $\det(A) = 0$. On the other hand, since p minimizes f, $ZZf(p) \geq 0$, and, since the sectional curvature is strictly positive, by part (b) of Exercise 2, if $Z \neq 0$ then

$$\langle A(Z), A(Z) \rangle_p = \frac{1}{2} ZZf(p) + \langle R(X, Z)X, Z \rangle_p > 0$$

Thus, $A(Z) \neq 0$, and so A is injective. This is a contradiction. ■

Exercise 7

Since each term is a tensor, by multilinearity, it suffices to show for a geodesic frame $\{e_i\}$ at p that

$$\nabla R(e_i, e_j, e_k, e_l, e_h) + \nabla R(e_i, e_j, e_l, e_h, e_k) + \nabla R(e_i, e_j, e_h, e_k, e_l) = 0. \tag{1}$$

Since $\{e_i\}$ is a geodesic frame, each $\nabla_{e_i} e_j = 0$, and so

$$\begin{aligned} \nabla R(e_i, e_j, e_k, e_l, e_h) &= e_h \langle R(e_i, e_j) e_k, e_l \rangle = e_h \langle R(e_i, e_j) e_k, e_l \rangle \\ &= \langle \nabla_{e_h} \nabla_{e_l} \nabla_{e_k} e_i - \nabla_{e_h} \nabla_{e_k} \nabla_{e_l} e_i + \nabla_{e_h} \nabla_{[e_k, e_l]} e_i, e_j \rangle. \end{aligned}$$

Since

$$R(e_h, [e_k, e_l]) e_i = \nabla_{[e_k, e_l]} \nabla_{e_h} e_i - \nabla_{e_h} \nabla_{[e_k, e_l]} e_i + \nabla_{[e_h, [e_k, e_l]]} e_i,$$

$$\nabla_{e_h} \nabla_{[e_k, e_l]} e_i = \nabla_{e_h} \nabla_{[e_k, e_l]} e_i - \nabla_{[e_k, e_l]} \nabla_{e_h} e_i - R(e_h, [e_k, e_l]) e_i.$$

If we add the three terms together, by the Jacobi Identity, the terms

$$\nabla_{[e_h, [e_k, e_l]]} + \nabla_{[e_h, [e_k, e_l]]} + \nabla_{[e_h, [e_k, e_l]]} = 0$$

and, since

$$R(e_h, [e_k, e_l]) e_i = \nabla_{e_h} \nabla_{e_l} \nabla_{e_k} e_i - \nabla_{e_h} \nabla_{e_l} \nabla_{e_k} e_i + \nabla_{[e_h, e_l]} \nabla_{e_k} e_i,$$

all terms cancel when we add the three terms in (1). ■

Exercise 9

Let $\{e_i\}$ be an orthonormal basis of $T_p\mathcal{M}$ such that, for $x = \sum_{i=1}^n x_i e_i$,

$$\mathrm{Ric}_p(x) = \sum_i \lambda_i x_i^2, \quad \lambda_i \in \mathbb{R}.$$

(Why does such a basis exist?) Let B_n denote the unit ball at the origin in $\mathbb{R}^n$ (so that $S^{n-1} = \partial B_n$). A well known recurrence relating the volume and surface area of the n-dimensional sphere unit ball (easily checked by taking a derivative of the volume with respect to the radius) gives $\mathrm{vol}(B_n) = n\omega_n$.

Thus, for $V : \mathbb{R}^n \to \mathbb{R}^n$ defined by $V = (\lambda_1 x_1, \ldots, \lambda_n x_n)$ for all $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
K(p) &= \frac{1}{n}\sum_i \mathrm{Ric}_p(e_i) = \frac{1}{n}\sum_i \lambda_i = \frac{1}{\omega_{n-1}}\sum_i \lambda_i \int_{B^n} dB^n \\
&= \frac{1}{\omega_{n-1}}\int_{B^n} \mathrm{div}\, V \, dB^n \\
&= \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \langle V, \nu\rangle \, dS^{n-1} && \text{(by the Divergence Theorem)} \\
&= \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \sum_i \lambda_i x_i^2 \, dS^{n-1} \\
&= \frac{1}{\omega_{n-1}}\int_{S^{n-1}} \mathrm{Ric}_p(x) \, dS^{n-1},
\end{aligned}$$

where $\nu = (x_1, \ldots, x_n)$ is a unit normal vector to S^{n-1}.

Exercise 10

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a) Let $\{e_i\}$ be a geodesic frame at $p \in \mathcal{M}$. We first show a few calculations:

$$\begin{aligned}
\sum_{ikjh} \delta_{hj}\delta_{ik} e_s(R_{hijk}) &= e_s\left(\sum_{ikjh} \delta_{hj}\delta_{ik} R_{hijk}\right) \\
&= e_s\left(\sum_{hj} \delta_{hj} R_{hj}\right) = e_s\left(\sum_{hj} \delta_{hj}(\lambda\delta_{hj})\right) = n e_s(\lambda).
\end{aligned}$$

Also,

$$\sum_{ikjh} \delta_{hj}\delta_{ik} e_s(R_{hiks}) = \sum_{ikjh} \delta_{hj} e_j\left(\sum_{ik} -\delta_{ik} R_{hijk}\right) = \sum_{ikjh} \delta_{hj} e_j(\lambda\delta_{hs}) = -e_s(\lambda),$$

and, by an essentially identical calculation,

$$\sum_{ikjh} \delta_{hj}\delta_{ik} e_k(R_{hiks}) = -e_s(\lambda).$$

By the second Bianchi identity (Exercise 7),

$$e_s(R_{hijk}) + e_j(R_{hiks}) + e_k(R_{hish}) = 0.$$

Thus,

$$(n-2)e_s(\lambda) = n e_s(\lambda) - e_s(\lambda) - e_s(\lambda) = 0,$$

proving the result for $n \geq 3$, since M^n is connected and s was arbitrary. ∎

b) I wasn't able to finish this part.