Lecture Notes for Week 8 (First Draft)

Semigroups of Linear Operators

Many linear evolution problems can be cast in the form

$$\dot{u}(t) = Au(t),\ t \geq 0,\ u(0) = u_0, \tag{IVP}$$

where X is a Banach space, $\mathcal{D}(A) \subset X$, $A : \mathcal{D}(A) \to X$ is a linear operator and $u_0 \in X$ is given. We "know" that the solution to (IVP) should be given by

$$u(t) = e^{tA}u_0,\ t \geq 0.$$

The crucial questions are: "What do we mean by e^{tA} if A is unbounded?"; and "How can we "construct" e^{tA} if A is unbounded?". To address these questions, let us recall some approaches to constructing e^{tA} when A is an $N \times N$ matrix.

Matrix Exponentials

Let A be an $N \times N$ (real or complex) matrix. The following standard methods can be used to produce e^{tA}.

I. *Series*: We can represent e^{tA} as a power series:

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{1}$$

This construction will work fine in an infinite-dimensional Banach space X provided that $A \in \mathcal{L}(X; X)$. However, it seems doomed to failure if A is unbounded.

II. *Explicit Euler Scheme*: Let $n \in \mathbb{N}$ and $t > 0$ be given. We can use the explicit Euler method to approximate $u(t)$ where u is the solution to (IVP). For $k \in \{0, 1, 2, \cdots, n-1\}$, we have

$$\begin{aligned} u\left(\frac{(k+1)t}{n}\right) &\approx u\left(\frac{kt}{n}\right) + \frac{t}{n}Au\left(\frac{kt}{n}\right) \\ &\approx \left(I + \frac{t}{n}A\right)u\left(\frac{kt}{n}\right). \end{aligned}$$

If we start with $u(0) = u_0$ and iterate the scheme above we find that

$$u(t) \approx \left(I + \frac{tA}{n}\right)^n u_0.$$

This leads to the formula

$$e^{tA} = \lim_{n\to\infty} \left(I + \frac{tA}{n} \right)^n .$$

As before, this will work in an infinite-dimensional Banach space X if $A \in \mathcal{L}(X;X)$, but runs into serious difficulties if A is unbounded.

III. *Implicit Euler Method*: Let $t > 0$ and $n \in \mathbb{N}$. The difference between this approach and the explicit Euler method is that the derivative is approximated by using the right endpoint of each time interval rather than the left endpoint. For $k \in \{0, 1, 2, \cdots, n-1\}$ we have

$$u\left(\frac{(k+1)t}{n}\right) \approx u\left(\frac{kt}{n}\right) + \frac{t}{n} A u \left(\frac{(k+1)t}{n}\right),$$

which gives

$$\left(I - \frac{t}{n}A\right) u \left(\frac{(k+1)t}{n}\right) \approx u\left(\frac{kt}{n}\right).$$

Starting with $u(0) = u_0$ and iterating the formula above gives

$$u(t) \approx \left(I - \frac{t}{n}\right)^{-n} u_0.$$

This leads to the formula

$$e^{tA} = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n},$$

where

$$\begin{aligned} \left(I - \frac{t}{n}A\right)^{-n} &= \left(\left(I - \frac{t}{n}A\right)^{-1}\right)^n \\ &= \left(\frac{n}{t}\right)^n R\left(\frac{n}{t}; A\right)^n . \end{aligned}$$

This approach appears promising for unbounded operators A because it involves high powers of the resolvent operator, and the resolvent operator is an everywhere-defined bounded operator.

IV. *Laplace Transforms*: The Laplace transform is a powerful tool for studying (IVP). Recall that for a scalar-valued function $x : [0, \infty) \to \mathbb{K}$, the Laplace transform $\hat{x}$ of x is defined by

$$\hat{x}(\lambda) = \int_0^\infty e^{-\lambda t} x(t)\, dt.$$

In particular, for the scalar exponential function

$$y(t) = e^{at},$$

we have

$$\hat{y}(\lambda) = \frac{1}{\lambda - a}, \quad \mathrm{Re}(\lambda) > \mathrm{Re}(a).$$

For a matrix exponential

$$X(t) = e^{tA},$$

we have

$$\hat{X}(\lambda) = \int_0^\infty e^{-\lambda t} e^{tA}\, dt = (\lambda I - A)^{-1} = R(\lambda; A).$$

We can recover e^{tA} form its Laplace transform via the inversion formula

$$e^{tA} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} R(\lambda; A)\, d\lambda.$$

Here γ is a sufficiently large positive number. Once again, this looks promising for unbounded operators A because it involves the resolvent operator.

V. *Spectral Decomposition*: Suppose that there is an orthonormal basis $(e^{(k)}|k = 1, 2, \cdots n)$ for N-space such each $e^{(k)}$ is an eigenvector for A, i.e.

$$Ae^{(k)} = \lambda_k e^{(k)}.$$

Then for every vector v we have

$$e^{tA}v = \sum_{k=1}^{n} e^{\lambda_k t}(v, e^{(k)})e^{(k)}.$$

This approach will work for some unbounded operators of interest. In particular, if A is an unbounded self-adjoint operator on a Hilbert space X then A has a spectral decomposition

$$A = \int_{-\infty}^{\infty} dP(\lambda).$$

If A is bounded above, then we can define

$$e^{tA} = \int_{-\infty}^{\infty} e^{\lambda t} dP(\lambda), \quad t \geq 0.$$

Properties of Matrix Exponentials

Let A be an $N \times N$ matrix and put

$$S(t) = e^{tA},\ t \geq 0.$$

Then the mapping $S : [0, \infty) \to \mathbb{K}^{N \times N}$ satisfies

(i) $S(0) = I$.

(ii) $S(t+s) = S(t)S(s)$ for all $s, t \geq 0$,

(iii) $\lim_{t \downarrow 0} S(t) = I$.

Moreover, we can recover A from S via the formula

$$A = \lim_{h \downarrow 0} \frac{S(h) - I}{h}. \tag{2}$$

Linear C_0-Semigroups

In order to extend the notion of matrix exponential to general linear operators, we shall start with mappings $T : [0, \infty) \to \mathcal{L}(X; X)$ satisfying conditions analogous to (i) through (iii) above and construct A through a formula analogous to (2).

Definition 8.1: Let X be a Banach space. A *linear C_0-semigroup* is a mapping $T : [0, \infty) \to \mathcal{L}(X; X)$ satisfying (i) thru (iii) below.

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$ for all $s, t \in [0, \infty)$.

(iii) $\forall x \in X$, we have $\lim_{t \downarrow 0} T(t)x = x$.

Remark 8.2: It follows immediately from (ii) that

$$T(t)T(s) = T(s)T(t) \quad \text{for all } s, t \in [0, \infty).$$

Definition 8.3: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. The *infinitesimal generator* of A is the linear operator $A : \mathcal{D}(A) \to X$ defined by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{h \downarrow 0} \frac{T(h)x - x}{h} \text{ exists} \right\},$$

$$Ax = \lim_{h \downarrow 0} \frac{T(h)x - x}{h} \quad \text{for all } x \in \mathcal{D}(A).$$

If T is a linear C_0-semigroup with infinitesimal generator A then in some very broad sense we can "think of $T(t)$ as e^{tA}".

Remark 8.4: Let X be a Banach space and assume $T : [0, \infty) \to \mathcal{L}(X; X)$ satisfies (i) and (ii) of Definition 8.1.

(a) If
$$\lim_{t\downarrow 0} x^*(T(t)x) = x^*(x) \ \text{ for all } x \in X,\ x^* \in X^*,$$
then T is a linear C_0-semigroup. In other words, if (i) and (ii) of Definition 8.1 hold then T is continuous from the right at 0 in the weak operator topology if and only if it is continuous from the right at 0 in the strong operator topology. [This result will be a homework problem.] However, continuity from the right at 0 in the uniform operator topology implies that the infinitesimal generator is bounded.

(b) If
$$\lim_{t\downarrow 0} \|T(t) - I\| = 0$$
then there exists $B \in \mathcal{L}(X;X)$ such that
$$T(t) = \sum_{n=0}^{\infty} \frac{(tB)^n}{n!} \quad t \geq 0.$$
In this case B is the infinitesimal generator of T.

Example 8.5: Let $X = BUC(\mathbb{R})$, the space of all bounded uniformly continuous functions $f : \mathbb{R} \to \mathbb{K}$, equipped with the supremum norm, i.e.
$$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}.$$
Consider the mapping $T : [0,\infty) \to \mathcal{L}(X;X)$ given by
$$(T(t)f)(x) = f(x+t) \ \text{ for all } u \in X,\ x \in \mathbb{R},\ t \geq 0.$$
It is straightforward to check that T is a linear C_0-semigroup. [Uniform continuity of the functions in X is essential for this.] It is called a *translation semigroup*. In order for a function $f \in X$ to be in the domain of the infinitesimal generator A, it is necessary (but not sufficient) that
$$\forall x \in \mathbb{R},\ \lim_{h\downarrow 0} \frac{f(x+h) - f(x)}{h} \text{ exists.}$$
Thus in order for a function to belong to $\mathcal{D}(A)$ it must be continuously right differentiable, and consequently it must be everywhere differentiable. Therefore we must have $Af = f'$ for all $f \in \mathcal{D}(A)$. It follows that
$$\mathcal{D}(A) = BUC^1(\mathbb{R}),$$
the set of all functions in $BUC(\mathbb{R})$ having first derivatives that belong to $BUC(\mathbb{R})$.

Let $u_0 \in \mathcal{D}(A)$ be given and define $u : [0,\infty) \times \mathbb{R} \to \mathbb{K}$ by
$$u(t,x) = (T(t)u_0)(x) = u_0(x+t) \ \text{ for all } t \geq 0,\ x \in \mathbb{R}.$$

Then u is a solution of an initial-value problem for a first-order partial differential equation:

$$u_t(t,x) = u_x(t,x), \quad t \geq 0,\ x \in \mathbb{R}; \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}.$$

[Here u_t and u_x are the partial derivatives of u with respect to the first and second arguments.]

Lemma 8.6: Let X be a Banach space and assume that $T : [0,\infty) \to \mathcal{L}(X;X)$ is a linear C_0-semigroup. Then there exist $M, \omega > 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

Proof: I claim that we may choose $\eta > 0$ such that

$$\sup\{\|T(t)\| : t \in [0,\eta]\} < \infty.$$

To validate the claim, suppose that no such η exists. Then we may choose a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n > 0$ for all $n \in \mathbb{N}$, $t_n \to 0$ as $n \to \infty$, and $\{\|T(t_n)\|\}_{n=1}^{\infty}$ is unbounded. On the other hand

$$\forall x \in X,\ \lim_{n\to\infty} \|T(t_n)x\| \text{ exists},$$

so we have

$$\forall x \in X,\ \sup\{\|T(t_n)x\| : n \in \mathbb{N}\} < \infty.$$

The Principle of Uniform Boundedness implies that

$$\sup\{\|T(t_n)\| : n \in \mathbb{N}\} < \infty,$$

and this is a contradiction.

Put

$$M = \sup\{\|T(t)\| : t \in [0,\eta]\},$$

and

$$\omega = \frac{1}{\eta} \log M.$$

Let $t > 0$ be given. Then we may choose $n \in \mathbb{N} \cup \{0\}$ and $\alpha \in [0,\eta)$ such that

$$t = n\eta + \alpha.$$

Then we have

$$\begin{aligned} T(t) &= T(n\eta+\alpha) = T(n\eta)T(\alpha) \\ &= (T(\eta))^n T(\alpha). \end{aligned}$$

We see that

$$\|T(t)\| \leq \|T(\eta)\|^n \|T(\alpha)\| = M^{n+1}.$$

Since $\omega t \geq n \log M$, we have
$$e^{\omega t} \geq M^n,$$
and we conclude that
$$\|T(t)\| \leq M e^{\omega t}. \quad \square$$

Uniformly Bounded and Contraction Semigroups

Definition 8.7: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. We say that T is

(i) *uniformly bounded* if there exists $M \in \mathbb{R}$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.

(ii) a *contraction semigroup* provided $\|T(t)\| \leq 1$ for all $t \geq 0$.

(iii) a *quasicontraction semigroup* provided there exists $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$.

Let $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup and choose $M, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$. Put
$$S(t) = e^{-\omega t} T(t) \quad \text{for all } t \geq 0.$$
It is easy to see that S is a linear C_0-semigroup satisfying
$$\|S(t)\| \leq M \quad \text{for all } t \geq 0.$$
Let B be the infinitesimal generator of S. It is straightforward to check that $\mathcal{D}(B) = \mathcal{D}(A)$ and
$$Bx = -\omega x + Ax \quad \text{for all } x \in \mathcal{D}(B).$$
Moreover, we can construct an equivalent norm $|||\cdot|||$ on X such that S becomes a contraction semigroup. Indeed, if we put
$$|||x||| = \sup\{\|T(s)\| : s \in [0, \infty)\} \quad \text{for all } x \in X,$$
then
$$\|x\| \leq |||x||| \leq M\|x\| \quad \text{for all } x \in X,$$
and
$$|||S(t)x||| \leq |||x||| \quad \text{for all } x \in X,\ t \geq 0.$$
Observe that T becomes a quasicontraction semigroup under $|||\cdot|||$.

Remark 8.8: The equivalent norm $|||\cdot|||$ constructed above need not preserve important geometric properties of the original norm on X. In fact, if X is a Hilbert space and $T : [0, \infty) \to \mathcal{L}(X; X)$ is a linear C_0-semigroup, it might happen that there is no norm that is equivalent to the original one, satisfies the parallelogram law, and has the property that the semigroup is quasicontractive.

Lemma 8.9: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. Let $x \in X$ be given. Then the mapping $t \to T(t)x$ is continuous on $[0, \infty)$.

Proof: For right continuity, let $t \geq 0$ and $h > 0$ be given. Then $T(t + h)x = T(h)T(t)x$, so we have

$$\lim_{h\downarrow 0} T(t+h)x = \lim_{h\downarrow 0} T(h)(T(t)x) = T(t)x.$$

To establish left continuity, we first choose $M, \omega \in \mathbb{R}$ such that $\|T(s)\| \leq Me^{\omega s}$ for all $s \geq 0$. Let $t > 0$ and $h \in (0, t)$ be given. Then we have

$$\begin{aligned} \|(T(t-h)x - T(t)x\| &= \|T(t-h)[I - T(h)]x\| \\ &\leq \|T(t-h)\| \cdot \|T(h)x - x\| \\ &\leq Me^{\omega(t-h)}\|T(h)x - x\| \\ &\to 0 \text{ as } h \downarrow 0. \quad \square \end{aligned}$$