Homework 8

21-236 Mathematical Studies Analysis II
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
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Problem 1

Let $F := f^{-1}((0,\infty))$ and let $G := f^{-1}(-\infty,0))$. Since f^+ and f^- are Riemann integrable in the improper sense over E, there exist exhausting sequences $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ of E such that f^+ is Riemann integrable over each A_i and f^- is Riemann integrable over each B_i. $\forall i \in \mathbb{N}$, define $F_i = A_i \cap F$ and $G_i = B_i \cap G$. Note that, since f^+ and f^- are Riemann integrable over A_i and B_i respectively, f^+ and f^- are bounded over A_i and B_i, respectively, and, furthermore, each A_i and each B_i is Peano-Jordan measurable. Thus, for each $i \in \mathbb{N}$, if R is a rectangle with $A_i \cup B_i \subseteq R$, then

$$\int_{A_i} f^+ \, d\mathbf{x} = \int_R f^+ \chi_{A_i} \, d\mathbf{x} = \int_R f \chi_{A_i \cap F} \, d\mathbf{x} = \int_R f \chi_{F_i} \, d\mathbf{x} = \int_{F_i} f \, d\mathbf{x}$$

and

$$\int_{B_i} f^- \, d\mathbf{x} = \int_R f^- \chi_{B_i} \, d\mathbf{x} = \int_R -f \chi_{B_i \cap G} \, d\mathbf{x} = \int_R -f \chi_{G_i} \, d\mathbf{x} = \int_{G_i} -f \, d\mathbf{x},$$

so that f is integrable over each F_n and each G_n. Note that, since $\int_E f^+ = \infty$, $\forall i \in \mathbb{N}$, $\exists j_i \in \mathbb{N}$ such that $\int_{F_{j_i}} f^+ \geq 1 + \int_{G_i} f^-$, so that, since $f = f^+ - f^-$, $\forall i \in \mathbb{N}$,

$$\int_{F_{j_i} \cup G_i} f \geq 1.$$

Thus, since $\{F_{j_i} \cup G_i\}_{i\in\mathbb{N}}$ is an exhausting sequence, if f were Riemann integrable in the improper sense over E, taking the limit as $i \to \infty$ then $\int_E f \, d\mathbf{x} \geq 1$.

Similarly, since $\int_E f^+ = \infty$, $\forall i \in \mathbb{N}$, $\exists k_i \in \mathbb{N}$ such that $\int_{G_i} f^- \geq 1 + \int_{F_i} f^+$, $\forall i \in \mathbb{N}$,

$$\int_{F_i \cup G_{k_i}} f \leq -1.$$

Thus, since $\{F_i \cup G_{k_i}\}_{i\in\mathbb{N}}$ is an exhausting sequence, if f were Riemann integrable in the improper sense over E, taking the limit as $i \to \infty$ then $\int_E f \, d\mathbf{x} \leq -1$. Therefore, f cannot be Riemann integrable over E in the improper sense.

Problem 2

Suppose k with $1 \leq k < N$, nonempty $M \subseteq \mathbb{R}^N$, $m \in \mathbb{N}$, $\mathbf{x}_0 \in M$, $U \subseteq \mathbb{R}^N$ and $\mathbf{g} : U \to \mathbb{R}^{N-k}$ satisfy (ii) in Proposition 219.

Since rank $J_g = N - k$, we can relabel the components of g such that, for some $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^{N-k}$, $\mathbf{x}_0 = (\mathbf{a}, \mathbf{b})$ and

$$\det \frac{\partial \mathbf{g}}{\partial \mathbf{y}}(\mathbf{a}, \mathbf{b}) \neq 0.$$

Since $\mathbf{x}_0 \in M \cap U$, so that $\mathbf{g}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, by the Implicit Function Theorem, there exist nonempty open balls $W = B_N(\mathbf{a}, r_0) \subseteq \mathbb{R}^k$ and $V = B_M(\mathbf{b}, r_1)$ with $W \times V \subseteq U$ and a function $\mathbf{h} : W \to$ of class C^m such that $\forall \mathbf{x} \in W$, $\mathbf{g}(\mathbf{x}, \mathbf{h}(\mathbf{x})) = \mathbf{0}$ and $\mathbf{h}(\mathbf{a}) = \mathbf{b}$.

Therefore, for the open set $U_2 = W \times V$ with $\mathbf{x}_0 \in U_2$, if we define $\boldsymbol{\varphi} : W \to M \cap U_2$ so that, $\forall \mathbf{x} \in W$, $\boldsymbol{\varphi}(\mathbf{x}) = (\mathbf{x}, \mathbf{h}(\mathbf{x}))$, $\boldsymbol{\varphi}$ is invertible with continuous inverse $(\boldsymbol{\varphi}^{-1}(\mathbf{x}, \mathbf{h}\mathbf{x})) = \mathbf{x})$, so that $\boldsymbol{\varphi}$ is a homeomorphism from W to $M \cap U_2$, and rank $J_{\boldsymbol{\varphi}} = k$, since $J_{\boldsymbol{\varphi}}$ contains I_k as a submatrix. The regularity of $\boldsymbol{\varphi}$ is that of $\mathbf{h}$, so that $\boldsymbol{\varphi}$ is of class C^m.

Thus, $\boldsymbol{\varphi}$ serves as a class C^m local chart for M near $\mathbf{x}_0$, so that M is a k-dimensional surface of class C^m, and thus (ii) implies (i) in Proposition 219. ■

Problem 3

(a) Let $\mathbf{x}_0 \in M$, and let $U = M$, so that $\mathbf{x}_0 \in U$. Since the components of $\boldsymbol{\varphi}$ are all polynomials, $\boldsymbol{\varphi}$ is of class C^∞. Furthermore, $\boldsymbol{\varphi}^{-1}$ is the function $(x, y, z) \mapsto (z, x - z)$, which is also clearly of class C^∞. Therefore, $\boldsymbol{\varphi}$ is a homeomorphism from V to $M \cap U$. Since U is inverse image of V under $\boldsymbol{\varphi}^{-1}$ and $\boldsymbol{\varphi}^{-1}$ is continuous, since V is open (since it contains none of its boundary), U is open. Thus, it remains only to show that rank $J_{\boldsymbol{\varphi}}(u, v) = 2$, $\forall (u, v) \in V$.

$$J_{\boldsymbol{\varphi}}(u, v) = \begin{bmatrix} 1 & 1 \\ 0 & 2v \\ 1 & 0 \end{bmatrix},$$

which always has rank 2, since the first and last rows are clearly linearly independent. Therefore, by definition, M is a 2-dimensional surface of class C^∞. ■

(b) Note that $\boldsymbol{\varphi} : V \to M$ is global chart for M, so that by definition of the Surface Integral, if f is the function $(x, y, z) \mapsto (u, v)$,

$$\int_M z \, d\mathcal{H}^2 = \int_V f(\boldsymbol{\varphi}(\mathbf{y})) \sqrt{\sum_{\boldsymbol{\alpha} \in \Lambda_{N,2}} \left[\det \frac{\partial(\varphi_{\alpha_1}, \varphi_{\alpha_2})}{(y_1, y_2)}(\mathbf{y}) \right]^2} \, d\mathbf{y}.$$

It follows from the computation of $J_{\boldsymbol{\varphi}}$ in part (a) above, that, $\forall (u, v) \in V$,

$$\sum_{\boldsymbol{\alpha} \in \Lambda_{N,2}} \left[\det \frac{\partial(\varphi_{\alpha_1}, \varphi_{\alpha_2})}{(y_1, y_2)}(\mathbf{y}) \right]^2 = (-1)^2 + (2v)^2 + (-2v)^2 = 1 + 8v^2.$$

Therefore, noting that $f \circ \boldsymbol{\varphi}$ is the function $(u, v) \mapsto u$, by Theorem 160 (Repeated Integration)

$$\begin{aligned}
\int_M z \, d\mathcal{H}^2 &= \int_V u\sqrt{1+8v^2} \, d\mathbf{y} \\
&= \int_0^1 \left(\int_0^{\sqrt{v}} u\sqrt{1+8v^2} \, du \right) \, dv \\
&= \int_0^1 \frac{1}{2} v\sqrt{1+8v^2} \, dv \\
&= \frac{1}{48}(1+8v^2)^{3/2} \Big|_{v=0}^{v=1} \\
&= \boxed{\frac{13}{24}.}
\end{aligned}$$

Problem 4

Suppose, for sake of contradiction, that M is a 2-dimensional surface of class C^1, so that, by Proposition 219 (the alternative definition of the manifold), $\forall \mathbf{x}_0 \in M$, $\exists$ an open set $U \subseteq \mathbb{R}^3$ containing $\mathbf{x}_0$ and $g : U \to \mathbb{R}$ of class C^1, such that

$$M \cap U = \{\mathbf{x} \in U : g(\mathbf{x}) = \mathbf{0}\},$$

and $\operatorname{rank}(J_g(\mathbf{x})) = 1$, $\forall \mathbf{x} \in M \cap U$; in particular, we take the open set U and the function g for $\mathbf{x}_0 = \mathbf{0}$. Note that, since g is a scalar function, $J_g = \nabla g$, so that it suffices to show that $\nabla g(\mathbf{0}) = \mathbf{0}$.

By Theorem 10, since U is open and $g \in C^1(U)$, all the directional derivatives of g exist at $\mathbf{0}$, and furthermore, for any direction $\mathbf{v}$,

$$\frac{\partial g}{\partial \mathbf{v}_1}(\mathbf{0}) = \nabla g(\mathbf{0}) \cdot \mathbf{v}.$$

Define the directions $\mathbf{v}_1 = (\sqrt{2}, 0, \sqrt{2})$, $\mathbf{v}_2 = (0, \sqrt{2}, \sqrt{2})$, $\mathbf{v}_3 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$. It is easily checked that, $\forall t \geq 0$, $t\mathbf{v}_1, t\mathbf{v}_2, t\mathbf{v}_3 \in M$, and, since U is open, for some $r > 0$, $\forall$ positive $t < r$, $t\mathbf{v}_1, t\mathbf{v}_2, t\mathbf{v}_3 \in U$. Therefore, since since g is identically 0 on $M \cap U$, it follows from the definition of directional derivative that

$$\frac{\partial g}{\partial \mathbf{v}_1} = \frac{\partial g}{\partial \mathbf{v}_2} = \frac{\partial g}{\partial \mathbf{v}_3} = 0.$$

Therefore, if $\nabla g = (x, y, z)$,

$$\sqrt{2}x + \sqrt{2}z = \nabla g(\mathbf{0}) \cdot (\sqrt{2}, 0, \sqrt{2}) = \frac{\partial g}{\partial \mathbf{v}_1}(\mathbf{0}) = 0,$$

$$\sqrt{2}y + \sqrt{2}z = \nabla g(\mathbf{0}) \cdot (0, \sqrt{2}, \sqrt{2}) = \frac{\partial g}{\partial \mathbf{v}_2}(\mathbf{0}) = 0,$$

and

$$x/2 + y/2 + z/\sqrt{2} = \nabla g(\mathbf{0}) \cdot (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}) = \frac{\partial g}{\partial \mathbf{v}_1}(\mathbf{0}) = 0.$$

However, the only solution to this system of equations is $x = y = z = 0$, so that $\nabla g = \mathbf{0}$, giving the desired contradiction. $\blacksquare$