Homework 5

21-740 Introduction to Functional Analysis II

Name: Shashank Singh[1]

Due: Friday, Decemeber 6, 2013

Problem 1

[I'm not sure I exactly analyzed the right properties of the semigroup here, or that I was sufficiently rigorous in these results...]

Let $X = H^2(\mathbb{R}^n)$, and let $\mathcal{D}(A) = X$. Define $A : \mathcal{D}(A) \to X$ by

$$(Af)(x) = \Delta f(x) - V(x)f(x), \quad \forall f \in X, x \in \mathbb{R}^n$$

(noting $u_t = iAu$). A is self-adjoint on $H^2(\mathbb{R}^n)$, since, using integration by parts,

$$(Af, g) = \int_{\mathbb{R}^n} (\Delta f)g - (Vf)g = \int_{\mathbb{R}^n} \Delta f(\Delta g) - f(Vg) = (f, Ag), \quad \forall f, g \in H^2(\mathbb{R}^n).$$

Since we are in a Hilbert space, A generates a semigroup $T : [0, \infty) \to \mathcal{L}(X; X)$ defined by

$$T(t) := \int_{-\infty}^{\infty} e^{\lambda t A} \, dP(\lambda).$$

Then, we can define the semigroup $U : [0, \infty) \to \mathcal{L}(X; X)$ generated by iA as

$$U(t) := \int_{-\infty}^{\infty} e^{i\lambda t} \, dP(\lambda).$$

Note that $\forall t \geq 0$,

$$U^*(t) = \int_{-\infty}^{\infty} e^{-i\lambda t} dE_\lambda = U^{-1}(t),$$

and so U is unitary. A consequence should be that the Schrodinger equation is time-reversible. Another immediate consequence is that U is a contraction semigroup. To gain more information, we should study the spectrum of A.

Problem 2

[I didn't have time to do this question.]

[1] sss1@andrew.cmu.edu

Problem 3

Suppose $\exists N \in \mathbb{N} \cup \{0\}, K \in \mathbb{R}$ such that

$$\|L\phi\|_X \leq L|||\phi|||_N, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n),$$

and suppose $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\{\phi_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{S}(\mathbb{R}^n)$ with $\phi_k \to \phi$ as $k \to \infty$. By Remark 13.13, $\forall \alpha, \beta \in M_n$ with $|\alpha|, |\beta| \leq N$,

$$\|P_\beta D^\alpha \phi_k - P_\beta D^\alpha \phi\|_\infty \to 0$$

as $k \to \infty$. Then, since $S := |\{\alpha, \beta \in M_n : |\alpha|, |\beta| \leq N\}|$ is finite,

$$m_k := \sup_{|\alpha|,|\beta| \leq N} \|P_\beta D^\alpha \phi_k - P_\beta D^\alpha \phi\|_\infty \to 0$$

as $k \to \infty$. Thus,

$$\begin{aligned} \|L\phi_k - L\phi\|_X &= \|L(\phi_k - \phi)\|_X \leq K|||\phi_k - \phi|||_N \\ &= K \sum_{|\alpha|,|\beta| \leq N} \|P_\beta D^\alpha \phi_k - P_\beta D^\alpha \phi\|_\infty \leq KSm_k \to 0. \end{aligned}$$

as $k \to \infty$, and so L is continuous. I didn't time to do the converse.

Problem 4

Such a tempered distribution does indeed exist. Define the functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ by

$$u(\phi) := (L\check{\phi})(0), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

If $\{\phi_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{S}(\mathbb{R}^n)$ converging to $\phi \in \mathcal{S}(\mathbb{R}^n)$ in the usual metric ρ on $\mathcal{S}(\mathbb{R}^n)$, then, since L is continuous, $\rho(L\phi_k, L\phi) \to 0$ as $k \to \infty$. By definition of ρ, this implies $\|L\phi_k - L\phi\|_\infty \to 0$, and hence $u(\phi_k) \to u(\phi)$, as $k \to \infty$. Thus, u is continuous and hence $u \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore, $\forall \phi \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n$,

$$(L\phi)(x) = (\tau_{-x}(L\phi))(0) = (L(\tau_{-x}\phi))(0) = u(\tau_x\check{\phi}) = (u * \phi)(x). \quad \blacksquare$$

Problem 5

[I ended up able to prove necessary conditions on r for the embedding, which I give below. I wasn't able to show sufficient conditions, as the question asked.]

$\forall \lambda > 0$, define the dilation operator $\delta_\lambda u(x) := u(\lambda x)$. Then,

$$\|\delta_\lambda u\|_r = \frac{1}{\lambda^{n/r}} \|u\|_r.$$

Also, by properties of the Fourier Transform and a Change of Variables, for $\lambda \geq 1$

$$\begin{aligned}
\|\delta_\lambda u\|_{s,2}^2 &= \int_{\mathbb{R}^n} Q_s(\xi) |\widehat{\delta_\lambda u(\xi)}|^2 \, d\xi = \int_{\mathbb{R}^n} Q_s(\xi) \left| \frac{1}{\lambda^n} \delta_{1/\lambda} \widehat{u(\xi)} \right|^2 \, d\xi \\
&= \frac{\lambda^n}{\lambda^{2n}} \int_{\mathbb{R}^n} Q_s(\lambda \xi) \left| \widehat{u(\xi)} \right|^2 \, d\xi = \frac{\lambda^{2s}}{\lambda^n} \int_{\mathbb{R}^n} (\lambda^{-2} + |\xi|^2)^s \left| \widehat{u(\xi)} \right|^2 \, d\xi \leq \frac{\lambda^{2s}}{\lambda^n} \|u\|_{s,2}^2.
\end{aligned}$$

If we have a constant $C > 0$ such that

$$\|\delta_\lambda u\|_r \leq C \|\delta_\lambda u\|_{s,2}^2, \quad \forall u \in H^s(\mathbb{R}^n),$$

then, $\forall \lambda \geq 1$,

$$\frac{1}{\lambda^{n/r}} \|u\|_r \leq C \frac{\lambda^s}{\lambda^{n/2}} \|u\|_{s,2}.$$

Thus, in order to avoid a contradiction when taking $\lambda \to \infty$, we must have

$$s > n/2 - n/r.$$

Solving this inequality for r gives

$$r \leq \frac{2n}{n - 2s}.$$

Due to tail behavior of functions in H^s, $H^s(\mathbb{R}^n) \not\hookrightarrow L^r(\mathbb{R}^n)$ for $r < 2$, Thus, if $H^s(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$,

$$r \in \left[2, \frac{2n}{n - 2s} \right].$$