Homework 1

21-470 Calculus of Variations
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Due: Friday, February 7, 2014

Problem 1

For any $y \in \mathscr{Y}$, integrating by parts and using the boundary condition $y(1) = 1$,

$$\int_0^1 xy(x)^4 = \frac{x^2}{2}y(x)^4\Big|_{x=0}^{x=1} - \int_0^1 2x^2y(x)^3y'(x)\,dx = \frac{1}{2} - \int_0^1 2x^2y(x)^3y'(x)\,dx.$$

Consequently, $\forall y \in \mathscr{Y}$, $J(y) = 1/2$, and so each $y \in \mathscr{Y}$ is both a minimizer and a maximizer.

Problem 2

First note that J is unbounded above on $\mathscr{Y}$ (it is straightforward to construct a sequence $\{p_n\}_{n=1}^{\infty}$ of second-order polynomials in $\mathscr{Y}$ with $J(p_n) \to +\infty$ as $n \to \infty$).

For $f : [1, 2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$f(x, y, z) := x^2z^2 + 2y^2 \quad \forall x \in [1, 2], y, z \in \mathbb{R},$$

$J(y) = \int_1^2 f(x, y(x), y'(x))\,dx$, for all $y \in \mathscr{Y}$. Since

$$f_{,2}(x, y, z) = 4y \quad \text{and} \quad f_{,3}(x, y, z) = 2x^2z, \quad \forall x \in [0, 1], y, z \in \mathbb{R},$$

if y minimizes J on $\mathscr{Y}$, the 1^{st} Euler-Lagrange Equation gives

$$4y(x) = \frac{d}{dx}2x^2y'(x) = 4xy'(x) + 2x^2y''(x).$$

Since $x \mapsto x^{-2}$ and $x \mapsto x$ are independent solutions of this linear second-order differential equation,

$$y(x) = \frac{c_2}{x^2} + c_1x,$$

for some $c_1, c_2 \in \mathbb{R}$. Plugging in the boundary conditions and solving the resulting linear system of equations gives $c_1 = 3, c_2 = -4$, so that $y(x) = -4x^{-2} + 3x, \forall x \in [1, 2]$. I wasn't able to show that this minimizes J, but I think a convexity argument should suffice.

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Problem 3

First note that, J is unbounded below on $\mathscr{Y}$. For $n \in \mathbb{N}$, define $y_n \in \mathscr{Y}$ by $y_n(x) = \sin(nx), \forall x \in [0, \pi]$. Then, since $\int_0^\pi \sin(x)^2\,dx = \int_0^\pi \cos(x)^2\,dx = \pi/2$,

$$J(y_n) = \int_0^\pi \sin(nx)^2 - n^2\cos^2(nx)\,dx = (1-n^2)\frac{\pi}{2} \to -\infty$$

as $n \to \infty$. It is also apparently the case, although I was unable to show this, that J is non-positive (i.e., that $\|y\|_2 \leq \|y'\|_2$ for all $y \in \mathscr{Y}$).

For $f : [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$f(x, y, z) := y^2 - z^2 \quad \forall x \in [0, \pi], y, z \in \mathbb{R},$$

$J(y) = \int_0^\pi f(x, y(x), y'(x))\,dx$, for all $y \in \mathscr{Y}$. Since

$$f_{,2}(x, y, z) = 2y \quad \text{and} \quad f_{,3}(x, y, z) = -2z, \quad \forall x \in [0, 1], y, z \in \mathbb{R},$$

if y minimizes J on $\mathscr{Y}$, the 1^{st} Euler-Lagrange Equation gives

$$2y(x) = \frac{d}{dx} - 2y'(x) = -2y''(x).$$

Since cos and sin are independent solutions of this linear second-order differential equation,

$$y(x) = c_1\cos(x) + c_2\sin(x), \quad \forall x \in [0, \pi]$$

for some $c_1, c_2 \in \mathbb{R}$. The boundary conditions immediately imply $c_1 = 0$. On the other hand

$$J(c_2\sin) = \int_0^\pi c_2^2\sin^2(x) - c_2^2\cos^2(x)\,dx = 0,$$

so that any multiple of sin maximizes J on $\mathscr{Y}$.

Problem 4

As noted in Problem 3, J is unbounded below on $\mathscr{Y}$. Without the boundary condition $y(0) = 0$, J is also unbounded above. For $n \in \mathbb{N}$, define $y_n \in \mathscr{Y}$ by $y_n(x) = n(\pi - x), \forall x \in [0, \pi]$. Then,

$$J(y_n) = \int_0^\pi n^2(\pi - x)^2 - n^2\,dx = n^2\frac{\pi^3}{3} - n^2\pi \to +\infty$$

as $n \to \infty$.

Problem 5

First note that, J is unbounded above on $\mathscr{Y}$. For $n \in \mathbb{N}$, define $y_n \in \mathscr{Y}$ by $y_n(x) = nx+1, \forall x \in [0,1]$.

$$J(y_n) = \int_0^1 (n-x)^2 + 2x(nx+1)\,dx \to +\infty$$

as $n \to \infty$. For $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$f(x,y,z) := (z-x)^2 + 2xy \quad \forall x \in [0,1], y, z \in \mathbb{R},$$

$J(y) = \int_0^1 f(x, y(x), y'(x))\,dx$, for all $y \in \mathscr{Y}$. Since

$$f_{,2}(x,y,z) = 2x \quad \text{ and } \quad f_{,3}(x,y,z) = 2z - 2x, \quad \forall x \in [0,1], y, z \in \mathbb{R},$$

if y minimizes J on $\mathscr{Y}$, the 1^{st} Euler-Lagrange Equation gives

$$0 = 2x - \frac{d}{dx}(2y'(x) - 2x) = x - y''(x) + 1,$$

and so $y''(x) = x + 1$. Integrating with respect to x twice gives

$$y(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + c_1 x + c_2,$$

for some $c_1, c_2 \in \mathbb{R}$. Since $y(0) = 1$, $c_2 = 1$. The second boundary condition derived for the free right endpoint is $0 = f_{,3}(x, y(x), y'(x))\big|_{x=1} = 2y'(1) - 2$, and it follows that $c_1 = -1/2$. I wasn't able to show that this minimizes J, but I think a convexity argument should suffice.

Problem 6

Note that J is unbounded above on $\mathscr{Y}$ (it is straightforward to construct a sequence $\{p_n\}_{n=1}^{\infty}$ of second-order polynomials in $\mathscr{Y}$ with $J(p_n) \to +\infty$ as $n \to \infty$). For $f : [1,8] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$f(x,y,z) := xz^4 \quad \forall x \in [1,8], y, z \in \mathbb{R},$$

$J(y) = \int_1^8 f(x, y(x), y'(x))\,dx$, for all $y \in \mathscr{Y}$. Since

$$f_{,2}(x,y,z) = 0 \quad \text{ and } \quad f_{,3}(x,y,z) = 4xz^3, \quad \forall x \in [1,8], y, z \in \mathbb{R},$$

if y minimizes J on $\mathscr{Y}$, the 1^{st} Euler-Lagrange Equation gives

$$0 = -\frac{d}{dx}4xy'(x)^3 = -4y'(x)^3 - 12xy'(x)^2y''(x) = y'(x) + 3xy''(x)$$

Since $x \mapsto x^{2/3}$ and any non-zero constant function are independent solutions of this linear second-order differential equation, $y(x) = c_1 x^{2/3} + c_2$, for some $c_1, c_2 \in \mathbb{R}$. The boundary conditions give $c_1 = 1, c_2 = 3$. I wasn't able to show that this minimizes J, but I think a convexity argument should suffice.

Problem 7

We conclude that g is constant on $[a,b]$. Suppose, for sake of contradiction, that $\exists x, y \in (a,b)$ with $g(x) \neq g(y)$ (without loss of generality, $x < y$ and $g(x) < g(y)$). Since g is continuous, $\exists \delta > 0$ with

$$a \leq x - \delta < x + \delta \leq y - \delta < y + \delta \leq b,$$

such that, for some $\varepsilon > 0$,

$$\inf\{g(z) : z \in (y-\delta, y+\delta)\} - \sup\{g(z) : z \in (x-\delta, x+\delta)\} \geq \varepsilon.$$

Define $v : [a,b] \to \mathbb{R}$ for all $x \in [a,b]$ by

$$v(z) := \begin{cases} -\exp\left(-\frac{1}{1-((z-x)/\delta)^2}\right) & : z \in (x-\delta, x+\delta) \\ \exp\left(-\frac{1}{1-((z-y)/\delta)^2}\right) & : z \in (y-\delta, y+\delta) \\ 0 & \text{else} \end{cases}.$$

Since the bump function $z \mapsto \exp\left(-\frac{1}{1-z^2}\right) 1_{(-1,1)}$ (where $1_{(-1,1)}$ denotes the indicator function of $(-1,1)$) is in $C^\infty(\mathbb{R})$, $v \in C^\infty([a,b])$. Furthermore, $v(a) = v(b) = 0$, and

$$\int_a^b v(z)\,dz = \int_{y-\delta}^{y+\delta} \exp\left(-\frac{1}{1-((z-y)/\delta)^2}\right) dz - \int_{x-\delta}^{x+\delta} \exp\left(-\frac{1}{1-((z-x)/\delta)^2}\right) dz = 0,$$

so that $v \in \overline{\mathscr{V}}$. However, a translating change of variables gives

$$\begin{aligned} \int_a^b g(z)v(z)\,dzi &= \int_{y-\delta}^{y+\delta} (g(z) - g(z+x-y)) \exp\left(-\frac{1}{1-((z-y)/\delta)^2}\right) dz \\ &\geq \varepsilon \int_{y-\delta}^{y+\delta} \exp\left(-\frac{1}{1-((z-y)/\delta)^2}\right) dz > 0, \end{aligned}$$

giving a contradiction. ■

Problem 8

At any $y \in \mathscr{Y}$, the set of admissible variations at y is

$$\mathscr{V} := \left\{ v \in C^2([a,b]) : v(a) = v(b) = \int_a^b v(x)\,dx = 0 \right\}.$$

Thus, for any extremum $y \in \mathscr{Y}, v \in \mathscr{V}$, the Gâteaux variation satisfies

$$\begin{aligned} 0 = \delta J(y;v) &= \int_a^b f_{,2}(x, y(x), y'(x))v(x) + f_{,3}(x, y(x), y'(x))v'(x)\,dx \\ &= \int_a^b \left[f_{,2}(x, y(x), y'(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x)) \right] v(x)\,dx, \end{aligned}$$

via integration by parts and $v(a) = v(b) = 0$. By the result of Problem 7, $\exists C \in \mathbb{R}$ such that

$$C = f_{,2}(x, y(x), y'(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x)), \quad \forall x \in [a,b]. \quad ■$$

Problem 9

Multiplying the 1^{th} Euler-Lagrange Equation by $y'(x)$ on both sides, $\forall x \in [a,b]$,

$$y'(x)f_{,2}(x,y(x),y'(x)) = y'(x)\frac{d}{dx}f_{,3}(x,y(x),y'(x)). \tag{1}$$

The Chain Rule gives

$$\frac{d}{dx}f(x,y(x),y'(x)) = f_{,1}(x,y(x),y'(x)) + y'(x)f_{,2}(x,y(x),y'(x)) + y''(x)f_{,3}(x,y(x),y'(x))$$

$$\Rightarrow y'(x)f_{,2}(x,y(x),y'(x)) = \frac{d}{dx}f(x,y(x),y'(x)) - f_{,1}(x,y(x),y'(x)) - y''(x)f_{,3}(x,y(x),y'(x)).$$

Plugging this into Equation (1) and rearranging gives

$$\begin{aligned}\frac{d}{dx}f(x,y(x),y'(x)) - f_{,1}(x,y(x),y'(x)) &= y'(x)\frac{d}{dx}f_{,3}(x,y(x),y'(x)) + y''(x)f_{,3}(x,y(x),y'(x))\\ &= \frac{d}{dx}y'(x)f_{,3}(x,y(x),y'(x)).\end{aligned}$$

By the product rule. Rearranging again gives

$$\frac{d}{dx}\left(f(x,y(x),y'(x)) - y'(x)f_{,3}(x,y(x),y'(x))\right) = f_{,1}(x,y(x),y'(x)),$$

and so integrating with respect to x gives, for some $c \in \mathbb{R}$,

$$f(x,y(x),y'(x)) - y'(x)f_{,3}(x,y(x),y'(x)) = c + \int_a^x f_{,1}(t,y(t),y'(t))\,dt. \quad \blacksquare$$