21-484 Notes

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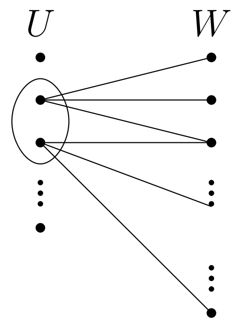
→ Show that a nonempty regular bipartite graph contains a perfect matching.

→ Matchings + bipartite ⇒ Hall's Thm.



→ Let $G = (U, W, E)$ be a $d > 0$ regular graph. Need to show

$$X \subseteq U \Rightarrow |X| \leq |N(X)|$$

→ Indeed, the sum of degrees of vertices in X is $d \cdot |X|$

→ If $|N(X)| < |X|$, then the sum of degrees of vertices in $N(X)$ is $d \cdot |N(X) < d \cdot |X|$ ↯ since all edges leaving X are going in $N(X)$.

→ The same for W

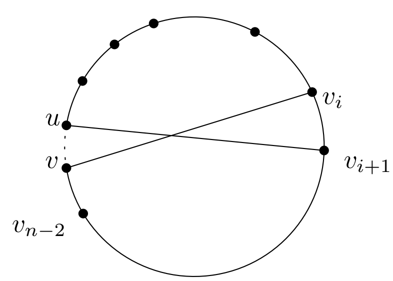
→ u and v are two nonadjacent vertices in G, such that $d(u) + d(v) \geq \underset{\text{number of vertices}}{\underset{\uparrow}{n}}$

$$G + uv \text{ is Hamiltonian} \iff G \text{ is Hamiltonian}$$

→ if G contains a Hamiltonian cycle C, then $G + uv$ also contains C.

→ Assume that $G + uv$ contains a Hamiltonian cycle C.

→ If C does not use uv, we are done.



→ Let $C = (v_1, \ldots, v_{n-2}, v, u, v_1)$.

→ we want to find two vertices v_i, v_{i+1} such that

$$vv_i \in E(G) \text{ and } uv_{i+1} \in E(G)$$

→ Let I be the set of indices of neighbors of v.

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$\rightarrow$ Let $J = \{i+1 | i \in I, i < n-2\}$

$|J| = |I| - 1 = d(v) - 1$

$\rightarrow$ u must have a neighbor with index in J, since there are $n-2-|J| = n-2-d(v)+1 = n-1-d(v)$ vertices out of J (and u and v) but the degree of u is $d(u) \geq n - d(v)$

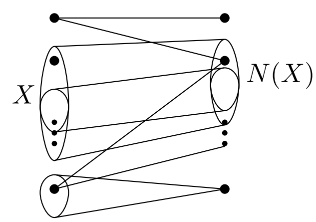
$\rightarrow$ So there are v_i and v_{i+1} as required.

$\rightarrow$ the cycle $u, v_1, \ldots, v_i, v, v_{n-2}, v_{n-3}, \ldots, v_{i+1}, u$ is Hamiltonian.

G is bipartite $G = (U \cup W, E)$. The size of a maximal matching is

$$|U| - \max_{X \subseteq U}(|X| - |N(X)|) \quad *$$

$\rightarrow$ There is no matching of size bigger than, because $*$, because if X is the maximal set then we can match all vertices of $U \setminus X$ plus $N(X)$ vertices from X: $|U \setminus X| + |N(X)| = |U| - |X| + |N(X)|$



$\rightarrow$ Let G be such a graph and let x be a maximal set.

$\rightarrow$ We want to match all the vertices in $U \setminus X$ with vertices from $W \setminus N(X)$.

$\rightarrow$ Need to show: $\forall Y \subseteq U \setminus X$, $|Y| \leq |N(Y) \setminus N(X)|$

indeed, if $|Y| > |N(Y) \setminus N(X)|$, consider $X \cup Y$

$$|X \cup Y| - |N(X \cup Y)| = |X| + |Y| - |N(X)| - |N(Y) \setminus N(X)| = |X| - |N(X)| + |Y| - \underbrace{|N(Y) \setminus N(X)|}_{>0}$$

$\lightning$ maximality of X.

$\rightarrow$ Have a matching that matches all vertices of $U \setminus X$ outside of $N(X)$.

$\rightarrow$ We want to show: $Z \subseteq N(X) \Rightarrow |Z| \leq |N(Z)|$. Assume $|Z| > |N(Z)|$, consider $X \setminus N(Z)$.

$$|X \setminus N(Z)| - |N(X \setminus N(Z))| = |X| - |N(Z)| - N(X \setminus X \setminus N(Z))|$$

<u>claim:</u> $|N(X \setminus N(Z))| \leq |N(X)| - |Z|$ a vertex of Z can not be in $N(X \setminus N(Z))$.

$$|X| - |N(Z)| - |N(X)| + |Z|. \lightning \text{ maximality of } X.$$

Q_k $\kappa(Q_k) = \lambda(Q_k) = k$.