Homework 3

21-759 Differential Geometry

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I would be willing to present solutions to problems 3, 4, and 5.

Problem 1

First, let $g : T_eG \times T_eG \to \mathbb{R}$ be an inner product (on the Lie algebra associated with G). We now extend this to a Riemann metric tensor on G by defining

$$g_p(v, w) := g_e(DL_{g^{-1}}v, DL_{g^{-1}}v).$$

$\forall p \in G, v, w \in T_pG$. Let

$$\omega := \sqrt{|g|}dx_1 \wedge \cdots \wedge dx_n,$$

be the usual volume form induced by g. Then, for any open set A of G, define

$$\mu(A) := \int_A$$

For any $p \in G$, since L_p is an isometry with respect to g, $L_p^*\omega = \omega$, so

$$\mu(L_p(A)) = \int_{L_p(A)} \omega = \int_A L_p^*\omega = \int_A \omega = \mu(A),$$

and we have left-invariance, as desired. $\blacksquare$

If μ_2 is another left-invariant Borel measure on G, define, for any open set A,

$$\phi(A) := \frac{\mu(A)}{\nu(A)}$$

(note, $\mu(A), \nu(A) \neq 0$). Then, $\forall p \in G$,

$$\phi(L_p(A)) = \frac{\mu(pA)}{\nu(pA)} = \frac{\mu(A)}{\nu(A)} = \phi(A),$$

and so ϕ is constant on G. From construction of the Borel σ-algebra, it follows that $\mu = c\nu$, for some constant $c \in \mathbb{R}$, on all Borel sets, so μ is unique up to constant multiples. $\blacksquare$

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Problem 2

(i) If X is a vector field and Φ_t is the associated flow,

$$\begin{aligned} L_X(\alpha \wedge \beta) &= \frac{d}{dt}\bigg|_{t=0} \Phi_t^*(\alpha \wedge \beta) \\ &= \frac{d}{dt}\bigg|_{t=0} \Phi_t^*(\alpha) \wedge \Phi_t^*(\beta) \\ &= \frac{d}{dt}\bigg|_{t=0} \Phi_t^*(\alpha) \wedge \beta + \alpha \wedge \frac{d}{dt}\bigg|_{t=0} \Phi_t^*(\beta) \\ &= (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta), \end{aligned}$$

where the third equality follows from the product rule and equation (4) in the notes (definition of the wedge product). ■

(ii)

(iii) If X is a vector field, since $d^2 = 0$, by Cartan's formula,

$$L_X \circ d = d \circ i_X \circ d + i_X \circ d \circ d = d \circ d \circ i_X + d \circ i_X \circ d = d \circ L_X. \quad ■$$

Problem 3

Fix any $p \in U$. If $q \in U$ and $\gamma : [0,1] \to U$ is a smooth curve from p to q (i.e., $\gamma(0) = p, \gamma(1) = q$), define

$$f(q) := \int_\gamma \phi^* \omega,$$

where $\phi^*\omega$ denotes the pullback of ω by ϕ^*.

Lemma 1 This f is well defined on U.

Proof: Let $q \in U$. Since U is simply connected, it is pathwise connected, and hence there is a smooth curve from p to q.

Suppose now that $\gamma_1, \gamma_2 : [0,1] \to U$ are smooth curves from p to q. Since U is simply connected, there is a smooth homotopy $H : [0,1]^2 \to U$ between γ_1 and γ_2 (i.e., $H(0,t) = \gamma_1(t)$ and $H(1,t) = \gamma_2(t)$, and $H(s,0) = p$ and $H(s,1) = q$, $\forall s,t \in [0,1]$.) Note that the image $\mathcal{M}_2 := H([0,1]^2)$ is a manifold whose boundary is the image of the "concatenation" γ of γ_1 and γ_2, where γ_2 is oriented backwards. Thus, by Stokes' Theorem, and properties of the pullback,

$$\int_{\gamma_1} \phi^*\omega - \int_{\gamma_2} \phi^*\omega = \int_\gamma \phi^*\omega = \int_{\mathcal{M}_2} d(\phi^*\omega) = \int_{\mathcal{M}_2} \phi^*(d\omega) = \int_{\mathcal{M}_2} \phi^*(0) = 0. \quad \square$$

Then, $\forall x \in \mathcal{M}$,

$$d(f \circ \phi^{-1})_x \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial x_i} (f \circ \phi^{-1} \circ \phi)(\phi^{-1}(x)) = \frac{\partial}{\partial x_i} f(\phi^{-1}(x)) = \phi^*\omega \Big|_{\phi^{-1}(x)} = \omega_x \left(\frac{\partial}{\partial x_i} \right). \quad \blacksquare$$

Problem 4

$$\int_\mathcal{M} \omega = \int_\mathcal{M} d\alpha = \int_{\partial \mathcal{M}} \alpha = \int_\emptyset \alpha = 0. \quad \blacksquare$$

Define $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ by

$$f(x,y) := \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right),$$

and let ω denote the 1-form on S^1 corresponding to f. Then,

$$\int_{S^1} \omega = 2\pi \neq 0,$$

and so, by the previous result, ω cannot be exact, since S^1 is a compact manifold.

Problem 5

Note that a parametrization of the catenoid is given by

$$(\cosh(z)\cos(t), \cosh(z)\sin(t), z) \quad z \in \mathbb{R}, t \in (0, 2\pi)$$

and a parametrization of the helicoid is given by

$$(t\cos(z), t\sin(z), z) \quad z, t \in \mathbb{R}.$$

Define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi(t, z) := (t, \sinh z).$$

The Jacobian of Φ is

$$J_\Phi = \begin{bmatrix} 1 & 0 \\ 0 & \cosh v \end{bmatrix},$$

so $\det(J_\Phi)$ is non-zero everywhere and hence Φ is a local diffeomorphism. It is easily checked that, under this reparametrization of the helicoid, for both the catenoid and the helicoid, that

$$\left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right\rangle = \cosh^2(t),$$
$$\left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right\rangle = 0,$$
$$\left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle = \cosh^2(t).$$

The catenoid and helicoid are not globally isometric, as they are not even homeomorphic. For example, the helicoid is clearly simply connected, whereas the catenoid is not. ■

Problem 6

I wasn't able to finish this problem.