Homework 7

21-651 General Topology
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Problem 1

Suppose that $(C_b(X,Y), d_\infty)$ is complete, and let $\{y_n\}_{n=1}^\infty$ be a Cauchy sequence in (Y, d_Y). Consider the sequence of constant functions in $C_b(X,Y)$ defined for each $n \in \mathbb{N}$ by $f_n = y_n \forall x \in X$ ($f_n \in C_b(X,Y)$ because any constant function is clearly continuous and bounded). Then, for any $i, j \in \mathbb{N}$, $d_\infty(f_i, f_j) = d_Y(y_i, y_j)$, so that, since $\{y_n\}_{n=1}^\infty$ is a Cauchy sequence in (Y, d_Y), $\{f_n\}_{n=1}^\infty$ must be a Cauchy sequence in $(C_b(X,Y), d_\infty)$, and, since the latter is complete, $\{f_n\}_{n=1}^\infty$ converges to some $f \in (C_b(X,Y), d_\infty)$.

If f were not constant (say, $f(x) \neq f(y)$, for some $x, y \in X$, then, for $\epsilon = \frac{1}{2}d_Y(f(x), f(y))$, no constant f_i could have both $d_\infty(f(x), f_i(x)) < \epsilon$ and $d_\infty(f(y), f_i(y)) < \epsilon$, contradicting the fact that $f_i \to f$ with respect to d_∞ as $i \to \infty$. Thus, we can pick $y \in Y$ with $f(x) = y\ \forall x \in X$, and it follows from the fact that $d_\infty(f, f_i) = d_Y(y, y_i)$ for each $i \in \mathbb{N}$ that $y_i \to y$ as $i \to \infty$. Therefore, (Y, d_Y) is complete.

Suppose, on the other hand, that (Y, d_Y) is complete, and let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(C(X,Y), d_\infty)$. Since (Y, d_Y) is complete for any $x \in X$, $f_i(x) \to f(x)$, for some $f(x) \in Y$. It remains only to show that f is continuous and bounded, so that $f \in C_b(X,Y)$. Given $\varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $d_\infty(f, f_n) < \varepsilon/3$. Thus, for any $x \in X$, since f_n is continuous, $\exists \delta > 0$ such that, $f(B_X(x,\delta)) \subseteq B_Y(f(x), \varepsilon/3)$. Then, for any $y \in B_X(x,\delta)$, by the Triangle Inequality,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

so that f is continuous. Finally, if $M_n > 0$ bounds f_n, then $M_n + \varepsilon/3$ clearly bounds f. ■

Problem 2

Let (X, τ) be a locally compact Hausdorff space, and let $U_1, U_2, \ldots \in \tau$ be a sequence of sets dense in X. Let $U := \bigcap_{n=1}^\infty U_n$. Let $V \in \tau$. Since U_1 is dense in X, $\exists x_1 \in V \cap U$. By Lemma 155, since $\{x_1\}$ is trivially compact, $\exists W_1 \in \tau$ with $\{x_1\} \subseteq W \subseteq \overline{W_1} \subseteq V \cap U$. Similarly, by picking $x_n \in W_{n-1} \cap U_n$, we can recursively find, $\forall n \in \mathbb{N}$, some nonempty $W_n \in \tau$ with $\overline{W} \subseteq W_{n-1} \cap U_n$. $\overline{W_1} \supseteq \overline{W_2} \supseteq \ldots$ is a decreasing sequence of closed sets, the intersection $W := \bigcap_{n=1}^\infty \overline{W_n} \subseteq V \cap \bigcap_{n=1}^\infty U_n$ is nonempty. Thus, since every open set intersects U, U is dense in X, so (X, τ) is Baire. ■

Problem 3

(a) Let $f \in C([0,1])$, and let $\varepsilon > 0$. Since f is continuous and has a compact domain, by Theorem 216, f is uniformly continuous, so that $\exists \delta_n > 0$ such that, $\forall x, y \in [0,1]$ with $|x-y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Let $g \in C([0,1])$ be defined $\forall x \in [0,1]$ by $g(x) = f(x) + \varepsilon \cos(ax)$, where $a := \max\left\{\frac{2\pi n}{\varepsilon}, \frac{2\pi}{\delta_n}\right\}$. Then, for $x \in [0,1]$, if x_2 is the second-nearest multiple of $\frac{\pi}{a}$ in $[0,1]$ to x (if two such multiples are equidistant, pick either one; then $|x - x_2| < \frac{\pi}{a}$), by the geometry of the cosine curve, $|(\cos(ax) - \cos(ax_2)| \geq 1$. Then, since $|f(x) - f(x_2)| < \delta_n$ (by choice of a and x_2,

$$
\begin{aligned}
|g(x) - g(x_2)| &= |\varepsilon(\cos(ax) - \cos(ax_2)) + f(x) - f(x_2)| \\
&> \varepsilon - \frac{\varepsilon}{2} = \varepsilon \geq \frac{n\pi}{a} = n|x - x_2|,
\end{aligned}
$$

and thus $g \notin X_n$. However, $d_\infty(f,g) < \varepsilon$. ■

(b) Let $U \subseteq C([0,1])$ be open and nonempty, and let $f \in U$. Since $B(f,\varepsilon) \subseteq U$ for some $\varepsilon > 0$, we can construct $g \in U$ in terms of f as in part (a), with $g \notin X_{2n}$, $d_\infty(f,g) < \varepsilon$. Then, by the construction of g, for any $h \in B(g, \varepsilon/2)$, $h \notin X_n$ (as h oscillates with frequency at least that of g and amplitude at least half that of g). Thus, $B(g,\varepsilon/2) \cap U$ is a nonempty open subset of U that does not intersect X_n. ■

(c) Note first that, by a result of Problem 1, since $\mathbb{R}$ is a complete metric space, $C([0,1])$ is also complete ($C([0,1]) = C_b([0,1],\mathbb{R})$, as continuous functions on compact domains are bounded). Then, by the Baire Category Theorem, $C([0,1]$ is a Baire space.

Let τ be the topology induced by d_∞ on $C([0,1])$. In part (b), we showed that, $\forall$ nonempty $U \in \tau, n \in \mathbb{N}$, $\exists$ a nonempty open set $V_{U,n} \subseteq U$ $V_{U,n}$ such that $V_{U,n} \cap X_n = \emptyset$. For each $n \in \mathbb{N}$, let $W_n := \bigcup_{U \in \tau} V_{U,n}$, so that, as a union of open sets, each W_n is open. Then, the intersection $W := \bigcap_{n \in \mathbb{N}} W_n$ is a G_δ set. By construction, any open set has non-empty intersection with each W_n, so that each W_n is dense in $C([0,1])$. Since $C([0,1])$ is a Baire space, V is dense in $C([0,1])$.

It remains then only to show that any $f \in V$ is nowhere differentiable. Note that, by construction of V, $\forall n \in \mathbb{N}$, $V \cap X_n = \emptyset$, so that it suffices to show that, if f is differentiable at some point x, then $f \in X_n$ for some $n \in \mathbb{N}$. If f is differentiable at x, then

$$
\exists D := \lim_{y \to x} \frac{f(x) - f(y)}{|x - y|} \in \mathbb{R},
$$

Thus, $\exists \delta > 0$ such that $\left|\frac{|f(x)-f(y)|}{|x-y|} - D\right| < \varepsilon$ on $B(x,\delta)$, and, on the compact set $[0,1] \backslash B(x,\delta)$, $\frac{|f(x)-f(y)|}{|x-y|}$ is bounded, since it is continuous. Thus, some $n \in \mathbb{N}$ bounds f, so $f \in X_n$. ■

Problem 4

(a) If $x \in E$, then $f(x) = f(x) + L(d(x,x))^\alpha \in \{f(y) + L(d(x,y))^\alpha : y \in E\}$, so that, since $h(x)$ is an infimum, $h(x) \leq f(x)$. $\forall y \in E$, since $|f(x) - f(y)| \leq L(d(x,y))^\alpha$, $f(x) \leq f(y) + L(d(x,y))^\alpha$, so that, taking the infimum over $y \in E$, $f(x) \leq h(x)$. ■

(b) It follows immediately from part (a) that $\inf_{x \in X} h(x) \leq \int_{y \in E} f(y)$. $\forall \varepsilon > 0, x \in X$, since $h(x)$ is an infimum, $\exists y \in E$ such that $f(y) \leq f(y) + L(d(x,y))^\alpha \leq h(x) + \varepsilon$. Thus, $\forall x \in X$, $\inf_{y \in E} f(y) \leq h(x)$, so, taking the infimum over $x \in X$, $\inf_{y \in E} f(y) \leq \inf_{x \in X} h(x)$. ■

(c) By definition of h, $\forall \varepsilon > 0, x, y \in X$, $\exists v, w \in E$ such that $|f(v) + L(d(v,x))^\alpha - h(x)|, |f(w) + L(d(w,y))^\alpha - h(y)| < \varepsilon$. Since $v, w \in E$, $|f(v) - f(w)| \leq L(d(v,w))^\alpha$. By the Triangle Inequality,

$$\begin{aligned}
|h(x) - h(y)| &\leq |f(v) + L(d(v,x))^\alpha - h(x)| + |f(v) - f(w)| + |f(w) + L(d(w,y))^\alpha - h(y)| \\
&< L(d(v,w))^\alpha + 2\varepsilon \leq L(d(x,y))^\alpha + 2\varepsilon
\end{aligned}$$

Taking $\varepsilon \to 0$ gives $|h(x) - h(y)| \leq L(d(x,y))^\alpha$. ■

Problem 5

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined $\forall x, y \in \mathbb{R}^2$ by

$$f(x,y) := \begin{bmatrix} \frac{x^4}{3}\sin(xy) - \frac{7x-y}{8} + x \\ \frac{y^2\cos(y)}{8} + \frac{e^x}{24}. \end{bmatrix}.$$

Note that any fixed point of f is a solution to the given system. Since f is continuous, by Brouwer's Fixed Point Theorem, to show that f has a fixed point, it suffices to show that $f(\overline{B(0,1)}) \subseteq \overline{B(0,1)}$. It is apparent that, $\forall (x,y) \in [0,1]^2$, $|f_1(x,y)| \leq \frac{7}{12}$, and $f_2(x,y) \leq \frac{1}{4}$. Thus, $f(x,y) \in B(0,1)$, since

$$\|f(x,y)\| \leq \sqrt{\left(\frac{7}{12}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{58}{144} < 1. \quad ■$$

Problem 6

Since K is continuous and has compact domain, $|K|$ is bounded by some $M \in \mathbb{R}$. Then, $\forall Tf \in \mathcal{F}, x \in [0,1]$,

$$|Tf(x)| = \left| \int_0^1 K(x-y)f(y)\,dy \right| \leq \int_0^1 |K(x-y)||f(y)|\,dy \leq \int_0^1 M\,dy = M,$$

so that $\mathcal{F}$ is pointwise bounded.

Since K is continuous and has compact domain, K is uniformly continuous, so, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that, $\forall x_1, x_2$ with $|x_1 - x_2| < \delta$, $|K(x_1) - K(x_2)| < \frac{\varepsilon}{2}$. Then, $\forall Tf \in \mathcal{F}, x_1, x_2 \in [0,1]$, by the Triangle Inequality and since $|f| \leq 1$ on $[0,1]$,

$$\begin{aligned} |Tf(x_1) - Tf(x_2)| &= \left| \int_0^1 K(x_1-y)f(y)\,dy - \int_0^1 K(x_1-y)f(y)\,dy \right| \\ &= \left| \int_0^1 (K(x_1-y) - K(x_2-y))f(y)\,dy \right| \\ &\leq \int_0^1 |(K(x_1-y) - K(x_2-y))|\,|f(y)|\,dy \leq \int_0^1 \frac{\varepsilon}{2}\,dy < \varepsilon. \end{aligned}$$

Thus, $\mathcal{F}$ is equicontinuous.

Since $[0,1]$ is separable and compact, by the Ascoli-Arzelà Theorem, $\mathcal{F}$ is sequentially compact. $\blacksquare$

Problem 7

Let $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$, so that, by the Ascoli-Arzelà Theorem, since $[0,1]$ is separable and compact, to show that every sequence in $\mathcal{F}$ has a convergent subsequence, it suffices to show that $\mathcal{F}$ is pointwise bounded and equicontinuous.

We show inductively that each f_n is in $C([0,1],\mathbb{R})$, that $\mathcal{F}$ is pointwise bounded by 1, and that, for any $\varepsilon > 0$, for $\delta = \frac{\varepsilon}{2}$, $\forall x, y \in [0,1]$, $y \in B(x,\delta)$ implies $f(y) \in B(f(x),\varepsilon)$ (implying in turn that $\mathcal{F}$ is equicontinuous). f_0 is clearly continuous and bounded by 1 on $[0,1]$, and f_0 satisfies the equicontinuity requirement, since it is differentiable on $[0,1]$ with derivative bounded by 2. Suppose, as an inductive hypothesis, that, for some $n \in \mathbb{N}$, f_n is in $C([0,1],\mathbb{R})$ and is bounded by 1 on $[0,1]$. Then, $\forall x \in [0,1]$,

$$f_{n+1}(x) = \int_0^x (f_n(s))^{1/3}\,ds \leq \int_0^x 1\,ds = x \leq 1,$$

so that f_{n+1} is bounded by 1 on $[0,1]$, and, by the Fundamental Theorem of Calculus, f_{n+1} is in $C([0,1],\mathbb{R})$. Also by the Fundamental Theorem of Calculus, $\forall x \in [0,1]$, f_{n+1} is differentiable at x with $f'(x) = (f_n(x))^{1/3} \leq 1$, so that f_{n+1} satisfies the equicontinuity requirement. $\blacksquare$