21-373, Algebraic Structures, Department of Mathematical Sciences, Carnegie Mellon University
Fall 2011: (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.
Luc Tartar, University Professor of Mathematics, Wean Hall 6212, tartar@cmu.edu

Assignment 3 - Saturday September 24, 2011. Due Friday September 30

Exercise 15: Prove that D_{12} and S_4 are not isomorphic.

Exercise 16: Write the cycle decompositions of all the elements of order 4 in S_4, and of all the elements of order 2 in S_4.

Exercise 17: Let σ the 8-cycle $(1\,2\,3\,4\,5\,6\,7\,8)$, τ the 12-cycle $(1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12)$, and ω the 14-cycle $(1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14)$. For which positive integer i is σ^i an 8-cycle? For which positive integer j is τ^j a 12-cycle? For which positive integer k is ω^k a 14-cycle?

Exercise 18: Show that in the three following cases, the centralizer of H is H, and the normalizer of H is G:

i) $G = S_3$ and $H = \{e, (123), (132)\}$,
ii) $G = D_4$ and $H = \{e, a^2, b, a^2 b\}$,
iii) $G = D_5$ and $H = \{e, a, a^2, a^3, a^4\}$.

[In a group G, for any subset $X \subset G$, the centralizer of X is $C_G(X) = \bigcap_{x \in X} C_G(x)$ (where the centralizer $C_G(x)$ is the stabilizer of x for the action of conjugation, i.e. $\{g \in G \mid g\,x = x\,g\}$). In D_n, a denotes an element of order n and b an element of order 2, satisfying $b\,a = a^{-1}b$.]

Exercise 19: For $m \geq 1$ and $q_1, \ldots, q_m \in \mathbb{Q}^*$, prove that the (finitely generated) subgroup $H = \langle q_1, \ldots, q_m \rangle$ of $\mathbb{Q}$ is a subgroup of $K = \left\langle \frac{1}{D} \right\rangle$, where D is the least common multiplier of the denominators of $q_1, \ldots, q_m$. Show that H is cyclic (hence $\mathbb{Q}$ is not finitely generated).

Exercise 20: A non trivial Abelian group G is called *divisible* if for each $a \in G$ and each positive integer k there exists $b \in G$ with $k\,b = a$. Show that $\mathbb{Q}$ is divisible, that no finite Abelian group is divisible, and that $G_1 \times G_2$ is divisible if and only if both G_1 and G_2 are divisible.

Exercise 21: Show that the group of rigid motion symmetries of a platonic solid (tetrahedron, cube, octahedron, dodecahedron, icosahedron) have respectively orders 12, 24, 24, 60, 60, i.e. $2E$, where E is the number of edges. Show that for the tetrahedron this group is isomorphic to a subgroup of S_4, and that for the cube or the octahedron this group is isomorphic to S_4.
[A Platonic solid is a convex polyhedron which is regular, so that its faces all are regular polygons with k sides, and ℓ edges arrive at each vertex, so that the number of faces F, of edges E, and of vertices V satisfy $k\,F = \ell\,V = 2E$; using $k, \ell \geq 3$ (which implies $k, \ell \leq 5$) and the relation $F - E + V = 2$ (that the Euler characteristic of the sphere $\mathbb{S}^2$ is 2), one finds there are five such regular polyhedron: the tetrahedron (4 triangular faces), the hexahedron = cube (6 square faces), the octahedron (8 triangular faces), the dodecahedron (12 pentagonal faces), and the icosahedron (20 triangular faces).]