15-359: Probability and Computing

Assignment 1 | Due: January 27, 2012

Problem 1: Chain gang (5 pts.) Let $E_1, \ldots, E_n$ be n events, each with positive probability. Prove that:

$$P(E_1 \cap \cdots E_n) = P(E_1) \cdot P(E_2 \mid E_1) \cdot P(E_3 \mid E_2 \cap E_1) \cdots P(E_n \mid E_1 \cap \cdots \cap E_{n-1})$$

Problem 2: Me and you (10 pts.) We say that events A and B are *positively dependent* if

$$P(A \mid B) > P(A)$$

Does that imply the following?

$$P(B \mid A) > P(B)$$

If so, prove it. If not, give a counterexample.

Problem 3: Fool me once, shame on you. Fool me twice... (20 pts.)

A pharmaceutical company has developed a potential vaccine against the H1N1 flu virus. Prior to any testing of the vaccine, the developers assume that with probability 0.5 their vaccine will be effective and with probability 0.5 it will be ineffective. The developers do an initial laboratory test on the vaccine. This initial lab test is only partially indicative of the effectiveness of the vaccine, with an accuracy of 0.6. Specifically, if the vaccine is effective, then this laboratory test will return "success" with probability 0.6, whereas if the vaccine is ineffective, then this laboratory test will return "failure" with probability 0.6.

A. What is the probability that the laboratory test returns success?

B. What is the probability that the vaccine is effective, given that the laboratory test returned success?

C. The developers decide to add a second *independent* experiment (this one on human beings) which is more indicative than the original lab test and has an accuracy of 0.8. Specifically, if the vaccine is effective, then the human being test will return "success" with probability 0.8. If the vaccine is ineffective, then the human being test will return "failure" with probability 0.8. What is the probability that the vaccine is effective, given that both the lab test and the human being test came up "success?" How useful was it to add this additional test?

Problem 4: Wrapping up Miller-Rabin (20 pts.) The Miller-Rabin test for primality is one-sided correct in the sense that if it returns NO, then the input n is guaranteed to be composite. The algorithm is also guaranteed to return YES when n is prime. However, the algorithm may produce an error: it may return YES? when the input is in fact composite.

We may safely assume that input to the algorithm is odd. Let C be the event: "input n is composite" and let Y be the event: "the Miller-Rabin test returns YES?". Assume that $P(Y \mid C) \leq 1/2$ (this may be proved by analyzing the set of bad choices for the number a described in the notes.) Recall that the Prime Number theorem states that the number of primes less than n is approximately $n/\ln n$.

A. Suppose we run the Miller-Rabin test m times on the same input n. Give a bound for the likelihood that the answer is always YES? even though n is composite.

B. Give an approximate value for $P(C)$ based on the Prime Number theorem.

C. Suppose we repeat the test m times. Use the value from part (B) to determine $P(C \mid Y_m)$ where Y_m denotes that the answer is 'yes' m times in a row.

Problem 5: Last die (20 pts.) I roll 5 fair standard dice. What is the probability that their sum is divisible by 6? (Hint: condition on something.)

Problem 6: If ya like it, ya shoulda put a probability mass on it (25 pts.) You will choose a spouse from among n prospects. Naturally, each one has a (unique) score, and you seek to marry the best one. (Let E be the event that you marry the best one.) Suppose Fate introduces you to these prospects in random order; after you date, you must decide either to break up (and never speak again) or marry. You decide to use the following two-phase algorithm to find a spouse:

- Play the field: date m prospects with no intention of marriage. Remember the best score, s, that you encountered.
- Seal the deal: marry the first prospect with score greater than s.

Let E_i be the event that you marry your i'th date (chronologically) and that he/she is the best.

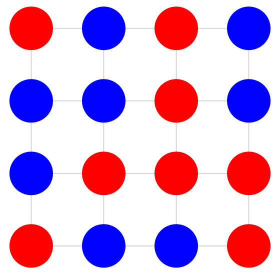
A. When $i \leq m$, what is $P(E_i)$?

B. When $i > m$, what is $P(E_i)$? (Hint: two properties.)

C. Conclude that $P(E) = \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1}$.

D. Extra credit: non-trivially lower bound $P(E)$ and find the maximizing value of m.

Problem 7: Connect four (extra credit) (10 pts.) Consider an $n \times n$ grid (lattice) of dots. For simplicity, assume n is even. Suppose we independently color each dot either red or blue with equal probability. For example, if $n = 4$, we might end up with the coloring below.



What's the probability that every 'unit square' of 4 dots consists of 2 red dots and 2 blue dots? (Hint: the problem is straightforward if you condition on the right event.)