21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
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Remark 11.1: The tensor product of two E-vector spaces V_1, V_2 appears as the solution of a universal problem: one would like to factorize any bilinear mapping B from $V_1 \times V_2$ into any E-vector space W by a particular bilinear mapping β (independent of B) from $V_1 \times V_2$ into a particular E-vector space Ω, followed by a linear mapping ℓ_B (dependent of B) from Ω into W. One solution is to take for β the mapping sending (v_1, v_2) to $v_1 \otimes v_2$, i.e. $\Omega = V_1 \otimes V_2$, and for defining ℓ_B to use a basis $\{e_i, i \in I\}$ of V_1, a basis $\{f_j, j \in J\}$ of V_2, and to define $\ell_B(e_i \otimes f_j) = B(e_i, v_j)$ for all $(i,j) \in I \times J$, and then extend ℓ_B to the whole $V_1 \otimes V_2$ by linearity. Since this definition depends upon the bases chosen, one may start by proving that two possible solutions are necessarily isomorphic.

One may give an abstract construction which does not mention a choice of bases for V_1 and V_2: in the E-vector space with basis $\{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$ (i.e. the finite sums $\sum_j \lambda_j (a_j, b_j)$ with $\lambda_j \in E, a_j \in V_1, b_j \in V_2$) one takes the quotient by the subspace spanned by the linear relations $\lambda\,(a,b) = (\lambda\,a, b) = (a, \lambda\,b)$, $(a_1 + a_2, b) = (a_1, b) + (a_2, b)$, $(a, b_1 + b_2) = (a, b_1) + (a, b_2)$, for all $\lambda \in E$, $a, a_1, a_2 \in V_1$, and $b, b_1, b_2 \in V_2$.

In principle, one should write $\otimes_E$ in order to see which field of scalars is used. Actually, if V_E is an E-vector space, and F is a field extension of E, one way to create the natural extension V_F of V_E when the field of scalars is F is to consider $V_E \otimes_E F$, and to observe that it is an F-vector space by defining $f'(v \otimes f) = v \otimes (f' f)$ for all $f, f' \in F$, $v \in V_E$.

Remark 11.2: One says that the indices at the top (like for vectors of V) are *contravariant*, because when using a new basis defined by a matrix P (whose column i is $P\,e_i$), a vector $v \in V$ is represented by a column vector X in the initial basis and a column vector X' in the new basis, with $X' = P^{-1} X$. On the contrary, one says that the indices at the bottom (like for vectors of V^*) are *covariant*, a vector $v_* \in V^*$ is represented by a row vector Y in the initial basis and a row vector Y' in the new basis, with $Y' = P\,Y$.

There are natural objects which are not tensors: for example, the force exerted by an electromagnetic field on a "particle" of charge q is given by the so called Lorentz force,[1] although it first appeared in an article by MAXWELL,[2] given by the formula $f = q\,(E + v \times B)$, where v is the velocity of the particle, E the electric field, and B the magnetic induction field. One should restrict oneself to matrices of change of basis P which send an orthonormal basis on an orthonormal basis, i.e. P belongs to the *orthogonal group* $\mathbb{O}(3)$ (of matrices satisfying $P^T P = I$), but then $det(P)$ (which is ± 1) appears, because B is not a tensor,[3] since it is replaced by $-B$ if one uses an orthonormal matrix of determinant -1; one must then restrict attention to P belonging to the *special orthogonal group* $S\mathbb{O}(3)$ (of matrices satisfying $P^T P = I$ and $det(P) = +1$).

Definition 11.3: A *topological group* G is a group with a topology, such that multiplication $(g_1, g_2) \mapsto g_1 g_2$ is continuous from $G \times G$ into G, and $g \mapsto g^{-1}$ is continuous from G into itself; G is a *Lie group* if moreover it is a *differentiable manifold*, i.e. the identity e has a basis of neighbourhoods U_i homeomorphic to open sets $V_i \subset \mathbb{R}^d$ for $i \in I$, and the charts ψ_i mapping V_i onto U_i are such that the change of charts are continuously differentiable, i.e. for $U_{i,j} = \psi_i(V_i) \cap \psi_j(V_j)$ the mapping $\psi_j^{-1} \circ \psi_i$ is C^1 from $\psi_i^{-1}(U_{i,j})$ into $\psi_j^{-1}(U_{i,j})$, for all i, j, and then it makes sense to add that multiplication and inversion are differentiable mappings; the dimension of the Lie group is d .

Remark 11.4: Because of the group property, the description of a Lie group near e permits to describe it near every point. We shall consider as example the orthogonal group $\mathbb{O}(n)$ and the special orthogonal

[1] Hendrik Antoon LORENTZ, Dutch physicist, 1853–1928. He received the Nobel Prize in Physics in 1902, jointly with Pieter ZEEMAN, in recognition of the extraordinary service they rendered by their researches into the influence of magnetism upon radiation phenomena. He worked in Leiden, The Netherlands. The Institute for Theoretical Physics in Leiden, The Netherlands, is name after him, the Lorentz Institute; the *Lorentz* force is also named after him.

[2] James CLERK MAXWELL, Scottish-born physicist, 1831–1879. He worked in Aberdeen, Scotland, and then in London, and in Cambridge, England, where he held the first Cavendish professorship of physics (1871–1879). Maxwell equation, which I call Maxwell–Heaviside equation, is named after him.

[3] Both E and B form the coefficients of a differential form (a 2-form) in $\mathbb{R}^3 \times \mathbb{R}$ (space-time).

group $S\mathbb{O}(n)$, and show that they are Lie groups of dimension $\frac{n\,(n-1)}{2}$. They are embedded in the space of matrices, which has dimension n^2, and it looks like imposing $P^T P = I$ gives n^2 equations, but since $P^T P$ is always symmetric, there are only $\frac{n\,(n+1)}{2}$ constraints. One way to prove that one has a structure of manifold is to apply the *implicit function theorem*,[4] and it applies here after writing $P = I + A + B$ with A symmetric and B skew symmetric,[5] and the equation becomes $2A + (A - B)\,(A + B) = 0$, giving $A = \psi(B)$ near 0.

Remark 11.5: We shall prove in a more analytic way that the tangent space at I for $S\mathbb{O}(n)$ (and for $\mathbb{O}(n)$, which coincides with $S\mathbb{O}(n)$ near I) is the space of skew-symmetric matrices, which has dimension $\frac{n\,(n-1)}{2}$, by considering the *exponential mapping*, and prove that for B skew symmetric $e^{t\,B}$ belongs to $S\mathbb{O}(n)$ for $t \in \mathbb{R}$, and actually that not only elements of $S\mathbb{O}(n)$ near I can be recovered in this way, but that every $P \in S\mathbb{O}(n)$ is equal to e^B for a skew-symmetric B.

For a general Lie group G, it is clear that a neighbourhood of e can be embedded in $\mathbb{R}^d$ by using one of the charts from the definition, but it is not so obvious that the whole G can be considered a manifold embedded in $\mathbb{R}^N$ for some N. However, the idea is to take a vector b in the tangent space $T_e G$ of G at e, and consider a curve in G which for $|s|$ small has the form $\psi(s) = e + s\,b + o(|s|) \in G$ and notice that for any $g \in G$ one has $g\,\psi(s) \in G$; the mapping M_h of multiplication by h (on the left), i.e. $g \mapsto h\,g$, induces for each $g \in G$ a linear mapping DM_h from the tangent space $T_g G$ to the tangent space $T_{h\,g} G$, which is actually an isomorphism, since $M_h \circ M_{h^{-1}} = M_{h^{-1}} \circ M_h = id_G$; having chosen a particular direction b at e (in $T_e G$), there is then at each $g \in G$ a transported direction $DM_g b$ at g (in $T_g G$), and the exponential mapping consists in solving a differential equation, more precisely of finding a curve $g(t) \in G$ parametrized by $t \in \mathbb{R}$, which for $|s|$ small has the form $e + s\,b + o(|s|)$, and for any t has the property that $g(t+s) = g(t) + s\,DM_{g(t)} b + o(|s|)$ for $|s|$ small.

Remark 11.6: One has $S\mathbb{O}(n) \subset GL(\mathbb{R}^N)$, which is also a Lie group, and since it has equation $det(M) \neq 0$, it is actually an open set of $L(\mathbb{R}^N, \mathbb{R}^N)$ and it then has dimension n^2; the tangent space to $GL(\mathbb{R}^N)$ at I being $L(\mathbb{R}^N, \mathbb{R}^N)$, it is natural to start by the exponential of an arbitrary matrix. The preceding analysis consists in taking an arbitrary $B \in L(\mathbb{R}^N, \mathbb{R}^N)$ and looking for a curve $M(t)$ parametrized by $t \in M$ such that $\frac{dM(t)}{dt} = M(t)B$ for all $t \in \mathbb{R}$ and $M(0) = I$, so that $M(t) = I + t\,B + o(|t|)$ for $|t|$ small. Of course, one may apply general results about existence of solutions for differential equations, like the Cauchy–Lipschitz theory,[6] but here it is easy to write the solution as a power series: $M(t) = e^{t\,B} = I + t\,B + \ldots + \frac{t^n}{n!}\,B^n + \ldots$, which has an infinite radius of convergence because $||B^k|| \leq ||B||^k$, so that $||e^{t\,B}|| \leq e^{|t|\,||B||}$, but one has to find more properties for deducing what it is when B is skew-symmetric.

Additional footnotes: CAVENDISH,[7] HEAVISIDE,[8] ZEEMAN.[9]

[4] If $(x, y) \mapsto \Phi(x, y)$ is a C^1 mapping from a neighbourhood of $(0, 0)$ in $\mathbb{R}^a \times \mathbb{R}^b$ into $\mathbb{R}^a$, and if the restriction of $D\Phi(0, 0)$ on $\mathbb{R}^a \times \{0\}$ is invertible, then the equation $\Phi(x, y) = \Phi(0, 0)$ can be parametrized near $(0, 0)$ by $x = \Psi(y)$ for a C^1 mapping Ψ.

[5] For guessing what the tangent space at I is, one writes $P = I + Q$ with $||Q||$ small, so that $P^T P = I + (Q^T + Q) + Q^T Q$, but since the norm of $Q^T Q$ is $||Q||^2$ (because for a symmetric matrix like $M = Q^T Q$, the norm is the supremum for $||x|| \leq 1$ of $(M\,x, x)$, which is $||Q\,x||^2$, and the supremum of $||Q\,x||$ for $||x|| \leq 1$ is $||Q||$ by definition) the equation at order 1 is $Q^T + Q = 0$, i.e. Q skew-symmetric.

[6] Rudolf Otto Sigismund LIPSCHITZ, German mathematician, 1832–1903. He worked in Breslau (then in Germany, now Wrocław, Poland) and in Bonn, Germany.

[7] Henry CAVENDISH, English physicist and chemist (born in Nice, not yet in France then), 1731–1810. He lived in London, England. He founded the Cavendish professorship of physics at Cambridge, England.

[8] Oliver HEAVISIDE, English engineer, 1850–1925. He worked as a telegrapher, in Denmark, in Newcastle upon Tyne, England, and then did research on his own, living in the south of England. He transformed equations written by MAXWELL into the system which one uses now under the name Maxwell equation, which I call Maxwell–Heaviside equation.

[9] Pieter ZEEMAN, Dutch physicist, 1865–1943. He received the Nobel Prize in Physics in 1902, jointly with Hendrik LORENTZ, in recognition of the extraordinary service they rendered by their researches into the influence of magnetism upon radiation phenomena. He worked in Leiden, and in Amsterdam, The Netherlands.