36 - 226 Introduction to Statistical Inference

Homework assignment 3

Due: Wednesday, February 6, 2013

- Write your full name, the course number, and the homework number at the top of each page.
- **STAPLE** your entire assignment together with a staple.
- Write clearly. Electronic submission of homework assignments is not accepted.

1. Wackerly 7.15. Suppose that $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples, with the variables X_i normally distributed with mean μ_1 and variance σ_1^2 and the variables Y_i normally distributed with mean μ_2 and σ_2^2. The difference between the sample means, $\bar{X} - \bar{Y}$, is then a linear combination of $m + n$ normally distributed random variables and, by Theorem 6.3, is itself normally distributed.
   (a) Find $E(\bar{X} - \bar{Y})$.
   (b) Find $Var(\bar{X} - \bar{Y})$.
   (c) Suppose that $\sigma_1^2 = 2$, $\sigma_2^2 = 2.5$, and $m = n$. Find the sample sizes so that $(\bar{X} - \bar{Y})$ will be within 1 unit of $(\mu_1 - \mu_2)$ with probability 0.95. *Hint: you are looking for the sample size such that* $P\left(|\bar{X} - \bar{Y} - (\mu_1 - \mu_2)| < 1\right) = 0.95$.

2. Wackerly 8.3. Suppose that $\hat{\theta}$ is an estimator for a parameter θ and $E(\hat{\theta}) = a\theta + b$ for some nonzero constants a and b.
   (a) In terms of a, b, and θ, what is $B(\hat{\theta})$?
   (b) Find a function of $\hat{\theta}$ - say, $\hat{\theta}^*$ – that is an unbiased estimator for θ.

3. Wackerly 8.4. Refer to Exercise 8.1. *[Note: exercise 8.1 defines the MSE($\hat{\theta}$).]*
   (a) If $\hat{\theta}$ is an unbiased estimator for θ, how does $MSE(\hat{\theta})$ compare to $V(\hat{\theta})$?
   (b) If $\hat{\theta}$ is an biased estimator for θ, how does $MSE(\hat{\theta})$ compare to $V(\hat{\theta})$?

4. Wackerly 8.11. Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a population with mean 3. Assume that $\hat{\theta}_2$ is an unbiased estimator of $E(Y^2)$ and that $\hat{\theta}_3$ is an unbiased estimator of $E(Y^3)$. Give an unbiased estimator for the third central moment of the underlying distribution. *Hint: the third central moment is defined as* $E[(Y - \mu)^3]$.

5. Wackerly 8.12. The reading on a voltage meter connected to a test circuit is uniformly distributed over the interval $(\theta, \theta + 1)$, where θ is the true but unknown voltage of the circuit. Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample of such readings.
   (a) Show that $\bar{Y}$ is a biased estimator of θ and compute the bias.
   (b) Find a function of $\bar{Y}$ that is an unbaised estimator of θ.
   (c) Find the $MSE(\bar{Y})$ when $\bar{Y}$ is used as an estimator of θ.

6. Wackerly 9.1. In Exercise 8.8, we considered a random sample of size 3 from an exponential distribution with density function given by

$$f(y) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

and determined that

$$\hat{\theta}_1 = Y_1, \qquad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \qquad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \text{ and,} \qquad \hat{\theta}_5 = \bar{Y}$$

are all unbiased estimators for θ. Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_5$, of $\hat{\theta}_2$ relative to $\hat{\theta}_5$, and $\hat{\theta}_3$ relative to $\hat{\theta}_5$.

7. Wackerly 9.3. Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the uniform distribution on the interval $(\theta, \theta + 1)$. Let

$$\hat{\theta}_1 = \bar{Y} - \frac{1}{2}, \text{ and } \hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}.$$

   (a) Show that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ.
   (b) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$. *Hint: for $\hat{\theta}_2$, consider the Beta distribution.*