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Problem Set 1

1.1. Consider the Lebesgue outer measure of the following sets:

(a). Any countable set. Given a countable set of points $\{x_n\} \subset \mathbb{R}^d$, we construct a cell covering of total outer measure ϵ as follows. Surround x_1 with a cell with side lengths $\sqrt[d]{\epsilon 2^{-1}}$, then surround x_2 with a cell with side lengths $\sqrt[d]{\epsilon 2^{-2}}$, and in general surround x_n with a cell with side lengths $\sqrt[d]{\epsilon 2^{-n}}$. Then since the sum is countable, we see that the total outer measure of this cover is at most: 2/2

$$\sum_{n=1}^{\infty} \left(\sqrt[d]{\frac{\epsilon}{2^n}} \right)^d = \epsilon \sum_{n=1}^{\infty} 2^{-n} = \epsilon$$

Therefore we can construct a cell covering of any countable set which has arbitrarily small total outer measure, so taking the infimum over such covers, we see that the measure can be no more than 0, but it must be non-negative, so it is 0.

(b). The Cantor set. Here again we contruct a sequence of cell coverings. This time we simply let our sequence be the usual iterative construction of the Cantor set. Namely $F_0 = [0,1]$, $F_1 = [0,1/3] \cup [2/3,1]$, and in general F_n is the interval after removing middle thirds n times. Clearly each F_n covers the entire cantor set, and is itself a union of cells. Now we see that at each time step n, the number of intervals doubles, while the length of the intervals goes down by a factor of 1/3. Therefore the total outer measure of F_n is $2^n/3^n$. Now taking the infimum over the outer measures of the F_n's, we see that in the large n limit, their total length approaches 0, so that the Lebesgue outer measure of the Cantor set is 0. 2/2

(c). The set $S = \{x \in [0,1] | x \notin \mathbb{Q}\}$. First note that by (a), we have that the Lebesgue outer measure of $T = \{\mathbb{Q} \cap [0,1]\}$ is 0. Then by sub-additivity, we know that:

$$m^*(S \cup T) = m^*([0,1]) = 1 \leq m^*(S) + m^*(T) = m^*(S)$$

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Then by monotonicity, since we have $S \subset [0,1]$, $m^*(S) \leq m^*([0,1]) = 1$. Therefore we have $m^*(S) = 1$.

what is H?

1.2.

(a). Consider a set $V \subset \mathbb{R}^d$ with $dim(V) < d$. Then we claim that $\lambda(V) = 0$. To see this, first tile $\mathbb{R}^d$ with countably many disjoint unit hypercubes. Then by additivity, we have $\lambda(V) = \lambda(V \cap H) + \lambda(V \cap H^C)$. Of course, the measure of $\lambda(V \cap H^C)$ can be re-written by intersecting the remainder with another hypercube, ad infinitum, so that if $\{H_i\}$ is the set of hypercubes, we claim that:

$$\lambda(V) = \sum_{i=1}^{\infty} \lambda(\underline{H \cap V})$$; should be in here

Therefore, we restrict ourselves to the bounded case. Because if each bounded intersection has measure 0, then the sum must be 0. Then taking a given bounded sub-space with dimension $< d$, we note that we will conclude that λ is invariant

(c) For notational convenience, let $S = \{x \in [0,1] | x \notin \mathbb{Q}\}$. Since $\mathbb{Q}$ is countable, it follows from the result of part (a) that $m^*(\mathbb{Q}) = 0$. Since $[0,1]$ is a cell, $m^* = 1 - 0 = 1$. By sub-additivity of the Lebesgue outer measure,

$$1 = m^*([0,1]) \leq m^*([0,1] \cap \mathbb{Q}) + m^*(S) = m^*(S).$$

By monotonicity, $m^*(S) \leq m^*([0,1]) = 1$. Therefore, $\boxed{m^*(S) = 1.}$

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Problem 2

(a) Any subspace V of $\mathbb{R}^d$ is the image of the subspace W whose basis is the set of the first $n := \dim(V)$ canonical basis vectors of $\mathbb{R}^d$, under some rotation. Thus, since all rotations are orthonormal transformations, by the result of part (b) of problem 3, it suffices to show that $\lambda(W) = 0$.

Let $\epsilon > 0$. Since $\mathbb{N}^n$ is countable, let $\mathbf{f} : \mathbb{N} \to \mathbb{N}^n$ be a bijection.

Then, $\forall i \in \mathbb{N}$, let

$$\begin{aligned} I_i =& (f_1(i), f_1(i) + 2) \times (f_2(i), f_2(i) + 2) \times \cdots \times (f_n(i), f_n(i) + 2) \\ & \times \left(-\frac{\epsilon}{2^{n+1+i}}, \frac{\epsilon}{2^{n+1+i}}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq \mathbb{R}^d, \end{aligned}$$

where $f_1(i), \ldots, f_n(i)$ are the components of $\mathbf{f}(i)$.

Since each I_i is a cell, $\lambda(I_i) = \ell(I_i) = \frac{\epsilon}{2^i}$. Thus, by monotonicity and then by subadditivity,

$$\lambda(V) \leq \lambda\left(\bigcup_{i=1}^{\infty} I_i\right) \leq \sum_{i=1}^{\infty} \lambda(I_i) = \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = \epsilon.$$

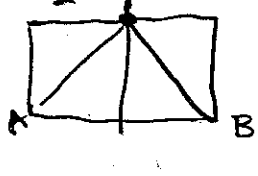
Since this holds for all $\epsilon > 0$, $\lambda(V) = 0$. ■

2/2

(b) Since each segment of the boundary ∂P is a subset of a line, which is the translation of 1-dimensional subspace of $\mathbb{R}^2$, by monotonicity, translational invariance of λ, and the result of part (a), each segment of ∂P has Lebesgue measure 0. Since there finitely many boundary segments, by subadditivity, $\lambda(\partial P) = 0$.

Good!

Since any triangle is the image of some triangle having at least 1 side parallel to the x-axis, under a rotation, by the result of part (b) of problem 3, to show the desired result in the case that P is a triangle, we can assume that P has some side parallel to the x-axis. Let I be the smallest open cell containing P, so that I has the same height and base length as P. Let v be the vertex of P that is not on the side parallel to the x-axis, and let I_1 and I_2 be the two cells into which I is split by the vertical line going through v. Then, $P_1 := P \cap I_1$ is a rotation of $R_1 := (P^c \cap I_1) \cancel{\setminus} \cup (\partial P)$, and $P_2 := P \cap I_2$ is a rotation of $R_2 := (P^c \cap I_2) \cancel{\setminus} \cup (\partial P)$. Therefore,



This is your picture and you're assuming A and B are acute

Note that question (2a) immediately gives the useful:

G-d!

Corollary 1: Any line segment $s \subseteq \mathbb{R}^2$ has $\lambda(s) = 0$.
Proof: Any line segment s in $\mathbb{R}^2$ is a subset of a line l_s, a one dimensional subspace of $\mathbb{R}^2$. Thus $0 \leq \lambda(s) \leq \lambda(l_s) = 0$. □

Let $P \subseteq \mathbb{R}^2$ be a polygon. By convention we'll take P to be closed; thus P is measurable. Note that all subsets of $\mathbb{R}^2$ I will mention in this problem will be either open or closed, so each is measurable.
Claim: area(P)=$\lambda(P)$.
Proof:

Let $\{T_1, \ldots, T_n\}$ denote a finite decomposition of P into closed triangles with non-intersecting interiors, such that $\bigcup_{i=1}^n T_i = P$ and area(T_i)+area(T_j)=area($T_i \cup T_j$) for all $i \neq j$. The existence of such a decomposition is easy to prove inductively; we'll take that as a given here.

Now decompose each triangle T_i into two closed right triangles with non-interesecting interiors (drop an altitude to the longest side), yielding a finite decomposition of P into right triangles $\{R_1, \ldots, R_{2n}\}$ such that area(P)=$\sum_{i=1}^{2n}$area(R_i).

Let for $i = 1, \ldots, 2n$, let O_i denote the (open) interior of R_i. The excess E of P that is not covered by any O_i is then given by $E = P \setminus (\bigcup_{i=1}^{2n} O_i) = \bigcup_{i=1}^{2n} (R_i \setminus O_i)$. As $R_i \setminus O_i$ is the union of three line segments, by Corollary 1 $\lambda(R_i - O_i) = 0$. Hence $\lambda(E) = \sum_{i=1}^{2n} \lambda(R_i - O_i) = 0$. Again by the additivity of λ we have $\lambda(P) = (\sum_{i=1}^{2n} \lambda(O_i)) - \lambda(E) = \sum_{i=1}^{2n} \lambda(O_i)$. So all that remains to show is that $\lambda(O_i)$=area(R_i).

Let $R \subseteq \mathbb{R}^2$ be an arbitrary closed right triangle, and let O denote its interior. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(R)$ is R reflected about its hypotenuse. Then T is an orthogonal linear transformation, so by question (3a), $\lambda(T(O)) = \lambda(O)$. Now $R \cup T(R)$ is a closed cell, and area($R \cup T(R)$)=2area(R). Since $R \cup T(R)$ is a cell, its Lebesgue measure is equal to its area. Finally, define the excess $E_R := R \cup T(R) \setminus (O \cup T(O))$. It is the union of five line segments, so $\lambda(E_R) = 0$ by Corollary 1. Now we have

$$\text{area}(R) = \frac{1}{2}\text{area}(R \cup T(R)) = \frac{1}{2}(\lambda(O) + \lambda(T(O)) + \lambda(E_R)) = \frac{1}{2}(2\lambda(0) + 0) = \lambda(O)$$

2/2

Hence

$$\lambda(P) = \sum_{i=1}^{2n} \lambda(O_i) = \sum_{i=1}^{2n} \text{area}(R_i) = \text{area}(P)$$

as desired. ■

✓

Problem 3

(a)

Suppose μ is a *translation invariant* measure on $(\mathbb{R}^d, \mathcal{L})$, and that μ is finite on bounded sets. For all $r \in \mathbb{R}$ with $r \geq 0$, let $C_r = [0, r]^d$ be the closed d-dimensional hypercube of side length r. By

By translation invariance, we now know that $\lambda(V_w) = 0$ for all $w \in \mathbb{Z}^k$. Since $V = \bigcup_{w \in \mathbb{Z}^k} V_w$ and $\mathbb{Z}^k$ is countable, it follows that V is measurable and $\lambda(V) \leq \sum_{w \in \mathbb{Z}^k} \lambda(V_w) = 0$.

(b) First, we note that we can freely disregard the boundary of a polygon. Each polygon has a finite number of line segments as its boundary. Each of these is a subset of a one-dimensional subspace of $\mathbb{R}^2$; by 2(a), such a subset has λ-measure zero. It also follows from this that all polygons are measurable, since they are the union of an open set and a set of measure zero.

Good

Every polygon can be written as a finite disjoint union of triangles, so it suffices to show that area$(P) = \lambda(P)$ for all triangles P. By 3, we can apply translations and rotations without changing measure, so that it suffices to show that area$(P) = \lambda(P)$ where P is a triangle with one point at $(0,0)$, one at $(a,0)$ where $a > 0$, and one at (b,c) where $b, c > 0$. Set $s = \max\{a,b\}$. Then P is contained in the cell $I = [0,s] \times [0,c]$, which has area sc as well as measure sc. P, on the other hand, has measure $\frac{1}{2}sc$. By copying P, splitting it into two disjoint pieces, and applying rotations and translations (or, if P is a right triangle, without splitting), we can cover the remainder of I exactly. It follows from the fact that P is measurable that $2\lambda(P) = \lambda(I)$. Thus $\lambda(P) = \frac{1}{2}\lambda(I) = \frac{1}{2}sc = \text{area}(P)$.

Show this

1/2

3. (a) Let a translation invariant measure μ on $(\mathbb{R}^d, \mathcal{L})$ be given. Define $c := [0,1)^d$, so that $\mu([0,1)^d) = c\lambda([0,1)^d)$, and assume that c is finite.

Inductively, we can see that $\mu\left([0, \frac{1}{2^n})^d\right) = c\lambda\left([0, \frac{1}{2^n})^d\right)$ for all $n \in \mathbb{N}$. Assume the cube $[0, \frac{1}{2^n})^d$ satisfies the property. $[0, \frac{1}{2^n})^d$ can be formed from 2^d disjoint translations of the cube $[0, \frac{1}{2^{n+1}})^d$, so etc

$$2^d c\lambda\left(\left[0, \frac{1}{2^{n+1}}\right)\right) = c\lambda\left(\left[0, \frac{1}{2^n}\right)\right) = \mu\left(\left[0, \frac{1}{2^n}\right)\right) = 2^d \mu\left(\left[0, \frac{1}{2^{n+1}}\right)\right)$$

It follows that $c\lambda\left([0, \frac{1}{2^{n+1}})\right) = \mu\left([0, \frac{1}{2^{n+1}})\right)$. We can perform the same inductive process for cubes of the form $[0, 2^n)$; since $[0, 2^{n+1})$ is composed of 2^d disjoint translated cubes $[0, 2^n)$, it follows from $c\lambda\left([0, 2^n)\right) = \mu\left([0, 2^n)\right)$ that $c\lambda\left([0, 2^{n+1})\right) = \mu\left([0, 2^{n+1})\right)$.

If $c = 0$, then it follows that $\mu(A) = 0$ for all $A \in \mathcal{L}$, since then $\mu(\mathbb{R}^d) \leq \sum_{n \in \mathbb{N}} \mu([0, 2^n)) = 0$ and $\mu(A) \leq \mu(\mathbb{R}^d)$ for all $A \in \mathcal{L}$. Then $\mu(A) = c\lambda(A)$ for all $A \in \mathcal{L}$.

Consider then the case where $c > 0$. Let a bounded set $A \in \mathcal{L}$ be given; it follows that $A \subseteq I$ for some half-open cube I of side length 2^n for some $n \in \mathbb{N}$. By Lemma 1, we know that

$$\lambda(A) = \inf\left\{\sum_1^\infty \lambda(Q_i) : Q_i \in \mathcal{Q} \text{ and } A \subseteq \bigcup_1^\infty Q_i\right\}$$

where $\mathcal{Q} := \left\{a + [0, \frac{1}{2^n})^d : a \in \mathbb{R}^d, n \in \mathbb{N}\right\}$. From the above proof, we then have that

$$c\lambda(A) = \inf\left\{\sum_1^\infty \mu(Q_i) : Q_i \in \mathcal{Q} \text{ and } A \subseteq \bigcup_1^\infty Q_i\right\}$$

By subadditivity and monotonicity of μ, it follows that $c\lambda(A) \geq \mu(A)$. Now, consider the set $I - A$. By the same argument, we have that $c\lambda(I - A) \geq \mu(I - A)$. By additivity we then find that

$$c(\lambda(I) - \lambda(A)) \geq \mu(I) - \mu(A)$$
$$c\lambda(I) - c\lambda(A) \geq c\lambda(I) - \mu(A)$$
$$c\lambda(A) \leq \mu(A)$$

Thus we have $\mu(A) = c\lambda(A)$ for all bounded sets $A \in \mathcal{L}$. Finally, every $A \in \mathcal{L}$ can be written as a countable union of the disjoint bounded sets $A_k = A \cap [k_1, k_1 + 1) \times \cdots \times [k_d, k_d + 1)$ for $k \in \mathbb{Z}^d$. Since each A_k satisfies $\mu(A_k) = c\lambda(A_k)$, we see that $\mu(A) = \sum_{k \in \mathbb{Z}^d} \mu(A_k) = \sum_{k \in \mathbb{Z}^d} c\lambda(A_k) = c\lambda(A)$. 2/2

(b) First, we show that $T(A)$ is measurable. By Lemma 2 (the conditions of which $(\mathbb{R}^d, \mathcal{L}, \lambda)$ satisfies), there exists a Borel set B such that $B \subseteq A$ and $\lambda(A - B) = 0$. Since T is a homeomorphism, $T(B)$ is also a Borel set. We show that $\lambda(T(A - B)) = 0$, from which it follows that $T(A) = T(B) \cup T(A - B)$ is measurable by Lemma 2. Let $\epsilon > 0$ be given. By Lemma 1, there exists a covering $\{Q_n\}_1^\infty$ of $A - B$ where each Q_n is a cube. Since T is an isometry, $\mathrm{diam}(T(Q_n)) = \mathrm{diam}(Q_n)$. Thus, for every Q_n there exists a cube R_n such that $R_n \supseteq T(Q_n)$ and $\ell(R_n) = \mathrm{diam}(T(Q_n)) = \ell(Q_n)\sqrt{d}$. Since $\bigcup_1^\infty T(Q_n)$ covers $T(A - B)$, $\bigcup_1^\infty R_n$ covers $T(A_B)$, so $\lambda T(A - B) \leq \sum_1^\infty \ell(R_n) = \sum_1^\infty \ell(Q_n)\sqrt{d} = \epsilon\sqrt{d}$. Since $\epsilon > 0$ was arbitrary, $\lambda(A - B)$ must be zero. Hence $T(A)$ is measurable.

Now we show that $\lambda(T(A)) = \lambda(A)$. Define the measure λ_T on $\mathcal{L}$ by $\lambda_T(A) = \lambda(T(A))$. Since $A \in \mathcal{L}$ implies $T(A) \in \mathcal{L}$, λ_T is trivially a measure. We note that for any $A \in \mathcal{L}$ and $x \in \mathbb{R}$ we have that $T(x + A) = y + T(A)$ for some $y \in \mathbb{R}$ by linearity. From the fact that $\lambda_T(A) = \lambda(T(A))$ and the translation invariance of Lebesgue measure, we see that λ_T is translation invariant. Then by 3(a), we have that $\lambda_T = c\lambda$ for some $c \geq 0$. In fact, $c > 0$, since $\lambda_T(\mathbb{R}^d) = \lambda(T(\mathbb{R}^d)) = \lambda(\mathbb{R}^d) = \infty$.

Assume for sake of contradiction that $c > 1$. Then there exists some $n \in \mathbb{N}$ such that $c^n > (\sqrt{d})^d$. Consider $T^n([0,1]^d)$. We have $\mathrm{diam}([0,1]^d) = \sqrt{d}$, and since T is an isometry it follows that $\mathrm{diam}(T^n([0,1]^d)) = \sqrt{d}$. Then $T^n([0,1]^d)$ can be contained in a box I of side length $\sqrt{d}$. But then I has measure $(\sqrt{d}^d)^d$, while $\lambda(T^n([0,1]^d)) = c^n\lambda([0,1]^d) = c^n > (\sqrt{d})^d$, so it is contradictory that $T^n([0,1]^d) \subseteq I$ by monotonicity. Therefore it must be that $c \leq 1$.

By applying the above argument to T^{-1}, which satisfies $\lambda(T^{-1}(A)) = \frac{1}{c}\lambda(A)$, we can also find that $\frac{1}{c} \geq 1$. Hence $c = 1$, and $\lambda(A) = \lambda(T(A))$ for all $A \in \mathcal{L}$. 1/2

4. (a) First, since $\emptyset \in \mathcal{E}$, we have

$$\mu^*(\emptyset) \leq \sum_1^\infty \rho(\emptyset) = 0$$

and since $\mu^*(\emptyset) \geq 0$, we conclude that $\mu^*(\emptyset) = 0$.

Second, let $A, B \subseteq X$ be given with $A \subseteq B$. Let a set $\{F_i\}_1^\infty \subseteq \mathcal{E}$ be given which covers B. Then it covers A, so

$$\mu^*(A) = \inf\left\{\sum_1^\infty \rho(E_i) : E_i \in \mathcal{E} \text{ and } A \subseteq \bigcup_1^\infty E_i\right\} \leq \sum_1^\infty \rho(F_i)$$

defintion, C_r is a cell for all r.

Claim: $\exists c \geq 0$ such that $\mu(A) = c\lambda(A)$ for all $A \in \mathcal{L}$.
Proof:

Consider C_1, the closed unit cell in $\mathbb{R}^d$. Cells are measurable, so $C_1 \in \mathcal{L}$, and $\lambda(C_1) = (1-0)^d = 1$. Define $c = \mu(C_1)$. We'll show $\mu(A) = c\lambda(A)$ for all $A \in \mathcal{L}$. If $c = 0$, this is guaranteed by subadditivty, so assume c is positive.

Let $n \in \mathbb{N}$ be arbitrary and consider $C_{\frac{1}{n}}$. We can translate n^d copies of this cell to generate a cover C for C_1. (Note that overlap between adjacent cells in this cover can be effectively ignored, either by invoking the corollary proven in question 2 to show that the overlap has measure zero, or by selectively making faces of hypercubes open to eliminate the overlap entirely. We'll do the latter and assume C is pairwise disjoint and its union is exactly C_1.) As μ is translation invariant, each element of C has the same Lebesgue measure, and by the additivity of μ, we have

Only know for Lebesgue measure!

$$\mu(C_1) = \sum_{i=1}^{n^d} \mu(C_{\frac{1}{n}}) = n^d C_{\frac{1}{n}} \implies \mu(C_{\frac{1}{n}}) = \frac{c}{n^d} = c\lambda(C_{\frac{1}{n}})$$

Now let I be a closed cell with rational side lengths defined by $I = [0, \frac{p_1}{q_1}] \times [0, \frac{p_2}{q_2}], \ldots, [0, \frac{p_d}{q_d}]$ where $p_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ for all i. Let $q = \frac{1}{\prod_{i=1}^d q_i}$ and $p = \prod_{i=1}^d p_i$. Then q divides q_i for all i, so we can translate p copies of C_q to cover C_1, yielding

$$\mu(I) = \sum_{i=1}^{p} \mu(C_q) = \sum_{i=1}^{p} \frac{c}{q} = c(\frac{p}{q}) = c\lambda(I)$$

Note that by translation invariance this covers every cell with rational side lengths. Now suppose I is a closed cell with arbitrary side lengths defined by $I = [0, r_1] \times \ldots \times [0, r_d]$ where $r_i \in \mathbb{R}$. Let $\epsilon > 0$. Then we can choose rationals $q_1 \ldots, q_d$ and $s_1 \ldots, s_d$ such that (1) $q_i \leq r_i \leq s_i$ for all i and (2) $\prod r_i - \frac{\epsilon}{c} < \prod q_i \leq \prod r_i \leq \prod s_i < \prod r_i + \epsilon$. Define cells $I_q = [0, q_1] \times \ldots \times [0, q_d]$ and $I_s = [0, s_1] \times \ldots \times [0, s_d]$. By monotonicity, $c \prod q_i = \mu(I_q) \leq \mu(I) \leq \mu(I_s) = c \prod s_i$; thus we have $c \prod r_i - \epsilon \leq \mu(I) \leq c \prod r_i + \epsilon$. As ϵ was arbitary, $\mu(I) = c \prod r_i = c\lambda(I) = cl(I)$. Moreover, translation invariance extends this to all cells.

Suppose A is any member of $\mathcal{L}$. Then for any $\epsilon > 0$, there is a cover of A by cells $(I_k)_{k \in \mathbb{N}}$ such that $c\lambda(A) - \epsilon \leq c(\sum_k l(I_k)) - \epsilon = (\sum_k \mu(I_k)) - \epsilon < \mu(A) \leq \sum_k \mu(I_k) = (c \sum_k l(I_k)) + \epsilon = c\lambda(A) + \epsilon$. Hence as ϵ was arbitrary, $\mu(A) = c\lambda(A)$ as desired. ■

$\not$ 2/2

(b)

Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal linear transformation, and $A \in \mathcal{L}$.

Claim: $T(A) \in \mathcal{L}$ and $\lambda(T(A)) = \lambda(A)$.
Proof:

If T is orthogonal, then it represents the composition of rotations and reflections, so it suffices to show that the claim holds for an arbitrary rotation R and reflection F. Then inductively it holds for any composition of finitely many such transformations.

Let $A \in \mathcal{L}$ and $\epsilon > 0$. Then there is a sequence of pairwise disjoint open balls $(B_i)_{i \in \mathbb{N}}$ whose union covers A, and $\sum_{i=1}^{\infty} \lambda(B_i) < \lambda(A) + \frac{\epsilon}{2}$.

Any rotation or reflection applied to a ball in $\mathbb{R}^d$ is equivalent to a translation. Since λ is translation invariant, $\lambda(T(B_i)) = \lambda(B_i)$ for all balls B_i in the cover. Thus we have $\lambda(T(A)) \leq \sum_{i=1}^{\infty} \lambda(T(B_i)) \leq \lambda(T(A)) + \frac{\epsilon}{2}$, so $\lambda(T(A)) \leq \sum_{i=1}^{\infty} \lambda(B_i) \leq \lambda(T(A)) + \frac{\epsilon}{2}$.

So both $\lambda(A)$ and $\lambda(T(A))$ are within $\frac{\epsilon}{2}$ of $\sum_{i=1}^{\infty} \lambda(B_i)$; thus $|\lambda(A) - \lambda(T(A))| < \epsilon$. As ϵ was arbitrary, $\lambda(A) = \lambda(T(A))$. ∎ 7/2

Problem 4

(a)

Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho(\emptyset) = 0$. For any $A \subseteq X$ define

$$S_A = \{(E_i)_{i \in \mathbb{N}} \mid E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j\}$$

$$T_A = \{\sum_{i=1}^{\infty} \rho(E_i) \mid (E_i)_{i \in \mathbb{N}} \in S_A\}$$

$$\mu^*(A) = \inf T_A$$

Claim: μ^* is an outer measure.
Proof:

We simply check each requisite property for μ^*.

(1) *Null Empty Set:*

Since $Y = \emptyset$ is the only member of X (and thus of $\mathcal{E}$) with $\emptyset \subseteq Y$, the only member of $S_\emptyset$ is $(E_i)_{i \in \mathbb{N}}$ where $E_i = \emptyset$ for all i. Thus since $\rho(\emptyset) = 0$, we have $\sum_{i=1}^{\infty} \rho(E_i) = 0$. Hence $T_\emptyset = \{0\}$, so $\mu^*(\emptyset) = \inf\{0\} = 0$. □

(2) *Monotonicity:*

Suppose $A, B \in \mathcal{P}(X)$ and $A \subseteq B$, and suppose $E := (E_i)_{i \in \mathbb{N}} \in S_B$. Then

$$\bigcup_{i=1}^{\infty} E_i \supseteq B \supseteq A \implies E \in S_A$$

Hence $S_A \supseteq S_B$, so $T_A \supseteq T_B$. Thus any lower bound for T_B is also a lower bound for T_A, so

$$\mu^*(B) = \inf T_B \geq \inf T_A = \mu^*(A)$$

as desired. □

(3) *Countable Subadditivity:*

Suppose $(A_1, A_2, \ldots)$ is a countable sequence of subsets of X. Let $A = \bigcup_{i=1}^{\infty} A_i$. We'll show $\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

4. (a) Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty)$ be such that $\emptyset \in \mathcal{E}, X \in \mathcal{E}$, and $\rho(\emptyset) = 0$. For any $A \subseteq X$ define

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) : E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_{i=1}^{\infty} E_i\right\}.$$

We will show that μ^* is an outer measure. Since $\emptyset \subseteq \emptyset$ and $\rho(\emptyset) = 0$ it follows that $\mu^*(\emptyset) \leq 0$ so $\mu^*(\emptyset) = 0$. If $A \subseteq B$ then any cover of B is also a cover of A, so

$$\left\{\sum_{i=1}^{\infty} \rho(E_i) : E_i \in \mathcal{E}, \text{ and } B \subseteq \bigcup_{i=1}^{\infty} E_i\right\} \subseteq \left\{\sum_{i=1}^{\infty} \rho(E_i) : E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_{i=1}^{\infty} E_i\right\}$$

and so

$$\mu^*(B) \geq \mu^*(A).$$

Finally we show countable subadditivity. Let $(E_k)_k \subseteq X$. If $\mu^*(E_i) = \infty$ for any i then subadditivity is trivial, so we assume not. Fix $\varepsilon > 0$. For $n \in \mathbb{N}$ pick $(I_{n,k})_k \subseteq \mathcal{E}$ with $E_n \subseteq \bigcup_k I_{n,k}$ and

$$\sum_{k=1}^{\infty} \rho(I_{n,k}) \leq \mu^*(E_n) + \frac{\varepsilon}{2^k}.$$

We may pick such $(I_{n,k})_k$ since at least $X \in \mathcal{E}$ so the set of possible covers is nonempty. Then we have

$$\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k}$$

and so we have by definition of infimum that

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n,k=1}^{\infty} \rho(I_{n,k}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho(I_{n,k}) \leq \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon.$$

Letting $\varepsilon \to 0$ then gives

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$$

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and so μ^* is an outer measure. $\square$

(b) *(Hausdorff measure)* Let (X, d) be a metric space, and $\mathcal{E}_\delta = \{B(x, r) : x \in X, r \in (0, \delta)\} \cup \{\emptyset, X\}$. Given $\alpha > 0$ define $\rho(B(x, r)) = c_\alpha r^\alpha$, where c_α is a normalization constant defined by $c_\alpha = \pi^{\alpha/2}/\Gamma(1 + \alpha/2)$. Let $H^*_{\alpha,\delta}$ be the outer measure obtained with this choice of ρ and the collection of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta\to 0} H^*_{\alpha,\delta}$. We will show that H^*_α is an outer measure and restricts to a measure H_α on a $\sigma-$algebra that contains all Borel sets.

Clearly $\rho \geq 0$, and $\emptyset, X \in \mathcal{E}_\delta$, and $\rho(\emptyset) = 0$ for all $\delta, \alpha > 0$, so $H^*_{\alpha,\delta}$ is indeed an outer measure by the previous part. Restricting the radius of coverings can only increase the infimum of the sums, so if $\delta_1 < \delta_2$ then $H^*_{\alpha,\delta_1} \geq H^*_{\alpha,\delta_2}$ so that the limit $\lim_{\delta\to 0} H^*_{\alpha,\delta}$ makes sense. We check that H^*_α is an outer measure. Clearly

$$H^*_\alpha(\emptyset) = \lim_{\delta\to 0} H^*_{\alpha,\delta} = \lim_{\delta\to 0} 0 = 0.$$

Also, by monotonicity of the approximating measures, if $E \subseteq F \subseteq X$ then

$$H^*_{\alpha,\delta}(E) \leq H^*_{\alpha,\delta}(F)$$

and taking the limit as $\delta \to 0$ gives
$$H_\alpha^*(E) \leq H_\alpha^*(F).$$
Similarly, for any sequence $(A_i)_i$ of subsets of X we have
$$H_{\alpha,\delta}^*\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty H_{\alpha,\delta}^*(A_i) \leq \sum_{i=1}^\infty H_\alpha^*(A_i)$$
and taking the limit as $\delta \to 0$ gives
$$H_\alpha^*\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty H_\alpha^*(A_i)$$
so H_α^* is an outer measure. We finish the proof in three parts.

(1) We claim H_α^* is additive on separated sets. Suppose $A, B \subseteq X$ have positive distance D. We get subadditivity for free. For all $\delta < D$ any $\delta-$cover $(K_k)_k$ of $A \cup B$ can be paritioned into a cover $(I_i)_i$ of A and a cover of $(J_j)_j$ of B so that $H_{\alpha,\delta}^*(A) \leq \sum_{i=1}^\infty \rho(I_i)$ and $H_{\alpha,\delta}^*(B) \leq \sum_{j=1}^\infty \rho(J_j)$ so that
$$H_{\alpha,\delta}^*(A) + H_{\alpha,\delta}^*(B) \leq \sum_{i=1}^\infty \rho(I_i) + \sum_{j=1}^\infty \rho(J_j) = \sum_{k=1}^\infty \rho(K_k) \leq H_{\alpha,\delta}^*(A \cup B)$$
and so $H_{\alpha,\delta}^*(A) + H_{\alpha,\delta}^*(B) = H_{\alpha,\delta}^*(A \cup B)$. Taking the limit as $\delta \to 0$ then gives
$$H_\alpha^*(A) + H_\alpha^*(B) = H_\alpha^*(A \cup B)$$
as desired.

(2) We claim that if $(A_n)_n \subseteq X$ is an increasing sequence of sets with $\operatorname{dist}(A_i, A_{i+1}^c) > 0$ for all i, then
$$H_\alpha^*\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} H_\alpha^*(A_n).$$
The $\geq$ inequality is trivial by monotonicity since $\bigcup_n A_n \supseteq A_k$ for all k. It suffices to show $\leq$ and we may assume that $\lim_{n\to\infty} H_\alpha^*(A_n) < \infty$ otherwise the inequality holds trivially. Let $B_1 := A_1$ and for $n > 1$ define $B_n := A_n \setminus A_{n-1}$. Then for all $m \geq n+2$ we have by assumption that $H_\alpha^*(B_n \cup B_m) = H_\alpha^*(B_n) + H_\alpha^*(B_m)$. In particular for all $N > 0$
$$\sum_{1\leq n\leq N, n \text{ odd}} H_\alpha^*(B_n) = H_\alpha^*\left(\bigcup_{1\leq n\leq N, n \text{ odd}} B_n\right) \leq H_\alpha^*(A_n) \leq \lim_{n\to\infty} H_\alpha^*(A_n) < \infty.$$
It follows that
$$\sum_{1\leq n, n \text{ odd}} H_\alpha^*(B_n) < \infty$$
and in particular,
$$\lim_{k\to\infty} \sum_{k\leq n, n \text{ odd}} H_\alpha^*(B_n) = 0.$$
Analagous statements hold for the series with even terms, so that
$$\lim_{k\to\infty} \sum_{n=k}^\infty H_\alpha^*(B_n) = 0.$$

Then we note that for all $k > 0$

$$H_\alpha^*\left(\bigcup_{n=1}^{\infty} A_n\right) = H_\alpha^*\left(A_k \cup \bigcup_{n=k}^{\infty} B_n\right) \leq H_\alpha^*(A_k) + H_\alpha^*\left(\bigcup_{n=k}^{\infty} B_n\right)$$

$$\leq H_\alpha^*(A_k) + \sum_{n=k}^{\infty} H_\alpha^*(B_n)$$

and so taking the limit as $k \to \infty$ gives

$$H_\alpha^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \lim_{k\to\infty} H_\alpha^*(A_k)$$

as desired.

(3) We consider Σ to be the set of H_α^*-measurable subsets of X, i.e. those that satisfy the Carathèodory condition. We know Σ is a $\sigma-$algebra and we must show that $\Sigma \supseteq \mathcal{B}$. It suffices to show that Σ contains the closed sets. Let $C \subseteq X$ closed, and $A \subseteq X$ be given. Define for all $n > 0$ the set

$$B_n := \left\{x \in A \cap C^c : d(x, C) > \frac{1}{n}\right\}.$$

Since C is closed, we know C^c is open, and so $\bigcup_n B_n = A \cap C^c$ since every point in $A \cap C^c \subseteq C^c$ has strictly positive distance to C. Note that $(B_n)_n$ is increasing and $\operatorname{dist}(B_n, B_{n+1}^c) \geq \frac{1}{n} - \frac{1}{n+1} > 0$ for all $n > 0$ so we are in a position to apply the previous step. We find

$$H_\alpha^*(A \cap C^c) = H_\alpha^*\left(\bigcup_n^{\infty} B_n\right) = \lim_{n\to\infty} H_\alpha^*(B_n).$$

Then we know that

$$H_\alpha^*(A) = H_\alpha^*((A\cap C) \cup (A \cap C^c)) \geq H_\alpha^*((A \cap C) \cup B_n) = H_\alpha^*(A \cap C) + H_\alpha^*(B_n)$$

since $\operatorname{dist}(A \cap C, B_n) > 0$. Taking the limit as $n \to \infty$ then gives

$$H_\alpha^*(A) = H_\alpha^*(A \cap C) + H_\alpha^*(A \cap C^c)$$

as desired. It follows that $C \in \Sigma$, and so $\Sigma \supseteq \mathcal{B}$. □

(c) If $X = \mathbb{R}^d$, and $\alpha = d$ we will show that H_d is the Lebesgue measure. Note that in the proof of 3. (a) we didn't use any fact about $\mathcal{L}$ except that λ and μ are measures on $\mathcal{L}$. Hence we could replace $\mathcal{L}$ with $\mathcal{B}$ and the theorem still holds. Clearly H_d is translation invariant since none of the approximating measures involve anything but the radii of the covering balls, and since a translates of covers are covers of translates. For α a positive integer, we have that c_α corresponds to the Lebesgue measure of the α dimensional sphere of radius 1. Note that $B(0,1)$ may be written as the countable union of nonoverlapping balls of radius at most $\delta/2$,

$$B(0,1) = \bigcup_{i=1}^{\infty} B_i(x_i, r_i)$$

so by extending the radii by a factor of $t > 1$ and using open balls we have

$$H_{d,\delta}^*(B(0,1)) \leq H_{d,\delta}^*\left(\bigcup_{i=1}^{\infty} B_i(x_i, tr_i)\right) \leq \sum_{i=1}^{\infty} c_d(tr_i)^d = t^d \sum_{i=1}^{\infty} c_d r_i^d = \underbrace{t^d \sum_{i=1}^{\infty} \lambda(B_i(x_i, r_i)) = t^d \lambda\left(\bigcup_{i=1}^{\infty} B_i(x_i, r_i)\right)}_{\text{For } \textit{open} \text{ balls nonoverlapping means disjoint}}$$

this needs $tr_i \leq \delta$

(c) Prove if $X=\mathbb{R}^d$ and $\alpha=d$ that H_d is a non-zero finite constant multiple of λ.

proof: First we show that H_d is translation invariant by showing that $H^*_{d,\delta}$ is translation invariant for all $\delta>0$. Let $A\subseteq\mathbb{R}^d$ and $A\subseteq\bigcup_{i=1}^{\infty}B(x_i,r_i)$ where $r_i<\delta$ for all i. Also let $x\in\mathbb{R}^d$ and consider $x+A$. So $x+A\subseteq\bigcup_{i=1}^{\infty}B(x+x_i,r_i)$ and so

$H^*_{d,\delta}(x+A)\leq\sum_i\rho(B(x+x_i,r_i))=\sum_i\rho(B(x_i,r_i))$ so taking the infimum over all covers $\{B(x_i,r_i)\}$ we get $H^*_{d,\delta}(x+A)\leq H^*_{d,\delta}(A)$. Switching roles we get the reverse inequality and so $H^*_{d,\delta}$ is translation invariant. Thus by taking the limit $\delta\to0$ we see H^*_d is translation invariant and so H_d is a translation invariant measure on $\mathbb{R}^d$.

Thus by problem 3, if there exists a cube with finite measure then $H_d=c\cdot\lambda$ for some $\infty>c\geq0$.

We prove $H_d([0,1)^d)<\infty$. Let $\delta>0$. Then there exists $k\in\mathbb{N}$ s.t. $\sqrt{d}\cdot\frac{1}{k}<\delta$. So since any d-cube with side length $\frac{1}{k}$ is can be compactly contained in a ball with radius $\sqrt{d}/k$ and $[0,1)^d$ can be broken up into k^d d-cubes with side length $\frac{1}{k}$, we have $H^*_{d,\delta}([0,1)^d)\leq k^d C_d\left(\frac{\sqrt{d}}{k}\right)^d=C_d(\sqrt{d})^d$. Thus taking $\lim\delta\to0$ we see: $H_d([0,1)^d)\leq C_d(\sqrt{d})^d<\infty$. Thus $H_d=c\cdot\lambda$ for some $0\leq c<\infty$.

Also note that $\rho(B(x,r))=C_d r^d$ is the d-dimensional volume of $B(x,r)$. Thus for any covering of $[0,1)^d$ by balls $\{B(x_i,r_i)\}$, $r_i<\delta$ for all i we have $\sum_{i=1}^{\infty}\rho(B(x_i,r_i))\geq\text{volume}([0,1)^d)=1$ and so $H_{d,\delta}([0,1)^d)\geq1$. Taking ~~the infimum and~~ $\lim\delta\to0$ we have $H_d([0,1)^d)\geq1$ and so $c\geq1$. Thus H_d is a non-zero finite constant multiple of λ.

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(d) **Lemma:** If, for some $\alpha, \beta \in [0, \infty)$, $\alpha < \beta$ and $H_\alpha(S) < \infty$, then $H_\beta(S) = 0$.

Proof of Lemma: Let $\delta > 0$. Because $H_{\alpha,\delta}$ is an infimum, we can find a sequence of balls $B(x_1, r_1), B(x_2, r_2), \ldots \in \mathcal{E}_\delta$, with $S \subseteq \bigcup_{i=1}^\infty B(x_i, r_i)$, such that

$$\sum_{i=1}^\infty c_\alpha r_i^\alpha = \sum_{i=1}^\infty \rho(B(x_i, r_i)) \leq H_{s,\delta+1}.$$

Furthermore, since $H_{\alpha,\delta}$ is an infimum and $\mathcal{E}_\delta$ becomes smaller as δ decreases, $H_{\alpha,\delta}$ increases as δ decreases. Thus, taking the limit as $\delta \to 0^+$,

$$\sum_{i=1}^\infty c_\alpha r_i^\alpha \leq H_{\alpha,\delta}(S) + 1 \leq H_\alpha(S) + 1.$$

Therefore,

$$\begin{aligned} H_{\beta,\delta}(S) &\leq \sum_{i=1}^\infty c_\beta r_i^\beta \leq \frac{c_\beta}{c_\alpha} \sum_{i=1}^\infty c_\alpha r_i^\alpha r_i^{\beta-\alpha} \\ &\leq \frac{c_\beta}{c_\alpha} \delta^{\beta-\alpha} \sum_{i=1}^\infty c_\alpha r_i^\alpha \qquad (\text{each } r_i \leq \delta) \\ &\leq \frac{c_\beta}{c_\alpha} \delta^{\beta-\alpha} (H_\alpha(S) + 1). \end{aligned}$$

Since $H_\alpha(S) < \infty$ and $\beta > \alpha$, taking the limit as $\delta \to 0$ proves the lemma.

Let $d = \sup\{\alpha \in [0, \infty] | H_\alpha^*(S) = \infty\}$. Suppose $\alpha \in (d, \infty)$. By choice of d, for $\beta = \frac{\alpha - d}{2} > d$, $H_\beta(S) < \infty$, so that, by the above lemma, $H_\alpha(S) = 0$. On the other hand, suppose $\alpha \in (0, d)$. By the above lemma, if $H_\alpha(S) \neq \infty$, then, $\forall \beta \in (\alpha, d]$, $H_\beta(S) = 0$, contradicting the choice of d as the supremum. Thus, d has the desired properties. Note d is unique, as, if $d' \neq d$ (without loss of generality, $d' > d$), also had the desired properties, then, for $\alpha \in (d, d'$, $\alpha = 0$ and $\alpha = \infty$, which is impossible. ■

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(e) **Lemma:** $\forall A \subseteq \mathbb{R}$, $c \in \mathbb{R}$, if cA is the dilation of A by c, then, $H_\alpha(cA) = c^\alpha H_\alpha(A)$.

Proof of Lemma: Note that, if $B_1, B_2, \ldots \in \mathcal{E}_\delta$ with $A \subseteq \bigcup_{i=1}^\infty B_i$, then $cB_1, cB_2, \ldots \in \mathcal{E}_\delta$ with $cA \subseteq \bigcup_{i=1}^\infty cB_i$. Also, for any ball $B(x, r)$, $\rho(cB(x, r)) = \rho(B(cx, cr)) = c^\alpha \rho(B(x, r))$. Thus,

$$H_\alpha(cA) = \lim_{\delta \to 0+} \inf \left\{ \sum_{i=1}^\infty c^\alpha \rho(B_i) \middle| B_i \in \mathcal{E}_\delta, cA \subseteq \bigcup_{i=1}^\infty B_i \right\} = c^\alpha H_\alpha(A),$$

proving the lemma.

The Cantor set C has the property that

$$C = \frac{1}{3} (C \cup (C + 2)),$$

where addition denotes translation and multiplication denotes dilation.

It was shown in the proof of part (c) that H_d is translation invariant. Note also that, since $C \subseteq [0,1]$, $\operatorname{dist}(C, C+2) > 0$, and thus that, by the Lemma shown in in part (b), $H_\alpha(C \cup (C+2)) = H_\alpha(C) + H_\alpha(C+2)$. Therefore, by the above lemma, $\forall \alpha \in [0, \infty]$,

$$
\begin{aligned}
H_\alpha(C) &= H_\alpha\left(\frac{1}{3}(C \cup (C+2))\right) \\
&= \left(\frac{1}{3}\right)^\alpha H_\alpha(C \cup (C+2)) && \text{by above lemma} \\
&= \left(\frac{1}{3}\right)^\alpha H_\alpha(C) + H_\alpha(C+2) && \text{since } \operatorname{dist}(C, C+2) > 0 \\
&= \left(\frac{1}{3}\right)^\alpha 2H_\alpha(C). && \text{(by translation invariance)}
\end{aligned}
$$

Suppose, then, that there exists some $\alpha \in (0, \infty)$ such that $H_\alpha(C) \in (0, \infty)$. Then, for that value of α, we can divide both sides of the above equation by $H_\alpha(C)$, so that

$$2 = 3^\alpha.$$

Then,

$$\alpha = \log_3(2) = \boxed{\frac{\ln(2)}{\ln(3)}}.$$

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