Final Study Guide
21-236 Mathematical Studies Analysis II
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Exam Date: Tuesday, May 8, 2012

Integration

1. **Theorem 160 (Repeated Integration)** Let $S \subseteq \mathbb{R}^N$, $T \subseteq \mathbb{R}^M$ be rectangles, let $f : S \times T \to \mathbb{R}$ be Riemann integrable, and suppose that, $\forall \mathbf{x} \in S$, $\mathbf{y} \in T \mapsto f(\mathbf{x}, \mathbf{y})$ is Riemann Integrable. Then, the function $\mathbf{x} \in S \mapsto \int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ is Riemann integrable and

$$\int_{S\times T} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) = \int_S \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x}.$$

Proof: Construct a partition $\mathcal{P} = \{R_1, \dots, R_k\}$ (by refinement of partitions $\mathcal{P}$ and $\mathcal{Q}$) of $R := S \times T$ such that $R_i = S_i \times T_j$, where $\mathcal{P}_N = \{S_1, \dots, S_m\}$ and $\mathcal{P}_M = \{T_i, \dots, T_l\}$ are partitions of S and T, respectively. Using the fact that

$$\operatorname{meas}_{N+M} R_k = \operatorname{meas}_{N+M}(S_i \times T_j) = \operatorname{meas}_N S_i \operatorname{meas}_M T_j,$$

$$L(f,\mathcal{P}) \le \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le U(f,\mathcal{Q}).$$

Taking the supremum over all partitions $\mathcal{P}$ of R gives

$$\underline{\int_{S\times T}} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) \le \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x}. \le \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le U(f,\mathcal{Q}).$$

Taking the infimum over all partitions $\mathcal{Q}$ of R gives

$$\underline{\int_{S\times T}} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) \le \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x}. \le \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \le \overline{\int_{S\times T}} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}).$$

Therefore, since f is Riemann integrable,

$$\int_{S\times T} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) = \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} = \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \right) d\mathbf{x}. \quad \blacksquare$$

Peano-Jordan Measure

1. **Theorem 177 (Peano-Jordan Measurability of a Set and its Boundary)** A bounded set $E \subset \mathbb{R}^N$ is Peano-Jordan measurable if and only if ∂E is Peano-Jordan measurable and $\operatorname{meas} \partial E = 0$.

Proof: Note that, if P is a pluri-rectangle, then $\operatorname{meas} \partial P = 0$, so that $\operatorname{meas} P = \operatorname{meas} \overline{P} = \operatorname{meas} P^\circ$.

Step 1: Suppose E is Peano-Jordan measurable and let R be a rectangle containing E. Then, $\forall \epsilon > 0$, there exist pluri-rectangles P_1 and P_2 with $P_1 \subseteq E \subseteq P_2$ such

$$0 \leq \operatorname{meas} P_2 - \operatorname{meas} P_1 \leq \epsilon.$$

Then,

$$\operatorname{meas}(\overline{P_2} \backslash P_2^\circ) = \operatorname{meas} \overline{P_2} - \operatorname{meas} P_1^\circ = \operatorname{meas} P_2 - \operatorname{meas} P_1 \leq \epsilon.$$

Since $\overline{P_2} \backslash P_1^\circ$ is also a pluri-rectangle and $P_1^\circ E^\circ \subseteq \overline{E} \subseteq \overline{P_2}$,

$$\partial E = \overline{E} \backslash E^\circ \subseteq \overline{P_2} \backslash P^\circ.$$

Thus,

$$\begin{aligned} 0 &\leq \sup\{\operatorname{meas} P : P \text{ pluri-rectangle, } \partial E \supseteq P\} \\ &\leq \inf\{\operatorname{meas} P : P \text{ pluri-rectangle, } \partial E \subseteq P\} \\ &\leq \epsilon, \end{aligned}$$

so that, letting $\epsilon \to 0^+$,

$$\begin{aligned} 0 &= \sup\{\operatorname{meas} P : P \text{ pluri-rectangle, } \partial E \supseteq P\} \\ &= \inf\{\operatorname{meas} P : P \text{ pluri-rectangle, } \partial E \subseteq P\} = 0 \end{aligned}$$

Therefore, ∂E is Peano-Jordan measurable with $\operatorname{meas} E = 0$. ■

Step 2: Suppose $\partial E \subset \mathbb{R}^N$ is Peano-Jordan measurable with $\operatorname{meas} \partial E = 0$. Since E is bounded, there exists a rectangle R containing $\overline{E}$. Since $\operatorname{meas} \partial E = 0$, for $\epsilon > 0$, there exists a pluri-rectangle P containing ∂E with $\operatorname{meas} P \leq \epsilon$. Then, $R \backslash P$ is also a pluri-rectangle, so that we can decompose it into a disjoint union of rectangles $R_1, \ldots, R_k$. Let

$$P_1 = \cup\{R_i : R_i \subseteq E\},$$

and let $P_2 = P \cup P_1$, so that $E \subseteq P_2$. Thus,

$$0 \leq \operatorname{meas} P_2 - \operatorname{meas} P_1 \leq \epsilon,$$

so that E is Peano-Jordan measurable. ■

2. **Theorem 178 (Riemann Integrability and Peano-Jordan Measurability)** Let $R \subseteq \mathbb{R}^N$ be a rectangle and let $f : R \to [0, \infty)$ be a bounded function. Then, f is Riemann integrable over R if and only if the set

$$S_f := \{(\mathbf{x}, y) \in R \times [0, \infty) : 0 \leq y \leq f(\mathbf{x})\}.$$

is Peano-Jordan measurable in $\mathbb{R}^{N+1}$, in which case

$$\operatorname{meas}_{N+1} S_f = \int_R f(\mathbf{x})\, d\mathbf{x}.$$

Proof: Assume f is Riemann integrable over R. Then, For $\epsilon > 0$, there exists a partition $\mathcal{P} = \{R_1, \ldots, R_k\}$ such that

$$0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq \epsilon.$$

Define

$$T_i := R_i \times \left[0, \inf_{R_i} f\right], U_i := R_i \times \left[0, \sup_{R_i} f\right],$$

and

$$P_1 := \cup_{i=1}^k T_i, \quad P_2 := \cup_{i=1}^k U_i.$$

Then, P_1 and P_2 are pluri-rectangles, $P_1 \subseteq S_f \subseteq P_2$, and

$$\begin{aligned}
\operatorname{meas}_{N+1} P_2 - \operatorname{meas} P_1 &= \sum_{i=1}^k \operatorname{meas}_{N+1} U_i - \sum_{i=1}^k T_{N+1} \\
&= \sum_{i=1}^k \left(\sup_{R_i} f - 0\right) \operatorname{meas}_N R_i - \sum_{i=1}^k \left(\inf_{R_i} f - 0\right) \operatorname{meas}_N R_i \\
&= U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq \epsilon.
\end{aligned}$$

Thus, S_f is Peano-Jordan measurable with

$$\operatorname{meas}_{N+1} S_f = \int_R f(\mathbf{x})\, d\mathbf{x}.$$

Suppose, conversely, that S_f is Peano-Jordan measurable and let $R \times [0, M]$ be a rectangle containing S_f.

3. **Corollary 182 (Riemann Integration over the Region Between Two Functions)** Let $R \subset \mathbb{R}^N$ be a rectangle, let $\alpha, \beta : R \to \mathbb{R}$ be Riemann integrable with $\alpha \leq \beta$, and let $f : E \to \mathbb{R}$ be a bounded, continuous function. Then, f is Riemann integrable over E, and

$$\int_E f(\mathbf{x}, y)\, d(\mathbf{x}, y) = \int_R \left(\int_{\alpha(\mathbf{x})}^{\beta(\mathbf{x})} f(\mathbf{x}, y)\, dy \right) d\mathbf{x}.$$

Proof: Consider a rectangle $R \times [a, b]$ containing E and let

$$g(\mathbf{x}, y) := \begin{cases} f(\mathbf{x}, y) & (\mathbf{x}, y) \in E \\ 0 & (\mathbf{x}, y) \in (R \times [a, b]) \backslash E \end{cases}$$

Let $c \leq \alpha(\mathbf{x})$ in R. Then, for

$$T_\beta := \{(\mathbf{x}, y) \in R \times [c, \infty) : c \leq y \leq \beta(\mathbf{x})\},$$

$$T_\alpha := \{(\mathbf{x}, y(\in R \times [c, \infty) : c \leq y \leq \alpha(\mathbf{x}), \}$$

if $E := T_\beta \backslash T_\alpha$, $\partial E \subseteq \partial T_\beta \cup \partial T_\alpha$, so that, since $\operatorname{meas}_{N+1} T_\beta = \operatorname{meas}_{N+1} T_\alpha = 0$, $\operatorname{meas}_{N+1} \partial E = 0$. Therefore, since g is continuous in E, the set of discontinuity points of g (which must be in ∂E) has measure zero, so that g is Riemann integrable in $R \times [a, b]$.

$\forall \mathbf{x} \in R$, the function $y \in [a, b] \mapsto g(\mathbf{x}, y)$ is Riemann integrable since it is continuous except at most at the points $y = \alpha(\mathbf{x})$ and $y = \beta(\mathbf{x})$. Then, by Theorem 160 (Repeated Integration), the function $\mathbf{x} \in R \mapsto \int_{[a,b]} g(\mathbf{x}, y)\, dy$ is Riemann integrable and

$$\int_{R \times [a,b]} g(\mathbf{x}, y)\, d(\mathbf{x}, y) = \int_R \left(\int_{[a,b]} g(\mathbf{x}, y)\, dy \right) d\mathbf{x} = \int_R \left(\int_{\alpha(\mathbf{x})}^{\beta(\mathbf{x})} f(\mathbf{x}, y)\, dy \right) d\mathbf{x}.$$

Improper Integrals

1. **Theorem 191 (Improper Riemann Integrability of Non-negative Functions)** Let $E \subseteq \mathbb{R}^N$, and let $f : E \to [0, \infty)$. If there exists *one* exhausting sequence $\{E_n\}$ such that f is Riemann integrable over each E_n and

$$\lim_{n\to\infty} \int_{E_n} f(\mathbf{x})\, d\mathbf{x}.$$

   Proof:

2. **Theorem 193 (Comparison for Improper Riemann Integrals)** Let $E \subseteq \mathbb{R}^N$ and let $f, g : E \to \mathbb{R}$, and assume f is Riemann integrable over every subset $F \subseteq E$ where g is integrable. Then,

   (a) If $|f| \leq g$ and g is Riemann integrable in the improper sense over E with $\int_E g(\mathbf{x})\, d\mathbf{x} < \infty$, then f and $|f|$ are Riemann integrable in the improper sense over E, and

$$\left| \int_E f(\mathbf{x})\, d\mathbf{x} \right| \leq \int_E |f(\mathbf{x})|\, d\mathbf{x} \leq \int_E g(\mathbf{x})\, d\mathbf{x}.$$

   (b) If $f \geq g \geq 0$ and g is Riemann integrable in the improper sense over E with $\int_E g(\mathbf{x})\, d\mathbf{x} = \infty$, then f is Riemann integrable in the improper sense over E and

$$\int_E f(\mathbf{x})\, d\mathbf{x} = \infty.$$

3. **Theorem 197 (Improper Riemann Integrals of f and $|f|$)** Let $E \subseteq \mathbb{R}^N$ and $f : E \to \mathbb{R}$ be Riemann integrable in the improper sense Riemann integral. Then $|f|$ is Riemann integrable in the improper sense over E and, furthermore,

$$\int_E |f|\, d\mathbf{x} < \infty.$$

Change of Variables

1. **Theorem 204 (Change of Variables)** Let $U \subseteq \mathbb{R}^N$ be an open set, and let $\mathbf{g} : U \to \mathbb{R}^N$ be injective and of class C^1, such that $\det J_{\mathbf{g}}(\mathbf{x}) \neq 0$ in U. Let $E \subset \mathbb{R}^N$ be Peano-Jordan measurable with $\overline{E} \subseteq U$ and let $f : \mathbf{g}(E) \to \mathbb{R}$ be Riemann integrable. Then, the function $\mathbf{x} \in E \mapsto f(\mathbf{g}(\mathbf{x}))|\det J_{\mathbf{g}}(\mathbf{x})|$ is Riemann integrable and, furthermore,

$$\int_{\mathbf{g}(E)} f(\mathbf{y})\, d\mathbf{y} = \int_E f(\mathbf{g}(\mathbf{x}))|\det J_{\mathbf{g}}(\mathbf{x})|\, d\mathbf{x}.$$

2. **Lemma 207 (One-Dimensional Change of Variables)** Let $g : [a, b] \to \mathbb{R}$ be an injective, differentiable function continuous derivative g' such that $g'(x) \neq 0$ in $[a, b]$, and let $f : g([a, b]) \to \mathbb{R}$ be bounded. Then,

$$\overline{\int_{g([a,b])}} f(y)\, dy = \overline{\int_a^b} f(g(x))g'(x)|\, dx.$$

Spherical Coordinates in $\mathbb{R}^N$

1. **Lemma 209 (Measure of the n-Dimensional Unit Ball)** Let $N \geq 1$. Then, $\forall \mathbf{x}_0 \in \mathbb{R}^N$,

$$\operatorname{meas} B_N(\mathbf{x}_0, r) = \frac{\pi^{N/2}}{\Gamma(1 + N/2)} r^N.$$

2. **Corollary 211 (Convenient Criteria for finite Riemann Integrals)** Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$ be continuous, and let $\mathbf{x}_0 \in \mathbb{R}^N \backslash E$. Then, if there exist $a, C > 0$ such that, in E

$$|f(\mathbf{x})| \leq \frac{C}{\|\mathbf{x} - \mathbf{x}_0\|^a},$$

then

(i) If E is Peano-Jordan measurable and $a < N$, then f is Riemann integrable in the improper sense over E with finite improper Riemann integral.

(ii) If $E \subseteq \mathbb{R}^N \backslash B(\mathbf{x}_0, r)$ admits an exhausting sequence and $a > N$, then f is Riemann integrable in the improper sense over E with finite improper Riemann integral.

Proof: Since f is continuous and

$$|f(\mathbf{x})| \leq g(\mathbf{x}) := \frac{C}{\|\mathbf{x} - \mathbf{x}_0\|^a},$$

f is bounded wherever g is bounded, and therefore f is Riemann integrable on any $F \subseteq E$ such that g is Riemann integrable over F. The desired consequences then follow from Theorem 193 (Comparisson for Improper Riemann Integrals). ■

3. **Corollary 212 (Convenient Criteria for infinite Riemann Integrals)** Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$ be continuous, and let $\mathbf{x}_0 \in \mathbb{R}^N \backslash E$. Then, if there exist $a, C > 0$ such that, in E

$$|f(\mathbf{x})| \geq \frac{C}{\|\mathbf{x} - \mathbf{x}_0\|^a},$$

then

(i) If E is Peano-Jordan measurable with $(B(\mathbf{x}_0, r) \backslash \{\mathbf{x}_0\}) \subseteq E$, $a < N$, and f is bounded on each $E_n := E \backslash B(\mathbf{x}_0, \frac{1}{n})$, then f is Riemann integrable in the improper sense over E with infinite improper Riemann integral.

(ii) If E admits an exhausting sequence and $\{E_n\}$ such that f is bounded on each E_n, $(\mathbb{R}^N \backslash B(\mathbf{x}_0, r)) \subseteq E$, and $a > N$, then f is Riemann integrable in the improper sense over E with infinite improper Riemann integral.

Proof: Since f is continuous and f is bounded on each E_n, f is Riemann integrable over each E_n. The rest of the proof is identical to the proof of (ii) in Theorem 193 (Comparisson for Improper Riemann Integrals). ■

Differential Surfaces

1. **Definition 214 (Definition of a Manifold)** For $1 \leq k < N$, a nonempty set $M \subseteq \mathbb{R}^N$ is a k-dimensional manifold of class C^m if and only if, $\forall \mathbf{x}_0 \in M$, there exists an open set U with $\mathbf{x}_0 \in U$ and a function $\boldsymbol{\varphi} : W \subseteq \mathbb{R}^k \to \mathbb{R}^N$ of class C^m such that $\boldsymbol{\varphi} : W \to M \cap U$ is a homeomorphism and $\operatorname{rank} J_{\boldsymbol{\varphi}}(\mathbf{y}) = k$ in W.

2. **Proposition 219 (Equivalent Definition of a Manifold)** Suppose $1 \leq k < N$, $M \subseteq \mathbb{R}^N$ is nonempty, and $m \in \mathbb{N}$. Then, the following are equivalent:

   (i) M is a k-dimensional manifold of class C^m.

   (ii) $\forall \mathbf{x}_0 \in M$, there exists an open set $U \subseteq \mathbb{R}^N$ with $\mathbf{x}_0 \in U$ and $\mathbf{g} : U \to \mathbb{R}^{N-k}$ of class C^m such that
   $$M \cap U = \{\mathbf{x} \in U : \mathbf{g}(\mathbf{x}) = \mathbf{0}\},$$
   and $\operatorname{rank} J_{\mathbf{g}}(\mathbf{x}) = N - k$ in $M \cap U$.

   Proof: That (ii) $\Rightarrow$ (i) is an exercise. We show that (i) implies (ii). Given $\mathbf{x}_0 \in M$, let U,W, and $\boldsymbol{\varphi}$ be as in the definition of the manifold. Let $\mathbf{y}_0 \in W$ be such that $\boldsymbol{\varphi}(\mathbf{y}_0) = \mathbf{x}_0$. Since $\operatorname{rank} J_{\boldsymbol{\varphi}}(\mathbf{y}_0) = k$, there is a $k \times k$ submatrix A of $J_{\boldsymbol{\varphi}}(\mathbf{y}_0)$ with $\det A \neq 0$; without loss of generality,
   $$\det \begin{bmatrix} \frac{\partial \varphi_1}{\partial y_1}(\mathbf{y}_0) & \cdots & \frac{\partial \varphi_1}{\partial y_k}(\mathbf{y}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_k}{\partial y_1}(\mathbf{y}_0) & \cdots & \frac{\partial \varphi_k}{\partial y_k}(\mathbf{y}_0) \end{bmatrix} \neq 0.$$
   Consider $\Pi : \mathbb{R}^N \to \mathbb{R}^k$ defined by $\Pi(\mathbf{x}) := (x_1, \ldots, x_k)$, and let $\mathbf{f} : W \to \mathbb{R}^k$ be defined by
   $$\mathbf{f}(\mathbf{y}) := \Pi(\boldsymbol{\varphi}(\mathbf{y})) = (\varphi_1(\mathbf{y}), \ldots, \varphi(\mathbf{y})),$$
   so that $\det J_{\mathbf{f}}(\mathbf{y}) \neq 0$. Then, by the Inverse Function Theorem, $\exists r > 0$ such that $B_k(\mathbf{y}_0, r) \subseteq W$, $\mathbf{f}(B_k(\mathbf{y}_0, r))$ is open, $\mathbf{f} : B_k(\mathbf{y}_0, r) \to \mathbf{f}(B_k(\mathbf{y}_0, r))$ is invertible, with inverse $\mathbf{f}^{-1}$ of class C^m. Let $\mathbf{z} = (x_1, \ldots, x_k)$; then, we can write $\mathbf{y}$ as a function of $\mathbf{z}$ ($\mathbf{y} = \mathbf{f}^{-1}(\mathbf{z})$). Let $\mathbf{w} := (x_{k+1}, \ldots, x_N)$, so that $\mathbf{x} = (\mathbf{z}, \mathbf{w})$. Since $\boldsymbol{\varphi}$ is a homeomorphism, $\boldsymbol{\varphi}(B_k(\mathbf{y}, r))$ is relatively open in M, so that it can be written as some
   $$\boldsymbol{\varphi}(B_k(\mathbf{y}, r)) = M \cap U_1$$
   for some open set $U_1 \in \mathbb{R}^N$. Then,
   $$M \cap U_1 = \{\boldsymbol{\varphi}(\mathbf{y}) : \mathbf{y} \in B_k(\mathbf{y}_0, r)\} = \{(\mathbf{z}, \varphi_{k+1}(\mathbf{f}^{-1}(\mathbf{z})), \ldots, \varphi_n(\mathbf{f}^{-1}(\mathbf{z})) : \mathbf{z} \in U_1\},$$
   so that $M \cap U_1$ is the graph of the function
   $$\mathbf{z} \in U_1 \mapsto \varphi_{k+1}(\mathbf{f}^{-1}(\mathbf{z})), \ldots, \varphi_n(\mathbf{f}^{-1}(\mathbf{z})).$$
   Let $\mathbf{g} : U_1 \to \mathbb{R}^{N-k}$ be the class C^m function defined by
   $$g_i(\mathbf{x}) := x_{k+i} - \varphi_{k+i}\mathbf{f}^{-1}(x_1, \ldots, x_k).$$

Then, $M \cap U_1 = \{\mathbf{x} \in U_1 : \mathbf{g}(\mathbf{x}) = \mathbf{0}\}$. Moreover, since, for $i, j \geq k+1$,

$$\frac{\partial g_i}{\partial x_j}(\mathbf{x}) = \frac{\partial}{\partial x_j}(x_i - \varphi_i(\mathbf{f}^{-1}(x_1, \ldots, x_k))) = \delta_{i,j} - 0.$$

$J_{bfg}(\mathbf{x})$ contains I_{N-k} as a submatrix, so that rank $J_{\mathbf{g}}(\mathbf{x}) = N - k$. ■

Surface Integrals

1. **Definition of the Surface Integral** Given a k-dimensional manifold M of class C^m, $m \in \mathbb{N}$, if $\boldsymbol{\varphi} : V \to M$ is a local chart for M with $E \subseteq \boldsymbol{\varphi}(V)$ such that $\boldsymbol{\varphi}^{-1}(E)$ is Peano-Jordan measurable and $f : E \to \mathbb{R}$ is bounded, then we define the *surface integral of f over E* as

$$\int_E d\mathcal{H}^k := \int_{\boldsymbol{\varphi}(E)} f(\boldsymbol{\varphi}(\mathbf{y})) \sqrt{\sum_{\alpha \in \Lambda_{N,k}} \left[\det \frac{\partial(\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_k})}{\partial(y_1, \ldots, y_k)}(\mathbf{y})\right]^2} \, d\mathbf{y},$$

provided the latter exists.

Divergence Theorem

1. **Definition 227 (Regular Set)** An open and bounded set $U \subset \mathbb{R}^N$ is *regular* if and only if there exists a function $\mathbf{g} \in C^1(\mathbb{R}^N)$ with $\nabla g(\mathbf{x}) = 0$ in ∂U, such that

$$U = \{\mathbf{x} \in \mathbb{R}^N : g(\mathbf{x}) = 0\},$$

$$\partial U = \{\mathbf{x} \in \mathbb{R}^N : \nabla g(\mathbf{x}) = 0\}.$$

2. **Theorem 230 (Divergence Theorem)** Let $U \subset \mathbb{R}^N$ be regular, and let $\mathbf{f} : \overline{U} \to \mathbb{R}^N$ be bounded and continuous in $\overline{U}$ such that all partial derivatives of $\mathbf{f}$ exist and are bounded and continuous in U.

$$\int_U \operatorname{div} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = \int_{\partial U} \mathbf{f}(\mathbf{x}) \cdot \nu(\mathbf{x}) \, d\mathcal{H}^{N-1}(\mathbf{x}).$$

Proof: Step 1: First, prove the Theorem in the case that U is the rectangle $R = (a_1, b_1) \times \cdots (a_N, b_N)$. Let $R' = (a_2, b_2) \times \cdots (a_N, b_N)$, and let $\mathbf{x}' = (x_2, \ldots, x_N)$. Then, by Theorem 160 (Repeated Integration) and the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_R \frac{\partial f_1}{\partial x_1}(\mathbf{x}) \, d\mathbf{x} &= \int_{R'} \left(\int_{a_1}^{b_1} \frac{\partial f_1}{\partial x_1}(x_1, \mathbf{x}') \, dx_1 \right) \, d\mathbf{x}' \\
&= \int_{R'} \left(f_1(b_1, \mathbf{x}') - f_1(a_1, \mathbf{x}') \right) \, d\mathbf{x}' \\
&= \int_{R'} \mathbf{f}(b_1, \mathbf{x}') \cdot \mathbf{e}_1 \, d\mathbf{x}' + \int_{R'} \mathbf{f}(a_1, \mathbf{x}') \cdot (-\mathbf{e}_1) \, d\mathbf{x}' \\
&= \int_{\{b_1\} \times R'} \mathbf{f}(\mathbf{x}) \cdot \nu \, d\mathcal{H}^{N-1}(\mathbf{x}) + \int_{\{a_1\} \times R'} \mathbf{f}(\mathbf{x}) \cdot \nu \, d\mathcal{H}^{N-1}(\mathbf{x})
\end{aligned}$$

Applying this argument to each component and summing the resulting identities give the desired result.

Step 2: Next, prove the result in the case that U is of the form

$$U = \{(x_1, \mathbf{x}') \in \mathbb{R} \times \mathbb{R}^{N-1} : h(\mathbf{x}') < x_1 < b_1, \mathbf{x}' \in R'\},$$

by using the Change of Variables

$$y_1 := x_1 - h(\mathbf{x}'), \quad \mathbf{y}' := \mathbf{x}'.$$

A similar argument works in the case that x_1 is replaced by some other x_i, and in the case that b_i is switched approriately by a_i.

Step 3: Now, prove that, for any $\mathbf{x} \in \partial U$, there exists $r_{\mathbf{x}} > 0$ such that $U \cap Q(\mathbf{x}, r_{\mathbf{x}})$ is of a form addressed in Step 2.

Step 4: Finally, we "stitch" together the results of Step 1, 2, and 3 to prove the result for general U. Since $\overline{U}$ is closed and bounded, it is compact, so that it has a finite open cover $\{Q(\mathbf{x}, r_{\mathbf{x}})\} = Q_1, \ldots, Q_k$. Note that, for each $\mathbf{x}$, either $Q(\mathbf{x}, r_{\mathbf{x}}) \subseteq U$ or $Q(\mathbf{x}, r_{\mathbf{x}}) \cap U$ is of the form covered by Step 2 (by the result of part 3). Thus, we construct φ_k, the partition of unity subordinated to the family of open sets $\{Q_i\}$. Note that, since in each Q_i, φ_i is constant,

$$\sum_{i=1}^{k} \frac{\partial \varphi}{\partial x_i}(\mathbf{x}) = \frac{\partial}{\partial x_i}(1) = 0$$

in U.

By the results of Steps 1 and 2, since $\varphi_i \mathbf{f}$ is nonzero only inside Q_i,

$$\int_U \operatorname{div}(\varphi_k \mathbf{f}) \, d\mathbf{x} = \int_{\partial U} \varphi \mathbf{f} \cdot \nu \, d\mathcal{H}^{N-1}.$$

Since $\sum_{i=1}^{k} \varphi_i = 1$ in $\overline{U}$, taking the sum over all i gives

$$\begin{aligned}
\int_U \operatorname{div} \mathbf{f} \, d\mathbf{x} &= \int_U \operatorname{div}\left(\sum_{i=1}^{k} \varphi_i \mathbf{f}\right) d\mathbf{x} = \sum_{i=1}^{k} \int_U \operatorname{div}(\varphi_i \mathbf{f}) \, d\mathbf{x} \\
&= \sum_{i=1}^{k} \int_{\partial U} \varphi_i \mathbf{f} \cdot \nu \, d\mathcal{H}^{N-1} = \int_{\partial U} \sum_{i=1}^{k} \varphi_i \mathbf{f} \cdot \nu \, d\mathcal{H}^{N-1} = \int_{\partial U} \mathbf{f} \cdot \nu \, d\mathcal{H}^{N-1}. \quad \blacksquare
\end{aligned}$$

3. **Corollary 234 (Integration by Parts)** Let $U \subset \mathbb{R}^N$ be regular, and let $f, g : \overline{U} \to \mathbb{R}$ be bounded and continuous, such that all partial derivates of f and g exist and are bounded and continuous in U. Then, $\forall i \in \{1, 2, \ldots, N\}$,

$$\int_U f(\mathbf{x}) \frac{\partial g}{\partial x_i}(\mathbf{x}) \, d\mathbf{x} = -\int_U g(\mathbf{x}) \frac{\partial f}{\partial x_i}(\mathbf{x}) \, d\mathbf{x} + \int_{\partial U} f(\mathbf{x}) g(\mathbf{x}) \nu_i(\mathbf{x}) \, d\mathcal{H}^{N-1}(\mathbf{x}).$$

Proof: Apply the Divergence Theorem to the function $\mathbf{f} : \overline{U} \to \mathbb{R}^N$ defined by

$$f_j(\mathbf{x}) = \begin{cases} f(\mathbf{x}) g(\mathbf{x}) & j = i, \\ 0 & j \neq i. \end{cases}$$

Then, since

$$\operatorname{div}\mathbf{f} = \sum_{j=1}^{N} \frac{\partial f_i}{\partial x_j} = \frac{\partial (fg)}{\partial x_j} = f\frac{\partial g}{\partial x_j} + g\frac{\partial f}{\partial x_j},$$

$$\int_U \left(f\frac{\partial g}{\partial x_j} + g\frac{\partial f}{\partial x_j} \right) d\mathbf{x} = \int_U \operatorname{div}\mathbf{f}\, d\mathbf{x} = \int_{\partial U} \mathbf{f}\cdot\nu\, d\mathcal{H}^{N-1} = \int_{\partial U} fg\nu_i\, d\mathcal{H}^{N-1}. \quad \blacksquare$$

4. **Proposition 237 (Mollifiers)** Given open sets $A, D \subset \mathbb{R}^N$ with D bounded and $\operatorname{dist}(A, D) \geq 3d > 0$ for some $d > 0$, there exists a function $f \in C^1(\mathbb{R}^N)$ such that $0 \leq f \leq 1$, $f(\mathbf{x}) = 0$ in A, $f(\mathbf{x}) = 1$ in D, and $\|\nabla f\| \leq \frac{C}{d}$, where $C > 0$ depends only on N.

   Proof: Construct a nonnegative function $g \in C^1(\mathbb{R}^N)$ with $g(\mathbf{x}) = 0$ in $\mathbb{R}^N \backslash B(\mathbf{0}, 1)$, $g(\mathbf{x}) > 0$ in $B(\mathbf{0}, \frac{1}{2})$, and

$$\int_{B(\mathbf{0},1)} g(\mathbf{x})\, d\mathbf{x} = 1.$$

5. **Theorem 242 (Generalization of the Divergence Theorem)** The Divergence Theorem continues to hold if $U \subset \mathbb{R}^N$ is open and bounded, and its boundary consists of sets E_1, E_2, where $\operatorname{meas}_{N-1} E_1 = 0$ and, $\forall \mathbf{x}_0 \in E_2$, for $B := B(\mathbf{x}_0, r)$, there exists $g \in C^1(B)$ with $\nabla g(\mathbf{x}) = 0$ in $B \cap \partial U$, and

$$B \cap U = \{\mathbf{x} \in B : g(\mathbf{x}) < 0\},$$

$$B \backslash \overline{U} = \{\mathbf{x} \in B : g(\mathbf{x}) > 0\},$$

$$B \cap \partial U = \{\mathbf{x} \in B : g(\mathbf{x}) = 0\}.$$

   Proof:

6. **Lemma 243 (Lemma for Generalization of the Divergence Theorem)** Let $K \subset \mathbb{R}^N$ be a compact set with $\operatorname{meas}_{N-1} K = 0$. Then, $\forall r > 0$, if

$$A_r := \{\mathbf{x} \in \mathbb{R}^N : \operatorname{dist}(\mathbf{x}, K) < r\},$$

$$\lim_{r\to 0^+} \frac{\operatorname{meas}_o(A_r)}{r} = 0.$$

   Proof: Since $\operatorname{meas}_{N-1} K = 0$, $\forall \epsilon > 0$, $\exists r_\epsilon > 0$ such that, for $0 < r \leq r_\epsilon$, there exists a finite family $\{Q(\mathbf{x}_{n,r}, r)\}$ of open cubes covering K and

$$\sum_n \left(\sqrt{N} r\right)^{N-1} = \sum_n (\operatorname{diam} Q(\mathbf{x}_{n,r}, r))^{N-1} \leq \frac{N^{(N-1)/2}}{2^N}\epsilon.$$

   By choice of A_r, if $\mathbf{x} \in A_r$, then $\exists \mathbf{y} \in K$ such that $\|\mathbf{x} - \mathbf{y}\| < r$. Since $\{Q(\mathbf{x}_{n,r}, r)\}$ covers K, $\mathbf{y}$ is in some $Q(\mathbf{x}_{n,r}, r)$, so that $\mathbf{x} \in Q(\mathbf{x}_{n,r}, 2r)$, and therefore $\{Q(\mathbf{x}_{n,r}, 2r)\}$ covers A_r. Then,

$$\frac{1}{r}\operatorname{meas}_o(A_r) \leq \frac{1}{r}\sum_n \operatorname{meas} Q(\mathbf{x}_{n,r}, 2r) = \frac{1}{r}\sum_n (2r)^N = \frac{2^N}{N^{(N-1)/2}}\sum_n \left(\sqrt{N} r\right)^{N-1} \leq \epsilon,$$

   and therefore, $\lim_{r\to 0} \frac{1}{r}\operatorname{meas}_o(A_r) = 0$. $\blacksquare$

Stokes' Theorem

1. **Theorem 244 (Stokes' Theorem)** Let $U \subseteq \mathbb{R}^3$ be an open set, and let $\mathbf{f} : U \to \mathbb{R}^3$ be of class C^1. Let $M \subseteq \mathbb{R}^2$ be a 2-dimensional manifold of class C^2, with boundary Γ, of positive orientation. Then,
$$\int_M \operatorname{curl} \mathbf{f} \cdot \nu \, d\mathcal{H}^2 = \int_\Gamma \mathbf{f}.$$
Proof: Don't need to know this one (just know how to apply the theorem).