**Math 21-236, Mathematical Studies Analysis II, Spring 2012**
**Assignment 1**

**The due date for this assignment is Monday January 30.**

1. Given $E \subseteq \mathbb{R}^N$, we recall that a function $f : E \to \mathbb{R}$ is *Lipschitz continuous* in $E$ if there exists a constant $L \geq 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq L \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in E$.

   (a) Let $U \subseteq \mathbb{R}^N$ be open and convex and let $f : U \to \mathbb{R}$ be differentiable in $U$. Prove that $f$ is Lipschiz continuous if and only if there exists a constant $M > 0$ such that

$$\|\nabla f(\mathbf{x})\| \leq M$$

   for all $\mathbf{x} \in U$.

   (b) Let $U := \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$ and consider the function

$$f(x, y) := \sqrt{x^2 + y^2} \operatorname{arccot} \frac{y}{\sqrt{x^2 + y^2} - x}.$$

   Prove that $f$ is differentiable in $U$, that there exists a constant $M > 0$ such that

$$\|\nabla f(x, y)\| \leq M$$

   for all $(x, y) \in U$ but that $f$ is not Lipschitz continuous.

2. Let $f : B(\mathbf{x}_0, r) \to \mathbb{R}$ be Lipschitz continuous.

   (a) Assume that all the directional derivatives of $f$ at $\mathbf{x}_0$ exist and that $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(\mathbf{x}_0) v_i$ for every direction $\mathbf{v}$. Prove that $f$ is differentiable at $\mathbf{x}_0$.

   (b) Assume that there exist countably many directions $\mathbf{v}^{(n)}$, with $E := \{\mathbf{v}^{(n)} : n \in \mathbb{N}\}$ dense in $\partial B(\mathbf{0}, 1)$, such that $\frac{\partial f}{\partial \mathbf{v}^{(n)}}(\mathbf{x}_0) = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \mathbf{v}_i^{(n)}$ for every direction $\mathbf{v}^{(n)}$. Prove that $f$ is differentiable at $\mathbf{x}_0$.

3. Construct a function $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}^2$ and $(0, 0) \in E$, with the properties that $f$ is continuous in $E$, differentiable in $(0, 0)$, but $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ do not exist.[1]

4. Study the continuity, the existence of directional and partial derivatives, and the differentiability of the following functions:

[1] The set $E$ should be the "natural" domain of the function, that is, the largest set where your function $f$ is defined. Taking a differentiable function and then restricting its domain would be too cheap.

(a) $f(x,y) = \begin{cases} \frac{x^m y^n}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$ where $m, n \in \mathbb{N}$,

(b) $f(x,y) = \begin{cases} \frac{x^m y^n}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$ where $m, n \in \mathbb{N}$,

(c) $f(x,y) = \begin{cases} \frac{x^m y^n}{x^2-y^2} & \text{if } x^2 - y^2 \neq 0, \\ 0 & \text{otherwise}, \end{cases}$ where $m, n \in \mathbb{N}$,

(d) $f(x,y) = \begin{cases} x^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y} & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{otherwise } y = 0. \end{cases}$