Math 21-236, Mathematical Studies Analysis II, Spring 2012
Assignment 4

The due date for this assignment is Monday March 5.

1. **A global implicit function theorem.** Let $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1. Assume that for every $x \in (a, b)$ there exists $\alpha_x > 0$ such that
$$\left|\frac{\partial f}{\partial y}(x, y)\right| \geq \alpha_x$$
for all $y \in \mathbb{R}$. Prove that there exists a unique function $g : (a, b) \to \mathbb{R}$ of class C^1 such that
$$f(x, g(x)) = 0$$
for all $x \in (a, b)$.

2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, let $x, y \in X$, with $x \neq y$, let S be the segment of endpoints x and y, that is,
$$S = \{tx + (1-t)y : t \in [0, 1]\},$$
and let $f : S \to R$ be such that f is continuous in S and there exists the directional derivative $\frac{\partial f}{\partial v}(z)$ for all $z \in S$ except at most x and y, where $v := \frac{x-y}{\|x-y\|_X}$. Prove that
$$\|f(x) - f(y)\|_Y \leq \sup_{w \in S} \left\|\frac{\partial f}{\partial v}(w)\right\|_Y \|x - y\|_X.$$

3. Let $f : [a, b] \to \mathbb{R}$ be a function of class C^2 with $f = 0$ in $[a, a + \varepsilon] \cup [b - \varepsilon, b]$. Let
$$E := \{(x, f(x)) : x \in [a, b]\}$$
and consider the function
$$g(x, y) := \operatorname{dist}((x, y), E), \quad (x, y) \in \mathbb{R}^2.$$
Let
$$U_\delta := \{(x, y) : x \in (a, b), f(x) < y < f(x) + \delta\}.$$
   (a) Prove that there exists $\delta > 0$ small with the property that for every $(x, y) \in U_\delta$ there exists a unique point $(t, s) \in E$ such that
$$\operatorname{dist}((x, y), E) = \|(x, y) - (t, s)\|.$$
   (b) Find a precise formula in terms of f and f' that relates (t, s) to (x, y).
   (c) Prove that the function g is of class C^1 in U_δ.

4. Given the equation

$$\alpha \log(1+xy) + \alpha^2 xy - 2\sin x + y - 1 = 0,$$

where $\alpha \in \mathbb{R}$ is a fixed parameter.

(a) Prove that in a neighborhood of the point $P = (0, 1)$ the equation implicitly determines a function $y = g(x)$.

(b) Determine, if they exist, those values of α for which the function g has a maximum at the point $x = 0$.