Lecture Notes for Week 3

The Principle of Uniform Boundedness

In practice, pointwise bounds are generally much easier to obtain than uniform bounds, but uniform bounds are much more useful. The *Banach-Steinhaus Theorem* (also know as the *Principle of Uniform Boundedness*) tells us that in certain important situations, pointwise bounds for a family bounded linear operators actually imply uniform bounds on the family of operator norms. Dunford & Schwartz list the Principle of Uniform Boundedness as one of *Three Basic Principles of Linear Analysis* (the other two being the *Hahn-Banach Theorem* and the *Open Mapping Theorem*).

Theorem 3.1 (Banach-Steinhaus): Let X be a Banach space and Y be a NLS. Let $(T_\alpha|\alpha \in I)$ be a family in $\mathcal{L}(X;Y)$. (Here, I can be any index set.) Assume that for every $x \in X$, we have

$$\sup\{\|T_\alpha x\| : \alpha \in I\} < \infty.$$

Then we also have

$$\sup\{\|T_\alpha\| : \alpha \in I\} < \infty.$$

Proof: For each $k \in \mathbb{N}$ let

$$A_k = \{x \in X : \|T_\alpha x\| \leq k \text{ for all } \alpha \in I\}.$$

Each A_k is a closed set because it is the intersection over α of the closed sets $\{x \in X : \|T_\alpha x\| \leq k\}$. The hypotheses of the theorem imply that

$$X = \bigcup_{k=1}^{\infty} A_k.$$

Since X is complete, the Baire Category theorem allows us to choose $N \in \mathbb{N}$ such that

$$\text{int}(A_N) \neq \emptyset.$$

Therefore, we may choose $x_0 \in A_N$, $\delta > 0$ such that $B_\delta(x_0) \subset A_N$. Then we have

$$\|T_\alpha x\| \leq N \;\text{ for all } \alpha \in I,\; x \in B_\delta(x_0).$$

Let $y \in X \backslash \{0\}$ be given and put

$$\gamma = \frac{\delta}{2\|y\|}, \quad z = x_0 + \gamma y.$$

Observe that $z \in B_\delta(x_0)$ so that

$$\|T_\alpha z\| \leq N \;\text{ for all } \alpha \in \; I.$$

Since $y = \gamma^{-1}(z - x_0)$ we have

$$T_\alpha y = \frac{1}{\gamma}\left(T_n z - T_n x_0\right) \quad \text{for all } \alpha \in I.$$

We conclude that for all $\alpha \in I$,

$$\|T_\alpha y\| \leq \frac{1}{\gamma}\left(\|T_\alpha z\| + \|T_\alpha x_0\|\right) \leq \frac{2}{\gamma} N = \frac{4N\|y\|}{\delta}.$$

Taking the supremum over all $y \in X$ with $\|y\| \leq 1$, we arrive at

$$\|T_\alpha\| \leq \frac{4N}{\delta} \quad \text{for all } \alpha \in \mathbb{N}. \quad \Box$$

Remark 3.2: The Banach-Steinhaus Theorem is also called the *Principle of Uniform Boundedness*.

Completeness of X is essential in the Banach-Steinhaus Theorem, as the following example illustrates.

Example 3.3: Let X be the set of all $\mathbb{K}$-valued sequence having finite support (i.e., the set of all sequences $x : \mathbb{N} \to \mathbb{K}$ such that $\{k \in \mathbb{N} : x_k \neq 0\}$ is finite), equipped with the norm $\|\cdot\|$ defined by

$$\|x\| = \sup\{|x_k| : k \in \mathbb{N}\},$$

and let $Y = \mathbb{K}$ equipped with $|\cdot|$. Notice that Y is complete, but X is not.

For each $n \in \mathbb{N}$ define the linear mapping $T_n : X \to Y$ by

$$T_n x = n x_n \quad \text{for all } x \in X.$$

Observe that

$$|T_n x| \leq n\|x\| \quad \text{for all } x \in X,\ n \in \mathbb{N},$$

so each $T_n \in \mathcal{L}(X, Y)$. For every $x \in X$, the sequence $(n x_n : n \in \mathbb{N})$ converges to 0 (since it has finite support), and consequently

$$\sup\{|T_n x| : n \in \mathbb{N}\} < \infty.$$

Consider the sequence $\{x^{(n)}\}_{n=1}^{\infty}$ in X defined by

$$x_k^{(n)} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

Notice that $\|x^{(n)}\| = 1$ and $|T_n x^{(n)}| = n$ for all $n \in \mathbb{N}$. It follows that $\|T_n\| \geq n$ for all $n \in \mathbb{N}$ and consequently

$$\sup\{\|T_n\| : n \in \mathbb{N}\} = \infty. \quad \Box$$

Let X be a linear space over $\mathbb{K}$. Since $\mathbb{K}$ is a linear space over itself, it is possible to study linear mappings from X to $\mathbb{K}$. Such mappings are called *linear functionals*.

Definition 3.4: The set of all linear functionals on X is called the *algebraic dual* of X and is denoted by $X^{\#}$.

Remark 3.5: It is clear that $X^{\#}$ is a linear space over $\mathbb{K}$. Other notations are frequently used for algebraic duals.

The algebraic dual of an infinite-dimensional normed linear space is generally "too large" to be useful in analysis. We will not use the algebraic dual very much in this course. We introduce a smaller dual space, known as the topological dual, or simply the dual space, which will be used extensively.

Definition 3.6: Let $(X, \|\cdot\|)$ be a normed linear space. The set of all bounded linear functionals on X is called the *dual* of X and is denoted by X^*. We equip X^* with the norm $\|\cdot\|_*$ defined by

$$\|x^*\|_* = \sup\{|x^*(x)| : x \in X,\ \|x\| \leq 1\}.$$

Remark 3.7: Since $(\mathbb{K}, |\cdot|)$ is complete, it follows from Proposition 2.23 that $(X^*, \|\cdot\|_*)$ is complete, even if $(X, \|\cdot\|)$ is not.

Remark 3.8: If there is any danger of confusion with the algebraic dual, we shall refer to X^* as the *topological dual* of X. The notation X' is frequently used to denote the topological dual of X.

Definition 3.9: The mapping $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{K}$ defined by

$$\langle x^*, x \rangle = x^*(x) \text{ for all } x \in X, x^* \in X^*$$

is called the *duality pairing* for X.

For each $x \in X$ the mapping from X^* to $\mathbb{K}$ that carries x^* into $\langle x^*, x \rangle$ is linear in x^*. This observation leads to a canonical linear injection J of X into $X^{**} = (X^*)^*$. More precisely, we define $J : X \to X^{**}$ by

$$(J(x))(x^*) = \langle x^*, x \rangle \text{ for all } x \in X, x^* \in X^*.$$

The mapping J is called the *canonical embedding of X into X^{**}*.

Remark 3.10: In order to show that J is injective we need to know that given $x \in X \backslash \{0\}$, there exists $x^* \in X^*$ such that $\langle x^*, x \rangle \neq 0$. Although the existence of such functionals may seem obvious, the proof is not completely straightforward. The Hahn-Banach theorem (to be discussed later) ensures the existence of linear functionals having many useful properties. In particular, it is a consequence of the Hahn-Banach Theorem that $\|J(x)\|_{**} = \|x\|$ for all $x \in X$.

Definition 3.11: A normed linear space $(X, \|\cdot\|)$ is said to be *reflexive* if the canonical embedding of X into X^{**} is surjective.

Reflexivity is a very important and subtle concept. We shall explore it extensively later on in the course.

Sequence Spaces

We now give a systematic introduction to some standard spaces of sequences of elements of $\mathbb{K}$. The spaces are very useful for constructing interesting examples.

We denote by $\mathbb{K}^{\mathbb{N}}$ the set of all sequences $x : \mathbb{N} \to \mathbb{K}$. It is clear that $\mathbb{K}^{\mathbb{N}}$ is a linear space over $\mathbb{K}$. There is no standard norm for this space.

We denote by $\mathbb{K}^{(\mathbb{N})}$ the set of all members of $\mathbb{K}^{\mathbb{N}}$ having finite support, i.e. the set of all $x \in \mathbb{K}^{\mathbb{N}}$ such that $\{n \in \mathbb{N} : x_n \neq 0\}$ is finite. It is clear that $\mathbb{K}^{(\mathbb{N})}$ is a linear space over $\mathbb{K}$. For each $n \in \mathbb{N}$ we define $e^{(n)} \in \mathbb{K}^{(\mathbb{N})}$ by

$$e_k^{(n)} = \begin{cases} 1 & \text{if } k = n \\ \\ 0 & \text{if } k \neq n. \end{cases}$$

Remark 3.12: It is easy to see that $\{e^{(n)}\}_{n=1}^{\infty}$ is a Hamel basis for $\mathbb{K}^{(\mathbb{N})}$. It follows that there is no norm that will render $\mathbb{K}^{(\mathbb{N})}$ complete.

The space l^∞: We denote by l^∞ the set of all bounded members of $\mathbb{K}^{\mathbb{N}}$. It is clear that l^∞ is a linear space over $\mathbb{K}$. We equip l^∞ with the norm defined by

$$\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}.$$

It is easy to check that $\|\cdot\|_\infty$ is a norm and it is straightforward to show that $(l^\infty, \|\cdot\|_\infty)$ is complete.

The space c: We denote by c the set of all convergent members of $\mathbb{K}^{\mathbb{N}}$ and we equip c with $\|\cdot\|_\infty$. It is clear that c is a linear space over $\mathbb{K}$. Notice that $c \subset l^\infty$. It is straightforward to show that $(c, \|\cdot\|_\infty)$ is complete. Consequently c is a closed subspace of $(l^\infty, \|\cdot\|_\infty)$.

The space c_0: We denote by c_0 the set of all $x \in \mathbb{K}^{\mathbb{N}}$ such that $x_n \to 0$ as $n \to \infty$. It is clear that c_0 is a linear space. Notice that $c_0 \subset c$. It is straightforward to show that $(c_0, \|\cdot\|_\infty)$ is complete. Consequently c_0 is a closed subspace of $(l^\infty, \|\cdot\|_\infty)$ and of $(c, \|\cdot\|_\infty)$.

The spaces l^p, $1 \le p < \infty$: For each $p \in [1, \infty)$ we denote by l^p the set of all $x \in \mathbb{K}^{\mathbb{N}}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty,$$

and we define $\|\cdot\|_p : l^p \to \mathbb{R}$ by

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

It is clear that l^1 is a linear space over $\mathbb{K}$ and that $\|\cdot\|_1$ is a norm on l^1. It is not difficult to show that $(l^1, \|\cdot\|_1)$ is complete. It is true that l^p is a linear space over $\mathbb{K}$ and that $\|\cdot\|_p$ is a norm on l^p for each $p \in [1, \infty)$, but this is not completely obvious and a proof will be given shortly. These spaces are also complete. Notice that for $1 \le p_1 \le p_2 < \infty$ we have

$$l^{p_1} \subset l^{p_2} \subset c_0 \subset c \subset l^{\infty}.$$

Remark 3.13: Let $p \in [1, \infty]$ and $x \in l^p$ be given. It is straightforward to show that

$$\|x\|_p \ge \|x\|_{\infty}.$$

If we allow for the norms to take infinite values, then this inequality is valid for all $x \in \mathbb{K}^{\mathbb{N}}$. Moreover, if there exists $q \ge 1$ such $x \in l^p$ for all $p > q$ then

$$\|x\|_p \to \|x\|_{\infty} \quad \text{as } p \to \infty.$$

Before proving that l^p is in fact a linear space and that $\|\cdot\|$ is a norm, we record some important properties of the sequence spaces introduced above.

Remark 3.14: From now on, when we talk about the spaces c, c_0, and l^p, $1 \le p \le \infty$ it is to be understood that they are equipped with the norms described above.

Properties of the Sequence Spaces

Completeness: c, c_0, and l^p, $1 \le p \le \infty$ are complete.

Separability: c, c_0, and l^p, $1 \le p < \infty$ are separable. l^{∞} is not separable.

Schauder Bases: The sequence $\{e^{(n)}\}_{n=1}^{\infty}$ is a Schauder basis for c_0 and for l^p, $1 \le p < \infty$. The space c has Schauder bases; you are asked to find one in Assignment 3.

Reflexivity: The spaces l^p, $1 < p < \infty$ are reflexive. The spaces c, c_0, l^1, and l^{∞} are not reflexive.

Remark 3.15:

(a) Let X denote one of the spaces $c, c_0, l^p, 1 \leq p \leq \infty$. To prove that X is complete, one takes a Cauchy sequence $\{x^{(n)}\}_{n=1}^{\infty}$ in X and observes that for each $k \in \mathbb{N}$, $\{x_k^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{K}$; since $\mathbb{K}$ is complete, one may choose $x_k \in \mathbb{K}$ such that $x_k^{(n)} \to x_k$ as $n \to \infty$. One then shows that the sequence $x \in \mathbb{K}^{\mathbb{N}}$ satisfies $x \in X$ and $\|x^{(n)} - x\|_X \to 0$ as $n \to \infty$.

(b) Separability of c, c_0, and l^p, $1 \leq p < \infty$ follows from the existence of Schauder bases.

(c) To see that l^∞ is not separable, let $S = \{x \in l^\infty : x_n \in \{0,1\}$ for all $n \in \mathbb{N}\}$ and observe that S is uncountable. Observe further that $\|x - y\|_\infty = 1$ for all $x, y \in S$ with $x \neq y$. Consequently, each (open) ball of radius 1 contains at most one element of S and we conclude that no countable collection of open balls of radius 1 can cover l^∞. Proposition M8 implies that l^∞ is not separable.

In order to prove that the spaces l^p, $1 < p < \infty$ are linear spaces and that $\|\cdot\|_p$ is a norm on l^p, it suffices to prove that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \text{ for all } x, y \in l^p.$$

The other properties are immediate. The inequality above is called Minkowski's inequality. In order to establish this inequality, we start with an important algebraic inequality, known as Young's inequality, and then prove a fundamental inequality, called Holder's inequality, for l^p spaces. Holder's inequality will be used to establish Minkowski's inequality.

Proposition 3.16 (Young's Inequality): Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \geq 0$ be given. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

(with equality if and only if $b = a^{p-1}$).

Remark 3.17: For each $p \in (1, \infty)$ there is exactly one $q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, namely

$$q = \frac{p}{p-1}.$$

Proof of Young's Inequality: We assume that $a, b > 0$. (If $a = 0$ or $b = 0$, there is nothing to prove.) Put $\gamma = p - 1$. In the $x - y$ plane, consider the rectangle with vertices $(0,0), (a,0), (a,b), (0,b)$ and sketch the graph of $y = x^\gamma$ for $0 \leq x \leq \max\{a, b^{1/\gamma}\}$. By looking at areas we see that

$$ab \leq \int_0^a x^\gamma dx + \int_0^b y^{1/\gamma} dy,$$

(with equality if and only if $a = b^{1/\gamma}$.) Evaluating the integrals we obtain

$$ab \leq \frac{a^{\gamma+1}}{\gamma+1} + \frac{b^{\frac{1}{\gamma}+1}}{\frac{1}{\gamma}+1}.$$

Substituting $p = \gamma + 1$ and $q = \frac{p}{p-1}$ gives the desired result. $\square$

Notice that for $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$, Young's Inequality implies the existence of constants $C_p, K_q > 0$ such that

$$ab \leq C_p a^p + K_q b^q \text{ for all } a, b \geq 0.$$

In particular the standard form of Young's Inequality uses $C_p = p^{-1}$ and $K_q = q^{-1}$. If either one of these constants is reduced, without increasing the other one, then the inequality above will fail. In applications it is sometimes essential to reduce one of the constants, say C_p, at the expense of increasing the other. There is a simple scaling argument to see how this works.

Remark 3.18: Given $\epsilon > 0$, $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$, and $a, b \geq 0$, observe that

$$\begin{aligned} ab &= [(\epsilon p)^{\frac{1}{p}} a][(\epsilon p)^{-\frac{1}{p}} b] \\ &\leq \epsilon a^p + \frac{b^q}{q(\epsilon p)^{\frac{q}{p}}}. \end{aligned}$$

Proposition 3.19 (Holder's Inequality): Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in l^p, y \in l^q$ be given. Then

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q.$$

Proof: We may assume that $x \neq 0$ and $y \neq 0$. Let us put

$$u = \frac{x}{\|x\|_p}, \quad v = \frac{y}{\|y\|_q}.$$

Notice that $u \in l^p$, $v \in l^q$, and $\|u\|_p = \|v\|_q = 1$, from which we conclude that

$$\frac{\|u\|_p^p}{p} + \frac{\|v\|_q^q}{q} = 1.$$

It suffices to show that

$$\sum_{i=1}^{\infty} |u_i v_i| \leq 1.$$

By Young's Inequality, we have

$$|u_i v_i| \leq \frac{|u_i|^p}{p} + \frac{|v_i|^q}{q} \text{ for all } i \in \mathbb{N}.$$

Let $N \in \mathbb{N}$ be given. Then we have

$$\sum_{i=1}^{N} |u_i v_i| \leq \frac{1}{p} \sum_{i=1}^{N} |u_i|^p + \frac{1}{q} \sum_{i=1}^{N} |v_i|^q \leq \frac{\|u\|_p^p}{p} + \frac{\|v\|_q^q}{q} = 1.$$

Letting $N \to \infty$, we conclude that

$$\sum_{i=1}^{\infty} |u_i v_i| \leq 1 \quad \square.$$

Remark 3.20: Holder's inequality remains valid for $p = 1$ and $q = \infty$. The proof is immediate.

Proposition 3.21 (Minkowski's Inequality): Let $p \in [1, \infty]$ be given. Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \text{ for all } x, y \in l^p.$$

Proof: The cases $p = 1$ and $p = \infty$ are immediate, so we assume that $p \in (1, \infty)$. Also, if $x + y = 0$ there is nothing to prove, so we assume that $x + y \neq 0$. Let $N \in \mathbb{N}$ be given, sufficiently large so that

$$\sum_{i=1}^{N} |x_i + y_i|^p > 0.$$

Choose $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have

$$\begin{aligned}
\sum_{i=1}^{N} |x_i + y_i|^p &= \sum_{i=1}^{N} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\
&\leq \sum_{i=1}^{N} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{N} |y_i||x_i + y_i|^{p-1} \\
&\leq \left(\sum_{i=1}^{N} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{N} |x_i + y_i|^p \right)^{1/q} + \left(\sum_{i=1}^{N} |y_i|^p \right)^{1/p} \left(\sum_{i=1}^{N} |x_i + y_i|^p \right)^{1/q}.
\end{aligned}$$

Here we have used Holder's inequality and the fact that $(p-1)q = p$. Dividing both sides by

$$\left(\sum_{i=1}^{N} |x_i + y_i|^p \right)^{1/q},$$

we obtain

$$\left(\sum_{i=1}^N |x_i + y_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{i=1}^N |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^N |y_i|^p\right)^{1/p}.$$

Since $1 - \frac{1}{q} = \frac{1}{p}$ we conclude that

$$\left(\sum_{i=1}^N |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^N |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^N |y_i|^p\right)^{1/p}.$$

Letting $N \to \infty$ we obtain the desired result. $\square$

The Open Mapping Theorem

The main tool used in the proof of the Principle of Uniform Boundedness was the Baire Category Theorem. Another very important consequence of the Baire Category Theorem is the so-called Open Mapping Theorem. This theorem says that if a bounded linear mapping between two Banach spaces is surjective, then it maps open sets to open sets.

Theorem 3.22 (Open Mapping Theorem): Let X, Y be Banach spaces, $T \in \mathcal{L}(X;Y)$ be given and assume that is surjective. Let $\mathcal{O}$ be an open subset of X. Then $\{Tx : x \in \mathcal{O}\}$ is an open subset of Y.

Before proving this theorem, we record a very useful (and immediate) corollary and make a few observations.

Corollary 3.23: (Bounded Inverse Theorem) Let X, Y be Banach spaces and assume that $T \in \mathcal{L}(X;Y)$ is bijective. Then T^{-1} is bounded.

By combining Remark 2.14 with Corollary 3.23, we obtain the following result.

Corollary 3.24: Let X be a linear space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ both are complete. Assume that there exists $M \in \mathbb{R}$ such that

$$\|x\|_2 \leq \|x\|_1 \quad \text{for all } x \in X.$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

In Corollary 3.24, it is essential that X be complete under both norms, as the following example shows.

Example 3.25: Let $\mathbb{K} = \mathbb{R}$ and $X = C([0,1];\mathbb{R})$ (i.e., the set of continuous functions from $[0,1]$ to $\mathbb{R}$). Consider the norms on X given by

$$\|f\|_\infty = \max\{|f(x)| : x \in [0,1]\}, \quad \|f\|_1 = \int_0^1 |f(x)|\,dx, \quad \text{for all } f \in X.$$

It is easy to see that $(X, \|\cdot\|_\infty)$ is complete because this norm corresponds to uniform convergence. The space $(X, \|\cdot\|_1)$ is incomplete. Observe that

$$\|f\|_1 \leq \|f\|_\infty \ \text{ for all } f \in X.$$

However, it is easy to see that these two norms are not equivalent. Consider the sequence $\{f_n\}_{n=1}^\infty$ of continuous functions defined by

$$f_n(x) = \begin{cases} n - n^2 x & \text{for } 0 \leq x \leq \frac{1}{n} \\ \\ 0 & \text{for } \frac{1}{n} \leq x \leq 1, \end{cases}$$

and observe that $\|f_n\|_\infty = n$ and $\|f_n\|_1 = 1$ for all $n \in \mathbb{N}$.

Remark 3.26: Let $X = C([0,1];\mathbb{R}$ equipped with $\|\cdot\|_\infty$, $Y = C([0,1];\mathbb{R})$ equipped with $\|\cdot\|_1$, and define $T \in \mathcal{L}(X;Y)$ by $Tf = f$ for all $f \in X$. Clearly T is bijective. Example 3.25 shows that T does not map every open set in X to an open set in Y because T^{-1} is not continuous. This shows that completeness of the space Y is essential in the Open Mapping Theorem. You will be asked what happens to the Open Mapping Theorem in Y is complete but X is incomplete in Assignment 3.

It is easy to see that the conclusion of the Banach-Steinhaus can fail if T is not surjective – even if X and Y are both complete.

Example 3.27: Let X and Y be Banach spaces and assume that $Y \neq 0$. Define $T : X \to Y$ by $Tx = 0$ for all $x \in X$. Clearly T is linear and continuous, but $\{Tx : x \in X\} = \{0\}$ which is not open, even though X is open.