Homework 3

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36-705 Intermediate Statistics

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1. For all $n \in \mathbb{N}$, $F \in \mathcal{F}_n$, since $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, $\{C \cap F : C \in \mathcal{C}\} = \{A \cap F : A \in \mathcal{A}\} \cup \{B \cap F : B \in \mathcal{B}\}$, and hence $s(\mathcal{C}, F) \leq s(\mathcal{A}, F) + s(\mathcal{B}, F)$. Thus,

$$s_n(\mathcal{C}) = \sup_{F \in \mathcal{F}_n} s(\mathcal{C}, F) \leq \sup_{F \in \mathcal{F}_n} s(\mathcal{A}, F) + s(\mathcal{B}, F) \leq \sup_{F \in \mathcal{F}_n} s(\mathcal{A}, F) + \sup_{F \in \mathcal{F}_n} s(\mathcal{B}, F) = s_n(\mathcal{A}) + s_n(\mathcal{B}).$$

2. For all $n \in \mathbb{N}$, $F \in \mathcal{F}_n$, $\{C \cap F : C \in \mathcal{C}\} = \{(A \cap F) \cup (B \cap F) : C \in \mathcal{C}\}$, and hence $s(\mathcal{C}, F) \leq s(\mathcal{A}, F) s(\mathcal{B}, F)$. Thus,

$$s_n(\mathcal{C}) = \sup_{F \in \mathcal{F}_n} s(\mathcal{C}, F) \leq \sup_{F \in \mathcal{F}_n} s(\mathcal{A}, F) s(\mathcal{B}, F) \leq \left(\sup_{F \in \mathcal{F}_n} s(\mathcal{A}, F) \right) \left(\sup_{F \in \mathcal{F}_n} s(\mathcal{B}, F) \right) = s_n(\mathcal{A}) s_n(\mathcal{B}).$$

3. For all $n, m \in \mathbb{N}$, $F \in \mathcal{F}_{n+m}$, there exist disjoint sets $G_F \in \mathcal{F}_n$ and $H_F \in \mathcal{F}_m$ such that $F = G_F \cup H_F$, and hence $C \cap F = (C \cap G_F) \cup (C \cap H_F)$, for all $C \in \mathcal{C}$. Hence,

$$\begin{aligned} s_{n+m}(\mathcal{C}) = \sup_{F \in \mathcal{F}_{n+m}} s(\mathcal{C}, F) &= \sup_{F \in \mathcal{F}_{n+m}} s(\mathcal{C}, G_F) s(\mathcal{C}, H_F) \\ &\leq \sup_{G \in \mathcal{F}_n} s(\mathcal{C}, G) \sup_{H \in \mathcal{F}_m} s(\mathcal{C}, H) = s_n(\mathcal{C}) s_m(\mathcal{C}). \end{aligned}$$

4. The VC dimension of $\mathcal{A}$ is 4.

   Suppose $n = 4$, and suppose $x_1, \dots, x_n \in \mathbb{R}$ with $x_1 < x_2 < x_3 < x_4$. It is clear that $\mathcal{A}$ picks any subset of $F := \{x_1, x_2, x_3, x_4\}$ of cardinality 0, 1, 2, or 4. In any subset of cardinality 3, at least two points must be consecutive. Thus, $\mathcal{A}$ also picks out any subset of cardinality 3, and hence $2^n \geq s_n(\mathcal{A}) \geq s(\mathcal{A}, F) = 2^n$, so that $s_n(\mathcal{A}) = 2^n$ $d \geq 4$.

   Suppose, on the other hand, that $n \geq 5$. Then, $\forall x_1, \dots, x_n \in \mathbb{R}$ with $x_1 < \cdots < x_n$, $\mathcal{A}$ cannot pick out the subset $\{x_1, x_3, x_5\}$. Hence, $s_5(\mathcal{A}) < 2^n$, and $d < 5$. ■

5. Problem removed.

6. Let $\mu_n := \mathbb{E}[X_n]$. Note that

$$\begin{aligned} \mathbb{E}[(X_n - b)^2] &= \mathbb{E}[(X_n - \mu_n + \mu_n - b)^2] \\ &= \mathbb{E}[(X_n - \mu_n)^2] + 2\mathbb{E}[(X_n - \mu_n)(\mu_n - b)] + \mathbb{E}[(\mu_n - b)^2] = \operatorname{Var}[X_n] + (\mu_n - b)^2. \end{aligned}$$

   All terms above are non-negative, and hence, as $n \to \infty$, the left-hand side vanishes if and and only if both terms on the right-hand side vanish. ■

7. Since L_2 convergence implies convergence in probability, it suffices to show that $n^{-1} \sum_{i=1}^n X_i^2 \to p$ in L_2 as $n \to \infty$. Hence, by the previous problem, it suffices to observe that, as $n \to \infty$,

$$\mathbb{E}\left[n^{-1} \sum_{i=1}^n X_i^2 \right] = \mathbb{E}[X_1^2] = p \quad \text{and that} \quad \operatorname{Var}\left[n^{-1} \sum_{i=1}^n X_i^2 \right] = n^{-1} \operatorname{Var}[X_1^2] \to 0. \quad ■$$

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8. (a) For any ε, as $n \to \infty$,

$$\mathbb{P}[X_n \geq \varepsilon] \leq 1 - \mathbb{P}[X_n = 0] = 1 - e^{-1/n}\frac{1/n^0}{0!} = 1 - e^{-1/n} \to 0. \quad \blacksquare$$

(b) For any ε, as $n \to \infty$, as above,

$$\mathbb{P}[nX_n \geq \varepsilon] \leq 1 - \mathbb{P}[nX_n = 0] \leq 1 - \mathbb{P}[X_n = 0] \to 0. \quad \blacksquare$$

9. Suppose X_n approaches X in distribution. For all integers k, since X is integer valued, F is continuous at $k + 1/2$ and $k - 1/2$, and so $F_n(k + 1/2) \to F(k + 1/2)$ and $F_n(k + 1/2) \to F(k + 1/2)$, as $n \to \infty$. Since $\mathbb{P}[X_n = k] = F(k + 1/2) - F(k - 1/2)$ and each $\mathbb{P}[X_n = k] = F_n(k + 1/2) - F_n(k - 1/2)$, $\mathbb{P}[X_n = k] \to \mathbb{P}[X_n = k]$.

   Suppose now that, $\forall k \in \mathbb{Z}$, $\mathbb{P}[X_n = k] \to \mathbb{P}[X_n = k]$ as $n \to \infty$. Since, $\forall x \in \mathbb{R}$, $F(x) = \sum_{i=1}^{\lfloor x \rfloor} \mathbb{P}[X = k]$ and each $F_n(x) = \sum_{i=1}^{\lfloor x \rfloor} \mathbb{P}[X_n = k]$, $F_n(x) \to F(x)$ as $n \to \infty$.

10. Note that, by the Central Limit Theorem,

$$\begin{bmatrix} \sqrt{n}(\overline{X}_1 - \mu_1) \\ \sqrt{n}(\overline{X}_2 - \mu_2) \end{bmatrix} \to \mathcal{N}(0, \Sigma),$$

in distribution, and that, for $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x_1, x_2) = x_1/x_2$, $\nabla_\mu := \nabla g(\mu_1, \mu_2) = (1/\mu_2, -\mu_1/\mu_2^2)^T$. Hence, assuming $\mu_1, \mu_2 \neq 0$, by the Multivariate Delta Method,

$$\sqrt{n}(Y_n - \mu_1/\mu_2) = \sqrt{n}(g(\overline{X}_1, \overline{X}_2) - g(\mu_1, \mu_2)) \to \mathcal{N}(0, \nabla_\mu^T \Sigma \nabla_\mu).$$