Lecture Notes for Week 3 (First Draft)

Invariant and Reducing Subspaces

Let X be a Hilbert space, M be a closed subspace of X, and $A \in \mathcal{L}(X;X)$. The Projection Theorem says that

$$X = M \oplus M^{\perp}.$$

In order to understand the behavior of the operator A it is useful break x and Ax down into their components in M and $M^{\perp}$:

$$\begin{pmatrix} P_M(Ax) \\ \\ P_{M^{\perp}}(Ax) \end{pmatrix} = \begin{pmatrix} B & C \\ \\ D & F \end{pmatrix} \begin{pmatrix} P_M x \\ \\ P_{M^{\perp}} x \end{pmatrix}, \tag{1}$$

where $B \in \mathcal{L}(M;M)$,$C \in \mathcal{L}(M^{\perp};M)$, $D \in \mathcal{L}(M;M^{\perp})$, and $F \in \mathcal{L}(M^{\perp};M^{\perp})$.

Definition 3.1 Let X be a Hilbert space, M be a closed subspace of X, and $A \in \mathcal{L}(X;X)$ be given. We say that

(a) M is invariant under A provided $A[M] \subset M$.

(b) M reduces A provided that M and $M^{\perp}$ both are invariant under A.

Notice that with regard to the decomposition (1) we have

- M is invariant under $A \Leftrightarrow D = 0$,
- $M^{\perp}$ is invariant under $A \Leftrightarrow C = 0$,
- M reduces $A \Leftrightarrow (C = 0$ and $D = 0)$.

Proposition 3.2: Let X be a Hilbert space, M be a closed subspace of X, and $A \in \mathcal{L}(X;X)$ be given. Then M is invariant under A if and only if $M^{\perp}$ is invariant under A^*.

Proof: Let $x \in M,\ y \in M^{\perp}$ be given. Since

$$(x, A^*y) = (Ax, y),$$

we have

$$(x, A^*y) = 0 \ \Leftrightarrow\ (Ax, y) = 0,$$

and the result follows. $\square$

Proposition 3.3: Let X be a Hilbert space, M be a closed subspace of X, and $A \in \mathcal{L}(X;X)$ be given. Then M is invariant under A if and only if $P_M A P_M = A P_M$.

Proof: Assume first that M is invariant under A and let $x \in X$ be given. Then $P_M x \in M$. Since M is invariant under A we have $A P_M x \in M$ so $P_M A P_M x = A P_M x$. Assume now that $P_M A P_M = A P_M$. Let $x \in M$ be given. Then

$$Ax = AP_M x = P_M A P_M x \in M,$$

so that M is invariant under A. $\square$

Theorem 3.4: Let X be a Hilbert space, M be a closed subspace of X, and $A \in \mathcal{L}(X;X)$ be given. The following three statements are equivalent.

(i) M reduces A.

(ii) $P_M A = A P_M$.

(iii) M is invariant under both A and A^*

Proof: It follows from Proposition 3.2 that (i) holds if and (iii) holds, so we only need to prove (i) $\Leftrightarrow$ (ii). Assume that (i) holds and let $x \in X$ be given. Then we have

$$x = P_M x + (I - P_M)x,$$

and consequently

$$Ax = AP_M x + A(I - P_M)x. \tag{2}$$

Since M and $M^\perp$ are both invariant under A, we have

$$AP_M x \in M, \quad A(I - P_M)x \in M^\perp.$$

On the other hand, we also have

$$Ax = P_M(Ax) + (I - P_M)(Ax). \tag{3}$$

Since there is exactly one decomposition of Ax into a sum of an element of M and an element of $M^\perp$ we conclude that

$$AP_M x = P_M Ax \;\text{ for all } x \in X,$$

and consequently (ii) holds.

Assume now that (ii) holds. Then we have

$$P_M A P_M = A P_M^2 = A P_M.$$

Since $(P_M)^* = P_M$ and $(AP_M)^* = (P_M)^* A^*$ and $(P_M A)^* = A^*(P_M)^*$ it follows from (ii) that

$$P_M A^* = A^* P_M.$$

Consequently we have

$$P_M A^* P_M = A^* P_M.$$

It follows from Proposition 3.3 that M is invariant under A and A^* Using Proposition 3.2, we conclude that M reduces A. $\square$

Spectral Theory

Definition 3.5: Let X be a Hilbert space over $\mathbb{K}$ and $A \in \mathcal{L}(X; X)$ be given.

(a) The *resolvent set* of A, denoted $\rho(A)$ is defined by

$$\rho(A) = \{\lambda \in \mathbb{K} : \lambda I - A \text{ is bijective}\}.$$

(b) The *spectrum* of A, denoted $\sigma(A)$ is defined by

$$\sigma(A) = \mathbb{K} \backslash \rho(A).$$

The definition of resolvent set given above is appropriate only for bounded linear operators from a complete space to itself. (It is appropriate for Banach spaces as well as Hilbert spaces.) We will give a more general definition of resolvent set later on in the course that applies to incomplete normed linear spaces and to linear operators that need not be continuous.

Observe that by the Bounded Inverse Theorem, if $\lambda \in \rho(A)$ then $(\lambda I - A)^{-1}$ is bounded.

Definition 3.6: Let X be a Hilbert space over $\mathbb{K}$ and $A \in \mathcal{L}(X; X)$ be given.

(a) A number $\lambda \in \mathbb{K}$ is called and *eigenvalue* for A provided that $\mathcal{N}(\lambda I - A) \neq \{0\}$.

(b) If λ is an eigenvalue for A, then the nonzero elements of $\mathcal{N}(\lambda I - A)$ are called *eigenvectors* associated with λ.

(c) The set of all eigenvalues of A is called the *point spectrum* of A and is denoted by $\sigma_p(A)$.

Notice that

$$\sigma_p(A) \subset \sigma(A).$$

If X is finite dimensional then $\sigma_p(A) = \sigma(A)$. However, it is easy to give examples in infinite-dimensional Hilbert spaces where the spectrum of a bounded linear operator contains elements that are not eigenvalues.

Definition 3.7: Let X be a Hilbert space over $\mathbb{K}$ and $A \in \mathcal{L}(X; X)$ be given. A number $\lambda \in \mathbb{K}$ is called a generalized eigenvalue for A provided that

$$\inf\{\|(\lambda I - A)x\| : x \in X,\ \|x\| = 1\} = 0.$$

The same definitions of eigenvalue, eigenvector, point spectrum, and generalized eigenvector apply to bounded linear operators from a Banach space X to itself.

Remark 3.8: Notice that $\lambda \in \mathbb{K}$ is a generalized eigenvalue for $A \in \mathcal{L}(X; X)$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $(\lambda I - A)x_n \to 0$ as $n \to \infty$.

Every generalized eigenvalue for A belongs to $\sigma(A)$. If X is finite dimensional then every generalized eigenvalue is an eigenvalue. However, it is easy to give examples in infinite dimensions of a generalized eigenvalue that is not an eigenvalue.

Example 3.9: Let $X = l^2$ and define $A \in \mathcal{L}(L^2; l^2)$ by

$$Ax = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_k}{k}, \cdots) \text{ for all } x \in X.$$

It is straightforward to check that

$$\sigma_p(A) = \{\frac{1}{k} : k \in \mathbb{N}\}.$$

If $\{e^{(n)}\}_{n=1}^{\infty}$ denotes the standard orthonormal basis for l^2, then

$$\|Ae^{(n)}\| = \frac{1}{n} \to 0 \text{ as } n \to \infty,$$

and consequently 0 is a generalized eigenvalue, but not an eigenvalue.

Compact Operators

Compact operators have special properties with regard to spectral analysis.

Proposition 3.10: Let X be a Hilbert space, $\lambda \in \mathbb{K}\backslash\{0\}$, and $A \in \mathcal{C}(X; X)$ be given. Then $\mathcal{N}(\lambda I - A)$ is finite dimensional.

Proof: Suppose that $\mathcal{N}(\lambda I - A)$ is infinite dimensional. Then we may choose an orthonormal sequence $\{z_n\}_{n=1}^{\infty}$ such that $z_n \in \mathcal{N}(\lambda I - A)$ for all $n \in \mathbb{N}$. Then $z_n \rightharpoonup 0$ (weakly) as $n \to \infty$. Since A is compact, we can conclude that $Az_n \to 0$ (strongly) as $n \to \infty$. Since $\lambda z_n = Az_n$ for all $n \in \mathbb{N}$ and $\lambda \neq 0$ we see that $z_n \to 0$ (strongly) as $n \to \infty$, which is impossible since $\|z_n\| = 1$ for all $n \in \mathbb{N}$. $\square$

Proposition 3.11: Let X be a Hilbert space, $\lambda \in \mathbb{K}\backslash\{0\}$, and $A \in \mathcal{C}(X; X)$ be given. Then $\mathcal{R}(\lambda I - A)$ is closed.

Proof: Let $y \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in X be given such that

$$(\lambda I - A)x_n \to y \ \text{ as } n \to \infty.$$

Put $M = \mathcal{N}(\lambda I - A)^{\perp}$ and

$$z_n = P_M x_n \ \text{ for all } n \in \mathbb{N}.$$

Since $x_n - z_n \in \mathcal{N}(\lambda I - A) = M^{\perp}$ for all $n \in \mathbb{N}$ we see that

$$(\lambda I - A)z_n \to y \ \text{ as } n \to \infty.$$

I claim that $\{z_n\}_{n=1}^{\infty}$ is bounded. Indeed, suppose $\{z_n\}_{n=1}^{\infty}$ is unbounded and choose a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that

$$\|z_{n_k}\| > k \ \text{ for all } k \in \mathbb{N}.$$

Put

$$w_k = \frac{z_{n_k}}{\|z_{n_k}\}|} \ \text{ for all } k \in \mathbb{N},$$

and observe that

$$\|w_k\| = 1 \text{ for all } k \in \mathbb{N} \ \text{ and } (\lambda I - A)w_k \to 0 \text{ (strongly) as } n \to \infty.$$

Since A is compact, we may choose a sequence $\{w_{k_j}\}_{j=1}^{\infty}$ converges strongly as $k \to \infty$. Since $\lambda w_{k_j} = A w_{k_j}$ for all $j \in \mathbb{N}$ and $\lambda \neq 0$ we conclude that $\{w_{k_j}\}_{j=1}^{\infty}$ is strongly convergent; put

$$w = \lim_{j \to \infty} w_{k_j}.$$

Then $\|w\| = 1$, $w \in \mathcal{N}(\lambda I - A)^{\perp}$, and $(\lambda I - A)w = 0$, which is impossible. We conclude that $\{z_n\}_{n=1}^{\infty}$ is bounded.

Since $\{z_n\}_{n=1}^{\infty}$ is bounded, we may choose a subsequence $\{z_{n_j}\}_{j=1}^{\infty}$ such that $\{Az_{n_j}\}_{j=1}^{\infty}$ is strongly convergent. Since $\lambda \neq 0$ we conclude that $\{z_{n_j}\}_{j=1}^{\infty}$ is strongly convergent to some $z \in X$ and consequenty $(\lambda I - A)z = y$ and $y \in \mathcal{R}(\lambda I - A)$. $\square$

Proposition 3.12: Let X be a Hilbert space, $\lambda \in \mathbb{K}\backslash\{0\}$, and $A \in \mathcal{C}(X;X)$ be given and assume that λ is a generalized eigenvalue for A. Then $\lambda \in \sigma_p(A)$.

Proof: Choose a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $(\lambda I - A)x_n \to 0$ as $n \to \infty$. Since A is compact, we may choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\{Ax_{n_k}\}_{k=1}^{\infty}$ is strongly convergent. Since $\lambda \neq 0$ and $\{(\lambda I - A)x_{n_k}\}_{k=1}^{\infty}$ is strongly convergent, we conclude that $\{x_{n_k}\}_{k=1}^{\infty}$ is strongly convergent; put

$$x = \lim_{k \to \infty} x_{n_k}.$$

Then, we have $\|x\| = 1$ and $(\lambda I - A)x = 0$, so that $\lambda \in \sigma_p(A)$. $\square$

Corollary 3.13: Let X be a Hilbert space, $\lambda \in \mathbb{K}\backslash\{0\}$, and $A \in \mathcal{C}(X;X)$ be given. Assume that $\lambda \notin \sigma_p(A)$ and $\overline{\lambda} \notin \sigma_p(A^*)$. Then $\lambda \in \rho(A)$.

Proof: Since $\mathcal{R}(\lambda I - A)$ is closed, we have

$$\mathcal{R}(\lambda I - A) = \mathcal{N}(\overline{\lambda} I - A^*)^\perp = \{0\}^\perp = X,$$

so that $(\lambda I - A)$ is surjective. Since $\lambda \notin \sigma_p(A)$, we also have $(\lambda I - A)$ is also injective. $\square$

The assumption that $\overline{\lambda} \notin \sigma_p(A^*)$ in Corollary 3.13 is actually redundant. We shall prove this later on. However, for developing spectral prroperties of compact normal operators, Corollary 3.13 in the form given above will be sufficient.

Some Spectral Properties of Normal Operators

Proposition 3.14: Let X be a Hilbert space, $A \in \mathcal{L}(X;X)$ and $\lambda \in \mathbb{K}$ be given. Assume that A is normal. Then

$$\mathcal{N}(\lambda I - A) = \mathcal{N}(\overline{\lambda} I - A^*) \tag{4}$$

and $\mathcal{N}(\lambda I - A)$ reduces A.

Proof: Since $\lambda I - A$ is normal and $(\lambda I - A)^* = \overline{\lambda} I - A^*$, Corollary 1.11 implies that (4) holds. We want to show that $\mathcal{N}(\lambda I - A)$ is invariant under A and A^*. Let

$$x \in \mathcal{N}(\lambda I - A) = \mathcal{N}(\overline{\lambda} I - A^*)$$

be given.Then we have

$$Ax = \lambda x \in \mathcal{N}(\lambda I - A), \quad A^* x = \overline{\lambda} x \in \mathcal{N}(\lambda I - A),$$

and $\mathcal{N}(\lambda I - A)$ is invariant under both A and A^*. $\square$

Proposition 3.15: Let X be a Hilbert space, $A \in \mathcal{L}(X;X)$ be given and assume that A is normal. Let $\lambda, \mu \in \sigma_p(A)$ with $\lambda \neq \mu$ be given. Then

$$\mathcal{N}(\lambda I - A) \perp \mathcal{N}(\mu I - A).$$

Proof:Let $x \in \mathcal{N}(\lambda I - A)$, $y \in \mathcal{N}(\mu I - A)$ be given. Then $y \in \mathcal{N}(\overline{\mu} I - A^*)$ and consequently

$$\begin{aligned} \lambda(x,y) &= (\lambda x, y) = (Ax, y) \\ &= (x, A^* y) = (x, \overline{\mu} y) \\ &= \mu(x,y). \end{aligned}$$

Since $\lambda \neq \mu$ we conclude that $(x, y) = 0$. $\square$

Corollary 3.16: Let X be a Hilbert space and assume that $A \in \mathcal{L}(X;X)$ is self-adjoint. Then $\sigma_p(A) \subset \mathbb{R}$.

Proof: Let $\lambda \in \sigma_p(A)$ be given and choose a unit vector $x \in \mathcal{N}(\lambda I - A) = \mathcal{N}(\overline{\lambda} I - A)$. Then we have

$$\lambda x = Ax = \overline{\lambda} x,$$

which gives

$$\lambda(x, x) = \overline{\lambda}(x, x).$$

Since $\|x\| = 1$ we have $\lambda = \overline{\lambda}$. $\square$

Spectral Decomposition of Compact Self-Adjoint Operators

We are going to prove that every compact self-adjoint operator can be diagonalized. The key step is to show that every compact self-adjoint operator (on a nontrivial Hilbert space) has at least one eigenvalue.

Proposition 3.17: Let X be a Hilbert space and assume that $X \neq \{0\}$. Let $A \in \mathcal{C}(X;X)$ be given and assume that $A = A^*$. Then at least one of $\|A\|$, $-\|A\|$ is an eigenvalue of A.

Proof: If $A = 0$ we are done, so assume that $A \neq 0$. Recall that

$$\|A\| = \sup\{|(Ax, x)| : x \in X, \|x\| = 1\}.$$

Choose a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and

$$|(Ax_n, x_n)| \to \|A\| \text{ as } n \to \infty.$$

Notice that $(Ax_n, x_n) \in \mathbb{R}$ for all $n \in \mathbb{N}$ since A is self-adjoint. We may choose $\lambda \in \mathbb{R}$ such that $|\lambda| = \|A\|$ and a subsequence $\{x_{n_k}\}_{n=1}^{\infty}$ such that

$$(Ax_{n_k}, x_{n_k}) \to \lambda \text{ as } n \to \infty.$$

Observe that

$$\begin{aligned} \|(\lambda I - A)x_{n_k}\|^2 &= \lambda^2 \|x_{n_k}\|^2 - 2\lambda(Ax_{n_k}, x_{n_k}) = \|Ax_{n_k}\|^2 \\ &\leq \lambda^2 - 2\lambda(Ax_{n_k}, x_{n_k}) + \lambda^2 \to 0 \text{as } k \to \infty. \end{aligned}$$

We conclude that λ is a generalized eigenvalue for A. Since $\lambda \neq 0$, it follows from Proposition that λ is an eigenvalue for A. $\square$

Lemma 3.18: Let X be a Hilbert space and $A \in \mathcal{C}(X;X)$. Assume that A is self-adjoint. Then there is an orthonormal basis $(e_i | i \in J)$ for X such that

$$\forall j \in I, \quad e_i \text{ is an eigenvector for } A.$$

Remark 3.19: Sometimes unnecessary assumptions regarding separability of X are made when discussing the spectral decomposition of compact self-adjoint opeartors. Notice that we care only about

$$A\Big|_{\mathcal{N}(A)^\perp}.$$

If A is compact and self-adjoint then $\mathrm{cl}(\mathcal{R}(A))$ is separable, and consequently $\mathcal{N}(A)^\perp = \mathcal{N}(A^*)^\perp$ is separable.

Proof of Lemma 3.19: If $A = 0$ the result is immediate, so assume that $A \neq 0$. By Proposition 3.17, we may choose an eigenvector x with $\|x\| = 1$ corresponding to an eigenvalue $\pm\|A\|$. Let $\mathcal{E}$ be the collection of all orthonormal sets of eigenvectors for A, partially ordered by set inclusion. Since every chain has an upper bound (just take the union), Zorn's Lemma implies that $\mathcal{E}$ has a maximal element $\mathcal{O}$. Let us put

$$W = \mathrm{cl}(\mathrm{span}(\mathcal{O})).$$

We want to show that $W = X$, i.e. that $W^\perp = \{0\}$. Notice that $A[W] \subset W$ and $A[W^\perp] \subset W^\perp$. Furthermore

$$A\Big|_{W^\perp}$$

is self-adjoint. Suppose $W^\perp \neq \{0\}$. Then there is an eigenvector $y \in W^\perp$ with $\|y\| = 1$ for $A\Big|_{W^\perp}$. Since y is also an eigenvector for A and y is orthogonal to $\mathcal{O}$, this contradicts the maximality of $\mathcal{O}$. It follows that $W^\perp = \{0\}$. $\square$

Combining results above with a few additional observations, we obtain the following very important theorem.

Theorem 3.20 (Spectral Theorem for Compact Self-Adjoint Operators): Let X be a (real or complex) Hilbert space and let $A \in \mathcal{C}(X;X)$ be given. Assume that $A^* = A$. Then we have

(i) $\sigma(A) \subset \sigma_p(A) \cup \{0\}$.

(ii) $\sigma_p(A) \subset \mathbb{R}$, $\sigma_p(A) \subset [-\|A\|, \|A\|]$, at least one $-\|A\|, \|A\|$ belongs to $\sigma_p(A)$.

(iii) $\sigma_p(A)$ is countable and 0 is the only possible accumulation point.

(iv) A has finite rank if and only if $\sigma_p(A)$ is finite.

(v) $\forall \lambda, \mu \in \mathbb{K}$ with $\lambda \neq \mu$ we have $\mathcal{N}(\lambda I - A) \perp \mathcal{N}(\mu I - A)$.

(vi) $\forall \lambda \in \mathbb{K}\backslash\{0\}$, $\mathcal{N}(\lambda I - A)$ is finite dimensional and $\mathcal{R}(\lambda I - A)$ is closed.

(vii) There is an orthonormal basis $(e_i | i \in J)$ for X such that for every $i \in J$, e_i is an eigenvector for A.

Proof: We have already established (i), (v), (vi), (vii).

To establish (ii), it remains only to show that $|\lambda| \leq \|A\|$ for all $\lambda \in \sigma_P(A)$. Let $\lambda \in \sigma_p(A)$ be given and choose $x \in \mathcal{N}(\lambda I - A)$ with $\|x\| = 1$. Then we have

$$|\lambda| = \|\lambda x\| = \|Ax\| \leq \|A\|.$$

In order to prove (iii), we shall show that for every $\epsilon > 0$, the set

$$\{\lambda \in \sigma_p(A) : |\lambda| \geq \epsilon\}$$

is finite. Suppose that there is some $\epsilon_0 > 0$ such that

$$\{\lambda \in \sigma_p(A) : |\lambda| \geq \epsilon_0\}$$

is infinite. Then we may choose an injective sequence $\{\lambda_i\}_{i=1}^{\infty}$ such that

$$\lambda_i \in \sigma_p(A), \quad |\lambda_i| \geq \epsilon_0 \;\text{ for all } i \in \mathbb{N}.$$

Using the fact that eigenvectors corresponding to distinct eigenvalues are orthogonal, we may choose an orthonormal sequence $\{e_i\}_{i=1}^{\infty}$ such that

$$Ae_i = \lambda_i e_i \;\text{ for all } i \in \mathbb{N}.$$

Notice that for $i \neq j$ we have

$$\|Ae_i - Ae_j\|^2 \geq |\lambda_i|^2 + |\lambda_j|^2 \geq 2\epsilon_0^2.$$

This is impossible since A is compact which implies that $\{Ae_i\}_{i=1}^{\infty}$ must have a strongly convergent subsequence.

Finally, (iv) follows easily from (vii) and the fact that the eigenspaces corresponding to distinct nonzero eigenvalues are finite dimensional and orthogonal. $\square$