Homework 6
36-705
Due: Thursday October 16 by 3:00

1. Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$. Let $\theta = \mathbb{P}(X_i = 0)$. Note that $\theta = g(\lambda)$ for some function g.

   (a) Find the mle $\widehat{\theta}$ for θ.

   (b) Find the limiting distribution for $\widehat{\theta}$ (appropriately normalized).

   (c) Show that $\widehat{\theta}$ is consistent.

2. Let $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$. Let's compare the method of moments estimator $\widehat{\theta}_1$ and the maximum likelihood estimator $\widehat{\theta}_2$.

   (a) Find both estimators.

   (b) Show that they are both consistent.

   (c) Find their limiting distributions.

   (d) Show that $\widehat{\theta}_1 - \theta = O_P(1/\sqrt{n})$ and that $\widehat{\theta}_2 - \theta = O_P(1/n)$. Which estimator should be preferred?

3. Let $X_1, \ldots, X_n \sim N(\mu, 1)$. Consider testing

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

   Construct the Wald test and show that the p-value has a Unif(0,1) under H_0. Find the distribution of the p-value under H_1.

4. Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ where both μ and σ^2 are unknown.

   (a) Find the Wald test for

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

   State the rejection rule explicitly.

   (b) Find the LRT for

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0.$$

   State the rejection rule explicitly.

   (c) Find the Wald test for

$$H_0 : \sigma = \sigma_0 \quad \text{versus} \quad H_1 : \sigma \neq \sigma_0.$$

   State the rejection rule explicitly.

   (d) Find the LRT for

$$H_0 : \sigma = \sigma_0 \quad \text{versus} \quad H_1 : \sigma \neq \sigma_0.$$

   State the rejection rule explicitly.