Graph Theory — Exercise 1

Due Wednesday, February 1st, 2012

1. Let G is a graph with at least three vertices.
   (a) Assume that G contains two distinct vertices u and v such that both $G - \{u\}$ and $G - \{v\}$ are connected graphs. Prove that G is a connected graph.
   (b) Assume that G is connected. Prove that there are two distinct vertices u and v such that both $G - \{u\}$ and $G - \{v\}$ are connected graphs.
   (c) Every connected graph G with at least four vertices contains three distinct vertices u, v and w such that $G - \{u\}$, $G - \{v\}$ and $G - \{w\}$ are connected graphs. Prove or disprove this statement.
2. The *complement* of a graph G is the graph $\overline{G} = (V(G), \{u \neq v \in V \mid uv \notin E(G)\})$. That is, $V(\overline{G}) = V(G)$, and every pair of distinct vertices forms an edge in $\overline{G}$ iff it is not an edge of G. Prove: For every graph G either G or $\overline{G}$ is connected (or both).
3. We have seen in class that if for any two vertices u, v of a graph G with n vertices one has $\deg(u) + \deg(v) \geq n - 1$ then G is connected. Show that this is sharp by finding a disconnected graph with n vertices such that for any two vertices u, v one has $\deg(u) + \deg(v) \geq n - 2$.
4. Let G be a graph and let R be a binary relation defined by uRv if and only if u is connected to v in G. We have seen that R is an equivalence relation. Show that the equivalence classes of R are the connected components of G.
5. Which of the following sequences are graphical? Prove your claim.
   (a) 5,3,3,3,3,2,2,2,1
   (b) 6,3,3,3,3,2,2,2,2,1,1
   (c) 7,5,4,4,4,3,2,1
   (d) 7,6,5,4,4,3,2,1
6. Prove that any sequence $d_1, \ldots, d_n$ of integers satisfing the conditions below is graphical
   (a) $d_1 + \cdots + d_n$ is even,
   (b) $0 \leq d_i \leq n - 1$ for all $1 \leq i \leq n$, and
   (c) $|d_i - d_j| \leq 1$ for every pair of indices $1 \leq i, j \leq n$.