Department of Mathematical Sciences
Carnegie Mellon University

21-640 **Functional Analysis** **Spring 2013**

Take-Home Midterm Exam
Due on Friday, March 29 at 5:00 PM

1. Let $\mathbb{K} = \mathbb{R}$ for definiteness. Determine whether or not $\mathcal{L}(l^2; l^2)$ is separable.

2. *Prove or Disprove*: Assume that X and Y are normed linear spaces and let a linear mapping $T : X \to Y$ be given. Assume further that for every sequence $\{x_n\}_{n=1}^{\infty}$ in X with $x_n \to 0$ as $n \to \infty$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded in Y. Then T is continuous.

3. Let X be a Banach space and Y, Z be closed subspaces of X. Assume that for every $x \in X$ there is a unique pair $(y, z) \in Y \times Z$ such that $x = y + z$. Define $T, L : X \to X$ by
$$\forall x \in X, \quad x = Tx + Lx, \; Tx \in Y, \; Lx \in Z.$$
Show that $T, L \in \mathcal{L}(X; X)$.

4. Let X be a Banach space with dual space X^*, $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ be sequences in X and X^*, respectively, and let $x \in X$, $x^* \in X^*$ be given.
   (a) Show that if $x_n^* \to x^*$ (strongly) as $n \to \infty$ and $x_n \rightharpoonup x$ (weakly) as $n \to \infty$, then $\langle x_n^*, x_n \rangle \to \langle x^*, x \rangle$ as $n \to \infty$.
   (b) Show that if $x_n^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$, and $x_n \to x$ (strongly) as $n \to \infty$, then $\langle x_n^*, x_n \rangle \to \langle x^*, x \rangle$ as $n \to \infty$.

5. Let X be a Banach space and $T : X \to X$ be a linear mapping such that $T^2 = T$. Show that T is continuous if and only if $\mathcal{N}(T)$ and $\mathcal{R}(T)$ both are closed.

6. Let $(X, ||\cdot||)$ be a Banach space and let $|||\cdot|||$ be a norm on X such that there exists $K \in \mathbb{R}$ for which
$$|||x||| \leq K||x|| \;\text{ for all }\; x \in X.$$
(Notice that $(X, |||\cdot|||)$ may be incomplete.) Let $M > 0$ be given, put
$$Z = \{x \in X \;:\; ||x|| \leq M\}$$
and define the metric ρ on Z by
$$\rho(x, y) = |||x - y||| \;\text{ for all }\; x, y \in Z.$$

(a) Show that if X is reflexive then (Z, ρ) is complete.

(b) Show, by giving an example, that (Z, ρ) can be incomplete if X is not reflexive.

7. Give an example of a Banach space X and a sequence $\{K_n\}_{n=1}^{\infty}$ of bounded subsets of X satisfying the following conditions

(i) $\forall n \in \mathbb{N},\ K_n \neq \emptyset,\ K_n$ is closed, K_n is convex,

(ii) $\forall n \in \mathbb{N},\ K_{n+1} \subset K_n$,

(iii) $\displaystyle\bigcap_{n=1}^{\infty} K_n = \emptyset$.

8. Let X and Y be Banach spaces and assume that $T : X \to Y$ is a continuous linear surjection. Show that there exists $M \in \mathbb{R}$ with the following property: For every convergent sequence $\{y_n\}_{n=1}^{\infty}$ in Y there is a convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

$$\forall n \in \mathbb{N}, \text{ we have } y_n = Tx_n \text{ and } \|x_n\| \leq M\|y_n\|.$$

9. Let X be a normed linear space and K be a convex absorbing subset of X. Show that the Minkowski functional p^K is continuous if and only if 0 is an interior point of K.

10. Prove or Disprove: Let X be a Banach space and $x^{**} \in X^{**}$ be given. Let $\{x_n^*\}_{n=1}^{\infty}$ be a sequence in X^* and $x^* \in X^*$ be given. Assume that $x_n^* \xrightarrow{*} x^*$ (weakly*) as $n \to \infty$. Then $x^{**}(x_n^*) \to x^{**}(x^*)$ as $n \to \infty$.