Math 21-236, Mathematical Studies Analysis II, Spring 2012
Assignment 2

The due date for this assignment is Monday February 6.

1. Let $C \subseteq \mathbb{R}^N$ be a nonempty closed set and let $f : \mathbb{R}^N \to [0, \infty)$ be the *distance function from* C, that is,

$$f(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, C) = \inf \{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in C\}$$

for $\mathbf{x} \in \mathbb{R}^N$.

   (a) Prove that for every $\mathbf{x} \in \mathbb{R}^N$ there exists $\mathbf{y}_\mathbf{x} \in C$ such that $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}_\mathbf{x}\|$.

   (b) Prove that if $\mathbf{x} \in \mathbb{R}^N \setminus C$, then $f(\mathbf{z}) = \|\mathbf{z} - \mathbf{y}_\mathbf{x}\|$ for all $\mathbf{z}$ in the segment of endpoints $\mathbf{x}$ and $\mathbf{y}_\mathbf{x}$.

   (c) Prove that f is Lipschitz continuous with Lipschitz constant at most 1 and deduce that if f is differentiable at $\mathbf{x} \in \mathbb{R}^N$, then

$$\|\nabla f(\mathbf{x})\| \leq 1.$$

   (d) Assume that f is differentiable at $\mathbf{x} \in \mathbb{R}^N \setminus C$ and find $\nabla f(\mathbf{x})$.

   (e) Given $\mathbf{x} \in \mathbb{R}^N \setminus C$, prove that if there exist $\mathbf{y}, \mathbf{z} \in C$ with $\mathbf{y} \neq \mathbf{z}$ such that $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x} - \mathbf{z}\|$, then f is not differentiable at $\mathbf{x}$.

   (f) Construct a set C for which f is not always differentiable.

2. Let $E \subseteq \mathbb{R}^N$ be a nonempty set. What is the relation between the following two properties?

   (a) There exist two disjoint open sets U and V such that

$$E \subseteq U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset.$$

   (b) There exist two open sets U and V such that

$$E \subseteq U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset, \quad E \cap U \cap V = \emptyset.$$

3. Assume that $g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and that all its partial derivatives exist and are continuous. Given the normed space $C^1([a, b])$ with the norm

$$\|f\| := \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |f'(x)|,$$

consider the functional $G : C^1([a, b]) \to \mathbb{R}$ defined by

$$G(f) := \int_a^b g(x, f(x), f'(x))\, dx, \quad f \in C^1([a, b]).$$

(a) Prove that G is continuous.

(b) Prove that for every $f \in C^1([a,b])$ and every direction $v \in C^1([a,b])$, there exists the directional derivative $\frac{\partial G}{\partial v}(f)$ and that

$$\frac{\partial G}{\partial v}(f) = \int_a^b \left[\frac{\partial g}{\partial y}(x, f(x), f'(x))\, v(x) + \frac{\partial g}{\partial z}(x, f(x), f'(x))\, v'(x) \right] dx,$$

where $g = g(x,y,z)$.

(c) Given $\alpha, \beta \in \mathbb{R}$, let $X = \left\{ f \in C^1([a,b]) : f(a) = \alpha,\ f(b) = \beta \right\}$. Prove that a necessary condition for $f_0 \in X$ to minimize G over X, that is,

$$\min_{f \in X} G(f) = G(f_0)$$

is that

$$\int_a^b \left[\frac{\partial g}{\partial y}(x, f_0(x), f_0'(x))\, v(x) + \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))\, v'(x) \right] dx = 0$$

for all $v \in C^1([a,b])$ such that $v(a) = v(b) = 0$.

(d) Given a function $h \in C([a,b])$ such that

$$\int_a^b h(x)\, v(x)\ dx = 0$$

for all $v \in C^1([a,b])$ such that $v(a) = v(b) = 0$, prove that $h = 0$.

4. Find the mimimum of the following functionals

(a) $G(f) = \int_a^b (f'(x))^2\ dx$, $X = \left\{ f \in C^1([a,b]) : f(a) = 0,\ f(b) = L > 0 \right\}$,

(b) $G(f) = \int_0^1 \left[(f'(x))^2 + 2xf(x) \right] dx$, $X = \left\{ f \in C^1([0,1]) : f(0) = f(1) = 0 \right\}$,

(c) $G(f) = \int_0^1 \left[(f'(x))^2 - 2xf(x)\, f'(x) + e^x f(x) \right] dx$, $X = \left\{ f \in C^1([0,1]) : f(0) = f(1) = 0 \right\}$.