Homework 2
15-423 Digital Signal Processing for CS
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Due: Wednesday, February 13, 2013

1 Proof

By definition of the convolution, $\forall n \in \mathbb{Z}$,

$$\begin{aligned}
((x \otimes h_1) \otimes h_2)[n] &= \sum_{j=-\infty}^{\infty} (x \otimes h_1)[n-j]h_2[j] \\
&= \sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]h_1[n-j-k] \right) h_2[j] \\
&= \sum_{k=-\infty}^{\infty} x[k] \sum_{j=-\infty}^{\infty} h_1[n-j-k]h_2[j] \\
&= \sum_{k=-\infty}^{\infty} x[k](h_1 \otimes h_2)[n-k] = (x \otimes (h_1 \otimes h_2))[n]. \quad \blacksquare
\end{aligned}$$

2 Convolutions

1. By definition of the convolution, $\forall n \in \mathbb{Z}$,

$$(x \otimes h)[n] = \sum_{k=-\infty}^{\infty} u[n-k]u[k]\alpha^k = \sum_{k=0}^{n} \alpha^k = \boxed{\begin{cases} \frac{1-\alpha^{n+1}}{1-\alpha} & : n \geq 0 \\ 0 & : n < 0 \end{cases}}.$$

   See Figure 1 for graph.

2. By definition of the convolution, $\forall t \in \mathbb{R}$,

$$(x \otimes h)(t) = \int_{-\infty}^{\infty} u(t-s)u(s)e^{\alpha s}\,ds = \int_0^t e^{\alpha s}\,ds = \frac{e^{\alpha s}}{\alpha}\bigg|_{s=0}^{s=t} = \boxed{\begin{cases} \frac{e^{\alpha t}-1}{\alpha} & : t \geq 0 \\ 0 & : t < 0 \end{cases}}.$$

   See Figure 2 for graph.

3. By definition of the convolution, $\forall n \in \mathbb{N}$,

$$(x \otimes h)[n] = \sum_{k=-\infty}^{\infty} u[n]u[n-k] = \sum_{k=0}^{n} 1 = \boxed{\begin{cases} n & : n \geq 0 \\ 0 & : n < 0 \end{cases}}.$$

   See Figure 3 for graph.

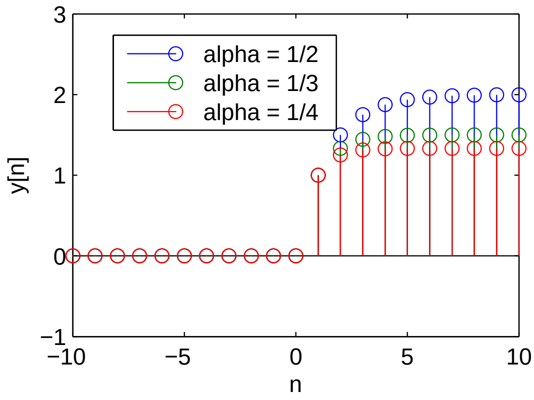

Figure 1: $y[n] = (x \otimes h)[n]$ plotted for $n \in [-10, 10]$ and for various values of α.

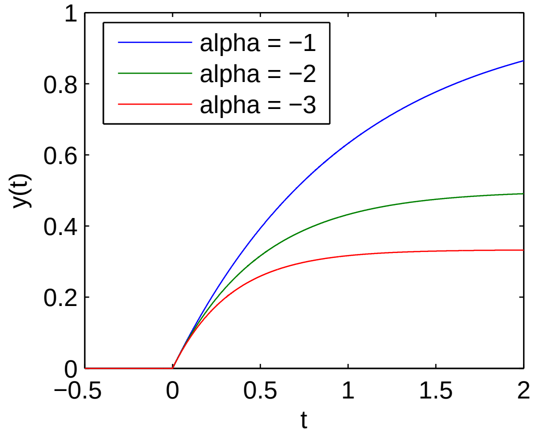

Figure 2: $y(t) = (x \otimes h)(t)$ plotted for $t \in [-0.5, 2]$ and for various values of α.

4. Note that, $\forall n \in \mathbb{N}$, $x[n] = u[n] - u[n - n_0]$ and $h[n] = u[n] - u[n - n_1]$.

   Linearity allows us to distribute the convolution over the differences, and then shift-invariance allows us to compute all of the resulting convolutions as shifted versions of the convoluion computed in Problem 3. Thus, $\forall n \in \mathbb{N}$, we have

$$(x \otimes h)[n] = \boxed{\left\{ \begin{array}{cl} 0 & : n \leq 0 \\ n & : 0 < n \leq n_1 \\ n - (n - n_1) = n_1 & : n_1 < n \leq n_2 \\ n_1 - (n - n_2) = n_1 + n_2 - n & : n < 0 \end{array} \right.} .$$

   See Figure 4 for graph.

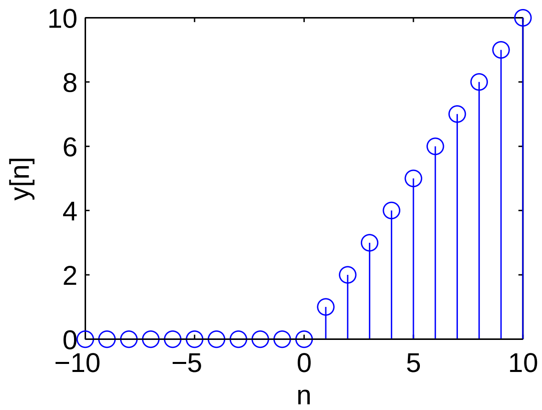

Figure 3: $y[n] = (x \otimes h)[n]$ plotted for $n \in [-10, 10]$.

5. Note that, $\forall n \in \mathbb{N}$, $x[n] = u[n] - u[n - n_0]$. By definition of convolution, $\forall n \in \mathbb{N}$,

$$(x \otimes h)[n] = \sum_{k=-\infty}^{\infty} (u[n-k] - u[n-n_0-k])u[k]\alpha^k = \sum_{k=0}^{n} \alpha^k - \sum_{k=0}^{n-n_0} \alpha^k$$

$$= \boxed{\begin{cases} \frac{\alpha^{n+1} - \alpha^{n-n_0+1}}{\alpha - 1} & : n_0 < n \\ \frac{\alpha^{n+1} - 1}{\alpha - 1} & : 0 < n \leq n_0 \\ 0 & : n \leq 0 \end{cases} .}$$

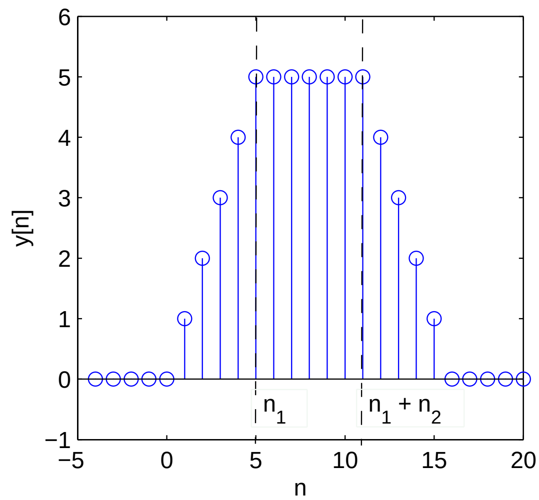

Figure 4: $y[n] = (x \otimes h)[n]$ plotted for $n \in [-5, 20]$, with $n_1 = 5$ and $n_2 = 6$.