Solutions to Practice Problems

1. $$|X(t,\lambda) - X(t,\beta)| = \Big| h(t,\lambda) - h(t,\beta)$$
$$+ \int_0^t \big[f(s, X(s,\lambda)) - f(s, X(s,\beta))\big] ds \int_0^t g(s, X(s,\lambda)) ds$$
$$+ \int_0^t f(s, X(s,\beta)) ds \int_0^t \big[g(s,X(s,\lambda)) - g(s,X(s,\beta))\big] ds \Big|$$
$$\le L|\lambda-\beta| + \int_0^t L|X(s,\lambda) - X(s,\beta)| ds \int_0^t B ds$$
$$+ \int_0^t B ds \int_0^t L|X(s,\lambda) - X(s,\beta)| ds$$
$$= L|\lambda-\beta| + 2LBt\int_0^t |X(s,\lambda) - X(s,\beta)| ds.$$

On $0 \le t \le T$

$$|X(t,\lambda) - X(t,\beta)| \le L|\lambda-\beta| + 2LBT\int_0^t |X(s,\lambda) - X(s,\beta)| ds$$

so

$$|X(t,\lambda) - X(t,\beta)| \le L|\lambda-\beta| e^{2LBTt}.$$

In particular

$$|X(T,\lambda) - X(T,\beta)| \le L e^{2LBT^2} |\lambda-\beta|.$$

2. Let $\mathcal{C}_B = \{U : [0,1] \to [-B, B]$ with U continuous$\}$ and for $U \in \mathcal{C}_B$

$$\mathcal{F}[U](x) = \int_0^1 G(x,y)\left(U^4(y) + f(y)\right)dy.$$

Then since

$$0 \leq G(x,y) \leq 1 \qquad \forall\, x, y$$

we have

$$\left|\mathcal{F}[U](x)\right| \leq \int_0^1 \left(U^4(y) + |f(y)|\right)dy \leq \|U\|^4 + 1/4$$

so for $U \in \mathcal{C}_B$

$$\|\mathcal{F}[U]\| \leq B^4 + 1/4.$$

Also for $U, W \in \mathcal{C}_B$

$$\left|\mathcal{F}[U](x) - \mathcal{F}[W](x)\right| \leq \int_0^1 G(x,y)\left|U^4(y) - W^4(y)\right| dy$$

$$\leq \int_0^1 \left|U^3 + U^2 W + U W^2 + W^3\right| |U - W| dy$$

$$\leq \int_0^1 4B^3 \|U - W\| dy = 4B^3 \|U - W\|,$$

so

$$\|\mathcal{F}[U] - \mathcal{F}[W]\| \leq 4B^3 \|U - W\|.$$

We may use contraction mapping provided

$$B^4 + 1/4 \leq B \quad \text{and} \quad 4B^3 < 1,$$

which holds for $B = 1/2$. So $\exists\, U \in \mathcal{C}_{1/2}$ s.t.

$$U(x) = \mathcal{F}[U](x) = \int_0^1 G(x,y)\left(U^4(y) + f(y)\right)dy.$$

Comment: U satisfies

$$\begin{cases} U''(x) = U^4(x) + f(x) \\ U(0) = U(1) = 0. \end{cases}$$

3. A) $f(x,y) = \begin{pmatrix} -x + 4y + x^3y^2 \\ -2y + xy \end{pmatrix}$

so

$$Df(0,0) = \left.\begin{pmatrix} -1+3x^2y^2 & 4+2x^3y \\ y & -2+x \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} -1 & 4 \\ 0 & -2 \end{pmatrix}$$

The eigenvalues are -1 & -2, both negative, so $(0,0)$ is asymptotically stable.

B) $\frac{d}{dt} X^6 = 6X^5y^3$

$\frac{d}{dt} y^4 = 4y^3(-X^5)$

so

$$\frac{d}{dt}(4X^6 + 6y^4) = 0.$$

Let $w(x,y) = 4x^6 + 6y^4$. w is pos. def, & $D_* w$ is neg. semidef. so $(0,0)$ is stable.

Also $(X,y) \underset{t\to\infty}{\rightarrow} (0,0) \Rightarrow w(X,y) \to w(0,0) = (0,0)$

$\Rightarrow w(X,y) = w(X(0), y(0)) = 0 \Rightarrow X \equiv y \equiv 0$

so $(0,0)$ is not asymptotically stable.

c) $f(x,y) = \begin{pmatrix} x - 2y + x^3 y \\ -y + xy^3 \end{pmatrix}$

so

$$Df(0,0) = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}.$$

There is a positive eigenvalue so $(0,0)$ is unstable.

4. Let $w(x,y) = x^2+y^2$.

A) $D_* w = 2x(-xy^2 + x^2y^3) + 2y(-x^2y - x^3y^2) = -4x^2y^2$

w is pos. def. & $D_* w$ is neg. semidef. $(x,0)$ is a critical point $\forall x$ so $(0,0)$ can not be asymptotically stable.

B) $D_* w = 2x(-xy^2+y) + 2y(-x^2y - x) = -4x^2y^2$

and again $(0,0)$ must be stable. For $\eta > 0$ let $H_\eta = \{(x,y) : w(x,y) \le \eta\}$ and let M be the largest positively invariant subset of $H_\eta \cap \{(x,y) : D_* w(x,y) = 0\}$

$= \{(x,y) : x^2+y^2 \le \eta \text{ and } (x=0 \text{ or } y=0)\}$.

Let x, y be a solution with $(x(0), y(0)) \in M$. Then $(x,y) \in M \ \forall t \ge 0$. If $x(0) \ne 0 \ \exists \delta > 0$ s.t. $x(t) \ne 0$ on $[0,\delta]$ and hence $y(t) = 0$ on $[0,\delta]$ and

$$0 = \dot y(t) = -x^2 y - x = -x \ne 0, \text{ contradiction.}$$

$y(0) \neq 0$ leads to a contradiction in a similar way. Hence $(X(0), Y(0)) = (0,0)$ and

$$M = \{(0,0)\}.$$

Now if $(X(0), Y(0)) \in H_2$ then

$$\text{dist}((X,Y), M) = \sqrt{X^2+Y^2} \to 0.$$

Hence $(0,0)$ is asymptotically stable and in fact all solutions $\to (0,0)$ as $t \to +\infty$.

5. Convert to polar coordinates:

$$\begin{aligned}\dot{r} &= \cos\theta\left(r\cos\theta + r^5\cos\theta - 4r^3\cos^3\theta - 8r^3\sin^2\theta\cos\theta - r\sin\theta\right)\\ &\quad + \sin\theta\left(r\sin\theta + r^5\sin\theta - 4r^3\cos^2\theta\sin\theta - 8r^3\sin^3\theta + r\cos\theta\right)\\ &= r + r^5 - 4r^3\cos^2\theta - 8r^3\sin^2\theta\end{aligned}$$

and

$$\begin{aligned}\dot{\theta} &= \frac{\cos\theta}{r}\left(r\sin\theta + r^5\sin\theta - 4r^3\cos^2\theta\sin\theta - 8r^3\sin^3\theta + r\cos\theta\right)\\ &\quad - \frac{\sin\theta}{r}\left(r\cos\theta + r^5\cos\theta - 4r^3\cos^3\theta - 8r^3\sin^2\theta\cos\theta - r\sin\theta\right)\\ &= 1.\end{aligned}$$

Note that $(x,y) = (0,0)$ is the only critical point. Also

$$r + r^5 - 8r^3 \leq \dot{r} \leq r + r^5 - 4r^3,$$

A) $r=\frac{1}{4} \Rightarrow \dot{r} \geq \frac{1}{4}+\frac{1}{4^5}-8\frac{1}{4^3} > \frac{1}{8}$

and

$$r=1 \Rightarrow \dot{r} \leq 1+1-4=-2$$

so

$$S=\{(x,y): \tfrac{1}{4} \leq r \leq 1\} \text{ is pos. inv.}$$

By Poincaré Bendixson there is a periodic solution whose orbit is in S.

B) Let (X,y) be a solution and consider

$$(\overline{X}(t), \overline{y}(t)) := (X(-t), y(-t)).$$

Taking $\overline{r}(t)=r(-t)$ & $\overline{\theta}(t)=\theta(-t)$ we have

$$\dot{\overline{r}} = -\overline{r}-\overline{r}^5+4\overline{r}^3\cos^2\overline{\theta}+8\overline{r}^3\sin^2\overline{\theta}$$

$$\dot{\overline{\theta}} = -1.$$

Now

$$\overline{r}=1 \Rightarrow \dot{\overline{r}} \geq -1-1+4=2$$

and

$$\overline{r}=2 \Rightarrow \dot{\overline{r}} \leq -2-32+8(2)^3=-2$$

so

$$\overline{S}=\{(\overline{x},\overline{y}): 1 \leq \overline{r} \leq 2\} \text{ is pos. inv. for}$$

the $\overline{x},\overline{y}$ system. By Poincaré Bendixson there is a periodic solution $(\overline{X},\overline{y})$ with

$$C^+(\overline{X}(0),\overline{y}(0)) \subset \overline{S}.$$

But now (X,y) is also periodic and distinct from the solution found in part A.