Homework 10; Due Wednesday, 11/23 (but handed in on 11/28)

(Real Analysis, 1 – Schwab)

Answer these questions and prove these lemmas. All problems require proof/justification, as noted by the style of proof used in the examples in class.

Lemma 0.1. *Let f_1 and f_2 be real and bounded functions on a domain, X. Then*

$$\sup_{x\in X}(f_1(x)+f_2(x)) \leq \sup_{x\in X}(f_1(x)) + \sup_{x\in X}(f_2(x))$$

and

$$\inf_{x\in X}(f_1(x)+f_2(x)) \geq \inf_{x\in X}(f_1(x)) + \inf_{x\in X}(f_2(x))$$

Question 0.2. *Chp 4 #8*

Question 0.3. *Provide a counter example to the outcome of thm 5.10 if the assumption that f is continuous on the closed interval $[a,b]$ is dropped. That is, assume only that f is continuous on (a,b) instead of $[a,b]$. (PLEASE try to keep your example as simple as possible.)*

Question 0.4. *Chp 5 #3*

Question 0.5. *Suppose that f and g are real valued functions on the domain (a,b) and that they obey the equality*

$$f'(x) = g'(x)$$

for all $x \in (a,b)$. Determine and prove the relationship between f and g.

Lemma 0.6. *Suppose that f and g are both integrable and that $f \leq g$ on $[a,b]$. Then*

$$\int_a^b f dx \leq \int_a^b g dx.$$

Lemma 0.7. *Suppose that f is continuous on $[a,b]$, $f \geq 0$ on $[a,b]$, and $f(x_0) > 0$ at some $x_0 \in [a,b]$. Then*

$$\int_a^b f dx > 0.$$

Here are two steps to prove the previous lemma:

step 0 : if $f(x_0) > 0$, can you find some interval, $(x_0-\alpha, x_0+\alpha)$ such that

$$f(x) > f(x_0)/2$$

for all $x \in (x_0-\alpha, x_0+\alpha)$?

step 1 : use the additivity property of the integral to rewrite the integral as

$$\int_a^{x_0-\alpha} f dx + \int_{x_0-\alpha}^{x_0+\alpha} f dx + \int_{x_0+\alpha}^b f dx,$$

and use the fact that $f \geq 0$ combined with step 0 and Lemma 0.6 to conclude.

Question 0.8. *Chp 6 #2 (hint: use Lemma 0.7)*

Question 0.9. *Chp 6 #4*

(The remaining problems form a group with common assumptions and common themes)

Lemma 0.10. *Suppose that f and g are two functions both from $\mathbb{R}$ to $\mathbb{R}$, both differentiable on $\mathbb{R}$, both $f > 0$ and $g > 0$, and both satisfying the equation for all $x \in \mathbb{R}$,*

$$f'(x) = f(x) \quad \text{and} \quad g'(x) = g(x).$$

Then there exists a constant , C_0 (depending on f and g), such that $f(x) = C_0 g(x)$ for all $x \in \mathbb{R}$.

(Notice that the goal of showing $f(x) = C_0 g(x)$ is equivalent to showing that $(f/g)(x) = C_0$. So maybe you can check what kind of information you have for $(f/g)'$ and appeal to thm 5.11.)

Lemma 0.11. *Let f and g be as in the previous Lemma. Suppose that $f(x_0) = g(x_0)$ for at least one value of x_0. Prove that $f(x) = g(x)$ for all x.*

The previous lemma indicates that as soon as you have specified one value for f as above, then you have uniquely determined all other values of f (via the special relationship $f' = f$).

Lemma 0.12. *Let f be the unique function with the assumptions as above and the additional assumption that $f(0) = 1$. Prove that f satisfies*

$$f(x + y) = f(x)f(y).$$

(First, fix y and consider the new function $g(x) = f(x+y)$. Does g also satisfy the special equation $g' = g$? If so, then how does it compare to f? What must be the value of $g(0)$?)