MATH 651: PROBLEM SET 3 SOLUTIONS ARE IN CLASS ON WED. OCT 10.

1. (10 points) Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. Let $\Gamma = \{(x, y) \in X \times Y \ : \ y = f(x)\}$ be the graph of f.

   (i) Show that if Y is Hausdorff then f is continuous if and only if Γ is closed in $(X \times Y, \tau_X \times \tau_Y)$.

   (ii) Assume $X = Y$. Show that if X is not Hausdorff then for the identity function, $f(x) = x$ for all $x \in X$, the graph is not closed.

2. (10 points) Show that if a space is $T3$ then it is $T2$. Also show that there exists a space which is normal and $T0$ but not $T4$.

3. (5 points) Give an example of a topological space which is normal but not regular.

4. (15 points) Consider the following topological space. Let $I = [0, 1]$ and let $X = I \times \{1, 2\}$. Let

   $$\mathcal{B} = \{(B(x, r) \times \{1, 2\}) \backslash \{(x, 2)\} \ \ : \ x \in I, r > 0\} \cup \{\{(x, 2)\} \ : \ x \in I\} \cup \{\emptyset\}.$$

   Here $B(x, r)$ is the interval $(x - r, x + r) \cap I$. Show that

   (i) $\mathcal{B}$ is a basis of topology; call it τ.

   (ii) (X, τ) is compact.

   (iii) (X, τ) is T4.

   (iv) (X, τ) is first countable, but not second countable.

   (v) (X, τ) is not separable.

5. (5 points) Let (X, τ) be a topological space. Prove that a set X is compact if and only if for every family $\{C_\alpha\}_{\alpha \in \Lambda}$ of closed subsets of X with the finite intersection property,

   $$\bigcap_{\alpha \in \Lambda} C_\alpha \neq \emptyset.$$

6. (10 points) Let (X, τ) be a topological space and let (X^∞, τ_∞) where $X^\infty = X \cup \{\infty\}$ be its one-point compactification.

   (i) Prove that if $v \in C(X^\infty)$ then $u := (v - v(\infty))|_X \in C_0(X)$.

   (ii) Conversely, show that if $u \in C_0(X)$ then the extension:

   $$v(x) = \begin{cases} u(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}$$

   satisfies $v \in C(X^\infty)$.

Here the set $C_0(X)$ is the closure (with respect to topology of $C(X)$ with base of open balls $B(f,r) = \{g \in C(X) \ : \ \sup_X |f-g| < r\}$) of the set of compactly supported functions, $C_c(X)$.