Lecture Notes for Week 9 (First Draft)

Infinitesimal Generators

Lemma 9.1: Let X be a Banach space and $T : [0,\infty) \to \mathcal{L}(X;X)$ be a linear C_0-semigroup with infinitesimal generator A. Let $x \in X$ be given.

(i) $\forall t \geq 0$, we have $\displaystyle\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(s)x\,ds = T(t)x$ (right limit if $t = 0$).

(ii) $\forall t \geq 0$, we have $\displaystyle\int_0^t T(s)x\,ds \in \mathcal{D}(A)$ and $\displaystyle A\int_0^t T(s)x\,ds = T(t)x - x$.

Proof: Part (i) is a standard result from calculus.

If $t = 0$ then (ii) is immediate, so assume $t > 0$. For $h > 0$ we have

$$(T(h) - I)\int_0^t T(s)x\,ds = \int_0^t T(s+h)x\,ds - \int_0^t T(s)x\,ds. \tag{1}$$

Putting $\tau = s + h$ we see that

$$\int_0^t T(s+h)x\,ds = \int_h^{t+h} T(\tau)x\,d\tau. \tag{2}$$

Moreover, we have

$$\int_0^t T(s)x\,ds = \int_0^h T(s)x\,ds + \int_h^t T(s)x\,ds. \tag{3}$$

We also have

$$\int_h^{t+h} T(\tau)x\,d\tau - \int_h^t T(s)x\,ds = \int_t^{t+h} T(s)x\,ds. \tag{4}$$

Combining (1), (2), (3), and (4) we find that

$$\left(\frac{T(h) - I}{h}\right)\int_0^t T(s)x\,ds = \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds. \tag{5}$$

Using part (i), we can take the limit as $h \downarrow 0$ of the right side of (5) to conclude that

$$\lim_{h\downarrow 0}\left(\frac{T(h) - I}{h}\right)\int_0^t T(s)x\,ds = T(t)x - x. \quad \square$$

Lemma 9.2: Let X be a Banach space and let $T : [0,\infty) \to \mathcal{L}(X;X)$ be a linear C_0-semigroup with infinitesimal generator A. Let $x \in \mathcal{D}(A)$ be given. Then

(a) $T(t)x \in \mathcal{D}(A)$ for all $t \geq 0$.

(b) $T(t)Ax = AT(t)x$ for all $t \geq 0$.

(c) Put $u(t) = T(t)x$ for all $t \geq 0$. Then $u \in C^1([0,\infty); X)$ and

$$\dot{u}(t) = Au(t), \quad \text{for all } t \geq 0.$$

Proof: Let $t \geq 0$ and $h > 0$ be given. Then we have

$$\begin{aligned} \left(\frac{T(h)-I}{h}\right) T(t)x &= T(t)\left(\frac{T(h)-I}{h}\right) x \\ &\to T(t)Ax \text{ as } h \downarrow 0. \end{aligned}$$

We conclude that $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T(t)Ax$.

This proves (a) and (b) and shows that u is right differentiable with

$$(D^+u)(t) = Au(t) \ \text{ for all } t \geq 0.$$

To establish left differentiability, let $t > 0$ and $h \in (0,t)$ be given. Then we have

$$\begin{aligned} \frac{u(t-h)-u(t)}{h} &= T(t-h)\left(\frac{I-T(h)}{h}\right) x \\ &\to T(t)(-Ax) \text{ as } h \downarrow 0. \end{aligned}$$

We see that

$$(D^-u)(t) = (D^+u)(t) = Au(t) = T(t)Ax.$$

Since the mapping $t \to T(t)Ax$ is continuous, we are done. $\square$

The following result is an immediate consequence of Lemma 9.2 and the fundamental theorem of calculus.

Lemma 9.3: Let X be a Banach space and $T : [0,\infty) \to \mathcal{L}(X;X)$ be a linear C_0-semigroup with infinitesimal generator A. Let $x \in \mathcal{D}(A)$ be given. Then

$$\forall \tau, t \geq 0, \quad T(t)x - T(\tau)x = \int_\tau^t AT(s)x\,ds = \int_\tau^t T(s)Ax\,ds.$$

Theorem 9.4: Let X be a Banach space and $T : [0,\infty) \to \mathcal{L}(X;X)$ be a linear C_0-semigroup with infinitesimal generator A. Then $\mathcal{D}(A)$ is dense in X and A is closed.

Proof: Let $x \in X$ be given. For $h > 0$ put

$$x_h = \frac{1}{h}\int_0^h T(s)x\,ds.$$

Then $x_h \in \mathcal{D}(A)$ for all $h > 0$ and $x_h \to x$ as $h \downarrow 0$ by Lemma 9.1.

To show that A is closed, let $x, y \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{D}(A)$ be given. Assume that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. Let $h > 0$ be given. Then, by Lemma 9.3, we have

$$T(h)x_n - x_n = \int_0^h T(s)Ax_n \, ds \tag{6}$$

Letting $n \to \infty$ in (6) we obtain

$$T(h)x - x = \int_0^h T(s)y \, ds. \tag{7}$$

(We can pass to the limit under the integral because the integrand in (6) converges uniformly on $[0, h]$ to the integrand in (7). It follows immediately from (7) that

$$\frac{T(h)x - x}{h} = \frac{1}{h}\int_0^h T(s)y \, ds. \tag{8}$$

The right-hand side of (8) converges to y as $h \downarrow 0$. It follows that $x \in \mathcal{D}(A)$ and $Ax = y$ and consequently A is closed. $\square$

Lemma 9.5: Let X be a Banach space and $S, T : [0, \infty) \to \mathcal{L}(X; X)$ be linear C_0-semigroups having the same infinitesimal generator A. Then $S(t) = T(t)$ for all $t \geq 0$.

Proof: Let $x \in \mathcal{D}(A)$ and $t > 0$ be given. Define $u : [0, t] \to X$ by

$$u(s) = T(t-s)S(s)x \quad \text{for all } s \in [0, t].$$

Let $s \in [0, t]$ and $h \in \mathbb{R}\backslash\{0\}$ be given with $s + h \in [0, t]$. Then we have

$$\begin{aligned}
\frac{u(s+h) - u(s)}{h} &= \frac{1}{h}[T(t-s-h)S(s+h)x - T(t-s)S(s)x] \\
&= \frac{1}{h}[T(t-s-h)(S(s+h) - S(s))x + (T(t-s-h) - T(t-s))S(s)x] \\
&= T(t-s-h)\left[\frac{S(s+h)x - S(S)x}{h}\right]x + \left[\frac{T(t-s-h) - T(t-s)}{h}\right]S(s)x \\
&\to T(t-s)AS(s)x - T(t-s)AS(s)x = 0 \text{ as } h \to 0
\end{aligned}$$

It follows that u is constant on $[0, t]$; in particular

$$T(t)x = u(0) = u(t) = S(t)x.$$

Since $\mathcal{D}(A)$ is dense, we can conclude that $T(t)x = S(t)x$ for all $x \in X$. $\square$

Let $a \in \mathbb{K}$ and $n \in \mathbb{N}$ be given and define $f : [0, \infty) \to \mathbb{K}$ by

$$f(t) = t^{n-1}e^{at}, \quad \text{for all } t \geq 0. \tag{9}$$

The Laplace transform $\hat{f}$ of f is given by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t}t^{n-1}e^{at}\,dt = \frac{(n-1)!}{(\lambda - a)^n} \quad \text{for} \lambda \in \mathbb{K} \text{ with } \mathrm{Re}(\lambda) > \mathrm{Re}(a). \tag{10}$$

Evaluation of the integral in (10) is discussed in most elementary textbooks on differential equations.

If A is an $N \times N$ (real or complex) matrix and we put

$$F(t) = t^{n-1}e^{tA}, \quad \text{for all } t \geq 0, \tag{11}$$

then we have

$$\hat{F}(\lambda) = \int_0^\infty e^{-\lambda t}t^{n-1}e^{tA}\,dt = (n-1)!R(\lambda; A)^n, \tag{12}$$

for $\mathrm{Re}(\lambda)$ sufficiently large. Here, $R(\lambda; A) = (\lambda I - A)^{-1}$.

It is natural to conjecture that if $T : [0, \infty) \to \mathcal{L}(X; X)$ is a linear C_0-semigroup and n is a positive integer then

$$R(\lambda; A)^n = \frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}T(t)\,dt$$

for $\mathrm{Re}(\lambda)$ suitably large. This is, in fact, correct. We begin by providing a proof when $n = 1$.

Lemma 9.6: Let X be a Banach space. Let $M, \omega \in \mathbb{R}$ and $\lambda \in \mathbb{K}$ with $\mathrm{Re}(\lambda) > \omega$ be given. Assume that $T : [0, \infty) \to \mathcal{L}(X; X)$ is a linear C_0-semigroup satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and having infinitesimal generator A. Then $\lambda \in \rho(A)$ and

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t}T(t)\,dt \quad \text{for all } x \in X. \tag{13}$$

Proof: Define $\Phi(\lambda) \in \mathcal{L}(X; X)$ by

$$\Phi(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt \quad \text{for all } x \in X.$$

We need to show that $\lambda \in \rho(A)$ and $R(\lambda; A) = \Phi(\lambda)$. Let $x \in \mathcal{D}(A)$ be given.

Then we have

$$
\begin{aligned}
\Phi(\lambda)Ax &= \int_0^\infty e^{-\lambda t} T(t)Ax\, dt \\
&= \int_0^\infty e^{-\lambda t} \frac{d}{dt}(T(t)x)\, dt \quad \text{(integration by parts)} \\
&= -x + \lambda \int_0^\infty e^{-\lambda t} T(t)x\, dt \\
&= \lambda\Phi(\lambda)x - x.
\end{aligned}
$$

Now let $x \in X$ be given. We need to show that $\Phi(\lambda)x \in \mathcal{D}(A)$ and that

$$A\Phi(\lambda)x = \lambda\Phi(\lambda)x - x.$$

For this purpose, let $h > 0$ be given. Then we have

$$
\begin{aligned}
\left(\frac{T(h)-I}{h}\right)\Phi(\lambda)x &= \frac{1}{h}\int_0^\infty e^{-\lambda t}(T(t+h)x - T(t)x)\, dt \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t+h)x\, dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\, dt \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda(s-h)} T(s)x\, ds - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\, dt \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda(s-h)} T(s)x\, ds - \frac{1}{h}\int_0^h e^{-\lambda(s-h)} T(s)x\, ds \\
&\quad -\frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\, dt \\
&= \int_0^\infty \left[\frac{e^{-\lambda(t-h)} - e^{-\lambda t}}{h}\right] T(t)x\, dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda(t-h)} T(t)x\, dt \\
&\to \lambda\Phi(\lambda)x - x \quad \text{as } h \downarrow 0.
\end{aligned}
$$

We conclude that $\Phi(\lambda)x \in \mathcal{D}(A)$ and $A\Phi(\lambda)x = \lambda\Phi(\lambda)x - x$. It follows that $\lambda \in \rho(A)$ and $R(\lambda; A) = \Phi(\lambda)$. $\square$

Lemma 9.7: Let $M, \omega \in \mathbb{R}$ and $\lambda \in \mathbb{K}$ with $\mathrm{Re}(\lambda) > \omega$ and $n \in \mathbb{N}$ be given. Assume that $T : [0,\infty) \to \mathcal{L}(X; X)$ is a linear C_0-semigroup satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and having infinitesimal generator A. Then

$$R(\lambda; A)^n x = \frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t} t^{n-1} T(t)\, dt \quad \text{for all } x \in X.$$

Proof: We know that the mapping $\mu \to R(\mu; A)$ is analytic and that

$$\frac{R^{(n-1)}(\lambda; A)}{(n-1)!} = (-1)^{n-1} R(\lambda; A)^n, \tag{14}$$

where $R^{(n-1)}$ is the $(n-1)^{st}$ derivative of R with respect to the first argument. By Lemma 9.8, we have

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt. \tag{15}$$

Combining (14) and (15) we arrive at

$$\begin{aligned} R(\lambda; A)^n &= \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty e^{-\lambda t} (-t)^{n-1} T(t)x\, dt \\ &= \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x\, dt. \quad \square \end{aligned}$$

Theorem 9.8 (Hille-Yosida, 1948): Let X be a Banach space and $M, \omega \in \mathbb{R}$ be given. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to X$ is linear. Then A is the infinitesimal generator of a linear C_0-semigroup satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ if and only if (i) and (ii) below hold:

(i) $\mathcal{D}(A)$ is dense in X and A is closed.

(ii) $\rho(A) \supset \{\lambda \in \mathbb{R} : \lambda > \omega\}$ and

$$\|R(\lambda; A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all } n \in \mathbb{N} \text{ and all } \lambda \in \mathbb{R} \text{ with } \lambda > \omega.$$

Remark 9.9: The inequality in (ii) of Theorem 9.8 can be quite complicated to check in practice. Observe that if

$$\|R(\lambda; A)\| \leq \frac{M}{\lambda - \omega},$$

then

$$\|R(\lambda; A)^n\| \leq \frac{M^n}{(\lambda - \omega)^n}.$$

Consequently, if $M = 1$ and the inequality in (i) holds when $n = 1$ then it automatically holds for all $n \in \mathbb{N}$.

Proof of the Hille-Yosida Theorem: (Necessity) It follows from Theorem 9.4 that (i) holds. Also, it follows from Lemma 9.6 that

$$\rho(A) \supset \{\lambda \in \mathbb{R} : \lambda > \omega\}.$$

Let $x \in X$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ with $\lambda > \omega$ be given. By Lemma 9.7, we have

$$\|R(\lambda; A)^n\|x = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x \, dt.$$

It follows that

$$\|R(\lambda; A)^n\| \leq \frac{M}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} e^{\omega t} \|x\| \, dt. \tag{16}$$

Combining (15) with (16) we obtain

$$\|R(\lambda; A)^n x\| \leq \frac{M\|x\|}{(\lambda - \omega)^n}.$$

(Sufficiency) We shall approximate A by bounded linear operators, construct semigroups generated by the approximating operators by using the standard exponential series, and then pass to the limit to obtain a semigroup generated by A.

Let $x \in \mathcal{D}(A)$ be given. Then for all $\lambda > \omega$ we have

$$(\lambda I - A)R(\lambda; A)x = x$$

and consequently

$$\begin{aligned} \lambda R(\lambda; A)x - x &= AR(\lambda; A)x \\ &= R(\lambda; A)Ax. \end{aligned} \tag{17}$$

It follows that

$$\|\lambda R(\lambda; A)x - x\| \leq \frac{M\|Ax\|}{(\lambda - \omega)} \to 0 \text{ as } \lambda \to \infty.$$

Since $\mathcal{D}(A)$ is dense in X we have established that

$$\forall x \in X, \ \lambda R(\lambda; A)x \to x \text{ as } \lambda \to \infty.$$