Homework 6

21-470 Calculus of Variations
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Problem 1

The 1^{st} Euler-Lagrange Equation gives

$$-\frac{1}{2}\sqrt{\frac{1+y'(x)^2}{(\gamma+y(x))^3}} = \frac{d}{dx}\frac{y'(x)}{\sqrt{(\gamma+y(x))(1+y'(x)^2}}.$$

Making the substitution $u(x) = \sqrt{\gamma+y(x)}$ (and noting that, since $y(x) = u(x)^2 - \gamma, y'(x) = 2u(x)u'(x)$),

$$-\frac{1}{2}\sqrt{\frac{1+4u(x)^2u'(x)^2}{u(x)^3}} = \frac{d}{dx}\frac{2u(x)u'(x)}{u(x)\sqrt{(1+4u(x)^2u'(x)^2}} = \frac{d}{dx}\frac{2u'(x)}{\sqrt{(1+4u(x)^2u'(x)^2}}.$$

[I guess I wasn't able to see the consequence of the u-substitution, as I wasn't really sure how to proceed from here.]

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Problem 2

We show that, $\forall y \in \mathscr{Y}$, $J(y) \geq \alpha$, for $\alpha := \left(\frac{4}{7}\right)^{9/2}\left(\frac{1}{2}\right)^{21/8}$.

Let $y \in \mathscr{Y}$. Since $y \in C^1[0,1]$ and $[0,1]$ is compact, $\exists M > 1$ such that $|y(x) - y(z)| < M|x - z|$, $\forall x, z \in [0,1]$. Hence, since $y(0) = 0$, for $x \in [0, (2M^3)^{-1/2}]$, $y(x)^3 \leq M^3 x^3 \leq x/2$ and thus $|y(x)| \leq (x/2)^{1/3}$. Since $y(1) = 1$, by the Intermediate Value Theorem, $\exists \beta \in (0,1)$ such that $y(\beta) = (\beta/2)^{1/3}$ (in particular, we choose the smallest such β).

For all $x \in [0, \beta]$, since $2|y(x)|^3 \leq x$,

$$|y(x)|^3 \leq x - |y(x)|^3 \leq x - y(x)^3 \Rightarrow y(x)^6 \leq (y(x)^3 - x)^2. \tag{1}$$

Hence, by (a special case of) Jensen's Inequality,

$$\begin{aligned}
J(y) &\geq \int_0^\beta (y(x)^3 - x)|y'(x)|^{9/2}\,dx && \text{(non-negative integrand)}\\
&\geq \int_0^\beta y(x)^6 |y'(x)|^{9/2}\,dx && \text{(by (1))}\\
&\geq \frac{1}{\beta^{7/2}}\left(\int_0^\beta \left|y(x)^{4/3} y'(x)\right|\,dx\right)^{9/2} && \text{(Jensen's Inequality)}\\
&= \frac{1}{\beta^{7/2}}\left(\frac{4}{7} y(x)^{7/4}\Big|_0^\beta\right)^{9/2} && \text{(Integration)}\\
&= \frac{1}{\beta^{7/2}}\left(\frac{4}{7}(\beta/2)^{7/12}\right)^{9/2} && (y(0) = 0, y(\beta) = (\beta/2)^{1/3})\\
&= \frac{1}{\beta^{28/8}}\left(\frac{4}{7}\right)^{9/2}(\beta/2)^{21/8} \geq \left(\frac{4}{7}\right)^{9/2}\left(\frac{1}{2}\right)^{21/8}. && (\beta \in (0,1))
\end{aligned}$$

Problem 3

As usual, define $P, Q : (\alpha, \beta) \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} & P(x,y) = f(x,y,\Phi(x,y)) - \Phi(x,y) f_{,3}(x,y,\Phi(x,y)) \\ \text{and} \quad & Q(x,y) = f_{,3}(x,y,\Phi(x,y)), \qquad \forall x \in (\alpha,\beta), y \in \mathbb{R}. \end{aligned}$$

Since we are working over $\mathbb{R}$, we trivially have that $(Q_{,2}(x,y))^T = Q_{,2}(x,y)$. By Remark 8.4 and the Chain Rule, it suffices to show that

$$P_{,2}(x,y) = f_{,3,1}(x,y,\Phi(x,y)) + f_{,3,3}(x,y,\Phi(x,y))\Phi_{,1}(x,y) = \frac{d}{dx} f_{,3}(x,y,\Phi(x,y)) = Q_{,1}(x,y).$$

Applying the Chain Rule and observing that some terms cancel (again, since we work in $\mathbb{R}$ and so multiplication commutes),

$$\begin{aligned} P_{,2}(x,y) &= \frac{d}{dy} f(x,y,\Phi(x,y)) - \Phi(x,y) f_{,3}(x,y,\Phi(x,y)) \\ &= f_{,2}(x,y,\Phi(x,y)) + f_{,3}(x,y,\Phi(x,y))\Phi_{,2}(x,y) - \Phi_{,2}(x,y) f_{,3}(x,y,\Phi(x,y)) \\ &- \Phi(x,y) \left(f_{,3,2}(x,y,\Phi(x,y)) + f_{,3,3}(x,y,\Phi(x,y))\Phi_{,2}(x,y) \right) \\ &= f_{,2}(x,y,\Phi(x,y)) - \Phi(x,y) \left(f_{,3,2}(x,y,\Phi(x,y)) + f_{,3,3}(x,y,\Phi(x,y))\Phi_{,2}(x,y) \right). \end{aligned} \tag{2}$$

Since Φ is a stationary field for f, if $y'(x) = \Phi(x, y(x))$, we have

$$\begin{aligned} f_{,2}(x,y(x),\Phi(x,y(x))) &= \frac{d}{dx} f_{,3}(x,y(x),\Phi(x,y(x))) \\ &= f_{,3,1}(x,y(x),\Phi(x,y(x))) + f_{,3,2}(x,y(x),\Phi(x,y(x)))\Phi(x,y(x)) \\ &+ f_{,3,3}(x,y(x),\Phi(x,y(x))) \left(\Phi_{,1}(x,y(x)) + \Phi_{,2}(x,y(x))\Phi(x,y(x)) \right). \end{aligned}$$

Plugging this into Equation (2) gives, after cancelling terms,

$$P_{,2}(x,y) = f_{,3,1}(x,y,\Phi(x,y)) + f_{,3,3}(x,y,\Phi(x,y))\Phi_{,1}(x,y). \quad \blacksquare$$

Problem 4

I wasn't able to complete this question.