Homework 4

21-236 Mathematical Studies Analysis II
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Problem 1

Let $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ be of class C^1, such that, $\forall x \in (a,b), \exists \alpha_x > 0$ such that, $\forall y \in \mathbb{R}$,

$$\left| \frac{\partial f}{\partial y}(x,y) \right| \geq \alpha_x.$$

Suppose there existed $\mathbf{x}_1, \mathbf{x}_2 \in (a,b) \times \mathbb{R}$ such that $f(\mathbf{x}_1) < 0 < f(\mathbf{x}_2)$. Let $h : [0,1] \to \mathbb{R}$ such that, $\forall t \in [0,1]$, $h(t) = \frac{\partial f}{\partial y}(x_1 + t(x_2 - x_1)$. Since f is C^1, $\frac{\partial f}{\partial y}$ is continuous, and therefore h is continuous. Thus, by the Intermediate Value Theorem, there exists $t_0 \in [0,1]$ such that $h(t_0) = 0$. However, then, $\exists (x,y) \in (a,b) \times \mathbb{R}$ such that

$$\left| \frac{\partial f}{\partial y}(x,y) \right| = 0,$$

contradicting the given constraint. Therefore, $\frac{\partial f}{\partial y}$ is either everywhere positive or everywhere negative. Without loss of generality, take it to be everywhere postive (since taking $(-f)$ does not change $\{(x,y) \in (a,b) \times \mathbb{R} | f(x,y) = 0\}$). Let $x \in (a,b)$. Then, since

$$\frac{\partial f}{\partial f} \geq \alpha_x > 0,$$

by the Mean Value Theorem, for $y = \frac{-f(0)}{\alpha_x}$, if $f(0) < 0$, then $f(x,y) \geq f(x,0) + \alpha_x(y-0) = 0$, and, if $f(x,0) \geq 0$, then $f(x,y) \leq f(x,0) + \alpha_x(y-0) = 0$; in any case, $f(y)$ and $f(0)$ have different sign, so, by the intermediate value theorem, there exists $y_0 \in [f(0), y] \cup [y, f(0)]$, such that $f(x, y_0) = 0$. Since f is strictly increasing in y, y_0 is unique. Therefore, $\forall x \in (a,b)$, there exists a unique function $g : (a,b) \to \mathbb{R}$ (defined by $g := (x \mapsto y_0)$) such that, $\forall x \in (a,b)$, $f(x, g(x)) = 0$. ■

Let $x \in (a,b)$. Then, since $f(x, g(x)) = 0$ and $\frac{\partial f}{\partial y}(x, g(x)) > \alpha_x$, by the implicit function theorem, within open balls $B_1 = B(x, r_1) \subseteq (a,b)$, $B_2 = B(y, r_2) \subseteq \mathbb{R}$, such that, $\forall x \in B_1$, there is a unique function $g_x : B_1 \to B_2$ mapping x to y such that $f(x,y) = 0$, and, furthermore, since f is of class C^1, g_x is also of class C^1. However, the restriction of g to B_1 has this property, so that, since g_x is the unique function with this property, $g|_{B_1} = g_x$, and consequently g is of class C^1 in a ball around x. Since this is true for all $x \in (a,b)$, g is of class C^1 on its entire domain. ■

Problem 2

Let $g : [0,1] \to Y$ such that, $\forall t \in [0,1]$, $g(t) = f(x + t(y-x))$. Let $y_0 = f(y) - f(x) = g(1) - g(0) \in Y$. By the Corollary of the Hahn-Banach Theorem proven in the notes, there exists a linear function $L : Y \to \mathbb{R}$ such that, for $y_0 = g(1) - g(0)$, $L(y_0) = \|y_0\|$, and, $\forall y \in Y$, $L(y) \leq \|y\|$.

By the Mean Value Theorem, $L(g(1)) - L(g(0)) \leq \sup(\frac{d(L \circ g)}{dt}(1-0)$.

Because L is linear, so that, $\forall t_1, t_2 \in [0,1]$ $\frac{L(g(t_1)) - L(g(t_2))}{t_1 - t_2} = L\left(\frac{g(t_1) - g(t_2)}{t_1 - t_2}\right)$. Thus, $\forall t \in [0,1]$, $(L \circ g)'(t) = L(g'(t))$.

By the Chain Rule, $\forall w \in S$, $\frac{dg}{dt}(w) = \frac{\partial f}{\partial v}(w)\|y - x\|_X$.

$$\begin{aligned}
\|f(y) - f(x)\|_Y &= \|g(1) - g(0)\|_Y \\
&= \|L(g(1) - g(0))\|_Y \\
&= \|(L \circ g)(1) - (L \circ g)(0)\|_Y \\
&\leq \sup_{t \in [0,1]} \left\|\left(L \circ g)'(t)\right)\right\|_Y |1 - 0| \\
&= \sup_{t \in [0,1]} \left\|L(g'(t))\right\|_Y \\
&= \sup_{w \in S} \left\|L\left(\frac{\partial f}{\partial v}(w)\right)\|y - x\|_X\right\|_Y \\
&\leq \sup_{w \in S} \left\|\frac{\partial f}{\partial v}(w)\right\|_Y \|y - x\|_X
\end{aligned}$$

Problem 3

Let

(a) Let $(x, y) \in \{(x, y) : f(x) < y\}$. By the result of part (a) of Problem 1 on Assignment 2, there exists at least one point $(t, s) \in E$ such that $\text{dist}((x, y), E) = \|(x, y) - (t, s)\|$. Suppose there exist distinct $x_1, x_2 \in [a, b]$ such that $\text{dist}((x, y), E) = \|(x, y) - (x_1, f(x_1))\|$ and $\text{dist}((x, y), E) = \|(x, y) - (x_2, f(x_2))\|$. Without loss of generality, $x_1 < x_2$. Let $B = B((x, y), g(x, y))$. By definition of the distance function, there cannot be $t \in [a, b]$ such that $\|(x, y) - (t, f(t))\| < g(x, y)$, so that, at $(x_1, f(x_1))$ and $(x_2, f(x_2))$, B is tangent to the graph of f (insofar as $B \cap \{(x, f(x)) : x \in [a, b]\} = \emptyset$ and $(x_1, f(x_1)), (x_2, f(x_2)) \in \partial B \cap \{(x, f(x)) : x \in [a, b]\}$). Thus, the curvature of f at some point must be at least that of the boundary of B, so that, since the curvature of a circle of radius δ is $\frac{1}{\delta}$, $f'' \geq \frac{1}{\delta}$.

On the other hand, since f is of class C^2, f'' is continuous, so that, by the Weierstrass Theorem, f'' has an upper bound M on $[a, b]$. Therefore, letting $\delta < \frac{1}{M}$, ensures that (t, s) is unique, $\forall (x, y) \in U_\delta$. ■

(b) Let $(x, y) \in U_\delta$, for δ as in part (a), and let (t, s) be the unique point shown to exist in part (a). Since the tangent vector of f at (t, s) is $(t, f'(t))$, the normal vector of f at (t, s) is $(-f', t)$. Since $(x, y) - (t, s)$ is in the direction of the normal vector of f at (t, s), we have

$$(x, y) = (t, s) + \|(x, y) - (t, s)\| \frac{(t, -\frac{1}{f'(t)})}{\|(t, -\frac{1}{f'(t)})\|}.$$

$\blacksquare$

(c) Let $L : \mathbb{R}^2 \to \mathbb{R}$ such that, $\forall (x, y) \in \mathbb{R}^2$, $L(x, y) = x^2 + y^2$. Then, Letting $h : \mathbb{R} \to \mathbb{R}$ such that, $\forall (x, y) \in \mathbb{R}^2$,

$$h(x, y) = L((t, s) + \|(x, y) - (t, s)\| \frac{(t, -\frac{1}{f'(t)})}{\|(t, -\frac{1}{f'(t)})\|} - (x, y)),$$

by the result of part (b) above, fixing $(L \circ h)(x, y) = 0$, since the derivative of $L \circ h$ is non-zero, by the Implicit Function Theorem, in some ball around (t, s), there exists a unique function g_x such that $h(x, g_x(x)) = 0$, and, furthermore, g_x is of class C^1. However, $(L \circ h)(x, y) = 0$ if and only if $g(x, y) = \|(x, y) - (t, s)\|$, so that g also has this property. Therefore, since g_x is unique $g = g_x$, so that g is of class C^1. $\blacksquare$

Problem 4

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that, $\forall (x, y) \in \mathbb{R}^2$,

$$f(x, y) = \alpha \log(1 + xy) + \alpha^2 xy - 2 \sin x + y - 1.$$

Evaluating f at $(0, 1)$ gives $f(0, 1) = 0$. Evaluating

$$\frac{\partial f}{\partial y}(x, y) = \frac{\alpha x}{1 + xy} + \alpha^2 x + 1$$

at $(0, 1)$ gives

$$\frac{\partial f}{\partial y}(0, 1) = 1 \neq 0.$$

Therefore, by the Implicit Function Theorem, fixing $f(x, y) = 0$ determines, in a ball $B_x = B(0, r_1)$, a unique function $g : B_x \to B_y = B(1, r_2)$ such that, $\forall x \in B_x$, $f(x, g(x)) = 0$. $\blacksquare$