Lecture Notes for Week 15 (Preliminary Draft)

Orthogonal Projections

Theorem 15.1: Let K be a nonempty closed convex subset of a Hilbert space X and let $x \in X$ be given. Then there exists exactly one point $y_0 \in K$ such that

$$\|x - y_0\| = \inf\{\|x - y\| : y \in K\}.$$

Proof: Without loss of generality, we may assume that $x = 0$. (Indeed, the set $-x + K$ is nonempty closed and convex.) Let us put

$$\gamma = \inf\{\|y\| : y \in K\}.$$

Choose a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in K$ for all $n \in \mathbb{N}$ and such that $\|y_n\| \to \gamma$ as $n \to \infty$. By the parallelogram law, we have

$$\|\frac{1}{2}(y_n - y_m)\|^2 = \frac{1}{2}(\|y_n\|^2 + \|y_m\|^2) - \|\frac{1}{2}(y_n + y_m)\|^2. \tag{1}$$

Since K is convex, $\frac{1}{2}(y_n + y_m) \in K$ and consequently

$$\|\frac{1}{2}(y_n + y_m)\|^2 \geq \gamma^2. \tag{2}$$

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$\|y_n\|^2 < \gamma^2 + \frac{1}{4}\epsilon^2 \text{ for all } n \geq N. \tag{3}$$

Combining (1), (2), and (3) we find that

$$\|y_n - y_m\|^2 < \epsilon^2 \text{ for all } m, n \geq N,$$

and consequently $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since X is complete, we may choose $y_0 \in X$ such that $y_n \to y_0$ as $n \to \infty$. We deduce that $y_0 \in K$ because K is closed. Since the mapping $x \mapsto \|x\|$ is continuous, we infer that $\|y_0\| = \gamma$.

To prove that there is only one such point y_0, suppose that $\hat{y}_0 \in K$ satisfies $\|\hat{y}_0\| = \gamma$. Then, since K is convex, we have $\frac{1}{2}(y_0 + \hat{y}_0) \in K$. This implies that

$$\gamma \leq \|\frac{1}{2}(y_0 + \hat{y}_0)\| \leq \frac{1}{2}(\|y_0\| + \|\hat{y}_0\|) = \gamma,$$

and consequently

$$\|\frac{1}{2}(y_0 + \hat{y}_0)\| = \gamma. \tag{4}$$

By the parallelogram law, we have

$$\|\frac{1}{2}(y_0 + \hat{y}_0)\|^2 + \|\frac{1}{2}(y_0 - \hat{y}_0)\|^2 = \gamma^2. \tag{5}$$

Combining (4) and (5) we deduce that $\|y_0 - \hat{y}_0\| = 0$. $\square$

Theorem 15.2: Let X be a Hilbert space and M be a closed subspace of X. Let $x \in X$ be given and let y_0 be the unique element of K such that $\|x - y_0\| \leq \|x - y\|$ for all $y \in M$. Then $x - y_0 \in M^\perp$. Conversely, if $\hat{y}_0 \in M$ and $x - \hat{y}_0 \in M^\perp$ then $\|x - \hat{y}_0\| \leq \|x - y\|$ for all $y \in M$.

Proof: Let $y \in M$ and $\alpha \in \mathbb{K}$ be given. Then $y_0 + \alpha y \in M$ so that

$$\begin{aligned}\|x - y_0\|^2 &\leq \|x - (y_0 + \alpha y)\|^2 = \|(x - y_0) - \alpha y\|^2 \\ &\leq \|x - y_0\|^2 + |\alpha|^2\|y\|^2 - 2\mathrm{Re}(x - y_0, \alpha y).\end{aligned}$$

It follows that

$$2\mathrm{Re}(x - y_0, \alpha y) \leq |\alpha|^2\|y\|^2. \tag{6}$$

Putting $\alpha = t > 0$ in (6) yields

$$2t\mathrm{Re}(x - y_0, y) \leq t^2\|y\|^2. \tag{7}$$

Dividing (7) by t and letting $t \downarrow 0$ we arrive at

$$\mathrm{Re}(x - y_0, y) \leq 0.$$

Since $-y \in M$ we also have $\mathrm{Re}(x - y_0, y) \geq 0$ and consequently

$$\mathrm{Re}(x - y_0, y) = 0. \tag{8}$$

If $\mathbb{K} = \mathbb{R}$, then $(x - y_0, y) = 0$ and $x - y_0 \in M^\perp$.

Suppose that $\mathbb{K} = \mathbb{C}$. Then we can replace y with iy in (8) and use the fact that

$$\mathrm{Re}(u, iv) = \mathrm{Im}(u, v) \;\; \text{for all } u, v \in X$$

to deduce that

$$\mathrm{Im}(x - y_0, y) = 0.$$

We conclude that $(x - y_0, y) = 0$ and that $x - y_0 \in M^\perp$.

To prove the second claim in the theorem, suppose that $\hat{y}_0 \in M$ and $x - \hat{y}_0 \in M^\perp$. Then, for all $y \in M$ we have

$$\begin{aligned}\|x - y\|^2 &= \|(x - \hat{y}_0) + (\hat{y}_0 - y)\|^2 \\ &= \|x - \hat{y}_0\|^2 + \|\hat{y}_0 - y\|^2 \geq \|x - \hat{y}_0\|^2,\end{aligned}$$

since $\hat{y}_0 - y \in M$. $\square$

Remark 15.3: Dealing with inequalities involving inner products and norms that hold for all elements of a certain subspace is, of course, very important. For this reason, we point out a slightly different way of obtaining the first claim in Theorem 15.2. If $y = 0$, then trivially $(x - y_0, y) = 0$, so we may assume that $y \neq 0$. If we put

$$\alpha = \frac{(x - y_0, y)}{\|y\|^2}$$

in (6) then we get

$$\frac{|(x - y_0, y)|^2}{\|y\|^2} \leq 0,$$

which implies that $(x - y_0, y) = 0$. This argument is a bit shorter than the one given above, but probably a bit trickier to discover.

An important consequence of Theorem 15.2 is that every element of X can be written in precisely one way as the sum of an element of M and an element of $M^\perp$. This result is known as the Projection Theorem.

Corollary 15.4 (Projection Theorem): Let X be a Hilbert space, M be a closed subspace of X and $x \in X$ be given. Then there exists exactly one pair $(y, z) \in M \times M^\perp$ such that $x = y + z$.

For a fixed closed subspace M, the mapping that carries $x \in X$ to the unique point y_0 in M that minimizes the distance to x is linear. (You should verify this fact as a simple exercise for yourself.) We refer to this mapping as the orthogonal projection onto M and denote it by P_M. Notice that $P_{M^\perp} = I - P_M$.

Definition 15.5: Let M be a closed subspace of a Hilbert space X. For each $x \in X$ we define $P_M x$ to be the unique element of M such that

$$\|x - P_M x\| \leq \|x - y\| \text{ for all } y \in M.$$

Duality in Hilbert Spaces

Lemma 15.6: Let X be a Hilbert space and M be a closed subspace of X. Assume that $M \neq X$. Then there exists $z \in M^\perp$ such that $z \neq 0$.

Proof: Choose $x \in X \backslash M$. By the Projection Theorem, we may choose $y \in M, z \in M^\perp$ such that $x = y + z$. Since $x \notin M$, we know that $x - y \neq 0$ and consequently $z \neq 0$. $\square$

One of the most important results concerning Hilbert spaces, is that a Hilbert space can be identified with its own dual.

Theorem 15.7 (Riesz Representation Theorem): Let X be a Hilbert space and $x^* \in X^*$ be given. Then there exists exactly one $y \in X$ such that

$$x^*(x) = (x, y) \text{ for all } x \in X.$$

Moreover $\|x^*\|_* = \|y\|$.

Proof: Put $M = \mathcal{N}(x^*)$ and observe that M is a closed subspace of X. If $M = X$ we are done because $x^* = 0$ and $y = 0$ is the unique element of X that does the job. Assume that $M \neq X$. Then, by Lemma 15.6, we may choose $y_0 \in M^\perp \backslash \{0\}$. Let us put $\alpha = x^*(y_0)$.

Let $x \in X$ be given and observe that

$$x - \frac{x^*(x)}{\alpha} y_0 \in \mathcal{N}(x^*) = M.$$

Since $y_0 \in M^\perp$, we conclude that

$$(x - \frac{x^*(x)}{\alpha} y_0, y_0) = 0,$$

and consequently

$$x^*(x)(\alpha^{-1} y_0, y_0) = (x, y_0). \tag{9}$$

Let us put

$$y = \frac{\overline{\alpha} y_0}{(y_0, y_0)}.$$

Substitution into (9) gives

$$x^*(x) = (x, y).$$

Suppose that $z \in X$ is such that $(x, y) = (x, z)$ for all $x \in X$. Then $(x, y - z) = 0$ for all $x \in X$, so that $(y - z, y - z) = 0$ and $z = y$.

It remains only to show that $\|x^*\|_* = \|y\|$. Observe that

$$\begin{aligned} \|x^*\|_* &= \sup\{|x^*(x)| : x \in X, \|x\| \leq 1\} \\ &= \sup\{|(x, y)| : x \in X, \|x\| \leq 1\} \\ &\leq \|y\|, \end{aligned}$$

by virtue of the Cauchy-Schwarz inequality. To establish the reverse inequality, we assume that $y \neq 0$. (If $y = 0$ there is nothing to prove.) Then we have

$$\|y\|^2 = (y, y) = x^*(y) \leq \|x^*\|_* \|y\|,$$

which gives $\|y\| \leq \|x^*\|_*$. $\square$

A very important consequence of the Riesz Representation Theorem is that **every Hilbert space is reflexive**. (You should verify the details of a proof for yourself).

Another important consequence of the Riesz Representation Theorem is the following simple remark concerning weak convergence.

Remark 15.8: Let X be a Hilbert space, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. The following three statements are equivalent.

(i) $x_n \rightharpoonup x$ (weakly) as $n \to \infty$.

(ii) $\forall y \in X, \quad (x_n, y) \to (x, y)$ as $n \to \infty$.

(iii) $\forall y \in X, \quad (y, x_n) \to (y, x)$ as $n \to \infty$.

In applications, one frequently constructs a sequence of approximate solutions to some problem and shows that this sequence (or a subsequence) is weakly convergent. In order to pass to the limit and obtain a solution to the original problem, it is very helpful if it can be shown the weakly convergent sequence is actually strongly convergent. The following proposition gives a simple criterion that can sometimes be used to show that a weakly convergent sequence is actually strongly convergent.

Proposition 15.9: Let X be a Hilbert space, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. The following two statements are equivalent.

(i) $x_n \to x$ (strongly) as $n \to \infty$.

(ii) $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ and $\|x_n\| \to \|x\|$ as $n \to \infty$.

Proof: The implication (i) $\Rightarrow$ (ii) is clear. Assume that (ii) holds. Then for all $n \in \mathbb{N}$ we have

$$\|x_n - x\|^2 = (x_n - x, x_n - x) = \|x_n|^2 + \|x\|^2 - 2\mathrm{Re}(x_n, x). \tag{10}$$

As $n \to \infty$, $\|x_n\|^2 \to \|x\|^2$ and $\mathrm{Re}(x_n, x) \to \mathrm{Re}(x, x) = \|x\|^2$, and consequently, the right-hand side of (10) tends to 0. $\square$

Remark 15.10: Proposition 15.9 is valid for *uniformly convex* Banach spaces. (See Assignment 6 for the definition of uniform convexity.)

Hilbert Adjoints

Let X be a Hilbert space and $A \in \mathcal{L}(X; X)$ be given. Since X^* can be identified with X via the Riesz Representation Theorem, it is useful to modify the definition of adjoint so that the domain of the adjoint is the space X itself, rather than the dual space X^*. We shall temporarily use the notation A_H^* for this new type of adjoint, so that we can define it in terms of A^* and deduce its properties from previous results in a simple and clear-cut manner. After we have developed the basic properties of Hilbert adjoints, we shall drop the subscript "H".

We begin by defining the mapping $R : X \to X^*$ by

$$(R(y))(x) = (x, y) \quad \text{for all } x, y \in X \tag{11}$$

Observe that R is *conjugate linear* (i.e. $R(y+z) = R(y) + R(z)$ and $R(\alpha y) = \overline{\alpha} R(y)$ for all $y, z \in X, \alpha \in X$) and isometric (i.e. $\|R(y)\|_* = \|y\|$ for all $y \in X$).

Definition 15.11: We define the *Hilbert adjoint* A_H^* of A by the formula

$$A_H^* = R^{-1} A^* R. \tag{12}$$

It is straightforward to check that $A_H^* \in \mathcal{L}(X;X)$ and that

$$(Ax, y) = (x, A_H^*, y) \quad \text{and} \quad (A_H^* x, y) = (x, Ay) \quad \text{for all } x, y \in X. \tag{13}$$

Some of the basic properties of Hilbert adjoints are summarized in the following theorem.

Theorem 15.12: Let X be a Hilbert space and let $A, B \in \mathcal{L}(X;X)$ and $\alpha \in \mathbb{K}$ be given. Then:

(a) $(\alpha A + B)_H^* = \overline{\alpha} A_H^* + B_H^*$,

(b) $(AB)_H^* = B_H^* A_H^*$,

(c) $(A_H^*)_H^* = A$,

(d) A is bijective if and only if A_H^* is bijective; in this case we have $(A_H^*)^{-1} = (A^{-1})_H^*$,

(e) $\|A_H^*\| = \|A\|$,

(f) If there exists $c > 0$ such that $\|A_H^* x\| \geq c\|x\|$ for all $x \in X$ then A is surjective,

(g) $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A_H^*)$ is closed,

(h) A is compact if and only if A_H^* is compact.

Proposition 15.13: Let X be a Hilbert space and $A \in \mathcal{L}(X;X)$. Then

$$\|A\|^2 = \|A_H^* A\|. \tag{14}$$

Proof: Let $x \in X$ with $\|x\| \leq 1$ be given. Then we have

$$\|Ax\|^2 = (Ax, Ax) = (A_H^* Ax, x) \leq \|A_H^* Ax\|\|x\| \leq \|A_H^* A\|.$$

Taking the supremum over all $x \in X$ with $\|x\| \leq 1$ we find that $\|A\|^2 \leq \|A_H^* A\|$. On the other hand, we have $\|A_H^* A\| \leq \|A_H^*\|\|A\| = \|A\|^2$. $\square$

Proposition 15.14 Let X be a Hilbert space $A \in \mathcal{L}(X;X)$ and $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}(X;Y)$.

(a) If $A_n \to A$ in the uniform operator topology as $n \to \infty$ then $(A_n)_H^* \to A_H^*$ in the uniform operator topology.

(b) If $A_n \to A$ in the weak operator topology as $n \to \infty$ then $(A_n)_H^* \to A_H^*$ in the weak operator topology.

In order to translate the results concerning the relationship between the ranges and null spaces of operators and their adjoints over to Hilbert adjoints, we need to examine the relationship between annihilators, pre-annihilators, and orthogonal complements. For this purpose, we temporarily put a subscripted "H" on the orthogonal complement of a set.

Let X be a Hilbert space, $M \subset X$ and $Z \subset X^*$. We put

$$M_H^{\perp} = \{y \in X : (x, y) = 0 \text{ for all } x \in X\},$$

$$M^{\perp} = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in X\},$$
$$^{\perp}Z = \{x \in X : x^*(x) = 0 \text{ for all } x^* \in Z\}.$$

With these definitions we have

$$M^{\perp} = R[M_H^{\perp}],$$

$$^{\perp}Z = (R^{-1}[Z])_H^{\perp},$$

where R is defined by (11).

Theorem 15.15: Let X be a Hilbert space and $A \in \mathcal{L}(X;X)$ be given. Then

(a) $\mathcal{N}(A) = (\mathcal{R}(A_H^*))_H^{\perp}$,

(b) $\mathcal{N}(A_H^*) = (\mathcal{R}(A))_H^{\perp}$,

(c) $\mathrm{cl}(\mathcal{R}(A)) = (\mathcal{N}(A_H^*))_H^{\perp}$,

(d) $\mathrm{cl}(\mathcal{R}(A_H^*)) = (\mathcal{N}(A))_H^{\perp}$.

Orthonormal Families and Bases

Throughout this section, X is a given Hilbert space. Families of nonzero vectors that are pairwise orthogonal have many nice properties. It is convenient to normalize the vectors in such a family so that they each have norm one.

Definition 15.16: A family $(e_i|i \in I)$ of elements of X is said to be *orthonormal* provided that $e_i \perp e_j$ for all $i, j \in I$ with $i \neq j$ and $\|e_i\| = 1$ for all $i \in I$.

Definition 15.17: An orthonormal family $(e_i|i \in I)$ is said to be *maximal* provided that

$$\forall x \in X, \quad (x \perp e_i \text{ for all } i \in I) \Rightarrow x = 0.$$

A maximal orthonormal family is called an *orthonormal basis*.

Remark 15.18: We say that a set $\mathcal{E} \subset X$ is orthonormal, or is an orthonormal basis, provided that the family $(e|e \in \mathcal{E})$ has the desired property. Notice that an orthonormal set $\mathcal{E}$ is an orthonormal basis if and only if it is a maximal orthonormal set with respect to set inclusion.

Proposition 15.19: Let $\mathcal{O}$ be an orthonormal subset of X. Then there is an orthonormal basis $\mathcal{E} \subset X$ such that $\mathcal{O} \subset \mathcal{E}$.

The proof of Proposition 15.19 is a straightforward application of Zorn's Lemma. It is not difficult to show that all orthonormal bases for X have the same cardinality.

A major advantage of orthonormal families is that there is a simple formula for the coefficients appearing in linear combinations. More specifically, suppose that $(e_i|i \in I)$ is an orthonormal family, F is a finite subset of I, and that

$$x = \sum_{i \in F} \alpha_i e_i. \tag{15}$$

For each $j \in F$, we take the inner product with e_j in (15) to obtain

$$(x, e_j) = \alpha_j. \tag{16}$$

It follows easily that every orthonormal family is linearly independent. Indeed, suppose that (15) holds with $x = 0$. Then (16) implies that $\alpha_j = 0$ for all $j \in F$.

Proposition 15.20: Let $(e_i|i = 1, 2, \cdots, N)$ be an orthonormal family in X and let $M = \mathrm{span}(e_1, e_2, \cdots, e_n)$. Then

$$P_M x = \sum_{k=1}^{N} (x, e_k) e_k \quad \text{for all } x \in X. \tag{17}$$

Here P_M is the orthogonal projection onto M.

Proof: Let us put

$$Qx = \sum_{k=1}^{N} (x, e_k) e_k \quad \text{for all } x \in X.$$

We need to show that $Q = P_M$. Clearly $Qx \in M$ for all $x \in X$. Therefore, it suffices to show that $x - Qx \in M^\perp$ for all $x \in X$. Let $x \in X$ be given. For each $j = 1, 2, \cdots, N$ we have

$$(x - Qx, e_j) = (x, e_j) - (x, e_j) = 0,$$

and consequently $(x - Qx, y) = 0$ for all $y \in M$. $\square$

Proposition 15.21 (Gram-Schmidt): Let $(w_i | i \in \mathbb{N})$ be a linearly independent family of elements of X. Then there is an orthonormal family $(e_i | i \in \mathbb{N})$ such that

$$\mathrm{span(e_1, e_2, \cdots, e_n)} = \mathrm{span}(w_1, w_2, \cdots, w_n) \quad \text{for all } n \in \mathbb{N}.$$

Proposition 15.22 (Bessel's Inequality): Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal sequence. Then

$$\sum_{k=1}^{\infty} |(x, e_k)|^2 \leq \|x\|^2 \quad \text{for all } x \in X. \tag{18}$$

Proof: Let $n \in \mathbb{N}$ be given and put

$$y_n = x - \sum_{k=1}^{n} (x, e_k) e_k.$$

Observe that $(y, e_k) = 0$ for all $k = 1, 2, \cdots, n$. By the Pythagorean Theorem, we have

$$\begin{aligned} \|x\|^2 &= \|y_n\|^2 + \left\| \sum_{k=1}^{n} (x, e_k) e_k \right\|^2 \\ &= \|y_n\|^2 + \sum_{k=1}^{n} |(x, y_k)|^2 \\ &\geq \sum_{k=1}^{n} |(x, e_k)|^2. \end{aligned}$$

Letting $n \to \infty$ gives the desired result. $\square$

Corollary 15.23: Let $(e_i | i \in I)$ be an orthonormal family and let $x \in X$ be given. Then $\{i \in I : (x, e_i) \neq 0\}$ is countable.

The proof of the corollary follows from the observation that for each $n \in \mathbb{N}$, $\{i \in I : |(x, e_i)| \geq \frac{1}{n}\}$ is finite, by virtue of Bessel's Inequality.

Corollary 15.24: Let $(e_i|i \in I)$ be an orthonormal family and let $x \in X$ be given. Then

$$\sum_{i \in I} |(x, e_i)|^2 < \infty.$$

There is, of course, no difficulty in interpreting the sum in the last corollary because all of the terms are nonnegative (and only countably many are nonzero). Now it is natural to ask if we can make sense of the sum

$$\sum_{i \in I} (x, e_i) e_i, \tag{19}$$

when $(e_i|i \in I)$ is an orthonormal family and $x \in X$. The answer is yes, but we must be a bit careful in assigning a precise meaning to the sum. If I is finite, there is nothing to worry about. If I is infinite, then there are only countably many nonzero terms in the sum, but we need to worry about whether the order in which the terms are arranged could matter. We begin by giving a result when the index set is $\mathbb{N}$.

Theorem 15.25: Let X be a Hilbert space, $(e_i|i \in \mathbb{N})$ be an orthonormal family, and put

$$M = \operatorname{span}(\operatorname{cl}(e_i|i \in I)).$$

Then for every $x \in X$ we have

$$\sum_{i=1}^{\infty} (x, e_i) e_i = P_M x.$$

Definition 15.26: Let Y be a normed linear space and $(y_i|i \in I)$ be a family of elements of Y. We say that $(y_i|i \in I)$ is *summable* with sum $S \in Y$, and we write

$$\sum_{i \in I} y_i = S,$$

provided that for every $\epsilon > 0$ there exists a finite set $F \subset I$ such that for every finite set J with $F \subset J \subset I$ we have

$$\left\| \sum_{i \in J} y_i - S \right\| < \epsilon.$$

Remark 15.27: A family $(a_i|i \in I)$ of real numbers is summable if and only if it is absolutely summable; in this case there can be only a countable of nonzero terms.

Theorem 15.28: Let X be a Hilbert space and $(e_i|i \in I)$ be an orthonormal family. The following 5 statements are equivalent.

(i) $(e_i|i \in I)$ is an orthonormal basis,

(ii) $\mathrm{cl}(\mathrm{span}(e_i|i \in I)) = X$,

(iii) $\forall x \in X, \quad x = \sum_{i\in I}(x, e_i)e_i$,

(iv) $\forall x, y \in X, \quad (x, y) = \sum_{i\in I}(x, e_i)\overline{(y, e_i)}$,

(v) $\forall x \in X, \quad \|x\|^2 = \sum_{i\in I} |(x, e_i)|^2$

Proposition 15.29: Let X be a Hilbert space. X is separable if and only if it has a countable orthonormal basis.

Theorem 15.30: Let X be a Hilbert space and $A \in \mathcal{C}(X; X)$ be given. Then there is a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{L}(X; X)$ such that each A_n has finite rank and $A_n \to A$ in the uniform operator topology as $n \to \infty$.

Sketch of Proof: Let $Y = \mathrm{cl}(\mathcal{R}(A))$. If Y is finite dimensional, then we may take $A_n = A$ for all $n \in \mathbb{N}$. Assume that Y is infinite dimensional. By Theorem $\mathcal{R}(A)$ is separable, and consequently Y is also separable. By Prop. 41.5, we may choose an orthonormal basis $(e_i|i \in \mathbb{N})$ for Y. For each $n \in \mathbb{N}$, let us put

$$M_n = \mathrm{span}(e_1, e_2, \cdots, e_n), \quad A_n = P_{M_n}A.$$

Clearly, each A_n has finite rank. Using the fact that A is compact, it is not too difficult to show that

$$\|A_n - A\| \to 0 \text{ as } n \to \infty.$$

You should check the details as an exercise for yourself. $\square$

Remark 15.31: Theorem 15.30 is valid for any Banach space X having a Schauder basis.

Spectral Theory for Compact Self-Adjoint Operators

Let X be a Hilbert space.

Definition 15.32: Let $A \in \mathcal{L}(X; X)$ be given.

(a) The *resolvent set* of A, denoted $\rho(A)$ is the set of all $\alpha \in \mathbb{K}$ such that $(\alpha I - A)$ is bijective.

(b) The *spectrum* of A, denoted $\sigma(A)$ is defined by $\sigma(A) = \mathbb{K}\backslash\rho(A)$.

(c) Let $\lambda \in \mathbb{K}$ be given. λ is said to be an *eigenvalue* of A provided that $\mathcal{N}(\lambda I - A) \neq \{0\}$. The set of all eigenvalues of A is called the *point spectrum* of A and is denoted by $\sigma_p(A)$.

(d) Let $\lambda \in \sigma_p(A)$ be given. The nonzero members of $\mathcal{N}(A)$ are called *eigenvectors* of A.

Proposition 15.33: Let $A \in \mathcal{L}(X;X)$ and $\lambda \in \mathbb{K}$ be given. If $\lambda > \|A\|$ then $\lambda \in \rho(A)$.

Proposition 15.34: Let $A \in \mathcal{C}(X;X)$ be given and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then $\mathcal{N}(\lambda I - A)$ is finite dimensional.

Proposition 15.35: Let $A \in \mathcal{C}(X;X)$ be given. Then $\sigma(A) \subset \sigma_p(A) \cup \{0\}$. Moreover $\sigma_p(A)$ is countable and 0 is the only possible accumulation point.

Theorem 15.36: Let $A \in \mathcal{C}(X;X)$ be given and assume that A is self-adjoint, i.e. $A_H^* = A$. Then

(a) $\sigma(A) \subset \mathbb{R}$

(b) there exists $\lambda \in \sigma_p(A)$ such that $|\lambda| = \|A\|$.

(c) $\mathcal{N}(\lambda - a) \perp \mathcal{N}(\mu - A)$ for all $\lambda, \mu \in \mathbb{K}$ with $\lambda \neq \mu$.

(d) there is an orthonormal basis $(e_i | i \in I)$ such e_i is an eigenvector of A for every $i \in I$. Moreover for every such basis we have

$$\forall x \in X, \quad Ax = \sum_{i \in I} \lambda_i (x, e_i) e_i, \quad \text{where } Ae_i = \lambda_i e_i \text{ for all } i \in I.$$