21-484 Notes

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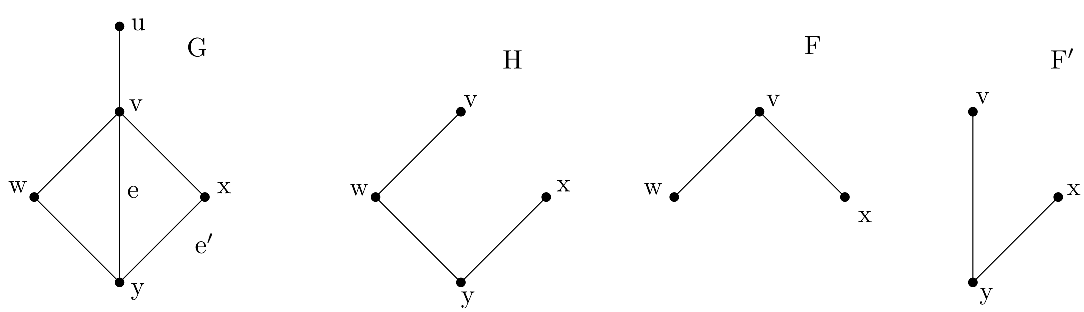
recall:
- $\deg(v) = d(v)$
- claim (Theorem 2.1): If G is a (multi)graph then $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

Def:

- A graph H is called a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$. Also say "G contains H as a subgraph".
- If $H \subseteq G$ and $H \neq G$ then H is a proper subgraph of G.
- If $H \subseteq G$ and $V(H) = V(G)$ then H is a spanning subgraph of G.
- If $H \subseteq G$ and $E(H) = E(G)|_{V(H)} = \{uv \in E(G) | u, v \in V(H)\}$ then H is an induced subgraph of G.
- Let $\varnothing \neq S \subseteq V(G)$ be a set of verticies and let $\varnothing \neq X \subseteq E(G)$ be a set of edges.
  - Then $G[S] = \langle S \rangle$ is the induced subgraph over S $G[S] = (S, E(G)|_S)$
  - And $G[X] = \langle X \rangle$ is the induced subgraph over X, $G[X] = \left(\bigcup_{e \in X} e, X \right)$

Example (1.15)

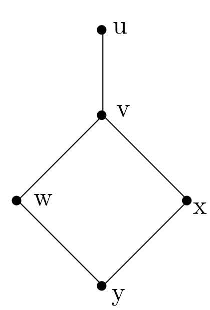


here: $H \subseteq G$
$F \not\subseteq H$

let $S = \{x, v, w\}$ then $F = G[S]$

let $X = \{e, e'\}$ then $F' = G[X]$

$F = G - \{y, u\}$



= G - e

Def. (page 11)

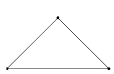
- A walk in a graph G is a sequence of verticies $v_0, v_1, \dots, v_\ell$ such that $\forall 1 \leq i \leq \ell, v_{i-1}v_i \in E(G)$
- If G is a multigraph then a walk is a sequence $v_0, e_1, v_1, e_2, v_2, \dots, e_\ell, v_\ell$ s.t. $\forall 1 \leq i \leq \ell$ $e_i = \{v_{i-1}, v_i\} \in E(G)$
- also called a v_0-v_ℓ walk
- If $v_0 = v_\ell$ then the walk is closed, otherwise it's open.
- length is measured by edges, so in the definition above the length is ℓ.
- a walk is a trail if no edge is tranversed more than once.
- a walk is a path if no vertex is visited more than once.

claim (Theorem 1.6): If G contains a u-v walk of length ℓ, then it contains a path from u to v of length $\leq \ell$.

Proof:

- Assume that G has no u-v path
- let P be the shortest u-v walk, $P = v_0, v_1, \dots, v_k$
- P is not a path, so $\exists i < j$ s.t. $v_i = v_j$
- $P' = v_0, \dots, v_i, v_{j+1}, \dots, v_k$ is a shorter u-v walk ■

more Def. (p. 13):

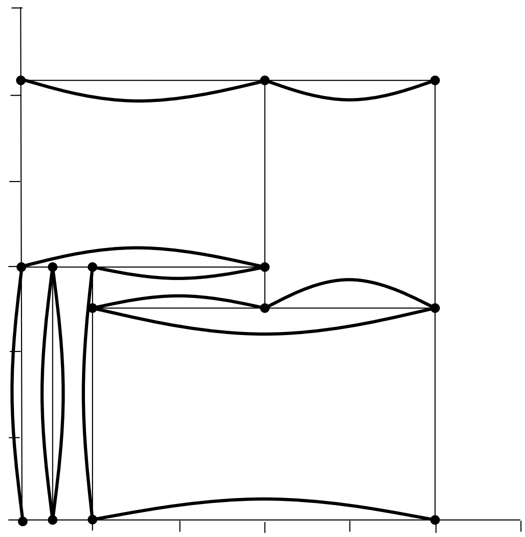
- A circuit is a closed trail of length ≥ 3.
- A cycle is closed trail in which no vertex appears twice except for the first and last.
- A k-cycle is a cycle of length k.
- similarly: even cycle, odd cycle
- 3-cycle: 

Example: You are given a rectangle divided into smaller rectangles s.t. each small rectangle has at least on side of integer length.

Show that the big rectangle also has at least one side of integer length.

standard solution:

- integrate $e^{2\pi i(x+y)}$ over the plane
- notice that the integral over a rectangle is 0 iff one side is integer



Define a graph G:

V - the corners of all rectangle

E - Pick two parallel "integer sides" from each small rectangle. Make the endpoints of the integer sides adjacent in G.

<u>Notice:</u>

1. the corners of the big rectangle are of degree 1.
2. the other verticies have degree 2 or 4 (because every such vertex is the corner of 2 or 4 small rectangle.