Homework 3

21-759 Differential Geometry
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I would be willing to present solutions to Exercises 9 and 12.

Problem 1

Since the extension into $\overline{\mathcal{M}}$, the connection on $\overline{\mathcal{M}}$, Π_T, and $(df)^{-1}$ are all linear (in X), $\nabla_X Y$ is linear in X. Also, if h is a smooth function on $\overline{\mathcal{M}}$, then

$$\begin{aligned}
\nabla_X(hY)|_p &= (df)^{-1}\left(\Pi_T \nabla_{\overline{X}} h\overline{Y}\right) = (df)^{-1}\left(\Pi_T dh(\overline{X})\overline{Y} + h\nabla_{\overline{X}}\overline{Y}\right) \\
&= (df)^{-1}\left(dh(\overline{X})Y + h\nabla_{\overline{X}}Y\right) = dh(X)Y + h\nabla_X Y,
\end{aligned}$$

so we have the Leibnitz Rule.

$$\begin{aligned}
g(\nabla_X Y, Z) + g(Y, \nabla_X Z) &= \overline{g}(\Pi_T \nabla_{\overline{X}}\overline{Y}, df Z) + \overline{g}(df Y, \Pi_T \nabla_{\overline{X}}\overline{Z}) \\
&= \overline{g}(\nabla_{\overline{X}} df Y, df Z) + \overline{g}(df Y, \nabla_{\overline{X}} df Z) \\
&= \overline{X}\,\overline{g}(df Y, df Z) = X g(Y, Z),
\end{aligned}$$

so we have symmetry. Finally, the connection is torsion-free since

$$\begin{aligned}
\nabla_X Y - \nabla_Y X &= (df)^{-1}\left(\Pi_T \nabla_{\overline{X}}\overline{Y} - \nabla_{\overline{Y}}\overline{X}\right) \\
&= (df)^{-1}\left(\Pi_T[\overline{X}, \overline{Y}]\right) \\
&= (df)^{-1}[\Pi_T \overline{X}, \Pi_T \overline{Y}] = [X, Y]. \quad \blacksquare
\end{aligned}$$

Exercise 5

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a) The flow is the solution to the differential equation

$$\begin{cases} \frac{d}{dt}\varphi(t,p) = A\varphi(t,p) \\ \varphi(0,p) = p \end{cases}$$

Since A is linear and constant, we have from ODEs that $\varphi(t,p) = \exp(tA)p$.

Let $p \in \mathcal{M}, w, v \in T_{\varphi(t,p)}\mathcal{M}$. Then, since $\exp(tA)$ is linear,

$$g(d\exp(tA)w, d\exp(tA)v) = g(\exp(tA)w, \exp(tA)v) = g(w, \exp(tA^T)\exp(tA)v).$$

Since $[\exp(tA^t)]^{-1} = \exp(-tA)$, $\exp(tA)$ is an isometry if and only if $A^T = -A$. $\blacksquare$

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b) Since $\frac{d}{dt}\varphi(t,p) = X(p) = 0$, $\varphi(t,p) = p$ for all $t \in \mathbb{R}$. Let $d : \mathcal{M}^2 \to \mathbb{R}$ denote the metric induced by g. Then, since $q \mapsto \varphi(t,q)$ is an isometry for $t \in (-\varepsilon, \varepsilon)$,

$$d(\varphi(t,q),p) = d(\varphi(0,q),p).$$

Thus, $\frac{d}{dt}d(\varphi(t,q),p)\big|_{t=0} = 0$, and hence $X(\varphi(t,q)) = \frac{d}{dt}\varphi(t,q)$ is tangent to the geodesic sphere centered at p.

c) Since $f^{-1} : M \to N$ is an isometry and, $\forall q \in Y$, $X(f^{-1}(q)) = df_q^{-1}(Y(q))$, one direction suffices.

Assume X is a Killing field, and let $\varepsilon > 0$ such that the flow $\varphi_X : (-\varepsilon,\varepsilon) \times U \to \mathcal{M}$ is an isometry. By local uniqueness, a solution to

$$\begin{cases} \frac{d}{dt}\varphi_Y(t,q) = Y(\varphi_Y(t,q)) \\ \varphi_Y(0,q) = q \end{cases}$$

is precisely $\varphi_Y(t, f(p)) = f(\varphi_X(t,p))$, since

$$\frac{d}{dt}\varphi_Y(t,f(p)) = df_{\varphi_X(t,p)}\frac{d}{dt}\varphi_X(t,p) = df_{\varphi_X(t,p)}X(\varphi_X(t,p)) = Y(f(X(\varphi_X(t,p)))) = Y(\varphi_Y(t,f(p))).$$

Also, if $t \in (-\varepsilon,\varepsilon)$, since f is an isometry,

$$\begin{aligned} d(\varphi_Y(t,f(p)),\varphi_Y(t,f(q))) &= d(\varphi_X(t,p),\varphi_X(t,q)) \\ &= d(\varphi_X(0,p),\varphi_X(0,q)) = d(\varphi_Y(0,f(p)),\varphi_Y(0,f(q))), \end{aligned}$$

so $q \mapsto \varphi(t,q)$ is an isometry.

d) ($\Rightarrow$) Suppose, first, that $q \in U$ with $X(q) \neq 0$. If $X(q) \neq 0$, then we can let S be a submanifold of U passing through q and normal to $X(q)$, with $\dim S = \dim M - 1$. We can choose coordinates $(x_1, \ldots, x_{n-1})$ in a neighborhood $V \subseteq U$ of q such that $(x_1, \ldots, x_{n-1}, t)$ are coordinates in a neighborhood $V \times (-\varepsilon,\varepsilon) \subseteq U$ and $X = \frac{\partial}{\partial t}$. Defining $X_i = \frac{\partial}{\partial x_i}$, we have

$$\langle \nabla_{X_j} X, X_i \rangle + \langle \nabla_{X_i} X, X_j \rangle = X\langle X_i, X_j \rangle - \langle [X,X_i], X_j \rangle - \langle [X,X_j], X_i \rangle = \frac{\partial}{\partial t}\langle X_i, X_j \rangle = 0,$$

where the last equality uses the fact that X is a Killing field.

Now, if $X(q) = 0$, then either $q \in \overline{\{q \in U : X(q) \neq 0\}}$ or there is a neighborhood q on which $X \equiv 0$. In the first case, we have the desired equation by continuity, and, in the second case, the equation holds trivially.

($\Leftarrow$) Under the same setup,

$$\frac{\partial}{\partial t}\langle X_i, X_j \rangle = X\langle X_i, X_j \rangle - \langle [X,X_i], X_j \rangle - \langle [X,X_j], X_i \rangle = \langle \nabla_{X_j} X, X_i \rangle + \langle \nabla_{X_i} X, X_j \rangle = 0,$$

so that $\langle X_i, X_j \rangle$ is constant and hence X is Killing. ■

e) Using the same coordinates as in part d, since

$$0 = \frac{\partial}{\partial t}\langle X_i, X_j \rangle = \frac{\partial}{\partial t}g_{i,j} = \frac{\partial}{\partial x_n}g_{i,j},$$

$g_{i,j}$ does not depend on x_n.

Exercise 7

Let $U := \exp_p((-\varepsilon, \varepsilon))$, and let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $T_p\mathcal{M}$. If $q \in U$, then there is a geodesic $\exp_q(v)$ from p to q. Since exp is a local isometry, $\{d(\exp_q)e_i\}_{i=1}^n$ is an orthonormal basis of $T_q\mathcal{M}$, so define $E_i(q) := d(\exp_q)e_i$. Since $\nabla_{E_i}E_j(p) = 0$ and the Riemann connection is a function of the metric (and hence is invariant under isometries) $\nabla_{E_i}E_j \equiv 0$ on U.

Exercise 8

a) Since $\{E_i(p)\}_{i=1}^n$ is an orthonormal basis of $T_p\mathcal{M}$,

$$\operatorname{grad} f(p) = \sum_{i=1}^n \langle \operatorname{grad} f(p), E_i(p)\rangle E_i(p) = \sum_{i=1}^n df_p(E_i(p))E_i(p) = \sum_{i=1}^n (E_i(f))E_i(p).$$

It follows from the Levi-Civita formula for the Riemann connection that, in a geodesic frame, the trace of the mapping $Y(p) \mapsto \nabla_Y X(p)$ is

$$\sum_{i=1}^n E_i(f_i)(p).$$

b) If $\mathcal{M} = \mathbb{R}^n$, then $(E_i(f)) = \frac{\partial f}{\partial x_i}$ and $E_i(p) = e_i$, $\forall p \in \mathcal{M}$, so it follows from part a) that

$$\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i \quad \text{and} \quad \operatorname{div} X = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}.$$

Exercise 9

a) By part a) of Problem 8,

$$\Delta f(p) = (\operatorname{div}\operatorname{grad} f)(p) = \operatorname{div}\left(\sum_{i=1}^n (E_i(f))E_i(p)\right) = \sum_{i=1}^n (E_i(E_i(f))(p).$$

By part b) of Problem 8, it follows that, if $M = \mathbb{R}^n$, then

$$\Delta f(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} e_j\right)_i = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_i} = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}.$$

b) Using the normal product rule for derivatives,

$$\begin{aligned}\Delta(fg)(p) &= \sum_{i=1}^{n}(E_i(E_i(fg))(p) = \sum_{i=1}^{n}(E_i(fE_i(g)+gE_i(f)))(p)\\ &= \sum_{i=1}^{n} fE_i(E_i(g))(p) + gE_i(E_i(f))(p) + E_i(f)E_i(g)\\ &= (f\Delta g)(p) + (g\Delta f)(p) + 2\left\langle \operatorname{grad} f(p), \operatorname{grad} g(p)\right\rangle. \quad \blacksquare\end{aligned}$$

Exercise 11

Let $p \in \mathcal{M}$ and let E_i be a geodesic frame at p and let ω_i denote the differential 1-form defined on the same neighborhood of p as the geodesic frame by $\omega_i(E_j) = \delta_{i,j}$. Also, define the n-form $\nu := \wedge_{i=1}^n \omega_i$. Then, for any $v_1, \ldots, v_n \in T_p\mathcal{M}$,

$$(\nu_p(v_1,\ldots,v_n))^2 = \det\left([\omega_i(v_j)]^2\right) = \det\left([\langle E_i(p), v_j\rangle)]^2\right) = \det\left([\langle v_i, v_j\rangle]\right),$$

so ν is the volume element on M. Also, if $\theta_i := \omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \omega_n$ and $X = \sum_{i=1}^n f_i E_i$, then

$$\begin{aligned}(i(X)\nu)_p(v) &= \nu_p(X(p), v)\\ &= \sum_{\sigma\in S_n}\operatorname{sign}(\sigma)\prod_{i=1}^{n}\omega_i(v_{\sigma(i)}) = \sum_{i=1}^{n}(-1)^{i+1}\omega_i(X(p))\theta_i(v) = \sum_{i=1}^{n}(-1)^{i+1}f_i\theta_i(v)\end{aligned}$$

(where $v = (v_2, \ldots, v_n)$ and $v_1 = X(p)$). It follows from properties of the exterior derivative that

$$d(i(X)\nu) = \sum_{i=1}^{n}(-1)^{i+1}(df_i \wedge \theta_i + f_i \wedge d\theta_i) = \left(\sum_{i=1}^{n}E_i(f_i)\right)\nu + \sum_{i=1}^{n}(-1)^{i+1}f_i \wedge d\theta_i.$$

Since

$$d\omega_k(E_i, E_j) = E_i\omega_k(E_j) - E_j\omega_k(E_i) - \omega_k([E_i, E_j]) = \omega_k(\nabla_{E_i}E_j - \nabla_{E_j}E_i) = \omega_k(0) = 0,$$

$(d\theta_i)_p = 0$, and so

$$d(i(X)\nu)(p) = \left(\sum_{i=1}^{n}E_i(f_i)\right)\nu. \quad \blacksquare$$

Exercise 12

By the result of Exercise 11 and Stokes' Theorem, if ν is a volume form, then

$$\int_M \Delta f\nu = \int_M \operatorname{div}\operatorname{grad} f\nu = \int_M d(i(\operatorname{grad} f)\nu) = 0.$$

Since $\nabla f \geq 0$, $\nabla f = 0$. Then, applying Stokes Theorem and the result of part b) of Exercise 9,

$$0 = \int_M d(i(\operatorname{grad} f^2/2)\nu) = \int_M \Delta(f^2/2)\nu = \int_M \|\operatorname{grad} f\|^2\nu,$$

so $0 = \operatorname{grad} f = df$. By Problem 5 from Assignment 1, then, $f \equiv C$ on $\mathcal{M}$, for some $C \in \mathbb{R}$. $\blacksquare$