Homework 4
21-470 Calculus of Variations
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Problem 1

Suppose y minimizes J on $\mathscr{Y}$. Define $C := \int_0^1 y(x)\,dx$ and $\mathscr{V} := \{v \in C^1[0,1] : v(0) = v(1) = 0\}$. Then, for any $v \in \mathscr{V}$,

$$\begin{aligned} 0 = \delta J(y, v) &= \int_0^1 f_{,3}(x, y(x), y'(x))v'(x) + \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\int_0^1 y(x) + \varepsilon v(x)\,dx\right)^2 \\ &= \int_0^1 2y'(x)v'(x)\,dx + 2\left(\int_0^1 y(x)\,dx\right)\left(\int_0^1 v(x)\,dx\right) = \int_0^1 y'(x)v'(x) + Cv(x)\,dx, \end{aligned}$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $f(x, y, z) = z^2$ (since $f_{,2} = 0$). Integrating by parts gives

$$0 = \int_0^1 (y'(x) - Cx)v'(x)\,dx.$$

since $v(0) = v(1) = 0$, and it follows by the du Bois-Reymond Lemma that, for some $c_1 \in \mathbb{R}$, $y'(x) = Cx + c_1, \forall x \in [0,1]$. Hence, there exists $c_0 \in \mathbb{R}$ such that

$$y(x) = \frac{C}{2}x^2 + c_1 x + c_0, \forall x \in [0,1].$$

The boundary condition $y(0) = 0$ implies $c_0 = 0$. The conditions

$$C = \int_0^1 y(x)\,dx = \int_0^1 \frac{C}{2}x^2 + c_1 x\,dx = \frac{C}{6} + \frac{c_1}{2} \quad \text{and} \quad 1 = y(1) = \frac{C}{2} + c_1$$

together give $C = \frac{6}{13}, c_1 = \frac{10}{13}$, and so

$$y(x) = \frac{3}{13}x^2 + \frac{10}{13}x, \quad \forall x \in [0,1].$$

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Problem 2

(a) Put $\mathscr{Y}_B := \{y \in \mathscr{Y} : y(b) = B\}$ for any $B \in \mathbb{R}$. Then, finding each optimizer y_B of J over $\mathscr{Y}_B$ is simply the basic problem in the Calculus of Variations. If we are able to solve this problem using the standard machinery, then, optimizing the function $B \mapsto J(y_B)$ is a one-dimensional problem, which may again be solved with standard machinery in many cases.

(b) First note that J is unbounded above on $\mathscr{Y}$. For $n \in \mathbb{N}$, defining $y_n \in \mathscr{Y}$ by $y_n(x) = nx+1, \forall x \in [0,1]$, we clearly have $J(y_n) \to +\infty$ as $n \to \infty$.

For $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$f(x,y,z) := y^2 + z^2 \quad \forall x \in [0,1], y, z \in \mathbb{R},$$

$J(y) = \int_0^1 f(x, y(x), y'(x))\, dx + y(1)^2$, for all $y \in \mathscr{Y}$. Since

$$f_{,2}(x,y,z) = 2y \quad \text{and} \quad f_{,3}(x,y,z) = 2z, \quad \forall x \in [0,1], y, z \in \mathbb{R},$$

if y minimizes J on $\mathscr{Y}$, as discussed in part (a), the 1^{st} Euler-Lagrange Equation gives

$$y(x) = \frac{d}{dx} y'(x) = y''(x), \quad \forall x \in [0,1],$$

so that $y = c_1 \cosh + c_2 \sinh$ for some $c_1, c_2 \in \mathbb{R}$. Since $y(0) = 1$, $c_1 = 1$. Hence,

$$\begin{aligned} J(y) &= \int_0^1 (\cosh(x) + c_2 \sinh(x))^2 + (\sinh(x) + c_2 \cosh(x))^2\, dx + (\cosh(1) + c_2 \sinh(1))^2 \\ &= \frac{1}{2}(c_2^2 + 1)\sinh(2) + c_2(\cosh(2) - 1) + (\cosh(1) + c_2 \sinh(1))^2 \end{aligned}$$

This is simply a second-order polynomial in c_2, with a positive leading coefficient, and hence can be minimized by differentiating with respect to c_2.

Problem 3

First note that J is unbounded above on $\mathscr{S}$. For $n \in \mathbb{N}$, define $y_n \in \mathscr{S}$ by

$$y_n(x) := \frac{2}{e-1} \sin\left(\frac{2\pi n}{e-1}(x-1)\right).$$

Then,

$$J(y_n) = \int_1^e x^2 y_n'(x)^2 \, dx = \int_1^e x^2 \left(\frac{4\pi n}{(e-1)^2} \cos\left(\frac{2\pi n}{e-1}(x-1)\right)\right)^2 dx \to +\infty$$

as $n \to \infty$.

Put $\mathscr{Y} := \{y \in C^1[1,e] : y(1) = y(e) = 0\}$ and $\mathscr{V} := \mathscr{Y}$, and let $G : \mathscr{Y} \to \mathbb{R}$ defined by

$$G(y) = \int_1^e y(x)^2 \, dx, \quad \forall y \in \mathscr{Y}.$$

Then, for $\forall y \in \mathscr{Y}, v \in \mathscr{V}$,

$$\delta G(y;v) = \int_1^e 2y(x)v(x) \, dx \quad \text{and} \quad \delta J(y;v) = \int_1^e 2x^2 y'(x)v'(x) \, dx.$$

$\delta G(y;v) = 0$ for all $v \in \mathscr{V}$ only if $y = 0 \notin \mathscr{S}$ and hence, if y minimizes J on $\mathscr{S}$, $\exists \lambda \in \mathbb{R}$ such that, $\forall v \in \mathscr{V}$,

$$2\int_1^e x^2 y'(x)v'(x) \, dx = J(y;v) = \lambda G(y;v) = 2\lambda \int_1^e y(x)v(x) \, dx$$

Let $Y \in C^2[0, 2\pi]$ such that $Y' = y$. Rearranging and integrating by parts gives

$$0 = \int_1^e \left(x^2 y'(x) + \lambda Y(x)\right) v'(x) \, dx, \quad \forall v \in \mathscr{V}.$$

Hence, by the du Bois-Reymond Lemma, $x^2 y'(x) + \lambda Y(x)$ is a constant in x. Since $x \in [1,e]$, it follows that $y' \in C^2[1,e]$, and, differentiating, we have that

$$x^2 y''(x) + 2xy'(x) + \lambda y(x) = 0.$$

General solutions to this equation are of the form

$$y(x) = c_1 x^{C-1/2} + c_2 x^{-C-1/2}, \quad \forall x \in [1,e],$$

for some $c_1, c_2, C \in \mathbb{R}$ ($C = \sqrt{1-4\lambda}$). The condition $y(1) = y(e) = 0$ gives

$$\begin{aligned} 0 &= c_1 + c_2 \quad \Rightarrow \quad c_2 = -c_1 \\ 0 &= c_1 e^{C-1/2} + c_2 e^{-C-1/2} = c_1 \left(e^{C-1/2} - e^{-C-1/2}\right), \end{aligned}$$

so that either $c_1 = c_2 = 0$ or $C - 1/2 = -C - 1/2$ (since the exponential function is injective). In the first case, clearly $y = 0$. In the second case, $C = 0$, and hence, again, $\forall x \in [1,e]$, $y(x) = (c_1 + c_2)x^{-1/2} = 0$. Then, however, $G(y) = 0$, and so J has no minimizer on $\mathscr{S}$.

Problem 4

First note that J is unbounded above on $\mathscr{S}$. For $n \in \mathbb{N}$, define $y_n \in \mathscr{S}$ by $y_n(x) := \sin(nx)$. Then,

$$J(y_n) = \int_0^{2\pi} y_n'(x)^2 - y_n(x)^2 \, dx = \int_0^{2\pi} n^2 \cos^2(nx) - \sin^2(nx) \, dx = (n^2 - 1)\pi \to +\infty$$

as $n \to \infty$.

Put $\mathscr{Y} := \{y \in C^1[0, 2\pi] : y(0) = y(2\pi) = 0\}$ and $\mathscr{V} := \mathscr{Y}$, and let $G : \mathscr{Y} \to \mathbb{R}$ defined by

$$G(y) = \int_0^{2\pi} y(x) \, dx, \quad \forall y \in \mathscr{Y}.$$

Then, for $\forall y \in \mathscr{Y}, v \in \mathscr{V}$,

$$\delta G(y; v) = \int_0^{2\pi} v(x) \, dx \quad \text{and} \quad \delta J(y; v) = \int_0^{2\pi} 2y'(x)v'(x) - 2y(x)v(x) \, dx.$$

Since it is not the case that $\delta G(y; v) = 0$ for all $v \in \mathscr{V}$, if y minimizes J on $\mathscr{S}$, $\exists \lambda \in \mathbb{R}$ such that, $\forall v \in \mathscr{V}$,

$$2 \int_0^{2\pi} y'(x)v'(x) - y(x)v(x) \, dx = J(y; v) = \lambda G(y; v) = \lambda \int_0^{2\pi} v(x) \, dx$$

Let $Y \in C^2[0, 2\pi]$ such that $Y' = y$. Rearranging and integrating by parts gives

$$0 = \int_0^{2\pi} y'(x)v'(x) - \left(y(x) + \frac{\lambda}{2}\right) v(x) \, dx = \int_0^{2\pi} \left(y'(x) + Y(x) + \frac{\lambda}{2}x\right) v'(x) \, dx$$

since $v(0) = v(2\pi) = 0$. Hence, by the du Bois-Reymond Lemma, $y'(x) + Y(x) + \frac{\lambda}{2}x$ is a constant in x. It follows that $y \in C^2[0, 2\pi]$ and $y''(x) + y(x) + \frac{\lambda}{2} = 0$, $\forall x \in [0, 2\pi]$. Solutions to this differential equation are of the form

$$y(x) = c_1 \cos(x) + c_2 \sin(x) - \frac{\lambda}{2}$$

for some $c_1, c_2 \in \mathbb{R}$. The constraint $\int_0^{2\pi} y(x) \, dx = 0$ implies $\lambda = 0$, and hence the boundary condition $y(0) = 0$ implies $c_1 = 0$. Thus, y is any multiple of sin, since

$$J(y) = c_2^2 \int_0^{2\pi} \cos^2(x) - \sin^2(x) \, dx = 0.$$

Problem 5

(a) Let $\mathscr{Y} := \{y \in C^1[-1,1] : y(-1) = y(1) = 0\}$ and $\mathscr{V} := \mathscr{Y}$, and let $G : \mathscr{Y} \to \mathbb{R}$ defined by

$$G(y) = \int_{-1}^{1} \sqrt{1 + y'(x)^2}\, dx, \quad \forall y \in \mathscr{Y}.$$

Then, for $\forall y \in \mathscr{Y}, v \in \mathscr{V}$,

$$\delta G(y; v) = \int_{-1}^{1} \frac{y'(x)v'(x)}{\sqrt{1 + y'(x)^2}}\, dx \quad \text{and} \quad \delta J(y; v) = \int_{-1}^{1} \sqrt{1 + y'(x)^2} v(x) + \frac{y(x)y'(x)v'(x)}{\sqrt{1 + y'(x)^2}}\, dx.$$

If, $\forall v \in \mathscr{V}$, $\delta G(y; v) = 0$, then the du Bois-Reymond Lemma gives that $\frac{y'(x)}{\sqrt{1+y'(x)^2}}$ is constant in x. Since this is a strictly increasing function of $y'(x)$, $y'(x)$ must be constant in x, and hence y is an affine function. The boundary conditions then imply that $y(x) = 0, \forall x \in [-1, 1]$, which breaks the constraint $G(y) = 4$. Thus, we have that, if y minimizes J on $\mathscr{S}$, $\exists \lambda \in \mathbb{R}$ such that, $\forall v \in \mathscr{V}$,

$$\int_{-1}^{1} \sqrt{1 + y'(x)^2} v(x) + \frac{y(x)y'(x)v'(x)}{\sqrt{1 + y'(x)^2}}\, dx = J(y; v) = \lambda G(y; v) = \lambda \int_{-1}^{1} \frac{y'(x)v'(x)}{\sqrt{1 + y'(x)^2}}\, dx.$$

Let $Y \in C^2[-1, 1]$ such that $Y'(x) = -\sqrt{1 + y'(x)^2}$ for all $x \in [-, 1, 1]$. Integrating the first term on the left by parts and rearranging gives

$$\int_{-1}^{1} \left(Y(x) + \frac{y(x)y'(x)}{\sqrt{1 + y'(x)^2}} - \lambda \frac{y'(x)}{\sqrt{1 + y'(x)^2}} \right) v'(x)\, dx = 0,$$

since $v(-1) = v(1) = 0$. By the du Bois-Reymond Lemma, $\exists C \in \mathbb{R}$ such that, $\forall x \in [-1, 1]$,

$$C = Y(x) + \frac{y(x)y'(x)}{\sqrt{1 + y'(x)^2}} - \lambda \frac{y'(x)}{\sqrt{1 + y'(x)^2}} = Y(x) + (y(x) - \lambda) \frac{y'(x)}{\sqrt{1 + y'(x)^2}}.$$

Since the function $z \mapsto \frac{z}{\sqrt{1+z^2}}$ has a differentiable inverse, it follows that, if $y(x) \neq \lambda$, y' is differentiable at x. Differentiating gives

$$0 = -\sqrt{1 + y'(x)^2} + \frac{y'(x)^2}{\sqrt{1 + y'(x)^2}} + \frac{(y(x) - \lambda)y''(x)}{(1 + y'(x)^2)^{3/2}}.$$

Multiplying by $\sqrt{1 + y'(x)^2}$ and simplifying gives

$$1 + y'(x)^2 = (y(x) - \lambda)y''(x).$$

I wasn't sure how to proceed further, as I couldn't characterize solutions to this ODE in a useful manner.

(b) I wasn't able to finish this part.