Make-up Assignment
21-484A Graph Theory
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Problem 1

Since Q_0 is K_1 and Q_1 is K_2, and, by convention, $\forall i \in \mathbb{N}$, $\kappa(K_i) = i - 1$, $\kappa Q_0 = 0$ and $\kappa Q_1 = 1$. Since, by convention, $\lambda(K_1) = 0$, $\lambda(Q_0) = 0$. Finally, since removing the only edge in Q_1 creates a disconnected graph, $\lambda(Q_1) = 1$. Therefore, the claim in question holds for $k \in \{0, 1\}$.

Lemma: $\forall k \in \mathbb{N}\backslash\{0, 1\}$, $\forall u, v \in V(Q_k)$ there exists a set of k internally disjoint paths from u to v,

Proof: The proof goes by induction on k. For $k = 2$, the Lemma is trivial, since there are two internally disjoint paths between any two vertices of a square. Suppose, as an inductive hypothesis, that the lemma holds for some $k \in \mathbb{N}\backslash\{0, 1\}$. Let $u = (u_1, u_2, \dots, u_{k+1})$ and $v = (v_1, v_2, \dots, v_{k+1})$ be two vertices in Q_{k+1}. We split into two cases:

Case 1: There exists some $i \in \{1, 2, \dots, k + 1\}$ such that $u_k = v_k$. Then, the subgraph of Q_{k+1} induced by the set of vertices with i^{th} component $u_i = v_i$ is Q_k, so that, by the inductive hypothesis, there are k internally disjoint paths from u to v going only through vertices with i^{th} component $u_i = v_i$. There also exists another from u to v path through $(u_1, u_2, \dots, 1 - u_i, \dots, u_k)$ and $(v_1, v_2, \dots, 1 - v_i, \dots, v_k)$, and going through only vertices with i^{th} component $1 - u_k = 1 - v_k$, so that adding this to the set of k internally disjoint paths from u to v creates a set of $k+1$ internally disjoint paths from u to v. Therefore, there exists a set of $k + 1$ internally disjoint paths from u to v, concluding this case.

Case 2: $\forall i \in \{1, 2, \dots, k + 1\}$, $u_i \neq v_i$ (i.e., u and v are 'opposite' vertices in Q_{k+1}). Consider constructing $k+1$ paths $P_1, P_2, \dots, P_{k+1}$ from u to v in the following fashion: $\forall i, j \in \{1, 2, \dots, k+1\}$, the j^{th} vertex of the i^{th} path is

$$(u_1, u_2, \dots, u_{i-1}, 1 - u_i, 1 - u_{i+1}, \dots, 1 - u_{i+j-1}, u_{i+j}, u_{i+j+1}, \dots, u_{k+1}k).$$

(i.e., the i^{th} path proceeds by 'flipping' consecutive component bits on each step of the path, starting with the i^{th} component. Then, $\{P_1, P_2, \dots, P_{k+1}\}$ is an internally disjoint set of paths from u to v, concluding the proof of the claim.

By Menger's Theorem, it follows from this claim that, $\forall u, v \in V(Q_k)$, any minimal u-v separating set has size at least k. Thus, any vertex-cut of Q_k has size at least k, and therefore $\kappa(Q_k) \geq k$. Furthermore, by Theorem 5.11 (Whitney), $\lambda(Q_k) \geq \kappa(Q_k) \geq k$.

Suppose that, for some $k \in \mathbb{N}\backslash\{0, 1\}$, with $k \geq 2$, $v \in (Q_k)$. Since there are exactly k sequences of ones and zeros differing from v in exactly one coordinate, the degree of v in Q_k is k. Since there are clearly vertices that are not adjacent to v, removing the set of neighbors of v from G results in a disconnected graph, so that $\kappa(Q_k) \leq k$. Removing each of the k edges incident to v from Q_k clearly results in a disconnected graph, so that $\lambda(Q_k) \leq k$.

Therefore, $\kappa(Q_k) = \lambda(Q_k) = k$. ■