ASSIGNMENT NUMBER 3, 21.630 Spring 2013

Due Wednesday, February 6, 2013

1. Let $T \in [0, \infty)$, $g : [0, T] \to \mathbb{R}$ be continuous, and $f : \mathbb{R} \to \mathbb{R}$ be continuous and bounded.

A) For each positive integer, n, define $X^{(n)}$ by $X^{(n)}(t) = g(0)$ if $t < 0$ and

$$X^{(n)}(t) = g(t) + t \int_0^t f(X^{(n)}(s - 1/n))ds$$

if $0 \leq t \leq T$. Prove that there is a subsequence, $X^{(k_n)}$, that converges uniformly on $[0, T]$. Define it's limit to be X.

B) Prove that $X^{(k_n)}(t - 1/k_n)$ converges uniformly to X on $[0, T]$.

C) Prove that

$$X(t) = g(t) + t \int_0^t f(X(s))ds$$

for all $t \in [0, T]$.

2. Let $t_0 \in \mathbb{R}$, $A \in \mathbb{R}$, and $B \geq 0$. Define $\mathcal{F} : \mathcal{C}[t_0, \infty) \to \mathcal{C}[t_0, \infty)$ by

$$\mathcal{F}[X](t) = A + B \int_{t_0}^t X(s)ds.$$

Also define

$$\overline{X}(t) = Ae^{B(t-t_0)}.$$

A) Show that if $X \leq Y$ (meaning $X(t) \leq Y(t)$ for all $t \in [t_0, \infty)$), then $\mathcal{F}[X] \leq \mathcal{F}[Y]$.

B) Show that

$$\mathcal{F}[\overline{X}] = [\overline{X}].$$

C) Assume that $X \in \mathcal{C}[t_0, \infty)$ satisfies

$$X \leq \mathcal{F}[X].$$

Define a sequence, $\{X^{(n)}\}$, by $X^{(0)} = X$ and $X^{(n+1)} = \mathcal{F}[X^{(n)}]$. Show that $X^{(n+1)} \geq X^{(n)}$ for all $n \geq 1$. Then show that $X \leq \overline{X}$. You may assume that $X^{(n)}$ converges pointwise to $\overline{X}$ to do this. Note: this is an important result.

We'll prove it in class another way.

3. Assume that f is continuous on $[t_0, \infty) \times \mathbb{R}^N$ and satisfies

$$|f(t,x)| \leq a(t) + b(t)|x|$$

for all $(t,x) \in [t_0, \infty) \times R^N$ where a and b are continuous functions on $[t_0, \infty)$. Show that every solution of $\frac{dX}{dt}(t) = f(t, X(t))$ may be extended to the interval $[t_0, \infty)$. Hint: Use problem 2.

4. Suppose that $p \in (0,1]$ and $X : [0, \infty) \to \mathbb{R}$ is continuous and satisfies

$$|X(t)| \leq \int_0^t |X(s)|^p ds$$

for all $t \in [0, \infty)$. Show that if $p = 1$ then $X(t) = 0$ for all $t \in [0, \infty)$ (using problem 2 this is very short). Now consider $p \in (0,1)$. Does X have to vanish? Either prove it does or give a counterexample to show it does not.