Homework 7

21-260 Differential Equations
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Section 3.5, Problem 12

Taking inspiration from the given hint, we suppose that a solution to the given differential equation is of the form $Y(t) = Ae^{2t} + Be^{-2t}$, for some $A, B \in \mathbb{R}$. Then,

$$Y' = 2Ae^{2t} - 2Be^{-2t} \quad \text{and} \quad Y'' = 4Ae^{2t} + 4Be^{-2t},$$

so that plugging Y into the differential equation gives $(1+2+4)A = \frac{1}{2}$ and $(1-2+4)B = \frac{1}{2}$. Thus,

$$Y(t) = \boxed{\frac{1}{14}e^{2t} + \frac{1}{6}e^{-t}.}$$

The characteristic equation of the homogeneous equation corresponding to the given differential equation is $r^2 - r - 2 = 0$, whose roots are $r \in \{1, 2\}$. Thus, the solutions to the given differential equation are of the form

$$y = \boxed{c_1e^t + c_2e^{2t} + Y(t),} \quad c_1, c_2 \in \mathbb{R}.$$

Section 3.5, Problem 16

Suppose that some particular solution of the given differential equation is of the form

$$Y(t) = (A_1t + A_0)e^{2t}.$$

Then, $Y' = A_1e^{2t} + 2Y(t)$ and $Y'' = 2A_1e^{2t} + 2Y'(t)$. Thus, plugging this solution into the given differential equation and simplifying gives $(2A_1 - 3A_0)e^{2t} - 3A_1te^{2t} = 3te^{2t}$, so that $-3A_1 = 3$ and $2A_1 - 3A_0 = 0$, and thus $A_1 = -1$ and $A_0 = -\frac{2}{3}$.

The characteristic equation of the homogeneous equation associated with the given differential equation is $r^2 - 2r - 3 = 0$, whose roots are $r \in \{-1, 3\}$. Therefore, the general solution of the given equation is of the form

$$y = \boxed{c_1e^{-t} + c_2e^{3t} + Y(t),} \quad c_1, c_2 \in \mathbb{R},$$

$$Y(t) = \boxed{-\left(t + \frac{2}{3}\right)e^{2t}.}$$

Solving for c_1 and c_2 using the givens $y(0) = 1$ and $y'(0) = 0$ gives $\boxed{c_1 = \frac{2}{3}}$ and $\boxed{c_2 = 1.}$

Section 3.6, Problem 16

Since, for $y = y_1$,

$$(1-t)y'' + ty' - y = (1-t)e^t + te^t - e^t = (1-t)e^t - (1-t)e^t = 0,$$

and, for $y = y_2$,

$$(1-t)y'' + ty' - y = (1-t)\cdot 0 + t\cdot 1 - t = t - t = 0,$$

y_1 and y_2 indeed satisfy the homogeneous differential equation corresponding to the given differential equation. Furthermore, since the Wronskian of y_1 and y_2 is $(1-t)e^t$, which is 0 only when $t = 1$, $\{y_1, y_2\}$ is a fundamental set of solutions of this corresponding homogeneous differential equation on the interval $t \in (0,1)$.

Thus, by Theorem 3.6.1, a particular solution of the given differential equation is, for $t_0 = 0$,

$$\begin{aligned}
Y(t) &:= -e^t \int_{t_0}^{t} \frac{2s(s-1)^2 e^{-s}}{(1-s)e^s}\, ds + t \int_{t_0}^{t} \frac{2e^s(s-1)^2 e^{-s}}{(1-s)e^s}\, ds \\
&= -e^t \int_{t_0}^{t} 2s(1-s)e^{-2s}\, ds + t \int_{t_0}^{t} 2(1-s)e^{-s}\, ds \\
&= -e^t \left(s^2 e^{-2s}\right)\Big|_{s=t_0}^{s=t} + t\left(2se^{-s}\right)\Big|_{s=t_0}^{s=t} \\
&= -t^2 e^{-t} + 2t^2 e^{-t} = \boxed{t^2 e^{-t}.}
\end{aligned}$$

Thus, the general solution to the given differential equation is of the form

$$y = \boxed{c_1 e^t + c_2 t + Y(t),} \quad c_1, c_2 \in \mathbb{R}.$$

Section 3.7, Problem 18

As derived in the context of the mass-on-a-spring system, a system obeying the differential equation

$$ay'' + by' + cy = 0$$

(where $a, b, c \in \mathbb{R}$) is critically damped when $b = 2\sqrt{ac}$. Thus, since the given series circuit obeys the differential equation

$$LQ'' + RQ' + \frac{1}{C}Q = 0,$$

the circuit is critically damped when

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{0.2\,H}{0.8\times 10^{-6}\,F}} = \boxed{1000\,\Omega.}$$

Section 3.8, Problem 12

This system obeys the differential equation

$$mu'' + \gamma' + ku = 3\cos(3t)N - 2\sin(3t)N,$$

where $m = 2\,kg$, $\gamma = 1\,kg/s$, and $k = 3N/m$. Suppose that a particular solution of this system is of the form

$$Y(t) = A\cos(3t) + B\sin(3t), \quad A, B \in \mathbb{R}.$$

Then, plugging Y into the given differential equation gives $-15A + 3B = 3$ and $-3A - 15B = 2$, so that $A = -17/78$ and $B = -7/78$, and thus

$$Y(t) = \boxed{-\frac{17}{78}\cos(3t) + -\frac{7}{78}\sin(3t)}.$$

The characteristic equation of the homogeneous differential equation associated with the given differential equation is $2r^2 + r + 3 = 0$, whose roots are $r = -\frac{1}{4}(1 \pm \sqrt{23}i)$, Thus, the general solution the given differential equation are of the form

$$y = c_1 e^{-\frac{1}{4}t}\cos(\sqrt{23}t) + c_2 e^{-\frac{1}{4}t}\sin(\sqrt{23}t) + Y(t),$$

for some $c_1, c_2 \in \mathbb{R}$.

Since the real parts of both roots of the above characteristic equation are negative, both solutions to the homogeneous equation are transient, and the steady state response of the system is identical to the particular solution Y of the given nonhomogeneous equation.