Midterm 1

21-640 Introduction to Functional Analysis
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Due: Friday, March 29, 2013

Problem 1

We show $\mathcal{L}(l^2, l^2)$ is not separable by finding an uncountable subset with discrete induced topology. Let $\mathcal{P}(\mathbb{N})$ denote the set of subsets of $\mathbb{N}$. $\forall K \in \mathcal{P}(N)$, define $T_K : l^2 \to l^2$ by

$$(T_K(x))_i = \begin{cases} x_i & : \text{ if } i \in K \\ 0 & : \text{ else} \end{cases}, \forall i \in \mathbb{N}, x \in l^2.$$

Let $S := \{T_K : K \in \mathcal{P}(\mathbb{N}) \backslash \{\emptyset\}\}$. Clearly, S is uncountable. $\forall T \in S$, linearity is clear, and, furthermore, $\forall x \in l^2$, $\|T(x)\|_2 \leq \|x\|_2$, so that T is bounded. Thus, $S \subseteq \mathcal{L}(l^2, l^2)$. Also, for each $T \in S$, it is easy to find $x \in l^2$ with $T_K(x) = x$, and so $\|T_K\| \geq 1$.

If $K_1, K_2 \in \mathcal{P}(\mathbb{N})$ are distinct, then, since $K_3 := (K_1 \backslash K_2) \backslash (K_2 \backslash K_1) \neq \emptyset$, $\|T_{K_1} - T_{K_2}\| = \|T_{K_3}\| \geq 1$. Hence, for $T, R \in S$ distinct, $B_{\frac{1}{2}}(T) \cap B_{\frac{1}{2}}(S) = \emptyset$.

If there were a dense set $A \subseteq \mathcal{L}(l^2, l^2)$, then, $\forall T \in S$, $B_{\frac{1}{2}}(T) \cap A \neq \emptyset$. However, a countable set cannot have non-empty intersection with uncountably many disjoint sets, so A is uncountable. ■

Problem 2

We prove the contrapositive statement. Suppose T is discontinuous and hence unbounded on $B_1(0)$. Then, there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that, $\forall n \in \mathbb{N}$, $\|T(x_n)\| \geq n^2$. $\forall n \in \mathbb{N}$, $\left\|\frac{x_n}{n}\right\| \leq n$, so that, as $n \to \infty$, $x_n \to 0$, but $\left\|T\left(\frac{x_n}{n}\right)\right\| \geq n \to \infty$. ■

Problem 3

If $x_1, x_2 \in X$ with $x_1 = y_1 + z_1, x_2 = y_2 + z_2$, $y_1, y_2 \in Y$, $z_1, z_2 \in Z$, then, $\forall a, b \in \mathbb{K}$,

$$ax_1 + bx_2 = ay_1 + by_2 + az_1 + bz_2.$$

Since Y is a linear manifold, $ay_1 + by_2 \in Y$ so that

$$T(ax_1 + bx_2) = ay_1 + by_2 = aT(x_1) + bT(x_2).$$

Thus, T is linear. Since $\forall x \in X, T(x) \in Y$, $T^2 = T$. Thus, by the result of Problem 5, to show that T is continuous, it suffices to show that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are closed.

Since $x = 0 + x$, it follows from the uniqueness assumption that $T(x) = 0$ if and only if $x \in Z$, and so $\mathcal{N}(T) = Z$, which is closed. Also, clearly, $\mathcal{R}(T) = Y$, which is closed.

Since T is linear and continuous, $T \in \mathcal{L}(X, X)$. The proof that $L \in \mathcal{L}(X, X)$ is identical. ■

Problem 4

(a) By Theorem 7.15, $\|x_n\|$ is bounded by some $B \in \mathbb{R}$. Since $\|x_n^* - x^*\|_*, \|x^*(x_n) - x^*(x)\| \to 0$,

$$\|x_n^*(x_n) - x^*(x)\| \leq \|x_n^*(x_n) - x^*(x_n)\| + \|x^*(x_n) - x^*(x)\| \leq \|x_n^* - x^*\|_* B + \|x^*(x_n) - x^*(x)\| \to 0$$

as $n \to 0$. ■

(b) By Theorem 7.24, x_n^* is bounded by some $B \in \mathbb{R}$. Since $\|x_n^*(x) - x_n^*(x)\|, \|x_n - x\| \to 0$,

$$\|x_n^*(x_n) - x^*(x)\| \leq \|x_n^*(x_n) - x_n^*(x)\| + \|x_n^*(x) - x^*(x)\| \leq B\|x_n - x\| + \|x_n^*(x) - x^*(x)\| \to 0$$

as $n \to 0$. ■

Problem 5

We first note that, since $T^2 = T$, $\forall y \in \mathcal{R}(T), y = T(y)$.

($\Rightarrow$) Suppose T is continuous. Since the singleton $\{0\}$ is closed, $\mathcal{N}(T) = T^{-1}[\{0\}]$ is closed. Suppose $\exists y_n \in \mathcal{R}(T)$ with $\lim_{n\to\infty} y_n = y \in X$. Then,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} T(y_n) = T\left(\lim_{n\to\infty} y_n\right) = T(y) \in \mathcal{R}(T),$$

since T is continuous. Thus, $\mathcal{R}(T)$ is closed. ■

($\Leftarrow$) Suppose that $\mathcal{N}(T), \mathcal{R}(T)$ are closed. $\forall n \in \mathbb{N}$, let $(x_n, y_n) \in \mathrm{Gr}(T)$ with $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Since $\mathcal{R}(T)$ is closed, $y = \lim_{n\to\infty} y_n \in \mathcal{R}(T)$, so that $y = T(y)$. Since each $y_n = T(y_n)$, $x_n - y_n \in \mathcal{N}(T)$, so that, since $\mathcal{N}(T)$ is closed, $x - y = \lim_{n\to\infty}(x_n - y_n) \in \mathcal{N}(T)$, and so $T(x) = T(y) = y$. Thus, $(x, y) \in \mathrm{Gr}(T)$, so $\mathrm{Gr}(T)$ is closed. Then, by the Closed Graph Theorem, T is continuous. ■

Problem 6

(a) Since $|||x||| \leq K\|x\|$, $\forall x \in X$, $\mathcal{L}((X, |||\cdot|||), \mathbb{K}) \subseteq \mathcal{L}((X, \|\cdot\|), \mathbb{K})$, and, similarly, $\mathcal{L}((X^*, |||\cdot|||_*), \mathbb{K}) \subseteq \mathcal{L}((X^*, \|\cdot\|_*), \mathbb{K})$, where X^* is defined in terms of the associated norm. Thus, since the canonical embedding J of X into X^{**} under $\|\cdot\|$, it is also a surjection under $|||\cdot|||$. Hence, $(X, |||\cdot|||)$ is also reflexive, and thus complete. It follows from Corollary 3.24 that $\|\cdot\|$ and $|||\cdot|||$ are in fact equivalent norms on X.

Thus, since Z is closed under the topology induced by $\|\cdot\|$, it is also closed under the topology $|||\cdot|||$. Then, since any closed subspace of a complete metric space is itself complete, (Z, ρ) is complete. ■

(b) I wasn't able to come up with a counterexample for this one. Assuming a counterexample lies in one of the spaces we've discussed, I did make the following observations that should narrow the space of possible counterexamples to very few possibilites:

(a) The only non-reflexive Banach spaces we've discussed are l^1, c_0, c, l^∞.

(b) Of these spaces, c_0, c, l^∞ have only one well-defined norm ($\|\cdot\|_\infty$, up to scaling), so the counterexample should be in l^1.

(c) In order for $(l^1, \|\cdot\|)$ to be a Banach space, $\|\cdot\| = \|\cdot\|_1$.

(d) The only norm that $\|\cdot\|_1$ bounds by a constant multiple is $\|\cdot\|_\infty$, so $|||\cdot||| = \|\cdot\|_\infty$.

I wasn't able to find a counterexample beyond this though...

Problem 7

Let $X := (C[0,1], \|\cdot\|_\infty)$ (so that X is a Banach space), and define, $\forall n \in \mathbb{N}$,

$$K_n := K \cap B_n, \qquad \text{where} \qquad K := \left\{ f \in X : f(0) = 0 \text{ and } \int_0^1 f(x)\,dx \geq 1 \right\}$$

and $B_n = \{f \in X : \|f\|_\infty \leq 1 + 1/n\}$. B_n is clearly bounded, convex and closed.

I showed in my solution to Problem 9 on Assignment 4 that K is convex and closed, so that, as the intersection of two convex and closed sets, each K_n is also convex and closed. I also demonstrated a family $\{f_n\} \in K$ with each $f_n \in B_n$, so that $f_n \in K_n$. Thus, $\{K_n\}_{n=1}^\infty$ satisfies condition (i).

I also showed that, $\forall f \in K$, $\|f\|_\infty > 1$, so that

$$\bigcap_{n=1}^\infty K_n = K \cap \bigcap_{n=1}^\infty B_n = K \cap \{f \in X : \|f\|_\infty \leq 1\} = \emptyset,$$

and so $\{K_n\}_{n=1}^\infty$ satisfies condition (iii).

Finally, $\forall n \in \mathbb{N}$, since $K_{n+1} = K \cap B_{n+1} \subseteq K \cap B_n = K_n$, $\{K_n\}_{n=1}^\infty$ satisfies condition (ii). ■

In principle, K could be any set satisfying the properties in Problem 9 on Assignment 4.

Problem 8

We construct a closed subspace $Z \subseteq X$ such that the restriction of T to Z bijection into Y, allowing us to use the Bounded Inverse Theorem to obtain the desired result.

By Proposition 1.21, we can construct a Hamel basis $(x_i | i \in I)$ for X such that, for some $J \subseteq I$, $(x_i | i \in J)$ is a Hamel basis for $\mathcal{N}(T)$. $\forall i \in I$, let α_i denote the projection onto x_i.

Define

$$Z := \{x \in X \mid \forall j \in J, \alpha_j(x) = 0\} = \bigcap_{j \in J} \mathcal{N}(\alpha_j).$$

By definition of the product topology, each α_i is continuous. Then, since projections are continuous, each $\mathcal{N}(\alpha_i)$ is closed, so that Z is a closed linear manifold in X, and hence Z is a Banach space.

Let $T_Z : Z \to Y$ denote the restriction of T to Z. It follows from the construction of J and Z that $Z \cap \mathcal{N}(T) = \{0\}$, so that T_Z is injective.

Let $x \in X$, and choose finite sets $J_x \subseteq J, I_x \subseteq I \backslash J$ by

$$x = \sum_{j \in J_x} \alpha_j(x) x_j + \sum_{i \in I_x} \alpha_i(x) x_i \quad \text{and} \quad \alpha_i(x) \neq 0, \forall i \in J_x \cup I_x.$$

Then,

$$T(x) = T\left(\sum_{j \in J_x} \alpha_j(x) x_j + \sum_{i \in I_x} \alpha_i(x) x_i\right) = \sum_{j \in J_x} \alpha_j(x) T(x_j) + T\left(\sum_{i \in I_x} \alpha_i(x) x_i\right) = T(x'),$$

for $x' := \sum_{i \in I_x} \alpha_i(x) x_i$. Furthermore, $x' \in Z$, and it follows that T_Z is surjective.

By the Bounded Inverse Theorem, T_Z has a bounded linear inverse T_Z^{-1}. Thus, given a convergent sequence $\{y_n\}_{n=1}^{\infty}$ in Y for $x_n := T_Z^{-1}(y_n)$, by continuity of T_Z, $\{x_n\}_{n=1}^{\infty}$ is convergent, and, furthermore,

$$\forall n \in \mathbb{N}, \text{ we have } y_n = T(x_n) \text{ and } \|x_n\| \leq \left\|T_Z^{-1}\right\|_* \|y_n\|. \quad \blacksquare$$

Problem 9

($\Rightarrow$) If p^K is continuous, then, since $p^K(0) = 0$, for some $\delta > 0$, $p^K < 1$ on $B_\delta(0)$. If $x \in B_\delta(0)$, then by definition of p^K, $\exists s \in (0,1)$ with $s^{-1}x \in K$. Then, since K is convex and $0 \in K$, $x = s(s^{-1}x) + (1-s)(0) \in K$, and so $B_\delta(0) \subseteq K$. $\blacksquare$

($\Leftarrow$) Suppose $B_\delta(0) \subseteq K$, for some $\delta > 0$. Note that, $\forall x \in X$, since $\frac{\delta x}{2\|x\|} \in B_\delta(0)$, by definition of p^K, $p^K(x) \leq 2\delta^{-1}\|x\|$. By part (c) of Lemma 5.32, $\forall x, h \in X$,

$$\begin{gathered} p^K(x+h) \leq p^K(x) + p^K(h) \leq p^K(x) + 2\delta^{-1}\|h\| \\ \text{and } p^K(x) = p^K(x+h-h) \leq p^K(x+h) + p^K(-h) \leq p^K(x+h) + 2\delta^{-1}\|h\|. \end{gathered}$$

Thus, $|p^K(x+h) - p^K(x)| \leq 2\delta^{-1}\|h\| \to 0$ as $h \to 0$, and so p^K is (Lipschitz) continuous at x. $\blacksquare$

Problem 10

We disprove the given statement.

Let $x_n^* = x^{(n)}$ and $x^* = x$, as defined in part (b) of Example 7.23 of the notes. As shown in the example, $x_n^* \xrightarrow{*} x^*$ (weakly*), but x_n^* does not converge weakly to x^* as $n \to \infty$, so that there exists $x^{**} \in X^{**}$ such that $x^{**}(x_n^*)$ does not converge to $x^{**}(x^*)$ as $n \to \infty$. $\blacksquare$