Assignment 6

15-359 Probability and Computing
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Section: B
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Problem 1: How we get them

Since $-\ln X < 0$ if and only if $X > 1$, $P(-lnX < 0) = 0$. Let $x \in \mathbb{R}$ with $0 \leq x$. Then, since $0 \leq e^{-x} \leq 1$,

$$P(0 \leq -\ln X \leq x) = P(e^{-x} \leq X \leq 1) = \int_{e^{-x}}^{1} 1\ dt = 1 - e^{-x}.$$

Differentiating gives the probability density function $f : \mathbb{R} \to \mathbb{R}$ such that, $\forall x \in \mathbb{R}$,

$$f_{-\ln X}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases},$$

which is that of an exponential distribution, where $\lambda = 1$. $\blacksquare$

Problem 2: The rain in Spain

Let $\lambda = 1/25$.

A. By definition of the conditional probability of a continuous random variable and the exponential distribution,

$$E[X|X > 10] = \frac{\int_{10}^{\infty} x f_X(x)\ dx}{\int_{10}^{\infty} f_X(x)\ dx} = \frac{\int_{10}^{\infty} xe^{-\lambda x})\ dx}{\int_{10}^{\infty} e^{-\lambda x}\ dx} = \frac{\int_{10}^{\infty} xe^{-\lambda x})\ dx}{\frac{1}{\lambda}(e^{-10\lambda})}.$$

Integrating by parts gives

$$\int_{10}^{\infty} xe^{-\lambda x})\ dx = \frac{10e^{-10\lambda}}{\lambda} - \int_{10}^{\infty} \frac{e^{-\lambda x}}{\lambda}\ dx = \left(\frac{10}{\lambda} + \frac{1}{\lambda^2}\right) e^{-10\lambda},$$

so that $E[X|X > 10] = 10 + \frac{1}{\lambda} = \boxed{35}$.

B. Since the exponential distribution is memoryless, $E[X|X > 10] = 10 + E[X] = 10 + \frac{1}{\lambda} = \boxed{35}$.

C. Integration by parts gives

$$\int_{10}^{\infty} x^2 e^{-\lambda x}\ dx = \frac{100e^{-10\lambda}}{\lambda} - \int_{10}^{\infty} \frac{-2xe^{-\lambda x}}{\lambda}\ dx = \frac{100e^{-10\lambda}}{\lambda} + \frac{2}{\lambda}\int_{10}^{\infty} xe^{-\lambda x}\ dx,$$

so that, by the integral computed in part A.,

$$\int_{10}^{\infty} x^2 e^{-\lambda x}\ dx = \frac{100e^{-10\lambda}}{\lambda} + \frac{2}{\lambda}\left(\frac{10}{\lambda} + \frac{1}{\lambda^2}\right) e^{-10\lambda} = \left(\frac{100}{\lambda} + \frac{2}{\lambda^2} + \frac{2}{\lambda^3}\right) e^{-10\lambda}.$$

Thus,

$$E[X^2|X>10] = \frac{\left(\left(\frac{100}{\lambda}+\frac{2}{\lambda^2}+\frac{2}{\lambda^3}\right)e^{-10\lambda}\right.}{\frac{1}{\lambda}(e^{-10\lambda})} = 100+\frac{2}{\lambda}+\frac{2}{\lambda^2} = \boxed{1400.}$$

Therefore, $\mathrm{Var}(X|X>10) = E[X^2|X>10] - E[X|X>10]^2 = 1400 - 35^2 = \boxed{175.}$ Since X is distributed exponentially with parameter λ, $\mathrm{Var}(X) = \frac{1}{\lambda^2} = 625 > \mathrm{Var}(X|X>10)$.

Problem 3: Failure rate

A. Since X is distributed exponentially, for some $\lambda > 0$, $\forall x \in \mathbb{R}$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases},$$

and

$$\overline{F}_X(x) = \begin{cases} e^{-\lambda x} & x \geq 0 \\ 1 & x < 0 \end{cases}.$$

Thus, for $x \geq 0$,

$$r(x) = \frac{f_X(x)}{\overline{F}_X(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda,$$

so that r is constant on $[0,\infty)$. ■

B. Suppose that $r = \lambda$, for some constant $\lambda > 0$ (noting that $r(x)$ is necessarily positive for some $x \in [0,\infty)$). Then, $\forall x \in [0,\infty)$,

$$\frac{f_X(x)}{1-F_X(x)} = \lambda,$$

so that $\lambda - \lambda F_X(x) = f_X(x)$. By definition of F, $\lambda - \int_{-\infty}^{x} f_X(t)\,dt = f_X(x)$. Thus, by the Fundamental Theorem of Calculus, differentiation gives

$$-\lambda f_X(x) = \frac{d}{dx} f_X(x).$$

The unique solution of this well-known differential equation is the exponential $f_X(x) = e^{-\lambda x}, \forall x \in [0,\infty)$. Thus, the exponential is the unique probability distribution with a constant failure rate. ■

Problem 4: λ

A. By definition of half-life, if $t_{1/2}$ is the half-life of the isotope in question, $\frac{1}{2} = e^{-\lambda t_{1/2}}$. Solving for $t_{1/2}$ gives $\boxed{t_{1/2} = \frac{\ln 2}{\lambda}.}$

B. Since $X_1, X_2, \ldots, X_n$ are independent,

$$F_Y(x) = P(Y < x) = \prod_{i=1}^{n} P(X < x) = (1 - e^{-\lambda x})^n.$$

Differentiating gives $f_Y(x) = n(1 - e^{-\lambda x})^{n-1}$. Thus, $E[Y] = n \int_0^\infty x(1 - e^{-\lambda x})^{n-1}\, dx$.

Problem 5: Sparse selection

A. Suppose that, in step 6., $a^- \leq b^- \leq b^+ \leq a^+$. By definition, A_0 is the set of elements in A in between b^- and b^+, so that $|A_0| \leq \delta(b^+, b^-) \leq \delta(a^+, a^-)$, where, $\forall x, y \in \mathcal{A}$, $\delta(x, y)$ denotes the number of elements of A between x and y (when A is sorted). Thus, since, by definition, $\lambda^- \geq n/2 - 2n^{3/4}$ and $\lambda^+ \leq n/2 + 2n^{3/4}$, $\delta(a^+, a^-) \leq n/2 + 2n^{3/4} - (n/2 - 2n^{3/4}) = 4n^{3/4}$, so that $|A_0| \leq 4n^{3/4}$, the desired result.

Problem 6: Bayes of our lives

A.

$$f_P(p|N > 47) = \sum_{i=48}^{\infty} \frac{P(N = i|P = p) f_P(p)}{P(N = i)} \cdot P(N = i) = \sum_{i=48}^{\infty} P(N = i|P = p) f_P(p)$$

$$= \sum_{i=48}^{\infty} (1 - p)^{i-1} p = (1 - p)^{47} p \sum_{i=0}^{\infty} (1 - p)^i = (1 - p)^{47} p \frac{1}{p} = \boxed{(1 - p)^{47}.}$$

B. By the result of part A., $E[P|N > 47] = \int_0^1 p(1 - p)^{47}\, dp = \boxed{\frac{1}{2352} \approx 0.00043.}$