Final Exam
15-423 Digital Signal Processing for CS
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1 Signals

1. See Figure 1 for graphs.

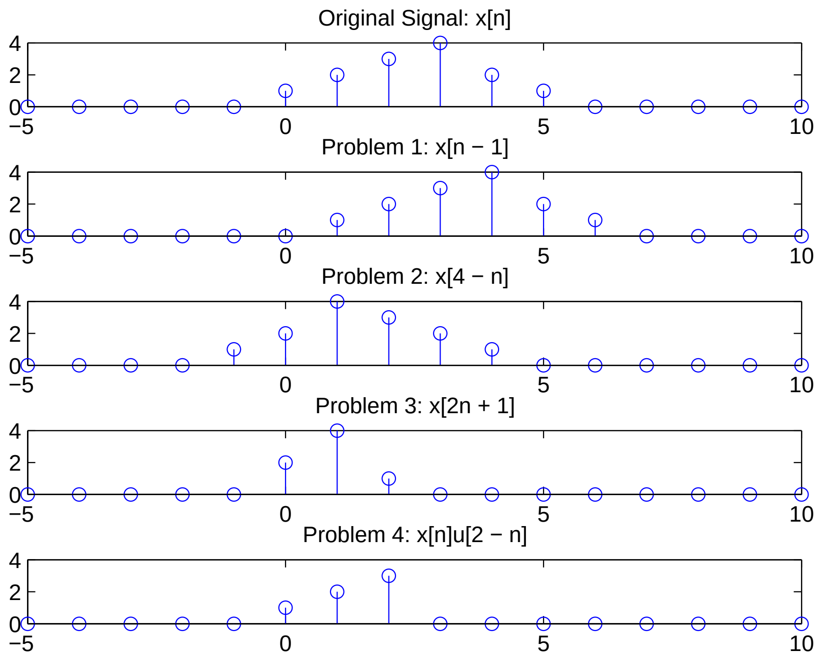

Figure 1: A signal $x[n]$ and some of its variants.

2. 1. $x[n]$ is not periodic.
   2. $x[n]$ is periodic with period $T = 7$, since $\exp\left(j\frac{8\pi t}{7} + \phi\right)$ has period $7/4$.
   3. $x(t)$ is periodic with period $T = \frac{2\pi}{7}$.
   4. $x(t)$ is not periodic.

2 Systems

1. The properties of each signal are given in the table below:

Problem	System	Memoryless	Shift-Invariant	Linear	Causal	Stable
1	$y[n] = x[n]x[n-1]$	No	Yes	No	Yes	Yes
2	$y[n] = nx[n]$	Yes	No	Yes	Yes	No
3	$y[n] = x[2n] - 0.1y[n-1]$	No	No	Yes	No	Yes
4	$y(t) = \sin(\omega t)x(t)$	Yes	No	Yes	Yes	Yes
5	$y(t) = \begin{cases} x[n] & n > 0 \\ 0 & n = 0 \\ -x[n] & n < 0 \end{cases}$	Yes	No	Yes	Yes	Yes

2. 1. By bilinearity and associativity of the convolution, the entire system response is to an input x is

$$(x * h_1 + x * h_2) * h_3 = x * ((h_1 + h_2) * h_3).$$

   It follows from bilinearity and shift-invariance of the convolution that the entire system is linear and shift-invariant. ■

   2. The output of the system is $x * ((h_1 + h_2) * h_3)$. The following MATLAB computation gives the result displayed in Figure 2:

```
>> h1 = [zeros(1,4) 0:4 zeros(1,7)];
>> h2 = [zeros(1,9) 3:-1:0 zeros(1,3)];
>> h3 = [zeros(1,4) ones(1,7) zeros(1,5)];
>> x = h3;
>> y = conv(x,conv((h1 + h2),h3));
```

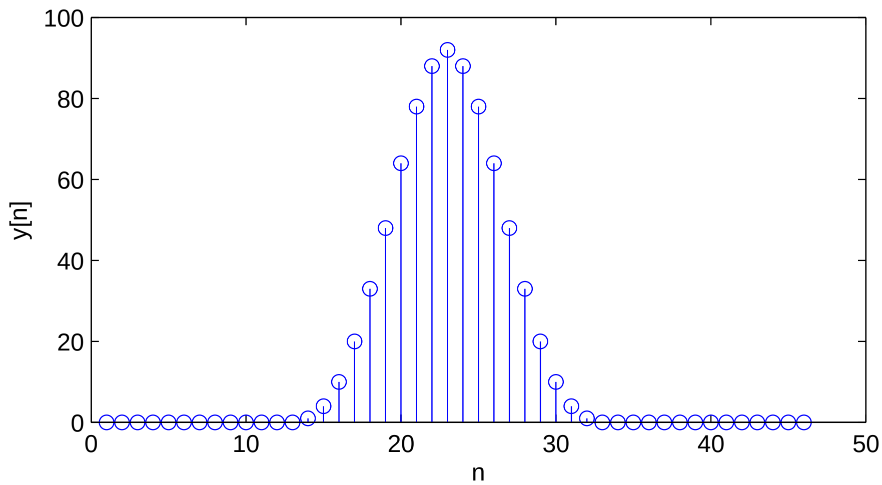

Figure 2: The response y of the composite system h to the input x.

3 Transforms

3.1 Fourier Transforms

1.
   - The following MATLAB code computes the discrete Fourier series coefficients:

```
>> for k = 0:11
     for n=0:11
       a(k+1,n+1) = (sin(2*pi*n/3)*cos(pi*n/2))*exp(-i*k*2*pi*n/12);
     end
   end
>> sum(a,2)/12

ans =

   0.0000
   0.0000 - 0.2500i
  -0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 0.0000i
  -0.0000 + 0.2500i
   0.0000 - 0.0000i
  -0.0000 - 0.2500i
  -0.0000 - 0.0000i
   0.0000 - 0.0000i
   0.0000 - 0.0000i
  -0.0000 + 0.2500i
```

   - The following MATLAB code computes the discrete Fourier series coefficients:

```
>> for k = 0:5
     for n=-2:3
       a(k+1,n+3) = ((1/2)^n)*exp(-i*k*2*pi*n/6);
     end
   end
>> sum(a,2)/6

ans =

   1.3125
   0.0000 + 0.7578i
  -0.3750 - 0.3248i
   0.4375 - 0.0000i
  -0.3750 + 0.3248i
   0.0000 - 0.7578i
```

- ```
  >> for k = 0:7
       for n=0:7
         a(k+1,n+1) = (x(n+1))*exp(-i*k*2*pi*n/8);
       end
     end
  >> sum(a,2)/8

  ans =

     0.5335
     0.0000 - 0.1875i
     0.1083 - 0.2500i
     0.0000 + 0.0000i
     0.3583 + 0.0000i
    -0.0000 + 0.1875i
    -0.2165 + 0.2500i
     0.0000 - 0.1875i
    -0.0000 - 0.0000i
     0.0000 - 0.0000i
     0.0000 - 0.0000i
    -0.0000 + 0.3750i
  ```

2. I chose not to do this problem.

3. - I didn't have time to finish this problem.
   - I didn't have time to finish this problem.
   - Letting
     $$I(t) := \sum_{k \in \mathbb{Z}} \delta(t-k)$$
     denote a unit impulse train, note that $x(t) = I(t) + I(t/2)$. Recalling that $\mathcal{F}\{I(t)\} = 2\pi I\left(\frac{\omega}{2\pi}\right)$, by linearity and the time-scaling property of Fourier Transforms,
     $$\boxed{\mathcal{F}\{x(t)\} = 2\pi\left(I\left(\frac{\omega}{2\pi}\right) + 2I\left(\frac{\omega}{\pi}\right)\right)}.$$

4. Writing x and y^* as Inverse Fourier Transforms of Fourier Transforms and recognizing that
   $$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, dt, \quad \forall \omega \in \mathbb{R},$$
  ```

gives

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)y^*(t)\,dt &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega_1)e^{i\omega_1 t}\,d\omega_1\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} Y^*(\omega_2)e^{-i\omega_2 t}\,d\omega_2\right)\,dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega_1)Y^*(\omega_2)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega_1-\omega_2)t}\,dt\,d\omega_1\,d\omega_1 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega_1)\int_{-\infty}^{\infty} Y^*(\omega_2)\delta(\omega_1-\omega_2)\,d\omega_2\,d\omega_1 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega_1)Y^*(\omega_1)\,d\omega_2
\end{aligned}$$

since convolution with a $\delta(x)$ function results in translation by x. ■

3.2 Z-Transforms

1. - I didn't have time to finish this problem.
   - $$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{-1} (z/2)^{-n} + \sum_{n=0}^{\infty}(2z)^{-n} = \frac{1}{1-z/2} + \frac{1}{1-(2z)^{-1}}.$$
   - $$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{9} z^{-n} = \frac{1-z^{-10}}{1-z^{-1}}.$$
2. - I didn't have time to finish this problem.
   - The poles are $z = 2$ and $z = 1/2$. The ROC is $\{1/2 < |z| < 2\}$, and hence the sequence has a Fourier transform.
   - The only pole is at the origin. The ROC is the entire plane, and hence the sequence has a Fourier transform.
3. I chose not to do this problem.
4. I didn't have time to finish this problem.

4 Filters

I.

II.