1 Mastery set [25 points] (Aaditya)

A1 [2] $\forall k \in 1, \ldots, n$, define $S_k := \sum_{i=1}^k \theta_i$ and $y_k := \displaystyle\sum_{i=1}^{k} \frac{\theta_i x_i}{S_k} \in C$. Suppose that, for some $k \in 1, \ldots, n-1$, $y_k \in C$. Then,

$$\begin{aligned} y_{k+1} := \sum_{i=1}^{k+1} \frac{\theta_i x_i}{S_{k+1}} &= \frac{\theta_{k+1} x_{k+1}}{S_{k+1}} + \sum_{i=1}^{k} \frac{\theta_i x_i}{S_{k+1}} = \frac{\theta_{k+1} x_{k+1}}{S_{k+1}} + \frac{S_k}{S_{k+1}} \sum_{i=1}^{k} \frac{\theta_i x_i}{S_k} \\ &= \frac{\theta_{k+1} x_{k+1}}{S_{k+1}} + \left(1 - \frac{\theta_{k+1}}{S_{k+1}}\right) \sum_{i=1}^{k} \frac{\theta_i x_i}{S_k} \in C, \end{aligned}$$

since C is convex. Since $y_1 = x_1 \in C$, by induction on k, $y = y_n \in C$. ■

A2 [3] We showed in class that $conv_2(M)$ is convex. Since each point in M is a convex combination of points in M, $M \subseteq conv_2(M)$, so $conv_1(M) \subseteq conv_2(M)$. If $C \supseteq M$ is convex, then, by part A1, any convex combination of points in M is in C. Thus, $conv_2(M) \subseteq conv_1(M)$. ■

B1 [2+2] $HP(a, b)$ is convex. If $\theta \in [0, 1]$ and $x_1, x_2 \in HP(a, b)$, then

$$a^T(\theta x_1 + (1-\theta)x_2) = \theta a^T x_1 + (1-\theta) a^T x_2 = \theta b + (1-\theta) b = b. \quad ■$$

If $x_1 \in HP(a, b_1)$ and $x_2 \in HP(a, b_2)$, then, by Cauchy-Schwarz,

$$\|x_1 - x_2\| \geq \left| \frac{a}{\|a\|}(x_1 - x_2) \right| = \boxed{\frac{|b_1 - b_2|}{\|a\|},}$$

and it is easily checked that $x_1 = \frac{b_1}{\|a\|^2} a$ and $x_2 = \frac{b_2}{\|a\|^2} a$ achieve this bound.

B2 [2+2] $HS(a, b)$ is convex. If $\theta \in [0, 1]$ and $x_1, x_2 \in HS(a, b)$, then

$$a^T(\theta x_1 + (1-\theta)x_2) = \theta a^T x_1 + (1-\theta) a^T x_2 \leq \theta b + (1-\theta) b = b. \quad ■$$

$HS(a_1, b_1) \subseteq HS(a_2, b_2)$ if and only if $\exists c \in \mathbb{R}$ with $a_1 = c a_2$ and $b_1 \leq c b_2$.

B3 [2] $\forall x \in \mathbb{R}^d$,

$$\begin{aligned} & \|u - x\|_2^2 \leq \|v - x\|_2^2 \\ \Leftrightarrow & \|u\|_2 - 2u^T x + \|x\|_2 \leq \|v\|_2 - 2v^T x + \|x\|_2 \\ \Leftrightarrow & \|u\| - \|v\| \leq 2(u - v)^T x. \end{aligned}$$

Thus, $\{x \in \mathbb{R}^d \mid \|u - x\| \leq \|v - x\|\} = HS(2(u - v), \|u\| - \|v\|)$, and is thus convex. ■

C [2+3] $\forall\theta \in [0,1], x, y \in \mathbb{R}_+$,

$$f(s(\theta x + (1-\theta)y) = f(\theta sx + (1-\theta)sy) \leq \theta f(sx) + (1-\theta)f(sy). \quad \blacksquare$$

Note that, via the change of variables $u = t/x$,

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\int_0^1 f(xu)x\,du = \int_0^1 f(xu)\,du.$$

Thus, $\forall\theta \in [0,1], x, y \in \mathbb{R}_+$, by convexity of the function $u \mapsto f(xu)$,

$$\begin{aligned} F(\theta x + (1-\theta)y) &= \int_0^1 f((\theta x + (1-\theta)y)u)\,du \\ &\leq \int_0^1 \theta f(xu) + (1-\theta)f(yu)\,du = \theta F(x) + (1-\theta)F(y). \quad \blacksquare \end{aligned}$$

D [3+2] The LP can be written in standard form as an LP over 6 variables:

$$0 \leq u = \begin{bmatrix} x_2 \\ y_2 \\ z_1 \\ z_2 \\ s_1 \\ s_2 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

The optimum occurs at $\boxed{(x, y, z) = (1, -1, 1),}$ when $\boxed{3x - y + z = 5.}$