a Do you expect the values of $\overline{Y}_n$ to cluster around any particular value? What value?

b If the results of 50 sample paths are plotted, how do you expect the variability of the estimates to change as a function of sample size?

c Click the button "New Sequence" several times. Did you observe what you expected based on your answers to parts (a) and (b)?

9.15 Refer to Exercise 9.3. Show that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are consistent estimators for θ.

9.16 Refer to Exercise 9.5. Is $\hat{\sigma}_2^2$ a consistent estimator of σ^2?

9.17 Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that $\overline{X} - \overline{Y}$ is a consistent estimator of $\mu_1 - \mu_2$.

9.18 In Exercise 9.17, suppose that the populations are normally distributed with $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Show that

$$\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2 + \sum_{i=1}^{n}(Y_i - \overline{Y})^2}{2n - 2}$$

is a consistent estimator of σ^2.

9.19 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y) = \begin{cases} \theta y^{\theta - 1}, & 0 < y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta > 0$. Show that $\overline{Y}$ is a consistent estimator of $\theta/(\theta + 1)$.

9.20 If Y has a binomial distribution with n trials and success probability p, show that Y/n is a consistent estimator of p.

9.21 Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from a normal population with mean μ and variance σ^2. Assuming that $n = 2k$ for some integer k, one possible estimator for σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{2k}\sum_{i=1}^{k}(Y_{2i} - Y_{2i-1})^2.$$

a Show that $\hat{\sigma}^2$ is an unbiased estimator for σ^2.

b Show that $\hat{\sigma}^2$ is a consistent estimator for σ^2.

9.22 Refer to Exercise 9.21. Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from a Poisson-distributed population with mean λ. Again, assume that $n = 2k$ for some integer k. Consider

$$\hat{\lambda} = \frac{1}{2k}\sum_{i=1}^{k}(Y_{2i} - Y_{2i-1})^2.$$

a Show that $\hat{\lambda}$ is an unbiased estimator for λ.

b Show that $\hat{\lambda}$ is a consistent estimator for λ.

9.23 Refer to Exercise 9.21. Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from a population for which the first four moments are finite. That is, $m_1' = E(Y_1) < \infty$, $m_2' = E(Y_1^2) < \infty$, $m_3' = E(Y_1^3) < \infty$, and $m_4' = E(Y_1^4) < \infty$. (*Note:* This assumption is valid for the normal and Poisson distributions in Exercises 9.21 and 9.22, respectively.) Again, assume

In the next section (optional), we summarize some of the convenient and useful large-sample properties of MLEs.

Exercises

9.80 Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the Poisson distribution with mean λ.

- **a** Find the MLE $\hat{\lambda}$ for λ.
- **b** Find the expected value and variance of $\hat{\lambda}$.
- **c** Show that the estimator of part (a) is consistent for λ.
- **d** What is the MLE for $P(Y = 0) = e^{-\lambda}$?

9.81 Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from an exponentially distributed population with mean θ. Find the MLE of the population variance θ^2. [*Hint:* Recall Example 9.9.]

9.82 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the density function given by

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{\theta}\right) r y^{r-1} e^{-y^r/\theta}, & \theta > 0, y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where r is a known positive constant.

- **a** Find a sufficient statistic for θ.
- **b** Find the MLE of θ.
- **c** Is the estimator in part (b) an MVUE for θ?

9.83 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a uniform distribution with probability density function

$$f(y \mid \theta) = \begin{cases} \dfrac{1}{2\theta + 1}, & 0 \le y \le 2\theta + 1, \\ 0, & \text{otherwise.} \end{cases}$$

- **a** Obtain the MLE of θ.
- **b** Obtain the MLE for the *variance* of the underlying distribution.

9.84 A certain type of electronic component has a lifetime Y (in hours) with probability density function given by

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{\theta^2}\right) y e^{-y/\theta}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

That is, Y has a gamma distribution with parameters $\alpha = 2$ and θ. Let $\hat{\theta}$ denote the MLE of θ. Suppose that three such components, tested independently, had lifetimes of 120, 130, and 128 hours.

- **a** Find the MLE of θ.
- **b** Find $E(\hat{\theta})$ and $V(\hat{\theta})$.
- **c** Suppose that θ actually equals 130. Give an approximate bound that you might expect for the error of estimation.
- **d** What is the MLE for the variance of Y?

9.85 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the density function given by

$$f(y \mid \alpha, \theta) = \begin{cases} \left(\dfrac{1}{\Gamma(\alpha)\theta^\alpha}\right) y^{\alpha-1}e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$ is known.

a Find the MLE $\hat{\theta}$ of θ.

b Find the expected value and variance of $\hat{\theta}$.

c Show that $\hat{\theta}$ is consistent for θ.

d What is the best (minimal) sufficient statistic for θ in this problem?

e Suppose that $n = 5$ and $\alpha = 2$. Use the minimal sufficient statistic to construct a 90% confidence interval for θ. [*Hint:* Transform to a χ^2 distribution.]

9.86 Suppose that $X_1, X_2, \ldots, X_m$, representing yields per acre for corn variety A, constitute a random sample from a normal distribution with mean μ_1 and variance σ^2. Also, Y_1 $Y_2, \ldots, Y_n$, representing yields for corn variety B, constitute a random sample from a normal distribution with mean μ_2 and variance σ^2. If the X's and Y's are independent, find the MLE for the common variance σ^2. Assume that μ_1 and μ_2 are unknown.

9.87 A random sample of 100 voters selected from a large population revealed 30 favoring candidate A, 38 favoring candidate B, and 32 favoring candidate C. Find MLEs for the proportions of voters in the population favoring candidates A, B, and C, respectively. Estimate the difference between the fractions favoring A and B and place a 2-standard-deviation bound on the error of estimation.

9.88 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y \mid \theta) = \begin{cases} (\theta+1)y^\theta, & 0 < y < 1,\ \theta > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the MLE for θ. Compare your answer to the method-of-moments estimator found in Exercise 9.69.

9.89 It is known that the probability p of tossing heads on an unbalanced coin is either 1/4 or 3/4. The coin is tossed twice and a value for Y, the number of heads, is observed. For each possible value of Y, which of the two values for p (1/4 or 3/4) maximizes the probability that $Y = y$? Depending on the value of y actually observed, what is the MLE of p?

9.90 A random sample of 100 men produced a total of 25 who favored a controversial local issue. An independent random sample of 100 women produced a total of 30 who favored the issue. Assume that p_M is the true underlying proportion of men who favor the issue and that p_W is the true underlying proportion of women who favor of the issue. If it actually is true that $p_W = p_M = p$, find the MLE of the common proportion p.

***9.91** Find the MLE of θ based on a random sample of size n from a uniform distribution on the interval $(0, 2\theta)$.

***9.92** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with density function

$$f(y \mid \theta) = \begin{cases} \dfrac{3y^2}{\theta^3}, & 0 \le y \le \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

In Exercise 9.52, you showed that $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ is sufficient for θ.

a Find the MLE for θ. [*Hint:* See Example 9.16.]

b Find a function of the MLE in part (a) that is a pivotal quantity. [*Hint:* see Exercise 9.63.]

c Use the pivotal quantity from part (b) to find a $100(1-\alpha)\%$ confidence interval for θ.