21-484 Notes

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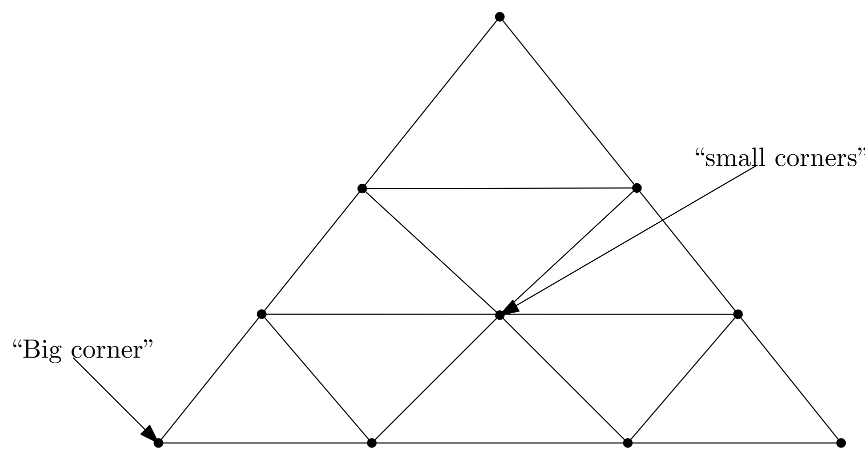
February 3, 2012

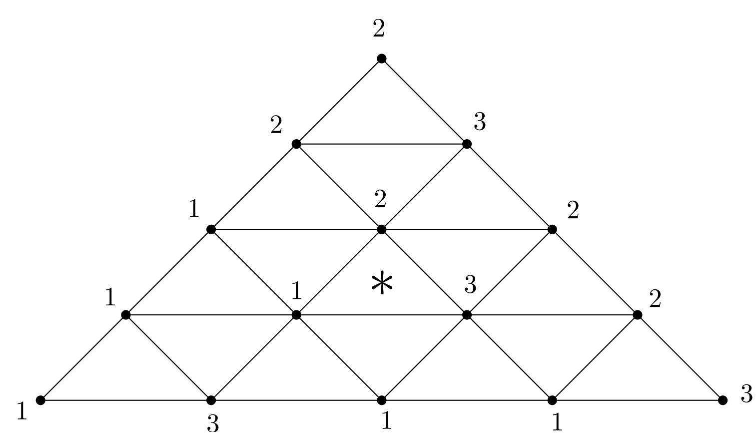
Def:

- A triangulation of a triangle is a subdivision of the triangle into smaller triangles.



- A Sperner labeling of a triangulation is a labeling of the corners by 1,2,3 such that
  - → The big corners are labeled 1,2,3
  - → A small corner lying on the line connecting two Big corners labeled i, j can only be labeled i or j.

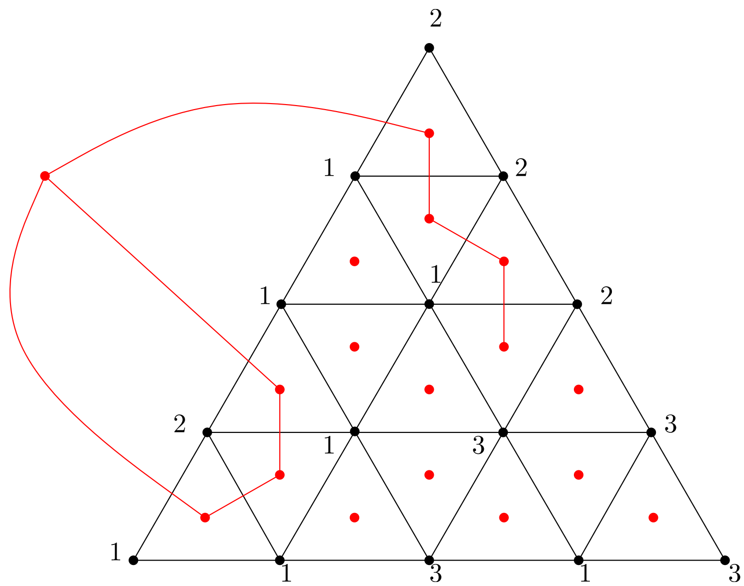
Example:



Lemma: (Sperner's lemma) In every Sperner's labeling there is a small triangle lableed 1,2,3.

Proof: Define the following Graph G.

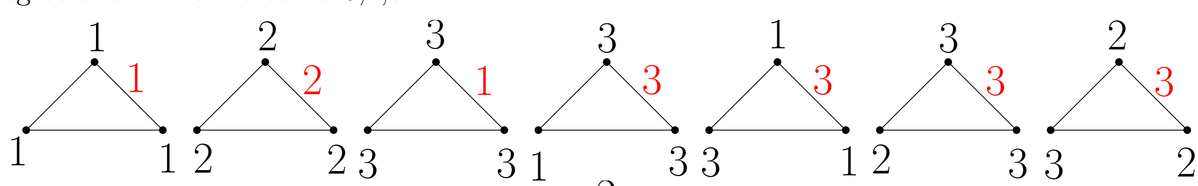
- The vertex set is the set of small triangles plus another vertex representing the outer face.
- There is an edge between two vertices if there is a side who's endpoints are labeled 1,2.



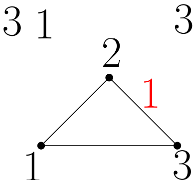
<u>Notice:</u>

1. the degree of an inner vertex is 0,1,or 2

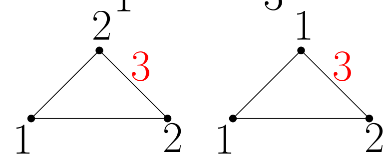
0:



1:



2:



2. the degree of an inner vertex is 1 iff it is labeled 1,2,3

3. the degree of the outer vertex is odd because we start with 1 and end with 2. Let x be the number of lines moving from $1 \to 2$. Let y be the number of lines moving from $2 \to 1$. $x - y = 1$ so $x + y$ is odd.

$\to$ since the sum of degrees in a graph is even, we must have an inner vertex with odd degree. Actualy, we proved that there is an odd number of such triangles.

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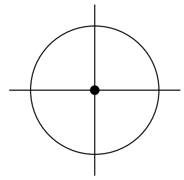
Application: Proving Brouwer's Fixed point Thm.

Thm: Every continuous function t from $D = \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\right\}$ to itself has a fixed point x_0 such that $f(x_0) = x_0$



Proof: - having a fixed point is a topological property

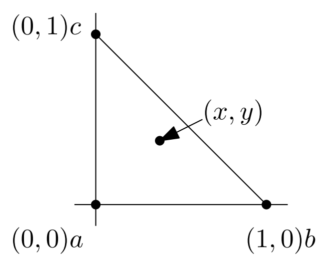
- If $f : G \to G$ is continuous and we know that the FP theorem holds in H, and there is $h : G \to H$ continuous and bijective

$$(h \circ f \circ h^{-1})(x_0) = x_0$$
$$f \circ (h^{-1}(x_0)) = h^{-1}(x_0) \quad \text{Can prove on triangles}$$

$\to$ Use Barycentric coordinates

$\to$ write (x,y) as a convex combination of a, b, c

$$(x,y) \mapsto (1-x-y, x, y)$$



$\to$ let F be a continuous function from $\triangle abc$ to itself, assume $f(x,y,z) = (x', y', z')$ label (x,y,z)

1 if $x' < x$

2 if $x' \geq x$ bu $y' < y$

3 if $x' \geq x, y' \geq y$ but $z' < z$

Notice: $\to$ if a point can not be labeled, then $a' \geq a, b' \geq b, c' \geq c \Rightarrow a' = a, b' = b, c' = c \Rightarrow$ found a fixed point

$\to$ a is labeled 1 (or it is a fixed point)

$\to$ b is labeled 2 (or it is a fixed point)

$\to$ c is labeled 3 (or it is a fixed point)

$\to$ if (x,y) is on the a–b line, then $y = 0$, so the Barycentric coordinates $(1-x, x, 0)$ in particular, the 3^{rd} coordinate will not become smaller. So such (x,y) will be labeled 1 or 2 (or be a fixed point)

$\to$ true for all sides

$\to$ can apply Sperner's lemma