Homework 8
21-630 Ordinary Differential Equations
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Due: Wednesday, March 27, 2013

Problem 1

Given $\varepsilon > 0$, define $\delta := \min \left\{ \frac{\sigma - \beta}{2B^3} e^{-\beta t_0}, \varepsilon / B \right\}$, $T := \sup \left\{ t > t_0 : |Y(s)| e^{\beta s} < \frac{\sigma - \beta}{B^2}, \forall s \in [t_0, t] \right\}$.
Assume $|Y(t_0)| < \delta$, and note that, by continuity of $|Y(s)|e^{\beta s}$, $T > 0$.
By Variation of Parameters, $\forall t \in [t_0, \infty)$,

$$\begin{aligned}
|Y(s)| = \left| e^{A(t-t_0)} Y(t_0) + \int_{t_0}^t e^{A(t-s)} F(s, Y(s))\, ds \right| &\leq \left| e^{At} \right| |Y(t_0)| + \int_{t_0}^t \left| e^{A(t-s)} \right| |F(s, Y(s))|\, ds \\
&\leq B e^{-\sigma t} |Y(t_0)| + B^2 \int_{t_0}^t e^{-\sigma(t-s)} e^{\beta s} |Y(s)|^2\, ds, \\
&\leq e^{-\sigma t} \left(B |Y(t_0)| + B^2 \int_{t_0}^t e^{(\sigma + \beta)s} |Y(s)|^2\, ds \right),
\end{aligned}$$

and so, for $t \in [t_0, T]$, by definition of T,

$$|Y(t)| e^{\sigma t} \leq B\delta + B^2 \int_{t_0}^t e^{(\sigma+\beta)s} |Y(s)|^2\, ds \leq B\delta + (\sigma - \beta) \int_{t_0}^t |Y(s)| e^{\sigma s}\, ds.$$

Applying the simple version of Gronwall's Inequality gives, $\forall t \in [t_0, T)$,

$$|Y(t)| e^{\sigma t} \leq B\delta e^{(\sigma-\beta)t} \leq \frac{\sigma - \beta}{2B^2} e^{-\beta t_0} e^{(\sigma - \beta)t} \leq \frac{\sigma - \beta}{2B^2} e^{(\sigma-\beta)t}. \tag{1}$$

Consequently, if T were finite,

$$|Y(T)| e^{\beta T} \leq \frac{1}{2} \frac{\sigma - \beta}{B^2}$$

which would contradict the continuity of $|Y(t)|e^{\beta t}$. Thus, $\forall t \geq t_0$, by the first inequality in (1),

$$|Y(t)| \leq B\delta e^{-\beta t} \to 0 \quad \text{as} \quad t \to \infty,$$

and, by choice of δ, $|Y(t)| \leq \varepsilon$, so that 0 is stable and hence asymptotically stable. ■

Problem 2

The set of critical points of the system is

$$\boxed{C := \{(0,0),(1,0),(0,3),(2,-1)\}.}$$

Also,

$$Df = \begin{bmatrix} 1-2X-Y & -X \\ -2Y & 3-2X-2Y \end{bmatrix}.$$

Since

$$Df\big|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix},$$

which has as an eigenvalue $1 > 0$, by Theorem 5.3, $\boxed{(0,0) \text{ is unstable.}}$ Since

$$Df\big|_{(1,0)} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

which has as an eigenvalue $1 > 0$, by Theorem 5.3, $\boxed{(1,0) \text{ is unstable.}}$ Since

$$Df\big|_{(0,3)} = \begin{bmatrix} -2 & 0 \\ -6 & -3 \end{bmatrix},$$

which has as an eigenvalues $-2,-3 < 0$, by Theorem 5.2, $\boxed{(0,3) \text{ is (asymptotically) stable.}}$ Since

$$Df\big|_{(2,-1)} = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix},$$

whose eigenvalues have real part $-1/2 < 0$, by Theorem 5.2, $\boxed{(2,-1) \text{ is (asymptotically) stable.}}$

Problem 3

Define $Y(t) : \mathbb{R} \to \mathbb{R}^2$ by $Y_1 = X$ and $Y_2 = \dot{X}$, let $\varepsilon > 0$ and choose $\delta := \min\{\varepsilon/2, 1\}$. Then,

$$\dot{Y} = f(Y(t)) := Y_2(t) = \begin{bmatrix} Y_2(t) \\ -Y_1(t) - 2Y_1^3(t) \end{bmatrix}.$$

Define $v : \mathbb{R}^2 \to \mathbb{R}$ for all $(y_1, y_2) \in \mathbb{R}^2$ by $v(y_1, y_2) := y_1^2 + y_1^4 + y_2^2$. Then,

$$\frac{d}{dt}v(Y(t)) = D_*v(Y(t)) = \begin{bmatrix} Y_2(t) \\ -Y_1(t) - 2Y_1^3(t) \end{bmatrix} \cdot \begin{bmatrix} 2Y_1(t) + 4Y_1^3(t) \\ 2Y_2(t) \end{bmatrix} = 0,$$

so that $v(Y(t))$ is constant. Then, $\forall t \geq 0$, if $|Y(0)| < \delta$ (so that $Y_1^2(0) + Y_1^4(0) \leq 2Y_1^2(0)$),

$$|Y(t)| = \sqrt{Y_1^2(t) + Y_2^2(t)} \leq \sqrt{v(Y(t))} = \sqrt{v(Y(0))} \leq \sqrt{2(Y_1^2(0) + Y_2^2(0))} < 2\delta \leq \epsilon. \quad \blacksquare$$

Problem 4

Define $Y(t) : \mathbb{R} \to \mathbb{R}^2$ by $Y_1 = X$ and $Y_2 = \dot{X}$, let $\varepsilon > 0$ and choose $\delta := \min\{\varepsilon, 1\}/4$. Then,

$$\dot{Y} = f(Y(t)) := Y_2(t) = \begin{bmatrix} Y_2(t) \\ -Y_1(t) + 2Y_1^3(t) \end{bmatrix}.$$

Suppose $|Y(0)| < \delta$, and define

$$T := \sup\{t \geq 0 : |Y(s)| < \min\{\varepsilon, 1\}/2, \forall s \in [0, t]\},$$

supposing, for sake of contradiction, that T is finite (since $|Y(0)| < \delta$, $T \geq 0$).
Define $v : \mathbb{R}^2 \to \mathbb{R}$ for all $(y_1, y_2) \in \mathbb{R}^2$ by $v(y_1, y_2) := y_1^2 - y_1^4 + y_2^2$. Then,

$$\frac{d}{dt} v(Y(t)) = D_* v(Y(t)) = \begin{bmatrix} Y_2(t) \\ -Y_1(t) + 2Y_1^3(t) \end{bmatrix} \cdot \begin{bmatrix} 2Y_1(t) - 4Y_1^3(t) \\ 2Y_2(t) \end{bmatrix} = 0,$$

so that $v(Y(t))$ is constant. $\forall t \in [0, T]$, since $|Y(t)| < 1/2$,

$$Y_1^2(t) - Y_1^4(t) = Y_1^2(t)(1 - Y_1^2(t)) \geq \frac{1}{2} Y_1^2(t),$$

and it follows that

$$|Y(t)| = \sqrt{Y_1^2(t) + Y_2^2(t)} \leq \sqrt{2v(Y(t))} = \sqrt{2v(Y(0))} \leq \sqrt{4(Y_1^2(0) + Y_2^2(0))} < 2\delta \leq \min\{\epsilon, 1\}/2.$$

In particular, $|Y(T)| < \min\{\varepsilon, 1\}/2$, contradicting the continuity of Y (by definition of T). $\blacksquare$