Lecture Notes for Week 8 (First Draft)

Weak and Weak Convergence (Continued)*

Theorem 8.1: Let X be a reflexive Banach space and $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in X. Then $\{x_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence.

Proof: Let $Z = \mathrm{cl}(\mathrm{span}(\{x_n : n \in \mathbb{N}\}))$ and observe that Z is closed and separable. It follows from Theorem 7.1 that Z is reflexive. Since Z^{**} is isomorphic to Z and Z is separable, we conclude that Z^{**} is separable. It follows from Theorem 7.4 that Z^* is separable.

Choose a dense sequence $\{z_n^*\}_{n=1}^{\infty}$ in Z^*. We shall construct a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ by a diagonalization argument. For consistency in indexing, let us put $x_{n,0} = x_n$ for all $n \in \mathbb{N}$. Then the sequence $\{z_1^*(x_{n,0})\}_{n=1}^{\infty}$ is bounded in $\mathbb{K}$, so we may choose a subsequence $\{x_{n,1}\}_{n=1}^{\infty}$ of $\{x_{n,0}\}_{n=1}^{\infty}$ such that $\{z_1^*(x_{n,1})\}_{n=1}^{\infty}$ is convergent. Then $\{z_2^*(x_{n,1})\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{K}$, so we may choose a subsequence $\{x_{n,2}\}_{n=1}^{\infty}$ of $\{x_{n,1}\}_{n=1}^{\infty}$ such that $\{z_2^*(x_{n,2})\}_{n=1}^{\infty}$ is convergent. Proceeding by induction, we obtain for each $k \in \mathbb{N}$ a sequence $\{x_{n,k}\}_{n=1}^{\infty}$ such that

- $\{x_{k+1,n}\}_{n=1}^{\infty}$ is a subsequence of $\{x_{n,k}\}_{n=1}^{\infty}$,
- $\{z_k^*(x_{n,k})\}_{n=1}^{\infty}$ is convergent.

Put $y_n = x_{n,n}$ for all $n \in \mathbb{N}$ and observe that $\{y_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$. Observe further that for each $k \in \mathbb{N}$ the sequence $\{z_k^*(y_n)\}_{n=1}^{\infty}$ is convergent. Since the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded in Z, and the sequence $\{z_n^*\}_{n=1}^{\infty}$ is dense in Z^*, we conclude that for every $z^* \in Z^*$, the sequence $\{z^*(y_n)\}_{n=1}^{\infty}$ is convergent. A few details have been omitted here, but they can easily be filled in by writing

$$z^*(y_n) - z^*(y_m) = z^*(y_n) - z_k^*(y_n) + z_k^*(y_n) - z_k^*(y_m) + z_k^*(y_m) - z^*(y_m).$$

For each $z^* \in Z^*$, define

$$y^{**}(z^*) = \lim_{n\to\infty} z^*(y_n)$$

and observe that $y^{**} \in Z^{**}$. Since Z is reflexive, we may choose $y \in Z$ such that $J_Z(y) = y^{**}$. Then we have

$$\lim_{n\to\infty} z^*(y_n) = z^*(y) \ \text{ for all } z^* \in Z^*.$$

Let $x^* \in X^*$ be given and put $z^*(x) = x^*(x)$ for all $x \in Z$. It follows that

$$\langle x^*, y_n\rangle = \langle z^*, y_n\rangle \to \langle z^*, y\rangle = \langle x^*, y\rangle \ \text{ as } n \to \infty.$$

We conclude that $y_n \rightharpoonup y$ (weakly) as $n \to \infty$. $\square$

Definition 8.2: A sequence $\{x_n\}_{n=1}^{\infty}$ in a NLS X is said to be a *weak Cauchy sequence* provided that for every $x^* \in X^*$, $\{\langle x^*, x_n\rangle\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{K}$.

Definition 8.3: A NLS is said to be *weakly sequentially complete* provided that every weak Cauchy sequence is weakly convergent.

Remark 8.4: Since $\mathbb{K}$ is complete, a sequence $\{x_n\}_{n=1}^{\infty}$ in a NLS X is a weak Cauchy sequence if and only if the sequence $\{\langle x^*, x_n\rangle\}_{n=1}^{\infty}$ is convergent for each $x^* \in X^*$.

Theorem 8.5: Let X be a reflexive Banach space. Then X is weakly sequentially complete.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a weak Cauchy sequence. Then for each $x^* \in X^*$, $\{(J(x_n))(x^*)\}_{n=1}^{\infty}$ is bounded. By the Principle of Uniform Boundedness, $\{\|J(x_n)\|\}_{n=1}^{\infty}$ is bounded, and consequently $\{x_n\}_{n=1}^{\infty}$ is bounded. By Theorem 8.1, we may choose $x \in X$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \rightharpoonup x$ (weakly) as $k \to \infty$. Let $x^* \in X^*$ be given. Since $\{\langle x^*, x_n\rangle\}_{n=1}^{\infty}$ is convergent (by completeness of $\mathbb{K}$) and $\langle x^*, x_{n_k}\rangle \to \langle x^*, x\rangle$ as $k \to \infty$, we conclude that

$$\langle x^*, x_n\rangle \to \langle x^*, x\rangle \ \text{ as } n \to \infty,$$

and $\{x_n\}_{n=1}^{\infty}$ is weakly convergent. $\square$

Theorem 8.6 Let X be a separable NLS and $\{x_n^*\}_{n=1}^{\infty}$ be a bounded sequence in X^*. Then $\{x_n^*\}_{n=1}^{\infty}$ has a weakly* convergent subsequence.

Proof: Choose a dense sequence sequence $\{x_n\}_{n=1}^{\infty}$ in X. We shall again use a diagonalization argument to construct the desired subsequence. For consistency in indexing, let us put $x_{n,0}^* = x_n^*$ for all $n \in \mathbb{N}$. Then the sequence $\{\langle x_{n,0}^*, x_1\rangle\}_{n=1}^{\infty}$ is bounded in $\mathbb{K}$. We may extract a subsequence $\{x_{n,1}^*\}_{n=1}^{\infty}$ of $\{x_{n,0}^*\}_{n=1}^{\infty}$ such that $\{\langle x_{n,1}^*, x_1\rangle\}_{n=1}^{\infty}$ is convergent. Since $\{\langle x_{n,1}^*, x_2\rangle\}_{n=1}^{\infty}$ is bounded, we may extract a subsequence $\{x_{n,2}^*\}_{n=1}^{\infty}$ of $\{x_{n,1}^*\}_{n=1}^{\infty}$ such that $\{\langle x_{n,2}^*, x_2\rangle\}_{n=1}^{\infty}$ is convergent. Proceeding by induction, we obtain, for each $k \in \mathbb{N}$ a sequence $\{x_{n,k}^*\}_{n=1}^{\infty}$ such that

- $\{x_{n,k+1}^*\}_{n=1}^{\infty}$ is a subsequence of $\{x_{n,k}^*\}_{n=1}^{\infty}$,
- $\{\langle x_{n,k}^*, x_k\rangle\}_{n=1}^{\infty}$ is convergent.

Put $y_n^* = x_{n,n}^*$ for all $n \in \mathbb{N}$. Observe that $\{y_n^*\}_{n=1}^{\infty}$ is a subsequence of $\{x_n^*\}_{n=1}^{\infty}$. Observe further that for each $k \in \mathbb{N}$, the sequence $\{\langle y_n^*, x_k\rangle\}_{n=1}^{\infty}$ is convergent. Since $\{y_n^*\}_{n=1}^{\infty}$ is bounded in X^* and $\{x_n\}_{n=1}^{\infty}$ is dense in X we conclude that $\{\langle y_n^*, x\rangle\}_{n=1}^{\infty}$ is convergent for every $x \in X$. Now, define $y^* \in X^*$ by

$$\langle y^*, x\rangle = \lim_{n\to\infty} \langle y_n^*, x\rangle \ \text{ for all } x \in X.$$

It follows that $y_n^* \overset{*}{\rightharpoonup} y^*$ (weakly *) as $n \to \infty$. $\square$

Definition 8.7: A subset S of an NLS X is said to be *total* provided that $\mathrm{span}(S)$ is dense in X.

Proposition 8.8: Let X be a NLS, S be a total subset of X^*, $\{x_n\}_{n=1}^\infty$ be a sequence in X and $x \in X$ be given. Then $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ if and only if $\{x_n\}_{n=1}^\infty$ is bounded and

$$\forall y^* \in S, \quad \langle y^*, x_n \rangle \to \langle y^*, x \rangle \ \text{ as } n \to \infty. \tag{1}$$

Proof: The necessity of the conditions follows from the definition of weak convergence and part (i) of Theorem 7.15. Assume that $\{x_n\}_{n=1}^\infty$ is bounded and that (1) holds. Observe that

$$\forall y^* \in \mathrm{span}(S), \quad \langle y^*, x_n \rangle \to \langle y^*, x \rangle \ \text{ as } n \to \infty. \tag{2}$$

Let $x^* \in X^*$ and $\epsilon > 0$ be given. Choose $M > 0$ such that

$$\|x_n - x\| \le M \ \text{ for all } n \in \mathbb{N}. \tag{3}$$

Since $\mathrm{span}(S)$ is dense in X^*, we may choose $y^* \in \mathrm{span}(S)$ such that

$$\|x^* - y^*\| < \frac{\epsilon}{2M}. \tag{4}$$

Since (2) holds we may choose $N \in \mathbb{N}$ such that

$$|\langle y^*, x_n - x \rangle| < \frac{\epsilon}{2} \ \text{ for all } n \ge N. \tag{5}$$

Then for all $n \in \mathbb{N}$ with $n \ge N$ we have

$$|\langle x^*, x_n - x \rangle| \le |\langle x^* - y^*, x_n - x \rangle| + |\langle y^*, x_n - x \rangle| < \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

by virtue of (3), (4), and (5). $\square$

Proposition 8.9: Let X be a Banach space, S be a total subset of X, $\{x_n^*\}_{n=1}^\infty$ be a sequence in X^* and $x^* \in X^*$ be given. Then $x_n^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$ if and only if $\{x_n^*\}_{n=1}^\infty$ is bounded and

$$\forall y \in S, \quad \langle x_n^*, y \rangle \to \langle x^*, y \rangle \ \text{ as } n \to \infty. \tag{6}$$

The proof of Proposition 8.9 is very similar to the proof of Proposition 8.8 and will be omitted.

For the sake of completeness, we formulate Remark 7.9 as a propositon and provide a proof.

Proposition 8.10: Let X be a finite-dimensional NLS, $\{x^{(n)}\}_{n=1}^\infty$ be a sequence in X and let $x \in X$ be given. Then $x^{(n)} \rightharpoonup x$ (weakly) as $n \to \infty$ if and only if $x^{(n)} \to x$ (strongly) as $n \to \infty$.

Remark 8.11: Since every finite-dimensional NLS is reflexive, it follows from Proposition 8.10 that a sequence in the dual of a finite-dimensional NLS is weakly* convergent if and only if it is strongly convergent.

Proof of Proposition 8.10: We already know that strong convergence implies weak convergence in any NLS. Assume that $x^{(n)} \rightharpoonup x$ (weakly) as $n \to \infty$. Choose a basis $(b_j | j = 1, 2, \cdots, N)$ for X and define $b_k^* \in X^*$ for $k = 1, 2, \cdots, N$ by

$$b_k^*(b_j) = \begin{cases} 1 & \text{if } j = k \\ \\ 0 & \text{if } j \neq k. \end{cases}$$

($(b_j^* | j = 1, 2, \cdots, N)$ is called the dual basis for $(b_j | j = 1, 2, \cdots, N)$.)

Choose $\alpha_j^{(n)},\ \alpha_j,\ n \in \mathbb{N}, j = 1, 2, \cdots, N$ such that

$$x^{(n)} = \sum_{j=1}^{N} \alpha_j^{(n)} b_j, \quad x = \sum_{j=1}^{N} \alpha_j b_j.$$

Since $\langle b_k^*, x^{(n)} \rangle \to \langle b_k^*, x \rangle$ as $n \to \infty$ for $k = 1, 2, \cdots, N$, we conclude that

$$\alpha_k^{(n)} \to \alpha_k \text{ as } n \to \infty \text{ for } k = 1, 2, \cdots, N.$$

It follows that

$$\|x^{(n)} - x\| \leq \sum_{k=1}^{N} |\alpha_k^{(n)} - \alpha_k| \cdot \|b_k\| \to 0 \text{ as } n \to \infty. \quad \square$$

Weak convergence is an extremely important tool in applications. In many situations, an existence theorem is proved by first constructing a sequence of "approximate solutions" and showing that the sequence is bounded in a reflexive Banach space. Then a weakly convergent subsequence can be extracted. The final step is to show that the weak limit of the subsequence is actually a solution of the original problem. A key tool in this process is often the following theorem which says that every closed convex sets is sequentially weakly closed (i.e. contains the limit of each weakly convergent sequence of points from the set).

Theorem 8.12: Let X be a NLS and K be a closed convex subset of X. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$ be given. Assume that $x_n \in K$ for all $n \in \mathbb{N}$ and that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. Then $x \in K$.

The proof of Theorem 8.12 is based on the following separation lemma for convex sets.

Lemma 8.13: Let X be a NLS, K be a nonempty closed convex subset of X, and $x_0 \in X \backslash K$ be given. Then there exists $x^* \in X^*$ such that

$$\mathrm{Re}(x^*(x_0)) < \inf\{\mathrm{Re}(x^*(y)) : y \in K\}.$$

Proof of Lemma 8.13: Without loss of generality, we may assume that $x_0 = 0$. Since K is closed and $0 \notin K$ we may choose $\delta > 0$ such that $B_\delta(0) \cap K = \emptyset$. By Theorem 6.6, we may choose $x^* \in X^*$ such that $x^* \neq 0$ and

$$\mathrm{Re}(x^*(x)) \le \mathrm{Re}(x^*(y)) \quad \text{for all } x \in B_\delta(0), y \in K. \tag{7}$$

Let us put

$$\alpha = \inf\{\mathrm{Re}(x^*(y)) : y \in K\}.$$

Then we have

$$\mathrm{Re}(x^*(x)) \le \alpha \le \mathrm{Re}(x^*(y)) \quad \text{for all } x \in B_\delta(0), y \in K. \tag{8}$$

Since $\mathrm{Re}(x^*(0)) = 0$, it remains only to show that $\alpha > 0$.

Since the range of a nonzero linear functional is all of $\mathbb{K}$, we may choose $z \in X$ such that $\mathrm{Re}(x^*(z)) > 0$. Let

$$w = \frac{\delta z}{2\|z\|},$$

and notice that $w \in B_\delta(0)$ and $\mathrm{Re}(x^*(w)) > 0$. We conclude from the first inequality in (8) that $\alpha > 0$ and the proof is complete. $\square$

Proof of Theorem 8.12: Suppose that $x \notin K$. Then, by Lemma 8.13, we may choose $x^* \in X^*$ such that

$$\mathrm{Re}(x^*(x)) < \inf\{\mathrm{Re}(x^*(x_n) : n \in \mathbb{N}\} \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

On the other hand, we know that

$$\lim_{n\to\infty} x^*(x_n) = x^*(x) \quad \text{as } n \to \infty, \tag{10}$$

because $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. Equation (10) is not possible because of (9). We conclude that $x \in K$. $\square$

Lemma 8.14: Let X be a reflexive Banach space and K be a nonempty closed convex subset of X. Then there exists $\hat{x} \in K$ such that

$$\|\hat{x}\| \le \|x\| \quad \text{for all } x \in K.$$

In view of Problem 9 on Assignment 4, we have the following interesting consequence of Lemma 8.14.

Corollary 8.15: $C[0,1]$ (equipped with the maximum norm) is not reflexive.

Proof of Lemma 8.14: Let

$$\gamma = \inf\{\|x\| : x \in K\}.$$

Choose a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in K$ for all $n \in \mathbb{N}$ and

$$\|x_n\| \to \gamma \quad \text{as } n \to \infty. \tag{11}$$

It follows from (11) that $\{x_n\}_{n=1}^{\infty}$ is bounded. By Theorem 8.1, we may choose $\hat{x} \in X$ and a subsequence $\{x_{n_k}\}_{n=1}^{\infty}$ such that

$$x_{n_k} \rightharpoonup \hat{x} \ \text{ (weakly) as } k \to \infty.$$

Theorem 8.12 tells us that $\hat{x} \in K$. By part (iii) of Theorem 7.15, we have

$$\|\hat{x}\| \le \liminf_{k\to\infty} \|x_{n_k}\| = \gamma \le \|x\| \ \text{ for all } x \in K. \ \square$$

The proof of Lemma 8.14 is a simple instance of the *Direct Method of the Calculus of Variations*. The calculus of variations is concerned with minimizing (or maximizing) functionals (typically defined by integrals) on subsets of infinite-dimensional NLS. The subject has a rich history beginning with Johann Bernoulli's announcement of the *brachsitochrone* problem in 1696. Early emphasis was placed on finding necessary conditions for minima. In the 20^{th} century, starting with the work of Tonelli, it was discovered that methods of functional analysis can be used to prove existence theorems by a direct method. This method makes crucial use of weak convergence and results about convex sets and convex functionals. You will have some homework exercises concerning this topic. The method can be summarized as follows: To prove the existence of a minimizer for a functional $G : K \to \mathbb{R}$, where K is a subset of a reflexive Banach space one carries out the steps below.

- Step 1: Put $\gamma = \inf\{G(y) : y \in K\}$.
- Step 2: Choose a *minimizing sequence*, i.e. a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in K$ for all $n \in \mathbb{N}$ and $G(y_n) \to \gamma$ as $n \to \infty$.
- Step 3: Show that $\{y_n\}_{n=1}^{\infty}$ is bounded.
- Step 4: Extract a weakly convergent subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ and let y denote the weak limit of this subsequence.
- Step 5: Show that $y \in K$. (This will be automatic if K is closed and convex.)
- Step 6: Show that $\liminf_{k\to\infty} G(y_{n_k}) \ge G(y)$, which implies that $G(y) = \gamma$. (This will be automatic if G is convex and lower semicontinuous in the norm topology.)

As another application of Theorem 8.12, we mention the following result.

Lemma 8.16: Let X be a reflexive Banach space and let $\{K_n\}_{n=1}^{\infty}$ be a sequence of nonempty, closed, bounded, convex subsets of X such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

The proof of this lemma is left as an exercise.

Weak and Weak Convergence in Sequence Spaces*

Before giving some simple criteria for weak and weak* convergence in sequence spaces, it is useful to state a result identifying the dual of c_0 with l^1

Proposition 8.17: Let $x^* \in (c_0)^*$ be given. Then there exists exactly one $y \in l^1$ such that

$$x^*(x) = \sum_{k=1}^{\infty} y_k x_k \quad \text{for all } x \in c_0; \tag{12}$$

moreover, $\|y\|_1 = \|x^*\|$.

The proof of Proposition 8.17 will be a homework exercise.

Remark 8.18: It follows from from Proposition 8.17 that the dual space of c_0 can be identified with l^1 through the isometric isomorphism given in (12). Even though l^1 is not strictly speaking a dual space, it is standard to talk about weak* convergence in l^1 and to say that a sequence $\{x^{(n)}\}_{n=1}^{\infty}$ in l^1 converges weakly* to $x \in l^1$ provided that

$$\forall z \in c_0, \quad \sum_{k=1}^{\infty} x_k^{(n)} z_k \to \sum_{k=1}^{\infty} x_k z_k \quad \text{as } n \to \infty.$$

The following simple definition is useful for stating very simple criteria for weak and weak* convergence in sequence spaces.

Definition 8.19: Let $x \in \mathbb{K}^{\mathbb{N}}$ be given and let $\{x^{(n)}\}_{n=1}^{\infty}$ be a sequence of elements of $\mathbb{K}^{\mathbb{N}}$. We say that $x^{(n)} \to x$ *componentwise* or *pointwise* as $n \to \infty$ provided that

$$\forall k \in \mathbb{N}, \text{ we have } x_k^{(n)} \to x_k \text{ as } n \to \infty.$$

Remark 8.20: Consider the sequence $\{e^{(n)}\}_{n=1}^{\infty}$ of elements of $\mathbb{K}^{\mathbb{N}}$ defined by

$$e_k^{(n)} = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{for } k \neq n. \end{cases}$$

(a) The set $\{e^{(m)} : m \in \mathbb{N}\}$ is total in l^p for $1 \leq p < \infty$, but it is not total in l^∞.

(b) The set $\{e^{(m)} : m \in \mathbb{N}\}$ is total in c_0, but it is not total in c.

Remark 8.21: Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a sequence in $\mathbb{K}^{\mathbb{N}}$ and let $x \in \mathbb{K}^{\mathbb{N}}$ be given and notice that for every $m \in \mathbb{N}$ we have

$$\sum_{k=1}^{\infty} x_k^{(n)} e_k^{(m)} = x_m^{(n)}, \quad \sum_{k=1}^{\infty} x_k e_k^{(m)} = x_m,$$

and consequently $x^{(n)} \to x$ componentwise as $n \to \infty$ if and only if

$$\forall m \in \mathbb{N}, \quad \sum_{k=1}^{\infty} x_k^{(n)} e_k^{(m)} \to \sum_{k=1}^{\infty} x_k e_k^{(m)} \quad \text{as } n \to \infty.$$

We can use Definition 8.19 and Remarks 8.20, 8.21 together with Propositions 8.8 and 8.9 to give simple and useful criteria for weak and weak* convergence in sequence spaces.

Proposition 8.22: Let X be one of the spaces l^p, $1 \leq p \leq \infty$ or c_0 and let $\{x^{(n)}\}_{n=1}^{\infty}$ be a sequence of elements of X and let $x \in X$ be given.

(a) If $X = c_0$ then $x^{(n)} \rightharpoonup x$ (weakly) as $n \to \infty$ if and only if $\{x^{(n)}\}_{n=1}^{\infty}$ is bounded in X and $x^{(n)} \to x$ componentwise as $n \to \infty$.

(b) If $X = l^p$ and $1 < p < \infty$ then $x^{(n)} \rightharpoonup x$ (weakly) as $n \to \infty$ if and only if $\{x^{(n)}\}_{n=1}^{\infty}$ is bounded in X and $x^{(n)} \to x$ componentwise as $n \to \infty$.

(c) If $X = l^1$ or $X = l^\infty$ then $x^{(n)} \overset{*}{\rightharpoonup} x$ as $n \to \infty$ if and only if $\{x^{(n)}\}_{n=1}^{\infty}$ is bounded in X and $x^{(n)} \to x$ componentwise as $n \to \infty$.

The space l^1 has a very unusual property, namely that it is an infinite-dimensional Banach space in which every weakly convergent sequence is strongly convergent. (Consequently, we cannot extend Proposition 8.22(b) to the case $p = 1$.)

Proposition 8.22: Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a sequence in l^1 and $x \in l^1$ be given. Then $x^{(n)} \rightharpoonup x$ (weakly) as $n \to \infty$ if and only if $x^{(n)} \to x$ (strongly) as $n \to \infty$.

The proof of Proposition 8.22 will be a homework exercise.