General Topology Lecture Notes

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Contents

Lecture 2, August 29, 2012.

1. Sets and Ordering

1.1. Finite and infinite sets.

Definition 1. Sets A and B are *e*quipotent, and we write $A \sim B$, if there exists a bijection which maps A to B.

The terms *equipollence* and *equinumerability*, or saying that two sets *have the same cardinality* are also used to name the "relation" above.

It is straightforward to check that equipotence is reflexive, symmetric, and transitive.

We denote the set of natural numbers (positive integers) by $\mathbb{N}$.

Definition 2. A set is finite if it is equipotent to $\{1, 2, \ldots, n\}$ for some $n \in N$. Otherwise the set is infinite.

Exercise 3. Show that $\{1, 2, \ldots, n\} \sim \{1, 2, \ldots, m\}$ if and only if $n = m$. Hint: Use induction on the number of elements.

Theorem 4 (Cantor-Bernstein-Schröder)**.** *If there exist injections $f : A \to B$ and $g : B \to A$ then $A \sim B$.*

Proof. We wish to use f and g^{-1} to define a bijection, h, from A to B. The question is on which elements of A should we set $h = f$ and on which $h = g^{-1}$.

Let $C_0 = A \backslash g(B)$. Note that on C_0 we must use f to define h. Let $C_1 = g(f(C_0))$. Note that on C_1 one again must use f to define h, since if g^{-1} was used the function h would not be injective.

Continuing that reasoning we define recursively $C_{n+1} = g(f(C_n))$. Let $C = \cup_{n=0}^{\infty} C_n$. Then define

$$h(a) = \begin{cases} f(a) & \text{if } a \in C \\ g^{-1}(a) & \text{if } a \in A \backslash C. \end{cases}$$

Note that h is well defined since $A \backslash C \subset g(B)$.

Let us show that h is injective. Since h is injective both on C and on $A \backslash C$ the only way it can fail to be injective is if there exist $a_1 \in C$ and $a_2 \in A \backslash C$ such that $h(a_1) = h(a_2)$. That is if $f(a_1) = g^{-1}(a_2)$. But then $a_2 = g(f(a_1)$. Consequently $a_2 \in C$ which contradicts the assumption about a_2. Thus h is injective.

To show that h is surjective consider $b \in B$. If $b \in f(C)$ then $b = f(a) = h(a)$ for some $a \in C$. If $b \notin f(C)$ then consider $a = g(b)$. Note that $a \notin C_0$. If $a \notin C$ then $b = h(a)$ which is the desired conclusion. So let us show that $a \notin C$. Assume that $a \in C$. Then there exists $n > 0$ such that $a \in C_n$. By definition of C_n there exists $\bar{a} \in C_{n-1}$ such that $a = g(f(\bar{a}))$. Since g is injective, $f(\bar{a}) = b$ and thus $b \in f(C)$ which contradicts the assumption we made on b. ■

Definition 5. A set is countable if it is equipotent to $\mathbb{N}$. A set is uncountable if it is infinite, but not countable.

For a set X, the power set of X, $\mathcal{P}(X)$, is the set of all subsets of X.

Lemma 6 (Cantor)**.** *Let X be a set. There is no surjection (i.e. 'onto' mapping) from X to $\mathcal{P}(X)$. In particular $X \not\sim \mathcal{P}(X)$.*

Proof. If $X = \emptyset$ then $\mathcal{P}(X) = \{\emptyset\}$, so there is no surjection.

Let us consider a nonempty set X. Assume that there exists a surjection $\Phi : X \to \mathcal{P}(X)$. Consider the set $A \in \mathcal{P}(X)$ defined as follows:

$$A = \{x \in X \,:\, x \notin \Phi(x)\}.$$

Since Φ is a surjection there exists $a \in X$ such that $\Phi(a) = A$. If $a \in A$ then $a \in \Phi(a)$ and thus, by definition of A, $a \notin A$. So it must be that $a \notin A$. But then $a \notin \Phi(a)$ which implies that $a \in A$. Contradiction. ∎

Exercise 7. Let X be a nonempty set. Show that $\mathcal{P}(X) \sim \{0,1\}^X$ (the set of all functions from X to $\{0,1\}$).

Lemma 8. *The set of real numbers* $\mathbb{R}$ *is uncountable.*

Proof. Note that for any real number there is a unique decimal expansion provided that the expansions that end with an infinite sequence of $9'$s are excluded. The mapping Ψ from $\mathbb{R}$ to $\{0,1\}^{\mathbb{N}}$ (the set of all sequences with values in $\{0,1\}$) defined by

$$\Psi(x) = \begin{cases} a_1, a_2, \ldots & \text{if } x = 0.a_1a_2\ldots \text{ and } \forall i \in N \quad a_i \in \{0,1\} \\ 0, 0, \ldots & \text{else} \end{cases}$$

is a surjection.

Thus if there existed surjection from $\mathbb{N}$ to $\mathbb{R}$ then (considering the composition) there would exist a surjection from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$ which would contradict Lemma 6. ∎

Lecture 3, August 31, 2012.

1.2. Ordering.

Definition 9 (Partial Ordering)**.** Consider an arbitrary set X and a binary relation $\preccurlyeq$ on X. We call $(X, \preccurlyeq)$ a *partial ordering* if the following properties hold:

(i) (reflexivity) $x \preccurlyeq x$ for every $x \in X$.
(ii) (antisymmetry) For all $x, y \in X$, if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$.
(iii) (transitivity) For all $x, y, z \in X$, if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$.

The word partial indicates that given $x, y \in X$, in general we cannot always say that $x \preccurlyeq y$ or $y \preccurlyeq x$.

Associated to this relation is a *strict partial ordering*:

$$\text{for all } x, y \in X \text{ we have that } x \prec y \text{ if and only if } x \preccurlyeq y \text{ and } x \neq y.$$

Example 10. Let $X = \mathcal{P}(\mathbb{R})$. Given $E, F \in X$, we say that $E \preccurlyeq F$ if $E \subseteq F$. Then $\preccurlyeq$ is a partial ordering. Note that given the sets $\{1,2,3\}$ and $\{2,3,4\}$ one is not contained in the other.

Definition 11 (Linear (Total) Ordering)**.** Consider an arbitrary set X and a binary relation $\preccurlyeq$ on X. We call $(X, \preccurlyeq)$ a *linear ordering* if it is partial ordering and the dichotomy holds:

$$\text{for all } x, y \in X \text{ we have that } x \preccurlyeq y \text{ or } y \preccurlyeq x.$$

Given a partially ordered set $(X, \preccurlyeq)$, a subset $E \subseteq X$ is a *chain* if for all $x, y \in E$, $x \preccurlyeq y$ or $y \preccurlyeq x$, that is if $(E, \preccurlyeq)$ is a totally ordered set.

In the previous example $E = \{\{1,2,3\}, \{1,2\}, \{2\}\}$ is a chain.

Given a partially ordered set $(X, \preccurlyeq)$, and a set $E \subset X$, an element $x \in X$ is an *upper bound of* E if $y \preccurlyeq x$ for all $y \in E$. A set E may not have any upper bounds. An element $x \in E$ is a *maximal element* of E if there does not exist $y \in E$ such that $x \prec y$. A set E may not have maximal elements or it may have maximal elements that are not upper bounds (it can happen that a maximal element cannot be compared with all the elements of E).

Axiom 12 (Axiom of Choice)**.** *Given a collection of nonempty disjoint sets, $\mathcal{A}$, there exists a set C such that*

(i) $C \subset \cup \mathcal{A}$
(ii) *For all $A \in \mathcal{A}$, $C \cap A$ has exactly one element.*

Lemma 13 (Zorn's lemma)**.** *Let $(X, \preccurlyeq)$ be a partially ordered set. If every totally ordered subset of X has an upper bound, then X has a maximal element.*

2. Topological Spaces

Definition 14. Let X be a nonempty set. A collection $\tau \subset \mathcal{P}(X)$ is a *topology* if the following hold.

(i) $\emptyset, X \in \tau$.
(ii) If $U_i \in \tau$ for $i = 1, \ldots, M$, then $U_1 \cap \ldots \cap U_M \in \tau$.
(iii) If $\{U_\alpha\}_{\alpha \in \Lambda}$ is an arbitrary collection of elements of τ, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$.

The pair (X, τ) is called a *topological space* and the elements of τ are *open sets*. For simplicity, we often apply the term topological space only to X.

Example 15.

(i) Let X be a nonempty set. $\tau = \{\emptyset, X\}$ is the so-called *trivial* topology
(ii) Let X be a nonempty set. $\tau = \mathcal{P}(X)$ is the *discrete* topology.
(iii) Let $\tau_r = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Then τ_r is a topology on $\mathbb{R}$. Analogously so is $\tau_l = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.
(iv) Let $\tau = \{A \subset \mathbb{R} : (\forall x \in A)(\exists \varepsilon > 0)\ (x - \varepsilon, x + \varepsilon) \subset A\}$. The topology τ is called the standard topology on $\mathbb{R}$.
(v) (Sorgenfrey line) Let $\tau_{Sorg} = \{A \subset \mathbb{R} : (\forall x \in A)(\exists \varepsilon > 0)\ [x, x + \varepsilon) \subset A\}$. τ_{Sorg} is a topology on $\mathbb{R}$.

If τ_1 and τ_2 are two topologies on X, we say that τ_1 is *coarser*, than τ_2 if $\tau_1 \subset \tau_2$. We also say that τ_2 is *finer* that τ_1. Note that the trivial topology is always the coarsest one and the discrete topology is the finest one.

A point x of the topological space (X, τ) is called an *isolated point* if $\{x\}$ is an open set. Note that every point in the discrete topology is an isolated point. On the other point $\mathbb{R}$ with standard topology has no isolated points.

Remark 16. Given a family of topologies $\{\tau_\alpha\}_{\alpha \in \Lambda}$ on a set X, the family of sets

$$\bigcap_{\alpha \in \Lambda} \tau_\alpha := \{U \subset X : U \in \tau_\alpha \text{ for every } \alpha \in \Lambda\}$$

is still a topology on X, while in general $\bigcup_{\alpha \in \Lambda} \tau_\alpha$ is not. For example for τ_l and τ_r defined in the previous example $\tau_r \cup \tau_l$ is not a topology on $\mathbb{R}$.

In view of the previous remark, we can construct topologies starting from any family of subsets of X.

Proposition 17. *Let X be a set and let $\mathcal{F}$ be a family of subsets of X whose union is X. Then there exists a unique, smallest topology τ containing $\mathcal{F}$. Moreover, τ consists of $\emptyset$, finite intersections of elements of $\mathcal{F}$ and arbitrary unions of finite intersections of elements of $\mathcal{F}$.*

The family $\mathcal{F}$ is called a *subbase* for τ and τ is said to be *generated* by $\mathcal{F}$.

Proof. Let $\{\tau_\alpha\}_{\alpha\in\Lambda}$ be the family of all topologies that contain $\mathcal{F}$. Note that this family is nonempty since $\mathcal{P}(X)$ is one such topology. Then

$$\tau := \bigcap_{\alpha\in\Lambda} \tau_\alpha$$

is still a topology, contains $\mathcal{F}$, and it is the smallest such topology. Moreover, it is unique in view of its definition. By the properties of a topology, we have that τ contains X, $\emptyset$, all finite intersections of elements of $\mathcal{F}$ and all arbitrary unions of finite intersections of elements of $\mathcal{F}$. On the other hand, let τ' be the family that consists of X, $\emptyset$, all finite intersections of elements of $\mathcal{F}$ and arbitrary unions of finite intersections of elements of $\mathcal{F}$. Then τ' is a topology. ■

Example 18. (i) In $\mathbb{R}$ we can consider the family $\mathcal{F} = \tau_l \cup \tau_r$. The smallest topology containing $\mathcal{F}$ is the standard one.

Lecture 4, September 5, 2012.

2.1. Basis of topology. Let (X,τ) be a topological space. Given a point $x\in X$, a $U\subset X$ is a *neighborhood* of x if $U\in\tau$ and U contains x. [1] Given a set $E\subset X$, a *neighborhood* of E is an open set $U\in\tau$ that contains E.

Definition 19. A family β of open sets of X is a *base* for the topology τ if every open set $U\in\tau$ may be written as the union of elements of β. Given a point $x\in X$, a family β_x of neighborhoods of x is a *local base at x* if every neighborhood of x contains an element of β_x.

Example 20. The following families β represent a base of the topology listed

(i) $(\mathbb{R},\tau_{standard})$, $\beta = \{(a,b) \,:\, a,b\in\mathbb{Q}\}$.
(ii) $(\mathbb{R},\tau_{Sorg})$, $\beta = \{[x,b) \,:\, x\in\mathbb{R},\ b\in\mathbb{Q}\}$.
(iii) $(X,\tau_{discrete})$, $\beta = \{\{x\} \,:\, x\in X\}\cup\{\emptyset\}$.

Proposition 21. *Let X be a nonempty set and let $\beta\subset\mathcal{P}(X)$ be a family of sets. Then β is a base for a topology on X if and only if*

(i) *it contains the empty set;*
(ii) *for every $x\in X$ there exists $B\in\beta$ such that $x\in B$,*
(iii) *for every $B_1,B_2\in\beta$ with $B_1\cap B_2\neq\emptyset$ and for every $x\in B_1\cap B_2$ there exists $B_3\in\beta$ such that $x\in B_3$ and $B_3\subset B_1\cap B_2$.*

Proof. Assume that τ is a topology and that β is a base for τ. Then every open set is a union of sets of β. In particular, the empty set and X can be written

[1] In some texts the definition of neighborhood is different.

as union of sets of β, and so (i) and (ii) hold. To prove (iii), note that if $B_1, B_2 \in \beta$, then $B_1 \cap B_2$ is open, and so it can be written as union of elements of β, say

$$B_1 \cap B_2 = \bigcup_{\gamma} B_\gamma$$

Hence, if $x \in B_1 \cap B_2$, then there is a B_γ such that $x \in B_\gamma \subset B_1 \cap B_2$.

Conversely, let $\beta = \{B_\alpha\}_{\alpha\in\Lambda}$ be a family of sets satisfying (i)-(iii) and let τ be given by arbitrary unions of elements of β. We need to show that τ is a topology. By property (i), we have that the empty set belongs to τ, while by property (ii) we have that

$$X = \bigcup_{\alpha\in\Lambda} B_\alpha,$$

and so $X \in \tau$.

If $U_i \in \tau$ for $i = 1, \ldots, M$, then we can write

$$U_i = \bigcup_{\alpha\in\Lambda_i} B_\alpha$$

for some $\Lambda_i \subset \Lambda$. Then

$$U_1 \cap \ldots \cap U_M = \bigcap_{i=1}^{M} \bigcup_{\alpha\in\Lambda_i} B_\alpha.$$

If $x \in U_1 \cap \ldots \cap U_M$, then there exist $\alpha_i \in \Lambda_i$ such that $x \in B_{\alpha_1} \cap \ldots \cap B_{\alpha_M}$. By property (iii) and an induction argument we may find $B_x \in \beta$ such that $x \in B_x$ and $B_x \subset B_{\alpha_1} \cap \ldots \cap B_{\alpha_M}$. Hence

$$U_1 \cap \ldots \cap U_M = \bigcup_{x\in U_1\cap\ldots\cap U_M i} B_x \in \tau.$$

Finally, given an arbitrary collection $\{U_\gamma\}_{\gamma\in\Xi}$ of elements of τ, since each U_γ is a union of elements of β, we have that $\bigcup_{\gamma\in\Xi} U_\gamma$ is a union of elements of β, and so it belongs to τ.

Thus, τ is a topology. The fact that β is a base for τ follows from the definition of τ. ■

2.2. Countability Axioms.

Definition 22. Let (X, τ) be a topological space.

(i) The space X is *first countable* (i.e. satisfies the *first axiom of countability*) if every $x \in X$ admits a countable local base.

(ii) The space X is *second countable* (i.e. satisfies the *second axiom of countability*) if it has a countable base.

Example 23. (i) $(\mathbb{R}, \tau_{standard})$ is second countable, with the base of open intervals with rational endpoints.

(ii) $(\mathbb{R}, \tau_{Sorg})$ is first countable; $\beta_x = \{[x, x + \frac{1}{n}) \ : \ n \in \mathbb{N}\}$ represents a countable local base at x. On the other hand it does not have a countable base. In particular we claim that if β is a basis then the mapping $\phi : \beta \to \mathbb{R}$ defined by $\phi(B) = \inf B$ is a surjection and thus β must be uncountable. To show surjectivity consider $x \in \mathbb{R}$. Then $[x, x+1)$ is open and thus can be represented as a union of basis sets. In particular there exists $B_x \in \beta$ such that $x \in B_x$ and $B_x \subset [x, x+1)$. Therefore $\phi(B_x) = \inf B_x = x$.

(iii) Consider an uncountable set X with the discrete topology τ. Then X satisfies the first axiom of countability, since for every $x \in X$, the singleton $\{x\}$ is a local base. However, X does not have a countable base, since singletons must belong to any base and are uncountable.

(iv) Let X be uncountable. Let τ be the *cofinite* topology that is
$$\tau = \{X \backslash F \, : \, F - \text{ finite } \} \cup \{\emptyset\}.$$
Then (X, τ) is not first countable. To prove this, assume that for $x \in X$, there is a countable local base: $\beta_x = \{X \backslash F_n \, : \, n \in N\}$ for some finite sets F_n. Let $F = \cup_{n=1}^{\infty} F_n$. Then F is at most countable and thus there exists $y \in (X \backslash \{x\}) \backslash F$. Let $U = X \backslash \{y\}$. Then U is a neighborhood of x. However for all n, $y \in X \backslash F_n$ and thus $X \backslash F_n \not\subset U$. So β_x is not a local base. Contradiction.

Lecture 5, September 7, 2012.

2.3. Interior and Closure. Given a topological space (X, τ) and a set $E \subset X$, a point $x \in E$ is called an *interior point* of E if there exists a neighborhood U of x such that $U \subset E$. The *interior* E° of a set $E \subset X$ is the union of all its interior points.

The proof of following proposition is left as an exercise.

Proposition 24. *Let (X, τ) be a topological space and let $E \subset X$. Then*

(i) *E° is an open subset of E,*
(ii) *E° is given by the union of all open subsets contained in E; that is, E° is the largest (in the sense of union) open set contained in E,*
(iii) *E is open if and only if $E = E^\circ$,*
(iv) *$(E^\circ)^\circ = E^\circ$.*

A set $C \subset X$ is *closed* if its complement $X \setminus C$ is open. The next proposition follows from De Morgan's laws.

Proposition 25. *Let (X, τ) be a topological space. Then*

(i) *$\emptyset$ and X are closed.*
(ii) *If $C_i \subset X$, $i = 1, \ldots, n$, is a finite family of closed sets of X, then $C_1 \cup \cdots \cup C_n$ is closed.*
(iii) *If $\{C_\alpha\}_{\alpha \in \Lambda}$ is an arbitrary collection of closed sets of X, then $\bigcap_{\alpha \in \Lambda} C_\alpha$ is closed.*

The *closure* of a set $U \subset X$ is the set:
$$\overline{U} = X \backslash (X \backslash U)^o. \tag{1}$$
The following proposition follows directly from the Proposition 24.

Proposition 26. *Let (X, d) be a metric space and let $E \subset X$. Then*

(i) *$\overline{E}$ is closed and $E \subset \overline{E}$.*
(ii) *$\overline{E}$ is equal to the intersection of all closed sets which contain E.*
(iii) *E is closed if and only if $E = \overline{E}$,*
(iv) *$\overline{(\overline{E})} = \overline{E}$.*

We remark that the property (ii) can be taken as a definition of the closure, in which case (1) is a property that is not hard to verify.

Exercise 27. Let (X,d) be a metric space.

(i) Prove that if $E_1,\dots,E_n$ are subsets of X, then
$$\overline{E_1}\cap\cdots\cap\overline{E_n}\supset\overline{E_1\cap\cdots\cap E_n},$$
$$\overline{E_1}\cup\cdots\cup\overline{E_n}=\overline{E_1\cup\cdots\cup E_n}.$$

(ii) Show that in general $\overline{E_1}\cap\cdots\cap\overline{E_n}\neq\overline{E_1\cap\cdots\cap E_n}$.

(iii) Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary collection of sets of X. What is the relation, if any, between between $\bigcap_{\alpha\in\Lambda}\overline{U_\alpha}$ and $\overline{\bigcap_{\alpha\in\Lambda}U_\alpha}$? And between $\bigcup_{\alpha\in\Lambda}\overline{U_\alpha}$ and $\overline{\bigcup_{\alpha\in\Lambda}U_\alpha}$?

(iv) Think of some nontrivial condition on $\{E_\alpha\}_{\alpha\in\Lambda}$ that guarantees
$$\bigcup_{\alpha\in\Lambda}\overline{U_\alpha}=\overline{\bigcup_{\alpha\in\Lambda}U_\alpha}.$$

Lecture 6, September 10, 2012.

2.4. Topological constructions.

(i) (*Induced topology.*) Let (X,τ) be a topological space. Let $Y\subset X$. The *induced topology* on Y is the family $\tau_Y:=\{Y\cap U\,:\,U\in\tau\}$.

(ii) (*Inverse image.*) Given a topological space (Y,τ_Y) and $f:X\to Y$ then
$$\tau_X=\{f^{-1}(V)\,:\,V\in\tau_Y\}$$
is a topology on X. It is called the inverse image of τ_Y via f.

(iii) (*Direct image.*) Given a topological space (X,τ_X) and an onto function $f:X\to Y$, the family of sets
$$\tau_Y:=\left\{E\subset Y:\,f^{-1}(E)\in\tau_X\right\}$$
is a topology on Y. It is called the direct image of τ_X via the mapping f.

(iv) (*Quotient topology.*) Given a nonempty set X with topology τ, let $\sim$ be an equivalence relation. We define
$$Y=X\diagup\sim:=\{[x]:\,x\in X\}$$
and consider the projection of X onto Y
$$\begin{aligned}P:X&\to Y\\ x&\mapsto[x]\end{aligned}$$
The quotient topology on Y is the direct image of τ via P. That is
$$\tau_Y:=\left\{E\subset Y:\,P^{-1}(E)\in\tau_X\right\}.$$
Note that
$$P^{-1}(E)=\{x\in X:\,[x]\in E\}=\bigcup_{[x]\in E}[x],$$
that $P^{-1}(E)$ is is given by the union of the equivalence classes belonging to E. Thus, an open set in the quotient topology is a collection of equivalence classes whose union is an open set of X.

(v) (*Product topology*) Given two topological spaces (X,τ_X) and (Y,τ_Y), by considering the family $\mathcal{F}=\{U\times V:\,U\in\tau_X,\,V\in\tau_Y\}$, the smallest topology containing $\mathcal{F}$ on $X\times Y$ is called the *product topology*.

2.5. Metric Spaces (as examples of topological cases).

Definition 28. A *metric* on a nonempty set X is a map $d : X \times X \to [0, \infty)$ such that

(i) (positivity)
  (a) $d(x, x) = 0$
  (b) $d(x, y) > 0$ if $x \neq y$,

(ii) (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A *metric space* (X, d) is a set X endowed with a metric d. When the metric is clear from the context, we abbreviate by saying that X is a metric space.

A space satisfying (i)(a), (ii), and (iii) is called a *pseudometric* space.

For any $x \in X$ and $r \geq 0$ we consider the *ball* $B(x, r) = \{y \in X \,:\, d(x, y) < r\}$.

Lemma 29. *Let (X, d) be a pseudometric space. Then $\mathcal{B} = \{B(x, r) \,:\, x \in X, \text{ and } r \geq 0\}$ is a base of a topology.*

Given a metric space, the topology generated by $\mathcal{B}$ is the natural topology and will always be the one we consider.

Lecture 7, September 12, 2012.

Example 30. Here are some examples of metric and pseudometric spaces.

(i) In $\mathbb{R}$ the distance $d(x, y) := |x - y|$ is a metric.

(ii) Consider the interval $[a, b] \subset \mathbb{R}$ and take

$$C([a, b]) := \{f : [a, b] \to \mathbb{R} : \, f \text{ continuous}\}$$

with the metric

$$d(f, g) := \max_{x \in [a,b]} |f(x) - g(x)| \, .$$

Note that here we are using the Weierstrass theorem. More generally, if $K \subset \mathbb{R}^N$ is compact, then we can take

$$C(K) := \{f : K \to \mathbb{R} : \, f \text{ continuous}\}$$

with the metric

$$d(f, g) := \max_{x \in K} |f(x) - g(x)| \, .$$

(iii) Consider the interval $[a, b] \subset \mathbb{R}$ and take

$$C([a, b]) := \{f : [a, b] \to \mathbb{R} : \, f \text{ continuous}\}$$

with

$$d(f, g) := \int_a^b |f(x) - g(x)| \, dx,$$

where the integral is the Riemann integral. It is a metric.

If we consider the same distance on a larger set

$$\mathcal{R}([a, b]) := \{f : [a, b] \to \mathbb{R} : \, f \text{ bounded and Riemann integrable}\}$$

then the property that $d(f, g) = 0$ only if $f = g$ fails. The other properties of a metric are satisfied. So $(\mathcal{R}, d)$ is a pseudometric space.

(iv) In $\mathbb{R}^N$, $N \geq 1$, for $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$,

$$\begin{aligned} d_2(x,y) &:= \sqrt{(x_1 - y_1)^2 + \ldots + (x_N - y_N)^2} \\ d_1(x,y) &:= |x_1 - y_1| + \cdots + |x_N - y_N| \\ d_\infty(x,y) &:= \max_{i=1,\ldots,N} |x_i - y_i| \end{aligned}$$

each define a metric.

(v) The above is a special case of creating a metric on a product space. Namely if (X_i, d_i) is a metric space for $i = 1, \ldots, N$ then on $X = X_1 \times \cdots \times X_N$ one can define a metric in the following ways:

$$\begin{aligned} d(x,y) &:= \sqrt{d_1(x_1,y_1)^2 + \ldots + d_N(x_N,y_N)^2} \\ d(x,y) &:= d_1(x_1,y_1) + \cdots + d_N(x_N,y_N) \\ d(x,y) &:= \max_{i=1,\ldots,N} d_N(x_i,y_i). \end{aligned}$$

(vi) Given a nonempty set X, the metric

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

is called the *discrete metric.*

2.6. Quotient metric space. Given a pseudometric space (X, ρ), we define an relation $\sim$ on X as follows: For $x, y \in X$, we say that $x \sim y$ if $\rho(x,y) = 0$. It is straightforward to show that $\sim$ is an equivalence relation. Consider the quotient space $Y = X \diagup \sim$. We define a distance on Y as follows:

$$d([x],[y]) := \rho(x,y), \quad [x],[y] \in Y. \tag{2}$$

Note that d is indeed well defined since if $x \sim x_1$ then for all $y \in X$

$$\rho(x_1, y) \leq \rho(x_1, x) + \rho(x, y) = \rho(x, y).$$

Reversing the roles of x and x_1 implies that $\rho(x_1, y) = \rho(x, y)$. Consequently if $x_1 \sim x$ and $y_1 \sim y$ then

$$\rho(x_1, y_1) = \rho(x, y).$$

Lemma 31. *Let (X, ρ) be a pseudometric space. Then (Y, d) is a metric space.*

Exercise 32. Let (Y, d_Y) be a pseudometric space, and let τ_Y be the associated topology. Let $\sim$ be the equivalence relation defined above and let $X = Y/\sim$. Let (X, d_X) be the quotient metric space. Consider the mapping $P : Y \to X$ defined by $x \mapsto [x]$. Consider the following two topologies on X: let τ_D be the direct image of toplogy τ_Y via the mapping P and let τ_X be the topology of the metric space (X, d). Show that $\tau_D = \tau_X$.

Exercise 33. Let d_1 and d_2 be two metrics on X, and let τ_1 and τ_2 be the associated topologies. Show that if $d_1 \leq d_2$ then $\tau_1 \subseteq \tau_2$.

Lecture 8, September 14, 2012.

Example 34. (i) If (X,d) is a metric space, then

$$d_2(x,y) := \frac{d(x,y)}{1+d(x,y)}$$

is also a metric on X by Problem 4 on Set 1, since the function $f:[0,\infty)\to[0,\infty)$ defined by $f(s)=\frac{s}{1+s}$ is increasing and subadditive, because

$$f(a+b)=\frac{a+b}{1+a+b}\le\frac{a}{1+a+b}+\frac{b}{1+a+b}\le\frac{a}{1+a}+\frac{b}{1+b}=f(a)+f(b).$$

(ii) Consider $C\left((0,1)\right):=\{f:(0,1)\to\mathbb{R}:\ f \text{ is continuous}\}$. Consider $K_n:=\left[\frac{1}{n},1-\frac{1}{n}\right]$. Then

$$\bigcup_{n=1}^{\infty}K_n=(0,1)\,.$$

Define

(3)
$$d\left(f,g\right):=\max_n\frac{1}{2^n}\frac{\max_{x\in K_n}\left|f\left(x\right)-g\left(x\right)\right|}{1+\max_{x\in K_n}\left|f\left(x\right)-g\left(x\right)\right|}.$$

Then d is a metric (Problem 7, Set 1) and $d\left(f_n,f\right)\to 0\to 0$ if and only if $f_n\to f$ uniformly on compact sets.

More generally if $\Omega\subset\mathbb{R}^N$ is an open set, construct an increasing sequence of compact sets such $\bigcup_{n=1}^{\infty}K_n=\Omega$ and put on $C\left(\Omega\right)$ a metric as in (3).

(iii) Given a set X let

$$B_b\left(X\right)=\ell^{\infty}\left(X\right):=\{f:X\to\mathbb{R}:\ f \text{ is bounded}\}$$

with

$$d_{\infty}\left(f,g\right):=\sup_{x\in X}\left|f\left(x\right)-g\left(x\right)\right|.$$

(iv) (Problem 5, Set 1) Let (A,d_A) be a bounded metric space and let $X=\mathcal{P}(A)\backslash\emptyset$. For $U,V\in X$ let

(4)
$$d_H(U,V)=\max\{\sup_{y\in V}\inf_{x\in U}d_A(x,y),\sup_{x\in U}\inf_{y\in V}d_A(x,y)\}.$$

Then (X,d_H) is a pseudometric space.

(v) (Problem 6, Set 1) Given sets X and Y, their *disjoint union* is the set

$$X\sqcup Y=(X\times\{1\})\cup(Y\times\{2\}).$$

Let A be a nonempty set. Let

$$\mathcal{M}=\{(X,d_X)\ :\ X\subseteq A \text{ and } (X,d_X) \text{ is a metric space}\}$$

We would like to introduce a distance that compares elements of $\mathcal{M}$. For (X,d_X) and (Y,d_Y) in $\mathcal{M}$ let

$$\begin{aligned}\mathcal{C}(\,(X,d_X),(Y,d_Y)\,)=\{d\ :(X\sqcup Y,d)&\text{ is a metric space}\\ &\text{for all } x_1,x_2\in X\quad d_X(x_1,x_2)=d((x_1,1),(x_2,1)) \text{ and}\\ &\text{for all } y_1,y_2\in Y\quad d_Y(y_1,y_2)=d((y_1,2),(y_2,2))\}\end{aligned}$$

be the set of "couplings" of (X,d_X) and $(Y,d_Y)\,)$.

Consider the following function that measures how different two metric spaces are: For (X,d_X) and (Y,d_Y) in $\mathcal{M}$ let

$$D(\,(X,d_X),(Y,d_Y)\,)=\inf_{d\in\mathcal{C}}d_H(X\times\{1\},Y\times\{2\})$$

where d_H is a metric on $\mathcal{P}(X \sqcup Y)\backslash\emptyset$ corresponding to d as defined by (4). Then $(\mathcal{M}, D)$ is a pseudometric space.

Exercise 35. Consider $X = C((0,1))$ and define for $f, g \in X$

$$\tilde{d}(f,g) = \sup_{x\in(0,1)} \frac{|f(x) - g(x)|}{1 + |f(x) + g(x)|}.$$

Show that $\tilde{d}$ is a metric. Compare the topologies on X corresponding to $\tilde{d}$ and the metric defined by (3). Show that one is strictly coarser than the other.

Lemma 36. *Let (X, d_i) be a pseudometric space for all $i \in I$. Define d by $d(x,y) = \sup_{i\in I} d_i(x,y)$ for all $x, y \in X$. Assume that $d(x,y) < \infty$ for all $x, y \in X$. Then d is a pseudometric on X.*

Proof. To check if $d(x,x) = 0$ note that $d(x,x) = \sup_{i\in I} d_i(x,x) = \sup_{i\in I} 0 = 0$. To check symmetry note that $d(x,y) = \sup_{i\in I} d_i(x,y) = \sup_{i\in I} d_i(y,x) = d(y,x)$. Finally consider $x, y, z \in X$ for any $\varepsilon > 0$ there exists $j \in I$ such that

$$d(x,z) \leq d_j(x,z) + \varepsilon \leq d_j(x,y) + d_j(x,z) + \varepsilon \leq d(x,y) + d(y,z) + \varepsilon.$$

Taking the infimum over $\varepsilon > 0$ gives

$$d(x,z) \leq d(x,y) + d(y,z).$$

■

2.7. Infinite sums.

Definition 37. Given a set X and a function $f : X \to [0,\infty]$ the *infinite sum*

$$\sum_{x\in X} f(x)$$

is defined as

$$\sum_{x\in X} f(x) := \sup\left\{\sum_{x\in Y} f(x) : \, Y \subset X, \, Y \text{ finite}\right\}.$$

Proposition 38. *Given a set X and a function $f : X \to [0,\infty]$, if*

$$\sum_{x\in X} f(x) < \infty,$$

then the set $\{x \in X : \, f(x) > 0\}$ is at most countable. That is

$$\sum_{x\in X} f(x) = \sum_{n\in I} f(x_n),$$

where $I = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$ or $I = \mathbb{N}$. Moreover, f does not take the value ∞.

Proof. Define

$$M := \sum_{x\in X} f(x) < \infty.$$

For $k \in \mathbb{N}$ set $X_k := \left\{x \in X : \, f(x) > \frac{1}{k}\right\}$ and let Y be a finite subset of X_k. Then

$$\frac{1}{k}\text{number of elements of } Y \leq \sum_{x\in Y} f(x) \leq M,$$

which shows that Y cannot have more than $\lfloor kM \rfloor$ elements, where $\lfloor \cdot \rfloor$ is the integer part. In turn, X_k has a finite number of elements, and so

$$\{x \in X : f(x) > 0\} = \bigcup_{k=1}^{\infty} X_k$$

is countable. ■

Exercise 39. Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Prove that the set of discontinuity points of f is countable.

Example 40. Given a set X and $1 \le p < \infty$, we define the space

$$\ell^p(X) := \left\{ f : X \to \mathbb{R} : \sum_{x \in X} |f(x)|^p < \infty \right\}.$$

Consider the metric

$$d_p(f,g) := \left(\sum_{x \in X} |f(x) - g(x)|^p \right)^{\frac{1}{p}}.$$

In the particular case in which $X = \mathbb{N}$, then a function

$$\begin{aligned} f : \mathbb{N} &\to \mathbb{R} \\ n &\mapsto f(n) =: a_n \end{aligned}$$

is just a sequence $\{a_n\}_n$ and so we have

$$\ell^p(\mathbb{N}) := \left\{ \{a_n\}_n \subset \mathbb{R} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}.$$

We usually write $\ell^p := \ell^p(\mathbb{N})$.

Given a number $1 \le p \le \infty$, the *Hölder conjugate exponent of* p is the number $1 \le q \le \infty$ defined as

$$q := \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ \infty & \text{if } p = 1, \\ 1 & \text{if } p = \infty. \end{cases}$$

Note that, with an abuse of notation, we have

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In the sequel, the Hölder conjugate exponent of p will often be denoted by p'.

Remark. Below we present a proof that ℓ^p spaces are metric spaces. This proof applies to all $p \in [1, \infty)$, while the proof in the lecture was given only for $p = 1, p = 2$, and $p = \infty$. You are not required to know the proof for p not in this set, in particular you are not required to know the proof of Young's inequality, Hölder's inequality and Minkowski's for $p \ne 2$. Note that for $p = 2$ these inequalities simplify. On the other hand these inequalities are important and useful in theory of integration and in functional analysis, so you may want to learn them.

Proposition 41 (Young's inequality)**.** *Let $1 < p < \infty$, and let q be its Hölder conjugate exponent. Then*

$$ab \le \frac{1}{p} a^p + \frac{1}{q} b^q$$

for all $a, b \ge 0$.

Proof. If $a = 0$ or $b = 0$, then there is nothing to prove. Thus, assume that $a, b > 0$. Since the function $t \in [0, \infty) \mapsto \ln t$ is concave and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \geq \frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q = \ln ab,$$

that is

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab.$$

■

Theorem 42 (Hölder's inequality)**.** *Let X be a set, let $1 \leq p \leq \infty$, and let q be its Hölder conjugate exponent. Given $f, g : X \to [-\infty, \infty]$, then*

$$\sum_{x \in X} |f(x)\, g(x)| \leq \left(\sum_{x \in X} |f(x)|^p\right)^{\frac{1}{p}} \left(\sum_{y \in X} |g(y)|^q\right)^{\frac{1}{q}}$$

if $1 < p < \infty$, while

$$\sum_{x \in X} |f(x)\, g(x)| \leq \left(\sum_{x \in X} |f(x)|\right)\left(\sup_{y \in X} |g(y)|\right)$$

if $p = 1$. In particular, if $f \in \ell^p(X)$ and $g \in \ell^q(X)$ then $fg \in \ell^1(X)$.

Proof. Assume that $1 < p < \infty$. If $\sum_{x \in X} |f(x)|^p = 0$ or $\sum_{y \in X} |g(y)|^q = 0$, then $f(x)\, g(x) = 0$ for all $x \in X$ and so there is nothing to prove. Thus assume that both sums are positive. If one of them is infinite, then the right-hand side is ∞ and so there is nothing to prove. Hence in what follows we consider the case in which both sums are finite belong to $(0, \infty)$.

If we apply Young's inequality with

$$a = \frac{|f(x)|}{\left(\sum_{y \in X} |f(y)|^p\right)^{\frac{1}{p}}} \quad \text{and} \quad b = \frac{|g(x)|}{\left(\sum_{y \in X} |g(y)|^q\right)^{\frac{1}{q}}},$$

we get

$$\frac{|f(x)\, g(x)|}{\left(\sum_{y \in X} |f(y)|^p\right)^{\frac{1}{p}} \left(\sum_{z \in X} |g(z)|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p}\frac{|f(x)|^p}{\sum_{y \in X} |f(y)|^p} + \frac{1}{q}\frac{|g(x)|^q}{\sum_{z \in X} |g(z)|^q}.$$

Upon summation, we obtain

$$\begin{aligned}\frac{\sum_{x \in X} |f(x)\, g(x)|}{\left(\sum_{y \in X} |f(y)|^p\right)^{\frac{1}{p}} \left(\sum_{z \in X} |g(z)|^q\right)^{\frac{1}{q}}} &\leq \frac{1}{p}\frac{\sum_{x \in X} |f(x)|^p}{\sum_{y \in X} |f(y)|^p} + \frac{1}{q}\frac{\sum_{x \in X} |g(x)|^q}{\sum_{z \in X} |g(z)|^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

This gives the desired result for $1 < p < \infty$.

If $p = 1$ and $q = \infty$, then

$$|f(x)\, g(x)| \leq |f(x)| \sup_{y \in X} |g(y)|,$$

and we can now sum both sides. ■

Theorem 43 (Minkowski's inequality)**.** *Let X be a set, let $1 \le p < \infty$, and let $f, g : X \to [-\infty, \infty]$ be two functions. Then,*

$$\left(\sum_{x\in X} |f(x)+g(x)|^p\right)^{\frac{1}{p}} \le \left(\sum_{x\in X} |f(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{x\in X} |g(x)|^p\right)^{\frac{1}{p}}.$$

In particular, if $f, g \in \ell^p(X)$, then $f + g \in \ell^p(X)$.

Proof. If $\sum_{x\in X} |f(x)|^p = \infty$ or $\sum_{x\in X} |g(x)|^p = \infty$ then the right-hand side of Minkowski's inequality is ∞, and so there is nothing to prove. Thus assume that both sums are finite.

We consider first the case $1 < p < \infty$. By the convexity of the function $t \in [0,\infty) \mapsto t^p$, for any a, $b > 0$, we have

$$(a+b)^p = 2^p \left(\frac{a+b}{2}\right)^p \le \frac{2^p}{2} a^p + \frac{2^p}{2} b^p = 2^{p-1}(a^p + b^p),$$

and so

$$\sum_{x\in X} |f(x)+g(x)|^p \le \sum_{x\in X} (|f(x)| + |g(x)|)^p \le 2^{p-1} \left(\sum_{x\in X} (|f(x)|^p + |g(x)|^p)\right),$$

which shows that $f + g \in \ell^p(X)$. To prove Minkowski's inequality, we observe that

$$\begin{aligned}\sum_{x\in X} |f(x)+g(x)|^p &= \sum_{x\in X} |f(x)+g(x)| \, |f(x)+g(x)|^{p-1} \\ &\le \sum_{x\in X} |f(x)| \, |f(x)+g(x)|^{p-1} + \sum_{x\in X} |g(x)| \, |f(x)+g(x)|^{p-1}.\end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned}\sum_{x\in X} |f(x)+g(x)|^p \le{}& \left(\sum_{x\in X} |f(x)|^p\right)^{\frac{1}{p}} \left(\sum_{x\in X} |f(x)+g(x)|^{(p-1)p'}\right)^{\frac{1}{p'}} \\ &+ \left(\sum_{x\in X} |g(x)|^p\right)^{\frac{1}{p}} \left(\sum_{x\in X} |f(x)+g(x)|^{(p-1)p'}\right)^{\frac{1}{p'}} \\ \le{}& \left(\left(\sum_{x\in X} |f(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{x\in X} |g(x)|^p\right)^{\frac{1}{p}}\right) \left(\sum_{x\in X} |f(x)+g(x)|^p\right)^{\frac{1}{p'}},\end{aligned}$$

where we have used the fact that $(p-1)p' = p$. If $\sum_{x\in X} |f(x)+g(x)|^p$, then there is nothing to prove, thus assume that $\sum_{x\in X} |f(x)+g(x)|^p \in (0,\infty)$. Hence, we may divide both sides of the previous inequality by $\left(\sum_{x\in X} |f(x)+g(x)|^p\right)^{\frac{1}{p'}}$ to obtain

$$\left(\sum_{x\in X} |f(x)+g(x)|^p\right)^{\frac{1}{p}} \le \left(\sum_{x\in X} |f(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{x\in X} |g(x)|^p\right)^{\frac{1}{p}},$$

where we have used the fact that $\frac{1}{p} + \frac{1}{p'} = 1$.

If $p = 1$, Minkowski's inequality follows from the triangle inequality. ■

Minkowski's inequality shows that $\ell^p(X)$ is a metric space.

Corollary 44. *Let X be a nonempty set and let $1 \leq p < \infty$. Then $\ell^p(X)$ is a metric space.*

Proof. Define

$$\|f\|_p := \left(\sum_{x \in X} |f(x)|^p\right)^{\frac{1}{p}}.$$

By Minkowski's inequality, we have that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, and so, since

$$d(f,g) = \|f-g\|_p,$$

it follows that

$$\begin{aligned} d(f,g) &= \|f-g\|_p = \|f \pm h - g\|_p \leq \|f-h\|_p + \|h-g\|_p \\ &= d(f,h) + d(h,g). \end{aligned}$$

■

Lecture 9, September 17, 2012.

2.8. Limit point (accumulation point).

Definition 45. Let (X,τ) be a topological space. Given $E \subseteq X$ we say that $p \in X$ is a *limit point* (a.k.a. *an accumulation point*) of E if for every neighborhood U of p

$$(U \cap E) \backslash \{p\} \neq \emptyset.$$

We denote the set of all limit points of E by E'.

Recall that a point p is a topological space is an isolated point if $\{p\}$ is an open set. [2]

Exercise 46. Let $E \subset X$. Consider the induced topology τ_E. Show that $p \in E \backslash E'$ if and only if $\{p\} \in \tau_E$, that is if p is an isolated point of (E, τ_E).

Exercise 47. Assume that X is a metric space. Show that if x is a limit point of E then every ball centered at x contains infinitely many points of E.

Recall that closure of a set was defined in (1)

Lemma 48. *Let E be a subset of the topological space X. Then*

$$E \cup E' = \overline{E}.$$

Proof. $(\subseteq)$. Let $p \in E \cup E'$. If $p \in E$ then $p \in \overline{E}$ by definition of $\overline{E}$. consider $p \in E'$.

Assume that $p \notin \overline{E}$. Then $p \in (X \backslash E)^o$. Thus there exists U an neighborhood of p such that $U \subset X \backslash E$. But then $U \cap E = \emptyset$ which contradicts the definition of $\overline{E}$. Thus $p \in \overline{E}$.

$(\supseteq)$. It is enough to show that $X \backslash (E \cup E') \subseteq (X \backslash E)^o$. Consider $p \notin E \cup E'$. Then there exists $U \in \tau$ such that $U \cap E = \emptyset$. Therefore p is an interior point of $X \backslash E$ and thus $p \in (X \backslash E)^o$. ■

Corollary 49. $E \subset X$ *is closed if and only if it contains all of its limit points.*

[2]Here I deviate a bit from what I did in the lecture. Namely regarding what was the definition and what is the claim to be proved.

2.9. Sequences. A sequence in X is a function $x : \mathbb{N} \to X$. We write x_i instead of $x(i)$ for $i \in \mathbb{N}$.

Definition 50. A sequence $\{x_n\}_{n=1,2,\dots}$ in a topological space X *converges to* p, and we write

$$x_n \to p \text{ as } n \to \infty,$$

if for every neighborhood U of p there exists n_0 such that for all $n \geq n_0$, $x_n \in U$.

Remark 51. The limit of a sequence is not in general unique. For example let $X = \mathbb{N}$ and let τ be the trivial topology, $\tau = \{\emptyset, X\}$. Let $p \in \mathbb{N}$ and let $\{x_n\}_n$ be an arbitrary sequence in $\mathbb{N}$. Then $x_n \to p$ as $n \to \infty$.

Lemma 52. *Let E be a subset of a topological space X, and let $\{x_n\}_{n=1,2,\dots}$ be a sequence in E. Assume that for some $p \in X$, $x_n \to p$ as $n \to \infty$. Then $p \in \overline{E}$.*

Proof. If $p \in E$ there is nothing to prove. So assume $p \notin E$. Let $\{x_n\}_{n=1,2,\dots}$ be a sequence in E converging to p. Let U be a neighborhood of p. Then there exists n_0 such that for all $n \geq n_0$, $x_n \in U$. consequently, using that $x_n \neq p$ for all n, $U \cap E \setminus \{p\}$ is nonempty. Therefore $p \in E' \subset \overline{E}$. ■

Lemma 53. *Let (X, τ) be a first countable topological space. Let E be a subset of a topological space X. Then for all $p \in \overline{E}$ there exists a sequence $\{x_n\}_{n=1,2,\dots}$ in E which converges to p.*

Proof. If $p \in E$ it suffices to consider the constant sequence $x_n = p$ for all n. If $p \in \overline{E} \setminus E$ then by Lemma 48, $p \in E' \setminus E$. Since X is first countable there exists an at most countable neighborhood base at p: $\{V_i \,:\, i \in I\}$. If I is finite then consider $V = \cap_{i \in I} V_i$ which is then a neighborhood of p too. Furthermore $\{V\}$ is a neighborhood base at p. Since $p \in E'$ there exists $x \in V \cap E \setminus \{p\}$. Let $x_n = x$ for all n. Then $x_n \to p$ and $n \to \infty$.

Let us consider now the case that I is countable. Then we can assume $I = \mathbb{N}$. Define $U_j = \cap_{i=1}^{j} V_i$. Note that $\{U_j \,:\, j \in \mathbb{N}\}$ is also a neighborhood base at p. Furthermore note that if $i < j$ then $U_i \supseteq U_j$. Since $p \in E'$ for all $j \in \mathbb{N}$ there exist $x_j \in U_j \cap E \setminus \{p\}$. We claim that $x_j \to p$ as $n \to \infty$. To see that consider W a neighborhood of p. Since U_i is a neighborhood base there exists $i \in \mathbb{N}$ such that $U_i \subseteq W$. Then $x_j \in U_j \subseteq U_i \subseteq W$ for all $j \geq i$. ■

Example 54. To show that the claim of the lemma above is not true in all topological spaces consider the co-countable topology on $\mathbb{R}$. That is let

$$\tau = \{X \setminus A \,:\, A - \text{ at most countable}\} \cup \{\emptyset\}.$$

It is straightforward to verify that τ is indeed a topology. Let $E = \mathbb{R} \setminus \{0\}$. Then $0 \in \overline{E}$. Namely if U is a neighborhood of 0 then $U = \mathbb{R} \setminus A$ where A is at most countable. Thus $U \cap E = \mathbb{R} \setminus (A \cup \{0\}) \neq \emptyset$. On the other hand we claim that there is no sequence in E which converges to 0. Namely consider any sequence $\{x_n\}_{n=1,2,\dots}$ in E. Define $A = \{x_1, x_2, \dots\}$ Since A is at most countable $\mathbb{R} \setminus A$ is an open neighborhood of 0. However there are no elements of the sequence in $\mathbb{R} \setminus A$. So x_n cannot converge to 0.

Lecture 10, Sep 19, 2012.

2.10. Separability.

Definition 55. A subset E of a topological space X is said to be *dense* if its closure is the entire space, i.e., $\overline{E} = X$. We say that a topological space is *separable* if it contains an at most countable dense subset.

Note that if (X, τ_2) is separable and τ_1 is coarser than τ_2 then (X, τ_1) is separable.

Exercise 56. Let (X, τ) be a topological space satisfying the second axiom of countability. Prove that X is separable.

Example 57. (i) $\mathbb{R}^N$ is separable, since $\mathbb{Q}^N$ is dense in $\mathbb{R}^N$.

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \ldots + (x_N - y_N)^2}.$$

(ii) $\mathbb{R}$ with Sorgenfrey topology is separable. In particular $\mathbb{Q}$ is dense since every neighborhood of every point contains an interval of the form $[a, b)$, which contains a rational point.

(iii) Given a nonempty set X with discrete metric

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

X is separable if and only if X is countable.

(iv) $\mathbb{R}$ with co-finite topology is separable. In particular $\mathbb{N}$ is a countable dense subset.

(v) $\mathbb{R}$ with co-countable topology is not separable. To see this note that if E is a countable subset then it is closed and thus $\overline{E} = E$. So E cannot be dense in $\mathbb{R}$.

(vi) $\mathbb{R}^2$ with topology of radially open sets is separable. In particular $E = \mathbb{Q}^2$ is a dense subset. To see this consider $A = \mathbb{Q} \times R$. We first show that $A \subset \overline{E}$. To see this consider $(p, y) \in A$ and U a radially open neighborhood of (p, y). Then there exists $\delta > 0$such that $\{p\} \times (y - \delta, y + \delta) \subset U$. There exists $q \in \mathbb{Q}$ such that $q \in (y, y + \delta)$. Therefore $(p, q) \in U \cap E$. So every neighborhood of (p, y) contains points in E and thus $(p, y) \in \overline{E}$.

Furthermore $\overline{A} \subset \overline{\overline{E}} = \overline{E}$. So it suffices to show that A is dense in $\mathbb{R}^2$. That argument is analogous to one above (only that horizontal line segment is considered instead of the vertical one).

Recall from Problem 2 on Set 1 that the induced topology on a circle C is the discrete one. Thus $(C, \tau_{rad,C})$ is not separable. So a separable space can have a subspace which is not separable.

(vii) Using uniform continuity, one can show that piecewise affine functions are dense in $C([a, b])$. By approximating a piecewise affine function with one with rational slopes and endpoints, it follow that $C([a, b])$ is separable.

(viii) $\ell^p = \ell^p(\mathbb{N})$, $1 \leq p < \infty$, is separable. Take

$$E = \{x = (r_1, \ldots, r_n, 0, \ldots) : r_i \in \mathbb{Q}, \, i = 1, \ldots, n, \, n \in \mathbb{N}\}.$$

If $x = \{x_n\} \in \ell^p$ and $\varepsilon > 0$, let $n_\varepsilon \in \mathbb{N}$ be so large that

$$\sum_{n=n_\varepsilon+1}^{\infty} |x_n|^p \leq \frac{\varepsilon^p}{2}.$$

Using the density of $\mathbb{Q}$ in $\mathbb{R}$ we may find $r_1, \dots, r_{n_\varepsilon}$ such that

$$\sum_{n=1}^{n_\varepsilon} |x_n - r_n|^p \le \frac{\varepsilon^p}{2}.$$

Then $y = (r_1, \dots, r_{n_\varepsilon}, 0, \dots)$ belongs to E and

$$d_p(x, y) = \left(\sum_{n=1}^{n_\varepsilon} |x_n - r_n|^p + \sum_{n=n_\varepsilon+1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \le \left(\frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} \right)^{\frac{1}{p}} = \varepsilon.$$

(ix) If X is uncountable, then $\ell^p(X)$, $1 \le p < \infty$, is not separable.

(x) $\ell^\infty = \ell^\infty(\mathbb{N})$ is not separable. Here is a sketch of a proof. Assume ℓ^∞ has a dense countable subset $E = \{x^1, x^2, \dots\}$ (x^n are sequences). Consider a sequence x such that for all n, $|x_n - x_n^n| > 1$ and $|x_n| < 2$. Then $d(x, x^n) > 1$ for all n, and thus $x \notin \overline{E}$.

A property of topological spaces is called *hereditary* if whenever a topological space has the property then so do all of its subsets with respect to the induced topology. Example (vi) above shows that separability is not hereditary.

On the other hand

Exercise 58. If (X, d) is a separable metric space then for every $A \subset X$, the space (A, d) is separable.

Lecture 11, September 21, 2012.

2.11. Connectedness.

Definition 59. A topoloical space (X, τ) is connected if the only open sets that are also closed are X and $\emptyset$.

A set $E \subset X$ is connected if (E, τ_E) is connected, where τ_E is the induced topology.

Exercise 60. Show that if (X, d) is a metric space, $E \subset X$ is not connected if and only if there exist U_1 and U_2 disjoint, open in X such that $E \subset U_1 \cup U_2$, $U_1 \cap E \neq \emptyset$, and $U_2 \cap E \neq \emptyset$. [Remark: This exercise is not quite as easy as it may seem. Furthermore the claim is not true in a general topological space. To see that consider (X, τ) with $X = \{1, 2, 3\}$ and $\tau = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ and $E = \{1, 3\}$.]

Theorem 61. *A set $C \subset \mathbb{R}$ is connected if and only if it is convex.*

Proof. Step 1: Assume that C is convex. We claim that C is connected. If not, then there exist two nonempty disjoint open sets $U, V \subset \mathbb{R}$ such that

$$C \subset U \cup V, \quad C \cap U \neq \emptyset, \quad C \cap V \neq \emptyset.$$

Let $x \in C \cap U$ and $y \in C \cap V$. Without loss of generality, we may assume that $x < y$. By convexity, the interval $[x, y]$ is contained in C. Let

$$z := \sup (U \cap [x, y]).$$

Then $x \le z \le y$. Since $x \in U$ and $y \in V$ and U and V are open, we can find $\delta > 0$ such that $x + \delta < z < y - \delta$. We will show that $z \notin U \cup V$. Indeed, if $z \in U$, then, since since U is open, we can find $r > 0$ such that $(z - r, z + r) \subset U$ and taking

$r < y - z$, we have that $[z, z + r) \subset U \cap [x, y]$, which contradicts the definition of z. Hence, $z \notin U$ but since

$$z \in [x, y] \subset C \subset U \cup V,$$

this implies that $z \in V$. Since V is open, we can find $r_1 > 0$ such that $(z - r_1, z + r_1) \subset V$ and taking $r_1 < z - x$, we have that $(z - r, z] \subset V \cap [x, y]$, which contradicts the definition of z. This shows that C is connected.

Step 2: Assume that C is connected, let $x, y \in C$, with, say, $x < y$. We claim that the interval $[x, y]$ is contained in C. If not, then there exists $x < z < y$ such that $z \notin C$. Define

$$U := (-\infty, z), \quad V := (z, \infty).$$

Then U and V are open, disjoint, both intersect C and their union covers C. This contradicts the fact that C is connected. ■

Proposition 62. *Let (X, d) be a metric space and let $E \subset X$. Assume that*

$$E = \bigcup_{\alpha \in \Lambda} E_\alpha,$$

where each E_α is a connected set. If $\bigcap_{\alpha \in \Lambda} E_\alpha$ is nonempty, then E is connected.

Proposition 63. *Let (X, τ) be a topological space and let $E \subset X$ be a connected set. Then $\overline{E}$ is connected.*

Proof. If not, then there exist two nonempty disjoint open sets $U, V \subset \overline{E}$ such that

$$\overline{E} \subset U \cup V, \quad U \neq \emptyset, \quad V \neq \emptyset.$$

Note that at least one of them must have an empty intersection with E; since otherwise E would not be connected. So assume $U \subset \overline{E} \backslash E = E' \backslash E$. Let $x \in U$. Since $x \in E'$ and U is an open neighborhood of x in $\overline{E}$ the set $E \cap U$ is not empty. But this contradicts the fact that $U \cap E = \emptyset$. ■

Exercise 64. Prove that the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected.

Exercise 65. Let (X, d) be a metric space and let $E_1, E_2 \subset X$ be two connected sets. Prove that if there exists $x \in E_1 \cap \overline{E_2}$, then $E_1 \cup E_2$ is connected.

We now introduce another notion of connectedness, which is simpler to verify. To do so we first define continuity in this special setting. An extensive treatment of continuity of mappings between topological spaces will follow later.

Let (X, τ) be a topological space. A function (path) $\gamma : [a, b] \to X$ is said to be continuous if for all $x \in [a, b]$ and any sequence $\{x_n\}_n$ in $[a, b]$ such that $x_n \to x$ as $n \to \infty$ it holds that $\gamma(x_n) \to \gamma(x)$ in X as $n \to \infty$.

Lemma 66. *If $\gamma : [a, b] \to X$ is a continuous path and $U \in \tau$ then $\gamma^{-1}(U)$ is open in $[a, b]$.*

Proof. Assume that there exists $U \in \tau$ such that $\gamma^{-1}(U)$ is not open. Then there exists $x \in \gamma^{-1}(U)$ which is not an interior point of $\gamma^{-1}(U)$. Then there exists a sequence $\{x_n\}_n$ in $[a, b] \backslash \gamma^{-1}(U)$ such that $x_n \to x$ as $n \to \infty$. Note that for all n, $f(x_n) \notin U$. Since U is an open neighborhood of $f(x)$ this implies that $f(x_n)$ does not converge to $f(x)$ as $n \to \infty$. ■

Definition 67. A topological space (X,τ) is *pathwise connected* if for any $x, y \in X$ there exists a continuous path $\gamma : [a,b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

Proposition 68. *Let (X,τ) be a pathwise connected topological space. Then X is connected.*

Proof. Assume that X is pathwise connected but not connected. Then there exist U, V disjoint, nonempty, open sets such that $U \cup V = X$. But then by Lemma 66 the sets $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are nonempty, disjoint, open subsets of $[a,b]$. Furthermore their union is $[a,b]$. So $[a,b]$ is not connected. Contradiction with Theorem 61 ■

Definition 69. In the Euclidean space $\mathbb{R}^N$, a *polygonal path* is a continuous path $\gamma : [a,b] \to \mathbb{R}^N$ for which there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ with the property that $\gamma : [x_{i-1}, x_i] \to \mathbb{R}^N$ is affine for all $i = 1, \ldots, n$, that is,

$$\gamma(t) = c_i + t d_i \quad \text{for } t \in [x_{i-1}, x_i],$$

for some $c_i, d_i \in \mathbb{R}^N$.

Exercise 70. Let $O \subset \mathbb{R}^N$ be open and connected and let $x_0 \in O$.

(i) Prove that the set

$$U := \{x \in O : \text{ there exists a polygonal path with} \\ \text{endpoints } x \text{ and } x_0 \text{ and range contained in } O\}$$

is open and nonempty.

(ii) Prove that the set

$$V := \{x \in O : \text{ there does not exist a polygonal path with} \\ \text{endpoints } x \text{ and } x_0 \text{ and range contained in } O\}$$

is open.

(iii) Prove that O is pathwise connected.

Lecture 12, September 24, 2012.

The next examples show that in $\mathbb{R}^N$ a connected set may fail to be pathwise connected.

Example 71. (Comb space) Consider the following subsets of $\mathbb{R}^2$: Let $H = [0,1] \times \{0\}$, $V_n = \frac{1}{2^n} \times [0,1]$ and

$$\tilde{E} = \bigcup_{n=0}^{\infty} (V_n \cup H).$$

Note that $\tilde{E}$ is connected by Proposition 62. Let $E = \tilde{E} \cup \{(0,1)\}$. Note that the closure of $\tilde{E}$ is equal to E and is thus connected by Proposition 63.

To show that E is not pathwise connected, assume that it is. Then there exists $\gamma \in C([0,1], E)$ such that $\gamma(0) = (0,1)$ and that $\gamma(1) = (1,0)$. Let $T = \sup\{s : \gamma(s) = (0,1)\}$. By continuity $\gamma(T) = (0,1)$. Also by continuity there exists $\delta > 0$ such that if $|t - T| < \delta$ then $|\gamma(t) - (0,1)| < \frac{1}{2}$. It follows that $\gamma(T + \delta/2) \in V_n$ for some n. Let $\tilde{\gamma}$ be the restriction of γ to $[0, z+\delta)$. Let $V = \tilde{\gamma}^{-1}(V_n) = \{t \in [0, z+\delta) : \gamma(t) \cdot e_1 = 2^{-n}\}$. Since $\tilde{\gamma}$ is continuous V is a closed subset of $[0, T+\delta)$. Also note that for $\varepsilon = 2^{-n-1}$, $V = \{t \in [0, z+\delta) : \gamma(t) \cdot e_1 \in (2^{-n} - \varepsilon, 2^{-n} + \varepsilon)\}$. By continuity of $\tilde{\gamma}$, V is an open subset of $[0, T+\delta)$. Furthermore note that $T + \delta/2 \in V$ so V is nonempty. Also $0 \notin V$.

The existence of such V implies that $[0, T+\delta)$ is not connected. Contradiction.

Definition 72. Let (X, τ) be a topological space. We define the relation $\sim$ as follows:

$$x \sim y \quad \text{if there exists } E \subset X \text{ connected for which } x, y \in E.$$

From Proposition 62 follows that $\sim$ is an equivalence relation. The equivalence classes of relation $\sim$ are called the *connected components* of X.

Let E_x be the connected component to which x belongs. We note that

$$E_x = \bigcup\{E \subseteq X \ : \ x \in E \text{ and } E \text{ is connected}\}.$$

By Proposition 62, E_x is connected and thus it can be characterized as the maximal connected subset of X containing x.

Proposition 73. *Let (X, τ) be a topological space and let $C \subset X$ be a closed set. Then the connected components of C are closed.*

Proof. Let C_α be a connected component of C. Then $C_\alpha \subset \overline{C_\alpha} \subset \overline{C} = C$. By Proposition 63, $\overline{C_\alpha}$ is connected, and so by the maximality of C_α, $\overline{C_\alpha} = C_\alpha$, i.e., C_α is closed. ■

Exercise 74. Prove that if $U \subset \mathbb{R}^N$ is open, then the connected components of U are open. Is this still true for open subsets of arbitrary topologic spaces?

A space is called *completely disconnected* if every connected component is a singleton.

Example 75. The set of rational numbers in $\mathbb{R}$ is completely disconnected. So is the set of irrational numbers. At the same time the induced topology on each of the sets is not the discrete topology.

Lecture 13, September 26, 2012.

2.12. Continuity I.

Definition 76. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. The function f is *continuous* if for every $V \in \tau_Y$, the inverse image, $f^{-1}(V)$, is an open subset of X.

Exercise 77. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. The following are equivalent.

(i) f is continuous.
(ii) For every $x \in X$ and every neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subseteq V$.
(iii) $f^{-1}(C)$ is closed for every closed set $C \subset Y$.

Exercise 78. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. Let $\mathcal{S}$ be a subbase of the topology τ_Y. Then f is continuous if and only if $f^{-1}(U) \in \tau_X$ for all $U \in \mathcal{S}$. [Hint: Consider first the case that $\mathcal{S}$ is a base.]

Definition 79. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. The function f is *sequentially continuous at* $a \in X$ if for every sequence $\{x_n\}_{n=1,2,\dots}$ in X such that $x_n \to a$ as $n \to \infty$ it holds that $f(x_n) \to f(a)$ as $n \to \infty$.

Exercise 80. Show that the composition of two (sequentially) continuous functions is (sequentially) continuous.

Exercise 81. Show that if $f, g : X \to \mathbb{R}$ are continuous functions then the function $h(x) = \max\{f(x), g(x)\}$ is continuous.

Proof. Note that $\mathcal{S} = \{(-\infty, b) \ : \ b \in \mathbb{R}\} \cup \{(a, \infty(: \ a \in \mathbb{R}\}$ is a subbase of the topology on $\mathbb{R}$. To show that h is continuous, by Exercise cont subbase, it is enough to check that $h^{-1}((-\infty, b))$ and $h^{-1}((a, \infty)$ are open. To verify that note that

$$h^{-1}((-\infty, b)) = \{x \in X \ : \ f(x) < b \text{ and } g(x) < b\} = f^{-1}((-\infty, b)) \cap g^{-1}((-\infty, b))$$
$$h^{-1}((a, \infty)) = \{x \in X \ : \ f(x) > a \text{ or } g(x) > a\} = f^{-1}((a, \infty)) \cup g^{-1}((a, \infty)).$$

Since f and g are continuous the sets on the right-hand side are open, as desired. ■

Lemma 82. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. If f is continuous then f is sequentially continuous.*

Proof. Assume f is continuous and $x_n \to a$ as $n \to \infty$. Let V be a neighborhood of $f(a)$ and let $U = f^{-1}(V)$. Since f is continuous U is a neighborhood of a. Since $x_n \to a$ as $n \to \infty$, there exists n_0 such that for all $n \geq n_0$, $x_n \in U$. Consequently for all $n \geq n_0$, $f(x_n) \in V$. Therefore $f(x_n) \to f(a)$ as $n \to \infty$. ■

Lemma 83. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$. Assume that (X, τ_X) is a first countable topological space. If f is sequentially continuous then f is continuous.*

Consequently, if the domain of a function is first countable then the notions of continuity and sequential continuity are equivalent.

Proof. Assume (X, τ_X) is first countable and $f : X \to Y$ is sequentially continuous. Assume that f is not continuous. then there exists $V \subset Y$ such that $U = f^{-1}(V)$ is not open. Then there exists $a \in U$ which is not an interior point of U. Let $\mathcal{B} = \{B_i \ : \ i \in \mathbb{N}\}$ be a neighborhood basis at a such that $B_{i+1} \subseteq B_i$ for all $i \in \mathbb{N}$. Since a is not an interior point for all $n \in \mathbb{N}$ there exists $x_n \in B_n \backslash U$. Therefore $x_n \to a$ as $n \to \infty$. However for all $n \in N$, $f(x_n) \notin V$ and hence $f(x_n) \not\to f(a)$ as $n \to \infty$. So f is not sequentially continuous. Contradiction. ■

Example 84. Let τ_{coco} be the co-countable topology on $\mathbb{R}$. The identity function mapping $(\mathbb{R}, \tau_{coco})$ to $(\mathbb{R}, \tau_{discrete})$ is sequentially continuous, but is not continuous.

Next we show that continuous functions preserve connectedness.

Proposition 85. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be continuous. If X is connected then $f(X)$ is a connected subspace of Y.*

Proof. assume that X is connected and assume by contradiction that $f(X)$ is disconnected. Then there exist two nonempty open sets in Y $U, V \subset Y$ such that $f(X) \subseteq U \cup V$ and that $f(X) \cap U \cap V = \emptyset$. By continuity, $f^{-1}(U)$ and $f^{-1}(V)$ are open,

$$X \subset f^{-1}(U) \cup f^{-1}(V), \quad f^{-1}(U) \neq \emptyset, \quad f^{-1}(V) \neq \emptyset,$$

and $f^{-}1(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(U \cap V \cap f(X)) = \emptyset$, which shows that X is disconnected. Contradiction. ■

2.13. Axioms of Separation. An important feature of topological spaces is how well they can separate points and other sets. Here are the definitions of such properties:

Definition 86. A topological space (X, τ) is

(i) T_0 if for all $x, y \in X$ and $x \neq y$ there exists an open set U such that exactly one of the points belongs to U.
(ii) T_1 if for all $x, y \in X$ and $x \neq y$ there exists an open set U such that $x \in U$ and $y \notin U$. (Note that this implies that there exists $V \in \tau$ such that $y \in V$ and $x \notin V$.)
(iii) T_2 **or Hausdorff** if for all $x, y \in X$ and $x \neq y$ there exists disjoint open sets U and V such that $x \in U$ and $y \in V$.
(iv) **completely Hausdorff** if for every $x, y \in X$, $x \neq y$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$.
(v) **regular** if for every $x \in X$ and every closed set $C \subset X$ that does not contain x there exist disjoint open sets U and V such that $C \subset U$ and $x \in V$.
(vi) T_3 if it is T_0 and regular.
(vii) **completely regular** if for every $x \in X$ and every closed set $C \subset X$ that does not contain x, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f = 0$ on C.
(viii) $T_{3\frac{1}{2}}$ **or Tikhonov** if it is T_0 and completely regular.
(ix) **normal** if for every pair of disjoint closed sets C_1, C_2 there exist disjoint open sets U_1 and U_2 such that $C_1 \subset U_1$ and $C_2 \subset U_2$.
(x) T_4 if it is T_1 and normal.

Lecture 14, September 28, 2012.

We spent a good part of the lecture discussing the solution to problem set 2.

Exercise 87. Show that if (X, τ) is a T_1 space then $\{x\}$ is a closed set for all $x \in X$.

Exercise 88. Show that if a space is $T3$ then it is $T2$. Also show that there exists a space which is normal and $T0$ but not $T4$.

Example 89. (i) Let X be a set with more than one element. Then the trivial topology is not T_0.
(ii) Let $X = \mathbb{R}$ and $\tau = \{(a, \infty) : a \in [-\infty, \infty]\}$. Then (X, τ) is T_0 but not T_1.
(iii) Let $X = \mathbb{N}$ and $\tau = \{\mathbb{N} \backslash I : I \subseteq N - \text{finite}\} \cup \{\emptyset\}$. Then (X, τ) is T_1, but not Hausdorff.
(iv) Let $X = \{(x, y) : x, y \in \mathbb{R}, y \geq 0\}$. Let

$$\mathcal{B} = \{B_r(x,y) : 0 < r \leq y\} \cup \Big\{ \{(a,0)\} \cup B_r(a,0) \cap \{(x,y) : y > 0\} : a \in \mathbb{R}, r > 0 \Big\} \cup \{\emptyset\}.$$

$\mathcal{B}$ is a basis of a topology on X. The topology generated is separable, first countable, but not second countable. The topology generated by $\mathcal{B}$ is Hausdorff but not regular. To see that note that any subset of $\mathbb{R} \times \{0\})$ is closed. Note that it is not possible to separate $(0, 0)$ and $(\mathbb{R} \backslash \{0\}) \times \{0\}$ via open sets.
(v) Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. Consider the topology τ on X determined by the following local base: If $(x, y) \in X$ with $y > 0$, take as a local base

the Euclidean balls $B((x,y),r)$, with $0 < r < y$, while if $(x,0) \in X$ take as a local base $B((x,r),r) \cup \{(x,0)\}$, with $r > 0$. Then (X,τ) is T_3 but not normal.

Exercise 90. Show that any metric space is T_4.

Lecture 15, October 1, 2012.

Showing that the topology of example (v) is not normal can be done via the following lemma:

Lemma 91. *Let (X,τ) be a separable topological space. If X has a closed subspace, D, eqiopotent to $\mathbb{R}$ and such that the induced topology is discrete then X is not normal.*

Proof. Assume (X,τ) is a separable normal space such that there exists a closed subset D, equipotent to $\mathbb{R}$ and such that the induced topology is discrete. Let E be a dense countable subset of X. Given $C \subseteq D$ note that C and $D \backslash C$ are closed subsets of X. Since X is normal there exist disjoint open sets U and V such that $C \subseteq U$ and $X \backslash C \subseteq V$. Let us denote by ψ the mapping $C \mapsto U \cap E$. Then $\psi : \mathcal{P}(D) \to \mathcal{P}(E)$. Note that $C \subseteq U \subseteq \overline{U \cap E}$. Furthermore $\overline{U \cap E} \subseteq X \backslash V$ and thus $\overline{U \cap E} \cap D = C$. Consequently the mapping ψ is an injection. Since D is equipotent to $\mathbb{R}$ and E is countable this implies that the power set of $\mathbb{R}$ is equipotent to a subset of $\mathbb{R}$ (since $\mathbb{R}$ is equipotent to $\mathcal{P}(E)$). Contradiction. ■

Exercise 92. (Songenfrey line) Consider the topology on $\mathbb{R}$ generated by $\mathcal{F} = \{[a,b) : a,b \in \mathbb{R}\}$. $\mathbb{R}^2$ endowed with the topology generated by $\mathcal{G} = \{[a,b) \times [c,d) : a,b,c,d \in \mathbb{R}\}$ is called the Sorgenfrey plane.

Show that Sorgenfrey line is normal, but that Sorgenfrey plane is not.

2.14. Compactness. Let (X,τ) be a topological space. An *open cover* of X is any collection of open sets whose union is X. A *subcover* is a subset of a cover which is still a cover.

Definition 93. Let (X,τ) be a topological space.

(i) (X,τ) is *compact* if for every open cover of X there exists a finite subcover.
(ii) (X,τ) is *sequentially compact* if for every sequence $\{x_n\}_{n=1,2,\ldots}$ in X, there exists a convergent subsequence.
(iii) A set $E \subset X$ is *relatively compact* (or *precompact*) if its closure $\overline{E}$ is compact with respect to the induced topology.

Remark 94. Consider a nonempty set X and let τ_1 and τ_2 be two topologies on X with $\tau_1 \subset \tau_2$. If $K \subset X$ is compact with respect to τ_2, then $K \subset X$ is compact with respect to τ_1. The less open sets we have, the easier it becomes for a set to be compact. This remark will be important in Functional Analysis.

Lemma 95. *Let (X,τ) be a Hausdorff space, let $K \subset X$ be a compact set and let $x_0 \in X \setminus K$. Then there exist two disjoint open sets U and V such that U contains K and $x_0 \in V$.*

Proof. Since X is Hausdorff, for every $x \in K$ there is a neighborhood U_x of x and a neighborhood V_x of x_0 such that $U_x \cap V_x = \emptyset$. Since $\{U_x\}_{x \in X}$ is an open cover of K, by the compactness of K there exist $x_1, \ldots, x_m \in K$ such that

$$U := U_{x_1} \cup \cdots \cup U_{x_m} \supset K.$$

The open set
$$V := V_{x_1} \cap \cdots \cap V_{x_m}$$
is a neighborhood of x_0 and does not intersect U. ∎

Lecture 16, October 3, 2012.

Proposition 96. *Let (X, τ) be a topological space.*

(i) *If X is compact and $C \subset X$ is closed, then C is compact.*
(ii) *If X is a Hausdorff space, and $K \subseteq X$ is compact then K is closed.*
(iii) *If X is Hausdorff and compact then X is regular.*
(iv) *If X is Hausdorff and compact then X is normal.*

Proof. Proof of (i). Assume C is closed. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of C. Then $\mathcal{U} \cup \{X \backslash C\}$ is an open cover of X. Since X is compact there exists an open subcover: $U_{\alpha_1}, \ldots, U_{\alpha_n}, X \backslash C$. Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite cover of C. So C is compact.

Proof of (ii). By the Lemma 95 for every $x_0 \in X \backslash K$ there exists a neighborhood U of x_0 that does not intersect K. This shows that x_0 is an interior point of $X \setminus K$ and, in turn, that $X \setminus K$ is open.

For the proof of (iii) consider C closed and $x_0 \notin C$. By (i) C is compact. Thus by the previous proposition there exist U and V open and disjoint such that $C \subset U$ and $x_0 \in V$.

For the proof of (iv) consider C, E closed. By (iii) for every $x \in E$ there exist, open and disjoint U_x and V_x such that $C \subset U_x$ and $x \in v_x$. Since $\{V_x : x \in E\}$ is an open cover of E and E is compact, there exists a finite subcover: $V_{x_1}, \ldots, V_{x_n}$. Let $U = U_{x_1} \cap \cdots \cap U_{x_n}$ and $V = V_{x_1} \cup \cdots \cup V_{x_n}$. Then $C \subset U$, $E \subset V$ and U and V are open and disjoint. ∎

Note that if the topology is not Hausdorff then compact sets may not be closed.

Example 97. Given a nonempty set X endowed with the trivial topology, any nonempty set strictly contained in X is compact but not closed.

Example 98. We use the fact from real analysis that $[a, b]$ is a compact subset of $\mathbb{R}$ (considered with Euclidean metric). The statement (i) of 96 implies that closed and bounded subsets of $\mathbb{R}$ are compact. Furthermore since $\mathbb{R}$ is Hausdorff compact subsets must be closed. To show that unbounded sets cannot be compact consider the cover $\{(-n, n) : n \in \mathbb{N}\}$. Thus subsets of $\mathbb{R}$ are compact if and only if they are closed and bounded.

Exercise 99. Let (X, d) be a compact metric space. Prove that X is separable and bounded. [Hint: Show that if the space is not separable then it has a countable subset where every point is isolated.]

Example 100. Closed and bounded sets do not have to be compact in general. Consider $l^2(\mathbb{N})$. Let $e_n \in l^2(\mathbb{N})$ be the sequence whose elements are all zero except for the n-th element which is equal to 1. Let $E = \{e_k : k \in \mathbb{N}\}$. Note that $d(e_k, e_n) = \sqrt{2}$ if $k \neq n$, so E is bounded. Each point of E is an isolated point, so it is closed. However E is not compact since $U_n = B(e_n, 1/2)$ is an open cover that has no finite subcover.

Exercise 101. Let $\overline{\mathbb{R}} := [-\infty, \infty]$ be the extended real line. Prove that there is a metric d on $\overline{\mathbb{R}}$ that makes $\overline{\mathbb{R}}$ compact.

Definition 102. A family $\{E_\alpha\}_{\alpha\in\Lambda}$ of subsets of a set X has the *finite intersection property* if every finite subfamily has nonempty intersection.

A decreasing sequence of nonempty sets has the finite intersection property.

Exercise 103. Let (X,τ) be a topological space. Prove that a set X is compact if and only if for every family $\{C_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X with the finite intersection property,

$$\bigcap_{\alpha\in\Lambda} C_\alpha \neq \emptyset.$$

Lemma 104. *Let (X,τ) be a compact topological space. Then every infinite subset of X has an accumulation point in X.*

Proof. Let $E\subset X$ be an infinite set and assume by contradiction that E has no accumulation points in X. Then E is closed and thus compact by Proposition 96. However $\mathcal{U}=\{\{x\} : x\in E\}$ is an open cover of E with no finite subcover. Contradiction. ■

Lemma 105. *Let (X,τ_X) be a compact T1 topological space satisfying the first axiom of countability. Then X is sequentially compact.*

Proof. Consider a sequence $\{x_n\}_{n=1,2,\ldots}$ in X. Let $E:=\{x_n : n\in\mathbb{N}\}$. By the previous lemma, either the set E is finite, in which case one element is repeated infinitely many times (so we have a convergent subsequence) or it has an accumulation point $x_0\in X$. Since (X,τ) satisfies the first axiom of countability, there exists a countable local base $\{B_k\}_k$ at x_0. As before, we may assume that $B_{k+1}\subset B_k$ for every $k\in\mathbb{N}$. Since x_0 is an accumulation point of E, there exists $n_1\in\mathbb{N}$ such that $x_{n_1}\in B_1$. Since the space is $T1$ there exists k_2 such that $x_i\notin B_{k_2}$ for all $i=1,2,\ldots,n_1$. Since x_0 is an accumulation point there exists $x_{n_2}\in B_{k_2}$. Note that $n_2>n_1$. Iterating this argument we obtain a subsequence $\{x_{n_j}\}_{j=1,2,\ldots}$ of $\{x_n\}_{n=1,2,\ldots}$ of distinct elements of E, which converges to x_0 as $n\to\infty$. ■

Next we show that continuous functions preserve compactness.

Proposition 106. *Consider two topological spaces (X,τ_X) and (Y,τ_Y), and a continuous function $f:X\to Y$. Then $f(K)$ is compact for every compact set $K\subset X$.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $f(K)$. By continuity, $f^{-1}(U_\alpha)$ is open for every $\alpha\in\Lambda$, and so $\left\{f^{-1}(U_\alpha)\right\}_{\alpha\in\Lambda}$ is an open cover of K. Since K is compact, we may find $U_{\alpha_1},\ldots,U_{\alpha_l}$ such that $\left\{f^{-1}(U_{\alpha_i})\right\}_{i=1}^{l}$ cover K. In turn, $U_{\alpha_1},\ldots,U_{\alpha_l}$ cover $f(K)$. Indeed, if $y\in f(K)$, then there exists $x\in K$ such that $f(x)=y$. Let $i=1,\ldots,m$ be such that $x\in f^{-1}(U_{\alpha_i})$. Then $y=f(x)\in U_{\alpha_i}$. ■

Lecture 17, October 5, 2012.

Definition 107. Let (X,τ_X) and (Y,τ_Y) be topological spaces. A function $f:X\to Y$ is a *homeomorphism* if f is a bijection and if $\tau_Y=\{f(U) : U\in\tau_X\}$.

Two spaces are *homeomorphic* if there exists a homeomorphism between them.

Exercise 108. Let (X,τ_X) and (Y,τ_Y) be topological spaces and let $f:X\to Y$. Show that f is a homeomorphism if and only if is a bijection and both f and f^{-1} are continuous.

Proposition 109. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be a bijection. If X is compact, Y is T2, and f is continuous, then f^{-1} is continuous. In particular, f is a homeomorphism.*

Proof. For every closed set $C \subset X$, we have that C is compact by Proposition 96, and so $f(C)$ is compact by Proposition 106. Again by Proposition 96 we have that $f(C)$ is closed. Let $g := f^{-1}$. We have shown that $g^{-1}(C)$ is closed for every closed set $C \subset X$. Thus, by Exercise 77, g is continuous. ■

The following examples show that if some of the conditions are removed the result above no longer holds.

Example 110. Let $X = (0, 1) \cup [2, 3]$ and define

$$f(x) = \begin{cases} x & \text{if } x \in (0,1), \\ x-1 & \text{if } x \in [2,3]. \end{cases}$$

Then $f : (0, 1) \cup [2, 3] \to (0, 2]$ is continuous, bijective, but the inverse function is discontinuous. The problem here is the fact that $(0, 1) \cup [2, 3]$ is not compact.

Example 111. Let $X = [0, 2\pi) \subset \mathbb{R}^2$. Let $f : X \to \mathbb{R}^2$ be given by $f(\theta) = (\cos\theta, \sin\theta)$. Let $Y = f(X)$. Then f^{-1} is not continuous at, for example, $(0, 1)$. Remark: If $[0, 2\pi)$ is replaced by $(0, 2\pi)$ the inverse function is continuous, although not uniformly so.

Example 112. Let $X = l^2(\mathbb{N})$ and $f(\{a_n\}_{n=1,2,\ldots}$ be the sequence $\{a_n/n\}_{n=1,2,\ldots}$. Let $Y = f(X) \subset l^2$. Then f^{-1} is not continuous at 0.

Example 113. Let $X = \{0, 1\}$. Consider the identity mapping $i_d : (X, \tau_{discrete}) \to (X, \tau_{trivial})$. Then both spaces are compact, i_d is a continuous bijection, but its inverse is not continuous.

We now show two forms of the Weierstrass theorem. One for compact spaces and one for sequentially compact spaces. Regarding the assumptions we note that in the second theorem one can assume continuity instead of sequential continuity, since by Lemma 82 and continuous function is sequentially continuous.

Theorem 114 (Weierstrass I)**.** *Let (X, τ) be a compact topological space and let $f : X \to \mathbb{R}$ be a continuous function. Then there exists $x_0 \in X$ such that*

$$f(x_0) = \inf_{x \in X} f(x).$$

and thus $f(x_0) = \min_{x \in X} f(x)$.

Proof. Let

$$t := \inf_{x \in X} f(x).$$

If the infimum is not attained, then for every $x \in X$ we may find $t < t_x < f(x)$. Then the family of open sets

$$U_x := \{y \in X : f(y) > t_x\}, \quad x \in X,$$

is an open cover for the compact set X, and so we may find a finite cover $U_{x_1}, \ldots, U_{x_l}$ of the set X. But then for all $x \in X$,

$$f(x) \geq \min_{i=1,\ldots,l} t_{x_i} > t = \inf_{w \in X} f(w),$$

which contradicts the definition of t. ■

Remark 115. Note that to prove the existence of a minimum we only used a weaker form of continuity, namely that the set $\{y \in X : f(y) > t\}$ is open for all $t \in \mathbb{R}$. A function satisfying this property is called *lower semicontinuous*.

Theorem 116 (Weierstrass II)**.** *Let (X, τ) be a sequentially compact topological space and let $f : X \to \mathbb{R}$ be a sequentially continuous function. Then there exists $x_0 \in X$ such that*

$$f(x_0) = \inf_{x \in X} f(x).$$

and thus $f(x_0) = \min_{x \in X} f(x)$.

Proof. As in the previous proof define $t := \inf_{x \in X} f(x)$ and assume that the infimum is not attained. Construct a sequence of real numbers $t < t_n$ such that $t_n \to t$ as $n \to \infty$ (if $t \in \mathbb{R}$ we can take $t_n = t + \frac{1}{n}$, while if $t = -\infty$, take $t_n = -n$). By the definition of infimum, for every $n \in \mathbb{N}$ we may find $x_n \in K$ such that

$$t < f(x_n) < t_n.$$

Letting $n \to \infty$, by the squeezing theorem, we get

$$\lim_{n \to \infty} f(x_n) = t. \tag{5}$$

Since $\{x_n\} \subset X$, and X is sequentially compact by, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in X$ such that $x_{n_k} \to x$ as $k \to \infty$. Using the continuity of f and (5), we get

$$t = \lim_{n \to \infty} f(x_n) = \lim_{k \to \infty} f(x_{n_k}) = f(x),$$

which shows that the infimum is a minimum. ■

The sequence $\{x_n\}$ constructed in the previous proof is called a *minimizing sequence.*

Remark 117. Note that to prove the existence of a minimum in the second proof we only used a weaker form of continuity, namely that the set

$$\liminf_{j \to \infty} f(x_j) \geq f(x)$$

for all sequences $\{x_j\}$ converging to $x \in \mathbb{R}$. A function satisfying this property is called *sequentially lower semicontinuous.*

Theorem 118. *The Cartesian product of two compact topological spaces is a compact topological space.*

Proof. Let (X, τ_X) and (Y, τ_Y) be two compact topological spaces. Assume by contradiction that $X \times Y$ is not compact. Then there is an open cover $\mathcal{W}$ of $X \times Y$ with the property that no finite subfamily of $\mathcal{W}$ covers $X \times Y$.

Step 1: We claim that there is $x_0 \in X$ such that for every neighborhood U of x_0, no finite subfamily of $\mathcal{W}$ covers $U \times Y$. If this is not the case, then for all $x \in X$ there is a neighborhood U_x of x and a finite subfamily of $\mathcal{W}$ that covers $U_x \times Y$. Since $\{U_x\}_{x \in X}$ is an open cover of X, by the compactness of X there exist $x_1, \dots, x_m$ such that

$$U_{x_1} \cup \cdots \cup U_{x_m} = X.$$

For each x_i, $i = 1, \dots, m$, find a finite subfamily $\mathcal{W}_i$ of $\mathcal{W}$ that covers $U_{x_i} \times Y$. Then the subfamily

$$\{W : W \in \mathcal{W}_i \text{ for some } i = 1, \dots, m\}$$

is finite and covers $X \times Y$, which is a contradiction. This proves the claim.
Step 2: We claim that there is $y_0 \in Y$ such that for every neighborhood $U \times V$ of (x_0, y_0), no finite subfamily of $\mathcal{W}$ covers $U \times V$. If this is not the case, then for all $y \in Y$ there is a neighborhood $U_y \times V_y$ of (x_0, y) and a finite subfamily of $\mathcal{W}$ that covers $U_y \times V_y$. Since $\{V_y\}_{y \in Y}$ is an open cover of Y, by the compactness of Y there exist $y_1, \ldots, y_\ell$ such that

$$V_{y_1} \cup \cdots \cup V_{y_\ell} = Y.$$

For each y_i, $i = 1, \ldots, \ell$, find a finite subfamily $\mathcal{W}_i'$ of $\mathcal{W}$ that covers $U_{y_i} \times V_{y_i}$. Then the subfamily

$$\{W : W \in \mathcal{W}_i' \text{ for some } i = 1, \ldots, m\}$$

is finite and covers $(U_{y_1} \cap \cdots \cap U_{y_\ell}) \times Y$, which contradicts Step 1. This proves the claim.
Step 3: Let $x_0 \in X$ and $y_0 \in Y$ be given as in Steps 1 and 2, respectively. Since $\mathcal{W}$ is an open cover of $X \times Y$, there exists $W \in \mathcal{W}$ such that $(x_0, y_0) \in W$. But then we can find neighborhoods U and V of x_0 and y_0 such that $U \times V \subset W$, which contradicts Step 2. This completes the proof. ■

Lecture 18, Oct 8, 2012.

2.15. Compactification. In studying a noncompact topological space X it is often useful to construct a space that contains X and that is compact. The extended real numbers are such an example.

Definition 119. Let (X, τ) be a topological space. We say that (Y, τ_Y) is a *compactification* of (X, τ) if Y is compact and X is homeomorphic to a dense subset of Y.

Exercise 120. Show that the extended line $Y = [-\infty, \infty]$ is a compactification of $\mathbb{R}$ with standard topology. Show that the function arctan, which is continuous on $\mathbb{R}$ can be extended to a continuous function on Y.

The simplest type of compactification is given by adding one point to X.

Theorem 121 (Alexandroff)**.** *Let (X, τ) be a topological space. Let ∞ denote a point not in X and consider the set $X^\infty := X \cup \{\infty\}$. Let τ_∞ be the collection of all subsets $U \subset X^\infty$ such that either U is an open set of X or $\infty \in U$ and $X \setminus U$ is a closed compact set of X. Then (X^∞, τ_∞) is a compact topological space. Moreover, (X^∞, τ_∞) is a Hausdorff space if and only if (X, τ) is Hausdorff and locally compact.*

We remark taking ϕ to be the inclusion mapping, $\phi(x) = x$ for all $x \in X$ shows that X^∞ is indeed a compactification of X.

Proof. Step 1: We prove that (X^∞, τ_∞) is a topological space. We begin by observing that if U belongs to τ_∞ if and only if

(i) $U \cap X$ belongs to τ
(ii) if $\infty \in U$, then $X \setminus U$ is compact set of X.

By (i), finite intersections and arbitrary unions of elements of τ_∞ intersect X in open sets. If $U_1, U_2 \in \tau_\infty$ and $\infty \in U_1 \cap U_2$, then by De Morgan's laws

$$X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2),$$

and since both $X \setminus U_1$ and $X \setminus U_2$ are closed and compact, so is their union. This shows that finite intersections of elements of τ_∞ are still in τ_∞. If $\{U_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary family of elements of τ_∞ and if

$$\infty \in \bigcup_{\alpha\in\Lambda} U_\alpha,$$

then $\infty \in U_\beta$ for some $\beta \in \Lambda$. Hence, $X \setminus U_\beta$ is closed and compact. Since $\bigcup_{\alpha\in\Lambda} U_\alpha \cap X$ is open, $X \setminus \bigcup_{\alpha\in\Lambda} U_\alpha$ is closed and since

$$X \setminus \bigcup_{\alpha\in\Lambda} U_\alpha \subset X \setminus U_\beta,$$

by Proposition 96, $X \setminus \bigcup_{\alpha\in\Lambda} U_\alpha$ is compact. Hence, $\bigcup_{\alpha\in\Lambda} U_\alpha$ belongs to τ_∞. The set X^∞ belongs to τ_∞, since $X^\infty \setminus X^\infty = \emptyset$ is closed and compact, while the empty set belongs to τ_∞ by (i). Thus, (X^∞, τ_∞) is a topological space.

Step 2: We prove that (X^∞, τ_∞) is compact. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a family of elements of τ_∞ such that

$$X^\infty = \bigcup_{\alpha\in\Lambda} U_\alpha.$$

Then $\infty \in U_\beta$ for some $\beta \in \Lambda$. Hence, $X \setminus U_\beta$ is closed and compact. Since

$$X \setminus U_\beta \subset \bigcup_{\alpha\in\Lambda} U_\alpha \cap X,$$

there exist $\alpha_1, \ldots, \alpha_m \in \Lambda$ such that

$$X \setminus U_\beta \subset \bigcup_{i=1}^{m} U_{\alpha_i} \cap X.$$

The finite family $\{U_{\alpha_1}, \ldots, U_{\alpha_m}, U_\beta\}$ covers X^∞.

Step 3: Finally, we show that (X^∞, τ_∞) is a Hausdorff space if and only if (X, τ) is Hausdorff and locally compact. Assume that (X^∞, τ_∞) is a Hausdorff space. Then (X, τ) is a Hausdorff space in view of property (i). To prove that it is locally compact, let $x \in X$. Since (X^∞, τ_∞) is a Hausdorff space, there exist two disjoint neighborhoods U and V of x and ∞, respectively. Then $X \setminus V$ is closed and compact and $U \subset X \setminus V$. Hence, $\overline{U} \subset X \setminus V$, and so $\overline{U}$ is compact by Proposition 96.

Conversely, assume that (X, τ) is Hausdorff and locally compact. Let $x, y \in X^\infty$ be two distinct points. If neither of them is ∞, then $x, y \in X$, and so since (X, τ) is a Hausdorff space, there exist two disjoint neighborhoods $U \subset X$ and $V \subset X$ of x and y, respectively. By (i), U and V belong to τ_∞. If $y = \infty$, since (X, τ) is locally compact, choose a neighborhood U of x such that $\overline{U}$ is compact. Then $X_\infty \setminus \overline{U}$ belongs to τ_∞. Hence, U and $X_\infty \setminus \overline{U}$ are two disjoint neighborhoods of x and ∞. ■

Exercise 122. Prove that the circle S^1 is the one-point compactification of $\mathbb{R}$. Note that, unlike in Exercise 120 the arctan function cannot be extended continuously to S^1.

Exercise 123. Describe the one-point compactification of the following sets of $\mathbb{R}^2$ with the usual topology.

(i) $\{(x, y) : x \in (0, 1], y = 0\}$
(ii) $\left\{\left(\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$
(iii) $\left\{(x, y) : x^2 + y^2 < 1\right\}$

(iv) $\{(x,y):\, x^2+y^2<1\}\cup\{(0,1)\}$
(v) $\{(x,y):\, -1\le x\le 1\}$.

Exercise 124. Let (X,τ) be a topological space and let (X^∞,τ_∞) where $X^\infty = X\cup\{\infty\}$ be its one-point compactification.

(i) Prove that if $v\in C(X^\infty)$ then $u:=(v-v(\infty))|_X\in C_0(X)$.
(ii) Conversely, show that if $u\in C_0(X)$ then the extension:

$$v(x)=\begin{cases} u(x) & \text{if } x\in X\\ 0 & \text{if } x=\infty\end{cases}$$

satisfies $v\in C(X^\infty)$.

Here the set $C_0(X)$ is the closure (with respect to topology of $C(X)$ with base of open balls $B(f,r)=\{g\in C(X)\ :\ \sup_X|f-g|<r\}$) of the set of compactly supported functions, $C_c(X)$.

Exercise 125. Note that if $\gamma:\mathbb{R}\to\mathbb{R}^2$ is a bounded continuous injection then the closure of $\gamma(\mathbb{R})$ in $\mathbb{R}^2$ is a compactification of f if $\gamma((a,b))$ is open in $\gamma(\mathbb{R})$ for all $a,b\in\mathbb{R}$.

Find γ such that the function $\sin\circ\gamma^{-1}:\gamma(\mathbb{R})\to\mathbb{R}$ has a continuous extension to $Y=\overline{(\gamma(\mathbb{R}))}$.

Lecture 19, Oct 10, 2012.

2.16. Product Topology. Given a collection $\{X_\alpha\}_{\alpha\in\Lambda}$ of sets, we define the product of the sets as

$$\prod_{\alpha\in\Lambda}X_\alpha := \left\{f:\Lambda\to\bigcup_{\alpha\in\Lambda}X_\alpha:\ f(\alpha)\in X_\alpha \text{ for every } \alpha\in\Lambda\right\}.$$

$f(\alpha)$ are called the coordinates of f and often denoted by f_α.

Remark 126. Note that, if the collection if finite, then $\prod_{\alpha\in\Lambda}X_\alpha$ reduces to the usual Cartesian product.

Another spacial case is when $X_\alpha=X$ for all $\alpha\in\Lambda$. Then $\prod_{\alpha\in\Lambda}X_\alpha$ is simply the space of all functions $f:\Lambda\to X$. It is denoted by X^Λ.

For each $\beta\in\Lambda$ the projection onto X_β is given by

$$\pi_\beta:\prod_{\alpha\in\Lambda}X_\alpha\to X_\beta$$
$$f\mapsto f(\beta)$$

Definition 127. Let (X_α,τ_α) with $\alpha\in\Lambda$ be a collection of topological spaces. We define the *product topology*, τ, on $\prod_{\alpha\in\Lambda}X_\alpha$ as the smallest topology that makes each projection continuous. More precisely, since π_α is continuous if and only if $\pi_\alpha^{-1}(V_\alpha)$ is open for every open set V_α of X_α, τ is the smallest topology that contains the family

$$\mathcal{F}=\left\{\pi_\alpha^{-1}(V_\alpha):\ V_\alpha\in\tau_\alpha,\ \alpha\in\Lambda\right\}. \tag{6}$$

Lemma 128. *Let $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of topological spaces. Then a base for the product topology is given by all sets of the form*

$$\bigcap_{\alpha\in\Lambda_0}\pi_\alpha^{-1}(V_\alpha),$$

where Λ_0 is a finite subset of Λ and V_α is an open set of X_α, or, equivalently, by all sets of the form

$$\prod_{\alpha\in\Lambda} V_\alpha,$$

where V_α is an open set of X_α and $V_\alpha = X_\alpha$ for all but finitely many $\alpha \in \Lambda$.

Proof. In view of Proposition 17 a base for the topology τ is given by finite intersections of elements of $\mathcal{F}$, precisely,

$$\bigcap_{\alpha\in\Lambda_0} \pi_\alpha^{-1}(V_\alpha)$$

where Λ_0 is a finite subset of Λ and V_α is an open set of X_α. Now, for $\beta \in \Lambda_0$,

$$\begin{aligned}\pi_\beta^{-1}(V_\beta) &= \left\{ f : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha : \ f(\alpha) \in X_\alpha \text{ for every } a \in \Lambda \setminus \{\beta\} \text{ and } f(\beta) \in V_\beta \right\} \\ &= \prod_{\alpha\in\Lambda,\, \alpha<\beta} X_\alpha \times V_\beta \times \prod_{\alpha\in\Lambda\, \alpha>\beta} X_\alpha.\end{aligned}$$

Hence,

$$\bigcap_{\alpha\in\Lambda_0} \pi_\alpha^{-1}(V_\alpha) = \prod_{\alpha\in\Lambda} V_\alpha,$$

where $V_\alpha := X_\alpha$ for all $a \in \Lambda \setminus \Lambda_0$. ■

Lemma 129. *Let (X_α, τ_α), $\alpha \in \Lambda$ be topological spaces and let $\{x^n\}_{n=1,2,...}$ be a sequence in the product space $\Pi_{\alpha\in\Lambda} X_\alpha$. Let y be an element of the product space. Then*

$$x^n \to y \text{ as } n \to \infty \quad \text{if and only if} \quad (\forall \alpha \in \Lambda) \quad x_\alpha^n \to y_\alpha \text{ as } n \to \infty.$$

Proof. To show that the convergence in the product topology implies the convergence of each of the coordinates consider $\alpha \in \Lambda$ and $U_\alpha \in \tau_\alpha$. Since $W = \pi_\alpha^{-1}(U_\alpha)$ is an open neighborhood of x in the product space there exists n_0 such that for all $n \geq n_0$, $x^n \in W$. Thus for all $n \geq n_0$, $x_\alpha^n \in U_\alpha$, as desired.

Now assume that each of the coordinates of the sequence $\{x^n\}_{n=1,2,...}$ converges to the appropriate coordinate of y. Let W be a neighborhood of x. By definition of the product topology there exist coordinates $\alpha_1, \ldots, \alpha_k \in \Lambda$ such that $U = \bigcap_{j=1}^k \pi_{\alpha_j}^{-1}(U_{\alpha_j}) \subseteq W$. The convergence of the coordinates implies that for each $j = 1, \ldots, k$ there exists n_j such that for all $n \geq n_j$, $x_{\alpha_j}^n \in U_{\alpha_j}$. Let $n_0 = \max\{n_1, \ldots, n_k\}$. Then for all $n \geq n_0$, $x^n \in U$. ■

One may wonder: Why do we consider this topology on the product of sets? Why not the smallest topology that contains all sets of the form $\prod_{\alpha\in\Lambda} V_\alpha$, where V_α is an open set of X_α? This topology is called the *box topology.* If Λ is finite, then the product topology and the box topology coincide. However, if Λ is infinite, then the box topology is finer. In fact one could say that it has too many open sets, since many of the properties that are true for the product topology fail for the box topology (compactness, connectedness, etc.). Let us also illustrate that the condition for the convergence of sequences in the box topology is too restrictive.

Example 130. Consider the box topology in $\mathbb{R}^\mathbb{N}$. A sequence x^n converges to 0 in the box topology if and only if x_k^n converges for all $k \in \mathbb{N}$ and if there exist n_0 and k_0 such that for all $n \geq n_0$ and all $k \geq k_0$, $x_k^n = 0$.

Proof. Assume that x^n converges to 0 but that for all n_0 and k_0 there exists $n \geq n_0$ and $k \geq k_0$ such that $x_k^n \neq 0$. Then there exist strictly increasing sequences $\{n_j\}_j$ and $\{k_j\}_j$ such that for all j, $x_{k_j}^{n_j} \neq 0$. Let $\delta_j = \frac{1}{2}|x_{k_j}^{n_j}|$. Let $W = \bigcap_{j\in\mathbb{N}} \pi_{k_j}^{-1}((-\delta_j, \delta_j))$. Note that W is an open neighborhood of zero. We claim that there is no $\bar{n}$ such that for all $n \geq \bar{n}$, $x^n \in W$. Assume that such $\bar{n}$ exists. Then there exists j such that $n_j > \bar{n}$. Note that $x_{k_j}^{n_j} \notin (-\delta_j, \delta_j)$ and hence $x^{n_j} \notin \pi_{k_j}^{-1}((-\delta_j, \delta_j))$ which implies that $x^{n_j} \notin W$. So x^n does not converge to 0.

The other implication is straightforward. ■

Lecture 20, Oct 12, 2012.

We spent most of the time reviewing the material for the Monday's exam. We did some fact on continuity, which are now listed as exercises 77,79, and 80 in the Continuity Section.

Lemma 131. *Let (X, τ) and (X_α, τ_α), $\alpha \in \Lambda$, be topological spaces and let*

$$f : X \to \prod_{\alpha\in\Lambda} X_\alpha.$$

Then f is continuous if and only if $\pi_\beta \circ f : X \to X_\beta$ is continuous for every $\beta \in \Lambda$.

Proof. Let f be continuous. Since π_β is continuous by definition of the product topology, $\pi_\beta \circ f$ is continuous as a composition of two continuous functions.

Conversely, assume that each $\pi_\beta \circ f$ is continuous. Since a subbase for the product topology is given by all sets of the form $\pi_\alpha^{-1}(V_\alpha)$, where V_α is an open set of X_α, by Exercise 78, it suffices to show that $f^{-1}\left(\pi_\alpha^{-1}(V_\alpha)\right)$ is open in X. But

$$f^{-1}\left(\pi_\alpha^{-1}(V_\alpha)\right) = (\pi_\alpha \circ f)^{-1}((V_\alpha)),$$

which is open in X, since $\pi_\alpha \circ f$ is continuous. Hence, f is continuous. ■

Example 132. Consider the set $\mathbb{R}^\mathbb{N} = \{g : \mathbb{N} \to \mathbb{R}\}$ and the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^\mathbb{N} \\ x &\mapsto (x, x, \ldots) \end{aligned}$$

For every $n \in \mathbb{R}^\mathbb{N}$ we have that $\pi_n \circ f : \mathbb{R} \to \mathbb{R}$ is the function $\pi_n \circ f(x) = x$, which is continuous. Hence, f is continuous with respect to the product topology. However, f is not continuous with respect to the box topology. Indeed, consider the open set

$$\begin{aligned} B &= (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{n}, \frac{1}{n}\right) \cdots \\ &= \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right). \end{aligned}$$

If f were continuous, then the set $f^{-1}(B)$ would be open. Note that $0 \in f^{-1}(B)$, and so $f^{-1}(B)$ should contain an interval $(-\delta, \delta)$ for some $\delta > 0$. Hence, $f((-\delta, \delta)) \subset B$, but then taking projections, we get

$$(\pi_n \circ f)((-\delta, \delta)) = (-\delta, \delta) \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$$

for all $n \in \mathbb{N}$, which is a contradiction.

We also discussed the Problem 6 on Set 3. The main point is that one needs to be careful when construction the continuous, compactly supported approximation (to make sure that it is continuous). One way to do so is to consider

$$f(x) = \begin{cases} u(x) - \varepsilon & \text{if } u(x) \geq \varepsilon \\ 0 & \text{if } u(x) \in (-\varepsilon, \varepsilon) \\ u(x) + \varepsilon & \text{if } u(x) \leq -\varepsilon. \end{cases}$$

Exercise 133. (Songenfrey plane) $\mathbb{R}^2$ endowed with the topology τ_{Sorg}, generated by $\mathcal{G} = \{[a,b) \times [c,d) \,:\, a,b,c,d \in \mathbb{R}\}$ is called the Sorgenfrey plane. Show that Songenfrey plane is

(i) 1st countable, but not 2nd countable.
(ii) completely disconnected.
(iii) T3 (in fact T3$\frac{1}{2}$)
(iv) not $T4$
(v) separable
(vi) not compact, and in fact $[0,1]^2$ is not compact.

In showing that the Sorgenfrey plane is not 2nd countable and not T4 it is useful to consider the anti-diagonal $A = \{(x,-x) : x \in \mathbb{R}\}$ and observe that the induced topology on A is discrete.

Lecture 21, Oct 17, 2012.

The following lemma is key to proving that the product of compact spaces is compact. It's proof requires the Axiom of Choice.

Lemma 134 (Alexander)**.** *Let (X,τ) be a topological space and let $\mathcal{S}$ be a subbase of the topology. If every subcover of $\mathcal{S}$ has a (further) finite subcover, then X is compact.*

[3]

Proof. Suppose that the assumptions of the lemma are satisfied, but that X is not compact. Let A be the collection of all open covers of X that do not have a finite subcover. We note that $(A, \subseteq)$ is a partially ordered set. Let $I \neq \emptyset$ and $\{\mathcal{V}_i \,:\, i \in I\}$ be a chain (i.e. a linearly ordered subset) in A. Let

$$\mathcal{V}_I = \bigcup_{i \in I} \mathcal{V}_i = \{V \in \tau \,:\, (\exists i \in I) \quad V \in \mathcal{V}_i\}.$$

We claim that $\mathcal{V}_I \in A$. First note that $\mathcal{V}_I$ is a cover. Furthermore if $\mathcal{V}_I$ had a finite subcover, say: $V_1 \in \mathcal{V}_{i_1}, \ldots, V_n \in \mathcal{V}_{i_n}$ then since $\{\mathcal{V}_i \,:\, i \in I\}$ is a chain with respect to set inclusion there exists $k \in \{1, \ldots, n\}$ such that for all $j = 1, \ldots, n$, $\mathcal{V}_{i_j} \subseteq \mathcal{V}_{i_k}$. Thus $V_1, \ldots V_n$ is a finite subcover of $\mathcal{V}_{i_k}$ which contradicts the definition of A. So $\mathcal{V}_I \in A$.

Therefore every chain in $(A, \subseteq)$ has an upper bound. Thus by Zorn lemma A has a maximal element. Let $\mathcal{W}$ be a maximal element. Let

$$\mathcal{U} = \{U \in \tau \,:\, U \notin \mathcal{W}\}.$$

Note that $X \notin \mathcal{W}$ since otherwise $\mathcal{W}$ would have a finite subcover: $\{X\}$. So $X \in \mathcal{U}$ which guarantees that $\mathcal{U}$ is nonempty.

Claim 1: If $U_1, U_2 \in \mathcal{U}$ then $U_1 \cap U_2 \in \mathcal{U}$.

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[3]October 10th: I changed the definition of *subbase* contained in Proposition 17 so that every subbase must be a cover. This is more standard.

To prove this claim note that since $\mathcal{W}$ is a maximal element of A and $U_1 \notin \mathcal{W}$, the cover $\mathcal{W} \cup \{U_1\}$ has a finite subcover: $U_1, U_{1,1}, \dots U_{1,k}$. Analogously $\mathcal{W} \cup \{U_2\}$ has a finite subcover $U_2, U_{2,1}, \dots U_{2,m}$. Then $U_1 \cap U_2, U_{1,1}, \dots U_{1,k}, U_{2,1}, \dots U_{2,m}$ is a finite subcover of $\mathcal{W} \cup \{U_1 \cap U_2\}$ and thus $U_1 \cap U_2 \notin \mathcal{W}$. So $U_1 \cap U_2 \in \mathcal{U}$.

Claim 2: If $U \in \mathcal{U}$, $U \subseteq V$, and $V \in \tau$ then $V \in \mathcal{U}$. To see this note that since $\mathcal{W} \cup \{U\}$ has a finite subcover of the form $\mathcal{W}_1 \cup \{U\}$ then $\mathcal{W}_1 \cup \{V\}$ is a finite subcover of $\mathcal{W} \cup \{V\}$ and thus $V \in \mathcal{U}$.

Now consider $\mathcal{S}$ an arbitrary subbase of τ. Let

$$\mathcal{W}_S = \{W \,:\, W \in \mathcal{W} \text{ and } W \in \mathcal{S}\}.$$

We claim that $\mathcal{W}_S \in A$. It suffices to show that W_S is a cover, for if is had a finite subcover so would $\mathcal{W}$. Let $x \in X$. Since $\mathcal{W}$ is a cover there exists $V \in \mathcal{W}$ such that $x \in V$. Since $V \in \tau$ and $\mathcal{S}$ is a subbase there exist $S_1, \dots, S_n \in \mathcal{S}$ such that for $B = S_1 \cap \dots \cap S_n$ it holds that $x \in B \subseteq V$. We claim that $B \in \mathcal{W}$, for if $B \notin \mathcal{W}$ then $\mathcal{W} \cup \{B\}$ has a finite subcover. But then $\mathcal{W} \cup \{V\} = \mathcal{W}$ has a finite subcover. Contradiction.

So for some $i \in \{1, \dots, n\}$, $S_i \in \mathcal{W}$, because otherwise by Claim 1, $B = S_1 \cap \dots \cap S_n \in \mathcal{U}$. In conclusion for some i, $x \in S_i \in \mathcal{W}_S$. Therefore $\mathcal{W}_S$ is a cover of X.

By the assumptions of the lemma it has a finite subcover. But then $\mathcal{W}$ has a finite subcover. Contradiction. ■

Lecture 22, Oct 22, 2012.

Theorem 135 (Tikhonov's theorem)**.** *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ be a collection of compact topological spaces. Then $\prod_{\alpha \in \Lambda} X_\alpha$ is compact.*

Proof. By definition of the product topology

$$\mathcal{S} = \left\{\pi_\alpha^{-1}(V_\alpha) : V_\alpha \in \tau_\alpha, \, \alpha \in \Lambda\right\}$$

is a subbase. By Alexander lemma 134 is suffices to show that every subcover of $\mathcal{S}$ has a finite subcover.

Let

$$\mathcal{U} = \{\pi_\alpha^{-1}(V_{\alpha,i}) \,:\, \alpha \in \Lambda, i \in I_\alpha\}$$

where $V_{\alpha,i} \in \tau_\alpha$, be a subcover. We claim that there exists α such that $\{V_{\alpha,i} \,:\, i \in I_\alpha\}$ is an open cover of X_α. For if that was the case then for each α there exists $x_\alpha \in X_\alpha \setminus \cup_{i \in I_\alpha} V_{\alpha,i}$. But then $x \in \Pi_{\alpha \in \Lambda} X_\alpha$ given by $\alpha \mapsto x_\alpha$ is not in $\bigcup \mathcal{U}$ which contradicts the assumption that $\mathcal{U}$ is a cover.

So for some α the collection $\{V_{\alpha,i} \,:\, i \in I_\alpha\}$ is an open cover of X_α. Since X_α is compact there exists a finite subcover $V_{\alpha,i_1}, \dots, V_{\alpha,i_k}$ Then $\pi_\alpha^{-1}(V_{\alpha,i_1}), \dots, \pi_\alpha^{-1}(V_{\alpha,i_k})$ is a finite subcover of $\mathcal{U}$; existence of which we wanted to establish. ■

Exercise 136. Prove that $[0,1]^{\mathbb{N}} = \{f : \mathbb{N} \to [0,1]\}$ with the box topology is not compact.

Lemma 137. *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in I}$ be a collection of topological spaces and let $E_\alpha \subset X_\alpha$ be nonempty sets for every $\alpha \in I$.*

(i)

$$\left(\prod_{\alpha \in I} E_\alpha\right)^\circ \subseteq \prod_{\alpha \in I} (E_\alpha)^\circ$$

where equality holds if and only if $E_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$.

(ii)

$$\overline{\prod_{\alpha\in I} E_\alpha} = \prod_{\alpha\in I} \overline{E_\alpha}.$$

Proof. (i) Assume $f \in \left(\prod_{\alpha\in I} E_\alpha\right)^\circ$. Then there exists a finite set $J \subseteq I$ and $Z_\alpha \in \tau_\alpha$ such that $Z_\alpha = X_\alpha$ for all $\alpha \in I\backslash J$ and that

$$f \in \prod_{\alpha\in I} Z_\alpha \subseteq \prod_{\alpha\in I} E_\alpha.$$

Thus $Z_\alpha \subseteq E_\alpha$ for all $\alpha \in I$. Since Z_α are open it follows that $Z_\alpha \subseteq E_\alpha^\circ$ for all $\alpha \in I$.

For equality to hold $\left(\prod_{\alpha\in I} E_\alpha\right)^\circ$ must be nonempty, which implies that it contains a set from the base of the topology. Thus $E_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$. On the other hand if $E_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$ then $\prod_{\alpha\in I} (E_\alpha)^\circ$ is open in the product topology which is enough to conclude.

(ii) Note that

$$\prod_{\alpha\in I} \overline{E_\alpha} = \bigcap_{\alpha\in I} \pi_\alpha^{-1}(\overline{E_\alpha})$$

and is thus closed as an intersection of closed sets. Thus $\overline{\prod_{\alpha\in I} E_\alpha} \subseteq \prod_{\alpha\in I} \overline{E_\alpha}$.

Now consider $f \in \prod_{\alpha\in I} \overline{E_\alpha}$. If $\prod_{\alpha\in I} Z_\alpha$ is any set from the base of product topology containing f then, since Z_α is open, $Z_\alpha \cap E_\alpha \neq \emptyset$ for all $\alpha \in I$. Thus $\Pi_{\alpha\in I} Z_\alpha \cap \Pi_{\alpha\in I} E_\alpha \neq \emptyset$. So $f \in \overline{\prod_{\alpha\in I} E_\alpha}$. ■

Exercise 138. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of nonempty topological spaces, and let $E_\alpha \subset X_\alpha$ be nonempty for every $\alpha \in \Lambda$. Fix $g \in \prod_{\alpha\in\Lambda} E_\alpha$ and consider the set

$$E := \left\{ f \in \prod_{\alpha\in\Lambda} E_\alpha : \, f(\alpha) = g(\alpha) \text{ for all but finitely many } \alpha \in \Lambda \right\}.$$

Prove that

$$\overline{E} = \overline{\prod_{\alpha\in\Lambda} E_\alpha}.$$

Next we discuss connectedness of product spaces.

Lemma 139. *Let (X, τ_X) and (Y, τ_Y) be two connected topological spaces. Then $X \times Y$ is a connected topological space.*

Proof. Fix two points $(x_1, y_1) \in X \times Y$ and $(x_2, y_2) \in X \times Y$. Note that the sets $X \times \{y_1\}$ and $\{x_2\} \times Y$ are connected since the functions $x \in X \mapsto (x, y_1)$ and $y \in Y \mapsto (x_2, y)$ are continuous. In view of Proposition 62, the set $(X \times \{y_1\}) \cup (\{x_2\} \times Y)$ is connected, since

$$(X \times \{y_1\}) \cap (\{x_2\} \times Y) = \{(x_2, y_1)\}.$$

Thus any two elements of $X \times Y$ lie in the same connected component (see Definition 72). Therefore $X \times Y$ is connected. ■

Theorem 140. *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of topological spaces. Then $\prod_{\alpha\in\Lambda} X_\alpha$ is connected if and only if each X_α is connected.*

Proof. Assume that $\prod_{\alpha\in\Lambda} X_\alpha$ is connected. By Proposition 85 for each $\beta\in\Lambda$

$$\pi_\beta\left(\prod_{\alpha\in\Lambda} X_\alpha\right) = X_\beta$$

is connected.

Conversely, assume that each X_α is connected. Fix $g\in\prod_{\alpha\in\Lambda} X_\alpha$ and consider the set

$$E := \left\{ f\in\prod_{\alpha\in\Lambda} X_\alpha : \, f(\alpha) = g(\alpha) \text{ for all but finitely many } \alpha\in\Lambda \right\}.$$

By the Exercise 138,

$$\overline{E} = \prod_{\alpha\in\Lambda} X_\alpha .$$

To show that $\overline{E}$ is connected it suffices, by Proposition 63 to show that E is connected.

Let $\mathcal{T}$ be the family of all finite subsets of Λ. For every $T\in\mathcal{T}$ consider the set

$$F_T := \prod_{\alpha\in\Lambda} F_\alpha ,$$

where $F_\alpha = \{g(\alpha)\}$ if $\alpha\notin T$ and $F_\alpha = X_\alpha$ if $\alpha\in T$. Write $T = \{\alpha_1,\ldots,\alpha_n\}$. Since the function

$$\Psi_T : X_{\alpha_1}\times\cdots\times X_{\alpha_n} \to \prod_{\alpha\in\Lambda} X_\alpha$$
$$(x_{\alpha_1},\ldots,x_{\alpha_n}) \mapsto f$$

where

$$f_T(\alpha) := \begin{cases} g(\alpha) & \text{if } \alpha\notin T, \\ x_\alpha & \text{if } \alpha\in T, \end{cases}$$

is continuous (why?) and the set $X_{\alpha_1}\times\cdots\times X_{\alpha_n}$ is connected by Lemma 139, the set $F_T = \Psi_T(X_{\alpha_1}\times\cdots\times X_{\alpha_n})$ is connected by Proposition 85. Using the fact that

$$\bigcap_{T\in\mathcal{T}} F_T$$

is nonempty, since it contains the function g the set

$$E = \bigcup_{T\in\mathcal{T}} F_T$$

is connected and the proof is completed. ■

Exercise 141. Prove that in $\mathbb{R}^{\mathbb{N}} = \{f:\mathbb{N}\to\mathbb{R}\}$ with the box topology, the set

$$U := \{f:\mathbb{N}\to\mathbb{R} : \, f \text{ is bounded}\}$$

is both open and closed, so that $\mathbb{R}^{\mathbb{N}}$ is not connected.

Exercise 142. Let $\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of Hausdorff topological spaces. Prove that $\prod_{\alpha\in\Lambda} X_\alpha$ is a Hausdorff space.

Lecture 23, Oct. 24, 2012.

2.17. Stone–Čech Compactification. Recall that (Y, τ_Y) is a *compactification* of (X, τ) if Y is compact and X is homeomorphic to a dense subset of Y. That is if there exists $\phi : X \to Y$ which is continuous and '1-1', $\phi^{-1} : \phi(X) \to X$ is continuous, and $\overline{\phi(X)} = Y$.

Here we construct a compactification which is larger and more flexible then the 1-point compactification. I particular it has the property that every bounded continuous function on X can be extended to a continuous function on Y, see Remark 144. However for the construction to work X needs to satisfy certain separability axioms which the uncover as the we carry the construction out.

Given a topological space (X, τ_X) consider the space $C_b(X)$ of all real-valued continuous bounded functions $f : X \to \mathbb{R}$. For every $f \in C_b(X)$ there exists $m_f > 0$ such that $f(x) \in [-m_f, m_f]$ for all $x \in X$. Consider

$$Y_0 := \prod_{f \in C_b(X)} [-m_f, m_f] = \{g : C_b(X) \to \mathbb{R} \, : \, (\forall f \in C_b(X)) \;\, g(f) \in [-m_f, m_f]\}.$$

By Tikhonov's theorem, this space is compact with the product topology τ_{Y_0}. Consider the evaluation map $e : X \to Y_0$ defined as follows: for every $x \in X$, $e(x) : C_b(X) \to \mathbb{R}$ is the function

$$e(x)(f) := f(x) \in [-m_f, m_f], \quad f \in C_b(X).$$

We claim that the function e is continuous. To see this, note that for every $f \in C_b(X)$ the projection π_f is given by

$$\begin{aligned} \pi_f : Y_0 &\to [-m_f, m_f] \\ g &\mapsto g(f) \end{aligned}$$

and so $\pi_f \circ e : X \to [-m_f, m_f]$ is the function f itself, since for every $x \in X$,

$$(\pi_f \circ e)(x) = \pi_f(e(x)) = e(x)(f) = f(x). \tag{7}$$

Since f is continuous, it follows from Theorem 131 that e is continuous. Define $\beta(X) := \overline{e(X)} \subset Y_0$. Since Y_0 is compact and $\beta(X)$ is closed, we have that $\beta(X)$ is compact (see Proposition 96). Thus we can consider the pair $(e, \beta(X))$. We want to see when this pair is a compactification of X. We are missing two properties. We need e to be one-to-one and we need $e^{-1} : e(X) \to X$ to be continuous.

The evaluation map e is one-to-one if for every $x, y \in X$ with $x \neq y$, we have that $e(x) \neq e(y)$. This is equivalent to say that there exists $f \in C_b(X)$ such $f(x) = e(x)(f) \neq e(y)(f) = f(y)$. That is that X is completely Hausdorff. Assume that this is the case.

Then we can consider the inverse function $e^{-1} : e(X) \to X$. Note that $e^{-1} : e(X) \to X$ is continuous if and only if $e : X \to e(X)$ is open.

Assume that X is completely regular. Fix an open set $U \subset X$ and let $x_0 \in U$. Since X is completely regular, there exists $f \in C_b(X)$ such that $f(x_0) = 1$ and $f = 0$ on $X \setminus U$. The set $V_f := f^{-1}((0, \infty))$ is open in X, contains x_0 and $V_f \subset U$. Moreover,

$$\begin{aligned} e(V_f) &= \{g \in Y_0 : \, g(f) > 0\} \cap e(X) \\ &= \pi_f^{-1}((0, \infty)) \cap e(X). \end{aligned}$$

Thus, $e(V_f)$ is an open neighborhood of $e(x_0)$ in $e(X)$. Conversely, if $e: X \to e(X)$ is a homeomorphism, then since Y_0 is compact, we have that Y_0 is completely regular, and since $e(X) \subset Y_0$, it follows that $e(X)$ is also completely regular. In turn, X is completely regular.

Let us also note that since $\{x\}$ are closed sets if X is Hausdorff (Exercise 87) and completely regular Hausdorff space is completely Hausdorff. Thus we have proved the following.

Theorem 143 (Stone–Čech)**.** *Let (X, τ) be a $T_{3\frac{1}{2}}$ topological space. Then $\beta(X)$ described above is a compactification of X.*

Remark 144. An important property of the Stone–Čech compactification of X is that every bounded continuous function $f_1 : X \to \mathbb{R}$ can be uniquely "extended" to a continuous function $F_1 : \beta(X) \to \mathbb{R}$. More precisely, $f_1 \circ e^{-1} : e(X) \to \mathbb{R}$ can be uniquely "extended" to a continuous function $F_1 : \beta(X) \to \mathbb{R}$. To see this, note that the projection mapping

$$\begin{aligned} \pi_{f_1} : \beta(X) &\to [-t_{f_1}, t_{f_1}] \\ g &\mapsto g(f_1) \end{aligned}$$

is continuous. Moreover, if $g \in e(X)$, then there exists $x \in X$ such that $g = e(x)$ and so

$$\left(f_1 \circ e^{-1}\right)(g) = \left(f_1 \circ e^{-1}\right)(e(x)) = f_1(x).$$

On the other hand, by (7),

$$\pi_{f_1}(g) = \pi_{f_1}(e(x)) = e(x)(f_1) = f_1(x).$$

Hence, $\left(f_1 \circ e^{-1}\right)(g) = \pi_{f_1}(g)$ for all $g \in e(X)$. Thus, if we identify X with $e(X)$, then π_{f_1} can be considered as a continuous extension of f_1.

Note that this extension is actually unique. Indeed, we have the following.

Exercise 145. Let (Y, τ_Y) and (Z, τ_Z) be topological spaces, with Z Hausdorff and let $f : E \to Z$ be a continuous function, where $E \subset Y$. Then there is at most one extension of f to $\overline{E}$.

Lecture 24, Oct. 26, 2012.

2.18. Lower semicontinuity I.

Definition 146. Let (X, τ) be a topological space. $f : X \to \mathbb{R}$ is lower semicontinuous (LSC) if $f^{-1}((a, \infty))$ is open for all $a \in \mathbb{R}$.

The function f is upper semicontinuous (USC) if $f^{-1}((-\infty, a))$ is open for all $a \in \mathbb{R}$.

Example 147. Let $E \subset X$. Consider the characteristic function of U:

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \backslash E. \end{cases}$$

The function χ_E is LSC if and only if E is open. Similarly, χ_E is USC if and only if E is closed.

Here are some useful properties of LSC and USC functions:

Lemma 148. *Let $f : X \to \mathbb{R}$.*

(i) f is LSC if and only if $-f$ is USC.

(ii) *f is continuous if and only if f is both LSC and USC.*
(iii) *If f_α are LSC for all $\alpha \in I$ and for all $x \in X$, $\sup_{\alpha\in I} f_\alpha(x) < \infty$ then the function $f(x) = \sup_{\alpha\in I} f_\alpha(x)$ is LSC. Likewise infimum of a family of USC functions is USC.*

Proof. Let us prove (iii). Assume f_α for $\alpha \in I$ satisfy the assumptions. Then $f^{-1}((a,\infty)) = \bigcup_{\alpha\in I} f_\alpha^{-1}((a,\infty))$ and is thus open as a union of a family of open sets. ■

2.19. Normal Spaces. Recall that a topological space (X,τ) is a *normal space* if for every pair of disjoint closed sets $C_1, C_2 \subset X$ there exist two disjoint open set U_1, U_2 such that $C_1 \subset U_1$ and $C_2 \subset U_2$.

The next theorem gives an important characterization of normal spaces.

Theorem 149 (Urysohn's lemma)**.** *A topological space (X,τ) is normal if and only if for all disjoint closed sets $C_1, C_2 \subset X$ there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C_1 and $f \equiv 0$ in C_2.*

Lemma 150. *Let (X,τ) be a topological space, X is normal if and only if for any C closed and U open such that $C \subset U$ there exists an open set $V \subset X$ such that*

$$C \subset V \subset \overline{V} \subset U.$$

Proof. Assume (X,τ) is normal. Since the sets C and $X \setminus U$ are closed and disjoint we may find disjoint open sets V and W such that $C \subset V$. Since $V \subset X\backslash W$ and $X\backslash W$ is closed $\overline{V} \subset X\backslash W$. Furthermore $X \setminus U \subset W$ implies $X \setminus W \subset U$ and thus $\overline{V} \subset U$.

To show the converse, note that if C_1 and C_2 are disjoint sets then for $U = X\backslash C_2$ there exists V open such that $C_1 \subset V \subset \overline{V} \subset U$. Define $U_1 = V$ and $U_2 = X \setminus \overline{V}$. then U_1 and U_2 are open, disjoint and $C_1 \subset U_1$ and $C_2 \subset U_2$. ■

We now turn to the proof of Urysohn's lemma.

Proof of Urysohn's lemma. Step 1: Assume that X is normal. Let $C_1, C_2 \subset X$ be two disjoint closed sets. Set $r_0 := 0$ and $r_1 := 1$ and let $\{r_n\}_{n=2}^\infty$ be an enumeration of the rationals numbers in $(0,1)$. By the previous lemma applied to C_1 and $X \setminus C_2$ there exists an open set $V_0 \subset X$ such that

$$C_1 \subset V_0 \subset \overline{V_0} \subset X \setminus C_2.$$

Again by the previous lemma, this time with C_1 and V_0, there exists an open set $V_1 \subset X$ such that

$$C_1 \subset V_1 \subset \overline{V_1} \subset V_0,$$

and so,

$$C_1 \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset X \setminus C_2.$$

Inductively, assume that given $n \in \mathbb{N}$ there exist open sets $V_{r_1}, \dots, V_{r_n} \subset X$ such that if $r_i < r_j$, then $\overline{V_{r_j}} \subset V_{r_i}$. Consider r_{n+1}. Since $r_0 = 0 < r_{n+1} < r_1 = 1$, one of the numbers $r_1, \dots, r_n$, say r_i, will be the largest below r_{n+1}, and one, say r_j, will be the smallest greater than r_{n+1}. Then $\overline{V_{r_j}} \subset V_{r_i}$, and so by the previous lemma we may find an open set $V_{r_{n+1}} \subset X$ such that

$$\overline{V_{r_j}} \subset V_{r_{n+1}} \subset \overline{V_{r_{n+1}}} \subset V_{r_i}.$$

Thus, by induction, we can construct a sequence of open sets $\{V_r\}_{r\in[0,1]\cap\mathbb{Q}}$ with the properties that $C_1 \subset V_r$, $\overline{V_r} \subset X \setminus C_2$, and for all $r, s \in [0,1]\cap\mathbb{Q}$ with $r < s$ we have $\overline{V_s} \subset V_r$. For $r, s \in [0,1]\cap\mathbb{Q}$ define the functions

$$f_r(x) := \begin{cases} r & \text{if } x \in V_r, \\ 0 & \text{otherwise,} \end{cases} \qquad g_s(x) := \begin{cases} 1 & \text{if } x \in \overline{V_s}, \\ s & \text{otherwise,} \end{cases}$$

and

$$f := \sup_{r\in[0,1]\cap\mathbb{Q}} f_r, \quad g := \inf_{s\in[0,1]\cap\mathbb{Q}} g_s.$$

Then f is lower semicontinuous, g is upper semicontinuous, and $0 \leq f \leq 1$.

We note that if $r \leq s$ then by definition $f_r(x) \leq g_s(x)$ for all $x \in X$. If $r > s$ then whenever $f_r(x) > 0$ it holds that $x \in V_r \subseteq V_s$ and hence $g_s(x) = 1$. In conclusion

$$(\forall r, s \in [0,1]\cap\mathbb{Q})(\forall x \in X) \qquad f_r(x) \leq g_s(x).$$

By taking first the supremum over all r and then the infimum over all s, we conclude that $f \leq g$.

If $x \in C_1 \subset V_1$, then, since $V_1 \subset V_r$ for all $r \in [0,1]\cap\mathbb{Q}$, we have that $x \in V_r$ for all $r \in [0,1]\cap\mathbb{Q}$, and so $f_r(x) = r$ and

$$f(x) = \sup_{r\in[0,1]\cap\mathbb{Q}} f_r(x) = \sup_{r\in[0,1]\cap\mathbb{Q}} r = 1,$$

while if $x \in C_2$, then, since $\overline{V_0} \subset X \setminus C_2$ and $\overline{V_r} \subset V_0$ for all $r \in [0,1]\cap\mathbb{Q}$, we have that $x \notin V_r$ for any $r \in [0,1]\cap\mathbb{Q}$, and so $f_r(x) = 0$ and

$$f(x) = \sup_{r\in[0,1]\cap\mathbb{Q}} f_r(x) = \sup_{r\in[0,1]\cap\mathbb{Q}} 0 = 0.$$

To conclude the first part of the proof, it remains to prove that f is continuous. It is enough to show that $f = g$. Now assume by contradiction that $f(x) < g(x)$ for some $x \in X$. Then by the density of the rationals numbers we may find $r, s \in [0,1]\cap\mathbb{Q}$ such that

$$f(x) < r < s < g(x).$$

Since $f(x) < r$, it follows that $x \notin V_r$. On the other hand, since $s < g(x)$, we have that $x \in \overline{V_s}$. This contradicts the fact that $\overline{V_s} \subset V_r$ and completes the first part of the proof.

Step 2: Assume that for all disjoint closed sets $C_1, C_2 \subset X$ there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C_1 and $f \equiv 0$ in C_2. We claim that X is normal. Indeed, let $C_1, C_2 \subset X$ be disjoint closed sets and let $f : X \to [0,1]$ be as above. Then the sets $f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ and $f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ are open, disjoint, and contain C_1 and C_2, respectively. This concludes the proof. ■

Lecture 25, Oct. 29, 2012.

Definition 151. Let (X, τ) be a topological space. A set $E \subset X$ is called an F_σ *set* if it is a countable union of closed sets, a G_δ *set* if it is a countable intersection of open sets.

Exercise 152. Let (X, τ) be a normal space.

(i) Prove that given a proper set closed set $C \subset X$, there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 0$ in C and $f > 0$ in $X \setminus C$ if and only if C is a G_δ set.

(ii) Let $C_1, C_2 \subset X$ be two disjoint closed sets. Prove that there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C_1, $f \equiv 0$ in C_2, and $0 < f < 1$ in $X \setminus (C_1 \cup C_2)$ if and only if C_1, C_2 are G_δ sets.

Theorem 153. *If (X, τ) is a locally compact Hausdorff space and $K \subset U \subset X$, with K compact and U open, $U \neq X$, then there exists a function $f \in C_c(X)$ such that $0 \leq f \leq 1$, $f \equiv 1$ on K, and $f \equiv 0$ on $X \setminus U$.*

The function f is usually referred to as a *cutoff function*. The proof uses a few preliminary results.

Exercise 154. Prove that if $\{K_\alpha\}_{\alpha \in \Lambda}$ is a collection of compact subsets of a Hausdorff space such that

$$\bigcap_{\alpha \in \Lambda} K_\alpha = \emptyset,$$

then there exists a finite subcollection of $\{K_\alpha\}_{\alpha \in \Lambda}$ whose intersection is still empty.

Lemma 155. *If X is a locally compact Hausdorff space and $K \subset U \subset X$, with K compact and U open, then there exists V open such that $\overline{V}$ is compact and $K \subset V \subset \overline{V} \subset U$.*

Proof. Since every point $x \in X$ has a neighborhood whose closure is compact and since K is covered by a finite union D of these neighborhoods, if $U = X$ it is enough to take V to be D.

If $U \neq X$, by Proposition 96, for every $x \in X \setminus U$ there exist two disjoint open sets U_x and V_x such that U_x contains K and $x \in V_x$. In particular, $x \notin \overline{U_x}$.

Hence, the family $\left\{(X \setminus U) \cap \overline{D} \cap \overline{U_x}\right\}_{x \in X \setminus U}$ is a collection of compact sets such that

$$\bigcap_{x \in X \setminus U} \left((X \setminus U) \cap \overline{D} \cap \overline{U_x}\right) = \emptyset.$$

By the previous theorem there exist $x_1, \ldots, x_m \in X \setminus U$ such that

$$(X \setminus U) \cap \overline{D} \cap \overline{U_{x_1}} \cap \cdots \cap \overline{U_{x_m}} = \emptyset.$$

The open set

$$V := D \cap U_{x_1} \cap \cdots \cap U_{x_m}$$

contains K, has compact closure, since $\overline{V} \subset \overline{D}$, and $\overline{V} \subset U$, since

$$\overline{V} \subset \overline{D} \cap \overline{U_{x_1}} \cap \cdots \cap \overline{U_{x_m}}.$$

■

We are now ready to prove the theorem.

Proof. (1st proof) [4] By Lemma 155, there exists W open such that $K \subset W \subset \overline{W} \subset U$. Let X^∞ be the one-point compactification of X. By Theorem 121 X^∞ is a compact Hausdorff space, and so by Proposition 96 it is a normal space.

Since K and $X^\infty \setminus W$ are disjoint closed sets, by Urysohn's lemma there exists a continuous function $f : X^\infty \to [0,1]$ such that $f \equiv 1$ in K and $f \equiv 0$ in $X^\infty \setminus W$. The function $f : X \to [0,1]$ has all the desired properties.

(2nd proof – uses the compactness of W, but not the compactification) By Lemma 155, there exists W open such that $\overline{W}$ is compact and $K \subset W \subset \overline{W} \subset U$.

[4] Indeed as Sebastien pointed out the compactness of W is not needed.

Since $\overline{W}$ is compact it is normal. Note that K and $\overline{W}\backslash W$ are disjoint closed subsets of $\overline{W}$. Thus there exists a function $f \in C(\overline{W}, [0,1])$ such that f takes value 1 on K and value 0 on $\overline{W}\backslash W$. Let us extend f to X as follows

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in \overline{W} \\ 0 & \text{else.} \end{cases}$$

Note that $\tilde{f}$ is 1 on K and zero on $X\backslash U$, so we only need to show that $\tilde{f}$ is continuous. Let A be a closed subset of $\mathbb{R}$. Then

$$\tilde{f}^{-1}(A) = \begin{cases} f^{-1}(A) & \text{if } 0 \notin A \\ f^{-1}(A) \cup (X\backslash W) & \text{if } 0 \in A. \end{cases}$$

In both cases $\tilde{f}^{-1}(A)$ is closed and thus $\tilde{f}$ is continuous. ∎

Lecture 26, Oct. 31, 2012.

Theorem 156 (Tietze's extension theorem)**.** *A topological space (X, τ) is normal if and only if for every closed set $C \subset X$ and every continuous function $f : C \to \mathbb{R}$ there exists a continuous function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in C$. Moreover, if $f(C) \subset [a, b]$, then F may be constructed so that*

$$F(C) \subset [a, b].$$

Proof. Step 1: Assume that X is normal, let $C \subset X$ be a closed set and let $f : C \to [-1, 1]$ be a continuous function. Then the sets $f^{-1}\left(\left[\frac{1}{3}, \infty\right)\right)$ and $f^{-1}\left(\left(-\infty, -\frac{1}{3}\right]\right)$ are disjoint closed subsets of C with respect to the induced topology τ_C but since C is closed, it follows that they are actually closed with respect to τ. Hence, we may apply Urysohn's lemma to find a continuous function $f_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $f_1 \equiv \frac{1}{3}$ in $f^{-1}\left(\left[\frac{1}{3}, \infty\right)\right)$ and $f_1 \equiv -\frac{1}{3}$ in $f^{-1}\left(\left(-\infty, -\frac{1}{3}\right]\right)$. We claim that

$$|f - f_1| \le \frac{2}{3} \quad \text{on } C.$$

Indeed, if $f(x) \in \left[-1, -\frac{1}{3}\right]$, then $f_1(x) = -\frac{1}{3}$; if $f(x) \in \left[\frac{1}{3}, 1\right]$, then $f_1(x) = \frac{1}{3}$; while if $f(x) \in \left[-\frac{1}{3}, \frac{1}{3}\right]$, then so $f_1(x) \in \left[-\frac{1}{3}, \frac{1}{3}\right]$.

Repeat this construction with $f - f_1$ in place of f and $(f - f_1)^{-1}\left(\left[\frac{2}{9}, \infty\right)\right)$ and $(f - f_1)^{-1}\left(\left(-\infty, -\frac{2}{9}\right]\right)$ in place of $f^{-1}\left(\left[\frac{1}{3}, \infty\right)\right)$ and $f^{-1}\left(\left(-\infty, -\frac{1}{3}\right]\right)$, respectively, to find a continuous function $f_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$ such that $f_2 \equiv \frac{2}{9}$ in $(f - f_1)^{-1}\left(\left[\frac{2}{9}, \infty\right)\right)$ and $f_2 \equiv -\frac{2}{9}$ in $(f - f_1)^{-1}\left(\left(-\infty, -\frac{2}{9}\right]\right)$. As before, we can prove that

$$|(f - f_1) - f_2| \le \left(\frac{2}{3}\right)^2 \quad \text{on } C.$$

Inductively for every $n \in \mathbb{N}$ we can construct a continuous function

$$f_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}\right] \tag{8}$$

such that

$$|f - f_1 - \cdots - f_n| \le \left(\frac{2}{3}\right)^n \quad \text{on } C. \tag{9}$$

Define

$$F(x) := \sum_{n=1}^{\infty} f_n(x), \quad x \in X.$$

Note that by (8),

$$\sum_{n=1}^{\infty} |f_n(x)| \leq \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 1, \tag{10}$$

and so F is well-defined and takes values in $[-1,1]$. In turn, since the series converges, it follows from (9) that for every $x \in C$,

$$\begin{aligned} |f(x) - F(x)| &= \left| f(x) - \lim_{m\to\infty} \sum_{n=1}^{m} f_n(x) \right| \\ &\leq \lim_{m\to\infty} \left| f(x) - \sum_{n=1}^{m} f_n(x) \right| \leq \lim_{m\to\infty} \left(\frac{2}{3}\right)^m = 0, \end{aligned}$$

and so $F = f$ on C. It remains to show that F is continuous. Fix $x \in X$ and $\varepsilon > 0$ and find n_ε so large that

$$\sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \leq \frac{\varepsilon}{2}. \tag{11}$$

Since $f_1, \ldots, f_{n_\varepsilon}$ are continuous at x, for every $n = 1, \ldots, n_\varepsilon$ there exists a neighborhood U_n of x such that if $y \in U_n$, then

$$|f_n(y) - f_n(x)| \leq \frac{\varepsilon}{2n_\varepsilon}. \tag{12}$$

Take $U := \bigcap_{n=1}^{n_\varepsilon} U_n$. Then by (10), (11), and (12), for every $y \in U$,

$$\begin{aligned} |F(y) - F(x)| &= \left| \sum_{n=1}^{n_\varepsilon} f_n(y) - \sum_{n=1}^{n_\varepsilon} f_n(x) \right| + \sum_{n=n_\varepsilon+1}^{\infty} |f_n(x)| + \sum_{n=n_\varepsilon+1}^{\infty} |f_n(y)| \\ &\leq \sum_{n=1}^{n_\varepsilon} |f_n(y) - f_n(x)| + \frac{2}{3} \sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \leq n_\varepsilon \frac{\varepsilon}{2n_\varepsilon} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which shows that F is continuous at x. Note that the proof continues to work if in place of $f : C \to [-1,1]$ we have $f : C \to [a,b]$, with the only change that in this case $F : X \to [a,b]$.

Step 2: Assume that X is normal, let $C \subset X$ be a closed set and let $f : C \to \mathbb{R}$ be a continuous function. Since $(-1,1)$ is homeomeorphic to $\mathbb{R}$, we can construct an heomeomorphism $g : \mathbb{R} \to (-1,1)$. Consider the function $h := g \circ f : C \to [-1,1]$. Since h is continuous, by Step 1 there exists a continuous function $H : X \to [-1,1]$ such that $H = h$ on C. The problem is that H can take values -1 and 1. To avoid this, let $C_1 := H^{-1}(\{-1,1\})$. Then C_1 and C are closed and disjoint, and so by Urysohn's lemma we may find a continuous function $h_1 : X \to [0,1]$ such that $h_1 \equiv 0$ in C_1 and $h_1 \equiv 1$ in C. Then $H_1 := Hh_1$ is continuous, $H_1 = H = h$ on C and $H_1 : X \to (-1,1)$. Since g^{-1} is continuous, the function $F := g^{-1} \circ H_1$ is continuous and real-valued and for $x \in C$,

$$F(x) = g^{-1}(H_1(x)) = g^{-1}(h(x)) = g^{-1}(g \circ f(x)) = f(x).$$

Step 3: Assume that or every closed set $C \subset X$ and every continuous function $f : C \to [a, b]$ there exists a continuous function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in C$ and $F(C) \subset [a, b]$. Let $C_1, C_2 \subset X$ be two disjoint closed sets. Then $C_1 \cup C_2$ is closed. Define $f := 1$ in C_1 and $f := 0$ in C_2. Then $f : C_1 \cup C_2 \to [0, 1]$ is continuous, and so by hypothesis there exists a continuous function $F : X \to [0, 1]$ such that $F(x) = f(x)$ for all $x \in C_1 \cup C_2$. Thus, we are in a position to apply Urysohn's lemma (or repeat Step 2 of its proof) to conclude that X is normal. ■

Lecture 27, Nov. 2, 2012.

2.20. Limit.

Definition 157. Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : E \to Y$, where $E \subset X$. Given $x_0 \in E'$, if there exists $y_0 \in Y$ with the property that for every neighborhood $V \subset Y$ of y_0, there exists a neighborhood $U \subset X$ of x_0 such that $f(x) \in V$ for all $x \in U \cap (E \setminus \{x_0\})$, we say that y_0 is a *limit* of f as x approaches x_0 within E.

Note that x_0 need not belong to E.

Remark 158. In applications it is enough to take V in a local base of y and U in a local base of x_0.

Proposition 159. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces with Y a Hausdorff space and let $f : E \to Y$, where $E \subset X$. Given $x_0 \in E'$, if y_1 and y_2 are both limits of f as x approaches x_0 within E then $y_1 = y_2$.*

In other words if limit exists it is unique. That allows us to write

$$\lim_{x \to x_0, x \in E} f(x) = y_1.$$

Also when E is clear from context we will omit it in writing the limit.

Proof. Assume by contradiction that there $y_1 \neq y_2$. Then there exist two disjoint neighborhoods V_1 and V_2 of y_1 and y_2, respectively. In turn, by the definition of limit, there exist two neighborhoods U_1 and U_2 of x_0 such that $f(x) \in V_1$ for all $x \in U_1 \cap (E \setminus \{x_0\})$ and $f(x) \in V_2$ for all $x \in U_2 \cap (E \setminus \{x_0\})$. Since $x_0 \in E'$ there exists $x \in U_1 \cap U_2 \cap (E \setminus \{x_0\})$. But then $f(x) \in V_1 \cap V_2$, which contradicts the fact that V_1 and V_2 are disjoint. ■

We have seen that for metric spaces, when working with limits, it is enough to work with sequences. For topological spaces, this is no longer true, although we have one implication.

Proposition 160. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : E \to Y$, where $E \subset X$. Given $x_0 \in E'$, if there exists* $\lim_{x \to x_0} f(x) = y_0$, *then $f(x_n) \to y_0$ for every sequence $\{x_n\} \subset E \setminus \{x_0\}$ that converges that to x_0.*

Proof. Let $\{x_n\} \subset E \setminus \{x_0\}$ converge to x_0. Given a neighborhood $V \subset Y$ of y_0, there exists a neighborhood $U \subset X$ of x_0 such that

$$f(x) \in V$$

for all $x \in U \cap (E \setminus \{x_0\})$. Since $x_n \to x_0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_\varepsilon$ and so

$$f(x_n) \in V$$

for all $n \geq n_\varepsilon$, which shows that $f(x_n) \to f(x_0)$. ■

Proposition 161. *Let (X,τ_X) be a topological space satisfying the first axiom of countability, let (Y,τ_Y) be a topological space, and let $f:E\to Y$, where $E\subset X$. Given $x_0\in E\cap E'$, if there exists $y_0\in Y$ such that $f(x_n)\to y_0$ for every sequence $\{x_n\}\subset E\setminus\{x_0\}$ that converges that to x_0, then there exists $\lim_{x\to x_0}f(x)=y_0$.*

Proof. We claim that $\lim_{x\to x_0}f(x)=y_0$. If not, then there exists a neighborhood $V\subset Y$ of y_0 such that for every neighborhood $U\subset X$ of x_0 there exists $x\in U\cap(E\setminus\{x_0\})$ such that $f(x)\notin V$. Since (X,τ) satisfies the first axiom of countability, there exists a countable local base $\{B_n\}_n$ at x_0. By selecting a subsequence, we may assume that $B_{n+1}\subset B_n$ for every $n\in\mathbb{N}$. Then for every n we may find $x_n\in B_n\cap(E\setminus\{x_0\})$ such that $f(x_n)\notin V$. Since $\{B_n\}_n$ is a decreasing local base at x_0, the sequence $\{x_n\}$ converges to x_0. By hypothesis, $f(x_n)\to y_0$, and so $f(x_n)\in V$ for all n sufficiently large, which is a contradiction. ■

Remark 162. When $Y=\mathbb{R}$ we also allow for the limit to be infinity. More precisely by saying that

$$f(x)\to\infty\quad\text{ as }\quad x\to x_0,$$

we have in mind the limit in the sense defined above but with $\mathbb{R}$ replaced by the extended set of real numbers $\overline{\mathbb{R}}=\mathbb{R}\cup[-\infty,\infty]$ endowed with the topology that has the base:

$$\mathcal{B}=\{(a,b)\,:\,a,b\in\mathbb{R}\}\cup\{[-\infty,b)\,:\,b\in\mathbb{R}\}\cup\{(a,\infty]\,:\,a\in\mathbb{R}\}.$$

Convergence to $-\infty$ is defined analogously.

If $X=\mathbb{R}$, we can also consider limit at $x_0\in\{-\infty,\infty\}$ by replacing $\mathbb{R}$ by $\overline{\mathbb{R}}$.

If $X=\mathbb{R}^n$ for $n\geq 2$ then by $f(x)\to y_0$ as $|x|\to\infty$ we mean the limit as above, but considered on $\mathbb{R}^n\cup\{\infty\}$ the one-point compactification of $\mathbb{R}^n$.

In all of the cases above all the standard theorems about the sum, product, quotient of limits continue to hold with the standard modifications. We omit the details.

Lecture 28, Nov. 7, 2012.

Definition 163. Let (X,τ) be a topological space, let $f:E\to\mathbb{R}$, where $E\subset\mathbb{R}$, and let $x_0\in\operatorname{acc}E$. The *limit inferior* of f as x tends to x_0 is defined as

$$\liminf_{x\to x_0,x\in E}f(x):=\sup_{U\in\tau(x_0)}\inf_{x\in U\cap(E\setminus\{x_0\})}f(x),$$

while the *limit superior* of f as x tends to x_0 is defined as

$$\limsup_{x\to x_0,x\in E}f(x):=\inf_{U\in\tau(x_0)}\sup_{x\in U\cap(E\setminus\{x_0\})}f(x),$$

where $\tau(x_0)$ stands for the collection of all neighborhoods of x_0.[5]

[5]In several books, $\sup_{A\in\tau(x_0)}\inf_{x\in A\cap E}f(x)$ is used as a definition for the limit inferior $\liminf_{x\to x_0}f(x)$. The definition we use here is in accordance with the definition of limit, in which the value of the function at the point x_0 plays no role. In particular, with our definition we recover the fact that the limit exists at x_0 if and only if the limit inferior and superior coincide, while with the other definition one would get that the limit inferior and superior at x_0 coincide if and only if the function f is continuous at f_0.

Remark 164. If U and V are two neighborhoods of x_0 with $U \subset V$, then

$$\inf_{x \in V \cap (E \setminus \{x_0\})} f(x) \le \inf_{x \in U \cap (E \setminus \{x_0\})} f(x),$$

and since we are interested in the supremum over all neighborhoods of x_0, we can neglect V. Thus, we can focus on "small" neighborhoods of x_0 (this is the analogous of $\varepsilon > 0$ small in the standard definition of limits). In particular, we could replace $\tau(x_0)$ with a local base at x_0 in the definition of $\liminf_{x \to x_0}$. A similar reasoning works for $\limsup_{x \to x_0}$. This is why in a metric space we only consider balls.

Theorem 165. *Let (X, τ) be a topological space, let $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}$, and let $x_0 \in \operatorname{acc} E$. Then*

$$\liminf_{x \to x_0} f(x) \le \limsup_{x \to x_0} f(x). \tag{13}$$

Moreover there exists $\lim_{x \to x_0} f(x)$ if and only if equality holds in (13), and in this case the limit coincides with the common value in (13).

Proof. Let U and V be two neighborhoods of x_0. Then

$$\begin{aligned}\inf_{x \in U \cap (E \setminus \{x_0\})} f(x) &\le \inf_{x \in U \cap V \cap (E \setminus \{x_0\})} f(x) \\ &\le \sup_{x \in U \cap V \cap (E \setminus \{x_0\})} f(x) \le \sup_{x \in V \cap (E \setminus \{x_0\})} f(x).\end{aligned}$$

Taking the supremum over all $U \in \tau(x_0)$ gives

$$\liminf_{x \to x_0} f(x) \le \sup_{x \in V \cap (E \setminus \{x_0\})} f(x).$$

Taking the infimum over all $V \in \tau(x_0)$ gives (13).

To prove the second part of the theorem assume that there exists $\lim_{x \to x_0} f(x) = \ell$. I will consider only the case $\ell \in \mathbb{R}$ and leave the cases $\ell = \infty$ and $\ell = -\infty$ as an exercise. By the definition of limit, for every $\varepsilon > 0$ there exists a neighborhood U_ε of x_0 such that

$$\ell - \varepsilon \le f(x) \le \ell + \varepsilon$$

for all $x \in U_\varepsilon \cap (E \setminus \{x_0\})$. Hence,

$$\ell - \varepsilon \le \inf_{x \in U_\varepsilon \cap (E \setminus \{x_0\})} f(x) \le \sup_{U \in \tau(x_0)} \inf_{x \in U \cap (E \setminus \{x_0\})} f(x) = \liminf_{x \to x_0} f(x)$$

$$\limsup_{x \to x_0} f(x) \le \inf_{U \in \tau(x_0)} \sup_{x \in U \cap (E \setminus \{x_0\})} f(x) \le \sup_{x \in U_\varepsilon \cap (E \setminus \{x_0\})} f(x) \le \ell + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we conclude that

$$\liminf_{x \to x_0} f(x) = \limsup_{x \to x_0} f(x) = \ell.$$

Conversely, assume that

$$\liminf_{x \to x_0} f(x) = \limsup_{x \to x_0} f(x) = L$$

for some $L \in [-\infty, \infty]$. Again we consider the case $L \in \mathbb{R}$ and leave the cases $L = \infty$ and $L = -\infty$ as an exercise. Fix $\varepsilon > 0$. By the definition of $\liminf_{x \to x_0} f(x)$

and $\limsup_{x\to x_0} f(x)$ there exist two neighborhoods U_ε and V_ε of x_0 such that

$$L-\varepsilon \le \inf_{x\in U_\varepsilon \cap (E\setminus\{x_0\})} f(x),$$
$$\sup_{x\in V_\varepsilon \cap (E\setminus\{x_0\})} f(x) \le L+\varepsilon.$$

Taking $U := U_\varepsilon \cap V_\varepsilon$, we have that for all $x \in U \cap (E \setminus \{x_0\})$,

$$L-\varepsilon \le f(x) \le L+\varepsilon,$$

which shows that there exists $\lim_{x\to x_0} f(x) = L$. ■

Exercise 166. Let (X,τ) be a topological space, let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$, where $E \subset \mathbb{R}$, and let $x_0 \in E'$. Assume that one of the two functions is bounded. Prove that

$$\begin{aligned}\liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x) &\le \liminf_{x\to x_0} (f(x)+g(x)) \le \limsup_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x)\\ &\le \limsup_{x\to x_0} (f(x)+g(x)) \le \limsup_{x\to x_0} f(x) + \limsup_{x\to x_0} g(x)\end{aligned}$$

and that in general all inequalities may be strict. Prove that if there exists $\lim_{x\to x_0} f(x) = \ell \in \mathbb{R}$, then

$$\liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x) = \liminf_{x\to x_0} (f(x)+g(x)),$$
$$\limsup_{x\to x_0} (f(x)+g(x)) = \limsup_{x\to x_0} f(x) + \limsup_{x\to x_0} g(x).$$

The proof of the following corollary is left as an exercise.

Corollary 167 (Cauchy)**.** *Let (X,τ) be a topological space, let $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}$, and let $x_0 \in E'$. Then a necessary and sufficient condition for $\lim_{x\to x_0} f(x)$ to exist in $\mathbb{R}$ is that for every $\varepsilon > 0$ there exists a neighborhood U_ε of x_0 such that*

$$|f(x_1) - f(x_2)| \le \varepsilon$$

for all $x_1, x_2 \in U_\varepsilon \cap (E \setminus \{x_0\})$.

Lecture 29, Nov. 9, 2012.

2.21. Partition of unity.

Definition 168. Let X be a topological space and let $\mathcal{F}$ be a collection of subsets of X. Then

(i) $\mathcal{F}$ is *point finite* if every $x \in X$ belongs to only finitely many $U \in \mathcal{F}$,
(ii) $\mathcal{F}$ is *locally finite* if every $x \in X$ has a neighborhood that has nonempty intersection with only finitely many $U \in \mathcal{F}$,
(iii) $\mathcal{F}$ is *σ-locally finite* if

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n,$$

where each $\mathcal{F}_n$ is a locally finite collection in X.

(iv) $\mathcal{G} \subset \mathcal{P}(X)$ is a *refinement* of $\mathcal{F}$ if

$$\bigcup_{G\in\mathcal{G}} G = \bigcup_{F\in\mathcal{F}} F$$

and every element of $\mathcal{G}$ is contained in some element of $\mathcal{F}$.

Definition 169. If (X, τ) is a topological space, a *partition of unity* on X is a family $\{\varphi_i\}_{i\in\Lambda}$ of continuous functions $\varphi_i : X \to [0,1]$ such that

$$\sum_{i\in\Lambda} \varphi_i(x) = 1$$

for all $x \in X$. A partition of unity is *locally finite* if for every $x \in X$ there exists a neighborhood U of x such that the set $\{i \in \Lambda : U \cap \operatorname{supp}\varphi_i \neq \emptyset\}$ is finite. If $\{U_j\}_{j\in\Xi}$ is an open cover of X, a partition of unity *subordinated to the cover* $\{U_j\}_{j\in\Xi}$ is a partition of unity $\{\varphi_i\}_{i\in\Lambda}$ such that for every $i \in \Lambda$, $\operatorname{supp}\varphi_i \subset U_j$ for some $j \in \Xi$.

We begin by proving that in a separable metric space every open cover admits a locally finite partition of unity subordinated to it.

Exercise 170. Let (X, d) be a separable metric space and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. Prove that there exists a countable family $\{V_n\}_n$ of open sets with the properties that

$$X = \bigcup_n V_n$$

and that for every n there exists $\alpha \in \Lambda$ such that $V_n \subset U_\alpha$.

Theorem 171 (Partition of unity)**.** *Let (X, d) be a separable metric space and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. Then there exists a locally finite partition of unity subordinated to it.*

Proof. Step 1: We assume first that Λ is finite, say $\Lambda = \{1, \dots, n\}$. For each $i = 1, \dots, n$ define the continuous function

$$f_i(x) := \operatorname{dist}(x, X \setminus U_i), \quad x \in X,$$

and set $f := \sum_{i=1}^n f_i$. Note that $f > 0$, since $U_1, \dots, U_n$ is a cover of X and $f_i > 0$ in U_i, $i = 1, \dots, n$. Next, for each $i = 1, \dots, n$ we define the continuous function

$$g_i(x) := \max\left\{f_i(x) - \frac{1}{n+1} f(x), 0\right\}, \quad x \in X.$$

We claim that $\operatorname{supp} g_i \subset U_i$ for each $i = 1, \dots, n$. Indeed,

$$\operatorname{supp} g_i = \overline{\left\{x \in X : f_i(x) > \frac{1}{n+1} f(x)\right\}} \subset \left\{x \in X : f_i(x) \geq \frac{1}{n+1} f(x)\right\}.$$

Since $f > 0$, the closed set $\left\{x \in X : f_i(x) \geq \frac{1}{n+1} f(x)\right\}$ is contained in $U_i = \{x \in X : f_i(x) > 0\}$.

Next we show that $\sum_{i=1}^n g_i > 0$. For all $x \in X$ we have

$$\sum_{i=1}^n g_i(x) \geq \sum_{i=1}^n \left(f_i(x) - \frac{1}{n+1} f(x)\right) = f(x) - \frac{n}{n+1} f(x) = \frac{1}{n+1} f(x) > 0.$$

It now suffices to define for each $i = 1, \ldots, n$,

$$\varphi_i(x) := \frac{g_i(x)}{\sum\limits_{j=1}^{n} g_j(x)}, \quad x \in X.$$

Step 2: Assume next that $\Lambda = \mathbb{N}$. Without loss of generality we can assume that $d(x, y) < 1$ for all $x, y \in X$ (if not define new metric $\bar{d} = d/(1 + d)$). For each $i \in \mathbb{N}$ define the continuous function $f_i : X \to \left[0, \frac{1}{2^i}\right]$ by

$$f_i(x) := \min\left\{\operatorname{dist}(x, X \setminus U_i), \frac{1}{2^i}\right\}, \quad x \in X.$$

Then $f_i > 0$ in U_i, $f_i = 0$ outside U_i, and $f_i \le \frac{1}{2^i}$. Hence the function $f := \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} f_i$ is continuous and $f > 0$ since $\{U_i\}_{i \in \mathbb{N}}$ is a cover of X. Next, for $i \in \mathbb{N}$ we define the continuous function $g_i : X \to [0, 1]$ by

$$g_i(x) := \max\left\{f_i(x) - \frac{1}{3} f(x), 0\right\}, \quad x \in X.$$

As in the previous step we have that $\operatorname{supp} g_i \subset U_i$. We claim that $\{g_i\}_{i \in \mathbb{N}}$ is locally finite. For any fixed $x \in X$, since f is positive and continuous there exist $\varepsilon, r > 0$ such that $f(y) > \varepsilon$ for all $y \in B(x, r)$. Let $i_0 \in \mathbb{N}$ be so large that $\frac{1}{2^{i_0}} < \frac{\varepsilon}{3}$. From the definition of g_i and the fact that $f_i \le \frac{1}{2^i}$ it follows that $g_i(y) = 0$ for all $y \in B(x, r)$ and $i \ge i_0$.

Next we show that $\sum_{i=1}^{\infty} g_i > 0$. Fix $x \in X$. Since $f_i(x) > 0$ for some $i \in \mathbb{N}$ and $f_n \le \frac{1}{2^n} < \frac{1}{2^i}$ for all $n \ge i$, it follows that there exists $i_0 \in \mathbb{N}$ such that

$$f_{i_0}(x) = \sup_{i \in \mathbb{N}} f_i(x),$$

and so

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} f_i(x) \le f_{i_0}(x) \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = 2 f_{i_0}(x).$$

Hence

$$g_{i_0}(x) \ge f_{i_0}(x) - \frac{1}{3} f(x) \ge f_{i_0}(x) - \frac{2}{3} f_{i_0}(x) = \frac{1}{3} f_{i_0}(x) > 0.$$

It now suffices to define for each $i \in \mathbb{N}$,

$$\varphi_i(x) := \frac{g_i(x)}{\sum\limits_{j=1}^{\infty} g_j(x)}, \quad x \in X.$$

Step 3: Finally, if $\{U_\alpha\}_{\alpha \in \Lambda}$ is an arbitrary open cover of X, by the previous exercise there exists a countable refinement $\{V_n\}_n$ of $\{U_\alpha\}_{\alpha \in \Lambda}$. Apply Step 2 to $\{V_n\}_n$ (defining $\varphi_\alpha \equiv 0$ if U_α does not contain any V_n). ■

When the metric space X is not separable, the result is still true but relies on Axiom of Choice.

Lecture 30, Nov. 12, 2012.

Lemma 172. *A topological space (X, τ) is normal if and only if for every point finite open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X there exists another locally finite open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of X with the property that $\overline{V_\alpha} \subset U_\alpha$.*

The proof of the lemma relies on the Axiom of Choice and transfinite induction. When the set Λ is at most countable this reduces to ordinary induction. This was the case considered in the lecture. Below the full proof is presented, but you are not required to know the general proof.

Proof. Assume that (X,τ) is normal and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a point finite open cover of X. By the axiom of choice, we may assume that Λ is well-ordered; that is, there is an order relation $\le$ such that every subset of Λ has a smallest element. Let $\alpha_0 \in \Lambda$ be the least element of Λ. We claim that there exists a collection of open sets V_β such that

$$C_\beta \subset V_\beta \subset \overline{V_\beta} \subset U_\beta,$$

where

$$C_\beta := X \setminus \left(\left(\bigcup_{\alpha<\beta} V_\alpha \right) \cup \left(\bigcup_{\alpha>\beta} U_\alpha \right) \right).$$

To construct V_β we use transfinite induction on Λ. Define

$$C_{\alpha_0} := X \setminus \bigcup_{\alpha>\alpha_0} U_\alpha.$$

Then $C_{\alpha_0} \subset U_{\alpha_0}$ and is closed. By Lemma 150 there exists an open set V_{α_0} such that

$$C_{\alpha_0} \subset V_{\alpha_0} \subset \overline{V_{\alpha_0}} \subset U_{\alpha_0}.$$

Suppose that V_α has been chosen for every $\alpha \in \Lambda$ with $\alpha < \beta$ and define

$$C_\beta := X \setminus \left(\left(\bigcup_{\alpha<\beta} V_\alpha \right) \cup \left(\bigcup_{\alpha>\beta} U_\alpha \right) \right).$$

To prove that $C_\beta \subset U_\beta$, fix $x \in C_\beta$. Then

$$x \notin V_\alpha \text{ for any } \alpha < \beta \text{ and } x \notin U_\alpha \text{ for any } \alpha > \beta. \tag{14}$$

Since $\{U_\alpha\}_{\alpha\in\Lambda}$ is a point finite cover, x belongs to only finitely many U_α, say, $U_{\alpha_1}, \ldots, U_{\alpha_m}$. Without loss of generality, we may assume that $\alpha_m = \max\{\alpha_1, \ldots, \alpha_m\}$. Then $x \notin U_\alpha$ for $\alpha > \alpha_m$. Hence, by (14), we have that $\alpha_m \le \beta$. We claim that $\alpha_m = \beta$. Indeed, if $\alpha_m < \beta$, then by (14), $x \notin V_{\alpha_m}$ but

$$x \in C_{\alpha_m} = X \setminus \left(\left(\bigcup_{\alpha<\alpha_m} V_\alpha \right) \cup \left(\bigcup_{\alpha>\alpha_m} U_\alpha \right) \right),$$

which contradicts the fact that $C_{\alpha_m} \subset V_{\alpha_m}$. This shows that $\alpha_m = \beta$, so that $x \in U_\beta$.

By Lemma 150 there exists an open set V_β such that

$$C_\beta \subset V_\beta \subset \overline{V_\beta} \subset U_\beta.$$

Thus, by Proposition **??** we have constructed a family of open sets $\{V_\alpha\}_{\alpha\in\Lambda}$.

It remains to show that it covers X. Fix $x \in X$. Since $\{U_\alpha\}_{\alpha\in\Lambda}$ is a point finite cover, x belongs to only finitely many U_α, say, $U_{\alpha_1}, \ldots, U_{\alpha_m}$. As before assume that $\alpha_m = \max\{\alpha_1, \ldots, \alpha_m\}$. Then $x \notin U_\alpha$ for $\alpha > \alpha_m$. There are now two cases. If $x \in \bigcup_{\alpha<\alpha_m} V_\alpha$, then there is nothing to prove. If $x \notin \bigcup_{\alpha<\alpha_m} V_{\alpha_m}$, then $x \in C_{\alpha_m}$ and so $x \in V_{\alpha_m}$. This shows that $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover of X.

Conversely, assume that every point finite open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X there exists another open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X with the property that $\overline{V_\alpha} \subset U_\alpha$. We claim that (X,τ) is normal. Let $C_1, C_2 \subset X$ be two disjoint closed sets. Then $\{X \setminus C_1, X \setminus C_2\}$ is a point finite open cover of X. Hence, there exist two open sets V_1 and V_2 such that

$$X = V_1 \cup V_2, \quad \overline{V_1} \subset X \setminus C_1, \quad \overline{V_2} \subset X \setminus C_2.$$

The sets $X \setminus \overline{V_1}$ and $X \setminus \overline{V_2}$ are open, disjoint, and contain C_1 and C_2, respectively. ■

As a corollary of the previous theorem we can show that in a normal space every point finite open cover admits a partition of unity subordinated to it.

Theorem 173. *Let (X,τ) be a normal space and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a point finite open cover of X. Then there exists a partition of unity subordinated to it.*

Proof. By Lemma 172 there exists another locally finite open cover $\{W_\alpha\}_{\alpha\in\Lambda}$ of X with the property that $\overline{W_\alpha} \subset U_\alpha$.

Using the Lemma 172 one more time, implies the existence of open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ such that

$$\overline{V_\alpha} \subset W_\alpha \subset \overline{W_\alpha} \subset U_\alpha.$$

Since the closed sets sets $\overline{V_\alpha}$ and $X \setminus W_\alpha$ are disjoint, there exists a continuous function $f_\alpha : X \to [0,1]$ such that $f_\alpha \equiv 1$ in $\overline{V_\alpha}$ and $f_\alpha \equiv 0$ in $X \setminus W_\alpha$. Hence,

$$\{x \in X : f_\alpha(x) > 0\} \subset W_\alpha$$

and so,

$$\operatorname{supp} f_\alpha \subset \overline{W_\alpha} \subset U_\alpha.$$

Define

$$f(x) := \sum_{\alpha\in\Lambda} f_\alpha(x), \quad x \in X.$$

Since $\{V_\alpha\}_{\alpha\in\Lambda}$ covers X, for every $x \in X$ there exists $\alpha \in \Lambda$ such that $x \in V_\alpha$, and so $f_\alpha(x) = 1$. Thus, $f > 0$. Moreover, since $\left\{\overline{V_\alpha}\right\}_{\alpha\in\Lambda}$ is locally finite for every $x \in X$ there exists a neighborhood U of x that intersects only finitely many $\overline{V_\alpha}$. Thus, f reduces to a finite sum in U. In particular, $f < \infty$ in U and f is continuous in x. By the arbitrariness of x, we have that f is continuous. Hence the function

$$\varphi_\alpha(x) := \frac{f_\alpha(x)}{f(x)}, \quad x \in X,$$

is well-defined and continuous. Since $\operatorname{supp}\varphi_\alpha = \operatorname{supp} f_\alpha \subset U_\alpha$, the family $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a locally finite partition of unity subordinated to $\{U_\alpha\}_{\alpha\in\Lambda}$. ■

2.22. Paracompact Spaces and Partition of Unity. *The proofs for this section were not provided in class. You are not required to know them.*

Theorem 173 leaves open the question of the existence of a partition of unity subordinated to an arbitrary open cover of X. This problem brings us to the definition of paracompact spaces.

Definition 174. A topological space (X,τ) is *paracompact* if it is Hausdorff and if for every an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X, there exists an open cover of X that is a locally finite refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$.

An important class of paracompact spaces is given by metric spaces. The next theorem was first proved by Stone. The present proof is due to Ornstein.

Theorem 175. *A metric space (X,d) is paracompact and has a σ-locally finite base.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By the axiom of choice we may assume that Λ is well-ordered. Fix $\alpha\in\Lambda$. A *chosen* ball (with respect to α) is a ball $B\left(x,\frac{1}{2^{n_x+1}}\right)$ such that

(i) $B\left(x,\frac{1}{2^{n_x}}\right)\subset U_\alpha$,
(ii) n_x is the smallest integer for which (i) holds,
(iii) $B\left(x,\frac{1}{2^{n_x}}\right)\subset U_\beta$ for some $\beta<\alpha$.

Step 1: Let $\mathcal{B}_\alpha:=\left\{B\left(x,\frac{1}{2^{n_x+1}}\right):\,B\left(x,\frac{1}{2^{n_x+1}}\right)\text{ is a chosen ball}\right\}$ and define the open set

$$V_\alpha:=U_\alpha\setminus\overline{\bigcup_{B\in\mathcal{B}_\alpha}B}.$$

We claim that $\{V_\alpha\}_{\alpha\in\Lambda}$ is still an open cover of X. Indeed, assume by contradiction that there exists $x\in X$ that is not covered by $\{V_\alpha\}_{\alpha\in\Lambda}$. Let U_α be the first element that contains x (recall that the set $\{\beta\in\Lambda:\,x\in U_\beta\}$ has a minimum). Then $B(x,r)\subset U_\alpha$ for some $r>0$. Since $x\notin V_\alpha$ and x is not in any chosen ball (with respect to α) in view of (iii), it follows that x must be an accumulation point of chosen balls, namely, there exist two sequences $\left\{B\left(x_k,\frac{1}{2^{n_{x_k}+1}}\right)\right\}\subset\mathcal{B}_\alpha$ and $y_k\in B\left(x_k,\frac{1}{2^{n_{x_k}+1}}\right)$ such that $y_k\to x$ as $k\to\infty$ (the balls could be repeated). Note that n_{x_k} cannot approach infinity along a subsequence, not relabeled, since this would imply that

$$B\left(x_k,\frac{1}{2^{n_{x_k}-1}}\right)\subset B(x,r)\subset U_\alpha$$

for infinitely many k and this would contradict (ii). Hence,

$$\min_k\frac{1}{2^{n_{x_k}+1}}=\frac{1}{2^{n_0+1}}$$

for some $n_0\in\mathbb{N}$. Let $k\in\mathbb{N}$ be so large that $d(x,y_k)<\frac{1}{2^{n_0+1}}$, then

$$d(x,x_k)\le d(x,y_k)+d(y,x_k)<\frac{1}{2^{n_0+1}}+\frac{1}{2^{n_{x_k}+1}}\le\frac{2}{2^{n_{x_k}+1}}=\frac{1}{2^{n_{x_k}}};$$

that is, $x\in B\left(x_k,\frac{1}{2^{n_{x_k}}}\right)$. But then by (iii), it follows that x must belong to U_β for some $\beta<\alpha$, which contradicts the choice of α and proves the claim.

Step 2: Next we prove that for every $x\in X$ there exists a finite number of V_α that contain x. By the previous step there exists V_β such that $x\in V_\beta$. By construction, this means that U_β is the first element to contain some ball $B\left(x,\frac{1}{2^m}\right)$. Note that if U_β is the first element to contain some ball $B\left(x,\frac{1}{2^m}\right)$ and U_γ is the first element to contain some ball $B\left(x,\frac{1}{2^n}\right)$, and if, say, $n>m$, then $\gamma\le\beta$. Hence, the family α such that V_α contain x form a descending sequence. Since Λ is well-ordered, only a finite number are distinct.

Step 3: Finally, we construct a refinement of $\{V_\alpha\}_{\alpha\in\Lambda}$ that is locally finite. For every $x\in X$ let

$$r_x:=\frac{1}{2}\sup\{r>0:\,B(x,r)\subset V_\alpha\text{ for some }\alpha\in\Lambda\}.$$

If $r_x=\infty$ for some x, then one could construct a refinement of $\{V_\alpha\}_{\alpha\in\Lambda}$ given by sequence of balls $\{B(x,n)\}_{n\in\mathbb{N}}$, which is a locally finite. Thus, assume that $r_x<\infty$

for all $x \in X$. For every $\beta \in \Lambda$, let W_β be the union of all balls $B\left(x, \frac{r_x}{2}\right)$ such that V_β is the first open set in the cover $\{V_\alpha\}_{\alpha\in\Lambda}$ to contain $B(x, r_x)$. By construction, $\{W_\alpha\}_{\alpha\in\Lambda}$ is still an open cover of X. Moreover, $W_\alpha \subset V_\alpha \subset U_\alpha$ for every $\alpha \in \Lambda$, and so $\{W_\alpha\}_{\alpha\in\Lambda}$ is a refinement of $\{V_\alpha\}_{\alpha\in\Lambda}$ and, in turn, of $\{U_\alpha\}_{\alpha\in\Lambda}$. It remains to show that $\{W_\alpha\}_{\alpha\in\Lambda}$ is locally finite.

We claim that if $W_\alpha \cap B\left(x, \frac{r_x}{8}\right) \neq \emptyset$, then $x \in V_\alpha$. Indeed, assume the contrary for some $x \in X$. Then there exists $y \in W_\alpha$ such that $B\left(y, \frac{r_y}{2}\right) \cap B\left(x, \frac{r_x}{8}\right) \neq \emptyset$ such that $x \notin V_\alpha$. Since $y \in W_\alpha$, we have that $B(y, r_y) \subset V_\alpha$. In particular, $x \notin B(y, r_y)$, and so

$$r_y < d(x, y) < \frac{r_y}{2} + \frac{r_x}{8},$$

which implies that $\frac{r_y}{2} < \frac{r_x}{8}$. Hence,

$$d(x, y) < \frac{r_y}{2} + \frac{r_x}{8} < \frac{r_x}{8} + \frac{r_x}{8} = \frac{r_x}{4};$$

that is, $y \in B\left(x, \frac{r_x}{4}\right)$. In turn, $B\left(y, \frac{5r_y}{2}\right) \subset B(x, r_x)$. By the definition of r_x, this implies that $B(y, 5r_y)$ belongs to some V_β, which contradicts the definition of r_y. Hence, the claim holds. Thus, for every $x \in X$, the only open sets W_α that intersect $B\left(x, \frac{r_x}{8}\right)$ are those for which V_α contains x. By the previous step these V_α are finite and the proof of paracompactness is complete.

Step 4: We prove that X has a σ-locally finite base. For every $x \in X$ and $n \in \mathbb{N}$ consider the ball $B\left(x, \frac{1}{n}\right)$. Then $\left\{B\left(x, \frac{1}{n}\right)\right\}_{x\in X, n\in\mathbb{N}}$ is a base. Fix $n \in \mathbb{N}$ and consider the open cover of X, $\left\{B\left(x, \frac{1}{n}\right)\right\}_{x\in X}$. Since X is paracompact, there exists a locally finite open refinement $\mathcal{V}_n$ of $\left\{B\left(x, \frac{1}{n}\right)\right\}_{x\in X}$. This completes the proof. ■

Next we show that paracompact spaces are normal spaces.

Proposition 176. *Let (X, τ) be a paracompact space. Then (X, τ) is a normal space.*

Proof. Step 1: Let $C \subset X$ be a closed set and let $x \in X \setminus C$. We claim that there exist two disjoint neighborhoods of x and C. Since X is a Hausdorff space, for every $y \in C$, there exist two disjoint neighborhoods V_x and V_y of x and y. Note that since $V_x \cap V_y = \emptyset$, we have that $x \notin \overline{V_y}$. Then the sets V_y, $y \in C$, and $X \setminus C$ form an open cover of X. Since X is paracompact, there exists a locally finite refinement $\{U_\alpha\}_{\alpha\in\Lambda}$. Define

$$U := \bigcup_{\alpha\in\Lambda:\, U_\alpha\cap C\neq\emptyset} U_\alpha.$$

Then U is open and contains C. Moreover, by the previous lemma,

$$\overline{U} = \bigcup_{\alpha\in\Lambda:\, U_\alpha\cap C\neq\emptyset} \overline{U_\alpha}.$$

Since by construction each U_α such that $U_\alpha \cap C \neq \emptyset$ is contained in some V_y, it follows that $\overline{U_\alpha} \subset \overline{V_y}$, and so $x \notin \overline{U_\alpha}$. In turn, $x \notin \overline{U}$. The sets $X \setminus \overline{U}$ and U are open, disjoint, and contain x and C, respectively.

Step 2: Let $C_1, C_2 \subset X$ be two closed disjoint sets. By the previous step, for each $y \in C_2$ we may find an open set V_y such that $V_y \cap C_1 = \emptyset$. Define U as in the previous step with C replaced with C_2. Then exactly as before we can show that

$C_1 \cap \overline{U} = \emptyset$, and so the sets $X \setminus \overline{U}$ and U are open, disjoint, and contain C_1 and C_2, respectively. This shows that X is normal. ■

Next we present several characterizations of paracompact spaces.

Theorem 177 (Michael)**.** *Let (X,τ) be a normal space. Then the following are equivalent.*

(i) *(X,τ) is paracompact.*
(ii) *For every open cover of X there exists a locally finite refinement (not necessarily open).*
(iii) *For every open cover of X there exists a closed, locally finite refinement.*
(iv) *For every open cover of X there exists a σ-locally finite open refinement.*

Proof. (i)$\Longrightarrow$(ii) There is nothing to prove.

(ii)$\Longrightarrow$(iii) Assume (ii) and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By Theorem 172 there exists an open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X with the property that $\overline{V_\alpha} \subset U_\alpha$ for every $\alpha \in \Lambda$. Now apply (ii) to $\{V_\alpha\}_{\alpha\in\Lambda}$ to find a locally finite refinement $\mathcal{E}$ of $\{V_\alpha\}_{\alpha\in\Lambda}$. Let $\mathcal{C} := \left\{\overline{E} : E \in \mathcal{E}\right\}$. Then $\mathcal{C}$ is still a refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. It remains to show that $\mathcal{C}$ is still locally finite. Let $x \in X$. Since $\mathcal{E}$ is locally finite, there exists a neighborhood U of x that intersects only finitely many elements of $\mathcal{E}$, say, $E_1, \dots, E_m$. If $\overline{E} \cap U \neq \emptyset$, then by Proposition **??**, $E \cap U \neq \emptyset$, and so E is one of the $E_1, \dots, E_m$. Thus, $\mathcal{C} := \left\{\overline{E} : E \in \mathcal{E}\right\}$ is locally finite.

(iii)$\Longrightarrow$(i) Assume (iii) and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By (iii) there exists a closed locally finite refinement $\mathcal{C}$ of $\{U_\alpha\}_{\alpha\in\Lambda}$. Hence, for every $x \in X$ there exists a neighborhood V_x of x that intersects only finitely many elements of $\mathcal{C}$. Since $\{V_x\}_{x\in X}$ is an open cover of X, we can apply (iii) once more to find a closed locally finite refinement $\mathcal{K}$ of $\{V_x\}_{x\in X}$. For every $C \in \mathcal{C}$ define $\mathcal{K}_C := \{K \in \mathcal{K} : K \cap C = \emptyset\}$ and set

$$D_C := X \setminus \bigcup_{K\in\mathcal{K}_C} K.$$

Since $\mathcal{K}_C$ is locally finite, by Lemma 181 the set $\bigcup_{K\in\mathcal{K}_C} K$ is closed, and thus D_C is open. Moreover, since $\mathcal{K}$ is a cover of X, it follows from the definition of $\mathcal{K}_C$ that D_C contains C. Thus, the family $\{D_C\}_{C\in\mathcal{C}}$ is an open cover of X. Moreover, if $K \in \mathcal{K}$, then K intersects D_C if and only if K intersects C. We claim that $\{D_C\}_{C\in\mathcal{C}}$ is locally finite. To see this, fix $x \in X$. Since $\mathcal{K}$ is locally finite there exist a neighborhood U of x that intersects only finitely many elements of $\mathcal{K}$, say, $K_1, \dots, K_m$. On the other hand, if $U \cap D_C \neq \emptyset$ for some $C \in \mathcal{C}$, then there is $y \in U$ such that $y \notin \bigcup_{K\in\mathcal{K}_C} K$. Since $\mathcal{K}$ is a cover, y belongs to some $K \notin \mathcal{K}_C$. Thus, $U \cap K \neq \emptyset$ for some $K \notin \mathcal{K}_C$, which implies that one of the $K_1, \dots, K_m$ intersects $C \cap U$. This means that one of the $K_1, \dots, K_m$ intersects D_C. Since each K_i is contained in some V_z and V_z intersects only finitely many C in $\mathcal{C}$, we have that $U \cap D_C \neq \emptyset$ for finitely many $C \in \mathcal{C}$, which shows that $\{D_C\}_{C\in\mathcal{C}}$ is locally finite.

Since $\mathcal{C}$ is refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$, for each $C \in \mathcal{C}$ we may find $\alpha_C \in \Lambda$ such that $C \subset U_{\alpha_C}$. Finally, define

$$\mathcal{V} := \{D_C \cap U_{\alpha_C} : C \in \mathcal{C}\}.$$

We claim that is locally finite open refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. Since the open set $D_C \cap U_{\alpha_C}$ contains C and $\mathcal{C}$ is a cover of X, the family $\mathcal{V}$ is an open refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. Since $\{D_C\}_{C\in\mathcal{C}}$ is locally finite, then so is $\mathcal{V}$.

It remains to show that (iv)$\Longrightarrow$(ii). Assume (iv) and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By (iv) there exists an open refinement of the form

$$\mathcal{V}=\bigcup_{n=1}^{\infty}\mathcal{V}_n,$$

where each $\mathcal{V}_n$ is locally finite. For every $n\in\mathbb{N}$ and for every $V\in\mathcal{V}_n$ define

$$\mathcal{V}_V:=\{U\in\mathcal{V}:\,U\in\mathcal{V}_k\text{ for some }k<n\}$$

and set

$$E_V:=V\setminus\bigcup_{U\in\mathcal{V}_V}U.$$

Then $E_V\subset V$ and $\{E_V\}_{V\in\mathcal{V}}$ is a cover of X. Note that the family $\{E_V\}_{V\in\mathcal{V}}$ is locally finite. Indeed, given $x\in X$, let $n\in\mathbb{N}$ be the first integer such that x belongs to some V in the family $\mathcal{V}_n$. Then V is a neighborhood of x that does not intersect any E_U for $U\in\mathcal{V}_k$ with $k>n$ (since V has been removed from each such E_U). Thus, V can only intersect sets E_U such that $U\in\mathcal{V}_k$ with $k\le n$. Since each $\mathcal{V}_k$ is locally finite for $k\le n$, we may find a neighborhood W_k of x that intersects only finitely many U in $\mathcal{V}_k$., and, in turn, only finitely many of the corresponding E_U. The neighborhood

$$V\cap\bigcap_{k=1}^{n}W_k$$

of x intersects finitely many E_U. This shows that $\{E_V\}_{V\in\mathcal{V}}$ is locally finite and concludes the proof. ■

The importance of paracompact spaces comes from the following theorem.

Theorem 178 (Michael)**.** *Let (X,τ) be a normal space. Then the following are equivalent.*

(i) (X,τ) *is paracompact.*
(ii) *For every open cover of X there exists a locally finite partition of unity subordinated to it.*
(iii) *For every open cover of X there exists a partition of unity subordinated to it.*

Proof. Step 1: Assume that (X,τ) is paracompact and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. Since (X,τ) is paracompact there exists a locally finite refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$ and (X,τ) is a normal space, by the previous proposition. We are now in a position to apply Theorem 173 to find locally finite partition of unity subordinated to the refinement, and in particular, to $\{U_\alpha\}_{\alpha\in\Lambda}$.
Step 2: Assume that every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X admits a locally finite partition of unity $\{\varphi_i\}_{i\in I}$ subordinated to it. Fix any two such $\{U_\alpha\}_{\alpha\in\Lambda}$ and $\{\varphi_i\}_{i\in I}$. For every $n\in\mathbb{N}$ and every $i\in I$ consider the set

$$V_{i,n}:=\left\{x\in X:\,\varphi_i(x)>\frac{1}{n}\right\}.$$

Then $V_{n,i}$ is open. We claim that $\{V_{i,n}\}_{i\in I,n\in\mathbb{N}}$ is an open cover of X that is a locally finite refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. Indeed, if $x\in X$, then

$$\sum_{i\in I}\varphi_i(x)=1,$$

and so there exists $i \in I$ such that $\varphi_i(x) \in (0,1]$. It follows that $\varphi_i(x) > \frac{1}{n}$ for all $n \in \mathbb{N}$ sufficiently large, and so $x \in V_{i,n}$ for all $n \in \mathbb{N}$ sufficiently large. This shows that $\{V_{i,n}\}_{i\in I, n\in\mathbb{N}}$ is an open cover of X. Moreover, since $V_{i,n} \subset \operatorname{supp}\varphi_i$ and $\{\varphi_i\}_{i\in I}$ is subordinated to $\{U_\alpha\}_{\alpha\in\Lambda}$, each $\operatorname{supp}\varphi_i$ (and in turn each $V_{i,n}$) is contained in some U_α. Thus, $\{V_{i,n}\}_{i\in I,n\in\mathbb{N}}$ is a refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. We claim that $\{V_{i,n}\}_{i\in I,n\in\mathbb{N}}$ is σ-locally finite. Fix $x_0 \in X$ and $n \in \mathbb{N}$. Write

$$1 = \sum_{i\in I} \varphi_i(x_0) = \sum_{i\in I_0} \varphi_i(x_0),$$

where $I_0 := \{i \in I : \varphi_i(x_0) > 0\}$ is countable. Since the series $\sum_{i\in I_0} \varphi_i(x_0)$ is convergent, there exists a finite subset $I_1 \subset I_0$ such that

$$\sum_{i\in I_1} \varphi_i(x_0) > 1 - \frac{1}{2n}.$$

By continuity and the fact that I_1 is finite, we may find an open neighborhood U of x_0 such that

$$\sum_{i\in I_1} \varphi_i(x) > 1 - \frac{1}{n}$$

for all $x \in U$. Note that if i does not belong I_1, then U cannot intersect $V_{i,n}$ (the sum would be greater than one). Hence, U intersect only finitely many $V_{i,n}$ (recall that n is fixed). The result now follows from the previous theorem. ■

Lecture 31, Nov. 14, 2012.

2.23. Metrization. A topological space (X,τ) is *metrizable* if its topology can be determined by a metric. The metrizability and the normability of a given topology depend on the properties of a base (see Theorems 179 and 180 below).

In view of Exercise 90 in order for a topological space (X,τ) to be metrizable it is necessary that (X,τ) be T_4. Thus in the next two theorems, without loss of generality, we will assume that these two properties are satisfied.

Theorem 179 (Urysohn's metrization theorem)**.** *A topological space (X,τ) is metrizable and separable if and only if X is T_4 and has a countable base.*

Proof. If (X,d) is a metric space then it is T_4 (Exercise 90). Moreover, if (X,d) is separable, then there exists a sequence $\{x_n\} \subset X$ which is dense in X. The countable family of balls $\left\{B\left(x_n, \frac{1}{k}\right)\right\}_{k,n\in\mathbb{N}}$ is a base for the topology τ determined by d.

Conversely, assume that (X,τ) is a T_4 space with a countable base $\mathcal{B} = \{B_n\}_n$. We will show that X is homeomorphic (in the topological sense) to a subset of ℓ^2, which is separable.

Step 1: We claim that every closed set is a G_δ, or, equivalently, that every open set is an F_σ set. Fix an open set $U \subset X$. Fix $x \in U$. Since X is Hausdorff, the singleton $\{x\}$ is closed, and $\{x\} \subset U$. By Lemma 150 there exists an open set $V \subset X$ such that

$$\{x\} \subset V \subset \overline{V} \subset U.$$

Since $\mathcal{B}$ is a base, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset V$, and so

$$\{x\} \subset B_x \subset \overline{B_x} \subset \overline{V} \subset U.$$

This shows that

$$U = \bigcup_{\overline{B_n} \subset U} \overline{B_n};$$

that is, that U is an F_σ set.

Step 2: By the previous step and Exercise 152 for each element B_n in the base $\mathcal{B}$ there exists a continuous function $\varphi_n : X \to [0,1]$ with the property that

$$\varphi_n(x) = 0 \text{ for } x \in X \setminus B_n, \quad \varphi_n(x) > 0 \text{ for } x \in B_n. \tag{15}$$

Define

$$\psi_n(x) := \frac{1}{n} \frac{\varphi_n(x)}{\sqrt{1 + (\varphi_n(x))^2}}, \quad x \in X.$$

Then ψ_n is continuous. Moreover,

$$\sum_n (\psi_n(x))^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{(\varphi_n(x))^2}{1 + (\varphi_n(x))^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

which shows that for every fixed $x \in X$, $\{\psi_n(x)\}_n$ belongs ℓ^2. Hence, the map

$$\begin{aligned} f : X &\to \ell^2 \\ x &\mapsto \{\psi_n(x)\}_n \end{aligned}$$

is well-defined.

We claim that f is one-to-one. To see this, let $x, y \in X$ with $x \neq y$. Since X is Hausdorff, there exists B_n such that $x \in B_n$ and $y \in X \setminus B_n$. It follows by (15) that $\psi_n(x) > 0$, while $\psi_n(y) = 0$. Hence, $f(x) \neq f(y)$ and the claim is proved.

Next we claim that f is continuos. Fix $x_0 \in X$ and $\varepsilon > 0$ and find $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty} \frac{1}{n^2} \le \frac{\varepsilon^2}{8}.$$

Since each ψ_n, $n = 1, \ldots, n_\varepsilon$, is continuous at x_0 and we have a finite number of them, there exists a neighborhood $V \subset U$ of x_0 such that

$$|\psi_n(x) - \psi_n(x_0)| \le \frac{\varepsilon}{\sqrt{2n_\varepsilon}}$$

for all $x \in V$ and $n = 1, \ldots, n_\varepsilon$. Then for $x \in V$,

$$\sum_{n=1}^{n_\varepsilon} (\psi_n(x) - \psi_n(x_0))^2 \le n_\varepsilon \frac{\varepsilon^2}{2n_\varepsilon} = \frac{\varepsilon^2}{2},$$

while

$$\begin{aligned} \sum_{n=n_\varepsilon+1}^{\infty} (\psi_n(x) - \psi_n(x_0))^2 &\le 2 \sum_{n=n_\varepsilon+1}^{\infty} \left[(\psi_n(x))^2 + (\psi_n(x_0))^2 \right] \\ &\le 2 \sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n^2} \right) \le \frac{4\varepsilon^2}{8} = \frac{\varepsilon^2}{2}. \end{aligned}$$

Hence,

$$d_2(f(x), f(x_0)) \le \varepsilon$$

for all $x \in V$, which shows continuity at x_0.

Finally, we prove that $f^{-1} : f(X) \to X$ is continuous. Let $y_0 \in f(X)$, then there exists $x_0 \in X$ such that $f(x_0) = y_0$. Consider a neighborhood U of x_0. Since

$\mathcal{B}$ is a base, there exists B_n in $\mathcal{B}$ such that $B_n \subset U$ so that $\psi_n(x_0) > 0$. Let $\delta := \psi_n(x_0)$. If $d_2(f(x), f(x_0)) < \delta$, then

$$\begin{aligned}|\psi_n(x) - \psi_n(x_0)| &\le \left(\sum_{n=1}^{\infty} (\psi_n(x) - \psi_n(x_0))^2\right)^{\frac{1}{2}} \\ &= d_2(f(x), f(x_0)) < \delta = \psi_n(x_0),\end{aligned}$$

axwhich implies that $\psi_n(x) > 0$, and so, by (16), that $x \in B_n \subset U$. This shows that f^{-1} is continuous at $y_0 = f(x_0)$. ■

Next we drop the separability requirement.

Theorem 180 (Nagata–Smirnov's metrization theorem)**.** *A topological space* (X, τ) *is metrizable if and only if it is* T_4 *and has a* σ*-locally finite base.*

Lemma 181. *Let* $\{E_\alpha\}_{\alpha\in\Lambda}$ *be a locally finite family of sets. Then*

$$\overline{\bigcup_{\alpha\in\Lambda} E_\alpha} = \bigcup_{\alpha\in\Lambda} \overline{E_\alpha}.$$

In particular, the union of a locally finite family of closed sets is closed.

The proof of the lemma is a problem on Set 6.

We now turn to the proof of Nagata–Smirnov's metrization theorem.

Proof. If (X, d) is a metric space, then by Exercise 90 it is T_4. In Theorem 175, we will see that X has a σ-locally finite base.

Conversely, assume that X is T_4 and has a σ-locally finite base, that is, a base of the form

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n,$$

where each $\mathcal{B}_n = \{B_{n,\alpha}\}_{\alpha\in\Lambda_n}$ is locally finite, then X is metrizable. We will show that X is homeomorphic (in the topological sense) to a subset of an ℓ^2 metric space.

Step 1: We claim that every closed set is a G_δ, or, equivalently, that every open set is an F_σ set. Fix an open set $U \subset X$. As in Step 1 of the proof of Urysohn's metrization theorem, for every $x \in U$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset \overline{B_x} \subset U$. For every $n \in \mathbb{N}$, let

$$C_n := \bigcup_{B_x\in\mathcal{B}_n} \overline{B_x}.$$

By Lemma 181, C_n is closed. Moreover, $C_n \subset U$. Since

$$U = \bigcup_{n=1}^{\infty} C_n,$$

it follows that U is an F_σ set.

Step 2: By the previous step and Exercise 152 for each element $B_{n,\alpha}$ in the base $\mathcal{B}$ there exists a continuous function $\varphi_{\alpha,n} : X \to [0,1]$ with the property that

$$\varphi_{\alpha,n}(x) = 0 \text{ for } x \in X \setminus B_{\alpha,n}, \quad \varphi_{\alpha,n}(x) > 0 \text{ for } x \in B_{\alpha,n}. \tag{16}$$

Define

$$\psi_{\alpha,n}(x) := \frac{1}{n} \frac{\varphi_{\alpha,n}(x)}{\sqrt{1 + \sum_\beta (\varphi_{\beta,n}(x))^2}}, \quad x \in X.$$

Note that since each $\mathcal{B}_n$ is locally finite, for every $x \in X$ there exists a neighborhood U of x such that $\varphi_{\beta,n} = 0$ in U for all β except at most finitely many. Thus, the infinite sum $\sum_\beta (\varphi_{\beta,n})^2$ reduces to a finite sum in U. This shows that $\psi_{\alpha,n}$ is continuous. Moreover,

$$\sum_{\alpha,n} (\psi_{\alpha,n}(x))^2 = \sum_{n=1}^\infty \frac{1}{n^2} \sum_\alpha \frac{(\varphi_{\alpha,n}(x))^2}{1 + \sum_\beta (\varphi_{\beta,n}(x))^2} \le \sum_{n=1}^\infty \frac{1}{n^2} < \infty,$$

which shows that for every fixed $x \in X$, $\{\psi_{\alpha,n}(x)\}_{\alpha,n}$ belongs to the ℓ^2 space

$$\ell^2 := \left\{ \{a_{\alpha,n}\}_{\alpha,n} : \sum_{\alpha,n} a_{\alpha,n}^2 < \infty \right\}$$

with metric

$$d_2(a,b) := \left(\sum_{\alpha,n} (a_{\alpha,n} - b_{\alpha,n})^2 \right)^{\frac{1}{n}},$$

where $a = \{a_{\alpha,n}\}_{\alpha,n}$ and $b = \{a_{\alpha,n}\}_{\alpha,n}$. Hence, the map

$$\begin{aligned} f : X &\to \ell^2 \\ x &\mapsto \{\psi_{\alpha,n}(x)\}_{\alpha,n} \end{aligned}$$

is well-defined.

We claim that f is one-to-one. To see this, let $x, y \in X$ with $x \ne y$. Then there exist $B_{\alpha,n}$ such that $x \in B_{\alpha,n}$ and $y \in X \setminus B_{\alpha,n}$. It follows by (16) that $\psi_{\alpha,n}(x) > 0$, while $\psi_{\alpha,n}(y) = 0$. Hence, $f(x) \ne f(y)$ and the claim is proved.

Next we claim that f is continuos. Fix $x_0 \in X$ and $\varepsilon > 0$ and find $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^\infty \frac{1}{n^2} \le \frac{\varepsilon^2}{8}.$$

By local finiteness, for every $n = 1, \ldots, n_\varepsilon$ we may find a neighborhood of x_0 that intersects finitely many $B_{\alpha,n}$ in $\mathcal{B}_n$. By intersecting this *finite* number of neighborhoods, we obtain a neighborhood U of x that intersects finitely many $B_{\alpha,n}$ for $n = 1, \ldots, n_\varepsilon$, say, say, $B_{\alpha_1,n_1}, \ldots, B_{\alpha_m,n_m}$. Since each ψ_{α_i,n_i} is continuous at x_0 and we have a finite number of them, there exists a neighborhood $V \subset U$ of x_0 such that

$$|\psi_{\alpha_i,n_i}(x) - \psi_{\alpha_i,n_i}(x_0)| \le \frac{\varepsilon}{\sqrt{2m}}$$

for all $x \in V$ and $i = 1, \ldots, m$. Then for $x \in V$,

$$\sum_{n=1}^{n_\varepsilon} \sum_\alpha (\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0))^2 = \sum_{i=1}^m (\psi_{\alpha_i,n_i}(x) - \psi_{\alpha_i,n_i}(x_0))^2 \le m \frac{\varepsilon^2}{2m} = \frac{\varepsilon^2}{2},$$

while

$$\begin{aligned} \sum_{n=n_\varepsilon+1}^\infty \sum_\alpha (\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0))^2 &\le 2 \sum_{n=n_\varepsilon+1}^\infty \sum_\alpha \left[(\psi_{\alpha,n}(x))^2 + (\psi_{\alpha,n}(x_0))^2 \right] \\ &\le 2 \sum_{n=n_\varepsilon+1}^\infty \left(\frac{1}{n^2} + \frac{1}{n^2} \right) \le \frac{4\varepsilon^2}{8} = \frac{\varepsilon^2}{2}. \end{aligned}$$

Hence,

$$d_2\left(f\left(x\right), f\left(x_0\right)\right) \le \varepsilon$$

for all $x \in V$, which shows continuity at x_0.

Finally, we prove that $f^{-1} : f\left(X\right) \to X$ is continuous. Let $y_0 \in f\left(X\right)$, then there exists $x_0 \in X$ such that $f\left(x_0\right) = y_0$. Consider a neighborhood U of x_0. Since $\mathcal{B}$ is a base, there exists $B_{\alpha,n}$ in $\mathcal{B}$ such that $B_{\alpha,n} \subset U$ so that $\psi_{\alpha,n}\left(x_0\right) > 0$. Let $\delta := \psi_{\alpha,n}\left(x_0\right)$. If $d_2\left(f\left(x\right), f\left(x_0\right)\right) < \delta$, then

$$\begin{aligned}\left|\psi_{\alpha,n}\left(x\right) - \psi_{\alpha,n}\left(x_0\right)\right| &\le \left(\sum_{n=1}^{\infty}\sum_{\alpha}\left(\psi_{\alpha,n}\left(x\right) - \psi_{\alpha,n}\left(x_0\right)\right)^2\right)^{\frac{1}{2}} \\ &= d_2\left(f\left(x\right), f\left(x_0\right)\right) < \delta = \psi_{\alpha,n}\left(x_0\right),\end{aligned}$$

which implies that $\psi_{\alpha,n}\left(x\right) > 0$, and so, by (16), that $x \in B_{\alpha,n} \subset U$. This shows that f^{-1} is continuous at $y_0 = f\left(x_0\right)$. ■

3. Metric Spaces

Lecture 32, Nov. 16, 2012.

3.1. Completion of a Metric Space.

Definition 182. A metric space (X, d) is complete if every Cauchy sequence converges.

In this section we show that every metric space can be completed.

Exercise 183. Let (Y, d) be a metric space and let $E \subset X$ be a dense set with the property that every Cauchy sequence $\{y_n\} \subset E$ converges to some point in Y. Prove that Y is complete.

Theorem 184. *Given a metric space (X, d_X), there exists a complete metric space (Y, d_Y) and a distance preserving map $f : X \to Y$ such that $f\left(X\right)$ is dense in Y. The space Y is unique up to isometries, that is, if $(Y', d_{Y'})$ is a complete metric space having a dense subset isometric to X, then Y and Y' are isometric.*

Proof. Let (X, d_X) be a metric space.

Step 1: Consider the set Z of all Cauchy sequences in X. We define $d_Z : Z \times Z \to [0, \infty)$ by

$$d_Z(\{x_n\}, \{z_n\}) = \lim_{n\to\infty} d_X(x_n, z_n).$$

We claim that d_Z is well-defined, that is that the limit exists. By the triangle inequality applied twice,

$$d_X\left(x_n, z_n\right) \le d_X\left(x_n, x_m\right) + d_X\left(x_m, z_m\right) + d_X\left(z_m, z_n\right),$$

and so (interchanging the roles of x_n and y_n)

$$\left|d_X\left(x_n, z_n\right) - d_X\left(x_m, z_m\right)\right| \le d_X\left(x_n, x_m\right) + d_X\left(z_m, z_n\right).$$

Letting $n, m \to \infty$ and using the fact that $\{x_n\}, \{z_n\}$ are Cauchy sequences in X gives

$$\lim_{n,m\to\infty}\left|d_X\left(x_n, z_n\right) - d_X\left(x_m, z_m\right)\right| = 0,$$

which shows that $\{d_X\left(x_n, z_n\right)\}$ is a Cauchy sequences in $\mathbb{R}$. Thus, there exists $\lim_{n\to\infty} d_X\left(x_n, z_n\right) \in [0, \infty)$.

Step 2: Checking that (Z, d_Z) is a pseudometric space is straightforward. We consider the standard quotient metric space associated to the pseudometric space Z, as was discussed in Subsection 2.6. More precisely relation $\sim$ on Z defined by

$$\{x_n\} \sim \{y_n\} \quad \text{if} \quad d_Z(\{x_n\}, \{y_n\}) = 0$$

is an equivalence relation. We consider the quotient space $Y := Z/\sim$ and the associated metric d_Y:

$$d_Y\left([\{x_n\}], [\{z_n\}]\right) := \lim_{n\to\infty} d_X\left(x_n, z_n\right).$$

Step 3: We construct an isometry $f : X \to Y$. For every $x \in X$ define $\widehat{x} := [\{x_n\}]$, where $x_n := x$ for all $n \in \mathbb{N}$. Define

$$\begin{aligned} f : X &\to Y \\ x &\mapsto \widehat{x} \end{aligned}$$

We claim that f is an isometry. To see this, note that

$$\begin{aligned} d_Y\left(f(x), f(z)\right) &= d_Y\left(\widehat{x}, \widehat{z}\right) = d_Y\left([\{x, \ldots, x, \ldots\}], [\{z, \ldots, z, \ldots\}]\right) \\ &= \lim_{n\to\infty} d_X(x, z) = d_X(x, z). \end{aligned}$$

Step 4: We prove that $f(X)$ is dense in Y. Let $[\{x_n\}] \in Y$ and let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_X(x_n, x_m) \le \varepsilon$$

for all $n, m \in \mathbb{N}$ with $n, m \ge n_\varepsilon$. Consider $f(x_{n_\varepsilon}) = \widehat{x_{n_\varepsilon}} \in Y$. Then

$$d_Y\left([\{x_n\}], \widehat{x_{n_\varepsilon}}\right) = \lim_{n\to\infty} d_X\left(x_n, x_{n_\varepsilon}\right) \le \varepsilon,$$

which proves that $f(X)$ is dense in Y.

Step 5: We prove that Y is a complete metric space. In view of the previous exercise and Step 4, it suffices to show that every Cauchy sequence $\{y_k\}$ contained in $f(X)$ converges in Y. Let $\{y_k\}$ be a Cauchy sequence in $f(X)$. Then for each $k \in \mathbb{N}$ we have that

$$y_k = \widehat{x_k} = [\{x_k, \ldots, x_k, \ldots\}]$$

for some $x_k \in X$. By Step 3, we have that

$$\lim_{l,k\to\infty} d_X(x_l, x_k) = \lim_{l,k\to\infty} d_Y\left(\widehat{x_l}, \widehat{x_k}\right) = 0$$

and so the sequence $\{x_n\} \subset X$ is a Cauchy sequence. Hence, $[\{x_n\}] \in Y$. We claim that $\{y_k\}$ converges to $[\{x_n\}]$. Fix $\varepsilon > 0$. Since $\{y_k = \widehat{x_k}\}$ is a Cauchy sequence, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$d_Y\left(\widehat{x_l}, \widehat{x_k}\right) = d_X(x_l, x_k) \le \varepsilon$$

for all $l, k \in \mathbb{N}$ with $l, k \ge k_\varepsilon$. Then for every $k \ge k_\varepsilon$,

$$d_Y\left([\{x_n\}], \widehat{x_k}\right) = \lim_{n\to\infty} d_X(x_n, x_k) \le \varepsilon,$$

which proves that $\{y_k\}$ converges to $[\{x_n\}]$.

Step 6: (sketch) Finally, we prove uniqueness. Assume that $(Y', d_{Y'})$ is a complete metric space having a dense subset X' isometric to X. Given $y', w' \in Y'$ by the density of X' in Y', we can find two sequences $\{y'_n\}, \{w'_n\} \subset X'$ such that

$d_{Y'}(y_n', y') \to 0$ and $d_{Y'}(w_n', w') \to 0$. In particular, $\{y_n'\}, \{w_n'\}$ are Cauchy sequences in X'. Since X' is isometric to X and X is isometric to $f(X)$, there exist $\{y_n\}, \{w_n\} \subset X$ corresponding to $\{y_n'\}, \{w_n'\}$ such that

$$\begin{aligned} d_Y(f(y_n), f(y_m)) &= d_X(y_n, y_m) = d_{Y'}(y_n', y_m'), \\ d_Y(f(w_n), f(w_m)) &= d_X(w_n, w_m) = d_{Y'}(w_n', w_m'), \\ d_Y(f(y_n), f(w_n)) &= d_X(y_n, w_n) = d_{Y'}(y_n', w_n'), \end{aligned}$$

which implies that $\{f(y_n)\}, \{f(w_n)\}$ are Cauchy sequences in Y and so, by the completeness of Y, they converge to some $y, w \in Y$, respectively. Define $g(y') := y$, $g(w') := w$.

By the triangle inequality,

$$\begin{aligned} |d_{Y'}(y', z') - d_{Y'}(y_n', w_n')| &\le d_{Y'}(y_n', y') + d_{Y'}(w_n', w'), \\ |d_Y(y, z) - d_Y(f(y_n), f(w_n))| &\le d_Y(f(y_n), y) + d_{Y'}(f(w_n), w) \end{aligned}$$

and so

$$d_{Y'}(y', z') = \lim_{n\to\infty} d_{Y'}(y_n', w_n') = \lim_{n\to\infty} d_Y(f(y_n), f(w_n)) = d_Y(y, z).$$

Thus, Y and Y' are isometric. ■

Example 185. If we take $\mathbb{Q}$ with the metric $d(x, y) := |x - y|$ and we complete it, we obtain $\mathbb{R}$.

Given a metric space (X, d) and a function $f : X \to \mathbb{R}$, the *support* of f is the closed set

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}.$$

Exercise 186. Consider an open interval $I \subset \mathbb{R}$ and consider the space

$$C_c(I) := \{f : I \to \mathbb{R} \text{ continuous, } \operatorname{supp} f \text{ is a compact set of } I\}$$

with the metric

$$d(f, g) := \max_{x\in I} |f(x) - g(x)|.$$

Prove that $C_c(I)$ is not complete and characterize its completion. That is show that the completion of $C_c(\mathbb{R})$ is isometric to a subset of $C(\overline{I})$ with appropriate metric.

Exercise 187. Show that both metric spaces considered in (30)[iii] are not complete.

Given metric spaces X and Y let

$$C_b(X, Y) = \{f \in C(X, Y) : \sup_{x,z\in X} d_Y(f(x), f(z)) < \infty\}.$$

For $f, g \in C_b(X, Y)$ we consider

$$d_\infty(f, g) = \sup_{x\in X} d_Y(f(x), g(x)).$$

It is straightforward to check that $(C_b(X, Y), d_\infty)$ is a metric space.

Exercise 188. Show that the metric space $(C_b(X; Y), d_\infty)$ is complete if and only if (Y, d_Y) is complete.

Lecture 33, November 19, 2012.

3.2. Baire Spaces.

Definition 189. A topological space (X, τ) is a *Baire space* is intersection of any countable family of dense open sets in X is dense in X.

Definition 190. A set $E \subset X$ is called

(i) *nowhere dense* if the interior of its closure is empty.
(ii) *meager* if it can be written as a countable union of nowhere dense sets.

Note that if E is nowhere dense then $(X \backslash E)^o$ is open and dense.

Exercise 191. (X, τ) is a Baire space if and only if every meager subset of X has empty interior.

Exercise 192. If (X, τ) is a Baire space and M is a meager subset then $X \backslash M$ is dense.

Theorem 193 (Baire category theorem)**.** *Let (X, d) be a complete metric space. Then it is a Baire space.*

Proof. Let $\{U_n\} \subset X$ be a countable family of dense open sets. We consider here only the case in which the family $\{U_n\}$ is infinite. The case in which the family is finite is simpler and is left as an exercise. Let

$$E := \bigcap_{n=1}^{\infty} U_n.$$

We claim that $\overline{E} = X$. Fix $x_0 \in X$. We claim that $x_0 \in \overline{E}$. It is enough to show that for

$$B(x_0, r) \cap E \neq \emptyset$$

for every $r > 0$. Fix $r > 0$. Since U_1 is dense, $x_0 \in X = \overline{U_1}$, and so the open set $B(x_0, r) \cap U_1$ is nonempty. Let $x_1 \in B(x_0, r) \cap U_1$. Since $B(x_0, r) \cap U_1$ is open, there exists $0 < r_1 < 1$ such that

$$B(x_1, 2r_1) \subset B(x_0, r) \cap U_1. \tag{17}$$

Inductively, assume that $x_n \in X$ and $0 < r_n < \frac{1}{n}$ have been chosen. Since U_{n+1} is dense, $x_n \in X = \overline{U_{n+1}}$, and so the open set $B(x_n, r_n) \cap U_{n+1}$ is nonempty, and so there exist $x_{n+1} \in X$ and $0 < r_{n+1} < \frac{1}{n+1}$ such that

$$B(x_{n+1}, 2r_{n+1}) \subset B(x_n, r_n) \cap U_{n+1}. \tag{18}$$

By induction we can construct two sequences $\{x_n\}$ and $\{r_n\}$ such that (18) holds and $0 < r_n < \frac{1}{n}$ for all $n \geq 1$. Note that, by construction (see (17) and (18)), for every $n \in \mathbb{N}$,

$$B(x_{n+1}, 2r_{n+1}) \subset B(x_n, r_n) \subset B(x_n, 2r_n) \subset \cdots \subset B(x_1, 2r_1) \subset B(x_0, r) \cap U_1. \tag{19}$$

Hence, if $n, m > k$, then $x_n, x_m \in B(x_k, r_k)$, so that

$$d(x_n, x_m) \leq d(x_n, x_k) + d(x_k, x_m) < r_k + r_k < \frac{2}{k}.$$

By letting $k \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. We claim that $x \in B(x_0, r) \cap U_\ell$ for every $\ell \in \mathbb{N}$. Indeed, fix $k \in \mathbb{N}$. Then for all $n > k$ we have that $x_n \in B(x_k, r_k)$, and so,

$$d(x_k, x) \leq d(x_k, x_n) + d(x_n, x) < r_k + d(x_n, x).$$

Letting $n \to \infty$ we conclude that $d(x_k, x) \leq r_k < 2r_k$. It follows that $x \in B(x_k, 2r_k) \subset B(x_{k-1}, r_{k-1}) \cap U_k$ for all $k \in \mathbb{N}$ (where we set $r_0 := r$). In turn, by (17) and (18), we have that $x \in B(x_0, r) \cap E$ holds and the proof is complete. ■

Corollary 194. *Let (X, d) be a complete metric space. Then the intersection of a countable family of dense G_δ sets in X is still a dense G_δ set.*

Proof. Let $\{G_n\} \subset X$ be a sequence of dense G_δ sets. Each G_n is given by the intersection of countably many open sets,

$$G_n = \bigcap_k U_{n,k}.$$

Since G_n is dense in X, any subset that contains G_n is also dense in X. Hence, $U_{n,k}$ is dense in X. The family of open sets $\{U_{n,k}\}_{n,k}$ is countable, and so by the Baire theorem its intersection G is dense in X, but

$$G = \bigcap_{k,n} U_{n,k} = \bigcap_n \bigcap_k U_{n,k} = \bigcap_n G_n,$$

which shows that G is dense and it is a G_δ set. ■

Exercise 195. Show that every locally compact Hausdorff space is a Baire space.

Theorem 196 (Weierstrass)**.** *Let $0 < a < 1$ and let $b \in \mathbb{N}$ be an odd integer such that $ab > 1 + \frac{3}{2}\pi$. Then the function*

$$u(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R},$$

is continuous, but nowhere differentiable.

Theorem 197. *There exists a continuous $f : [0,1] \to \mathbb{R}$ that is nowhere monotone.*

Proof. Let I be a closed interval of $[0,1]$ and let

$$\mathcal{I}_I := \{f \in C([0,1]) : f \text{ is increasing in } I\},$$
$$\mathcal{D}_I := \{f \in C([0,1]) : f \text{ is decreasing in } I\}$$

The sets $\mathcal{I}_I$ and $\mathcal{D}_I$ are closed. Define

$$\mathcal{M}_I := \mathcal{I}_I \cup \mathcal{D}_I.$$

Then $\mathcal{M}_I$ is closed. Moreover, $\mathcal{M}_I$ has empty interior (why?). Consider the sequence $\{I_n\}_n$ of closed intervals $I_1 = \left[0, \frac{1}{2}\right]$, $I_2 = \left[\frac{1}{2}, 1\right]$, $I_3 = \left[0, \frac{1}{3}\right]$, $I_4 = \left[\frac{1}{3}, \frac{2}{3}\right]$, $I_5 = \left[\frac{2}{3}, 1\right]$, $I_6 = \left[0, \frac{1}{4}\right]$, etc... and let $\mathcal{M}_n := \mathcal{M}_{I_n}$. Then

$$\mathcal{M} := \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

is a meager set. By the previous corollary

$$C([0,1]) \neq \bigcup_{n=1}^{\infty} \mathcal{M}_n.$$

Any function $f \in C([0,1]) \setminus \left(\bigcup_{n=1}^{\infty} \mathcal{M}_n\right)$ is nowhere monotone. ∎

Exercise 198 (Nowhere differentiable functions)**.** Consider the metric space $C([0,1])$ with the sup metric and for every $n \in \mathbb{N}$ let

$$X_n := \{f \in C([0,1]) : \text{ there is } x \in [0,1] \text{ such that } |f(x) - f(y)| \le n|x-y| \text{ for all } y \in [0,1]\}.$$

(i) Fix $n \in \mathbb{N}$ and prove that each $f \in C([0,1])$ can be approximated by a zigzag function $g \in C([0,1])$ with very large slopes so that it does not belong to X_n and such that $d_\infty(f,g)$ is small.
(ii) Fix $n \in \mathbb{N}$ and prove that every open set $U \subset C([0,1])$ contains an open set that does not intersect X_n.
(iii) Prove that there exists a dense G_δ set in $C([0,1])$ that consists of nowhere differentiable functions.

Lecture 34, November 26, 2012.

3.3. Uniform Continuity.

Definition 199. Consider two metric spaces (X, d_X) and (Y, d_Y) and a function $f : E \to Y$, where $E \subset X$. The function f is said to be *uniformly continuous* if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_Y(f(x), f(x_0)) < \varepsilon$$

for all $x, x_0 \in E$ with $d_X(x, x_0) < \delta$.

Remark 200. To negate uniform continuity it is enough to find two sequences $\{x_n\}, \{z_n\} \subset E$ such that

$$\lim_{n\to\infty} d_X(x_n, z_n) = 0$$

and $d_Y(f(x_n), f(z_n)) \nrightarrow 0$ (so either the limit does not exist or it exists but it is not zero).

Example 201. The function $f(x) = x$, $x \in \mathbb{R}$, is uniformly continuous, while the function $g(x) = x^2$, $x \in \mathbb{R}$, is not. To see this, take $\varepsilon = \delta$ for the function f. To prove that g is not uniformly continuous, consider the two sequences $x_n = n + \frac{1}{n}$ and $z_n = n$. Then $x_n - z_n = \frac{1}{n} \to 0$, while

$$f(x_n) - f(z_n) = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n} \to 2 \neq 0,$$

which implies that g is not uniformly continuous, by the previous remark.

Definition 202. A function $\omega : [0,\infty) \to [0,\infty]$ is a modulus of continuity for function $f : X \to Y$ if

(i) $\omega(0) = 0$ and $\lim_{s\to 0} \omega(s) = 0$
(ii) For all $x, z \in X$

$$d_Y(f(x), f(z)) \le \omega(d_X(x,z)).$$

Exercise 203. Show that $f : X \to Y$ is uniformly continuous if and only if there exists a modulus of continuity for f.

Simple examples of uniformly continuous functions are Lipschitz and Hölder's continuous functions.

Definition 204. Consider two metric spaces (X, d_X) and (Y, d_Y) and a function $f : E \to Y$, where $E \subset X$.

(i) The function f is said to be *Lipschitz continuous* if there exists $L > 0$ such that
$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2)$$
for all $x_1, x_2 \in E$. The number
$$\operatorname{Lip}(f; E) := \sup_{x_1, x_2 \in E,\, x_1, \neq x_2} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \leq L$$
is called the *Lipschitz constant* of f. It is also denoted $\operatorname{Lip} f$. The function f is called a *contraction* if $\operatorname{Lip} f < 1$.

(ii) The function f is said to be *Hölder continuous with exponent* $\alpha \in (0, 1)$ if there exists $L > 0$ such that
$$d_Y(f(x_1), f(x_2)) \leq L (d_X(x_1, x_2))^a$$
for all $x_1, x_2 \in E$. Note that f is Hölder continuous if $\omega(s) = Ls^\alpha$ is a modulus of continuity.

Remark 205. Consider two metric spaces (X, d_X) and (Y, d_Y) and a function $f : E \to Y$, where $E \subset X$.

(i) If f is Lipschitz continuous with Lipschitz constant L, then to see that it is uniformly continuous, given $\varepsilon > 0$, it is enough to take $\delta = \frac{\varepsilon}{L}$. The function $f(x) = \sqrt{x}$, $x \in [0, 1]$, is uniformly continuous, but not Lipschitz. Indeed, in the case $X = Y = \mathbb{R}$, if f is Lipschitz, then
$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq L$$
for all $x, y \in E$, $x \neq y$. In particular, if f is differentiable at x, then letting $y \to x$ in the previous inequality gives,
$$|f'(x)| \leq L.$$
Hence, the derivative is bounded (where it exists). For $f(x) = \sqrt{x}$, $x \in [0, 1]$, we have that $f'(x) = \frac{1}{2\sqrt{x}}$ for $x \in (0, 1)$, which is not bounded. Similarly, if $X = \mathbb{R}^N$ and $Y = \mathbb{R}$, then if f is Lipschitz and admits a partial derivative $\frac{\partial f}{\partial x_i}$ at some point x, then $\left| \frac{\partial f}{\partial x_i}(x) \right| \leq L$.

(ii) If f is Hölder continuous with exponent $\alpha \in (0, 1)$ and constant $L > 0$, then to see that it is uniformly continuous, given $\varepsilon > 0$, it is enough to take $\delta = \left(\frac{\varepsilon}{L}\right)^{\frac{1}{\alpha}}$. Indeed, if $x_1, x_2 \in E$ with $d_X(x_1, x_2) < \left(\frac{\varepsilon}{L}\right)^{\frac{1}{\alpha}}$, then
$$d_Y(f(x_1), f(x_2)) \leq L (d_X(x_1, x_2))^a < L \left(\frac{\varepsilon}{L}\right)^{\frac{\alpha}{\alpha}} = \varepsilon.$$
The Weierstrass nowhere differentiable function is an example of a uniformly continuous function that is not Hölder continuous of *any* $\alpha \in (0, 1)$.

(iii) If f is Lipschitz continuous with Lipschitz constant $L > 0$ and if E is bounded, then f is Hölder continuous of any exponent $\alpha \in (0, 1)$. To see this, let $E \subset B_X(x_0, r)$. Then for all $x_1, x_2 \in E$, we have
$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq L d_X(x_1, x_2) = L (d_X(x_1, x_2))^a (d_X(x_1, x_2))^{1-a} \\ &\leq L (d_X(x_1, x_2))^a (2r)^{1-a}, \end{aligned}$$

where in the last inequality we have used the fact that $d_X(x_1, x_2) \leq d_X(x_1, x_0) + d_X(x_0, x_2) < r + r$, since $E \subset B_X(x_0, r)$. If E is unbounded, then this is no longer true. Indeed, the function $f(x) = x$, $x \in \mathbb{R}$, cannot be Hölder continuous of any exponent $\alpha \in (0, 1)$. To see this, take $x_1 = x > 0$ and $x_2 = 0$, then we cannot have an inequality of the type

$$x = |f(x) - f(0)| \leq Lx^a,$$

because as $x \to \infty$, x goes faster than x^a.

Proposition 206. *Let (X, d_X) be a metric spaces, let (Y, d_Y) be a complete metric spaces, and let $f : E \to Y$ be uniformly continuous, where $E \subset X$. Then f can be extended uniquely to a uniformly continuous function $g : \overline{E} \to Y$.*

Proof. Step 1: We begin by showing that if $\{x_n\} \subset E$ is a Cauchy sequence, then $\{f(x_n)\}$ is a Cauchy sequence in Y. Fix $\varepsilon > 0$. By the uniform continuity of f there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_Y(f(x'), f(x'')) < \varepsilon \tag{20}$$

for all $x', x'' \in E$ with $d_X(x', x'') < \delta$. Since $\{x_n\} \subset X$ is a Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_X(x_n, x_m) < \delta$$

for all $n, m \geq n_\varepsilon$, and so

$$d_Y(f(x_n), f(x_m)) < \varepsilon \tag{21}$$

for all $n, m \geq n_\varepsilon$, which shows that $\{f(x_n)\}$ is a Cauchy sequence in Y.
Step 2: For $x \in E$ define $g(x) := f(x)$. Fix $x \in \overline{E} \setminus E$. By Lemma 53 there exists a sequence $\{x_n\} \subset E$ with $x_n \neq x$ for all $n \in \mathbb{N}$ such that $\{x_n\}$ converges to x. In particular, $\{x_n\}$ is a Cauchy sequence, and so by the previous step $\{f(x_n)\}$ is a Cauchy sequence in Y. Since Y is complete, $\{f(x_n)\}$ converges to some element $y \in Y$. Note that for any continuous extension $h : \overline{E} \to Y$ of f to $\overline{E}$ we must have $h(x) = y$ (this will show uniqueness). Thus, we define $g(x) := y$. To make sure that g is well-defined, we need to verify that $g(x)$ does not depend on the particular sequence $\{x_n\}$ converging to x. Thus, let $\{z_n\} \subset E$ be another sequence converging to x. Then by the triangle inequality we have

$$d_X(x_n, z_n) \leq d_X(x_n, x) + d_X(x, z_n) < \frac{\delta}{2} + \frac{\delta}{2}$$

for all $n \in \mathbb{N}$ sufficiently large, say $n \geq n_1$. Hence, by (20),

$$d_Y(f(x_n), f(z_n)) < \varepsilon$$

for all $n \geq n_1$. Then, since $\{f(x_n)\}$ converges to y,

$$d_Y(y, f(z_n)) \leq d_Y(y, f(x_n)) + d_Y(f(x_n), f(z_n)) \leq \varepsilon + \varepsilon$$

for all sufficiently large, which shows that $\{f(x_n)\}$ converges to y. Thus, $g(x)$ is well-defined.
Step 3: It remains to show that g is uniformly continuous. Let $x', x'' \in \overline{E}$ be such that $d_X(x', x'') < \delta$ and consider two sequences $\{x'_n\}, \{x''_n\} \subset E$ converging to x' and x'', respectively. Then for all n sufficiently large we have that

$$d_X(x'_n, x''_n) \leq d_X(x'_n, x') + d_X(x', x'') + d_X(x'', x''_n) < \delta,$$

and so by (20),

$$d_Y\left(f\left(x_n'\right), f\left(x_n''\right)\right) < \varepsilon$$

for all n sufficiently large. Letting $n \to \infty$, we obtain

$$d_Y\left(g\left(x'\right), g\left(x''\right)\right) \leq \varepsilon,$$

which shows that g is uniformly continuous. ■

Let (X, d) be a metric space. If $x \in X$ and $E \subset X$ nonempty, the *distance* of x from the set E is defined by

$$\operatorname{dist}(x, E) := \inf\{d(x, y) : y \in E\},$$

while the distance between two sets E_1, $E_2 \subset X$ is defined by

$$\operatorname{dist}(E_1, E_2) := \inf\{d(x, y) : x \in E_1, y \in E_2\}.$$

Remark 207. While above we were able to extend g to $\overline{E}$ it may not be possible to extend g further. In particular it may not be possible to extend g to a continuous function on X. Can you think of an example?

Exercise 208. Let (X, d) be a metric space and let $E \subset X$ be a nonempty set.

(i) Fix $x_0 \in X$. Prove that the function

$$x \in X \mapsto d(x, x_0)$$

is Lipschitz continuous with Lipschitz constant one.

(iii) Prove that the distance function

$$x \in X \mapsto \operatorname{dist}(x, E)$$

is Lipschitz continuous with Lipschitz constant one.

(ii) Characterize the points $x \in X$ such that $\operatorname{dist}(x, E) = 0$.

Lecture 35, November 28, 2012.

3.4. Fixed Point Theorems. Let $\Phi : X \to X$. An element $x \in X$ is a *fixed point* of Φ is $\Phi(x) = x$.

Theorem 209 (Banach's contraction principle)**.** *Let (X, d) be a nonempty complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point; that is, there is a unique $x \in X$ such that $f(x) = x$.*

Proof. Step 1: Let us first prove uniqueness. Assume that x_1 and x_2 are fixed points of f. Then

$$d_X(x_1, x_2) = d_X(f(x_1), f(x_2)) \leq L d_X(x_1, x_2),$$

which implies that

$$(1 - L)\, d_X(x_1, x_2) \leq 0.$$

Since $L < 1$, we have that $d_X(x_1, x_2) = 0$, and so $x_1 = x_2$.

Step 2: To prove existence, fix $x_0 \in X$ and define inductively

$$x_1 := f(x_0), \quad x_{n+1} := f(x_n).$$

We claim that $\{x_n\}$ is a Cauchy sequence. Indeed, note that

$$d_X(x_1, x_2) = d_X(f(x_0), f(x_1)) \leq L d_X(x_0, x_1)$$

and by induction

$$d_X(x_n, x_{n+1}) = d_X(f(x_{n-1}), f(x_n)) \leq L^n d_X(x_0, x_1).$$

Hence, for every $m, n \in \mathbb{N}$, by the triangle inequality

$$\begin{aligned} d_X(x_n, x_{n+m}) &\le \sum_{i=n}^{n+m-1} d_X(x_i, x_{i+1}) \le d_X(x_0, x_1) \sum_{i=n}^{n+m-1} L^i \\ &\le d_X(x_0, x_1) \sum_{i=n}^{\infty} L^i = d_X(x_0, x_1) \frac{L^n}{1-L}. \end{aligned}$$

Letting $n \to \infty$, we have that $\{x_n\}$ is a Cauchy sequence. Since the space is complete, there exists $x \in X$ such that $\{x_n\}$ converges to x. But by the continuity of f,

$$x \leftarrow x_{n+1} = f(x_n) \to f(x),$$

which shows that $f(x) = x$. ■

An important application of Banach's contraction principle is the existence of solutions of ODE. Consider the initial value problem

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(0) &= u_0. \end{aligned}$$

Let $I = [0, \tau]$ for some $\tau > 0$. Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and that f satisfies the following Lipschitz condition

$$|f(t, z_1) - f(t, z_2)| \le L |z_1 - z_2|$$

for all $t \in I$, $z_1, z_2 \in \mathbb{R}$. Then we can prove short time existence of solutions. Consider the space $X = C([0, T])$, where T will be chosen later and consider the operator

$$F : C([0, T]) \to C([0, T])$$

given by

$$F(g)(t) = u_0 + \int_0^t f(s, g(s))\, ds$$

for $g \in C([0, T])$ and $t \in [0, T]$. It is clear that F is well-defined, since the function on the right-hand side is continuous.

Let's prove that F is a contraction. Take $g_1, g_2 \in C([0, T])$. Then

$$\begin{aligned} |F(g_1)(t) - F(g_2)(t)| &= \left| \int_0^t [f(s, g_1(s)) - f(s, g_2(s))]\, ds \right| \\ &\le \int_0^t |f(s, g_1(s)) - f(s, g_2(s))|\, ds \\ &\le L \int_0^t |g_1(s) - g_2(s)|\, ds \le L \max_{y \in [0, 0+T]} |g_1(y) - g_2(y)| \int_0^t ds \\ &\le LT d_\infty(g_1, g_2), \end{aligned}$$

and so taking the maximum over all $t \in [0, T]$, we get

$$d_\infty(F(g_1), F(g_2)) \le LT d_\infty(g_1, g_2).$$

If we take T so small that $LT < 1$ and $T \le \tau$, then F is a contraction. By Banach's contraction principle there exists a unique fixed point $u \in C([0, T])$, that is,

$$u(t) = F(u)(t) = u_0 + \int_0^t f(s, u(s))\, ds$$

for all $t \in [0, T]$. Since u is continuous, the right-hand side is of class C^1, and so u is actually of class C^1. By differentiating both sides, we get that u is a solution of the ODE. Moreover, $u(0) = u_0$. Since any other solution of the initial value problem is a fixed point of F, we have uniqueness.

To show existence on I (if $T < \tau$), note that T does not depend on the initial data. Thus we can apply the argument above to the problem on the time interval $[T, \min\{2T, \tau\}]$ with initial data $u(T)$. This argument can be iterated if needed until τ is reached. Note furthermore that argument can be extended to $I = [0, \infty)$.

Lecture 36, November 30, 2012.

Theorem 210 (Brouwer's fixed point theorem)**.** *Let $E \subset \mathbb{R}^N$ be a set homeomorphic to $\overline{B} = \overline{B}(0, 1)$. Let $\Phi : E \to E$ be a continuous function. Then there exists $x \in E$ such that $\Phi(x) = x$.*

Exercise 211. Show that the Brouwer's fixed point theorem holds if E is taken to be a bounded closed convex set. [Hint: By induction on the dimension. If $E = \overline{E^\circ}$ then show that E is homeomorphic to a ball. Otherwise show that E is contained in a plane and thus that the dimension of the problem can be reduced.]

Proof. Since E is homeomorphic to a ball there exists a homeomorphism $h : E \to \overline{B}$. Consider $\Psi : \overline{B} \to \overline{B}$ given by $\Psi(x) = h(\Phi(h^{-1}(x)))$. Note that Ψ is continuous and that it suffices to show that Ψ has a fixed point.

Step 1. Assume that Ψ does not have a fixed point. Let us construct a smooth perturbation on Ψ that also has no fixed points. We first extend Ψ to all of $\mathbb{R}^n$ by

$$\Psi(x) = \begin{cases} \Psi(x) & \text{if } x \in \overline{B} \\ \Psi(x/|x|) & \text{if } |x| > 1. \end{cases}$$

Note that the extended Ψ is still continuous. Let $L = \min_{\overline{B}} |\Psi(x) - x| > 0$. Since Ψ is uniformly continuous on $\overline{B}(0, 2)$ there exists $\delta \in (0, 1)$ such that for all $x, y \in \overline{B}(0, 2)$ if $|x - y| < \delta$ then $\Psi(x) - \Psi(y) < L/2$.

Let $\eta : \mathbb{R}^n \to [0, \infty)$ be smooth, such that its support is contained in $B(0, \delta)$ and that $\int_{R^n} \eta(x)dx = 1$. Let $\tilde{\Psi} = \Psi * \eta$. Note that $\tilde{\Psi}$ is smooth by the properties of the convolution. Since $\overline{B}$ is convex, $\tilde{\Psi}(x) = \int_{B(0,\delta)} \Psi(x - z)\eta(z)dz$ belongs to $\overline{B}$ for all x. Furthermore notice that for all $x \in \overline{B}$ and $z \in B(0, \delta)$, $\Psi(x - z) \in B(\Psi(x), L/2)$ therefore $\tilde{\Psi}(x) \in B(\Psi(x), L/2)$. Hence for all $x \in \overline{B}$, $|\tilde{\Psi}(x) - x| \geq |\Psi(x) - x| - L/2 \geq L/2$. We conclude that $\tilde{\Psi}$ has no fixed points.

Step 2.[Constructing the retraction.] For any $x \in \overline{B}$ consider the line passing through x and $\tilde{\Psi}(x)$. It intersects $\partial B(0, 1)$ at two points: y_1 and y_2. Let y be the one of the two for which $(y - x) \cdot (\tilde{\Psi}(x) - x) \leq 0$. Let $f : \overline{B} \to \partial B$ be the mapping that maps x to y. It is elementary to verify that f is smooth and that for $x \in \partial B$ $f(x) = x$.

Step 3.[The argument via the Stokes theorem] Note that for all $x \in B$ and all $j = 1, \dots, n$ taking partial derivative of $f \cdot f = 1$ gives $\partial_j f \cdot f = 0$ which implies that $\partial_j f$ belong to the hyperplane orthogonal to $f(x)$. Thus $\partial_1 f, \dots, \partial_n f$ are not linearly independent. Hence $det Df(x) = 0$ for all $x \in B$ Therefore

$$0 = \int_B det Df(x)dx. \tag{22}$$

For simplicity we only carry out the rest of the proof for $n = 2$. Let $f = (f^1, f^2)$. By divergence theorem

$$\int_B detDf(x)dx = \int_B \partial_1 f^1 \partial_2 f^2 - \partial_2 f^1 \partial_1 f^2 dx = \int_{\partial B} (f^1 \partial_2 f^2 \nu_1 - f^1 \partial_1 f^2 \nu_2) dS$$

where $\nu = (\nu_1, \nu_2)$ is the unit outside normal vector to ∂B. Note that the expression on the right hand side depends on f^1 only via its restriction to ∂B. That is the right-hand side would not change if one replaced f^1 by some other function having the same value on ∂B. By symmetry (exchange the indexes 1 and 2) the same is true for f^2. Thus for $\tilde{f}(x) = x$, which satisfies that $\tilde{f} = f$ on ∂B it holds that

$$\int_{\partial B} (f^1 \partial_2 f^2 \nu_1 - f^1 \partial_1 f^2 \nu 2) dS = \int_{\partial B} (\tilde{f}^1 \partial_2 \tilde{f}^2 \nu_1 - \tilde{f}^1 \partial_1 \tilde{f}^2 \nu_2) dS.$$

Using the divergence theorem as above we conclude that

$$\int_{\partial B} (\tilde{f}^1 \partial_2 \tilde{f}^2 \nu_1 - \tilde{f}^1 \partial_1 \tilde{f}^2 \nu_2) dS = \int_B \det D\tilde{f} dx = \int_B 1 dx > 0.$$

This contradicts (22). ■

Lecture 37, Dec. 3, 2012.

3.5. Compactness in metric spaces.

Definition 212. Metric space (X, d) is *totally bounded* if for every $\varepsilon > 0$ there exist $x_1, \ldots, x_m \in X$ such that

$$X = \bigcup_{i=1}^{m} B(x_i, \varepsilon).$$

A set $E \subset X$ is bounded if $\sup\{d(x,y) \ \ x, y \in E\} < \infty$.

Exercise 213. Let (X, d) be a compact metric space. Prove that X is separable, complete and bounded. [Hint: Show that if the space is not separable then it has a countable subset where every point is isolated.]

Example 214. Closed and bounded sets do not have to be compact in general. Consider $l^2(\mathbb{N})$. Let $e_n \in l^2(\mathbb{N})$ be the sequence whose elements are all zero except for the n-th element which is equal to 1. Let $E = \{e_k \ : \ k \in \mathbb{N}\}$. Note that $d(e_k, e_n) = \sqrt{2}$ if $k \neq n$, so E is bounded. Each point of E is an isolated point, so it is closed. However E is not compact since $U_n = B(e_n, 1/2)$ is an open cover that has no finite subcover.

The following theorem is one of the main results of this subsection.

Theorem 215. *Let (X, d) be a metric space. The the following are equivalent.*

(i) *X is sequentially compact.*
(ii) *X is complete and totally bounded.*
(iii) *X is compact.*

Proof of Theorem 215. (i)⇒(ii) Assume that X is sequentially compact. We claim that (X, d) is complete. To see this, let $\{x_n\} \subset X$ be a Cauchy sequence. Since X is sequentially compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in X$ such that $x_{n_k} \to x$ as $k \to \infty$. Since $\{x_n\} \subset X$ is Cauchy sequence, for every fixed $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x_m) \leq \frac{\varepsilon}{2} \tag{23}$$

for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$. On the other hand, since $x_{n_k} \to x$ as $k \to \infty$, there exists exists $k_\varepsilon \in \mathbb{N}$ such that

$$d\left(x_{n_k}, x\right) \leq \frac{\varepsilon}{2} \tag{24}$$

for all $k \in \mathbb{N}$ with $k \geq k_\varepsilon$. Fix $k \in \mathbb{N}$ so large that $n_k \geq \max\{n_\varepsilon, n_{k_\varepsilon}\}$. Then for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ we have that

$$d\left(x_n, x\right) \leq d\left(x_n, x_{n_k}\right) + d\left(x_{n_k}, x\right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

which implies that $d\left(x_n, x\right) \to 0$ as $n \to \infty$. Hence, (X, d) is complete.

Next we show that X is totally bounded. Assume by contradiction that X is not totally bounded. Then there exists $\varepsilon_0 > 0$ such that X cannot be covered by a finite number of balls of radius ε_0. Fix $x_1 \in X$. Then there exists $x_2 \in X$ such that $d\left(x_1, x_2\right) \geq \varepsilon_0$ (otherwise $B\left(x_1, \varepsilon_0\right)$ would cover X). Similarly, we can find $x_3 \in X$ such that $d\left(x_1, x_3\right) \geq \varepsilon_0$ and $d\left(x_2, x_3\right) \geq \varepsilon_0$ (otherwise $B\left(x_1, \varepsilon_0\right)$ and $B\left(x_2, \varepsilon_0\right)$ would cover X). Inductively, construct a sequence $\{x_n\} \subset X$ such that $d\left(x_n, x_m\right) \geq \varepsilon_0$ for all $n, m \in \mathbb{N}$ with $n \neq m$. The sequence $\{x_n\}$ cannot have a convergent subsequence, which contradicts the fact that X is sequentially compact.

(ii)$\Rightarrow$(i) Assume that X is complete and totally bounded. Let $\{x_n\} \subset X$. We want to prove that a subsequence of $\{x_n\}$ is a Cauchy sequence and then use the completeness of X. For every $k \in \mathbb{N}$ let $\mathcal{B}_k$ be a finite cover of X with balls of radius $\frac{1}{2^k}$ and centers in X. Since $\mathcal{B}_1$ covers X there exists a ball $B_1 \in \mathcal{B}_1$ such that $x_n \in B_1$ for infinitely many $n \in \mathbb{N}$. Since $\mathcal{B}_2$ covers B_1 there exists a ball $B_2 \in \mathcal{B}_2$ such that $x_n \in B_1 \cap B_2$ for infinitely many $n \in \mathbb{N}$. Inductively, for every $k \in \mathbb{N}$ we may find a ball $B_k \in \mathcal{B}_k$ such that $x_n \in B_1 \cap \cdots \cap B_k$ for infinitely many $n \in \mathbb{N}$.

Let $n_1 \in \mathbb{N}$ be the first of the integers $n \in \mathbb{N}$ such that $x_n \in B_1$, let $n_2 \in \mathbb{N}$ be the first of the integers $n \in \mathbb{N}$ such that $n > n_1$ and $x_n \in B_1 \cap B_2$. Inductively, for every $k \in \mathbb{N}$ let $n_k \in \mathbb{N}$ be the first of the integers $n \in \mathbb{N}$ such that $n > n_{k-1}$ and $x_n \in B_1 \cap \cdots \cap B_k$. We claim that the subsequence $\{x_{n_k}\}$ is a Cauchy sequence. Indeed, if $k, \ell \in \mathbb{N}$ with $k, \ell \geq m$, then $x_{n_k}, x_{n_\ell} \in B_m$, and so

$$d\left(x_{n_k}, x_{n_\ell}\right) \leq \frac{1}{2^m} \to 0$$

as $m \to \infty$. Thus, $\{x_{n_k}\}$ is a Cauchy sequence and since X is complete, $\{x_{n_k}\}$ converges to a point in X.

(i)+(ii)$\Rightarrow$(iii) Assume that X is complete and totally bounded. Let $\{U_\alpha\}$ be a collection of open sets such that $\bigcup_\alpha U_\alpha = X$. As in the previous part, for every $k \in \mathbb{N}$ let $\mathcal{B}_k$ be a finite cover of X with balls of radius $\frac{1}{2^k}$ and centers in X. We want to prove that there exists $\overline{k} \in \mathbb{N}$ such that every ball in $\mathcal{B}_{\overline{k}}$ is contained in some U_α. Note that this would conclude the proof. Indeed, for every $B \in \mathcal{B}_{\overline{k}}$ fix one U_α containing B. Since $\mathcal{B}_{\overline{k}}$ is a finite family and covers X, the subcover of $\{U_\alpha\}$ just constructed has the same properties.

To find $\overline{k}$, assume by contradiction that for every $k \in \mathbb{N}$ there exists a ball $B\left(x_k, \frac{1}{2^k}\right) \in \mathcal{B}_k$ that is not contained in any U_α. Since $\{x_k\} \subset X$, there exist a subsequence $\left\{x_{k_j}\right\}$ of $\{x_k\}$ and $x \in X$ such that $x_{k_j} \to x$ as $j \to \infty$. Since $\bigcup_\alpha U_\alpha = X$, there exists a such that $x \in U_\alpha$. Since U_α is an open set, there exists $r > 0$ such that $B\left(x, r\right) \subset U_\alpha$. Let $j \in \mathbb{N}$ be so large that $d\left(x_{k_j}, x\right) < \frac{r}{2}$ and

$\frac{1}{2^{k_j}} < \frac{r}{2}$. Then

$$B\left(x_{k_j}, \frac{1}{2^{k_j}}\right) \subset B(x, r) \subset U_\alpha,$$

which is a contradiction.

(iii)⇒(i) This follows from Lemma 105. ■

Lecture 38, Dec. 5, 2012.

Theorem 216. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Assume X is compact, and let $f : X \to Y$ be a continuous function. Then f is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. By continuity for every $x \in X$ there exists $\delta_x = \delta_x(\varepsilon) > 0$ such that

$$d_Y(f(x), f(z)) < \varepsilon \tag{25}$$

for all $z \in X$ with $d_X(x, z) < \delta_x$. The family $\left\{B\left(x, \frac{\delta_x}{2}\right)\right\}_{x \in X}$ is an open cover for the compact set X, and so we may find a finite cover $x_1, \ldots, x_m$ such that

$$X \subset \bigcup_{i=1}^{m} B\left(x_i, \frac{\delta_{x_i}}{2}\right).$$

Let

$$\delta := \min_{i=1,\ldots,m} \frac{\delta_{x_i}}{2} > 0.$$

Let $x, z \in X$ be such that $d_X(x, z) < \delta$. Since $X \subset \bigcup_{i=1}^{m} B\left(x_i, \frac{\delta_{x_i}}{2}\right)$, there exists we may find x_i such that $x \in B\left(x_i, \frac{\delta_{x_i}}{2}\right)$, and so by the triangle inequality

$$d_X(z, x_i) \le d_X(z, x) + d_X(x, x_i) < \delta + \frac{\delta_{x_i}}{2} \le \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i},$$

which shows that $z \in B(x_i, \delta_{x_i})$. Hence, by (25),

$$d_Y(f(x), f(z)) \le d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(z)) < 2\varepsilon.$$

■

3.6. The Ascoli–Arzela Theorem. Next we show that in an arbitrary metric space, closed and bounded sets are not compact.

Example 217. Let $X := C([0,1])$. The sequence of functions

$$f_n(x) = x^n, \quad x \in [0,1]$$

is bounded in $C([0,1])$, but no subsequence converges uniformly to a continuous function. This shows that $\overline{B_X(0,1)}$ is closed and bounded but not compact. Hence, Bolzano–Weierstrass theorem fails for infinite dimensional metric spaces.

Lecture 21, Oct. 24, 2012.

Definition 218. Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{F}$ of functions $f : X \to Y$ is said to be *equicontinuous at a point* $x_0 \in X$ if for every $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$d_Y(f(x), f(x_0)) \le \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x \in X$ with $d(x, x_0) \leq \delta$. The family $\mathcal{F}$ of functions $f : X \to Y$ is said to be *(uniformly) equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(x), f(y)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d(x, y) \leq \delta$.

Definition 219. Let (X, d) be a metric space. A family $\mathcal{F}$ of functions $f : X \to \mathbb{R}$ is said to be *pointwise bounded* if for every $x \in X$ there exists $M_x > 0$ such that

$$|f(x)| \leq M_x$$

for all $f \in \mathcal{F}$.

Example 220. The sequence of functions

$$f_n(x) = x^n, \quad x \in [0, 1],$$

is pointwise bounded but not equicontinuous at $x = 1$. To see this, fix $0 < \varepsilon < 1$. We want to find $\delta > 0$ such that $1 - x^n \leq \varepsilon$ for all $1 - \delta \leq x < 1$. We have $(1-\varepsilon)^{1/n} \leq x$. So for each n the best δ is $1 - \delta_n = (1-\varepsilon)^{1/n}$, that is,compact $\delta_n = 1 - (1-\varepsilon)^{1/n} \to 0$ as $n \to \infty$. Hence, no δ works for all n.

Example 221. Consider two metric spaces (X, d_X) and (Y, d_Y) and a family $\mathcal{F}$ of functions from X into Y. If there exist $\alpha \in (0, 1]$ if there exists $L > 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq L(d_X(x_1, x_2))^a$$

for all $x_1, x_2 \in X$ and for all $f \in \mathcal{F}$, then the family $\mathcal{F}$ is equicontinuous. The sequence of functions

$$f_n(x) = \frac{x^n}{n}, \quad x \in [0, 1],$$

is pointwise bounded and equicontinuous at $x = 1$. Indeed,

$$f_n'(x) = x^{n-1}, \quad x \in [0, 1],$$

so that $\max_{x \in [0,1]} \left|x^{n-1}\right| = 1$, which shows that the sequence $\{f_n\}$ is equi-Lipschitz (take $L = 1$). Hence, it is equicontinuous.

Theorem 222 (Ascoli–Arzelà)**.** *Let (X, d) be a separable metric space and let $\mathcal{F} \subset C(X)$ be a family of functions. Assume that $\mathcal{F}$ is pointwise bounded and equicontinuous. Then every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of X to a continuous function $g : X \to \mathbb{R}$.*

Proof. Without loss of generality, we may assume that $\mathcal{F}$ has infinite many elements, otherwise there is nothing to prove. Since X is separable, there exists a countable set $E \subset X$ such that $X = \overline{E}$.

Step 1: Let $\mathcal{G} \subset \mathcal{F}$ be an infinite set. We claim that $\mathcal{G}$ contains a sequence $\{f_n\}$ such that the limit $\lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$ for all $x \in E$. The proof makes use of the *Cantor diagonal argument*. Write $E = \{x_k\}_k$. Since the set

$$\{f(x_1) : f \in \mathcal{G}\}$$

is bounded in $\mathbb{R}$, by the Bolzano–Weierstrass theorem we can find a sequence $\{f_{n,1}\}_n \subset \mathcal{G}$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,1}(x_1) = \ell_1 \in \mathbb{R}.$$

Since the set

$$\{f_{n,1}(x_2) : n \in \mathbb{N}\}$$

is bounded in $\mathbb{R}$, again by the Bolzano–Weierstrass theorem we can find a sequence $\{f_{n,2}\}_n \subset \{f_{n,1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,2}(x_2) = \ell_2 \in \mathbb{R}.$$

By induction for every $k \in \mathbb{N}$, $k > 1$, we can find a subsequence $\{f_{n,k}\}_n \subset \{f_{n,k-1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,k}(x_k) = \ell_k \in \mathbb{R}.$$

We now consider the diagonal elements of the infinite matrix, that is, the sequence $\{f_{n,n}\}_n$. For every fixed $x_k \in E$ we have that the sequence $\{f_{n,n}(x_k)\}_{n=k}^{\infty}$ is a subsequence of $\{f_{n,k}(x_k)\}_n$, and thus it converges to ℓ_k as $n \to \infty$. This completes the proof of the claim. Set $f_n := f_{n,n}$ and define $g : E \to \mathbb{R}$ by

$$g(x) := \lim_{n\to\infty} f_n(x) \in \mathbb{R}, \quad x \in E. \tag{26}$$

Step 2: Let $K \subset X$ be compact and fix $\varepsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \varepsilon \tag{27}$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d(x,y) \le \delta$. Since K is compact, we may cover it with a finite number of balls $B\left(y_1, \frac{\delta}{2}\right), \ldots, B\left(y_M, \frac{\delta}{2}\right)$. Since E is dense, for every $i = 1, \ldots, M$ there exists $z_i \in B\left(y_i, \frac{\delta}{2}\right) \cap E$. Using (26), we have that there exists an integer $n_\varepsilon \in \mathbb{N}$ such that

$$|f_n(z_i) - f_m(z_i)| \le \varepsilon \tag{28}$$

for all $i = 1, \ldots, M$ and for all $n, m \in \mathbb{N}$ with $n, m \ge n_\varepsilon$. Fix $x \in K$. Then x belongs to $B\left(y_i, \frac{\delta}{2}\right)$ for some i. In particular,

$$d(x, z_i) \le d(x, y_i) + d(y, z_i) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Using (27) and (28), we have that

$$|f_n(x) - f_m(x)| \le |f_n(x) - f_n(z_i)| + |f_n(z_i) - f_m(z_i)| + |f_m(z_i) - f_m(x)| \le \varepsilon + \varepsilon + \varepsilon$$

for all $n, m \in \mathbb{N}$ with $n, m \ge n_\varepsilon$, which shows that the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence, there exists

$$g(x) := \lim_{n\to\infty} f_n(x) \in \mathbb{R}.$$

Moreover, since

$$|f_n(x) - f_m(x)| \le 3\varepsilon$$

for all $x \in K$ and all $n, m \in \mathbb{N}$ with $n, m \ge n_\varepsilon$, letting $m \to \infty$, we conclude that

$$|f_n(x) - g(x)| \le 3\varepsilon$$

for all $x \in K$ and all $n \in \mathbb{N}$ with $n \ge n_\varepsilon$, or, equivalently,

$$\sup_{x\in X} |f_n(x) - g(x)| \le 3\varepsilon$$

for all $n \in \mathbb{N}$ with $n \ge n_\varepsilon$, which shows that $\{f_n\}$ converges to g uniformly on X. In turn, g restricted to X is continuous.

Step 3: Is g defined everywhere? Yes, for every $x \in X$, take K to be the singleton $\{x\}$. Is g continuous? Yes, this follows from (27). ■