Homework 2

21-630 Ordinary Differential Equations
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Problem 1

A) For $T > 1$, let $X, Y \in \mathcal{C}[0,T]$ be the constant functions 1 and 0, respectively. Then,

$$\|\mathcal{F}[X] - \mathcal{F}[Y]\|_{\mathcal{C}} \geq |\mathcal{F}[X](T) - \mathcal{F}[Y](T)| = \left| \int_0^T 1\, ds \right| = T > 1 > C\|X - Y\|_{\mathcal{C}},$$

for any $C \in (0,1)$. Thus, $\mathcal{F}$ is not a contraction. $\blacksquare$

B) We first show, by induction on n, that, $\forall n \in \mathbb{N}, t \in [0,T]$,

$$|X^{(n+1)}(t) - X^{(n)}(t)| \leq \|g\| \frac{t^n}{n!}.$$

For $n = 0$, $\forall t \in [0,T]$,

$$\left| X^{(n+1)}(t) - X^{(n)}(t) \right| = \left| X^{(1)} \right| = \left| g(t) + \int_0^t 0\, ds \right| \leq \|g\| = \|g\| \frac{t^n}{n!}.$$

since $t^0 = 0! = 1$. Supposing now that the conclusion holds for some $n \in \mathbb{N}$, $\forall t \in [0,T]$,

$$\left| X^{(n+2)}(t) - X^{(n+1)}(t) \right| \leq \int_0^t \left| X^{(n+1)} - X^{(n)} \right| ds \leq \int_0^t \|g\| \frac{s^n}{n!}\, ds = \|g\| \frac{t^{n+1}}{(n+1)!},$$

concluding the proof by induction. Thus, by the Triangle Inequality, $\forall n, k \in \mathbb{N}, t \in [0,T]$,

$$|X^{(n+k)} - X^{(n)}| = \left| \sum_{l=n}^{n+k-1} X^{(n+l+1)} - X^{(n+l)} \right| \leq \sum_{l=n}^{n+k-1} \left| X^{(n+l+1)} - X^{(n+l)} \right| \leq \sum_{l=n}^{\infty} \|g\| \frac{t^l}{l!}.$$

$\sum_{l=0}^{\infty} \frac{t^l}{l!}$ converges, so $X^{(n)}$ is uniformly Cauchy and thus uniformly convergent on $[0,T]$. $\blacksquare$

Problem 2

We first show by induction on n that, $\forall n \in \mathbb{N}$, $\forall t \in [0, \infty)$,

$$X^{(2n+1)}(t) = t^2$$
$$\text{and } X^{(2n+2)}(t) = -t^2.$$

For $n = 0$, $\forall t \in [0, \infty)$,

$$X^{(2n+1)}(t) = \int_0^t f(s, X^{(0)}(s))\, ds = \int_0^t f(s, 0)\, ds = \int_0^t 2s\, ds = s^2 \bigg|_{s=0}^{s=t} = t^2,$$

$$X^{(2n+2)}(t) = \int_0^t f(s, X^{(1)}(s))\, ds = \int_0^t f(s, s^2)\, ds = \int_0^t -2s\, ds = -s^2 \bigg|_{s=0}^{s=t} = -t^2,$$

as desired. If we suppose now that the conclusion holds for some $n \in \mathbb{N}$, then, $\forall t \in [0, \infty)$,

$$X^{(2(n+1)+1)}(t) = \int_0^t f(s, X^{(2n+2)}(s))\, ds = \int_0^t f(s, -s^2)\, ds = \int_0^t 2s\, ds = s^2 \bigg|_{s=0}^{s=t} = t^2,$$

$$X^{(2(n+1)+2)}(t) = \int_0^t f(s, X^{(2(n+1)+1)}(s))\, ds = \int_0^t f(s, s^2)\, ds = \int_0^t -2s\, ds = -s^2 \bigg|_{s=0}^{s=t} = -t^2,$$

concluding the proof by induction.

$\{X^{(n)}\}_{n=0}^{\infty}$ has two constant, and thus convergent, subsequences: $\{X^{(2n+1)}\}_{n=0}^{\infty} = \{t \mapsto t^2\}_{n=0}^{\infty}$ and $\{X^{(2n+2)}\}_{n=0}^{\infty} = \{t \mapsto -t^2\}_{n=0}^{\infty}$, with limits $X, Y : [0, \infty) \to \mathbb{R}$ defined $\forall t \in [0, \infty)$ by $X(t) = t^2$ and $Y(t) = -t^2$, respectively. However, neither X nor Y satisfies the differential equation: $\forall t \in (0, \infty)$,

$$\frac{dX}{dt}(t) = 2t > -2t = f(t, X(t)),$$
$$\text{and } Y(t) = -2t < 2t = f(t, Y(t)). \quad \blacksquare$$

Problem 3

A) Let $B = 1/4 + \|g\| < (0, 1/2)$, define $\mathcal{C}_B := \{X \in \mathcal{C} : \|X\|_{\mathcal{C}} \leq B\}$, and define $\mathcal{F} : \mathcal{C}_B \to \mathcal{C}$ by $\mathcal{F}[X](t) = g(t) + \int_0^1 X^2(s)\,ds$, $\forall X \in t \in [0,1]$. We show first that the image $\mathcal{F}[\mathcal{C}_B] \subseteq \mathcal{C}_B$ and then that $\mathcal{F}$ is a contraction on $\mathcal{C}_B$. Then, the existence of $X \in \mathcal{C}_b$ with the desired property (i.e., being a fixed point of $\mathcal{F}$) follows from the Contraction Mapping Theorem (noting that any limit of a sequence of functions bounded by B is itself bounded by B, so that $\mathcal{C}_B$ is closed).

Suppose $X \in \mathcal{C}_b$. Clearly, $\mathcal{F}[X] \in \mathcal{C}$, so it suffices to show that $\|\mathcal{F}[X]\|_{\mathcal{C}} \leq B$:

$$\begin{aligned}
\|\mathcal{F}[X]\|_{\mathcal{C}} &= \sup_{t\in[0,1]} \left| g(t) + \int_0^1 X^2(s)\,ds \right| \\
&\leq \sup_{t\in[0,1]} |g(t)| + \int_0^1 \left|X^2(s)\right|\,ds \\
&= \|g(t)\| + \int_0^1 \|X\|_{\mathcal{C}}^2\,ds \leq \|g\| + B^2 \leq B,
\end{aligned}$$

since $B^2 < 1/4$. We now show that $\mathcal{F}$ is a contraction. Suppose $X, Y \in \mathcal{C}_B$. Then,

$$\begin{aligned}
\|\mathcal{F}[X] - \mathcal{F}[Y]\|_{\mathcal{C}} &= \left| \int_0^1 X^2(s) - Y^2(s)\,ds \right| \leq \int_0^1 |(X(s) + Y(s))(X(s) - Y(s))|\,ds \\
&\leq \int_0^1 \|X + Y\|\|X - Y\|\,ds \leq \|X + Y\|\|X - Y\|\,ds \\
&\leq (\|X\| + \|Y\|)\,\|X - Y\|\,ds \leq 2B\|X - Y\|,
\end{aligned}$$

so that, since $2B \in (0,1)$, $\mathcal{F}$ is a contraction, as desired. ■

B) Note that, $\forall t \in [0,1]$, $\boxed{X(t) = g(t) + c,}$ where $c = \int_0^1 X^2(s)\,ds$ does not vary with t. Thus,

$$c = \int_0^1 X^2(s)\,ds = \int_0^1 (g(t) + c)^2\,ds = k_2 + 2ck_1 + c^2,$$

where $k_1 = \int_0^1 g(s)\,ds$ and $k_2 = \int_0^1 g^2(s)\,ds$ are constants. The quadratic formula then gives

$$\boxed{c = \frac{1}{2} - k_1 \pm \sqrt{k_1^2 - k_1 + \frac{1}{4} - k_2}.}$$

Note that, since $k_2 \leq k_1^2$ and $k_1 < 1/4$, $0 < k_1^2 - k_1 + \frac{1}{4} - k_2$, and thus the possible values of c give rise to $\boxed{\text{two distinct real solutions.}}$

Problem 4

A) $\forall n \in \mathbb{N}$, $X^{(n)}(0) = 0 < 1$ and, $\forall t \in (0,1]$, $X^{(n)}(t) = \frac{t^2}{t^2+(1-nt)^2} \leq \frac{t^2}{t^2} = 1$, so $X^{(n)}$ is uniformly bounded on $[0,1]$. $\blacksquare$

B) Suppose, for sake of contradiction, that $X^{(n)}$ is equicontinuous on $[0,1]$, so that, for $\varepsilon = 1/2$, $\exists \delta > 0$ such that, $\forall n \in \mathbb{N}, s, t \in [0,1]$ with $|t-s| < \delta$, $|X^{(n)}(t) - X^{(n)}(s)| < \varepsilon$. Then, however, for $s = 0$, $t = \left(2\lceil \delta^{-1} \rceil\right)^{-1} \in (0,\delta)$, $n = 1/t \in \mathbb{N}$, $|t-s| < \delta$, but

$$|X^{(n)}(t) - X^{(n)}(s)| = \left| \frac{t^2}{t^2+(1-nt)^2} - 0 \right| = \left| \frac{t^2}{t^2} \right| = 1 \geq 1/2 = \varepsilon,$$

which is a contradiction. $\blacksquare$

Problem 5

Let $\varepsilon > 0$ be given, and choose $\delta := \left(\frac{\varepsilon}{3000}\right)^3$. Then, $\forall n \in \mathbb{N}$, since $X^{(n)}$ is continuously differentiable, $\forall t, s \in [0,1]$ with $|t-s| < \delta$ (without loss of generality, $s \leq t$),

$$\begin{aligned}
\left| X^{(n)}(t) - X^{(n)}(s) \right| &= \left| \int_s^t \frac{dX^{(n)}}{dx}(x)\, dx \right| && \text{(Fundamental Theorem of Calculus)} \\
&\leq \int_s^t \left| \frac{dX^{(n)}}{dx}(x) \right| dx && \text{(Triangle Inequality)} \\
&\leq \int_s^t 1000x^{-2/3}\, dx && \text{(given bound)} \\
&= 3000x^{1/3} \Big|_{x=s}^{x=t} \\
&\leq 3000(t-s)^{1/3} && \text{(concavity of cube root)} \\
&< 3000\delta^{1/3} = \varepsilon. \quad \blacksquare
\end{aligned}$$