Questions for the Final Exam

1. π-systems and uniqueness of extensions of σ-finite measures (Dynkin's lemma).
2. Carathéodory's Extension Theorem. Lebesgue measure on $((0,1], \mathcal{B}(0,1])$.
3. First Borel-Cantelli Lemma.
4. Distribution function. Existence of RV with given DF. Skorohod's construction.
5. Monotone-Class Theorem.
6. Lemma on independence of σ-algebras, generated by families of independent events.
7. Second Borel-Cantelli Lemma.
8. Tail σ-algebra. Kolmogorov's 0-1 law.
9. Definition of $\mu(f) := \int f d\mu$.
10. Convergence theorems (Monotone, Fatou, Dominated, Scheffe, Bounded).
11. Markov's inequality, Chebyshev's inequality, Jensen's inequality for convex functions.
12. $\mathcal{L}^p$ space for $p \geq 1$: monotonicity of norms, completeness, Hölder inequality, Minkowski's inequality.
13. $\mathcal{L}^2$ space: Schwartz inequality, Covariation and correlation, Parallelogram law, Pythagorean theorem, Orthogonal Projection.
14. Product σ-algebra. Product measure. Fubini's Theorem. Formula for expected value as integral of $\mathbb{P}[X \geq x]$.
15. Canonical model for sequence of IRVs.
16. Conditional expectation in $\mathcal{L}_2$ as best least square predictor. Conditional expectation in $\mathcal{L}_1$.

17. Properties of conditional expectation: Linearity, Positivity, (cMON), (cFATOU), (cDOM), (cJENSEN), Tower Property, (Taking out what is known), Rôle of independence.

18. Regular conditional probabilities.

19. Martingale, submartingale, supermartingale. Examples: sums and products of IRVs.

20. Previsible or predictable process. Martingale transform or discrete stochastic integral. Lemma: martingale transform of a martingale is a martingale.

21. Stopping time. Stopped supermartingales are supermartingales. Doob's Optional Stopping Theorem. Hitting times for simple random walk.

22. Doob's Upcrossing Lemma and "Forward" Convergence Theorem for submartingales.

23. Convergence of $\mathcal{L}^2$-martingales.

24. Sums of IRVs in $\mathcal{L}^2$.

25. Kolmogorov's Three-Series Theorem.

26. SLLN for IRV's in $\mathcal{L}^2$.

27. SLLN for IIDRV's in $\mathcal{L}^1$.

28. Doob's decomposition of an adapted process in $\mathcal{L}^1$. The angle-brackets process for $\mathcal{L}^2$-martingale.

29. Sets of a.s. convergence of martingales in $\mathcal{L}^2$. Strong law of LN for martingales in $\mathcal{L}^2$.

30. Levy's extension of the Borel-Cantelli Lemmas.

31. Definition of UI. Sufficient conditions in terms of $\mathcal{L}^p$ boundedness and dominance by an integrable RV.

32. UI of conditional expectations.

33. Convergence in probability + UI = $\mathcal{L}^1$-convergence.

34. Levi's Theorem on Upward Convergence of Martingales. Martingale proof of Kolmogorov's 0-1 law.

35. Levi's Theorem on Downward Convergence. Martingale proof of Kolmogorov's SLLN.

36. Doob's submartingale inequality.

37. Law of iterated logarithm for Gaussian RVs.

38. Doob's maximal $\mathcal{L}^p$ inequalities.

39. Kukutani's theorem on convergence of multiplicative martingales generated by IRVs.

40. The Radon-Nykodim Theorem.

41. Characteristic Function, Levi's Inversion Formula.

42. Definition of weak convergence.

43. Skorohod's Representation Theorem on $\mathbf{R}$ for weak convergence.

44. Helly's Lemma on Sequential Compactness of Distribution Functions on $\mathbf{R}$.

45. Theorem on equivalence between weak convergence and convergence of characteristic functions.

46. Central Limit Theorem for IID RVs in $\mathcal{L}_2$.