Assignment 10

15-359 Probability and Computing
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Section: B
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Problem 9.2 Practice with Balance Equations and Time-Reversibility Equations

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(a) The given Markov chains are as follows:

(b) It is fairly clear by symmetry that, if $\vec{\pi}$ is the limiting distribution of $\mathbf{P}^{(1)}$, then

$$\vec{\pi} = \boxed{\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}.$$

$\mathbf{P}^{(2)}$ is time reversible, with $x_1 = \frac{1}{15}$, $x_2 = \frac{2}{15}$, $x_3 = \frac{4}{15}$, $x_4 = \frac{8}{15}$, so that, if $\vec{\pi}$ is the limiting distribution of $\mathbf{P}^{(2)}$, then

$$\vec{\pi} = \boxed{\begin{bmatrix} 1/15 \\ 2/15 \\ 4/15 \\ 8/15 \end{bmatrix}}.$$

(c) If $\mathbf{P}^{(1)}$ were time reversible, then there would have to exist $x_1, x_2, x_3, x_4 \in [0, 1]$ such that $\frac{2}{3}x_1 = \frac{1}{3}x_2$, $\frac{2}{3}x_2 = \frac{1}{3}x_3$, $\frac{2}{3}x_3 = \frac{1}{2}x_4$, $\frac{2}{3}x_4 = \frac{1}{2}x_1$. Then, however, $x_1 = 16x_1$, $x_2 = 16x_2$, $x_3 = 16x_3$, $x_4 = 16x_4$, in which case $x_1 + x_2 + x_3 + x_4 = 0 \neq 1$, so that $\mathbf{P}^{(1)}$ cannot be time reversible. However, as shown in part (b) above, $\mathbf{P}^{(2)}$ is time reversible. ■

(d) The rate of transitions from a state i to a state j is the probability of being in state i times the probability of transitioning to state j given that one is at state i. Thus, if a Markov chain is time reversible, if $\vec{\pi}$ is the limiting distribution of the Markov chain, since $x_i = \pi_i$ so that $\pi_i P_{ij} = x_i P_{ij} = x_j P_{ji} = \pi_j P_{ji}$, and, therefore the rates are the same. ■

Problem 9.6 Expected time until k failures

Consider the following Markov chain:

This Markov chain accurately models the failure rate because, for $i < k$, after i consecutive failures, another failure occurs with probability p, and a success occurs with probability $1-p$ (in which case we restart the count), and we are not concerned with what occurs after k consecutive failures.

Let $\vec{\pi}$ be the limiting distribution of this Markov chain. Then, $\pi_0 = \pi_k + (1-p)\sum_{i=0}^{k-1} \pi_i$, and, $\forall i \in \{1, 2, \ldots, k\}$, $\pi_i = p\pi_{i-1}$, so that $\pi_i = p^i \pi_0$. Thus, since

$$1 = \sum_{i=0}^{k} \pi_i = \sum_{i=0}^{k} p^i \pi_0 = \frac{1-p^{k+1}}{1-p}\pi_0,$$

$\pi_0 = \frac{1-p}{1-p^{k+1}}$, and so, $\forall i \in \{1, 2, \ldots, k\}$, $\pi_i = p^i \frac{1-p}{1-p^{k+1}}$.

Note that, since, from state k, one moves to state 0 with probability 1, if $m_{k,k}$ is the expected time to return to state k from state k and $m_{0,k}$ is the expected time to reach state k from state 0 (the quantity in question), $m_{0,k} = m_{k,k} - 1$. Since $m_{k,k} = \frac{1}{\pi_k}$,

$$m_{0,k} = \frac{1}{\pi_k} - 1 = \boxed{\frac{1-p^{k+1}}{p^k(1-p)}.}$$

Problem 9.10 Symmetric Random Walk

(a) $m_{00} = E[T_{00}]$.

(b) Clearly, n must be even. Since we must stay to the right of 0, $\forall k \in \{1, 2, \ldots, n-2\}$, for the first k moves, the number of moves to the right must be at least the number of moves to the left.

(c) As a result of the properties described in part (b) above, for even n, the set of valid sequences of $n-2$ middle steps can be mapped bijectively to the set of length $n-2$ strings with $(n-1)/2$ 0's and $(n-2)/2$ 1's such that no prefix of the string contains more 0's than 1's by identifying moves to the left with 0's and moves to the right with 1's. By observation that $P\{T_{00} = n\} =$

$P\{T_{00} = n | \text{First step is Right}\}$, we may assume that the first move of any valid path is to the right. Thus, since the number of all possible walks beginning with a step to the right is 2^{n-2},

$$P\{T_{00} = n\} = \boxed{\frac{C\left(\frac{n-2}{2}\right)}{2^{n-2}}\ .}$$

(d) By Misha's Lemma, $\forall n \in \mathbb{N}$,

$$\frac{4^n}{2n+1} < \binom{2k}{k} < 4^n.$$

Thus, by the result of part (c) above, $\forall n \in \mathbb{N}$ with n even, for $k = \frac{n-2}{2}$,

$$\frac{1}{2k+1} = \frac{4^k}{(2k+1)(4^k)} < P\{T_{00} = n\} < \frac{4^k}{4^k} = 1.$$

(e) By the results of parts (a) and (d) above,

$$m_{00} = E[T_{00}] = \sum_{i=0}^{\infty} 2iP\{T_{00} = 2i\} > \sum_{i=0}^{\infty} \frac{2i}{2i-3} > \sum_{i=0}^{\infty} 1 = \infty. \quad \blacksquare$$

Problem 9.14 Finite State DTMCs

(a) Suppose i is a state in a finite, aperiodic, irreducible DTMC. Since the Markov chain is irreducible, for every state j there is some $n_j > 0$ such that $P_{ji}^{n} > 0$. Let $n = \prod_{j \in S} n_j$, where S is the set of states in the Markov chain. Then, regardless of the start state of the Markov chain, there is some $p > 0$ such that p is the probability of visiting state i after at most n transitions. Then, the probability of not visiting state i after kn transitions is at most $(1-p)^k$, so that the probability of visiting state i after kn transitions is at least $1-(1-p)^k \to 1$ as $k \to\ infty$. $\blacksquare$

Problem 10.1 Caching

(a) The state of the cache and the current web page can is modeled by the following Markov chain (where the numbers are the cached pages, and the current page is listed first): Note that the

states with pages 1 and 2 cached are not accessible from any other states, so that the Markov chain can be reduced to the following: If $\vec{\pi}$ is the limiting distribution of this Markov chain,

then

$$\pi_{23} = \pi_{32} + \pi_{31} + x\pi_{13},$$

$$\pi_{32} = (1 - y)\pi_{23},$$

$$\pi_{13} = y\pi_{23},$$

and

$$\pi_{13} + \pi_{23} + \pi_{31} + \pi_{32} = 1.$$

The solution

(i) The probability of having 1 and 2 in the cache is 0.

(ii) The probability of having 2 and 3 in the cache is π_{23} (see above).

(iii) The probability of having 1 and 3 in the cache is π_{13} (see above).

(b) Conditioning on the pages currently cached shows that the probability that a requested page is cached is

$$\pi_{13}(1 - x) + \pi_{23}(1 - y) + \pi_{32},$$

where π_{13}, π_{23}, and π_{32} are as defined above.

Problem 10.3 Time to Empty

(a) Let $T_{2,0}$ be a random variable denoting the time to travel from state 2 to state 0. Then, by Linearity of Expectation and symmetry, $E[T_{2,0}] = 2E[T_{1,0}]$. Thus, conditioning of whether the first move is to the left or to the right,

$$E[T_{1,0}] = 0.6 \cdot 1 + 0.4 \cdot (1 + E[T_{2,0}]) = 0.6 \cdot 1 + 0.4 \cdot (1 + 2E[T_{1,0}]) = 1 + 0.8 \cdot E[T_{1,0}],$$

so that

$$E[T_{1,0}] = \boxed{5.}$$

(b) Let T_{10}, T_{21} be as defined in part (a).

Conditioning in whether the first transition is to the right or to the left, by Linearity of Expectation, and the result of part (a) above,

$$\begin{aligned}
E[T_{10}^2] &= 0.6 \cdot 1 + 0.4 \cdot (E[(1 + T_{20})^2] \\
&= 0.6 + 0.4 + 0.8E[T_{20}] + 0.4E[T_{20}^2] \\
&= 1 + 0.8(10) + 0.4E[T_{20}^2] \\
&= 9 + 0.4E[(T_{10} + T_{21})^2] \\
&= 9 + 0.4(E[T_{10}^2] + 2E[T_{10}T_{21}] + E[T_{21}^2]) \\
&= 9 + 0.8(E[T_{10}^2] + E[T_{10}T_{21}]).
\end{aligned}$$

Since T_{10} and T_{21} are independent, this gives

$$E[T_{10}^2] = 9 + 0.8(E[T_{10}^2] + E[T_{10}]^2) = 9 + 0.8(E[T_{10}^2] + 25) = 29 + 0.8E[T_{10}^2],$$

so that $E[T_{10}] = 145$. Therefore, by the result of part (a) above,

$$\mathrm{Var}(T_{10}) = E[T_{10}^2] - E[T_{10}]^2 = 145 - 25 = \boxed{120\ .}$$