Homework 4

21-621 Introduction to Lebesgue Integration
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Chapter 2, Page 91, Problems 11, 15, 21

Problem 11

Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable, with $\int_E f \geq 0$ on every measurable $E \subseteq \mathbb{R}^d$, and suppose, for sake of contradiction that, that $E^- := f^{-1}((-\infty, 0))$ has $m(E^-) > 0$. Then, if, $\forall n \in \mathbb{N}$, we define $E_n := f^{-1}((-\infty, -1/n))$, by countable additivity of the Lebesgue measure,

$$0 < m(E^-) = m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n)$$

(since f is measurable, each E_n is measurable). Thus, for some $n > 0$, $m(E_n) > 0$. Therefore,

$$\int_{E_n} f \leq \int_{E_n} -\frac{1}{n} = -\frac{m(E_n)}{n} < 0,$$

contradicting the fact that f has non-negative integral over all measurable sets. ■

A nearly identical proof shows that, if f has non-positive integral over all measurable sets, then, $f \leq 0$ almost everywhere. It follows that, if f has zero integral on all measurable sets, $f \leq 0$ almost everywhere, and $f \geq 0$ almost everywhere, so that $f = 0$ almost everywhere. ■

Problem 15

$\forall n \in \mathbb{N}$, since f is continuous on $\left(\frac{1}{n}, 1\right)$, since Riemann and Lebesgue integrals of f agree on $\left(\frac{1}{n}, 1\right)$,

$$\lim_{a \to 0} \int_a^1 f = \lim_{a \to 0} \int_a^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0} 2\sqrt{x}\Big|_{x=a}^{x=1} = 2 - 2\sqrt{a} = 2.$$

Thus, by the Lebesgue Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f = \int_0^1 \lim_{n \to \infty} f\chi_{(\frac{1}{n}, 1]} = \lim_{n \to \infty} \int_{\frac{1}{n}}^1 f = 2.$$

Then, again, by the Lebesgue Monotone Convergence Theorem, since translating a function does not change its integral over $\mathbb{R}$,

$$\int_{\mathbb{R}} F = \sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{f(x - r_n)}{2^n}\, dx. = \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{\mathbb{R}} f(x - r_n)\, dx. = 2\sum_{n=1}^{\infty} \frac{1}{2^n} = 2 < \infty,$$

so that F is integrable, and thus that F is finite almost everywhere, so that the series by which F is defined converges almost everywhere. ■

Suppose $\tilde{F} = F$ almost everywhere, let I be an interval, and let $M \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $r_n \in \mathbb{Q} \cap I$, for some $n \in \mathbb{N}$. Note that $f > 2^n M$ on the interval $I_M := \left(r_n, r_n + \frac{1}{2^n M}\right)$. Since $I \cap I_M$ is an nonempty open interval, $m(I \cap I_M) > 0$, so that, since $\tilde{F} = F$ almost everywhere, $\exists x \in I \cap I_M$ with $\tilde{F}(x) = F(x) \geq \frac{f(x)}{2^n} > M$. Thus, $\tilde{F}$ is unbounded on I. ■

Problem 21

The following lemma is used in proofs of parts (b) and (d) of this problem:

Lemma 1: If $f, g \in L^1(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)| \, dy \, dx = \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}.$$

Proof: By Fubini's Theorem and linearity of the integral,

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)| \, dy \, dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)| \, dx \, dy, \\
&= \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x-y)| \, dx \, dy, \\
&= \int_{\mathbb{R}^d} |g(y)| \|f\|_{L^1(\mathbb{R}^d)} \, dy \\
&= \|f\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} |g(y)| \, dy = \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

The third equality holds because translating a function does not change its L^1 norm. ■

(a) The function $(x, y) \in \mathbb{R}^{2d} \mapsto x - y$ is measurable, so that, since any composition or product of measurable functions is measurable, the function $(x, y) \in \mathbb{R}^{2d} \mapsto f(x-y)g(y)$ is measurable. ■

(b) By Fubini's Theorem and Lemma 1, if $\|f\|_{L^1(\mathbb{R}^d)}\|, \|g\|_{L^1(\mathbb{R}^d)}\| < \infty$, then

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)| \, dx \, dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)| \, dy \, dx = \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)} < \infty. \quad ■$$

(c) This follows immediately from part (b) and Fubini's Theorem. ■

(d) By the Triangle Inequality and then by Lemma 1,

$$\|f * g\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)g(y) \, dy \right| \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)| \, dy \, dx, = \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}.$$

with equality when $f, g \geq 0$. ■

(e) Since f is integrable, it follows from Proposition 4.1 that $\hat{f}$ is continuous and bounded. ■