Homework 8

21-260 Differential Equations
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Section 6.4, Problem 16ab

(a) Figure 1 below shows the graph of $g(t)$.

(b) As shown in Section 2 of Chapter 6, for $Y(s) = \mathcal{L}\{u(t)\}$, since $\mathcal{L}\{ku_{3/2}(t) - ku_{5/2}(t)\} = \frac{k}{s}\left(e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}\right)$,

$$Y(t) = k\left(\frac{e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}}{s\left(s^2 + \frac{1}{4}s + 1\right)}\right)$$

Partial fraction decomposition shows that

$$\begin{aligned}
Y(t) &= k\left(\frac{1}{s} - \frac{s + 1/4}{s^2 + \frac{1}{4}s + 1}\right)\left(e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}\right) \\
&= k\left(\frac{1}{s} - \frac{s + 1/8}{s^2 + \frac{1}{4}s + 1} - \frac{1}{8}\frac{1}{s^2 + \frac{1}{4}s + 1}\right)\left(e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}\right) \\
&= k\left(\frac{1}{s} - \frac{s + 1/8}{(s + 1/8)^2 + 63/64} - \frac{64}{504}\frac{63/64}{(s + 1/8)^2 + 63/64}\right)\left(e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}\right)
\end{aligned}$$

Thus, Table 6.2.1 shows that

$$\begin{aligned}
y(t) &= ku_{3/2}(t)\left(1 - e^{-\frac{1}{8}t}\cos\left(\frac{63}{64}\left(t - \frac{3}{2}\right)\right) - \frac{64}{504}e^{-\frac{1}{8}t}\sin\left(\frac{63}{64}\left(t - \frac{3}{2}\right)\right)\right) \\
&- ku_{5/2}(t)\left(1 - e^{-\frac{1}{8}t}\cos\left(\frac{63}{64}\left(t - \frac{5}{2}\right)\right) - \frac{64}{504}e^{-\frac{1}{8}t}\sin\left(\frac{63}{64}\left(t - \frac{5}{2}\right)\right)\right).
\end{aligned}$$

Section 6.5, Problem 6a

(a) As shown in Section 2 of Chapter 6, for $Y(s) = \mathcal{L}\{y(t)\}$, since $\mathcal{L}\{\delta(t-4\pi)\} = e^{-4\pi s}$,

$$Y(s) = \frac{s/2 + e^{-4\pi s}}{s^2+4} = \frac{1}{2}\frac{s}{s^2+4} + \frac{1}{2}e^{-4\pi s}\frac{2}{s^2+4}.$$

Table 6.2.1 shows that

$$\frac{s}{s^2+4} = \mathcal{L}\{\cos(2t)\} \quad \text{and that} \quad e^{-4\pi s}\frac{s}{s^2+4} = \mathcal{L}\{u_{4\pi}(t)\sin(2(t-4\pi))\}.$$

Thus, by linearity of the inverse Laplace transform,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{\frac{1}{2}\left(\cos(2t) + u_{4\pi}(t)\sin(2(t-4\pi))\right).}$$

Section 6.5, Problem 16abc

(a) As shown in Section 2 of Chapter 6, for $Y_k(s) = \mathcal{L}\{\phi(t,k)\}$, since $\mathcal{L}\{f_k(t)\} = \frac{1}{2ks}\left(e^{-(4-k)s} - e^{-(4+k)s}\right)$,

$$Y_k(s) = \frac{1}{2ks}\left(e^{-(4-k)s} - e^{-(4+k)s}\right)\left(\frac{1}{s^2+1}\right).$$

Table 6.2.1 shows that $\frac{1}{s} = \mathcal{L}\{1\}$ and that $\frac{s}{s^2+1} = \mathcal{L}\{\cos(t)\}$. Thus, by linearity of the inverse Laplace transform,

$$\phi(t,k) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{\frac{1}{2k}\left(u_{4-k}(t)(1-\cos(4-k-t)) - u_{4+k}(t)(1-\cos(4+k-t))\right).}$$

(b)

$$\lim_{k\to 0}\left(\phi(t,k)\right) = \boxed{u_4(t)\sin(t-4).}$$

(c) As shown in Section 2 of Chapter 6, for $Y(s) = \mathcal{L}\{\phi_0(t)\}$, since $\mathcal{L}\{\delta(t-4)\} = e^{-4s}$,

$$Y(s) = \frac{e^{-4s}}{s^2+1}.$$

Table 6.2.1 shows that

$$\phi_0(t) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{u_4(t)\sin(t-4),}$$

so that $\phi_0(t) = \lim_{k\to 0}\left(\phi(t,k)\right)$.

Section 6.6, Problem 16

As shown in Section 2 of Chapter 6, for $Y(s) = \mathcal{L}\{y(t)\}$,

$$Y(s) = \frac{s + \mathcal{L}\{1 - u_\pi(t)\}}{s^2 + s + \frac{5}{4}} = \frac{s + \mathcal{L}\{1 - u_\pi(t)\}}{\left(s + \frac{1}{2}\right)^2 + 1}.$$

By linearity of the inverse Laplace transform, then,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + 1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s + \frac{1}{2}\right)^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{1 - u_\pi(t)\}}{\left(s + \frac{1}{2}\right)^2 + 1}\right\}.$$

Table 6.2.1 shows that $\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} = \mathcal{L}\{e^{-\frac{1}{2}t}\cos(t)\}$ and that $\frac{1}{\left(s+\frac{1}{2}\right)^2+1} = \mathcal{L}\{e^{-\frac{1}{2}t}\sin(t)\}$. Thus, by Theorem 6.6.1,

$$y(t) = \boxed{e^{-\frac{1}{2}t}\cos(t) - \frac{1}{2}e^{-\frac{1}{2}t}\sin(t) + (1 - u_\pi(t)) * e^{-\frac{1}{2}t}\sin(t),}$$

where $*$ denotes convolution.