Homework 4

21-630 Ordinary Differential Equations

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Collaborators: None

Problem 1

I wasn't able to solve this problem. I did notice that the solution to the equation

$$Y(t) = A + \int_{t_0}^{t} b(s)\sqrt{Y(s)}\,ds,$$

has the form of the desired bound:

$$Y(t) = \left(\sqrt{A} + \int_{t_0}^{t} b(s)\,ds\right)^2.$$

I tried working along the lines of the proof given in class of Gronwall's Inequality with this solution in mind, but wasn't able to get the desired result.

Problem 2

A) We suppose that R is continuous. It follows, as discussed in class, that, since $R(0) = 1 > 0$, R is positive and differentiable on $[0,\infty)$. Differentiating the given equation gives, $\forall t \in [0,\infty)$, $\frac{dR(t)}{dt} = \frac{1}{R(t)}$. Separation of variables and then integration give, $\forall t \in [0,\infty)$,

$$\frac{R^2(t)}{2} = t + C,$$

for some constant $C \in \mathbb{R}$, and so, since $R(0) = 1$, $\boxed{R(t) = \sqrt{2t+1}.}$

B) The function $X : [0,\infty) \to \mathbb{R}$ defined $\forall t \in [0,\infty)$ by

$$X(t) = \begin{cases} 0.1 & : t \in [0, 1.01) \\ 10(t-1) & : t \in [1.01, 2) \\ 10 & : t \in [2, \infty) \end{cases}.$$

is continuous and positive on $[0,\infty)$. $\forall t \in [0, 1.01)$, $X(t) \leq 1 \leq 1 + \int_0^t \frac{1}{X(s)}\,ds$, and $\forall t \in [1.01,\infty)$,

$$X(t) \leq 10 \leq 1 + \int_0^1 10\,ds \leq 1 + \int_0^1 \frac{1}{X(s)}\,ds \leq 1 + \int_0^t \frac{1}{X(s)}\,ds.$$

Furthermore, $X(2) = 10 > \sqrt{5} = R(2)$. Thus, X is a counterexample. ■

Problem 3

By orthogonality and then by linearity of R,

$$f(t, Rx) = F(t, |Rx|)Rx = F(t, |x|)Rx = R(F(t, |x|)x) = Rf(t, x). \tag{1}$$

Since solutions to the given system are unique, it suffices to show that

$$\begin{aligned} \frac{d(RX)}{dt} &= f(t, RX(t, t_0, x_0)) \\ RX(t_0, t_0, x_0) &= Rx_0. \end{aligned}$$

The latter equation trivially holds. Furthermore, for all t, by (1) and the fact that R is linear,

$$f(t, RX(t, t_0, x_0)) = Rf(t, X(t, t_0, x_0)). = R\frac{dX}{dt} = \frac{d(RX)}{dt}. \quad \blacksquare$$

Problem 4

Suppose $f(t, x) = x^2(1 - x)$, $\forall x \in \mathbb{R}$. Since f satisfies a Lipschitz condition in x on any bounded subset of the domain, any solution X to the given system is unique. However, for any $x_0 \in \mathbb{R}$, if $x_0 \leq 0$, then $\lim_{t\to+\infty} X(t, t_0, x_0) = 0$, whereas, if $x_0 > 0$, then $\lim_{t\to+\infty} X(t, t_0, x_0) = 1$. Thus, the function $x_0 \mapsto \lim_{t\to+\infty} X(t, t_0, x_0)$ is discontinuous, and so the given statement is false. $\blacksquare$