ASSIGNMENT NUMBER 1, 21.630 Spring 12

Due Wednesday, January 23

1. Let $p > 0$. For each $n \in \mathbb{N}$ define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(t) = \frac{t^p}{1 + nt^2}.$$

Note that f_n converges pointwise to zero on $[0, \infty)$ (you don't have to show this). For what values of p does f_n converge uniformly to zero? Prove your answer to be correct. For what values of p does f_n not converge uniformly to zero? Prove your answer to be correct.

2. Let $p \in (0, 1)$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if $x \leq 0$, $f(x) = x^p$ if $0 < x < 1$, and $f(x) = 1$ if $x \geq 1$. Find explicitly an infinite family of solutions to the initial value problem

$$\frac{dX}{dt} = f(X)$$

$$X(0) = 0.$$

3. Define $f(t, x) = x \ln(x)$ if $x > 0$ and $f(t, 0) = 0$. Show that f does not satisfy a Lipschitz condition in x on $D = \{(t, x) : t \text{ is real and } 0 \leq x \leq e^{-1}\}$. Then show that for every $\alpha \in (0, 1)$, f does satisfy a Holder condition in x with exponent α on D. You may use the following fact without proof: For any $p \in (0, 1)$ there exists $C_p > 0$ such that $|\ln(z)| \leq C_p z^{-p}$ for all $z \in (0, 1)$.

4. Define $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by $f(t, x) = 2t$ if $x \leq 0$, $f(t, x) = 2t - 4x/t$ if $0 < x < t^2$ and $f(t, x) = -2t$ if $t^2 \leq x$. Show that f does not satisfy a Lipschitz condition in x on $[0, \infty) \times \mathbb{R}$. Show that f does satisfy a Holder condition in x with exponent one half on $[0, \infty) \times \mathbb{R}$. Here is one approach to showing the Holder condition: Do the case $0 \leq x \leq y \leq t^2$ first. Then (taking $x \leq y$) there are five other cases. Two are trivial and the remaining three may be done by comparison with the first.

5. Define

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k \sin(e^{kx})}{k^k}$$

for all $x \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$.