MATH 651: PROBLEM SET 5 SOLUTIONS ARE IN CLASS ON MON. NOV 5.

1. (10 points) Let (X, τ) be a $T_{3\frac{1}{2}}$ space. Show that X is homeomorhpic to a subset of $Y = [0,1]^I$ for some index set I. The topology on Y is the product topology where on $[0,1]$ we consider the standard topology.

2. (10 points) Let (X, τ) be a normal space.

   (i) Prove that given a proper set closed set $C \subset X$, there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 0$ in C and $f > 0$ in $X \setminus C$ if and only if C is a G_δ set (i.e. a countable intersection of open sets).

   (ii) Let $C_1, C_2 \subset X$ be two disjoint closed sets. Prove that there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C_1, $f \equiv 0$ in C_2, and $0 < f < 1$ in $X \setminus (C_1 \cup C_2)$ if and only if C_1, C_2 are G_δ sets.

3. (10 points) Let (Y, τ_Y) and (Z, τ_Z) be topological spaces, with Z Hausdorff and let $f : E \to Z$ be a continuous function, where $E \subset Y$. Then there is at most one extension of f to $\overline{E}$.

4. (10 points) A function $f : X \to \mathbb{R}$ is *sequentially lower semicontinuous (sLSC)* if for every convergent sequence $x_n \to a$ in X as $n \to \infty$ it holds that

$$\liminf_{n\to\infty} f(x_n) \geq f(a).$$

   Show that if f LSC then f is sLSC. Find an example of a topological space and a function which is sLSC, but not LSC.

5. (10 points) Let (X, τ) be a topological space satisfying the first axiom of countability and let $f : E \to \mathbb{R}$ be a function, where $E \subset X$.. If f is sequentially lower semicontinuous on E, then f is lower semicontinuous on E.

6. (10 points) Let (X, τ) be a topological space and let $f : X \to \mathbb{R}$. The *epigraph* of f is the set

$$\operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} : \ f(x) \leq t\}.$$

   Show that f is LSC if and only $\operatorname{epi} f$ is closed.