Department of Mathematical Sciences
Carnegie Mellon University

21-740 **Functional Analysis II** **Fall 2013**

Assignment 4

Due on Monday, November 25

1. Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup with infinitesimal generator A. Let $X_A = \mathcal{D}(A)$ equipped with the norm

$$\|x\|_A = \|x\| + \|Ax\| \quad \text{for all } x \in X_A.$$

Let $\tau > 0$ and $F \in C^1([0, \tau]; X)$ and $G \in C([0, \tau] : X_A)$ be given. Put

$$f(t) = F(t) + G(t), \quad v(t) = \int_0^t T(t-s) f(s)\, ds \quad \text{for all } t \in [0, \tau].$$

Show that

$$v \in C^1([0, \tau]; X) \cap C([0, \tau]; X_A)$$

and

$$\dot{v} = Av(t) + f(t) \quad \text{for all } t \in [0, \tau].$$

2. Let X be a complex Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be an analytic semigroup with infinitesimal generator A. Let $L \in \mathcal{L}(X; X)$ be given. Show that $A + L$ generates an analytic semigroup.

3. Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup with infinitesimal generator A. For $h > 0$ put

$$\mathcal{A}_h = \frac{T(h) - I}{h}.$$

Show that for every $t \geq 0$ and every $x \in X$ we have

$$e^{t\mathcal{A}_h} x \to T(t)x \quad \text{as } h \downarrow 0.$$

4. Let X be a complex Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense and that A is linear and closed. Assume further that there exists $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 0$ such that $\lambda_0 \in \rho(A)$. Let $[\cdot, \cdot]$ be a semi-inner product that is compatible with the norm on X. Let $\beta \in (0, \frac{\pi}{2})$ be given and assume that

$$\{[Ax, x] : x \in \mathcal{D}(A),\ \|x\| \leq 1\} \subset \{0\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \geq \frac{\pi}{2} + \beta\}.$$

Show that A generates an analytic semigroup. (In fact, this semigroup will also be uniformly bounded.)

Definition: Let X be a Banach space and $\tau > 0$, $\theta \in (0,1)$ be given. By $C^{0,\theta}([0,\tau];X)$ we mean the set of all functions $g : [0,\tau] \to X$ for which there exists a constant C (depending on g) such that

$$\|g(t) - g(s)\| \leq |t-s|^{\theta}, \quad \text{for all } s,t \in [0,\tau].$$

5. Let X be a Banach space and $T : [0,\infty) \to \mathcal{L}(X;X)$ be an analytic semigroup with ifinitesimal generator A. Let $\tau > 0$, $\theta \in (0,1)$, and $f \in C^{0,\theta}([0,\tau];X)$ be given. Define $w : [0,\tau] \to X$ by

$$w(t) = \int_0^t T(t-s)(f(s) - f(t))\, ds \quad \text{for all } t \in [0,\tau].$$

Show that $w(t) \in \mathcal{D}(A)$ for all $t \in [0,\tau]$ and $Aw \in C^{0,\theta}([0.\tau];X)$.