Homework 4
21-484A Graph Theory
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Problem 1

Let C be the set of columns of the board, and let R be the set of rows of the board. Construct an undirected graph G (with vertex set $C \cup R$) such that there is an edge between two vertices u and v if and only if there is a rook on a square shared by u and v.

Since no pair of rows and no pair of columns can share a square, G is bipartite. Since each row and each column contains k rooks, G is k-regular. Thus, since all regular, bipartite graphs have perfect matchings, G has a perfect matching M. Let S be the set of rooks corresponding to edges in M. Since each row is paired with exactly one column and each column is paired with exactly one row, no two rooks in S are in the same row or column. Since $|C \cup R| = 2n$, $|S| = n$. Therefore, S is a set of n rooks with the desired properties. ■

Problem 3

Suppose that, for some $k \in \mathbb{N}$, $\|G\| < \binom{k}{2}$, and consider a proper k coloring of G. Since each edge connects at most 2 vertices (so that $\binom{k}{2}$ edges are required to have an edge between vertices of each possible pair of colors), there exist two colors b and r such that there is no edge between any vertex colored b and any vertex colored r. Thus, all the vertices of color b can be recolored with color r to give a proper $k-1$ coloring, so that $\chi(G) < k$. Therefore, if $\|G\| < \binom{k}{2}$, then $\chi(G) < k$, so that the contrapositive, that, if $\chi(G) \geq k$, then $\|G\| \geq \binom{k}{2}$, also holds. ■

Problem 4

Note that we discuss only connected graphs here, since disconnected graphs can be colored one component at a time. Let G be a graph which neither an odd cycle nor a complete graph. Suppose G is not $\Delta(G)$-regular, so that there exists a vertex v of degree $d < \Delta(G)$. Consider a breadth-first search tree rooted at v (since G is connected, this tree includes every vertex in G). Color the tree from the bottom up by repeatedly coloring the lowest uncovered vertex in the tree, until v has been colored. This can be done with $\Delta(G)$ colors, since each vertex is colored before its parent (so that it has at most $\Delta(G) - 1$ colored neighbors (it children)), with the exception of v, which, by choice, has degree at most $\Delta(G) - 1$, so that it can be colored with one of the $\Delta(G)$ colors. Therefore, any graph that is not $\Delta(G)$-regular is $\Delta(G)$-colorable, so that Brooks' theorem holds for all graphs that are not $\Delta(G)$-regular, or are $\Delta(G)$-regular and 2-connected, leaving only those graphs which are k-regular but not 2-connected.

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Let K be a graph which is neither an odd cycle nor a complete graph. By the previous result, we can color each biconnected component of K with $\Delta(K)$ colors. Let σ be a cyclic permutation of the the $\Delta(G)$ colors. Let A and B be two biconnected components thus colored. As distinct biconnected components, A and B share at most 1 vertex v or are connected by at most one edge e. In the first case, we can construct a proper coloring of $A \cup B$ (the subgraph of K induced by those vertices in either A or B) by using colorings of A and B, and, in the case that v is colored differently in the two colorings, repeatedly permuting the colors of the vertices in A by σ until v is the same color in both the coloring of A and the coloring of B. In the second case, we can construct a proper coloring of $A \cup B$ by using the colorings of A and B, and permuting the colors of A once if the vertices to which e is incident are the same color. Furthermore, we can connect an arbitrary number of biconnected components of K in this manner, so that we can construct a proper $\Delta(K)$-coloring of K in this manner. Thus, Brooks' Theorem holds for graphs which are not 2-connected.

Therefore, Brooks' theorem holds for all graphs. ∎

Problem 6

Let K be the graph whose vertices are the edges of G, with edges between vertices in K if and only if those vertices are adjacent edges in G. Since G contains $4k + 1$ vertices, no independent set of edges in G contains more than $2k$ edges (since 2 vertices are required for each independent edge). Thus, since a set of edges in G is independent if and only the corresponding set of vertices in K is independent, is $\alpha(K)$ is the maximum size of an independent of vertices in K, $\alpha(K) \leq 2k$. The number of edges in H is half the sum of the degrees of the vertices in H, $\frac{1}{2}\frac{2k}{4k+1} = 4k^2 + k$. Thus, the number of edges in G is $4k^2 + k - (k - 1) = 4k^2 + 1$, so that the number of vertices in K is $4k^2 + 1$. By Theorem 10.5, then,

$$\chi(K) \geq \frac{|V(K)|}{\alpha(K)} = \frac{4k^2 + 1}{2k} > \frac{4k^2}{2k} = 2k.$$

Thus, since a proper vertex coloring of K corresponds exactly to a proper edge coloring of G, $\chi_1(G) > 2k$. Thus, since Vizing's Theorem implies that $\chi_1(G) = \Delta(G)$ or $\chi_1(G) = \Delta(G) + 1$ and $\Delta(G) = 2k$, $\chi_1(G) = 2k + 1 = \Delta(G) + 1$. ∎

Problem 7

Suppose, for sake of contradiction that G is planar. Let H be a plane drawing of G. Suppose v and u are vertices in H that are also in K_5 or $K_{3,3}$ (as appropriate). Replace the vertices and edges in the path from u to v (i.e., the path that was added in the subdivision) with a single edge that traverses that path. Doing this for each pair of vertices u and v that were in K_5 or $K_{3,3}$ (as appropriate) gives a plane drawing of either K_5 or $K_{3,3}$, respectively. However, since both K_5 and $K_{3,3}$ are not planar, this is a contradiction. ∎