Skew-Symmetric Matrices in Linear Systems of Differential Equations

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Consider an equation of the form

$$\frac{dx}{dt} = ax, \tag{1}$$

where $a \in \mathbb{R}$.

Rewriting equation (1) and then integrating gives

$$\begin{aligned} \frac{1}{x}\,dx &= a\,dt \\ \ln(x) &= at + C \\ x &= e^{at+C} = e^C e^{at} \end{aligned}$$

(we can "absorb" the integration constants from both sides into C).

Suppose now that we have n equations with n independent variables:

$$\begin{aligned} \frac{dx_1}{dt} &= a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \\ \frac{dx_1}{dt} &= a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \\ &\vdots \\ \frac{dx_1}{dt} &= a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n}x_n. \end{aligned}$$

We can write this more compactly as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{2}$$

where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Inspired by the solution to the 1 variable equation, we'll assume that the solution is an exponential. Since we want a vector solution, we just multiply by a vector $\boldsymbol{\chi}$, so that we assume a solution is of the form $\mathbf{x} = \boldsymbol{\chi}e^{rt}$, for some $r \in \mathbb{R}$. Plugging this into equation (2) gives

$$r\boldsymbol{\chi}e^{rt} = A\boldsymbol{\chi}e^{rt}.$$

Since $e^{rt} \neq 0$,

$$(A - rI)\boldsymbol{\chi} = \mathbf{0}.$$

Notice that r and $\boldsymbol{\chi}$ solve this equation if and only if $\boldsymbol{\chi}$ is an eigenvector of A and r is the associated eigenvalue.

For reasons that I won't go into here, the solutions of an $n \times n$ system of linear differential equations form an n-dimensional vector space, so that we can summarize the solution set as a linear combination of a basis set of solutions:

$$\mathbf{x} = \sum_{k=1}^{n} c_k \boldsymbol{\chi}_k e^{r_k t},$$

where $\boldsymbol{\chi}_k$ is the i^{th} eigenvector and r_k is the associated eigenvalue. (This gets more complicated when A there are fewer than n linearly independent eigenvalues, but this doesn't happen when A is skew-symmetric, which is what we care about here.)

Now what happens when some eigenvalues are in $\mathbb{C}$ but not in $\mathbb{R}$? Then, each $r_k = a_k + b_k i$, where $a, b \in \mathbb{R}$. Thus,

$$\mathbf{x} = \sum_{k=1}^{n} c_k \boldsymbol{\chi}_k e^{a_k t} e^{b_k i t}. \tag{3}$$

Now it's clear that a_k, the real part of the each eigenvalue acts as a dilaiton factor. But how do we interpret this complex component? We use Euler's identity:

$$e^{it} = \cos(t) + i\sin(t).$$

(if you haven't seen this before, the proof is one line; examine the Taylor expansions of sine, cosine, and the exponential function.) Rewriting equation (3) using Euler's Identity gives us

$$\mathbf{x} = \sum_{k=1}^{n} c_k \boldsymbol{\chi}_k e^{a_k t} \left(\cos(b_k t) + i\sin(b_k t)\right).$$

Splitting up $\boldsymbol{\chi}_k = \mathbf{c}_k + i\mathbf{d}_k$ into its real and complex components and doing some algebra shows that, for

$$\mathbf{u}_k = e^{a_k t}(\mathbf{c}_k \cos(b_k t) - \mathbf{d}_k \sin(b_k t)$$

$$\mathbf{v}_k = e^{a_k t}(\mathbf{c}_k \sin(b_k t) + \mathbf{d}_k \cos(b_k t)$$

(since the complex eigenvectors of A come in complex conjugate pairs and $\mathbf{u}_k$ and $\mathbf{v}_k$ are the same for each pair, we're replacing each pair with the pair $\{\mathbf{u}_k, \mathbf{v}_k\}$, which can be shown to be linearly independent, so that we still have a basis.)

As it happens, when A is skew-symmetric, *all* (nonzero) eigenvalues of A are *purely* complex; that is, each $a_k = 0$. Therefore, we get the solution

$$\mathbf{x} = \sum_{k=1}^{n} c_k \boldsymbol{\chi}_k \left(\cos(b_k t) + i\sin(b_k t)\right),$$

which is purely oscillatory.

Thus, if we fit a skew-symmetric matrix M_{skew} to the equation

$$\mathbf{x}' = M_{skew}\mathbf{x},$$

we get a matrix that captures the oscillatory motion of the dynamical system.