*Proof.* Let $c$ be the number given in Lemma 4.1. Choose $\delta > 0$ and $W$ in Theorem 3.7 in such a way that $\delta < \frac{c}{2}$. Take $\beta < \delta$ such that $\overline{B_\beta(p)} \subset W$. We shall prove that $B_\beta(p)$ is strongly convex. Let $q_1, q_2 \in \overline{B_\beta(p)}$ and let $\gamma$ be the (unique) geodesic of length $< 2\delta < c$ joining $q_1$ to $q_2$. It is clear that $\gamma$ is contained in $B_c(p)$ (Fig. 5).

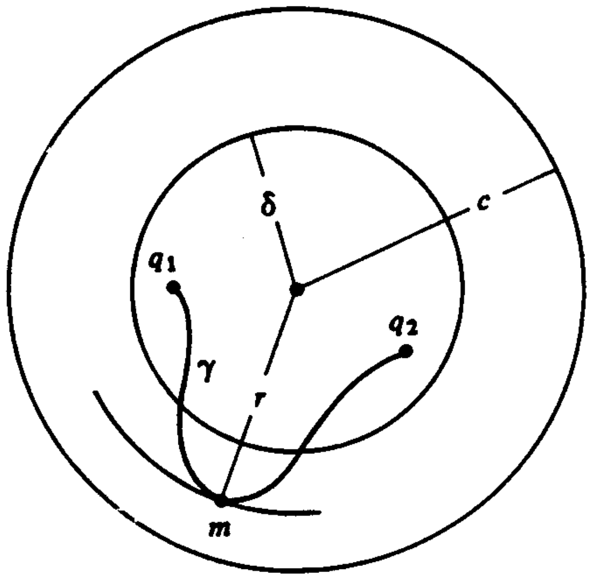


**Figure 5**

If the interior of $\gamma$ is not contained in $B_\beta(p)$, then there exists a point $m$ in the interior of $\gamma$ where the maximum distance $r$ from $p$ to $\gamma$ is attained. The points of $\gamma$ in a neighborhood of $m$ remain in the closure of $B_r(p)$. Since $m \in B_c(p)$ this contradicts Lemma 4.1 and proves the proposition. □

## *EXERCISES*

1. (*Geodesics of a surface of revolution*). Denote by $(u, v)$ the cartesian coordinates of $\mathbf{R}^2$. Show that the function $\varphi: U \subset \mathbf{R}^2 \to \mathbf{R}^3$ given by $\varphi(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$,
$$U = \left\{(u, v) \in \mathbf{R}^2 : u_o < u < u_1; v_o < v < v_1\right\},$$
where $f$ and $g$ are differentiable functions, with $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$, is an immersion. The image $\varphi(U)$ is

the surface generated by the rotation of the curve $(f(v), g(v))$ around the axis $0z$ and is called a *surface of revolution* $S$. The image by $\varphi$ of the curves $u =$ constant and $v =$ constant are called *meridians* and *parallels*, respectively, of $S$.

a) Show that the induced metric in the coordinates $(u, v)$ is given by

$$g_{11} = f^2, \quad g_{12} = 0, \quad g_{22} = (f')^2 + (g')^2.$$

b) Show that local equations of a geodesic $\gamma$ are

$$\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0,$$

$$\frac{d^2v}{dt^2} - \frac{ff'}{(f')^2 + (g')^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f'^2 + (g')^2}\left(\frac{dv}{dt}\right)^2 = 0.$$

c) Obtain the following geometric meaning of the equations above: the second equation is, except for meridians and parallels, equivalent to the fact that the "energy" $|\gamma'(t)|^2$ of a geodesic is constant along $\gamma$; the first equation signifies that if $\beta(t)$ is the oriented angle, $\beta(t) < \pi$, of $\gamma$ with a parallel $P$ intersecting $\gamma$ at $\gamma(t)$, then

$$r \cos \beta = \text{const.},$$

where $r$ is the radius of the parallel $P$ (the equation above is called *Clairaut's relation*).

d) Use Clairaut's relation to show that a geodesic of the paraboloid

$$(f(v) = v, g(v) = v^2, 0 < v < \infty, -\varepsilon < u < 2\pi + \varepsilon),$$

which is not a meridian, intersects itself an infinite number of times(Fig. 6).

2. It is possible to introduce a Riemannian metric in the tangent bundle $TM$ of a Riemannian manifold $M$ in the following manner. Let $(p, v) \in TM$ and $V, W$ be tangent vectors in $TM$ at $(p, v)$. Choose curves in $TM$

$$\alpha: t \to (p(t), v(t)), \ \ \beta: s \to (q(s), w(s)),$$

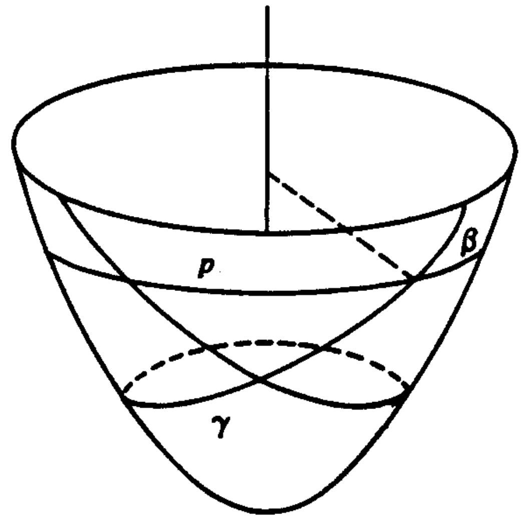


**Figure 6.** *Geodesics of a paraboloid.*

with $p(0) = q(0) = p$, $v(0) = w(0) = v$, and $V = \alpha'(0)$, $W = \beta'(0)$. Define an inner product on $TM$ by

$$\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\rangle_p,$$

where $d\pi$ is the differential of $\pi: TM \to M$.

a) Prove that this inner product is well-defined and introduces a Riemannian metric on $TM$.

b) A vector at $(p, v) \in TM$ that is orthogonal (for the metric above) to the fiber $\pi^{-1}(p) \approx T_pM$ is called a *horizontal vector*. A curve
$$t \to (p(t), v(t))$$
in $TM$ is *horizontal* if its tangent vector is horizontal for all $t$. Prove that the curve

$$t \to (p(t), v(t))$$

is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in $M$.

c) Prove that the geodesic field is a horizontal vector field (i.e., it is horizontal at every point).

d) Prove that the trajectories of the geodesic field are geodesics on $TM$ in the metric above.

*Hint*: Let $\bar{\alpha}(t) = (\alpha(t), v(t))$ be a curve in $TM$. Show that $\ell(\bar{\alpha}) \geq \ell(\alpha)$ and that the inequality is verified if $v$ is parallel along $\alpha$. Consider a trajectory of the geodesic flow passing through $(p,v)$ which is locally of the form $\bar{\gamma}(t) = (\gamma(t), \gamma'(t))$, where $\gamma(t)$ is a geodesic on $M$. Choose convex neighborhoods $W \subset TM$ of $(p,v)$ and $V \subset M$ of $p$ such that $\pi(W) = V$. Take two points $Q_1 = (q_1, v_1)$, $Q_2 = (q_2, v_2)$ in $\bar{\gamma} \cap W$. If $\bar{\gamma}$ is not a geodesic, there exists a curve $\bar{\alpha}$ in $W$ passing through $Q_1$ and $Q_2$ such that $\ell(\bar{\alpha}) < \ell(\bar{\gamma}) = \ell(\gamma)$. Let $\alpha = \pi(\bar{\alpha})$; since $\ell(\alpha) \leq \ell(\bar{\alpha})$, this contradicts the fact that $\gamma$ is a geodesic.

e) A vector at $(p,v) \in TM$ is called *vertical* if it is tangent to the fiber $\pi^{-1}(p) \approx T_pM$. Show that:

$$\langle W, W\rangle_{(p,v)} = \langle d\pi(W), d\pi(W)\rangle_p, \quad \text{if } W \text{ is horizontal},$$
$$\langle W, W\rangle_{(p,v)} = \langle W, W\rangle_p, \quad \text{if } W \text{ is vertical},$$

where we are identifying the tangent space to the fiber with $T_pM$.

3. Let $G$ be a Lie group, $\mathcal{G}$ its Lie algebra and let $X \in \mathcal{G}$ (see Example 2.6, Chap. 1). The trajectories of $X$ determine a mapping $\varphi: (-\varepsilon, \varepsilon) \to G$ with $\varphi(0) = e$, $\varphi'(t) = X(\varphi(t))$.

a) Prove that $\varphi(t)$ is defined for all $t \in \mathbf{R}$ and that $\varphi(t+s) = \varphi(t) \cdot \varphi(s)$, ($\varphi: \mathbf{R} \to G$ is then called a *1-parameter subgroup* of $G$).

*Hint*: Let $\varphi(t_o) = y$, $t_o \in (-\varepsilon, \varepsilon)$. Show that, from the left invariance, $t \to y^{-1}\varphi(t)$, $t \in (-\varepsilon, \varepsilon)$, is also an integral curve of $X$ passing through $e$ for $t = t_o$. By uniqueness, $\varphi(t_o)^{-1}\varphi(t) = \varphi(t - t_o)$, hence $\varphi$ can be extended out from $t_o$ in an interval of radius $\varepsilon$. This shows that $\varphi(t)$ is defined for all $t \in \mathbf{R}$. In addition $\varphi(t_o)^{-1} = \varphi(-t_o)$ and, since $t_o$ is arbitrary, we obtain $\varphi(t+s) = \varphi(t) \cdot \varphi(s)$.

b) Prove that if $G$ has a bi-invariant metric $\langle\ ,\ \rangle$ then the geodesics of $G$ that start from $e$ are 1-parameter subgroups of $G$.

*Hint*: Use the relation (see Eq. (9) of Chap. 2)

$$\begin{aligned} 2\langle X, \nabla_Z Y\rangle = {} & Z\langle X, Y\rangle + Y\langle X, Z\rangle - X\langle Y, Z\rangle \\ & + \langle Z, [X,Y]\rangle + \langle Y, [X,Z]\rangle - \langle X, [Y,Z]\rangle \end{aligned}$$

and the fact that the metric is left invariant to prove that $\langle X, \nabla_Y Y\rangle = \langle Y, [X,Y]\rangle$, where $X, Y$ and $Z$ are left invariant fields. Use also the fact that the bi-invariance of $\langle\, ,\, \rangle$ implies that

$$\langle [U,X],V\rangle = -\langle U,[V,X]\rangle, \qquad X,U,V \in \mathcal{G}.$$

It follows that $\nabla_Y Y = 0$, for all $Y \in \mathcal{G}$. Thus 1-parameter subgroups are geodesics. By uniqueness, geodesics are 1-parameter subgroups.

4. A subset $A$ of a differentiable manifold $M$ is *contractible* to a point $a \in A$ when the mapping $\mathrm{id}_A$ (identity on $A$) and $k_a : x \in A \to a \in A$ are homotopic (with base point $a$). $A$ is *contractible* if it is contractible to one of its points.
   a) Show that a convex neighborhood in a Riemannian manifold $M$ is a contractible subset (with respect to any of its points).
   b) Let $M$ be a differentiable manifold. Show that there exists a covering $\{U_\alpha\}$ of $M$ with the following properties:
      i) $U_\alpha$ is open and contractible, for each $\alpha$.
      ii) If $U_{\alpha_1}, \ldots, U_{\alpha_r}$ are elements of the covering, then $\bigcap_1^r U_{\alpha_i}$ is contractible

5. Let $M$ be a Riemannian manifold and $X \in \mathcal{X}(M)$. Let $p \in M$ and let $U \subset M$ be a neighborhood of $p$. Let $\varphi\colon (-\varepsilon,\varepsilon)\times U \to M$ be a differentiable mapping such that for any $q \in U$ the curve $t \to \varphi(t,q)$ is a trajectory of $X$ passing through $q$ at $t = 0$ ($U$ and $\varphi$ are given by the fundamental theorem for ordinary differential equations, Cf. Theorem 2.2). $X$ is called a *Killing field* (or an *infinitesimal isometry*) if, for each $t_o \in (-\varepsilon,\varepsilon)$, the mapping $\varphi(t_o, \ )\colon U \subset M \to M$ is an isometry.
   Prove that:
   a) A vector vield $v$ on $\mathbf{R}^n$ may be seen as a map $v : \mathbf{R}^n \to \mathbf{R}^n$; we say that the field is linear if $v$ is a linear map. A linear field on $\mathbf{R}^n$, defined by a matrix $A$, is a Killing field if and only if $A$ is anti-symmetric.
   b) Let $X$ be a Killing field on $M$, $p \in M$, and let $U$ be a normal neighborhood of $p$ on $M$. Assume that $p$ is a unique point of $U$ that satisfies $X(p) = 0$. Then, in $U$, $X$ is tangent to the geodesic spheres centered at $p$.
   c) Let $X$ be a differentiable vector field on $M$ and let $f\colon M \to N$ be an isometry. Let $Y$ be a vector field on $N$ defined

by $Y(f(p)) = df_p(X(p))$, $p \in M$. Then $Y$ is a Killing field if and only if $X$ is also a Killing vector field.

d) $X$ is Killing $\Leftrightarrow \langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all $Y, Z \in \mathcal{X}(M)$ (the equation above is called the *Killing equation*).

*Hint for* $\Rightarrow$: By continuity, it suffices to prove the equation above for points $q \in U$ where $X(q) \neq 0$. If this is the case, let $S \subset U$ be a submanifold of $U$, passing through $q$, normal to $X(q) \neq 0$ at $q$, with $\dim S = \dim M - 1$. Let $(x_1, \ldots, x_{n-1})$ be coordinates in a neighborhood $V \subset S$ of $q$ such that $(x_1, \ldots, x_{n-1}, t)$ are coordinates in a neighborhood $V \times (-\varepsilon, \varepsilon) \subset U$ and $X = \frac{\partial}{\partial t}$. Putting $X_i = \frac{\partial}{\partial x_i}$, obtain

$$\langle \nabla_{X_j} X, X_i\rangle + \langle \nabla_{X_i} X, X_j\rangle = X\langle X_i, X_j\rangle - \langle [X, X_i], X_j\rangle \\ - \langle [X, X_j], X_i\rangle = \frac{\partial}{\partial t}\langle X_i, X_j\rangle = 0,$$

where in the last equality the fact was used that $X$ is a Killing field.

e) Let $X$ be a Killing field on $M$ with $X(q) \neq 0$, $q \in M$. Then there exists a system of coordinates $(x_1, \ldots, x_n)$ in a neighborhood of $q$, so that the coefficients $g_{ij}$ of the metric in this system coordinates do not depend on $x_n$.

6. Let $X$ be a Killing field (Cf. Exercise 5) on a connected Riemannian manifold $M$. Assume that there exists a point $q \in M$ such that $X(q) = 0$ and $\nabla_Y X(q) = 0$, for all $Y(q) \in T_qM$. Prove that $X \equiv 0$.
*Hint*: Show that, for all $t$, the local isometry $\varphi(t, \ ): U \subset M \to M$ generated by the field $X$ (Cf. Exercise 5) leaves the point $q$ fixed and its differential at $q$ , as a linear map of $T_qM$, is the identity. For this, observe that $d\varphi_t : T_qM \to T_qM$ for all $t$. In addition, $[X, Y](q) = (\nabla_X Y - \nabla_Y X)(q) = 0$, by hypothesis. Since

$$0 = [Y, X](q) = \lim_{t \to 0} \frac{1}{t}[d\varphi_t - \mathrm{Id}](Y) = \frac{d}{dt}(d\varphi_t)\Big|_{t=0}$$

and $d\varphi_{s+t} = d\varphi_s \cdot d\varphi_t$, conclude that $d\varphi_t$ does not depend on $t$, and it is equal to Id. Now use the exponential map to show that such an isometry is the identity on $M$.

7. (*Geodesic frame*). Let $M$ be a Riemannian manifold of dimension $n$ and let $p \in M$. Show that there exists a neighborhood $U \subset M$ of $p$ and $n$ vector fields $E_1, \dots, E_n \in \mathcal{X}(U)$, orthonormal at each point of $U$, such that, at $p$, $\nabla_{E_i} E_j(p) = 0$.
Such a family $E_i$, $i = 1, \dots, n$, of vector fields is called a *(local) geodesic frame* at $p$.

8. Let $M$ be a Riemannian manifold. Let $X \in \mathcal{X}(M)$ and $f \in \mathcal{D}(M)$. Define the *divergence* of $X$ as a function $\operatorname{div} X \colon M \to \mathbf{R}$ given by $\operatorname{div} X(p) =$ trace of the linear mapping $Y(p) \to \nabla_Y X(p)$, $p \in M$, and the *gradient* of $f$ as a vector field $\operatorname{grad} f$ on $M$ defined by

$$\langle \operatorname{grad} f(p), v \rangle = df_p(v), \quad p \in M, \ v \in T_pM.$$

a) Let $E_i$, $i = 1, \dots, n = \dim M$, be a geodesic frame at $p \in M$ (See Exercise 7). Show that:

$$\operatorname{grad} f(p) = \sum_{i=1}^{n} (E_i(f)) E_i(p),$$

$$\operatorname{div} X(p) = \sum_{i=1}^{n} E_i(f_i)(p), \quad \text{where} \quad X = \sum_i f_i E_i.$$

b) Suppose that $M = \mathbf{R}^n$, with coordinates $(x_1, \dots, x_n)$ and $\frac{\partial}{\partial x_i} = (0, \dots, 1, \dots, 0) = e_i$. Show that:

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i,$$

$$\operatorname{div} X = \sum_i \frac{\partial f_i}{\partial x_i}, \quad \text{where} \quad X = \sum_i f_i e_i.$$

9. Let $M$ be a Riemannian manifold. Define an operator $\Delta \colon \mathcal{D}(M) \to \mathcal{D}(M)$ (the *Laplacian* of $M$) by

$$\Delta f = \operatorname{div} \operatorname{grad} f, \qquad f \in \mathcal{D}(M).$$

a) Let $E_i$ be a geodesic frame at $p \in M$, $i = 1, \dots, n = \dim M$ (see Exercise 7). Prove that

$$\Delta f(p) = \sum_i E_i(E_i(f))(p).$$

Conclude that if $M = \mathbf{R}^n$, $\triangle$ coincides with the usual Laplacian, namely, $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$.

b) Show that

$$\Delta(f \cdot g) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle.$$

10. Let $f\colon [0,1] \times [0,a] \to M$ be a parametrized surface such that for all $t_o \in [0,a]$, the curve $s \to f(s,t_o)$, $s \in [0,1]$, is a geodesic parametrized by arc length, which is orthogonal to the curve $t \to f(0,t)$, $t \in [0,a]$, at the point $f(0,t_o)$. Prove that, for all $(s_o,t_o) \in [0,1] \times [0,a]$, the curves $s \to f(s,t_o)$, $t \to f(s_o,t)$ are orthogonal.
*Hint*: Differentiate $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle$ with respect to $s$, obtaining

$$\begin{aligned}\frac{d}{ds}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle &= \langle \frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\rangle + \langle \frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\rangle \\ &= \frac{1}{2}\frac{d}{dt}\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\rangle = 0,\end{aligned}$$

where we used the symmetry of the connection and the fact that $\frac{D}{ds}\frac{\partial f}{\partial s} = 0$.

†11. Let $M$ be an oriented Riemannian manifold. Let $\nu$ be a differential form of degree $n = \dim M$ defined in the following way:

$$\begin{aligned}\nu(v_1,\dots,v_n)(p) &= \pm\sqrt{\det(\langle v_i, v_j\rangle)} \\ &= \text{orient. vol. } \{v_1,\dots,v_n\}, \quad p \in M,\end{aligned}$$

where $v_1,\dots,v_n \in T_p(M)$ are linearly independent, and the oriented volume is affected by the sign + or - depending on whether or not the basis $\{v_1,\dots,v_n\}$ belongs to the orientation of $M$; $\nu$ is called the *volume element* of $M$. For a vector field $X \in \mathcal{X}(M)$ define the *interior product* $i(X)\nu$ of $X$ with $\nu$ as the $(n-1)$-form:

$$i(X)\nu(Y_2,\dots,Y_n) = \nu(X,Y_2,\dots,Y_n),\ Y_2,\dots,Y_n \in \mathcal{X}(M).$$

Prove that

$$d(i(X)\nu) = \operatorname{div} X\,\nu.$$

*Hint*: Let $p \in M$ and let $E_i$ be a geodesic frame at $p$. Write $X$ as a sum, $X = \sum f_i E_i$ and let $\omega_i$ be differential forms of degree one defined on a neighborhood of $p$ by $\omega_i(E_j) = \delta_{ij}$. Show that $\omega_i \wedge \ldots \wedge \omega_n$ is a volume form $\nu$ on $M$. Next put $\theta_i = \omega_1 \wedge \ldots \wedge \hat{\omega}_i \wedge \ldots \wedge \omega_n$, where $\hat{\omega}_i$ signifies that the factor $\hat{\omega}_i$ is not present. Prove that $i(X)\nu = \sum_i (-1)^{i+1} f_i \theta_i$. It then follows that

$$\begin{aligned} d(i(X)\nu) &= \sum_i (-1)^{i+1} df_i \wedge \theta_i + \sum_i (-1)^{i+1} f_i \wedge d\theta_i \\ &= (\sum_i E_i(f_i))\nu + \sum_i (-1)^{i+1} f_i \wedge d\theta_i. \end{aligned}$$

But $d\theta_i = 0$ at $p$, since

$$\begin{aligned} d\omega_k(E_i, E_j) &= E_i \omega_k(E_j) - E_j \omega_k(E_i) - \omega_k([E_i, E_j]) \\ &= \omega_k(\nabla_{E_i} E_j - \nabla_{E_j} E_i). \end{aligned}$$

Therefore

$$d(i(X)\nu)(p) = (\sum_i E_i(f_i)(p))\nu = \operatorname{div} X(p)\nu$$

and since $p$ is arbitrary, this completes the proof.
*Remark*. The result obtained implies that the notion of the divergence of $X$ makes sense on an oriented differentiable manifold on which a "volume element" has been chosen, that is, an $n$-form $\nu$ which takes positive values on positive bases.

†12. (*Theorem of E. Hopf*). Let $M$ be a compact orientable Riemannian manifold which is also connected. Let $f$ be a differentiable function on $M$ with $\Delta f \geq 0$. Then $f = \text{const}$. In particular, the harmonic functions on $M$, that is, those for which $\Delta f = 0$, are constant.
*Hint*: Take $\operatorname{grad} f = X$. Using Stokes theorem and the result of exercise 11, obtain

$$\int_M \Delta f \nu = \int_M \operatorname{div} X \nu = \int_M d(i(X)\nu) = \int_{\partial M} i(X)\nu = 0.$$

Since $\Delta f \geq 0$, we have $\Delta f = 0$. Using again Stokes theorem on $f^2/2$, and the result of exercise 9(b), we obtain

$$0 = \int_M \Delta(f^2/2)\nu = \int_M f\Delta f\nu + \int_M |\mathrm{grad}\, f|^2\,\nu$$

$$= \int_M |\mathrm{grad}\, f|^2\,\nu,$$

which together with the connectedness of $M$, implies that $f =$ const..

†13. Let $M$ be a Riemannian manifold and $X \in \mathcal{X}(M)$. Let $p \in M$ such that $X(p) \neq 0$. Choose a coordinate system $(t, x_2, \ldots, x_n)$ in a neighborhood $U$ of $p$ such that $\frac{\partial}{\partial t} = X$. Show that if $\nu = g\ dt \wedge dx_2 \wedge \ldots \wedge dx_n$ is a volume element of $M$, then

$$i(X)\nu = g\, dx_2 \wedge \ldots \wedge dx_n.$$

Conclude from this, using the result of Exercise 11, that

$$\mathrm{div} X = \frac{1}{g}\frac{\partial g}{\partial t}.$$

This proves that div$X$ intuitively measures the degree of variation of the volume element of $M$ along the trajectories of $X$.

14. (*Liouville's Theorem*). Prove that if $G$ is the geodesic field on $TM$ then div$G = 0$. Conclude from this that the geodesic flow preserves the volume of $TM$.
*Hint*: Let $p \in M$ and consider a system $(u_1, \ldots, u_n)$ of *normal coordinates* at $p$. Such coordinates are defined in a normal neighborhood $U$ of $p$ by considering an orthonormal basis $\{e_i\}$ of $T_pM$ and taking $(u_1, \ldots, u_n)$ , $q = \exp_p(\sum_i u_i e_i)$, $i = 1, \ldots, n$, as coordinates of $q$. In such a coordinate system, $\Gamma^k_{ij}(p) = 0$, since the geodesics that pass through $p$ are given by linear equations. Therefore if $X = \sum x_i \frac{\partial}{\partial u_i}$, then div $X(p) = \sum \frac{\partial x_i}{\partial u_i}$.
Now let $(u_i)$ be normal coordinates in a neighborhood $U \subset M$ around $p \in M$ and let $(u_i, v_j)$, $v = \sum_j v_j \frac{\partial}{\partial u_j}$, $i, j = 1, \ldots, n$, be coordinates on $TM$. Calculate the volume element

of the natural metric of $TM$ at $(q, v)$, $q \in U$, $v \in T_qM$, and show that it is the volume element of the product metric on $U \times U$ at the point $(q, q)$ (See Exercise 2(e)). Since the divergence of $G$ only depends on the volume element (see Exercise 11), and $G$ is horizontal, we can calculate div$G$ in the product metric. Observe that in the coordinates $(u_i, v_j)$ we have

$$G(u_i) = v_i, \quad G(v_j) = -\sum_{ik} \Gamma^j_{ik} v_i v_k, \quad k = 1, \ldots, n.$$

Since the Christoffel symbols of the product metric on $U \times U$ vanish at $(p, p)$, we obtain finally, at $p$,

$$\mathrm{div} G = \sum_i \frac{\partial v_i}{\partial u_i} - \sum_j \frac{\partial}{\partial v_j} (\sum_{ik} \Gamma^j_{ik} v_i v_k) = 0.$$