Homework 7

21-630 Ordinary Differential Equations
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Problem 1

We first show that 0 is asymptotically stable for the homogeneous

$$\dot{X}(t) = AX, \tag{1}$$

and then use Variation of Parameters to show that stability also holds for Y.

By the comment on page 90 of the notes, it suffices to show that 0 is asymptotically stable in the case that A is in Jordan form. Let $J_0, J_1, \ldots J_k$ be the Jordan blocks of A (with only J_0 diagonal). Since the operator norm of a matrix is at least the operator norm of any submatrix, $\forall t \geq 0, i \in \{0, \ldots, k\}$, if J_i is a $k_i \times k_i$ block associated with generalized eigenvalue λ_i and σ is the eigenvalue with the greatest real part,

$$Ce^{-\lambda t} \geq \left|e^{At}\right| \geq |e^{J_i t}| \geq |e^{J_i t}|_\infty \geq \begin{cases} e^{\sigma t} : \text{ if } i = 0 \\ e^{\lambda_i t} \frac{t^{k_i - 1}}{(k_i - 1)!} : \text{ else} \end{cases}$$

Since the above inequality holds for all $t \geq 0$, $\sigma, \lambda_i < 0$. Then, by Theorem 5.1, 0 is asymptotically stable for equation (1).

Pick δ such that, for all solutions ϕ to equation (1) with initial condition $|\phi(0)| < \delta$, $\phi(t) \to \infty$ as $t \to \infty$, and suppose ϕ is a solution to equation (1) (with $|\phi(0) > 0|$). Then, since solutions to a homoeneous linear system form a vector space, $\frac{\delta \phi(t)}{2\phi(0)} \to 0$ as $t \to \infty$, so that $\phi(t) \to 0$ as $t \to 0$ (i.e., all solutions of equation (1) converge to 0).

By Theorem 4.3 (Variation of Parameters), for any fundamental matrix solution ϕ to equation (1), then, $\forall t \geq 0$,

$$|Y(t)| \leq |\phi(t)\phi^{-1}(0)Y(0)| + \int_0^t |\phi(t)\phi^{-1}(s)||b(s)|\, ds.$$

I wasn't able to finish this solution, but I think that if I were to establish a stronger statement about equation (1), such as that solutions decay *expontially*, then the result could easily be deduced from the above inequality; the fact that $\phi(t) \to 0$ as $t \to \infty$ doesn't seem quite sufficient, as the integral is still positive and non-decreasing with t. ∎

Problem 2

(A) For $P = \begin{pmatrix} i/\omega & -i/\omega \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix}$, so that $P^{-1} = \frac{1}{2}\begin{pmatrix} -i\omega & 1 \\ i\omega & 1 \end{pmatrix}$, $A = PDP^{-1}$.

Then, since D is diagonal,

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= P\begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} P^{-1} \\ &= P\begin{pmatrix} \cos(\omega t) - i\sin(\omega t) & 0 \\ 0 & \cos(\omega t) + i\sin(\omega t) \end{pmatrix} P^{-1} \\ &= P\left(\cos(\omega t) I + \omega^{-1}\sin(\omega t) D\right) P^{-1} \\ &= \cos(\omega t) PIP^{-1} + \omega^{-1}\sin(\omega t) PDP^{-1} = \cos(\omega t) I + \omega^{-1}\sin(\omega t) A. \quad \blacksquare \end{aligned}$$

(B) We first show, by induction on n, that, $\forall n \in \mathbb{N}$,

$$A^{2n} = (-\omega^2)^n I \quad \text{and} \quad A^{2n+1} = (-\omega^2)^n A.$$

For $n = 0$, this clearly holds. Suppose now that the formula holds for some $n \in \mathbb{N}$. Then,

$$\begin{aligned} A^{2(n+1)} &= A^2 A^{2n} = ((-\omega^2)I)((-\omega^2)^n I) = (-\omega^2)^{n+1} I, \\ \text{and so } A^{2(n+1)+1} &= A^{2(n+1)}A = (-\omega^2)^{n+1} A, \end{aligned}$$

completing the induction. Using this formula for A^k and the Taylor series of sine and cosine,

$$\begin{aligned} e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (-\omega^2)^k I + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (-\omega^2)^k A \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\omega t)^{2k}}{(2k)!} I + \omega^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k (\omega t)^{2k+1}}{(2k+1)!} A = \cos(\omega t) I + \frac{\sin(\omega t)}{\omega^{-1}} A. \quad \blacksquare \end{aligned}$$

Problem 3

We first show that 0 is stable for the homogeneous equation

$$\dot{X}(t) = AX, \tag{2}$$

and then use Variation of Parameters to show that stability also holds for Y.

By the comment on page 90 of the notes, it suffices to show that 0 is stable in the case that A is in Jordan form. Let $J_0, J_1, \dots J_k$ be the Jordan blocks of A (with only J_0 diagonal). Since the operator norm of a matrix is at least the operator norm of any submatrix, $\forall t \geq 0, i \in \{0, \dots, k\}$, if J_i is a $k_i \times k_i$ block associated with generalized eigenvalue λ_i and σ is the eigenvalue with the greatest real part,

$$B \geq \left|e^{At}\right| \geq |e^{J_i t}| \geq |e^{J_i t}|_\infty \geq \begin{cases} e^{\sigma t} : \text{ if } i = 0 \\ e^{\lambda_i t} \frac{t^{k_i - 1}}{(k_i - 1)!} : \text{ else} \end{cases}$$

Since the above inequality holds for all $t \geq 0$, $\sigma \leq 0$ and each $\lambda_i < 0$ (as $k_i \geq 2$). Then, by Theorem 5.1, 0 is stable for equation (2).

Let $\varepsilon > 0$, and define $\beta := \int_0^\infty |b(s)|\, ds$ (we assume the Lebesgue integral, so that $|b|$ has finite integral). By Theorem 4.3 (Variation of Parameters), for any fundamental matrix solution ϕ to equation (2), $\forall t \geq 0$,

$$\begin{aligned} |Y(t)| &\leq |\phi(t)\phi^{-1}(0)Y(0)| + \int_0^t |\phi(t)\phi^{-1}(s)||F(s, Y(s))|\, ds \\ &\leq |\phi(t)\phi^{-1}(0)Y(0)| + \int_0^t |\phi(t)\phi^{-1}(s)||b(s)||Y(s)|\, ds. \end{aligned}$$

Since $\phi(t)\phi^{-1}(s)$ is a solution of equation (2) with initial condition $\phi(s)\phi^{-1}(s) = I$, $\delta\phi(t)\phi^{-1}(s)$ is also a solution, with initial condition δI. Thus, since 0 is stable for equation (2), by choosing δ sufficiently small, $|\delta\phi(t)\phi^{-1}(s)| \leq 1$, and thus $\phi(t)\phi^{-1}(s)| \leq \delta^{-1}$, for all $s, t \geq 0$, so that

$$|Y(t)| \leq |Y(0)|\delta^{-1} + \int_0^t \delta^{-1}|b(s)||Y(s)|\, ds.$$

Then, by the full version of Gronwall's Inequality, $\forall t \geq 0$,

$$\begin{aligned} |Y(t)| &\leq |Y(0)|\delta^{-1} + \int_0^t |Y(0)|\delta^{-1}\delta^{-1}|b(s)|e^{\int_s^t \delta^{-1}|b(\tau)|\, d\tau}\, ds \\ &\leq |Y(0)| \left(\delta^{-1} + \delta^{-2}\int_0^t |b(s)|e^{\delta^{-1}\beta}\right)\, ds \leq |Y(0)| \left(\delta^{-1} + \delta^{-2}\beta e^{\delta^{-1}\beta}\right) \leq \varepsilon, \end{aligned}$$

for $|Y(0)| \leq \left(\delta^{-1} + \delta^{-2}\beta e^{\delta^{-1}\beta}\right)^{-1}$, so that 0 is stable for the non-homogeneous equation. ■

Problem 4

Let $\delta > 0$, and suppose $Y(0) = \delta/2$. Solving by separation of variables gives, for some $C \in \mathbb{R}$,

$$\log(Y(t)) = \int \frac{1}{Y(t)}\,dY = \int \frac{1}{1+t}\,dt = \log(1+t) + C,$$

so that $Y(t) = e^C(1+t)$, and thus, using the initial condition $Y(0) = \delta/2$, $Y(t) = (\delta/2)(1+t) \to \infty$ as $t \to \infty$. It follows that 0 is not stable. $\blacksquare$