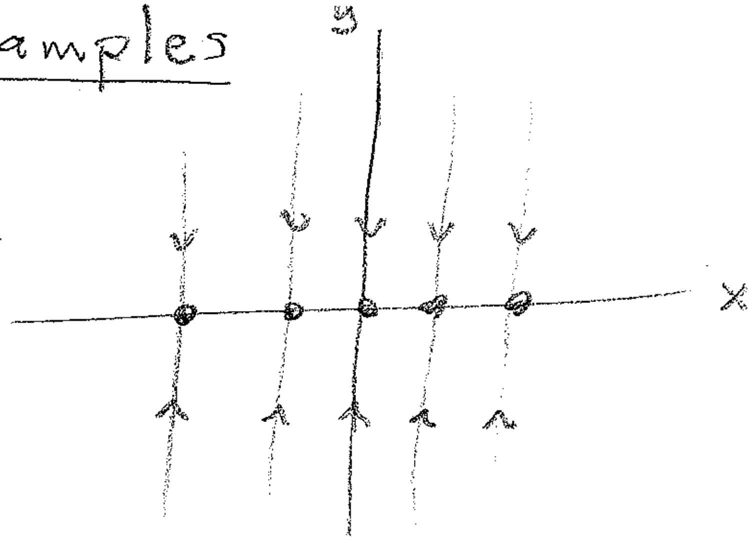
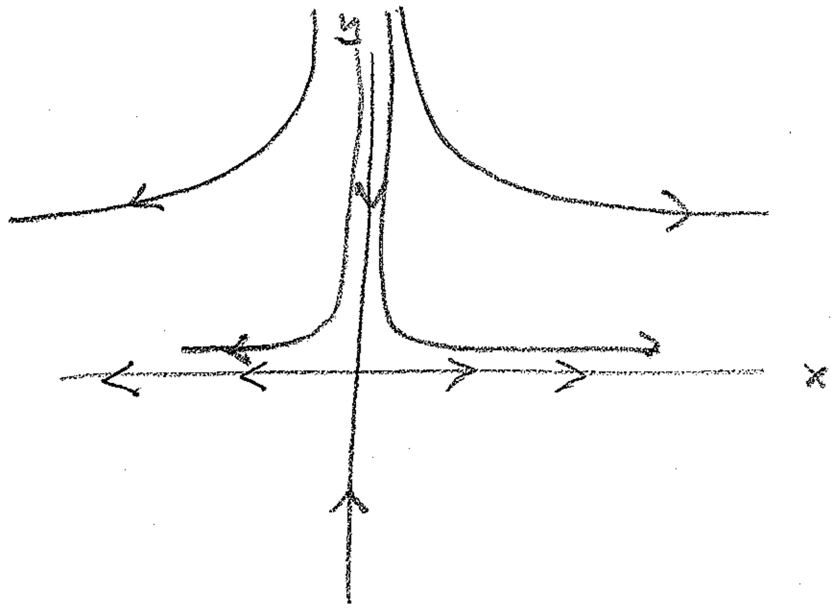
The Center Manifold Theorem

Examples

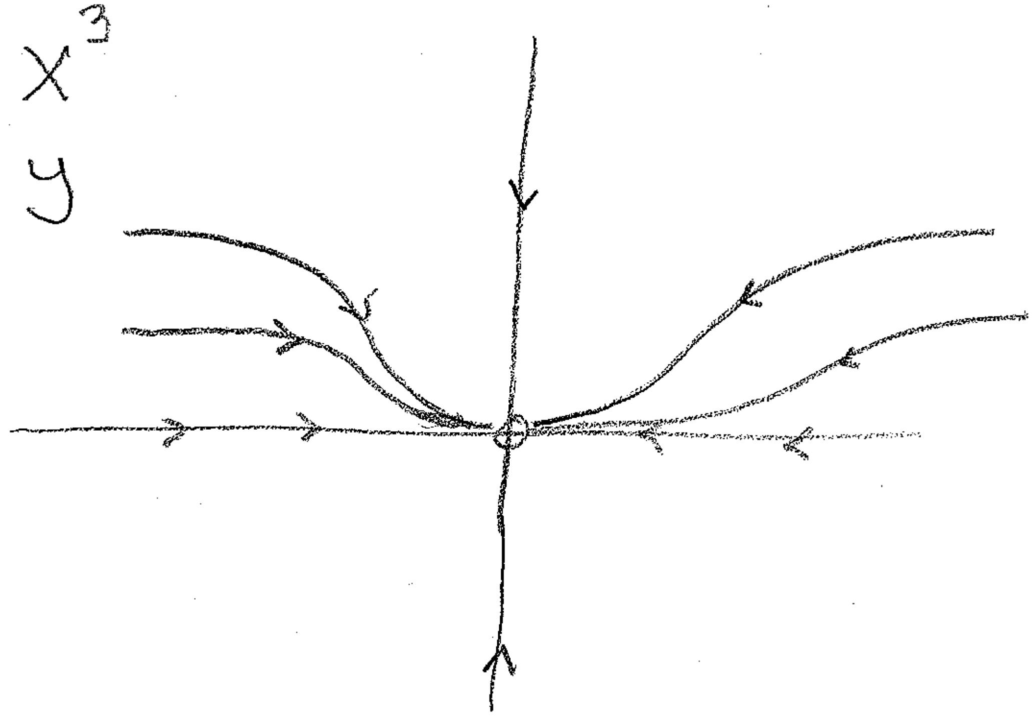
1. $\dot{x} = 0$
   $\dot{y} = -y$



2. $\dot{x} = x^3$
   $\dot{y} = -y$



3. $\dot{x} = -x^3$
   $\dot{y} = -y$



<u>Theorem</u> Assume $F : (x,y) \in \mathbb{R}^c \times \mathbb{R}^s \to \mathbb{R}^c$ and $G : (x,y) \mapsto \mathbb{R}^s$ are C^r near $(0,0)$ with $r \geq 1$,

$$F(0,0) = 0 \ , \ DF(0,0) = 0$$
$$G(0,0) = 0 \ , \ DG(0,0) = 0.$$

Let C be a c by c matrix with c eigenvalues with real part 0. Let S be an s by s matrix with s eigenvalues with real part < 0. Consider

$$(FS) \quad \begin{cases} \dot{X} = CX + F(X,y) \\ \dot{y} = Py + G(X,y). \end{cases}$$

Then there is $\delta > 0$ and $h : x \to \mathbb{R}^s$ which is C^r on $\{x : |x| < \delta\}$, $h(0) = 0$, $Dh(0) = 0$,

$$CM = \{(x,y) : |x| < \delta \text{ and } y = h(x)\}$$

is "locally" invariant.

<u>Comments</u>

1. Suppose $(X(0), y(0)) \in CM$. Let

$$T = \sup\{t > 0 : (X(s), y(s)) \in CM \ \forall s \in [0,t]\}.$$

Then T finite $\Rightarrow |X(T)| = \delta.$

2. For $(X(t), y(t)) \in CM \quad \forall t \in [0,T)$

$$y(t) = h(X(t)),$$

$$\dot{X}(t) = CX(t) + F(X(t), y(t)),$$

$$\text{(RS)} \qquad \dot{X} = CX + F(X, h(X)).$$

Also

$$\dot{y} = Sy + G(X,y) = Sh(X) + G(X, h(X))$$

$$= \frac{d}{dt}(h(X)) = Dh(X)\dot{X}$$

$$= Dh(X)(CX + F(X, h(X)))$$

so

$$Dh(x)(Cx + F(x, h(x))) = Sh(x) + G(x, h(x))$$

$$h(0) = 0 \qquad Dh(0) = 0$$

3. h is not always unique.

4. If the origin is asymptotically stable (unstable) for RS then the origin is asymptotically stable for (FS).

Examples

1. $$\dot{x} = xy \qquad C=0 \quad F = xy$$
$$\dot{y} = -y - x^2 \qquad S=-1 \quad G = -x^2$$

$$h'(x)\big(Cx + F(x,h(x))\big) = h'(x)\,x\,h(x)$$
$$= Sh(x) + G(x,h(x)) = -h(x) - x^2$$

$$\begin{cases} h'(x)\,x\,h(x) = -h(x) - x^2 \\ h(0) = h'(0) = 0 \end{cases}$$

Attempt $h(x) = ax^2 + bx^3 + O(x^4)$:

$$\big(2ax + 3bx^2 + O(x^3)\big)\,x\,\big(ax^2 + bx^3 + O(x^4)\big)$$
$$= O(x^4)$$
$$= -\big(ax^2 + bx^3 + O(x^4)\big) - x^2$$

so

$$a = -1 \quad \text{and} \quad b = 0,$$
$$h(x) = -x^2 + O(x^4).$$

Now on CM

(RS) $$\dot{X} = CX + F(X, h(X)) = X\,h(X)$$
$$= -X^3 + O(X^5).$$

0 is asympt. stable for (RS) so

$(0,0)$ is asympt. stable for (FS).

2. $\dot{x}_1 = -x_2 + x_1 y$

$\dot{x}_2 = x_1 + x_2 y$

$\dot{y} = -y - x_1^2 - x_2^2 + y^2$

$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} x_1 y \\ x_2 y \end{pmatrix}$

$S = -1 \qquad G = -x_1^2 - x_2^2 + y^2$

$Dh(x)(Cx + F(x,h(x))) = (h_{x_1}(x), h_{x_2}(x)) \begin{pmatrix} -x_2 + x_1 h(x) \\ x_1 + x_2 h(x) \end{pmatrix}$

$= h_{x_1}(x)(-x_2 + x_1 h(x)) + h_{x_2}(x)(x_1 + x_2 h(x))$

$= Sh(x) + G(x,h(x)) = -h(x) - x_1^2 - x_2^2 + h^2(x)$

Attempt $h(x) = ax_1^2 + bx_1x_2 + cx_2^2 + O_3(x)$:

$(2ax_1 + bx_2)(-x_2 + O_3(x)) + (bx_1 + 2cx_2)(x_1 + O_3(x))$

$= bx_1^2 + 2(c-a)x_1x_2 - bx_2^2 + O_3(x)$

$= -ax_1^2 - bx_1x_2 - cx_2^2 - x_1^2 - x_2^2 + O_3(x)$

$b = -a - 1$

$2(c-a) = -b$

$-b = -c - 1$

$a = c = -1$

$b = 0$

$h(x) = -(x_1^2 + x_2^2) + O_3(x).$

On CM

(RS) $\dot{X} = CX + F(X, h(X)) = \begin{pmatrix} -X_2 + h(X)X_1 \\ X_1 + h(X)X_2 \end{pmatrix}$

$$= \begin{pmatrix} -X_2 - (X_1^2+X_2^2)X_1 \\ X_1 - (X_1^2+X_2^2)X_2 \end{pmatrix} + O_4(X)$$

so

$$\dot{r} = \cos\theta(-\cancel{r\sin\theta} - r^3\cos\theta)$$
$$+ \sin\theta(\cancel{r\cos\theta} - r^3\sin\theta) + O_4(r)$$
$$= -r^3 + O_4(r).$$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymp. stable for (RS) so $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is asymp. stable for (FS).

3. $\dot{x} = -x^3$ $\quad C = 0 \quad F = -x^3$

$\dot{y} = -y$ $\quad S = -1 \quad G = 0$

$$h'(x)(Cx + F(x, h(x))) = -h'(x)x^3$$

$$= Sh(x) + G(x, h(x)) = -h(x)$$

$$0 = (h'(x) - x^{-3}h(x))\, e^{\frac{1}{2x^2}} = \left(h(x)\, e^{\frac{1}{2x^2}}\right)'$$

$$h(x) = C\, e^{-\frac{1}{2x^2}}$$

Note this satisfies $h(0) = h'(0) = 0 \ \forall C$.

4, $\dot{u} = w$
$\dot{w} = -w + u^2$

Note in the right form, since this needs to be $Cu + F(u,w)$.

Consider linear system:

$$\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = M \begin{pmatrix} u \\ w \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad M \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Recall: if M is diagonalizable then

$$M v^{(i)} = \lambda_i v^{(i)} \qquad i = 1, \dots, N$$

$$P = \left(v^{(1)}, \dots, v^{(N)} \right)$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

then

$$P^{-1} M P = D$$

since (letting $e^{(i)}_j = \delta_{ij}$)

$$P^{-1} M P e^{(i)} = P^{-1} M v^{(i)} = P^{-1} \lambda_i v^{(i)}$$
$$= \lambda_i P^{-1} v^{(i)} = \lambda_i e^{(i)} = D e^{(i)} \quad \forall i.$$

Now in this example

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

let

$$\begin{pmatrix} x \\ y \end{pmatrix} = P^{-1}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u+w \\ -w \end{pmatrix}$$

then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = P^{-1}\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = P^{-1}M\begin{pmatrix} u \\ w \end{pmatrix}$$

$$= P^{-1}MP\begin{pmatrix} x \\ y \end{pmatrix} = D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -y \end{pmatrix}.$$

Consider nonlinear system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = P^{-1}\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = P^{-1}\begin{pmatrix} w \\ -w+u^2 \end{pmatrix}$$

$$= P^{-1}\left(M\begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 \end{pmatrix}\right)$$

$$= P^{-1}MP\begin{pmatrix} x \\ y \end{pmatrix} + P^{-1}\begin{pmatrix} 0 \\ u^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -y \end{pmatrix} + P^{-1}\begin{pmatrix} 0 \\ u^2 \end{pmatrix}.$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ -y \end{pmatrix} \quad \text{so}$$

$$P^{-1}\begin{pmatrix} 0 \\ u^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ (x+y)^2 \end{pmatrix}$$

$$= \begin{pmatrix} (x+y)^2 \\ -(x+y)^2 \end{pmatrix}$$

Finally

$$(FS)\quad \begin{aligned} \dot{x} &= (x+y)^2 \\ \dot{y} &= -y-(x+y)^2 \end{aligned} \qquad \begin{aligned} C &= 0 \quad F = (x+y)^2 \\ S &= -1 \quad G = -(x+y)^2 \end{aligned}$$

and we may apply the theorem:

$$h(0) = h'(0) \qquad h(x) = O_2(x)$$

so,

$$(RS)\quad \begin{aligned} \dot{x} &= Cx + F(x, h(x)) \\ &= 0 + (x + h(x))^2 = x^2 + O_3(x). \end{aligned}$$

0 is unstable for (RS) so

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unstable for (FS) and the original system in u & w.