Zorn's Lemma

In order to prove the Hahn-Banach theorem, we shall make use of a result known as *Zorn's Lemma*.

Zorn's lemma is equivalent to the *axiom of choice* in the sense that the Zermelo-Fraenkel axioms of set theory, together with the axiom of choice imply Zorn's lemma and the Zermelo-Fraenkel axioms, together with Zorn's lemma imply the axiom of choice. Zorn's lemma is more convenient to use than the axiom of choice in a number of important situations. In particular, Zorn's lemma is well adapted to proving the existence of a Hamel basis in an arbitrary linear space. It is also useful for proving Tychonoff's theorem in topology, the existence of maximal ideals in rings, the existence of algebraic closures of fields, and the existence of maximal extensions of solutions of differential equations.

Before we can give a precise statement of the lemma, we need some definitions concerning *partially ordered sets*.

Definition Z.1: A *partially ordered set* is a pair $(\mathcal{M}, \leq)$ where $\mathcal{M}$ is a set and $\leq$ is a binary relation on $\mathcal{M}$ satisfying the following three conditions:

(i) $\forall x \in \mathcal{M}$, we have $x \leq x$,

(ii) $\forall x, y \in \mathcal{M}$, if $x \leq y$ and $y \leq x$ then $x = y$,

(iii) $\forall x, y, z \in \mathcal{M}$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

The relation $\leq$ is called a *partial order* on $\mathcal{M}$.

Definition Z.2: Let $(\mathcal{M}, \leq)$ be a partially ordered set and let $x, y \in \mathcal{M}$ be given. We say that x *precedes* y (or equivalently that y *follows* x) provided $x \leq y$. We say that x and y are *comparable* provided that x precedes y or y precedes x. If neither x precedes y nor y precedes x we say that x and y are *incomparable*.

An obvious example of a partially ordered set is $(\mathbb{R}, \leq)$, the real numbers together with the usual notion of less than or equal to. Here we have the additional property that for all $x, y \in \mathbb{R}$, x and y are comparable. It is easy to produce a partially ordered set with incomparable elements.

Example Z.3: Let S be any set and let $\mathcal{M} = \mathcal{P}(S)$, the *power set* of S (i.e., the collection of all subsets of S). Define the relation $\leq$ on $\mathcal{M}$ by $A \leq B$ if and only if $A \subset B$. It is easy to see that $(\mathcal{M}, \leq)$ is a partially ordered set. If S contains two or more elements, then there are incomparable elements of $\mathcal{M}$.

Definition: Let $(\mathcal{M}, \leq)$ be a partially ordered set. A set $\mathcal{C} \subset \mathcal{M}$ is said to be a *chain* (or a *totally ordered subset*) provided that for all $x, y \in \mathcal{C}$, x and y are comparable (i.e., for all $x, y \in \mathcal{C}$, we have $x \leq y$ or $y \leq x$).

Definition Z.4: Let $(\mathcal{M}, \leq)$ be a partially ordered set and $\mathcal{A} \subset \mathcal{M}$. An element $x \in \mathcal{M}$ is called an *upper bound* for $\mathcal{A}$ provided that $y \leq x$ for all $y \in \mathcal{A}$.

Definition Z.5: Let $(\mathcal{M}, \leq)$ be a partially ordered set. An element $z \in \mathcal{M}$ is called a *maximal element* provided that for all $x \in \mathcal{M}$ if $z \leq x$ then $z = x$. In other words, a maximal element is not followed by an element other than itself.

Remark Z.6: A maximal element of $(\mathcal{M}, \leq)$ need not be an upper bound for $\mathcal{M}$.

Lemma Z.7 (Zorn's Lemma): Let $(\mathcal{M}, \leq)$ be a partially ordered set and assume that every chain has an upper bound. Then $(\mathcal{M}, \leq)$ has a maximal element.