21-470 **Calculus of Variations** **Spring 2014**

Assignment 4
Due on Monday, March 3

1. Let
$$\mathscr{Y} = \left\{ y \in C^1[0,1] : y(0) = 0,\ y(1) = 1 \right\}$$
and define $J : \mathscr{Y} \to \mathbb{R}$ by
$$J(y) = \int_0^1 y'(x)^2 dx + \left(\int_0^1 y(x) dx \right)^2 \quad \text{for all } y \in \mathscr{Y}.$$
Discuss the problem of minimizing J on $\mathscr{Y}$.

2. (a) Let $a, b, A \in \mathbb{R}$ with $a < b$ and $g : \mathbb{R} \to \mathbb{R},\ f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given. Choose a value for k that is either 1 or 2 (completely up to you). Assume that f and g are of class C^k. Let
$$\mathscr{Y} = \left\{ y \in C^k[a,b] : y(a) = A \right\}$$
and define $J : \mathscr{Y} \to \mathbb{R}$ by
$$J(y) = \int_a^b f(x, y(x), y'(x)) dx + g(y(b)) \quad \text{for all } y \in \mathscr{Y}.$$
Discuss the problem of maximizing or minimizing J on $\mathscr{Y}$.

   (b) Apply your results from part (a) to find all possible maximizers and minimizers for
$$J(y) = \int_0^1 \left[y(x)^2 + y'(x)^2 \right] dx + y(1)^2 \text{ on } \mathscr{Y} = \left\{ y \in C^k[0,1] : y(0) = 1 \right\}.$$

   If possible, determine whether or not each candidate is a maximizer, minimizer, or neither.

3. Let
$$\mathscr{S} = \left\{ y \in C^1[1,e] : y(1) = y(e) = 0,\ \int_1^e y(x)^2 dx = 1 \right\},$$
and define $J : \mathscr{S} \to \mathbb{R}$ by
$$J(y) = \int_1^e x^2 y'(x)^2 dx \ \text{ for all } y \in \mathscr{S}.$$
Find all possible maximizers and minimizers for J on $\mathscr{S}$. If possible, determine whether each candidate is a maximizer, minimizer, or neither.

4. Let

$$\mathscr{S} = \{y \in C^1[0, 2\pi] : y(0) = y(2\pi) = 0, \int_0^{2\pi} y(x)\, dx = 0\},$$

and

$$J(y) = \int_0^{2\pi} [y'(x)^2 - y(x)^2]\, dx \quad \text{for all } y \in \mathscr{S}.$$

Find all possible maximizers and minimizers for J on $\mathscr{S}$. If possible, determine whether each candidate is a maximizer, minimizer, or neither.

5. Let

$$\mathscr{S} = \left\{ y \in C^1[-1, 1] : y(-1) = y(1) = 0, \int_{-1}^{1} \sqrt{1 + y'(x)^2} dx = 4 \right\},$$

and

$$J(y) = \int_{-1}^{1} y(x)\sqrt{1 + y'(x)^2} dx \quad \text{for all } y \in \mathscr{S}.$$

(a) Find all possible minimizers for J on $\mathscr{S}$ (You may need to solve for some constants numerically.)

(b) Use Mathematica, Maple, Matlab (or something similar) to plot the possible minimizers found in part (a).