Homework 7

21-236 Mathematical Studies Analysis II

Name: Shashank Singh

Email: sss1@andrew.cmu.edu

Problem 1

(a) Let Γ be the range of γ_1, and let $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ be a parametrization of γ_1. Since U is open, $\forall \mathbf{x} \in U$, $\exists r > 0$ such that $B(\mathbf{x}, r) \subseteq U$, so that there is a family $\{U_\alpha\}_\alpha$ of open balls around every point in Γ with
$$\bigcup \{U_\alpha\}_\alpha \subseteq U.$$
Clearly, $\{U_\alpha\}_\alpha$ covers Γ. Since γ_1 is continuous and $[a, b]$ is compact, Γ is compact. Thus, there exists some $\delta > 0$ (Lebesgue's number) such that, $\forall E \subseteq \Gamma$ with $\operatorname{diam} E \leq \delta$, $E \subseteq U_\alpha$ for some α. Note that, since γ_1 is continuous and $[a, b]$ is compact, $\boldsymbol{\varphi}$ is uniformly continuous, so that, $\exists \delta_2 > 0$ such that, $\forall x, y \in [a, b]$, $|x - y| < \delta_2$ implies $\|\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(y)\| < \delta$. Define $k = \lceil \frac{b-a}{\delta_2/2} \rceil$, and, $\forall i \in \{0, 1, \dots, k\}$, let $t_i = a + i\frac{b-a}{k}$. Then, $\forall i \in \{0, 1, \dots, k\}$, $t_i - t_{i-1} < \delta_2$, so that, by choice of δ_2, $\operatorname{diam} \boldsymbol{\varphi}([t_{i-1}, t_i]) < \delta$, and therefore, by choice of δ, $\boldsymbol{\varphi}([t_{i-1}, t_i]) \subseteq U_\alpha$ for some α. Define $\mathbf{h} : [a, b] \times [0, 1]$ such that, $\forall (s, t) \in [a, b] \times [0, 1]$, if $s \in [t_{i-1}, t_i]$,
$$\mathbf{h}(s, t) = \boldsymbol{\varphi}(s) + t(L_i(s) - \boldsymbol{\varphi}(s)),$$
where
$$L_i(s) := \boldsymbol{\varphi}(t_{i-1}) + \frac{s - t_{i-1}}{t_i - t_{i-1}}(\boldsymbol{\varphi}(t_i) - \boldsymbol{\varphi}(t_{i-1}))$$
is the point that is the same fraction along the line segment from $\boldsymbol{\varphi}(t_{i-1})$ to $\boldsymbol{\varphi}(t_i)$ as is s on the line segment from t_{i-1} to t_i.

Since each $\boldsymbol{\varphi}([t_{i-1}, t_i])$ is contained in some ball U_α, which must be convex, $\mathbf{h}([a, b] \times [0, 1]) \subseteq U$. $\forall s \in [a, b]$, $\mathbf{h}(s, 0) = \boldsymbol{\varphi}(s)$, and, $\forall t \in [0, 1]$, $\mathbf{h}(a, t) = \boldsymbol{\varphi}(a) = \boldsymbol{\varphi}(b) = h(b, t)$. Finally, each $\mathbf{h}([t_i, t_{i-1}], 1)$ is linear, so that $h([a, b], 1)$ is a closed polygonal path. Therefore, h is a homotopy from γ_1 to a closed polygonal path. ■

(b) Let $bfh_1 = \mathbf{h}$ as defined in part (a), and let $\boldsymbol{\psi} : [a, b] \to U$ such that, $\forall s \in [a, b]$, $\boldsymbol{\psi}(s) = \mathbf{h}_1(s, 1)$, so that $\boldsymbol{\psi}$ parametrizes a polygonal path to which γ_1 is homotopic. Since U is open, $\forall i \in \{1, 2, \dots, k\}$, $\exists r_i > 0$ such that $B_i := B(\boldsymbol{\psi}(t_i), r_i) \subseteq U$. Note that, since $\boldsymbol{\psi}$ is continuous, $\boldsymbol{\psi}^{-1}(B_i)$ is an open interval (a_i, b_i), and consider $\mathbf{h}_2 : [a, b] \times [0, 1]$ such that, $\forall t \in [0, 1]$, $\forall s \in (a_i, b_i)$,
$$\mathbf{h}_2(s, t) = L_i(s) + t\frac{s - a_i}{b_i - a_i}(L_{i+1}(s) - L_i(s)),$$
where L_i is as defined in part (a), and, for all other $s \in [a, b]$, $\mathbf{h}_2(s, t) = L_i(s)$.

Let $\mathbf{h} : [a, b] \times [0, 1]$ so that, $\forall (s, t) \in [a, b] \times [0, 1]$,
$$\mathbf{h}(s, t) = \begin{cases} \mathbf{h}_1(s, 2t) & t \in [0, \frac{1}{2}] \\ \mathbf{h}_2(s, 2t - 1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Then, $\forall s \in [a,b]$, $\mathbf{h}(s,0) = \boldsymbol{\varphi}(s)$ (where $\boldsymbol{\varphi}$ is the parametrization of γ_1 from part (a)) and $\mathbf{h}(s,1)$ parametrizes a closed C^1 curve, and $\forall t \in [0,1]$, $\mathbf{h}(a,t) = \mathbf{h}(b,t)$. Since each $B_i \subseteq U$ is convex and $\mathbf{h}_1([a,b] \times [0,1]) \subseteq U$, $\mathbf{h}([a,b] \times [0,1]) \subseteq U$. Therefore, $\mathbf{h}$ is a homotopy in U from γ_1 to a closed C^1 curve. $\blacksquare$

Problem 2

(a) **Lemma:** If $F = \cup_{i=1}^{\infty} F_i$ is a union of disjoint Peano-Jordan measurable sets and F is Peano-Jordan measurable, then

$$\operatorname{meas} F \leq \sum_{i=1}^{\infty} \operatorname{meas} F_i.$$

Proof: Suppose, for sake of contradiction, that $\sum_{i=1}^{\infty} \operatorname{meas} F_n < \operatorname{meas} F$. Note that, since $\cup_{i=2}^{n} F_i \subseteq F \backslash F_1$, $\sum_{i=1}^{n} \operatorname{meas} F_i \leq \operatorname{meas}(F \backslash F_1)$, so that

$$\sum_{i=1}^{\infty} \operatorname{meas} F_i \leq \operatorname{meas}(F \backslash F_1).$$

Thus,

$$\begin{aligned}
\operatorname{meas} F_1 &= \sum_{i=1}^{\infty} \operatorname{meas} F_i - \sum_{i=2}^{\infty} \operatorname{meas} F_i \\
&< \operatorname{meas} F - \sum_{i=2}^{\infty} \operatorname{meas} F_i \\
&\leq \operatorname{meas} F - \operatorname{meas}(F \backslash F_1) \\
&= \operatorname{meas}(F \backslash (F \backslash F_1)) = \operatorname{meas} F_1,
\end{aligned}$$

which is a contradiction, proving the Lemma.

Note that, since $\{\operatorname{meas} E_n\}$ is increasing (as $E_i \subseteq E_{i+1}$) and bounded above by $\operatorname{meas} E$ (as $E_i \subseteq E$), $\lim_{n\to\infty} \operatorname{meas} E_n$ exists and $\lim_{n\to\infty} E_n \leq \operatorname{meas} E$.

$\forall i \in \mathbb{N}^*$, let $F_i := E_i \backslash E_{i-1}$. For each $i \in \mathbb{N}^*$, since $\{E_n\}$ is exhausting, $E_i = \cup_{j=1}^{i} F_i$, so that $\operatorname{meas} E_i = \sum_{j=1}^{i} \operatorname{meas} F_i$. Since $E = \cup_{i=1}^{\infty} F_i$, by the Lemma, since the F_i's are Peano-Jordan measurable and disjoint,

$$\operatorname{meas} E \leq \sum_{i=1}^{\infty} \operatorname{meas} F_i = \lim_{n\to\infty} \sum_{i=1}^{n} \operatorname{meas} F_i = \lim_{n\to\infty} \operatorname{meas} E_i,$$

and, therefore, $\lim_{n\to\infty} \operatorname{meas} E_n = \operatorname{meas} E$. $\blacksquare$

(b) Since f is Riemann integrable, it is bounded above by some constant M and below by some constant m. Since E is Peano-Jordan measurable , E is bounded, so that it is contained in some rectangle R. Therefore, by the result of part (a) above,

$$\begin{aligned}
0 &= M \lim_{n\to\infty} \operatorname{meas}(E \backslash E_n) \\
&= M \lim_{n\to\infty} \int_R \chi_{E\backslash E_n}(\mathbf{x})\, d\mathbf{x} \\
&= M \lim_{n\to\infty} \int_R (\chi_E - \chi_{E_n})(\mathbf{x})\, d\mathbf{x} \\
&\geq \lim_{n\to\infty} \int_R f(\mathbf{x})(\chi_E - \chi_{E_n})(\mathbf{x})\, d\mathbf{x} \\
&= \lim_{n\to\infty} \left(\int_R f(\mathbf{x})\chi_E(\mathbf{x})\, d\mathbf{x} - \int_R f(\mathbf{x})\chi_{E_n}(\mathbf{x})\, d\mathbf{x} \right).
\end{aligned}$$

Subtracting $\int_R f(\mathbf{x})\chi_E(\mathbf{x})\, d\mathbf{x}$ shows that

$$\int_R f(\mathbf{x})\chi_E(\mathbf{x})\, d\mathbf{x} \leq \int_R f(\mathbf{x})\chi_{E_n}(\mathbf{x})\, d\mathbf{x}.$$

A similar proof with m shows that

$$\int_R f(\mathbf{x})\chi_E(\mathbf{x})\, d\mathbf{x} \geq \lim_{n\to\infty} \int_R f(\mathbf{x})\chi_{E_n}(\mathbf{x})\, d\mathbf{x}.$$

Thus, $\lim_{n\to\infty} \int_R f(\mathbf{x})\chi_{E_n}(\mathbf{x})\, d\mathbf{x}$ exists and

$$\int_R f(\mathbf{x})\chi_E(\mathbf{x})\, d\mathbf{x} = \lim_{n\to\infty} \int_R f(\mathbf{x})\chi_{E_n}(\mathbf{x})\, d\mathbf{x},$$

as desired. ■

Problem 3

For $k \in \{1, 2, \ldots, N\}$, let i_k be the smallest number such that $(\nabla \mathbf{g}_k)_{i_k} \neq 0$ and $i_k \neq i_1, i_2, \ldots, i_{k-1}$ (such i_k must exist, since $\det J_{\mathbf{g}}(\mathbf{x}) \neq 0$).

Let $\mathbf{h} = \mathbf{f}_1 \circ \mathbf{f}_2 \circ \cdots \circ \mathbf{f}_N$, where each $\mathbf{f}_j$ is a flip switching j with i_j.

Then, $\forall k \in \{1, 2, \ldots, N\}$, let

$$\mathbf{h}_i(\mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, \mathbf{g}_i((\mathbf{h}_{k-1} \circ \mathbf{h}_{k-2} \circ \ldots \circ \mathbf{h}_1)^{-1}(\mathbf{x})), x_{i+1}, \ldots, x_N)$$

(noting that, by the inverse function theorem and the fact that the composition of invertible functions is invertible, $\mathbf{h}_{k-1} \circ \mathbf{h}_{k-2} \circ \ldots \circ \mathbf{h}_1$ is invertible in some ball U_k). Then,

$$\mathbf{g} = \mathbf{h}_N \circ \mathbf{h}_{N-1} \circ \ldots \circ \mathbf{h}_1 \circ \mathbf{h},$$

and, $\det J_{\mathbf{h}} = 1$, and, since $\det J_{\mathbf{g}} \neq 0$, $\det_{\mathbf{h}_i} \neq 0$. $\blacksquare$

Problem 4

(a) E appears as follows:

We show that E is Peano-Jordan measurable by computing the integral of its characteristic function (using a Change of Variables into polar coordinates):

$$\begin{aligned}
\int_E 1 \, d\mathbf{x} &= \int_{\pi/4}^{\pi/2} \int_{\frac{\alpha}{\sin\theta}}^{1} r \, dr \, d\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{2} - \frac{\alpha^2}{2\sin^2\theta} \, d\theta \\
&= \frac{1}{2}(\pi/2) + \frac{\alpha^2 \cot(\pi/2)}{2} \\
-\frac{1}{2}(\pi/4) + \frac{\alpha^2 \cot(\pi/4)}{2} &= \boxed{\frac{\pi + 4\alpha^2}{8}.}
\end{aligned}$$

(b) We show f is integrable over E by computing its integral (using a Change of Variables into polar coordinates):

(c)

(d)