Lecture Notes: Calculus of Variations

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Notation

$\forall$	for all
$\exists$	there exists
$A \subset B$	A is a subset of B (possibly with $A = B$)
$A \backslash B$	$\{x \in A \mid x \notin B\}$
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \cdots\}$
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^n$	with $n \in \mathbb{N}$, the set of all ordered n-tuples. In general, a point $\mathbf{x} \in \mathbb{R}^n$ can be represented by $(x_1, x_2, \cdots, x_n)$. For points in $\mathbb{R}^2$ and $\mathbb{R}^3$, however, we typically use (x, y) and (x, y, z) respectively.
$C([a, b]; \mathbb{R}^n)$	the set of all continuous mappings from the interval $[a, b]$ to $\mathbb{R}^n$
$C^k([a, b]; \mathbb{R}^n)$	with $k \in \mathbb{N}$, the set of all k-times continuously differentiable mappings from the interval $[a, b]$ to $\mathbb{R}^n$
$C[a, b]$	the set of all continuous functions from the interval $[a, b]$ to $\mathbb{R}$
$C^k[a, b]$	with $k \in \mathbb{N}$, the set of all k-times continuously differentiable functions from $[a, b]$ to $\mathbb{R}$
$f_{,k}$	with f a function from $\mathbb{R}^n$ to $\mathbb{R}$ and $1 \leq k \leq n$, we generally denote the partial derivative of f with respect to its k-th argument by $f_{,k}$ when it exists.
∇	the gradient operator. If f is a function from $\mathbb{R}^n$ to $\mathbb{R}$ and f has continuous partial derivatives, then $\nabla f = (f_{,1}, f_{,2}, \cdots, f_{,n})$.
$\delta J(y; v)$	the Gâteaux variation of J at y in the direction v
$\mathscr{V}_y$	the class of admissible variations at y
V	an alternative notation for a class of admissible variations
$\mathscr{Y}$	a class of admissible functions
Y	an alternative notation for a class of admissible functions
$\mathfrak{X}$	a real linear space

Chapter 1

Introduction

Many problems that arise in a wide variety of applications lead to mathematical formulations that require maximizing or minimizing an integral involving an unknown function and one or more of its derivatives. Typically, the unknown function is required to satisfy certain constraints, e.g. its values on the boundary of its domain may be prescribed. The branch of applied mathematics that deals with such problems is generally referred to as the *Calculus of Variations*.

The aim of this course is to provide an introduction to the calculus of variations with emphasis on basic principles rather than on recipes to solve particular problems. The main prerequisites are calculus (including functions of several variables), linear algebra, a bit of exposure to differential equations, and some mathematical maturity (i.e., the ability to follow and construct proofs).

We begin with a simple example.

1.1 Shortest Path Between Two Points

Working in the x-y plane, we seek a curve of shortest length joining the points (a, A) and (b, B), where a, b, A, B are given real numbers with $a < b$. For now, we consider only those curves that are graphs of continuously differentiable functions $y : [a, b] \to \mathbb{R}$ (see Figure 1.1). Later on we shall consider problems with parametric curves. By the arclength formula, the length of such a curve is given by

$$\int_a^b \sqrt{1 + y'(x)^2}\, dx. \tag{1.1}$$

The expression in (1.1) takes a continuously differentiable function as input and returns a real number as output. Our problem is to minimize the value of this expression over all continuously differentiable functions $y : [a, b] \to \mathbb{R}$ satisfying $y(a) = A$ and $y(b) = B$.

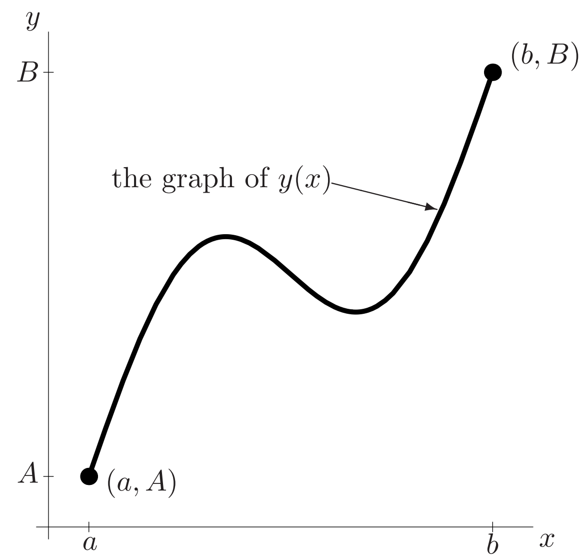

Figure 1.1: A typical curve that we consider

For this purpose it is convenient to put[1]

$$\mathscr{Y} := \{y \in C^1[a,b] \,:\, y(a) = A \text{ and } y(b) = B\}$$

and define the functional $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b \sqrt{1 + y'(x)^2}\, dx \quad \text{for all } y \in \mathscr{Y}.$$

(Here, $C^1[a,b]$ denotes the set of all continuously differentiable functions $y : [a,b] \to \mathbb{R}$. A more detailed, and precise, definition of this set of functions is given in Appendix .) The problem can now be stated very succinctly: Minimize J over $\mathscr{Y}$.

The term *functional* is traditionally used for a real (or complex) valued function whose domain is a set of functions. Moreover, we shall refer to $\mathscr{Y}$ as the *class of admissible functions*.

As is well known, the minimum for J over $\mathscr{Y}$ is attained at the line segment described by

$$y(x) := A + \frac{B - A}{b - a}(x - a) \quad \text{for all } x \in [a,b].$$

It is possible to prove directly that

$$J(y) \geq \sqrt{(b-a)^2 + (B-A)^2} \quad \text{for all } y \in \mathscr{Y}$$

[1] We use the symbol := for an equality in which the left-hand side is defined by the right-hand side.

using a standard inequality from vector calculus. (See Exercise .)

1.2 Example 1.2

For this example, we take the class of admissible functions to be

$$\mathscr{Y} := \left\{ y \in C^1[0,1] \,:\, y(0) = 0 \text{ and } y(1) = 1 \right\}.$$

(There is nothing special about the numbers 0 and 1; they were simply chosen for convenience.) We consider the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) := \int_0^1 \left[y(x)^2 + y'(x)^2 \right] \, dx \quad \text{for all } y \in \mathscr{Y}. \tag{1.2}$$

Our problem is to minimize J over $\mathscr{Y}$. It is not obvious what a minimizer for J should look like. It is not even clear that a minimizer exists. (We use the term *minimizer* for a function $y \in \mathscr{Y}$ that minimizes J.)

Notice that $J(y) \geq 0$ for all $y \in \mathscr{Y}$. One might first guess that $J(y)$ can be made very small in magnitude by taking $y(x)$ to be a large power of x. Indeed, if $y(x) = x^k$ with k very large, then $y(x)$ and $y'(x)$ are nearly zero over most of the interval $[0, 1]$. As x approaches 1, however, $y(x)$ must rise rapidly to 1, and consequently $y'(x)$ is very large when x is close to 1 (see Figure 1.2). It turns

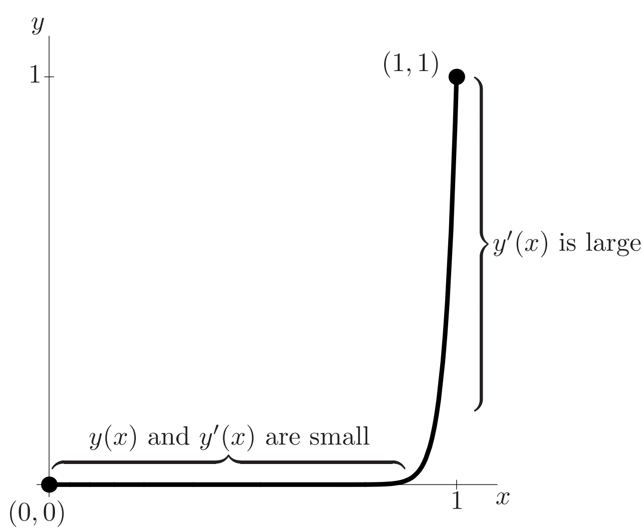

Figure 1.2: A $y \in \mathscr{Y}$ that rapidly increases to 1 as x nears 1

out that J charges a large "penalty" for this rapid rise and the value of $J(y)$ is quite large for such a function y, as we shall now see.

Let us compute the values of J for some functions in $\mathscr{Y}$ (see Figure 1.3). We look first at the function y_1 defined by $y_1(x) := x$ for all $x \in [0,1]$ and note

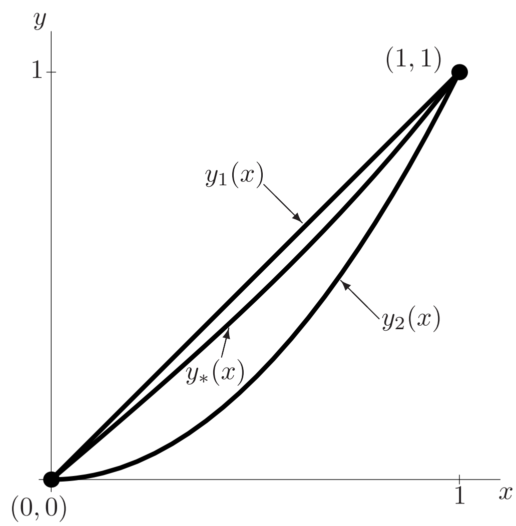

Figure 1.3: Some functions found in $\mathscr{Y}$

that $y_1'(x) = 1$ for all $x \in [0,1]$. Substituting y_1 and y_1' into (1.2), we find that

$$J(y_1) = \int_0^1 \left[y_1(x)^2 + y_1'(x)^2\right] dx = \int_0^1 \left[x^2 + 1\right] dx = \left[\frac{1}{3}x^3 + x\right]\Bigg|_0^1 = \frac{4}{3}.$$

Let us now consider $y_2(x) := x^2$. Then $y_2'(x) = 2x$ and

$$\begin{aligned} J(y_2) &= \int_0^1 \left[y_2(x)^2 + y_2'(x)^2\right] dx = \int_0^1 \left[x^4 + 4x^2\right] dx = \left[\frac{1}{5}x^5 + \frac{4}{3}x^3\right]\Bigg|_0^1 \\ &= \frac{1}{5} + \frac{4}{3} > \frac{4}{3} = J(y_1). \end{aligned}$$

More generally, for $y_\alpha(x) := x^\alpha$ with $\alpha \geq 1$, we have

$$\begin{aligned} J(y_\alpha) &= \int_0^1 \left[x^{2\alpha} + \alpha^2 x^{2\alpha-2}\right] dx = \left[\frac{1}{2\alpha+1}x^{2\alpha+1} + \frac{\alpha^2}{2\alpha-1}x^{2\alpha-1}\right]\Bigg|_0^1 \\ &= \frac{1}{2\alpha+1} + \frac{\alpha^2}{2\alpha-1}. \end{aligned}$$

It is easy to see that $J(y_\alpha) \to \infty$ as $\alpha \to \infty$. In particular, this shows that J does not attain a maximum on $\mathscr{Y}$. It can be shown that the function y_* given by $y_*(x) := \frac{e^x - e^{-x}}{e - e^{-1}}$ for all $x \in [0,1]$ (see Figure 1.3) minimizes J on $\mathscr{Y}$. For this function, we have $y_*'(x) = \frac{e^x + e^{-x}}{e - e^{-1}}$ and

$$
\begin{aligned}
J(y_*) &= \int_0^1 \left[y_*(x)^2 + y_*'(x)^2 \right] dx = \frac{1}{(e - e^{-1})^2} \int_0^1 \left[(e^x - e^{-x})^2 + (e^x + e^{-x})^2 \right] dx \\
&= \frac{2}{(e - e^{-1})^2} \int_0^1 \left[e^{2x} + e^{-2x} \right] dx = \frac{1}{(e - e^{-1})^2} \left[e^{2x} - e^{-2x} \right]\Big|_0^1 = \frac{e^2 - e^{-2}}{(e - e^{-1})^2} \\
&= \frac{e^2 + 1}{e^2 - 1} < \frac{4}{3} = J(y_1).
\end{aligned}
$$

Summarizing our computations, we have found that

$$\frac{e^2 + 1}{e^2 - 1} = J(y_*) < \frac{4}{3} = J(y_1) < \frac{23}{15} = J(y_2).$$

In this case, there was no obvious guess for a minimizer. In "ordinary calculus" one looks for minima and maxima by considering points where the derivative vanishes (and also places where the derivative fails to exist as well as boundary points of the domain). We shall see that it is possible to define an appropriate "derivative" of functionals like the one in (1.2). Setting such a derivative equal to zero leads to a differential equation. For the specific functional in (1.2) the differential equation is $y''(x) - y(x) = 0$. Functions y in $\mathscr{Y}$ are required to satisfy $y(0) = 0$ and $y(1) = 0$, so we are led to the boundary value problem

$$\begin{cases} y''(x) - y(x) = 0; \\ y(0) = 0 \text{ and } y(1) = 1. \end{cases} \tag{1.3}$$

The reason why a minimizer for J on $\mathscr{Y}$ must satisfy (1.3) will be discussed later (Chapter 3). It is interesting to notice that although we only assumed the functions in $\mathscr{Y}$ have continuous first-order derivatives, the differential equation is a second-order equation. The theory to be developed will imply that for certain problems (such as this one), a minimizer must have additional smoothness properties beyond what seems "natural" from the definition the functional.

It is important to observe that although the functional has an especially simple form (and the boundary conditions are very simple), there is no "obvious" candidate for a minimizer. We need to develop some machinery that will help us to identify possible minimizers (and maximizers) and also some machinery to determine whether or not the functions that we find are actually minimizers.

1.3 The Basic Problem in the Calculus of Variations

We now formulate a much more general problem that includes the first two examples as special cases. Let $a, b, A, B \in \mathbb{R}$ with $a < b$ and $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

be given. For now, we assume that f is continuous. When we actually prove theorems, we will need to impose stronger assumptions on f. Let

$$\mathscr{Y} := \left\{ y \in C^1[a,b] \,:\, y(a) = A \text{ and } y(b) = B \right\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

The basic problem in the calculus of variations is to maximize or minimize J over $\mathscr{Y}$. Most of the time, we will talk about minimization problems. There is no loss of generality in focusing on minimization, because maximizing J is equivalent to minimizing $-J$.

There are many possible variants of the basic problem. Some of these are:

(a) Boundary conditions other than $y(a) = A$ and $y(b) = B$.

(b) Additional types of constraints on the admissible functions, e.g.

$$\int_a^b g(x, y(x), y'(x))\, dx = c,$$

where $g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ are given. Numerous of other types of constraints are possible. For example, in certain problems the graphs of the admissible functions are required to lie above (or below) some obstacle in the $x - y$ plane.

(c) The functional takes as input a parameter as well as a function, e.g.

$$J(y, \alpha) = \int_a^b f(x, y(x), y'(x), \alpha)\, dx \quad \text{for all } y \in \mathscr{Y}, \alpha \in \mathbb{R},$$

and the problem is to find a function $y \in \mathscr{Y}$ and a real number $\alpha \in \mathbb{R}$ so that the pair (y, α) minimizes J over $\mathscr{Y} \times \mathbb{R}$;

(d) $J(y)$ may involve higher-order derivatives of y, e.g.

$$J(y) = \int_a^b f(x, y(x), y'(x), y''(x))\, dx.$$

(e) The functions $y \in \mathscr{Y}$ may be vector-valued.

(f) The functions $y \in \mathscr{Y}$ may be functions of more than one variable.

(g) Combinations of (1)–(6).

1.4 The Brachistochrone Problem (1696)

1.4.1 Description

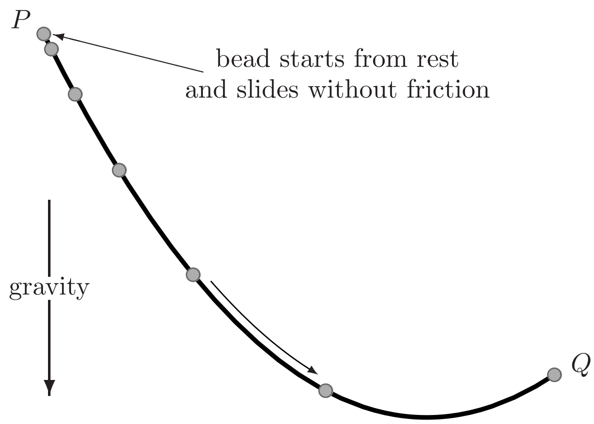

Figure 1.4: Illustration for the statement of the brachistochrone problem. The position of the bead is shown at equal time intervals.

The birth of the calculus of variations is often associated with the following mathematical challenge issued by JOHANN BERNOULLI in 1696: *Let two horizontally and vertically displaced points P and Q be given in a vertical plane with gravity acting downward. Find the smooth curve joining the two points such that a heavy particle (or bead) starting from rest at the higher point will slide without friction along the curve to reach the lower point in the shortest possible time* (Figure 1.4).

The desired curve is called a *brachistochrone*, from the Greek words $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$ (=shortest) and $\chi\rho o\nu o\varsigma$ (=time). Solutions to this problem were obtained by JOHANN BERNOULLI, his brother JAKOB BERNOULLI, L'HÔPITAL, LEIBNITZ, and NEWTON. NEWTON's solution was published anonymously. It is part of mathematical folklore that, upon reading the anonymous solution, JOHANN BERNOULLI immediately recognized it as the work of NEWTON and exclaimed: "*I can tell the lion by his claw*".

1.4.2 Mathematical Formulation of the Brachistochrone Problem

For definiteness, we assume that P lies above Q and that Q lies to the right of P. We begin by choosing a convenient coordinate system. We orient the y-axis so that the positive direction is downward, and take the x-axis to have the usual orientation. Without loss of generality, we assume that $P = (0, 0)$ and consequently $Q = (b, B)$ with $b, B > 0$ (see Figure 1.5).

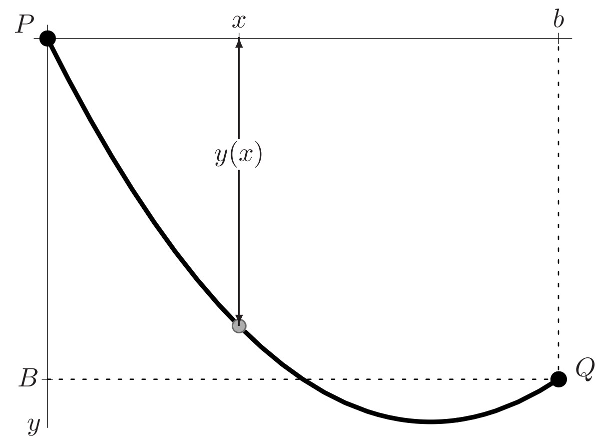

Figure 1.5: Setup for the brachistochrone problem

Next, we establish some notation. Let us denote the mass of the bead by m, the total time of transit from P to Q by T, and the speed of the bead by v. Also let t be the time variable and g be the acceleration due to gravity (which we take to be constant). We assume that the curve along which the bead travels is the graph of a smooth function y with $y(x) > 0$ for all $x \in (0, b]$.

We want to find an expression for T in terms of the function y. To this end, we let l be the total length of curve and $s : [0, b] \to \mathbb{R}$ be the arclength function for y, i.e.

$$s(x) = \int_0^x \sqrt{1 + y'(\tau)^2}\, d\tau \quad \text{for all } x \in [0, b].$$

Thus $s(0) = 0$, $s(b) = l$ and $ds = \sqrt{1 + y'(x)^2}dx$. Now the speed of the bead is $v = \frac{ds}{dt}$, or in terms of differentials, we have $dt = \frac{1}{v} ds$. Thus, the total time of transit for the bead is

$$T = \int_0^l \frac{1}{v}\, ds. \tag{1.4}$$

To express T in terms of the function y, we need to find a relationship between v and y. For this purpose we use the fact that the total energy of the bead is conserved: (Kinetic Energy)+(Potential Energy) is constant. The kinetic energy of the bead is given by $\frac{1}{2}mv^2$, while the potential energy is $-mgy$. (The minus sign in the potential energy is due to the fact that the y-axis is oriented downward.) Thus conservation of energy implies

$$\frac{1}{2}mv^2 - mgy = C \quad \text{for all } x \in [a, b],$$

where C is a constant. Since the bead starts from rest at the point $P = (0,0)$, we know that $y = v = 0$ when $x = 0$. Thus $C = 0$, and

$$\frac{1}{2}mv^2 - mgy = 0 \Rightarrow v^2 = 2gy \Rightarrow v = \sqrt{2gy(x)} \quad \text{for all } x \in [a,b]. \tag{1.5}$$

Here we have used the fact that $v \geq 0$. Substituting our expression for v and using the fact that $ds = \sqrt{1 + y'(x)^2}dx$ in (1.4) yields

$$T = \sqrt{\frac{1}{2g}} \int_0^b \frac{\sqrt{1 + y'(x)^2}}{\sqrt{y(x)}}\, dx. \tag{1.6}$$

This is our expression for the transit time of the bead in terms of the function y.

Since we are only considering curves that join $(0,0)$ to (b,B), we require the admissible functions y to satisfy $y(0) = 0$ and $y(b) = B$. We also want $y(x) > 0$ for all $x \in (0,b]$. There are some subtle issues involved with giving a precise mathematical formulation of the brachistochrone problem. In particular, since $y(0) = 0$, the integral in (1.6) is guaranteed to be singular at $x = 0$. Moreover, it turns out that the "solution" to this problem has a vertical tangent at the origin, so we cannot require the admissible functions to belong to $C^1[0,b]$. Therefore, we put[2]

$$\mathscr{Y} := \left\{ y \in C[0,b] \cap C^1(0,b] \,:\, y(0) = 0, y(b) = B \text{ and } y(x) > 0 \text{ for all } x \in (0,b] \right\}.$$

(The functions in $\mathscr{Y}$ are required to be continuous on $[0,b]$ and continuously differentiable on $(0,b]$. They are not assumed to be differentiable (from the right) at 0.) It is not difficult to show that there are functions $y \in \mathscr{Y}$ for which the integral in (1.6) diverges to $+\infty$. Consequently, we must either reduce the domain for T or allow T to take the value $+\infty$. We shall adopt the latter approach and define the functional $J : \mathscr{Y} \to \mathbb{R} \cup \{+\infty\}$ by

$$J(y) := \int_0^b \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Our problem is to minimize J over $\mathscr{Y}$. Notice that in the definition of J, we have dropped the constant factor $\sqrt{\frac{1}{2g}}$. Indeed if $c > 0$ is a constant, then minimizing J is equivalent to minimizing cJ.
(It is sometimes very convenient in minimization problems to have a simpler domain for the functional at the expense of allowing the functional to take the value $+\infty$.)

[2] We employ a standard "abuse" of notation here. Strictly speaking, the sets $C[0,b]$ and $C^1(0,b]$ are disjoint because the functions in these sets have different domains ($[0,b]$ versus $(0,b]$). By $C[0,b] \cap C^1(0,b]$ we mean the set of all functions in $C[0,b]$ whose restrictions to the smaller domain $(0,b]$ belong to $C^1(0,b]$.

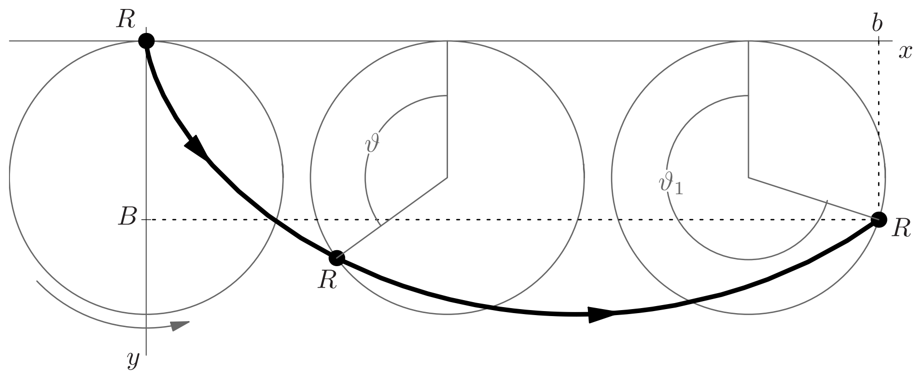

Figure 1.6: Generation of a cycloid

The minimizer for this problem is a cycloid with a cusp at the point $(0,0)$. A cycloid is the curve followed by a point R on the circumference of a circle as the circle is rolled along a horizontal line (see Figure 1.6). A useful way to represent the cycloid is by the parametric equations

$$x = \frac{c^2}{2}(\theta - \sin\theta) \;\text{ and }\; y = \frac{c^2}{2}(1 - \cos\theta) \quad \text{for all } \theta \in [0, \theta_1],$$

where $\theta_1 \in (0, 2\pi)$ and $c \in \mathbb{R}$. The quantity $\frac{c^2}{2}$ is the radius of the circle that is being rolled. The values for c and θ_1 should be chosen so that $x(\theta_1) = b$, $y(\theta_1) = B$. The condition $\theta_1 \in (0, 2\pi)$ ensures that the only cusp is at the origin, i.e. the solution involves only one "arch" of the cycloid.

1.4.3 Jakob Bernoulli's Brachistochrone Problem (1697)

In 1697, after Johann Bernoulli presented his solution to the brachistochrone problem, his brother, Jakob Bernoulli, issued a counter-challenge: *In a vertical plane with gravity acting downward, let P be a given point and let L be a given vertical line that does not contain P. Find the curve joining P and L such that a bead starting from rest at P will slide without friction along the curve and reach the line L in the least possible time.*

For definiteness, we assume that L lies to the right of P. We orient the coordinate axes in the same way as above and we assume that $P = (0,0)$. The mathematical formulation of this problem is essentially the same as that for Johann Bernoulli's brachistochrone problem, but with one important exception. The admissible class of functions is now

$$\mathscr{Y} := \left\{ y \in C[0,b] \cap C^1(0,b] \;:\; y(0) = 0 \text{ and } y(x) > 0 \text{ for all } x \in (0,b] \right\}.$$

With $J : \mathscr{Y} \to \mathbb{R} \cup \{+\infty\}$ again given by

$$J(y) := \int_0^b \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, dx \quad \text{for all } y \in \mathscr{Y},$$

the problem is to minimize J over $\mathscr{Y}$. Since the minimizer for Johann Bernoulli's problem is a cycloid, it follows immediately that the minimizer for Jakob Bernoulli's problem (if one exists) is also a cycloid. (Indeed, the solution to Jakob Bernoulli's problem is a solution to Johann Bernoulli's problem for *some* value of B.) It turns out that the there is exactly one solution which is a cycloid with a cusp at $(0, 0)$ and a horizontal tangent at $x = b$. This is an example of a problem with *one end free* because there is no boundary condition imposed on the admissible functions at $x = b$. As we shall see in Chapter 3, when we do not impose a boundary condition at an endpoint, the minimization procedure will yield a boundary condition, called a *natural boundary condition* at the free end. In this particular problem, the natural boundary condition is $y'(b) = 0$.

1.5 The Hanging Cable Problem

We wish to find the shape that an inextensible thin cable will assume hanging under its own weight when it is pinned at both ends. We assume that the cable is homogeneous and has a uniform cross-section. Let ρ be the density of the cable (mass per unit length) and let g be the acceleration due to gravity (which we assume to be constant and acting downward). The shape of the cable should minimize the potential energy. Let us assume that the endpoints of the cable are pinned at (a, A) and (b, B) with $a < b$ and that the configuration of the cable can be represented as the graph of a function $y \in C^1[a, b]$ (see Figure 1.7).

We need to express the potential energy of the cable in terms of the function y. Let l denote the (fixed) length of the cable and let s be the arclength function for the graph of y. We think of subdividing the cable into small pieces of length Δs. The mass of a portion of the cable having length Δs is $\rho \Delta s$. Thus the potential energy for this portion of the cable is approximately $gy\rho\Delta s$, and the total potential energy for the cable is approximately

$$\sum \rho g y \Delta s.$$

Taking the limit as Δs tends to zero, we can express the total energy as

$$E = \int_0^l \rho g y\, ds = \int_a^b \rho g y(x) \sqrt{1 + y'(x)^2}\, dx = \rho g \int_a^b y(x) \sqrt{1 + y'(x)^2}\, dx. \quad (1.7)$$

The problem is to minimize the potential energy over an appropriate class of admissible functions. Since the cable is inextensible and has length l, the total length of the graph of an admissible function y must be l. Hence y must satisfy

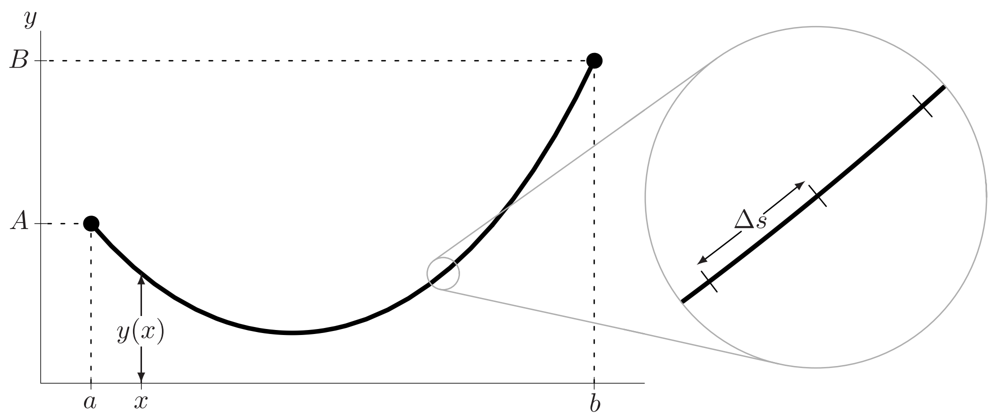

Figure 1.7: Setup for the hanging cable problem

$$\int_a^b \sqrt{1+y'(x)^2}\, dx = l. \tag{1.8}$$

The cable is also pinned at its ends, so we must have

$$y(a) = A \text{ and } y(b) = B.$$

An appropriate class of admissible functions is therefore

$$\mathscr{Y} := \left\{ y \in C^1[a,b] \,:\, y(a) = A,\; y(b) = B \text{ and } \int_a^b \sqrt{1+y'(x)^2}\, dx = l \right\}.$$

Using (1.7), we define the functional $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b y(x)\sqrt{1+y'(x)^2}\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Our problem is to minimize J over $\mathscr{Y}$. (Again, we have dropped a positive multiplicative constant in the definition of J.)

1.6 Some Other Classical Problems

In this section we briefly describe several other classical problems that fall within the domain of the calculus of variations.

1.6.1 Minimal Surface of Revolution

Given $a, b, A, B \in \mathbb{R}$ with $a < b$ and $A, B > 0$, we seek a smooth curve in the $x - y$ plane joining the points (a, A) and (b, B) such that the surface generated by rotating the curve about the x-axis will have the smallest possible area. For now, we consider only curves that are graphs of smooth functions $y : [a, b] \to \mathbb{R}$. (However as we shall discuss later, it is very important in this problem to consider parametric curves as well.) In order to avoid certain technical complications, we will not allow curves that cross (or even touch) the x-axis. The surface area of such a curve is given by

$$S = 2\pi \int_a^b y(x)\sqrt{1 + y'(x)^2}\, dx.$$

Dropping the constant factor 2π, we wish to minimize over

$$\mathscr{Y} := \left\{ y \in C^1[a, b] | y(a) = A, y(b) = B \text{ and } y(x) > 0 \text{ for all } x \in [a, b] \right\}$$

the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) = \int_a^b y(x)\sqrt{1 + y'(x)^2}\, dx.$$

This problem has many very interesting features and will be discussed in much more detail in Section 3.7.

1.6.2 Geodesic Problems

It is often crucial to know the shortest path joining two given locations. If a straight-line path is feasible, then this will provide the desired route. However, if a straight-line path is not possible, then the most efficient route may not be obvious. For example, the paths may be constrained to lie on a surface, such as the surface of a sphere. The curves of shortest length on a given surface are called *geodesics* on the surface. The geodesics on a sphere are arcs of great circles. Calculus of variations plays an important role in the study of geodesics. Geodesics are discussed briefly in Section and there are several exercises concerning special types of geodesics in Chapter . However, we will not devote a great deal of attention to geodesics in this book. The interested reader should consult [ ] for more information.

1.6.3 The Isoperimetric Problem

The so-called isoperimetric problem is concerned with finding planar simple closed curves of fixed length that enclose the maximum possible area. It has been known since ancient times that such curves are circles, but a satisfactory proof of this fact is not easy. We shall discuss a version of this problem in Section 4.5.

1.7 Overview of the Text

The notes are organized as follows. In Chapter 2, we introduce our notation for partial derivatives, review some material concerning minimization problems in $\mathbb{R}^n$, and discuss a general necessary condition for extrema of functionals defined on subsets of a real linear space. This condition is based on an extension of the idea of directional derivatives. Chapter 3 is devoted to applying the ideas from Chapter 2 to problems from the calculus of variations. We begin by using ideas of Lagrange to develop a theory that applies to situations in which the admissible functions y and the integrand f have continuous second-order derivatives. We then employ ideas of du-Bois Reymond to study situations in which the admissible functions and the integrand are assumed only to have continuous first-order derivatives. We shall see that minimizers (and maximizers) must satisfy certain differential equations called Euler-Lagrange equations.

Certain types of constraints that arise naturally (such as the isoperimetric constraint (1.8) cannot be treated adequately by the methods of Chapters 2 and 3. In Chapter 4, we introduce the method of Lagrange Multipliers which can often help in treating such constraints.

Chapter 5 is concerned with convexity and its role in establishing that the candidates for minima found by applying the necessary conditions are indeed minimizers. Chapters 6 and 7 are concerned with extending the ideas and techniques of the first five chapters to problems in which the admissible functions are vector-valued and problems with functionals involving second-order derivatives of the admissible functions. In Chapter 8 we briefly investigate sufficient conditions for minimality that are based on constructing fields of solutions to the Euler-Lagrange equations. Although this approach can be very difficult to implement in practice, it is helpful in certain problems where the approach of Chapter 5 does not apply.

Finally, in Chapter 9 we consider situations in which the admissible functions are not required to be continuously differentiable. We study the classical Weierstrass-Erdmann corner conditions as well a phenomenon discovered by Lavrentiev in which there can be a gap between the minimum value of J over an enlarged class of admissible functions and the values of J that can be obtained by using continuously differentiable functions.