21-484 Notes

JD Nir
jnir@andrew.cmu.edu
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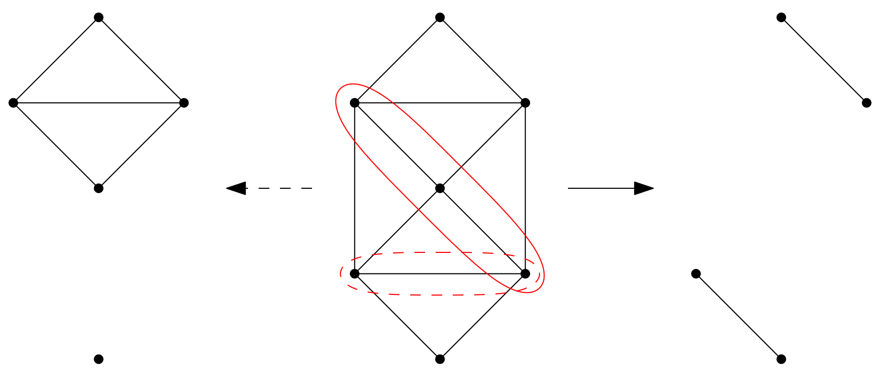
(Havel-Hakimi)

Def: (p. 115-116):

- A vertex-cut in a graph G is a set U such that $G - U$ is disconnected.
- A vertex-cut of minimal cardinality is called a minimal vertex cut.
- A vertex-cut U such that no proper subset of U is a vertex-cut is called a minimal vertex-cut.



→ Every minimum vertex-cut is minimal.

→ A graph contains a vertex-cut iff it is not complete.

→ The vertex-connectivity of a graph G, denoted by $\kappa(G)$, is the size of a smallest set U such that $G = U$ is disconnected or trivial.

→ The number of connected components in G will be denoted $k(G)$.

→ A graph is said to be k-connected if $\kappa(G) \geq k$.

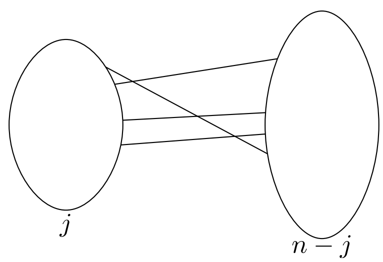
Def (p. 116-117):

- An edge-cut $X \subseteq E(G)$ is a set of edges such that $G - X$ is disconnected.
- An edge-cut with minimal size is called a minimum edge-cut.
- An edge-cut for which no proper subset is an edge-cut is called a minimal edge-cut.
- $\lambda(G)$, the edge-connectivity of G, is the size of a minimal $X \subseteq E(G)$ such that $G - X$ is disconnected or trivial.
- G is k-edge-connected if $\lambda(G) \geq k$.

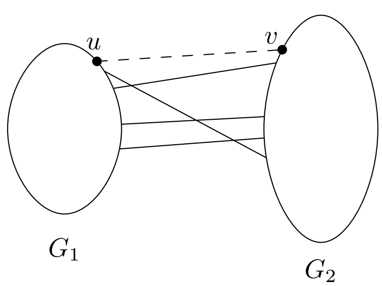
Property (theorem 5.11, Whitney): For all G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$

Proof:

- $\rightarrow$ If G is disconnected or trivial, $\kappa(G) = \lambda(G) = 0$ ✓
- $\rightarrow$ If G is complete graph then $\kappa(K_n) = n - 1 = \lambda(K_n)$
  - $\rightarrow$ removing all $n-1$ edges incident with one vertex disconnects the graph.
  - $\rightarrow$ Let X be an edge-cut and assume that $G - X$ has two components of size j and $n - j$.
  - $\rightarrow$ $|X| = j(n-j)$
  - $\rightarrow$ both components are not empty, so $j \geq 1, n - j \geq 1$
  - $\rightarrow$ $0 \leq (j-1)(n-j-1) = j(n-j) - j - n + j + 1 = j(n-j) - n + 1$
  - $\Rightarrow$ $|X| = j(n-j) \geq n - 1$
  - $\Rightarrow$ $\lambda(K_n) = n - 1 = \delta(K_n)$.



- $\rightarrow$ Assume that G is non-trivial, connected, not complete and with at least three vertices.
- $\rightarrow$ $\lambda(G) \leq \delta(G)$ ✓(removing all edges incident with a vertex of minimum degree disconnects the graph).
- $\rightarrow$ Let X be a minimum edge-cut, and let G_1 and G_2 be the two components of $G - X$.



- $\rightarrow$ If all of the edges between G_1 and G_2 are in G, then $|X| = j(n-j)$ where j is the number of vertices in G_1. Then $j \geq 1$ and $n - j \geq 1 \Rightarrow j(n-j) \geq n-1$ contradicting the facts that $\delta(G \ncong K_n) < n - 1$ and $\lambda(G) \leq \delta(G)$.
- $\rightarrow$ Since not all the edges between G_1 and G_2 are in G, then we have $u \in G_1$ and $v \in G_2$ such that $uv \notin E(G)$.
- $\rightarrow$ Define U as follows. For every $e \in X$, pick a vertex incident with e as follows.
  - $\rightarrow$ if $u \in e$, pick $e \cap V(G_2)$
  - $\rightarrow$ otherwise, pick $e \cap V(G_2)$

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jnir@andrew.cmu.edu

- $|U| \leq |X|$
- $X \cap E(G-U) = \varnothing$

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