21-484 Notes

JD Nir
jnir@andrew.cmu.edu
January 25, 2012

Def. (p. 15-16) Let G be a graph and let u, v be two vertices of G.

- The distance between u and b is the length of a shortest path connecting u and v, if such a path exists. If there is no u–v path in G, then the distance is undefinted (sometimes it is ∞).

  notation: $\mathrm{dist}_G(u, v)$ or $\mathrm{dist}(u, v)$ ($d_G(u, v)$ or $d(u, v)$)
- The maximal distance between any two verticies in G is the diameter of G, denoted $\mathrm{diam}(G)$

Example: Seen: If G has n verticies and for every $u, v \in V(G)$ we have

$$\deg(u) + \deg(v) \geq n - 1$$

then G is connected.

In fact, $\mathrm{diam}(G) \leq 2$.

Proof: Same proof: need to show: $\forall u, v \in V(G).\mathrm{dist}(u, v) \leq 2$

- $u = v$ ✓
- $uv \in E(G)$ ✓
- $uv \notin E(G)$ – we've seen that this implies $\exists w \in V(G).uw, wv \in E(G)$ ✓

Def: (p. 43): Given a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the degree sequence of G to be $\deg(v_1), \deg(v_2), \ldots, \deg(v_n)$.

(p.31): an isolated vertex is a vertex of degree zero.

an end point (or a leaf) is a vertex of degree one.

the minimal degree of G is $\min_{v \in V(G)} \deg(v)$, denoted by $\delta(G)$.

the maximal degree of G is $\max_{v \in V(G)} \deg(v)$, denoted by $\Delta(G)$.

Claim: The degree sequence of any nontrivial graph has repetitions.

Proof: Let G be a graph with n verticies. Then $\delta(G) \geq 0$ and $\Delta(G) \leq n - 1$.

- notice that if G has an isolated vertex, then $\Delta(G) \leq n - 2$
- If $\Delta(G) = n - 1$, then G does not contain an isolated vertex ($\delta(G) \geq 1$).

$\Rightarrow$ For any graph $\Delta(G) - \delta(G) \leq n - 2$, so the range of possible degrees is of size $\leq n - 1$.

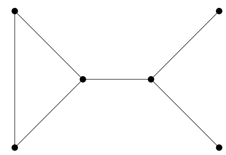
–Pigeon-Hole Principle

JD Nir
jnir@andrew.cmu.edu

Def. (p. 43): A finite sequence of non-negative integers is called graphical if it is the degree sequence of some graph.

Example: (2.9) : Which of the following is graphical?

1. 3,3,2,2,1,1



2. 6,5,5,4,3,3,3,2,2 X sum of the degrees is odd
3. 7,6,4,4,3,3,3 X max degree $\leq n-1$
4. 3,3,3,1 X Each of the verticies of degree 3 must be connected to each other vertex, but the vertex of degree 1 can only be connected to one of them.

Lemma: (Theorem 2.10): A non-increasing sequence $S = d_1, d_2, \ldots, d_n$ ($n \geq 2$) of non-negative integers, where $d_1 \geq 1$ is graphical if and only if the sequence

$$S_1 = d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphical.

Proof: If S_1 is graphical, then there is a graph G with degree sequence s_1. Assume that $V(G) = \{v_2, \ldots, v_n\}$ and that $\deg(v_i) = \begin{cases} d_i - 1 & 2 \leq i \leq d_1 + 1 \\ d_i & d_1 + 2 \leq i \leq n \end{cases}$

Construct G' by adding a vertex v_1 and the edges

$$v_i v_j \ 2 \leq j \leq d_1 + 1$$

Assume that s is graphical.

If G has a vertex such that $d(v_1) = d_1$ and the degrees of the neightbors of v_1 are $d_2, \ldots, d_{d_1+1}$, then removing v_1 yields a graph with degree sequene s_1.

$*$ Assume that there is no G such that G has a vertex v of degree d_1 and the degrees of the neighbors of v are $d_2, \ldots, d_{d_1+1}$.

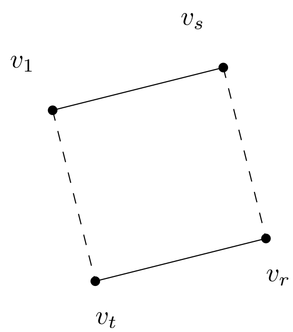
Let G be a graph such that

- the degree sequence of G is S
- the maximal sum (over verticies of degree d_1) of the degrees of neighbors of a vertex of degree d_1 is maximal (over all graphs with degree sequence S).

Let $V(G) = \{v_1, \ldots, v_n\}$, and assume that $\deg(v_1) = d_1$ and

$$\sum_{u \in N(v_1)} \deg(u) \text{ is maximal (over all such graphs and verticies of degree } d_1\text{)}$$

- by $(*)$ the degrees of the neighbors of v_1 are <u>not</u> $d_2, \ldots, d_{d_1+1}$
  - $\rightarrow$ v_1 has a neighbor v_s such that there is a non neighbor of v_1, v_t, such that $\deg(v_t) > \deg(v_s)$.
  - $\rightarrow$ $\exists v_r$ such that $v_r v_t \in E(G)$ but $v_r v_s \notin E(G)$.



def G' by