21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
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Definition 2.1: The *external direct sum* of two E-vector spaces V and W, denoted $V \oplus W$, is the product $V \times W$, with coordinate-wise operations (i.e. $(v_1,w_1)+(v_2,w_2) = (v_1+v_2,w_1+w_2)$, and $\lambda\,(v,w) = (\lambda\,v,\lambda\,w)$ for all $v_1,v_2,v \in V, w_1,w_2,w \in W, \lambda \in E$). The external direct sum of a family V_i, $i \in I$, of E-vector spaces, denoted $\oplus_{i\in I}V_i$, is the subset of $\prod_{i\in I} V_i$ consisting of $\{v = (v_i, i \in I) \mid v_i \neq 0$ for only a finite number of indices$\}$, with coordinate-wise operations.

If V is an E-vector space, and X,Y are two subpaces, the *internal direct sum* of X and Y, also denoted $X \oplus Y$, is only defined if $X \cap Y = \{0\}$, as $X+Y$; then $(x,y) \mapsto x+y$ is an isomorphism of $X \oplus Y$ onto $X+Y$. If X_i, $i \in I$, is a family of subspaces of V, the internal direct sum of the X_i, also denoted $\oplus_{i\in I}X_i$, is only defined if for every $i \in I$, X_i intersects the finite sums of elements of X_j for $j \neq i$ at $\{0\}$, and it is the set of finite sums $x = \sum_{i\in I} x_i$ with $x_i \in X_i$ for all $i \in I$, so that each such x has only one decomposition. In that case, $\oplus_{i\in I}X_i$ is the smallest subspace containing all the X_i.

If V is an E-vector space, and X is a subspace of V, then a *complement* of X in V is a subspace Y of V such that $V = X \oplus Y$, i.e. satisfying $X \cap Y = \{0\}$ and $X + Y = V$.

Remark 2.2: This definition also applies to left R-modules. However, the next result, that every subspace of a vector space has a complement, is not true for all left R-modules, and one then defines a *semi-simple* left R-module as one for which every submodule has a complement.

For example $\mathbb{Z}$ is a $\mathbb{Z}$-module, of course, but it is not a simple $\mathbb{Z}$-module since its submodules have the form $n\,\mathbb{Z}$ for $n \in \mathbb{N}$ (because the notion coincides with that of subgroups in this case), hence it has other submodules than $\{0\}$ and itself; it is not a semi-simple $\mathbb{Z}$-module either, since $m\,\mathbb{Z}$ does not have a complement for $m \geq 2$, because for $n \neq 0$ the intersection $m\,\mathbb{Z} \cap n\,\mathbb{Z}$ is not $\{0\}$ (it is $r\,\mathbb{Z}$ for the least common multiple of m and n).

Lemma 2.3: In an E-vector space V, every subspace X has (at least) one complement Y; every vector $v \in V$ has a unique decomposition $v = x + y$ with $x \in X, y \in Y$, and the mapping P defined by $P\,v = x$, called the *projection onto X parallel to Y*, is an idempotent endomorphism of V;[1] conversely, every idempotent endomorphism Q of V defines a projection onto $im(Q)$ parallel to $ker(Q)$.

Proof: The existence of Y follows from the same maximality argument (based on "Zorn"'s lemma) used for proving the existence of a *Hamel basis* in any vector space,[2] and the argument applies for R-modules if R is a division ring, but not for general rings:[3] one chooses a basis $\{e_i, i \in I\}$ of X, and by completing this family for obtaining a basis of V one adds a family $\{f_j, j \in J\}$, which spans a subspace Y which then is a complement of X.

Once a complement exists, the rest of the argument is valid for left R-modules. If $v \in V$ has two decompositions $v = x_1 + y_1 = x_2 + y_2$ with $x_1,x_2 \in X, y_1,y_2 \in Y$, then it implies that $x_1 - x_2 = y_2 - y_1$; since $x_1 - x_2 \in X$ and $y_2 - y_1 \in Y$, and $X \cap Y = \{0\}$, one deduces that $x_2 = x_1$ and $y_2 = y_1$. Defining the mapping P by $P(v) = x$ then makes sense. P is linear, because $v_1 = x_1 + y_1$ and $v_2 = x_2 + y_2$ with $x_1,x_2 \in X, y_1,y_2 \in Y$ imply $v_1 + v_2 = (x_1 + x_2) + (y_1 + y_2)$, and since $x_1 + x_2 \in X, y_1 + y_2 \in Y$ one has $P(v_1 + v_2) = x_1 + x_2 = P(v_1) + P(v_2)$; similarly for $\lambda \in E$ one has $\lambda\,v_1 = \lambda\,x_1 + \lambda\,x_2$, and since $\lambda\,x_1 \in X, \lambda\,y_1 \in Y$ one deduces that $P(\lambda\,v_1) = \lambda\,x_1 = \lambda\,P(v_1)$.

By definition $P\,v \in X$ for all v, so that $im(P) \subset X$, but since for every $x \in X$ one has $x = x + 0$, hence $P\,x = x$, one has $im(P) = X$, which is the eigen-space of P for the eigen-value 1, and $P^2 = P$ on X. If

[1] P idempotent means $P^2 = P$, and such a term is used in any ring, and here the ring is $L(V,V)$, the E-vector space of linear mappings from V into itself, and since such a mapping is also called an endomorphism of V, one also denotes it $End(V)$.

[2] Georg Karl Wilhelm Hamel, German mathematician, 1877–1954. He worked in Aachen and in Berlin, Germany. Hamel bases for vector spaces are named after him.

[3] If an integral domain R has an element $r_0 \neq 0$ with no inverse for multiplication, then R is a R-module with $\{r_0\}$ linearly independent and generating a submodule which does not contain 1, but $\{1, r_0\}$ is linearly dependent.

$v \in ker(P)$, it means that v has the form $0 + y \in Y$, and conversely for every $y \in Y$ one has $y = 0 + y$ so that $P\,y = 0$, hence $ker(P) = Y$, which is the eigen-space of P for the eigen-value 0, and $P^2 = P$ on Y.

Finally, if $Q^2 = Q$, one lets $X = im(Q)$ and $Y = ker(Q)$, and each $v \in V$ can be written as $v = Q\,v + (v - Q\,v)$, and $x = Q\,v \in X$, while $y = v - Q\,v \in Y$ since $Q\,y = Q\,v - Q^2 v = 0$, showing that $X + Y = V$; then, if $w \in X \cap Y$ one has $w = Q\,v$ for some $v \in V$ and then $0 = Q\,w = Q^2 v = Q\,v = w$, so that $X \cap Y = \{0\}$.

Definition 2.4: For an E-vector space V, the *dual* V^* is $L(V, E)$, the space of linear *forms*, i.e. linear mappings from V into the field of scalars E.[4] In tensor analysis, the elements of V^* are called *covectors*.

Remark 2.5: If A is a linear mapping from an E-vector space V into an E-vector space W, then for each choice $\{e_i, i \in I\}$ of a basis of V and each choice $\{f_j, j \in J\}$ of a basis of W, one associates to A a (possibly infinite) *matrix*, whose ith column contains the vector $A\,e_i$, which one then decomposes as $A\,e_i = \sum_{j \in J} A_{j,i} f_j$, with only a finite number of entries $A_{j,i} \neq 0$, so that $A_{j,i}$ is the entry in row $\#j$ and column $\#i$; each column only contains a finite number of non-zero entries, but if I is infinite a row may contain infinitely many non-zero entries.

In the case of an endomorphism, i.e. $W = V$, it is natural to choose for W the same basis as for V, and a linear mapping $A \in End(V) = L(V, V)$ is then represented by a matrix for each choice of the basis: naturally, we shall have to study how one passes from one matrix for a first basis to the matrix for a second basis.

Remark 2.6: If, as in the preceding remark, the vectors of W are represented as column-vectors, it is then usual to consider that the vectors of W^* are represented as row-vectors.

If $\{e_i, i \in I\}$ is a basis of V, an element A of V^* is then defined by a family $\alpha_i \in E$ indexed with $i \in I$, and such that $A\,e_i = \alpha_i$ for all $i \in I$.

As we shall see later, when introducing *tensors*, indices will appear either in lower or in upper position, as in the following definition, and there will be some rules to follow, and one may use a convention due to EINSTEIN of not writing the sum sign $\sum_i$ when the index i appears both in a lower position and in an upper position in a formula.[5]

Definition 2.7: If $\{e_i, i \in I\}$ is a basis of an E-vector space V, then for each $i \in I$ one denotes e^i the element of V^* defined by $e^i(e_j) = \delta^i_j$, for all $j \in I$, where δ^i_j is the *Kronecker symbol*,[6] equal to 1 if $j = i$ and to 0 if $j \neq i$.[7]

Lemma 2.8: If $\{e_i, i \in I\}$ is a basis of an E-vector space V, then the decomposition of a vector $v \in V$ on the basis is $v = \sum_{i \in I} v^i e_i$ (written $v = v^i e_i$ if one applies Einstein's convention) with $v^i = e^i(v)$ for all $i \in I$. If I is finite, then $\{e^i, i \in I\}$ is a basis of V^*, called the *dual basis*, so that V^* has the same dimension than V. If I is infinite, then the (co)vectors $\{e^i, i \in I\}$ are linearly independent, but they do not span V^*, hence $\{e^i, i \in I\}$ is *not* a basis of V^*.

Proof: v has a unique decomposition $v = \sum_{i \in I} \lambda_i e_i$, and then for each $k \in I$, e^k is a linear mapping, so that one has $e^k(v) = \sum_{i \in I} e^k(\lambda_i e_i) = \sum_{i \in I} \lambda_i e^k(e_i) = \sum_{i \in I} \lambda_i \delta^k_i = \lambda_k$.

If a linear combination $\sum_{i \in I} \mu_i e^i$ is 0, then for each $\ell \in I$ one has $0 = \sum_{i \in I} \mu_i e^i(e_\ell) = \sum_{i \in I} \mu_i \delta^i_\ell = \mu_\ell$, so that the $\{e^i, i \in I\}$ are linearly independent. Since a (co)vector $A \in V^*$ is characterized by a family $\{\alpha_i, i \in I\}$ of scalars, the linear combinations of the family $\{e^i, i \in I\}$ give precisely those A for which only

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[4] In analysis, one puts a topology on V, and what one calls the dual of V is the space $V' = \mathcal{L}(V; E)$ of linear *continuous* forms, and V^* is then called the *algebraic dual* of V.

[5] Albert EINSTEIN, German-born physicist, 1879–1955. He received the Nobel Prize in Physics in 1921, for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect. He worked in Bern, in Zürich, Switzerland, in Prague, now capital of the Czech Republic, at ETH (Eidgenössische Technische Hochschule), Zürich, Switzerland, in Berlin, Germany, and at IAS (Institute for Advanced Study), Princeton, NJ. The Max Planck Institute for Gravitational Physics in Potsdam, Germany, is named after him, the Albert Einstein Institute.

[6] Leopold KRONECKER, German mathematician, 1823–1891. He worked in Berlin, Germany.

[7] The Kronecker symbol has the same definition whatever the position of the indices, i.e. $\delta_{i,j} = \delta^j_i = \delta^i_j = \delta^{i,j} = 0$ if $j \neq i$, and $\delta_{i,i} = \delta^i_i = \delta^i_i = \delta^{i,i} = 1$ (without applying Einstein's convention).

a finite number of α_i are $\neq 0$, hence if I is infinite there exist many (co)vectors $A \in V^*$ which are not in the span of $\{e^i, i \in I\}$.

Additional footnotes: Planck.[8]

[8] Max Karl Ernst Ludwig Planck, German physicist, 1858–1947. He received the Nobel Prize in Physics in 1918, in recognition of the services he rendered to the advancement of Physics by his discovery of energy quanta. He worked in Kiel and in Berlin, Germany. There is a Max Planck Society for the Advancement of the Sciences, which promotes research in many institutes, mostly in Germany (I spent my sabbatical year 1997–1998 at the Max Planck Institute for Mathematics in the Sciences in Leipzig, Germany).