21-470 **Calculus of Variations** **Spring 2014**

Take-Home Midterm Exam
Due on Friday, March 28

In Problems 1-4, find all possible maximizers and minimizers for J on $\mathcal{Y}$. If possible, determine whether each of your candidates is a maximizer, minimizer, or neither.

1. $\mathcal{Y} = \left\{y \in C^1[0,1] : y(0) = 0,\ y(1) = 1\right\},\ J(y) = \displaystyle\int_0^1 e^{x^2} e^{y'(x)} dx.$

2. $\mathcal{Y} = \left\{y \in C^1[1,2] : y(1) = 0\right\},\ J(y) = \displaystyle\int_1^2 [2y(x) - x^2 y'(x)^2] dx.$

3. $\mathcal{Y} = \left\{y \in C^1[0,1] : y(0) = 0,\ y(1) = 1,\ \displaystyle\int_0^1 x y(x) dx = 1\right\},\ J(y) = \displaystyle\int_0^1 y'(x)^2 dx.$

4. $\mathcal{Y} = \left\{y \in C^1\left[0, \frac{\pi}{2}\right] : y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 0\right\},\ J(y) = \displaystyle\int_0^{\pi/2} \left[y'(x)^2 - y(x)^2 + 2e^x y(x)\right] dx.$

5. Let $a, b, \alpha, \beta, \gamma \in \mathbb{R}$ with $a < b$ and $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given. Assume that f is of class C^1. Let
$\mathcal{Y} = \{y \in C^1[a,b] : \alpha y(a) + \beta y(b) = \gamma\}$ and define $J : \mathcal{Y} \to \mathbb{R}$ by
$$J(y) = \int_a^b f(x, y(x), y'(x)) dx.$$
Discuss the problem of minimizing J on $\mathcal{Y}$.

6. Let $\mathcal{V} = \left\{v \in C^1[0,1] : v(0) = v(1) = 0,\ \displaystyle\int_0^1 v(x) dx = 0\right\}$ and let $g \in C[0,1]$ be given. Assume that
$$\int_0^1 g(x) v'(x) dx = 0 \quad \text{for all } v \in \mathcal{V}.$$
What can you deduce about g? Prove your conclusions.

7. Let $a, b, B \in \mathbb{R}$ with $a < b$ and $f \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$ be given. Let
$$\mathcal{Y} = \{y \in C^1[a,b] : y(a) = \int_a^b y(x) dx,\quad y(b) = B\}$$
and define $J : \mathscr{Y} \to \mathbb{R}$ by
$$J(y) = \int_a^b f(x, y(x), y'(x)) dx \quad \text{for all } y \in \mathscr{Y}.$$

Discuss the problem of minimizing J on $\mathcal{Y}$.

8*. (Extra Credit) Let $\mathcal{Y} = \{y \in C^1[0,1] : y(0) = 0,\ y(1) = 1\}$ and define $J : \mathcal{Y} \to \mathbb{R}$ by

$$J(y) = \int_0^1 \left[e^{-y'(x)} + y(x)^2 \right] dx \quad \forall y \in \mathcal{Y}.$$

Show that J has no minimizer in $\mathcal{Y}$.