Department of Mathematical Sciences
Carnegie Mellon University

21-740 **Functional Analysis II** **Fall 2013**

Assignment 1

Due on Friday September 20

Please hand in solutions to all six problems..

1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces with X compactly embedded in Y and assume that X is reflexive. Let K be a closed convex subset of X having the property that
$$\alpha x \in K \quad \text{for all } x \in K, \alpha \geq 0.$$
Let $A \in \mathcal{L}(X;Y)$ be given and assume that
$$\mathcal{N}(A) \cap K = \{0\},$$
$$\|x\|_X \leq \|x\|_Y + \|Ax\|_Y \quad \text{for all } x \in X.$$
Show that there exists a constant C such that
$$\|x\|_Y \leq C\|Ax\|_Y \quad \text{for all } x \in K.$$

2. Let X be a Banach space. A closed subspace M of X is said to be complemented provided that there exists a closed subspace N such that
$$X = M + N, \quad \text{and} \quad M \cap N = \{0\}.$$
Every closed subspace of a Hilbert space is complemented by the Projection Theorem. However, closed subspaces of Banach spaces need not be complemented. (In particular, c_0 is not complemented in l^∞.) Show that in a Banach space X, every finite-dimensional subspace is complemented.

3. Give an example of two closed subspaces M and N of a Hilbert space X such that
$$M \cap N = \{0\} \quad \text{and} \quad M + N \text{ is dense in } X,$$
but $M + N \neq X$.

4. Let X be a Hilbert space and assume that $A \in \mathcal{L}(X;X)$ is normal. Show that A is injective if and only if $\mathcal{R}(A)$ is dense.

5. Let X be a complex Hilbert space and $A \in \mathcal{L}(X;X)$ be given. Show that A is compact if and only if $(Ax_n, x_n) \to 0$ for every sequence $\{x_n\}_{n=1}^\infty$ such that $x_n \rightharpoonup 0$ (weakly) as $n \to 0$.

6. Let X be a complex Hilbert space and $A \in \mathcal{L}(X;X)$ be given. Assume that A is self-adjoint. Put
$$U = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}.$$
Show that U is unitary.