21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
Spring 2012: Monday, Wednesday, Friday, 10:30 am, Doherty Hall 1211.
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Assignment 1 - Sunday February 5, 2012. Due Friday February 10

Exercise 1: (Putnam 1969-B6) Let A and B be matrices of size 3×2 and 2×3 respectively. Suppose that their product in the order $A\,B$ is given by

$$A\,B = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}.$$

Show that the product $B\,A$ is given by

$$B\,A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

Exercise 2: Let V be an E-vector space of dimension n. One wants to find linear mappings A, B from E into itself such that $A^2 = B^2 = I$ (where I is the identity mapping), and $A\,B + B\,A = 0$.
i) Show that there is no solution if n is odd.
ii) Describe the structure of all solutions if $n = 2m$.

Exercise 3: Let A, B be $n \times n$ matrices with entries in a field E, such that A has distinct eigenvalues $a_1, \ldots, a_n$, B has distinct eigenvalues $b_1, \ldots, b_n$, and $a_i + b_j \neq 0$ for all $i, j = 1, \ldots, n$.
i) Show that for any $n \times n$ matrix C, there is a unique $n \times n$ matrix X such that $A\,X + X\,B = C$.
ii) Show that the linear mapping $X \mapsto A\,X + X\,B$ (from $L(E^n, E^n)$ into itself) is diagonalizable, and find its eigenvalues.

Exercise 4: Let W be a non-negative continuous function on $[\alpha, \beta] \subset \mathbb{R}$ which is not identically 0.
i)Show that for $n \geq 3$ the exists a unique quadrature formula with n points $a_1, \ldots, a_n$ with $a_1 = \alpha$ and $a_n = \beta$ and weights $w_1, \ldots, w_n$ such that $\int_\alpha^\beta P(x)\,W(x)\,dx = \sum_{i=1}^n w_i P(a_i)$ for all $P \in \mathcal{P}_{2n-3}[x]$, and prove that the weights are > 0.
ii) In the case $W = 1$ and $(\alpha, \beta) = (-1, +1)$, show that $a_2, \ldots, a_{n-1}$ are the roots of P'_{n-1}, where P_m denotes the Legendre polynomial of degree m.

Exercise 5: (Putnam 1996-B4) For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: There exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$