

Lecture 13: Pareto and Normal Distributions

Contents

1	Memorylessness and Failure rate	1
2	The Pareto distribution	4
2.1	Definition of Pareto	4
2.2	Pareto in job lifetimes	4
2.3	Properties of the Pareto	7
2.4	Pareto distributions are everywhere!	10
3	Normal distribution	11
3.1	Linear transformation property	13
3.2	Central limit theorem	16

1 Memorylessness and Failure rate

A random variable X is said to be **memoryless** if

$$\mathbf{P}\{X > s + t \mid X > s\} = \mathbf{P}\{X > t\} \quad \forall s, t \geq 0$$

Question: Prove that $X \sim \text{Exp}(\lambda)$ is memoryless.

Answer:

$$\begin{aligned}
 \mathbf{P}\{X > s + t \mid X > s\} &= \frac{\mathbf{P}\{X > s + t\}}{\mathbf{P}\{X > s\}} \\
 &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\
 &= e^{-\lambda t} \\
 &= \mathbf{P}\{X > t\}
 \end{aligned}$$

To understand this, think of X as being the lifetime of say a lightbulb. The above says that the probability that the lightbulb survives for at least another t seconds

before burning out given that the lightbulb has survived for s seconds already, is the same as the probability that the lightbulb survives at least t seconds, *independent of* s .

Question: Does this seem realistic for a lightbulb?

Answer: Who knows.

Question: What are some real-life examples whose lifetimes can be modeled by an X such that $\mathbf{P}\{X > s + t \mid X > s\}$ goes down as s goes up?

Answer: A car's lifetime. The older a car is, the less likely that it will survive another, say, $t = 6$ years.

Distributions for which $\mathbf{P}\{X > s + t \mid X > s\}$ goes down as s goes up are said to have **increasing failure rate**. The device is more and more likely to fail as time goes on.

Question: What are some real-life examples whose lifetimes can be modeled by an X such that $\mathbf{P}\{X > s + t \mid X > s\}$ goes up as s goes up?

Answer: We will soon see that one example is UNIX job CPU lifetimes, see [9, 10]. The more CPU a job has used up so far, the more CPU it is likely to use up. Another example is computer chips. If they're going to fail, they'll do it early. That's why chip manufacturers test them for a long while.

Distributions for which $\mathbf{P}\{X > s + t \mid X > s\}$ goes up as s goes up are said to have **decreasing failure rate**. The device is less likely to fail as time goes on.

More precisely, the **failure rate function $r(t)$** , (a.k.a. hazard rate function), is defined as follows:

Let X be a continuous random variable with probability density function $f(t)$ and cumulative distribution function $F(t) = \mathbf{P}\{X < t\}$. Then $r(t)$ is formally defined as:

$$r(t) \equiv \frac{f(t)}{\overline{F}(t)}$$

To interpret this expression, consider the probability that a t -year old item will fail during the next dt seconds.

$$\begin{aligned} \mathbf{P}\{X \in (t, t + dt) \mid X > t\} &= \frac{\mathbf{P}\{X \in (t, t + dt)\}}{\mathbf{P}\{X > t\}} \\ &\approx \frac{f(t) \cdot dt}{\overline{F}(t)} \\ &= r(t) \cdot dt \end{aligned}$$

Thus $r(t)$ represents the instantaneous failure rate of a t -year old item.

Definition 1 When $r(t)$ is strictly decreasing in t we say that the distribution $f(t)$ has **decreasing failure rate**; if $r(t)$ is strictly increasing in t , we say that the distribution has **increasing failure rate**.

Observe that $r(t)$ is not necessarily going to always decrease with t or increase with t – it might behave differently for different t .

Question: Suppose $r(t)$ is constant. What do you know about $f(t)$?

Answer: We will prove in the next homework that $f(t)$ *must* be the Exponential p.d.f.

Bank example

Question: Suppose that the time a customer spends in a bank is Exponentially distributed with mean 10 minutes. What is $\mathbf{P}\{\text{Customer spends} > 5 \text{ min in bank}\}$

Answer: $e^{-5 \cdot 1/10} = e^{-1/2}$

Question: What is

$\mathbf{P}\{\text{Customer spends} > 15 \text{ min in bank total} \mid \text{he's there after 10 min}\}$

Answer: Same as above answer.

The reason why the Exponential distribution is so convenient to work with is that history doesn't matter!

Question: Suppose $X \sim \text{Exp}(\lambda)$. What is $\mathbf{E}\{X|X > 10\}$?

Answer: The Exponential distribution “starts over” at 10, or any other point. Hence $\mathbf{E}\{X|X > 10\} = 10 + \mathbf{E}\{X\} = 10 + \frac{1}{\lambda}$.

Post office example

Suppose that a post office has 2 clerks. Customer B is being served by one clerk, and customer C is being served by the other clerk, when customer A walks in. All service times are Exponentially distributed with mean $\frac{1}{\lambda}$.

Question: What is $\mathbf{P}\{A \text{ is the last to leave}\}$?

Answer: $\frac{1}{2}$. Note that one of B and C will leave first. WLOG, Let us say B leaves first. Then C and A will have the same distribution on their remaining service time. It doesn't matter that C has been serving for a while.

It can be proven that the Exponential distribution is the *only* continuous-time memoryless distribution.

Question: What's the only discrete-time memoryless distribution?

Answer: The Geometric distribution.

We will later delve more deeply into the exact relationship between the Exponential and the Geometric. For now, we move to another distribution.

2 The Pareto distribution

2.1 Definition of Pareto

Pareto(α) is a distribution with a power-law tail, meaning that its density decays as a polynomial in $1/x$ rather than exponentially, as in $\text{Exp}(\lambda)$. The parameter α is often referred to as the “tail parameter.” It is generally assumed that $0 < \alpha < 2$. If $X \sim \text{Pareto}(\alpha)$, then:

$$f_X(x) = \begin{cases} \alpha x^{-\alpha-1} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} F_X(x) &= 1 - x^{-\alpha} \\ \overline{F}_X(x) &= x^{-\alpha} \end{aligned}$$

While the Pareto distribution has a ski-slope shape, like that of the Exponential, its tail decreases much more slowly (compare $\overline{F}(x)$ for the two distributions). The Pareto distribution is said to have a “heavy tail,” or “fat tail,” where a lower α corresponds to a fatter tail because the decrease is more gradual.

2.2 Pareto in job lifetimes

A large part of my PhD research was spent on understanding the CPU requirements of UNIX jobs, see [9, 10]. I was interested in the *distribution* of the CPU requirements of UNIX jobs – this is called the CPU lifetime distribution. This is important for understanding the impact of different load balancing schemes.

Some terminology: When we talk about a job's **size** we mean its total CPU requirement. When we talk about a job's **age** we mean its total CPU usage thus far. A job's **lifetime** refers to its total CPU requirement (same thing as size). A job's **remaining lifetime** refers to its remaining CPU requirement. Observe that at any point in time, you don't know the job's remaining lifetime, just its current CPU age.

At the time that I was doing my PhD (the Dark Ages of 1994), it was believed that the CPU lifetimes of jobs were Exponentially-distributed. Refusing to believe that UNIX job lifetimes were Exponentially-distributed, I decided to measure the distribution, $F(\cdot)$, of job lifetimes and see if I could find a closed-form distribution which fit it.

I collected the CPU lifetime of millions of jobs on a wide range of different machines, including instructional, research, and administrative machines, over the course of many months. Figure 1 below shows the fraction of jobs whose size exceeds x for all jobs whose size is greater than one second.

At a first glance this plot looks like an Exponential distribution, $\bar{F}(x) = e^{-\lambda x}$. But on a closer examination you can see that it is not Exponential.

Question: How can you tell that it's not Exponential?

Answer: For an Exponential distribution, the fraction of jobs remaining should drop by a constant factor with each unit increase in x (constant failure rate). In Figure 1, we see that the fraction of jobs remaining decreases by a slower and slower rate as we increase x (decreasing failure rate). In fact, looking at the graph, we see that if we start with jobs of CPU age 1 second, half of them make it to 2 seconds. Of those that make it to 2 seconds, half of those make it to 4 seconds. Of those that make it to 4 seconds, half of those make it to 8 seconds. Of those that make it to 8 seconds, half of those make it to 16 seconds, and so on.

To see the distribution more easily it helps to view it on a log-log plot as shown in Figure 2. The bumpy line shows the data and the straight line is the best curve-fit.

To see that the measured distribution is *not* an Exponential distribution, consider Figure 3 which shows the best-fit Exponential distribution in juxtaposition with the measured distribution from Figure 2.

From Figure 2 it is apparent that the tail of the distribution of jobs with lifetimes longer than 1 second decays like $\frac{1}{x}$. That is,

$$\mathbf{P}\{\text{Job size} > x \mid \text{Job size} > 1\} = \frac{1}{x}$$

At the time (mid-90's), I didn't recognize this distribution and was suspicious of

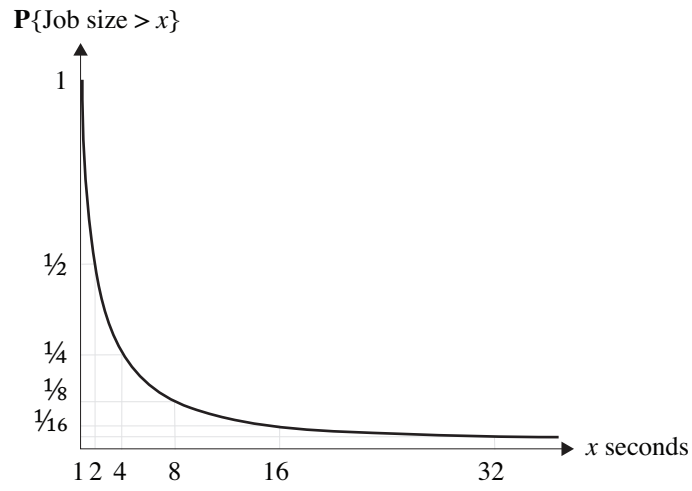


Figure 1: Plot of measured distribution, $\overline{F}(x) = \mathbf{P}\{\text{Job size} > x\}$.

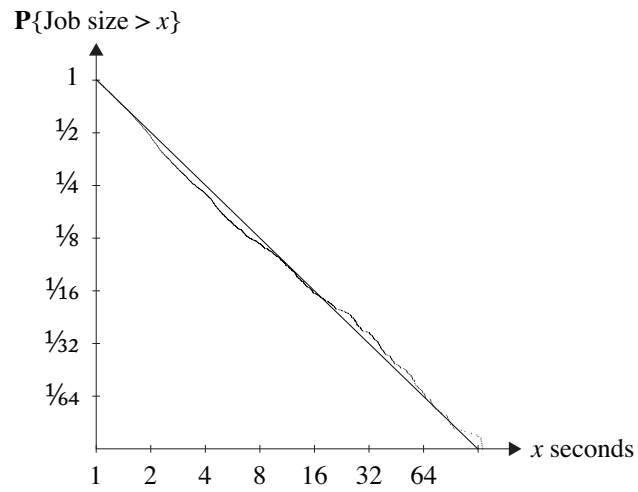


Figure 2: Log-log plot of measured distribution, $\overline{F}(x) = \mathbf{P}\{\text{Job size} > x\}$.

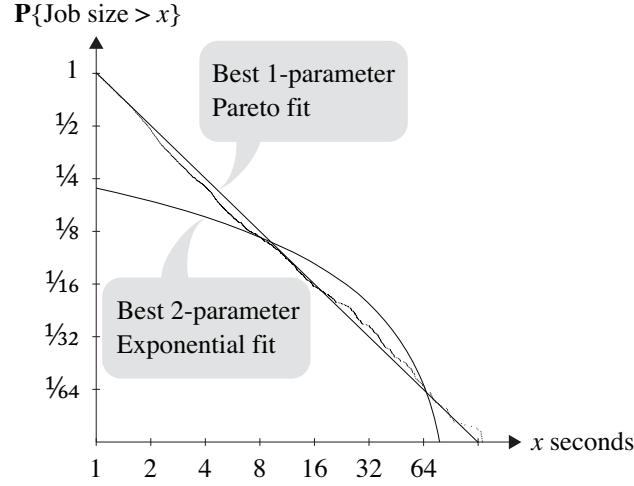


Figure 3: Plot of measured distribution on log-log axes along with best-fit Exponential distribution.

its simplicity, so I tried measuring different types of UNIX workloads. I also tried removing shells and daemon processes, as well as removing very short jobs. No matter what I tried, in all cases, I saw a straight line on a log-log plot, indicating the following distribution:

$$\overline{F}(x) = \frac{1}{x^\alpha}, \quad x \geq 1$$

where α ranged from about 0.8 to about 1.2, in my measurements, across machines. Most commonly α was very close to 1, and the regression showed goodness of fit (R^2) values of over 0.96.

2.3 Properties of the Pareto

It turns out that the distribution that I had measured has a name in economic theory. It is called the Pareto distribution, or “power-law distribution.”

Definition 2 A distribution, $\overline{F}_X(x)$, such that

$$\overline{F}_X(x) = \mathbf{P}\{X > x\} = x^{-\alpha}, \quad \text{for } x \geq 1$$

where $0 < \alpha < 2$ is called a **Pareto distribution**.

We will primarily be concentrating on the Pareto distribution with $\alpha \approx 1$.

Question: What is the failure rate of the Pareto distribution?

Answer:

$$\begin{aligned}\overline{F}(x) &= \mathbf{P}\{X > x\} = x^{-\alpha}, \quad x \geq 1 \\ \Rightarrow F(x) &= \mathbf{P}\{X < x\} = 1 - x^{-\alpha}, \quad x \geq 1 \\ \Rightarrow f(x) &= \frac{dF(x)}{dx} = \alpha x^{-\alpha-1}, \quad x \geq 1 \\ \Rightarrow r(x) &= \frac{f(x)}{\overline{F}(x)} = \frac{\alpha x^{-\alpha-1}}{x^{-\alpha}} = \frac{\alpha}{x}, \quad x \geq 1\end{aligned}$$

Notice that $\int_1^\infty f(x)dx = \int_1^\infty \alpha x^{-\alpha-1}dx = 1$, so $f(x)$ is a valid probability distribution.

Since $r(x) = \frac{\alpha}{x}$ decreases with x , the Pareto distribution has decreasing failure rate (DFR). Thus the older a job is (the more CPU it has used up so far), the greater its probability of using another second of CPU.

Question: What are the implications of DFR for process migration?

Answer: Older jobs (those with higher age) are likely to last longer. Thus it is often worth migrating those jobs, even though they have a lot of additional state, because they will use up more resources in the future.

Question: For a Pareto with $\alpha \leq 1$, what is the mean and variance of the distribution?

Answer: The calculations are straightforward, by integration over the density function. If $0 < \alpha \leq 1$,

$$\begin{aligned}\mathbf{E}\{\text{Lifetime}\} &= \infty \\ \mathbf{E}\{i\text{th moment of Lifetime}\} &= \infty, \quad i = 2, 3, \dots \\ \mathbf{E}\{\text{Remaining Lifetime} \mid \text{age} = a > 1\} &= \infty.\end{aligned}$$

Question: How do the above answers change when α is above 1?

Answer: Both the expected lifetime and the expected remaining lifetime are now finite. Higher moments of lifetime are still infinite.

Question: Under the Pareto distribution with $\alpha = 1$, what's the probability that a job of CPU age a lives to CPU age b , where $b > a$?

Answer:

$$\mathbf{P}\{\text{Life} > b \mid \text{Life} \geq a > 1\} = \frac{1/b}{1/a} = \frac{a}{b}$$

Using the above expression, for the Pareto($\alpha = 1$) distribution, we can interpret the distribution in the following way:

- Of all the jobs currently of age 1 sec, half of those will live to age ≥ 2 sec.
- The probability that a job of age 1 sec uses $> T$ sec of CPU is $\frac{1}{T}$.
- The probability that a job of age T sec lives to be age $\geq 2T$ sec is $\frac{1}{2}$.

The following are three properties of the Pareto distribution:

Decreasing Failure Rate (DFR) — The more CPU you have used so far, the more you will continue to use.

Infinite or near-infinite Variance

“Heavy-Tail Property” — A minuscule fraction of the very largest jobs comprise half of the total system load. For example, when $\alpha = 1.1$, the largest 1% of the jobs comprise about $\frac{1}{2}$ of the load. (Note that this is stronger than the often quoted 80-20 rule.)

The last property, which we call the “heavy-tail property,” comes up in many other settings. For example, in economics, when studying people's wealth, it turns out that the richest 1% of all people have more money between them than all the remaining 99% of us combined. The heavy-tailed property is often referred to as “a few big elephants (big jobs) and many, many mice (little jobs),” as illustrated in Figure 4. For comparison, in an Exponential distribution with the same mean, the largest 1% of the jobs comprise only about 5% of the total demand.

The parameter α can be interpreted as a measure of the variability of the distribution and the heavy-tailedness: $\alpha \rightarrow 0$ yields the most variable and most heavy-tailed distribution, while $\alpha \rightarrow 2$ yields the least variable, and least heavy-tailed distribution. These properties are explored in more depth in the exercises.



Figure 4: *Heavy-tailed property: “Elephants and mice.”*

2.4 Pareto distributions are everywhere!

It’s not just UNIX jobs that fit heavy-tailed Pareto distribution. Pareto job size distributions are everywhere! Here are some more practical and interesting stories:

Around 1996-98, Mark Crovella, Azer Bestavros, and Paul Barford at Boston University were measuring the sizes of files at web sites. They found that these sizes had a Pareto distribution with $\alpha \approx 1.1$. They also found similar results for the sizes of files requested from web sites. Their SURGE web workload generator is based on these findings [2, 7, 6].

Around this same time, the three Faloutsos brothers were observing a similar distribution when looking at the Internet topology. They observed, for example, that most nodes have low out-degree, but a very few nodes have very high out-degree, and the distribution of the degrees follows a Pareto distribution. This and other observations were published in their beautiful 1999 paper which won the Sigcomm Test of Time award, [8].

In 1999, Jennifer Rexford, Anees Shaikh, and Kang Shin at AT&T were working on routing IP flows to create better load balancing. They didn’t want to have to re-route all flows because the overhead would be too high. Ideally, they wanted to just have to re-route only 1% of the IP flows. Would that be enough? Fortunately, their measurements showed that the number of packets in IP flows follows a heavy-tailed Pareto distribution. Consequently, the 1% largest IP flows (those with the most packets) contain about 50% of the bytes in all flows. Thus by re-routing only 1% of the flows, they were able to redistribute half the load. Their paper appeared in Sigcomm 99 [14] and generated a large group of followup papers dealing with sampling methods for how to detect which flows are large.

Around this same time, my students and I, in collaboration with colleagues at Boston University, started a project called SYNC (Scheduling Your Network Connections). The goal was to improve the performance of web servers by changing the order in which they scheduled their jobs to favor requests for small files over requests for large files. Clearly favoring requests for small files over large ones would decrease mean response time. However people hadn’t tried this in the past because

they were afraid that the requests for large files would “starve” or at least be treated unfairly compared to requests for small files. Using the heavy-tailed property of web file sizes, we were able to prove analytically and in implementation that this fear is unfounded for the distribution of web files. The crux of the argument is that although short requests do go ahead of long requests, all those short requests together make up very little load (more than half the load is in the top 1% of long requests) and hence don’t interfere noticeably with the long requests, [1, 5, 11]. In 2004, Ernst Biersack, Idris Rai, and Guillaume Urvoy-Keller extended the SYNC results to TCP flow scheduling, by exploiting the DFR property of the Pareto distribution to discern which flows had short remaining duration [13, 12].

There are many, many more examples of the Pareto distribution in measured distributions involving jobs created by humans. Wireless session times have been shown to follow a Pareto distribution, [3]. Phone call durations have been shown to follow a distribution similar to a Pareto. Human wealth follows a Pareto distribution. Natural phenomenon too follow Pareto distributions. For example, John Doyle at Caltech has shown that the damage caused by forest fires follows a Pareto distribution, with most forest fires causing little damage, but the top few forest fires causing the majority of the damage. The same property holds for earthquakes and other natural disasters.

Given the prevalence of the Pareto distribution, there has been a large amount of research interest into **why** the Pareto distribution comes up everywhere. Ideally, we would like to prove something similar in nature to the Central Limit Theorem, which explains the ubiquity of the Normal distribution, but this time for the Pareto distribution. We do not have room to delve into the many theories proposed for the origin of the Pareto distribution, e.g., the HOT theory [4]. To date, this is still an open research problem of great practical importance.

3 Normal distribution

A very important and ubiquitous continuous distribution is the Normal distribution.

Definition 3 A continuous r.v. X is said to be **Normal**(μ, σ^2) or **Gaussian**(μ, σ^2) if it has p.d.f. $f_X(x)$ of the form:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

where $\sigma > 0$. The parameter μ is called the **mean** and the parameter σ is called the **standard deviation**.

Definition 4 A $Normal(0, 1)$ r.v. Y is said to be a **standard Normal**. Its c.d.f. is denoted by

$$\Phi(y) = F_Y(y) = \mathbf{P}\{Y \leq y\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-t^2/2} dt$$

The $Normal(\mu, \sigma^2)$ p.d.f. has a “bell” shape, and is clearly symmetric around μ , as shown in Figure 5. The fact that $f_X(x)$ in Definition 3 is actually a density function can be seen by integrating it via a change into polar coordinates (trust me, you don’t want to see the gory details [16]).

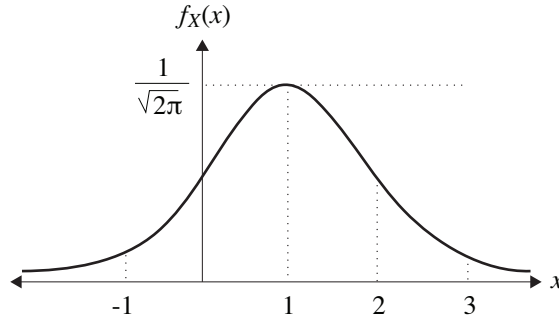


Figure 5: Normal p.d.f. with mean $\mu = 1$ and variance $\sigma^2 = 1$.

Theorem 5 Let $X \sim Normal(\mu, \sigma^2)$, then $\mathbf{E}\{X\} = \mu$ and $\mathbf{Var}(X) = \sigma^2$.

Proof: Since $f_X(x)$ is symmetric around μ , it is obvious that the mean of X must be μ .

$$\begin{aligned} \mathbf{Var}(X) &= \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-\frac{1}{2}((x-\mu)/\sigma)^2} dx \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy \quad \text{let } y = (x - \mu)/\sigma \text{ and } dx = \sigma dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \cdot (ye^{-y^2/2}) dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \left(-ye^{-y^2/2} \right) \Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \quad (\text{integration by parts}) \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy \\ &= \sigma^2 \end{aligned}$$

The last line was obtained by using the fact that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = 1$$

since the integrand is the density function of the standard Normal. ■

3.1 Linear transformation property

The Normal distribution has a very particular property known as the “linear transformation property,” which says that if X is a Normal r.v., and you take a linear function of X then that new r.v. will also be distributed as a Normal. Note that this property is *not* true for other distributions that we have seen, like the Exponential.

Theorem 6 *Let $X \sim \text{Normal}(\mu, \sigma^2)$. Let $Y = aX + b$, where $a > 0$ and b are scalars. Then $Y \sim \text{Normal}(a\mu + b, a^2\sigma^2)$.*

Proof: It’s easy to show that $\mathbf{E}\{Y\} = a\mathbf{E}\{X\} + b = a\mu + b$ and $\mathbf{Var}(Y) = a^2\mathbf{Var}(X) = a^2\sigma^2$. All that remains is to show that $f_Y(y)$ is Normally distributed. We relate the c.d.f. of Y to the c.d.f. of X as follows:

$$F_Y(y) = \mathbf{P}\{Y \leq y\} = \mathbf{P}\{aX + b \leq y\} = \mathbf{P}\left\{X \leq \frac{y-b}{a}\right\} = F_X\left(\frac{y-b}{a}\right)$$

We now differentiate both sides with respect to y :

$$\begin{aligned} \frac{d}{dy} F_Y(y) &= \frac{d}{dy} \int_{-\infty}^y f_Y(t) dt = f_Y(y) \\ \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) &= \frac{d}{dy} \int_{-\infty}^{\frac{y-b}{a}} f_X(t) dt = f_X\left(\frac{y-b}{a}\right) \cdot \frac{d}{dy} \left(\frac{y-b}{a}\right) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} \end{aligned}$$

Thus we have shown that:

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Evaluating this, we have:

$$\begin{aligned}
 f_Y(y) &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \\
 &= \frac{1}{a\sqrt{2\pi}\sigma} e^{-\left(\frac{y-b}{a}-\mu\right)^2/2\sigma^2} \\
 &= \frac{1}{\sqrt{2\pi}(a\sigma)} e^{-(y-b-a\mu)^2/2a^2\sigma^2} \\
 &= \frac{1}{\sqrt{2\pi}(a\sigma)} e^{-(y-(b+a\mu))^2/2a^2\sigma^2}
 \end{aligned}$$

So $f_Y(y)$ is a Normal p.d.f. with mean $a\mu + b$ and variance $a^2\sigma^2$. ■

Unfortunately, it is not known how to integrate the density of the Normal from 0 to y symbolically. To compute the c.d.f. of a Normal distribution we must therefore use a table of numerically integrated results for $\Phi(y)$, such as that given in [17]¹. A subset of the table is given below for reference:

y	$\Phi(y)$
0.5	0.6915
1.0	0.8413
1.5	0.9332
2.0	0.9772
2.5	0.9938
3.0	0.9987

Question: Looking at the table you will see, for example, that $\Phi(1) = 0.8413$. What does this tell us about the probability that the standard Normal is within one standard deviation of its mean?

Answer: We are given that $\mathbf{P}\{Y < 1\} = 0.84$. We want to know $\mathbf{P}\{-1 < Y < 1\}$.

¹In practice no one ever goes to the table anymore, since there are approximations that allow you to compute the values in the table to within 7 decimal places, see for example [15].

$$\begin{aligned}
\mathbf{P}\{-1 < Y < 1\} &= \mathbf{P}\{Y < 1\} - \mathbf{P}\{Y < -1\} \\
&= \mathbf{P}\{Y < 1\} - \mathbf{P}\{Y > 1\} \quad (\text{by symmetry}) \\
&= \mathbf{P}\{Y < 1\} - (1 - \mathbf{P}\{Y < 1\}) \\
&= 2\mathbf{P}\{Y < 1\} - 1 \\
&= 2\Phi(1) - 1 \\
&= 2 \cdot 0.84 - 1 \\
&= 0.68
\end{aligned}$$

So with probability 68%, we are within one standard deviation of the mean.

Likewise, we can use the same argument to show that with probability 95%, we are within two standard deviations of the mean, and with probability 99.7%, we are within three standard deviations of the mean, etc.

Question: The above results were expressed for a standard Normal. What if we don't have a standard Normal?

Answer: We can convert a non-standard Normal into a standard Normal using the Linear Transformation Property. Here's how it works:

$$X \sim \text{Normal}(\mu, \sigma^2) \iff Y = \frac{X - \mu}{\sigma} \sim \text{Normal}(0, 1)$$

So

$$\mathbf{P}\{X < k\} = \mathbf{P}\left\{\frac{X - \mu}{\sigma} < \frac{k - \mu}{\sigma}\right\} = \mathbf{P}\left\{Y < \frac{k - \mu}{\sigma}\right\} = \Phi\left(\frac{k - \mu}{\sigma}\right)$$

Theorem 7 *If $X \sim \text{Normal}(\mu, \sigma^2)$, then the probability that X deviates from its mean by less than k standard deviations is the same as the probability that the standard Normal deviates from its mean by less than k .*

Proof: Let $Y \sim \text{Normal}(0, 1)$. Then:

$$\mathbf{P}\{-k\sigma < X - \mu < k\sigma\} = \mathbf{P}\left\{-k < \frac{X - \mu}{\sigma} < k\right\} = \mathbf{P}\{-k < Y < k\}$$

■

The above theorem illustrates why it's often easier to think in terms of standard deviations rather than absolute values.

Question: Proponents of IQ testing will tell you that human intelligence (IQ) has been shown to be Normally distributed with mean 100 and standard deviation 15. What fraction of people have IQ greater than 130 (“the gifted cutoff”), given that $\Phi(2) = 0.9772$?

Answer: We are looking for the fraction of people whose IQ is more than two standard deviations *above* the mean. This is the same as probability that the standard normal exceeds its mean by more than two standard deviations, which is $1 - \Phi(2) = 0.023$. Thus only about 2% of people have an IQ above 130.

3.2 Central limit theorem

Next time!

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