Math 21-236, Mathematical Studies Analysis II, Spring 2012
Assignment 5

The due date for this assignment is Monday April 2.

Given n closed continuous oriented curves $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_N$, the family $\Gamma := \{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_N\}$ is called a *cycle*. The *range* of Γ is given by the union of the ranges of $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_N$. Given a point $\mathbf{x} \in \mathbb{R}^2$ not contained in the range of Γ, we define the *winding number* of Γ around $\mathbf{x}$ to be the integer

$$\operatorname{ind}_\Gamma(\mathbf{x}) := \sum_{k=1}^{n} \operatorname{ind}_{\boldsymbol{\gamma}_k}(\mathbf{x}).$$

1. **On the definition of absolute continuity.** Let $f : [a, b] \to \mathbb{R}$.

   (a) Prove that f belongs to $AC([a, b])$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

   $$\left| \sum_{k=1}^{\ell} (f(b_k) - f(a_k)) \right| \le \varepsilon$$

   for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subseteq [a, b]$ and

   $$\sum_{k=1}^{\ell} (b_k - a_k) \le \delta.$$

   (b) Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

   $$\left| \sum_{k=1}^{\ell} (f(b_k) - f(a_k)) \right| \le \varepsilon$$

   for every finite number of intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subseteq [a, b]$ and

   $$\sum_{k=1}^{\ell} (b_k - a_k) \le \delta.$$

   Prove that f is Lipschitz.

2. Let $U \subseteq \mathbb{R}^2$ be an open set and let $K \subset U$ be a compact set. Construct a cycle Γ with range contained in $U \setminus K$ such that

   $$\operatorname{ind}_\Gamma(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in K, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^2 \setminus U. \end{cases}$$

3. Determine if following sets are simply connnected:[1]

[1] If they are simply connected, you need to prove it rigorously: constructing the homotopy and/or using theorems proved in class, while if they are not, again you need to prove it rigorously.

(a) $\mathbb{R}^2 \setminus ([1,\infty) \times [-1,1])$,

(b) $\mathbb{R}^3 \setminus \{(0,0,0)\}$,

(c) $\mathbb{R}^3\backslash$line.

4. Solve the differential equation

$$y' = -\frac{2\ln(xy)+1}{\frac{x}{y}}.$$