Homework 2

21-260 Differential Equations
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Section 2.3, Problem 6

(a) The gravitational potential energy of a particle with mass m at height h is mgh, whereas the kinetic energy of such a particle after having fallen from height h is $\frac{1}{2}mv^2$, where v is the speed of the particle; equating the two by Conservation of Energy and solving for v gives (by Torricelli's principle) the desired quantity:

$$v = \sqrt{\frac{2mgh}{m}} = \sqrt{2gh}. \quad \blacksquare$$

(b) The instantaneous rate of change of the volume V of liquid in the tank is the product of the cross-sectional area of the tank (as seen from above) and the instantaneous rate of change in the height of the liquid in the tank: $\frac{dV}{dt} = A(h)\frac{dh}{dt}$.

Assuming the density of the water in the tank is uniform, since the mass is conserved, this is the same as the volumetric rate of outflow from the tank, which is the product of the cross-sectional area of the outflow stream and the outflow velocity v: $\frac{dV}{dt} = -\alpha a v$.

Thus, by the result of part (a), as desired,

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}. \quad \blacksquare \tag{1}$$

(c) Since the tank is a right cylinder, for $0\,m < h < 3\,m$, $A = \pi\,m^2$ in equation (1) above is constant with respect to h. Thus, rewriting equation (1) as

$$Ah^{-1/2}\frac{dh}{dt} = -\alpha a\sqrt{2g}$$

shows that the equation is separable, so that, as shown in class,

$$2A\sqrt{h} = \int Ah^{-1/2}\,dh = \int -\alpha a\sqrt{2g}\,dt = (-\alpha a\sqrt{2g})t + C,$$

for some $C \in \mathbb{R}$. Since the tank is initially full, $h(0) = 3\,m$, giving $C = 2\sqrt{3}A$. Thus, when $h = 0$,

$$t = \frac{2\sqrt{3}A}{\alpha a\sqrt{2g}} \approx \boxed{130\,s.}$$

Section 2.3, Problem 8

(a) As derived in example 2 in section 2.3 of the textbook,

$$S(t) = \boxed{\frac{k}{r}\left(e^{rt} - 1\right)\big).} \tag{2}$$

(b) Solving equation (2) above in the case $S = 10^6$, $r = 0.075$, $t = 40$ for k gives

$$k = \frac{0.075 \cdot 10^6}{e^{0.075 \cdot 40} - 1} \approx \boxed{3930}$$

(c) Solving equation (2) above in the case $S = 10^6$, $k = 2000$, $t = 40$ for r gives

$$r \approx \boxed{0.098.}$$

Section 2.4, Problem 30

The differential equation can be re-written in the form

$$y^{-3}y' - \epsilon y^{-2} = -\sigma. \tag{3}$$

For $v = y^{1-n} = y^{-2}$, $v' = -2y^{-3}y'$ (where differentiation is with respect to the independent variable). Thus, substituting v and v' appropriately into equation (1) gives:

$$-\frac{1}{2}v' - \epsilon v = -\sigma,$$

which is a linear first-order ODE, whose standard form is

$$v' + 2\epsilon v = 2\sigma.$$

Let $\mu = e^{\int 2\epsilon\, dt} = e^{2\epsilon t}$. Then, multiplying by μ gives

$$v'\mu + 2\epsilon v\mu = 2\sigma\mu,$$

so that

$$v\mu = \int \left(v'\mu + 2\epsilon v\mu\right)\, dt = \int 2\sigma\mu\, dt = \frac{2\sigma e^{2\epsilon t}}{2\epsilon} + C,$$

for some $C \in \mathbb{R}$. Thus,

$$v = \frac{2\sigma e^{2\epsilon t}}{2\epsilon\mu} + \frac{C}{\mu} = \frac{2\sigma}{2\epsilon} + Ce^{-2\epsilon t},$$

so that

$$y = v^{-1/2} = \boxed{\left(\frac{2\sigma}{2\epsilon} + Ce^{-2\epsilon t}\right)^{-1/2}.}$$

Section 2.5, Problem 10

Figure 1 (see last page) shows sketch of graph of f. $f(y)$ has as roots $y \in \{-1, 0, 1\}$. Furthermore, $\forall y \in (-\infty, -1), f(y) > 0$, $\forall y \in (-1, 0), f(y) < 0$, $\forall y \in (0, 1), f(y) > 0$, and, $\forall y \in (1, \infty), f(y) < 0$. $\boxed{\text{Thus, } y = -1 \text{ and } y = 1 \text{ are stable equilibrium points, and } y = 0 \text{ is an unstable equilibrium point.}}$

Section 2.5, Problem 21

(a) The quadratic formula gives that $\frac{dy}{dt} = 0$ if and only if

$$y = \boxed{\frac{r \pm \sqrt{r^2 - 4rh/K}}{2r/K}.}$$

If $h < rK/4$, then the discriminant $r^2 - 4rh/K$ above is strictly positive, so that there exist 2 distinct values of y such that $\frac{dy}{dt} = 0$, as determined by the above equation. ■

(b) Since $\frac{dy}{dt}$ is quadratic polynomial in terms of y and it first coefficient is negative, for $y < y_1$, $\frac{dy}{dt} < 0$, and, for $y_1 < y < y_2$, $\frac{dy}{dt} > 0$, so that y_1 is an unstable equilibrium point. Similarly, since, for $y_1 < y < y_2$, $\frac{dy}{dt} > 0$, and, for $y > y_2$, $\frac{dy}{dt} < 0$, y_2 is an asymptotically stable equilibrium point. ■

(c) Since, as shown in Figure 2 (see last page), $\frac{dy}{dt} > 0$ when $y_1 < y < y_2$ and $\frac{dy}{dt} < 0$ when $y > y_2$, when $y_1 < y < y_2$, y increases over time, and, when $y > y_2$, y decreases over time, in either case approaching y_2. Similarly, since $\frac{dy}{dt} < 0$ when $y < y_1$, when $y < y_1$ decreases over time, approaching $-\infty$. ■

(d) For $h > rK/4$, the discriminant found in part (a) above is negative, so that the quadratic $(\frac{dy}{dt})$ has no zeros. Since the leading coefficient of the quadratic is negative, this is possible only if it takes only negative values, so that, $\forall y \in \mathbb{R}$, $\frac{dy}{dt} < 0$, and thus y always decreases over time. ■

(e) For $h = rK/4$, the discriminant fround in part (a) above is 0, so that $\frac{dy}{dt}$ has exactly 1 root, $y = K/2$, and so $y = K/2$ is the single equilibrium point. Since, for $y \neq K/2$, $\frac{dy}{dt} < 0$, this equilibrium point is semistable (it resists increase in y, but not decrease in y). ■