21-484 Notes

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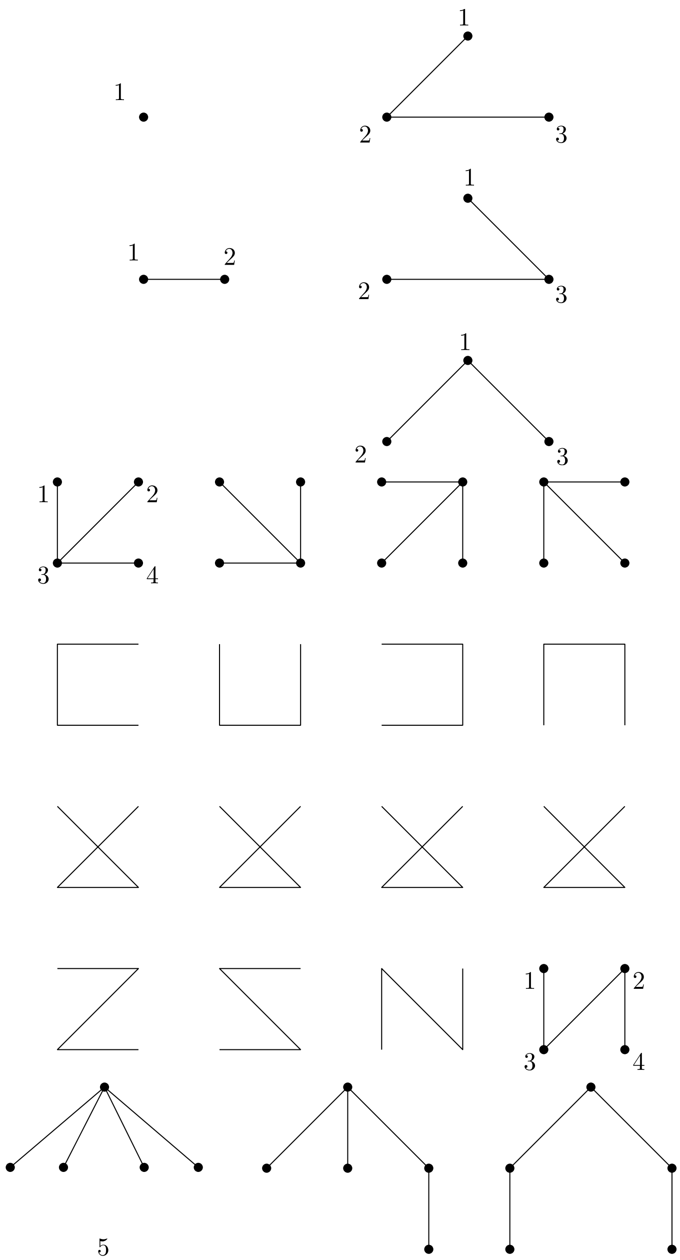
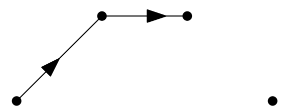


5 $\quad \frac{5!}{2} = 60 \quad \frac{5!}{2} = 60$

$1, 1, 3, 16, 125$

n^{n-2}

Theorem (Cayley's Formula, Thm. 4.15): The number of labeled trees with vertex set $[n] = \{1, 2, \ldots, n\}$ is n^{n-2}.



Def: A directed graph is a graph in which the set of edges is a set of order pairs (instead of 2-sets)

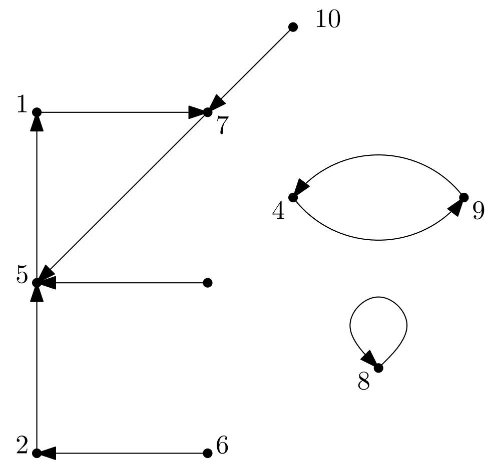
- Generally, in a directed graph we allow loops.

Proof: (Joyal) Let $\mathcal{T}_n$ be the set of all trees with vertex set $[n]$ and two marking $\bigcirc, \square$. Let T_n be the number of trees with vertex set $[n]$. Clearly $|\mathcal{T}_n| = n^2 \cdot T_n$. We are going to show that $|\mathcal{T}_n| = n^n$, by showing a bijection between $\mathcal{T}_n$ and $[n]^{[n]}$.

→ Let $f : [n] \to [n]$ be any function from $[n]^{[n]}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$$

→ Let $\vec{G_f}$ be the directed graph $([n], \{(i, f(i)) | i \in [n]\})$



→ The outdegree of every vertex (the number of edges going out of the vertex) is 1 (since f is a function)

→ In every connected component, the number of edges is the same as the number of vertices.

→ Every conponent is unicyclic (a tree + one edge)

→ This cycle is a directed cycle (otherwise, we will have a vertex with outdegree 2).

→ Let M be the set of all vertices in cycles

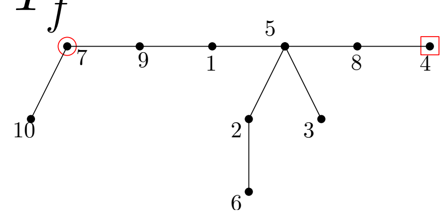
$$M = \{1, 4, 5, 7, 8, 9\}$$

→ notice: M is the largest subset of $[n]$ such that $f|_m$ is a bijection.

→ Define T_f: - $V[t_f] = [n]$
- write $f|_m = f(m_1), f(m_2), \ldots, f(m_{|M|})$
- create a path $f(m_1), f(m_2), \ldots, f(m_{|M|})$

$$f|_m = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}$$

T_f



→ mark $f(m_1)$ by a $\bigcirc$
mark $f(m_{|M|})$ by a $\square$

→ for any vertex i out of m add $\{i, f(i)\}$

→ To complete the proof we need to show that the maaping $f \to T_f$ is a bijection, by describing the inverse map.

→ write the elements of the $\bigcirc \to \square$ path
write them again sorted to get $f|_m$.
any element not in M is mapped to the next vertex in the path connecting it to the $\bigcirc \to \square$ path