Math 21-236, Mathematical Studies Analysis II, Spring 2012
Assignment 7

The due date for this assignment is Monday, April 23.

1. Let $U \subseteq \mathbb{R}^N$ be an open set and let $\boldsymbol{\gamma}_1$ be a continuous, closed, oriented curve with range contained in U.

   (a) Prove that$\boldsymbol{\gamma}_1$ is homotopic in U to a closed polygonal path.

   (b) Prove that$\boldsymbol{\gamma}_1$ is homotopic in U to a C^1, closed, oriented curve.

2. Let $E \subseteq \mathbb{R}^N$ be a Peano–Jordan measurable set and let $f : E \to \mathbb{R}$ be Riemann integrable.

   (a) Let $\{E_n\}$ be an exhausting sequence of E. Prove that
   $$\lim_{n\to\infty} \operatorname{meas} E_n = \operatorname{meas} E.$$

   (b) Prove that f is Riemann integrable in the improper sense and the Riemann integral of f over E coincides with the improper Riemann integral of f over E.

3. Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^N$ be a function of class C^1 such that $\det J_{\mathbf{g}}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in U$. Given $\mathbf{x}_0 \in U$, prove that there exist $B(\mathbf{x}_0, r) \subseteq U$ and n bounded functions of class C^1, $\mathbf{g}_i : U_i \to \mathbb{R}^N$, $i = 1, \dots, n$, such that $U_i \subset \mathbb{R}^N$ is open and bounded, $U_n = B(\mathbf{x}_0, r)$, $\det J_{\mathbf{g}_i}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in U_i$, $\mathbf{g}_i$ is either elementary or a flip, $\mathbf{g}_i(U_i) \subseteq U_{i-1}$ for all $i = 2, \dots, n$, and
   $$\mathbf{g} = \mathbf{g}_1 \circ \cdots \circ \mathbf{g}_n \quad \text{in } B(\mathbf{x}_0, r).$$

4. Given the set
   $$E = \left\{(x, y) \in \mathbb{R}^2 : y \geq x \geq 0, \quad x^2 + y^2 \leq 1, \quad y > \alpha\right\},$$
   where $\alpha \in \left(0, \frac{\sqrt{2}}{2}\right)$,

   (a) sketch E, prove that it is Peano–Jordan measurable, and find its measure;

   (b) prove that the function $f(x, y) = \frac{x}{(x^2+y^2)^{\alpha/2}}$ is Riemann integrable in the improper sense over E and find $\int_E \frac{x}{(x^2+y^2)^{\alpha/2}} dxdy$;

   (c) calculate $\lim_{\alpha\to 0^+} \int_E \frac{x}{(x^2+y^2)^{\alpha/2}} dxdy$;

   (d) if $1 > \alpha > \frac{\sqrt{2}}{2}$, sketch E, prove that the function $f(x, y) = \frac{x}{(x^2+y^2)^{\alpha/2}}$ is Riemann integrable in the improper sense over E, calculate $\int_E \frac{x}{(x^2+y^2)^{\alpha/2}} dxdy$ and find the limit $\lim_{\alpha\to 1^-} \int_E \frac{x}{(x^2+y^2)^{\alpha/2}} dxdy$.