Math 720: Homework.

Do, but don't turn in optional problems. *There is a firm 'no late homework' policy.*

Assignment 1: Assigned Wed 09/05. Due Wed 09/12

Following the notation of Cohn, I use λ to denote the Lebesgue measure.

1. For each of the following sets, compute the Lebesgue outer measure.
   (a) Any countable set. (b) The Cantor set. (c) $\{x \in [0,1] \mid x \notin \mathbb{Q}\}$.
2. (a) If $V \subseteq \mathbb{R}^d$ is a subspace with $\dim(V) < d$, then show that $\lambda(V) = 0$.
   (b) If $P \subseteq \mathbb{R}^2$ is a polygon show that $\operatorname{area}(P) = \lambda(P)$.
3. (a) Say μ is a *translation invariant* measure on $(\mathbb{R}^d, \mathcal{L})$ (i.e. $\mu(x + A) = \mu(A)$ for all $A \in \mathcal{L}$, $x \in \mathbb{R}^d$) which is finite on bounded sets. Show that $\exists c \geqslant 0$ such that $\mu(A) = c\lambda(A)$.
   (b) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal linear transformation, and $A \in \mathcal{L}$. Show that $T(A) \in \mathcal{L}$ and $\lambda(T(A)) = \lambda(A)$. [HINT: Express T in terms of elementary transformations.]
4. (a) Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \subseteq X$ define
   $$\mu^*(A) = \inf\Big\{\sum_1^\infty \rho(E_i) \;\Big|\; E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_1^\infty E_j\Big\}.$$
   Show that μ^* is an outer measure.
   (b) Let (X, d) be any metric space, $\delta > 0$ and define $\mathcal{E}_\delta = \{B(x, r) \mid x \in X, r \in (0, \delta)\}$. Given $\alpha > 0$ define $\rho(B(x, r)) = c_\alpha r^\alpha$, where $c_\alpha = \pi^{\alpha/2}/\Gamma(1+\alpha/2)$ is a normalization constant. Let $H^*_{\alpha,\delta}$ be the outer measure obtained with this choice of ρ and the collection of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta\to 0} H^*_{\alpha,\delta}$. Show H^*_α is an outer measure and restricts to a measure H_α on a σ-algebra that contains all Borel sets. The measure H_α is called the *Hausdorff measure of dimension* α. [Don't reprove Caratheodory.]
   (c) If $X = \mathbb{R}^d$, and $\alpha = d$ show that H_d is the Lebesgue measure.
   (d) Let $S \in \mathcal{B}(X)$. Show that there exists (a unique) $d \in [0, \infty]$ such that $H_\alpha(S) = \infty$ for all $\alpha \in (0, d)$, and $H_\alpha(S) = 0$ for all $\alpha \in (d, \infty)$. This number is called the *Hausdorff dimension* of the set S.
   (e) Compute the Hausdorff dimension of the Cantor set.

Details in class I left for you to check. (Do it, but don't turn it in.)

- We saw in class $\ell(I) = I$ for closed cells. Show it for arbitrary cells.
- Show that $m^*(a + E) = m^*(E)$ for all $a \in \mathbb{R}^d$, $E \subseteq \mathbb{R}^d$.
- Show that the arbitrary intersection of σ-algebras on X is also a σ-algebra.
- Verify that the counting measures and delta measures are measures.
- When proving Caratheodory, we proved in class Σ is a σ-algebra, and that $\mu^*|_\Sigma$ is *finitely* additive. Show that $\mu^*|_\Sigma$ is countably additive.

Assignment 2: Assigned Wed 09/12. Due Wed 09/19

1. Let (X, Σ, μ) be a measure space. For $A \in P(X)$ define $\mu^*(A) = \inf\{\mu(E) \mid E \supseteq A \,\&\, E \in \Sigma\}$, and $\mu_*(A) = \sup\{\mu(E) \mid E \subseteq A \,\&\, E \in \Sigma\}$.
   (a) Show that μ^* is an outer measure.
   (b) Let $A_1, A_2, \dots \in \mathcal{P}(X)$ be disjoint. Show that $\mu_*(\bigcup_1^\infty A_i) \geqslant \sum_1^\infty \mu_*(A_i)$. [The set function μ_* is called an *inner measure*.]
   (c) Show that for all $A \subseteq X$, $\mu^*(A) + \mu_*(A^c) = \mu(X)$.
   (d) Show that $\Sigma_\mu = \{A \in \mathcal{P}(X) \mid \mu_*(A) = \mu^*(A)\}$.
2. Here's an alternate (cleaner) approach to proving $\mathcal{L} = \mathcal{B}_\lambda$. We do it by proving a stronger statement than necessary.
   (a) If $A \in \mathcal{L}(\mathbb{R}^d)$ show that for any $\varepsilon > 0$ there exists two sets C, U such that $C \subseteq A \subseteq U$, C is closed, U is open and $\lambda(U - C) < \varepsilon$.
   (b) For $A \in \mathcal{L}(\mathbb{R}^d)$, show that that there exists an F_σ, F and a G_δ, G such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$. Conclude $\mathcal{B}_\lambda = \mathcal{L}$.
3. Let $A \in \mathcal{L}(\mathbb{R}^d)$. Prove every subset of A is Lebesgue measurable $\iff \lambda(A) = 0$.
4. (a) Prove $\mathcal{B}(\mathbb{R}^{m+n}) = \sigma(\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\})$.
   (b) Prove $\mathcal{L}(\mathbb{R}^{m+n}) \supsetneq \sigma(\{A \times B \mid A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$.
   (c) Show $\mathcal{L}(\mathbb{R}^2) \supsetneq \mathcal{B}(\mathbb{R}^2)$.
5. Find $E \in \mathcal{B}(\mathbb{R})$ so that for all $a < b$, we have $0 < \lambda(E \cap (a, b)) < b - a$.

We say $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if $\emptyset \in \mathcal{A}$, and $\mathcal{A}$ is closed under complements and *finite* unions. We say $\mu_0 : \mathcal{A} \to [0, \infty]$ is a (positive) *pre-measure* on $\mathcal{A}$ if $\mu_0(\emptyset) = 0$, and for any countable disjoint sequence of sets sequence $A_i \in \mathcal{A}$ *such that* $\bigcup_1^\infty A_i \in \mathcal{A}$, we have $\mu_0(\bigcup_1^\infty A_i) = \sum_1^\infty \mu_0(A_i)$.

Namely, a pre-measure is a finitely additive measure on an algebra $\mathcal{A}$, which is also countably additive for disjoint unions *that belong to the algebra.*

6. *(Caratheodory extension)* If $\mathcal{A}$ is an algebra, and μ_0 is a pre-measure on $\mathcal{A}$, show that there exists a measure μ defined on $\sigma(\mathcal{A})$ that extends μ_0.

Optional problems, and details in class I left for you to check.

- Prove any open subset of $\mathbb{R}^d$ is a countable union of cells. Conclude $\mathcal{L} \supseteq \mathcal{B}$.
- Show that the cardinality $\mathcal{B}(\mathbb{R})$ is the same as that of $\mathbb{R}$, however, the cardinality of $\mathcal{L}(\mathbb{R})$ is the same as that of $\mathcal{P}(\mathbb{R})$. Conclude $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$. [There are of course other ways to prove this.]
- If $A_i \in \Sigma$ are such that $A_i \supseteq A_{i+1}$, show that $\mu(\bigcap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$, provided $\mu(A_1) < \infty$. Given an example to show this is not true if $\mu(A_1) = \infty$.
- We saw in class $\lambda(A) = \sup\{\lambda(K) \mid K \subseteq A \,\&\, K \text{ is compact}\}$ for all bounded sets $A \in \mathcal{L}$. Prove it for arbitrary $A \in \mathcal{L}$.
- Show that there exists $A \subseteq \mathbb{R}$ such that if $B \subseteq A$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$, *and further*, if $B \subseteq A^c$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$.