Assignment 3: Assigned Wed 09/19. Due Wed 09/26

1. Let X be a topological space, and μ be a regular Borel measure on X. Show that X has a *maximal* open set of measure 0. Namely, show that there exists $U \subseteq X$, such that U open set, $\mu(U) = 0$ and further for any open set $V \subseteq X$ with $\mu(V) = 0$, we must have $V \subseteq U$. [The complement of U is defined to be the *support* of the measure μ, and denoted by $\operatorname{supp}(\mu)$.]

2. Let $\Sigma \supseteq \mathcal{B}(\mathbb{R}^d)$, and μ be a regular measure on $(\mathbb{R}^d, \Sigma)$. Suppose $A \in \Sigma$ is σ-finite (i.e. $A = \cup_1^\infty A_n$, and $\mu(A_n) < \infty$). Show that $\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ is compact}\}$. [This remains true if we replace $\mathbb{R}^d$ with any Hausdorff space.]

3. Let μ, ν be two measures on (X, Σ). Suppose $\mathcal{C} \subseteq \Sigma$ is a π-system such that $\mu = \nu$ on $\mathcal{C}$.
   (a) Suppose $\exists C_i \in \mathcal{C}$ such that $\bigcup_1^\infty C_i \in X$ and $\mu(C_i) = \nu(C_i) < \infty$. Show that $\mu = \nu$ on $\sigma(\mathcal{C})$.
   (b) If we drop the finiteness condition $\mu(C_i) < \infty$ is the previous subpart still true? Prove or find a counter example.

4. Let $\kappa \in (0, 1)$. Does there exist $E \in \mathcal{L}(\mathbb{R})$ such that for all $a < b \in \mathbb{R}$, we have $\kappa(b - a) \leqslant \lambda(I \cap (a, b)) \leqslant (1 - \kappa)(b - a)$? Prove or find a counter example. [I'm aware that this looks suspiciously like a homework problem you already did. Also, this problem has a short, elegant solution using only what we've seen in class so far.]

5. For $i \in \{1, 2\}$, let (X_i, Σ_i, μ_i) be two measure spaces with $\mu_i(X_i) < \infty$. Define $\Sigma_1 \otimes \Sigma_2 = \sigma\{A_1 \times A_2 \mid A_i \in \Sigma_i\}$.
   (a) Let $x_1 \in X_1$ and $A \in \Sigma_1 \otimes \Sigma_2$. Let $S_{x_1}(A) = \{x_2 \in X_2 \mid (x_1, x_2) \in A\}$, and $T_{x_2}(A) = \{x_1 \in X_1 \mid (x_1, x_2) \in A\}$. Show that $S_{x_1}(A) \in \Sigma_2$ and $T_{x_2}(A) \in \Sigma_1$.
   (b) If $A \in \mathcal{P}(X_1 \times X_2)$ is such that for all $x_i \in X_i$, $S_{x_1}(A) \in \Sigma_2$ and $S_{x_2}(A) \in \Sigma_1$. Must $A \in \Sigma_1 \otimes \Sigma_2$?
   (c) Show that there exists a measure ν on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ such that for all $A_i \in \Sigma_i$ we have $\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$.

6. *(An alternate approach to λ-systems.)* Let $\mathcal{M} \subseteq P(X)$. We say $\mathcal{M}$ is a *Monotone Class*, if whenever $A_i, B_i \in \mathcal{M}$ with $A_i \subseteq A_{i+1}$ and $B_i \supseteq B_{i+1}$ then $\bigcup_1^\infty A_i \in \mathcal{M}$ and $\bigcap_1^\infty B_i \in \mathcal{M}$. If $\mathcal{A} \subseteq P(X)$ is an algebra, then show that the *smallest* monotone class containing $\mathcal{A}$ is exactly $\sigma(A)$. [You should also address existence of a smallest monotone class containing $\mathcal{A}$.]

Optional problems, and details in class I left for you to check.

* Let X be a second countable locally compact Hausdorff space, and μ be a Borel measure on X that is finite on compact sets. Show that μ is regular.
* Is any σ-finite Borel measure on $\mathbb{R}^d$ regular?
* Show that any λ-system that is also a π-system is a σ-algebra.
* If Π is a π-system, then $\lambda(\Pi) = \sigma(\Pi)$. (We only proved $\lambda(\Pi) \subseteq \sigma(\Pi)$.)