Shashank Singh
sss1@andrew.cmu.edu
21-373 Algebraic Structures, Fall 2011
Assignment 3
Due: Friday, September 30

Exercise 15: As can be seen by enumerating the elements of each, S_4 has no elements of order 12, whereas D_{12} does have an element of order 12 (e.g, the element corresponding to rotation of the dodecahedron by $\frac{pi}{6}$). As discussed in Remark 5.4, if two groups are isomorphic, $\forall n \in \mathbb{N}$, each group must have the same number of elements of order n. Thus, $S_4 \not\cong D_12$. ■

Exercise 16: The elements of order 4 in S_4, denoted by their cycle decompositions, are

$$(1\,2\,3\,4), (1\,2\,4\,3), (1\,3\,2\,4), (1\,3\,4\,2), (1\,4\,2\,3), (1\,4\,3\,2).$$

The elements of order 2 in S_4, denoted by their cycle decompositions, are

$$(1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(3\,4).$$

Exercise 17: Suppose some element $a \in S_n$, for some $n \in \mathbb{N}$, can be written as a k-cycle. Then, a^i can be written as a k-cycle if and only if $(i, k) = 1$ (i.e., i and k are relatively prime). This follows directly from the fact that the order of any permutation is the least common multiple of the lengths of the cycles in its cycle decomposition.

As a consequence, for $\sigma = (1\,2\,3\,4\,5\,6\,7\,8)$, σ^i can be written as an 8-cycle if and only if $(i, 8) = 1$, for $\tau = (1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12)$, τ^i can be written as a 12-cycle if and only if $(i, 12) = 1$, and, for $\omega = (1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14)$, ω^i can be written as a 14-cycle if and only if $(i, 14) = 1$. ■

Exercise 19: Let $m \geq 1$, and let $q_1, q_2, \dots, q_m \in \mathbb{Q}$. By definition of H, H is a group. Suppose $b_1, b_2, \dots, b_m$, are the denominators of some $q_1, q_2, \dots, q_m$. Then, clearly, $q_1, q_2, \dots, q_m$ can be written as multiples of $\frac{1}{D}$. Furthermore, the sum and difference of any two q_i, q_j can be written as a multiple of $\frac{1}{D}$, so $H \subseteq K$. Thus, $H \leq K$. ■

Since $H \leq K$, and K can be generated by a single element, H can be generated by a single element (some multiple of $\frac{1}{D}$), so H is cyclic. ■

Exercise 20: Let $a \in \mathbb{Q}$, so that $a = \frac{p}{q}$, for some $p, q \in \mathbb{N}$. Suppose $k \in \mathbb{N}$. Then, since $\mathbb{N}$ is closed under multiplication, $kq \in \mathbb{N}$, and so $\frac{p}{kq} \in \mathbb{Q}$. Therefore, since $k\frac{p}{kq} = \frac{p}{q} = a$, $\mathbb{Q}$ is divisible. ■

Suppose G is a finite Abelian group. Then, if $|G| > 1$, G is not divisible. Let $k = |G|$. Since $|G| > 1$, $\exists a \in G$ with $a \neq e$. Suppose $b \in G$. Thenm $kb = e \neq a$, so no finite, Abelian group is divisible. ■

Let G_1, G_2 be Abelian groups, and let $G = G_1 \times G_2$.

Suppose G is divisible, and let $a_1 \in G_1, a_2 \in G_2, k \in \mathbb{N}$. Since G is divisible, $\exists (b_1, b_2) \in \mathbb{G}$ such that $k(b_1, b_2) = (g_1, g_2)$. Thus, $kb_1 = a_1$, and $kb_2 = a_2$, so G_1, G_2 are divisible.

Suppose G_1, G_2 are divisible, and let $a \in G$, so that $a = (a_1, a_2)$, for some $a_1 \in G_1, a_2 \in G_2$. Let $k \in \mathbb{N}$. Then, $\exists b_1 \in G_1, b_2 \in G_2$ such that $kb_1 = a_1, kb_2 = a_2$. Thus, $k(b_1, b_2) = (a_1, a_2) = a$, so G is divisible. ■

Exercise 21: Any symmetric rigid motion can be determined uniquely by mapping a fixed edge E_0 of the platonic solid to another (non necessarily distinct edge E_1 of the platonic solid (which can be done in E ways), and then choosing which vertex of E_1 edge to move to which vertex of E_2 (which can be done in 2 ways). Thus, there are $2E$ rigid motion symmetries of any platonic solid. ■

Label the vertices of the tetrahedron, any four non-adjacent vertices of the cube, and any four non-adjacent faces of the octahedron with the numbers $1, 2, 3$, and 4. Then, any permutation of $\{1, 2, 3, 4\}$ corresponds to a rigid motion symmetry of the tetrahedron, cube, or octahedron, respectively, (in particular, the permutation π corresponds to symmetry of the the rigid motion moving the vertex or face labelled i to the vertex or face, respectively, labelled $\pi(i)$). This can be shown formally by noting the correspondence for transpositions, and then observing that S_4 is generated by its subset of transpositions, and the group of rigid motion symmetries is generated by the motion symmetries corresponding to switching two vertices or faces in a solid. ■