# 21-484 Notes

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<u>Recall:</u> - 2-Cell region

- 2-Cell embedding

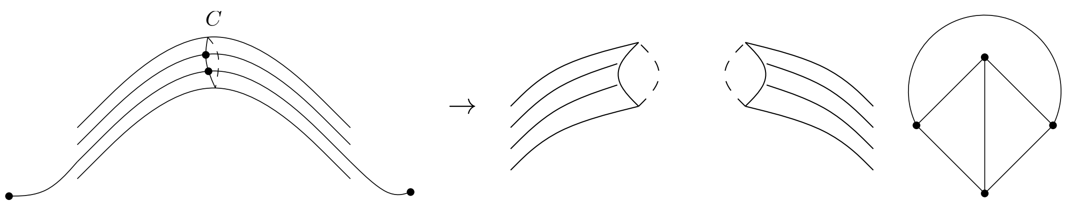
<u>Theorem:</u> (9.9): Let $G$ be a connected graph, 2-cell embedded on a surface of genus $k$, and $G$ has $n$ vertices, $m$ edges and $r$ regions.

Then $n - m + r = 2 - 2k$

<u>Proof ideas:</u> Induction on $k$

- $k = 0$ – Euler's identity

- $k > 0$

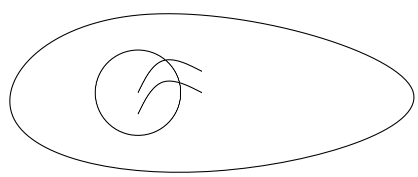



$n' = n + t + 2$
$m = \dots$

$n'' = n + 2t + 4$

<u>Claim:</u> Let $G$ be a connected graph, let $k = \delta(G)$. Any embedding of $G$ on a surface of genus $k$ is a 2-cell embedding.

<u>Proof:</u> Assume for the sake of contradiction that $G$ is embedded on $S_k$ and there is a region that is not a 2-cell region. There is a closed curve $C$ that is not continuously contractable in the region to a point.



This region contains a handle and there are no edges or vertices on the handle. Remove the handle to ger $S_{k-1}$ in which $G$ is embedded. $\lightning$ $\delta(G) = k$.

<u>Corollary:</u> (9.10) If $G$ is connected, embedded on a surface of genus $\delta(G)$ and $n, m, r$ are as usual, then

$$n - m + r = 2 - 2\delta(G).$$

$\rightarrow$ following the same proof that showed $m \leq 3n - 6$ for planar graphs give. If $G$ has $n$ vertices and $m$ edges then

$$m \leq 3n + 6(\delta(G) - 1)$$

$\rightarrow$ $G$ is embeddable on $S_0$ (the sphere) iff it is planar.

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→ If $G$ is planar, embed it in the plan, take a curve surrounding $G$ and contract it to a point to get an embedding of $G$ on $S_0$. For the other direction, start with a point not on an edge, and "tear" through it to get an embedding in the plane.

Def: (p.249-250)

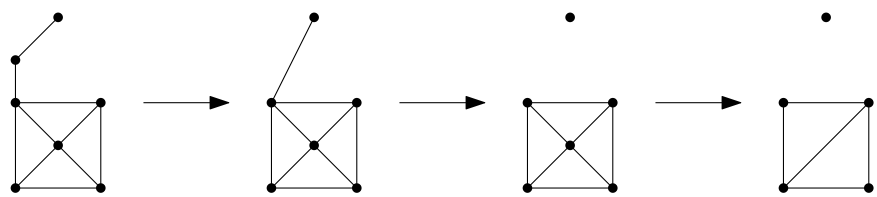
→ Let $G$ be a graph and assume $uv \in E(G)$

contracting the edge $uv$ measn the following:

- remove the vertices $u$ and $v$
- add a new vertex $w$
- add edges between $w$ and all the vertices in $N(u) \cup N(v)$.

→ A minor of a graph $G$ is a graph that can be obtained from $G$ by a sequence of vertex deletions, edge deletions and edge contractions.

Example:



fact: If $H$ is a subdivision of $G$ then $G$ is a minor of $H$.

→ contract the new paths back to an edge.

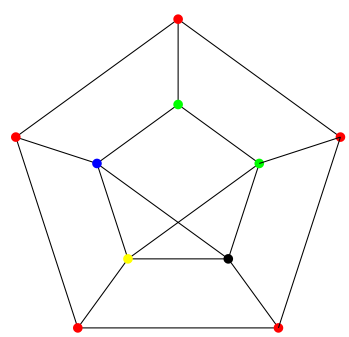
claim: If $H$ is a minor of $G$ then $\delta(H) \leq \delta(G)$.

→ contracting an edge does not increase the genus.

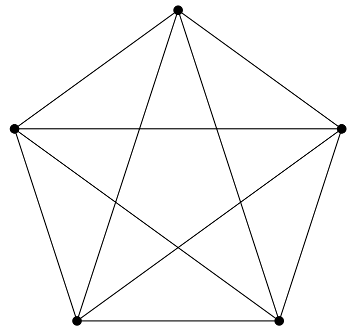
Thm (9.15, Wagner's thm)

A graph is planar iff it does not contain a $K_5$ or a $K_{3,3}$ minor.

Example: recall that the graph below is not planar. We showed that by finding a $K_{3,3}$ subdivision in it.



→ there is no $K_5$ subdivision in it.



There is a $K_5$ minor.