21-484 Notes

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<u>Def:</u> (p. 125): Let G be a graph, u, v are vertices in G.

- → a set $S \subseteq V(G)$ is called a <u>u–v separating set</u> if $G - S$ is disconnected and u and v are in different connected components of $G - S$.
- → <u>also:</u> "S separates u and v"
- → A minimal (by size) u–v separating set is called a minimal u–v separating set.
- → <u>Notice:</u> the size of a u–v separating set is at least $\kappa(G)$.

<u>Def:</u> → Let P be a u–v path in G. A vertex of p that is not u or v is called an <u>internal vertex</u> of P.

→ A set of u–v paths, $P_1, \ldots, P_k$ is called <u>internally disjoint</u> if there is no commmon internal vertex between any two paths of the set.

→ Theorem (Thm 5.16, Menger's Theorem)

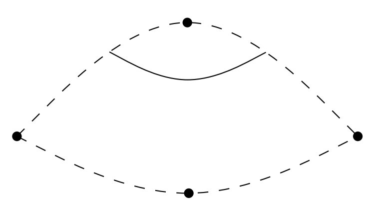
Let G be a graph, and let u and v be two nonadjacent vertices. Then the size of a minimum separating set equals the number of maximal internally disjoint u–v paths.

<u>Proof:</u> Let G be a graph and let u and v be two nonadjacent vertices.

→ Let S be a u–v separating set. Clearly every u–v path must contain a vertex from S.

→ therefore, the number of internally disjoint u–v paths is at most $|S|$.

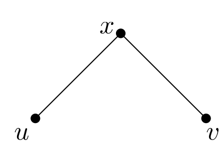
→ Let k be the size of a minimal u–v separating set.



→ By induction on the number of edges in G.

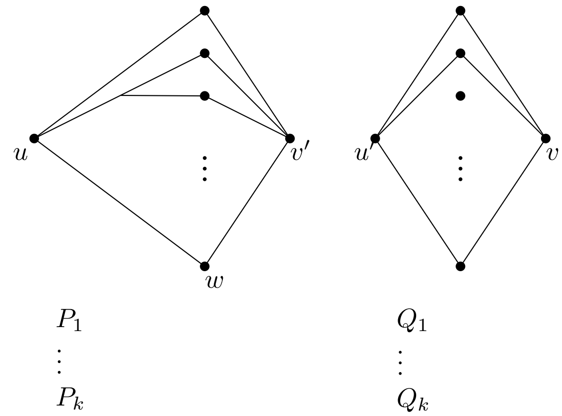
- → If G is an empty graph, everything is zero. ✓
- → Assume the theorem for all graphs with $< m$ edges.

  case 1: If there is a separating set S containing a vertex x adjacent to beoth u and v, let $G' = G - \{x\}$.



→ Notice that $S - \{x\}$ is a minimal u–v separating set in G' (Since $G' - (S - \{x\}) = G - S$.)
→ By the induction hypothesis we have $k - 1$ interally disjoint u–v paths in $G - \{x\}$. Adding the path uxv, we get a set of k internally disjoint u–v paths in G.

case 2: Assume there is a separating set W such that one vertex of W is not a neighbor of u and at least one vertex of W is not a neighbor of v.



→ Let V_u be the vertex set containing the component containing u in $G - W$. Let G_u be the graph spanned over $V_u \cup W$, $G_u = G[V_u \cup W]$. (G_u is a connected graph). → Define G'_u by adding another vertex v' and all the edges of the form $v'w_1, v'w_2, \ldots, v'w_k$.

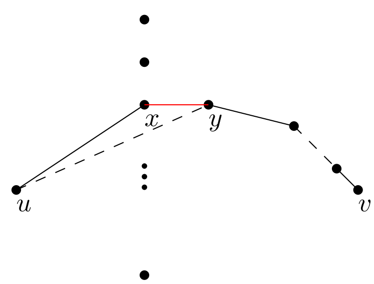
→ G'_u has fewer edges than G because in G u is not adjacent to at least to at least one member of w.

→ By the induction hypothesis, there are k internally disjoint u–v' paths $P_1, \ldots, P_k$, where $w_i \in P_i$.

⟶ Repeat the process with V_v, G_v, G'_v and u' to get k internally disjoint v–u' paths $Q_1, \ldots, Q_k$ when $w_i \in Q_i$.

→ The paths P_i without u' and Q' without v' are k internally disjoint u–v paths.

→ Assume that in every minimal u–v separating set all the vertices are adjacent to u or all of them are adjacent to v.



→ Let $P = u, x, y, \ldots, v$ be a geodesic u–v path.

→ Let $G' = G - \{e = xy\}$.

→ Let Z be a minimal u–v separating set in G'. Assume $|Z| < k$.

→ $Z \cup \{x\}$ is a minimal u–v separating set in G, because $G - (Z \cup \{x\}) = G' - Z$.

→ by our assumption, all the members of Z are adjacent to u.

→ $Z - \{y\}$ is also a minimal separating set in G.

→ y is also adjacent to u, but then there is a u–v path shorter than P. ↯

→ Therefore, $|Z| = k$, and there are k internally disjoint u–v paths in G'. ■