Homework 3
21-238 Mathematical Studies Algebra II
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Due: Wednesday, February 29, 2012

Exercise 11

---

i) Note that $\forall A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\forall v \in \mathbb{C}^n$, $j \in \{1, 2, \infty\}$,

$$\frac{\|Av\|_j}{\|v\|_j} = \frac{\|Av\|_j}{\|v\|_j} \cdot \frac{\|v\|_j}{\|v\|_j} = \frac{\|A\frac{v}{\|v\|_j}\|_j}{\|\frac{v}{\|v\|_j}\|_j},$$

so that, in computing $|||A|||_j$, we need only consider vectors $v \in B_j$, the unit ball determined by the norm $\| \cdot \|_j$.

Computing $|||A|||_1$:

Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, and let $v \in B_\infty$. Then, letting $M = \max_{i \in \{1,2,\dots,n\}} \sum_{j=1}^n |a_{j,i}|$,

$$\|Av\|_1 = \sum_{i=1}^n \sum_{j=1}^n |A_{j,i}v_i| = \sum_{i=1}^n |v_i| \sum_{j=1}^n |A_{i,j}| \leq \sum_{i=1}^n |v_i|M = M\|v\|_1 = M.$$

Furthermore, if $v \in B_1$ is such that, for $i = \arg\max_{i \in 1,2,\dots,n} \sum_{j=1}^n A_{j,i}$, $v_i = 1$ and, $\forall j \in \{1, 2, \dots, n\} \backslash \{i\}$, $v_j = 0$, then $AV = M$, so that

$$|||A|||_1 = \max_{i \in \{1,2,\dots,n\}} \left(\sum_{j=1}^n |a_{j,i}| \right). \qquad \blacksquare$$

Computing $|||A|||_\infty$:

Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, let $v \in B_\infty$, and let $b = Av$, with components $b_1, b_2, \dots, b_n$. Then, $\forall i \in \{1, 2, \dots, n\}$, $b_i = A_i \cdot v$, where A_i is the i^{th} row vector of A. b_i is maximized over B_∞ by the vector v with j^{th} component $v_i = \frac{A_{i,j}}{|A_{i,j}|}$, so that $\max_{v \in B_\infty} b_i = \sum_{j=1}^n |A_{i,j}|$. Thus, we find the induced matrix norm by maximizing this sum over $i \in \{1, 2, \dots, n\}$, so that

$$|||A|||_\infty = \boxed{\max_{i \in \{1,2,\dots,n\}} \left(\sum_{j=1}^n |a_{i,j}| \right).}$$

ii)

iii)

Exercise 12

Let V be a Euclidean space.

i) Let $n = \dim V$, and suppose $n \geq 2$.

Let $M_1, M_2 \in L(V,V)$ such that $M_1 = 0$ and

$$M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then, $\forall v = (v_1, v_2, \dots, v_n) \in V$, $(M_1 v, v) = 0 = v_1 v_2 - v_1 v_2 = (M_2 v, v)$, but $M_1 \neq M_2$. Thus, $\geq$ is not antisymmetric $L(V,V)$ when $\dim V \geq 2$, so it is not a partial order on $L(V,V)$.

Since $\forall M \in L_s(V,V)$, $\forall v \in V$ $(Mv, v) \geq (Mv, v)$, so $\geq$ is reflexive. For $M_1, M_2, M_3 \in L(V,V)$, if, $\forall v \in V$, $(M_1 v, v) \geq (M_2 v, v)$ and $(M_2 v, v) \geq (M_3 v, v)$, then, since $\geq$ is transitive on $\mathbb{R}$, $(M_1 v, v) \geq (M v_3, v)$, so that $\geq$ is transitive on $L_s(V,V)$. Let $M_1, M_2 \in L_s(V,V)$ such that $M_1 \geq M_2$ and $M_2 \geq M_1$. Then, $\forall v \in V$, since the inner product is linear in its first argument, $((M_1 - M_2)v, v) = (M_1 v, v) - (M_2 v, v) = 0$. Since M_1 and M_2 are symmetric, $(M_1 - M_2)$ is also symmetric, so that it has n eigenvalues, and is diagonalizable, so that $(M_1 - M_2) = SDS^{-1}$ for some invertible S and some diagonal matrix D. Furthermore, since, $\forall v \in V, ((M_1 - M_2)v, v) = 0$, all eigenvalues of $(M_1 - M_2)$ are 0. Since the non-zero entries of D are the eigenvalues of $(M_1 - M_2)$, $D = 0$. Therefore, $M_1 - M_2 = 0$, so $M_1 = M_2$. Therefore, $\geq$ is antisymmetric on $L_s(V,V)$, so that it is a partial order on $L_s(V,V)$. ■

ii)

iii)

Exercise 13

i) Since M is diagonalizable, it can be diagonalized on an orthogonal basis, so that $M = SDS^{-1} = SDS^T$, where D is diagonal and S is orthogonal. Let $v \in V$, and let $w = Sv$, so that $v = S^T w$. Then, $(Mv, v) = (SDS^T v, v) = v^T SDS^T v = w^T Dw$, and, since the identity commutes with all matrices, $(aIv, v) = v^T aIv = w^T SaIS^T w = aw^T SS^T w = aw^T w$. Thus, $aI \leq M$ if and only if $aI \leq D$. Similarly, $M \leq bI$ if and only if $D \leq bI$.

ii) Let n be the degree of P, and, $\forall i \in \{0, 1, \ldots, n\}$, Suppose $P_i(x) = x^i$, where $a_i \in \mathbb{R}$ is the coefficient of x^i in P. Then, for some $C \in L(V, V)$,

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{P_i(A + \epsilon B) - P_i(A)}{\epsilon} &= \lim_{\epsilon \to 0} \frac{A^i + \epsilon \sum_{j=1}^{n} A^j B A^{i-j} + \epsilon^2 C - A^i}{\epsilon} \\
&= \lim_{\epsilon \to 0} \sum_{j=1}^{n} A^j B A^{i-j} + \epsilon C \\
&= \sum_{j=1}^{n} A^j B A^{i-j}.
\end{aligned}$$

Since the limit is additive and multiplicative where it exists, and $P = \sum_{i=0}^{n} a_i P_i$ for some $a_i \in \mathbb{R}$, $\lim_{\epsilon \to 0} \frac{P(A+\epsilon B)+P(A)}{\epsilon}$ exists.

Exercise 14

---

i)

ii)

Exercise 15

Trivially, $B_0 \in L_s(V, V)$. Suppose, as an induction hypothesis, that, for some $n \in \mathbb{N}$, $B_n \in L_s(V, V)$. Since B_n must commute with itself, B_n^2 is symmetric, so that, since a sum of symmetric matrices and a multiple of symmetric matrices must both be symmetric,

$$B_{n+1} = \frac{A + B^2}{2}$$

must also be symmetric, since A is symmetric. Therefore, by induction on n, $\forall n \in \mathbb{N}$, $B_n \in L_s(V, V)$.