Homework 6

21-260 Differential Equations
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Due: Wednesday, July 25, 2012

Section 3.1, Problem 14

The characteristic equation of the given differential equation is $2r^2 + r - 4 = 0$, whose solutions are $r_1 = \frac{1}{4}\left(\sqrt{33} - 1\right)$ and $r_2 = \frac{1}{4}\left(-\sqrt{33} - 1\right)$. Thus, general solutions to the given differential equation are of the form

$$y = \boxed{c_1 e^{r_1 t} + c_2 e^{r_2 t},}$$

where, for $t_0 = 0$,

$$\begin{aligned} c_1 &= \frac{y'(t_0) - y(t_0) r_2}{r_1 - r_2} e^{-r_1 t_0} = \boxed{\frac{2}{\sqrt{33}}} \\ c_2 &= \frac{y(t_0) r_1 - y'(t_0)}{r_1 - r_2} e^{-r_2 t_0} = \boxed{-\frac{2}{\sqrt{33}}.} \end{aligned}$$

Section 3.2, Problem 18

If the Wronskian of f and g is $t^2 e^t$, then, $f(t)g'(t) - f'(t)g(t) = t^2 e^t$. Thus, since $f(t) = t$, so that $f' = 1$,

$$tg'(t) - g(t) = t^2 e^t.$$

Since this is a linear first-order differential equation, we can solve it by writing it in standard form and multiplying by an integration factor of $\mu := e^{-\int 1/t\,dt} = t^{-1}$, giving

$$\mu(t)g(t) = \int e^t\,dt = e^t + C,$$

for some $C \in \mathbb{R}$. Then,

$$g(t) = \frac{e^t + C}{\mu(t)} = \boxed{te^t + Ct}$$

(note that, without any initial value of g, we cannot further specify C).

Section 3.3, Problem 12

The characteristic equation of the given differential equation is $4r^2 + 9 = 0$, whose solutions are $r_1 = \frac{3}{2}i$ and $r_2 = -\frac{3}{2}i$.

Thus, the general solution of the given differential equation is

$$y = \boxed{c_1 \cos\left(\frac{3}{2}t\right) + c_2 \sin\left(\frac{3}{2}t\right).}$$

Section 3.4, Problem 12

The characteristic equation of the given differential equation is $r^2 - 6r + 9 = 0$, whose unique solution is $r = 3$. Thus, one solution to the differential equation is

$$y_1(t) = e^{3t}.$$

Employing d'Alembert's method, to find another linearly independent solution gives the solution

$$y_2(t) = te^{3t}.$$

Thus, the general solution of the given differential equation is

$$y = \boxed{c_1 e^{3t} + c_2 t e^{3t}},$$

for some $c_1, c_2 \in \mathbb{R}$. Since $y(0) = -1$, $\boxed{c_1 = -1,}$ and, since $y'(0) = 2$, $\boxed{c_2 = 5.}$

Section 3.4, Problem 28

Suppose another solution to the given differential equation is of the form $y_2(x) = v(x)y_1(x)$. Substituting this into the given differential equation and simplifying gives

$$e^x v''(x) + \left(2e^x + \frac{x}{1-x}e^x\right) v' = 0,$$

so that cancelling the e^x terms gives

$$v''(x) + \left(2 + \frac{x}{1-x}\right) v' = 0.$$

Letting $w = v'$ gives the first order linear differential equation

$$w' + \left(2 + \frac{x}{1-x}\right) w = 0,$$

which can be written in the form $\frac{1}{w}w' = -\left(2 + \frac{x}{1-x}\right)$, so that it is seperable. Then,

$$\ln|w| = \int \frac{1}{w}\,dw = \int -\left(2 + \frac{x}{1-x}\right) dx = \log(1-x) - x + C_1,$$

for some $C_1 \in \mathbb{R}$. Thus, $w = (1-x)e^{-x+C_1}$, so that $v = \int w\,dx = xe^{-x+C_1} + C_2$, for some $C_2 \in \mathbb{R}$. Therefore,

$$y_2 = \boxed{(xe^{-x+C_1} + C_2)e^x = xe^{C_1} + C_2 e^x.}$$