Lecture Notes for Week 7 (First Draft)

General Linear Operators (Continued)

Proposition 7.1: Let X and Y be normed linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. Assume further that $\mathcal{D}(A)$ is closed and that A is bounded. Then A is closed.

Proof: Since A is bounded, we may choose $K \in \mathbb{R}$ such that $\|Ax\| \leq K\|x\|$ for all $x \in X$. Let $x \in X$, $y \in Y$, and a sequence $\{x_n\}_{n=1}^{\infty}$ be given such that $x_n \in \mathcal{D}(A)$ for all $n \in \mathbb{N}$. Assume that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. Since $\mathcal{D}(A)$ is closed, we know that $x \in \mathcal{D}(A)$. We need to show that $y = Ax$. Since

$$\|Ax_n - Ax\| \leq K\|x_n - x\| \quad \text{for all } n \in \mathbb{N},$$

we see that $Ax_n \to Ax$ as $n \to \infty$. By uniqueness of limits, we have $Ax_n \to y$ as $n \to \infty$ and A is closed. $\square$

Proposition 7.2: Let X and Y be normed linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. Then $\mathcal{N}(A)$ is closed.

Proof: Let $x \in X$ be given and let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in \mathcal{N}(A)$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Since $x_n \in \mathcal{N}(A)$ for all $n \in \mathbb{N}$ we have $Ax_n \to 0$ as $n \to \infty$. Since A is closed, we have $x \in \mathcal{D}(A)$ and $Ax = 0$, i.e. $x \in \mathcal{N}(A)$ and $\mathcal{N}(A)$ is closed. $\square$

Proposition 7.3: Let X and Y be linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear, closed, and injective. Then $A^{-1} : \mathcal{R}(A) \to X$ is closed.

Proof: Let $y \in Y$, $x \in X$ and a sequence $\{y_n\}_{n=1}^{\infty}$ be given such that $y_n \in \mathcal{R}(A)$ for all $n \in \mathbb{N}$. Assume that $y_n \to y$ and $A^{-1}y_n \to x$ as $n \to \infty$.

Notice that $A^{-1}y_n \in \mathcal{D}(A)$ for all $n \in \mathbb{N}$ and $A(A^{-1}y_n) \to y$ as $n \to \infty$. Since A is closed, we conclude that $x \in \mathcal{D}(A)$ and $Ax = y$. This implies that $y \in \mathcal{R}(A)$ and $A^{-1}y = x$ and consequently A^{-1} is closed. $\square$

Definition 7.4: Let X and Y be normed linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. A is said to be *closable* provided that there exists a linear manifold $\mathcal{D}(\tilde{A}) \subset X$ and a closed linear operator linear operator $\tilde{A} : \mathcal{D}(\tilde{A}) \to Y$ such that $\mathrm{Gr}(A) \subset \mathrm{Gr}(\tilde{A})$, i.e. $\tilde{A}$ is a closed linear extension of A.

Remark 7.5: Let X and Y be normed linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. It is straightforward to show that A is closable if and only if

$$\forall y \in Y\backslash\{0\}, \quad \text{we have } (0, y) \notin \mathrm{cl}(\mathrm{Gr}(A)).$$

Moreover, if A is closable then the minimal closed linear extension of A is called the *closure* of A and is denoted $\overline{A}$.

Adjoints

We have seen that for an (everywhere-defined) bounded linear operator, it is frequently useful to work with the adjoint operator. Let $A : \mathcal{D}(A) \to Y$ be a linear operator with $\mathcal{D}(A) \subset X$, where X and Y are normed linear spaces. We want a linear operator $A^* : \mathcal{D}(A^*) \to X^*$ with $\mathcal{D}(A^*) \subset Y^*$ such that

$$(A^*(y^*))(x) = y^*(Ax) \quad \text{for all } y^* \in \mathcal{D}(A^*),\ x \in \mathcal{D}(A). \tag{1}$$

In order to ensure that A^*y^* is uniquely determined by (1) we need to have $\mathcal{D}(A)$ dense in X. The following simple remark will be employed in the definition of adjoint.

Remark 7.6: Let X be a normed linear space and $S \subset X$ be a dense linear manifold. Assume that $l : S \to \mathbb{K}$ is linear and bounded. Then there exists exactly one $x^* \in X^*$ such that $l(x) = x^*(x)$ for all $x \in S$.

Definition 7.7: Let X and Y be normed linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. Assume further that $\mathcal{D}(A)$ is dense in X. The *adjoint* of A, denoted A^* is the linear operator defined by

$$\mathcal{D}(A^*) = \{y^* \in Y^* : y^*A : \mathcal{D}(A) \to \mathbb{K} \text{ is bounded}\} \tag{2}$$

$$A^*y^* = z^*_{y^*} \quad \text{for all } y^* \in \mathcal{D}(A^*), \tag{3}$$

where for each $y^* \in \mathcal{D}(A^*)$, $z^*_{y^*}$ is the unique element of X^* such that

$$y^*(x) = z^*_{y^*}(x) \quad \text{for all } x \in \mathcal{D}(A). \tag{4}$$

Warning: Even though we have assumed that $\mathcal{D}(A)$ is dense in Definition 7.7, it can happen that $\mathcal{D}(A^*) = \{0\}$.

Definition 7.8: Let X be a Hilbert space. Let $\mathcal{D}(\mathcal{A}) \subset X$ and assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to X$ is linear. The *Hilbert adjoint* A^*_H of A is the linear operator defined by

$$\mathcal{D}(A^*) = \{y \in X : \exists \text{ exactly one } z \in X,\ (Ax, y) = (x, z) \text{ for all } x \in \mathcal{D}(A)\},$$

$$(Ax, y) = (x, A^*y) \quad \text{for all } x \in \mathcal{D}(A),\ y \in \mathcal{D}(A^*).$$

Example 7.9: Let $p \in [1, \infty)$ be given and choose $q \in (1, \infty]$ such that $p^{-1} + q^{-1} = 1$. Put $X = Y = l^p$ and identify X^* and Y^* with l^q in the usual way. Put $\mathcal{D}(A) = \mathbb{K}^{(\mathbb{N})}$ and

$$Ax = \left(\sum_{k=1}^{\infty} kx_k, x_2, x_3, x_4, \cdots \right) \quad \text{for all } x \in \mathcal{D}(A).$$

In order to determine the adjoint of A we need to find $z \in l^q$ satisfying

$$y_1 \sum_{k=1}^{\infty} k x_k + \sum_{k=2}^{\infty} x_k y_k = z_1 x_1 + \sum_{k=2}^{\infty} z_k x_k \quad \text{for all } x \in \mathbb{K}^{(\mathbb{N})}. \tag{5}$$

Substituting $x = e^{(n)}$ (the vector with 1 in the n^{th} slot and zeros elsewhere) we arrive at

$$y_1 = z_1 \quad \text{using } e^{(1)},$$

$$n y_1 + y_n = z_n \quad \text{using } e^{(n)} \text{ for } n \geq 2.$$

Since the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ must be bounded we must have $y_1 = 0$ and we must also have $z_k = y_k$ for all $k \geq 2$. We conclude that

$$\mathcal{D}(A^*) = \{y \in l^q : y_1 = 0\},$$

$$A^* y^* = (0, y_2, y_3, y_4, \cdots) \quad \text{for all } y \in \mathcal{D}(A^*).$$

Notice that $\mathcal{D}(A^*)$ fails to be dense in Y^*.

Example 7.10: Let $\mathbb{K} = \mathbb{R}$ and $X = Y = L^2[0,1]$. We identify X^* and Y^* with $L^2[0,1]$ in the usual way. Consider the linear operator A defined by

$$\mathcal{D}(A) = \{f \in AC[0,1] : f' \in L^2[0,1],\ f(0) = f(1) = 0\},$$

$$Af = f' \quad \text{for all } f \in \mathcal{D}(A).$$

Here $AC[0,1]$ is the set of all absolutely continuous functions $f : [0,1] \to \mathbb{R}$ and f' is the derivative of f. The domain of A is the same as the Sobolev space $W_0^{1,2}(0,1)$ which is also frequently denoted by $H_0^1(0,1)$. Notice that $\mathcal{D}(A)$ is dense. We want to determine the adjoint of A. Let $g \in L^2[0,1]$ be given. In order to have $g \in \mathcal{D}(A^*)$ it is necessary and sufficient for there exist to $h \in L^2[0,1]$ such that

$$\int_0^1 f' g = \int_0^1 f h \quad \text{for all } f \in \mathcal{D}(A). \tag{6}$$

Notice that if $g \in \mathcal{D}(A)$, then $g' \in L^2[0,1]$ and using integration by parts we have

$$\int_0^1 f' g = -\int_0^1 f g' \quad \text{for all } f \in \mathcal{D}(A). \tag{7}$$

It follows that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$. The condition $g(0) = g(1) = 0$ played no role in the derivation of (7). In fact if $g \in AC[0,1]$ and $g' \in L^2[0,1]$ then (7) holds. It can be shown (without too much difficulty) that if there exists $h \in L^2[0,1]$ such that (6) holds then $g \in AC[0,1]$ and $h = -g'$. Consequently

$$\mathcal{D}(A^*) = \{g \in AC[0,1] : g' \in L^2[0,1]\},$$

$$A^* g = -g' \quad \text{for all } g \in \mathcal{D}(A^*).$$

The domain of A^* is the same as the Soloblev space $W^{1,2}(0,1)$ which is also frequently denoted $H^1(0,1)$.

For differential operators, boundary conditions play crucial roles in the specification of domains of operators and their adjoints. One can show that with A as above we have $(A^*)^* = A$ (with equality of domains, i.e. $\mathcal{D}(A^{**}) = \mathcal{D}(A)$.)

You should convince yourself as an exercise that if we define the linear operator $B : \mathcal{D}(B) \to Y$ by

$$\mathcal{D}(B) = \{f \in AC[0,1] : f' \in L^2[0,1],\ f(0) = 0\},$$

$$Bf = f' \text{ for all } f \in \mathcal{D}(B),$$

then

$$\mathcal{D}(B^*) = \{g \in AC[0,1] : g' \in L^2,\ g(1) = 0\},$$
$$B^*g = -g' \text{ for all } g \in \mathcal{D}(B^*).$$

An important feature of adjoints is that they are always closed operators.

Theorem 7.11: Let X and Y be normed linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to y$ is linear. Then A^* is closed.

Proof: Let $x^* \in X^*$, $y^* \in Y^*$, and a sequence $\{y_n^*\}_{n=1}^{\infty}$ in $\mathcal{D}(A^*)$ be given. Assume that $y_n^* \to y^*$ and $A^*y_n^* \to x^*$ as $n \to \infty$. We need to show that $y^* \in \mathcal{D}(A^*)$ and $A^*y^* = x^*$.

Let $x \in \mathcal{D}(A)$ be given. Then we have

$$y_n^*(Ax) \to y^*(Ax) \text{ as } n \to \infty, \tag{8}$$

$$y_n^*(Ax) = (A^*y_n^*)(x) \to x^*(x). \tag{9}$$

It follows from (8), (9), and uniqueness of limits that

$$y^*(Ax) = x^*(x) \text{ for all } x \in \mathcal{D}(A). \tag{10}$$

We conclude from (10) that $y^* \in \mathcal{D}(A^*)$ and $A^*y^* = x^*$ and therefore A^*. It follows that A^* is closed. $\square$

Proposition 7.12: Let X and Y be normed linear spaces. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. Assume further that $\mathcal{D}(A)$ is dense in X. Then $\mathcal{D}(A^*) = Y^*$ if and only if A is bounded. Moreover if $\mathcal{D}(A^*) = Y^*$ then $A^* \in \mathcal{L}(Y^*; X^*)$ and

$$\|A^*\| = \sup\{\|Ax\| : x \in X,\ \|x\| \leq 1\}.$$

Proof: Assume first that A is bounded. Then $y^*A : \mathcal{D}(A) \to \mathbb{K}$ is bounded for every $y^* \in Y^*$ and consequently $\mathcal{D}(A^*) = Y^*$.

Assume that $\mathcal{D}(A^*) = Y^*$. Then, by the Closed Graph Theorem, we have $A^* \in \mathcal{L}(Y^*; X^*)$. Put

$$\mathcal{B} = \{x \in \mathcal{D}(A) : \|x\| \leq 1\},$$
$$|||A||| = \sup\{\|Ax\| : x \in \mathcal{B}\} \leq \infty.$$

Let $y^* \in Y^*$ be given. Then we have

$$\sup\{|y^*(Ax)| : x \in \mathcal{B}\} = \sup\{|(A^*y^*)(x)| : x \in \mathcal{B}\} \leq \|A^*\|\|y^*\|.$$

I claim that

$$|||A||| < \infty.$$

To see why the claim is true, let J denote the canonical injection of Y into Y^{**} and put $\mathcal{S} = \{Ax : x \in \mathcal{B}\}$. For every $y^* \in Y^*$ we have

$$\sup\{|J(y)y^*| : y \in \mathcal{S}\} < \infty,$$

so the Principle of Uniform Boundedness tells us that

$$\sup\{\|J(y)\| : y \in \mathcal{S}\} < \infty.$$

We conclude that $J[\mathcal{S}]$ is bounded in Y^{**} and consequently $\mathcal{S}$ is bounded in Y.

For all $y^* \in Y^*$ we have

$$\begin{aligned}
\|A^*y^*\| &\leq \sup\{|(A^*y^*)(x)| : x \in X, \|x\| \leq 1\} \\
&\leq \sup\{|(A^*y^*)(x)| : x \in \mathcal{B}\} \quad \text{since } B \text{ is dense} \\
&\leq \sup\{|y^*(Ax)| : x \in \mathcal{B}\} \\
&\leq \|y^*\| \cdot |||A|||.
\end{aligned}$$

It follows that $\|A^*\| \leq |||A|||$.

To establish the reverse inequality, observe that for all $x \in \mathcal{D}(A)$ we have

$$\begin{aligned}
\|Ax\| &= \sup\{|y^*(Ax)| : y^* \in Y^*, \|y^*\| \leq 1\} \\
&= \sup\{|(A^*y^*)(x)| : y^* \in Y^*, \|y^*\| \leq 1\} \\
&\leq \|A^*\| \cdot \|x\|.
\end{aligned}$$

It follows that $|||A||| \leq \|A^*\|$. $\square$

Since adjoints are always closed, a natural way to look for a closed extension of an operator A is to try to construct the second adjoint A^{**}. This, of course, will require that $\mathcal{D}(A)$ is dense in X and that $\mathcal{D}(A^*)$ is dense in Y^*.

Let X be a normed linear space and Y be a reflexive Banach space. Assume that $\mathcal{D}(A)$ is dense in X and that $A : \mathcal{D}(A) \to Y$ is linear. Assume further that $\mathcal{D}(A^*)$ is dense in Y^*. Then it is relatively straightforward to show that the operator $\tilde{A}$ defined by

$$\mathcal{D}(\tilde{A}) = \{x \in X : J_X(x) \in \mathcal{D}(A^{**})\},$$

$$\tilde{A}x = (J_Y)^{-1}(A^{**} J_X(x)) \quad \text{for all } x \in \mathcal{D}(\tilde{A})$$

is a closed extension of A. It is in fact the minimal closed extension of A.

Proposition 7.13: Let X be a normed linear space and Y be a reflexive Banach space. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. Then A is closable if and only if $\mathcal{D}(A^*)$ is dense in Y^*. Moreover, in this case the minimal closed extension of A is given by

$$\overline{A} = (J_Y)^{-1} A^{**} J_X.$$

See Section II.2 of *Unbounded Linear Operators* by Seymour Goldberg for the details.

Spectral Theory for General Linear Operators

We now extend some ideas from spectral theory to general linear operators. We assume here that X is a Banach space.

Definition 7.14: Let X be a Banach space. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to X$ is linear. The *resolvent set* for A, denoted by $\rho(A)$ is the set of all $\lambda \in \mathbb{K}$ satisfying

(i) $\lambda I - A$ is injective,

(ii) $\mathcal{R}(\lambda I - A)$ is dense,

(iii) $(\lambda I - A)^{-1} : \mathcal{R}(\lambda I - A) \to X$ is bounded.

Definition 7.15: Let X be a Banach space. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to X$ is linear. The *spectrum* of A, denoted by $\sigma(A)$ is defined by

$$\sigma(A) = \mathbb{K} \backslash \rho(A).$$

It is customary to partition the spectrum into several pairwise disjoint pieces. The most common way of doing this is contained in the next definition.

Definition 7.16: Let X be a Banach space. Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to X$ is linear.

(a) The *point spectrum* of A, denoted $\sigma_p(A)$ is the set of all $\lambda \in \mathbb{K}$ such that (i) in Definition 7.14 fails to hold.

(b) The *continuous spectrum* of A, denoted $\sigma_c(A)$, is the set of all $\lambda \in \mathbb{K}$ such that (i) and (ii) of Definition 7.14 hold, but (iii) fails.

(c) The *residual spectrum* of A, denoted $\sigma_r(A)$, is the set of all $\lambda \in \mathbb{K}$ such that (i) in Definition 7.14 holds, but (ii) fails.

Remark 7.17: The sets $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_r(A)$ are pairwise disjoint and we have

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

The notion of continuous spectrum and residual spectrum are relevant even for compact linear operators as the following example shows.

Example 7.18: Let $X = l^2$ and define $A, B, C \in \mathcal{C}(X; X)$ by

(a) $Ax = \left(0, \frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots\right)$ for all $x \in X$,

(b) $Bx = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots\right)$ for all $x \in X$,

(c) $Cx = \left(0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots\right)$ for all $x \in X$.

Observe that $0 \in \sigma_p(A)$, $0 \in \sigma_c(B)$, and $0 \in \sigma_r(C)$. Observe also that A, B, C all are compact, that A and B are self-adjoint, and that C fails to be normal. (In fact $C = RB$, where R is the right shift operator.)

it is not an accident that C is Example 7.18 is not normal. In fact, we have the following useful observation.

Remark 7.19 Let X be a Hilbert space and let $A \in \mathcal{L}(X; X)$ be given. Assume that A is normal. Let $\lambda \in \mathbb{K}$ be given. It follows from a homework problem that $\lambda I - A$ is injective if and only if $\mathcal{R}(\lambda I - A)$ is dense. This implies that $\sigma_r(A) = \emptyset$.

Remark 7.20: The elements of $\sigma_p(A)$ are called *eigenvalues* and for $\lambda \in \sigma_p(A)$, the nonzero elements of $\mathcal{N}(\lambda I - A)$ are called *eigenvectors* associated with λ.

Definition 7.21: Let X be a Banach space, $\mathcal{D}(A) \subset X$, and assume that $A : \mathcal{D}(A) \to X$ is linear. The approximate point spectrum of A, denoted $\sigma_{ap}(A)$ is the set of all $\lambda \in \mathbb{K}$ such that

$$\inf\{\|(\lambda I - A)x\| : x \in X, \|x\| = 1\} = 0.$$

The elements of $\sigma_{ap}(A)$ are called *generalized eigenvalues.*

An important feature of closed linear operators A is that if $\lambda \in \rho(A)$ then $\mathcal{R}(\lambda I - A) = X$.

Proposition 7.22: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D} \to X$ is linear and closed and let $\lambda \in \rho(A)$ be given. Then $(\lambda I - A)^{-1} \in \mathcal{L}(X; X)$.

Proof: Since A is closed, we see that $\lambda I - A$ is closed and consequently $(\lambda I - A)^{-1}$ is closed. Since $(\lambda I - A)^{-1}$ is bounded, we must have $\mathcal{D}((\lambda I - A)^{-1}) = \mathcal{R}(\lambda I - A)$ is closed. Since $\mathcal{R}(\lambda I - A)$ is dense, we conclude that $\mathcal{R}(\lambda I - A) = X$. $\square$

For unbounded closed operators with dense domains, it can happen that the resolvent set is empty or (even if $\mathbb{K} = \mathbb{C}$) that the spectrum is empty (or that neither of these sets is empty).

Example 7.23: Let $\mathbb{K} = \mathbb{C}$ and $X = L^2[0, 1]$. We want to find closed, densely defined linear operators $A : \mathcal{D}(A) :\to X$, $B : \mathcal{D}(B) \to X$ such that $\rho(A) = \emptyset$ and $\sigma(B) = \emptyset$. The key will be to look at the differential equation

$$f' - \lambda f = g.$$

We shall insist that

$$\mathcal{D}(A), \mathcal{D}(B) \subset \{f \in AC[0, 1] : f' \in L^2[0, 1]\}.$$

For $\lambda \in \mathbb{C}$, put

$$u_\lambda(x) = e^{\lambda x},$$

and observe that

$$u'_\lambda = \lambda u_\lambda,$$

so that

$$\lambda u_\lambda - u'_\lambda = 0 \ \text{ for all} \lambda \in \mathbb{C}.$$

We put

$$\mathcal{D}(A) = \{f \in AC[0, 1] : f' \in L^2[0, 1]\},$$

and define $A : \mathcal{D}(A) \to X$ by

$$Au = u' \ \text{ for all } u \in \mathcal{D}(A).$$

It is immediate that $\sigma(A) = \sigma_p(A) = \mathbb{C}$.

Given $g \in L^2[0, 1]$ and $\lambda \in \mathbb{C}$, put

$$u_{\lambda,g}(x) = -\int_0^x e^{\lambda(x-t)} g(t)\, dt \ \text{ for all } x \in [0, 1].$$

Then we see that $u_{\lambda,g}(0) = 0$ and

$$(u_{\lambda,g})'(x) = -g(x) + \lambda u_{\lambda,g}(x) \ \text{ for a.e. } x \in [0,1].$$

It is straightforward to check that

$$\|u_{\lambda,g}\|_X \leq \|g\|_X \ \text{ for all}\lambda \in \mathbb{C}.$$

We put

$$\mathcal{D}(B) = \{f \in AC[0,1] : f' \in L^2[0,1],\ f(0) = 0\},$$

and define $B : \mathcal{D}(B) \to X$ by

$$Bu = u' \ \text{ for all } u \in \mathcal{D}(B).$$

Then $\mathcal{D}(B)$ is dense, B is closed, and $\rho(B) = \mathbb{C}$.

The Resolvent Operator

Definition 7.24: Let X be a Banach space and $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to X$ is linear and closed. For $\lambda \in \rho(A)$ the *resolvent operator* of A at λ is defined by

$$R(\lambda; A) = (\lambda I - A)^{-1} \in \mathcal{L}(X;X).$$

Notice that $R(\lambda; A) : X \to \mathcal{D}(A)$ and for all $x \in \mathcal{D}(A)$ we have

$$x = (\lambda I - A)R(\lambda; A)x = R(\lambda; A)(\lambda I - A)x,$$

and consequently

$$AR(\lambda; A)x = R(\lambda; A)Ax.$$

Making minor modifications in the proofs of Propositions 5.6 and 5.7, we easily establish the following generaliztions for closed linear operators.

Proposition 7.25: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear and closed. Then

(i) $\rho(A)$ is open,

(ii) $\sigma(A)$ is closed,

(iii) for all $\lambda_0 \in \rho(T)$ and all $\lambda \in \mathbb{K}$ with $|\lambda - \lambda_0| \cdot \|R(\lambda_0; A)\| < 1$ we have $\lambda \in \rho(A)$ and

$$R(\lambda; A) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R(\lambda_0; A)^{n+1},$$

(iv) the mapping $\lambda \to R(\lambda; A)$ is analytic on $\rho(A)$,

(v) for all $\lambda_0 \in \rho(T)$ and all $n \in \mathbb{N}$ we have

$$R^{(n)}(\lambda_0; A) = (-1)^n n! R(\lambda_0; A)^{n+1}.$$

Here $R^{(n)}$ is the n^{th} derivative of R with respect to the first argument.

Proposition 7.26: Let X be a Banach space and let $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear and closed. Let $\mu, \lambda \in \rho(A)$ be given. Then we have

(i) $R(\lambda; A) - R(\mu; A) = (\mu - \lambda) R(\lambda; A) R(\mu; A)$,

(ii) $R(\lambda; A) R(\mu; A) = R(\mu; A) R(\lambda; A)$.