21-484 Notes

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Connectivity - Def: (p. 108): A vertex v in a graph G is a cut-vertex if the number of connected components in $G - \{v\}$ is greater than the number of connected components in G.

- notation: (p. 145): The number of connected components in G is denoted by $\kappa(G)$.

- Claim (Theorem 5.1): A vertex incident with a bridge is a cut-vertex if an only if its degree is at least two.

Proof: let $e = uv$ be a bridge. If v is a leaf, then $\kappa(G - \{v\}) = \kappa(G)$. If $\deg(v) \geq 2$, then let $w \neq u$ be another neighbor of v.

$\rightarrow$ In $G - \{v\}$, e is not an edge. Since e was a bridge then there is no w–u path in $G - \{v\}$ (uvw is a u–w path in G using e, so every u–w path uses e). So u and w are not connected in $G - \{v\}$, $\kappa(G - \{v\}) > \kappa(G)$.

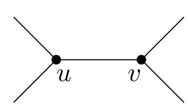
Def (p. 111): A non-trivial connected graph with no cut-vertices is called a nonseparable graph.

Remark: $K_2 =$ •——• is nonseparable.

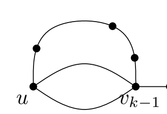
Proposition (Theorem 5.7): A graph with at least 3 vertices is nonseparable if and only if every two vertices lie on a common cycle.

Proof: Let G be a graph with at least 3 vertices.

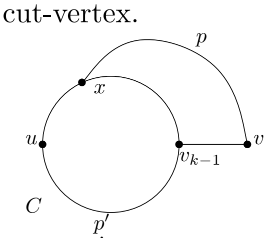
- $\rightarrow$ Assume that every two vertices lie on a common cycle. Assume for the sake of contradiction that v is a cut vertex.
  - G is connected (since there is a cycle between every two vertices). $\kappa(G) = 1$.
  - $\kappa(G - \{v\}) \geq 2$, so there are u and w in separate connected components in $G - \{v\}$.
  - In G, u and w lie on a common cycle, so there are $\geq$ two disjoint u–w paths.
  - v can lie in at most one of these paths.
  - $\Rightarrow$ There is a u–v path in $G - \{v\}$ $\lightning$.
- $\rightarrow$ Assume that G is a nonseparable. Assume for the sake of contradiction that not all pairs of vertices lie on a common cycle, and let u,v be two vertices such that $\text{dist}(u, v)$ is minimal.
  - $\rightarrow$ if $d(u, v) = 1$ then none of them are leaves since then the other vertex is a cut-vertex. Also, can't be that both are leaves.



  - $\Rightarrow$ $\deg(u)$, $\deg(v) \geq 2$. No common cycle containing both u and v means that uv is a bridge.
  - $\rightarrow$ By the claim, we get that both are cut-vertices $\lightning$.
- $\rightarrow$ Assume $d(u, v) \geq 2$ and let $v_0 = u, v_1, \ldots, v_k = v$ be a u–v path.
  - $\rightarrow$ u and v_{k-1} lie on a common cycle, $C = \{u_0 = u, u_1, \ldots, u_\ell = u\}$ (by minimality of u,v).



- $\rightarrow$ There is a u–v path p in $G - \{v_{k-1}\}$, otherwise v_{k-1} is a cut-vertex.
  - $\rightarrow$ Let $x = u_i$ be the last common vertex between c and p.



- $\rightarrow$ call the part of C connecting u and v_{k-1} and not containing x, p'.

We have found a cycle: u, p', go backwards on p until $x, u_{i-1}, \ldots, u_0 = u$ common to u and v. ↯

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