Chapter 7

Second-Order Problems

7.1 C^4-Theory

We now investigate minimization problems for functionals that involve the second derivative of the unknown function. We start with a C^4-theory which can be obtained a bit more simply than the corresponding C^2-theory. Moreover, we consider only the case in which the unknown function is scalar-valued.

Given $a, b \in \mathbb{R}$ with $a < b$, we consider boundary conditions that are any combination of the following

(1) $y(a) = A_0$ with $A_0 \in \mathbb{R}$ given;

(2) $y'(a) = A_1$ with $A_1 \in \mathbb{R}$ given;

(3) $y(b) = B_0$ with $B_0 \in \mathbb{R}$ given;

(4) $y'(b) = B_1$ with $B_1 \in \mathbb{R}$ given.

There are 16 possible sets of boundary conditions (including problems with completely free ends). We want to treat all of the various possibilities in a unified way. For this purpose let $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \{0, 1\}$ be given and put

$$\mathscr{Y} := \left\{ y \in C^4[a,b] \,:\, \alpha_0 y(a) = \alpha_0 A_0, \alpha_1 y'(a) = \alpha_1 A_1, \beta_0 y(b) = \beta_0 B_0, \beta_1 y'(b) = \beta_1 B_1 \right\}.$$

If $\alpha_0 = 0$, then the boundary value for y at a is unprescribed. On the other hand, if $\alpha_0 = 1$, then the boundary value of y at a must be A_0. Similar remarks hold for the other boundary conditions.

We now state our problem. Let $f : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function with continuous third-order partial derivatives. Define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

We wish to minimize J over $\mathscr{Y}$.

For each $y \in \mathscr{Y}$, the class of admissible variations at y is easily seen to be

$$\mathscr{V} := \left\{v \in C^4[a,b] \,:\, \alpha_0 v(a) = \alpha_1 v'(a) = \beta_0 v(b) = \beta_1 v'(b) = 0\right\}.$$

Define

$$\mathscr{V}_0 := \left\{v \in C^4[a,b] \,:\, v(a) = v'(a) = v(b) = v'(b) = 0\right\},$$

and notice that $\mathscr{V}_0 \subset \mathscr{V}$.

In order to find an appropriate analogue of the first Euler-Lagrange equation, we need to compute the Gâteaux variation of J at $y \in \mathscr{Y}$ in the direction $v \in \mathscr{V}$. Let $y \in \mathscr{Y}$ and $v \in \mathscr{V}$ be given. For each $\varepsilon \in \mathbb{R}$, we have

$$J(y+\varepsilon v) = \int_a^b f(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x), y''(x)+\varepsilon v''(x))\, dx.$$

Thus

$$\begin{aligned}\frac{d}{d\varepsilon}\left[J(y+\varepsilon v)\right] = \int_a^b \Big\{ & f_{,2}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x), y''(x)+\varepsilon v''(x))v(x) \\ & + f_{,3}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x), y''(x)+\varepsilon v''(x))v'(x) \\ & + f_{,4}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x), y''(x)+\varepsilon v''(x))v''(x)\Big\}\, dx.\end{aligned}$$

Evaluating the above expression at $\varepsilon = 0$ yields

$$\begin{aligned}\delta J(y; v) = \int_a^b \Big\{ & f_{,2}(x, y(x), y'(x), y''(x))v(x) + f_{,3}(x, y(x), y'(x), y''(x))v'(x) \\ & + f_{,4}(x, y(x), y'(x), y''(x))v''(x)\Big\}\, dx.\end{aligned}$$

Suppose that J attains a minimum over $\mathscr{Y}$ at y_*. Then, for each $v \in \mathscr{V}$, the Gâteaux variation $\delta J(y_*; v)$ is zero. Define $F, G, H : [a,b] \to \mathbb{R}$ by

$$F(x) := f_{,2}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b],$$

$$G(x) := f_{,3}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b]$$

and

$$H(x) := f_{,4}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b].$$

With these definitions, we may write

$$\delta J(y_*; v) = \int_a^b \Big\{F(x)v(x) + G(x)v'(x) + H(x)v''(x)\Big\} = 0 \quad \text{for all } v \in \mathscr{V}.$$

Notice that

$$\int_a^b H(x)v''(x)\,dx = H(x)v'(x)\Big|_a^b - \int_a^b H'(x)v'(x)\,dx.$$

Consequently, if $\delta J(y_*; v) = 0$ for every $v \in \mathscr{V}$, then

$$H(x)v'(x)\Big|_a^b + \int_a^b \Big\{F(x)v(x) + [G(x) - H'(x)]\,v'(x)\Big\} = 0 \quad \text{for all } v \in \mathscr{V}. \tag{7.1}$$

Now

$$\int_a^b [G(x) - H'(x)]\,v'(x)\,dx = [G(x) - H'(x)]\,v(x)\Big|_a^b - \int_a^b [G'(x) - H''(x)]\;dx,$$

and substitution into (7.1) yields

$$\Big\{H(x)v'(x) + [G(x) - H'(x)]\,v(x)\Big\}\Big|_a^b + \int_a^b \Big\{F(x) - G'(x) + H''(x)\Big\}v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}. \tag{7.2}$$

Since (7.2) must hold for each $v \in \mathscr{V}$, it must hold for each $v \in \mathscr{V}_0 \subset \mathscr{V}$. Whence

$$\int_a^b \Big\{F(x) - G'(x) + H''(x)\Big\}v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.3}$$

We state without proof the following

Lemma 7.1 *Let $g \in C[a,b]$ be given. Assume that*

$$\int_a^b g(x)v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0.$$

Then $g(x) = 0$ at each $x \in [a,b]$.

(The proof is very similar to the proof of Lemma 3.1. One can use $(x-\alpha)^6(x-\beta)^6$ instead of $(x-\alpha)^4(x-\beta)^4$ in the construction of $v_*(x)$.) Using Lemma 7.1, and equation(7.3) we find that

$$F(x) - G'(x) + H''(x) = 0 \quad \text{for all } x \in [a,b]. \tag{7.4}$$

We conclude that if y_* minimizes J over $\mathscr{Y}$, then y_* must satisfy

$$f_{,2}(x, y_*(x), y_*'(x), y_*''(x)) - \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x), y_*''(x))\right] + \frac{d^2}{dx^2}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right] = 0 \quad \text{for all } x \in [a,b]. \qquad (\text{E-L})_1$$

We turn now to the natural boundary conditions. Since we now know that y_* must satisfy $(\text{E-L})_1$, the condition in (7.2) reduces to

$$H(b)v'(b) + [G(b) - H'(b)]v(b) - H(a)v'(a) - [G(a) - H'(a)]v(a) = 0 \quad \text{for all } v \in \mathscr{V}. \tag{7.5}$$

First suppose that $\alpha_1 = 0$. Then we may choose $v \in \mathscr{V}$ such that $v(a) = v(b) = v'(b) = 0$ and $v'(a) = 1$. Thus (7.5) implies that $H(a) = 0$ and consequently

$$f_{,4}(a, y_*(a), y_*'(a), y_*''(a)) = 0.$$

If $\alpha_1 = 1$, then there is no associated natural boundary condition. We may express both of these situations simultaneously with the single condition

$$(1 - \alpha_1) f_{,4}(a, y_*(a), y_*'(a), y_*''(a)) = 0. \tag{7.6}$$

Now, we suppose that $\alpha_0 = 0$. Then we may choose $v \in \mathscr{V}$ such that $v'(a) = v(b) = v'(b) = 0$ and $v(a) = 1$. Thus (7.5) implies that $G(a) - H'(a) = 0$ and consequently

$$f_{,3}(a, y_*(a), y_*'(a), y_*''(a)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\Big|_{x=a} = 0.$$

As before, if $\alpha_0 = 1$, then there is no associated natural boundary condition, and we may express both possibilities together as

$$(1 - \alpha_0)\left\{f_{,3}(a, y_*(a), y_*'(a), y_*''(a)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\Big|_{x=a}\right\} = 0. \tag{7.7}$$

We conclude that the natural boundary conditions at $x = a$ are

$$\begin{cases} (1 - \alpha_0)\left\{f_{,3}(a, y_*(a), y_*'(a), y_*''(a)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\big|_{x=a}\right\} = 0; \\ (1 - \alpha_1) f_{,4}(a, y_*(a), y_*'(a), y_*''(a)) = 0. \end{cases} \qquad (\text{NBC})_a$$

Similarly, the natural boundary conditions at $x = b$ are

$$\begin{cases} (1 - \beta_0)\left\{f_{,3}(b, y_*(b), y_*'(b), y_*''(b)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\big|_{x=b}\right\} = 0; \\ (1 - \beta_1) f_{,4}(b, y_*(b), y_*'(b), y_*''(b)) = 0. \end{cases} \qquad (\text{NBC})_b$$

7.2 Example 7.2

For this example, we put

$$\mathscr{Y} := \left\{ y \in C^4[0,1] \ : \ y(0) = 0, y'(0) = 1 \text{ and } y'(1) = -1 \right\},$$

so that $\alpha_0 = \alpha_1 = \beta_1 = 1$ and $\beta_0 = 0$. Define $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, y, z, w) := y^2 + z^2 + w^2 \quad \text{for all } (x, y, z, w) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

Let the functional $J : \mathscr{Y} \to \mathbb{R}$ be given by

$$J(y) := \int_0^1 f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Computing the partial derivatives for f, we have at each $(x, y, z, w) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ that

$$f_{,2}(x, y, z, w) = 2y; \ f_{,3}(x, y, z, w) = 2z; \text{ and } f_{,4}(x, y, z, w) = 2w.$$

The Euler-Lagrange equation for J is thus

$$2y(x) - \frac{d}{dx}\left[2y'(x)\right] + \frac{d^2}{dx^2}\left[2y''(x)\right] = 0 \quad \text{for all } x \in [0,1]. \qquad \text{(E-L)}_1$$

To find solutions to (E-L)_1, we would need to solve the fourth order equation

$$y(x) - y''(x) + y^{(4)}(x) = 0 \quad \text{for all } x \in [0,1].$$

Let us look at the natural boundary condition at $x = 1$ (there is only one since $\alpha_0 = \alpha_1 = \beta_1 = 1$). We have

$$2y'(1) - \left.\frac{d}{dx}\left[2y''(x)\right]\right|_{x=1} = 0. \qquad \text{(NBC)}_1$$

Since $y \in \mathscr{Y}$ implies $y'(1) = -1$, the natural boundary condition is

$$y^{(3)}(1) = -1. \qquad \text{(NBC)}_1$$

So if $y \in \mathscr{Y}$ minimizes J over $\mathscr{Y}$, then y satisfies

$$\begin{cases} y(x) - y''(x) + y^{(4)}(x) = 0 \\ y(0) = 0, y'(0) = 1, y'(1) = -1, y^{(3)}(1) = -1. \end{cases}$$

7.3 C^2-Theory

We now look at what happens if the admissible functions are assumed to be of class C^2 rather than class C^4. We can still obtain the Euler-Lagrange quations under the more natural assumption that the admissible functions are only *twice* continuously differentiable, but we will need to work a little bit harder.

Let $a, b, A_0, A_1, B_0, B_1 \in \mathbb{R}$ with $a < b$ and $f : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \{0,1\}$ be given. We assume that f has continuous first-order partial derivatives.

We put

$$\mathscr{Y} := \left\{ y \in C^2[a,b] \,:\, \alpha_0 y(a) = \alpha_0 A_0, \alpha_1 y'(a) = \alpha_1 A_1, \beta_0 y(b) = \beta_0 B_0, \beta_1 y'(b) = \beta_1 B_1 \right\}.$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

For each $y \in \mathscr{Y}$, the class of admissible variations at y is easily seen to be

$$\mathscr{V} := \left\{ v \in C^2[a,b] \,:\, \alpha_0 v(a) = \alpha_1 v'(a) = \beta_0 v(b) = \beta_1 v'(b) = 0 \right\}.$$

Define

$$\mathscr{V}_0 := \left\{ v \in C^2[a,b] \,:\, v(a) = v'(a) = v(b) = v'(b) = 0 \right\},$$

and notice that $\mathscr{V}_0 \subset \mathscr{V}$

Let $y_* \in \mathscr{Y}$ be given and assume that y_* minimizes J over $\mathscr{Y}$. Then, for each $v \in \mathscr{V}$, the Gâteaux variation $\delta J(y_*; v)$ is zero. The previously derived formula for Gâteaux variations remains valid under the weaker smoothness assumptions of this section. As before, we define $F, G, H : [a,b] \to \mathbb{R}$ by

$$F(x) := f_{,2}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b],$$

$$G(x) := f_{,3}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b],$$

and

$$H(x) := f_{,4}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b].$$

With these definitions, we have

$$\delta J(y_*; v) = \int_a^b \Big\{ F(x)v(x) + G(x)v'(x) + H(x)v''(x) \Big\} = 0 \quad \text{for all } v \in \mathscr{V}.$$

In particular, since $\mathscr{V}_0 \subset \mathscr{V}$ we have

$$\delta J(y_*; v) = \int_a^b \Big\{ F(x)v(x) + G(x)v'(x) + H(x)v''(x) \Big\} = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.8}$$

The idea is to integrate by parts in such a way that we get

$$\int_a^b w(x)v''(x) = 0 \quad \text{for all } v \in \mathscr{V}_0$$

for some function w.

For this purpose, it will be convenient to define $\widetilde{F}, \widetilde{G}, \widehat{F}; [a,b] \to \mathbb{R}$ by

$$\widetilde{F}(x) := \int_a^x F(t)\,dt \qquad \widetilde{G}(x) := \int_a^x G(t)\,dt \quad \text{for all } x \in [a,b],$$

$$\widehat{F}(x) := \int_a^x \widetilde{F}(t)\,dt \quad \text{for all } x \in [a,b].$$

Observe that $\widetilde{F}, \widetilde{G} \in C^1[a,b]$ and $\widehat{F} \in C^2[a,b]$. Since $\widetilde{G}'(x) = G(x)$ for all $x \in [a,b]$ (and $v'(a) = v'(b) = 0$ for all $v \in \mathscr{V}_0$) we have

$$\int_a^b G(x)v'(x)\,dx = -\int_a^b \widetilde{G}(x)v''(x)\,dx \quad \text{for all } v \in \mathscr{V}_0.$$

In addition, we have $\widehat{F}''(x) = \widetilde{F}'(x) = F(x)$ for all $x \in [a,b]$ and consequently, after integrating by parts twice and using the boundary conditiona for v, we find that

$$\int_a^b F(x)v(x)\,dx = \int_a^b \widehat{F}(x)v''(x)\,dx \quad \text{for all } v \in \mathscr{V}_0.$$

Substituting these formulas into (7.8) we obtain

$$\int_a^b \Big\{\widehat{F}(x) - \widetilde{G}(x) + H(x)\Big\}v''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.9}$$

Lemma 7.2 *Let $w \in C[a,b]$ be given and assume that*

$$\int_a^b w(x)v''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0.$$

Then there exist $c_0, c_1 \in \mathbb{R}$ such that

$$w(x) = c_0 + c_1 x \quad \text{for all } x \in [a,b].$$

Proof. We shall find $d_0, d_1 \in \mathbb{R}$ such that $w(x) = d_0 + d_1(x-a)$ for all $x \in [a,b]$. Let $d_0, d_1 \in \mathbb{R}$ be given. Observe that

$$\int_a^b v''(x)\,dx = \int_a^b xv''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0,$$

and consequently

$$\int_a^b (w(x) - d_0 - d_1(x-a))v''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.10}$$

We want to construct $v_* \in \mathscr{V}_0$ such that

$$v_*''(x) = w(x) - d_0 - d_1(x-a) \quad \text{for all } x \in [a,b]. \tag{7.11}$$

In order for (7.11) to hold and also have $v_*'(a) = 0$ we need to have

$$\begin{aligned} v_*'(x) &= \int_a^x \{w(t) - d_0 - d_1 t\}\,dt \\ &= -d_0(x-a) - \tfrac{d_1}{2}(x-a)^2 + \int_a^x w(t)\,dt \quad \text{for all } x \in [a,b], \end{aligned} \tag{7.12}$$

and consequently, since we also want $v(a) = 0$ we need to have

$$\begin{aligned} v_*(x) &= \int_a^x \left\{ \int_a^t w(\tau)\,d\tau - d_0(t-a) - \tfrac{d_1}{2}(t-a)^2 \right\} dt \\ &= -\tfrac{d_0}{2}(x-a)^2 - \tfrac{d_1}{6}(x-a)^3 + \int_a^x \int_a^t w(\tau)\,d\tau \quad \text{for all } x \in [a,b]. \end{aligned} \tag{7.13}$$

Whether or not the expression in (7.13) gives a function in $\mathscr{V}_0$ depends on the values of d_0 and d_1. If v_* is defined by (7.13) then $v_*(a) = v_*'(a) = 0$ automatically. We need to choose d_0, d_1 (if possible) so that

$$v_*'(b) = -d_0(b-a) - \frac{d_1}{2}(b-a)^2 + \int_a^b w(x)\,dx = 0$$

and

$$v_*(b) = -\frac{d_0}{2}(b-a)^2 - \frac{d_1}{6}(b-a)^3 + \int_a^b \int_a^x w(t)\,dt\,dx = 0.$$

In other words, we should choose d_0 and d_1 (if possible) so that the linear system

$$\begin{pmatrix} (b-a) & \frac{(b-a)^2}{2} \\ \frac{(b-a)^2}{2} & \frac{(b-a)^3}{6} \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} = \begin{pmatrix} \int_a^b w(x)\,dx \\ \int_a^b \int_a^x w(t)\,dt\,dx \end{pmatrix} \tag{7.14}$$

is satisfied. The determinant of the coefficient matrix in (7.14) is

$$\frac{1}{6}(b-a)^3(b-a) - \left(\frac{1}{2}(b-a)^2\right)^2 = -\frac{1}{12}(b-a)^4 \neq 0,$$

and consequently there is exactly one choice for the pair (d_0, d_1) such that (7.14) is satisfied. If we make this choice for (d_0, d_1) and define v_* by (7.13) then $v_* \in \mathscr{V}_0$ and (7.11) holds so we have

$$\int_a^b (w(x) - d_0 - d_1(x-a))^2 \, dx = 0.$$

We conclude that $w(x) - d_0 - d_1(x-a) = 0$ for all $x \in [a,b]$ and the proof of Lemma 7.2 is complete. $\square$

Using Lemma 7.2 and equation (7.9) we may choose $c_0, c_1 \in \mathbb{R}$ such that

$$\widehat{F}(x) - \widetilde{G}(x) + H(x) = c_0 + c_1 x \quad \text{for all } x \in [a,b].$$

Since

$$H(x) = \widetilde{G}(x) - \widehat{F}(x) + c_0 + c_1 x \quad \text{for all } x \in [a,b],$$

and $\widehat{F}, \widetilde{G} \in C^1[a,b]$, we conclude that

$$H \in C^1[a,b] \text{ and } H'(x) = \widetilde{G}'(x) - \widehat{F}'(x) + c_1 = G(x) - \widetilde{F}(x) + c_1 \quad \text{for all } x \in [a,b].$$

Since

$$H'(x) - G(x) = c_1 - \widetilde{F}(x) \quad \text{for all } x \in [a,b]$$

and $\widetilde{F} \in C^1[a,b]$ with $\widetilde{F}' = F$ we conclude that

$$H' - G \in C^1[a,b] \text{ and } (H' - G)'(x) + F(x) = 0 \quad \text{for all } x \in [a,b].$$

(Notice that if $y_* \in C^4[a,b]$ and f has continuous third-order partial derivatives then the equation $(H' - G)' + F = 0$ reduces to $(\text{E-L})_1$ from Section 7.1.) With this additional information about F, G, H, we can return to the equation

$$\int_a^b \Big\{ F(x)v(x) + G(x)v'(x) + H(x)v''(x) \Big\} \, dx = 0 \quad \text{for all } v \in \mathscr{V}$$

and integrate by parts as we did in the C^4-theory to obtain

$$\Big\{ H(x)v'(x) + (G(x) - H'(x))v(x) \Big\} \Big|_a^b = 0 \quad \text{for all } v \in \mathscr{V}. \tag{7.15}$$

The same argument as in the C^4-theory can be used and we get exactly the same natural boundary conditions.

We summarize these results in a theorem.

Theorem 7.1 *Let $a, b, A_0, A_1, B_0, B_1 \in \mathbb{R}$ with $a < b$ and $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \{0, 1\}$ be given. Put*

$$\mathscr{Y} := \{ y \in C^2[a,b] \,:\, \alpha_0 y(a) = \alpha_0 A_0, \alpha_1 y'(a) = \alpha_1 A_1, \beta_0 y(b) = \beta_0 B_0, \beta_1 y'(b) = \beta_1 B_1 \}.$$

Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with continuous first partial derivatives be given and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Lrt $y_* \in \mathscr{Y}$ be given and assume that y_* minimizes J on $\mathscr{Y}$. Define $F, G, H : [a, b] \to \mathbb{R}$ by $F(x) = f_{,2}(x, y_*(x), y_*'(x), y_*''(x))$, $G(x) = f_{,3}(x, y_*(x), y_*'(x), y_*''(x))$, and $H(x) = f_{,4}(x, y_*(x), y_*'(x), y_*''(x))$ for all $x \in [a, b]$. Then, $H \in C^1[a, b]$, $(H' - G) \in C^1[a, b]$ and

$$F(x) + (H' - G)'(x) = 0 \quad \text{for all } x \in [a, b],$$

$$(1 - \alpha_0)\{G(a) - H'(a)\} = 0 \quad (1 - \alpha_1)H(a) = 0,$$

$$(1 - \beta_0)\{G(b) - H'(b)\} = 0 \quad (1 - \beta_1)H(b) = 0.$$

7.4 Example 7.4

Set
$$\mathscr{Y} := \left\{ y \in C^2[0, 1] \,:\, y(0) = y(1) = 0 \text{ and } y'(1) = 1 \right\}.$$
We take $\alpha_0 = \beta_0 = \beta_1 = 1$ and $\alpha_1 = 0$. Define $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, y, z, w) := 4y + z^2 + w^2 \quad \text{for all } x \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

We want to minimize the functional $J : \mathscr{Y} \to \mathbb{R}$ given by

$$J(y) := \int_0^1 f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}$$

over the class $\mathscr{Y}$.

We find that

$$f_{,2}(x, y, z, w) = 4; \;\; f_{,3}(x, y, z, w) = 2z; \;\; \text{and } f_{,4}(x, y, z, w) = 2w$$

for all $(x, y, z, w) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. If $y \in \mathscr{Y}$ is a minimizer, then $2y'' \in C^1[0, 1]$ and $2y''' - 2y' \in C^1[0, 1]$, from which we conclude that $y \in C^4[0, 1]$. The Euler-Lagrange equation for J is

$$4 - \frac{d}{dx}[2y'(x)] + \frac{d^2}{dx^2}[2y''(x)] = 0 \qquad \text{(E-L)}_1$$

The only natural boundary condition is

$$2y''(0) = 0. \qquad \text{(NBC)}_0$$

So, we seek those functions satisfying

$$\begin{cases} y^{(4)}(x) - y''(x) = -2 \\ y(0) = y''(0) = y(1) = 0 \text{ and } y'(1) = 1. \end{cases}$$

We first find general solutions to the homogeneous equation

$$y^{(4)}(x) - y''(x) = 0 \tag{H}$$

The roots to the characteristic equation $r^4 - r^2 = 0$ are $r = \pm 1, 0$, with zero being a root with multiplicity 2. So the general solution to (H) is

$$y_H(x) = c_1 \cosh x + c_2 \sinh x + c_3 + c_4 x.$$

We see that a particular solution to

$$y^{(4)}(x) - y''(x) = 2 \tag{7.16}$$

is

$$y_P(x) = x^2.$$

By summing the general solution to (H) and the particular solution y_P, we have

$$y(x) = c_1 \cosh x + c_2 \sinh x + c_3 + c_4 x + x^2,$$

which is the general solution to (7.16).

We now impose the condition $y(0) = 0$. We have

$$y(0) = c_1 + c_3 = 0 \Rightarrow c_3 = -c_1.$$

So

$$y(x) = c_1 \cosh x + c_2 \sinh x - c_1 + c_4 x + x^2.$$

For the condition $y''(0) = 0$, we have

$$y''(0) = c_1 + 2 = 0 \Rightarrow c_1 = -2.$$

Thus

$$y(x) = -2 \cosh x + c_2 \sinh x + c_4 x + 2 + x^2.$$

The condition $y(1) = 0$ implies

$$y(1) = -2 \cosh 1 + c_2 \sinh 1 + c_4 + 3 = 0,$$

while the condition $y'(1) = 1$ implies

$$y'(1) = -2 \sinh 1 + c_2 \cosh 1 + c_4 + 4 = 0.$$

Thus

$$2(\cosh 1 - \sinh 1) + c_2(\cosh 1 - \sinh 1) = -1 \Rightarrow c_2 = \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1},$$

and

$$c_4 = 2\cosh 1 - 3 - c_2 \sinh 1 = 2\cosh 1 - 3 - \sinh 1 \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1}.$$

The only possible minimizer for J over $\mathscr{Y}$ is

$$y(x) = -2\cosh x + \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1} \sinh x$$
$$+ \left[2\cosh 1 - 3 - \sinh 1 \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1} \right] x + 2 + x^2 \quad \text{for all } x \in [0,1].$$

7.5 Two Remarks Regarding Second-Order Problems

Remark 7.1 *If at each $x \in [a,b]$, the function $f(x,\cdot,\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is convex and $\mathscr{Y}$ is a convex set, then J is convex over $\mathscr{Y}$. In such a case, a solution to $(E\text{-}L)_1$ satisfying all boundary conditions is a minimizer for J over $\mathscr{Y}$.*

Remark 7.2 *Lagrange multiplier techniques can be used to handle constraints of the form*

$$\int_a^b g(x, y(x), y'(x), y''(x))\, dx = c,$$

where $g : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ are given.