MATH 651: PROBLEM SET 2 SOLUTIONS ARE IN CLASS ON FRI. SEP. 28.

1. (10 points)
   (a) Show that the topology of radially open sets in $\mathbb{R}^2$ is not first countable.
   (b) Show that the product of two first countable spaces is first countable.
   (c) Show that any subspace of a first countable space is first countable (that is being first countable is a hereditary property).
2. (5 points) Show that any second countable space is separable.
3. (10 points) Let (X, d) be a separable metric space. Let A be a nonempty subset of X. Show that (A, d) is separable.
4. (10 points) Consider the space of continuous functions on the real line, $C(\mathbb{R}, \mathbb{R})$, endowed with the metric

$$d(f, g) := \max_n \frac{1}{2^n} \frac{\max_{x \in [-n,n]} |f(x) - g(x)|}{1 + \max_{x \in [-n,n]} |f(x) - g(x)|}. \tag{1}$$

   Show that $(C(\mathbb{R}, \mathbb{R}), d)$ is separable. [You can use that $C([-n, n], \mathbb{R})$ is separable with respect to d_∞ metric.]
5. (10 points) In $\mathbb{R}^N$, a *polygonal path* is a continuous path $\gamma : [a, b] \to \mathbb{R}^N$ for which there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ with the property that $\gamma : [x_{i-1}, x_i] \to \mathbb{R}^N$ is affine for all $i = 1, \ldots, n$, that is,

$$\gamma(t) = c_i + td_i \quad \text{for } t \in [x_{i-1}, x_i],$$

   for some $c_i, d_i \in \mathbb{R}^N$.
   Let $O \subset \mathbb{R}^N$ be open and connected and let $x_0 \in O$.
   (i) Prove that the set

$$\begin{aligned} U := \{x \in O : &\text{ there exists a polygonal path with} \\ &\text{endpoints } x \text{ and } x_0 \text{ and range contained in } O\} \end{aligned}$$

   is open and nonempty.
   (ii) Prove that the set

$$\begin{aligned} V := \{x \in O : &\text{ there does not exist a polygonal path with} \\ &\text{endpoints } x \text{ and } x_0 \text{ and range contained in } O\} \end{aligned}$$

   is open.

6. (5 points) Let Y_i with $i \in I$ be a family of connected subsets of X, such that $\cap_{i\in I} Y_i \neq \emptyset$. Show that $Y = \cup_{i\in I} Y_i$ is connected.

7. (10 points) Prove that the graph of the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \in (0,1] \\ 1 & \text{if } x = 0. \end{cases}$$

   is connected, but not pathwise connected.