21-484 Notes

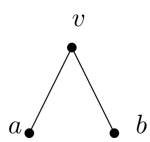
JD Nir
jnir@andrew.cmu.edu
April 6, 2012

→ <u>Brooks Theorem:</u> G is connected, not complete, not odd cycle $\chi(G) \leq \Delta(G)$.

→ Assume that G is 2-connected.

→ Assume that $\Delta(G) \geq 3$

→ Assume that G is Δ-regular

→ Find 

and $G - a - b$ is connected.

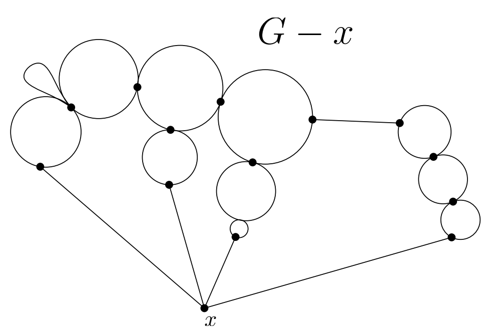
→ Once we have 3 vertices a, b, v such that $av, bv \in E(G)$, $ab \notin E(G)$ and $G - a - b$ is connected, color a and b by color 1. Find a spanning tree for $G - a - b$; root the tree at v. Color vertices according to their place in the tree from leaves towards the root. This can be done becasue every vertex has at most $\Delta - 1$ colored enighbors (its parent is not colored).

→ Whe we try to color v, it has at most $\Delta - 2$ colored neighbors besides a, b. But a and b are both colored 1.

→ Consider a vertex x that is not adjacent to all ofther vertices.

→ If $G - x$ is still 2-connected, find a vertex of distance 2 from x (call it y). Let v be a common neighbor of x and y. Letting x and y have the roles of a ad b works. Indeed, $G - x - y$ is connected because $G - x$ is 2-connected.

→ Assume that $G - x$ is not 2-connected. Consider the block decomposition of $G - x$.



→ We have a tree of blocks, there are at least two end blocks (because every tree has at least two leaves).

→ An end block B_i has a vertex j_i such that every other block B_k is either disjoint from B_i, or they have j_i as their only common vertex.

→ Let B_1 and B_2 be two end blocks. There are two vertices $b_1 \in B_2$, $b_2 \in B_2$ such that $b_1 \neq j_1$, $b_2 \neq j_2$, and $xb_1 \in E(G)$ and $xb_2 \in E(G)$.

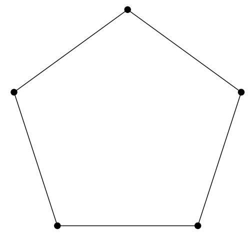
→ otherwise, if such b_1 does not exist, then j_1 is a cut vertex of G $\lightning$ G is 2-connected.

→ x, b_1, b_2 can have the roles of v, a, b.

→ b_1 and b_2 are neighbors of x, by the above. They are not adjacent because they are in different blocks in G.

→ $G - b_1 - b_2 - x$ is connected, because neither b_1 nor b_2 was a joint and $d_G(x) \geq 3$. ■

JD Nir
jnir@andrew.cmu.edu



$\chi(C_5) = 3$
$\omega(C_5) = 2$

Theorem 10.10: $\forall$ integer k there is a triangle free graph with chromatic number k.

$\rightarrow$ For every forest F, $\chi(F) \leq 2$.

Theorem (Erdős): For all integers k, ℓ, $\exists G$ such that girth$(G) > \ell$ and $\chi(G) > k$.

Proof: $\rightarrow$ set $0 < \theta < \frac{1}{\ell}$ constant

$\rightarrow$ define: $p = n^{-1+\theta}$

$\rightarrow$ Consider a graph on n vertices such that every possible edge is in G with probability P, independently of all other edges.

Let X be the number of short ($\leq \ell$) cycles in G. X is a random variable.

$$\mathbb{E}[x] = \sum_{i=3}^{\ell} (\# \text{ of } i\text{-cycles in } K_n) \cdot p^i = \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{\underset{\substack{\uparrow \\ \text{starting point} \\ + \text{ direction}}}{2 \cdot i}} \leq \sum_{i=3}^{\ell} n^i p^i \leq 2 \cdot (np)^\ell = 2n^{\theta\ell} \underset{\substack{n \text{ is large} \\ \text{enough}}}{<} \frac{n}{\log n}$$

$\Pr[X \geq {}^{n}/_{2}] \leq \frac{\mathbb{E}[X]}{n/2} \leq \frac{2n}{\log n \cdot n} = \frac{2}{\log n} \overset{n \to \infty}{\rightarrow} 0$

Markov's inequality: if X is a non-negative random variable with expectation then for any positive real a

$$\Pr[X > a] \leq \frac{\mathbb{E}[X]}{a}$$