Assignment 5

Due on Wednesday, April 10

Solutions to problems marked with an asterisk should be written up and handed in.

Def: Let X be a normed linear space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be

(i) *convex* provided that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \ \text{ for all } \ x, y \in X, \ t \in (0,1),$$

(ii) *lower semicontinuous* provided that for every $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) > \alpha\}$ is open,

(iii) *proper* provided that there exists $x_0 \in X$ such that $f(x_0) < \infty$,

(iv) *coercive* provided that

$$\forall \alpha \in \mathbb{R}, \ \exists M \in \mathbb{R} \ \text{ such that } f(x) > \alpha \ \text{ for all } \ x \in X \ \text{ with } \ ||x|| > M.$$

(This last condition just says that $f(x) \to +\infty$ as $||x|| \to \infty$.)

1.* Let X be a normed linear space and assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semicontinuous. Let $x \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in X be given. Assume that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. Show that

$$\liminf_{n \to \infty} f(x_n) \geq f(x).$$

2.* Let X be a reflexive Banach space and assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous, proper, and coercive. Show that f attains a minimum on X.

3. Let X be a normed linear space and S be a dense subset of X^*. Assume that $\{\langle y^*, x_n \rangle\}_{n=1}^{\infty}$ is convergent for every $y^* \in S$.

   (a) Show that if $\{x_n\}_{n=1}^{\infty}$ is bounded then $\{\langle x^*, x_n \rangle\}_{n=1}^{\infty}$ is convergent for every $x^* \in X^*$.

   (b) Show, by giving an example, that the conclusion of part (a) can fail if the sequence $\{x_n\}_{n=1}^{\infty}$ is unbounded.

4. Prove Theorem 9.2 from the notes: Define $T : l^1 \to (c_0)^*$ by

$$(Ty)(x) = \sum_{k=1}^{\infty} x_k y_k \ \text{ for all } y \in l^1, \ x \in c_0.$$

Then T is an isometric isomorphism of l^1 onto $(c_0)^*$.

5.* Let $X = c_0$ and identify X^* with l^1 in the usual way. Let $a \in \mathbb{K}^{\mathbb{N}^2}$ be given and assume that

$$\sup\left\{\sum_{n=1}^{\infty} |a_{mn}| : m \in \mathbb{N}\right\} < \infty$$

and that

$$\forall n \in \mathbb{N}, \quad a_{mn} \to 0 \text{ as } m \to \infty.$$

Show that the formula

$$(Tx)_m = \sum_{n=1}^{\infty} a_{mn} x_n, \quad m \in \mathbb{N}$$

defines a bounded linear mapping $T : X \to X$. Find a formula for $T^* : l^1 \to l^1$.

6. Prove Proposition 10.7 from the notes: Let X be a normed linear space and let $A \subset X$. Then

$$^{\perp}(A^{\perp}) = \mathrm{cl}(\mathrm{span}(A)).$$

7.* Give an example of a Banach space X and a closed subspace Z of X^* such that $Z \subsetneq \left(^{\perp}Z\right)^{\perp}$.

8.* Prove Lemma 10.13 from the notes: Let X, Y be Banach spaces and $S \in \mathcal{L}(X;Y)$ be given. Assume that there exists $c > 0$ such that

$$\|S^* y^*\| \geq c\|y^*\| \text{ for all } y^* \in Y^*.$$

Then S is surjective. (You are not allowed to use Lemma 10.14, because the proof of Lemma 10.14 made use of Lemma 10.13.)