Homework 2

21-759 Differential Geometry
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I would be willing to present a solution to problem 5 or the first part of problem 1 (showing the Lie algebra associated with $SL(n)$ is $sl(n)$.

Problem 1

Let I denote the $n \times n$ identity matrix. We first show that $T_I SL(n) = sl(n)$. Suppose $\gamma : (-\varepsilon, \varepsilon) \to SL(n)$ is differentiable, with $\gamma(0) = I$. We showed in class that,

$$\operatorname{trace}(\gamma'(0)) = \left(\frac{d}{dt} \det(\gamma(t)) \right) \Big|_{t=0} = \left(\frac{d}{dt} 1 \right) \Big|_{t=0} = 0,$$

and hence, by definition of the tangent space, $T_I SL(n) \subseteq sl(n)$.

Suppose $A \in sl(n)$. Since $\det(I) = 1$ and the function $t \mapsto \det(I + tA)$ is continuous, $\exists \varepsilon > 0$ such that $\det(I + tA) \neq 0$, $\forall t \in (-\varepsilon, \varepsilon)$. Thus, we define $\gamma : (-\varepsilon, \varepsilon) \to SL(n)$ for all $t \in (-\varepsilon, \varepsilon)$.

$$\gamma(t) = \frac{I + tA}{\det(I + tA)}.$$

Clearly the image of γ indeed lies in $SL(n)$, and $\gamma(0) = I$. Noting first that

$$\left(\frac{d}{dt} \det(I + tA) \right) \Big|_{t=0} = \operatorname{trace} \left(\frac{d}{dt} I + tA \right) \Big|_{t=0} = \operatorname{trace}(A) = 0,$$

we calculate

$$\gamma'(0) = \frac{\det(I + tA)A - (I + tA)\frac{d}{dt}\det(I + tA)}{(\det(I + tA))^2} \Bigg|_{t=0} = \frac{1 \cdot A - 0 \cdot I}{1^2} = A,$$

and hence $A \in T_I SL(n)$. ■

We now show $T_I SO(n) = so(n)$. Suppose $\gamma : (-\varepsilon, \varepsilon) \to SO(n)$ is differentiable, with $\gamma(0) = I$. Then,

$$0 = \left(\frac{d}{dt} I \right) \Big|_{t=0} = \left(\frac{d}{dt} \gamma(t)\gamma^T(t) \right) \Big|_{t=0} = \left(\gamma'(t)\gamma^T(t) + \gamma(t)(\gamma^T)'(t) \right) \Big|_{t=0} = \gamma'(0) + (\gamma^T)'(0).$$

Thus, $(\gamma'(0))^T = -\gamma'(0)$, and so $T_I SO(n) \subseteq so(n)$.

I didn't have time to finish this problem.

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Problem 2

Since that vector spaces over a field are isomorphic if and only if they have the same dimension, it suffices to show that the three spaces have the same dimension. First, note

$$\dim((V \otimes W)^*) = \dim(V \otimes W) = \dim(V)\dim(W) = \dim(V^*)\dim(W^*) = \dim(V^* \otimes W^*).$$

Since this problem isn't being graded, I didn't finish writing a thorough solution, but it is easy to show that a bilinear function $f \in L_2(V \otimes W)$ is defined by its values on each pair (v, w) for $v \in \mathcal{B}_V, w \in \mathcal{B}_W$, where $\mathcal{B}_V$ and $\mathcal{B}_W$ are bases of V and W, respectively. It follows that $\dim(L_2(V, W)) = \dim(V)\dim(W)$. ■

Problem 3

I wasn't able to finish this problem.

Problem 4

(i) Suppose S_1 and S_2 are $(0, s_1)$ and $(0, s_2)$ tensors, respectively. If $p \in \mathcal{M}$, $v_1, \ldots, v_{s_1+s_2} \in T_p\mathcal{M}$,

$$\begin{aligned}
\Phi^*(S_1 \otimes S_2)|_p(v_1, \ldots, v_{s_1+s_2}) &= (S_1 \otimes S_2)|_{\Phi(p)}(D\Phi v_1, \ldots, D\Phi v_{s_1+s_2}) \\
&= S_1|_{\Phi(p)}(D\Phi v_1, \ldots, D\Phi v_{s_1})S_2|_{\Phi(p)}(D\Phi v_{s_1+1}, \ldots, D\Phi v_{s_2}) \\
&= \Phi^* S_1|_{\Phi(p)}(v_1, \ldots, v_{s_2})\Phi^* S_2|_{\Phi(p)}(v_{s_1+1}, \ldots, v_{s_1+s_2}) \\
&= (\Phi^*(S_1) \otimes \Phi^*(S_2))_p(v_1, \ldots, v_{s_1+s_2}).
\end{aligned}$$

It follows that $\Phi^*(S_1 \otimes S_2) = \Phi^*(S_1) \otimes \Phi^*(S_2)$. ■

(ii) Suppose ω_1 and ω_2 are k- and l-forms on $\mathcal{M}$, respectively. If $p \in \mathcal{M}$, $v_1, \ldots, v_{s_1+s_2} \in T_p\mathcal{M}$,

$$\begin{aligned}
\Phi^*(\omega_1 \wedge \omega_2)|_p(v_1, \ldots, v_{s_1+s_2}) &= (S_1 \wedge S_2)|_{\Phi(p)}(D\Phi v_1, \ldots, D\Phi v_{s_1+s_2}) \\
&= S_1|_{\Phi(p)}(D\Phi v_1, \ldots, D\Phi v_{s_1})S_2|_{\Phi(p)}(D\Phi v_{s_1+1}, \ldots, D\Phi v_{s_2}) \\
&= \Phi^* S_1|_p(v_1, \ldots, v_{s_2})\Phi^* S_2|_p(v_{s_1+1}, \ldots, v_{s_1+s_2}) \\
&= (\Phi^*(S_1) \wedge \Phi^*(S_2))_p(v_1, \ldots, v_{s_1+s_2}).
\end{aligned}$$

It follows that $\Phi^*(S_1 \wedge S_2) = \Phi^*(S_1) \wedge \Phi^*(S_2)$. ■

(iii) If ω is a 0-form, $p \in \mathcal{M}, v \in T_p\mathcal{M}$, then

$$\Phi^*(d\omega)|_p(v) = d\omega|_{\Phi(p)}(D\Phi v) = d\omega|_{\Phi(p)}(D\Phi v) = D\Phi v[\omega] = v[\omega \circ \Phi] = (d\Phi^*\omega)(v)$$

Suppose that, for some $n \in \mathbb{N}$, $\forall k \leq n$, for all k-forms ω, $\Phi^*(d\omega) = d\Phi^*(\omega)$, and let ω be an $(n+1)$-form. Then, $\omega = \omega_1 \wedge \omega_2$, for some k_1- and k_2-forms ω_1 and ω_2, respectively, with $k_1, k_2 \leq n$. Thus, by part (ii) and since the pullback is clearly linear by its definition,

$$\begin{aligned}
\Phi^*(d\omega) = \Phi^*(d(\omega_1 \wedge \omega_2)) &= \Phi^*(d\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2) \\
&= \Phi^*(d\omega_1) \wedge \Phi^*(\omega_2) + (-1)^{k_1}\Phi^*(\omega_1) \wedge \Phi^*(d\omega_2) \\
&= d\Phi^*(d\omega_1) \wedge \Phi^*(\omega_2) + (-1)^{k_1}\Phi^*(\omega_1) \wedge d\Phi^*(\omega_2) \\
&= d\left(\Phi^*(\omega_1) \wedge \Phi^*(\omega_2)\right) \\
&= d\Phi^*(\omega_1 \wedge \omega_2) = d\Phi^*\omega
\end{aligned}$$

By induction on n, the exterior derivative and pullback commute for all n-forms. ■

Problem 5

Since ω is a 1-form, $\forall p \in S^2$, $\omega|_p \in (T_pS^2)^*$. Thus, we can choose a dual vector $v \in T_pS^2$ with the property that $v = 0$ implies $\omega|_p = 0$. In particular, since ω is smooth, we can choose v_p such that the map $p \mapsto (p, v_p)$ is smooth. Since this map defines a vector field on S^2 and one cannot "comb a coconut", $\exists p \in S^2$ with $v_p = 0$, and hence $\omega|_p = 0$.

Recall now that $SO(3)$ is the group of rotations in $\mathbb{R}^3$, and hence, $\forall q \in S^2$, $\exists \phi_q \in SO(3)$ such that $\phi(q) = p$. Then, for any $v \in T_qS^2$,

$$\omega|_q(v) = \phi^*\omega|_q(v) = \omega|_{\Phi(q)}(D\phi v) = 0,$$

and hence $\omega = 0$. ■