Homework 1

21-651 General Topology

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Problem 1

We show $\mathcal{B}$ is a basis by showing that it fulfills properties (i), (ii), and (iii) as given in proposition 21 of the lecture notes.

As corrected, $\emptyset \in \mathcal{B}$. Since $\mathbb{R} = (-\infty, 1) \cup (-2, 2) \cup (1, \infty)$, the sets of $\mathcal{B}$ cover $\mathbb{R}$. Thus, it remains only to show that $\mathcal{B}$ satisfies property (iii).

Let

$$\mathcal{B}_1 = \{(x - r, x + r) : x \in \mathbb{R}\backslash\{0\}, r \in (0, |x|)\}, \mathcal{B}_2 = \{(-\infty, -a) \cup (-r, r) \cup (a, \infty) : r, a \in (0, \infty)\},$$

so that $\mathcal{B} = \mathcal{B}_\infty \cup \mathcal{B}_\in$.

For

$$B_1 = (x_1 - r_1, x_1 + r_1), B_2 = (x_2 - r_2, x_2 + r_2) \in \mathcal{B}_1,$$

if $a = \max\{x_1 - r_1, x_2 - r_2\}$ and $b = \min\{x_1 + r_1, x_2 + r_2\}$, then, for $x = \frac{a+b}{2} \neq 0$, $r = x - a$, $B_1 \cap B_2 = (x - r, x + r) \in B_1$. Thus, property (iii) is clearly fulfilled in this case.

For

$$B_1 = (-\infty, -a_1) \cup (-r_1, r_1) \cup (a_1, \infty), B_2 = (-\infty, -a_2) \cup (-r_2, r_2) \cup (a_2, \infty) \in \mathcal{B}_2,$$

if $a = \max(a_1, a_2)$ and $r = \min(r_1, r_2)$, then, $B_1 \cap B_2 = (-\infty, a) \cup (-r, r) \cup (a, \infty) \in \mathcal{B}_2$. Thus, property (iii) is clearly fulfilled in this case as well.

Finally, suppose

$$B_1 = (x_1 - r_1, x_1 + r_1) \in \mathcal{B}_1, B_2 = (-\infty, -a) \cup (-r, r) \cup (a, \infty) \in \mathcal{B}_2.$$

Then, for

$$B_3 = (x_1 - r_1, -a) \cup (\max(x_1 - r_1, -r), \min(x_1 + r_1, r)) \cup (a, x_1 + r_1),$$

(where, if $y \leq x$, $(x, y) = \emptyset$). $B_3 = B_1 \cap B_3$, and B_3 is the union of (at most 3) basis sets in $\mathcal{B}$, so that property (iii) is fulfilled in all cases. ■

Clearly, any set $B \in \mathcal{B}$ can be written as a union of open intervals, so that τ is no finer than the standard topology. iFurthermore, if $0 \in B \in \mathcal{B}$, then B is unbounded, so that $(-1, 1)$ is not the union of sets in $\mathcal{B}$, and thus τ is strictly coarser than the standard topology.

Problem 2

(a) Suppose that $U_i \in \tau$, for all i in some index set I. If $x \in U := \bigcup_{i \in I} U_i$, then $x \in U_i$ for some $i \in I$, so that,in any given direction U_i contains an open segment S through x. Then, $S \subseteq U$, so that $U \in \tau$.

Suppose that $U_1, U_2, \dots, U_k \in \tau$, for some $k \in \mathbb{N}$. If $x \in U := \bigcap_{i=1}^{k} U_k$, then, for each i, $x \in U_i$, so that, in any given direction D, there is an open segment $S_i \subseteq U_i$ through x. Since the finite intersection of open segments in the same direction is an open segment, for $S := \bigcap_{i=1}^{k} S_i$, S is an open segment through x in direction D, with $S \subseteq U$. Thus, $U \in \tau$.

Since, trivially, $\emptyset, \mathbb{R} \in \tau$, and since τ is closed under arbitrary unions and finite intersections, τ is a topology on $\mathbb{R}^2$. ■

Since any open set U in the standard topology contains a ball of some radius $\delta > 0$ around x, so that U contains open the segments of length δ through x in every direction, every set that is open in the standard topology is radially open.

On the other hand, suppose $U = B((0,0),1) \backslash S \subseteq \mathbb{R}^2$, where

$$S = \{(x,y) : x > 0, x^2 = y\}.$$

Then, U is not open (any open ball around $(0,0)$ contains points in S), but U is radially open, since, in no direction D, are there points in S arbitrarily close to $(0,0)$ (indeed, no open segment through $(0,0)$ contains more that 1 point in S).

Thus, the topology of radially open sets is strictly stronger than the standard topology. ■

If $L \subseteq \mathbb{R}^2$ is a line, then, since every set that is open under the standard topology is open under τ, every open segment $S \subseteq L$ is in the topology τ_L induced on L. Furthermore, if $U \in \tau$, then $\forall x \in U \cap L$, U contains an open segment through x in L, so that τ_L contains only open "intervals" in L. τ_L is then homeomorphic to the standard topology on $\mathbb{R}$, where a homeomorphism is the composition of rotations and translations which maps L to the line $y = 0$. ■

If $C \subseteq \mathbb{R}^2$ is a (nondegenerate) circle, then, $\forall x \in C$, if $U_x = (U) \backslash C) \cup \{x\}$ (where U is any open set (under the standard topology) containing x), then U_x is radially open and $U_x \cap C = \{x\}$. Therefore, if τ_C is the topology induced on C by τ, then, $\forall x \in C$, $\{x\} \in \tau_C$. Thus, τ_C is the discrete topology on the circle. ■

(b) If C is the unit circle centered at the origin, then, as explained in part (a), the topology τ_C induced on C is the discrete topology on C. Furthermore, C is uncountable (the function $(x,y) \in C \backslash \{(0, \pm 1)\} \mapsto \tan(y/x)$ is a surjection from a subset of C to $\mathbb{R}$), and C is closed (since the Euclidean norm is a continuous function, any sequence of points in C converges to a point in C, and in $\mathbb{R}^2$, all sequentially closed sets are closed). Thus, C has the desired properties. ■

Since C is discrete, any base of (C, τ_C) must be countable. For suppose $\mathbb{R}^2, \tau)$ had a countable base $\mathcal{B}$, then, $\forall x \in C$, $\exists B_x \in \mathcal{B}$ such that $B_x \cap C = \{x\}$. Then, however, $x \mapsto B_x$ is an injection, implying C is countable, a contradiction. Thus, $(\mathbb{R}, \tau)$ is no second countable. ■

Problem 3

(i) By definition of the closure and some basic set arithmetic,

$$\overline{E}\backslash E^\circ = (X\backslash((X\backslash E)^\circ))\backslash E^\circ = X\backslash((X\backslash E)^\circ \cup E^\circ),$$

so that $\overline{E}\backslash E^\circ$ is the set of points in neither E° nor $(X\backslash E)^\circ$.

Suppose $x \in \partial E$. Then, $\nexists U \in \tau$ with $x \in U$ such that $U \cap (X\backslash E) = U\backslash E = \emptyset$, and $\nexists U \in \tau$ with $x \in U$ such that $U \cap E = \emptyset$. Therefore, $x \notin E^\circ$ and $x \notin (X\backslash E)^\circ$, so that $x \in \overline{E}\backslash E^\circ$.

Suppose, on the other hand, that $x \in \overline{E}\backslash E^\circ$. Then, $x \notin E^\circ$ and $x \notin (X\backslash E)^\circ$, so that, if $x \in U \in \tau$, then $U\backslash E = U \cap (X\backslash E) \neq \emptyset$ and $U \cap E \neq \emptyset$. Therefore, $x \in \partial E$. ■

(ii) Since E is arbitrary and $X\backslash(X\backslash E) = E$, it suffices to show that $\partial E \subseteq \partial(X\backslash E)$, as, then, $\partial(X\backslash E) \subset \partial(X\backslash(X\backslash E)) = \partial E$. Suppose $x \in \partial E$, and suppose $U \in \tau$ with $x \in U$. Since $x \in \partial E$,

$$U \cap (X\backslash E) = U\backslash E \neq \emptyset \qquad \text{and} \qquad U\backslash(X\backslash E) = U \cap E \neq \emptyset,$$

so that, since U is arbitrary, $x \in \partial(X\backslash E)$. ■

(iii) It is sufficient to show that $\partial(A \cup E) \subseteq \partial E \cup \partial A$ and $\partial(A \cap E) \subseteq \partial E \cup \partial A$.

Suppose $x \notin \partial A \cup \partial E$. Then, there exist sets U_E and U_A with $x \in U_A, U_E$ such that $U_A \cap A = \emptyset$ or $U_A\backslash A = \emptyset$, and $U_E \cap E = \emptyset$ or $U_E\backslash E = \emptyset$.

If $U_A\backslash A = \emptyset$ or $U_E\backslash E = \emptyset$, then $U_A\backslash(A \cup E) = \emptyset$ or $U_E\backslash(A \cup E) = \emptyset$, so that $x \notin \partial(A \cup E)$. Otherwise, $U_A \cap A = \emptyset$ and $U_E \cap E = \emptyset$, so that, for $U = U_A \cap U_E$, $U \cap (A \cup E) = \emptyset$. Thus, $x \notin \partial(A \cup E)$, so that $\partial(A \cup E) \subseteq \partial A \cup \partial E$.

Suppose $x \in \partial(A \cap E)$. Then, for $U \subseteq A \cap E$ with $x \in U$,

$$U \cap A, U \cap E \supseteq U \cap (A \cap E) \neq \emptyset,$$

and

$$(U\backslash A) \cup (U\backslash E) = U\backslash(A \cap E) \neq \emptyset,$$

so that $x \in \partial E$ or $x \in \partial A$, and thus $\partial(A \cap E) \subseteq \partial E \cup \partial A$. ■

Suppose $A = (0, 1]$ and $E = (1, 2)$ under the standard topology on $\mathbb{R}$. Then, $1 \in \partial A \cup \partial E$. However, since $A \cup E = (0, 2)$ and $A \cap E = \emptyset$, $1 \notin \partial(A \cup E) \cup \partial(A \cap E)$. Thus, equality does not hold in general. ■

Problem 4

It suffices to show that $d(x, y) = 0$ if and only if $x = y$, that d is symmetric in its arguments, and that d obeys the triangle inequality.

Since each d_i is a metric, $\forall x \in X$, $d_i(x_i, x_i) = 0$, so that $(d_1(x_1, x_1), d_2(x_2, x_2), \ldots, d_N(x_N, x_N)) = \mathbf{0}$. Thus, by property (i) of Φ, $d(x, x) = 0$. Also since each d_i is a metric, $\forall x, y \in X$ $(x \neq y)$, since some $x_i \neq y_i$, $d_i(x_i, y_i) \neq 0$, so that $(d_1(x_1, x_1), d_2(x_2, x_2), \ldots, d_N(x_N, x_N)) \neq \mathbf{0}$. Thus, by property i) of Φ, $d(x, x) \neq 0$.

Since each d_i is a metric, it is symmetric in its arguments. Thus, $\forall x, y \in X$,

$$\begin{aligned} d(x,y) &= \Phi(d_1(x_1,y_1), d_2(x_2,y_2), \ldots, d_N(x_N,y_N)) \\ &= \Phi(d_1(y_1,x_1), d_2(y_2,x_2), \ldots, d_N(y_N,x_N)) = d(y,x), \end{aligned}$$

so that d is symmetric in its arguments.

Suppose $x, y, z \in X$. Since each d_i is a metric, $d_i(x_i, z_i) \leq d_i(x_i, y_i) + d(y_i, z_i)$. Since Φ is nondecreasing in each of its variables,

$$\begin{aligned} d(x,z) &= \Phi(d_1(x_1,z_1), d_2(x_2,z_2), \ldots, d_N(x_N,z_N)) \\ &\leq \Phi(d_1(x_1,y_1) + d_1(y_1,z_1), d_2(x_2,y_2) + d_2(y_2,z_2), \ldots, d_n(x_N,y_N) + d_n(y_N,z_N)). \end{aligned}$$

Then, since Φ is subadditive,

$$\begin{aligned} d(x,z) &\leq \Phi(d_1(x_1,y_1), d_2(x_2,y_2), \ldots, d_N(x_N,y_N)) \\ &\quad + \Phi(d_1(y_1,z_1), d_2(y_2,z_2), \ldots, d_N(y_N,z_N)) = d(x,y) + d(y,z), \end{aligned}$$

so that d obeys the triangle inequality. Thus, d is a metric, so that (X, d) is a metric space. ■

Suppose $\Phi : [0,\infty)^N \to [0,\infty)$ is the function

$$(x_1, x_2, \ldots, x_N) \mapsto \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}.$$

Since, $\forall x_i \in [0,\infty)$, $x_i^2 \geq 0$ and $x_i^2 = 0$ if and only if $x_i = 0$, $\Phi(x) = 0$ if and only if $x = \mathbf{0}$. Since Φ can be written as the sum and composition of nondecreasing functions, Φ is nondecreasing.

Suppose $\Phi : [0,\infty)^N \to [0,\infty)$ is the function

$$(x_1, x_2, \ldots, x_N) \mapsto x_1 + x_2 + \ldots + x_N.$$

Since, $\forall x_i \in [0,\infty)$, $x_i \geq 0$, $\Phi(x) = 0$ if and only if $x = \mathbf{0}$. Clearly Φ is nondecreasing. Since Φ is linear, it is additive and thus subadditive.

Suppose $\Phi : [0,\infty)^N \to [0,\infty)$ is the function

$$(x_1, x_2, \ldots, x_N) \mapsto \max\{x_1, x_2, \ldots, x_N\}.$$

Clearly, $\forall x_i \in [0,\infty)$, $\Phi(x) \geq x_i \geq 0$ and $x_i = 0$ if and only if $x = \mathbf{0}$. Clearly Φ is nondecreasing and $\Phi(x+y) \leq \Phi(x) + \Phi(y)$.

Thus, it follows from the first part of this problem that, for each of the above definitions of d, d is a metric and thus (X, d) is a metric space. ■

Problem 5

Clearly, if $U = V \in X$, $d_H(U,V) = 0$, and clearly $d_H(U,V)$ is symmetric in U and V. Since d_A is a metric, if $U \subseteq V$, then $\sup_{x\in U} \inf_{y \in V} d_A(x,y) = 0$, and, if $V \subseteq U$, then $\sup_{y \in V} \inf_{x \in u} d_A(x,y) = 0$, so that, $d_H(U,U) = 0$. Thus, it remains only to show that d_H obeys the triangle inequality.

Note that, since (A, d_A) is bounded and $\emptyset \notin X$, all of the below suprema and infima are finite.

Let $U, V, W \in X$. Since d_A is a metric, $\forall x \in U, y \in V, z \in W$, $d_A(x,z) \leq d_A(x,y) + d_A(y,z)$. Taking the infimum over $z \in W$ gives

$$\begin{aligned}\inf_{z\in W} d_A(x,z) &\leq \inf_{z\in W} (d_A(x,y) + d_A(y,z)) \\ &\leq d_A(x,y) + \inf_{z\in W} d_A(y,z).\end{aligned}$$

Taking the infimum over $y \in V$ gives

$$\begin{aligned}\inf_{z\in W} d_A(x,z) = \inf_{y\in V} \inf_{z\in W} d_A(x,z) &\leq d_A(x,y) + \inf_{z\in W} d_A(y,z) \\ &\leq \inf_{y\in V}\left(d_A(x,y) + \inf_{z\in W} d_A(y,z)\right) \\ &\leq \inf_{y\in V} d_A(x,y) + \sup_{y\in V}\inf_{z\in W} d_A(y,z).\end{aligned}$$

Then, taking the supremum over $x \in U$,

$$\begin{aligned}\sup_{x\in W}\inf_{z\in W} d_A(x,z) &\leq \sup_{x\in W}\left(d_A(x,y) + \sup_{y\in V}\inf_{z\in W} d_A(y,z)\right) \\ &\leq \sup_{x\in W} d_A(x,y) + \sup_{y\in V}\inf_{z\in W} d_A(y,z).\end{aligned}$$

Since $\sup_{x\in U}\inf_{y\in V} d(x,y)$ is symmetric in U and V, this implies that

$$d_H(U,W) \leq d_H(U,V) + d_H(V,W),$$

so that d_H obeys the triangle inequality. Therefore, d_H is a pseudometric. ■

Suppose $A = [0,1]$. Then, if $V = [0,1], U = [0,1)$, $V \neq U$, but $d_H(U,V) = 0$ Thus, d_H is not always a metric.

Problem 6

As shown in problem 5, d_H is symmetric in its arguments. For every metric d_1 on $X \sqcup Y$, there exists another metric d_2 on $Y \sqcup X$ such that $d_1(x,y) = d_2(y,x)$. Thus, the term $\sup_{d\in\mathcal{C}}$ is symmetric in (X, d_X) and (Y, d_Y), and thus D is symmetric in (X, d_X) and (Y, d_Y).

Since, for any metric d over $x \sqcup Y$, $d(x,x) = 0$i, for any $d \in \mathcal{C}$, $d_H(U \times \{1\}, U \times \{2\}) = 0$. Therefore, $D(X \times \{1\}, X \times \{1\}) = 0$. Thus, it remains only to show that D obeys the triangle inequality.

For any metric $d \in \mathcal{C}$, as shown in problem 5, d_H obeys the triangle inequality, so that

$$\begin{aligned}d_H(X \times \{1\}, Z \times \{2\}) &\leq d_H(X \times \{1\}, Y \times \{2\}) + d_H(Y \times \{2\}, Z \times \{2\}) \\ &\leq d_H(X \times \{1\}, Y \times \{2\}) + d_H(Y \times \{1\}, Z \times \{2\}).\end{aligned}$$

Taking the infimum on both sides gives

$$\inf_{d\in\mathcal{C}} d_H(X \times \{1\}, Z \times \{2\}) \leq \inf_{d\in\mathcal{C}} d_H(X \times \{1\}, Y \times \{2\}) + \inf_{d\in\mathcal{C}} d_H(Y \times \{1\}, Z \times \{2\}),$$

so that

$$D((X, d_X), (Z, d_Z)) \leq D((X, d_X), (Y, d_Y)) + D((Y, d_Y), (Z, d_Z)). \quad \blacksquare$$

Problem 7

For notational convenience, $\forall n \in \mathbb{N}$, $f, g \in C((0,1))$, let $F_n(f, g) = \max_{x \in K_n} |f(x) - g(x)|$, so that

$$d(f, g) = \max_{n \in \mathbb{N}} \frac{1}{2^n} \frac{F_n(f, g)}{1 + F_n(f, g)}.$$

Note that, $\forall n \in \mathbb{N}$, F_n is everywhere non-negative, and that it obeys the triangle inequality.

Since, $\forall f, g \in C((0,1))$, $\forall x \in (0,1)$, $|f(x) - g(x)| = |g(x) - f(x)|$, d is symmetric.

If $f = g$, then, clearly, $\forall n \in \mathbb{N}$, $F_n(f, g) = 0$, so that $d(f, g) = 0$. If, $f \neq g$, then $|f(x) - g(x)| > 0$ for some $x \in (0,1)$. Since $\bigcup_{i=1}^{\infty} K_i = (0,1)$, $x \in K_i$ for some i, so that

$$d(f, g) = \max_n \frac{1}{2^n} \frac{F_n(f, g)}{1 + F_n(f, g)} \geq \frac{1}{2^i} \frac{F_i(f, g)}{1 + F_i(f, g)} \geq F_i(f, g) > 0.$$

Thus, it remains only to show that d obeys the triangle inequality. Note first that, $\forall x \in [0, \infty)$, if $x \leq y$, then

$$\frac{x}{1 + x} \leq \frac{y}{1 + y}.$$

Thus, $\forall f, g, h \in C((0,1)), \forall n \in \mathbb{N}$, since,

$$F_n(f, g) + F_n(g, h) + F_n(f, g)F_n(g, h) \geq F_n(f, g) + F_n(g, h) \geq F_n(f, h)$$

(because F_n is nonnegative and obeys the triangle inequality),

$$\begin{aligned}
\frac{F_n(f, g)}{1 + F_n(f, g)} + \frac{F_n(g, h)}{1 + F_n(g, h)} &= \frac{F_n(f, g) + F_n(g, h) + 2F_n(f, g)F_n(g, h)}{1 + F_n(f, g) + F_n(g, h) + F_n(f, g)F_n(g, h)} \\
&\geq \frac{F_n(f, g) + F_n(g, h) + F_n(f, g)F_n(g, h)}{1 + F_n(f, g) + F_n(g, h) + F_n(f, g)F_n(g, h)} \\
&\geq \frac{F_n(f, h)}{1 + F_n(f, h)}.
\end{aligned}$$

Dividing by 2^n and taking the max over all $n \in \mathbb{N}$ gives

$$d(f, g) + d(g, h) \geq d(f, h). \quad \blacksquare$$