Homework 10

21-630 Ordinary Differential Equations
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Problem 1

Note that $(0,0)$ is a critial point, and so any solution with initial condition at $(0,0)$ will be constant. Hence, we assume the initial condition is not $(0,0)$.

We first calculate

$$\begin{aligned}
\dot{r} &= \dot{X}\cos\theta + \dot{Y}\sin\theta \\
&= (1-r^2)r(\cos^2\theta - \sin\theta\cos\theta + \cos\theta\sin\theta + \sin^2\theta) \\
&= (1-r^2)r \\
\dot{\theta} &= \dot{Y}r^{-1}\cos\theta - \dot{X}r^{-1}\sin\theta \\
&= (1-r^2)r(\cos^2\theta + \sin\theta\cos\theta - \cos\theta\sin\theta + \sin^2\theta) \\
&= (1-r^2) = \dot{r}/r.
\end{aligned}$$

Thus, $\forall t \geq t_0$,

$$\theta(t) = \theta(t_0) + \int_{t_0}^{t} \dot{r}/r\,dt = \theta(t_0) + \int_{t_0}^{t} \frac{d}{dt}\ln(r(t))\,dt = \theta(t_0) + \ln(r(t)) - \ln(r(t_0)).$$

By choosing $\theta(t_0)$ appropriately, we ensure that $r(t_0) \geq 0$, for any initial conditions. Thus, it is clear from the above (autonomous) equation for $\dot{r}$ that

$$r(t) \to 1 \text{ as } t \to \infty.$$

Then, since $\log(r(t)) \to 0$ as $t \to \infty$, from the above equation for θ,

$$\theta(t) \to \theta(t_0) - \log(r(t_0)) \text{ as } t \to \infty.$$

It follows that

$$\boxed{\Omega(r(t_0), \theta(t_0)) = \{(1, \theta(t_0) - \log(r(t_0)))\}.}$$

Problem 2

Suppose $X(t) \in C^+(X(0))$, and define, $\forall k \in \mathbb{N}$, $t_k := t + kT$. Since T is a period of X, a trivial induction argument shows that $0 \leq t_k \to \infty$ and $X(t_k) \to X(t)$ as $k \to \infty$. Hence, $X(t) \in \Omega(X(0))$.

Suppose $\overline{x} \in \Omega(X(0))$, so that there is a sequence $0 \leq t_k \to \infty$ with $X(t_k) \to \overline{x}$ as $k \to \infty$. Since T is a period of X, a trivial induction argument shows that $C^+(X(0)) = \{X(t) : t \in [0, T]\}$. Therefore, $C^+(X(0))$ is the image of the compact set $[0, T]$ under the continuous function X, and so $C^+(X(0))$ is compact. Hence, since each $X(t_k) \in C^+(X(0))$, $\overline{x} \in C^+(X(0))$. ■

Problem 3

We first calculate

$$\begin{aligned} D_* w(x, y) &= \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} yg(x, y) - x(x - y)^2 \\ -xg(x, y) - y(x - y)^2 \end{bmatrix} \\ &= 2xy(g(x, y) - g(x, y)) - 2(x^2 + y^2)(x - y)^2 \\ &= -2(x^2 + y^2)(x - y)^2. \end{aligned}$$

It is clear, then, that $Z := \{(x, y) : D_* w(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 : x = y\}$.

Fix $\eta > 0$. From the choice of w, it is clear that H_η is the circle of radius $\sqrt{\eta}$ centered at the origin. By Theorem 5.8, it suffices to show that the largest positively invariant subset M of $H_\eta \cap Z$ is the singleton $\{(0, 0)\}$. Since $(0, 0)$ is a critical point, $(0, 0) \in M$.

Suppose $X(0) = Y(0) \neq 0$. Since $g(X(0), Y(0)) \neq 0$, without loss of generality,

$$\frac{d(X - Y)}{dt}(0) = 2X(0)g(X, Y) > 0.$$

By continuity of this derivative, $\exists \varepsilon, \delta > 0$ such that, $\forall t \in [0, \delta]$, $\frac{d(X-Y)}{dt}(t) > \varepsilon$. Hence,

$$(X - Y)(\delta) \geq \int_0^\delta \varepsilon \, dt = \delta\varepsilon > 0.$$

and hence $(X(\delta), Y(\delta)) \notin M$. By definition of positive invariance, $M = \{(0, 0)\}$. ■

Problem 4

A) By definition of v and u, the given system can be written as

$$\frac{dv}{dt} = -\left(v + u + \frac{\partial P}{\partial x}\right)$$
$$\frac{du}{dt} = -\left(v + u + \frac{\partial P}{\partial y}\right)$$

Thus,

$$D_* w(x, y, v, u) = \begin{bmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ v \\ u \end{bmatrix} \cdot \begin{bmatrix} v \\ u \\ \frac{dv}{dt} \\ \frac{du}{dt} \end{bmatrix} = v\frac{\partial p}{\partial x} + u\frac{\partial p}{\partial y} - v\left(v + u + \frac{\partial P}{\partial x}\right) - u\left(v + u + \frac{\partial P}{\partial y}\right)$$
$$= -(v^2 + 2vu + u^2) = \boxed{-(v+u)^2} \leq 0.$$

B) The origin is asymptotically stable. Let M be as in Theorem 5.8 (for an arbitrary $\eta > 0$). Since the origin is a critical point, it is in M. Then, by Theorem 5.8, it suffices to show that any initial condition aside from the origin causes the solution to leave M.

Since $D_* w = 0$ in M, $v + u = 0$ in M. Thus, in M,

$$\frac{d^2x}{dt^2} = -\frac{\partial P}{\partial x} = -4x^3$$
$$\frac{d^2y}{dt^2} = -\frac{\partial P}{\partial y} = -2y.$$

This system has no (non-zero) solution preserving $V(t) + U(t) = 0, \forall t \geq t_0$. Thus solutions with initial conditions not at the origin leave M. ■

C) Letting M be as in part (b), we have, for solutions lying in M,

$$\frac{d^2x}{dt^2} = -\frac{\partial P}{\partial x} = -4x^3$$
$$\frac{d^2y}{dt^2} = -\frac{\partial P}{\partial y} = -4y^3.$$

If, for any $\delta > 0$, we choose the initial condition $X(t_0) = \delta$, $Y(t_0) = -\delta$, $V(t_0) = U(t_0) = 0$, then, since the negation of any solution to each of the above equations is a solution, by uniqueness, the solution will preserve $V(t) + U(t) = 0$. However, the solution is periodic rather than converging to the origin. ■