Homework 11

21-630 Ordinary Differential Equations
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Problem 1

A) By definition of $\Omega(X(0))$, picking some $\overline{x} \in \Omega(X(0))$, there is a sequence $\{t_k'\}$ with $t_k' \to \infty$ as $k \to \infty$, such that $X(t_k') \to \overline{x}$ as $k \to \infty$. Consequently, $\operatorname{dist}(X(t_k'), \Omega(X(0))) \to 0$ as $k to \infty$. Thus, by choosing k_1 sufficiently large, the sequence $\{t_k\} = \{t_{k+k_1}'\}$ has the desired properties.

By definition of $C^+(X(0))$, since $C^+(X(0))$ is unbounded, there is a sequence $\{s_k'\}$ with $s_k' \to \infty$ as $k \to \infty$, such that $|X(s_k')| \to \infty$ as $k \to \infty$. Consequently, since $\Omega(X(0))$ is bounded, $\operatorname{dist}(X(s_k'), \Omega(X(0))) \to \infty$ as $k \to \infty$. Thus, for k_2 sufficiently large, the sequence $\{s_k\} = \{s_{k+k_1}'\}$ has the desired properties. ∎

B) Since $t_k, s_k \to \infty$ as $k \to \infty$, for each $k \in \mathbb{N}$, $\exists m, n \in \mathbb{N}$ with $t_k < s_m$ and $s_k < t_n$. Thus, we can inductively construct a sequence $\{(t_{k_n}, s_{k_n})\}$ such that, for each $k \in \mathbb{N}$, $t_k < s_k$ and $s_k < t_{k+1}$. By continuity of X and the distance function and the Intermediate Value Theorem, $\exists \tau_k$ with $t_k < \tau_k < s_k$ and $X(\tau_k) \in S_2$ (clearly $\tau_k \to \infty$ as $k \to \infty$). ∎

C) Since $\Omega(X(0))$ is bounded, S_2 is bounded, and, since the distance function is continuous, S_2 is closed. Thus, S_2 is compact, and so $\{X(\tau_k)\}$ has a subsequence $\{X(\tau_{k_n})\}$ converging to some $\overline{x} \in S_2$. It follows that $\overline{x} \in \Omega(X(0))$, contradicting the definition of S_2. ∎

Problem 2

If $r(0) > 1$, then, from the differential equation defining r, it is clear that $r(t) \to 1$. It follows that

$$\theta(t) = \int_0^t (r-1)^2 + \sin^2\theta\, dt = \int_0^t \frac{-\dot{r}}{(r-1)^2} + \sin^2\theta\, dt = \int_0^t \frac{d}{dt}\frac{1}{r-1} + \sin^2\theta\, dt$$
$$= \frac{1}{r-1} - \frac{1}{r(0)-1} + \int_0^t \sin^2\theta\, dt \geq \frac{1}{r-1} - \frac{1}{r(0)-1} \to \infty$$

as $t \to \infty$. Consequently, $\boxed{\Omega(X(0)) = \{(x,y) : x^2 + y^2 = 1\}}$ is the unit circle.

If $(X(0), Y(0)) = (0, 1)$, then $(r(0), \theta(0)) = (1, \pi/2)$. Thus, by uniqueness, $r(t) = 1, \forall t \geq 0$. Also,

$$\frac{d\theta}{dt} = \sin^2(\theta) \Rightarrow -\cot(\theta) = \int \csc^2\theta\, d\theta = t + C,$$

and hence, with the initial condition $\theta(0) = \pi/2$,

$$\theta = \cot^{-1}(-t - C) = \cot^{-1}(-t).$$

It follows that, as $t \to \infty$, $\theta = \cot^{-1}(-t) \to \pi$. Consequently,

$$\boxed{\Omega((0,1)) = \{(-1,0)\}.}$$

Problem 3

By definition of $C^+(X(0))$, we can choose $t_0 \geq 0$ with $X(t_0) \in \Omega(X(0))$. By Uniqueness, it suffices to show that $\exists t_1 > t_0$ such that $X(t_1) = X(t_0)$.

Suppose, for sake of contradiction, that no such time t_1 exists. Since X is non-constant, $X(t_0)$ is not a critical point. Thus, by the Comment on page 148, we can choose a transversal L with $X(t_0) \in L \backslash \{\text{end points}\}$. By Corollary 6.1, $\exists s_k \to \infty$ with $X(s_k) \in L$, $X(s_k) \to X(t_0)$, and $s_{k+1} > s_k, \forall k \in \mathbb{N}$, and, by our assumption, we may also assume $X(t_0) \neq X(s_k), \forall k \in \mathbb{N}$. Then, we may choose $j, k \in \mathbb{N}$ with $t_0 < s_j < s_k$, and $|X(s_k) - X(t_0)| < |X(s_j) - X(t_0)|$. However, for $S := X([0, s_k])$, this contradicts the monotonicity conclusion of Lemma 6.2. ■

Problem 4

We first calculate

$$\begin{aligned}
\dot{r} &= \dot{X}\cos\theta + \dot{Y}\sin\theta \\
&= \left(r\cos\theta + r^3\cos\theta\sin^2\theta - r^5\cos\theta + r^3\sin\theta\right)\cos\theta \\
&+ \left(r\sin\theta + r^3\sin^3\theta - r^5\sin\theta - r^3\cos\theta\right)\sin\theta, \\
&= r\left(1 + r^2\sin^2\theta - r^4\right), \\
\dot{\theta} &= \dot{Y}r^{-1}\cos\theta - \dot{X}r^{-1}\sin\theta \\
&= \left(\sin\theta + r^2\sin^3\theta - r^4\sin\theta - r^2\cos\theta\right)\cos\theta \\
&- \left(\cos\theta + r^2\cos\theta\sin^2\theta - r^4\cos\theta + r^2\sin\theta\right)\sin\theta = -r^2.
\end{aligned}$$

Since

$$1 - r^4 \leq 1 + r^2\sin^2\theta - r^4 \leq 1 + r^2 - r^4,$$

if $r \in (0, 1]$, then $\dot{r} > 0$, and, if $r > \sqrt{\frac{1}{2}(1+\sqrt{5})}$, then $\dot{r} < 0$. It follows, then, that the annulus

$$A = \left[1, \sqrt{\frac{1}{2}(1+\sqrt{5})}\right] \times \mathbb{R}$$

defined in polar coordinates is positively invariant. Hence, for any solution X with initial condition in A, since A is closed, $\Omega(X(0)) \subseteq A$. Furthermore, since $\dot{\theta} = -r^2 \leq -1$ in A, X has no critical points in A, and so $\Omega(X(0))$ contains no critical points. Thus, by the Poincaré-Bendixson Theorem, there is a periodic solution $\widetilde{X}$, and, moreover, $C^+(\widetilde{X}(0)) \subseteq A$, so that $\widetilde{X}$ is nonconstant. ■