Math 21-236, Mathematical Studies Analysis II, Spring 2012
Assignment 6

The due date for this assignment is Wednesday April 11.

Given a continuous oriented curve $\boldsymbol{\gamma}$ with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^3$, the *projection* $\Pi\boldsymbol{\gamma}$ of $\boldsymbol{\gamma}$ in $\mathbb{R}^2$ is the curve with parametric representation $(\varphi_1, \varphi_2) : [a,b] \to \mathbb{R}^2$.

1. Let $U \subseteq \mathbb{R}^2$ be an open set starshaped with respect to (x_0, y_0), let $\mathbf{g} : U \setminus \{(x_0, y_0)\} \to \mathbb{R}^2$ be an irrotational vector field of class C^1, and let $\boldsymbol{\gamma}$ be a continuous closed oriented curve with range contained in $U \setminus \{(x_0, y_0)\}$. Prove that
$$\int_{\boldsymbol{\gamma}} \mathbf{g} = \operatorname{ind}_{\boldsymbol{\gamma}}((x_0, y_0)) \int_{\boldsymbol{\gamma}_1} \mathbf{g},$$
where $\boldsymbol{\gamma}_1$ is the curve parametrized by $\boldsymbol{\psi}(s) = (x_0, y_0) + (r\cos s, r\sin s)$, $s \in [0, 2\pi]$ and $r > 0$ is so small that $\partial B((x_0, y_0), r) \subset U$.

2. Let $V = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \neq (0,0)\}$, let $\mathbf{g} : V \to \mathbb{R}^3$ be an irrotational vector field of class C^1. and let $\boldsymbol{\gamma}$ be a continuous closed oriented curve with range contained in V. Prove that
$$\int_{\boldsymbol{\gamma}} \mathbf{g} = \operatorname{ind}_{\Pi\boldsymbol{\gamma}}((0,0)) \int_{\boldsymbol{\gamma}_1} \mathbf{g},$$
where $\boldsymbol{\gamma}_1$ is the curve parametrized by $\boldsymbol{\psi}(s) = (\cos s, \sin s, 0)$, $s \in [0, 2\pi]$.

3. Given the function
$$\mathbf{g}(x,y,z) = \left(\frac{xz}{x^2+y^2}, \frac{yz}{x^2+y^2}, h(x,y,z) \right)$$
defined in $U = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \neq (0,0)\}$, $h \in C^1(U)$,
   (a) find h in such a way that $\mathbf{g}$ is irrotational in U;
   (b) prove that $\int_{\boldsymbol{\gamma}} \mathbf{g} = 0$, where $\boldsymbol{\gamma}$ is the closed curve parametrized by $\boldsymbol{\varphi}(t) = (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$,
   (c) prove that $\mathbf{g}$ is conservative (without using part (d)),
   (d) find f such that $\nabla f = \mathbf{g}$ and $f(1,1,1) = 0$.

4. Let $E \subset \mathbb{R}^N$ be a Peano–Jordan measurable set.
   (a) Let $\mathbf{g} : E \to \mathbb{R}^M$ be a Lipschitz function, with $N \leq M$. Moreover, if $N = M$, assume that E has measure zero. Prove that $\mathbf{g}(E)$ is Peano–Jordan measurable with measure zero.
   (b) Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^M$ be a function of class C^1. Assume that $\overline{E} \subseteq U$. Moreover, if $N = M$, assume that E has measure zero. Prove that $\mathbf{g}(E)$ is Peano–Jordan measurable with measure zero.

(c) Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^N$ be a function of class C^1. Assume that $\overline{E} \subseteq U$ and that $\det J_{\mathbf{g}}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E^\circ$.

i. Prove that $\partial \mathbf{g}(E) \subseteq \mathbf{g}(\partial E)$.

ii. Prove that $\mathbf{g}(E)$ is Peano–Jordan measurable.