Homework 1

21-484A Graph Theory
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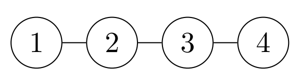
Problem 1

Let G be a graph with at least 3 vertices.

(a) Suppose G contains two distinct vertices u and v such that $G_u := G - \{u\}$ and $G_v - \{v\}$ are both connected. Since G contains at least 3 vertices, G contains some vertex $t \neq u, v$. Since G_u is connected, for every vertex $s \neq u$ in G, there is a $s - t$ path in G_u. Since G_u is a subgraph of G, that $s - t$ path is also in G. Since G_v is connected, there exists a $t - u$ path in G_v, and, since G_v is a subgraph of G, that $t - u$ path is also in G. Thus, for every vertex r in G, there is a $r - t$ path in G. Thus, for every pair of vertices (r, s) in G reversing a $r - t$ path, removing t, and appending it to a $s - t$ path gives a $s - r$ walk. Since, as shown in class, every $s - r$ walk contains a $s - r$ path, G has a $s - r$ path. Thus, since s and r are arbitrary nodes in G, G is connected. $\blacksquare$

(b) Suppose, for sake of contradiction, that there exists a connected graph G such that, for every two distinct vertices u and v in G, $G - \{u\}$ and $G - \{v\}$ are disconnected. In particular, let G be a minimal such graph, with respect to the number of nodes in G. It can be shown, by enumerating all graphs on 1 and 2 nodes that no such graphs exists on fewer than 3 nodes. Let t be a vertex in G, and let $H = G - \{t\}$. Since H is disconnected, it has (at least) two distinct connected components, K and L. Since K and L each have fewer vertices than G, they each must have 2 vertices, a and b in K, and c and d in L, such that $K - \{a\}$, $K - \{b\}$, $L - \{c\}$, and $L - \{d\}$ are each connected (we can exclude the case that K or L has only one vertex, as, in this case, removing that vertex would not cause G to be disconnected). Let $u = a$ if at is not an edge in G, and let $u = b$ otherwise, and let $v = c$ if ct is not an edge in G, and let $v = d$ otherwise. Then, $G - \{u\}$ and $G - \{v\}$ are both connected, contradicting the choice of G. Therefore, for every connected graph G, there exist two distinct vertices u and v in G such that $G - \{u\}$ and $G - \{v\}$ are connected. $\blacksquare$

(c) Let $V = \{1, 2, 3, 4\}$, let $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$, and let $G = (V, E)$ (the graph pictured below).



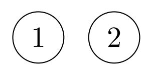
Then, G is a graph with at least four vertices, but, for any three distinct vertices u, v, and w, since either u, v, or w must be in $\{2, 3\}$, one of $G - \{u\}$, $G - \{v\}$, or $G - \{w\}$ is disconnected. Thus, the statement in question is false. $\blacksquare$

Problem 2

Suppose G is a disconnected graph. Let u and v be vertices in G. If u and v are in different connected components of G, then uv is not an edge in G, so that uv is an edge in $\overline{G}$ and thus there exists a uv path in $\overline{G}$. If u and v are in the same connected component of G, then there exists a vertex t in G such that t is in a different connected component than u and v (if this were not the case, then every vertex in G would be in the same connected component of G, so that G would be connected). Since t is in a different connected component of G than u and v, ut and tv are not edges in G, so that they are edges in $\overline{G}$. Therefore, u, t, v is a uv path in $\overline{G}$. Thus, if G is not connected, then $\overline{G}$ is connected, so that, for all graphs G, either G or $\overline{G}$ is connected. ■

Problem 3

Let $V = \{1, 2\}$, let $E = \emptyset$, and let $G = (V, E)$ (the graph pictured below).



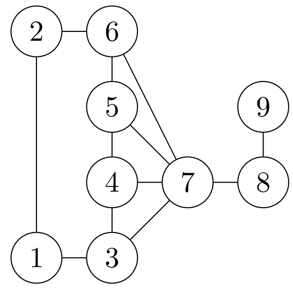
Then, for all vertices u and v in G, $\deg(u) + \deg(v) = 0 \geq n - 2$. However, G is disconnected. Thus, the condition in question is sharp. ■

Problem 4

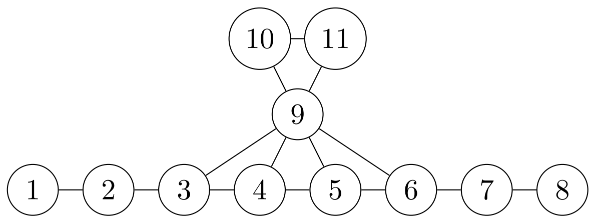
Let G be a graph, and let u and v be verticea in G. Suppose u and v are in the same connected component H if G. Then, if u and v are not connected, then H is not connected, contradicting the definition of H as a connected component. Thus, if u and v are in the same connected component of G, then uRv. Suppose, on the other hand, that u and v are not in the same connected component of G, but u and v are connected. Let H be a connected component of G containing u. If v is not in H, then, if K is the graph created by adding the vertex v and the edge uv to H, the H is a proper subgraph of H, which is a connected subgraph of G, contradicting the maximality of H as a connected component. Therefore, if uRv, then u and v are in the same connected component of G, then u and v are connected. Thus, two vertices in G are in the same connected component if and only if they are connected, so the equivalence classes of R are the connected components of G. ■

Problem 5

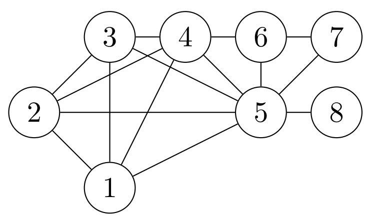
(a) The graph pictured below has the given degree sequence, so that the given degree sequence is graphical.



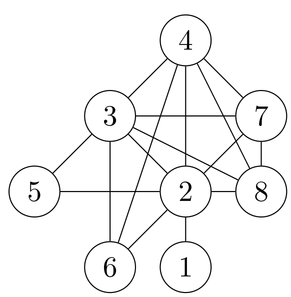
(b) The graph pictured below has the given degree sequence, so that the given degree sequence is graphical.



(c) The graph pictured below has the given degree sequence, so that the given degree sequence is graphical.



(d) The graph pictured below has the given degree sequence, so that the given degree sequence is graphical.



Problem 6

For $n = 1$, the only sequence of length n obeying property (b) is the degree sequence of the graph with one vertex and no edges, and is thus graphical. Suppose that, for some $n \in \mathbb{N}$, all sequences of legraphical. Let $D = \{d_k\}_{1\leq k\leq n+1}$ be a sequence of integers obeying properties (a), (b), and (c). Let $E = \{e_k\}_{1\leq k\leq n+1}$ be a sequence that results from sorting the terms of D in descending order, so that E also obeys properties (a), (b), and (c). Let F be the sequence $f_2, f_3, \ldots, f_n = e_2 - 1, e_3 - 1, \ldots, e_{e_1} - 1, e_{e_1+1} - 1, e_{e_1+2}, e_{e_1+3}, \ldots, e_n$, so that, as shown in class, E is graphical if and only if F is graphical. Note that

$$\sum_{i=1}^{n} f_i = \left(\sum_{i=1}^{n+1} e_i\right) - 2e_1,$$

so that, since $\sum_{i=1}^{n+1} d_i$ is even, $\sum_{i=1}^{n} f_i$ is as well, and thus F obeys property (a). Suppose, for sake of contradiction that F did not obey property (b), so that it had some term f_k such that either $f_k > n-1$ or $f_k < 0$. In the first case, since f_k is an element of D, which obeys property (b), $f_k = n$. However, since E is sorted in descending order, this would imply that $f_k \leq e_1 = n$, so that $f_k = e_k - 1 \leq n - 1$, which is a contradiction. In the second case, since D obeys property (b), $f_k = -1$, as $e_k = 0$. Since E obeys property (c), this implies that $0 \leq e_1 \leq 1$. If $e_1 = 0$, then $f_k = e_k \geq 0$, which is a contradiction. If $e_1 = 1$, then either $e_2 = 0$, which would contradict the fact that E obeys property (a), or $e_2 = 1$, in which case either $k = 2$ and $f_k = 0$, or $k > 2$, in which case $f_k = e_k \geq 0$; either is a contradiction. Thus, F obeys property (b). Suppose, for sake of contradiction, that F does not satisfy property (c), so that for some pair (i, j) with $1 \leq i, j \leq n$ (without loss of generality, $i < j$), $|f_i - f_j| > 1$. Since E is sorted in descending order and obey property (c), $0 \leq e_i - e_j \leq 1$. However, if $f_j \neq e_j$, then $f_j = e_j - 1$, in which case, by construction of F, $f_i = e_i - 1$, so that $f_i - f_j = e_i - e_j$, contradicting the fact that E obeys property (c), so that F must also obey property (c). Since F is a sequence of integers of length n obeying properties (a), (b), and (c), by the induction hypothesis, F is graphical, so that E and thus D are graphical. By the Principle of Mathematical Induction, then, any sequence of integers obeying properties (a), (b), and (c) is graphical. ■