21-640 **Intro to Functional Analysis** **Spring 2013**

Review Guide for Final Exam

Normed Linear Spaces

- Baire Category Theorem.
- Definition and basic properties of norms.
- Definition of Banach space.
- Linear manifolds and closed subspaces.
- Linear combination, linear independence, span, Hamel basis.
- Notions of finite dimensionality and infinite dimensionality.
- A NLS is complete if and only if every absolutely summable sequence is summable.
- Schauder basis; the existence of a Schuder basis implies that X is separable.
- Properties of finite-dimensional NLS.
- The Riesz Lemma (Lemma 2.1).
- The closed unit ball is compact if and only if X is finite dimensional.
- Basic properties of linear mappings.
- Boundedness and continuity of linear mappings.
- A linear functional $f : X \to \mathbb{K}$ is continuous if and only if $\mathcal{N}(f)$ is closed.
- Equivalence of norms.
- Spaces of bounded linear mappings; operator norm; $\mathcal{L}(X; Y)$ is complete if Y is complete.
- **Principle of Uniform Boundedness** (a.k.a Banach-Steinhaus Theorem).
- Topological dual and duality pairing.
- Second duals and the canonical embedding $J : X \to X^{**}$.
- The basic sequence spaces and their properties.
- Young's Inequality, Holder's Inequality, Minkowski's Inequality.
- Dual spaces of the basic sequence spaces.

- **Open Mapping Theorem** and Bounded Inverse Theorem.
- **Closed Graph Theorem**.
- **Hahn-Banach Theorems**; extension and separation forms; consequences of Hahn-Banach Theorems.
- Convex sets, absorbing sets, balanced sets.
- Minkowski functional and properties.
- Seminorms.
- Reflexivity; reflexivity implies completeness; closed subspaces of reflexive spaces are reflexive; a Banach space X is reflexive if and only if X^* is reflexive.
- Weak convergence; weakly convergent sequences are bounded, the norm is sequentially lower semicontinuous, closed convex sets are sequentially weakly closed; if X is reflexive then every bounded sequence has a weakly convergent subsequence.
- Weak* convergence; if X is a Banach space then every weakly convergent sequence is bounded; the norm is sequentially weakly* lower semicontinuous; weak* limits can escape from closed convex sets; if X is separable then every bounded sequence in X^* has a weakly* convergent subsequence.
- X^* separable implies X separable.
- If X is reflexive then every nonempty, closed, convex set has an element of minimum norm.
- In l^1 a sequence is weakly convergent if and only if it is strongly convergent.
- Adjoints; definition; $\|T\| = \|T^*\|$; T^{**} is an extension of T; adjoints of inverses and products.
- Annihilator; pre-annihilator; relationship between $\mathcal{R}(T)$ and $\mathcal{N}(T^*)$; relationship between $\mathcal{R}(T^*)$ and $\mathcal{N}(T)$.
- If X and Y are Banach spaces and $T \in \mathcal{L}(X;Y)$ then $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^*)$ is closed.
- Linear mappings applied to weakly and weakly* convergent sequences.
- Compact linear operators; definition; characterizations in terms of sequences.
- If T is compact then $\mathcal{R}(T)$ is separable.
- If X and Y are Banach spaces and $T \in \mathcal{L}(X;Y)$ then T is compact if and only if T^* is compact.

- Continuous and compact embeddings of NLS.
- Convergence of sequences of bounded linear operators in the weak, strong, and uniform operator topologies.
- The uniform limit of a sequence of compact linear operators is compact.

Topological Vector Spaces and Weak Topologies

- Definition of TVS; local base; constructions of special types of neighborhoods; local convexity.
- Topologically bounded sets.
- The interior and closure of a convex set are convex.
- Continuity of linear mappings.
- A TVS is metrizable if and only if it has a countable local base.
- If a locally convex TVS is metrizable, then there is a metric that induces the topology and such that all open balls are convex.
- A TVS is normable if and only if zero has a bounded convex neighborhood.
- A TVS is locally convex if and only if there is a separating family of seminorms that induces the topology.
- Weak topology of a NLS; construction of a local base; a set is weakly bounded if and only if it is norm bounded; the weak and strong closures of a convex set coincide.
- Weak* topology of X^* where X is a NLS; construction of a local base; if X is complete then a subset of X^* is weakly* bounded if and only if it is norm bounded.
- **Alaoglu's Theorem**
- If X is separable and K^* is a weakly* compact subset of X^* then $(K^*, \sigma(X^*, X))$ is metrizable.
- $J : (X, \sigma(X, X^*)) \to (J[X], \sigma(X^{**}, X^*))$ is a homeomorphism.
- Goldstine's Theorem.
- A NLS is reflexive if and only if the closed unit ball is weakly compact.
- If X is a separable NLS and K is a weakly compact subset of X then $(K, \sigma(X, X^*))$ is metrizable.

- Definition and basic properties of inner product; orthogonality.
- Cauchy-Schwarz inequality; polar identity; Pythagorean Theorem; Parallelogram Law.
- Definition of Hilbert space.
- In a Hilbert space, every nonempty, closed, convex set has a unique element of minimum norm.
- Projection Theorem; projection operator P_M where M is a closed subspace.
- Riesz Representation Theorem; Riesz mapping; weak convergence in Hilbert spaces.
- Hilbert adjoints.
- Orthonormal family; orthonormal basis; Bessel's inequality; Theorem 15.28 concerning basis expansions.
- If X is a Hilbert space and $A \in \mathcal{C}(X; X)$ then there is a sequence of continuous linear operators of finite rank that converges to A in the uniform operator topology.