Homework 1

21-759 Differential Geometry
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I would be willing to present solutions to problems 1,3,4, and 5.

Problem 1

(i) As a subspace of $\mathbb{R}^n$, $\mathcal{M}_\alpha$ is second countable and Hausdorff.

Let $p \in \mathcal{M}_\alpha$. Since α is regular, without loss of generality, $(DF)_1 \neq 0$. By the Implicit Function Theorem, there exist open neighborhoods $U \subseteq \mathbb{R}^{n-1}$ and $V \subseteq \mathbb{R}$ of $(p_2, \dots, p_n)$ and p_1, respectively, and a smooth function $g : U \to V$ such that $\{(q, g(q)) : q \in U\} = (U \times V) \cap F^{-1}(\alpha)$. The function $\phi(x) := (x, g(x))$ on U is clearly a diffeomorphism. Therefore, (U, ϕ) is a coordinate chart for $\mathcal{M}_\alpha$ in the neighborhood $U \times V$ of p, so $\mathcal{M}_\alpha$ is a manifold (of dimension $n-1$). $\blacksquare$

(ii) As shown in part (i), M_α admits a differentiable structure of the form $\{(U_\gamma, \phi_\gamma)\}$, where $\phi(x) = (x, g_\gamma(x))$ on U_γ. Thus, if (U_γ, ϕ_γ) and (U_β, ϕ_β) are charts with $W := \phi_\gamma(U_\gamma) \cap \phi_\beta(U_\beta) \neq \emptyset$, then, on W, $\phi_\beta^{-1}\phi_\gamma \equiv id_n$, the identity on $\mathbb{R}^n$, and hence

$$\det(D(\phi_\beta^{-1}\phi_\gamma)) = \det(id_n) = 1 > 0,$$

so that $\mathcal{M}_\alpha$ is orientable. $\blacksquare$

Problem 2

If $p_1 = p_2$ and $v_1 \neq v_2$ then, since $T_{p_1}\mathcal{M}$ is homeomorphic to $\mathbb{R}^n$ (which is Hausdorff), there are open sets $V_1, V_2 \subset T_{p_1}\mathcal{M}$ separating v_1 and v_2, and so the open sets $\mathcal{M} \times V_1$ and $\mathcal{M} \times V_2$ separate (p_1, v_1) and (p_2, v_2). If $p_1 \neq p_2$, then, since $\mathcal{M}$ is Hausdorff, there are open sets $U_1, U_2 \subset \mathcal{M}$ separating p_1 and p_2 and, and so the open sets $U_1 \times T_{p_1}\mathcal{M}$ and $U_2 \times T_{p_2}\mathcal{M}$ separate (p_1, v_1) and (p_2, v_2). Thus, $T\mathcal{M}$ is Hausdorff. Since $\mathcal{M}$ and $T_p\mathcal{M}$ have a countable bases $\mathcal{B}$ and $\mathcal{B}_p$, set $\{U \times V : U \in \mathcal{B}, V \in \mathcal{B}_p, p \in U\}$ is a countable base for $T\mathcal{M}$.

$\forall p \in \mathcal{M}$, since $T_p\mathcal{M}$ is a vector space for which the vectors $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ form a basis, there is a unique linear mapping $D_p : T_p\mathcal{M} \to \mathbb{R}^n$ such that, $\forall \in T_p\mathcal{M}$, $v = \sum_{i=1}^n D_i(v)\frac{\partial}{\partial x_i}$. Then, the function $(p, v) \mapsto (\phi^{-1}(p), D_p(v))$ is a diffeomorphism between $\mathcal{M}$ and $\mathbb{R}^{2d}$.

We first compute

$$v\left[\left(\phi^{-1}\right)_i\right] = \sum_{j=1}^n a^j \frac{\partial}{\partial x_j}(\phi^{-1} \circ \phi)_i(\phi^{-1}(p)) = \sum_{j=1}^n a^j \frac{\partial}{\partial x_j} x_i(\phi^{-1}(p)) = \sum_{j=1}^n a^j \delta_i^j = a^i.$$

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We now compute this in an alternate manner:

$$v\left[\left(\phi^{-1}\right)_i\right] = \sum_{j=1}^{n} b^j \frac{\partial}{\partial y_j}(\phi^{-1}\circ\psi)_i(\psi^{-1}(p)).$$

The result can be summarized in matrix notation:

$$\boxed{a = \left[D(\phi^{-1}\circ\psi)\right]\Big|_{\psi^{-1}(p)} b.}$$

Problem 3

Since $x_i = \left(\phi^{-1}\right)_i$, $df = \sum_{i=1}^{n} \alpha_i d\left(\phi^{-1}\right)_i$, and so

$$\frac{\partial}{\partial x_j}[f] = \frac{\partial}{\partial x_j}\sum_{i=1}^{n}\alpha_i\left(\left(\phi^{-1}\right)_i\circ\phi\right)\left(\phi^{-1}(p)\right) = \sum_{i=1}^{n}\alpha_i\frac{\partial}{\partial x_j}x_i\left(\phi^{-1}(p)\right) = \sum_{i=1}^{n}\alpha_i\delta_i^j = \alpha_j.$$

Also, since $y_i = \left(\psi^{-1}\right)_i$,

$$\frac{\partial}{\partial x_j}[f] = \frac{\partial}{\partial x_j}\sum_{i=1}^{n}\beta_i\left(\left(\psi^{-1}\right)_i\circ\phi\right)\left(\phi^{-1}(p)\right) = \sum_{i=1}^{n}\beta_i\frac{\partial}{\partial x_j}\left(\psi^{-1}\circ\phi\right)_i\left(\phi^{-1}(p)\right).$$

The result can be summarized in matrix notation:

$$\boxed{\alpha = \left[D(\psi^{-1}\circ\phi)\right]\Big|_{\phi^{-1}(p)} \beta.}$$

Problem 4

Suppose that, for some $\alpha\in\mathbb{R}^n$, $df := \displaystyle\sum_{i=1}^{n}\alpha_i dx_i = 0$. Then,

$$0 = df\left(\sum_{j=1}^{n}\alpha_j\frac{\partial}{\partial x_j}\right) = \sum_{j=1}^{n}\alpha_j\frac{\partial}{\partial x_j}\sum_{i=1}^{n}\alpha_i x_i = \sum_{j=1}^{n}\sum_{i=1}^{n}\alpha_j\alpha_i\delta_i^j = \sum_{i=1}^{n}\alpha_i^2,$$

and so $\alpha = 0$. Thus, $dx_1,\ldots,dx_n$ are linearly independent. As the dual of an n-dimensional vector space, $T_p\mathcal{M}^*$ has dimension n, and so the n independent vectors $dx_1,\ldots,dx_n$ span $T_p\mathcal{M}^*$. ■

Problem 5

Without loss of generality, we may assume that, for every chart (U_α, ϕ_α) of $\mathcal{M}$, U_α is connected, since, otherwise, we may replace the chart by identical charts on the connected components of U_α.

Let $p \in \mathcal{M}$, and let (U_α, ϕ_α) be a chart at p. Since $df = 0$, for $v_i = \frac{\partial}{\partial x_i}$ and g the identity on $\mathbb{R}$

$$0 = df_p(v_i)[g] = v_i[g \circ f] = v_i[f] = \frac{\partial}{\partial x_i}(f \circ \phi_\alpha)(\phi_\alpha^{-1}(p)),$$

And so $[D(f \circ \phi_\alpha)]\Big|_{\phi_\alpha^{-1}(p)} = 0$. By a theorem in multivariable calculus, since U_α is connected, $f \circ \phi_\alpha \equiv C_\alpha$ on U_α, for some constant $C_p \in \mathbb{R}$. Since ϕ is a surjection, it follows that $f \equiv C_p$ on $\phi(U)$.

Now let $U := \cup\{\phi_\beta(U_\beta) : C_\beta = C_\alpha\}$, and let $V := \cup\{\phi_\beta(U_\beta) : C_\beta \neq C_\alpha\}$. As unions of open sets, both U and V are open, and clearly $\mathcal{M} = U \cup V$ and $U \cap V = \emptyset$. Thus, since $\mathcal{M}$ is connected and $U \neq \emptyset$, $V = \emptyset$, and so f is constant. $\blacksquare$

Problem 6

To show that DF is injective (so that F is an immersion), it suffices to show that

$$DF = \begin{bmatrix} 2x & y & z & 0 \\ -2y & x & 0 & z \\ 0 & 0 & x & y \end{bmatrix}$$

satisfies $\operatorname{rank}(DF) \geq 2$, $\forall(x, y, z)$ with $x^2 + y^2 + z^2 = 1$. If x is non-zero, then the first and third columns are independent. If y is non-zero, then the first and fourth columns are independent. If $x = y = 0$, then $z = \pm 1$, and so the last two columns are independent. Thus, $\operatorname{rank}(DF) \geq 2$.

We now show F is injective. If one of (x_1, y_1, z_1) is zero, then, checking several cases and using the fact that $([0], [0], [0]) \notin P^2$, one can verify that $(x_1, y_1, z_1) = (\pm x_2, \pm y_2, \pm z_2)$. If all values are nonzero, then

$$\frac{x_1}{x_2} = \frac{y_2}{y_1} = \frac{z_1}{z_2} = \frac{x_2}{x_1},$$

so that $x_1^2 = x_2^2$, and hence $x_1 \sim x_2$ (similarly, $y_1 \sim y_2$ and $z_1 \sim z_2$). It follows that F is injective, and thus F is a diffeomorphism (it follows from the above work that F^{-1} is differentiable).

Problem 7

(i) Suppose the orientation induced on $\mathcal{M}$ by $\{(U_\alpha, \phi_\alpha)\}$ is preserved by all $g \in G$. Let $\{(V_\alpha, \pi_\alpha \circ x_\alpha)\}$ be the quotient differentiable structure on $\mathcal{M}/G$. Then, if $\pi_\alpha(x_\alpha(V_\alpha)) \cap \pi_\beta(x_\beta(V_\beta)) \neq \emptyset$, then, since $\pi_\beta^{-1}\pi_\alpha = \Phi_g$, for some $g \in G$, and Φ_g preserves orientation on $\mathcal{M}$,

$$\begin{aligned} \operatorname{sign}(\det(D((\pi_\beta \circ x_\beta)^{-1} \circ \pi_\alpha \circ x_\alpha))) &= \operatorname{sign}(\det(D(x_\beta^{-1} \circ \Phi_g \circ x_\alpha))) \\ &= \operatorname{sign}(\det(D(x_\beta^{-1} \circ x_\alpha))) = 1, \end{aligned}$$

and hence $\mathcal{M}/G$ is orientable.

(ii) Let G be the group $(\{-1, 1\}, \cdot)$. As discussed in class, $G \times S^n \to S^n$ is a properly discontinuous action. Since S^n is orientable, fix an oriented differentiable structure $\{(U_\alpha, \phi_\alpha)\}$.

Since Φ_{-1} is the restriction of the antipodal mapping $x \mapsto -x$ on $\mathbb{R}^{n+1}$ to S^n and ϕ_α and ϕ_β share orientation,

$$\det(D(\phi_\beta^{-1} \circ \Phi_{-1} \circ \phi_\alpha)) = \det(-D(\phi_\beta^{-1} \circ \phi_\alpha)) = (-1)^{n+1} \det(D\phi_\beta^{-1} \circ \phi_\alpha)),$$

has sign $(-1)^{n+1}$. Thus, Φ_{-1} is orientation preserving if and only if n is odd, and so, by the result of part (i), $P^n = S^n/G$ is orientable if and only if n is odd. ■

Problem 8

Suppose $\exists \gamma_0 \in C^\infty([0, t_0), \mathcal{M})$ with $\gamma_0(0) = p$. Consider an increasing sequence $t_n \in [0, t_0)$ converging to t_0. Since $\mathcal{M}$ is compact and $\gamma_0 \in C^\infty$, the sequence $\gamma_0(t_n)$ has a limit $p \in \mathcal{M}$. Thus, the solution γ_0 can be extended to a solution $\gamma_1 \in C^\infty([0, t_0], \mathcal{M})$ (i.e., γ_0 is not right maximal).

Similarly, any solution on a left-open interval can be left-extended. It follows by an extension theorem from ODE that there is a solution for all $t \in \mathbb{R}$.