Homework.

There is a firm 'no late homework' policy.

Due Wed 09/12

e the Lebesgue measure.

te the Lebesgue outer measure.

Cantor set. (c) $\{x \in [0,1] \mid x \notin \mathbb{Q}\}$.

n$(V) < d$, then show that $\lambda(V) = 0$.

t area$(P) = \lambda(P)$.

measure on $(\mathbb{R}^d, \mathcal{L})$ (i.e. $\mu(x+A) = \mu(A)$

finite on bounded sets. Show that $\exists c \geqslant 0$

al linear transformation, and $A \in \mathcal{L}$. Show

$\lambda(A)$. [Hint: Express T in terms of elementary

] be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\emptyset) = 0$.

$E_i) \Big| E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_1^\infty E_j \Big\}$.

re.

$\delta > 0$ and define $\mathcal{E}_\delta = \{B(x,r) \mid x \in X, r \in$

$x,r)) = c_\alpha r^\alpha$, where $c_\alpha = \pi^{\alpha/2}/\Gamma(1+\alpha/2)$

$H^*_{\alpha,\delta}$ be the outer measure obtained with

on of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta\to 0} H^*_{\alpha,\delta}$.

d restricts to a measure H_α on a σ-algebra

measure H_α is called the *Hausdorff mea-*

ve Caratheodory.]

H_d is the Lebesgue measure.

re exists (a unique) $d \in [0,\infty]$ such that

and $H_\alpha(S) = 0$ for all $\alpha \in (d,\infty)$. This

dimension of the set S.

on of the Cantor set.

Do it, but don't turn it in.)

ells. Show it for arbitrary cells.

l $a \in \mathbb{R}^d$, $E \subseteq \mathbb{R}^d$.

of σ-algebras on X is also a σ-algebra.

d delta measures are measures.

ed in class Σ is a σ-algebra, and that $\mu^*|_\Sigma$

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Assignment 2: Assigned Wed 09/12. Due Wed 09/19

F 1. Let (X, Σ, μ) be a measure space. For $A \in P(X)$ define $\mu^*(A) = \inf\{\mu(E) \mid E \supseteq A \,\&\, E \in \Sigma\}$, and $\mu_*(A) = \sup\{\mu(E) \mid E \subseteq A \,\&\, E \in \Sigma\}$.

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3 (a) Show that μ^* is an outer measure. Pm

(b) Let $A_1, A_2, \dots \in \mathcal{P}(X)$ be disjoint. Show that $\mu_*(\bigcup_1^\infty A_i) \geqslant \sum_1^\infty \mu_*(A_i)$.
3 [The set function μ_* is called an *inner measure*.]

(c) Show that for all $A \subseteq X$, $\mu^*(A) + \mu_*(A^c) = \mu(X)$. Miller

5 (d) Let $A \subseteq \mathcal{P}(X)$ with $\mu^*(A) < \infty$. Show that $A \in \Sigma_\mu \iff \mu_*(A) = \mu^*(A)$.

F/5 2. Here's an alternate (cleaner) approach to proving $\mathcal{L} = \mathcal{B}_\lambda$. We do it by proving a stronger statement than necessary. Murphy III

4 5 (a) If $A \in \mathcal{L}(\mathbb{R}^d)$ show that for any $\varepsilon > 0$ there exists two sets C, U such that $C \subseteq A \subseteq U$, C is closed, U is open and $\lambda(U - C) < \varepsilon$.

3 (b) For $A \in \mathcal{L}(\mathbb{R}^d)$, show that that there exists an F_σ, F and a G_δ, G such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$. Conclude $\mathcal{B}_\lambda = \mathcal{L}$.

5/5 3. Let $A \in \mathcal{L}(\mathbb{R}^d)$. Prove every subset of A is Lebesgue measurable $\iff \lambda(A) = 0$.

5 4. (a) Prove $\mathcal{B}(\mathbb{R}^{m+n}) = \sigma(\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\})$. 2nv if people just do R

5 (b) Prove $\mathcal{L}(\mathbb{R}^{m+n}) \supsetneq \sigma(\{A \times B \mid A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$. Cam lo

12 2 (c) Show $\mathcal{L}(\mathbb{R}^2) \supsetneq \mathcal{B}(\mathbb{R}^2)$.

5 5. Find $E \in \mathcal{B}(\mathbb{R})$ so that for all $a < b$, we have $0 < \lambda(E \cap (a,b)) < b - a$. Singh

People followed Rudin II, 8 ~~Murphy III~~

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We say $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if $\emptyset \in \mathcal{A}$, and $\mathcal{A}$ is closed under complements and *finite* unions. We say $\mu_0 : \mathcal{A} \to [0,\infty]$ is a (positive) *pre-measure* on $\mathcal{A}$ if $\mu_0(\emptyset) = 0$, and for any countable disjoint sequence of sets sequence $A_i \in \mathcal{A}$ *such that* $\bigcup_1^\infty A_i \in \mathcal{A}$, we have $\mu_0(\bigcup_1^\infty A_i) = \sum_1^\infty \mu_0(A_i)$.

Namely, a pre-measure is a finitely additive measure on an algebra $\mathcal{A}$, which is also countably additive for disjoint unions *that belong to the algebra.*

T 5 6. *(Caratheodory extension)* If $\mathcal{A}$ is an algebra, and μ_0 is a pre-measure on $\mathcal{A}$, show that there exists a measure μ defined on $\sigma(\mathcal{A})$ that extends μ_0. Murphy III

Optional problems, and details in class I left for you to check.

* Prove any open subset of $\mathbb{R}^d$ is a countable union of cells. Conclude $\mathcal{L} \supseteq \mathcal{B}$.
* Show that the cardinality $\mathcal{B}(\mathbb{R})$ is the same as that of $\mathbb{R}$, however, the cardinality of $\mathcal{L}(\mathbb{R})$ is the same as that of $\mathcal{P}(\mathbb{R})$. Conclude $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$. [There are of course other ways to prove this.] $\inf_i \mu(A_i) \geq \mu(\cap A_i)$
* If $A_i \in \Sigma$ are such that $A_i \supseteq A_{i+1}$, show that $\mu(\bigcap_{i=1}^\infty A_i) = \lim_{i\to\infty} \mu(A_i)$, provided $\mu(A_1) < \infty$. Given an example to show this is not true if $\mu(A_1) = \infty$.
* We saw in class $\lambda(A) = \sup\{\lambda(K) \mid K \subseteq A \,\&\, K \text{ is compact}\}$ for all bounded sets $A \in \mathcal{L}$. Prove it for arbitrary $A \in \mathcal{L}$.
* Show that there exists $A \subseteq \mathbb{R}$ such that if $B \subseteq A$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$, *and further*, if $B \subseteq A^c$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$.

1\. Proof: (a) 1) $\phi \in \Sigma$, so $\mu^*(\phi) = 0$

2\) If $A, B \in \mathcal{P}(X)$, $A \subset B$. Any set $E \in \Sigma$ s.t. $B \subset E$ also satisfies that $A \subset E$. So $\mu^*(A) \le \mu^*(B)$

3\) Let $\{A_n\}_{n=1}^\infty \in \mathcal{P}(X)$. $A_n \subset E_n \in \Sigma$. $\mu^*(A_n) \ge \mu(E_n) - \frac{\varepsilon}{2^n}$. Σ

Since $\bigcup_{n=1}^\infty E_n$ is measurable and $\bigcup_{n=1}^\infty A_n \subset \bigcup_{n=1}^\infty E_n$

we have $\mu^*(\bigcup_{n=1}^\infty A_n) \le \mu(\bigcup_{n=1}^\infty E_n) \le \sum_{n=1}^\infty \mu(E_n)$

$\le \sum_{n=1}^\infty (\mu^*(A_n) + \frac{\varepsilon}{2^n}) = \sum_{n=1}^\infty \mu^*(A_n) + \varepsilon$ 3/3

This is true for all $\varepsilon > 0$, so $\mu^*(\bigcup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty \mu^*(A_n)$

Hence μ^* is an outer measure

Math 720: Homework.

Do, but don't turn in optional problems. *There is a firm 'no late homework' policy.*

Assignment 1: Assigned Wed 09/05. Due Wed 09/12

Following the notation of Cohn, I use λ to denote the Lebesgue measure.

1. For each of the following sets, compute the Lebesgue outer measure.
   (a) Any countable set. (b) The Cantor set. (c) $\{x \in [0,1] \mid x \notin \mathbb{Q}\}$.
2. (a) If $V \subseteq \mathbb{R}^d$ is a subspace with $\dim(V) < d$, then show that $\lambda(V) = 0$.
   (b) If $P \subseteq \mathbb{R}^2$ is a polygon show that $\text{area}(P) = \lambda(P)$.
3. (a) Say μ is a *translation invariant* measure on $(\mathbb{R}^d, \mathcal{L})$ (i.e. $\mu(x + A) = \mu(A)$ for all $A \in \mathcal{L}$, $x \in \mathbb{R}^d$) which is finite on bounded sets. Show that $\exists c \geqslant 0$ such that $\mu(A) = c\lambda(A)$.
   (b) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal linear transformation, and $A \in \mathcal{L}$. Show that $T(A) \in \mathcal{L}$ and $\lambda(T(A)) = \lambda(A)$. [Hint: Express T in terms of elementary transformations.]
4. (a) Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \subseteq X$ define

$$\mu^*(A) = \inf\Big\{\sum_1^\infty \rho(E_i) \,\Big|\, E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_1^\infty E_j\Big\}.$$

   Show that μ^* is an outer measure.
   (b) Let (X, d) be any metric space, $\delta > 0$ and define $\mathcal{E}_\delta = \{B(x,r) \mid x \in X, r \in (0,\delta)\}$. Given $\alpha > 0$ define $\rho(B(x,r)) = c_\alpha r^\alpha$, where $c_\alpha = \pi^{\alpha/2}/\Gamma(1+\alpha/2)$ is a normalization constant. Let $H^*_{\alpha,\delta}$ be the outer measure obtained with this choice of ρ and the collection of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta\to 0} H^*_{\alpha,\delta}$. Show H^*_α is an outer measure and restricts to a measure H_α on a σ-algebra that contains all Borel sets. The measure H_α is called the *Hausdorff measure of dimension* α. [Don't reprove Caratheodory.]
   (c) If $X = \mathbb{R}^d$, and $\alpha = d$ show that H_d is the Lebesgue measure.
   (d) Let $S \in \mathcal{B}(X)$. Show that there exists (a unique) $d \in [0, \infty]$ such that $H_\alpha(S) = \infty$ for all $\alpha \in (0, d)$, and $H_\alpha(S) = 0$ for all $\alpha \in (d, \infty)$. This number is called the *Hausdorff dimension* of the set S.
   (e) Compute the Hausdorff dimension of the Cantor set.

Details in class I left for you to check. (Do it, but don't turn it in.)

* We saw in class $\ell(I) = I$ for closed cells. Show it for arbitrary cells.
* Show that $m^*(a + E) = m^*(E)$ for all $a \in \mathbb{R}^d$, $E \subseteq \mathbb{R}^d$.
* Show that the arbitrary intersection of σ-algebras on X is also a σ-algebra.
* Verify that the counting measures and delta measures are measures.
* When proving Caratheodory, we proved in class Σ is a σ-algebra, and that $\mu^*|_\Sigma$ is *finitely* additive. Show that $\mu^*|_\Sigma$ is countably additive.

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b. Let $A_1, A_2, \ldots \in P(X)$ be disjoint. Then we claim:

$$\mu_*(\bigcup_{i=1}^\infty A_i) \geq \sum_{i=1}^\infty \mu^*(A_i)$$

First, let $U = \bigcup_{i=1}^\infty A_i$ for ease of reference. Then suppose $S_i \in \Sigma$, $S_i \subseteq A_i$ $\forall i \in \mathbb{N}$. First note that the S_i's must be disjoint - if $x \in S_i \cup S_j$, then $x \in A_i \cup A_j$. Then if $S = \bigcup_{i=1}^\infty S_i$, certainly $S \subseteq U$. Therefore $\mu_*(U) \geq \mu(S)$ by definition, but for μ we know $\mu(S) = \sum_{i=1}^\infty \mu(S_i)$, so that $\mu_*(U) \geq \sum_{i=1}^\infty \mu(S_i)$. But again, the S_i's were arbitrary, so taking the supremum, we havw $\mu_*(U) \geq \sum_{i=1}^\infty \mu_*(A_i)$. □ 3/3

c. Let $A \in P(X)$. Then we claim $\mu(X) = \mu^*(A) + \mu_*(A^C)$. First take $E \in \Sigma$, $A \subseteq E$. Then by definition, $E^C \in \Sigma$, and we also have $E^C \subseteq A^C$ by DeMorgan's laws. Therefore, we see that the condition $A \subseteq E$ is identical to $E^C \subseteq A^C$. Therefore write:

$$\begin{aligned} \mu^*(A) + \mu_*(A^C) &= inf\{\mu(E)|A \subseteq E,\ E \in \Sigma\} + sup\{\mu(E)|A^C \supseteq E,\ E \in \Sigma\} \\ &= inf\{\mu(E)|A \subseteq E,\ E \in \Sigma\} + sup\{\mu(E^C)|A \subseteq E,\ E \in \Sigma\} \end{aligned}$$

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If one of the measures is infinite, then we are safe, because in this case $\mu(X) = \infty$, and if they are both finite then $\mu(X) < \infty$ and we may safely (in a well definied way) write $\mu(E^C) = \mu(X) - \mu(E)$, in which case the last line becomes:

$$\begin{aligned} & inf\{\mu(E)|A \subseteq E,\ E \in \Sigma\} + sup\{\mu(X) - \mu(E)|A \subseteq E,\ E \in \Sigma\} \quad \text{nice!} \\ = \ & inf\{\mu(E)|A \subseteq E,\ E \in \Sigma\} + \mu(X) - inf\{\mu(E)|A \subseteq E,\ E \in \Sigma\} \\ = \ & \mu(X) \end{aligned}$$

d. Suppose $A \in P(X)$, and $\mu^*(A) < \infty$. Then we claim $A \in \Sigma_\mu$ if and only if $\mu^*(A) = \mu_*(A)$. First, suppose $A \in \Sigma_\mu$. Then, $A \subseteq A$, so by monotonicity, $\mu^*(A) = \mu(A)$. We may write the exact same line, with a different emphasis, and conclude $\mu_*(A) = \mu(A)$, so that $\mu^*(A) = \mu_*(A)$. Now suppose $A \notin \Sigma_\mu$. Then suppose $\mu_*(A) = \mu^*(A)$. Then there must be $B, C \in \Sigma_\mu$ such that $B \subseteq A \subseteq C$, and $\mu(C - B) = 0$. But Σ_μ is complete, so every subset of $C - B$ is measurable, yet the set $A - B \subseteq C - B$, and $A - B \cup B = A$, so that we conclude that A is measurable, but this contradicts our assumption that $A \notin \Sigma_\mu$, so that we must have $\mu_*(A) \neq \mu^*(A)$. Then by the contrapositive, we see that if $\mu_*(A) = \mu^*(A)$, $A \in \Sigma_\mu$.

~~not assumed~~

5/5

2. We will show that $\mathcal{L} = \mathcal{B}_\lambda$ by proving a stronger statement.

(a) Suppose $A \in \mathcal{L}(\mathbb{R}^d)$. We will show that for any $\varepsilon > 0$ there exists two sets C, U such that $C \subseteq A \subseteq U$, C is closed, U is open, and $\lambda(U - C) < \varepsilon$.
Suppose $\lambda(A) < \infty$. Fix $\varepsilon > 0$. By regularity of λ we choose U open such that $\lambda(U) \leq \lambda(A) + \varepsilon/4 < \infty$, and we choose C closed such that $\lambda(C) \geq \lambda(A) - \varepsilon/4$. Then $C \subseteq A \subseteq U$ and since $\lambda(U) < \infty$ we have

$$\lambda(U - C) = \lambda(U) - \lambda(C) \leq \varepsilon/2 < \varepsilon.$$

nice

Now suppose $\lambda(A) = \infty$. Fix $\varepsilon > 0$. We write $A = \bigcup_{i=1}^{\infty} A_k$ for disjoint bounded A_k's. This can be done since $\mathbb{R}^d$ is the countable union of half-open cells of measure 1. For each k choose $U_k \supseteq A_k$ open with $\lambda(U_k - A_k) < \varepsilon/2^k$. Then $U := \bigcup_{k=1}^{\infty} U_k$ is open and $U \supseteq A$. Then

$$\lambda(U - A) = \lambda\left(\bigcup_{k=1}^{\infty}(U_k - A)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty}(U_k - A_k)\right) \leq \sum_{k=1}^{\infty} \lambda(U_k - A_k) \leq \varepsilon.$$

By what we just showed we choose V open coverering A^c with $\lambda(V - A^c) \leq \varepsilon$. Then we note that

$$\lambda(V - A^c) = \lambda(V \cap A) = \lambda(A \cap V) = \lambda(A - V^c).$$

Since V is open, choosing $C = V^c$ suffices so that $C \leq A$ and $\lambda(A - C) \leq \varepsilon$. Claim: we have $U - C \subseteq (U - A) \cup (A - C)$. Proof: If $U - C = \emptyset$ we have nothing to show, otherwise let $x \in U - C$; if $x \in A$ then $x \in A - C$, if $x \notin A$ then $x \in U - A$. Hence by monotonicity we have

$$\lambda(U - C) \leq \lambda((U - A) \cup (A - C)) \leq \lambda(U - A) + \lambda(A - C) \leq 2\varepsilon.$$

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Since $\varepsilon > 0$ was arbitrary the conclusion follows. □

(b) Suppose $A \in \mathcal{L}(\mathbb{R}^d)$. We will show that there exists an F_σ, F, and a G_δ, G, such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$.
By part (a) we choose $(U_k)_k$ open and $(C_k)_k$ closed such that $C_k \subseteq A \subseteq U_k$ and $\lambda(U_k - C_k) < \frac{1}{k}$ for all k. Define

$$G := \bigcap_{k=1}^{\infty} \left(\cap_{i=1}^{k} U_i\right), \quad F := \bigcup_{k=1}^{\infty} \left(\cup_{i=1}^{k} C_i\right).$$

The finite intersection of open sets is open, and the finite union of closed sets is closed, so G is a G_δ and F is an F_σ. Furthermore, all the U_i contain A, so all the $\cap_{i=1}^{k}$ contain A, so G contains A. All the C_i are contained in A, so all the $\cup_{i=1}^{k}$ are contained in A, so F is contained in A. That is, $F \subseteq A \subseteq G$. Then we have

$$\begin{aligned}
\lambda(G - F) &= \lambda\left(\left(\bigcap_{k=1}^{\infty}\left(\cap_{i=1}^{k} U_i\right)\right) - \left(\bigcup_{k=1}^{\infty}\left(\cup_{i=1}^{k} C_i\right)\right)\right) \\
&= \lambda\left(\left(\bigcap_{k=1}^{\infty}\left(\cap_{i=1}^{k} U_i\right)\right) \cap \left(\bigcap_{k=1}^{\infty}\left(\cap_{i=1}^{k} C_i^c\right)\right)\right) \\
&= \lambda\left(\left(\bigcap_{k=1}^{\infty}\left(\cap_{i=1}^{k}(U_i \cap C_i^c)\right)\right)\right) \\
&= \lambda\left(\left(\bigcap_{k=1}^{\infty}\left(\cap_{i=1}^{k}(U_i - C_i)\right)\right)\right).
\end{aligned}$$

By construction $\left(\cap_{i=1}^{k}(U_i - C_i)\right)_k$ is a decreasing sequence of sets, the first of which has finite measure $\lambda(U_1 - C_1) < \frac{1}{1}$. Hence,

$$\begin{aligned}
&= \lambda\left(\left(\bigcap_{k=1}^{\infty}\left(\cap_{i=1}^{k}(U_i - C_i)\right)\right)\right) = \lim_{k\to\infty} \lambda\left(\cap_{i=1}^{k}(U_i - C_i)\right) \\
&\leq \limsup_{k\to\infty} \lambda\left(\cap_{i=1}^{k}(U_i - C_i)\right) \\
&\leq \limsup_{k\to\infty} \lambda\left(U_k - C_k\right) \\
&\leq \limsup_{k\to\infty} \frac{1}{k} \\
&= 0.
\end{aligned}$$

Hence we have shown

$$\lambda(G - F) = 0$$

as desired. □

Finally we will conclude that $\mathcal{B}_\lambda = \mathcal{L}$. Since F_σ and G_δ sets are Borel, the previous step shows that $\mathcal{L}(\mathbb{R}^d) \subseteq \mathcal{B}_\lambda(\mathbb{R}^d)$. For the other inclusion, suppose we have $B \in \mathcal{B}_\lambda(\mathbb{R}^d)$. That is, $A \in \mathcal{P}(\mathbb{R}^d)$ and we choose $E, F \in \mathcal{B}$ such that $E \subseteq A \subseteq F$ and $\lambda(F - E) = 0$. We claim that $A \in \mathcal{L}(\mathbb{R}^d)$. Indeed, since $E \subseteq A \subseteq F$ we have

$$A = E \cup (A - E).$$

We know E is measurable since it is Borel, and $A - E \subseteq F - E$ so that

$$\lambda^*(A - E) \leq \lambda^*(F - E) = 0$$

3/3

and we conclude that $A - E$ is measurable since it has outer measure zero. Hence A is the union of two measurable sets and therefore is measurable. It follows that $A \in \mathcal{L}$, and that $\mathcal{B}_\lambda = \mathcal{L}$. □

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3. Let $A \in \mathcal{L}(\mathbb{R}^d)$. We will show that every subset of A is Lebesgue measurable if and only if $\lambda(A) = 0$.

   ($\Longleftarrow$) Suppose that $\lambda(A) = 0$. Let $S \subseteq A$ be given. By monotonicity we have $\lambda^*(S) \leq \lambda^*(A) = 0$ so $\lambda^*(S) = 0$ and so S is measurable.

   ($\Longrightarrow$) We proceed by contrapositive, supposing $\lambda(A) > 0$, we will show there is a nonmeasurable subset of A. First we construct a nonmeasurable set in $\mathbb{R}^d$, closely following the construction in class, then we prove the proposition. We have that $(\mathbb{R}^d, +)$ is a group, and $(\mathbb{Q}^d, +) \lhd (\mathbb{R}^d, +)$ so we may consider the quotient group $\mathbb{R}^d/\mathbb{Q}^d$. By the Axiom of Choice we may choose $W \subseteq \mathbb{R}^d$ which contains exactly one representative from each coset. We claim that W is nonmeasurable. Suppose for the sake of contradiction that W is measurable. Since $\mathbb{Q}^d$ is countable we may write $\mathbb{Q}^d = (q_i)_i$, so that

$$\mathbb{R}^d = \bigcup_{i=1}^{\infty}(q_i + W).$$

We know that $q_i + W$ is measurable for all i also. By translation invariance and countable subadditivity we have

$$\infty = \lambda(\mathbb{R}^d) = \lambda\left(\bigcup_{i=1}^{\infty}(q_i + W).\right) \leq \lambda\left(\bigcup_{i=1}^{\infty}(q_i + W).\right) \leq \sum_{i=1}^{\infty}\lambda(q_i + W) = \sum_{i=1}^{\infty}\lambda(W)$$

Let $p \in \mathbb{Q}$ and $K \subset (p+E) \cap A$ compact be given. Take

$$B = \bigcup_{q \in \mathbb{Q}\cap[0,1]} (q+K)$$

By assumption, $K \subset A$ is measurable. Then $\lambda(B) = \sum_1^\infty \lambda(K)$. Since B is bounded, $\lambda(B)$ must be finite. This can only be true if $\lambda(K) = 0$, so this must be the case.

By regularity of $\mathbb{R}$, there is for every $\epsilon > 0$ a compact set $K \subset (p+E) \cap A$ such that $\lambda((p+E) \cap A) < \lambda(K) + \epsilon$. But every compact subset of $(p+E) \cap A$ has λ-measure zero, so $(p+E) \cap A$ must have measure zero. Now, we have that

$$A = A \cap \mathbb{R} = A \cap \bigcup_{p \in \mathbb{Q}} (p+E) = \bigcup_{p \in \mathbb{Q}} ((p+E) \cap A)$$

Hence $\lambda(A) = \sum_{p\in\mathbb{Q}} \lambda((p+E) \cap A) = \sum_{p\in\mathbb{Q}} 0 = 0$.

For the reverse direction, assume that $\lambda(A) = 0$. Since the Lebesgue measure is complete, every subset of A is measurable.

5/5

4. (a) First, we show that

$$\begin{aligned}\Sigma &:= \sigma(\{A \times B : A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\}) \\ &= \sigma(\{A \times B : A \in \mathbb{R}^m \text{ open} \,\&\, B \in \mathbb{R}^n \text{ open}\}) =: \Sigma'\end{aligned}$$

That the $\Sigma \supseteq \Sigma'$ is trivial. We show the opposite inclusion.

Let $\mathcal{A} \in \mathcal{P}(\mathbb{R}^m)$ be the set of $A \in \mathbb{R}^m$ such that $\{A \times B : B \in \mathbb{R}^n \text{ open}\} \subseteq \Sigma'$. Then

what is B?

(i) Let $A_1, A_2, \ldots \in \mathcal{A}$ be given. Let $B \in \mathbb{R}^n$ open be given. For all $n \geq 1$, we have $A_n \times B \subseteq \Sigma'$. Since Σ' is closed under countable unions and intersections, $(\bigcup_1^\infty A_n) \times B = \bigcup_1^\infty (A_n \times B) \in \Sigma'$ and $(\bigcap_1^\infty A_n) \times B = \bigcap_1^\infty (A_n \times B) \in \Sigma'$. Since B open was arbitrary, we conclude that $\bigcup_1^\infty A_n \in \mathcal{A}$ and $\bigcap_1^\infty A_n \in \mathcal{A}$. Thus $\mathcal{A}$ is closed under countable unions and intersections.

(ii) $\mathcal{A}$ trivially contains all open sets. Let a closed set C be given. Define the open sets $A_n = \{x \in \mathbb{R}^m : \operatorname{dist}(x, C) < \frac{1}{n}\}$; then $C = \bigcap_1^\infty A_n$. Since each A_n is in $\mathcal{A}$ and $\mathcal{A}$ is closed under countable intersections, we have $C \in \mathcal{A}$. Thus $\mathcal{A}$ contains all closed sets.

By Lemma 1, it follows that $\mathcal{A} \supseteq \mathcal{B}(\mathbb{R}^m)$. Thus

$$\begin{aligned}\Sigma'' &:= \sigma(\{A \times B : A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathbb{R}^n \text{ open}\}) \\ &\subseteq \sigma(\{A \times B : A \in \mathcal{A} \,\&\, B \in \mathbb{R}^n \text{ open}\}) \\ &\subseteq \Sigma'\end{aligned}$$

since $\sigma(\{A \times B : A \in \mathcal{A} \,\&\, B \in \mathbb{R}^n \text{ open}\})$ is the smallest σ-algebra containing $\{A \times B : A \in \mathcal{A} \,\&\, B \in \mathbb{R}^n \text{ open}\}$ and $\{A \times B : A \in \mathcal{A} \,\&\, B \in \mathbb{R}^n \text{ open}\} \subseteq \Sigma'$. Obviously $\Sigma'' \supseteq \Sigma'$, so $\Sigma'' = \Sigma'$.

Now, let $\mathcal{D} \in \mathcal{P}(\mathbb{R}^n)$ be the set of $B \in \mathbb{R}^n$ such that $\{A \times B : A \in \mathcal{B}(\mathbb{R}^m)\} \subseteq \Sigma'$. Then, analogously to the proof with $\mathcal{A}$,

(i) Let $B_1, B_2, \dots \in \mathcal{D}$ be given. Let $A \in \mathcal{B}(\mathbb{R}^m)$ be given. For all $n \geq 1$, we have $A \times B_n \subseteq \Sigma''$. Since Σ'' is closed under countable unions and intersections, $A \times (\bigcup_1^\infty B_n) = \bigcup_1^\infty (A \times B_n) \in \Sigma'$ and $A \times (\bigcap_1^\infty B_n) = \bigcap_1^\infty (A \times B_n) \in \Sigma'$. Since $A \in \mathcal{B}(\mathbb{R}^m)$ was arbitrary, we conclude that $\bigcup_1^\infty B_n \in \mathcal{D}$ and $\bigcap_1^\infty B_n \in \mathcal{D}$. Thus $\mathcal{D}$ is closed under countable unions and intersections.

(ii) $\mathcal{D}$ trivially contains all open sets. Let a closed set C be given. Define the open sets $B_n = \{x \in \mathbb{R}^n : \operatorname{dist}(x, C) < \frac{1}{n}\}$; then $C = \bigcap_1^\infty B_n$. Since each B_n is in $\mathcal{D}$ and $\mathcal{D}$ is closed under countable intersections, we have $C \in \mathcal{D}$. Thus $\mathcal{D}$ contains all closed sets.

Then $\mathcal{D} \supseteq \mathcal{B}(\mathbb{R}^n)$ by Lemma 1, so

$$\begin{aligned}
\Sigma &:= \sigma(\{A \times B : A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\}) \\
&\subseteq \sigma(\{A \times B : A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{D}\}) \\
&\subseteq \Sigma''
\end{aligned}$$

It then follows that $\Sigma = \Sigma''$, so $\Sigma = \Sigma'$. Now, we show that $\Sigma = \mathcal{B}(\mathbb{R}^{m+n})$. Since every element of $\{A \times B : A \in \mathbb{R}^m \text{ open} \,\&\, B \in \mathbb{R}^n \text{ open}\}$ is open in $\mathbb{R}^{m+n}$, we have $\Sigma = \Sigma' \subseteq \mathcal{B}(\mathbb{R}^{m+n})$.

We now show that the countable set $E := \{(a, b)^{m+n} : a, b \in \mathbb{Q}\}$ is a basis for the standard topology on $\mathbb{R}^{m+n}$. Let $U \subseteq \mathbb{R}^{m+n}$ and $x \in U$ be given. Then there exists a ball $B(x, r) \subseteq U$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ with $0 < q < \frac{r}{\sqrt{2}}$. Then $x \in (x - q, x + q)^{m+n} \subseteq B(x, r) \subseteq U$. Thus E is a basis. Since E is countable, this implies that every open set in $\mathbb{R}^{m+n}$ can be written as the countable union of elements of E. Now, since $E \subseteq \{A \times B : A \in \mathbb{R}^m \text{ open} \,\&\, B \in \mathbb{R}^n \text{ open}\}$, it follows that $\Sigma = \Sigma' = \sigma(\{A \times B : A \in \mathbb{R}^m \text{ open} \,\&\, B \in \mathbb{R}^n \text{ open}\})$, a σ-algebra containing E, contains all open sets in $\mathbb{R}^{m+n}$. From this we deduce that $\mathcal{B}(\mathbb{R}^{m+n}) \subseteq \Sigma$.

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Hence $\mathcal{B}(\mathbb{R}^{m+n}) = \Sigma = \sigma(\{A \times B : A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\})$.

(b) First, we show that $\sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\}) \subseteq \mathcal{L}(\mathbb{R}^{m+n})$. Let $A, B \in \mathcal{L}(\mathbb{R}^n)$ be given. By 2(b), there exist F_σ's F_A, F_B and G_δ's G_A, G_B such that $F_A \subseteq A \subseteq G_A$, $F_B \subseteq B \subseteq G_B$, and $\lambda(G_A - F_A) = \lambda(G_B - F_B) = 0$. Then $G_A \times G_B \subseteq (F_A \times F_B) \cup (G_A \times (G_B - F_B)) \cup ((G_A - F_A) \times G_B)$, so

$$\lambda(G_A \times G_B) \leq \lambda(F_A \times F_B) + \lambda(G_A \times (G_B - F_B)) + \lambda((G_A - F_A) \times G_B)$$

By Lemma 2, $\lambda(G_A \times (G_B - F_B)) \leq \lambda(G_A)\lambda(G_B - F_B) = \lambda(G_A)(0) = 0$ and $\lambda((G_A - F_A) \times G_B) \leq \lambda(G_A - F_A)\lambda(G_B) = (0)\lambda(G_B) = 0$. Hence $\lambda(G_A \times G_B) \leq \lambda(F_A \times F_B)$. Since $F_A \times F_B \subseteq A \times B \subseteq G_A \times G_B$, $(A \times B) - (F_A \times F_B)$ is a null set. Since $F_A \times F_B$ is measurable and the Lebesgue measure is complete, this implies that $A \times B = (F_A \times F_B) \cup ((A \times B) - (F_A \times F_B))$ is measurable. Thus $\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\} \subseteq \mathcal{L}(\mathbb{R}^{m+n})$, so $\sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\}) \subseteq \mathcal{L}(\mathbb{R}^{m+n})$.

Define $\pi : \mathcal{P}(\mathbb{R}^m \times \mathbb{R}^n) \to \mathbb{R}^n$ by $\pi(A) = \{(a, b) \in A : b = 0\}$. We show that $\pi(A)$ is λ-measurable for all $A \in \sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$.

(i) Clearly $\pi(A)$ is λ-measurable for every $A \in \{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\}$.

(ii) If $\pi(A)$ is λ-measurable, then $\pi(A^c) = \pi(A)^c$ is λ-measurable.

(iii) If $A_1, A_2, \ldots$ are such that $\pi(A_n)$ is λ-measurable for every $n \geq 1$, then $\pi(\bigcup_1^\infty A_n) = \bigcup_1^\infty \pi(A_n)$ is λ-measurable.

Then the set of $A \subseteq \mathbb{R}^{m+n}$ such that $\pi(A)$ is λ-measurable is a σ-algebra that contains $\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\}$, so it contains $\sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$.

By 3, there is a non-measurable set $N \subsetneq [0,1]^m$. Define $M := N \times \{0^n\}$. We have $M \subsetneq [0,1]^m \times \{0^n\}$, which is closed and therefore λ-measurable. By Lemma 2, $\lambda(M) \leq 1^m 0^n = 0$. Hence M is a null set, and since the Lebesgue measure is complete it follows that $M \in \mathcal{L}(\mathbb{R}^{m+n})$. On the other hand, $\pi(M) = N$ is not λ-measurable, so $M \notin \sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$. Hence

Good!

$$\mathcal{L}(\mathbb{R}^{m+n}) \supsetneq \sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$$

(c) Since $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathcal{R})$, we have

$$\sigma(\{A \times B : A \in \mathcal{B}(\mathbb{R}) \,\&\, B \in \mathcal{B}(\mathbb{R})\}) \subseteq \sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}) \,\&\, B \in \mathcal{L}(\mathbb{R})\})$$

Then by (a) and (b),

$$\begin{aligned}\mathcal{B}(\mathbb{R}^2) &= \sigma(\{A \times B : A \in \mathcal{B}(\mathbb{R}) \,\&\, B \in \mathcal{B}(\mathbb{R})\}) \\ &\subseteq \sigma(\{A \times B : A \in \mathcal{L}(\mathbb{R}) \,\&\, B \in \mathcal{L}(\mathbb{R})\}) \subsetneq \mathcal{L}(\mathbb{R}^2)\end{aligned}$$

by the result of part (a), $A \times B \in \mathcal{B}(\mathbb{R}^{m+n}) \subseteq \mathcal{L}(\mathbb{R}^{m+n})$. Otherwise, either $A \in \mathcal{N}_m$ or $B \in \mathcal{N}_n$; without loss of generality, suppose $A \in \mathcal{N}_m$ (the case $B \in \mathcal{N}_n$ is identical). Then, there is some $E \in \mathcal{B}(\mathbb{R}^m)$ such that $A \subseteq E$ and $\lambda_m(E) = 0$. But then, $\lambda_{m+n}(E \times B) = 0$, so that, since $A \times B \subseteq E \times B$, by the result of problem 3, $A \times B \in \mathcal{L}(\mathbb{R}^{m+n})$.

We now show that $\mathcal{L}(\mathbb{R}^{m+n}) \neq \sigma(\{A \times B | A \in \mathcal{L}(\mathbb{R}^m), B \in \mathcal{L}(\mathbb{R}^n)\})$. We showed in class the existence of a non-measurable set $E_1 \in \mathcal{P}(\mathbb{R}^m)$. Let $E_2 = E_1 \times \{(0, 0, \ldots, 0)\} \in \mathcal{P}(\mathbb{R}^{m+n})$. By the result of problem 3, $E_2 \in \mathcal{L}(\mathbb{R}^{m+n})$, since $\lambda_{m+n}(E_2) = 0$. (This follows from the result of part (a) of problem 2 on Assignment 1, since $E_2 \subseteq V$, for some subspace V of $\mathbb{R}^{m+n}$ with $\dim(V) \leq m < m + n$). However, $E_1 \notin \mathcal{L}(\mathbb{R}^m)$. ■

why isn't $E_1 \in LHS$ 3/5

(c) Since, $\mathcal{L}(\mathbb{R}) = \mathcal{B}_\lambda(\mathbb{R}) \supseteq \mathcal{B}(\mathbb{R})$, it follows from the results of parts (a) and (b) that

$$\mathcal{L}(\mathbb{R}^2) \supsetneq \sigma(\{A \times B | A, B \in \mathcal{L}(\mathbb{R})) \supseteq \sigma(\{A \times B | A, B \in \mathcal{B}(\mathbb{R})) = \mathcal{B}(\mathbb{R}^2). \quad ■$$

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Problem 5

This solution was inspired by Example 8 in Chapter 2 of Rudin's *Real and Complex Analysis*.

Since $\mathbb{Q}$ is countable, there is a bijection $f : \mathbb{N} \to \mathbb{Q}$. $\forall n \in \mathbb{N}$, let $R_i := (f(i) - \frac{1}{3^{i+1}}, f(i) + \frac{1}{3^{i+1}})$,

$$S_i := R_i \setminus \left(\bigcup_{k=i+1}^{\infty} R_k \right).$$

Then, $\forall i \in \mathbb{N}$, let $T_i = (f(i) - \frac{1}{2 \cdot 3^{i+1}}, f(i) + \frac{1}{2 \cdot 3^{i+1}})$ (noting that $\lambda(T_i) = \lambda(R_i)/2$), and let

$$V = \bigcup_{i=1}^{\infty} S_i \cap T_i.$$

~~Why is this~~ nonempty?

We claim that, for all $a < b$, $0 < \lambda(V \cap (a, b)) < b - a$. Note that, since any interval (a, b) has some $R_n \subseteq (a, b)$ such that since each $R_i \in \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(R)$, $b - a = \lambda(R_i) + \lambda(R_i^c \cap (a, b))$, it is sufficient to show that $0 < \lambda(R_i \cap V) < \lambda(R_i)$.

We first show that $0 < \lambda(R_i \cap V)$. Since, for any $i \in \mathbb{N}$, there must be some $k > i$ such that $f(k) \notin R_i$, $\bigcup_{k=i+1} R_k$ contains some interval with no points in S_i, and thus, by countable additivity of λ,

$$\lambda(R_i) - \sum_{k=i+1}^{\infty} \lambda(R_k) < \lambda(S_i).$$

Therefore, since each R_i is an interval, so that $\lambda(R_i) = 3^i$,

$$\frac{\lambda(R_i)}{2} = \lambda(R_i) \left(1 - \sum_{k=1}^{\infty} \frac{1}{3^i} \right) = \lambda(R_i) - \sum_{k=i+1}^{\infty} \frac{1}{3^k} \lambda(R_i) < \lambda(S_i). \tag{1}$$

Therefore, since $S_i, T_i \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
\lambda(S_i \cup T_i) &\leq \lambda(R_i) = \frac{1}{2}\lambda(R_i) + \frac{1}{2}\lambda(R_i) && \text{(by monotonicity } (S_i \cup T_i \subseteq R_i)\text{)} \\
&< \lambda(S_i) + \lambda(T_i) \quad \text{why this?} && \text{(by (1), since } \lambda(T_i) = \lambda(R_i)/2\text{)} \\
&= \lambda(S_i \backslash T_i) + \lambda(T_i \backslash S_i) + 2\lambda(S_i \cap T_i) \\
&= \lambda(S_i \cup T_i) + \lambda(S_i \cap T_i).
\end{aligned}$$

Subtracting $\lambda(S_i \cup T_i)$ then gives, by monotonicity, $0 < \lambda(S_i \cap T_i) \leq \lambda(R_i \cap V)$, as desired.

We now show that $\lambda(R_i \cap V) < \lambda(R_i)$. By countable additivity of λ and the definition of V,

$$\begin{aligned}
\lambda(R_i \cap V) &= \lambda\left(\bigcup_{k=i}^{\infty} R_i \cap S_k \cap T_k\right) && (R_i \cap S_k = \emptyset \text{ for } k < i) \\
&\leq \sum_{k=i}^{\infty} \lambda(R_i \cap S_k \cap T_k) \leq \sum_{k=i}^{\infty} \lambda(R_i \cap T_k) && \text{countable additivity, monotonicity of } \lambda \\
&< \sum_{k=i}^{\infty} \lambda(T_k) && \text{monotonicity, } \exists T_k \text{ with } T_k \cap V_n = \emptyset, \lambda(T_k) > 0 \\
&= \sum_{k=i}^{\infty} \frac{\lambda(R_k)}{2} = \sum_{k=i}^{\infty} \frac{\lambda(R_i)}{2^{k+1}} = \lambda(R_i). \quad \blacksquare
\end{aligned}$$

✓ 5/5

Problem 6

Let $\Sigma = \sigma(A)$, and define $\mu : \Sigma \to [0, \infty]$ by

$$\mu(A) = \inf\left\{\sum_{i=1}^{\infty} \mu_0(E_i) \,\middle|\, E_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^{\infty} E_i\right\}.$$

By the result of part (a) of Problem 4 on Assignment 1, μ is an outer measure on X. ✓

We first show that $\mu = \mu_0$ on $\mathcal{A}$, so that it is in fact an extension μ_0 to Σ. Note that, since $\mathcal{A}$ is closed under complements and finite unions, by De Morgan's laws, $\mathcal{A}$ is also closed under finite intersections. Suppose that $A \in \mathcal{A}$. Clearly, since $A \subseteq A$ and μ is an infimum, $\mu(A) \leq \mu_0(A)$. Suppose $A_1, A_2, \ldots \in \mathcal{A}$ with $A \in \bigcup_{i=1}^{\infty} A_i$. $\forall i \in \mathbb{N}$, let

$$B_i = A \cap A_i \cap \left(\bigcup_{k=1}^{i} A_k\right)^c,$$

so that the B_i's are pairwise disjoint, $A = \bigcup_{i=1}^{\infty} B_i$, and $B_i \in \mathcal{A}$. Thus,

$$\mu_0(A) = \mu_0\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu_0(B_i),$$

6. *(Carathéodory extension)* If $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra, and μ_0 is a pre-measure on $\mathcal{A}$, then there exists a measure μ defined on $\sigma(\mathcal{A})$ that extends μ_0.

Proof: We define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu^*(S) := \inf \left\{ \sum_{i=1}^{\infty} \mu_0(A_i) : A_i \in \mathcal{A}, S \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

We have $\emptyset, \mathcal{A} \in \mathcal{P}(X)$, and $\mu_0(\emptyset) = 0$ by definition, so by problem 4. (a) we know that μ^* is an outer measure. Fix $A \in \mathcal{A}$. Since $A \supseteq A$ we have $\mu^*(A) \leq \mu_0(A)$. Suppose $(A_i)_i$ in $\mathcal{A}$ are such that $A \subseteq \bigcup_{i=1}^\infty A_i$. Then for all i define

$$B_i := (A \cap A_i) \setminus \bigcup_{k=1}^{i-1} A_k \in \mathcal{A}.$$

The B_i's are disjoint by construction, and furthermore

$$\bigcup_{i=1}^\infty B_i = A \in \mathcal{A}$$

so we may use the *countable* additivity of μ_0 to find

$$\mu_0(A) = \sum_{i=1}^\infty \mu_0 \left((A \cap A_i) \setminus \bigcup_{k=1}^{i-1} A_k \right).$$

Then we note that if $S \subseteq T, S, T \in \mathcal{A}$ we have $\mu_0(T) = \mu_0(S) + \mu_0(T \setminus S) \geq \mu_0(S)$ so we have monotonicity. Then since $(A \cap A_i) \setminus \bigcup_{k=1}^{i-1} A_k \subseteq A_i$ and all sets involved are elements of $\mathcal{A}$, it follows that

$$\sum_{i=1}^\infty \mu_0 \left((A \cap A_i) \setminus \bigcup_{k=1}^{i-1} A_k \right) \leq \sum_{i=1}^\infty \mu_0 (A_i)$$

so we have shown

$$\mu_0(A) \leq \sum_{i=1}^\infty \mu_0 (A_i).$$

Taking the infimum over all covers gives $\mu_0(A) \leq \mu^*(A)$. Thus we have shown that $\mu_0(A) = \mu^*(A)$ for all $A \in \mathcal{A}$. That is, μ^* extends μ_0. We now apply the Carathéodory criterion and define $\mu := \mu^*|_\Sigma$ where Σ is the σ-algebra of all μ^*-measurable sets. Then μ is a measure on Σ which extends μ_0. All that is left to show is that $\Sigma \supseteq \sigma(\mathcal{A})$. It is enough to show that $\Sigma \supseteq \mathcal{A}$, i.e. that all elements of $\mathcal{A}$ are μ^* measurable. Fix $A \in \mathcal{A}$ and $S \subseteq X$. It suffices to show $\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \cap A^c)$. By definition of infimum we choose $(A_i)_i$ in $\mathcal{A}$ which cover S such that

$$\sum_{i=1}^\infty \mu_0(A_i) \leq \mu^*(S) + \varepsilon.$$

Note that $S \cap A \subseteq \bigcup_{i=1}^\infty A_i \cap A$ and $S \cap A^c \subseteq \bigcup_{i=1}^\infty A_i \cap A^c$. Hence, by monotonicity, we have

$$\mu^*(S \cap A) + \mu^*(S \cap A^c) \leq \sum_{i=1}^\infty \mu^*(A_i \cap A) + \sum_{i=1}^\infty \mu^*(A_i \cap A^c).$$

We have that $A_i \cap A \in \mathcal{A}$ and $A_i \cap A^c \in \mathcal{A}$ for all i, so that $\mu_0 = \mu^*$ on these quantityies. That is,

$$\sum_{i=1}^\infty \mu^*(A_i \cap A) + \sum_{i=1}^\infty \mu^*(A_i \cap A^c) = \sum_{i=1}^\infty \mu_0(A_i \cap A) + \sum_{i=1}^\infty \mu_0(A_i \cap A^c)$$

$$= \sum_{i=1}^\infty (\mu_0(A_i \cap A) + \mu_0(A_i \cap A^c))$$

$$= \sum_{i=1}^\infty \mu_0(A_i)$$

$$\leq \mu^*(S) + \varepsilon.$$

Letting $\varepsilon \to 0$ gives

$$\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \cap A^c)$$

so that we have shown A is μ^*-measurable. It follows that $\Sigma \supseteq \sigma(\mathcal{A})$ and so μ is a measure that is an extension of μ_0 which is defined on $\sigma(\mathcal{A})$. □

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