Homework 1

21-630 Ordinary Differential Equations
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Problem 1

If $p > 2$, then

$$f_n(t) = \frac{t^p}{1 + nt^2} = \frac{1}{t^{-p} + nt^{2-p}} \geq \frac{1}{nt^{2-p}}.$$

Since $2 - p < 0$, $\forall n \in \mathbb{N}$, $\exists t \in [0, \infty)$ such that $t^{2-p} < 1/n$, so that $f_n(t) > 1$, and thus f_n does not converge uniformly to 0.

If $p = 2$, then

$$f_n(t) = \frac{t^2}{1 + nt^2} = \frac{1}{t^{-2} + n} \leq \frac{1}{n},$$

so that f_n clearly converges uniformly to zero.

I wasn't able to show the case $0 < p < 2$.

Problem 2

Consider the infinite family of functions

$$\mathcal{F} := \left\{ X : \mathbb{R} \to \mathbb{R} \,\middle|\, X(t) = \begin{cases} 0 & \text{if } t \leq c \\ ((1-p)(t-c))^{\frac{1}{1-p}} & \text{if } c < t < \frac{1}{1-p} + c \\ t + 1 - \left(\frac{1}{1-p} + c\right) & \text{if } \frac{1}{1-p} + c \leq t \end{cases}, \text{ for some } c \in [0, \infty) \right\}.$$

Suppose $X \in \mathcal{F}$.
If $X(t) \leq 0$, then $t \leq c$, so $\frac{dX}{dt}(t) = 0$, as desired.
If $0 < X(t) < 1$, then $c < t < \frac{1}{1-p} + c$, so $\frac{dX}{dt}(t) = ((1-p)(t-c))^{\frac{p}{1-p}} = X^p$, as desired.
If $1 \leq X(t)$, $\frac{1}{1-p} + c \leq t$, so $\frac{dX}{dt}(t) = 1$, as desired.
Finally, $X(0) = 0$.

Thus, $\mathcal{F}$ is an infinite family of solutions to the given initial value problem. ■

Problem 3

Suppose, for sake of contradiction, that f satisfies a Lipschitz condition in x on D, so that, $\exists C > 0$ such that, $\forall (t, x), (t, y) \in D$,

$$|f(t, x) - f(t, y)| \leq C|x - y|.$$

Then, for $y = 0$, $x = e^{-(C+1)} \in (0, e^{-1}]$,

$$|x \ln(x)| = |f(t, x) - f(t, y)| \leq C|x - y| = C|x|,$$

implying $C + 1 = |\ln(x)| \leq C$, which is a contradiction. $\blacksquare$

By the given fact, $\forall \alpha \in (0, 1)$, $\exists C_{1-\alpha}$ such that, $\forall x \in (0, 1)$, $|\ln(x)| \leq C_{1-\alpha} x^{\alpha-1}$. Multiplying both sides by x gives $|x \ln(x)| \leq C_{1-\alpha} x^{\alpha}$. I wasn't able to get further in showing the Holder condition.

Problem 4

Suppose, for sake of contradiction, that f satisfies a Lipschitz condition in x on D, so that $\exists C > 0$ such that, $\forall (t, x), (t, y) \in D$,

$$|f(t, x) - f(t, y)| \leq C|x - y|.$$

Then, for $t = 1/C, x = t^2, y = 0$,

$$4/C = |4t| = |f(t, x) - f(t, y)| \leq C|x - y| = C|t^2| = 1/C,$$

which is impossible, since $C > 0$.

$\forall t \in [0, \infty)$, if $0 \leq x \leq y \leq t^2$, then,

$$|f(x) - f(y)| = 4\left|\frac{x-y}{t}\right| \leq 4\left|\frac{x-y}{\sqrt{xy}}\right| \leq 4\left|\sqrt{x} - \sqrt{y}\right| \leq C_1\sqrt{x-y}, \text{ for } C_1 = 4.$$

$\forall t \in [0, \infty)$, if $x \leq 0 \leq t^2 \leq y$, then $|f(t, x) - f(t, y)| = 4|t| \leq C_2\sqrt{x - y}$, for $C_2 = 4$.

$\forall t \in [0, \infty)$, if $x, y \leq 0$ or $t^2 \leq x, y$, then $|f(t, x) - f(t, y)| = 0 \leq C_3\sqrt{x - y}$, for $C_3 = 1$.

$\forall t \in [0, \infty)$, if $x \leq 0 \leq y \leq t^2$, then $|f(t, x) - f(t, y)| = 4|y/t| \leq 4\sqrt{y} \leq C_4\sqrt{y - x}$, for $C_4 = 4$.

$\forall t \in [0, \infty)$, if $0 \leq x \leq t^2 \leq y$, then $|f(t, x) - f(t, y)| = 4|t - x/t| \leq C_5\sqrt{y - x}$, for $C_5 = 4$.

It follows that f satisfies a Holder condition in x on D, with exponent $\alpha = 1/2$ and constant $C = \max\{C_1, C_2, C_3, C_4, C_5\} = 4$. $\blacksquare$

Problem 5

$\forall n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k \sin(e^{kx})}{k^k}, \forall x \in \mathbb{R}.$$

Since continuity is defined pointwise, it suffices to show that, $\forall B > 0$, f is continuous when restricted to $[-B, B]$. To do this, it suffices to show that the sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions converges uniformly to f when restricted to $[-B, B]$, $\forall B > 0$. $\forall m \in \mathbb{N}, x \in [-B, B]$,

$$\begin{aligned}
|f(x) - f_{n-1}(x)| &= \left| \sum_{k=m}^{\infty} \frac{x^k \sin(e^{kx})}{k^k} \right| \\
&\leq \sum_{k=m}^{\infty} \left| \frac{x^k \sin(e^{kx})}{k^k} \right| \leq \sum_{k=m}^{\infty} \left| \frac{B}{k} \right|^k \\
&\leq \sum_{k=m}^{\infty} \left| \frac{B}{B+1} \right|^k && \text{(assuming } m > |x|\text{, since we will take } m \to \infty\text{)} \\
&= \frac{\left| \frac{B}{B+1} \right|^m}{1 - \left| \frac{B}{B+1} \right|} \to 0, && \text{(geometric series)}
\end{aligned}$$

as $m \to \infty$, so that $\{f_n\}_{n=1}^{\infty}$ indeed converges uniformly to f. ■