21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
Spring 2012: Monday, Wednesday, Friday, 10:30 am, Doherty Hall 1211.
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5- Wednesday January 25, 2012.

Remark 5.1: The term *quadrature* is borrowed from French, and has its origin in the Latin word for square (quadratum), and the "quadrature of the circle" was the problem of constructing (with straightedge and compass) a square having the same area than a circle,[1] which was only shown to be impossible by the end of the 19th century, when LINDEMANN proved that π is transcendental, by improving an argument of HERMITE, who had shown that e is transcendental.

However, the term quadrature came to be used with the general meaning of computing an area, while computing a length was called rectification, and before the development of infinitesimal calculus around 1700, very few areas could actually be computed by (old or new) geometrical methods.

ARCHIMEDES had computed the area below a parabola, without having a Cartesian equation of a parabola, of course, and the generalization to curves of equation $y = c\,x^m$ is attributed to FERMAT. Among the debates which flourished in the 17th century, a few questions were asked about the *cycloid*: if a disc of radius R rolls on an horizontal line without slipping, a material point on the circumference moves along a cycloid, which is not an algebraic curve.[2] The name of the curve was coined by Galileo in 1599, but the curve had been studied before him by CUSA,[3] and by MERSENNE, who was unable to compute the area under the cycloid, and it was ROBERVAL who computed it in 1634,[4] showing that it is three times the area of its generating circle.[5] The area was also found ten years later by TORRICELLI,[6] apparently independently, despite a controversy by ROBERVAL. The length of one arch of a cycloid was computed in 1658 by WREN,[7] and is four times the diameter of its generating circle. With the tools of differential calculus, these results became easy to prove.

Remark 5.2: A quadrature formula consists in approximating an integral $\int_I f(x)w(x)\,dx$ by a finite sum $\sum_i f(a_i)\,w_i$ for some distinct points $a_i \in I$ and well chosen weights w_i, and there is a well developed one-dimensional theory, i.e. when I is an interval $\subset \mathbb{R}$, and for a reason related to estimates of the error made, one wants the formula to be exact for polynomials in $\mathcal{P}_k[x]$ for some k as large as possible. As shown by Lemma 5.3, this question is then intimately related with the question of interpolation polynomials. For the interval $I = (-1,+1)$ with $w = 1$, classical quadrature formulas approximate $\int_{-1}^{+1} f(x)\,dx$ by $f(-1)+f(+1)$, called the trapezoidal rule, or $\frac{f(-1)+4f(0)+f(+1)}{3}$, called Simpson's rule,[8] although SIMPSON learned it from

[1] One should say a disc instead of a circle. The disc $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$ has area π, and its boundary is the circle $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$, whose length is 2π. In three dimension, the ball $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\}$ has volume $\frac{4\pi}{3}$, and its boundary is the sphere $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$, whose area is 4π.

[2] It is parametrized by $x = R\,(t - \cos t), y = R\,(1 - \sin t)$.

[3] Nicholas of CUSA (Nicholas KRYFFS or KREBS, or Nikolaus CUSANUS), German-born mathematician, 1401–1464.

[4] Gilles PERSONNE de Roberval, French mathematician, 1602–1675. He held a chair at Collège de France in Paris (mathématiques, 1634–1675). He invented the Roberval balance in 1669.

[5] Galileo had cut a cycloid and a disc out of a sheet of metal, and he had found the ratio of their weights to be near 3, but he did not think that the ratio of areas could be exactly 3.

[6] Evangelista TORRICELLI, Italian mathematician and physicist, 1608–1647. He worked in Firenze (Florence), Italy. He built the first mercury barometer around 1644, and the torr is a unit of pressure named after him, corresponding to the pressure of 1 millimeter of mercury, 1760 of a standard atmospheric pressure.

[7] Sir Christopher Michael WREN, English architect, astronomer, and mathematician, 1632–1723. He taught astronomy at Gresham College, London, and then became Savilian Professor of Astronomy (1661–1672) at Oxford, England, and after that was an architect.

[8] Thomas SIMPSON, English mathematician, 1710–1761. He worked at the Royal Military Academy in Woolwich, England. He learned what one calls Simpson's rule from NEWTON, but he is the author of the improvement of the Newton–Raphson method which one uses now.

Newton, or by the so-called Newton–Cotes formulas,[9] which correspond to Lemma 5.3 for equidistant points a_i.

Lemma 5.3: One uses $E = \mathbb{R}$, and distinct points $a_1, \ldots, a_n \in [\alpha, \beta] \subset \mathbb{R}$. Then, for any continuous real function w on $[\alpha, \beta]$, there exist unique weights $w_1, \ldots, w_n$ such that $\int_\alpha^\beta P\, w\, dx = \sum_{i=1}^n w_i P(a_i)$ for all $P \in \mathcal{P}_{n-1}[x]$.
Proof: Let $P_1, \ldots, P_n$ be the interpolation polynomials at the points $a_1, \ldots, a_n$, i.e. $P_i(a_j) = \delta_{i,j}$ for $i, j = 1, \ldots, n$, then if the formula is exact on $\mathcal{P}_{n-1}[x]$, one must have $\int_\alpha^\beta P_i w\, dx = w_i$ for $i = 1, \ldots, n$; one then uses the fact that for all $P \in \mathcal{P}_{n-1}[x]$ one has $P = \sum_{i=1}^n P(a_i)\, P_i$, so that $\int_\alpha^\beta P\, w\, dx = \sum_{i=1}^n P(a_i) \int_\alpha^\beta P_i w\, dx = \sum_{i=1}^n w_i P(a_i)$.

Remark 5.4: Using interpolation polynomials can be done for any field E, and the reason why one has assumed that $E = \mathbb{R}$ in Lemma 5.3 is related to the definition of the integral.

One may consider a continuous function w from $[0, 1]$ into $\mathbb{R}$, and such that for every rational $q \in [0, 1] \cap \mathbb{Q}$ one has $w(q) \in \mathbb{Q}$, for example $w(x) = 1 - x^2$ for $x \in [0, \sqrt{a}]$ and $w(x) = 1 - a$ for $x \in [\sqrt{a}, 1]$, for a choice $a \in \mathbb{Q}$ with $a \in (0, 1)$; one finds that $\int_0^1 w(x)\, dx = \big(x - \frac{x^3}{3}\big)\big|_0^{\sqrt{a}} + (1 - \sqrt{a})\,(1 - a) = \frac{2a\,\sqrt{a}}{3} + 1 - a$, so that if $\sqrt{a} \not\in \mathbb{Q}$ the integral does not belong to $\mathbb{Q}$. The reason is that an integral is defined as a limit, for example the limit as n tends to ∞ of $\frac{1}{n} \sum_{k=1}^n w\big(\frac{k}{n}\big)$, but since $\mathbb{Q}$ is not a complete metric space, the limit of a sequence of rationals may be irrational.

Remark 5.5: If Gauss wondered where he should take the points $a_1, \ldots, a_n$ for the quadrature formula to be exact for polynomials in $\mathcal{P}_k[x]$ for k as large as possible, it was because he had to compute integrals by hand, and he wanted to devise efficient algorithms for avoiding to spend too much time in such computations; the reason why he had to compute integrals was that he worked as an astronomer in Göttingen, and he had to compute trajectories of planets, and he needed some integrals of elliptic functions, which were probably not yet tabulated at the time.[10]

The reason why it is useful to have a formula exact for polynomials of degree $\le k$ for k as large as possible is related to the error estimate in the case $w = 1$: assuming that $\int_0^1 P(x)\, dx = \sum_{i=1}^n w_i P(a_i)$ for all polynomials P of degree $\le k$, one can show that there exists a constant C_* such that $\big|\int_0^1 f(x)\, dx - \sum_{i=1}^n w_i f(a_i)\big| \le C_* \max_{x \in (0,1)} |f^{(k+1)}(x)|$ for all smooth functions f (at least of class C^{k+1}, i.e. with their first $k+1$ derivatives continuous). Then, one transports the quadrature formula on any interval (α, β) by an affine transformation, and one approximates $\int_\alpha^\beta g(x)\, dx$ by $(\beta - \alpha) \sum_{i=1}^n w_i g\big(\alpha + a_i(\beta - \alpha)\big)$, and using f defined by $f(x) = g\big(\alpha + x\,(\beta - \alpha)\big)$, one deduces that $\big|\int_\alpha^\beta g(x)\, dx - (\beta - \alpha) \sum_{i=1}^n w_i g\big(\alpha + a_i(\beta - \alpha)\big)\big| \le C_* |\beta - \alpha|^{k+2} \max_{x \in (\alpha,\beta)} |g^{(k+1)}(x)|$ for all smooth functions g at least of class C^{k+1}. By decomposing $[0, 1]$ into the union of m intervals of size $\frac{1}{m}$ and using the quadrature formula on each subinterval, the total error is the bounded by $\frac{C_*}{m^{k+1}} \max_{x \in (0,1)} |g^{(k+1)}(x)|$, hence the importance to have k as large as possible.

Remark 5.6: For finding the position of the n Gauss points, which give a formula exact on $\mathcal{P}_{2n-1}[x]$, we shall assume that $w(x) \ge 0$ for $x \in (\alpha, \beta)$, and $w \ne 0$ (i.e. w is not identically 0), and we shall use an Euclidean structure on polynomials, defined by $(f, g) = \int_\alpha^\beta f(x)\, g(x)\, w(x)\, dx$, and it will be crucial that $(f, f) > 0$ for every polynomial $f \ne 0$, and indeed $f^2 w \ge 0$ gives an integral ≥ 0, and if it was 0 one would deduce that $f(x)^2 w(x) = 0$ for all $x \in (\alpha, \beta)$, and because f vanishes only at a finite number of points, one deduces that w is 0 except possibly at these points, but since w is continuous, it is also 0 at these points, contrary to our assumption that $w \ne 0$.

Additional footnotes: Plume.[11]

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[9] Roger Cotes, English mathematician, 1682–1716. He was the first Plumian Professor of Astronomy and Experimental Philosophy at Cambridge, England.

[10] The tables of logarithms at his time may have been more precise than those first published by Briggs in 1617, following the suggestion of Napier, who had just died, but was duly given credit. I do not know who first published tables for trigonometric functions.

[11] Thomas Plume, English churchman and philanthropist, 1630–1704. He founded the chair of Plumian professor of astronomy and experimental philosophy in 1704 in Cambridge, England.