Department of Mathematical Sciences
Carnegie Mellon University

21-740 **Functional Analysis II** **Spring 2013**

Assignment 2

Due on Wednesday, October 9

* Please hand solutions to problems marked with an asterisk.

Throughout these problems X is a complex Hilbert space.

Definition: Let $A \in \mathcal{L}(X;X)$ be given. The set of all generalized eigenvalues of A is called the *approximate point spectrum* of A and is denoted by $\sigma_{ap}(A)$.

1.* Let $\{A_n\}_{n=1}^\infty$ be a sequence in $\mathcal{L}(X;X)$ such that $A_m A_n = A_n A_m$ for all $m, n \in \mathbb{N}$. Assume that

$$A_n^* = A_n, \quad A_n \leq A_{n+1} \leq I \text{ for all } n \in \mathbb{N}.$$

Show that there exists $L \in \mathcal{L}(X;X)$ with $L^* = L$ such $A_n \to L$ in the strong operator topology as $n \to \infty$. (Suggestion: Put $T_n = I - A_n$ and for $m < n$, look at $(T_m - T_n)T_m$ and $(T_m - T_n)T_n$. Then look at $\|T_m x - T_n x\|^2$.)

2.* Let $A, C \in \mathcal{L}(X;X)$ be given. Assume that $A^* = A$ and that $A \geq 0$. Show that there is exactly one $B \in \mathcal{L}(X;X)$ such that $B^* = B$, $B \geq 0$ and $B^2 = A$. Moreover, if $AC = CA$ then $BC = CB$. (Suggestion: Assume first that $A \leq I$. Put $B_0 = 0$ and $B_{n+1} = B_n + \dfrac{1}{2}(A - B_n^2)$ for $n = 0, 1, 2, \cdots$. Then apply your result to the operator $A/\|A\|$. It may prove useful to let $D = I - A$ and $F_n = I - B_n$.)

3. Let $T \in \mathcal{C}(X;X)$ be given. Show that there exist $A \in \mathcal{C}(X;X)$ and $U \in \mathcal{L}(X;X)$ such that $A^* = A$, $A \geq 0$, U is unitary and $T = AU = UA$.

4.* Prove or Disprove: Let $\alpha \in (-1, 1)$, and a sequence $\{x_n\}_{n=1}^\infty$ in X be given. Assume that

$$\forall m, n \in \mathbb{N}, \quad (x_m, x_n) = \begin{cases} 1 & \text{if } m = n \\ \alpha & \text{if } n \neq n. \end{cases}$$

Then $\{x_n\}_{n=1}^\infty$ is weakly convergent

5. Assume that $\mathbb{K} = \mathbb{C}$ and let $R, L \in \mathcal{L}(l^2; l^2)$ be the right and left shift operators, respectively. Find $\sigma(R)$ and $\sigma(L)$.

6.* Assume that $\mathbb{K} = \mathbb{C}$ and consider the operator $A \in \mathcal{L}(l^2; l^2)$ defined by

$$Ax = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots \frac{x_k}{k} \cdots\right) \quad \text{for all } x \in X.$$

Put

$$L(\lambda) = A - \lambda I \quad \text{for all } \lambda \in \mathbb{R}.$$

Describe explicitly the subspaces $\mathcal{N}(L(\lambda)^+)$ for $\lambda \in \mathbb{R}$.

7. Prove or Disprove: Let $A, B \in \mathcal{L}(X; X)$ be given. Assume that $A - B$ is compact and let $\lambda \in \sigma(A) \backslash \sigma_p(A)$ be given. Then $\lambda \in \sigma(B)$.

8.* Let $\mathbb{K} = \mathbb{C}$ and $R \in \mathcal{L}(l^2; l^2)$ be the right shift operator. Does there exist a compact operator $T \in \mathcal{C}(X; X)$ such that $R + T$ is normal? Explain.

9. Let $A, B \in \mathcal{L}(X; X)$ be given. What is the relationship between $\sigma(AB)$ and $\sigma(BA)$? Explain.

10. Prove or Disprove: Let $A \in \mathcal{L}(X; X)$ be given. Then $\sigma_{ap}(A)$ is closed.

11.* Prove or Disprove: Let $A \in \mathcal{L}(X; X)$ be given. Then

$$\text{bdry}(\sigma(A) \subset \sigma_{ap}(A).$$