21-484 Notes

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<u>Thm:</u> $\forall k, \ell \,.\, \exists G$ such that $\text{girth}(G) > \ell$ and $\chi(G) > k$.

<u>Tools:</u> 1. Markov's inequality

If X is a nonnegative random variable with expectation, then

$$\Pr[X > a] \leq \frac{\mathbb{E}[X]}{a}$$

<u>Proof:</u>

$$\begin{aligned} \mathbb{E}[X] &:= \sum_{x=0}^{\infty} x \cdot \Pr[X = x] = \sum_{0 \leq x \leq a} x \cdot \Pr[X = x] + \sum_{x > a} x \cdot \Pr[X = x] \geq \\ &\geq 0 + a \sum_{x>a} \Pr[X = x] = a \cdot \Pr[X > a] \end{aligned}$$

2. Stirling's Approximation

$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(1 + O(\frac{1}{n}))$

$n! \geq \left(\frac{n}{e}\right)^n$

$\rightarrow$ Set $0 < \theta < 1/\ell$

$\rightarrow$ Set $p = n^{-1+\theta} = \frac{n^\theta}{n}$

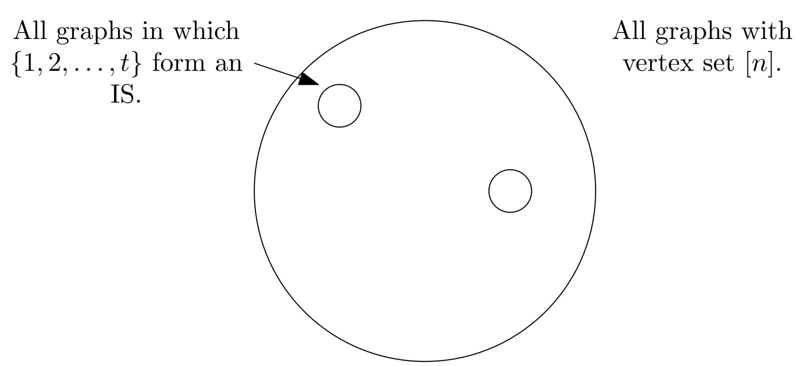
$\rightarrow$ Consider a graph G with n vertices in which every edge is in the graph with probability p.

$\rightarrow$ X = number of cycles of length $\leq \ell$ in the random graph.

$\rightarrow$ $\mathbb{E}[X] = \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} \cdot p^i \leq \cdots \leq \frac{n}{\log n}$

$\rightarrow$ Apply Markov's inequality $\Pr[X \geq n/2] \leq \dfrac{\frac{n}{\log n}}{\frac{n}{2}} = \dfrac{2}{\log n} \overset{n\to\infty}{\longrightarrow} 0$

$\rightarrow$ <u>Def:</u> $t = \left\lceil \frac{3 \ln n}{p} \right\rceil \approx \frac{n^{1-\theta}}{3 \ln n}$



$\rightarrow$

$$\begin{aligned} &\Pr[\alpha(G) \geq t] \leq \binom{n}{t}(1-p)^{\binom{t}{2}} \leq \left(\frac{ne}{t}\right)^t e^{-p\binom{t}{2}} = \\ &\left(\frac{en}{t}\right)^t e^{-pt(t-1)\frac{1}{2}} = \left(\frac{en}{t} e^{-\frac{1}{2}p(t-1)}\right)^t \leq ene^{-\frac{1}{2}\,3\ln n} \,. \\ &\leq ene^{-1.4\ln n} = enn^{-1.4} = en^{-0.4} \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

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$\to$ The probability that $X \geq n/2$ or $\alpha(G) \geq t$ is tending to 0 when $n \to \infty$.

$\to$ There is a graph G such that $X < n/2$ and $\alpha(G) < t$.

$\to$ Delete one vertex from every short cycle. Let G' be the graph spanned on the remaining vertices.

$\to$ $|V(G')| \geq n/2$

$\to$ G' contains no cycles of length $\leq \ell$.

$\to$ $\alpha(G') \leq \alpha(G) < t \Rightarrow \chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{t} = \frac{\frac{n}{2}}{\frac{3\ln n}{p}} = \frac{np}{6\ln n} = \frac{n^\theta}{6\ln n} \underset{\substack{\uparrow \\ n \text{ large} \\ \text{enough}}}{\geq} k$ ∎