Lecture Notes for Week 10 (First Draft)

Proof of the Hille-Yosida Theorem (Sufficiency Continued)

We assume that (i) and (ii) of Theorem 9.8 hold.

Last time we showed that

$$\forall x \in X, \quad \lambda R(\lambda; A)x \to x \quad \text{as } \lambda \to \infty. \tag{1}$$

This suggests a nice way of approximating A by bounded linear operators. For (real) $\lambda > \omega$ put

$$\begin{aligned} A_\lambda x &= \lambda A R(\lambda; A)x \\ &= \lambda^2 R(\lambda; A)x - \lambda x \quad \text{for all } x \in X. \end{aligned} \tag{2}$$

Observe that $A_\lambda \in \mathcal{L}(X; X)$ for all $\lambda > \omega$.

The operators A_λ are called the *Yosida approximates* of A. Using (1) and (2) the fact that $AR(\lambda; A)x = R(\lambda; A)Ax$ for $x \in \mathcal{D}(A)$ we see that

$$\forall x \in \mathcal{D}(A), \quad A_\lambda x \to Ax \quad \text{as } \lambda \to \infty. \tag{3}$$

In order to complete the proof, we need some basic results concerning semigroups generated by bounded linear operators. These will be stated in the form of a claim. Proof of the claim will be a homework exercise.

Claim: Let $B \in \mathcal{L}(X; X)$ be given. For each $t \geq 0$, put

$$e^{tB} = \sum_{n=0}^{\infty} \frac{(tB)^n}{n!}.$$

[The above definition is also valid for negative values of t as well.] Then we have

(i) The mapping $t \to e^{tB}$ is a linear C_0-semigroup with infinitesimal generator B.

(ii) If $C \in \mathcal{L}(X; X)$ and $CB = BC$ then $Ce^{tB} = e^{tB}C$ for all $t \geq 0$.

(iii) $\lim_{t \downarrow 0} \|e^{tB} - I\| = 0$.

(iv) For all $\lambda \in \mathbb{K}$ we have

$$e^{t(B-\lambda I)} = e^{-\lambda t} e^{tB} \quad \text{for all } t \geq 0.$$

Using (iv) of the claim and (2) we see that for all $\lambda > \omega$ we have

$$e^{tA_\lambda} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^{2n} t^n R(\lambda; A)^n}{n!} \quad \text{for all } t \geq 0. \tag{4}$$

Using (ii) from Theorem 9.8 and (4), we find that for all $\lambda > \omega$

$$\begin{aligned} \|e^{tA_\lambda}\| &\leq Me^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda^{2n} t^n}{(\lambda - \omega) n!} \quad \text{for all } t \geq 0, \\ &\leq Me^{-\lambda t} \exp\left(\frac{\lambda^2 t}{\lambda - \omega}\right) \quad \text{for all } t \geq 0, \qquad (5) \\ &\leq M \exp\left(\frac{\lambda \omega t}{\lambda - \omega}\right) \quad \text{for all } t \geq 0. \end{aligned}$$

Let $\omega_1 > \omega$ be given. Then, in view of (5) we may choose $\lambda_1 > \omega$ such that

$$\|e^{tA_\lambda}\| \leq Me^{\omega_1 t} \quad \text{for all } t \geq 0,\ \lambda > \lambda_1. \tag{6}$$

Put

$$T_\lambda(t) = e^{tA_\lambda} \quad \text{for all } \lambda > \omega,\ t \geq 0.$$

Observe that

$$A_\mu A_\lambda = A_\lambda A_\mu, \quad A_\lambda T_\mu(t) = T(t) A_\mu \quad \text{for all } \lambda, \mu > \omega,\ t \geq 0. \tag{7}$$

Let $x \in \mathcal{D}(A)$ and $\lambda, \mu > \omega$ be given. Then we have

$$\begin{aligned} T_\lambda(t)x - T_\mu(t)x &= \int_0^t \frac{d}{ds}(T_\mu(t-s) T_\lambda(s) x)\, ds \\ &= \int_0^t T_\mu(t-s) A_\lambda T_\lambda(s) - T_\mu(t-s) A_\mu T_\lambda(s) x\, ds \qquad (8) \\ &= \int_0^t (T_\mu(t-s) T_\lambda(s))(A_\lambda x - A_\mu x)\, ds. \end{aligned}$$

Using (6) and (8) we find that

$$\begin{aligned} \|T_\lambda(t)x - T_\mu(t)x\| &\leq M^2 e^{\omega_1 t} t \|A_\lambda x - A_\mu x\| \\ &\leq t M^2 e^{\omega_1 t} (\|A_\lambda x - Ax\| + \|A_\mu x - Ax\|) \quad \text{for all } \lambda, \mu > \lambda_1. \end{aligned} \tag{9}$$

For $x \in \mathcal{D}(A)$, it follows from (9) that $\{T_\lambda(t)x\}_{\lambda > \lambda_1}$ has the Cauchy property in λ, uniformly for t in bounded sets. Using (6), (9), and the fact that $\mathcal{D}(A)$ is dense in

X, we see that for all $x \in X$, $\{T_\lambda(t)x\}_{\lambda>\lambda_1}$ has the Cauchy property in λ, uniformly for t in bounded sets. Now define

$$T(t)x = \lim_{\lambda\to\infty} T_\lambda(t)x \ \text{ for all } x \in X,\ t \geq 0.$$

It is immediate that

- $T(0) = I$,
- $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$,
- $\|T(t)\| \leq Me^{\omega_1 t}$ for all $t \geq 0$.

Since the last inequality holds (with the same M) for every $\omega_1 > \omega$ we conclude that

- $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

Since the convergence of $T_\lambda(t)x$ to $T(t)x$ is uniform for t in bounded sets, we conclude that for every $x \in X$ the mapping $t \to T(t)x$ is continuous. It follows that T is a linear C_0-semigroup.

It remains to show that the infinitesimal generator is A. For this purpose, let us denote the infinitesimal generator of T by $\hat{A}$. We shall first show that $\hat{A}$ is an extension of A and then use a resolvent argument to show that $\mathcal{D}(\hat{A}) = \mathcal{D}(A)$.

Let $x \in \mathcal{D}(A)$ be given and observe that

$$\begin{aligned}\|T_\lambda(t)A_\lambda x - T(t)Ax\| &\leq \|T_\lambda(t)(A_\lambda x - Ax)\| + \|(T_\lambda(t) - T(t)Ax\| \\ &\leq Me^{\omega_1 t}\|A_\lambda x - Ax\| + \|(T_\lambda(t) - T(t))Ax\| \qquad (10) \\ &\to 0 \ \text{ as } \lambda \to \infty.\end{aligned}$$

Since the convergence of $T_\lambda(t)A_\lambda x$ to $T(t)Ax$ is uniform for t in bounded intervals we have

$$\begin{aligned}T(t)x - x &= \lim_{\lambda\to\infty} T_\lambda(t)x - x \\ &= \lim_{\lambda\to\infty} \int_0^t T_\lambda(s)A_\lambda x\, ds = \int_0^t T(s)Ax\, ds.\end{aligned} \qquad (11)$$

For $h > 0$ we have

$$\frac{T(h)x - x}{h} = \frac{1}{h}\int_0^h T(s)Ax\, ds \to Ax \ \text{ as } h \downarrow 0.$$

We conclude that $x \in \mathcal{D}(\hat{A})$ and $\hat{A}x = Ax$. In other words, $\hat{A}$ is an extension of A. In order to complete the proof, it suffices to show that $\mathcal{D}(\hat{A}) \subset \mathcal{D}(A)$.

Since $\hat{A}$ is an infinitesimal generator, it is a closed operator. Moreover, A is closed by assumption. Recall that for a closed linear operator $C : \mathcal{D}(C) \to X$ and $\lambda \in \rho(C)$ the operator $\lambda I - C$ is surjective, i.e. $(\lambda I - C)[\mathcal{D}(C)] = X$. By Lemma 9.6, $\rho(\hat{A}) \supset (\omega, \infty)$ and by assumption we have $\rho(A) \supset (\omega, \infty)$. Therefore we may choose $\lambda \in \rho(A) \cap \rho(\hat{A})$.

Since $\hat{A}$ and A are closed and $\lambda \in \rho(A) \cap \rho(\hat{A})$

$$(\lambda I - \hat{A})[\mathcal{D}(\hat{A})] = X,$$

$$(\lambda I - A)\mathcal{D}(A)] = X,$$

and since $\hat{A}$ extends A, we also have

$$(\lambda I - \hat{A})[\mathcal{D}(A)] = X.$$

It follows that

$$\mathcal{D}(A) = R(\lambda; \hat{A})[X] = \mathcal{D}(\hat{A}). \quad \square$$

Corollary 10.1: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense in X and that $A : \mathcal{D}(A) \to X$ is linear and closed. Then A generates a linear C_0 contraction semigroup. if and only if $\rho(A) \supset (0, \infty)$ and

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda} \quad \text{for all } \lambda > 0.$$

Remark 10.2: *Historical Comments* TO BE FILLED IN

Contraction Semigroups

Assume that $T : [0, \infty) \to \mathcal{L}(X; X)$ is a linear C_0 contraction semigroup with infinitesimal generator A. Then for all $t, h \geq 0$ and all $x \in X$ we have

$$\|T(t+h)x\| \leq \|T(h)\| \cdot \|T(t)x\| \leq \|T(t)x\|$$

and

$$\|T(t+h)\| \leq \|T(h)\| \cdot \|T(t)\| \leq \|T(t)\|.$$

In other words, the mappings $t \to \|T(t)x\|$ and $t \to \|T(t)\|$ both are nonincreasing on $[0, \infty)$.

Suppose that X is a Hilbert space. Let $x \in \mathcal{D}(A)$ be given and put

$$u(t) = \|T(t)x\|^2 = (T(t)x, T(t)x) \quad \text{for all } t \geq 0.$$

Then u is nonincreasing and differentiable, so we have

$$\dot{u}(t) = (T(t)x, T(t)Ax) + (T(t)Ax, T(t)x) = 2\mathrm{Re}(T(t)Ax, x) \leq 0 \quad \text{for all } t \geq 0.$$

Putting $t = 0$ we find that

$$\operatorname{Re}(Ax, x) \leq 0 \quad \text{for all } x \in \mathcal{D}(A).$$

We will prove the following remarkable result: If X is a Hilbert space, $\mathcal{D}(A) \subset X$, and $A : \mathcal{D}(A) \to X$ is linear then A generates a linear C_0 contraction C_0-semigroup if and only if (i) and (ii) below hold:

(i) $\operatorname{Re}(Ax, x) \leq 0$ for all $x \in \mathcal{D}(A)$,

(ii) There exists $\lambda_0 > 0$ such that $\lambda_0 I - A$ is surjective.

[Density of $\mathcal{D}(A)$, closedness of A, and the required resolvent properties of A all follow from (i) and (ii) above!]

We shall actually prove some extensions of this type of result for semigroups on Banach spaces. For this purpose we shall introduce the notion of semi-inner product on a Banach space. We shall present only those concepts that we need to prove a few basic results concerning contraction semigroups.

Semi-Inner Products on Banach Spaces

Definition 10.3: Let X be a linear space over $\mathbb{K}$. By a *semi-inner product* on X, we mean a mapping $[\cdot, \cdot] : X \times X \to \mathbb{K}$ such that

(i) $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in X$,

(ii) $[\alpha x, y] = \alpha[x, y]$ for all $x, y \in X$, $\alpha \in \mathbb{K}$,

(iii) $[x, x] > 0$ for all $x \in X \backslash \{0\}$,

(iv) $|[x, y]|^2 \leq [x, x] \cdot [y, y]$ for all $x, y \in X$.

Observe that every inner product is a semi-inner product. It follows easily from Definition 10.3 that if $[\cdot, \cdot]$ is a semi-inner product on X then the function $x \to \sqrt{[x, x]}$ is a norm on X. It turns out that every norm on X is induced by a semi-inner product. This, of course, means that there are semi-inner products that are not inner products. The notion of semi-inner product was introduced by Lumer in 1961 and developed further by Giles in 1967. Giles showed that one can include the homogoneity property $[x, \alpha y] = \overline{\alpha}[x, y]$ for all $x, y \in X$, $\alpha \in \mathbb{K}$ to the definition of semi-inner product without introducing any significant complications.

Definition 10.4: Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot, \cdot]$ be a semi-inner product on X. We say that $[\cdot, \cdot]$ is *compatible* with $\|\cdot\|$ provided that $[x, x] = \|x\|^2$ for all $x \in X$.

Proposition 10.5: Let $(X, \|\cdot\|)$ be a normed linear space. Then there is at least one semi-inner product $[\cdot,\cdot]$ on X compatible with $\|\cdot\|$.

Proof: For every $x \in X$, put

$$\mathcal{F}(x) = \{x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|^2\}. \tag{12}$$

By the Hahn-Banach Theorem, $\mathcal{F}(x) \neq \emptyset$ for every $x \in X$, so we may choose $F(x) \in \mathcal{F}(x)$. Define $[\cdot,\cdot] : X \times X \to \mathbb{K}$ by

$$[x, y] = (F(y))(x) \text{ for all } x, y \in X. \tag{13}$$

It is straightforward to check that $[\cdot,\cdot]$ is a semi-inner product compatible with $\|\cdot\|$. $\square$

Remark 10.6: There is exactly one semi-inner product on X compatible with $\|\cdot\|$ if and only if for every $x \in X$ the set $\mathcal{F}(x)$ in (12) is a singleton. A convenient sufficient condition for this property to hold is that X^* be strictly convex.

Remark 10.7: If $[\cdot,\cdot]$ is a semi-inner product compatible with the norm on a normed linear space $(X, \|\cdot\|)$ then for each $y \in X$, the mapping $x \to [x, y]$ is a continuous linear functional. As an aside, it is interesting to note that if X is reflexive and $x^* \in X^*$, then there exists a semi-inner product on X and $y \in Y$ such that $x^*(x) = [x, y]$ for all $x \in X$.

Example 10.8: Let $p \in (1, \infty)$ be given and let $X = L^p[0, 1]$ with the usual norm

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

It is straightforward to check that the mapping $[\cdot,\cdot] : X \times X \to \mathbb{R}$ defined by $[f, 0] = 0$ and

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int_0^1 f(x)\overline{g(x)}|g(x)|^{p-2}\operatorname{sgn}(g(x)) \, dx \text{ for } \|g\|_p \neq 0$$

is a semi-inner product compatible with $\|\cdot\|_p$.

Definition 10.9: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear. We say that A is *dissipative* provided that there exists a semi-inner product $[\cdot,\cdot]$ compatible with the norm on X such that

$$\operatorname{Re}[Ax, x] \leq 0 \text{ for all } x \in \mathcal{D}(A).$$

Lemma 10.10: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear. Then A is dissipative if and only if

$$\|(\lambda I - A)x\| \geq \lambda\|x\| \text{ for all } x \in \mathcal{D}(A), \lambda > 0.$$

Proof: Assume that A is dissipative and choose a semi-inner product $[\cdot,\cdot]$ compatible with the norm on X such that $\mathrm{Re}[x,x] \leq 0$ for all $x \in \mathcal{D}(A)$. Let $x \in \mathcal{D}(A)$ and $\lambda > 0$ be given. Then we have

$$\mathrm{Re}[(\lambda I - A)x, x] = \lambda\|x\|^2 - \mathrm{Re}[Ax, x] \geq \lambda\|x\|^2. \tag{14}$$

On the other hand we have

$$\mathrm{Re}[(\lambda I - A)x, x] \leq |[(\lambda I - A)x, x]| \leq \|(\lambda I - A)x\| \cdot \|x\|. \tag{15}$$

Combining (14) with (15) we get

$$\lambda\|x\|^2 \leq \|(\lambda I - A)x\| \cdot \|x\|,$$

which yields the desired conclusion.

Assume now that

$$\|(\lambda I - A)x\| \geq \lambda\|x\| \quad \text{for all } x \in calD(A),\ \lambda > 0. \tag{16}$$

In order to avoid some technical complications we assume here that X is either separable or reflexive. (To handle the general case, one can proceed as below using nets instead of sequences.) As before, for all $z \in X$, put

$$\mathcal{F}(z) = \{x^* \in X^* : x^*(z) = \|z\|^2 = \|x^*\|^2\}.$$

Let $x \in \mathcal{D}(A)\backslash\{0\}$ be given and observe that

$$\|(nI - A)x\| \geq n\|x\| > 0 \quad \text{for all } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$ we choose $y_n^* \in \mathcal{F}(nx - Ax)$ and notice that

$$y_n^*(nx - Ax) = \|nx - Ax\|^2 = \|y_n^*\|^2 > 0.$$

Now put

$$z_n^* = \frac{1}{\|y_n^*\|} y_n^* \quad \text{for all } n \in \mathbb{N}.$$

Then we have

$$\begin{aligned} \|nx - Ax\| &= \frac{\|y_n^*\|^*}{y\ \ _n}(nx - Ax) \\ &= z_n^*(nx - Ax) \\ &= n\mathrm{Re} z_n^*(x) - \mathrm{Re} z_n^*(Ax). \end{aligned}$$

Since $\|z_n^*\| = 1$ for all $n \in \mathbb{N}$ we find that

$$n\|x\| \leq \|nx - Ax\| = n\mathrm{Re} z_n^*(x) - \mathrm{Re} z_n^*(Ax) \leq n\|x\| - \mathrm{Re} z_n^*(Ax) \tag{17}$$

We choose a subsequence $\{z_{n_k}^*\}_{k=1}^{\infty}$ and $x^* \in X^*$ such that

$$z_{n_k}^* \overset{*}{\rightharpoonup} z^* \text{ (weakly}^*\text{) as } k \to \infty.$$

Then we have $\|z^*\| \leq 1$, $\mathrm{Re} z^*(Ax) \leq 0$, and $\mathrm{Re} z^*(x) \geq \|x\|$. It follows that $z^*(x) = \|x\|$.

Define $F : X \to X^*$ by

$$F(x) = \begin{cases} 0 & x = 0 \\ z^*\|x\| & x \in \mathcal{D}(A)\backslash\{0\} \\ \text{any element of } \mathcal{F}(x) & x \in X\backslash\mathcal{D}(A). \end{cases}$$

If we define the semi-inner product $[\cdot, \cdot]$ by

$$[x, y] = (F(y))(x) \text{ for all } x, y \in X,$$

then $\mathrm{Re}[Ax, x] \leq 0$ for all $x \in \mathcal{D}(A)$. $\square$

Lemma 10.11: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear and dissipative. Let $\lambda_0 > 0$ be given and assume that $\lambda_0 I - A$ is surjective. Then A is closed, $\rho(A) \supset (0, \infty)$ and

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda} \text{ for all } \lambda > 0.$$

Proof: The key points to prove are that A is closed and that $\lambda I - A$ is surjective for every $\lambda > 0$. (Injectivity of $\lambda I - A$ and the inequality for $\|R(\lambda; A)\|$ follow easily from Lemma 10.9.)

Since $\|(\lambda_0 - A)x\| \geq \lambda_0\|x\|$ for all $x \in \mathcal{D}(A)$ by Lemma 10.9, we conclude that $\lambda_0 I - A$ is injective and

$$\|(\lambda_0 I - A)^{-1}y\|\frac{1}{\lambda_0}\|y\| \text{ for all } y \in X.$$

It follows that $(\lambda_0 I - A)^{-1} \in \mathcal{L}(X; X)$ and consequently this operator is closed. We conclude that $\lambda_0 I - A$ is closed and this implies that A is closed.

To show that $\rho(A) \supset (0, \infty)$, put

$$\Lambda = \{\lambda \in (0, \infty) : \lambda \in \rho(A)\}.$$

We know that $\Lambda \neq \emptyset$ (because it contains λ_0). Observe that Λ is open in $(0, \infty)$. We shall show that Λ is also closed in $(0, \infty)$ (with the relative topology). Since $(0, \infty)$ is connected, this will ensure that $\Lambda = (0, \infty)$.

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in Λ converging to $\lambda_* \in (0, \infty)$. To show that $\lambda_* \in \Lambda$ it suffices to show that $\lambda_* I - A$ is surjective. Let $y \in X$ be given and for every $n \in \mathbb{N}$, put

$$x_n = R(\lambda_n; A)y.$$

We shall show that that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent to some $x \in X$ and that $(\lambda_* I - A)x = y$. Before developing the relevant inequalities, we observe that the sequence $\{1/\lambda_n\}_{n=1}^{\infty}$ is bounded because it converges to $1/\lambda_*$.

Let $m, n \in \mathbb{N}$ be given. Then we have

$$\begin{aligned} \|x_n - x_m\| &= \|R(\lambda_n; A)y - R(\lambda_m; A)y\| \\ &= \|(\lambda_n - \lambda_m)R(\lambda_n; A)R(\lambda_m; A)\| \\ &= |\lambda_n - \lambda_m| \cdot \|R(\lambda_n; A)\| \cdot \|R(\lambda_m; A)\| \\ &\leq \|\lambda_n - \lambda_m\| \frac{y}{\lambda_n \lambda_m}. \end{aligned}$$

It follows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Put

$$x = \lim_{n \to \infty} x_n.$$

Since $x_n \in \mathcal{D}(A)$ for all $n \in \mathbb{N}$ and $Ax_n \to \lambda_* x - y$ and the operator A is closed we infer that $x \in \mathcal{D}(A)$ and $Ax = \lambda_* x - x$. This completes the proof. $\square$