Shashank Singh[1]

2 Projections and proximal operators [25 points] (Adona)

Q1. [4+4]

a) For a symmetric matrix A we want to find positive semidefinite matrix B minimizing the sum of the absolute values of the eigenvalues of $B - A$. Clearly, if $A = U\Sigma U^{-1}$ is the singular value decomposition of A, this is achieved by

$$B = U\Sigma^+ U^{-1}.$$

where Σ^+ is the diagonal matrix with entries $\Sigma^+_{ii} = \max\{0, \Sigma_{ii}\}$.

b) Let $x^* \in \mathbb{R}^n$ such that $Ax^* + b$ is the projection of y onto $\{Ax + b : x \in \mathbb{R}^n\}$. Then, Ax^* is the projection of $y - b$ onto the column space $\mathcal{C}(A)$, so $(y - b) - Ax^*$ is orthogonal to every vector in $\mathcal{C}(A)$ and is hence in the null space $\mathcal{N}(A^T)$. Then,

$$A^T((y-b) - Ax^*) = 0, \quad \text{and so} \quad A^T(y-b) = A^T Ax^*.$$

Since A has full column rank, $A^T A$ is an invertible (square) matrix, so

$$(A^T A)^{-1} A^T (y - b) = x^*,$$

and hence the projection p of y onto $\{Ax + b : x \in \mathbb{R}^n\}$ is

$$p = Ax^* + b = \boxed{A(A^T A)^{-1} A^T (y - b) + b.}$$

Q2. [5+7+5]

a) Let

$$x^* := \operatorname*{argmin}_{x \in \mathbb{R}^n} \frac{1}{2}\|x - z\|_2^2 + \lambda \sum_{i=1}^n (x_i)_+.$$

Clearly, if $z_i \le 0$, then $x_i^* = z_i$. If $z_i > 0$, then $x_i^* \ge 0$. If $x_i^* > 0$, we have

$$0 = \frac{\partial}{\partial x_i^*} \frac{1}{2} \sum_{z_i > 0} (x_i^* - z_i)^2 + \lambda x_i^* = x_i^* - z_i + \lambda,$$

so that $x_i^* = z_i - \lambda$ when this is positive. Thus, we have

$$\boxed{x_i^* = \begin{cases} z_i & : \text{ if } z_i \le 0 \\ 0 & : \text{ if } 0 < z_i \le \lambda \\ z_i - \lambda & : \text{ if } \lambda < z_i \end{cases}.}$$

[1] sss1@andrew.cmu.edu

b) Let

$$x^* := \operatorname*{argmin}_{x \in \mathbb{R}^n} \frac{1}{2}\|x - z\|_2^2 + \lambda\|x\|_2$$

Clearly, $x^* = cz$, for some $c \in [0, 1]$. Thus, $c = 0$ or

$$0 = \frac{d}{dc}\frac{1}{2}\|(c-1)z\|_2^2 + \lambda\|cz\|_2 = (c-1)\|z\|_2^2 + \lambda\|z\|_2,$$

so $c = 1 - \frac{\lambda}{\|z\|_2}$. Thus,

$$\boxed{x^* = \begin{cases} \left(1 - \frac{\lambda}{\|z\|_2}\right) z & \text{if } \lambda \leq \|z\|_2 \\ 0 & \text{else} \end{cases}.}$$

c) Let

$$x^* := \operatorname*{argmin}_{x \in \mathbb{R}^n} \frac{1}{2}\|x - z\|_2^2 + \lambda\|x\|_\infty$$

and $\rho := \|x\|_\infty$. Clearly, by definitions of the 2- and ∞-norms, in each coordinate of x^*,

$$x_i^* = \begin{cases} \rho & : \text{ if } \rho < z_i \\ z_i & : \text{ if } -\rho \leq z_i \leq \rho \\ -\rho & : \text{ if } z_i < -\rho \end{cases}.$$

Then, we have

$$0 = \frac{d}{d\rho}\frac{1}{2}\|x^* - z\|_2^2 + \lambda\|x\|_\infty = \sum_{|z_i| > \rho} \rho - |z_i| + \lambda = -\|x^* - z\|_1 + \lambda,$$

so $\|x^* - z\|_1 = \lambda$.