Homework 3

21-740 Introduction to Functional Analysis II

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Problem 1

A composition of closed linear operators is not, in general, closed.

Consider the Banach space $X := C([0,1];\mathbb{R})$ of continuous function from $[0,1]$ to $\mathbb{R}$, under the supremum norm $\|f\| := \sup_{x\in[0,1]} |f(x)|$.

Let $\mathcal{D}(B) = C^1([0,1];\mathbb{R}) \subseteq X$ and let $B : \mathcal{D}(B) \to X$ be the differentiation operator $Bf := f'$, $\forall f \in \mathcal{D}(B)$. It is a well-known result that if a sequence $\{f_n\}_{n=1}^\infty$ of differentiable functions converges uniformly to f and $\{f_n'\}_{n=1}^\infty$ converges uniformly to g, then $f \in C^1([0,1];\mathbb{R})$ and $g = f'$. It follows that B is closed.

Now define $A : X \to X$ by $(Af)(x) := f(0)$, $\forall f \in X, x \in [0,1]$. A is continuous and hence closed.

For all $n \in \mathbb{N}$, define $f_n \in \mathcal{D}(B)$ by

$$f_n(x) := \frac{e^{-nx}}{n}, \quad \forall x \in [0,1].$$

Then, $\|f_n\| = f_n(0) = 1/n$ and $f'(0) = -1$, $\forall n \in \mathbb{N}$, so that, as $n \to \infty$, $f_n \to 0$ and $ABf_n \to -1 \neq 0 = AB(0)$. Thus, AB is not closed.

Problem 2

($\Rightarrow$) Suppose A is closable and $(0,y) \in cl(\mathrm{Gr}(A))$. If $\bar{A}$ denotes the closure of A, then $(0,y) \in \mathrm{Gr}(\bar{A})$. Since $\bar{A}$ is a linear operator $(0,0) \in \mathrm{Gr}(\bar{A})$, and it follows that $y = 0$.

($\Leftarrow$) Suppose that

$$\forall (x,y) \in cl(\mathrm{Gr}(A)), \quad x = 0 \Rightarrow y = 0.$$

First note that $cl(\mathrm{Gr}(A))$ is a linear manifold. Define

$$\mathcal{D}(\bar{A}) := \{x : (x,y) \in cl(\mathrm{Gr}(A)) \text{ for some } y \in X\}.$$

It is easily checked that $\mathcal{D}(\bar{A})$ is a linear manifold. If $(x,y_1),(x,y_2) \in cl(\mathrm{Gr}(A))$, then

$$(0, y_1 - y_2) = (x - x, y_1 - y_2) \in cl(\mathrm{Gr}(A)),$$

and hence $y_1 = y_2$ by the given condition. Thus, we can define an operator $\bar{A} : \mathcal{D}(\bar{A}) \to Y$ by $(x, \bar{A}x) \in cl(\mathrm{Gr}(A))$ for each $x \in cl(\mathcal{D}(A))$ and have that $cl(\mathrm{Gr}\, A) = \mathrm{Gr}(\bar{A})$, so that $\bar{A}$ is closed. Thus, A is closable (and $\bar{A}$ is the closure of A). ■

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Problem 3

(a) A is not closed. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{K}^{(\mathbb{N})}$ by

$$x_n^{(k)} = \begin{cases} k^{-3} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}, \quad \forall n, k \in \mathbb{N}.$$

Then, defining $x = (1^{-3}, 2^{-3}, 3^{-3}, \dots)$, $x_n \to x$ as $n \to \infty$. Also,

$$(Ax_n)^{(k)} = \begin{cases} \sum_{i=1}^{n} i^{-2} & \text{if } k = 1 \\ k^{-3} & \text{if } 1 < k \leq n \\ 0 & \text{if } k > n \end{cases}, \quad \forall n, k \in \mathbb{N},$$

so $Ax_n \to (\pi^2/6, 2^{-3}, 3^{-3}, \dots)$ as $n \to \infty$. Since $x \notin \mathbb{K}^{(\mathbb{N})}$, $\mathrm{Gr}(A)$ is not closed. ■

(b) A is not closable. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{K}^{(\mathbb{N})}$ by

$$x_n^{(k)} = \begin{cases} (nk)^{-1} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}, \quad \forall n, k \in \mathbb{N}.$$

Then, $\|x_n\|_p \leq \frac{H_n}{n} \to 0$ as $n \to \infty$ (where H_n denotes the n^{th} harmonic number). Also, noting $\sum_{i=1}^{n} k(nk)^{-1} = 1$ for all $n \in \mathbb{N}$,

$$(Ax_n)^{(k)} = \begin{cases} 1 & \text{if } k = 1 \\ (nk)^{-1} & \text{if } 1 < k \leq n \\ 0 & \text{if } k > n \end{cases}, \quad \forall n, k \in \mathbb{N},$$

so, $Ax_n \to (1, 0, 0, \dots) \neq 0$ as $n \to \infty$. By the result of Problem 2, A is not closable. ■

Problem 4

As the following translation semigroup shows, it may be the case that $T(t) = 0$ for some $t \geq 0$.

Let

$$X := \{f \in C([0,1]; \mathbb{R}) : f(1) = 0\}$$

and consider the mapping $T : [0, \infty) \to \mathcal{L}(X; X)$ given by

$$(T(t)f)(x) = \begin{cases} 0 & \text{if } x + t > 1 \\ f(x+t) & \text{else} \end{cases}, \forall f \in X, t \geq 0, x \in [0,1]$$

Since T simply applies a left-shift, clearly $T(0) = I$ and, $\forall s, t \geq 0$, $T(s+t) = T(s)T(t)$. Since $[0,1]$ is compact, every $f \in X$ is uniformly continuous, and hence (since each $f(1) = 0$), T is strongly continuous, so that T is a linear C_0-semigroup. Furthermore, it is clear that, $\forall t \geq 1$, $T(t) = 0$.

It's probably worth noting that this is not possible with T continuous in the uniform operator topology, since T is then the exponential of its generator and is therefore invertible.

Problem 5

(a) By definition of T, $\forall t \geq 0, x \in X_0$, $T(t)x \in X_0$. Suppose that, for some $n \in \{1, \dots, N\}$, for all $x \in X_{n-1}$, $T(t)x \in X_{n-1}$ for all $t \geq 0$. Let $t \geq 0, x \in X_n$. Since $X_n \subseteq X_{n-1}$, $T(t)x \in X_{n-1}$. Since $Ax \in X_{n-1}$, by part (b) of Lemma 9.2, $AT(t)x = T(t)Ax \in X_{n-1}$, so $T(t) \in X_n$, and the desired result follows by induction. ∎

(b) Suppose that, for some $n \in \{1, \dots, N\}$, T is continuous in the strong operator topology on $(X_{n-1}, \|\cdot\|_{n-1})$. If $x \in X_n, t \geq 0$, then by part (b) of Lemma 9.2

$$\|T(t)x - x\|_n = \|T(t)x - x\|_{n-1} + \|A(T(t)x - x)\|_{n-1} = \|T(t)x - x\|_{n-1} + \|T(t)Ax - Ax\|_{n-1} \to 0$$

as $t \downarrow 0$, since $Ax \in X_{n-1}$. Thus, by induction, T is continuous in the strong operator topology on each $(X_n, \|\cdot\|_n)$.

Since T is continuous in the strong operator topology on $(X_n, \|\cdot\|_n)$, $u \in C^0([0,\infty); X_N)$.

Suppose that, for some $n \in \{1, \dots, N\}$, we have $u \in C^{n-1}([0,\infty); X_{N-(n-1)})$ and $u^{(n-1)}(t) = T(t)A^{n-1}x$. Then, for $t \geq 0$, $x \in X_{N-n}$.

$$\lim_{h \downarrow 0} \frac{u^{(n-1)}(t+h) - u^{(n-1)}(t)}{h} = T(t) \lim_{h \downarrow 0} \frac{T(h)A^{n-1}x - A^{n-1}x}{h} = T(t)A^n x,$$

so $u \in C^n([0,\infty); X_{N-n})$ and $u^{(n)} = T(t)A^n x$. The desired result follows by induction. ∎

Problem 6

By the Hille-Yosida Theorem, we have, for some $M > 0, \omega \in \mathbb{R}$,

(i) $\mathcal{D}(A)$ is dense in X and A is closed.

(ii) $\rho(A) \supseteq (\omega, \infty)$.

(iii) $\|R(\lambda; A)^n\| \leq \dfrac{M}{(\lambda - \omega)^n}$, $\forall n \in \mathbb{N}, \lambda > \omega$.

Furthermore, also by the Hille-Yosida Theorem, it suffices to show

(iv) $\mathcal{D}(A + L)$ is dense in X and $A + L$ is closed.

(v) $\rho(A + L) \supseteq (\omega + \|L\|, \infty)$.

(vi) $\|R(\lambda; A + L)^n\| \leq \dfrac{M}{(\lambda - (\omega + M\|L\|))^n}$, $\forall n \in \mathbb{N}, \lambda > \omega$.

Since L is continuous, $\mathcal{D}(A+L)=\mathcal{D}(A)$ and $A+L$ is closed.

Also, since L is bounded, it can shift the spectrum by at most $\|L\|$, so $(\omega+\|L\|,\infty)\subseteq\rho(A+L)$.

It follows from (iii) that, for $\lambda>\omega+M\|L\|$,

$$\|LR(\lambda;A)\|\leq\frac{M\|L\|}{\lambda-\omega}<\frac{M\|L\|}{M\|L\|}=1.$$

Lemma 1 For $\lambda>\omega+M\|L\|$ (so $\lambda\in\rho(A+L)$),

$$R(\lambda;A+L)=R(\lambda;A)\left(I-LR(\lambda;A)\right)^{-1}=\sum_{k=0}^{\infty}R(\lambda;A)\left(LR(\lambda;A)\right)^{k}.$$

Proof: Since

$$\left(\lambda I-(A+L)\right)R(\lambda;A)=(\lambda I-A)R(\lambda;A)-LR(\lambda;A)=\left(I-LR(\lambda;A)\right),$$

$$\left(\lambda I-(A+L)\right)R(\lambda;A)\left(I-LR(\lambda;A)\right)^{-1}=I.$$

Since $\lambda\in\rho(A+L)$, $\lambda I-(A+L)$ is bijective, and hence its right inverse is its inverse. $\square$

Taking norms and reindexing terms in terms of powers of $\|L\|$,

$$\|R(\lambda;A+L)^n\|=\left\|\left[\sum_{k=0}^{\infty}R(\lambda;A)\left(LR(\lambda;A)\right)^{k}\right]^{n}\right\|\leq\sum_{k=0}^{\infty}\binom{n}{k}\frac{M^{k+1}\|L\|^{k}}{(\lambda-\omega)^{n+k}}$$

Then, since $\lambda>\omega+M\|L\|$, using the identity $\frac{1}{(1-x)^n}=\sum_{k=0}^{\infty}\binom{n}{k}x^k$.

$$\begin{aligned}\|R(\lambda;A+L)^n\|&\leq\sum_{k=0}^{\infty}\binom{n}{k}\frac{M^{k+1}\|L\|^{k}}{(\lambda-\omega)^{n+k}}\\&=\frac{M}{(\lambda-\omega)^n}\sum_{k=0}^{\infty}\binom{n}{k}\left(\frac{M\|L\|}{\lambda-\omega}\right)^{k}\\&=\frac{M}{(\lambda-\omega)^n}\cdot\left(1-\frac{M\|L\|}{\lambda-\omega}\right)^{-n}\\&=\frac{M}{(\lambda-(\omega+M\|L\|))^n}.\quad\blacksquare\end{aligned}$$

Problem 7

I wasn't able to prove this on my own. This proof is based on the proof of Theorem 1.6 in Engel and Nagel's *A Short Course on Operator Semigroups*.

By the Principle of Uniform Boundedness, $\forall t \geq 0$, $T(t) \in \mathcal{L}(X;X)$, and hence, again by the Principle of Uniform Boundedness, $\exists M, \delta > 0$ such that $\|T(t)\| \leq M$ for all $t \in [0, \delta]$. Thus, it suffices to show that the set

$$E := \left\{ x \in X : \lim_{t \downarrow 0} T(t)x = x \right\}$$

is dense in X. In particular, since E is convex, it suffices to show E is weakly dense in X.

$\forall x \in X, r > 0$, define $x_r \in X^{**}$ by

$$x_r(x^*) := \frac{1}{r} \int_0^r x^*(T(t)x)\, dt, \quad \forall x^* \in X^*.$$

Let $x \in X$ and suppose $r > 0$. Since $[0, r]$ is compact and the mapping $s \to T(s)x$ is weakly continuous, $S := \{T(s)x : s \in [0, r]\}$ is weakly compact. Thus, the closure of the convex hull of S, $cl(co(S))$, is weakly compact. Since x_r is a limit of Riemann sums, $x_r \in cl(co(S))$, and so $x_r \in X$. Since the set $D := \{x_r : r > 0, x \in X\}$ is weakly dense in X, it suffices to show $D \subseteq E$.

$$\begin{aligned}
\|T(t)x_r - x_r\| &= \sup_{\|x^*\| \leq 1} \frac{1}{r} \left| \int_t^{t+r} x^*(T(s)x)\, ds - \int_0^r x^*(T(s)x)\, ds \right| \\
&\leq \sup_{\|x^*\| \leq 1} \frac{1}{r} \left(\left| \int_r^{r+t} x^*(T(s)x)\, ds \right| + \left| \int_0^r x^*(T(s)x)\, ds \right| \right) \\
&\leq \frac{2t}{r} \|x\| \sup_{0 \leq s \leq r+t} \|T(s)\| \to 0 \quad \text{as } t \downarrow 0. \quad \blacksquare
\end{aligned}$$

Problem 8

We first note that, $\forall u \in X, t > 0$, $T(t)u = k_t * u$, where $k_t(x) = (\pi t)^{-1/2} e^{-x^2/t}$, $\forall x \in \mathbb{R}$. Then, $\{k_t\}_{t \geq 0}$ is an approximate identity (that is, $\forall t \geq 0$, $\int_{\mathbb{R}} k_t = 1$ and $\forall \delta > 0$, $\int_{\mathbb{R} \setminus (-\delta, \delta)} k_t(y)\, dy \to 0$ as $t \to 0$). We showed in Measure Theory that this implies $\|k_t * f - f\|_2 \to 0$ as $t \to 0$, $\forall f \in L^2(\mathbb{R})$. Thus, T is continuous in the strong operator topology. If $s, t \geq 0$, changing coordinates,

$$\begin{aligned}
(T(s)T(t)u)(x) &= (\pi st)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{ty^2 + sz^2}{st}} u(x - y - z)\, dy\, dz \\
&= \left(\frac{st}{s+t} \right)^{1/2} (\pi st)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{y^2}{s+t}} u(x - y)\, dy = (T(s+t)u)(x),
\end{aligned}$$

so we have the semigroup property. From reading Engel and Nagel's *A Short Course on Operator Semigroups*, I understand that the infinitesimal generator of T is the second derivative $A : C^2(\mathbb{R}) \cap L^2(\mathbb{R}) \to X$ defined by $Au = u''$, $\forall u \in C^2(\mathbb{R}) \cap L^2(\mathbb{R})$, but I wasn't able to show this. It seems to require Fourier transforms, which I'm not too familiar with.