Lecture Notes for Week 2

Linear Spaces and Norms (Continued)

Lemma 2.1 (Riesz): Let $(X, \|\cdot\|)$ be a NLS and let Y, Z be linear manifolds in X such that Y is closed, $Y \subset Z$, and $Y \neq Z$. Let $\theta \in (0,1)$ be given. Then there exists $z \in Z$ such that $\|z\| = 1$ and $\|z - y\| \geq \theta$ for all $y \in Y$.

Proof: Choose $v \in Z \backslash Y$. Put

$$m = \inf\{\|y - v\| : y \in Y\},$$

and notice that $m > 0$ since Y is closed. By the definition of m, and since $\theta < 1$, we may choose $\hat{y} \in Y$ such that

$$m \leq \|\hat{y} - v\| \leq \frac{m}{\theta}.$$

Put $c = \|\hat{y} - v\|^{-1}$, $z = c(\hat{y} - v)$, and notice that $\|z\| = 1$. Let $y \in Y$ be given. Since $\hat{y} - c^{-1}y \in Y$ we have

$$\begin{aligned}
\|z - y\| &= \|c(\hat{y} - v) - y\| \\
&= c\|\hat{y} - c^{-1}y - v\| \\
&= c\|v - (\hat{y} + c^{-1}y)\| \\
&\geq cm = \frac{m}{\|\hat{y} - v\|} \geq \frac{m}{m/\theta} = \theta. \quad \square
\end{aligned}$$

Proposition 2.2: Let $(X, \|\cdot\|)$ be a NLS and put $B = \{x \in X : \|x\| \leq 1\}$. If B is compact then X is finite dimensional.

Proof: We prove the contrapositive implication. Assume that X is infinite dimensional. We shall use induction to construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|x_m - x_n\| \geq \frac{1}{2}$ for all $m, n \in \mathbb{N}$ with $m > n$. The existence of such a sequence implies that B is not compact, because each $x_n \in B$, but there can be no subsequence that is a Cauchy sequence. Choose any $x_1 \in X$ with $\|x_1\| = 1$. Let $n \in \mathbb{N}$ with $n \geq 2$ be given and assume that we have constructed $x_1, x_2, \cdots, x_n \in X$ with $\|x_i\| = 1$ for $i = 1, 2, \cdots, n$ such that

$$\|x_n - y\| \geq \frac{1}{2} \quad \text{for all } y \in \operatorname{span}(x_1, x_2, \cdots, x_{n-1}).$$

Since $\text{span}(x_1, x_2, \cdots, x_n)$ is finite dimensional, it is a closed subspace. By Riesz's lemma, we may choose $x_{n+1} \in X$ with $\|x_{n+1}\| = 1$ such that

$$\|x_{n+1} - y\| \geq \frac{1}{2} \text{ for all } y \in \text{span}(x_1, x_2, \cdots, x_n).$$

By induction, this procedure produces a sequence with the desired properties. $\square$

Remark 2.3: It is an immediate consequence of Proposition 2.2 that if S is a compact set in an infinite-dimensional NLS then $\text{int}(S) = \emptyset$ and consequently (since S is closed) we can conclude that S is nowhere dense.

Remark 2.4: The proof of Proposition 2.2 also shows that if $(X, \|\cdot\|)$ is infinite dimensional, then the set $S = \{x \in X : \|x\| = 1\}$ is not compact.

Linear Mappings

Let X, Y, Z be linear spaces over the same field which we assume is either $\mathbb{R}$ or $\mathbb{C}$. Recall that a mapping $T : X \to Y$ is said to be *linear* provided that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \text{ for all } \alpha, \beta \in \mathbb{K}, \; x, y \in X.$$

For a linear mapping T it is traditional to write Tx in place of $T(x)$. Linear mappings are frequently referred to as *linear operators*. When we talk about linear mappings between two different linear spaces it should be understood that the fields for the linear spaces are the same.

Definition 2.5: The *null space* of a linear mapping $T : X \to Y$ is defined by

$$\mathcal{N}(T) = \{x \in X : Tx = 0\}.$$

The term *kernel* is frequently used as a synonym for null space.

Definition 2.6: The *range* of a linear mapping $T : X \to Y$ is defined by

$$\mathcal{R}(T) = \{Tx : x \in X\}.$$

Some Basic Facts about Linear Mappings

We record some simple facts about linear mappings that are assumed to be familiar. The proofs of these facts for general linear spaces are the same as typically given in Linear Algebra for finite-dimensional spaces.

Let X, Y, and Z be linear spaces over the same field and assume that $T : X \to Y$ and $L : Y \to Z$ are linear mappings. Then we have

(a) $\mathcal{N}(T)$ is a linear manifold in X.

(b) $\mathcal{R}(T)$ is a linear manifold Y.

(c) T is injective if and only if $\mathcal{N}(T) = \{0\}$.

(d) If T is bijective then $T^{-1} : Y \to X$ is linear.

(e) If X is finite dimensional then $\mathcal{R}(T)$ is finite dimensional and $\dim(\mathcal{R}) \leq \dim(X)$. (In fact, if X is finite dimensional then $\dim(\mathcal{R}(T)) + \dim(\mathcal{N}(T)) = \dim(X)$.

(f) The mapping $LT : X \to Z$ defined by $LT(x) = L(T(x))$ for all $x \in X$ is linear.

(g) If T and L are bijective then LT is bijective and $(LT)^{-1} = T^{-1}L^{-1}$.

Definition 2.7: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS and $T : X \to Y$ be linear. We say that T is *bounded* provided that there is a constant M such that

$$\|Tx\|_Y \leq M\|x\|_X \text{ for all } x \in X.$$

Convention: In dealing with linear operators between different NLS it is customary to drop the subscripts on the norms and simply write $\|Tx\|$ and $\|x\|$ when there is no danger of confusion.

Prop 2.8: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS and $T : X \to Y$ be linear. The following four statements are equivalent.

(i) There exists a point $x_0 \in X$ such T is continuous at x_0.

(ii) T is continuous (on X).

(iii) T is uniformly continuous (on X).

(iv) T is bounded.

Proof: The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are immediate, so it suffices to prove (i) $\Rightarrow$ (iv). Let $x_0 \in X$ be given and assume that T is continuous at x_0. We may choose $\delta > 0$ such that $\|Tx - Tx_0\| < 1$ for all $x \in X$ with $\|x - x_0\| < \delta$. Let $y \in X\backslash\{0\}$ be given and put

$$x = x_0 + \frac{\delta}{2\|y\|}y.$$

Since $\|x - x_0\| < \delta$, we have $\|Tx - Tx_0\| < 1$. On the other hand, we have

$$\|Tx - Tx_0\| = \frac{\delta}{2\|y\|}\|Ty\|.$$

We conclude that

$$\|Ty\| \leq \frac{2}{\delta}\|y\|. \quad \square$$

Remark 2.9: The scalar field $\mathbb{K}$ is a (one-dimensional) linear space over itself. It therefore makes sense to talk about linear mappings from a linear space X to $\mathbb{K}$.

Proposition 2.10: Let $(X, \|\cdot\|)$ be a NLS and assume that $L : X \to \mathbb{K}$ is linear. Then L is continuous if and only if $\mathcal{N}(L)$ is closed.

The proof of Proposition 2.10 is Problem 12 on Assignment 2.

Corollary 2.11: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS and $T : X \to Y$ be a linear operator. Assume that X is finite dimensional. Then T is continuous.

Proof: Choose a (Hamel) basis $(x_i | i = 1, \cdots, N)$ for X and let $(\alpha_i | i = 1, \cdots, N)$ be the family of coefficient mappings for this basis. (See Definition 1.28.) For every $i = 1, \cdots, N$ we have that $\mathcal{N}(\alpha_i)$ is a finite-dimensional linear manifold and is therefore closed by Remark 1.37. It follows from Corollary 2.11 that α_i is continuous for each $i = 1, \cdots, N$. By the linearity of T, we have

$$Tx = \sum_{i=1}^{N} \alpha_i(x) T x_i.$$

Since each α_i is continuous, we conclude that T is continuous. $\square$

Equivalence of Norms

In many situations the given norm on a linear space is inconvenient for certain purposes. This leads to a natural question: *Can we replace the norm with another one that is more convenient and is such that the space retains its important properties when we change to the new norm?*

Definition 2.12: Let X be a linear space over $\mathbb{K}$ and $\|\cdot\|_a$ and $\|\cdot\|_b$ be norms on X. We say that these norms are equivalent provided that there are constants $M, m > 0$ such that

$$m\|x\|_a \le \|x\|_b \le M\|x\|_a \;\text{ for all } x \in X.$$

Remark 2.13: Notice that if the equation above holds then

$$M^{-1}\|x\|_b \le \|x\|_a \le m^{-1}\|x\|_b \;\text{ for all } x \in X,$$

which shows that Definition 2.12 is symmetric in $\|\cdot\|_a$ and $\|\cdot\|_b$.

Using Proposition 2.8 (and the fact that the identity mapping $I : X \to X$ is linear) we obtain the following two remarks.

Remark 2.14: Let X be a linear space with two norms $\|\cdot\|_a$ and $\|\cdot\|_b$. Then these norms are equivalent if and only if the identity operator is continuous from $(X, \|\cdot\|_a)$ to $(X, \|\cdot\|_b)$ and from $(X, \|\cdot\|_b)$ to $(X, \|\cdot\|_a)$.

Remark 2.15: If a linear space is equipped with two equivalent norms, then the associated metric spaces are uniformly homeomorphic.

Proposition 2.16: Let X be a finite-dimensional linear space over $\mathbb{K}$. Then all norms on X are equivalent.

Proposition 2.16 follows immediately from Corollary 2.11 and Remark 2.14. It can also be established using Lemma 1.38.

Spaces of Bounded Linear Mappings

Definition 2.17: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS over the same field $\mathbb{K}$. We denote the set of all bounded linear mappings from X to Y by $\mathcal{L}(X;Y)$. We define the function $\|\cdot\|_{\mathcal{L}(X;Y)} : \mathcal{L}(X;Y) \to \mathbb{R}$ by

$$\|T\|_{\mathcal{L}(X;Y)} = \sup\{\|Tx\|_Y : x \in X,\ \|x\|_X \leq 1\}.$$

Remark 2.18: It is clear that $\mathcal{L}(X;Y)$ is a linear space.

The following result is an easy consequence of the definition of $\|\cdot\|_{\mathcal{L}(X;Y)}$.

Proposition 2.19: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be NLS. Then $\|\cdot\|_{\mathcal{L}(X;Y)}$ is a norm on $\mathcal{L}(X,Y)$.

Remark 2.20: When there is no danger of ambiguity, it is customary to omit subscripts on the norms for X, Y and $\mathcal{L}(X;Y)$.

Remark 2.21: If $X \neq \{0\}$ and $T \in \mathcal{L}(X;Y)$ then

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in X,\ x \neq 0\right\} = \sup\{\|Tx\| : x \in X,\ \|x\| = 1\}.$$

Remark 2.22: If $(Z, \|\cdot\|_Z)$ is a third NLS over the same field as X and Y and $T \in \mathcal{L}(X;Y),\ L \in \mathcal{L}(Y;Z)$ then $LT \in \mathcal{L}(X;Z)$ and $\|LT\| \leq \|L\| \cdot \|T\|$.

Proposition 2.23: Assume that $(Y, \|\cdot\|)$ is complete. Then $\mathcal{L}(X;Y)$ is complete under $\|\cdot\|_{\mathcal{L}(X;Y)}$.

Proof: Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{L}(X;Y)$. Let $x \in X$ be given. Then $\{T_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in Y; since Y is complete this sequence is convergent. Define $L : X \to Y$ by

$$Lx = \lim_{n\to\infty} T_n x.$$

It is clear that L is linear. We need to show that L is bounded and that $\|T_n - L\| \to 0$ as $n \to \infty$. Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $\|T_n - T_m\| < \frac{\epsilon}{2}$ for all

$m, n \geq N$. Let $x \in X$ be given. Then for all $n \geq N$ we have

$$\|T_n x\| \leq \|T_N x\| + \|(T_n - T_N)x\| \leq \left(\|T_N\| + \frac{\epsilon}{2}\right)\|x\|.$$

Letting $n \to \infty$ and using continuity of the norm we obtain

$$\|Lx\| \leq \left(\|T_N\| + \frac{\epsilon}{2}\right)\|x\| \quad \text{for all } x \in X,$$

so that L is bounded.

To show that $\|T_n - L\| \to 0$ as $n \to \infty$, let $x \in X$ and $n \geq N$ be given. Then we have

$$\begin{aligned} \|T_n x - Lx\| &= \lim_{m\to\infty} \|T_n x - T_m x\| \\ &\leq \limsup_{m\to\infty} \|T_n - T_m\| \cdot \|x\| \\ &\leq \frac{\epsilon}{2}\|x\|. \end{aligned}$$

Taking the supremum over all $x \in X$ with $\|x\| \leq 1$, we obtain

$$\|T_n - L\| \leq \frac{\epsilon}{2} < \epsilon \quad \text{for all } n \geq N. \quad \square$$