Homework 1

15-359 Probability and Computing
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Section: B
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Problem 1: Chain Gang

Let $E_1, E_2, E_3, \ldots$ be events, each with positive probability. For $n = 1$, clearly

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)$$

$$= P(E_1) \cdot P(E_2|E_1) \cdot \ldots \cdot P(E_n|E_1 \cap E_2 \cap \ldots \cap E_{n-1}).$$

Suppose that, for some $n \in \mathbb{N}\backslash\{0\}$,

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1) \cdot P(E_2|E_1) \cdot \ldots \cdot P(E_n|E_1 \cap E_2 \cap \ldots \cap E_{n-1}).$$

Then, by definition of conditional probability, for any event E_{n+1},

$$P(E_1 \cap E_2 \cap \ldots \cap E_n \cap E_{n+1}) = P(E_1 \cap E_2 \cap \ldots \cap E_n) \cdot P(E_{n+1}|E_1 \cap E_2 \cap \ldots \cap E_n)$$

$$= P(E_1) \cdot P(E_2|E_1) \cdot \ldots \cdot P(E_n|E_1 \cap E_2 \cap \ldots \cap E_{n-1}) \cdot P(E_{n+1}|E_1 \cap E_2 \cap \ldots \cap E_n).$$

Thus, by the Principle of Mathematical Induction, the identity holds for all $n \in \mathbb{N}$. ■

Problem 2: Me and you

The implication holds.

Since $P(A|B) > P(A)$,

$$\frac{P(A|B)P(B)}{P(A)} > P(B).$$

Thus, by Bayes' Theorem,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} > P(B).$$

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Problem 3: Fool me once, shame on you. Fool me twice...

A. Let S be the event that the laboratory test returns "success," and let V be the event that the vaccine is effective. The law of total probability gives

$$P(S) = P(S|V)P(V) + P(S|V^c)P(V^c) =$$

$$P(S|V)P(V) + P(S|V^c)(1 - P(V)) = 0.6 * 0.5 + 0.4 * 0.5 = \boxed{0.5.}$$

B. Let S and V be as in the solution to part A. above. Then, by the result of part A., Bayes' Theorem gives

$$P(V|S) = \frac{P(S|V)P(V)}{P(S)} = \frac{0.6 * 0.5}{0.6 + 0.4} = \boxed{0.6.}$$

C. Let S and V be as in the solution to part A. above, and let T be the event that the second test returns "success." Then, since S and T are independent,

$$P(V|S \cap T) = \frac{P(S \cap T|V)P(V)}{P(S \cap T)} = \frac{P(S \cap T|V)P(V)}{P(S \cap T|V)P(V) + P(S \cap T|V^c)P(V^c)}$$

$$= \frac{0.6 * 0.8 * 0.5}{0.6 * 0.8 * 0.5 + 0.4 * 0.2 * 0.5} \approx \boxed{0.86.}$$

Thus, adding the second test was fairly useful.

Problem 4: Wrapping up Miller-Rabin

A. Let E be the event that the Miller-Rabin test always returns YES? after m iterations, and, for $i \in \{1, \dots, m\}$, let E_i denote the event that the i^{th} iteration of the Miller-Rabin test returns YES?, so that $E = E_1 \cap \dots \cap E_m$. Assuming, the result of each iteration of the Miller-Rabin test is conditionally independent given of the results of all previous iterations of the test given C,

$$P(E|C) = P(E_1 \cap E_2 \cap \dots \cap E_m|C) = P(E_1|C)P(E_2|C)\dots P(E_m|C) \le \boxed{\frac{1}{2^m}.}$$

B. $P(C) \approx 1 - \frac{\ln n}{n}$.

C. By Bayes' Theorem,

$$P(C|Y_m) = \frac{P(Y_m|C)P(C)}{P(Y_m)},$$

where the event Y_m is identical to the event E used in the solution to part A. Thus, Bayes' Theorem and the result of part B.,

$$P(C|Y_m) \le \frac{1}{2^m\left(1 - \frac{\ln n}{n}\right)} = \boxed{\frac{n}{2^m(n - \ln n)}.}$$

Problem 5: Last Die

Choose 4 of the 5 die rolls and call them "the first four die rolls"; let the remaining die roll be the "fifth die roll".

Let A denote the event that the sum of the five die rolls is divisible by 6, let E denote the set of all possible rolls of the five dice (i.e., the entire sample space), and, for $k \in \{0, 1, 2, 3, 4, 5\}$, let E_k denote the event that the sum of the first four die rolls is congruent to $k \pmod 6$ and let A_k denote the probability that the fifth die roll is $6 - k$. By the Law of Total Probability, since $\{E_0, E_1, \ldots, E_5\}$ partitions E,

$$P(A) = \sum_{k=0}^{5} P(A|E_k)P(E_k).$$

Clearly, for $k \in \{0, 1, \ldots, 5\}$, $P(A|E_k) = P(A_k)$, and, furthermore, $P(A_0) = P(A_1) = \ldots = P(A_5) = \frac{1}{6}$. Thus,

$$P(A) = \sum_{k=0}^{5} \frac{1}{6} P(E_k) = \frac{1}{6} \sum_{k=0}^{5} P(E_k) = \frac{1}{6} \sum_{k=0}^{5} P(E)P(E_k),$$

so that, by the Law of Total Probability,

$$P(A) = \frac{1}{6} P(E) = \boxed{\frac{1}{6}.}$$

Problem 6: If ya like it, ya shoulda put a probability mass on it

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A. Since, using the prescribed algorithm, you will never, marry any of the first m prospects, for $i \leq m$, $P(E_i) = \boxed{0.}$

B. For $i > m$, let A_i be the event that you marry the i^{th} prospect, and let B_i be the event that the i^{th} prospect is the best, so that $P(E_i) = P(A_i \cap B_i) = P(A_i|B_i)P(B_i)$, by definition of conditional probability. Clearly, $P(B_i) = \frac{1}{i}$. Since, if the i^{th} prospect is the best, you will marry the i^{th} prospect if you have not already married a previous prospect, $P(A_i|B_i)$ is the probabilty that s is greater than the $(i - 1 - m)$ prospects between the first m prospects and the i^{th} prospect, or, equivalently, the probability that the best of the first $(i - 1)$ propects is among the first m prospects, or $\frac{m}{i-1}$. Thus, $\boxed{P(E_i) = \frac{m}{n(i-1)}.}$

C. By the Law of Total Probability,

$$P(E) = \sum_{i=0}^{n} P(E|E_i)P(E_i).$$

Since $E_i \subseteq E$, $P(E|E_i) = 1$. Thus, by the results of parts A. and B. above,

$$P(E) = \sum_{i=m+1}^{n} \frac{m}{n(i-1)} = \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{(i-1)}. \qquad \blacksquare$$