Homework 2

21-740 Introduction to Functional Analysis II
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Problem 1

For any $m, n \in \mathbb{N}$ with $m > n$, the define the operator $A_{m,n} := A_m - A_n$. Using the fact that $A_{m,n} \geq 0$, it is easy to adapt the proof of the Cauchy-Schwarz Inequality to show, $\forall x, y \in X$,

$$(A_{m,n}x, y)^2 \leq (A_{m,n}x, x)(A_{m,n}y, y). \tag{1}$$

Plugging $y = A_{m,n}x$ into this inequality gives

$$\begin{aligned} \|(A_m - A_n)x\|^4 = (A_{m,n}x, A_{m,n}y)^2 &\leq (A_{m,n}x, x)(A_{m,n}^2 x, A_{m,n}x) \\ &\leq (A_{m,n}x, x)\|A_{m,n}\|^3\|x\|^2 \\ &\leq (A_{m,n}x, x)\|I - A_1\|^3\|x\|^2. \end{aligned} \tag{2}$$

where the last line follows from $A_1 \leq A_n \leq A_m \leq I$. Since the sequence $\{(A_n x, x)\}_{n=1}^{\infty}$ is increasing and bounded by 1, it converges and is hence Cauchy. It follows from (2) that the sequence $\{A_n x\}_{n=1}^{\infty}$ is Cauchy. Since X is complete, we can define $L : X \to X$ by

$$Lx = \lim_{n \to \infty} A_n x.$$

Clearly, L is linear and self-adjoint, and $A_1, \leq L \leq I$. Thus, for any $x \in X$, $-\|A\| \leq (A_1 x, x) \leq (Lx, x) \leq \|x\|^2$, and so, since $\|L\| = \sup\{(Lx, x) : x \in X, \|x\| = 1\}$, L is bounded. ∎

I didn't use the hypothesis that $A_m A_n = A_n A_m$, but the proof *seems* correct to me.

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Problem 2

Note that, if $A = 0$, the operator $B = 0$ clearly has the desired properties. Thus, we assume $A \neq 0$. We first prove the result in the case $A \leq I$, and then generalize this.

Step 1 Assume $A \leq I$. Define $B_0 = 0$ and

$$B_{n+1} = B_n + \frac{1}{2}(A - B_n^2) \quad \text{for all } n \in \mathbb{N}.$$

It is easily checked by induction that each B_n is as a linear combination of powers of A, and it follows that, $\forall n, m \in \mathbb{N}$, $B_n B_m = B_m B_n$ (since powers of A commute), $B_n^* = B_n$. If for some $n \in \mathbb{N}$, $0 \leq B_n \leq A$, $\forall x \in X$, then

$$B_{n+1} = B_n + \frac{1}{2}(A - B_n^2) \geq B_n$$

and, since $B_n \leq I$, so that $B_n^2 \leq B_n$,

$$2B_{n+1} = 2B_n - B_n^2 + A \leq B_n + A \leq 2A,$$

and hence $B_{n+1} \leq A$. By induction, then, $\forall n \in \mathbb{N}$, $0 \leq B_n \leq B_{n+1} \leq I$. It follows from Problem 1 that $\{B_n\}_{n=1}^{\infty}$ has a limit $B \in \mathcal{L}(X;X)$ (in the strong operator topology) with $B^* = B$. It is also clear from the construction of this limit in Problem 1 that $B \geq 0$ (since each $B_n \geq 0$), and that if $CA = AC$, so that each $CB_n = B_n C$, then $CB = BC$. Also, $\forall x \in X$,

$$Bx = \lim_{n\to\infty} B_{n+1}x = \lim_{n\to\infty}\left(B_n + \frac{1}{2}(A - B_n^2)x\right) = Bx + \frac{1}{2}(A - B^2)X,$$

and it follows that $Ax - B^2x = 0$, so that $B^2 = A$.

Step 2: ($A \not\leq I$) Define $A_1 := \frac{A}{\|A\|}$. For all $x \in X$, applying Cauchy-Schwarz

$$0 \leq \frac{1}{\|A\|}(Ax, x) = (A_1 x, x) \leq \|A_1\|\|x\|^2 = \|x\|^2 = (Ix, x),$$

and hence $0 \leq A_1 \leq I$. Also, $A_1^* = A_1$ Thus, as already shown, $\exists B_1 \in \mathcal{L}(X;X)$ such that $B_1^* = B_1$, $B_1 \geq 0$, $B_1^2 = A_1$. Moreover, if $AC = CA$, then clearly $A_1 C = CA_1$, and so $B_1 C = CB_1$.

It follows that the operator $B := \sqrt{\|A\|}B_1$ has the desired properties: $B^* = B$, $B \geq 0$, $B^2 = \sqrt{\|A\|}^2 B_1^2 = A$, and, if $AC = CA$ then $BC = AC$.

Finally, we show that the positive operator B with $B^2 = A$ is unique. Suppose $B^2 = C^2 = A$. Define $D := B - C$ and let $x \in X$, $y := Dx$. Then,

$$(By, y) + (Cy, y) = ((B + C)y, y) = ((B + C)(B - C)x, y) = ((A - A)x, y) = 0.$$

Since B and C are positive, it follows from (1) that $By = Cy = 0$. Thus,

$$\|(B - C)x\|^2 = \|Dx\|^2 = (Dx, Dx) = (D^2x, x) = (Dy, x) = 0,$$

and hence, since x was chosen arbitrarily $B = C$. ■

Problem 4

If $\alpha = 0$, then, $\forall x \in X$, by Bessel's Inequality,

$$\sum_{n=1}^{\infty} |(x_n, x)|^2 \leq \|x\|^2,$$

and so $(x_n, x) \to 0$ as $n \to \infty$. Therefore, $x_n \rightharpoonup 0$ (weakly) as $n \to \infty$.

I believe that, if $|\alpha| \in (0, 1)$, then $\{x_n\}_{n=1}^{\infty}$ need not converge weakly, but somehow, I wasn't able to produce a counterexample.

I managed to show that, if $\{x_n\}_{n=1}^{\infty}$ converges weakly to $x \in X$, then the sequence

$$\{y_n\}_{n=1}^{\infty} := \left\{ \frac{1}{n} \sum_{k=1}^{n} \right\}_{n=1}^{\infty}$$

has a subsequence converging strongly to x, which might be helpful. In particular, as $n \to \infty$ $\|y_n\|^2 \to \alpha$, implying $\|x\|^2 = \alpha$, which I think might help produce a contradiction.

Problem 6

For any $\lambda \in \mathbb{R}$, define $B_\lambda \in \mathcal{L}(X; X)$ by

$$(B_\lambda x)_i = \left| \frac{1}{i} - \lambda \right| x_i, \quad \forall i \in \mathbb{N}, x \in l^2.$$

Then,

$$(B_\lambda^2 x)_i = \left(\frac{1}{i} - \lambda \right)^2 x_i = \left((A - \lambda I)^2 x \right)_i, \quad \forall i \in \mathbb{N}, x \in l^2,$$

and hence $B_\lambda = \sqrt{L(\lambda)^2} = |L(\lambda)|$. Noting that $B_\lambda x_i = L(\lambda) x_i$ if $i < 1/\lambda$ and $(B_\lambda x)_i = -(L(\lambda)x)_i$ otherwise, we then have

$$L(\lambda)^+ = \frac{1}{2}(B_\lambda + L(\lambda)) = \begin{cases} \left(\frac{1}{i} - \lambda\right) x_i & \text{if } i < \frac{1}{\lambda} \\ 0 & \text{else} \end{cases}, \qquad \forall i \in \mathbb{N}, \lambda \in \mathbb{R}, x \in l^2,$$

It follows immediate that $\forall \lambda \in \mathbb{R}$,

$$\mathcal{N}(L(\lambda)^+) = \{x \in l^2 : x_1 = x_2 = \cdots = x_{i-1} = 0\} = cl(\text{span}\{e_j : j \geq i\}),$$

where $i = \lceil 1/\lambda \rceil$ is the least integer with $i \geq 1/\lambda$. $\blacksquare$

Problem 8

Suppose, for sake of contradiction, that, for some $T \in \mathcal{C}(X;X)$, $B := R+T$ is normal. We first show that $\sigma(R+T)$ is uncountable, and then show that $\sigma(B)$ must be countable, giving a contradiction.

Let $L = R^*$ denote the left-shift operator. If $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $x = (1, \lambda, \lambda^2, \dots) \in l^2$, then $(L - \lambda I)x = 0$, and it follows that $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq \sigma_p(L)\}$. Since $(L - \lambda I) = (R - \bar{\lambda} I)^*$ is bijective if and only if $(R - \bar{\lambda} I)$ is bijective, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq \sigma(R)$, and hence $\sigma(R)$ is uncountable. In particular, since R has no eigenvalues, $\sigma(R) \backslash \sigma_p(R)$ is uncountable, and so, by the result of Problem 7, $\sigma(R+T)$ is uncountable.

Lemma 1 If $T \in \mathbb{C}(X;X)$, then $\sigma(T)$ is countable.

*P*roof: For all $n \in \mathbb{N}$, define $\Lambda_n := \{\lambda \in \sigma(T) : |\lambda| \geq 1/n\}$. Then,

$$\sigma(T) = \bigcup_{n \in \mathbb{N}} \Lambda_n.$$

Since 0 is the only accumulation point of $\sigma(T)$, each Λ_n is finite, and hence $\sigma(T)$ is countable. □

Now note that

$$I = LR = R^*R = (B-T)^*(B-T) = B^*B - B^*T - T^*B + T^*T,$$

and hence

$$B^*B = I + B^*T + T^*B - T^*T.$$

Let $C := B^*T + T^*B - T^*T$, and note that, since any finite sum of compact operators or composition of compact and bounded operators is compact, C is compact. Also, $I + C - \lambda I = C - (1+\lambda)I$, and so $\sigma(I+C) = \{\lambda - 1 : \lambda \in \sigma(C)\}$. By Lemma 1, then, $\sigma(B^*B) = \sigma(I+C)$ is countable. Since B is normal, it should follow that $\sigma(B)$ is countable, although I wasn't able to show this clearly. ■

Problem 11

The inclusion $\mathrm{bdry}(\sigma(A)) \subseteq \sigma_{ap}(A)$ holds.

Suppose $\lambda \in \mathrm{bdry}(\sigma_p(A))$. By definition of the boundary, there is a sequence $\{\lambda_n\}_{n=1}^{\infty}$ with each $\lambda_n \in \mathbb{K} \backslash \sigma(A) = \rho(A)$. Since $\sigma(A)$ is closed, $\lambda \in \sigma(A)$. By (the contrapositive of) part (iii) of Proposition 5.6, $\forall i \in \mathbb{N}$,

$$\|(\lambda_n I - A)^{-1}\| \geq \frac{1}{|\lambda_n - \lambda|} \to \infty \quad \text{as } n \to \infty.$$

It follows that there is a sequence $\{y_n\}_{n=1}^{\infty}$ in Y such that each $y_n \to 0$ as $n \to \infty$ and each $x_n := (A - \lambda_n I)^{-1} y_n$ has $\|x_n\| = 1$. Thus,

$$\lim_{n \to \infty} \|(\lambda I - A)x_n\| \to 0,$$

and hence $\lambda \in \sigma_{ap}(A)$. ■