21-630 Assignment #9 Due Wed., April 3, 201

1. Consider

   (*) $$\dot{X} = f(t, X)$$

   with f continuous and $f(t,0) = 0\,\forall t$. Assume there are positive constants, C_1, C_2, C_3, and $v(t,x)$ C^1 with

   $$\begin{aligned} C_1|x|^2 &\le v(t,x) \le C_2|x|^2 \\ D_* v(t,x) &\le -C_3|x|^2 \end{aligned}$$

   for all (t,x). Show that

   $$|X(t)| \le |X(t_0)| \sqrt{\frac{C_2}{C_1}} \exp\left(-\frac{C_3(t-t_0)}{2C_2}\right)$$

   for $t \ge t_0$ for every solution of (*). Be careful in using Gronwall's inequality here. To apply Gronwall to

   $$Z(t) \le A + B\int_0^t Z(s)ds$$

   we must have $B \ge 0$.

2. Determine the stability of the zero solution for the following systems:

   $$\begin{aligned} \dot{X}_1 &= -2X_2 + X_2X_3 \\ \dot{X}_2 &= X_1 - X_1X_3 \\ \dot{X}_3 &= X_1X_2 \end{aligned}$$

   and

   $$\begin{aligned} \dot{X}_1 &= -2X_2 + X_2X_3 - X_1^3 \\ \dot{X}_2 &= X_1 - X_1X_3 - X_2^3 \\ \dot{X}_3 &= X_1X_2 - X_3^3 \end{aligned}$$

3. Consider

   $$\begin{aligned} \dot{X} &= -X - Y^2 \\ \dot{Y} &= -Y - X^2. \end{aligned}$$

Let $v(x,y) = x^2 + y^2$ and show that on the set $\{(x,y) : x^2 + y^2 \leq C\}$ with $C \in (0,2)$ there is $C_2 > 0$ such that

$$D_* v \leq -C_2(x^2 + y^2).$$

Show that this fails if $C = 2$.

4. Find all critical points of the system

$$\begin{aligned} \dot{X} &= X - XY \\ \dot{Y} &= -Y + XY \end{aligned}$$

and determine the stability of each.

Hint: For one of them the function $\tilde{v}(x,y) = xe^{-x}ye^{-y}$ is useful. Note: work is needed to derive a Lyapunov function from this. In particular a Lyapunov finction must be zero at the critical point. Although in the statements of the theorems on Lyapunov functions presented in class the critical point was taken to be the origin, you may use the versions of these theorems obtained by translation.