Homework 1

21-238 Mathematical Studies Algebra II
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Exercise 1

Since the determinant of AB is 0, $\text{rank}(AB) < 3$. Furthermore, no row of AB is a multiple of another, so that $\text{rank}(AB) > 2$, and thus $\dim(\text{Im}(AB)) = 2$. Note that $(AB)^2 = 9AB$. Therefore, the $\dim(\text{Im}(A(BA)A)) \geq 2$. Since the dimension of the image of a composition of linear functions is at most the minimum of the dimensions of the images of those functions, $\dim(\text{Im}(BA)) \geq 2$. Since BA is a 2×2 matrix of rank 2, BA is invertible. Note that $(AB)^2 = 9AB$. Therefore, $B(AB)^2A = 9BABA$, so that, right-multiplying by $(BABA)^{-1}$ gives

$$BA = 9I_2 = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}. \qquad \blacksquare$$

Exercise 2

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i. Since the determinant is multiplicative and $AB = -BA = -IBA$, $|AB| = |-I||B||A|$. Since $(-I)$ is a diagonal matrix, its determinant is the product of the elements in its diagonal, so that $|-I| = (-1)^n$. Therefore, since $|BA| = |A||B| = |AB|$, n must be even. $\blacksquare$

ii. Since the polynomial $x^2 - 1$ over E has all of its roots in E and $A^2 - I_n$ is the minimal polynomial of A over E, A is diagonalizable. Thus, for some invertible P and some diagonal matrix S, $A = PSP^{-1}$, so that $A = ABB = -BAB^{-1} = (BP)S(BP)^{-1}$.

    Since the elements along the diagonals of both S and $(-S)$ are the eigenvalues of A, A must have the same number of positive and negative eigenvalues. Since the eigenvalues of A^2 are the squares of the eigenvalues of A, and $A^2 = I_n$, the eigenvalues of A are 1 and (-1), each with multiplicity m.

Exercise 5

Suppose, for sake of contradiction that there existed a square matrix A such that

$$\sin A = \begin{bmatrix} 1 & 1996 \\ 0 & 1 \end{bmatrix}.$$

As with sin, we can define $\cos A$ by the usual power series:

$$\cos A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Then,

$$\cos^2 A = I_2 - \sin^2 A = \begin{bmatrix} 0 & -3992 \\ 0 & 0 \end{bmatrix}.$$

Let a, b, c, d be the elements of $\cos A$, so that

$$\cos^2 A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + cd & bc + d^2 \end{bmatrix}.$$

This gives the system of simultaneous equations:

$$\begin{aligned} a^2 &= -bc \\ ab + bd &= -3992 \\ ca &= -cd \\ cb &= -d^2. \end{aligned}$$

If $c = 0$, then $a = d = 0$, so that $ab + bd = 0 \neq -3992$, which is a contradiction. Similarly, if $c \neq 0$, then $a = -d$, so that $ab + bd = 0 \neq -3992$. Therefore, there does not exist a matrix A such that

$$\sin A = \begin{bmatrix} 1 & 1996 \\ 0 & 1 \end{bmatrix}. \qquad \blacksquare$$