Department of Mathematical Sciences
Carnegie Mellon University

21-640 **Functional Analysis** **Spring 2013**

Assignment 3

Due on Friday, February 15

Solutions to problems marked with an asterisk should be written up and handed in.

1. Let $y \in \mathbb{K}^{\mathbb{N}}$ be given and assume that $\sum_{n=1}^{\infty} x_n y_n$ is convergent for every $x \in c_0$. Show that $y \in l^1$.

2. Find a Schauder basis for $(c, \|\cdot\|_\infty)$.

3.* Let X and Y be Banach spaces and $T \in \mathcal{L}(X;Y)$ show that either $\mathcal{R}(T)$ is of the first category in Y or $\mathcal{R}(T) = Y$.

4.* Let X be a linear space and $\|\cdot\|_1$, $\|\cdot\|_2$ be norms on X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ both are complete, but $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent. Show that there are points $y, z \in X$ with $y \neq z$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $||x_n - y||_1 \to 0$ and $||x_n - z||_2 \to 0$ as $n \to \infty$.

5. Let X and Y be Banach spaces and $T \in \mathcal{L}(X, Y)$ be given. Show that there exists $c > 0$ such that $||Tx|| \geq c||x||$ for all $x \in X$ if and only if $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed.

6.* Give an example of Banach spaces X and Y and a continuous linear injection $T : X \to Y$ such that $\mathcal{R}(T)$ is not closed.

7.* Let X, Y, Z be Banach space and $U : X \to Y$ and $V : Y \to Z$ be linear mappings and define $T : X \to Z$ by $Tx = VUx$ for all $x \in X$ (i.e. $T = V \circ U$). Assume that T is continuous and that V is injective and continuous. Show that U is continuous.

8.* Let $p, q \in [1, \infty)$ and $\alpha, \beta \in \mathbb{R}^{\mathbb{N}}$ be given and assume that $\alpha_n > 0$, $\beta_n > 0$ for all $n \in \mathbb{N}$ Let

$$X := \left\{ x \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} \alpha_n |x_n|^p < \infty \right\}$$

$$Y := \left\{ y \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} \beta_n |y_n|^q < \infty \right\}$$

and put

$$||x||_X := \left(\sum_{n=1}^{\infty} \alpha_n |x_n|^p\right)^{1/p} \qquad \forall x \in X$$

$$||y||_Y := \left(\sum_{n=1}^{\infty} \beta_n |y_n|^q\right)^{1/q} \qquad \forall y \in Y.$$

You may take it for granted that X and Y are Banach spaces.

Show that if $X \subset Y$ then there exists $K \in \mathbb{R}$ such that

$$||x||_Y \leq K||x||_X \text{ for all } x \in X.$$

9*. Show that there is normed linear space X, a Banach space Y, a linear mapping $T \in \mathcal{L}(X;Y)$, and an open set $\mathcal{O} \subset X$ such T is surjective, but $T[\mathcal{O}]$ is not open.

10.* Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence with $|\alpha_n| \to \infty$ as $n \to \infty$. Show that there is a continuous 2π-periodic function $g : \mathbb{R} \to \mathbb{R}$ such that the sequence $\left\{\alpha_n \int_0^{2\pi} g(x) \sin nx \, dx\right\}_{n=1}^{\infty}$ is unbounded.