Homework 6

21-651 General Topology
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Problem 1

(i) Suppose (X, τ) is a normal space, $A \subseteq X$ is closed in τ, and $f : A \to \mathbb{R}^I$ is continuous. $\forall i \in I$, let $f_i : A \to \mathbb{R}$, be the i^{th} coordinate of f, (i.e, $\forall x \in A$, $f_i(x) = f(x)(i)$). By Tietze's Extension Theorem, $\forall i \in I$, there is an extension $F_i : X \to \mathbb{R}$, of f_i to X. Let $F : X \to \mathbb{R}^I$ be the function whose i^{th} coordinate is F_i (so that, $F_i = \pi_i \circ F$, where π_i is the projection map into $\mathbb{R}$). Then, since each F_i is continuous, by Lemma 131, F is contiuous. Thus, f has a continuous extension F to X, so that $\mathbb{R}^I$ has the universal extension property. ■

(ii) If $X = \mathbb{R}$ and τ is the standard topology, then $A := \{0, 1\} \subseteq \mathbb{R}$ is closed under the standard topology. If $f : A \to A$ is the identity function, f is continuous (as the topology induced on A by τ is discrete), but it follows from the Intermediate Value Theorem that no extension of f to X is continuous. Thus, $\{0, 1\}$ does not have the universal extension property. ■

Problem 2

Since any closed set containing $\bigcup_{\alpha \in \Lambda} E_\alpha$ contains each E_α and thus each $\overline{E_\alpha}$, $\bigcup_{\alpha \in \Lambda} \overline{E_\alpha} \subseteq \overline{\bigcup_{\alpha \in \Lambda} E_\alpha}$.

Suppose, for sake of contradiction, that, for some $x \in \overline{\bigcup_{\alpha \in \Lambda} E_\alpha}$, $x \notin \bigcup_{\alpha \in \Lambda} \overline{E_\alpha}$. Since $\{E_\alpha\}_{\alpha \in \Lambda}$ is locally finite, x has a neighborhood $U \in \tau$ such that, for some finite $\Lambda_0 \subseteq \Lambda$, U intersects E_α only for $\alpha \in \Lambda_0$. Since $x \notin \bigcup_{\alpha \in \Lambda} \overline{E_\alpha}$, for each $\alpha \in \Lambda$, x has a neighborhood $U_\alpha \in \tau$ such that $U_\alpha \cap E_\alpha = \emptyset$. Then however, since Λ_0 is finite, $V := U \cap \bigcap_{\alpha \in \Lambda_0} U_\alpha \in \tau$ is a neighborhood of x with $V \cap \bigcup_{\alpha \in \Lambda} E_\alpha = \emptyset$, contradicting the fact that $x \in \overline{\bigcup_{\alpha \in \Lambda} E_\alpha}$. ■

Problem 3

Since the Sorgenfrey topology τ is finer than the standard topology on $\mathbb{R}$ (under which $\mathbb{R}$ is Hausdorff), $(\mathbb{R}, \tau)$, is Hausdorff.

I didn't have time to complete the proof that an open cover of the Sorgenfrey line has a locally finite refinement.

Problem 4

Note: I got this counterexample from Wikipedia, after being unable to find a proof or counterexample on my own. The proof that the Long Line is locally compact and Hausdorff is my own. However, I wasn't able to prove that the Long Line is not paracompact.

We construct the Long Line topology as follows:

Consider the lexicographical ordering $\prec$ on $X := \mathbb{R} \times [0,1)$ (so that $(x_0, y_0) \prec (x_1, y_1)$ if and only if $x_0 < x_1$ or $x_0 = x_1$ and $y_0 < y_1$, and consider the order topology τ on X generated by $\prec$ (the topology generated by the subbase $\{\{x \in X : a \prec x\} : a \in X\} \cup \{\{x \in X : x \prec b\} : b \in X\}$ (such sets can be written (a, ∞) and $(-\infty, b)$).

Suppose $(x_0, y_0), (x_1, y_1) \in X$ are distinct (so that, since $\prec$ is a total order, without loss of generality, $(x_0, y_0) \prec (x_1, y_1)$). Then, either $\left(-\infty, \left(\frac{|x_0 - x_1|}{2}, y_0\right)\right)$ and $\left(\left(\frac{|x_0 - x_1|}{2}, y_0\right), \infty\right)$ (if $x_0 < x_1$) or $\left(-\infty, \left(x_0, \frac{|y_0 - y_1|}{2}\right)\right)$ and $\left(x_0, \left(\frac{|y_0 - y_1|}{2}\right), \infty\right)$ (if $x_0 = x_1$) are disjoint open sets which separated (x_0, y_0) and (x_1, y_1). Thus, (X, τ) is Hausdorff.

For any $x \in \mathbb{R}$, $\{x\} \times [0,1) \subseteq X$ under the topology induced by τ is clearly homeomorphic to $[0,1) \subseteq R$ under the standard topology. Thus, since $\mathbb{R}$ is locally compact under the standard topology, (X, τ) is locally compact.

Problem 5

I was unable to come up with a counterexample for this one. However, Wikipedia claims that the following topology is a counterexample.

Let $X := \mathbb{Z}^+$, the set of positive integers, and, $\forall a, b \in X$, define $U_a(b) := \{b + na : n \in X \cup \{0\}\} \subseteq X$. Then, let τ be the topology generated by the subbase $\mathcal{B} := \{U_a(b) : a, b \in X, b \text{ is prime}\}$. Then, τ is second countable and Hausdorff, but not normal, and thus not metrizable.

Problem 6

Consider $\mathbb{R}$ under the Sorgenfrey topology τ. As shown in part (ii) of Example 23, $(\mathbb{R}, \tau)$ is first countable, as shown in part (ii) of Example 57, $(\mathbb{R}, \tau)$ is separable, and, as shown in Exercise 92, $(\mathbb{R}, \tau)$ is normal. Clearly, $(\mathbb{R}, \tau)$ is T_1, since, for any $x, y \in \mathbb{R}$, $\left[x, x + \frac{|x-y|}{2}\right)$ contains x but not y, and thus, $(\mathbb{R}, \tau)$ is T_4.

Since the Cartesian product of two second countable spaces is second countable and the Sorgenfrey plane is not second countable (as second countability is hereditary, and the anti-diagonal in $\mathbb{R}^2$ gives an uncountable subspace in which the induced topology is discrete), the Sorgenfrey line is not second countable.

Since $(\mathbb{R}, \tau)$ is separable but not second countable, it follows from Urysohn's Metrization Theorem that $(\mathbb{R}, \tau)$ is not metrizable, as desired. $\blacksquare$