Math 21-236, Mathematical Studies Analysis II, Spring 2012
Assignment 3

The due date for this assignment is Monday February 13.

A set $E \subseteq \mathbb{R}^N$ is *convex* if $\theta \mathbf{x} + (1-\theta)\,\mathbf{y}$ belongs to E for all $\mathbf{x}, \mathbf{y} \in E$ and all $\theta \in (0,1)$. Given a convex set $E \subseteq \mathbb{R}^N$, a function $f : E \to \mathbb{R}$ is *convex* if

$$f(\theta \mathbf{x} + (1-\theta)\,\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta)\, f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in E$ and all $\theta \in (0,1)$.

1. Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a convex function.

   (a) Prove that for every $x \in I$ there exist the left and right derivatives $f'_-(x)$, $f'_+(x)$.

   (b) Prove that for every $x, y \in I$, with $x < y$,

   $$f'_-(x) \leq \frac{f(y) - f(x)}{y - x} \leq f'_-(y) \leq f'_+(y)\,.$$

   (c) Prove that

   $$f(x) \geq f(y) + f'_-(x)\,(x - y)$$

   for all $x, y \in I$.

   (d) Let $U \subseteq \mathbb{R}^N$ be open and convex and let $g : U \to \mathbb{R}$ be a differentiable convex function. Prove that

   $$g(\mathbf{x}) \geq g(\mathbf{y}) + \nabla g(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$$

   for all $\mathbf{x}, \mathbf{y} \in U$.

2. Assume that $g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and that all its partial derivatives exist and are continuous. Given the normed space $C^1([a,b])$ with the norm

   $$\|f\| := \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |f'(x)|\,,$$

   consider the functional $G : C^1([a,b]) \to \mathbb{R}$ defined by

   $$G(f) := \int_a^b g(x, f(x), f'(x))\, dx, \quad f \in C^1([a,b])\,.$$

   (a) Given a function $h \in C([a,b])$ such that

   $$\int_a^b h(x)\, v'(x)\, dx = 0$$

   for all $v \in C^1([a,b])$ such that $v(a) = v(b) = 0$, prove that h is constant.

(b) Given two functions $p, q \in C([a,b])$ such that

$$\int_a^b [p(x)\, v(x)\ dx + q(x)\, v'(x)]\ dx = 0$$

for all $v \in C^1([a,b])$ such that $v(a) = v(b) = 0$, prove that q is of class $C^1([a,b])$ with $q' = p$.

(c) Given $\alpha, \beta \in \mathbb{R}$, let $X = \left\{f \in C^1([a,b]) :\ f(a) = \alpha,\ f(b) = \beta\right\}$. Prove that a necessary condition for $f_0 \in X$ to minimize G over X, that is,

$$\min_{f \in X} G(f) = G(f_0)$$

is that the function $q(x) := \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))$ is of class $C^1([a,b])$ with

$$\frac{d}{dx}\left(\frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))\right) = \frac{\partial g}{\partial y}(x, f_0(x), f_0'(x)). \qquad (1)$$

(d) Prove that if for every $(x,y) \in [a,b] \times \mathbb{R}$ the function $z \in \mathbb{R} \mapsto g(x,y,z)$ is convex and if $f_0 \in X$ satisfies (1), then f_0 is a minimizer of G over X.

3. Prove that the minimum of the following functionals does not exist.

(a) $G(f) = \int_0^1 e^{-\left(f'(x)\right)^2}\, dx$, $X = \left\{f \in C^1([0,1]) :\ f(0) = f(1) = 0\right\}$,

(b) $G(f) = \int_0^1 \left[\left(f'(x)\right)^2 - 1\right]^2 dx$, $X = \left\{f \in C^1([0,1]) :\ f(0) = f(1) = 0\right\}$,

(c) $G(f) = \int_0^1 \left[x\left(f'(x)\right)^2\right] dx$, $X = \left\{f \in C^1([0,1]) :\ f(0) = 1,\ f(1) = 0\right\}$.

4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) := \sum_{j=0}^{4} a_j \frac{x^j y^{4-j}}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0),$$
$$f(0,0) := 0,$$

where a_0, a_1, a_2, a_3, $a_4 \in \mathbb{R}$.

(a) Calculate the Hessian matrix

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}$$

for all $(x,y) \in \mathbb{R}^2$ and find a necessary and sufficient condition on a_0, a_1, a_2, a_3, a_4 for H_f to be symmetric.

(b) Find a necessary and sufficient condition on a_0, a_1, a_2, a_3, a_4 for ∇f to be everywhere differentiable.

(c) Prove that if $n \in \mathbb{N}$ is sufficiently large, then the function

$$g(x,y) := f(x,y) + n\left(x^2 + y^2\right)$$

is convex, but for appropriate values of a_0, a_1, a_2, a_3, a_4, $H_g(0,0)$ is not symmetric or ∇g is not everywhere differentiable.