21-630 Assignment #8 Due Wed., March 27, 2013

1. Consider $\dot{Y}(t) = AY(t) + F(t, Y(t))$ with

$$\left|e^{At}\right| \leq Be^{-\sigma t} \qquad B \geq 1, \sigma > 0$$

and

$$|F(t,y)| \leq Be^{\beta t}\,|y|^2 \qquad 0 < \beta < \sigma.$$

Show 0 is asymptotically stable.

Suggestion: For $t_0 \geq 0$ and $|Y(t_0)| < \dfrac{\sigma-\beta}{2B^3}\, e^{-\beta t_0}$ consider

$$T = \sup\left\{t > t_0 : |Y(s)|\, e^{\beta s} < \frac{\sigma - \beta}{B^2} \text{ on } [t_0, t]\right\}.$$

Then adapt the proof of the theorem on page 93 of the notes.

2. Find all critical points for

$$\begin{aligned}\dot{X} &= X(1 - X - Y)\\ \dot{Y} &= Y(3 - 2X - Y).\end{aligned}$$

Determine the stability of each.

3. For any $\varepsilon > 0$ find $\delta > 0$ such that $\ddot{X} + X + 2X^3 = 0$ and $\sqrt{X^2(0) + \dot{X}^2(0)} < \delta \Rightarrow \sqrt{X^2(t) + \dot{X}^2(t)} < \varepsilon$ for all $t \geq 0$.

4. For any $\varepsilon > 0$ find $\delta > 0$ such that $\ddot{X} + X - 2X^3 = 0$ and $\sqrt{X^2(0) + \dot{X}^2(0)} < \delta \Rightarrow \sqrt{X^2(t) + \dot{X}^2(t)} < \varepsilon$ for all $t \geq 0$.

Note: In problems 3 and 4, 0 may be shown to be stable by use of theorem 5.4 on page 108 of the notes. However, problems 3 and 4 ask you to find δ explicitly, given ϵ, not just show that δ exists. Problem 3 is quite easy, 4 is harder. Since theorem 5.4 applies, study of the proof of theorem 5.4 might yield insight into working problem 4.