Homework 5

21-640 Introduction to Functional Analysis
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Problem 1

We prove the contrapositive statement.

Suppose $\liminf_{n\to\infty} f(x_n) < f(x)$. Then, there exist $\varepsilon > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that each $f(x_{n_k}) \leq f(x) - \varepsilon$. Since f is lower semi-continuous, $B := \{y : f(y) \leq f(x) - \varepsilon\}$ is closed. Furthermore, B is convex, since, if $y_1, y_2 \in B, t \in (0,1)$, then

$$f(ty_1 + (1-t)y_2) \leq tf(y_1) + (1-t)f(y_2) \leq f(x) - \varepsilon.$$

By Theorem 8.12, since $x \notin B$, $x_{n_k} \not\rightharpoonup x$ as $k \to \infty$. But then $x_n \not\rightharpoonup \infty$ as $n \to \infty$. ∎

Problem 2

Since f is proper, $\exists x_0 \in X$ with $f(x_0) < \infty$. Define $B := \{x \in X : f(x) \leq f(x_0)\}$, and let $m := \inf f[B]$ (*a priori*, m may be $-\infty$). Note that it suffices to show f achieves m.

Since m is an infimum, there is a sequence $\{x_n\}_{n=1}^{\infty}$ with each $x_n \in B$ and $f(x_n) \to m$ as $n \to \infty$.

Since f is coercive, B is bounded, and so, by Theorem 8.1, $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging weakly to some $x \in X$. Since $f(x_{n_k}) \to m$ as $k \to \infty$, by the result of problem 1,

$$m = \liminf_{k\to\infty} f(x_{n_k}) \geq f(x) \quad ∎.$$

Problem 5

Linearity of T is clear, since each coordinate of Tx is a sum of coordinates of x. Define

$$M := \sup\left\{\sum_{n=1}^{\infty} |a_{mn}| : m \in \mathbb{N}\right\} \in \mathbb{R}.$$

If $x \in c_0$ with $\|x\|_\infty = 1$, then

$$\|Tx\| = \sup_{m\in\mathbb{N}} \left|\sum_{n=1}^{\infty} a_{mn}x_n\right| \leq \sup_{m\in\mathbb{N}} \sum_{n=1}^{\infty} |a_{mn}| = M,$$

and hence $T \in \mathcal{L}(c_0, c_0)$.

Define L for all $n \in \mathbb{N}, y^* \in l^1$ by

$$(Ly^*)_n = \sum_{m=1}^{\infty} a_{mn} y_m^*.$$

We first check that $L \in \mathcal{L}(l^1, l^1)$. Since L is clearly linear, it suffices to observe that, if $y^* \in l^1$,

$$\|Ly^*\|_1 = \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} a_{mn} y_m^* \right| \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn} y_m^*| \leq \sum_{m=1}^{\infty} |y_m^*| \sum_{n=1}^{\infty} |a_{mn}| \leq M \sum_{m=1}^{\infty} |y_m^*| = M\|y^*\|_1$$

(we can switch the order of summation, since each term is non-negative).

Then, by the usual identification of l^1 with c_0^*, if $y^* \in l^1, x \in c_0$,

$$\langle Ly^*, x \rangle = \sum_{n=1}^{\infty} (Ly^*)_n \, x_n = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{mn} \right) x_n$$

Hence, $L = T^*$. $\blacksquare$

Problem 7

We already mentioned in Remark 10.6 that $Z \subseteq \left({}^{\perp}Z\right)^{\perp}$.

Let $X = (l^1, \|\cdot\|_1)$, and let $Z = c_0$. Identifying the dual of X with l^∞ in the usual way, we note $Z \subseteq X^*$, and, since $(c_0, \|\cdot\|_\infty)$ is complete, Z is closed in X^*. If $x \in X$ is non-zero, then (noting $l^1 \subseteq c_0$), $\langle x, x \rangle \neq 0$, and hence $x \notin^{\perp} Z$. Thus, ${}^{\perp}Z = \{0\}$, and so $\left({}^{\perp}Z\right)^{\perp} = l^\infty$. But $c_0 \subsetneq l^\infty$. $\blacksquare$

Problem 8

For first step of this proof (showing $cl(\mathcal{R}(T))$ contains a ball in Y), I roughly followed the proof of Theorem 4.13 in Rudin's *Functional Analysis*.

If $y_0 \in Y \backslash cl(T[B_1(0)])$, then, by Theorem 8.13 and the fact that $T[B_1(0)]$ is balanced, there exists $y^* \in Y^*$ such that $\|y^*(y)\| \leq \|y^*(y_0)\|, \forall y \in T[B_1(0)]$. Note that $\forall x \in X$,

$$\|\langle T^* y^*, x \rangle\| = \|\langle y^*, Tx \rangle\| < \|y^*(y_0)\|.$$

Thus, $\|T^* y^*\| \leq \|y^*(y_0)\|$, and so

$$\|y_0\| \geq \frac{\|y_0\| \|T^* y^*\|}{\|y^*(y_0)\|} \geq \frac{\|y_0\| c \|y^*\|}{\|y^*(y_0)\|} \geq \frac{c\|y^*(y_0)\|}{\|y^*(y_0)\|} = c.$$

Thus, if $\|y\| \leq c$, then $y \in cl(T[B_1(0)])$, and so $B_{c/2}(0) \subseteq T[B_1(0)]$.

Then, by a proof identical to the proof of Lemma 4.2 (note that, in Lemma 4.2, surjectivity is only used to cite Lemma 4.1, whose result we already have), we have $B_{c/2}(0) \subseteq T[B_2(0)]$.

It now follows from linearity that, $\forall x \in X$, $B_{c/2}(Tx) \subseteq \mathcal{R}(T)$. Thus, it suffices to show that $\mathcal{R}(T)$ is dense in Y. Since S^* is clearly injective, $\mathcal{N}(S^*) = \{0\}$, and hence

$$cl(\mathcal{R}(T)) =^{\perp} (\mathcal{N}(S^*) = Y. \quad \blacksquare$$