21-484 Notes

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Wagner's Theorem: G is planar iff no K_5 or $K_{3,3}$ minor.

Def: (p. 252): A graph G is minimally nonembeddable in S_k if

- $\rightarrow$ G is not embeddable in S_k.
- $\rightarrow$ removing any vertex or any edge or contracting any edge results in a graph embeddable in S_k.

$\rightarrow$ Wagner's Theorem: K_5 and $K_{3,3}$ are the only minimally nonembeddable graphs in S_0.

$\rightarrow$ G is either not planar or has a K_5 or $K_{3,3}$ minor.

$\rightarrow$ Theorem (Seymor and Robertson, 1983-2004, graph minor theorem)

For any infinite set of graphs, there are two graphs such that one is a minor of the other.

Corollary: Every family of graphs closed under taking minors can be defined by a finite set of forbidden minors.

$\rightarrow$ Otherwise we have an infinite set of forbidden minors $\lightning$ the theorem above.

Corollary: (Cor 9.7) For all $k \geq 0$, the set of minimally nonembeddable graphs in S_k is finite.

Corollary 9.8 $\forall k \geq 0$ there is a finite set of graphs S such that G is embeddable in S_k iff it does not have an H minor for every H in S.

$\rightarrow$ For S_1, the set of forbidden minors is of size at least 800.

Recall: For every planar G, $\delta(G) \leq 5$.

Corollary: If G is planar, then $\chi(G) \leq 6$.

Proof: Given a graph $G = G_n$ we can find a vertex of degree ≤ 5, call it v_n and remove it to get G_{n-1}.

$\rightarrow$ G_{n-1} is also planar, so we can repeat this process.

$\rightarrow$ Color the vertices in order. Notice that when coloring a vertex, it has at most 5 colored neighbors. So one of the 6 colors is available.

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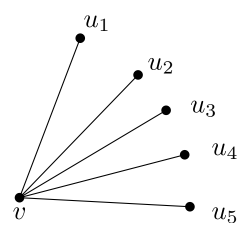
Theorem: Every planar graph is 5 colorable.

Proof: By induction on the number of vertices. If G has a vertex of degree ≤ 4, remove it, color the resulting graph with 5 colors, and add it back.

$\rightarrow$ Since $\delta(G) \leq 5$, there is a vertex of degree 5. call it v.

$\rightarrow$ Draw G in the plane to get a plane graph and name the neighbors of v according to the order in which they appear in the plane graph.

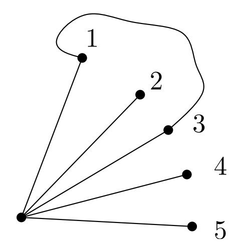


$\rightarrow$ Color $G \setminus \{v\}$ with 5 colors.

$\rightarrow$ If 2 of the u_i's are colored in the same color, we're done.

$\rightarrow$ Assume u_i is colored by color i.

$\rightarrow$ Let G_{13} be the graph spanned by vertices colored 1 or 3.



$\rightarrow$ If u_1 and u_3 are not in the same component, switch colors in the component containing u_3.

$\rightarrow$ If u_1 and u_3 are in the same component, there is a u_1–u_3 path in which all vertices are colored 1 or 3.

$\rightarrow$ Repeat for G_{24}. If they are in the same component there is a u_2–u_4 path in which all vertices are colored 2 or 4. ↯ planarity