Lecture Notes for Week 1

Roughly speaking *functional analysis* is the study of algebra, topology, and geometry of linear spaces that need not be finite dimensional, and of mappings between such spaces. In classical functional analysis, emphasis is usually placed on linear mappings.

The prerequisites for this course are a thorough understanding of linear algebra and topology (especially metric spaces), and familiarity and comfort with the Lebesgue integral. Students are (of course) expected to be proficient at writing careful and complete mathematical proofs. I have posted a set of notes that contains all of the results concerning metric spaces that I anticipate needing during the course.

I want to present a certain significant body of knowledge and also to familiarize you with a number of tools and standard arguments that are useful for proving results in functional analysis. In order to balance these objectives, I will skip the proofs of certain theorems (and simply give references instead) if I feel that the techniques needed for the proof have been (or will be) well exposed through other results whose proofs are discussed in detail elsewhere in the course.

The study of functional analysis was motivated by the fact that certain equations (such as Volterra integral equations) involving unknown functions have a structural similarity with systems of ordinary systems of (algebraic) linear equations. In order to exploit this structural similarity, one is led to consider linear spaces (or vector spaces) whose elements are functions rather than "ordinary vectors". Such *function spaces* are inherently infinite dimensional (meaning that there is no finite set of elements whose linear combinations span the entire space).

We shall study linear spaces (over $\mathbb{R}$ or $\mathbb{C}$), equipped with various topologies that are naturally adapted to the linear structure. We shall begin by looking at topologies that are induced by norms.

In finite-dimensional linear spaces (over $\mathbb{R}$ or $\mathbb{C}$), all norms are equivalent, all linear mappings are continuous, and the closed unit ball is compact. The situation is dramatically different in infinite dimensions. Moreover, every finite-dimensional linear space (over $\mathbb{R}$ or $\mathbb{C}$) is complete (in the sense that every Cauchy sequence is convergent). Infinite-dimensional normed linear spaces need not be complete. A number of important results concerning infinite-dimensional normed linear spaces require one or more of the spaces in question to be complete.

Baire Category

Many of the deepest and most important consequences of completeness can be obtained from the following result of Baire.

Theorem 1.1 (Baire): Let (X, ρ) be a complete metric space and $\{U_n\}_{n=1}^{\infty}$ be a sequence of subsets of X such that for every $n \in \mathbb{N}$, U_n is open and dense in X. Then

$$\bigcap_{n=1}^{\infty} U_n \text{ is dense in } X.$$

For the sake of completeness (no pun intended) we give a proof of Baire's Theorem. The following simple comment may make it easier to follow the proof. Let S be a subset of X. If $z \in \mathrm{int}(S)$ then we may choose $\eta > 0$ such that $\mathrm{cl}(B_\eta(z)) \subset S$. The validity of the comment follows immediately from the observation that

$$\mathrm{cl}(B_\eta(z)) \subset \{y \in X : \rho(y, z) \leq \eta\} \subset B_{2\eta}(z).$$

Proof of Baire's Theorem: Let $x \in X$ and $\delta > 0$ be given. It suffices to show that $B_\delta(x)$ contains a point that belongs to each of the sets U_n. We shall accomplish this by constructing a "shrinking" sequence $\{B_{\delta_n}(x_n)\}_{n=1}^{\infty}$ of balls. We show that the sequence $\{x_n\}_{n=1}^{\infty}$ of centers converges to a point l that belongs to $B_\delta(x)$ and to each of the U_n. For convenience, we choose the radii to satisfy $\delta_n \leq \frac{1}{n}$.

Since U_1 is dense in X, we may choose a point $x_1 \in U_1 \cap B_\delta(x)$. Since $U_1 \cap B_\delta(x)$ is open and $x_1 \in U_1 \cap B_\delta(x)$, we may choose $\delta_1 \in (0, 1]$ such that

$$B_{\delta_1}(x_1) \subset U_1 \cap B_\delta(x).$$

Since U_2 is dense, we may choose a point $x_2 \in U_2 \cap B_{\delta_1}(x_1)$. Moreover, since $U_2 \cap B_{\delta_1}(x_1)$ is open, we may choose $\delta_2 \in (0, \frac{1}{2}]$ such that

$$\mathrm{cl}(B_{\delta_2}(x_2)) \subset U_2 \cap B_{\delta_1}(x_1).$$

Proceeding by induction, we obtain, for each $n \in \mathbb{N}$ a point $x_n \in X$ and $\delta_n > 0$ such that

- $B_{\delta_n}(x_n) \subset U_n$,
- $\mathrm{cl}(B_{\delta_{n+1}}(x_{n+1})) \subset B_{\delta_n}(x_n)$,
- $\delta_n \leq \frac{1}{n}$.

Let $m, n, N \in \mathbb{N}$ with $m, n \geq N$ be given. Then $x_m, x_n \in B_{\delta_N}(x_n)$, so that

$$\rho(x_m, x_n) < 2\delta_N \leq \frac{2}{N},$$

which implies that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, we may choose $l \in X$ such that $x_n \to l$ as $n \to \infty$. Since $x_n \in B_{\delta_{N+1}}(x_{N+1})$ for all $n \geq N+1$, it follows that

$$l \in \mathrm{cl}(B_{\delta_{N+1}}(x_{N+1})) \subset B_{\delta_N}(x_N) \subset U_N.$$

Since $N \in \mathbb{N}$ was arbitrary, we conclude that

$$l \in \bigcap_{n=1}^{\infty} U_n.$$

Since $B_{\delta_1}(x_1) \subset B_\delta(x)$ and $l \in B_{\delta_1}(x_1)$, we conclude that $l \in B_\delta(x)$. $\square$

Definition 1.2: Let (X, ρ) be a metric space and $S \subset X$. We say that S is

(i) *nowhere dense* provided $\mathrm{int}(cl(S)) = \emptyset$.

(ii) *meager* if S can be expressed as a countable union of nowhere dense sets.

(iii) *residual* if S^c is meager.

Remark 1.3: It is traditional to say that meager sets are of the *first category* and non-meager sets are of the *second category*.

The following result is an easy consequence of Baire's Theorem.

Theorem 1.4 (Baire Category Theorem): Assume that $X \neq \emptyset$ and that (X, ρ) is complete. Then X cannot be expressed as a countable union of nowhere dense sets.

The way that we shall typically use this theorem in practice is as follows. We express

$$X = \bigcup_{n=1}^{\infty} S_n,$$

where each set S_n is closed. If $X \neq \emptyset$ and (X, ρ) is complete, then we can conclude that $\mathrm{int}(S_N) \neq \emptyset$ for some $N \in \mathbb{N}$.

Linear Spaces and Norms

We shall use the symbol $\mathbb{K}$ to denote a field that is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.5: Let X be a linear space over $\mathbb{K}$. By a *norm* on X we mean a function $\|\cdot\| : X \to \mathbb{R}$ satisfying

(i) $\forall x \in X, \quad \|x\| \geq 0,$

(ii) $\forall x \in X, \quad \|x\| = 0 \;\Leftrightarrow\; x = 0,$

(iii) $\forall x \in X, \alpha \in \mathbb{K}, \quad \|\alpha x\| = |\alpha| \cdot \|x\|$,

(iv) (triangle inequality) $\forall x, y \in X, \quad \|x+y\| \leq \|x\| + \|y\|$.

Definition 1.6: By a *normed linear space* (abbreviated *NLS*) we mean a pair $(X, \|\cdot\|)$ where X is a linear space over $\mathbb{K}$ and $\|\cdot\|$ is a norm on X.

Remark 1.7: It is useful to note that the triangle inequality (iv) can be reformulated as

(iv′) (reverse triangle inequality) $\forall x, y \in X, \quad |\|x\| - \|y\|| \leq \|x - y\|$.

In other words, if (i), (ii), and (iii) hold, then (iv) holds if and only if (iv′) holds. (You should verify this yourself as a simple exercise.)

In a NLS $(X, \|\cdot\|)$, the function $\rho : X \times X \to \mathbb{R}$ defined by

$$\rho(x, y) = \|x - y\| \text{ for all } x, y \in X$$

is a metric on X. It is called the *metric induced by the norm*. Metric and topological properties of $(X, \|\cdot\|)$ (such as completeness, continuity, compactness, etc.) are understood to be defined in terms of the metric induced by the norm unless stated otherwise. It follows immediately from (iv′) that the mapping $x \to \|x\|$ is continuous from (X, ρ) to $\mathbb{R}$. We note there are metrics on linear spaces that do not come from any norm.

Definition 1.8: A complete normed linear space is called a *Banach space* (or *B-space* for short).

There is sometimes confusion over terminology regarding subspaces of Banach spaces. It can happen that $(X, \|\cdot\|)$ is a Banach space, Y is a linear subspace of X, but $(Y, \|\cdot\|)$ is incomplete and hence is not a Banach space.

It seems useful to have a term to indicate that a subset of a linear space is itself a linear space, but need not have any special topological properties. Therefore we make the following definition.

Definition 1.9: Let X be a linear space. By a *linear manifold* in X we mean a nonempty subset of X that is closed under addition and scalar multiplication.

Proposition 1.10: Let $(X, \|\cdot\|)$ be a Banach space and Y be a linear manifold in X. Then $(Y, \|\cdot\|)$ is a Banach space if and only if Y is a (topologically) closed subset of X.

Proof: Assume first that Y is a closed subset of X and let $\{y_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(Y, \|\cdot\|)$. Then $\{y_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $(X, \|\cdot\|)$. Since

$(X, \|\cdot\|)$ is complete, we may choose $z \in X$ such that $\|y_n - z\| \to 0$ as $n \to \infty$. Since Y is closed, we conclude that $z \in Y$ and consequently $\{y_n\}_{n=1}^{\infty}$ is convergent (to z) in $(Y, \|\cdot\|)$. It follows that $(Y, \|\cdot\|)$ is complete.

Conversely, assume that $(Y, \|\cdot\|)$ is complete and let $\{y_n\}_{n=1}^{\infty}$ be a sequence such that $y_n \in Y$ for all $n \in \mathbb{N}$ and that $\{y_n\}_{n=1}^{\infty}$ is convergent in $(X, \|\cdot\|)$. Let $z = \lim_{n\to\infty} y_n$. We need to show that $z \in Y$. Since $(Y, \|\cdot\|)$ is complete, we may choose $w \in Y$ such that $\|y_n - w\| \to 0$ as $n \to \infty$. Since

$$\|z - w\| \le \|y_n - z\| + \|y_n - w\| \quad \text{for all } n \in \mathbb{N},$$

we conclude that $z = w$ and consequently $z \in Y$. $\square$

Remark 1.11: The proof of Proposition 1.10 shows that if $(X, \|\cdot\|)$ is a normed linear space and Y is a linear manifold in X such that $(Y, \|\cdot\|)$ is complete, then Y is a closed subset of $(X, \|\cdot\|)$.

In finite-dimensional NLS, every linear manifold is a closed set. If $(X, \|\cdot\|)$ is an infinite-dimensional NLS and Y is a linear manifold in X, then Y may or may not be a closed subset of $(X, \|\cdot\|)$. In the infinite-dimensional case, many important results concerning linear manifolds require topological closedness. Since a topologically closed linear manifold in a Banach space $(X, \|\cdot\|)$ is itself a Banach space when equipped with $\|\cdot\|$, the following definition seems reasonable. (In, general a substructure of a mathematical structure should itself satisfy the defining properties of the structure in question. In particular, a subspace of a Banach space should be a Banach space. The definition below is consistent with this idea and does not conflict with any conventions in the literature that I am aware of.)

Definition 1.12: Let $(X, \|\cdot\|)$ be a normed linear space. By a *closed subspace* of $(X, \|\cdot\|)$ we mean a linear manifold Y in X such that Y is a closed subset of X.

Example 1.13: Let $\mathbb{K} = \mathbb{R}$ and X be the set of all convergent sequences $x = (x_k | k \in \mathbb{N})$ of real numbers, equipped with the norm defined by

$$\|x\| = \sup\{|x_k| : k \in \mathbb{N}\}.$$

Sequences in X will be indicated by the notation $\{x^{(n)}\}_{n=1}^{\infty}$. You should verify as an exercise that $(X, \|\cdot\|)$ is complete. Let V denote the linear manifold consisting of all real sequences having only finitely many nonzero terms, i.e. the set of all sequences $(x_k | k \in \mathbb{N})$ such that $\{k \in \mathbb{N} : x_k \neq 0\}$ is finite. Consider the sequence $\{y^{(n)}\}_{n=1}^{\infty}$ in V defined by

$$y_k^{(n)} = \begin{cases} \frac{1}{k} & \text{if } k \le n \\ \\ 0 & \text{if } k > n. \end{cases}$$

It is straightforward to show that $\{y^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $(X, \|\cdot\|)$ and converges to the element $y \in X$ given by $y_k = \frac{1}{k}$ for all $k \in \mathbb{N}$. Since $y \notin V$, we conclude that V is not closed in $(X, \|\cdot\|)$ and consequently $(V, \|\cdot\|)$ is not complete.

Let W denote the set of all real sequences that converge to 0. It is straightforward to show that W is the (topological) closure of V in $(X, \|\cdot\|)$. It follows that $(W, \|\cdot\|)$ is complete. $\square$

Linear Independence and Bases

The situation with NLS is especially interesting in cases where the underlying linear space X is "infinite dimensional". We need to talk about concepts such as "linear combination", "linear independence", "basis", and "dimension" in a way that is suited to infinite-dimensional as well as finite-dimensional spaces.

Let X be a linear space over $\mathbb{K}$ and let $(x_i | i \in I)$ be a *family* of elements of X. Here I can be any *index set* and we allow for the possibility that $x_i = x_j$ for distinct $i, j \in I$.

Definition 1.14: By a *linear combination* of $(x_i | i \in I)$ we mean an expression of the form

$$\sum_{i \in J} \alpha_i x_i,$$

where J is a finite subset of I and $\alpha_i \in \mathbb{K}$ for all $i \in J$.

Observe that in the definition of linear combination, only *finite* sums are allowed.

Definition 1.15: The set of all linear combinations of $(x_i | i \in I)$ is called the *span* of $(x_i | i \in I)$ and is denoted $\text{span}(x_i | i \in I)$.

Definition 1.16: The family $(x_i | i \in I)$ is said to be *linearly independent* provided that for every finite set $J \subset I$ and for every family $(\alpha_i | i \in J)$ of elements of $\mathbb{K}$ the condition

$$\sum_{i \in J} \alpha_i x_i = 0$$

implies that $\alpha_i = 0$ for all $i \in J$.

In other words, to say that a family is linearly independent means that no nontrivial linear combination can be equal to zero.

Remark 1.17: If $(x_i | i \in I)$ is linearly independent then it is injective, i.e., if $i, j \in I$ with $i \neq j$ then $x_i \neq x_j$.

Definition 1.18: The family $(x_i | i \in I)$ is said to be *linearly dependent* if it is not linearly independent.

Clearly, then, a family is linearly dependent if and only if there is a nontrivial linear combination that is equal to zero.

Definition 1.19: The family $(x_i|i \in I)$ is said to be a *Hamel basis* for X provided that $(x_i|i \in I)$ is linearly independent and $span(x_i|i \in I) = X$.

We note that a family $(x_i|i \in I)$ is a Hamel basis if and only if every element of X can be expressed as a linear combination of members of the basis in precisely one way.

Remark 1.20: The notions of linear combination, span, linear independence, linear dependence, and Hamel basis apply to subsets $S \subset X$ as well as families by regarding a set S as a family $(x|x \in S)$ through self-indexing.

Using Zorn's lemma, it is possible to show that every linear space has a Hamel basis. We give a slightly more useful result.

Proposition 1.21: Let S be a linearly independent subset of X. Then there is a Hamel basis B for X such that $S \subset B$.

Proof: Let $\mathcal{S}$ denote the collection of all linearly independent sets $A \subset X$ such that $S \subset A$, partially ordered by set inclusion. Let $\mathcal{C}$ be a chain (i.e., a totally ordered subset of $\mathcal{S}$). Then $\cup\mathcal{C}$ is an upper bound for $\mathcal{C}$. (See Problem 1 of Assignment 2.) It follows from Zorn's lemma that $\mathcal{S}$ has a maximal element B. To see that $\text{span}(B) = X$, suppose that $\text{span}(B)$ is a proper subset of X. Then we may choose $x \in X \backslash \text{span}(B)$. Since $x \notin \text{span}(B)$, it follows that $B \cup \{x\}$ is linear independent and this contradicts the maximality of B. $\square$

We shall have an extended discussion of Zorn's lemma when we prove the Hahn-Banach theorem.

Since $\emptyset$ is linearly independent, we have the following corollary.

Corollary 1.22: Every linear space over $\mathbb{K}$ has a Hamel basis.

Remark 1.23: It can be shown that every Hamel basis for a given linear space has the same cardinality. (See, for example, Goffman & Pederick.)

Definition 1.24: We say that X is *finitely generated* if there is a finite set $S \subset X$ such that $\text{span}(S) = X$.

Proposition 1.25: Assume that X is finitely generated. Then all Hamel bases for X are finite and have the same number of elements. (See, for example, *Linear Algebra* by Charles Curtis.)

Definition 1.26: If X is finitely generated, we say that X is *finite dimensional* and the number of elements in a Hamel basis for X is called the *dimension* of X. If X is not finitely generated, we say that X is *infinite dimensional*.

The following simple result is an immediate consequence of the definition of Hamel basis.

Proposition 1.27 : Let X be a linear space over $\mathbb{K}$ and $(x_i|i \in I)$ be a Hamel basis for X. Then, for every $i \in I$, there is a linear mapping $\alpha_i : X \to \mathbb{K}$ such that for every $x \in X$, (i) and (ii) below hold:

(i) $\{i \in I : \alpha_i(x) \neq 0\}$ is finite.

(ii) $x = \sum_{i \in I} \alpha_i(x) x_i$.

Definition 1.28: We refer to $(\alpha_i|i \in I)$ as the *family of coefficient mappings* for the basis $(x_i|i \in I)$.

Remark 1.29: It is interesting to note that no infinite-dimensional Banach space can have a countable Hamel basis.

To see why the remark is true, assume that $(X, \|\cdot\|)$ is an infinite-dimensional Banach space and suppose that $(x_i|i \in \mathbb{N})$ is a Hamel basis for X. For each $n \in \mathbb{N}$ let $V_n = \text{span}(x_1, x_2, \cdots, x_n)$. It is straightforward to show that $\text{int}(V_n) = \emptyset$. We shall see shortly that every finite-dimensional linear manifold is closed. It follows that each V_n is nowhere dense. On the other hand, we must have

$$X = \bigcup_{n=1}^{\infty} V_n,$$

which is impossible since X is complete. (Such a representation would violate the Baire Category Theorem.)

Remark 1.30: The preceding remark shows that there is no norm under which the space of all real sequences having only finitely many nonzero terms can be complete. (Indeed, the list $(e^{(i)}|i \in \mathbb{N})$, where $e_i^{(i)} = 1$ and $e_k^{(i)} = 0$ when $i \neq k$ is a Hamel basis for this space.) Similarly, there is no norm under which the space of all real polynomials can be complete.

Let $(X, \|\cdot\|)$ be a NLS and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. We associate with $\{x_n\}_{n=1}^{\infty}$ the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums given by

$$s_n = \sum_{k=1}^{n} x_k \text{ for all } n \in \mathbb{N}.$$

Definition 1.31: We say that the sequence $\{x_n\}_{n=1}^{\infty}$ is *summable*, or equivalently, that the series $\sum_{n=1}^{\infty} x_n$ is *convergent* provided that the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums is convergent. In this case we write

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n.$$

Definition 1.32: We say that the sequence $\{x_n\}_{n=1}^{\infty}$ is *absolutely summable*, or equivalently that the series $\sum_{n=1}^{\infty} x_n$ is *absolutely convergent* provided that the series $\sum_{n=1}^{\infty} \|x_n\|$ is convergent to a real number.

A very useful characterization of completeness is contained in the following simple proposition, whose proof is part of Assignment 2.

Prop. 1.33: A normed linear space $(X, \|\cdot\|)$ is complete if and only if every absolutely summable sequence is summable.

Hamel bases are of limited use in the study of infinite-dimensional NLS. No infinite-dimensional Banach space has a countable Hamel basis. Moreover, since the notion of a Hamel basis is a purely algebraic concept, the coefficients in the representation for a given vector x cannot be controlled in terms of $\|x\|$. There is another type of basis, known as a Schauder basis, that is very useful for certain types of computations in infinite-dimensional NLS.

Definition 1.34: A sequence $\{x_n\}_{n=1}^{\infty}$ is called a *Schauder basis* for $(X, \|\cdot\|)$ provided that for every $y \in X$ there is exactly one sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{K}$ such that

$$\sum_{n=1}^{\infty} \alpha_n x_n = y.$$

Proposition 1.35: If $(X, \|\cdot\|)$ has a Schauder basis then $(X, \|\cdot\|)$ is separable.

The proof of Proposition 1.35 is also part of Assignment 2. Notice that a NLS that has a Schauder basis can (in some sense) be identified with a sequence space.

Completeness and Compactness in Finite-Dimensional NLS

The next two propositions express important and well-known properties of finite-dimensional NLS. I assume that students are already familiar with these results.

Prop. 1.36: Every finite-dimensional NLS is complete.

Remark 1.37: It follows from Proposition 1.36 that every finite-dimensional linear manifold in a NLS is closed. It is customary to refer to finite-dimensional linear manifolds as finite-dimensional subspaces. (Since they are automatically closed, there is no danger of ambiguity with the use of the term subspace.)

Another very important property of finite-dimensional NLS is that closed bounded sets are compact.

Proposition 1.38 (Heine-Borel Theorem): Let $(X, \|\cdot\|)$ be a finite-dimensional normed linear space, and $S \subset X$. Then S is compact if and only if S is both closed and bounded.

Proposition 1.38 has numerous important consequences. We shall soon show that in infinite-dimensional NLS, there are always closed bounded sets that fail to be compact. This lack of compactness in infinite dimensions is perhaps the most significant difference between finite-dimensional and infinite-dimensional NLS.

We complete this section with a lemma that can be used to prove proposition 1.36 and the difficult direction of proposition 1.38.

Lemma 1.39: Let $(X, \|\cdot\|)$ be a normed linear space and let $N \in \mathbb{N}$ be given. Assume that $(x_i | i = 1, 2, \cdots, N)$ is a linearly independent list of elements of X. Then there exists a constant $c > 0$ (depending on the list of vectors) such that

$$\|\sum_{i=1}^{N} \alpha_i x_i\| \geq c \sum_{i=1}^{N} |\alpha_i| \text{ for all } \alpha_1, \alpha_2, \cdots, \alpha_N \in \mathbb{K}.$$

If you have not seen this result before, I suggest that you work out a proof as an exercise for yourself. (Suggestion: If no such c exists then you can construct a sequence $\{\alpha^{(n)}\}_{n=1}^{\infty}$ in $\mathbb{K}^N$ such that

$$\sum_{i=1}^{N} |\alpha_i^{(n)}| = 1 \text{ for all } n \in \mathbb{N}$$

and

$$\|\sum_{i=1}^{N} \alpha_i^{(n)} x_i\| \to 0 \text{ as } n \to \infty.$$

This should lead to a contradiction.)