Monday, August 29, 2011

1 Real Numbers

There are two ways to introduce the real numbers. The first is to give them in an axiomatic way, the second is to construct them starting from the natural numbers. We will use the first method.

The *real numbers* are a set $\mathbb{R}$ with two binary operations, *addition*

$$\begin{aligned} + : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto x + y \end{aligned}$$

and *multiplication*

$$\begin{aligned} \cdot : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

and a relation $\leq$ such that

(A) $(\mathbb{R}, +) \leq$ is an *commutative group*, that is,

- (A_1) for every $a, b \in \mathbb{R}$, $a + b = b + a$,
- (A_2) for every $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$,
- (A_3) there exists a unique element in $\mathbb{R}$, called *zero* and denoted 0, such that $0 + a = a + 0 = a$ for every $a \in \mathbb{R}$,
- (A_4) for every $a \in \mathbb{R}$ there exists a unique element in $\mathbb{R}$, called the *opposite* of a and denoted $-a$, such that $(-a) + a = a + (-a) = 0$,

(M)

- (M_1) for every $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$,
- (M_2) for every $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- (M_3) there exists a unique element in $\mathbb{R}$, called *one* and denoted 1, such that $1 \neq 0$ and $1 \cdot a = a \cdot 1 = a$ for every $a \in \mathbb{R}$ with $a \neq 0$,
- (M_4) for every $a \in \mathbb{R}$ with $a \neq 0$ there exists a unique element in $\mathbb{R}$, called the *inverse* of a and denoted a^{-1}, such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$,

(O) $\leq$ is a *total order relation*, that is,

- (O_1) for every $a, b \in \mathbb{R}$ either $a \leq b$ or $b \leq a$,
- (O_2) for every $a, b, c \in \mathbb{R}$ if $a \leq b$ and $b \leq c$, then $a \leq c$,
- (O_3) for every $a, b \in \mathbb{R}$ if $a \leq b$ and $b \leq a$, then $a = b$,
- (O_4) for every $a \in \mathbb{R}$ we have $a \leq a$,

(AM) for every $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

(AO) for every $a, b, c \in \mathbb{R}$ if $a \leq b$, $a + c \leq b + c$,

(MO) for every $a, b \in \mathbb{R}$ if $0 \leq a$ and $0 \leq b$, then $0 \leq a \cdot b$,

(S) **(supremum property)**

Remark 1 *Properties (A), (M), (O), (AM), (AO), (MO), and (S) completely characterize the real numbers in the sense that if $(\mathbb{R}', \oplus, \odot, \preccurlyeq)$ satisfies the same properties, then there exists a bijection $T : \mathbb{R} \to \mathbb{R}'$ such that T is an isomorphism between the two fields, that is,*

$$T(a + b) = T(a) \oplus T(b), \qquad T(a \cdot b) = T(a) \odot T(b)$$

for all $a, b \in \mathbb{R}$, and $a \leq b$ if and only if $T(a) \preccurlyeq T(b)$. Hence, for all practical purposes, we cannot distinguish $\mathbb{R}$ from $\mathbb{R}'$.

If $a \leq b$ and $a \neq b$, we write $a < b$.

Exercise 2 *Using **only** the axioms (A), (M), (O), (AO), (AM) and (MO) of $\mathbb{R}$, prove the following properties of $\mathbb{R}$:*

(i) *if $a \cdot b = 0$ then either $a = 0$ or $b = 0$,*

(ii) *if $a \geq 0$ then $-a \leq 0$,*

(iii) *if $a \leq b$ and $c < 0$ then $ac \geq bc$,*

(iv) *for every $a \in \mathbb{R}$ we have $a^2 \geq 0$,*

(v) $1 > 0$.

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Definition 3 *Let $E \subseteq \mathbb{R}$ be a nonempty set.*

(i) *An element $L \in \mathbb{R}$ is called an* upper bound *of E if $x \leq L$ for all $x \in E$;*

(ii) *E is said to be* bounded from above *if it has at least an upper bound;*

(iii) *if E is bounded from above, the least of all its upper bounds, if it exists, is called the* supremum *of E and is denoted* $\sup E$.

(iv) *E has a* maximum *if there exists $L \in E$ such that $x \leq L$ for all $x \in E$. We write $L = \max E$.*

We are now ready to state the supremum property.

(S) **(supremum property)** every nonempty set $E \subseteq \mathbb{R}$ bounded from above has a supremum in $\mathbb{R}$.

The supremum property says that in $\mathbb{R}$ the supremum of a nonempty set bounded from above always exists in $\mathbb{R}$. We will see that this is not the case for the rationals numbers.

Remark 4 *(i) Note that if a set has a maximum L, then L is also the supremum of the set.*

(ii) If $E \subseteq \mathbb{R}$ is a set bounded from below, to prove that a number $L \in \mathbb{R}$ is the supremum of E, we need to show that L is an upper bound of E, that is, that $x \leq L$ for every $x \in E$, and that any number $s < L$ cannot be an upper bound of E, that is, that there exists $x \in E$ such that $s < x$.

Definition 5 *Let $E \subseteq \mathbb{R}$ be a nonempty set.*

(i) An element $\ell \in \mathbb{R}$ is called a lower bound *of E if $\ell \leq x$ for all $x \in E$;*

(ii) E is said to be bounded from below *if it has at least an lower bound;*

(iii) if E is bounded from above, the greatest of all its lower bounds, if it exists, is called the infimum *of E and is denoted* $\inf E$*;*

(iv) E has a minimum *if there exists $\ell \in E$ such that $\ell \leq x$ for all $x \in E$. We write $\ell = \min E$.*

Remark 6 *(i) Note that if a set has a minimum ℓ, then ℓ is also the infimum of the set.*

(ii) If $E \subseteq \mathbb{R}$ is a set bounded from above, to prove that a number $\ell \in \mathbb{R}$ is the infimum of E, we need to show that ℓ is a lower bound of E, that is, that $\ell \leq x$ for every $x \in E$, and that any number $\ell < s$ cannot be a lower bound of E, that is, that there exists $x \in E$ such that $x < s$.

Example 7 *Consider the set*

$$E = \left\{ y \in \mathbb{R} : \, y = \frac{n^2 + 2n}{n^2 + 2}, \, n \in \mathbb{N} \right\}.$$

Since $\frac{n^2+2n}{n^2+2} > 0$, the set E is bounded from below. To see if it is bounded from above, let's sketch the graph of the function

$$f(x) = \frac{x^2 + 2x}{x^2 + 2}$$

for $x \geq 1$. We have

$$\begin{aligned}
\lim_{x \to \infty} \frac{x^2 + 2x}{x^2 + 2} &= \lim_{x \to \infty} \frac{x^2 \left(1 + \frac{2}{x}\right)}{x^2 \left(1 + \frac{2}{x^2}\right)} \\
&= \lim_{x \to \infty} \frac{1 + \frac{2}{x}}{1 + \frac{2}{x^2}} = \frac{1 + \frac{2}{\infty}}{1 + \frac{2}{\infty^2}} = \frac{1 + 0}{1 + 0} = 1.
\end{aligned}$$

Moreover,

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+2x}{x^2+2}\right) = \frac{2\left(-x^2+2x+2\right)}{\left(x^2+2\right)^2} \geq 0$$

for all $-\sqrt{3}+1 \leq x \leq \sqrt{3}+1$. *Hence,* f *is increasing in* $\left[1, \sqrt{3}+1\right]$ *and decreasing for* $x \geq \sqrt{3}+1$. *Not that* $2 \leq \sqrt{3}+1 \leq 3$. *This implies that*

$$\sup E = \max E = \max\{f(2), f(3)\} = \max\left\{\frac{4}{3}, \frac{15}{11}\right\} = \frac{15}{11},$$

while

$$\inf E = \min\left\{f(1), \lim_{n\to\infty} f(n)\right\} = \min\{1, 1\} = 1,$$

so actually $\inf E = \min E = f(1) = 1$.

Example 8 *Consider the set*

$$E = \left\{t \in \mathbb{R} : \, t = \frac{xy}{x^2+y^2}, \, x, y \in \mathbb{R}, \, x < y\right\}.$$

The set E *is bounded since* $-\frac{1}{2} \leq \frac{xy}{x^2+y^2} \leq \frac{1}{2}$ *for all* $x, y \in \mathbb{R}$, *with* $x < y$. *Moreover, by taking* $x = -1$ *and* $y = 1$, *we get* $t = \frac{-1}{2}$, *so*

$$\inf E = \min E = -\frac{1}{2}.$$

Let's prove that

$$\sup E = \frac{1}{2}.$$

We need to show that any $s < \frac{1}{2}$ *is not an upper bound for the set* E. *If* $s \leq -\frac{1}{2}$, *then we can take* $x = -1$ *and* $y = 0$, *so that* $s < t = 0$. *Thus assume that* $-\frac{1}{2} < s < \frac{1}{2}$. *Take the sequence* $x_n = 1 - \frac{1}{n} < y_n = 1$. *Then*

$$t_n = \frac{x_n y_n}{x_n^2 + y_n^2} = \frac{1-\frac{1}{n}}{\left(1-\frac{1}{n}\right)^2+1} > s,$$

which gives

$$1 - \frac{1}{n} > s\left(\left(1-\frac{1}{n}\right)^2 + 1\right) \quad \text{or} \quad 2sn^2 - 2sn + s < n^2 - n$$
$$\text{or} \quad 0 = (1-2s)\, n^2 + (2s-1)\, n - s,$$

that is,

$$n > \frac{1 - 2s + \sqrt{-4s^2+1}}{2(1-2s)}$$

Note that $-4s^2 + 1 > 0$ *and* $1 - 2s > 0$ *for* $-\frac{1}{2} < s < \frac{1}{2}$. *Hence, for all* n *sufficiently large,* $t_n > s$, *which shows that* s *is not an upper bound of* E. *Thus,* $\frac{1}{2}$ *is the supremum of the set. Note that* $t = \frac{xy}{x^2+y^2} = \frac{1}{2}$ *only if* $x = y$, *which is not allowed, so the set does not have a maximum.*

2 Natural Numbers

Definition 9 *A set $E \subseteq \mathbb{R}$ is called an* inductive set *if it has the following properties*

(i) the number 1 belongs to E,

(ii) if a number x belongs to E, then $x+1$ also belongs to E.

Example 10 *The sets $[0,\infty) = \{x \in \mathbb{R} : 0 \leq x\}$, $[1,\infty) = \{x \in \mathbb{R} : 1 \leq x\}$, and $\mathbb{R}$ are all inductive sets.*

Definition 11 *The set of the* natural numbers $\mathbb{N}$ *is defined as the intersection of all inductive sets of $\mathbb{R}$.*

Note that $\mathbb{N}$ is nonempty, since 1 belongs to every inductive set, and so also to $\mathbb{N}$. We also define

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

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Proposition 12 *The set $\mathbb{N}$ is an inductive set.*

Proof. We already know that 1 belongs to $\mathbb{N}$. If x belongs to $\mathbb{N}$, then it belongs to every inductive set E but then, since E is an inductive set, it follows that $x+1$ belongs E. Hence, $x+1$ belongs to every inductive set, and so by definition of $\mathbb{N}$, we have that $x+1$ also belongs to $\mathbb{N}$. ■

The next result is very important.

Theorem 13 (Principle of mathematical induction) *Let $\{p_n\}$, $n \in \mathbb{N}$, be a family of propositions such that*

(i) p_1 is true,

(ii) if p_n is true for some $n \in \mathbb{N}$, then p_{n+1} is also true.

Then p_n is true for every *$n \in \mathbb{N}$.*

Proof. Let $E := \{n \in \mathbb{N} \text{ such that } p_n \text{ is true}\}$. Note that $E \subseteq \mathbb{N}$. It follows by (i) and (ii) that E is an inductive set, and so E contains $\mathbb{N}$ (since $\mathbb{N}$ is the intersection of all inductive sets). Hence, $E = \mathbb{N}$. ■

If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we define

$$x^n := \underbrace{x \cdot \cdots \cdot x}_{n \text{ times}}.$$

If $x \neq 0$, we define $x^0 := 1$. We do not define 0^0.

The following example and exercises will be used later on.

Example 14 *Let $x \geq -1$. Let's prove that*

$$(1+x)^n \geq 1+nx \tag{1}$$

for every $n \in \mathbb{N}$. For $n=1$, we have $(1+x)^1 \geq 1+1x$, which is true. Assume that for some $n \in \mathbb{N}$, the inequality (1) holds. We want to prove that $(1+x)^{n+1} \geq 1+(n+1)x$. To see this, observe that

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n \geq (1+x)(1+nx) \\ &= 1+(n+1)x+nx^2 \geq 1+(n+1)x+0 = 1+(n+1)x,\end{aligned}$$

where in the first inequality we have used the fact that $1+x \geq 0$. Hence, by the principle of mathematical induction, the inequality (1) holds for every $n \in \mathbb{N}$.

Remark 15 *For $x < -1$ the inequality (1) is false in general, take $x = -3$ and $n \in \mathbb{N}$. Then*

$$(1-3)^n = (-2)^n = (-1)^n 2^n \overset{??}{\geq} 1-3n.$$

For n odd, $(-1)^n = -1$, and so $-2^n \overset{??}{\geq} 1-3n$, or, equivalently, $2^n \overset{??}{\leq} 3n-1$,which is false for all n odd large. Take n = 4, you get $2^4 = 16 \leq 12-1$, which is false.

Example 16 *Prove that*

$$1+\cdots+n = \frac{n(n+1)}{2} \tag{2}$$

for every $n \in \mathbb{N}$

Exercise 17 *Let $x \neq 1$. Prove that*

$$1+x\cdots+x^n = \frac{x^{n+1}-1}{x-1}$$

for all $n \in \mathbb{N}$.

In what follows $0! := 1$, $1! := 1$ and $n! := 1 \cdot 2 \cdot \cdots \cdot n$ for all $n \in \mathbb{N}$. The number $n!$ is called the *factorial* of n. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

Theorem 18 (Binomial theorem) *Let $x, y \in \mathbb{R} \setminus \{0\}$ and let $n \in \mathbb{N}$. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. One possible proof is by induction (see the exercise below). We give another using partial derivatives. Since

$$(x+y)^n = (x+y) \cdot \cdots \cdot (x+y),$$

by expanding the right-hand side and an induction argument we find that

$$(x+y)^n = a_{0,n} x^0 y^n + a_{1,n-1} x^1 y^{n-1} + a_{2,n-2} x^2 y^{n-2} + \cdots + a_{n,0} x^n y^0,$$

where $a_{k,n-k}$ are natural numbers to be determined. If we consider the partial derivative $\frac{\partial^n}{\partial x^k \partial y^{n-k}}$ of $(x+y)^n$, we get

$$\frac{\partial^n}{\partial x^k \partial y^{n-k}} (x+y)^n = n!,$$

while

$$\frac{\partial^n}{\partial x^k \partial y^{n-k}} \left(x^j y^{n-j}\right) = \begin{cases} 0 & \text{if } k \neq j, \\ k!\,(n-k)! & \text{if } k = j. \end{cases}$$

Hence,

$$\begin{aligned} n! &= \frac{\partial^n}{\partial x^k \partial y^{n-k}} (x+y)^n \\ &= \frac{\partial^n}{\partial x^k \partial y^{n-k}} \left[a_{0,n} x^0 y^n + a_{1,n-1} x^1 y^{n-1} + a_{2,n-2} x^2 y^{n-2} + \cdots + a_{n,0} x^n y^0\right] \\ &= 0 + \cdots + 0 + k!\,(n-k)! a_{k,n-k} + 0 + \cdots + 0, \end{aligned}$$

which shows that

$$a_{k,n-k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}.$$

■

Exercise 19 *Let $j, k \in \mathbb{N}$ and $a \in \mathbb{R}$. Given the function $f(x) = (x+a)^j$, prove that*

$$\frac{d^k f}{dx^k}(x) = \begin{cases} 0 & \text{if } k > j, \\ j(j-1)\cdots(j-k+1)(x+a)^{j-k} & \text{if } k < j, \\ k! & \text{if } k = j. \end{cases}$$

Exercise 20 *Let $x, y \in \mathbb{R} \setminus \{0\}$ and let $n \in \mathbb{N}$.*

(i) Prove that for every $1 \leq k \leq n$,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

(ii) Prove the binomial theorem by induction on n.

Remark 21 *If in Theorem 13 we replace property (i) with* *(i)′ if p_{n_0} is true for some $n_0 \in \mathbb{N}$,*

then we can conclude that p_n is true for all $n \in \mathbb{N}$ with $n \geq n_0$. To see this, it is enough to define

$$E := \{n \in \mathbb{N} \text{ such that } p_{n+n_0-1} \text{ is true}\},$$

which is still an inductive set.

Exercise 22 *Prove that*

$$n^n > 2^n n!$$

for all $n > 6$. Hint: Use the binomial theorem.

3 The Rationals Numbers and the Supremum Property

In the previous section we have defined the natural numbers. Note that $(\mathbb{N}, +, \cdot, \leq)$ does not satisfy properties (A_3), (A_4), and (M_4). In particular, we cannot subtract two numbers $a, b \in \mathbb{N}$ unless, $a \geq b+1$. For this reason, we define the *set of integers* $\mathbb{Z}$ as follows

$$\mathbb{Z} := \{\pm n : n \in \mathbb{N}\} \cup \{0\}.$$

Now $(\mathbb{Z}, +, \cdot, \leq)$ satisfies properties (A_3), (A_4), but not (M_4). To resolve this issue, we introduce the *set of rational numbers* $\mathbb{Q}$ defined by

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\, q \neq 0 \right\},$$

where $\frac{p}{q} := p \cdot q^{-1}$. Then $(\mathbb{Q}, +, \cdot, \leq)$ satisfies properties (A), (M), (O), (AM), (AO), (MO). So, what's wrong?

Theorem 23 *There does not exist a rational number r such that $r^2 = 2$.*

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Labor day, no classes.

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Thus in the set of rational numbers the square root $\sqrt{r}$ is not defined, in general.

Theorem 24 *The rational numbers do not satisfy the supremum property.*

Proof. We need a nonempty set $E \subseteq \mathbb{Q}$ bounded from below but for which there exists no supremum in $\mathbb{Q}$. Define

$$E := \left\{ x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2 \right\}.$$

Then E is nonempty, since $1 \in E$. Moreover, E is bounded from below, since 2 is an upper bound.

Let's prove that if $y \in \mathbb{Q} \setminus E$ and $y > 0$, then y is an upper bound of E. Indeed, let $x \in E$. If $x > 0$, then $x^2 < 2 < y^2$, which, since $y > 0$, implies that $x < y$ (why?).

Assume by contradiction that there exists $L \in \mathbb{Q}$ such that $L = \sup E$. It cannot be $L \leq 0$, since $1 \in E$ and $1 > 0$. Hence, $L > 0$. Let's prove that it cannot be $L^2 < 2$. Choose $n \in \mathbb{N}$ so large that $n > \frac{2L+1}{2-L^2}$ (we will see later on that this can be done). Then

$$\left(L + \frac{1}{n}\right)^2 = L^2 + \frac{1}{n^2} + \frac{2L}{n} < L^2 + \frac{1}{n} + \frac{2L}{n} = L^2 + \frac{2L+1}{n} < 2,$$

by the choice of n. Hence, $L + \frac{1}{n}$ belongs to E, which contradicts the fact that L is an upper bound of E. Similarly, taking $L - \frac{1}{n}$, for n large, we can show that $\left(L - \frac{1}{n}\right)^2 > 2$ and $L - \frac{1}{n} > 0$, which, by what we proved before, shows that $L - \frac{1}{n}$ is an upper bound of E. This contradicts the fact that L is the least upper bound of E. Hence, it cannot be $L^2 > 2$, Thus, $L^2 = 2$, which is again a contradiction by Theorem 23. ■

The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational numbers*.

Theorem 25 *The set of irrational numbers is nonempty.*

Proof. Take
$$E := \left\{x \in \mathbb{R} : 0 < x \text{ and } x^2 < 2\right\}.$$
Exactly as in the previous proof, we have that E is nonempty and bounded from above. Hence, by the supremum property there exists $L \in \mathbb{R}$ such that $L = \sup E$. It follows as in the previous proof that $L^2 = 2$, and so L belongs to $\mathbb{R} \setminus \mathbb{Q}$. ■

The number L is denoted $\sqrt{2}$ and called *square root* of 2. Similarly, for every $n \in \mathbb{N}$ with n even and every $x \in \mathbb{R}$ with $x \geq 0$, we can show that there exists a unique $y \in \mathbb{R}$ with $y \geq 0$ such that $x^n = y$. On the other hand, for every $n \in \mathbb{N}$ with n odd and every $x \in \mathbb{R}$, we can show that there exists a unique $y \in \mathbb{R}$ such that $x^n = y$.

The number y is denoted $\sqrt[n]{x}$ and called *n-th root* of x.

Exercise 26 (The n-th root of a) *Given $a > 0$ and $n \in \mathbb{N}$, we want to define the n-th root of a.*

(i) Prove that the set

$$E := \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x > 0 \text{ and } x^n < a\}$$

is nonempty and bounded from above.

(ii) Let $L = \sup E$. Prove that $L^n = a$. Hint: Use the binomial theorem.

(iii) Prove that for every $x, y \in \mathbb{R}$, $x \neq 0$, $y \neq 0$, and $n \in \mathbb{N}$,

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right).$$

(iii) Prove that L is the only positive solution of the equation $x^n = a$.

In Theorem 24 we have used the fact that there exist arbitrarily large natural numbers. This follows from the Archimedean Property.

Proposition 27 (Archimedean Property) *If $a, b \in \mathbb{R}$ with $a > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$.*

Proof. If $b \leq 0$, then $n = 1$ will do. Thus, assume that $b > 0$. Assume by contradiction that $na \leq b$ for all $n \in \mathbb{N}$ and define the set

$$E = \{na : n \in \mathbb{N}\}.$$

Then the set E is nonempty and has an upper bound, b. By the supremum property, there exists $L = \sup E$. Hence, for every $m \in \mathbb{N}$, we have that $(m+1)\,a \leq L$, or, equivalently, $ma \leq L - a$ for all $m \in \mathbb{N}$. But this shows that $L - a$ is an upper bound of E, which contradicts the fact that L is the least upper bound. ■

The next result is left as an exercise.

Corollary 28 (The integer part) *Given a real number $x \in \mathbb{R}$, there exists an integer $k \in \mathbb{Z}$ such that $k \leq x < k + 1$.*

Definition 29 *Given a real number $x \in \mathbb{R}$, the integer k given by the previous corollary is called the* integer part *of x and is denoted $\lfloor x \rfloor$. The number $x - \lfloor x \rfloor$ is called the* fractional part *of x and is denoted $\operatorname{fr} x$ (or $\{x\}$). Note that $0 \leq \operatorname{fr} x < 1$.*

Exercise 30 *Prove that every nonempty subset of the natural numbers has a minimum.*

Corollary 31 (Density of the rationals) *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. By the Archimedean property (applied with 1 and $\frac{1}{b-a}$ in place of a and b) there exists $q \in \mathbb{N}$ such that $0 < \frac{1}{b-a} < q$. By the previous corollary there exists an integer $p \in \mathbb{Z}$ such that

$$p \leq qa < p + 1. \tag{3}$$

Note that since $1 < q\,(b - a)$,

$$p + 1 \leq qa + 1 < qa + q\,(b - a) = qb. \tag{4}$$

It follows by (3) and (4) that

$$qa < p + 1 < qb.$$

Multiplying by $\frac{1}{q} > 0$ gives

$$a < \frac{p+1}{q} < b.$$

It suffices to define $r := \frac{p+1}{q}$. ■

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Corollary 32 (Density of the irrationals) *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.*

Proof. Since $a < b$, we have that $\sqrt{2}a < \sqrt{2}b$. By the density of the rationals, there exists $r \in \mathbb{Q}$ such that $\sqrt{2}a < r < \sqrt{2}b$. Without loss of generality, we may assume that $r \neq 0$ (why?). Hence, $a < \frac{r}{\sqrt{2}} < b$. Since $\frac{r}{\sqrt{2}}$ is irrational (why?), the result is proved. ■

Given a number $x \in \mathbb{R}$, the *absolute value* of x is the number

$$|x| := \begin{cases} +x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The absolute value satisfies the following properties, which are left as as exercise.

Theorem 33 *Let $x, y, z \in \mathbb{R}$. Then the following properties hold.*

(i) $|x| \geq 0$ *for all* $x \in \mathbb{R}$, *with* $|x| = 0$ *if and only if* $x = 0$,

(ii) $|-x| = |x|$ *for all* $x \in \mathbb{R}$,

(iii) if $y \geq 0$ *and* $x \in \mathbb{R}$, *then* $|x| \leq y$ *if and only if* $-y \leq x \leq y$,

(iv) $-|x| \leq x \leq |x|$ *for all* $x \in \mathbb{R}$,

(iii) $|xy| = |x|\,|y|$ *for all* $x, y \in \mathbb{R}$,

(iv) $|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

4 The Euclidean Space

Definition 34 *A* vector space*, or* linear space*, over $\mathbb{R}$ is a nonempty set X, whose elements are called* vectors*, together with two operations,* addition *and* multiplication by scalars,

$$\begin{array}{ccc} X \times X \to X & & \mathbb{R} \times X \to X \\ (x, y) \mapsto x + y & \text{and} & (t, x) \mapsto tx \end{array}$$

with the properties that

(i) $(X, +)$ *is a commutative group, that is,*

(a) $x + y = y + x$ *for all* $x, y \in X$ *(commutative property),*

(b) $x + (y + z) = (x + y) + z$ *for all* $x, y, z \in X$ *(associative property),*

(c) there is a vector $0 \in X$*, called* zero*, such that* $x + 0 = 0 + x$ *for all* $x \in X$*,*

(d) for every $x \in X$ *there exists a vector in* X*, called the* opposite *of* x *and denoted* $-x$*, such that* $x + (-x) = 0$*,*

(ii) for all $x, y \in X$ *and* $s, t \in \mathbb{R}$*,*

(a) $s\,(tx) = (st)\,x$*,*

(b) $1x = x$*,*

(c) $s\,(x + y) = (sx) + (sy)$*,*

(d) $(s + t)x = (sx) + (tx)$*.*

Remark 35 *Instead of using real numbers, one can use* $\mathbb{C}$ *or a field* F*. For most or our purposes the real numbers will suffice.* From now on, whenever we don't specify, it is understood that a vector space is over $\mathbb{R}$.

Example 36 *Some important examples of vector spaces over* $\mathbb{R}$ *are the following.*

(i) The Euclidean space $\mathbb{R}^N$ *is the space of all* N*-tuples* $\mathbf{x} = (x_1, \dots, x_N)$ *of real numbers. The elements of* $\mathbb{R}^N$ *are called* vectors *or* points*. The Euclidean space is a vector space with the following operations*

$$\mathbf{x} + \mathbf{y} := (x_1 + y_1, \dots, x_N + y_N), \quad t\mathbf{x} := (tx_1, \dots, tx_N)$$

for every $t \in \mathbb{R}$ *and* $\mathbf{x} = (x_1, \dots, x_N)$ *and* $\mathbf{y} = (y_1, \dots, y_N)$ *in* $\mathbb{R}^N$*.*

(ii) The collection of all polynomials $p : \mathbb{R} \to \mathbb{R}$*.*

(iii) The space of continuous functions $f : [a, b] \to \mathbb{R}$*.*

Definition 37 *An* inner product*, or* scalar product*, on a vector space* X *is a function*

$$(\cdot, \cdot) : X \times X \to \mathbb{R}$$

such that

(i) $(x, x) \geq 0$ *for every* $x \in X$*,* $(x, x) = 0$ *if and only if* $x = 0$ *(positivity);*

(ii) $(x, y) = (y, x)$ *for all* $x, y \in X$ *(symmetry);*

(iii) $(sx + ty, z) = s\,(x, z) + t\,(y, z)$ *for all* $x, y, z \in X$ *and* $s, t \in \mathbb{R}$ *(bilinearity).*

Remark 38 *If X is a vector space over $\mathbb{C}$, then an inner product is a function $(\cdot,\cdot) : X \times X \to \mathbb{C}$ satisfying properties (i),*

(ii)′ $(x,y) = \overline{(y,x)}$ *for all $x, y \in X$ (skew-symmetry), where given $z \in \mathbb{C}$, the number $\overline{z}$ is the complex conjugate;*

(iii) $(sx + ty, z) = s(x,z) + t(y,z)$ *for all $x, y, z \in X$ and $s, t \in \mathbb{C}$ (bilinearity).*

Example 39 *Some important examples of inner products are the following.*

(i) Consider the Euclidean space $\mathbb{R}^N$, then

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \cdots + x_N y_N,$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$, is an inner product.

(ii) Consider the space of continuous functions $f : [a,b] \to \mathbb{R}$. Then

$$(f,g)_{L^2([a,b])} := \int_a^b f(x)\, g(x)\, dx$$

is an inner product.

Definition 40 *A* norm *on a vector space X is a map*

$$\|\cdot\| : X \to [0,\infty)$$

such that

(i) $\|x\| = 0$ *implies* $x = 0$;

(ii) $\|tx\| = |t|\,\|x\|$ *for all $x \in X$ and $t \in \mathbb{R}$;*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all $x, y \in X$.*

A *normed space* $(X, \|\cdot\|)$ is a vector space X endowed with a norm $\|\cdot\|$. For simplicity, we often say that X is a normed space.

Given an inner product $(\cdot,\cdot) : X \times X \to \mathbb{R}$ on a vector space X, it turns out that the function

$$\|x\| := \sqrt{(x,x)}, \quad x \in X, \tag{5}$$

is a norm. This follows from the following result.

Proposition 41 (Cauchy–Schwarz's inequality) *Given an inner product $(\cdot,\cdot) : X \times X \to \mathbb{R}$ on a vector space X,*

$$|(x,y)| \leq \|x\|\,\|y\|$$

for all $x, y \in X$.

Proof. If $y = 0$, then both sides of the previous inequality are zeros, and so there is nothing to prove. Thus, assume that $y \neq 0$ and let $t \in \mathbb{R}$. By properties (i)-(iii),

$$0 \leq (x + ty, x + ty) = \|x\|^2 + t^2 \|y\|^2 + 2t(x, y). \tag{6}$$

Taking

$$t := -\frac{(x, y)}{\|y\|^2}$$

in the previous inequality gives

$$0 \leq \|x\|^2 + \frac{(x, y)^2}{\|y\|^4} \|y\|^2 - 2\frac{(x, y)}{\|y\|^2},$$

or, equivalently,

$$(x, y)^2 \leq \|x\|^2 \|y\|^2.$$

It now suffices to take the square root on both sides. ■

Remark 42 *It follows from the proof that equality holds in the Cauchy–Schwarz inequality if and only if $x = sy$ for some $s \in \mathbb{R}$ or $y = 0$.*

Corollary 43 *Given a scalar product $(\cdot, \cdot) : X \times X \to \mathbb{R}$ on a vector space X, the function*

$$\|x\| := \sqrt{(x, x)}, \quad x \in X,$$

is a norm.

Proof. By property (i), $\|\cdot\|$ is well-defined and $\|x\| = 0$ if and only if $x = 0$. Taking $t = 1$ in (6) and using the Cauchy–Schwarz inequality gives

$$\begin{aligned} 0 \leq \|x + y\|^2 &= \|x\|^2 + \|y\|^2 + 2(x, y) \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\| \|y\| = (\|x\| + \|y\|)^2, \end{aligned}$$

which is the triangle inequality for the norm. Moreover, by properties (ii) and (iii) for every $t \in \mathbb{R}$,

$$\|tx\| = \sqrt{(tx, tx)} = \sqrt{t(x, tx)} = \sqrt{t(tx, x)} = \sqrt{t^2(x, x)} = |t| \|x\|.$$

Thus $\|\cdot\|$ is a norm. ■

Example 44 *Other important norms that one can put in $\mathbb{R}^N$ are*

$$\begin{aligned} \|\mathbf{x}\|_{\ell^\infty} &:= \max\{|x_1|, \dots, |x_N|\}, \\ \|\mathbf{x}\|_{\ell^1} &:= |x_1| + \cdots + |x_N|, \\ \|\mathbf{x}\|_{\ell^p} &:= (|x_1|^p + \cdots + |x_N|^p)^{1/p}, \end{aligned}$$

for $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and where $1 \leq p < \infty$.

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Proposition 45 (Parallelogram law) *Given an inner product* $(\cdot,\cdot) : X \times X \to \mathbb{R}$ *on a vector space* X,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$.

Proof. Taking $t = \pm 1$ in (6), we get

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2(x,y), \\ \|x-y\|^2 &= \|x\|^2 + \|y\|^2 - 2(x,y). \end{aligned}$$

By adding these identities, we obtain the desired result. ■

Theorem 46 *Let* $(X, \|\cdot\|)$ *be a normed space. Then there exists an inner product* $(\cdot,\cdot) : X \times X \to \mathbb{R}$ *such that* $\|x\| = \sqrt{(x,x)}$ *for all* $x \in X$ *if and only if* $\|\cdot\|$ *satisfies the parallelogram law*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in X$.

Example 47 *Using the previous theorem, we can prove that some normed spaces are not inner product spaces. The space* $\mathbb{R}^N$ *with the norm* $\|\cdot\|_{\ell^p}$ *for* $p \neq 2$ *is not an inner product . Take* $\mathbf{x} = (1,1,0,\ldots)$, $\mathbf{y} = (1,-1,0,\ldots)$. *Then* $\mathbf{x}+\mathbf{y} = (2,0,\ldots)$, $\mathbf{x}-\mathbf{y} = (0,2,0,\ldots)$. *Hence,*

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|_{\ell^p}^2 + \|\mathbf{x}-\mathbf{y}\|_{\ell^p}^2 &= (2^p)^{\frac{2}{p}} + (2^p)^{\frac{2}{p}} = 8 \\ &\neq 2\|\mathbf{x}\|_{\ell^p}^2 + 2\|\mathbf{y}\|_{\ell^p}^2 \\ &= 2(1^p + 1^p)^{\frac{2}{p}} + 2(1^p + 1^p)^{\frac{2}{p}} = 2^{2+\frac{2}{p}}. \end{aligned}$$

Definition 48 *A* metric *on a set* X *is a map* $d : X \times X \to [0,\infty)$ *such that*

(i) $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *(symmetry)*,

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$ *(triangle inequality)*.

A metric space (X,d) *is a set* X *endowed with a metric* d. *When there is no possibility of confusion, we abbreviate by saying that* X *is a metric space.*

Proposition 49 *Let* $(X, \|\cdot\|)$ *be a normed space. Then*

$$d(x,y) := \|x-y\|$$

is a metric.

Proof. By property (i) in Definition 40, we have that $0 = d(x, y) = \|x - y\|$ if and only if $x - y = 0$, that is, $x = y$.

By property (ii) in Definition 40, we obtain that

$$d(y, x) = \|y - x\| = \|-1(x - y)\| = |-1| \cdot \|x - y\| = \|x - y\| = d(x, y).$$

Finally, by property (ii) in Definition 40,

$$d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y).$$

■

Exercise 50 *Prove that in $\mathbb{R}$ the function*

$$d_1(x, y) := \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right| \tag{7}$$

is a metric.

5 Topological Properties of the Euclidean Space

Given a point $\mathbf{x}_0 \in \mathbb{R}^N$ and $r > 0$, the *ball* centered at $\mathbf{x}_0$ and of radius r is the set

$$B(\mathbf{x}_0, r) := \left\{ \mathbf{x} \in \mathbb{R}^N : \|\mathbf{x} - \mathbf{x}_0\| < r \right\}.$$

A subset $U \subseteq \mathbb{R}^N$ is *open* if for every $\mathbf{x} \in U$ there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U$.

Example 51 *Some simple examples of sets that are open and of some that are not.*

(i) The set $(a, \infty) = \{x \in \mathbb{R} : x > a\}$ is open. Indeed, if $x > a$, take $r := x - a > 0$. Then $B(x, r) \subset (a, \infty)$. Similarly, the set $(-\infty, a)$ is open.

(ii) The set $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is open. Indeed, given $a < x < b$, take $r := \min\{b - x, x - a\} > 0$. Then $B(x, r) \subseteq (a, b)$.

(iii) The set $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ is not open, since b belongs to the set but there is no ball $B(b, r)$ contained in $(a, b]$.

Example 52 *Consider the set*

$$U = \mathbb{R} \setminus \left(\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \right).$$

Let's prove that U is open. If $x < 0$, take $r = -x > 0$, then $B(x, r) = (-2x, 0) \subseteq U$. If $x > 1$, take $r = x - 1$, then $B(x, r) = (1, 2x - 1) \subseteq U$. If $\frac{1}{n+1} < x < \frac{1}{n}$, take $r = \min\left\{ \frac{1}{n} - x, x - \frac{1}{n+1} \right\} = \frac{1}{n+1}$, then $B(x, r) \subseteq U$. Hence, U is open.

Example 53 *Consider the set*

$$E = \mathbb{R} \setminus \left(\left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \right).$$

Let's prove that E is not open. The point $x = 0$ belongs to E, but for every $r > 0$, by the Archimedean principle we can find $n \in \mathbb{N}$ such that $n > \frac{1}{r}$, and so $0 < \frac{1}{n} < r$, which shows that $\frac{1}{n} \in (-r, r)$. Since $\frac{1}{n}$ does not belong to E, the ball $(-r, r)$ is not contained in E for any *$r > 0$. Hence, E is not open.*

The main properties of open sets are given in the next proposition.

In what follows by an arbitrary family of sets of $\mathbb{R}^N$ we mean that there exists a set Λ and a function

$$\begin{gathered} f : \Lambda \to \mathcal{P}\left(\mathbb{R}^N\right) \\ \alpha \in \Lambda \mapsto f(\alpha) = U_\alpha \end{gathered}$$

We write $\{U_\alpha\}$ or $\{U_\alpha\}_\alpha$ or $\{U_\alpha\}_{\alpha \in \Lambda}$ to denote the set $\{f(\alpha) : \alpha \in \Lambda\}$.

Proposition 54 *The following properties hold:*

(i) $\emptyset$ and $\mathbb{R}^N$ are open.

(ii) If $U_i \subseteq \mathbb{R}^N$, $i = 1, \ldots, n$, is a finite family of open sets of $\mathbb{R}^N$, then $U_1 \cap \cdots \cap U_n$ is open.

(iii) If $\{U_\alpha\}_\alpha$ is an arbitrary collection of open sets of $\mathbb{R}^N$, then $\bigcup_\alpha U_\alpha$ is open.

Proof. To prove (ii), let $\mathbf{x} \in U_1 \cap \cdots \cap U_M$. Then $\mathbf{x} \in U_i$ for every $i = 1, \ldots, n$, and since U_i is open, there exists $r_i > 0$ such that $B(\mathbf{x}, r_i) \subseteq U_i$. Take $r := \min\{r_1, \ldots, r_n\} > 0$. Then

$$B(\mathbf{x}, r) \subseteq U_1 \cap \cdots \cap U_n,$$

which shows that $U_1 \cap \cdots \cap U_n$ is open.

To prove (iii), let $\mathbf{x} \in U := \bigcup_\alpha U_\alpha$. Then there is α such that $\mathbf{x} \in U_\alpha$ and since U_α is open, there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U_\alpha \subseteq U$. This shows that U is open. ∎

Properties (i)–(iii) are used to define topological spaces.

Definition 55 *Let X be a nonempty set and let τ be a family of sets of X. The pair (X, τ) is called a* topological space *if the following hold.*

(i) $\emptyset, X \in \tau$.

(ii) If $U_i \in \tau$ for $i = 1, \ldots, M$, then $U_1 \cap \ldots \cap U_M \in \tau$.

(iii) If $\{U_\alpha\}_\alpha$ is an arbitrary collection of elements of τ, then $\bigcup_\alpha U_\alpha \in \tau$.

The elements of the family τ are called open sets.

Remark 56 *The intersection of infinitely many open sets is not open in general. Take* $U_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$ *for* $n \in \mathbb{N}$. *Then*

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

but $\{0\}$ *is not open. Indeed, for every* $r > 0$, *the ball* $(-r, r)$ *is not contained in* $\{0\}$.

Tuesday, September 13, 2011

Make-up class.

Given a set $E \subseteq \mathbb{R}^N$, a point $\mathbf{x} \in E$ is called an *interior point* of E if there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq E$. The *interior* E° of a set $E \subseteq \mathbb{R}^N$ is the union of all its interior points.

Remark 57 *For a topological space* (X, τ), *given a point* $x \in X$, *a* neighborhood *of* x *is an open set containing* x. *Neighborhoods play the role of balls in metric spaces. Thus, given a set* $E \subseteq X$, *a point* $x \in E$ *is called an* interior point *of* E *if there exists a neighborhood* U *of* x *such that* $U \subseteq E$.

The proof of following proposition is left as an exercise.

Proposition 58 *Let* $E \subseteq \mathbb{R}^N$. *Then*

(i) E° *is an open subset of* E,

(ii) E° *is given by the union of all open subsets contained in* E; *that is,* E° *is the largest (in the sense of union) open set contained in* E,

(iii) E *is open if and only if* $E = E^\circ$,

(iv) $(E^\circ)^\circ = E^\circ$.

Example 59 *Consider the set* $E = [0, 1)$. *Then* 0 *is not an in interior point of* E, *so* $E^\circ \subseteq (0, 1)$. *On the other hand, since* $(0, 1)$ *is open and contained in* E, *by part (ii) of the previous proposition,* $E^\circ \supseteq (0, 1)$, *which shows that* $E^\circ = (0, 1)$.

Exercise 60 *Some properties of the interior.*

(i) Prove that if E, F *are subsets of* $\mathbb{R}^N$, *then*

$$E^\circ \cap F^\circ = (E \cap F)^\circ,$$
$$E^\circ \cup F^\circ \subseteq (E \cup F)^\circ.$$

(ii) Show that in general $E^\circ \cup F^\circ \neq (E \cup F)^\circ$.

(iii) Let $\{E_\alpha\}_\alpha$ *be an arbitrary collection of sets of* $\mathbb{R}^N$. *What is the relation, if any, between* $\bigcap_\alpha (U_\alpha)^\circ$ *and* $(\bigcap_\alpha U_\alpha)^\circ$? *And between* $\bigcup_\alpha (U_\alpha)^\circ$ *and* $(\bigcup_\alpha U_\alpha)^\circ$?

A subset $C \subseteq \mathbb{R}^N$ is *closed* if its complement $\mathbb{R}^N \setminus C$.

The main properties of closed sets are given in the next proposition.

Proposition 61 *The following properties hold:*

(i) $\emptyset$ *and* $\mathbb{R}^N$ *are closed.*

(ii) If $C_i \subseteq \mathbb{R}^N$, $i = 1, \ldots, n$, *is a finite family of closed sets of* $\mathbb{R}^N$, *then* $C_1 \cup \cdots \cup C_n$ *is closed.*

(iii) If $\{C_\alpha\}_\alpha$ *is an arbitrary collection of closed sets of* $\mathbb{R}^N$, *then* $\bigcap_\alpha C_\alpha$ *is closed.*

The proof follows from Proposition 54 and De Morgan's laws. If $\{E_\alpha\}_\alpha$ is an arbitrary collection of subsets of a set $\mathbb{R}^N$, then *De Morgan's laws* are

$$\mathbb{R}^N \setminus \left(\bigcup_\alpha E_\alpha \right) = \bigcap_\alpha \left(\mathbb{R}^N \setminus E_\alpha \right),$$
$$\mathbb{R}^N \setminus \left(\bigcap_\alpha E_\alpha \right) = \bigcup_\alpha \left(\mathbb{R}^N \setminus E_\alpha \right).$$

Next we present a very useful proposition for identifying open an closed sets. We will prove it later on when we discuss about continuous functions.

Proposition 62 *Let* $D \subseteq \mathbb{R}^N$ *and let* $f : D \to \mathbb{R}$ *be continuous.*

(i) If D *is open, then* $f^{-1}(U)$ *is open for every open set* $U \subseteq \mathbb{R}$.

(ii) If D *is closed, then* $f^{-1}(C)$ *is closed for every closed set* $C \subseteq \mathbb{R}$.

Example 63 *Consider the set*

$$E = \left\{ x \in \mathbb{R} : \sin x > \frac{1}{2}, \cos x < 1 \right\}.$$

To see if this is open or closed, let's rewrite E *as follows*

$$E = \left\{ x \in \mathbb{R} : \sin x > \frac{1}{2} \right\} \cap \{ x \in \mathbb{R} : \cos x < 1 \}.$$

The function $f(x) = \sin x$ *is continuous and defined in* $\mathbb{R}$, *which is open. Note that*

$$\left\{ x \in \mathbb{R} : \sin x > \frac{1}{2} \right\} = f^{-1}\left(\left(\frac{1}{2}, \infty \right) \right)$$

and since $\left(\frac{1}{2}, \infty\right)$ *is open, by the previous proposition* $f^{-1}\left(\left(\frac{1}{2}, \infty\right)\right)$ *is open. Similarly, setting* $g(x) = \cos x$, *we have that the set*

$$\{ x \in \mathbb{R} : \cos x < 1 \} = g^{-1}((-\infty, 1))$$

is open. Thus E *is open, since intersection of two open sets.*

Remark 64 *Note that the majority of sets are neither open nor closed. The set $E = (0, 1]$ is neither open nor closed.*

Given a set $E \subseteq \mathbb{R}^N$, the *closure* of E, denoted $\overline{E}$, is the intersection of all closed sets that contain E; in other words, the closure of E is the smallest (with respect to inclusion) closed set that contains E. It follows by Proposition 61 that $\overline{E}$ is closed.

The proof of following proposition is left as an exercise.

Proposition 65 *Let $C \subseteq \mathbb{R}^N$. Then C is closed if and only if $C = \overline{C}$.*

Proposition 66 *Let $E \subseteq \mathbb{R}^N$, and let $\mathbf{x} \in \mathbb{R}^N$. Then $\mathbf{x} \in \overline{E}$ if and only if $B(\mathbf{x}, r) \cap E \neq \emptyset$ for every $r > 0$.*

Proof. Let $\mathbf{x} \in \overline{E}$ and assume by contradiction that there exists $r > 0$ such that $B(\mathbf{x}, r) \cap E = \emptyset$. Since $B(\mathbf{x}, r)$ is open and $B(\mathbf{x}, r) \cap E = \emptyset$, it follows that $\mathbb{R}^N \setminus B(x, r)$ is closed and contains E. By the definition of $\overline{E}$ we have that $\overline{E} \subseteq \mathbb{R}^N \setminus B(\mathbf{x}, r)$, which contradicts the fact that $\mathbf{x} \in \overline{E}$.

Conversely, let $\mathbf{x} \in \mathbb{R}^N$ and assume that $B(\mathbf{x}, r) \cap E \neq \emptyset$ for every $r > 0$. We claim that $\mathbf{x} \in \overline{E}$. Indeed, if not, then $\mathbf{x} \in \mathbb{R}^N \setminus \overline{E}$, which is open. Thus, there exists $B(\mathbf{x}, r) \subseteq \mathbb{R}^N \setminus \overline{E}$, which contradicts the fact that $B(\mathbf{x}, r) \cap E \neq \emptyset$.
■

The previous proposition leads us to the definition of accumulation points.

Definition 67 *Given a set $E \subseteq \mathbb{R}^N$, a point $\mathbf{x} \in \mathbb{R}^N$ is an* accumulation point*, or* cluster point *of E if for every $r > 0$ the ball $B(\mathbf{x}, r)$ contains at least one point of E different from $\mathbf{x}$.*

Note that $\mathbf{x}$ does not necessarily belong to the set E.

Example 68 *Consider the set*

$$E := \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{1 + \frac{1}{n}\right\}_{n\in\mathbb{N}}.$$

We want to prove that 0 and 1 are accumulation points of E. Note that $0 \notin E$, while $1 \in E$ (so accumulation points may or may not be in the set E). For $r > 0$, by taking a natural number $n > \frac{1}{r}$, we have that $0 < \frac{1}{n} < r$, and so $\frac{1}{n} \in B(0, r) \cap E$ (of course $\frac{1}{n} \neq 0$). This shows that 0 is an accumulation points of E.

Similarly, for $r > 0$, by taking a natural number $n > \frac{1}{r}$, we have that $0 < \frac{1}{n} < r$, and so $1 < 1 + \frac{1}{n} < 1 + r$, which shows that $1 + \frac{1}{n} \in B(1, r) \cap E$ (of course $1 + \frac{1}{n} \neq 1$). This shows that 1 is an accumulation points of E. Next we show that there are no other accumulation points of E.

Indeed, if $x < 0$, take $r = -x > 0$, then $B(x, r) = (-2x, 0)$ does not intersect E. If $x > 2$, take $r = x - 1$, then $B(x, r) = (1, 2x - 1)$ does not intersect E.

If $\frac{1}{n+1} < x < \frac{1}{n}$*, take* $r = \min\left\{\frac{1}{n} - x, x - \frac{1}{n+1}\right\} = \frac{1}{n+1}$*, then* $B(x, r)$ *does not intersect* E*. If* $x = \frac{1}{n}$*, with* $n > 1$*, take* $r = \min\left\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right\}$*. Then* $B(x, r)$ *intersects* E *only in* $\frac{1}{n}$*. Hence,* U *is open.*

If $1 + \frac{1}{n+1} < x < 1 + \frac{1}{n}$*, take* $r = \min\left\{1 + \frac{1}{n} - x, x - \left(1 + \frac{1}{n+1}\right)\right\} = \frac{1}{n+1}$*, then* $B(x, r)$ *does not intersect* E*. If* $x = 1 + \frac{1}{n}$*, take* $r = \min\left\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right\}$*. Then* $B(x, r)$ *intersects* E *only in* $1 + \frac{1}{n}$*.*

The set of all accumulation points of E is denoted $\operatorname{acc} E$.

Remark 69 *Note take if* $\mathbf{x} \in \mathbb{R}^N$ *is an accumulation point of* E*, then by taking* $r = \frac{1}{n}$*,* $n \in \mathbb{N}$*, there exists a sequence* $\{\mathbf{x}_n\} \subseteq E$ *with* $\mathbf{x}_n \neq \mathbf{x}$ *for all* $n \in \mathbb{N}$ *such that* $\|\mathbf{x}_n - \mathbf{x}\| < \frac{1}{n} \to 0$*. Thus* $\{x_n\}$ *converges to* x*. Conversely, if there exists* $\{\mathbf{x}_n\} \subseteq E$ *with* $\mathbf{x}_n \neq \mathbf{x}$ *for all* $n \in \mathbb{N}$ *such that* $\|\mathbf{x}_n - \mathbf{x}\| \to 0$*, then* $\mathbf{x}$ *is an accumulation point of* E*.*

It turns out that the closure of a set is given by the set and all its accumulations points.

Proposition 70 *Let* $E \subseteq \mathbb{R}^N$ *Then*

$$\overline{E} = E \cup \operatorname{acc} E.$$

In particular, a set $C \subseteq \mathbb{R}^N$ *is closed if and only if* C *contains all its accumulation points.*

Proof. Exercise. ■

Exercise 71 *(i) Prove that if* $E_1, \ldots, E_n$ *are subsets of* $\mathbb{R}^N$*, then*

$$\overline{E_1} \cap \cdots \cap \overline{E_n} \supseteq \overline{E_1 \cap \cdots \cap E_n},$$
$$\overline{E_1} \cup \cdots \cup \overline{E_n} = \overline{E_1 \cup \cdots \cup E_n}.$$

(ii) Show that in general $\overline{E_1} \cap \cdots \cap \overline{E_n} \neq \overline{E_1 \cap \cdots \cap E_n}$*.*

(iii) Let $\{E_\alpha\}_\alpha$ *be an arbitrary collection of sets of* $\mathbb{R}^N$*. What is the relation, if any, between between* $\bigcap_\alpha \overline{U_\alpha}$ *and* $\overline{\bigcap_\alpha U_\alpha}$*? And between* $\bigcup_\alpha \overline{U_\alpha}$ *and* $\overline{\bigcup_\alpha U_\alpha}$*?*

Wednesday, September 14, 2011

Definition 72 *A set* $E \subseteq \mathbb{R}^N$ *is* bounded *if it is contained in a ball.*

Theorem 73 (Bolzano–Weierstrass) *Every bounded set* $E \subseteq \mathbb{R}^N$ *with infinitely many elements has at least one accumulation point.*

The proof relies on a few preliminary results, which are of interest in themselves.

Lemma 74 *Let $\{[a_n, b_n]\}_n$ be a sequence of closed bounded intervals such that $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for all $n \in \mathbb{N}$. Then the intersection*

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

is nonempty.

Proof. Since

$$\cdots \subseteq [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \subseteq \cdots \subseteq [a_1, b_1],$$

we have that

$$a_1 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots, \tag{8}$$
$$b_1 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots. \tag{9}$$

Let

$$A := \{a_1, \ldots, a_n, \ldots\}.$$

By (8) and (9), for $n \in \mathbb{N}$,

$$a_n \leq b_n \leq b_1.$$

Hence, A is bounded from above, and so by the supremum property, there exists $x := \sup A \in \mathbb{R}$ and

$$a_n \leq x$$

for all $x \in \mathbb{N}$. We claim that $x \leq b_n$ for all $n \in \mathbb{N}$. If not, then there exists $m \in \mathbb{N}$ such that $b_m < x$. Since x is the least upper bound of A, there exists $n \in \mathbb{N}$ such that $b_m < a_n$. Find $k \geq m, n$. Then by (8) and (9),

$$a_n \leq a_k \leq b_k \leq b_m,$$

which is a contradiction. This proves the claim. Hence, $x \in [a_n, b_n]$ for all $n \in \mathbb{N}$, and so $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. ■

Given N bounded intervals $I_1, \ldots, I_N \subset \mathbb{R}$, a *rectangle* in $\mathbb{R}^N$ is a set of the form

$$R := I_1 \times \cdots \times I_N.$$

If all the intervals have the same length, we call R a cube.

Lemma 75 *Let $\{R_n\}_n$ be a sequence of closed bounded rectangles in $\mathbb{R}^N$ such that $R_n \supseteq R_{n+1}$ for all $n \in \mathbb{N}$. Then the intersection*

$$\bigcap_{n=1}^{\infty} R_n$$

is nonempty.

Proof. Each rectangle R_n has the form

$$R_n = [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,N}, b_{n,N}] .$$

Since $R_n \supseteq R_{n+1}$ for all $n \in \mathbb{N}$, for every fixed $k = 1, \ldots, N$, we have that $[a_{n,k}, b_{n;k}] \supseteq [a_{n+1,k}, b_{n+1,k}]$ for all $n \in \mathbb{N}$, and so by the previous lemma there exists $x_k \in \bigcap_{n=1}^{\infty} [a_{n,k}, b_{n;k}]$. Define $\mathbf{x} = (x_1, \ldots, x_N)$. Then $\mathbf{x} = (x_1, \ldots, x_N) \in [a_{n,1}, b_{n,1}] \times \cdots \times [a_{n,N}, b_{n,N}] = R_n$ for every $n \in \mathbb{N}$, and so $\mathbf{x} \in \bigcap_{n=1}^{\infty} R_n$. ■

We are now ready to prove the Bolzano–Weierstrass theorem.

Proof of the Bolzano–Weierstrass theorem. Since E is bounded, it is contained in ball, and in turn a ball is contained in a cube Q_1 of side-length ℓ. Divide Q_1 into 2^N two closed cubes of side-length $\frac{\ell}{2}$. Since E has infinitely many elements, at least one of these 2^N closed cubes contains infinitely many elements of E. Let's call this closed interval Q_1. Then $Q_2 \subset Q_1$, and Q_2 contains infinitely many elements of E.

Divide Q_2 into into 2^N two closed cubes of side-length $\frac{\ell}{2^2}$. Since E has infinitely many elements, at least one of these 2^N closed cubes contains infinitely many elements of E. Let's call this closed interval Q_3. By induction, we construct a sequence of closed cubes Q_n, $n \in \mathbb{N}$, with $Q_n \supseteq Q_{n+1}$, such that the side-length of Q_n is $\frac{\ell}{2^{n-1}}$ and Q_n contains infinitely many elements of E. By the previous lemma, there exists $\mathbf{x} \in \bigcap_{n=1}^{\infty} Q_n$. We claim that $\mathbf{x}$ is an accumulation point of E.

Fix $r > 0$ and consider the ball $B(\mathbf{x}, r)$. We claim that for n sufficiently large, $Q_n \subset B(\mathbf{x}, r)$. To see this, let $\mathbf{y} \in Q_n$. Then

$$\|\mathbf{y} - \mathbf{x}\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_1 - x_1)^2} < \sqrt{N \left(\frac{\ell}{2^{n-1}} \right)^2} = \frac{2\ell}{2^n} \sqrt{N}$$

By the Archimedean property, there exists $n \in \mathbb{N}$ such that

$$\frac{2\ell\sqrt{N}}{r} < 1 + n \leq 2^n,$$

and so $r > \frac{2\ell}{2^n}\sqrt{N}$, which proves the claim. Since Q_n contains infinitely many elements of E, the same holds for $B(\mathbf{x}, r)$ and so $\mathbf{x}$ is an accumulation point of E. ■

Definition 76 *Given a set $E \subseteq \mathbb{R}^N$, a point $\mathbf{x} \in \mathbb{R}^N$ is a* boundary point *of E if for every $r > 0$ the ball $B(\mathbf{x}, r)$ contains at least one point of E and one point of $\mathbb{R}^N \setminus E$. The set of boundary points of E is denoted ∂E.*

The following theorem is left as an exercise.

Theorem 77 *Let $E \subseteq \mathbb{R}^N$. Then*

(i) $\overline{E} = E \cup \partial E$,

(ii) E is closed if and only if it contains all its boundary points,

(iii) $\partial E = \partial\left(\mathbb{R}^N \setminus E\right)$,

(iv) $\partial E = \overline{(\mathbb{R}^N \setminus E)} \cap \overline{E}$.

Friday, September 16, 2011

6 Compactness

Exercise 78 *Let $K \subseteq \mathbb{R}^N$ be closed and bounded. Prove that if $E \subseteq K$ has infinitely many elements, then E has an accumulation point that belongs to K.*

Exercise 79 *Let $K_n \subset \mathbb{R}^N$ be nonempty, bounded, and closed. Assume that $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty} K_n$ is nonempty.*

Definition 80 *A set $K \subseteq \mathbb{R}^N$ is* compact *if for every open cover of K, i.e., for every collection $\{U_\alpha\}_\alpha$ of open sets such that $\bigcup_\alpha U_\alpha \supseteq K$, there exists a finite subcover (i.e., a finite subcollection of $\{U_\alpha\}_\alpha$ whose union still contains K).*

Example 81 *The set $(0, 1]$ is not compact, since taking $U_n := \left(\frac{1}{n}, 2\right)$, a finite number of U_n does not cover $(0, 1]$.*

Here the problem is that 0 does not belong to E. But what if E is closed?

Example 82 *The set $[0, \infty)$ is not compact, since taking $U_n := (-1, n)$, a finite number of U_n does not cover $[0, \infty)$.*

Here E is closed but the problem is that E is not bounded.

Theorem 83 *A compact set $K \subseteq \mathbb{R}^N$ is closed and bounded.*

Proof. To prove that K is closed, we show that $\mathbb{R}^N \setminus K$ is open. Fix $\mathbf{x} \in \mathbb{R}^N \setminus K$. For every $\mathbf{y} \in K$ consider $U_{\mathbf{y}} := B(\mathbf{y}, r_{\mathbf{y}})$ and $V_{\mathbf{y}} := B(\mathbf{x}, r_{\mathbf{y}})$, where $r := \frac{\|\mathbf{x}-\mathbf{y}\|}{4}$. Then $\{U_{\mathbf{y}}\}_{\mathbf{y}\in K}$ is an open cover of K, and so there exist $\mathbf{y}_1, \ldots, \mathbf{y}_m \in K$ such that

$$K \subseteq \bigcup_{i=1}^{m} U_{\mathbf{y}_i} =: U.$$

Let $V := \bigcap_{i=1}^{m} V_{\mathbf{y}_i}$. Then V is open and does not intersect U. In particular, $V \subseteq \mathbb{R}^N \setminus K$. This shows that every point of $\mathbb{R}^N \setminus K$ is an interior point, and so $\mathbb{R}^N \setminus K$ is open.

To prove that K is bounded, consider $U_n := B(\mathbf{0}, n)$. By compactness K is contained in a finite number of balls. Since the balls are one contained into the other, we have that K is contained in the ball of largest radius. Hence, K is bounded. ■

Remark 84 *The previous proof works for metric spaces. For a topological space (X, τ) we can still prove that a compact set $K \subseteq X$ is closed, provided the topological space X is a* Hausdorff space*, that is, for every x and $y \in X$, with $x \neq y$, there exist disjoint neighborhoods of x and y.*

A very simple example of a space that is not Hausdorff can be obtained by considering a nonempty set X and taking as topology $\tau := \{\emptyset, X\}$. If X has at least two elements, then any singleton $\{x\}$ is compact but not closed.

There is a way to define a notion of boundedness for special topological spaces, called topological vector spaces.

Theorem 85 *A closed and bounded set $K \subset \mathbb{R}^N$ is compact.*

Proof. Let $\{U_\alpha\}_\alpha$ be a family of open sets such that $\bigcup_\alpha U_\alpha \supseteq K$ and assume by contradiction that no finite subcover covers K. Since K is bounded, it is contained in ball, and in turn a ball is contained in a cube Q_1 of side-length ℓ. Divide Q_1 into 2^N two closed cubes of side-length $\frac{\ell}{2}$. If $K \cap Q'$ is contained in a finite subcover for every such subcube, then K would be contained in a finite subcover. Hence, there exists at least one subcube Q_1 such that $K \cap Q_1$ is not contained in a finite subcover of $\{U_\alpha\}_\alpha$.

By induction, we construct a sequence of closed cubes Q_n, $n \in \mathbb{N}$, with $Q_n \supseteq Q_{n+1}$, such that the side-length of Q_n is $\frac{\ell}{2^{n-1}}$ and $K \cap Q_n$ is not contained in a finite subcover of $\{U_\alpha\}_\alpha$. By Exercise 79, there exists $\mathbf{x} \in \bigcap_{n=1}^\infty Q_n \cap K$. Since $\{U_\alpha\}_\alpha$ covers K, there exists β such that $\mathbf{x} \in U_\beta$. On the other hand, U_β is open, and so there is a ball $B(\mathbf{x}, r)$ contained in U_β. As in the proof of the Bolzano–Weierstrass theorem, we have that for n sufficiently large, $Q_n \subset B(\mathbf{x}, r) \subseteq U_\beta$, which contradicts the fact that $K \cap Q_n$ is not contained in a finite subcover of $\{U_\alpha\}_\alpha$. ■

Remark 86 *The previous theorem fails for infinite dimensional normed spaces, and so, in general, for infinite dimensional metric spaces.*

Monday, September 19, 2011

7 Sequences

Definition 87 *Given a nonempty set X,* sequence *of elements of X is a function*

$$\begin{aligned} f : \mathbb{N} &\to X \\ n &\mapsto f(n) \end{aligned}$$

from the natural numbers to the set X.

If $f(n) = x_n$ for $n = 1, 2, \ldots$, usually we denote the sequence f by the symbol $\{x_n\}$ or $\{x_n\}_{n \in \mathbb{N}}$ or $x_1, x_2, \ldots$.

Definition 88 *We say that a sequence* $\{\mathbf{x}_n\} \subset \mathbb{R}^N$ converges *to* $\boldsymbol{\ell} \in \mathbb{R}^N$ *if for every* $\varepsilon > 0$ *there exists an integer* $N = N(\varepsilon)$ *such that*

$$\|\mathbf{x}_n - \boldsymbol{\ell}\| < \varepsilon$$

for all $n \geq N$*. In this case we say that* $\boldsymbol{\ell}$ *is the* limit *of* $\{\mathbf{x}_n\}$ *and we write*

$$\lim_{n\to\infty} \mathbf{x}_n = \boldsymbol{\ell} \quad or \quad \mathbf{x}_n \to \boldsymbol{\ell}.$$

Remark 89 *For a metric space* (X, d) *we say that a sequence* $\{x_n\} \subset X$ converges *to* $\ell \in X$ *if for every* $\varepsilon > 0$ *there exists an integer* $N = N(\varepsilon)$ *such that*

$$d(x_n, \ell) < \varepsilon$$

for all $n \geq N$*, while for a topological space* (X, τ) *we say that a sequence* $\{x_n\} \subset X$ converges *to* $\ell \in X$ *if for every neighborhood* U *of* ℓ *there exists an integer* $N = N(\varepsilon)$ *such that*

$$x_n \in U$$

for all $n \geq N$*.*

Remark 90 *We can replace the condition* $\|\mathbf{x}_n - \boldsymbol{\ell}\| < \varepsilon$ *with* $\|\mathbf{x}_n - \boldsymbol{\ell}\| \leq \varepsilon$ *and nothing changes (why?).*

Let's see some examples

Example 91 *Consider the limit* $\lim_{n\to\infty} \frac{n^2+\sin n}{n^2+2n-1}$*. We have*

$$\lim_{n\to\infty} \frac{n^2+\sin n}{n^2+2n-1} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{\sin n}{n^2}\right)}{n^2\left(1+\frac{2}{n}-\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1+\frac{\sin n}{n^2}}{1+\frac{2}{n}-\frac{1}{n^2}} = \frac{1+0}{1+0-0} = 1.$$

Let's prove it using the previous definition. Fix $\varepsilon > 0$*, we want to find* $N = N(\varepsilon) \in \mathbb{N}$ *such that*

$$\left|\frac{n^2+\sin n}{n^2+2n-1} - 1\right| \leq \varepsilon$$

for all $n \geq N$*. We have*

$$\begin{aligned}\left|\frac{n^2+\sin n}{n^2+2n-1} - 1\right| &= \left|\frac{n^2+\sin n - n^2 - 2n + 1}{n^2+2n-1}\right| \\ &= \frac{2n-1-\sin n}{n^2+2n-1} \leq \frac{4n}{n^2+0} = \frac{4}{n} \leq \varepsilon\end{aligned}$$

for all $n \geq \frac{4}{\varepsilon}$*, where we have used the fact that* $2n - 2 \geq 0$*. It is enough to take* $N = \left\lfloor \frac{4}{\varepsilon} \right\rfloor + 1$*.*

Next we discuss some important limits.

Example 92 *Let $x > 0$. We want to calculate*

$$\lim_{n\to\infty} \sqrt[n]{x}.$$

Consider first the case $x > 1$. Then $\sqrt[n]{x} > 1$ and so we may write

$$\sqrt[n]{x} = 1 + a_n,$$

where $a_n > 0$. By inequality (1),

$$x = (1 + a_n)^n \geq 1 + na_n,$$

which implies that

$$0 < a_n \leq \frac{x-1}{n}.$$

This implies that $a_n \to 0$ as $n \to \infty$. Indeed, given $\varepsilon > 0$, we have that $\frac{x-1}{n} \leq \varepsilon$ for all $n \geq n \geq \frac{x-1}{\varepsilon}$. It is enough to take $N = \left\lfloor \frac{x-1}{\varepsilon} \right\rfloor + 1$. In turn, $\sqrt[n]{x} = 1 + a_n \to 1 + 0$, which shows that

$$\lim_{n\to\infty} \sqrt[n]{x} = 1$$

for $x > 1$. If $x = 1$, then $\sqrt[n]{1} = 1 \to 1$ as $n \to \infty$.

Consider next the case $0 < x < 1$. Then $\sqrt[n]{x} < 1$ and so we may write

$$\sqrt[n]{x} = \frac{1}{1 + a_n},$$

where $a_n > 0$. Hence, $x = \frac{1}{(1+a_n)^n}$. By inequality (1),

$$(1 + a_n)^n \geq 1 + na_n,$$

which implies that

$$x = \frac{1}{(1 + a_n)^n} \leq \frac{1}{1 + na_n},$$

and so

$$0 < a_n \leq \frac{1 - \frac{1}{x}}{n}.$$

This implies that $a_n \to 0$ as $n \to \infty$. Indeed, given $\varepsilon > 0$, we have that $\frac{1-\frac{1}{x}}{n} \leq \varepsilon$ for all $n \geq \frac{1-\frac{1}{x}}{\varepsilon}$. In turn, $\sqrt[n]{x} = \frac{1}{1+a_n} \to \frac{1}{1+0}$, which shows that

$$\lim_{n\to\infty} \sqrt[n]{x} = 1.$$

Tuesday, September 20, 2011

Make-up class.

Let's introduce ∞ and $-\infty$. Consider two elements ∞ and $-\infty$ that do not belong to $\mathbb{R}$. The set

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$$

is called the extended real line. On the extended real line $\overline{\mathbb{R}}$ we can consider a total order relation $\leq$ by setting $-\infty \leq x \leq \infty$ for all $x \in \mathbb{R}$ and by keeping the usual order relation in $\mathbb{R}$. We can also extend the operations $+$ and $\cdot$ by defining for $x \in \mathbb{R}$,

$$x + \infty = \infty + x := \infty, \quad x + (-\infty) = (-\infty) + x := (-\infty)$$

and

$$x \cdot \infty = \infty \cdot x := \begin{cases} \infty & \text{if } x > 0, \\ -\infty & \text{if } x < 0, \end{cases} \qquad (-\infty) \cdot x = x \cdot (-\infty) := \begin{cases} \infty & \text{if } x < 0, \\ -\infty & \text{if } x > 0, \end{cases}$$

We DO NOT DEFINE $\infty + (-\infty)$, $\infty + (-\infty)$

If E is not bounded from above, then we set $\sup E := \infty$, while if E is empty, then we set $\sup E := -\infty$.

If E is not bounded from below, then we set $\inf E := -\infty$, while if E is empty, then we set $\inf E := \infty$.

Definition 93 *We say that a sequence $\{x_n\}$ of real numbers* diverges to plus infinity *if for any real number $M > 0$ there exists an integer $N = N(M)$ such that for all $n \geq N$ we have*

$$x_n > M. \tag{10}$$

In this case we write

$$\lim_{n\to\infty} x_n = \infty \quad \text{or} \quad x_n \to \infty.$$

Similarly we say that a sequence $\{x_n\}$ of real numbers diverges to minus infinity *if for any real number $M > 0$ there exists an integer $N = N(M)$ such that for all $n \geq N$ we have*

$$x_n < -M. \tag{11}$$

In this case we write

$$\lim_{n\to\infty} x_n = -\infty \quad \text{or} \quad x_n \to -\infty.$$

If $\{x_n\}$ does not converge and does not diverge to plus infinity or to minus infinity, we say that it oscillates.

Definition 94 *Given a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ and a strictly increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ (that is $n_1 < n_2 < \ldots$) the sequence $\{\mathbf{x}_{n_k}\}_{k\in\mathbb{N}}$ is called a* subsequence *of $\{\mathbf{x}_n\}$.*

Theorem 95 *A sequence $\{\mathbf{x}_n\} \subset \mathbb{R}^N$ of real numbers converges to $\boldsymbol{\ell}$ if and only if* every *subsequence $\{\mathbf{x}_{n_k}\}$ converges to $\boldsymbol{\ell}$.*

We will prove this theorem later on when we study the liminf and limsup of a sequence.

Example 96 *This theorem is very useful to show that a sequence diverges. It is enough to find two subsequences which converge to different numbers. As an example, the sequence* $\{(-1)^n\}$ *is divergent for both* $1,\ 1,\ldots$ *and* $-1,\ -1,\ \ldots$ *are subsequences and converge to different limits.*

Exercise 97 *Let* $0 < \theta < 1$ *be a rational number. Show that the sequence* $\{\sin(n\pi\theta)\}$ *oscillates.*

Example 98 *If* $x \in \mathbb{R}$ *then*

$$\lim_{n\to\infty} x^n = \begin{cases} \infty & \text{if } x > 1, \\ 1 & \text{if } x = 1, \\ 0 & \text{if } -1 < x < 1, \\ \text{oscillates} & \text{if } x \leq -1. \end{cases}$$

If $x > 1$*, then we can write* $x = 1 + a$*, where* $a > 0$*. By inequality (1),*

$$x^n = (1+a)^n \geq 1 + na.$$

Hence, for any $M > 0$*, by taking* $n \geq \frac{M-1}{a}$*, we have that* $x^n \geq M$*, which implies that* $\lim_{n\to\infty} x^n = 1$*.*

If $x = 1$*, then* $x^n = 1^n = 1 \to 1$ *as* $n \to \infty$*. If* $0 < x < 1$*, then we can write* $x = \frac{1}{1+a}$*, where* $a > 0$*, and so again by inequality (1),*

$$0 \leq x^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na}.$$

Hence, given $\varepsilon > 0$*, by taking* $n \geq \frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$*, we have that* $0 \leq x^n \leq \varepsilon$*, which implies that* $\lim_{n\to\infty} x^n = 0$*. The case* $-1 < x < 0$ *is similar and is left as an exercise.*

If $x = -1$*, we have already seen that the limit does not exist. If* $x < -1$*, then for* $n = 2k$*, we have that*

$$x^n = (x)^{2k} = (-x)^{2k} \to \infty$$

as $k \to \infty$*, since* $-x > 1$*. If instead we take* $n = 2k+1$*, then*

$$x^n = (x)^{2k+1} = -(-x)^{2k+1} \to -\infty$$

as $k \to \infty$*, since* $-x > 1$*. Thus, the limit* $\lim_{n\to\infty} x^n$ *does not exists.*

Exercise 99 *Prove that*

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Exercise 100 *Prove that*

$$\lim_{n\to\infty} \frac{n}{x^n} = \begin{cases} 0 & \text{if } |x| > 1, \\ \infty & \text{if } 0 < x \leq 1, \\ \text{does not exist} & \text{if } -1 \leq x < 0. \end{cases}.$$

Definition 101 *We say that a sequence* $\{\mathbf{x}_n\} \subset \mathbb{R}^N$ *of real numbers is* bounded *if it is contained in a ball.*

Theorem 102 *Let* $\{\mathbf{x}_n\} \subset \mathbb{R}^N$ *be a sequence.*

(i) If there exists $\lim_{n\to\infty} \mathbf{x}_n = \boldsymbol{\ell} \in \mathbb{R}^N$*, then* $\{\mathbf{x}_n\}$ *is bounded.*

(ii) If the limit of $\{\mathbf{x}_n\}$ *exists, it is unique.*

Proof. (i) Let $\varepsilon = 1$, then there exists an integer $N = N(1)$ such that

$$\|\mathbf{x}_n - \boldsymbol{\ell}\| < 1$$

for all $n \geq N$. Hence, if we take

$$R = \max\{\|\mathbf{x}_1 - \boldsymbol{\ell}\|, \ldots, \|\mathbf{x}_{N-1} - \boldsymbol{\ell}\|, 1\},$$

it follows that

$$\{\mathbf{x}_n\} \subset B(\boldsymbol{\ell}, R).$$

(ii) Let $\lim_{n\to\infty} x_n = \ell$ and $\lim_{n\to\infty} x_n = L$, with $L \neq \ell$. If one of the two is a real number, say $\ell \in \mathbb{R}$, then the sequence is bounded by part (i), and so also L must be a real number. Let $0 < \varepsilon < \frac{1}{2}|L - \ell|$. By the definition of limit, there exist an integer $N_1 = N_1(\varepsilon)$ such that

$$|x_n - \ell| \leq \varepsilon$$

for all $n \geq N_1$ and an integer $N_2 = N_2(\varepsilon)$ such that

$$|x_n - L| \leq \varepsilon$$

for all $n \geq N_2$. Then for $n \geq \max\{N_1, N_2\}$, we have that

$$|L - \ell| = |L \pm x_n - \ell| \leq |x_n - \ell| + |x_n - L| \leq \varepsilon + \varepsilon < |L - \ell|,$$

which is a contradiction. It remains the case in which one limit is $+\infty$ and the other $-\infty$, which again gives a contradiction (take $M = 1$ in (10) and (11)). ■

Remark 103 *The previous proof continues to hold for metric spaces. For topological spaces, the limit may not be unique. Indeed, if we consider* $\mathbb{R}$ *with the topology* $\tau = \{\emptyset, \mathbb{R}\}$*, then given a sequence* $\{x_n\} \subset \mathbb{R}$*, we have that* $\{x_n\}$ *converges to* every *real number. This does not happen if the topological space is Hausdorff.*

Theorem 104 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of real numbers such that there exist*

$$\lim_{n\to\infty} x_n = \ell_1 \in \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} y_n = \ell_2 \in \mathbb{R}.$$

Then

(i) there exists $\lim_{n\to\infty}(x_n+y_n)=\ell_1+\ell_2$*,*

(ii) there exists $\lim_{n\to\infty}x_ny_n=\ell_1\ell_2$*,*

(iii) if $y_n\neq 0$ *for all* $n\in\mathbb{N}$ *and* $\ell_2\neq 0$ *then there exists* $\lim_{n\to\infty}\frac{x_n}{y_n}=\frac{\ell_1}{\ell_2}$*.*

The proof is left as an exercise.
The following theorem is left as an exercise.

Theorem 105 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of real numbers.*

(i) If there exists $\lim_{n\to\infty}x_n=\infty$ *and* $\{y_n\}$ *is bounded from below, then there exists* $\lim_{n\to\infty}(x_n+y_n)=\infty$*.*

(ii) If there exists $\lim_{n\to\infty}x_n=-\infty$ *and* $\{y_n\}$ *is bounded from above, then there exists* $\lim_{n\to\infty}(x_n+y_n)=-\infty$*.*

(iii) If there exist $\lim_{n\to\infty}x_n=\infty$ *and* $\lim_{n\to\infty}y_n=\ell\in[-\infty,\infty]$ *with* $\ell\neq 0$*, then there exists* $\lim_{n\to\infty}(x_ny_n)=(\operatorname{sgn}\ell)\,\infty$*.*

(iv) If there exist $\lim_{n\to\infty}x_n=\infty$*, then there exists* $\lim_{n\to\infty}\frac{1}{x_n}=0$*.*

(v) If $x_n\neq 0$ *for all* $n\in\mathbb{N}$ *sufficiently large and there exist* $\lim_{n\to\infty}x_n=0$*, then .*

$$\lim_{n\to\infty}\frac{1}{x_n}=\begin{cases}+\infty & \text{if } x_n>0 \text{ for all } n \text{ sufficiently large,}\\ -\infty & \text{if } x_n<0 \text{ for all } n \text{ sufficiently large,}\\ \text{does not exist} & \text{otherwise.}\end{cases}$$

Remark 106 (Important) *In any of the following cases, nothing can be said without further investigation.*

(i) If there exist $\lim_{n\to\infty}x_n=\infty$ *and* $\lim_{n\to\infty}y_n=-\infty$*, then the limit* $\lim_{n\to\infty}(x_n+y_n)$ *needs further investigation.*

(ii) If there exist $\lim_{n\to\infty}x_n=\infty$ *(or* $-\infty$*) and* $\lim_{n\to\infty}y_n=0$*, then the limit* $\lim_{n\to\infty}(x_ny_n)$ *needs further investigation.*

(iii) If there exist $\lim_{n\to\infty}x_n=\infty$ *(or* $-\infty$*) and* $\lim_{n\to\infty}y_n=1$*, then the limit* $\lim_{n\to\infty}(y_n)^{x_n}$ *needs further investigation.*

(iv) If $y_n\neq 0$ *for all* $n\in\mathbb{N}$ *and there exist* $\lim_{n\to\infty}x_n=\infty$ *(or* $-\infty$*) and* $\lim_{n\to\infty}y_n=\infty$ *(or* $-\infty$*), then the limit* $\lim_{n\to\infty}\frac{x_n}{y_n}$ *needs further investigation.*

(v) If $y_n \neq 0$ for all $n \in \mathbb{N}$ and there exist $\lim\limits_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$, then the limit $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n}$ needs further investigation.

Example 107 *An important limit is*

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

*We will prove it later (see Example **??**).*

Example 108 *Let's calculate*

$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n.$$

We have

$$\left(\frac{n+1}{n+2}\right)^n = \left(\frac{1}{\frac{n+2}{n+1}}\right)^n = \left[\left(\frac{1}{1+\frac{1}{n+1}}\right)^{n+1}\right]^{\frac{n}{n+1}} \to e^1,$$

where we have used the previous limit and the fact that $\frac{n}{n+1} = \frac{n}{n\left(1+\frac{1}{n}\right)} = \frac{1}{1+\frac{1}{n}} \to 1$.

The proofs of the next results are left as exercise.

Theorem 109 (Squeeze) *Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences of real numbers such that*

$$x_n \leq y_n \leq z_n$$

for all $n \in \mathbb{N}$.

(i) If there exist $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = \ell \in \mathbb{R}$, then there exists $\lim\limits_{n\to\infty} y_n = \ell$.

(ii) If there exists $\lim\limits_{n\to\infty} x_n = \infty$, then there exists $\lim\limits_{n\to\infty} y_n = \infty$.

(iii) If there exists $\lim\limits_{n\to\infty} z_n = -\infty$, then there exists $\lim\limits_{n\to\infty} y_n = -\infty$.

Corollary 110 *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that there exists $\lim\limits_{n\to\infty} x_n = 0$ and $\{y_n\}$ is bounded. Then there exists $\lim\limits_{n\to\infty} (x_n y_n) = 0$.*

Example 111 *Consider*

$$\lim_{n\to\infty} \frac{\sin^4 n}{n^3}.$$

Since $0 \leq \sin^4 n \leq 1$ and $\frac{1}{n^3} \to 0$, by the previous corollary, $\lim_{n\to\infty} \frac{\sin^4 n}{n^3} = 0$.

The next theorem shows that $n!$ goes to infinity faster than b^n for $b > 1$ but slower than n^n. The proof makes use of the *Gamma function*

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx, \quad t > 0.$$

Exercise 112 *Let Γ be the Gamma function.*

(i) Prove that $\Gamma(t+1) = t\Gamma(t)$ for all $t > 0$.

(ii) Prove that for $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.

Theorem 113 (Stirling's Formula) *The following holds*

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

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Proof. By the previous exercise, we have that

$$n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx.$$

Consider the change of variable $x = n + y$. Then we have

$$\begin{aligned} n! &= \int_{-n}^\infty (n+y)^n e^{-n-y}\, dx = n^n e^{-n} \int_{-n}^\infty \left(1 + \frac{y}{n}\right)^n e^{-y}\, dy \\ &= n^n e^{-n} \left[\int_{-n}^0 e^{-y+n\log\left(1+\frac{y}{n}\right)}\, dy + \int_0^\infty e^{-y+n\log\left(1+\frac{y}{n}\right)}\, dy\right]. \end{aligned}$$

In the first integral we now make the change of variables

$$t = -\sqrt{y - n\log\left(1 + \frac{y}{n}\right)}$$

and in the second

$$t = \sqrt{y - n\log\left(1 + \frac{y}{n}\right)}.$$

In both cases, $t^2 = y - n\log\left(1 + \frac{y}{n}\right)$ and

$$2t\, dt = \left(1 - n\frac{1}{1+\frac{y}{n}}\frac{1}{n}\right) dy = \frac{y}{n+y}\, dy.$$

Consider the function $f(s) = \log(1+s)$. Then $f'(s) = \frac{1}{1+s}$ and $f''(s) = -\frac{1}{(1+s)^2}$. Now using Taylor's formula of second order applied to the function f at $s = 0$, we have that for every $s > 0$ there exists $\theta \in (0,1)$ such that

$$f(s) = f(0) + f'(0)(s-0) + \frac{1}{2}f''(\theta s)(s-0)^2,$$

that is,

$$\log(1+s) = 0 + s - \frac{1}{2}\frac{s^2}{(1+\theta s)^2}.$$

Taking $s = \frac{y}{n}$ we have that

$$\log\left(1+\frac{y}{n}\right) = \frac{y}{n} - \frac{1}{2}\frac{\left(\frac{y}{n}\right)^2}{\left(1+\theta\frac{y}{n}\right)^2},$$

so that

$$t^2 = y - n\log\left(1+\frac{y}{n}\right) = \frac{\frac{y^2}{2n}}{\left(1+\theta\frac{y}{n}\right)^2}.$$

In turn, $t\left(1+\theta\frac{y}{n}\right) = \frac{y}{\sqrt{2n}}$ or

$$\frac{1}{y} = -\frac{\theta}{n} + \frac{1}{t\sqrt{2n}}.$$

Hence,

$$dy = \left(\frac{n}{y}+1\right)2t\,dt = \left(2t(1-\theta)+\sqrt{2n}\right)\,dt.$$

We finally substitute to obtain

$$\begin{aligned}
n! &= n^n e^{-n}\left[\int_{-n}^{0} e^{-y+n\log\left(1+\frac{y}{n}\right)}\,dy + \int_0^\infty e^{-y+n\log\left(1+\frac{y}{n}\right)}\,dy\right] \\
&= n^n e^{-n}\left[\int_{-\infty}^{0} e^{-t^2}\left(2t(1-\theta)+\sqrt{2n}\right)\,dt + \int_0^\infty e^{-t^2}\left(2t(1-\theta)+\sqrt{2n}\right)\,dt\right] \\
&= n^n e^{-n}\sqrt{2n}\left[\int_{-\infty}^{\infty} e^{-t^2}\,dt + \frac{1}{\sqrt{2n}}\int_0^\infty e^{-t^2}2t(1-\theta)\,dt\right] \\
&= n^n e^{-n}\sqrt{2n}\left[\int_{-\infty}^{\infty} e^{-t^2}\,dt + o(1)\right].
\end{aligned}$$

Since $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$, we get Stirling's formula. ■

Remark 114 *Stirling's formula shows that $n!$ goes to infinity slower than n^n but faster that b^n. Indeed,*

$$\frac{n!}{n^n} = \frac{n!}{n^n}\frac{e^{-n}\sqrt{2\pi n}}{e^{-n}\sqrt{2\pi n}} = \frac{n!}{n^n e^{-n}\sqrt{2\pi n}}\frac{\sqrt{2\pi}n^{\frac{1}{2}}}{e^n} \to 1\cdot 0,$$

where we have used the fact that $\frac{n^{\frac{1}{2}}}{e^n} \to 0$ (since $e > 1$). On the other hand, if $b > 1$,

$$\frac{b^n}{n!} = \frac{b^n}{n!}\frac{n^n e^{-n}\sqrt{2\pi n}}{n^n e^{-n}\sqrt{2\pi n}} = \frac{n^n e^{-n}\sqrt{2\pi n}}{n!}\left(\frac{be}{n}\right)^n\frac{1}{n^{\frac{1}{2}}\sqrt{2\pi}} \to 1\cdot 0,$$

where we have used the fact that $\left(\frac{be}{n}\right)^n \leq 1$ for n large.

Remark 115 *More generally one can prove that*

$$\lim_{t\to\infty}\frac{\Gamma(t+1)}{t^t e^{-t}\sqrt{2\pi t}} = 1.$$

8 Monotone Sequences

Definition 116 *We say that a sequence* $\{x_n\}$ *of real numbers is*

(i) increasing *if* $x_n \le x_{n+1}$ *for all* $n \in \mathbb{N}$*;*

(ii) decreasing *if* $x_n \ge x_{n+1}$ *for all* $n \in \mathbb{N}$*;*

(iii) strictly increasing *if* $x_n < x_{n+1}$ *for all* $n \in \mathbb{N}$*;*

(iv) strictly decreasing *if* $x_n > x_{n+1}$ *for all* $n \in \mathbb{N}$*.*

We say that a sequence $\{x_n\}$ *of real numbers is a* monotone *sequence if it satisfies one of the four properties above.*

Theorem 117 *Let* $\{x_n\}$ *be a monotone sequence of real numbers and let*

$$E = \{x_n : n \in \mathbb{N}\}.$$

(i) If $\{x_n\}$ *is increasing, then there exists* $\lim_{n\to\infty} x_n = \sup E$*.*

(ii) If $\{x_n\}$ *is decreasing, then there exists* $\lim_{n\to\infty} x_n = \inf E$*.*

Proof. We prove (i). Let $L := \sup E \in (-\infty, \infty]$. Fix $t < L$. Since t is not an upper bound of E, there exists $N = N(t) \in \mathbb{N}$ such that $t < x_N \le L$. Since $\{x_n\}$ is increasing, for all $n \ge N$, we have that

$$t < x_N \le x_n \le L.$$

We now distinguish two cases. If $L \in \mathbb{R}$ then given $\varepsilon > 0$, we may take $t = L-\varepsilon$, to obtain that

$$L - \varepsilon \le x_n \le L \le L + \varepsilon$$

for all $n \ge N$, which implies that there exists $\lim_{n\to\infty} x_n = L$. On the other hand, if $L = \infty$, then we can take t to be any large number, and so $x_n \ge t$ for all $n \ge N$, which implies that there exists $\lim_{n\to\infty} x_n = \infty$. ■

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Next we study the behavior of sequences defined recursively.

Example 118 *Consider the sequence defined recursively as follows*

$$\begin{cases} a_1 = a \ge 0, \\ a_{n+1} = \sqrt[4]{a_n}, \end{cases}$$

where $n \in \mathbb{N}$*. If* $a = 0$*, then by induction, we have that* $a_n = 0$ *for all* $n \in \mathbb{N}$*. Similarly, if* $a = 1$*, then by induction, we have that* $a_n = 1$ *for all* $n \in \mathbb{N}$*. If*

$0 < a < 1$, then we claim that $0 < a_n < 1$ for all $n \in \mathbb{N}$. This can be proved by induction. Indeed, it is true for $n = 1$. Assume that $0 < a_n < 1$ for some $n \in \mathbb{N}$ and let's prove that $0 < a_{n+1} < 1$. We have that $0 < a_{n+1} = \sqrt[4]{a_n} < 1$. Hence, $0 < a_n < 1$ for all $n \in \mathbb{N}$. In turn, in this case we have that the sequence is increasing for all $n \in \mathbb{N}$. Indeed, $a_{n+1} = \sqrt[4]{a_n} \geq a_n$ for $0 < a_n < 1$. It follows from Theorem 117 that there exists

$$\lim_{n \to \infty} a_n = \ell \in [a, 1] .$$

Hence, $\ell \leftarrow a_{n+1} = \sqrt[4]{a_n} \to \sqrt[4]{\ell}$, which implies that $\ell = \sqrt[4]{\ell}$. It follows that $\ell = 1$.

Similarly, if $a > 1$, we can show that $a_n > 1$ for all $n \in \mathbb{N}$, that the sequence is decreasing, so that

$$\lim_{n \to \infty} a_n = \ell \in [1, a] .$$

As before, we conclude that $\ell = 1$.

9 Powers with Real Exponents

If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$x^n := \underbrace{x \cdot \cdots \cdot x}_{n \text{ times}}.$$

But what does it mean $x^{\sqrt{2}}$? Or more generally, x^a if $a \in \mathbb{R}$? To define this, we will assume that $x > 0$ (this is needed to preserve the properties of powers). If a is positive and rational, say $a = \frac{n}{m}$, where $m, n \in \mathbb{N}$, then we define

$$x^{\frac{n}{m}} := \left(\sqrt[m]{x}\right)^n .$$

Remark 119 *Note that $x^{\frac{n}{m}} = \sqrt[m]{x^n}$. Indeed, let $y = \sqrt[m]{x}$. Then*

$$(y^n)^m = (y^m)^n = x^n,$$

and so $y^n = \sqrt[m]{x^n}$, that is, $\left(\sqrt[m]{x}\right)^n = \sqrt[m]{x^n}$.

If a is rational and negative, say $a = -\frac{n}{m}$, where $m, n \in \mathbb{N}$, then we define

$$x^{-\frac{n}{m}} := \left(x^{-1}\right)^{\frac{n}{m}} .$$

Exercise 120 *Prove that if $x > 0$ and $r, q \in \mathbb{Q}$, then*

$$x^r x^s = x^{r+s},$$
$$(x^r)^s = (x^s)^r = x^{rs}.$$

Exercise 121 *Let $x > 1$ and $r, q \in \mathbb{Q}$.*

(i) Prove that if $r > 0$, then $x^r > 1$.

(ii) Prove that if $r < s$, then $x^r < x^s$.

(iii) Prove that if $\{t_n\} \subset \mathbb{Q}$ and $t_n \to 0$, then $x^{t_n} \to 1$. Hint: Use Example 92.

We are now ready to define x^a for a real. Assume that $x > 1$. We want to construct an increasing sequence $\{r_n\} \subset \mathbb{Q}$ such that $r_n \to a$. We proceed as follows. By the density of the rational numbers, given $a - 1$ and a there exists $r_1 \in \mathbb{Q}$ such that $a - 1 < r_1 < a$. Again by the density of the rational numbers, given $\max\left\{a - \frac{1}{2}, r_1\right\}$ and a there exists $r_2 \in \mathbb{Q}$ such that

$$\max\left\{a - \frac{1}{2}, r_1\right\} < r_2 < a.$$

Inductively, assume that rational numbers $r_1 < r_2 < \cdots < r_n < a$ have been constructed in such a way that

$$a - \frac{1}{i} < r_i < a$$

for all $i = 1, \ldots, n$. We need to construct r_{n+1}. Again by the density of the rational numbers, given $\max\left\{a - \frac{1}{n+1}, r_n\right\}$ and a there exists $r_{n+1} \in \mathbb{Q}$ such that

$$\max\left\{a - \frac{1}{n+1}, r_n\right\} < r_{n+1} < a.$$

Hence, by induction we have constructed an increasing sequence $\{r_n\}$ of rational numbers with

$$a - \frac{1}{n} < r_n < a.$$

Letting $n \to \infty$ in the previous inequality, it follows by the squeeze theorem that $r_n \to a$ as $n \to \infty$.

Since $\{r_n\}$ is increasing, by part (ii) of the previous exercise, it follows that the sequence $\{x^{r_n}\}$ is also increasing, and thus by Theorem 117, there exists

$$\lim_{n\to\infty} x^{r_n} = \ell \in (-\infty, \infty].$$

Since $r_n \leq a \leq \lfloor a \rfloor + 1$, again by part (ii) of the previous exercise, we have that $x^{r_n} \leq x^{\lfloor a \rfloor + 1}$, which implies that the sequence $\{x^{r_n}\}$ is bounded from above. Hence, $\ell \in \mathbb{R}$.

Next let $\{s_n\} \subset \mathbb{Q}$ be such that $s_n \to a$. Then by Exercise 120,

$$x^{s_n} - x^{r_n} = x^{r_n}\left(x^{s_n - r_n} - 1\right).$$

Since $x^{r_n} \to \ell \in \mathbb{R}$ and $x^{s_n - r_n} \to 1$ by part (iii) of the previous exercise, it follows that $x^{s_n} - x^{r_n} \to 0$. Thus, we have shown that there exists $\ell \in \mathbb{R}$ with the property that for *every* sequence $\{s_n\} \subset \mathbb{Q}$ such that $s_n \to a$, there exists

$$\lim_{n\to\infty} x^{s_n} = \ell.$$

Hence, we define $x^a := \ell$.

If $0 < x < 1$, we set

$$x^a := \left(x^{-1}\right)^{-a}.$$

Note that if $x > 0$ and $a, b \in \mathbb{R}$, then

$$x^a x^b = x^{a+b},$$

Indeed, let $\{r_n\} \subset \mathbb{Q}$ and $\{s_n\} \subset \mathbb{Q}$ be such that $r_n \to a$ and $s_n \to b$. Then by Exercise 120,

$$x^{r_n} x^{s_n} = x^{r_n + s_n},$$

and it is enough to let $n \to \infty$.

Exercise 122 *Let $x > 1$ and $a, b \in \mathbb{R}$.*

(i) Prove that if $a > 0$, then $x^a > 1$.

(ii) Prove that if $a < b$, then $x^a < x^b$.

(iii) Prove that if $\{a_n\} \subset \mathbb{Q}$ and $a_n \to 0$, then $x^{a_n} \to 1$.

(iv) Prove that if $\{a_n\} \subset \mathbb{Q}$ and $a_n \to a$, then $x^{a_n} \to x^a$.

Exercise 123 *Let $x > 0$ and $a, b \in \mathbb{R}$. Prove that*

$$(x^a)^b = \left(x^b\right)^a = x^{ab}.$$

Hint: It is enough to show $(x^a)^b = x^{ab}$. Consider first the case in which a is real and b is rational.

Exercise 124 *Let $x > 1$ and $a \in \mathbb{R}$. Prove that*

$$|x^a - 1| \le x^{|a|} - 1.$$

Example 125 *Let $x > 1$ and let $a > 0$. Let's prove that*

$$\lim_{n \to \infty} \frac{n^a}{x^n} = 0.$$

If $a = m \in \mathbb{N}$, then

$$\frac{n^a}{x^n} = \left(\frac{n}{\left[(x)^{\frac{1}{m}}\right]^n}\right)^m = \frac{n}{\left[(x)^{\frac{1}{m}}\right]^n} \times \cdots \times \frac{n}{\left[(x)^{\frac{1}{m}}\right]^n}$$

and since $b = (x)^{\frac{1}{m}} > 1$, we have that $\frac{n}{b^n} \to 0$ by Exercise 100. The result now follow from Theorem 104(ii). If $0 < a < 1$, then

$$0 \le \frac{n^a}{x^n} \le \frac{n^1}{x^n}$$

and the result follows from the squeeze theorem and Exercise 100. Finally. if $a > 1$, then $\lfloor a \rfloor \leq a \leq \lfloor a \rfloor + 1$, and so

$$\frac{n^{\lfloor a \rfloor}}{x^n} \leq \frac{n^a}{x^n} \leq \frac{n^{\lfloor a \rfloor + 1}}{x^n}$$

and the result follows from the squeeze theorem.

Exercise 126 *Let $\{a_n\}$ be a sequence of real numbers such that there exists $\lim_{n \to \infty} a_n = \infty$ and let $x > 1$.*

(i) Prove that

$$\lim_{n \to \infty} \frac{a_n}{x^{a_n}} = 0.$$

(ii) Prove that if $b > 0$, then

$$\lim_{n \to \infty} \frac{(a_n)^b}{x^{a_n}} = 0.$$

(iii) Use parts (i) and (ii) to prove that if $a > 0$ and $b > 0$, then

$$\lim_{n \to \infty} \frac{(\log n)^b}{n^a} = 0.$$

Example 127 *Let's calculate*

$$\lim_{n \to \infty} \frac{n^3 - n^2 + \log^2 n + 2^n - 3}{4^n - \log n + n^7}.$$

We have

$$\begin{aligned}
\frac{n^3 - n^2 + \log^2 n + 2^n - 3}{4^n - \log n + n^7} &= \frac{2^n \left(\frac{n^3}{2^n} - \frac{n^2}{2^n} + \frac{\log^2 n}{2^n} + 1 - \frac{3}{2^n} \right)}{4^n \left(1 - \frac{\log n}{4^n} + \frac{n^7}{4^n} \right)} \\
&= \frac{\left(\frac{n^3}{2^n} - \frac{n^2}{2^n} + \frac{\log^2 n}{2^n} + 1 - \frac{3}{2^n} \right)}{2^n \left(1 - \frac{\log n}{4^n} + \frac{n^7}{4^n} \right)} \to \frac{(0 - 0 + 0 + 1 - 0)}{\infty (1 - 0 + 0)} = \frac{1}{\infty} = 0
\end{aligned}$$

as $n \to \infty$, where we have used the facts that, by the previous exercise,

$$\frac{\log^2 n}{2^n} = \frac{\log^2 n}{n} \frac{n}{2^n} \to 0, \qquad \frac{\log n}{4^n} = \frac{\log n}{n} \frac{n}{4^n} \to 0.$$

Monday, September 26, 2011

No class, Oxford

Wednesday, September 28, 2011

No class, Oxford.

Friday, September 30, 2011

10 Limsup and Liminf

Given a sequence $\{x_n\}$ of real numbers, consider the set

$$E := \left\{ L \in [-\infty, \infty] : \text{ there is a subsequence } \{x_{n_k}\} \text{ such that } \lim_{k\to\infty} x_{n_k} = L \right\}.$$

We define the *limit superior* of the sequence $\{x_n\}$ to be

$$\limsup_{n\to\infty} x_n := \sup E$$

and we define the *limit inferior* of the sequence $\{x_n\}$ to be

$$\liminf_{n\to\infty} x_n := \inf E.$$

Example 128 *Consider the sequence $x_n = (-1)^n$. Then $E = \{-1, 1\}$, and so*

$$\liminf_{n\to\infty} (-1)^n = \min E = -1,$$
$$\limsup_{n\to\infty} (-1)^n = \max E = +1.$$

Theorem 129 *Given a sequence $\{x_n\}$ of real numbers, the set*

$$E := \left\{ L \in [-\infty, \infty] : \textit{ there is a subsequence } \{x_{n_k}\} \textit{ such that } \lim_{k\to\infty} x_{n_k} = L \right\}$$

is nonempty.

Proof. Let $F := \{x_n : n \in \mathbb{N}\}$. We distinguish three cases.
Case 1: If the set F has only a finite number of different elements, then this means that there exists $x \in \mathbb{R}$ such that $x_n = L$ for infinitely many n. Let $\{n_k\}$, $k \in \mathbb{N}$, be the sequence of all natural numbers such that $x_{n_k} = L$. Then

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} L = L$$

Thus, we have constructed a subsequence converging to L, that is, $L \in E$.
Case 2: If the set F has infinitely many different elements, then we have three further subcases.
Subcase 2A: Assume that the sequence $\{x_n\}$ is not bounded from above. In this case, we claim that $\infty \in E$. To see this, we construct a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} x_{n_k} = \infty.$$

Consider the number $M_1 = 1$. Since $\{x_n\}$ is not bounded from above, there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} > 1$. Consider the number

$$M_2 = \max \{2, x_1, x_2, \ldots, x_{n_1}\} < \infty.$$

Since $\{x_n\}$ is not bounded from above, there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} > M_2 \geq 2$. Note that, necessarily, $n_2 > n_1$.

Inductively, assume that k positive integers $n_1 < \cdots < n_k$ have been chosen in such a way that $x_{n_i} > i$ for all $i = 1, \ldots, k$ and let's choose n_{k+1}. Consider the number

$$M_k = \max \{k + 1, x_1, x_2, \ldots, x_{n_k}\} < \infty.$$

Since $\{x_n\}$ is not bounded from above, there exists $n_{k+1} \in \mathbb{N}$ such that $x_{n_{k+1}} > M_{k+1}$. Note that, necessarily, $n_{k+1} > n_k$. Thus we have constructed a subsequence $\{x_{n_k}\}$ of the original sequence $\{x_n\}$ such that $x_{n_k} > k$ for all $k \in \mathbb{N}$. Letting $k \to \infty$ and using the squeeze theorem, we get that $\lim\limits_{k\to\infty} x_{n_k} = \infty$. In turn $\infty \in E$, so that

$$\infty = \max E = \limsup_{n\to\infty} x_n.$$

Subcase 2B: Assume that the sequence $\{x_n\}$ is not bounded from below. Reasoning as in the first case, we may construct a subsequence $\{x_{n_k}\}$ of the original sequence $\{x_n\}$ such that $x_{n_k} < -k$ for all $k \in \mathbb{N}$. Letting $k \to \infty$ and using the squeeze theorem, we get that $\lim\limits_{k\to\infty} x_{n_k} = -\infty$, so that $-\infty \in E$ and

$$-\infty = \min E = \liminf_{n\to\infty} x_n.$$

Subcase 2C: The sequence $\{x_n\}$ is bounded. Then the set F is bounded and it has infinitely many different elements. By the Bolzano–Weierstrass theorem, F has an accumulation point L. Then by Remark 69 there exists a sequence in the set F that converges to L, namely there exists a subsequence $\{x_{n_k}\}$ converging to L. Thus $L \in E$ and the proof is complete. ■

The next theorem is important for the exercises.

Theorem 130 *Let $\{x_n\}$ be a sequence bounded from above and let $\ell \in \mathbb{R}$. Then the following are equivalent:*

(a) $\ell = \limsup\limits_{n\to\infty} x_n$*;*

(b) for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$x_n \leq \ell + \varepsilon \tag{12}$$

for all $n \geq n_\varepsilon$, and

$$x_n \geq \ell - \varepsilon \tag{13}$$

for infinitely many $n \in \mathbb{N}$.

Example 131 *Consider the sequence $x_n = (-1)^n \frac{2n}{n+1}$. To prove that*

$$2 = \limsup_{n\to\infty} (-1)^n \frac{2n}{n+1}$$

fix $\varepsilon > 0$. *We want to prove that*

$$(-1)^n \frac{2n}{n+1} \leq 2 + \varepsilon$$

for all n *sufficiently large. If* n *is odd, then there is nothing to prove, since a negative number is less than a positive. If* n *is even, we have that*

$$\frac{2n}{n+1} \leq 2 + \varepsilon,$$

that is, $2n \leq (2+\varepsilon)(n+1)$, *which gives* $0 \leq \varepsilon + n\varepsilon + 2$. *This is true for all* $n \in \mathbb{N}$. *Thus we can take* $n_\varepsilon = 1$.

Next we want to prove that

$$2 - \varepsilon \leq (-1)^n \frac{2n}{n+1} \tag{14}$$

for infinitely many n. *Note that if* n *is odd, then the previous inequality is false. Thus assume that* n *is even. Then*

$$2 - \varepsilon \leq \frac{2n}{n+1},$$

that is $(2-\varepsilon)(n+1) \leq 2n$, *which gives* $2n - \varepsilon - n\varepsilon + 2 \leq 2n$, *that is,* $2 - \varepsilon \leq n\varepsilon$. *Hence, the inequality (14) holds for all* n ***even*** *with* $n \geq \frac{2-\varepsilon}{\varepsilon}$. *There are infinitely many such* n.

We now turn to the proof of the theorem.

Proof. Assume that $\limsup\limits_{n\to\infty} x_n = \ell \in \mathbb{R}$. We need to prove (12) and (13). If (12) fails then there exist infinitely many many $n \in \mathbb{N}$ such that $x_n \geq \ell + \varepsilon$. So we can find a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \geq \ell + \varepsilon$ for all $k \in \mathbb{N}$. Applying the previous theorem to the sequence $\{x_{n_k}\}$, we have that the sequence $\{x_{n_k}\}_k$ admits a further subsequence $\left\{x_{n_{k_j}}\right\}_j$ such that $x_{n_{k_j}}$ has a limit, $x_{n_{k_j}} \to L$. Note that $L \in E$. Since $x_{n_{k_j}} \geq \ell + \varepsilon$ for all $j \in \mathbb{N}$ letting $j \to \infty$ we get that $L \geq \ell + \varepsilon$, which contradicts the fact that $\ell = \sup E$. Hence (12) holds.

To prove (13) note that, since $\ell - \varepsilon$ is not an upper bound of the set E there exist $L \in E$ such that $L > \ell - \varepsilon$. By the definition of E there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L$ as $k \to \infty$. Hence, taking $\varepsilon_1 = L - (\ell - \varepsilon) > 0$ there exists $K \in \mathbb{N}$ such that

$$|x_{n_k} - L| \leq \varepsilon_1 = L - (\ell - \varepsilon)$$

for all $k \geq K$, that is

$$-L + (\ell - \varepsilon) \leq x_{n_k} - L \leq L - (\ell - \varepsilon)$$

for all $k \geq K$, . In particular, $x_{n_k} \geq (\ell - \varepsilon)$ for all $k \geq K$. Hence (13) holds.

Conversely assume that (12) and (13) hold. Let's prove that $\ell+\varepsilon$ is an upper bound of E. Let $L \in E$. By the definition of E there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L$ as $k \to \infty$. Since

$$x_n \leq \ell + \varepsilon$$

for all $n \geq N$ it follows that

$$x_{n_k} \leq \ell + \varepsilon$$

for all k such that $n_k \geq N$. Letting $k \to \infty$ we conclude that $L \leq \ell + \varepsilon$. But since this is true for every $\varepsilon > 0$, letting $\varepsilon \to 0^+$ we get that $L \leq \ell$. Hence $L \leq \ell$ for all $L \in E$, which shows that ℓ is an upper bound of E.

Let's prove that $\ell - \varepsilon$ is not an upper bound of E. Take $\ell - \frac{\varepsilon}{2}$. By (13)

$$x_n \geq \ell - \frac{\varepsilon}{2}$$

for infinitely many $n \in \mathbb{N}$. So we can find a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \geq \ell - \frac{\varepsilon}{2}$ for all $k \in \mathbb{N}$. Applying the previous theorem to $\{x_{n_k}\}$, we have that the sequence $\{x_{n_k}\}_k$ admits a further subsequence $\left\{x_{n_{k_j}}\right\}_j$ such that $x_{n_{k_j}}$ has a limit, $x_{n_{k_j}} \to L$. Note that $L \in E$. Since $x_{n_{k_j}} \geq \ell - \frac{\varepsilon}{2}$ for all $j \in \mathbb{N}$, letting $j \to \infty$ we get that $L \geq \ell - \frac{\varepsilon}{2} > \ell - \varepsilon$, which shows $\ell - \varepsilon$ is not an upper bound of E. Hence $\ell = \sup E$. ■

Corollary 132 *Given a sequence $\{x_n\}$ of real numbers, then*

$$\limsup_{n\to\infty} x_n = \max E,$$

that is, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$.

Proof. If the sequence is not bounded from above, then we have already seen in Case 1 of the proof of Theorem 129 that $\max E = \infty$. If the sequence if bounded above, take $\varepsilon = 1$ in the previous theorem. Then by (12) and (13), there exists $n_1 \in \mathbb{N}$ such that $\ell - 1 \leq x_{n_1} \leq \ell + 1$. Inductively, taking $\varepsilon = \frac{1}{k}$, by (12) and (13) there exists $n_k \in \mathbb{N}$ with $n_k > n_{k-1}$ such that $\ell - \frac{1}{k} \leq x_{n_k} \leq \ell + \frac{1}{k}$. By the squeeze theorem, it follows that $\lim_{k\to\infty} x_{n_k} = \ell$. ■

A similar theorem holds for sequences which are not bounded from above.

Theorem 133 *Given a sequence $\{x_n\}$ of real numbers, the following are equivalent:*

(a) $\limsup_{n\to\infty} x_n = \infty$*;*

(b) The sequence $\{x_n\}$ is not bounded from above.

(c) For every $M > 0$ there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n \geq M.$$

Proof. We have already proved that (a) and (b) are equivalent. On the other hand, (b) and (c) are clearly equivalent, and so we are done. ■

We have similar theorems for the limit inferior.

Theorem 134 *Given a sequence $\{x_n\}$ of real numbers bounded from below and a number $\ell \in \mathbb{R}$, the following are equivalent:*

(a) $\liminf_{n\to\infty} x_n = \ell$*;*

(b) for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\ell - \varepsilon \leq x_n \quad \textit{for all } n \geq N$$

and there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n \leq \ell + \varepsilon.$$

Corollary 135 *Given a sequence $\{x_n\}$ of real numbers, then*

$$\liminf_{n\to\infty} x_n = \min E,$$

that is, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \liminf_{n\to\infty} x_n$.

Theorem 136 *Given a sequence $\{x_n\}$ of real numbers, the following are equivalent:*

(a) $\liminf_{n\to\infty} x_n = \infty$*;*

(b) The sequence $\{x_n\}$ is not bounded from below.

(c) For every $M > 0$ there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n \leq -M.$$

Monday, October 3, 2011

The relation between limit, limit superior and limit inferior is given by the following theorem.

Theorem 137 *Given a sequence $\{x_n\}$ of real numbers, then*

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n. \tag{15}$$

Moreover there exists $\lim_{n\to\infty} x_n$ if and only if equality holds in (15), and in this case the limit coincides with the common value in (15).

Proof. We have

$$\liminf_{n\to\infty} x_n = \inf E \leq \sup E = \limsup_{n\to\infty} x_n.$$

To prove the second part of the theorem assume that there exists $\lim_{n\to\infty} x_n = \ell$. I will consider only the case $\ell \in \mathbb{R}$ and leave the cases $\ell = \infty$ and $\ell = -\infty$ as an exercise. By definition of limit, for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\ell - \varepsilon \leq x_n \leq \ell + \varepsilon$$

for all $n \geq N$. Since properties (12) and (13) are satisfied, it follows from Theorem 130 that $\ell = \limsup_{n\to\infty} x_n$. Similarly, by Theorem 134, we have that $\ell = \liminf_{n\to\infty} x_n$.

Conversely, assume that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = L$$

for some $L \in [-\infty, \infty]$. Again we consider the case $L \in \mathbb{R}$ and leave the cases $L = \infty$ and $L = -\infty$ as an exercise. Fix $\varepsilon > 0$. By Theorems 130 there exists an integer $N_1 = N_1(\varepsilon)$ such that

$$x_n \leq L + \varepsilon$$

for all $n \geq N_1$ while by and Theorem 134 there exists an integer $N_2 = N_2(\varepsilon)$ such that

$$L - \varepsilon \leq x_n$$

for all $n \geq N_2$. Then for $n \geq \max\{N_1, N_2\}$, we have that

$$L - \varepsilon \leq x_n \leq L + \varepsilon,$$

which implies that there exists $\lim_{n\to\infty} x_n = L$. ■

As a corollary of this theorem, we can finally prove Theorem 95.

Proof of Theorem 95. Let $\{x_n\}$ be a sequence of real numbers. Assume that there exists

$$\lim_{n\to\infty} x_n = \ell \in [-\infty, \infty].$$

We want to prove that *every* subsequence $\{x_{n_k}\}$ converges to ℓ. Consider the set

$$E := \{L \in [-\infty, \infty] : \text{there is a subsequence of } \{x_n\} \text{ converging to } L\}.$$

Since $x_n \to \ell$, by the previous theorem,

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell.$$

Hence,

$$\inf E = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \sup E,$$

and so $\inf E = \sup E = \ell$. But this implies that $E = \{\ell\}$ (if E had two or more elements the infimum and the supremum would not coincide). By the definition of E and the previous theorem, we have that every subsequence must convergence to ℓ.

Conversely, if every subsequence of $\{x_n\}$ goes to $\ell \in [-\infty, \infty]$, then $E = \{\ell\}$, and so $\inf E = \sup E = \ell$, which implies that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell.$$

Again by the previous theorem, it follows that

$$\lim_{n\to\infty} x_n = \ell.$$

■

Exercise 138 *Consider a sequence $\{x_n\}$ of real numbers. Prove that*

$$\liminf_{n\to\infty} (-x_n) = -\limsup_{n\to\infty} x_n,$$
$$\limsup_{n\to\infty} (-x_n) = -\liminf_{n\to\infty} x_n.$$

Exercise 139 *Consider two sequences $\{x_n\}$ and $\{y_n\}$ of real numbers, and assume that one of them is bounded.*

(i) Prove that

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty} (x_n + y_n) \le \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$$
$$\le \limsup_{n\to\infty} (x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

and that in general all inequalities may be strict.

(ii) Prove that if there exists $\lim_{n\to\infty} x_n = \ell \in \mathbb{R}$, then

$$\lim_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = \liminf_{n\to\infty} (x_n + y_n),$$
$$\limsup_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

(iii) Prove that if there exists $\lim_{n\to\infty} x_n = \ell \in (0, \infty)$, then

$$\lim_{n\to\infty} x_n \liminf_{n\to\infty} y_n = \liminf_{n\to\infty} (x_n y_n),$$
$$\limsup_{n\to\infty} (x_n y_n) = \lim_{n\to\infty} x_n \limsup_{n\to\infty} y_n.$$

Example 140 *Consider the sequence $x_n = (-1)^n \frac{2n}{n+1}$. Since there exists $\lim_{n\to\infty} \frac{2n}{n+1} = 2$, by the previous exercise,*

$$\liminf_{n\to\infty} (-1)^n \frac{2n}{n+1} = \liminf_{n\to\infty} (-1)^n \lim_{n\to\infty} \frac{2n}{n+1} = -1 \cdot 2 = -2,$$
$$\limsup_{n\to\infty} (-1)^n \frac{2n}{n+1} = \limsup_{n\to\infty} (-1)^n \lim_{n\to\infty} \frac{2n}{n+1} = 1 \cdot 2 = 2.$$

Example 141 *Let* $x_n = \mathrm{fr}\,(\sqrt[3]{n})$*. Since for every number* $x \in \mathbb{R}$*,* $0 \leq \mathrm{fr}\,(x) < 1$*, we have that*

$$0 \leq \mathrm{fr}\left(\sqrt[3]{n}\right) < 1.$$

We claim that

$$\liminf_{n\to\infty} \mathrm{fr}\left(\sqrt[3]{n}\right) = 0.$$

Let's use Theorem 134. Fix $\varepsilon > 0$*. We want to prove that there exists a positive integer* $N = N\,(\varepsilon) \in \mathbb{N}$ *such that*

$$0 - \varepsilon \leq \mathrm{fr}\left(\sqrt[3]{n}\right) \quad \textit{for all } n \textit{ large.}$$

Since $0 \leq \mathrm{fr}\,(\sqrt[3]{n})$*, we have that the previous inequality holds for every* $n \in \mathbb{N}$*. Next we need to find infinitely many* $n \in \mathbb{N}$ *such that*

$$\mathrm{fr}\left(\sqrt[3]{n}\right) \leq 0 + \varepsilon.$$

Taking $n = k^3$ *for* $k \in \mathbb{N}$*, we have that*

$$x_{k^3} = \mathrm{fr}\left(\sqrt[3]{k^3}\right) = \mathrm{fr}\,(k) = 0 \leq 0 + \varepsilon$$

for all $k \in \mathbb{N}$*, as* $k \to \infty$*. This shows that* $0 = \liminf_{n\to\infty} \mathrm{fr}\,(\sqrt[3]{n})$*.*

Next, we claim that

$$\limsup_{n\to\infty} \mathrm{fr}\left(\sqrt[3]{n}\right) = 1.$$

Let's use Theorem 130. Fix $\varepsilon > 0$*. We want to prove that there exists a positive integer* $N = N\,(\varepsilon) \in \mathbb{N}$ *such that*

$$\mathrm{fr}\left(\sqrt[3]{n}\right) \leq 1 + \varepsilon \quad \textit{for all } n \textit{ large.}$$

Since $\mathrm{fr}\,(\sqrt[3]{n}) < 1$*, we have that the previous inequality holds for every* $n \in \mathbb{N}$*. Next we need to find infinitely many* $n \in \mathbb{N}$ *such that*

$$1 - \varepsilon \leq \mathrm{fr}\left(\sqrt[3]{n}\right). \tag{16}$$

Taking $n = k^3 - 1$ *for* $k \in \mathbb{N}$*, we have that*

$$k > \sqrt[3]{k^3 - 1} = k\sqrt[3]{1 - \frac{1}{k^3}} \geq k\left(1 - \frac{1}{k^3}\right) = k - \frac{1}{k^2}.$$

Hence, $\left\lfloor \sqrt[3]{k^3 - 1} \right\rfloor = k - 1$*, and so*

$$\begin{aligned} 1 \geq \mathrm{fr}\left(\sqrt[3]{k^3 - 1}\right) &= \sqrt[3]{k^3 - 1} - (k - 1) \\ &\geq k - \frac{1}{k^2} - (k - 1) = -\frac{1}{k^2} + 1 \to 1 \end{aligned}$$

as $k \to \infty$*. By the squeeze theorem* $\mathrm{fr}\left(\sqrt[3]{k^3 - 1}\right) \to 1$*. It follows from the definition of limit that there exists* $k_\varepsilon \in \mathbb{N}$ *such that*

$$\left|\mathrm{fr}\left(\sqrt[3]{k^3 - 1}\right) - 1\right| \leq \varepsilon$$

for all $k \geq k_\varepsilon$, which implies that

$$1-\varepsilon \leq \operatorname{fr}\left(\sqrt[3]{k^3-1}\right) \leq 1+\varepsilon$$

for all $k \geq k_\varepsilon$. Hence, we have found infinitely many n for which (16) holds. This shows that $1 = \limsup_{n\to\infty} \operatorname{fr}(\sqrt[3]{n})$.

Next we prove some important limits. Using Theorem 117, we can define the number e. Consider the sequence

$$s_n := \sum_{k=0}^{n} \frac{1}{k!},$$

where we recall that $0! := 1$ and $n! := 1\cdot 2\cdot\cdots\cdot n$. Note that $s_{n+1} = s_n + \frac{1}{(n+1)!} > s_n$, and so the sequence $\{s_n\}$ is strictly increasing. We leave it as an exercise to prove that $\{s_n\}$ is bounded. Since $\{s_n\}$ is bounded and increasing, there exists

$$\lim_{n\to\infty} s_n \in (0,\infty)\,.$$

We call this limit e.

Theorem 142 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$

Wednesday, October 5, 2011

Proof. Let

$$s_n := \sum_{k=0}^{n} \frac{1}{k!}, \qquad t_n := \left(1+\frac{1}{n}\right)^n.$$

By the binomial theorem,

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} 1^{n-k}.$$

Note that

$$\begin{aligned}\binom{n}{k}\frac{1}{n^k} &= \frac{n!}{k!\,(n-k)!}\frac{1}{n^k} = \frac{(n-k)!}{k!\,(n-k)!}\frac{n\,(n-1)\cdots(n-k+1)}{n\cdot\cdots\cdot n} \\ &= \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \leq \frac{1}{k!}.\end{aligned}$$

Hence, $t_n \leq s_n$, and so

$$\limsup_{n\to\infty} t_n \leq \limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n = e.$$

On the other hand, if $n \geq m$,

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \geq \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k} \\ &= \sum_{k=0}^{m} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right). \end{aligned}$$

Fixing m and letting $n \to \infty$ in the previous inequality gives

$$\begin{aligned} \liminf_{n\to\infty} t_n &\geq \liminf_{n\to\infty} \sum_{k=0}^{m} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \\ &\geq \sum_{k=0}^{m} \liminf_{n\to\infty} \left(\frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \right) \\ &= \sum_{k=0}^{m} \frac{1}{k!}. \end{aligned}$$

Note that in the last inequality it was important to have a finite sum $\sum_{k=0}^{m}$ rather than $\sum_{k=0}^{n}$. Finally, letting $m \to \infty$ gives

$$\liminf_{n\to\infty} t_n \geq \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} = e.$$

Thus, we have shown that

$$e \leq \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n \leq e,$$

and so by Theorem 137, there exists $\lim_{n\to\infty} t_n = e$. ■

11 Completeness

Definition 143 *Given a metric space (X, d) and a sequence $\{x_n\} \subseteq X$, we say that*

(i) $\{x_n\}$ *is a* Cauchy sequence *if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that*

$$d(x_n, x_m) < \varepsilon$$

for all $n, m \geq N_\varepsilon$,

(ii) $\{x_n\}$ converges to $x \in X$ *if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that*

$$d(x_n, x) < \varepsilon$$

for all $n \geq N_\varepsilon$.

Proposition 144 *Given a metric space (X,d) and a sequence $\{x_n\} \subseteq X$, if $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $\{x_n\}$ converges to $x \in X$, given $\varepsilon > 0$, consider $\frac{\varepsilon}{2}$ in the definition of convergence. Then there exists $N_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\varepsilon}{2}$$

for all $n \geq N_\varepsilon$. Hence, by the triangle inequality and symmetry of d, if $n, m \geq N_\varepsilon$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

■

The opposite is not true, that is, there are Cauchy sequences that do not have a limit.

Example 145 *Consider $X = (0,1)$ with the metric $d(x,y) = |x-y|$ and consider the sequence $x_n = \frac{1}{n}$. Then $x_n \to 0$ which does not belong to $X = (0,1)$, but $\{x_n\}$ is a Cauchy (just apply the previous proposition in the metric space $\mathbb{R}$).*

Exercise 146 *Let $\{x_n\}$ be a sequence in a metric space (X,d).*

(i) Prove that if $\{x_n\}$ is a Cauchy sequence and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to some $x \in X$, then $\{x_n\}$ converges to x.

(ii) Prove that if there exists $x \in X$ such that for every *subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a further subsequence $\left\{x_{n_{k_j}}\right\}$ that converges to x, then $\{x_n\}$ converges to x.*

A metric space (X,d) is said to be *complete* if every Cauchy sequence is convergent.

Example 147 *Let $X = (0,1)$ with the metric $d(x,y) = |x-y|$. The sequence $\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}}$ converges to 0 in $\mathbb{R}$, and so it is a Cauchy sequence in $\mathbb{R}$. In particular, it is a Cauchy sequence in X. However, it does not converge to an element of X, since $0 \notin X$.*

Friday, October 7, 2011

Theorem 148 $\left(\mathbb{R}^N, d\right)$ *is a complete metric space.*

Proof. Let $\{\mathbf{x}_n\}$ be a Cauchy sequence.
Step 1: We claim that $\{\mathbf{x}_n\}$ is bounded. Fix $\varepsilon = 1$. By the definition of Cauchy sequence, there is exists $N_1 \in \mathbb{N}$ such that

$$\|\mathbf{x}_n - \mathbf{x}_m\| < 1$$

for all $n, m \geq N_1$. In particular, taking $m = N_1$, we have that

$$\|\mathbf{x}_n - \mathbf{x}_{N_1}\| < 1,$$

for all $n \geq N_1$. Taking

$$R := \max\{1, \|\mathbf{x}_1 - \mathbf{x}_{N_1}\| + 1, \dots, \|\mathbf{x}_{N_1-1} - \mathbf{x}_{N_1}\| + 1\},$$

we have that $\mathbf{x}_n \in B(\mathbf{x}_{N_1}, R)$ for all $n \in \mathbb{N}$.
Step 2: Write

$$\mathbf{x}_n = (x_{1,n}, x_{2,n}, \dots, x_{N,n})$$

and consider the sequence of real numbers $\{x_{1,n}\}$. Since $\{x_{1,n}\}$ is bounded, by Theorem 129, there exists a subsequence $\{x_{1,n_k}\}$ of $\{x_{1,n}\}$ such that $\lim_{k\to\infty} x_{1,n_k} = \ell_1 \in \mathbb{R}$. Consider the subsequence

$$\mathbf{x}_{n_k} = (x_{1,n_k}, x_{2,n_k}, \dots, x_{N,n_k}).$$

Since $\{x_{2,n_k}\}$ is bounded, by Theorem 129, there exists a subsequence $\left\{x_{2,n_{k_j}}\right\}$ of $\{x_{2,n_k}\}$ such that $\lim_{j\to\infty} x_{2,n_{k_j}} = \ell_2 \in \mathbb{R}$. Not that x_{1,n_k} still converges to ℓ_1 by Theorem 95. Continuing in this way, we can find a subsequence $\{\mathbf{x}_{n_i}\}$ of $\{\mathbf{x}_n\}$ such that $\mathbf{x}_{n_i} \to \boldsymbol{\ell} \in \mathbb{R}^N$ as $i \to \infty$.

Given $\varepsilon > 0$, by the definition of Cauchy sequence, there is exists $N_\varepsilon \in \mathbb{N}$ such that

$$\|\mathbf{x}_n - \mathbf{x}_m\| \leq \varepsilon$$

for all $n, m \geq N_\varepsilon$. On the other hand, since $\mathbf{x}_{n_i} \to \boldsymbol{\ell}$, there is exists $K_\varepsilon \in \mathbb{N}$ such that

$$\|\mathbf{x}_{n_i} - \boldsymbol{\ell}\| \leq \varepsilon$$

for all $k \geq K_\varepsilon$. Since $\{n_i\}$ is strictly increasing, $n_i \to \infty$ as $i \to \infty$ and so $n_i \geq N_\varepsilon$ for all i large, say $i \geq K_1$. Fix $k \geq \max\{K_\varepsilon, K_1\}$. Then for all $n \geq N_\varepsilon$

$$\|\mathbf{x}_n - \boldsymbol{\ell}\| \leq \|\mathbf{x}_n - \mathbf{x}_{n_i}\| + \|\mathbf{x}_{n_i} - \boldsymbol{\ell}\| \leq \varepsilon + \varepsilon,$$

where we have used the facts that $n, n_k \geq N_\varepsilon$ and that $k \geq K_\varepsilon$.

This implies that $\{\mathbf{x}_n\}$ converges to $\boldsymbol{\ell}$. ■

Theorem 149 *Given a nonempty set X, consider the space $\ell^\infty\left(X; \mathbb{R}^M\right) := \left\{\mathbf{f} : X \to \mathbb{R}^M : \mathbf{f} \text{ is bounded}\right\}$ with the norm*

$$\|\mathbf{f}\|_\infty := \sup_{x\in X} \|\mathbf{f}(x)\|.$$

Then $\ell^\infty\left(X; \mathbb{R}^M\right)$ is complete.

Proof. To see this, let $\{\mathbf{f}_n\} \subseteq \ell^\infty\left(X; \mathbb{R}^M\right)$ be a Cauchy sequence. Let $\varepsilon > 0$ and find $n_\varepsilon \in \mathbb{N}$ so large that

$$\sup_{x\in X} \|\mathbf{f}_n(x) - \mathbf{f}_m(x)\| = d_\infty(\mathbf{f}_n, \mathbf{f}_m) \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$. This implies that for every fixed $x \in X$, the sequence of real numbers $\{\mathbf{f}_n(x)\}$ is a Cauchy sequence in $\mathbb{R}^M$ and so there exists

$$\lim_{n\to\infty} \mathbf{f}_n(x) = \mathbf{f}(x) \in \mathbb{R}^M.$$

Since

$$\|\mathbf{f}_n(x) - \mathbf{f}_m(x)\| \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$, letting $n \to \infty$ gives (why?)

$$\|\mathbf{f}(x) - \mathbf{f}_m(x)\| \leq \varepsilon$$

for all $m \geq n_\varepsilon$. This holds for every $x \in X$. Hence, taking the supremum over all $x \in X$ gives

$$\sup_{x\in X} \|\mathbf{f}(x) - \mathbf{f}_m(x)\| = d_\infty(\mathbf{f}, \mathbf{f}_m) \leq \varepsilon$$

for all $m \geq n_\varepsilon$; that is $d_\infty(\mathbf{f}, \mathbf{f}_m) \to 0$ as $m \to \infty$. Moreover, for every $x \in X$,

$$\|\mathbf{f}(x)\| \leq \|\mathbf{f}(x) + \mathbf{f}_{n_\varepsilon}(x) - \mathbf{f}_{n_\varepsilon}(x)\| \leq \sup_{x\in X} |\mathbf{f}(x) - \mathbf{f}_{n_\varepsilon}(x)| + \sup_{x\in X} |\mathbf{f}_{n_\varepsilon}(x)| \leq \varepsilon + \sup_{x\in X} |\mathbf{f}_{n_\varepsilon}(x)|,$$

which implies that $\mathbf{f} \in \ell^\infty\left(X; \mathbb{R}^M\right)$. ■

Exercise 150 *Let (X, d) be a metric space and let*

$$C_b(X) := \{f : X \to \mathbb{R} : f \text{ is continuous and bounded}\}$$

with the norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)|.$$

Prove that $C_b(X)$ is complete.

Monday, October 10, 2011

Definition 151 *Given a metric space (X, d), a set $E \subseteq X$ is* dense *if*

$$\overline{E} = X.$$

We have seen that the rationals and the irrationals are dense in $\mathbb{R}$ and in your homework you proved that if $x \in \mathbb{R}$ then the set $\{j + kx : j, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Theorem 152 (Baire category theorem) *Let (X, d) be a complete metric space. Then the intersection of a countable family of open dense sets in X is still dense in X.*

Proof. Let $\{U_n\} \subseteq X$ be a countable family of dense open sets. We consider here only the case in which the family $\{U_n\}$ is infinite. The case in which the family is finite is simpler and is left as an exercise. Let

$$E := \bigcap_{n=1}^{\infty} U_n.$$

We claim that $\overline{E} = X$. Fix $x_0 \in X$. We claim that $x_0 \in \overline{E}$. To see this, in view of Proposition 66 (which continues to work for metric spaces and topological spaces), it is enough to show that for

$$B(x_0, r) \cap E \neq \emptyset$$

for every $r > 0$. Fix $r > 0$. Since U_1 is dense, $x_0 \in X = \overline{U_1}$, and so by Proposition 66, the open set $B(x_0, r) \cap U_1$ is nonempty. Let $x_1 \in B(x_0, r) \cap U_1$. Since $B(x_0, r) \cap U_1$ is open, there exists $0 < r_1 < 1$ such that

$$B(x_1, 2r_1) \subseteq B(x_0, r) \cap U_1. \tag{17}$$

Inductively, assume that $x_n \in X$ and $0 < r_n < \frac{1}{n}$ have been chosen. Since U_{n+1} is dense, $x_n \in X = \overline{U_{n+1}}$, and so by Proposition 66, the open set $B(x_n, r_n) \cap U_{n+1}$ is nonempty, and so there exist $x_{n+1} \in X$ and $0 < r_{n+1} < \frac{1}{n+1}$ such that

$$B(x_{n+1}, 2r_{n+1}) \subseteq B(x_n, r_n) \cap U_{n+1}. \tag{18}$$

By induction we can construct two sequences $\{x_n\}$ and $\{r_n\}$ such that (18) holds and $0 < r_n < \frac{1}{n}$ for all $n \geq 1$. Note that, by construction (see (17) and (18)), for every $n \in \mathbb{N}$,

$$B(x_{n+1}, 2r_{n+1}) \subseteq B(x_n, r_n) \subseteq B(x_n, 2r_n) \subseteq \cdots \subseteq B(x_1, 2r_1) \subseteq B(x_0, r) \cap U_1. \tag{19}$$

Hence, if $n, m > k$, then $x_n, x_m \in B(x_k, r_k)$, so that

$$d(x_n, x_m) \leq d(x_n, x_k) + d(x_k, x_m) < r_k + r_k < \frac{2}{k}.$$

By letting $k \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. We claim that $x \in B(x_0, r) \cap U_\ell$ for every $\ell \in \mathbb{N}$. Indeed, fix $k \in \mathbb{N}$. Then for all $n > k$ we have that $x_n \in B(x_k, r_k)$, and so,

$$d(x_k, x) \leq d(x_k, x_n) + d(x_n, x) < r_k + d(x_n, x).$$

Letting $n \to \infty$ we conclude that $d(x_k, x) \leq r_k < 2r_k$. It follows that $x \in B(x_k, 2r_k) \subseteq B(x_{k-1}, r_{k-1}) \cap U_k$ for all $k \in \mathbb{N}$ (where we set $r_0 := r$). In turn, by (17) and (18), we have that $x \in B(x_0, r) \cap E$ holds and the proof is complete. ■

Definition 153 *Let (X, d) be a metric space. A set $E \subseteq X$ is called* *(i)* nowhere dense *if the interior of its closure is empty.*

(ii) meager *if it can be written as a countable union of nowhere dense sets.*

Note that if U is open and dense, then its complement is closed and nowhere dense. Hence, we have the following.

Corollary 154 *Let (X, d) be a nonempty complete metric space. If*

$$X = \bigcup_{n=1} C_n,$$

where C_n is closed for every $n \in \mathbb{N}$. Then at least one C_n has nonempty interior. In particular, every complete nonempty metric space is not meager.

Proof. If every C_n has empty interior, then C_n is nowhere dense. Hence, $U_n := X \setminus C_n$ is open and dense. By De Morgan's laws,

$$X = \bigcup_{n=1} C_n = \bigcup_{n=1} (X \setminus U_n) = X \setminus \bigcap_{n=1} U_n,$$

which implies that $\bigcap_{n=1} U_n$ is empty. This contradicts Baire's theorem. ■

Theorem 155 *There exists a continuous $f : [0, 1] \to \mathbb{R}$ that is nowhere monotone.*

Proof. Let I be a closed interval of $[0, 1]$ and let

$$\begin{aligned} \mathcal{I}_I &:= \{f \in C([0, 1]) : f \text{ is increasing in } I\}, \\ \mathcal{D}_I &:= \{f \in C([0, 1]) : f \text{ is decreasing in } I\} \end{aligned}$$

The sets $\mathcal{I}_I$ and $\mathcal{D}_I$ are closed. Define

$$\mathcal{M}_I := \mathcal{I}_I \cup \mathcal{D}_I.$$

Then $\mathcal{M}_I$ is closed. Moreover, $\mathcal{M}_I$ has empty interior (why?). ■

Wednesday, October 12, 2011

Proof. Consider the sequence $\{I_n\}_n$ of closed intervals $I_1 = \left[0, \frac{1}{2}\right]$, $I_2 = \left[\frac{1}{2}, 1\right]$, $I_3 = \left[0, \frac{1}{3}\right]$, $I_4 = \left[\frac{1}{3}, \frac{2}{3}\right]$, $I_5 = \left[\frac{2}{3}, 1\right]$, $I_6 = \left[0, \frac{1}{4}\right]$, etc... and let $\mathcal{M}_n := \mathcal{M}_{I_n}$. Then

$$\mathcal{M} := \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

is a meager set. By the previous corollary

$$C([0, 1]) \neq \bigcup_{n=1}^{\infty} \mathcal{M}_n.$$

Any function $f \in C([0, 1]) \setminus (\bigcup_{n=1}^{\infty} \mathcal{M}_n)$ is nowhere monotone. ■

12 Sequences and Topology

A set $C \subseteq \mathbb{R}^N$ is *sequentially closed* if for every sequence $\{\mathbf{x}_n\} \subseteq C$ such that $\{\mathbf{x}_n\}$ converges to some $\mathbf{x} \in \mathbb{R}^N$, then $\mathbf{x}$ belongs to C.

Proposition 156 *Let $C \subseteq \mathbb{R}^N$. Then C is closed if and only if C is sequentially closed.*

Proof. Step 1: Assume that C is closed and let $\{\mathbf{x}_n\} \subseteq C$ be such that $\{\mathbf{x}_n\}$ converges to some $\mathbf{x} \in \mathbb{R}^N$. We need to show that $\mathbf{x}$ belongs to C. If not, then $\mathbf{x} \in \mathbb{R}^N \setminus C$. Since $\mathbb{R}^N \setminus C$ is open, there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq \mathbb{R}^N \setminus C$. But then, taking $\varepsilon = r$ there exists $n_r \in \mathbb{N}$ such that $\|\mathbf{x}_n - \mathbf{x}\| < r$ for all $n \geq n_r$, which implies that $\mathbf{x}_n \in B(\mathbf{x}, r) \subseteq \mathbb{R}^N \setminus C$ for all $n \geq n_r$. This contradicts the fact that $\{\mathbf{x}_n\} \subseteq C$.
Step 2: Assume that C is sequentially closed. We need to show that $\mathbb{R}^N \setminus C$ is open. Let $\mathbf{x} \in \mathbb{R}^N \setminus C$. We claim that there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq \mathbb{R}^N \setminus C$. If not, then for every $r > 0$ we can find $\mathbf{y} \in C$ such that $\mathbf{y} \in B(\mathbf{x}, r)$. Taking $r = \frac{1}{n}$ we can find $\mathbf{x}_n \in C$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \frac{1}{n} \to 0$, which shows that $\{\mathbf{x}_n\}$ converges to $\mathbf{x}$. Since C is sequentially closed, it follows that $\mathbf{x} \in C$, which is a contradiction. ■

Remark 157 *The previous proposition works for metric spaces, while for topological spaces we can still adapt the first part of the proof to conclude that a closed set is sequentially closed.*

Definition 158 *A set $K \subseteq \mathbb{R}^N$*

(i) is sequentially compact *if for every sequence $\{\mathbf{x}_n\} \subseteq K$, there exist a subsequence $\{\mathbf{x}_{n_k}\}$ of $\{\mathbf{x}_n\}$ and $\mathbf{x} \in K$ such that $\mathbf{x}_{n_k} \to \mathbf{x}$ as $k \to \infty$,*

(ii) is totally bounded *if for every $\varepsilon > 0$ there exist $\mathbf{x}_1, \ldots, \mathbf{x}_m \in K$ such that*

$$K \subseteq \bigcup_{i=1}^{m} B(\mathbf{x}_i, \varepsilon).$$

The following theorem is one of the main results of this subsection.

Theorem 159 *Let $K \subseteq \mathbb{R}^N$ Then K is sequentially compact if and only if K is compact.*

Proof. Step 1: Assume that K is sequentially compact. We claim that K is totally bounded. Assume by contradiction that K is not totally bounded. Then there exists $\varepsilon_0 > 0$ such that K cannot be covered by a finite number of balls of radius ε_0. Fix $\mathbf{x}_1 \in K$. Then there exists $\mathbf{x}_2 \in K$ such that $\|\mathbf{x}_1 - \mathbf{x}_2\| \geq \varepsilon_0$ (otherwise $B(\mathbf{x}_1, \varepsilon_0)$ would cover K). Similarly, we can find $\mathbf{x}_3 \in K$ such that $\|\mathbf{x}_1 - \mathbf{x}_3\| \geq \varepsilon_0$ and $\|\mathbf{x}_2 - \mathbf{x}_3\| \geq \varepsilon_0$ (otherwise $B(\mathbf{x}_1, \varepsilon_0)$ and $B(\mathbf{x}_2, \varepsilon_0)$ would cover K). Inductively, construct a sequence $\{\mathbf{x}_n\} \subseteq K$ such that $\|\mathbf{x}_n - \mathbf{x}_m\| \geq$

ε_0 for all $n, m \in \mathbb{N}$ with $n \neq m$. The sequence $\{\mathbf{x}_n\}$ cannot have a convergent subsequence, which contradicts the fact that K is sequentially compact.

Next we prove that K is compact. Let $\{U_\alpha\}_\alpha$ be an open cover of K. Since K is totally bounded, for every $n \in \mathbb{N}$ let $\mathcal{B}_n$ be a finite cover of K with balls of radius $\frac{1}{n}$ and centers in K. We want to prove that there exists $\overline{n} \in \mathbb{N}$ such that every ball in $\mathcal{B}_{\overline{n}}$ is contained in some U_α. Note that this would conclude the proof. Indeed, for every $B \in \mathcal{B}_{\overline{n}}$ fix one U_α containing B. Since $\mathcal{B}_{\overline{n}}$ is a finite family and covers K, the subcover of $\{U_\alpha\}$ just constructed has the same properties.

To find $\overline{n}$, assume by contradiction that for every $n \in \mathbb{N}$ there exists a ball $B\left(\mathbf{x}_n, \frac{1}{n}\right) \in \mathcal{B}_n$ that is not contained in any U_α. By sequential compactness, there exist a subsequence $\{\mathbf{x}_{n_k}\}$ of $\{\mathbf{x}_n\}$ and $\mathbf{x} \in K$ such that $\mathbf{x}_{n_k} \to \mathbf{x}$ as $k \to \infty$. Since $\mathbf{x} \in K$, there exists β such that $\mathbf{x} \in U_\beta$. But U_β is open, and so there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U_\beta$. Since, $\mathbf{x}_{n_k} \to \mathbf{x}$, we have that $\|\mathbf{x}_{n_k} - \mathbf{x}\| < \frac{r}{2}$ for all k sufficiently large. In turn, if $\frac{1}{n_k} < \frac{r}{2}$, by the triangle inequality, $B\left(\mathbf{x}_{n_k}, \frac{1}{n_k}\right) \subseteq B(\mathbf{x}, r) \subseteq U_\beta$, which contradicts the fact that ball $B\left(\mathbf{x}_{n_k}, \frac{1}{n_k}\right)$ is not contained in any U_α. This shows that K is compact. ■

Friday, October 14, 2011

Proof. Step 2: Assume that K is compact. By Theorem 83, K is closed, and so sequentially closed by Proposition 156. We claim that K is sequentially compact. To see this, assume by contradiction that there exists a sequence $\{\mathbf{x}_n\} \subseteq K$ that has no subsequence converging in K. Then for every $m \in \mathbb{N}$ the number of $n \in \mathbb{N}$ such that $\mathbf{x}_n = \mathbf{x}_m$ is finite (otherwise, if $\mathbf{x}_n = \mathbf{x}_m$ for infinitely many $n \in \mathbb{N}$, then this would be a convergent subsequence). Moreover, the set $C := \{\mathbf{x}_n : n \in \mathbb{N}\}$ has no accumulation points. Indeed, if C had an accumulation point, then since K is sequentially closed, there would a subsequence of $\{\mathbf{x}_n\}$ converging to K. Since C has no accumulation point, it follows, in particular, that C is closed. Similarly, for every $m \in \mathbb{N}$ the sets $C_m := \{\mathbf{x}_n : n \in \mathbb{N}, n \geq m\}$ are closed. Moreover, $C_{m+1} \subseteq C_m$ and by what we said before,

$$\bigcap_{m=1}^{\infty} C_m = \emptyset. \tag{20}$$

For every $m \in \mathbb{N}$ the set $U_m := \mathbb{R}^N \setminus C_m$ is open, $U_{m+1} \supseteq U_m$ and by (20) and De Morgan's laws

$$\bigcup_{m=1}^{\infty} U_m = \bigcup_{m=1}^{\infty} \left(\mathbb{R}^N \setminus C_m\right) = \mathbb{R}^N \setminus \left(\bigcap_{m=1}^{\infty} C_m\right) = \mathbb{R}^N.$$

In particular, $\{U_m\}_m$ is an open cover of K. By compactness, it follows that there $\overline{m} \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{m=1}^{\overline{m}} U_m = U_{\overline{m}} = \mathbb{R}^N \setminus C_{\overline{m}},$$

which implies that $K \cap C_{\overline{m}} = \emptyset$. This is a contradiction, since $C_{\overline{m}}$ is nonempty and contained in K. ∎

Remark 160 *The previous theorem continues to hold for metric spaces but neither direction holds for topological spaces.*

13 Functions

Consider a function $\mathbf{f} : E \to \mathbb{R}^M$, where $E \subseteq \mathbb{R}^N$. The set E is called the *domain* of $\mathbf{f}$. If E is not specified, then E should be taken to be the largest set of $\mathbf{x}$ for which $\mathbf{f}(\mathbf{x})$ makes sense. This means that:

If there are even roots, their arguments should be nonnegative. If there are logarithms, their arguments should be strictly positive. Denominators should be different from zero. If a function is raised to an irrational number, then the function should be nonnegative.

Given a set $F \subseteq E$, the set $\mathbf{f}(F) = \left\{\mathbf{y} \in \mathbb{R}^M : \mathbf{y} = \mathbf{f}(\mathbf{x}) \text{ for some } \mathbf{x} \in F\right\}$ is called the *image* of F through $\mathbf{f}$. The function $\mathbf{f}$ is said to be bounded from above in F, bounded from below in F, bounded in F if the set $\mathbf{f}(F)$ is bounded from above, bounded from below, bounded, respectively.

Given a set $G \subseteq \mathbb{R}$, the set $\mathbf{f}^{-1}(G) = \{\mathbf{x} \in E : \mathbf{f}(\mathbf{x}) \in G\}$ is called the *inverse image* of F through $\mathbf{f}$. It has NOTHING to do with the inverse function. It is just one of those unfortunate cases in which we use the same symbol for two different objects.

The *graph* of a function is the set of $\mathbb{R}^N \times \mathbb{R}^M$ defined by

$$\operatorname{gr} \mathbf{f} = \{(\mathbf{x}, \mathbf{f}(\mathbf{x})) : \mathbf{x} \in E\}.$$

A function $\mathbf{f}$ is said to be

- *one-to-one* or *injective* if $\mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{z})$ for all $\mathbf{x}, \mathbf{z} \in E$ with $\mathbf{x} \neq \mathbf{z}$.

If $\mathbf{f} : E \to F$, where $E, F \subseteq \mathbb{R}$, then $\mathbf{f}$ is said to be

- *onto* or *surjective* if $\mathbf{f}(E) = F$,
- *bijective* or *invertible* if it is one-to-one and onto. The function $\mathbf{f}^{=1} : F \to E$, which assigns to each $\mathbf{y} \in F = \mathbf{f}(E)$ the unique $\mathbf{x} \in E$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$, is called the *inverse* function of $\mathbf{f}$.

If $N = M = 1$ a function $f : E \to \mathbb{R}$ is said to be

- *increasing* if $f(x) \leq f(y)$ for all $x, y \in E$ with $x < y$,
- *strictly increasing* if $f(x) < f(y)$ for all $x, y \in E$ with $x < y$,
- *decreasing* if $f(x) \geq f(y)$ for all $x, y \in E$ with $x < y$,
- *strictly decreasing* if $f(x) > f(y)$ for all $x, y \in E$ with $x < y$,
- *monotone* if one of the four property above holds.

14 Limits of Functions

Definition 161 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{x}_0 \in \mathbb{R}^N$ be an accumulation point of E, and let $\mathbf{f} : E \to \mathbb{R}^N$. We say that a number $\boldsymbol{\ell} \in \mathbb{R}^M$ is the* limit *of $\mathbf{f}(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{x}_0$ if for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, \mathbf{x}_0) > 0$ with the property that*

$$\|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| < \varepsilon$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. We write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \boldsymbol{\ell} \quad \text{or} \quad \mathbf{f}(\mathbf{x}) \to \boldsymbol{\ell} \text{ as } \mathbf{x} \to \mathbf{x}_0.$$

Note that even when $\mathbf{x}_0 \in E$, we cannot take $\mathbf{x} = \mathbf{x}_0$ since in the definition we require $0 < \|\mathbf{x} - \mathbf{x}_0\|$.

Remark 162 *If (X, d_X) and (Y, d_Y) are two metric spaces, $E \subseteq X$, $x_0 \in E$ is an accumulation point of E and $f : E \to Y$, we say that $\ell \in Y$ is the* limit *of $f(x)$ as x approaches x_0 if for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, x_0) > 0$ with the property that*

$$d_Y(f(x), \ell) < \varepsilon$$

for all $x \in E$ with $0 < d_X(x, x_0) < \delta$. We write

$$\lim_{x \to x_0} f(x) = \ell.$$

If (X, τ_X) and (Y, τ_Y) are two topological spaces, $E \subseteq X$, $x_0 \in E$ is an accumulation point of E and $f : E \to Y$, we say that $\ell \in Y$ is the limit *of $f(x)$ as x approaches x_0 if for every neighborhood V of ℓ there exists a neighborhood U of x_0 with the property that*

$$f(x) \in V$$

for all $x \in E$ with $x \in U \setminus \{x_0\}$. We write

$$\lim_{x \to x_0} f(x) = \ell.$$

Note that unless the space Y is Hausdorff, the limit may not be unique.

Definition 163 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{x}_0 \in \mathbb{R}^N$ be an accumulation point of E, and let $f : E \to \mathbb{R}$. We say that*

- *∞ is the* limit *of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{x}_0$ if for every $L > 0$ there exists a real number $\delta = \delta(L, \mathbf{x}_0) > 0$ with the property that*

  $$f(\mathbf{x}) > L$$

  for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. We write

  $$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \infty \quad \text{or} \quad f(\mathbf{x}) \to \infty \text{ as } \mathbf{x} \to \mathbf{x}_0,$$

- $-\infty$ *is the* limit *of* $f(\mathbf{x})$ *as* $\mathbf{x}$ *approaches* $\mathbf{x}_0$ *if for every* $L > 0$ *there exists a real number* $\delta = \delta(L, \mathbf{x}_0) > 0$ *with the property that*

$$f(\mathbf{x}) < -L$$

for all $\mathbf{x} \in E$ *with* $0 < \|\mathbf{x} - \mathbf{x}_0\| \leq \delta$. *We write*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = -\infty \quad \textit{or} \quad f(\mathbf{x}) \to -\infty \textit{ as } \mathbf{x} \to \mathbf{x}_0.$$

Example 164 *Let's study the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^m y}{x^2 + y^2},$$

where $m \in \mathbb{N}$. *In this case* $f(x,y) = \frac{x^m y}{x^2+y^2}$ *and the domain is* $\mathbb{R}^2 \setminus \{(0,0)\}$.

For $m \geq 2$, *we have that the limit is* 0. *Indeed, using the fact that* $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$, *we have*

$$\left| \frac{x^m y}{x^2 + y^2} - 0 \right| = \frac{|x|^m |y|}{x^2 + y^2} \leq \frac{\left(x^2 + y^2\right)^{m/2} \left(x^2 + y^2\right)^{1/2}}{x^2 + y^2} = \left(x^2 + y^2\right)^{(m-1)/2} \to 0$$

as $(x,y) \to (0,0)$. *On the other hand, if* $m = 1$, *taking* $y = x$, *we have that*

$$\lim_{x \to 0} f(x,x) = \lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2},$$

while taking $y = 0$, *we have that*

$$\lim_{x \to 0} f(x,0) = \lim_{x \to 0} \frac{0}{x^2 + 0} = 0,$$

and so the limit does not exist.

Exercise 165 *Study the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^m y}{x^2 - y^2},$$

where $m \in \mathbb{N}$.

Monday, October 17, 2011

Theorem 166 *Let* $E \subseteq \mathbb{R}^N$ *and let* $\mathbf{x}_0 \in \mathbb{R}^N$ *be an accumulation point of* E. *Given a function* $\mathbf{f} : E \to \mathbb{R}^M$ *there exists*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(x) = \boldsymbol{\ell} \in \mathbb{R}^M$$

if and only if

$$\lim_{n \to \infty} \mathbf{f}(\mathbf{x}_n) = \boldsymbol{\ell}$$

for ***every*** *sequence* $\{\mathbf{x}_n\} \subseteq E \setminus \{\mathbf{x}_0\}$ *converging to* $\mathbf{x}_0$. *In particular, if the limit exists, it is unique.*

Proof. Assume that there exists $\boldsymbol{\ell} = \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(x)$ and let $\{\mathbf{x}_n\} \subseteq E \setminus \{\mathbf{x}_0\}$ converge to $\mathbf{x}_0$. Fix $\varepsilon > 0$ and find $\delta = \delta(\varepsilon, \mathbf{x}_0) > 0$ such that

$$\|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| < \varepsilon$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. Since $\mathbf{x}_n \to \mathbf{x}_0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\|\mathbf{x}_n - \mathbf{x}_0\| < \delta$ for all $n \geq n_\varepsilon$ and so

$$\|f(\mathbf{x}_n) - \boldsymbol{\ell}\| < \varepsilon$$

for all $n \geq n_\varepsilon$, which shows that $\mathbf{f}(\mathbf{x}_n) \to \boldsymbol{\ell}$.

Conversely, assume that $\lim_{n\to\infty} \mathbf{f}(\mathbf{x}_n) = \boldsymbol{\ell}$ for every sequence $\{\mathbf{x}_n\} \subseteq E \setminus \{\mathbf{x}_0\}$ converging to $\mathbf{x}_0$. If either the limit $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ does not exist or it exists but it it different from $\boldsymbol{\ell}$, then there exists $\varepsilon > 0$ with the property that for every δ there exists $\mathbf{x} \in E \setminus \{\mathbf{x}_0\}$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ such that

$$\|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| \geq \varepsilon.$$

Take $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$ and find $\mathbf{x}_n \in E \setminus \{\mathbf{x}_0\}$ with $\|\mathbf{x}_n - \mathbf{x}_0\| < \frac{1}{n}$ such that

$$\|\mathbf{f}(\mathbf{x}_n) - \boldsymbol{\ell}\| > \varepsilon. \tag{21}$$

Then $\mathbf{x}_n \to \mathbf{x}_0$ but, by hypothesis $\mathbf{f}(\mathbf{x}_n) \to \boldsymbol{\ell}$ as $n \to \infty$, which contradicts (21). ■

Remark 167 *In view of the previous theorem, in order to show that the limit* $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ *does not exist, it is enough to find two sequences* $\{\mathbf{x}_n\} \subseteq E \setminus \{\mathbf{x}_0\}$ *and* $\{\mathbf{y}_n\} \subseteq E \setminus \{\mathbf{x}_0\}$ *both converging to* $\mathbf{x}_0$ *and such that*

$$\lim_{n\to\infty} \mathbf{f}(\mathbf{x}_n) \neq \lim_{n\to\infty} \mathbf{f}(\mathbf{y}_n).$$

Remark 168 *When* $M = 1$*, the previous theorem continues to hold if* $\ell \in [-\infty, \infty]$*.*

Example 169 *Consider the function* $f(x) = \cos\frac{1}{x}$ *defined in* $E = \mathbb{R} \setminus \{0\}$*. Note that* 0 *is an accumulation point of* E*. To prove that the limit*

$$\lim_{x\to 0} \cos\frac{1}{x}$$

does not exist, consider the sequences $x_n = \frac{1}{2n\pi} \to 0$ *and* $x_n = \frac{1}{n\pi} \to 0$*. Then*

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \cos(2n\pi) = 1 \neq \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \cos(n\pi) = -1.$$

Exercise 170 *Study the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}.$$

We now list some important operations for limits.

Theorem 171 *Let $E \subseteq \mathbb{R}^N$ and let $\mathbf{x}_0 \in \mathbb{R}^N$ be an accumulation point of E. Given three functions $f, g, h : E \to \mathbb{R}$, assume that there exist*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \ell_1 \in \mathbb{R}, \qquad \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \ell_2 \in \mathbb{R}.$$

Then

(i) there exists $\lim_{\mathbf{x}\to\mathbf{x}_0} (f+g)(\mathbf{x}) = \ell_1 + \ell_2$,

(ii) there exists $\lim_{\mathbf{x}\to\mathbf{x}_0} (fg)(\mathbf{x}) = \ell_1 \ell_2$,

(iii) if $\ell_2 \neq 0$ and $F := \{\mathbf{x} \in E : g(\mathbf{x}) \neq 0\}$, then $\mathbf{x}_0$ is an accumulation point of F and there exists $\lim_{\mathbf{x}\to\mathbf{x}_0} \left(\frac{f}{g}\Big|_F \right)(\mathbf{x}) = \frac{\ell_1}{\ell_2}$,

(iv) ***(Squeeze Theorem)*** *if $\ell_1 = \ell_2$ and $f(\mathbf{x}) \leq h(\mathbf{x}) \leq g(\mathbf{x})$ for every $\mathbf{x} \in E$, then there exists $\lim_{\mathbf{x}\to\mathbf{x}_0} h(\mathbf{x}) = \ell_1$.*

Proof. Parts (i) and (ii) follow from Theorems 104 and 166, part (iv) from Theorems 109 and 166. The fact that $\mathbf{x}_0$ is an accumulation point of F is left as an exercise. ■

Remark 172 *As in the case of sequences, the previous theorem continues to hold if $\ell_1, \ell_1 \in [-\infty, \infty]$, provided we avoid the cases $\infty - \infty$, 0∞, $\frac{0}{0}$, $\frac{\infty}{\infty}$.*

Theorem 173 *Let $E \subseteq \mathbb{R}^N$ and let $\mathbf{x}_0 \in \mathbb{R}$ be an accumulation point of E. Given two functions $f, g : E \to \mathbb{R}$, assume that there exists*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = 0,$$

and that g is bounded. Then there exists $\lim_{\mathbf{x}\to\mathbf{x}_0} (fg)(\mathbf{x}) = 0$.

Proof. This follows from Corollary 110 and Theorem 166. ■

Example 174 *The previous theorem can be used for example to show that for $a > 0$*

$$\lim_{x\to 0} x^a \sin \frac{1}{x} = 0.$$

We next study the limit of composite functions.

Theorem 175 *Let $E \subseteq \mathbb{R}^N$, $F \subseteq \mathbb{R}^M$ and let $\mathbf{x}_0 \in \mathbb{R}$ be an accumulation point of E. Given two functions $\mathbf{f} : E \to F$ and $g : F \to \mathbb{R}^P$, assume that there exist*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \boldsymbol{\ell} \in \mathbb{R}^M,$$

that $\boldsymbol{\ell}$ is an accumulation point of F, and that there exists

$$\lim_{\mathbf{y}\to\boldsymbol{\ell}} \mathbf{g}(\mathbf{y}) = \mathbf{L} \in \mathbb{R}^P.$$

Assume that either there exists $\delta_1 > 0$ such that $\mathbf{f}(\mathbf{x}) \neq \boldsymbol{\ell}$ for all $\mathbf{x} \in E$ with $0 < |\mathbf{x} - \mathbf{x}_0| \leq \delta_1$, or that $\boldsymbol{\ell} \in F$ and $\mathbf{g}(\boldsymbol{\ell}) = \mathbf{L}$. Then there exists $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{L}$.

Proof. Fix $\varepsilon > 0$ and find $\eta = \eta(\varepsilon, \boldsymbol{\ell}) > 0$ such that

$$\|\mathbf{g}(\mathbf{y}) - \mathbf{L}\| < \varepsilon \tag{22}$$

for all $\mathbf{y} \in F$ with $0 < \|\mathbf{y} - \boldsymbol{\ell}\| < \eta$.

Since $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \boldsymbol{\ell}$, there exists $\delta_2 = \delta_2(\mathbf{x}_0, \eta) > 0$ such that

$$\|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| < \eta$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$.

We now distinguish two cases.

Case 1: Assume that $\mathbf{f}(\mathbf{x}) \neq \boldsymbol{\ell}$ for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$. Then taking $\delta = \min\{\delta_1, \delta_2\}$, we have that for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$,

$$0 < \|\mathbf{f}(\mathbf{x}) - \boldsymbol{\ell}\| < \eta.$$

Hence, taking $\mathbf{y} = \mathbf{f}(\mathbf{x})$, by (22), it follows that

$$\|\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{L}\| < \varepsilon$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$. This shows that there exists $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{L}$.

Case 2: Assume that $\boldsymbol{\ell} \in F$ and $\mathbf{g}(\boldsymbol{\ell}) = \mathbf{L}$. Let $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$. If $\mathbf{f}(\mathbf{x}) = \boldsymbol{\ell}$, then $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{L}$, and so

$$\|\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{L}\| = 0 < \varepsilon,$$

while if $\mathbf{f}(\mathbf{x}) \neq \boldsymbol{\ell}$, then taking $\mathbf{y} = \mathbf{f}(\mathbf{x})$, by (22), it follows that

$$\|\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{L}\| < \varepsilon.$$

■

Example 176 *Let's prove that the previous theorem fails without the hypotheses that either $\mathbf{f}(\mathbf{x}) \neq \boldsymbol{\ell}$ for all $\mathbf{x} \in E$ near $\mathbf{x}_0$ or $\boldsymbol{\ell} \in F$, $L \in \mathbb{R}$ and $\mathbf{g}(\boldsymbol{\ell}) = L$. Consider the function*

$$g(y) := \begin{cases} 1 & \text{if } y \neq 0, \\ 2 & \text{if } y = 0. \end{cases}$$

Then there exists

$$\lim_{y\to 0} g(y) = 1.$$

So $L = 1$. Consider the function $f(x) := 0$ for all $x \in \mathbb{R}$. Then for every $x_0 \in \mathbb{R}$, we have that

$$\lim_{x\to x_0} f(x) = 0.$$

So $\ell = 0$. However, $g(f(x)) = g(0) = 2$ for all $x \in \mathbb{R}$. Hence,

$$\lim_{x\to x_0} g(f(x)) = \lim_{x\to x_0} 2 = 2 \neq 1,$$

which shows that the conclusion of the theorem is violated.

Example 177 *We list below some important limits that can be proved using Taylor's formula or De l'Hôpital's theorem.*

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\qquad \lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2},\qquad \lim_{x\to 0}\frac{\log(1+x)}{x}=1,$$
$$\lim_{x\to 0}\frac{(1+x)^a-1}{x}=a\quad \textit{for } a\in\mathbb{R},\qquad \lim_{x\to 0}\frac{e^x-1}{x}=1.$$

Note that the previous theorem can be used to change variables in limits.

Example 178 *Let's try to calculate*

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{x}.$$

For $\sin x\neq 0$*, we have*

$$\frac{\log(1+\sin x)}{x}=\frac{\log(1+\sin x)}{x}\frac{\sin x}{\sin x}=\frac{\log(1+\sin x)}{\sin x}\frac{\sin x}{x}.$$

Since $\lim_{x\to 0}\frac{\sin x}{x}=1$*, it remains to study*

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{\sin x}.$$

Consider the function $g(y)=\frac{\log(1+y)}{y}$ *and the function* $f(x)=\sin x$*. As* $x\to 0$*, we have that* $\sin x\to 0=\ell$*, while*

$$\lim_{y\to 0}\frac{\log(1+y)}{y}=1.$$

Moreover $\sin x\neq 0$ *for all* $x\in E:=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$*. Hence, we can apply the previous theorem to conclude that*

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{\sin x}=1.$$

In turn,

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{x}=1.$$

Wednesday, October 19, 2011

University of Puerto Rico, no class.

Friday, October 21, 2011

Midsemester break, no class.

Monday, October 24, 2011

Definition 179 *Let* $E\subseteq\mathbb{R}^N$ *and let* $\mathbf{f}:E\to\mathbb{R}^M$ *be a function. Given a subset* $F\subseteq E$ *we denote by* $\mathbf{f}|_F$ *the* restriction *of the function* $\mathbf{f}$ *to the set* F*, that is, the function* $\mathbf{f}:F\to\mathbb{R}^M$*.*

Exercise 180 *Let $E \subseteq \mathbb{R}^N$ be such that $E = F \cup G$ and let $\mathbf{x}_0 \in \mathbb{R}^N$ be an accumulation point of both F and G. Consider a function $\mathbf{f} : E \to \mathbb{R}^M$. Prove that there exists $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ if and only if there exist $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}|_F(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}|_G(\mathbf{x})$ and they are equal.*

An important special case is obtained by taking as sets F and G,

$$E^- := E \cap (-\infty, x_0], \quad E^+ := E \cap (x_0, \infty).$$

Whenever they exist, the limits $\lim_{x\to x_0} \mathbf{f}|_{E^-}(x)$ and $\lim_{x\to x_0} \mathbf{f}|_{E^+}(x)$ are called the *left* and *right limit* of $\mathbf{f}$ as $x \to x_0$ and they are denoted, respectively, by

$$\lim_{x\to x_0^-} \mathbf{f}(x) \quad \text{and} \quad \lim_{x\to x_0^+} \mathbf{f}(x).$$

Exercise 181 *Prove that there exists*

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}}.$$

Hint: Calculate the left and right limits separately.

Exercise 182 *Prove that*

$$\lim_{x\to 0^+} x^a \log^b x = 0$$

for $a > 0$ and $b \in \mathbb{R}$. Hint: Use Exercise 126.

15 Limits at Infinity

Definition 183 *Let $E \subseteq \mathbb{R}$. We say that ∞ is an* accumulation point *of E if for every $L > 0$,*

$$(L, \infty) \cap E \neq \emptyset.$$

Exercise 184 *Let $E \subseteq \mathbb{R}$. Prove that ∞ is an accumulation point of E if and only if there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \to \infty$ as $n \to \infty$.*

Definition 185 *Let $E \subseteq \mathbb{R}$ and let ∞ be an accumulation point of E. Given a function $\mathbf{f} : E \to \mathbb{R}^M$ we say that the* limit *of $\mathbf{f}(x)$ as x diverges to ∞ exists if there exists a number $\boldsymbol{\ell} \in \mathbb{R}^M$ such that, given $\varepsilon > 0$ there exists a real number $M = M(\varepsilon) > 0$ such that for all $x \in E$ with $x > M$ we have*

$$|\mathbf{f}(x) - \boldsymbol{\ell}| < \varepsilon.$$

In this case we say that $\boldsymbol{\ell}$ is the limit of $\mathbf{f}$ as x tends to ∞ and we write

$$\lim_{x\to\infty} \mathbf{f}(x) = \boldsymbol{\ell} \quad \text{or} \quad \mathbf{f}(x) \to \boldsymbol{\ell} \text{ as } x \to \infty.$$

With some obvious modifications we can define the limits

$$\lim_{x\to-\infty}\mathbf{f}(x)=\boldsymbol{\ell}.$$

Theorems 166, 171, 173, 175 continue to hold if we replace $x_0\in\mathbb{R}^N$ with ∞ or $-\infty$. We omit the details.

Exercise 186 *Calculate the following limits*

1. $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$,
2. $\displaystyle\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$,
3. $\displaystyle\lim_{x\to0^+}(1+x)^{\frac{1}{x}}$.

Exercise 187 *Using the previous exercise, prove that*

$$\lim_{x\to0}\frac{\log(1+x)}{x}=1$$

and that

$$\lim_{x\to0}\frac{e^x-1}{x}=1.$$

16 Continuity

Definition 188 *Let $E\subseteq\mathbb{R}^N$. A point $\mathbf{x}_0\in E$ is called an* isolated point *of E if there exists $\delta>0$ such that*

$$B(\mathbf{x}_0,\delta)\cap E=\{\mathbf{x}_0\}.$$

It is clear that if a point of the set E is not an isolated point of E then it is an accumulation point of E.

Definition 189 *Let $E\subseteq\mathbb{R}^N$ and let $\mathbf{x}_0\in E$. Given a function $\mathbf{f}:E\to\mathbb{R}^M$ we say that $\mathbf{f}$ is* continuous *at $\mathbf{x}_0$ if for every $\varepsilon>0$ there exists a real number $\delta=\delta(\varepsilon,\mathbf{x}_0)>0$ such that for all $\mathbf{x}\in E$ with $\|\mathbf{x}-\mathbf{x}_0\|<\delta$ we have*

$$\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)\|<\varepsilon.$$

If $\mathbf{f}$ is continuous at every point of E we say that $\mathbf{f}$ is continuous *on E and we write $\mathbf{f}\in C(E)$ or $\mathbf{f}\in C^0(E)$.*

Remark 190 *If (X,d_X) and (Y,d_Y) are two metric spaces, $E\subseteq X$, $x_0\in E$, and $f:E\to Y$, we say that f is* continuous *at x_0 if for every $\varepsilon>0$ there exists a real number $\delta=\delta(\varepsilon,x_0)>0$ with the property that*

$$d_Y(f(x),f(x_0))<\varepsilon$$

for all $x \in E$ *with* $d_X(x, x_0) < \delta$.

If (X, τ_X) *and* (Y, τ_Y) *are two topological spaces,* $E \subseteq X$, $x_0 \in E$, *and* $f : E \to Y$, *we say that* f *is* continuous *at* x_0 *if for every neighborhood* V *of* $f(x_0)$ *there exists a neighborhood* U *of* x_0 *with the property that*

$$f(x) \in V$$

for all $x \in E$ *with* $x \in U$.

Theorem 191 *Let* $E \subseteq \mathbb{R}^N$ *and let* $\mathbf{x}_0 \in E$. *Given a function* $\mathbf{f} : E \to \mathbb{R}^M$,

(i) if $\mathbf{x}_0$ *is an isolated point of* E *then* $\mathbf{f}$ *is continuous at* $\mathbf{x}_0$;

(ii) if $\mathbf{x}_0$ *is an accumulation point of* E *then* $\mathbf{f}$ *is continuous at* $\mathbf{x}_0$ *if and only if there exists* $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$.

Proof of part (i). If $\mathbf{x}_0$ is an isolated point of E then there exists $\delta_0 > 0$ such that

$$B(\mathbf{x}_0, \delta_0) \cap E = \{\mathbf{x}_0\}.$$

Fix $\varepsilon > 0$ and take $\delta := \delta_0$ in the definition of continuity. Clearly if $\mathbf{x} \in E$ and $\|\mathbf{x} - \mathbf{x}_0\| < \delta$ then necessarily $\mathbf{x} = \mathbf{x}_0$ so that we have $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| = 0$. ■

Exercise 192 *Prove that the functions* $\sin x$, $\cos x$, x^n, *where* $n \in \mathbb{N}$, *are continuous.*

The following theorems follows from the analogous results for limits.

Theorem 193 *Let* $E \subseteq \mathbb{R}^N$ *and let* $\mathbf{x}_0 \in E$. *Given two functions* $f, g : E \to \mathbb{R}$ *assume that* f *and* g *are continuous at* $\mathbf{x}_0$. *Then*

(i) $f + g$ *and* fg *are continuous at* $\mathbf{x}_0$;

(ii) if $g(\mathbf{x}_0) \neq 0$ *then* $\dfrac{f}{g}$ *restricted to the set* $F := \{\mathbf{x} \in E : g(\mathbf{x}) \neq 0\}$ *is continuous at* $\mathbf{x}_0$.

Example 194 *In view of Exercise 192 and the previous theorem, the functions* $\tan x = \frac{\sin x}{\cos x}$ *and* $\cot x = \frac{\cos x}{\sin x}$ *are continuous in their domain of definition.*

Theorem 195 *Let* E, $F \subseteq \mathbb{R}^N$ *and let* $\mathbf{x}_0 \in \mathbb{R}^N$ *be an accumulation point of* E. *Given two functions* $f : E \to F$ *and* $g : F \to \mathbb{R}$ *assume that* f *is continuous at* $\mathbf{x}_0$ *and that* g *is continuous at* $f(\mathbf{x}_0)$. *Then* $g \circ f : E \to \mathbb{R}$ *is continuous at* $\mathbf{x}_0$.

17 Discontinuities

Let $E \subseteq \mathbb{R}^N$ and let $\mathbf{f} : E \to \mathbb{R}^M$. Given $\mathbf{x}_0 \in E$, what happens when $\mathbf{f}$ is discontinuous at $\mathbf{x}_0$? Then $\mathbf{x}_0$ is an accumulation point of E. The following situations can arise. It can happen that there exists

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \boldsymbol{\ell} \in \mathbb{R}^M$$

but $\boldsymbol{\ell} \neq \mathbf{x}_0$. In this case, we say that $\mathbf{x}_0$ is a *removable discontinuity*. Indeed, consider the function $g : E \to \mathbb{R}^M$ defined by

$$g(\mathbf{x}) := \begin{cases} \mathbf{f}(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{x}_0, \\ \boldsymbol{\ell} & \text{if } \mathbf{x} = \mathbf{x}_0. \end{cases}$$

Then there exists

$$\lim_{\mathbf{x} \to \mathbf{x}_0} g(\mathbf{x}) = \boldsymbol{\ell} = g(\mathbf{x}_0),$$

and so the new function g is continuous at $\mathbf{x}_0$.

Another type of discontinuity is when x_0 is an accumulation point of $E^- := E \cap (-\infty, x_0]$ and of $E^+ := E \cap (x_0, \infty)$ and there exist

$$\lim_{x \to x_0^-} \mathbf{f}(x) = \boldsymbol{\ell} \in \mathbb{R}^M, \quad \lim_{x \to x_0^+} \mathbf{f}(x) = L \in \mathbb{R}^M$$

but $\boldsymbol{\ell} \neq L$. In this case the point x_0 is called a *jump discontinuity* of $\mathbf{f}$.

Example 196 *The integer and fractional part of x have jump discontinuity at every integer.*

An important class of functions that exhibit only jump discontinuities are monotone functions.

Definition 197 *A set $E \subseteq \mathbb{R}^N$ is* countable *if there exists a one-to-one function $f : E \to \mathbb{N}$.*

Remark 198 *It can be shown that $\mathbb{Q}$ is countable and that if $E_n \subseteq \mathbb{R}$, $n \in \mathbb{N}$, is countable, then*

$$E = \bigcup_{n=1}^{\infty} E_n$$

is countable. It can also be shown that $\mathbb{R}$ and the irrationals are NOT countable.

Definition 199 *A set $I \subseteq \mathbb{R}$ is an* interval *if for every $x, y \in I$, with $x < y$, we have that the interval $[x, y]$ is contained in I.*

Definition 200 *Given a set X and a function $f : X \to [0, \infty]$ the* infinite sum

$$\sum_{x \in X} f(x)$$

is defined as

$$\sum_{x\in X} f(x) := \sup\left\{\sum_{x\in Y} f(x) : Y \subset X,\, Y \text{ finite}\right\}.$$

Proposition 201 *Given a set X and a function $f : X \to [0,\infty]$, if*

$$\sum_{x\in X} f(x) < \infty,$$

then the set $\{x \in X : f(x) > 0\}$ is countable. Moreover, f does not take the value ∞.

Proof. Define

$$M := \sum_{x\in X} f(x) < \infty.$$

For $k \in \mathbb{N}$ set $X_k := \left\{x \in X : f(x) > \frac{1}{k}\right\}$ and let Y be a finite subset of X_k. Then

$$\frac{1}{k}\text{number of elements of } Y \le \sum_{x\in Y} f(x) \le M,$$

which shows that Y cannot have more than $\lfloor kM \rfloor$ elements, where $\lfloor\cdot\rfloor$ is the integer part. In turn, X_k has a finite number of elements, and so

$$\{x \in X : f(x) > 0\} = \bigcup_{k=1}^{\infty} X_k$$

is countable. ■

Exercise 202 *Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Prove that the set of discontinuity points of f is countable.*

Exercise 203 *Given a nonempty set X and two functions $f, g : X \to [0,\infty]$.*

(i) Prove that

$$\sum_{x\in X} (f(x) + g(x)) \le \sum_{x\in X} f(x) + \sum_{x\in X} g(x).$$

(ii) If $f \le g$, then

$$\sum_{x\in X} f(x) \le \sum_{x\in X} g(x).$$

Theorem 204 *Let $E \subseteq \mathbb{R}$, let $x_0 \in \mathbb{R}$ be an accumulation point of E, and let $f : E \to \mathbb{R}$ be a monotone function. If x_0 is an accumulation point of $E^- := E \cap (-\infty, x_0]$, then there exists*

$$\lim_{x\to x_0^-} f(x) = \ell \in [-\infty,\infty],$$

while if x_0 is an accumulation point of $E^+ := E \cap (x_0, \infty)$, then there exists

$$\lim_{x \to x_0^+} f(x) = L \in [-\infty, \infty].$$

Moreover, if f is increasing and x_0 is an accumulation point of E^- and E^+, then $\ell \le L$, while if f is decreasing and x_0 is an accumulation point of E^- and E^+, then $\ell \ge L$.

Proof. Assume that f is increasing and let

$$\ell := \sup \{f(x) : x \in E, x < x_0\}.$$

Assume that $\ell \in \mathbb{R}$ (the case $\ell = \infty$ is left as an exercise). Fix $\varepsilon > 0$. We need to find $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$\ell - \varepsilon \le f(x) \le \ell + \varepsilon$$

for all $x \in E$ with $x_0 - \delta \le x < x_0$. By the definition of supremum, we have that $f(x) \le \ell$ for all $x \in E$ with $x < x_0$. On the other hand, since $\ell - \varepsilon$ is not an upper bound, there exists $x_1 \in E$, with $x_1 < x_0$ such that $\ell - \varepsilon < f(x_1)$. Take $\delta := x_0 - x_1$. If $x \in E$ with $x_0 - \delta = x_1 \le x < x_0$, then, since f is increasing,

$$\ell - \varepsilon < f(x_1) \le f(x),$$

which gives the other inequality. ■

Theorem 205 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a monotone function. Then f has at most countably many discontinuity points. Conversely, given a countable set $E \subset \mathbb{R}$, there exists a monotone function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuity points is exactly E.*

Proof. Step 1: Assume that $I = [a, b]$ and, without loss of generality, that f is increasing. For every $x \in (a, b)$ there exist

$$\lim_{y \to x^+} f(y) =: f_+(x), \quad \lim_{y \to x^-} f(y) =: f_-(x).$$

Let $S(x) := f_+(x) - f_-(x) \ge 0$ be the jump of f at x. Then f is continuous at x if and only if $S(x) = 0$. Let $J \subseteq [a, b]$ be any finite subset, and write

$$J = \{x_1, \ldots, x_k\}, \quad \text{where } x_1 < \cdots < x_k.$$

Since f is increasing, we have that

$$\begin{aligned} f(a) &\le f_-(x_1) \le f_+(x_1) \le f_-(x_2) \le f_+(x_2) \\ &\le \cdots \le f_-(x_k) \le f_+(x_k) \le f(b), \end{aligned}$$

and so,

$$\sum_{x \in J} S(x) = \sum_{i=1}^{k} (f_+(x_i) - f_-(x_i)) \le f(b) - f(a),$$

which implies that

$$\sum_{x\in(a,b)} S(x) \leq f(b) - f(a).$$

By the previous proposition, it follows that the set of discontinuity points of f is at most countable.

Step 2: If I is an arbitrary interval, construct an increasing sequence of intervals $[a_n, b_n]$ such that

$$a_n \searrow \inf I, \quad b_n \nearrow \sup I.$$

Since the union of countable sets is countable and on each interval $[a_n, b_n]$ the set of discontinuity points of f is at most countable, by the previous step it follows that the set of discontinuity points of f in I is at most countable. ■

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To prove the second part of the theorem, we need the following two results.

Lemma 206 *Let $E \subseteq \mathbb{R}^N$ and $\mathbf{f}_n : E \to \mathbb{R}^M$ be such that*

$$\sum_{n=1}^{\infty} \|\mathbf{f}_n\|_\infty < \infty,$$

Then there exists a bounded function $\mathbf{f}_n : E \to \mathbb{R}^M$ such that the sequence of functions $\{\sum_{n=1}^m \mathbf{f}_n\}$ converges uniformly to $\mathbf{f}$ as $m \to \infty$.

$$\lim_{\ell\to\infty} \sup_{\mathbf{x}\in E} \|\mathbf{f}_n(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})\|$$

Proof. Set

$$L_n := \|\mathbf{f}_n\|_\infty = \sup_{\mathbf{x}\in E} \|\mathbf{f}_n(\mathbf{x})\|.$$

We claim that $\{\sum_{n=1}^m \mathbf{f}_n\}$ in the space $\ell^\infty(E; \mathbb{R}^M)$ of bounded functions with the norm $\|\cdot\|_\infty$. To see this, let $m > k$ and consider

$$\left\|\sum_{n=1}^m \mathbf{f}_n - \sum_{n=1}^k \mathbf{f}_n\right\|_\infty = \left\|\sum_{n=k+1}^m \mathbf{f}_n\right\|_\infty \leq \sum_{n=k+1}^m \|\mathbf{f}_n\|_\infty = \sum_{n=k+1}^m L_n. \tag{23}$$

Since the series $\sum_{n=1}^\infty L_n$ is convergent, there exists

$$\lim_{k\to\infty} \sum_{n=1}^k L_n = \ell \in \mathbb{R}.$$

But then the sequence $\{\sum_{n=1}^m L_n\}$ is a Cauchy sequence. Hence, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\left|\sum_{n=1}^m L_n - \sum_{n=1}^k L_n\right| \leq \varepsilon$$

for all $m, k > n_\varepsilon$. But if $m > k > n_\varepsilon$,

$$\sum_{n=k+1}^{m} L_n = \sum_{n=1}^{m} L_n - \sum_{n=1}^{k} L_n \leq \varepsilon,$$

and so by (23),

$$\left\| \sum_{n=1}^{m} \mathbf{f}_n - \sum_{n=1}^{k} \mathbf{f}_n \right\|_\infty \leq \varepsilon$$

for all $m > k > n_\varepsilon$. This proves that $\{\sum_{n=1}^{m} \mathbf{f}_n\}$ is a Cauchy sequence in $\ell^\infty(X; \mathbb{R}^M)$. The result now follows from the fact that the space $\ell^\infty(E; \mathbb{R}^M)$ is complete. ■

Lemma 207 *Let $E \subseteq \mathbb{R}^N$ and $\mathbf{g}_n : E \to \mathbb{R}^M$ be such that the sequence of functions $\{\mathbf{g}_n\}$ converges uniformly to some function $\mathbf{g} : E \to \mathbb{R}^M$ as $n \to \infty$. If all the functions $\mathbf{g}_n$ are continuous at some point $x_0 \in E$, then $\mathbf{g}$ is continuous at x_0.*

Proof. You essentially proved this in your homework. ■

We are now ready to continue the proof of Theorem 204.

Proof of Theorem 204, continued. Step 3: Conversely, let $E \subset I$ be a countable set. If E is finite, then an increasing function with discontinuity set E may be constructed by hand. Consider now the more interesting case in which E is denumerable, so that $E = \{x_n\}_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$ define the increasing function $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) := \begin{cases} 0 & \text{if } x < x_n, \\ \frac{1}{2^n} & \text{if } x \geq x_n. \end{cases}$$

Note that f_n is discontinuous only at the point x_n. Set

$$f(x) := \sum_{n=1}^{\infty} f_n(x) = \lim_{m \to \infty} \sum_{n=1}^{m} f_n(x), \quad x \in \mathbb{R}.$$

Since $|f_n(x)| \leq \frac{1}{2^n}$ for all $x \in \mathbb{R}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, it follows from Lemma 206 that the sequence $\{\sum_{n=1}^{m} f_n\}$ of functions converges uniformly convergent to f, and so by Lemma 207 (where $g_m = \sum_{n=1}^{m} f_n$), f is continuous at every point at which all the functions f_n are continuous. In particular, f is continuous in $\mathbb{R} \setminus E$.

We now prove that f is discontinuous at every point of E. Indeed, for every $k \in \mathbb{N}$ write

$$f = f_k + \sum_{n \neq k} f_n.$$

Then reasoning as before, $\sum\limits_{n \neq k} f_n$ is continuous at x_k while f_k is not. Hence, f is discontinuous at each $x \in E$. To conclude, observe that f is increasing, since it is the pointwise limit of a sequence of increasing functions. ■

By taking $E := \mathbb{Q}$, we obtain the following result.

Corollary 208 *There exists an increasing function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at all irrational points and discontinuous at all rational points.*

Finally, the last type of discontinuity is when at least one of the limits $\lim_{x \to x_0^-} \mathbf{f}(x)$ and $\lim_{x \to x_0^+} \mathbf{f}(x)$ is not finite or does not exist. In this case, the point x_0 is called an *essential discontinuity* of $\mathbf{f}$.

Example 209 *The function*

$$f(x) := \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

and

$$g(x) := \begin{cases} \log x & \text{if } x > 0, \\ 1 & \text{if } x = 0, \end{cases}$$

have an essential discontinuity at $x = 0$.

18 Important Theorems on Continuity

In this section we study some important consequences of continuity. The next theorem shows that continuity preserves the sign of a function.

Theorem 210 *Let $E \subseteq \mathbb{R}^N$,and let $f : E \to \mathbb{R}$ be continuous at some $\mathbf{x}_0 \in E$ with $f(\mathbf{x}_0) > 0$. Then there exists $\delta = \delta(\mathbf{x}_0) > 0$ such that for all $\mathbf{x} \in E$ with $|\mathbf{x} - \mathbf{x}_0| < \delta$ we have*

$$f(\mathbf{x}) > 0.$$

Proof. Assume that $\mathbf{x}_0$ is an accumulation point of E. Then $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$ and so taking $\varepsilon := \frac{1}{2} f(\mathbf{x}_0)$, there exists $\delta = \delta(\mathbf{x}_0) > 0$ such that for all $\mathbf{x} \in E$ with $|\mathbf{x} - \mathbf{x}_0| < \delta$ we have

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| < \frac{1}{2} f(\mathbf{x}_0),$$

that is,

$$0 < \frac{1}{2} f(\mathbf{x}_0) < f(\mathbf{x}) < \frac{3}{2} f(\mathbf{x}_0).$$

■

Remark 211 *A similar result continues to hold if $f(\mathbf{x}_0) < 0$.*

Example 212 *The previous theorem implies in particular that sets of the form*

$$\{x \in \mathbb{R} : 4 \sin x - \log(1 + |x|) > 0\}$$

are open. We used this in the exercises.

Definition 213 *Given a set $E \subseteq \mathbb{R}^N$, a set $F \subseteq E$ is said to be* relatively open *in E if there exists an open set $U \subseteq \mathbb{R}^N$ such that $F = U \cap E$. A set $G \subseteq E$ is said to be* relatively closed *in E if there exists a closed set $C \subseteq \mathbb{R}^N$ such that $G = C \cap E$.*

Theorem 214 *Let $E \subseteq \mathbb{R}^N$ and let $\mathbf{f} : E \to \mathbb{R}^M$.*

(i) Then $\mathbf{f}$ is continuous if and only if $\mathbf{f}^{-1}(U)$ is relatively open for every open set $U \subseteq \mathbb{R}^M$.

(ii) Then $\mathbf{f}$ is continuous if and only if $\mathbf{f}^{-1}(C)$ is relatively closed for every closed set $C \subseteq \mathbb{R}^M$.

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Proof. (i) Let $U \subseteq \mathbb{R}^M$ be open. Assume that $\mathbf{f}$ is continuous. If $\mathbf{f}^{-1}(U)$ is empty, then there is nothing to prove. Otherwise, let $\mathbf{x}_0 \in \mathbf{f}^{-1}(U)$. Since U is open and $\mathbf{f}(\mathbf{x}_0) \in U$, there exists $\varepsilon > 0$ such that $B(\mathbf{f}(\mathbf{x}_0), \varepsilon) \subseteq U$. Since $\mathbf{f}$ is continuous at $\mathbf{x}_0$ there exists $\delta_{\mathbf{x}_0} = \delta_{\mathbf{x}_0}(\varepsilon) > 0$ such that for all $\mathbf{x} \in E$ with $|\mathbf{x} - \mathbf{x}_0| < \delta_{\mathbf{x}_0}$, we have

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| < r.$$

Hence, for all $\mathbf{x} \in E$ with $|\mathbf{x} - \mathbf{x}_0| < \delta_{\mathbf{x}_0}$,

$$\mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{x}_0), r) \subseteq U,$$

and so $B(\mathbf{x}_0, \delta_{\mathbf{x}_0}) \cap E \subseteq \mathbf{f}^{-1}(U)$.

Take

$$V := \bigcup_{\mathbf{x} \in \mathbf{f}^{-1}(U)} B(\mathbf{x}, \delta_{\mathbf{x}}).$$

Then V is open and $\mathbf{f}^{-1}(U) \subseteq V$. Hence,

$$V \cap E = \mathbf{f}^{-1}(U),$$

which shows that $\mathbf{f}^{-1}(U)$ is relatively open.

The converse implication is left as an exercise.

(ii) Let $C \subseteq \mathbb{R}^M$ be closed. Then $U := \mathbb{R}^M \setminus C$ is open. Since $\mathbf{f}$ is continuous, by part (i), the set $\mathbf{f}^{-1}(U)$ is relatively open, and so there exists an open set $V \subseteq \mathbb{R}^N$ such that

$$V \cap E = \mathbf{f}^{-1}(U) = \mathbf{f}^{-1}\left(\mathbb{R}^M \setminus C\right) = E \setminus \mathbf{f}^{-1}(C).$$

Hence, if we consider the closed set $K := \mathbb{R}^N \setminus V$, we have that

$$K \cap E = \mathbf{f}^{-1}(C),$$

which shows that $\mathbf{f}^{-1}(C)$ is relatively closed. The converse implication is left as an exercise. ■

As a corollary, we get.

Corollary 215 *Let $D \subseteq \mathbb{R}^N$ and let $\mathbf{f} : D \to \mathbb{R}^M$.*

(i) If D is open, then $\mathbf{f}$ is continuous if and only if $\mathbf{f}^{-1}(U)$ is relatively open for every open set $U \subseteq \mathbb{R}^M$.

(ii) If D is closed, then $\mathbf{f}$ is continuous if and only if $\mathbf{f}^{-1}(C)$ is relatively closed for every closed set $C \subseteq \mathbb{R}^M$.

Remark 216 *The previous characterization of continuous functions is useful to define continuity in a topological space.*

Another important theorem on continuity is the following.

Theorem 217 (Zeros of a continuous function) *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a continuous function. If there exist $x_1, x_2 \in I$ such that $f(x_1) < 0$ and $f(x_2) > 0$, then there exists $x_0 \in I$ in the interval of endpoints x_1, x_2 such that $f(x_0) = 0$.*

Proof. Without loss of generality, we may assume that $x_1 < x_2$, the case $x_1 > x_2$ is similar. Then $[x_1, x_2] \subseteq I$, and so we may define the set

$$E := \{x \in [x_1, x_2] : \ f(x) < 0\}.$$

Let $x_0 := \sup E$. Then $x_0 \in [x_1, x_2]$ and $f(x) \geq 0$ for all $x \in (x_0, x_2]$.

We claim that $x_1 < x_0 < x_2$. Indeed, since $f(x_1) < 0$, by Theorem 210, we can find $\delta_1 > 0$ such that $f(x) < 0$ for all $x \in I \cap (x_1 - \delta_1, x_1 + \delta_1)$. Note that this imply that $\delta_1 \leq x_2 - x_1$. It follows that $x_0 \geq x_1 + \delta > x_1$. Similarly, since $f(x_2) > 0$, by Theorem 210, we can find $\delta_2 > 0$ such that $f(x) > 0$ for all $x \in I \cap (x_2 - \delta_2, x_2 + \delta_2)$, so that $\delta_2 \leq x_2 - x_1$ and $x_0 \leq x_2 - \delta_2 < x_2$.

Next, we claim that $f(x_0) = 0$. Indeed, if $f(x_0) < 0$, then by Theorem 210, we can find $\delta > 0$ such that $f(x) < 0$ for all $x \in I \cap (x_0 - \delta_0, x_0 + \delta_0)$. Since $x_0 < x_2$, taking $\delta_0 < x_2 - x_0$, this implies that $f(x) < 0$ for all $x \in [x_0, x_0 + \delta_0)$, which contradicts the fact that x_0 is the supremum of E.

On the other hand, if $f(x_0) > 0$, then again by Theorem 210, we can find $\delta > 0$ such that $f(x) > 0$ for all $x \in I \cap (x_0 - \delta_0, x_0 + \delta_0)$. Since $x_0 < x_2$, taking $\delta_0 < x_0 - x_1$, this implies that $f(x) > 0$ for all $x \in (x_0 - \delta_0, x_0]$. Together with the fact that $f(x) \geq 0$ for all $x \in (x_0, x_2]$, it follows that $E \subseteq [x_1, x_0 - \delta_0]$, and so its supremum cannot be x_0.This shows that $f(x_0) = 0$. ■

Remark 218 *By applying the previous theorem to the function $g(x) = f(x) - t$, we can show that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ a continuous function such that $f(x_1) < t$ and $f(x_2) > t$ for some $x_1, x_2 \in I$, then there exists $x_0 \in I$ in the interval of endpoints x_1, x_2 such that $f(x_0) = t$.*

Corollary 219 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a continuous function. Then f takes all the values between $\inf_I f$ and $\sup_I f$. Moreover, $f(I)$ is an interval (possibly degenerate).*

Proof. If $\inf_I f = \sup_I f$, then f is constant. In this case $f(I)$ is a singleton and there is nothing to prove.

Thus, in what follows we assume that $\inf_I f < \sup_I f$ and let $\inf_I f < t < \sup_I f$. By the definition of infimum and of supremum, there exist $x_1, x_2 \in I$ such that $f(x_1) < t$ and $f(x_2) > t$. By the previous remark, there exists $x_0 \in I$ in the interval of endpoints x_1, x_2 such that $f(x_0) = t$. This shows that $f(I) \supseteq (\inf_I f, \sup_I f)$. It remains to show that $f(I)$ is an interval. Let $y_1, y_2 \in f(I)$ with $y_1 < y_2$ and let $y_1 < y < y_2$. Since $\inf_I f \le y_1 < y_2 \le \sup_I f$, by what we just proved, there exists $x \in I$ such that $f(y) = x$. ■

Corollary 220 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a monotone function. Then f is continuous if and only if $f(I)$ is an interval.*

Proof. If f is continuous, then the result follows from the previous corollary. Assume that $f(I)$ is an interval. We claim that f is continuous. Let $x_0 \in I$ and assume that x_0 is an interior point (the case in which x_0 is an endpoint of I is similar). Assume that f is not continuous at x_0. Then by Theorem 204 there exist

$$\lim_{x \to x_0^-} f(x) = \ell \in [-\infty, \infty] < \lim_{x \to x_0^+} f(x) = L \in [-\infty, \infty].$$

Since f is increasing, we have that

$$\ell = \sup_{x < x_0} f(x), \quad L = \inf_{x > x_0} f(x),$$

and so $f(x) \le \ell$ for all $x < x_0$, while $f(x) \ge L$ for all x_0. This implies that $f(I)$ does not contain the set $(\ell, L) \setminus \{f(x_0)\}$, which contradicts the fact that $f(I)$ is an interval because f takes values less than or equal to ℓ and values greater than or equal to L. ■

Monday, October 31, 2011

The following theorem is important.

Theorem 221 (Weierstrass) *Let $K \subset \mathbb{R}^N$ be compact and let $f : K \to \mathbb{R}$ be a continuous function. Then there exists $\mathbf{x}_0, \mathbf{x}_1 \in K$ such that*

$$f(\mathbf{x}_0) = \min_{\mathbf{x} \in K} f(\mathbf{x}), \quad f(\mathbf{x}_1) = \max_{\mathbf{x} \in K} f(\mathbf{x}).$$

First proof. Let

$$t := \inf_{\mathbf{x} \in K} f(\mathbf{x}).$$

If the infimum is not attained, then for every $\mathbf{x} \in K$ we may find $t < t_{\mathbf{x}} < f(\mathbf{x})$. By Theorem 214 the set $\{y \in K : f(y) > t_{\mathbf{x}}\}$ is relatively open and so there exists an open set $U_{\mathbf{x}}$ such that

$$U_{\mathbf{x}} \cap K = \{y \in K : f(y) > t_{\mathbf{x}}\}, \quad \mathbf{x} \in K.$$

Note that $\mathbf{x} \in U_{\mathbf{x}}$. Hence, the family of open sets $\{U_{\mathbf{x}}\}_{\mathbf{x}\in K}$ is an open cover for the compact set K, and so we may find a finite cover $U_{\mathbf{x}_1}, \dots, U_{\mathbf{x}_l}$ of the set K. But then for all $\mathbf{x} \in K$,

$$f(\mathbf{x}) \geq \min_{i=1,\dots,l} t_{\mathbf{x}_i} > t = \inf_{w\in K} f(w),$$

which contradicts the definition of t. ∎

Remark 222 *Note that to prove the existence of a minimum, we only used a weaker form of continuity, namely that the set* $\{y \in K : f(y) > t\}$ *is relatively open for every* $t \in \mathbb{R}$. *A function satisfying this property is called* lower semicontinuous. *Note that the proof above also works for metric and topological spaces.*

First proof. Let

$$t := \inf_{\mathbf{x}\in K} f(\mathbf{x}).$$

If the infimum is not attained, then for every $\mathbf{x} \in K$ we may find $t < t_{\mathbf{x}} < f(\mathbf{x})$. By Theorem 214 the set $\{y \in K : f(y) > t_{\mathbf{x}}\}$ is relatively open and so there exists an open set $U_{\mathbf{x}}$ such that

$$U_{\mathbf{x}} \cap K = \{y \in K : f(y) > t_{\mathbf{x}}\}, \quad \mathbf{x} \in K.$$

Note that $\mathbf{x} \in U_{\mathbf{x}}$. Hence, the family of open sets $\{U_{\mathbf{x}}\}_{\mathbf{x}\in K}$ is an open cover for the compact set K, and so we may find a finite cover $U_{\mathbf{x}_1}, \dots, U_{\mathbf{x}_l}$ of the set K. But then for all $\mathbf{x} \in K$,

$$f(\mathbf{x}) \geq \min_{i=1,\dots,l} t_{\mathbf{x}_i} > t = \inf_{w\in K} f(w),$$

which contradicts the definition of t. ∎

Remark 223 *Note that to prove the existence of a minimum, we only used a weaker form of continuity, namely that the set* $\{y \in K : f(y) > t\}$ *is relatively open for every* $t \in \mathbb{R}$. *A function satisfying this property is called* lower semicontinuous. *Note that the proof above also works for metric and topological spaces.*

Second proof. Here we use the fact that K is sequentially compact. Let

$$t := \inf_{\mathbf{x}\in K} f(\mathbf{x}).$$

Note that $t \in [-\infty, \infty)$. Construct a sequence of real numbers $t < t_n$ such that $t_n \to t$ as $n \to \infty$ (if $t \in \mathbb{R}$ we can take $t_n = t + \frac{1}{n}$, while if $t = -\infty$, take $t_n = -n$). By the definition of infimum, for every $n \in \mathbb{N}$ we may find $\mathbf{x}_n \in K$ such that

$$t < f(\mathbf{x}_n) < t_n.$$

Letting $n \to \infty$, by the squeeze theorem, we get

$$\lim_{n\to\infty} f(\mathbf{x}_n) = t. \tag{24}$$

Since $\{\mathbf{x}_n\} \subset K$, and K is sequentially compact (see Theorem 159), there exist a subsequence $\{\mathbf{x}_{n_k}\}$ of $\{\mathbf{x}_n\}$ and $\mathbf{x} \in K$ such that $\mathbf{x}_{n_k} \to \mathbf{x}$ as $k \to \infty$. Using the continuity of f and (24), we get

$$t = \lim_{n\to\infty} f(\mathbf{x}_n) = \lim_{k\to\infty} f(\mathbf{x}_{n_k}) = f(\mathbf{x}),$$

which shows that the infimum is a minimum. ■

The sequence $\{\mathbf{x}_n\}$ constructed in the previous proof is called a *minimizing sequence.*

Remark 224 *Note that to prove the existence of a minimum, we only used a weaker form of continuity, namely that the set*

$$\liminf_{j\to\infty} f(\mathbf{x}_j) \geq f(\mathbf{x})$$

for all sequences $\{\mathbf{x}_j\}$ *converging to* $x \in K$. *A function satisfying this property is called* sequentially lower semicontinuous*.*

Remark 225 *A typical application of the Weierstrass theorem is the following. Let* $f : \mathbb{R}^N \to \mathbb{R}$ *be a continuous function. Assume that* f *is bounded from below, so that*

$$\ell = \inf_{\mathbf{x}\in\mathbb{R}^N} f(\mathbf{x}) > -\infty$$

and that

$$\lim_{|\mathbf{x}|\to\infty} f(\mathbf{x}) = \infty.$$

By the definition of limit, we can find $R > 0$ *such that* $f(\mathbf{x}) > \ell$ *for all* $\mathbf{x} \in \mathbb{R}^N$ *such that* $|\mathbf{x}| \geq R$. *Thus,*

$$\ell = \inf_{\mathbf{x}\in\mathbb{R}^N} f(\mathbf{x}) = \inf_{\mathbf{x}\in[-R,R]} f(\mathbf{x}).$$

By the Weierstrass theorem, f *has a minimum in* $[-R, R]$ *and we are done.*

Next we show that continuous functions preserve compactness.

Proposition 226 *Consider a continuous function* $\mathbf{f} : E \to \mathbb{R}^M$, *where* $E \subseteq \mathbb{R}^N$. *Then* $\mathbf{f}(K)$ *is compact for every compact set* $K \subseteq E$.

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $\mathbf{f}(K)$. By continuity, $\mathbf{f}^{-1}(U_\alpha)$ is relatively open for every $\alpha \in \Lambda$, and so there exists W_α open such that $\mathbf{f}^{-1}(U_\alpha) = E \cap W_\alpha$. The family $\{W_\alpha\}_{\alpha\in\Lambda}$ is an open cover of K. Since K is compact, we may find $U_{\alpha_1}, \ldots, U_{\alpha_l}$ such that $\{W_{\alpha_i}\}_{i=1}^{l}$ cover K. In turn, $U_{\alpha_1}, \ldots, U_{\alpha_l}$ cover $\mathbf{f}(K)$. Indeed, if $\mathbf{y} \in \mathbf{f}(K)$, then there exists $\mathbf{x} \in K$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Let $i = 1, \ldots, m$ be such that $\mathbf{x} \in \mathbf{f}^{-1}(U_{\alpha_i}) = E \cap W_{\alpha_i}$. Then $\mathbf{y} = \mathbf{f}(\mathbf{x}) \in U_{\alpha_i}$. ■

We now discuss the continuity of inverse functions and of composite functions. If a continuous function $\mathbf{f}$ is invertible its inverse function $\mathbf{f}^{-1}$ may not be continuous.

Example 227 *Let*

$$f(x) := \begin{cases} x & \text{if } 0 \le x \le 1, \\ x - 1 & \text{if } 2 < x \le 3. \end{cases}$$

Then $f^{-1} : [0,2] \to R$ *is given by*

$$f^{-1}(x) := \begin{cases} x & \text{if } 0 \le x \le 1, \\ x + 1 & \text{if } 1 < x \le 2, \end{cases}$$

which is not continuous at $x = 1$*.*

Wednesday, November 02, 2011

We will see that this cannot happen if E is an interval or a compact set.

Theorem 228 *Let* $K \subset \mathbb{R}^N$ *be a compact set and let* $\mathbf{f} : K \to \mathbb{R}^M$ *be one-to-one and continuous. Then the inverse function* $\mathbf{f}^{-1} : f(K) \to \mathbb{R}^N$ *is continuous.*

Lemma 229 *Let* $K \subset \mathbb{R}^N$ *be a compact set and let* $C \subseteq \mathbb{R}^N$ *be a closed set. Then* $C \cap K$ *is compact.*

First proof. This proof works only in $\mathbb{R}^N$. Since K is a compact set of $\mathbb{R}^N$, we have that K is closed and bounded. But since C is a subset of K, it follows that C is also bounded. Hence, C is closed and bounded, and since we are in $\mathbb{R}^N$, it follows that C is compact (this last statement fails in general for metric spaces and topological spaces). ■

Second proof. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $C \cap K$. Since C is closed, the set $U := \mathbb{R}^N \setminus C$ is open. Note that

$$K = (K \setminus C) \cup (C \cap K) \subseteq U \cup \bigcup_\alpha U_\alpha.$$

Since K is compact, there exist $U_{\alpha_1}, \ldots, U_{\alpha_l}$ such that

$$K \subseteq U \cup \bigcup_{i=1}^{l} U_{\alpha_i}.$$

But since $U = \mathbb{R}^N \setminus C$, it follows that

$$C \cap K \subseteq \bigcup_{i=1}^{l} U_{\alpha_i},$$

which shows that $C \cap K$ is compact. ∎

Proof of Theorem 228. Let $C \subseteq \mathbb{R}^N$ be a closed set. By the previous lemma $K \cap C$ is compact. By Proposition 226 we have that $\mathbf{f}(K \cap C)$ is compact. In particular, $\mathbf{f}(K \cap C)$ is closed by Theorem 83. Let $\mathbf{g} := \mathbf{f}^{-1}$. Then

$$\mathbf{f}(K \cap C) = \mathbf{g}^{-1}(C),$$

which shows that $\mathbf{g}^{-1}(C)$ is closed for every closed set $C \subseteq \mathbb{R}^N$. Thus, by Theorem 214, $\mathbf{g}$ is continuous. ∎

Remark 230 *Here we used the fact that a compact set is closed, so to extend this to a function $f : K \to Y$, where $K \subseteq X$ and X and Y are topological spaces, we need Y to be a Hausdorff topological space (see Remark 84).*

Theorem 231 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be one-to-one and continuous. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Proof. Step 1: Let's prove that if $x_0 \in I^\circ$ then for every $x \in I$ with $x > x_0$, $f(x) - f(x_0)$ always has the same sign. Fix $x_1 \in I$ with $x_1 > x_0$ and say that $f(x_1) - f(x_0) > 0$ (the case $f(x_1) - f(x_0) < 0$ is similar). We claim that $f(x) - f(x_0) > 0$ for all $x \in I$ with $x > x_0$. If not, then there exists $x_2 \in I$ with $x_2 > x_0$ such that $f(x_2) - f(x_0) < 0$. But then by Remark 218 (with $t = f(x_0)$) we would find $x_3 \in I$ in the interval of endpoints x_1 and x_2 such that $f(x_3) = f(x_0)$, which contradicts the fact that f is one-to-one.

In a similar way we can show that for every $x \in I$ with $x < x_0$, $f(x) - f(x_0)$ always has the same sign.

Step 2: Let's prove that f is monotone. Fix $a, b \in I^\circ$ with $a < b$ and assume that $f(b) > f(a)$ (the case $f(b) < f(a)$ is similar). We claim that f is strictly increasing. Let $x_1, x_2 \in I$ with $x_1 < x_2$ and let $b_1 = \max\{x_2, b\}$. Since $f(b) > f(a)$, by Step 1 (with $x_0 = a$), we have that $f(x) > f(a)$ for all $x \in I$ with $x > a$. In particular, $f(b_1) > f(a)$. Again by Step 1 (with $x_0 = b_1$), we have that $f(b_1) > f(x)$ for all $x \in I$ with $x < b_1$. In particular, $f(b_1) > f(x_1)$. By Step 1, once more, (with $x_0 = x_1$), we have that $f(x) > f(x_1)$ for all $x \in I$ with $x > x_1$. In particular, $f(x_2) > f(x_1)$, which proves that f is strictly increasing.

Step 3: Since f is continuous and I is an interval, by Corollary 219, $f(I)$ is an interval. Moreover, since f is monotone, it follows (exercise) that $f^{-1} : f(I) \to \mathbb{R}$ is also monotone. By Corollary 220 (applied to the function f^{-1}), it follows that f^{-1} is continuous. ∎

Example 232 *In view of the previous theorem and of Exercise 192, the functions* $\arccos x$, $\arcsin x$, $\arctan x$ *are continuous.*

Given $a > 0$, the function $\log_a x$ is continuous for $x > 0$, since it is the inverse of a^x.

Given $n \in \mathbb{N}$, the function $\sqrt[2n+1]{x}$, $x \in \mathbb{R}$, is continuous, since it is the inverse of x^{2n+1}. The function $\sqrt[2n]{x}$, $x \in [0, \infty)$, is continuous, since it is the inverse of x^{2n}.

Given $a > 0$, since e^x and $\log x$ are continuous in $(0, \infty)$, by writing

$$x^a = e^{\log x^a} = e^{a \log x},$$
$$x^x = e^{\log x^x} = e^{x \log x},$$

it follows from Theorems 193 and 195, that x^a and x^x are continuous in $(0, \infty)$.

19 Uniform Continuity

Next we introduce the notion of uniform continuity.

Definition 233 *Consider a function $\mathbf{f} : E \to \mathbb{R}^M$, where $E \subseteq \mathbb{R}^N$. The function $\mathbf{f}$ is said to be* uniformly continuous *if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)\| < \varepsilon$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in E$ with $\|\mathbf{x}_1 - \mathbf{x}_2\| < \delta$.

Remark 234 *To negate uniform continuity it is enough to fins two sequences $\{\mathbf{x}_n\}, \{\mathbf{z}_n\} \subseteq E$ such that*

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{z}_n\| = 0$$

and $\|\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{z}_n)\| \nrightarrow 0$ (so either the limit does not exist or it exists but it is not zero).

Remark 235 *We can give the same definition for metric spaces. Consider two metric spaces (X, d_X) and (Y, d_Y) and a function $f : E \to Y$, where $E \subseteq X$. The function f is said to be* uniformly continuous *if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$d_Y(f(x_1), f(x_2)) < \varepsilon$$

for all $x_1, x_2 \in E$ with $d(x_1, x_2) < \delta$.

Example 236 *The function $f(x) = x$, $x \in \mathbb{R}$, is uniformly continuous, while the function $g(x) = x^2$, $x \in \mathbb{R}$, is not. To see this, take $\varepsilon = \delta$ for the function f. To prove that g is not uniformly continuous, consider the two sequences $x_n = n + \frac{1}{n}$ and $z_n = n$. Then $x_n - z_n = \frac{1}{n} \to 0$, while*

$$f(x_n) - f(z_n) = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n} \to 2 \neq 0,$$

which implies that g is not uniformly continuous, by the previous remark.

Simple examples of uniformly continuous functions are Lipschitz continuous functions.

Definition 237 *Consider a function* $\mathbf{f} : E \to \mathbb{R}^M$*, where* $E \subseteq \mathbb{R}^N$*. The function* $\mathbf{f}$ *is said to be* Lipschitz continuous *if there exists* $L > 0$ *such that*

$$\|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)\| \leq L \|\mathbf{x}_1 - \mathbf{x}_2\|$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in E$*.*

Remark 238 *To negate Lipschitz continuity it is enough to find two sequences* $\{\mathbf{x}_n\}, \{\mathbf{z}_n\} \subseteq E$ *with* $\mathbf{x}_n \neq \mathbf{z}_n$ *such that*

$$\left\| \frac{\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{z}_n)}{\mathbf{x}_n - \mathbf{z}_n} \right\| \to \infty.$$

Proposition 239 *Let* $E \subseteq \mathbb{R}^N$ *and let* $\mathbf{f} : E \to \mathbb{R}^N$ *be Lipschitz continuous. Then* $\mathbf{f}$ *is uniformly continuous.*

Proof. Since $\mathbf{f}$ is Lipschitz continuous, there exists $L > 0$ such that

$$\|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)\| \leq L \|\mathbf{x}_1 - \mathbf{x}_2\|$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in E$.

Fix $\varepsilon > 0$, and take $\delta = \frac{\varepsilon}{L}$. If $\mathbf{x}, \mathbf{x}_0 \in E$ with $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, then

$$\|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2)\| \leq L \|\mathbf{x}_1 - \mathbf{x}_2\| \leq L\delta = L\frac{\varepsilon}{L} = \varepsilon.$$

■

Example 240 *The function* $f(x) = \sqrt{x}$*,* $x \in [0,1]$*, is uniformly continuous (we will see this later), but not Lipschitz. Indeed, let* $x_n = 0$ *and* $z_n = \frac{1}{n}$*. Then*

$$\left| \frac{f(x_n) - f(z_n)}{x_n - z_n} \right| = \left| \frac{0 - \sqrt{\frac{1}{n}}}{0 - \frac{1}{n}} \right| = \sqrt{n} \to \infty,$$

which shows that f *is not Lipschitz continuous.*

Proposition 241 *If* $I \subseteq \mathbb{R}$ *is an interval and* $f : I \to \mathbb{R}$ *is differentiable in* I *and the derivative* f' *is bounded, then* f *is Lipschitz continuous.*

Proof. By hypothesis, there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$. Let $x_1, x_2 \in I$, with, say $x_1 < x_2$. By the mean value theorem (we will prove this later), there exists $x_3 \in (x_1, x_2)$ such that

$$f(x_1) - f(x_2) = f'(x_3)(x_1 - x_2).$$

Hence,

$$|f(x_1) - f(x_2)| = |f'(x_3)| |x_1 - x_2| \leq M |x_1 - x_2|,$$

which shows that f is Lipschitz continuous. ■

Example 242 *The function $f(x) = |x|$ is Lipschitz continuous, since*

$$|f(x_1) - f(x_2)| = ||x_1| - |x_2|| \le |x_1 - x_2|,$$

but it is not differentiable in $x = 0$.

Remark 243 *Thus we have shown that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$, then*

$$\begin{aligned} f \text{ differentiable with bounded derivative } &\Longrightarrow f \text{ Lipschitz continuous} \\ &\Longrightarrow f \text{ uniformly continuous } \Longrightarrow f \text{ continuous} \end{aligned}$$

but none of the opposite implications is true.

Example 244 *The functions $f(x) = \cos x$, $f(x) = \sin x$, $f(x) = \arctan x$ all have bounded derivatives, and so they are Lipschitz continuous, and, in turn, uniformly continuous.*

Thursday, November 03, 2011

Make-up class

Simple examples of uniformly continuous functions are Lipschitz and Hölder's continuous functions.

Definition 245 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a function $f : E \to Y$, where $E \subseteq X$.*

(i) The function f is said to be Lipschitz continuous *if there exists $L > 0$ such that*

$$d_Y(f(x_1), f(x_2)) \le L d_X(x_1, x_2)$$

for all $x_1, x_2 \in E$. The number

$$\operatorname{Lip}(f; E) := \sup_{x_1, x_2 \in E,\, x_1, \neq x_2} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \le L$$

is called the Lipschitz constant *of f. It is also denoted* $\operatorname{Lip} f$*. The function f is called a* contraction *if* $\operatorname{Lip} f < 1$.

(ii) The function f is said to be Hölder continuous with exponent $\alpha \in (0,1)$ *if there exists $L > 0$ such that*

$$d_Y(f(x_1), f(x_2)) \le L (d_X(x_1, x_2))^a$$

for all $x_1, x_2 \in E$.

Remark 246 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a function $f : E \to Y$, where $E \subseteq X$.*

(i) If f is Hölder continuous with exponent $\alpha \in (0,1)$ and constant $L > 0$, then to see that it is uniformly continuous, given $\varepsilon > 0$, it is enough to take $\delta = \left(\frac{\varepsilon}{L}\right)^{\frac{1}{\alpha}}$. Indeed, if $x_1, x_2 \in E$ with $d_X(x_1, x_2) < \left(\frac{\varepsilon}{L}\right)^{\frac{1}{\alpha}}$, then

$$d_Y(f(x_1), f(x_2)) \leq L(d_X(x_1, x_2))^a < L\left(\frac{\varepsilon}{L}\right)^{\frac{\alpha}{\alpha}} = \varepsilon.$$

The Weierstrass nowhere differentiable function is an example of a uniformly continuous function that is not Hölder continuous of any $\alpha \in (0,1)$.

(ii) If f is Lipschitz continuous with Lipschitz constant $L > 0$ and if E is bounded, then f is Hölder continuous of any exponent $\alpha \in (0,1)$. To see this, let $E \subseteq B_X(x_0, r)$. Then for all $x_1, x_2 \in E$, we have

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\leq L d_X(x_1, x_2) = L(d_X(x_1, x_2))^a (d_X(x_1, x_2))^{1-a} \\ &\leq L(d_X(x_1, x_2))^a (2r)^{1-a}, \end{aligned}$$

where in the last inequality we have used the fact that $d_X(x_1, x_2) \leq d_X(x_1, x_0) + d_X(x_0, x_2) < r + r$, since $E \subseteq B_X(x_0, r)$. If E is unbounded, then this is no longer true. Indeed, the function $f(x) = x$, $x \in \mathbb{R}$, cannot be Hölder continuous of any exponent $\alpha \in (0,1)$. To see this, take $x_1 = x > 0$ and $x_2 = 0$, then we cannot have an inequality of the type

$$x = |f(x) - f(0)| \leq Lx^a,$$

because as $x \to \infty$, x goes faster than x^a.

Let (X, d) be a metric space. If $x \in X$ and $E \subseteq X$, the *distance* of x from the set E is defined by

$$\operatorname{dist}(x, E) := \inf\{d(x, y) : y \in E\},$$

while the distance between two sets E_1, $E_2 \subseteq X$ is defined by

$$\operatorname{dist}(E_1, E_2) := \inf\{d(x, y) : x \in E_1, y \in E_2\}.$$

Exercise 247 *Let (X, d) be a metric space and let $E \subseteq X$ be a nonempty set.*

(i) Fix $x_0 \in X$. Prove that the function

$$x \in X \mapsto d(x, x_0)$$

is Lipschitz continuous with Lipschitz constant one.

(ii) Prove that the distance function

$$x \in X \mapsto \operatorname{dist}(x, E)$$

is Lipschitz continuous with Lipschitz constant one.

(ii) Characterize the points $x \in X$ such that $\operatorname{dist}(x, E) = 0$.

Theorem 248 (Banach's contraction principle) *Let (X, d) be a nonempty complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point; that is, there is a unique $x \in X$ such that $f(x) = x$.*

Proof. Step 1: Let's first prove uniqueness. Assume that x_1 and x_2 are fixed points of f. Then

$$d_X(x_1, x_2) = d_X(f(x_1), f(x_2)) \leq L d_X(x_1, x_2),$$

which implies that

$$(1 - L) d_X(x_1, x_2) \leq 0.$$

Since $L < 1$, we have that $d_X(x_1, x_2) = 0$, and so $x_1 = x_2$.
Step 2: To prove existence, fix $x_0 \in X$ and define inductively

$$x_1 := f(x_0), \quad x_{n+1} := f(x_n).$$

We claim that $\{x_n\}$ is a Cauchy sequence. Indeed, note that

$$d_X(x_1, x_2) = d_X(f(x_0), f(x_1)) \leq L d_X(x_0, x_1)$$

and by induction

$$d_X(x_n, x_{n+1}) = d_X(f(x_{n-1}), f(x_n)) \leq L^n d_X(x_0, x_1).$$

Hence, for every $m, n \in \mathbb{N}$, by the triangle inequality

$$\begin{aligned} d_X(x_n, x_{n+m}) &\leq \sum_{i=n}^{n+m-1} d_X(x_i, x_{i+1}) \leq d_X(x_0, x_1) \sum_{i=n}^{n+m-1} L^i \\ &\leq d_X(x_0, x_1) \sum_{i=n}^{\infty} L^i = d_X(x_0, x_1) \frac{L^n}{1 - L}. \end{aligned}$$

Letting $n \to \infty$, we have that $\{x_n\}$ is a Cauchy sequence. Since the space is complete, there exists $x \in X$ such that $\{x_n\}$ converges to x. But by the continuity of f,

$$x \leftarrow x_{n+1} = f(x_n) \to f(x),$$

which shows that $f(x) = x$. ■

An important application of Banach's contraction principle is the existence of solutions of ODE. Consider the initial value problem

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(t_0) &= u_0. \end{aligned}$$

Here, $I \subseteq \mathbb{R}$ is an open interval, $t_0 \in I$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume that f satisfies the following Lipschitz condition

$$|f(t, z_1) - f(t, z_2)| \leq L |z_1 - z_2|$$

for all $t \in I$, $z_1, z_2 \in \mathbb{R}$. Then we can prove short time existence of solutions. Consider the space $X = C([t_0, t_0 + T])$, where T will be chosen later and consider the operator

$$F : C([t_0, t_0 + T]) \to C([t_0, t_0 + T])$$

given by

$$F(g)(t) = u_0 + \int_{t_0}^{t} f(s, g(s))\, ds$$

for $g \in C([t_0, t_0 + T])$ and $t \in [t_0, t_0 + T]$. It is clear that F is well-defined, since the function on the right-hand side is continuous. Let's prove that F is a contraction. Take $g_1, g_2 \in C([t_0, t_0 + T])$. Then

$$\begin{aligned}
|F(g_1)(t) - F(g_2)(t)| &= \left| \int_{t_0}^{t} [f(s, g_1(s)) - f(s, g_2(s))]\, ds \right| \\
&\leq \int_{t_0}^{t} |f(s, g_1(s)) - f(s, g_2(s))|\, ds \\
&\leq L \int_{t_0}^{t} |g_1(s) - g_2(s)|\, ds \leq L \max_{y \in [t_0, t_0 + T]} |g_1(y) - g_2(y)| \int_{t_0}^{t} ds \\
&\leq LT d_\infty(g_1, g_2),
\end{aligned}$$

and so taking the maximum over all $t \in [t_0, t_0 + T]$, we get

$$d_\infty(F(g_1), F(g_2)) \leq LT d_\infty(g_1, g_2).$$

If we take T so small that $LT < 1$ and $t_0 + T \in I$, then F is a contraction. By Banach's contraction principle there exists a unique fixed point $u \in C([t_0, t_0 + T])$, that is,

$$u(t) = F(u)(t) = u_0 + \int_{t_0}^{t} f(s, u(s))\, ds$$

for all $t \in [t_0, t_0 + T]$. Since u is continuous, the right-hand side is of class C^1, and so u is actually of class C^1. By differentiating both sides, we get that u is a solution of the ODE. Moreover, $u(t_0) = u_0$. Since any other solution of the initial value problem is a fixed point of F, we have uniqueness.

Friday, November 04, 2011

To prove that the function $f(x) = \sqrt{x}$, $x \in [0, 1]$, is uniformly continuous in $[0, 1]$, we apply the following theorem:

Theorem 249 *Let $K \subseteq \mathbb{R}^N$ be compact and let $\mathbf{f} : K \to \mathbb{R}$ be a continuous function. Then $\mathbf{f}$ is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. Since $\mathbf{f}$ is continuous, for every $\mathbf{x}_0 \in K$ there exists $\delta_{\mathbf{x}_0} = \delta_{\mathbf{x}_0}(\varepsilon) > 0$ such that for all $\mathbf{x} \in E$ with $\|\mathbf{x} - \mathbf{x}_0\| < \delta_{\mathbf{x}_0}$ we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\| < \varepsilon.$$

The family of open balls $\left\{B\left(\mathbf{x}, \frac{1}{2}\delta_{\mathbf{x}}\right)\right\}_{\mathbf{x}\in K}$ covers K. Hence, by compactness there exist $\mathbf{x}_1, \ldots, \mathbf{x}_n \in K$ such that

$$\bigcup_{i=i}^{n} B\left(\mathbf{x}_i, \frac{1}{2}\delta_{\mathbf{x}_i}\right) \supseteq K.$$

Let

$$\delta := \frac{1}{2}\min\left\{\delta_{\mathbf{x}_1}, \ldots, \delta_{\mathbf{x}_1}\right\} > 0.$$

If $\mathbf{x}, \mathbf{y} \in K$ with $\|\mathbf{x}-\mathbf{y}\| < \delta$, then there exists $i \in \{1, \ldots, n\}$ such that $\mathbf{x} \in B\left(\mathbf{x}_i, \frac{1}{2}\delta_{\mathbf{x}_i}\right)$. Moreover,

$$\|\mathbf{x}_i - \mathbf{y}\| \le \|\mathbf{x}_i - \mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\| < \frac{1}{2}\delta_{\mathbf{x}_i} + \frac{1}{2}\delta \le \delta_{\mathbf{x}_i}.$$

Hence, $\mathbf{x}, \mathbf{y} \in B\left(\mathbf{x}_i, \delta_{\mathbf{x}_i}\right)$, and so

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \le \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_i)\| + \|\mathbf{f}(\mathbf{x}_i) - \mathbf{f}(\mathbf{y})\| < \varepsilon + \varepsilon.$$

■

Example 250 *Since $[0,1]$ is sequentially compact and the function $f(x) = \sqrt{x}$, $x \in [0,1]$, is continuous, it follows by the previous theorem that it is uniformly continuous.*

Example 251 *We have seen that the continuous function $f(x) = x^2$, $x \in \mathbb{R}$, is not uniformly continuous in $\mathbb{R}$. However, by the previous theorem it is uniformly continuous in every set $[a,b]$.*

Monday, November 07, 2011

Example 252 *Next we study continuous functions of the type $f(x) = \sin\frac{1}{x}$. Consider the set $E = (0,1]$. This set is not sequentially compact (it is bounded but not closed), thus we cannot apply the previous theorem.*

Proposition 253 *Let $f : E \to \mathbb{R}$ be uniformly continuous, where $E \subseteq \mathbb{R}$. Then f can be extended uniquely to a uniformly continuous function $g : \overline{E} \to \mathbb{R}$.*

Remark 254 *In other words, we are saying that for every $x \in \overline{E} \setminus E$ there exists the limit*

$$\lim_{y\to x} f(y) \in \mathbb{R}$$

and that the function $g : \overline{E} \to \mathbb{R}$, defined by

$$g(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \lim_{y\to x} f(y) & \text{if } x \in \overline{E} \setminus E, \end{cases}$$

is uniformly continuous.

Proof. **Step 1:** We begin by showing that if $\{x_n\} \subseteq E$ is a Cauchy sequence, then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Fix $\varepsilon > 0$. By the uniform continuity of f there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x') - f(x'')| \leq \varepsilon \tag{25}$$

for all $x', x'' \in E$ with $|x' - x''| \leq \delta$. Since $\{x_n\} \subseteq \mathbb{R}$ is a Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_m| \leq \delta$$

for all $n, m \geq n_\varepsilon$, and so

$$|f(x_n) - f(x_m)| \leq \varepsilon \tag{26}$$

for all $n, m \geq n_\varepsilon$, which shows that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.
Step 2: Fix $x \in \overline{E} \setminus E$. We claim that there exists the limit

$$\lim_{y \to x} f(y) \in \mathbb{R}.$$

By Remark 69 there exists a sequence $\{x_n\} \subseteq E$ with $x_n \neq x$ for all $n \in \mathbb{N}$ such that $\{x_n\}$ converges to x. In particular, $\{x_n\}$ is a Cauchy sequence, and so by the previous step $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{f(x_n)\}$ converges to some element $\ell \in \mathbb{R}$. In view of Theorem 166, it remains to show that if $\{z_n\} \subseteq E$ is another sequence converging to x, then $\{f(z_n)\}$ converges to the same number $\ell \in \mathbb{R}$. By the triangle inequality we have

$$|x_n - z_n| \leq |x_n - x| + |x - z_n| \leq \frac{\delta}{2} + \frac{\delta}{2}$$

for all $n \in \mathbb{N}$ sufficiently large, say $n \geq n_1$. Hence, by (25),

$$|f(x_n) - f(z_n)| \leq \varepsilon$$

for all $n \geq n_1$. Then, since $\{f(x_n)\}$ converges to ℓ,

$$|y - f(z_n)| \leq |y - f(x_n)| + |f(x_n) - f(z_n)| \leq \varepsilon + \varepsilon$$

for all sufficiently large, which shows that $\{f(z_n)\}$ converges to ℓ. In view of Theorem 166, we have that there exists $\lim_{y \to x} f(y) = \ell$.
Step 3: It remains to show that the function $g : \overline{E} \to \mathbb{R}$, defined by

$$g(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \lim_{y \to x} f(y) & \text{if } x \in \overline{E} \setminus E, \end{cases}$$

is uniformly continuous. Let $x', x'' \in \overline{E}$ be such that $|x' - x''| < \delta$ and consider two sequences $\{x_n'\}, \{x_n''\} \subseteq E$ converging to x' and x'', respectively. Then for all n sufficiently large we have that

$$|x_n' - x_n''| \leq |x_n' - x'| + |x' - x''| + |x'' - x_n''| < \delta,$$

and so by (25),

$$|f(x'_n) - f(x''_n)| < \varepsilon$$

for all n sufficiently large. Letting $n \to \infty$, we obtain

$$|g(x') - g(x'')| \leq \varepsilon,$$

which shows that g is uniformly continuous. ■

Example 255 *The function $f(x) = \sin\frac{1}{x}$ is not uniformly continuous in $E = (0,1]$, since $0 \in \overline{E} = [0,1]$, but the limit*

$$\lim_{x\to 0}\sin\frac{1}{x}$$

does not exists.

Wednesday, November 09, 2011

20 The Ascoli–Arzela Theorem

Next we show that in an arbitrary metric space, closed and bounded sets are not compact.

Example 256 *Let $X := C([0,1])$. The sequence of functions*

$$f_n(x) = x^n, \quad x \in [0,1]$$

is bounded in $C([0,1])$, but no subsequence converges uniformly to a continuous function. This shows that $\overline{B_X(0,1)}$ is closed and bounded but not compact. Hence, Bolzano–Weierstrass theorem fails for infinite dimensional metric spaces.

Definition 257 *Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{F}$ of functions $f : X \to Y$ is said to be* equicontinuous at a point $x_0 \in X$ *if for every $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that*

$$d_Y(f(x), f(x_0)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x \in X$ with $d(x, x_0) \leq \delta$. The family $\mathcal{F}$ of functions $f : X \to Y$ is said to be (uniformly) equicontinuous *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d(x, y) \leq \delta$.

Definition 258 *Let (X, d) be a metric space. A family $\mathcal{F}$ of functions $f : X \to \mathbb{R}$ is said to be* pointwise bounded *if for every $x \in X$ there exists $M_x > 0$ such that*

$$|f(x)| \leq M_x$$

for all $f \in \mathcal{F}$.

Example 259 *The sequence of functions*

$$f_n(x) = x^n, \quad x \in [0,1],$$

is pointwise bounded but not equicontinuous at $x = 1$. To see this, fix $0 < \varepsilon < 1$. We want to find $\delta > 0$ such that $1 - x^n \leq \varepsilon$ for all $1 - \delta \leq x < 1$. We have $(1-\varepsilon)^{1/n} \leq x$. So for each n the best δ is $1 - \delta_n = (1-\varepsilon)^{1/n}$, that is, $\delta_n = 1 - (1-\varepsilon)^{1/n} \to 0$ as $n \to \infty$. Hence, no δ works for all n.

Example 260 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a family $\mathcal{F}$ of functions from X into Y. If there exist $\alpha \in (0,1]$ if there exists $L > 0$ such that*

$$d_Y(f(x_1), f(x_2)) \leq L(d_X(x_1, x_2))^a$$

for all $x_1, x_2 \in X$ and for all $f \in \mathcal{F}$, then the family $\mathcal{F}$ is equicontinuous. The sequence of functions

$$f_n(x) = \frac{x^n}{n}, \quad x \in [0,1],$$

is pointwise bounded and equicontinuous at $x = 1$. Indeed,

$$f_n'(x) = x^{n-1}, \quad x \in [0,1],$$

so that $\max_{x\in[0,1]} \left|x^{n-1}\right| = 1$, which shows that the sequence $\{f_n\}$ is equi-Lipschitz (take $L = 1$). Hence, it is equicontinuous.

Definition 261 *A metric space (X, d) is* separable *if there exists a sequence $\{x_n\} \subset X$ that is dense in X.*

Example 262 *We discuss separability of some of the examples introduced before.*

(i) *$\mathbb{R}^N$ is separable, since $\mathbb{Q}^N$ is dense in $\mathbb{R}^N$.*

(ii) *Given a nonempty set X with discrete metric*

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

X is separable if and only if X is countable. Why?

(iii) *Using uniform continuity, one can show that piecewise affine functions are dense in $C([a,b])$. By approximating a piecewise affine function with one with rational slopes and endpoints, it follow that $C([a,b])$ is separable.*

(iv) *$\ell^\infty = \ell^\infty(\mathbb{N})$ is not separable (exercise).*

(v) *The space $C_b(\mathbb{R})$ of continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ is not separable (exercise).*

Exercise 263 *Let (X,d) be a compact metric space. Prove that X is separable and complete.*

Exercise 264 *Let (X,d) be a separable metric space and let $E \subseteq X$. Prove that (E,d) is separable.*

The previous exercise fails for topological spaces.

Theorem 265 (Ascoli–Arzelà) *Let (X,d) be a separable metric space and let $\mathcal{F} \subseteq C(X)$ be a family of functions. Assume that $\mathcal{F}$ is pointwise bounded and equicontinuous. Then every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of X to a continuous function $g : X \to \mathbb{R}$.*

Proof. Without loss of generality, we may assume that $\mathcal{F}$ has infinite many elements, otherwise there is nothing to prove. Since X is separable, there exists a countable set $E \subseteq X$ such that $X = \overline{E}$.
Step 1: Let $\mathcal{G} \subseteq \mathcal{F}$ be an infinite set. We claim that $\mathcal{G}$ contains a sequence $\{f_n\}$ such that the limit $\lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$ for all $x \in E$. The proof makes use of the *Cantor diagonal argument*. Write $E = \{x_k\}_k$. Since the set

$$\{f(x_1) : f \in \mathcal{G}\}$$

is bounded in $\mathbb{R}$, by the Bolzano–Weierstrass theorem we can find a sequence $\{f_{n,1}\}_n \subseteq \mathcal{G}$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,1}(x_1) = \ell_1 \in \mathbb{R}.$$

Since the set

$$\{f_{n,1}(x_2) : n \in \mathbb{N}\}$$

is bounded in $\mathbb{R}$, again by the Bolzano–Weierstrass theorem we can find a sequence $\{f_{n,2}\}_n \subseteq \{f_{n,1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,2}(x_2) = \ell_2 \in \mathbb{R}.$$

By induction for every $k \in \mathbb{N}$, $k > 1$, we can find a subsequence $\{f_{n,k}\}_n \subseteq \{f_{n,k-1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,k}(x_k) = \ell_k \in \mathbb{R}.$$

We now consider the diagonal elements of the infinite matrix, that is, the sequence $\{f_{n,n}\}_n$. For every fixed $x_k \in E$ we have that the sequence $\{f_{n,n}(x_k)\}_{n=k}^{\infty}$ is a subsequence of $\{f_{n,k}(x_k)\}_n$, and thus it converges to ℓ_k as $n \to \infty$. This completes the proof of the claim. Set $f_n := f_{n,n}$ and define $g : E \to \mathbb{R}$ by

$$g(x) := \lim_{n\to\infty} f_n(x) \in \mathbb{R}, \quad x \in E. \tag{27}$$

■

Friday, November 11, 2011

Proof. Step 2: Let $K \subseteq X$ be compact and fix $\varepsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \leq \varepsilon \tag{28}$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d(x, y) \leq \delta$. Since K is compact, we may cover it with a finite number of balls $B\left(y_1, \frac{\delta}{2}\right), \ldots, B\left(y_M, \frac{\delta}{2}\right)$. Since E is dense, for every $i = 1, \ldots, M$ there exists $z_i \in B\left(y_i, \frac{\delta}{2}\right) \cap E$. Using (27), we have that there exists an integer $n_\varepsilon \in \mathbb{N}$ such that

$$|f_n(z_i) - f_m(z_i)| \leq \varepsilon \tag{29}$$

for all $i = 1, \ldots, M$ and for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$. Fix $x \in K$. Then x belongs to $B\left(y_i, \frac{\delta}{2}\right)$ for some i. In particular,

$$d(x, z_i) \leq d(x, y_i) + d(y, z_i) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Using (28) and (29), we have that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(z_i)| + |f_n(z_i) - f_m(z_i)| + |f_m(z_i) - f_m(x)| \leq \varepsilon + \varepsilon + \varepsilon$$

for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$, which shows that the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence, there exists

$$g(x) := \lim_{n \to \infty} f_n(x) \in \mathbb{R}.$$

Moreover, since

$$|f_n(x) - f_m(x)| \leq 3\varepsilon$$

for all $x \in K$ and all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$, letting $m \to \infty$, we conclude that

$$|f_n(x) - g(x)| \leq 3\varepsilon$$

for all $x \in K$ and all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$, or, equivalently,

$$\sup_{x \in K} |f_n(x) - g(x)| \leq 3\varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$, which shows that $\{f_n\}$ converges to g uniformly on K. In turn, g restricted to K is continuous.
Step 3: Is g defined everywhere? Yes, for every $x \in X$, take K to be the singleton $\{x\}$. Is g continuous? Yes, this follows from (28). ■

Corollary 266 *Let (X, d) be a compact metric space. Then $\mathcal{F} \subseteq C(X)$ is compact if and only if is closed, bounded, and equicontinuous.*

Proof. If $\mathcal{F}$ is closed, bounded, and equicontinuous, then by the previous theorem it follows that the closure of $\mathcal{F}$ is sequentially compact, and so by Theorem 159 the closure of $\mathcal{F}$ is compact. Conversely, assume that $\mathcal{F} \subseteq C(X)$ is compact. Then by Proposition **??**, $\mathcal{F}$ is bounded. It remains to show that

$\mathcal{F}$ is equicontinuous. Assume, by contradiction, that this is not the case. Then there exist $\varepsilon > 0$, $\{f_n\} \subseteq \mathcal{F}$, and $\{x_n\}, \{y_n\} \subseteq X$ such that

$$|f_n(x_n) - f_n(y_n)| > \varepsilon$$

and $d(x_n, y_n) \leq \frac{1}{n}$. Since X is compact (and so sequentially compact), there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_0 \in X$ such that $d(x_{n_k}, x) \to 0$ as $k \to \infty$. In turn, since $\{f_{n_k}\} \subseteq \mathcal{F}$, again by Theorem 159, there exist a subsequence $\left\{f_{n_{k_j}}\right\}$ of $\{f_{n_k}\}$ and $f_0 \in C(X)$ such that $d_{C(X)}\left(f_{n_{k_j}}, f_0\right) \to 0$ as $j \to \infty$. In particular, for all j sufficiently large, say, $j \geq j_0$,

$$\max_{x \in X} \left| f_{n_{k_j}}(x) - f_0(x) \right| < \frac{\varepsilon}{4}. \tag{30}$$

Using the continuity of f_0 at x_0, we may find $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$|f_0(x) - f_0(x_0)| < \frac{\varepsilon}{4} \tag{31}$$

for all $x \in X$ with $d(x, x_0) \leq \delta$. Since $d\left(x_{n_{k_j}}, x\right) \to 0$ and $d\left(x_{n_{k_j}}, x_{n_{k_j}}\right) \to 0$, by taking j_0 larger, if necessary, we may assume that $d\left(x_{n_{k_j}}, x_0\right) \leq \delta$ and $d\left(y_{n_{k_j}}, x_0\right) \leq \delta$ for all $j \geq j_0$. Hence, by (30) and (31), for all $j \geq j_0$,

$$\begin{aligned}\varepsilon < \left| f_{n_{k_j}}\left(x_{n_{k_j}}\right) - f_{n_{k_j}}\left(y_{n_{k_j}}\right) \right| &\leq \left| f_{n_{k_j}}\left(x_{n_{k_j}}\right) - f_0\left(x_{n_{k_j}}\right) \right| + \left| f_0\left(x_{n_{k_j}}\right) - f_0(x_0) \right| \\ &+ \left| f_0(x_0) - f_0\left(y_{n_{k_j}}\right) \right| + \left| f_0\left(y_{n_{k_j}}\right) - f_{n_{k_j}}\left(y_{n_{k_j}}\right) \right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,\end{aligned}$$

which is a contradiction. ∎

Monday, November 14, 2011

No class, SIAM San diego

Wednesday, November 16, 2011

No class, SIAM San diego

Thursday, November 17, 2011

Make-up class

21 Differentiation

Definition 267 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given a function $f : E \to \mathbb{R}$, if there exists in $[-\infty, \infty]$ the limit*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

then the limit is called the derivative *of f at x_0 and is denoted $f'(x_0)$ or $\frac{df}{dx}(x_0)$. The function f is* differentiable *at x_0 if $f'(x_0)$ exists in $\mathbb{R}$. If f is differentiable at every point of $E \cap \operatorname{acc} E$, we say that f is* differentiable *on E.*

Definition 268 *Given two functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ and a point $x_0 \in \operatorname{acc} E$, we say that the function f is a* little o *of g as $x \to x_0$, and we write $f = \mathrm{o}(g)$, if $g \neq 0$ in E and*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

Hence, a little o of g is simply a function that goes to zero faster than g as $x \to x_0$.

If f is differentiable at x_0, then by setting $x := x_0 + h$ we may write

$$f(x_0 + h) - f(x_0) = f'(x_0)\, h + o(h). \tag{32}$$

Note that (32) expresses $f(x_0 + h) - f(x_0)$ as the sum of the linear functional

$$\begin{aligned} L : \mathbb{R} &\to \mathbb{R} \\ h &\mapsto f'(x_0)\, h \end{aligned}$$

plus a small reminder. We can therefore regard the derivatives of f at x_0, not as a real number, but as a linear operator on $\mathbb{R}$ that takes h to $f'(x_0)\, h$. So an alternative definition of differentiability is the following

Definition 269 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given a function $f : E \to \mathbb{R}$ we say that f is* differentiable *at x_0 if there exists a linear functional $L_{x_0} : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - L_{x_0}(x - x_0)}{x - x_0} = 0.$$

This alternative form will be useful when we define differentiability for functions of several variables.

Theorem 270 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. If a function $f : E \to \mathbb{R}$ is differentiable at x_0, then it is continuous at x_0.*

Proof. For $x \in E$, $x \neq x_0$, write

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0).$$

Then by the product of limits

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \to f'(x_0) \cdot 0$$

as $x \to x_0$. ■

The converse of this theorem is not true.

Example 271 *(a) The functions $f(x) := |x|$ and $g(x) := \sqrt{x}$ are continuous but not differentiable at $x = 0$.*

(b) *By direct application of the definitions one easily proves that the derivative of any constant is clearly zero and that if* $f(x) := x$ *then* $f'(x) = 1$*. The basic elementary functions* $\sin x$, $\cos x$, e^x *are known to satisfy*

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (e^x)' = e^x.$$

The proof is left as an exercise.

Theorem 272 (Weierstrass) *Let* $0 < a < 1$ *and let* $b \in \mathbb{N}$ *be an odd integer such that* $ab > 1 + \frac{3}{2}\pi$*. Then the function*

$$f(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R},$$

is continuous, but nowhere differentiable.

Exercise 273 *For every nonnegative integer* n *and* $x \in (0,1)$ *let* $f_n(x)$ *denote the distance from* x *to the nearest number of the form* $\frac{m}{10^n}$*, where* m *is a nonnegative integer. Consider the function*

$$f(x) = \sum_{n=0}^{\infty} f_n(x).$$

Prove that f *is continuous but nowhere differentiable in* $(0,1)$*.*

We now list some elementary operations for derivatives.

Theorem 274 *Let* $E \subseteq \mathbb{R}$ *and let* $x_0 \in E$ *be an accumulation point of* E*. Given two functions* f, $g : E \to \mathbb{R}$ *assume that* f *and* g *are differentiable at* x_0*. Then*

(a) $f + g$ *is differentiable at* x_0 *and* $(f+g)'(x_0) = f'(x_0) + g'(x_0)$*;*

(b) fg *is differentiable at* x_0 *and* $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$*;*

(c) *if* $g(x_0) \neq 0$ *then* $\dfrac{f}{g}$ *restricted to the set* $F := \{x \in E : g(x) \neq 0\}$ *differentiable at* x_0 *and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Example 275 *Repeated application of (b) and (c) shows that if* $k \in \mathbb{Z}$ *then the function* $f(x) = x^k$ *is differentiable with* $f'(x) = kx^{k-1}$ *and that* $(\tan x)' = \tan^2 x + 1$*.*

Of course, if $k < 0$ we have to restrict ourselves to $x \neq 0$

We now study the differentiation of inverse and composite functions. We begin with composite functions.

Theorem 276 (Chain rule) *Let E, $F \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given two functions $f : E \to F$ and $g : F \to \mathbb{R}$ assume that f is differentiable at x_0, that $f(x_0)$ is an accumulation point of F and that g is differentiable at $f(x_0)$. Then $g \circ f : E \to \mathbb{R}$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0).$$

Proof. Consider the function

$$h(y) := \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)} & \text{if } y \neq f(x_0), \\ g'(f(x_0)) & \text{if } y = f(x_0). \end{cases}$$

Since g is differentiable at x_0, we have that $\lim_{y \to f(x_0)} h(y) = h(f(x_0)) = g'(f(x_0))$. For $x \in E$, $x \neq x_0$, write

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x)) \frac{f(x) - f(x_0)}{x - x_0}.$$

Since f is differentiable at x_0, it is continuous at x_0. But h is continuous at $f(x_0)$, thus the composition $h \circ f$ is continuous at x_0. It follows that $\lim_{x \to x_0} h(f(x)) = h(f(x_0)) = g'(f(x_0))$. Hence, by the product of limits

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x)) \frac{f(x) - f(x_0)}{x - x_0} \to g'(f(x_0)) f'(x_0)$$

as $x \to x_0$. ■

Example 277 *The function*

$$f(x) := \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous in $\mathbb{R}$ since by Example 174

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 = f(0).$$

Using Example 271 and Theorems 274 and 276 for $x \neq 0$ we have

$$\frac{d}{dx}\left(x^2 \sin \frac{1}{x}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

For $x = 0$ we cannot apply the previous theorems and so we have to use the definition. We have

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x} \to 0$$

as $x \to 0$ again by Example 174 and so $f'(0) = 0$. Note that

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0 \end{cases}$$

is not continuous at $x = 0$ since the limit

$$\lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

does not exist (exercise).

Theorem 278 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given a function $f : E \to \mathbb{R}$ assume that f is one-to-one and differentiable at x_0. Suppose also that the inverse function $f^{-1} : f(E) \to \mathbb{R}$ is continuous at $f(x_0)$. Then $f(x_0)$ is an accumulation point of $f(E)$ and f^{-1} is differentiable at $f(x_0)$ if and only if $f'(x_0) \neq 0$ and in this case*

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Proof. We first prove that $f(x_0)$ is an accumulation point of the set $f(E)$. Since x_0 is an accumulation point of E by Theorem **??** there exists a sequence $\{x_n\} \subseteq E \setminus \{x_0\}$ which converges to x_0. Define $y_n := f(x_n)$. Since f is one-to-one and $\{x_n\} \subseteq E \setminus \{x_0\}$ we have that $\{y_n\} \subseteq f(E) \setminus \{f(x_0)\}$. By the continuity of f in x_0 (which follows from Theorem 270) and Theorems 191 and 166 we have that $y_n = f(x_n) \to f(x_0)$. Hence $f(x_0)$ is an accumulation point of the set $f(E)$.

Next we show that $f^{-1} : f(E) \to \mathbb{R}$ is differentiable at $f(x_0)$ if and only if $f'(x_0) \neq 0$. Consider the quotient

$$\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)}$$

for $y \in f(E)$. For every $y \in f(E)$ there exists a unique $x \in E$ such that $f(x) = y$ and so we may write

$$\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{x - x_0}{f(x) - f(x_0)}.$$

By assumption the function f^{-1} is continuous at $f(x_0)$ and so $x = f^{-1}(y) \to x_0 = f^{-1}(f(x_0))$ as $y \to f(x_0)$. Hence since f is differentiable at x_0 we have that

$$\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \to \frac{1}{f'(x_0)}$$

as $y \to f(x_0)$. ■

Remark 279 *If f has a derivative (possibly infinite) at x_0, then the previous proof continuous to work provided we assume that f is continuous at x_0. In this case we would get that $\left(f^{-1}\right)'(f(x_0)) = 0$ if $f'(x_0) = \infty$ or $f'(x_0) = -\infty$.*

Example 280 *Using the previous theorem one can verify that*

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}},$$
$$(\arctan x)' = \frac{1}{x^2+1}, \quad (\log x)' = \frac{1}{x}.$$

Friday, November 18, 2011

Next we study local minima and maxima.

Definition 281 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}^N$, and let $\mathbf{x}_0 \in E$. We say that*

(i) f attains a local minimum *at $\mathbf{x}_0$ if there exists $r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E \cap B(\mathbf{x}_0, r)$,*

(ii) f attains a global minimum *at $\mathbf{x}_0$ if $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E$,*

(iii) f attains a local maximum *at $\mathbf{x}_0$ if there exists $r > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E \cap B(\mathbf{x}_0, r)$,*

(iv) f attains a global maximum *at $\mathbf{x}_0$ if $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E$.*

Theorem 282 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$. If f attains a local minimum (or maximum) at some interior point $x_0 \in E$ and if f has a derivative at x_0, then $f'(x_0) = 0$.*

Proof. Assume that f attains a local minimum (the case of a local maximum is similar). Then there exists $r > 0$ such that $f(x) \geq f(x_0)$ for all $x \in E \cap B(x_0, r)$. Since $x_0 \in E$ is an interior point, by taking r smaller, we can assume that $B(x_0, r) \subseteq E$ so that $f(x) \geq f(x_0)$ for all $x \in B(x_0, r)$. If $x > x_0$, then $f(x) - f(x_0) \geq 0$ and so

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Letting $x \to x_0^+$ and using the fact that f has a derivative at x_0, we get that $f'(x_0) \geq 0$.

If $x < x_0$, then $f(x) - f(x_0) \geq 0$ and so

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Letting $x \to x_0^-$ and using the fact that f has a derivative at x_0, we get that $f'(x_0) \leq 0$. This shows that $f'(x_0) = 0$. ■

Remark 283 *If $x_0 \in E$ is not an interior point, then it is on the boundary. In this case it is either an isolated point or an accumulation point. In the first case it makes not sense to talk about $f'(x_0)$, since we cannot take limits. On the other end, it $x_0 \in \operatorname{acc} E$ and f has a derivative at x_0, then if f attains a local minimum at x_0 we can adapt the previous proof to show that $f'(x_0) \geq 0$ if there exists a sequence $\{x_n\} \subseteq E \setminus \{x_0\}$ such that $x_n \to x_0^+$, while $f'(x_0) \leq 0$ if there exists a sequence $\{x_n\} \subseteq E \setminus \{x_0\}$ such that $x_n \to x_0^-$. Similar conclusions can be reached for local maximum (reversing the inequalities).*

Remark 284 *In view of the previous theorem, when looking for minima and maxima, we have to search among the following:*

- *Interior points at which f is differentiable and $f'(x) = 0$, these are called* critical points*. Note that if $f'(x_0) = 0$, the function f may not attain a local minimum or maximum at x_0. Indeed, consider the function $f(x) = x^3$. Then $f'(0) = 0$, but f is strictly increasing, and so f does not attain a local minimum or maximum at 0.*

- *Interior points at which f has no derivative. The function $f(x) = |x|$ attains a global minimum at $x = 0$, but f is not differentiable at $x = 0$.*

- *Boundary points.*

An important application of the previous theorem is given by the following result.

Theorem 285 (Rolle) *Assume that $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and has a derivative in (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since $[a, b]$ is compact, by the Weierstrass theorem, f has a global maximum and a global minimum in $[a, b]$. If

$$\max_{[a,b]} f = \min_{[a,b]} f,$$

then f is constant, and so $f'(x) = 0$ for all $x \in (a, b)$. If $\max_{[a,b]} f > \min_{[a,b]} f$, then since $f(a) = f(b)$, there f admits one of them at some interior point $c \in (a, b)$. By the previous theorem, $f'(c) = 0$. ■

Theorem 286 (Lagrange or Mean Value Theorem) *Assume that $f : [a, b] \to \mathbb{R}$ is continuous in $[a, b]$ and has a derivative in (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. The function

$$g(x) = f(x) - (x - a) \frac{f(b) - f(a)}{b - a}$$

is continuous in $[a, b]$, has a derivative in (a, b), and $g(a) = g(b) = f(a)$. Hence, we are in a position to apply Rolle's theorem to find $c \in (a, b)$ such that $g'(c) = 0$, or, equivalently,

$$0 = f'(x) - 1 \frac{f(b) - f(a)}{b - a}.$$

■

Corollary 287 *Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in (a, b) with f' bounded. Then f is Lipschitz continuous.*

Proof. Let $L > 0$ be such that $|f'(x)| \leq L$ for all $x \in (a, b)$. By the mean value theorem, for all $x, y \in [a, b]$ with $x < y$, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x).$$

Hence,

$$|f(y) - f(x)| \leq L(y - x),$$

which shows that f is Lipschitz continuous. ■

Corollary 288 *Let $f : (a, b) \to \mathbb{R}$ be differentiable in (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

Proof. By the mean value theorem, for all $x, y \in [a, b]$ with $x < y$, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x) = 0.$$

Hence, $f(y) - f(x) = 0$, which shows that f is constant. ■

Remark 289 *If $A \subseteq \mathbb{R}$ is open and $f : (a, b) \to \mathbb{R}$ be differentiable in A with $f'(x) = 0$ for all $x \in A$, then we can decompose A into a countable number of disjoint open intervals. By applying the previous result in each interval, we conclude that f is constant in every interval (the constant may change from interval to interval).*

Corollary 290 *Let $f : (a, b) \to \mathbb{R}$ have a derivative in (a, b). Then f is increasing if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.*

Proof. If f is increasing, then $f(x) \geq f(x_0)$ for $x > x_0$, and so

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Letting $x \to x_0^+$, we get $f'(x_0) \geq 0$.

Conversely, if $f' \geq 0$, by the mean value theorem, for all $x, y \in [a, b]$ with $x < y$, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x) \geq 0.$$

Hence, $f(y) \geq f(x)$. ■

Remark 291 *If $f' > 0$ is (a, b), then with the same proof we can show that f is strictly increasing, but the opposite is not true. Indeed, the function $f(x) = x^3$ is strictly increasing but $f'(x) = 3x^2$, which is zero for $x = 0$.*

Remark 292 *If $f'(x_0) > 0$, we cannot conclude that f is increasing near x_0 but only that $f(x) < f(x_0)$ for $x_0 - \delta < x < x_0$ and that $f(x) > f(x_0)$ for $x_0 < x < x_0 + \delta$ for some small $\delta > 0$. Indeed, consider the function*

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Let's look at the differentiability at $x = 0$. We have

$$\frac{f(x) - f(0)}{x - 0} = \frac{x + 2x^2 \sin \frac{1}{x} - 0}{x - 0} = 1 + 2x \sin \frac{1}{x} \to 1$$

as $x \to 0$, since $0 \le \left|2x \sin \frac{1}{x}\right| \le |2x| \to 0$. Hence,

$$f'(x) = \begin{cases} 1 + 4x \sin \frac{1}{x} - 2\cos \frac{1}{x^2} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Hence $f'(0) = 1$ but f is not increasing near $x = 0$. Indeed, if it were, then $f' \ge 0$ in $(-\delta, \delta)$ for some small $\delta > 0$. However, since $4x \sin \frac{1}{x} \to 0$ as $x \to 0$, we have that

$$f'(x) = 1 + 4x \sin \frac{1}{x} - 2\cos \frac{1}{x^2}$$

oscillates between -1 and 1 as $x \to 0$. Note that this is also an example of a differentiable function, whose derivative is discontinuous.

The previous example shows that derivatives can be discontinuous. However, derivatives are Darboux continuous, that is, the following theorem holds.

Theorem 293 *Let $f : [a, b] \to \mathbb{R}$ be differentiable. If f' takes two different values, then it takes all the values in between. Precisely, if $f'(c) < f'(d)$ for some $c, d \in [a, b]$, then for every*

$$f'(c) < s < f'(d)$$

there exists x_0 in the interval of endpoints c and d such that $f'(x_0) = s$.

Proof. Assume $c < d$ (the case $c > d$ is similar) and consider the function $g : [c, d] \to \mathbb{R}$, defined by

$$g(x) = f(x) - sx.$$

Then g is differentiable, and in particular continuous. By the Weierstrass theorem, the function g attains a global minimum at some point $x_0 \in [c, d]$. Since $g'(c) = f'(c) - s < 0$, we have that $g(x) < g(c)$ for $x > c$ near c, and so c cannot be x_0. On the other hand, since $g'(d) = f'(d) - s > 0$, we have that $g(x) < g(d)$ for $x < d$ near d, and so x_0 cannot be d. Thus, $x_0 \in (c, d)$, and so by Theorem 282, $0 = g'(x_0) = f'(x_0) - s$. ■

Remark 294 *The previous theorem is very useful when trying to determine if a given function g is the derivative of some function f, that is, if $g = f'$ for some function f.*

The functions

$$g_1(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

is not the derivative of a function f.

Exercise 295 *Consider the functions*

$$g_1(x) = \begin{cases} x & \text{if } x \neq \frac{1}{2}, \\ 0 & \text{if } x = \frac{1}{2}, \end{cases} \qquad g_2(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2}, \\ 1 & x = \frac{1}{2}, \\ x+2 & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

$$g_3(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1}{4} & x = \frac{1}{4}, \\ x - \frac{1}{2} & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \qquad g_4(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Determine which functions are derivatives of some function f and which are not.

Monday, November 21, 2011

Theorem 296 (Cauchy) *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and differentiable in (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Then there exists $c \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. The function

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

is continuous in $[a,b]$, differentiable in (a,b), and $h(a) = h(b) = f(a)g(b) - g(a)f(b)$. Hence, we are in a position to apply Rolle's theorem to find $c \in (a,b)$ such that $h'(c) = 0$, or, equivalently,

$$0 = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)).$$

■

Remark 297 *Why is $g(b) \neq g(a)$?*

Theorem 298 (De l'Hôpital) *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and assume that $f(x_0) = g(x_0) = 0$ for some $x_0 \in [a,b]$. Assume that f and g are differentiable in $(a,b) \setminus \{x_0\}$ with $g(x), g'(x) \neq 0$ for all $x \in (a,b) \setminus \{x_0\}$. If there exists*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell \in [-\infty, \infty],$$

then there exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

Proof. Let $\{x_n\} \subseteq (a,b) \setminus \{x_0\}$ be such that $x_n \to x_0$. Apply Cauchy's theorem in the interval of endpoints x_n and x_0 to find y_n between x_n and x_0 such that

$$\frac{f(x_n) - 0}{g(x_n) - 0} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(y_n)}{g'(y_n)}.$$

As $x_n \to x_0$, we have that $y_n \to x_0$, and so by hypothesis

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)} \to \ell.$$

Since this is true for every sequence $\{x_n\}$ converging to x_0, it follows that there exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

■

Example 299 *Let's calculate*

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

Consider the functions $f(x) = \sin x - x$ *and* $g(x) = x^3$. *Then* $f(0) = g(0) = 0$ *and both functions are differentiable. We have that* $f'(x) = \cos x - 1$ *and* $g'(x) = 3x^2$. *Let's calculate*

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}. \tag{33}$$

We get $\frac{0}{0}$. *Consider the functions* $f_1(x) = \cos x - 1$ *and* $g_1(x) = 3x^2$. *Then* $f_1(0) = g_1(0) = 0$ *and both functions are differentiable. We have that* $f_1'(x) = -\sin x$ *and* $g_1'(x) = 6x$. *Let's calculate*

$$\lim_{x \to 0} \frac{f_1'(x)}{g_1'(x)} = \lim_{x \to 0} \frac{-\sin x}{6x} = -\frac{1}{6}. \tag{34}$$

By (34) and De l'Hôpital's theorem, there exists

$$\lim_{x \to 0} \frac{f_1(x)}{g_1(x)} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = -\frac{1}{6}.$$

Finally, by (33) and De l'Hôpital's theorem, there exists

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

Remark 300 *The converse of De l'Hôpital's theorem does not hold, that is, if there exists* $\lim_{x\to x_0}\frac{f(x)}{g(x)}$, *we cannot conclude that there exists the limit* $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$. *To see this, take*

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

and $g(x)=x$. *Then* f *and* g *are continuous and for* $x\neq 0$, *have that* $f'(x)=2x\sin\frac{1}{x}-x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x}$ *and* $g'(x)=1$. *Then*

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}=\lim_{x\to 0}x\sin\frac{1}{x}=0,$$

since $0\leq\left|x\sin\frac{1}{x}\right|\leq|x|\to 0$, *but the limit*

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{2x\sin\frac{1}{x}+\cos\frac{1}{x}}{1}$$

does not exist (it oscillates between -1 *and* 1*).*

Another version of De l'Hôpital's theorem is the following:

Theorem 301 (De l'Hôpital) *Let* $f:[a,b]\setminus\{x_0\}\to\mathbb{R}$ *and* $g:[a,b]\setminus\{x_0\}\to\mathbb{R}$ *be continuous in* $[a,b]\setminus\{x_0\}$ *and assume that*

$$\lim_{x\to x_0}|f(x)|=\lim_{x\to x_0}|g(x)|=\infty$$

and that f *and* g *are differentiable in* $(a,b)\setminus\{x_0\}$ *with* $g(x),g'(x)\neq 0$ *for all* $x\in(a,b)\setminus\{x_0\}$. *If there exists*

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\ell\in[-\infty,\infty],$$

then there exists

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\ell.$$

The proof is left as an exercise.

Example 302 *Let's calculate*

$$\lim_{x\to 0^+}x^a\log x,$$

where $a>0$. *We get* $0\cdot(-\infty)$. *Let's write* $x^a\log x=\frac{\log x}{x^{-a}}$. *Consider the functions* $f(x)=\log x$ *and* $g(x)=x^{-a}$. *Then*

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\log x=-\infty,\qquad\lim_{x\to 0^+}g(x)=\lim_{x\to 0^+}x^{-a}=+\infty$$

and both functions are differentiable in $(0,\infty)$. *We have that* $f'(x)=\frac{1}{x}$ *and* $g'(x)=-ax^{-a-1}$. *Let's calculate*

$$\lim_{x\to 0^+}\frac{f'(x)}{g'(x)}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-ax^{-a-1}}=\lim_{x\to 0^+}\frac{x^a}{-a}=0,$$

and so there exists $\lim_{x\to 0^+}x^a\log x=0$.

Exercise 303 *Calculate*

$$\lim_{x\to 0^+} x^{\sin x}.$$

Another version of the previous theorem is the following.

Theorem 304 (De l'Hôpital) *Let* $f : [a,\infty) \to \mathbb{R}$ *and* $g : [a,\infty) \to \mathbb{R}$ *be continuous in* $[a,\infty)$ *and differentiable in* (a,∞)*. Assume that*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$$

and that $g(x), g'(x) \neq 0$ *for all* $x \in (a,\infty)$*. If there exists*

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \ell \in [-\infty,\infty],$$

then there exists

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell.$$

The proof is similar and we omit it.

Remark 305 *A similar result holds if* $\lim_{x\to\infty} |f(x)| = \lim_{x\to\infty} |g(x)| = \infty$*.*

Exercise 306 *Prove the following:*

$$\lim_{x\to\infty} \frac{x^a}{\log^b x} = 0,$$

where $a > 0$ *and* $b \in \mathbb{R}$*,*

$$\lim_{x\to\infty} \frac{x^a}{b^x} = 0,$$

where $a \in \mathbb{R}$ *and* $b > 1$*.*

Example 307 *Let's calculate*

$$\lim_{x\to\infty} \frac{\log x - x}{\log(1+e^x)}.$$

We get

$$\lim_{x\to\infty} \frac{\log x - x}{\log(1+e^x)} = \frac{\infty-\infty}{\infty},$$

and so we cannot use De l'Hôpital's theorem directly since the numerator does not approach ∞ *or* $-\infty$*, but it is of the type* $\infty - \infty$*, which is undetermined. To solve this problem, let's factor out* x*, which goes to infinity faster than* $\log x$*. We get*

$$\log x - x = x\left(\frac{\log x}{x} - 1\right) \to \infty(0-1) = -\infty,$$

and so now we know that we can apply De l'Hôpital's theorem. Let $f(x) = \log x - x$ *and* $g(x) = \log(1+e^x)$. *Then*

$$f'(x) = \frac{1}{x} - 1,$$

while

$$g'(x) = (\log(1+e^x))' = \frac{1}{1+e^x}(0+e^x),$$

and so

$$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{x}-1}{\frac{e^x}{1+e^x}} = \frac{\frac{1-x}{x}}{\frac{e^x}{1+e^x}} = \frac{(1-x)}{x}.\frac{(1+e^x)}{e^x} = \frac{1+e^x-x-xe^x}{xe^x}.$$

Let's factor out the fastest term, which is xe^x, *to get*

$$\frac{f'(x)}{g'(x)} = \frac{xe^x\left(\frac{1}{xe^x}+\frac{1}{x}-\frac{1}{e^x}-1\right)}{xe^x} = \left(\frac{1}{xe^x}+\frac{1}{x}-\frac{1}{e^x}-1\right) \to -1$$

and so by De l'Hôpital's theorem there exists

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\log x - x}{\log(1+e^x)} = -1.$$

Next we study Taylor's formula.

Definition 308 *Given an open set* $A \subseteq \mathbb{R}$ *and an integer* $n \in \mathbb{N}$, *a function* $f : A \to \mathbb{R}$ *is said to be of class* C^n *if* f *is differentiable up to order* n *with continuous derivatives* f', f'', f''', $f^{(4)},\cdots,f^{(n)}$. *The space of all functions of class* C^n *is denoted by* $C^n(A)$. *A function of class* C^n *for every* $n \in \mathbb{N}$ *is said to be of class* C^∞ *and the space of all functions of class* C^∞ *is denoted by* $C^\infty(A)$.

Theorem 309 (Taylor's Formula) *Let* $f \in C^{(n)}((a,b))$ *and let* $x_0 \in (a,b)$. *Then for every* $x \in (a,b)$,

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\ + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x,x_0), \end{aligned}$$

where the remainder $R_n(x,x_0)$ *satisfies*

$$\lim_{x\to x_0}\frac{R_n(x,x_0)}{(x-x_0)^n} = 0.$$

Lemma 310 *Let* $g \in C^{(n)}((a,b))$ *and let* $x_0 \in (a,b)$. *Then*

$$\lim_{x\to x_0}\frac{g(x)}{(x-x_0)^n} = 0 \tag{35}$$

if and only if

$$g(x_0) = g'(x_0) = \cdots = g^{(n)}(x_0) = 0. \tag{36}$$

Proof. Assume that (36) holds. By applying De l'Hôpital's theorem several times we get

$$\lim_{x\to x_0}\frac{g(x)}{(x-x_0)^n}=\lim_{x\to x_0}\frac{g'(x)}{n(x-x_0)^{n-1}}=\lim_{x\to x_0}\frac{g^{(2)}(x)}{n(n-1)(x-x_0)^{n-2}}$$
$$=\cdots=\lim_{x\to x_0}\frac{g^{(n-1)}(x)}{n!(x-x_0)}=\lim_{x\to x_0}\frac{g^{(n)}(x)}{n!1}=\frac{g^{(n)}(x_0)}{n!}=0.$$

Conversely, assume (35). If $g^{(k)}(x_0)\neq 0$ for some $0\leq k<n$, then by what we just proved (with k in place of n)

$$\lim_{x\to x_0}\frac{g(x)}{(x-x_0)^k}=\frac{g^{(k)}(x_0)}{k!}\neq 0.$$

On the other hand,

$$\frac{g(x)}{(x-x_0)^k}=\frac{g(x)}{(x-x_0)^k}\frac{(x-x_0)^{n-k}}{(x-x_0)^{n-k}}=\frac{g(x)}{(x-x_0)^n}(x-x_0)^{n-k}\to 0$$

as $x\to x_0$, which is a contradiction. ■

We now turn to the proof of Theorem 309.

Proof of Theorem 309. Note that given a polynomial of degree n,

$$p(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n=\sum_{i=0}^n a_i(x-x_0)^i,$$

we have that

$$p^{(k)}(x)=\sum_{i=k}^n i(i-1)\cdots(i-k+1)a_i(x-x_0)^{i-k},$$

so that

$$p^{(k)}(x_0)=k!a_k.$$

We apply the lemma to the function

$$g(x):=f(x)-p(x)$$

to conclude that

$$\lim_{x\to x_0}\frac{f(x)-p(x)}{(x-x_0)^n}=0$$

if and only if for all $k=0,\ldots,n$,

$$0=g^{(k)}(x_0)=f^{(k)}(x_0)-p^{(k)}(x_0)=f^{(k)}(x_0)-k!a_k,$$

that is

$$a_k=\frac{f^{(k)}(x_0)}{k!}.$$

Thus

$$g(x) = R_n(x, x_0) = f(x) - \left[f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right].$$

■

Definition 311 *Given two functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ and a point $x_0 \in \operatorname{acc} E$, we say that the function f is a* little o *of g as $x \to x_0$, and we write $f = \mathrm{o}(g)$, if $g \neq 0$ in E and*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

Hence, a little o of g is simply a function that goes to zero faster than g as $x \to x_0$. Hence, Taylor's formula can be written as

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 \\ + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + o\left((x - x_0)^n\right) \end{aligned}$$

as $x \to x_0$.

Wednesday, November 23, 2011

Thanksgiving, no classes

Friday, November 25, 2011

Thanksgiving, no classes

Monday, November 28, 2011

Theorem 312 (Taylor's Formula with Integral Remainder) *Let $f : [a, b] \to \mathbb{R}$ be $(n+1)$-times differentiable, $n \in \mathbb{N}$, with $f^{(n+1)}$ continuous in $[a, b]$ and let $x_0 \in [a, b]$. Then for every $x \in [a, b]$,*

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{n!}\int_{x_0}^{x} (x - t)^n f^{(n+1)}(t)\, dt. \end{aligned}$$

Proof. Step 1: For simplicity we do the case $n = 2$ first. Consider the function

$$g(x) := \int_{x_0}^{x} \int_{x_0}^{y} \int_{x_0}^{z} f^{(3)}(t)\, dt dz dy.$$

We compute g in two different ways. By applying the fundamental theorem of calculus three times, we get

$$\begin{aligned} g(x) &= \int_{x_0}^{x} \int_{x_0}^{y} \left(f''(z) - f''(x_0)\right) dz dy \\ &= \int_{x_0}^{x} \left(f'(y) - f'(x_0) - f^{(2)}(x_0)(y - x_0)\right) dy \\ &= f(x) - f(x_0) - f'(x_0)(x - x_0) - \frac{1}{2}f^{(2)}(x_0)(x - x_0)^2. \end{aligned}$$

On the other hand, by interchanging the order of integration,

$$\int_{x_0}^{y}\int_{x_0}^{z} f^{(3)}(t)\ dtdz = \int_{x_0}^{y}\int_{t}^{y} f^{(3)}(t)\ dzdt = \int_{x_0}^{y}(y-t)\, f^{(3)}(t)\ dt$$

Interchanging the order of integration again, gives

$$\begin{aligned} g(x) &= \int_{x_0}^{x}\int_{x_0}^{y}(y-t)\, f^{(3)}(t)\ dtdy = \int_{x_0}^{x}\int_{t}^{x}(y-t)\, f^{(3)}(t)\ dydt \\ &= \frac{1}{2}\int_{x_0}^{x}(y-t)^2 f^{(3)}(t)\ dt. \end{aligned}$$

By equating the two expressions for g we obtain

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f^{(2)}(x_0)(x-x_0)^2 + \frac{1}{2}\int_{x_0}^{x}(y-t)^2 f^{(3)}(t)\ dt.$$

Step 2: In the general case, one consider the function

$$g(x) := \int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_n} f^{(n+1)}(t)\ dtdx_n\cdots dx_1.$$

By applying the fundamental theorem of calculus $n+1$ times, we get

$$\begin{aligned} g(x) &= \int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{n-1}}\Big(f^{(n)}(x_n) - f^{(n)}(x_0)\Big)\ dx_n\cdots dx_1 \\ &= \int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{n-2}}\Big(f^{(n-1)}(x_{n-1}) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x_{n-1}-x_0)\Big)\ dx_{n-1}\cdots dx_1 \\ &= f(x) - f(x_0) - f'(x_0)(x-x_0) - \cdots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$

On the other hand, by interchanging the order of integration,

$$\int_{x_0}^{x_{n-1}}\int_{x_0}^{x_n} f^{(n+1)}(t)\ dtdx_n = \int_{x_0}^{x_{n-1}}\int_{t}^{x_{n-1}} f^{(n+1)}(t)\ dx_ndt = \int_{x_0}^{x_{n-1}}(x_{n-1}-t)\, f^{(n+1)}(t)\ dt.$$

Interchanging the order of integration again, gives

$$\begin{aligned} \int_{x_0}^{x_{n-2}}\int_{x_0}^{x_{n-1}}\int_{x_0}^{x_n} f^{(n+1)}(t)\ dtdx_ndx_{n-1} &= \int_{x_0}^{x_{n-2}}\int_{x_0}^{x_{n-1}}(x_{n-1}-t)\, f^{(n+1)}(t)\ dtdx_{n-1} \\ &= \int_{x_0}^{x_{n-2}}\int_{t}^{x_{n-2}}(x_{n-1}-t)\, f^{(n+1)}(t)\ dx_{n-1}dt \\ &= \frac{1}{2}\int_{x_0}^{x_{n-2}}(x_{n-2}-t)^2 f^{(n+1)}(t)\ dt. \end{aligned}$$

Continuing in this way, we get

$$g(x) = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\ dt.$$

By equating the two expressions for g we obtain

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$+ \frac{1}{n!}\int_{x_0}^{x}(x - t)^n f^{(n+1)}(t)\, dt.$$

■

Important Taylor's formulas with center $x = 0$

- $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n + o(x^n)$, hence the first order formula is
$$e^x = 1 + x + o(x),$$
while the second order formula is
$$e^x = 1 + x + \frac{1}{2!}x^2 + o\left(x^2\right).$$

- $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1}\frac{1}{n}x^n + o(x^n)$, hence the first order formula is
$$\log(1 + x) = x + o(x),$$
while the second order formula is
$$\log(1 + x) = x - \frac{1}{2}x^2 + o\left(x^2\right).$$

- $(1 + x)^a = 1 + ax + \frac{1}{2}a(a - 1)x^2 + \cdots + \frac{1}{n!}a(a - 1)(a - 2)(a - 3)\cdots(a - n + 1)x^n + o(x^n)$, hence the first order formula is
$$(1 + x)^a = 1 + ax + o(x),$$
while the second order formula is
$$(1 + x)^a = 1 + ax + \frac{1}{2}a(a - 1)x^2 + o\left(x^2\right).$$

- $\dfrac{1}{1 + x} = (1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^{n+1}x^n + o(x^n)$ hence the first order formula is
$$\frac{1}{1 + x} = (1 + x)^{-1} = 1 - x + o(x),$$
while the second order formula is
$$\frac{1}{1 + x} = (1 + x)^{-1} = 1 - x + x^2 + o\left(x^2\right).$$

- $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \cdots + (-1)^k \frac{1}{2k!}x^{2k} + o\left(x^{2k+1}\right)$, hence the third order formula is

$$\cos x = 1 - \frac{1}{2!}x^2 + o\left(x^3\right),$$

while the fifth order formula is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right),$$

- $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \cdots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + o\left(x^{2k+2}\right)$, hence the second order formula is

$$\sin x = x + o\left(x^2\right),$$

while the fourth order formula is

$$\sin x = x - \frac{1}{3!}x^3 + o\left(x^4\right),$$

- $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 \cdots + (-1)^k \frac{1}{(2k+1)}x^{2k+1} + o\left(x^{2k+2}\right)$, hence the second order formula is

$$\arctan x = x + o\left(x^2\right),$$

while the fourth order formula is

$$\arctan x = x - \frac{1}{3}x^3 + o\left(x^4\right).$$

Examples:

1. Let's calculate Taylor formula for $\log\left(1 + \sin^2(x)\right)$ of order 3 and center $x = 0$. Let's use the important Taylor's formulas given above. Taylor's formula of $\sin x$ of order four is given by

$$\sin x = x - \frac{1}{3!}x^3 + o\left(x^4\right) = x - \frac{1}{6}x^3 + o\left(x^4\right)$$

and so

$$\begin{aligned}
\sin^2 x = \left(x - \frac{1}{6}x^3 + o\left(x^4\right)\right)^2 &= x^2 + \frac{1}{36}x^6 + \left(o\left(x^4\right)\right)^2 \\
&+ 2x\left(-\frac{1}{6}x^3\right) + 2xo\left(x^4\right) - \frac{2}{6}x^3 o\left(x^4\right) \\
&= x^2 + \frac{1}{36}x^6 + o\left(x^8\right) - \frac{1}{3}x^4 + 2o\left(x^5\right) - \frac{2}{6}o\left(x^7\right) \\
&= x^2 + \frac{1}{36}x^6 + o\left(x^8\right) - \frac{1}{3}x^4 + o\left(x^5\right) + o\left(x^7\right) \\
&= x^2 - \frac{1}{3}x^4 + o\left(x^5\right)
\end{aligned}$$

where we have used the properties of the little o. Hence

$$\log\left(1+\sin^2 x\right) = \log\left(1+x^2-\frac{1}{3}x^4+o\left(x^5\right)\right),$$

Let's use now Taylor's formula

$$\log(1+t) = t-\frac{1}{2}t^2+o\left(t^2\right),$$

where for us $t = x^2-\frac{1}{3}x^4+o\left(x^5\right)$. We get

$$\begin{aligned}
&\log\left(1+x^2-\frac{1}{3}x^4+o\left(x^5\right)\right)\\
&=\left(x^2-\frac{1}{3}x^4+o\left(x^5\right)\right)-\frac{1}{2}\left(x^2-\frac{1}{3}x^4+o\left(x^5\right)\right)^2+o\left(\left(x^2-\frac{1}{3}x^4+o\left(x^5\right)\right)^2\right)\\
&=x^2-\frac{1}{3}x^4+o\left(x^5\right)-\frac{1}{2}\left(x^4+\frac{1}{9}x^8+\left(o\left(x^5\right)\right)^2+2x^2\left(-\frac{1}{3}x^4\right)+2x^2o\left(x^5\right)-\frac{2}{3}x^4o\left(x^5\right)\right)\\
&+o\left(x^4+\frac{1}{9}x^8+\left(o\left(x^5\right)\right)^2+2x^2\left(-\frac{1}{3}x^4\right)+2x^2o\left(x^5\right)-\frac{2}{3}x^4o\left(x^5\right)\right)\\
&=x^2-\frac{1}{3}x^4+o\left(x^5\right)-\frac{1}{2}x^4+o\left(x^4\right)=x^2-\frac{1}{3}x^4-\frac{1}{2}x^4+o\left(x^4\right)\\
&=x^2-\frac{5}{6}x^4+o\left(x^4\right),
\end{aligned}$$

where on line 5 of the previous formula all the powers of order bigger than 5 have been absorbed by $o\left(x^5\right)$ and in the last line $o\left(x^5\right)$ disappeared because there was $o\left(x^4\right)$.

2. Let's calculate the limit

$$\lim_{x\to 0}\frac{\sin^2 x+3\left(x-\sin^2\sqrt{x}\right)}{x^2}$$

Taylor's formula of $\sin x$ of order four

$$\sin x = x-\frac{1}{3!}x^3+o\left(x^4\right) = x-\frac{1}{6}x^3+o\left(x^4\right)$$

and so

$$\begin{aligned}
\sin^2 x &= \left(x-\frac{1}{6}x^3+o\left(x^4\right)\right)^2 = x^2+\frac{1}{36}x^6+\left(o\left(x^4\right)\right)^2\\
&+2x\left(-\frac{1}{6}x^3\right)+2xo\left(x^4\right)-\frac{2}{6}x^3o\left(x^4\right)\\
&=x^2+\frac{1}{36}x^6+o\left(x^8\right)-\frac{1}{3}x^4+2o\left(x^5\right)-\frac{2}{6}o\left(x^7\right)\\
&=x^2+\frac{1}{36}x^6+o\left(x^8\right)-\frac{1}{3}x^4+o\left(x^5\right)+o\left(x^7\right)\\
&=x^2-\frac{1}{3}x^4+o\left(x^5\right),
\end{aligned}$$

while (replacing x with $\sqrt{x}$)

$$\sin^2\sqrt{x} = \left(\sqrt{x}\right)^2 - \frac{1}{3}\left(\sqrt{x}\right)^4 + o\left(\left(\sqrt{x}\right)^5\right) = x - \frac{1}{3}x^2 + o\left(x^{5/2}\right),$$

which gives

$$\begin{aligned}\sin^2 x + 3\left(x - \sin^2\sqrt{x}\right) &= x^2 - \frac{1}{3}x^4 + o\left(x^5\right) + 3\left(x - \left(x - \frac{1}{3}x^2 + o\left(x^{5/2}\right)\right)\right)\\ &= x^2 - \frac{1}{3}x^4 + o\left(x^5\right) + 3\left(x - x + \frac{1}{3}x^2 + o\left(x^{5/2}\right)\right)\\ &= x^2 - \frac{1}{3}x^4 + o\left(x^5\right) + x^2 + o\left(x^{5/2}\right) = 2x^2 + o\left(x^{5/2}\right)\end{aligned}$$

and in turn

$$\begin{aligned}\frac{\sin^2 x + 3\left(x - \sin^2\sqrt{x}\right)}{x^2} &= \frac{2x^2 + o\left(x^{5/2}\right)}{x^2}\\ &= \frac{x^2\left(2 + o\left(x^{\frac{5}{2}-1}\right)\right)}{x^2} = 2 + o\left(x^{\frac{1}{2}}\right) \to 2 + 0 = 2\end{aligned}$$

Wednesday, November 30, 2011

22 Integration

Given N bounded intervals $I_1, \ldots, I_N \subset \mathbb{R}$, a *rectangle* in $\mathbb{R}^N$ is a set of the form

$$R := I_1 \times \cdots \times I_N.$$

The elementary measure of a rectangle is given by

$$\operatorname{meas} R := \operatorname{length} I_1 \cdot \cdots \cdot \operatorname{length} I_N,$$

where if I_n has endpoints $a_n \le b_n$, then we set $\operatorname{length} I_n := b_n - a_n$. To highlight the dependence on N, we will use the notation meas_N.

Remark 313 *Note that the intersection of two rectangles is still a rectangle. We will use this fact a lot in what follows.*

Given a rectangle R, by a *partition* $\mathcal{P}$ of R we mean a finite set of rectangles $R_1, \ldots, R_n$ such that $R_i \cap R_j = \emptyset$ if $i \neq j$ and

$$R = \bigcup_{i=1}^{n} R_i.$$

Exercise 314 *Let $R \subset \mathbb{R}^N$ be a rectangle and let $\mathcal{P} = \{R_1, \ldots, R_n\}$ be a partition of R. Prove that*

$$\operatorname{meas} R = \sum_{i=1}^{n} \operatorname{meas} R_i.$$

Hint: Use induction on N.

Given a rectangle R, consider a bounded function $f : R \to \mathbb{R}$ and a partition $\mathcal{P}$ of R. We define the *lower* and *upper sums* of f for the partition $\mathcal{P}$ respectively by

$$L(f, \mathcal{P}) := \sum_{i=1}^{n} \operatorname{meas} R_i \inf_{\mathbf{x} \in R_i} f(\mathbf{x}),$$
$$U(f, \mathcal{P}) := \sum_{i=1}^{n} \operatorname{meas} R_i \sup_{\mathbf{x} \in R_i} f(\mathbf{x}).$$

Since f is bounded, using the previous exercise, we have that

$$\begin{aligned}\operatorname{meas} R \inf_{\mathbf{x} \in R} f(\mathbf{x}) &= \sum_{i=1}^{n} \operatorname{meas} R_i \inf_{\mathbf{x} \in R} f(\mathbf{x}) \le L(f, \mathcal{P}) \qquad (37)\\ &\le U(f, \mathcal{P}) \le \sum_{i=1}^{n} \operatorname{meas} R_i \sup_{\mathbf{x} \in R} f(\mathbf{x}) \le \operatorname{meas} R \sup_{\mathbf{x} \in R} f(\mathbf{x}).\end{aligned}$$

The *lower* and *upper integrals* of f over R are defined respectively by

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} := \sup\{L(f, \mathcal{P}) : \mathcal{P} \text{ partition of } R\},$$
$$\overline{\int_R} f(\mathbf{x})\, d\mathbf{x} := \inf\{U(f, \mathcal{P}) : \mathcal{P} \text{ partition of } R\}.$$

We study some properties of the lower and upper integrals of f.

Given a rectangle R and two partitions $\mathcal{P}$ and $\mathcal{Q}$ of R, we say that $\mathcal{Q}$ is a *refinement* of $\mathcal{P}$, if each rectangle of $\mathcal{Q}$ is contained in some rectangle of $\mathcal{P}$.

Proposition 315 *Given a rectangle R, consider a bounded function $f : R \to \mathbb{R}$. Then*

$$\operatorname{meas} R \inf_{\mathbf{x} \in R} f(\mathbf{x}) \le \underline{\int_R} f(\mathbf{x})\, d\mathbf{x} \le \overline{\int_R} f(\mathbf{x})\, d\mathbf{x} \le \operatorname{meas} R \sup_{\mathbf{x} \in R} f(\mathbf{x}). \qquad (38)$$

Proof. Step 1: Let $\mathcal{P} = \{R_1, \ldots, R_n\}$ be a partition of R and consider a refinement $\mathcal{Q} = \{S_1, \ldots, S_m\}$ of $\mathcal{P}$. Fix $i \in \{1, \ldots, n\}$ and let $I_i := \{j \in \{1, \ldots, m\} : S_j \subset R_i\}$. Since $\mathcal{Q}$ is a refinement of $\mathcal{P}$, we have that

$$R_i = \bigcup_{j \in I_i} S_j$$

and so by Exercise, $\operatorname{meas} R_i = \sum_{j\in I_i} \operatorname{meas} S_j$. Moreover, since $S_j \subset R_i$ for all $j \in I_i$, we have that $\inf_{\mathbf{x}\in R_i} f(\mathbf{x}) \leq \inf_{\mathbf{x}\in S_j} f(\mathbf{x})$. Hence,

$$\begin{aligned}\operatorname{meas} R_i \inf_{\mathbf{x}\in R_i} f(\mathbf{x}) &= \sum_{j\in I_i} \operatorname{meas} S_j \inf_{\mathbf{x}\in R_i} f(\mathbf{x}) \\ &\leq \sum_{j\in I_i} \operatorname{meas} S_j \inf_{\mathbf{x}\in S_j} f(\mathbf{x})\end{aligned}$$

and so

$$\begin{aligned}L(f,\mathcal{P}) &= \sum_{i=1}^{n} \operatorname{meas} R_i \inf_{\mathbf{x}\in R_i} f(\mathbf{x}) \leq \sum_{i=1}^{n} \sum_{j\in I_i} \operatorname{meas} S_j \inf_{\mathbf{x}\in S_j} f(\mathbf{x}) \\ &= \sum_{j=1}^{m} \operatorname{meas} S_j \inf_{\mathbf{x}\in S_j} f(\mathbf{x}) = L(f,\mathcal{Q}).\end{aligned}$$

This shows that

$$L(f,\mathcal{P}) \leq L(f,\mathcal{Q}). \tag{39}$$

A similar argument shows that

$$U(f,\mathcal{Q}) \leq U(f,\mathcal{P}).$$

Step 2: Let $\mathcal{P} = \{R_1,\dots,R_n\}$ and $\mathcal{Q} = \{S_1,\dots,S_m\}$ be two partitions of R and consider the new partition $\mathcal{P}' := \{R_i \cap S_j : i = 1,\dots,n,\ j = 1,\dots,m\}$. Note that $\mathcal{P}'$ is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$. Thus, by what we just proved in Step 1,

$$L(f,\mathcal{P}) \leq L(f,\mathcal{P}') \leq U(f,\mathcal{P}') \leq U(f,\mathcal{Q}). \tag{40}$$

Hence, $L(f,\mathcal{P}) \leq U(f,\mathcal{Q})$ for all partitions $\mathcal{P}$ and $\mathcal{Q}$ of R. Taking the supremum over all partitions $\mathcal{P}$ of R, we get

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} = \sup_{\mathcal{P} \text{ partition of } R} L(f,\mathcal{P}) \leq U(f,\mathcal{Q})$$

for all partitions $\mathcal{Q}$ of R. Taking the infimum over all partitions $\mathcal{Q}$ of R, we get

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} \leq \inf_{\mathcal{Q} \text{ partition of } R} U(f,\mathcal{Q}) = \overline{\int_R} f(\mathbf{x})\, d\mathbf{x}.$$

The remaining inequalities in (38) follow from (37). ■

Proposition 316 *Given a rectangle R, consider two bounded functions $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$, with $f \leq g$. Then*

$$\underline{\int_R} f(\mathbf{x})\, d\mathbf{x} \leq \underline{\int_R} g(\mathbf{x})\, d\mathbf{x}, \quad \overline{\int_R} f(\mathbf{x})\, d\mathbf{x} \leq \overline{\int_R} g(\mathbf{x})\, d\mathbf{x}.$$

Proof. This follows from the fact that if $f \le g$ then for every set $E \subset R$,

$$\inf_E f \le \inf_E g, \quad \sup_E f \le \sup_E g.$$

■

Given a rectangle R and a bounded function $f : R \to \mathbb{R}$, we say that f is *Riemann integrable* over R if the lower and upper integral coincide. We call the common value the *Riemann integral* of f over R and we denote it by $\int_R f(\mathbf{x})\, d\mathbf{x}$. Thus, for a Riemann integrable function,

$$\int_R f(\mathbf{x})\, d\mathbf{x} := \underline{\int_R} f(\mathbf{x})\, d\mathbf{x} = \overline{\int_R} f(\mathbf{x})\, d\mathbf{x}.$$

Remark 317 *Note that the function $f = 1$ is Riemann integrable over a rectangle R and*

$$\int_R 1\, dx = \operatorname{meas} R.$$

We can also define the Riemann integral over bounded sets E. Given a bounded set $E \subset \mathbb{R}^N$, let R be a rectangle containing E. We say that function $f : E \to \mathbb{R}$ is *Riemann integrable over E* if the function

$$g(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in R \\ 0 & \text{if } \mathbf{x} \in R \setminus E \end{cases}$$

is Riemann integrable over R and we define the *Riemann integral of f over E* to be

$$\int_E f(\mathbf{x})\, d\mathbf{x} := \int_R g(\mathbf{x})\, d\mathbf{x}.$$

Exercise 318 *Prove the previous definition does not depend on the choice of the particular rectangle R containing E.*

Theorem 319 *Given a rectangle R, bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a partition $\mathcal{P}^\varepsilon$ of R such that for every refinement $\mathcal{P}$ of $\mathcal{P}^\varepsilon$,*

$$0 \le U(f, \mathcal{P}) - L(f, \mathcal{P}) \le \varepsilon. \tag{41}$$

Proof. Exercise. ■

To determine when a function is Riemann integrable we need to introduce the notion of sets of Lebesgue measure zero.

Definition 320 *A set $E \subset \mathbb{R}^N$ has* Lebesgue measure zero *if for every $\varepsilon > 0$ there exists a countable family of rectangles R_n such that*

$$E \subset \bigcup_n R_n \quad \textit{and} \quad \sum_n \operatorname{meas} R_n \le \varepsilon.$$

Example 321 *Let's see some examples.*

(i) *A singleton* $E = \{\mathbf{c}\}$ *has Lebesgue measure zero. Given* $\varepsilon > 0$, *take* R *to be the cube centered at* $\mathbf{c}$ *and of side-length* $\varepsilon^{1/N}$.

(ii) *If a set* E *contains an open rectangle* R, *then it cannot have Lebesgue measure zero. Indeed, for any countable family of rectangles* R_n, *we have*

$$R \subset E \subset \bigcup_n R_n,$$

and so

$$\operatorname{meas} R \leq \sum_n \operatorname{meas} R_n.$$

Taking $\varepsilon < \operatorname{meas} R$, *we obtain a contradiction.*

(iii) *A countable set* $E = \{\mathbf{x}_n\}_n$ *has Lebesgue measure zero. Given* $\varepsilon > 0$, *take* R_n *to be the cube centered at* $\mathbf{x}_n$ *and of side-length* $\left(\frac{\varepsilon}{2^n}\right)^{1/N}$. *Then*

$$\sum_n \operatorname{meas} R_n = \varepsilon \sum_n \frac{1}{2^n} \leq \varepsilon.$$

In particular, $\mathbb{N}$, $\mathbb{Q}$, *and* $\mathbb{Z}$ *all have Lebesgue measure zero.*

(iv) *If* $\{E_k\}_k$ *is a countable family of sets, each with Lebesgue measure zero, then their union*

$$E := \bigcup_k E_k$$

has Lebesgue measure zero. Indeed, given $\varepsilon > 0$ *fix* k. *Since* E_k *has Lebesgue measure zero, there exists a countable family of rectangles* $R_n^{(k)}$ *such that*

$$E_k \subset \bigcup_n R_n^{(k)} \quad \textit{and} \quad \sum_n \operatorname{meas} R_n^{(k)} \leq \frac{\varepsilon}{2^k}.$$

Consider the family $\left\{R_n^{(k)}\right\}_{n,k}$. *It is still countable,*

$$E = \bigcup_k E_k \subset \bigcup_k \bigcup_n R_n^{(k)}$$

and

$$\sum_{n,k} \operatorname{meas} R_n^{(k)} = \sum_k \sum_n \operatorname{meas} R_n^{(k)} \leq \sum_k \frac{\varepsilon}{2^k} \leq \varepsilon.$$

(v) *There are uncountable sets that have Lebesgue measure zero. One such example is given by the Cantor set.*

Exercise 322 *Prove that the boundary of a rectangle* $R \subset \mathbb{R}^N$ *has Lebesgue measure zero.*

Exercise 323 *Prove that in Definition 320, the rectangles R_n can be assumed to be open.*

Friday, December 02, 2011

The following theorem characterizes Riemann integrable functions.

Theorem 324 *Given a rectangle R, a bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if the set of its discontinuity points has Lebesgue measure zero.*

Definition 325 *Given a function $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}^N$, and a set $F \subset E$, the* oscillation *of f over the set F is the number*

$$\omega_f(F) := \sup_F f - \inf_F f \geq 0.$$

The oscillation of f at a point $\mathbf{x}_0 \in E$ is the number

$$\omega_f(\mathbf{x}_0) := \inf_{r>0} \omega_f(E \cap B(\mathbf{x}_0, r)).$$

Note that if $G \subset F$, then

$$\omega_f(F) = \sup_F f - \inf_F f \geq \sup_G f - \inf_G f = \omega_f(G). \tag{42}$$

Exercise 326 *Given a function $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}$, and a point $\mathbf{x}_0 \in E$, prove that f is continuous at $\mathbf{x}_0$ if and only if $\omega_f(\mathbf{x}_0) = 0$.*

Proof of Theorem 324. Step 1: Assume that f is Riemann integrable. By the previous exercise,

$$\begin{aligned} E :&= \{\mathbf{x} \in R : f \text{ is discontinuous at } \mathbf{x}\} \\ &= \{\mathbf{x} \in R : \omega_f(\mathbf{x}) > 0\}. \end{aligned}$$

For every $k \in \mathbb{N}$ write $E_k := \left\{\mathbf{x} \in R : \omega_f(\mathbf{x}) \geq \frac{1}{k}\right\}$. Then

$$E = \bigcup_{k=1}^{\infty} E_k.$$

Thus, by Example 321, to prove that E has Lebesgue measure zero, it is enough to show that every E_k has Lebesgue measure zero. Fix $k \in \mathbb{N}$ and let $\varepsilon > 0$. By Theorem 319 there exists a partition $\mathcal{P}^{\varepsilon,k}$ of R such that

$$0 \leq U\left(f, \mathcal{P}^{\varepsilon,k}\right) - L\left(f, \mathcal{P}^{\varepsilon,k}\right) \leq \frac{\varepsilon}{2k}.$$

Write $\mathcal{P}^{\varepsilon,k} = \{R_1, \ldots, R_n\}$ and set

$$F_k := E_k \cap \bigcup_{i=1}^{n} \partial R_i, \quad G_k := E_k \cap \bigcup_{i=1}^{n} R_i^\circ.$$

Since each ∂R_i has Lebesgue measure zero by Exercise 322, it follows from Example 321, that the set F_k has Lebesgue measure zero. Thus, to prove that E_k has Lebesgue measure zero, it is enough to show that G_k has Lebesgue measure zero.

Note that

$$U\left(f, \mathcal{P}^{\varepsilon,k}\right) - L\left(f, \mathcal{P}^{\varepsilon,k}\right) = \sum_{i=1}^{n} \operatorname{meas} R_i \left(\sup_{\mathbf{x}\in R_i} f(\mathbf{x}) - \inf_{\mathbf{x}\in R_i} f(\mathbf{x})\right)$$
$$= \sum_{i=1}^{n} \operatorname{meas} R_i \omega_f(R_i).$$

Let $F = \{i = 1, \ldots, n : R_i \cap G_k \neq \emptyset\}$. Note that if $i \in F$, then there is $\mathbf{x} \in R_i \cap G_k$ such that $\omega_f(\mathbf{x}) \geq \frac{1}{k}$. By the definition of G_k, we have that $\mathbf{x} \in R_i^\circ$, and so for all $r > 0$ sufficiently small, $R \cap B(\mathbf{x}, r) \subset R_i$. It follows from (42) that

$$\frac{1}{k} \leq \omega_f(\mathbf{x}) = \inf_{r>0} \omega_f(R \cap B(\mathbf{x}, r)) \leq \omega_f(R_i).$$

Thus

$$\frac{1}{k}\sum_{i\in F} \operatorname{meas} R_i \leq \sum_{i\in F} \operatorname{meas} R_i \omega_f(R_i)$$
$$\leq \sum_{i=1}^{n} \operatorname{meas} R_i \omega_f(R_i)$$
$$= U\left(f, \mathcal{P}^{\varepsilon,k}\right) - L\left(f, \mathcal{P}^{\varepsilon,k}\right) \leq \frac{\varepsilon}{2k}.$$

This shows that

$$\sum_{i\in F} \operatorname{meas} R_i \leq \frac{\varepsilon}{2}.$$

Hence, we can cover G_k with a finite number of rectangles of total measure less than or equal to ε. This shows that G_k has Lebesgue measure zero. ■

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Proof. Step 2: Next assume that the set E of discontinuity points of f has Lebesgue measure zero. We claim that f is Riemann integrable over R. Let $M > 0$ be such that $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in R$. Since ∂R has Lebesgue measure zero by Exercise 322, it follows from Example 321 that $E \cup \partial R$ has Lebesgue measure zero. Hence, also by Exercise 323, given $\varepsilon > 0$, there exists a countable family $\{R_n\}$ of open rectangles R_n such that

$$E \cup \partial R \subseteq \bigcup_n R_n \quad \text{and} \quad \sum_n \operatorname{meas} R_n \leq \frac{\varepsilon}{4M}.$$

If $\mathbf{x} \in R \setminus \bigcup_n R_n$, then f is continuous at $\mathbf{x}$ and thus there exists $\delta_{\mathbf{x}} > 0$ such that

$$|f(\mathbf{y}) - f(\mathbf{x})| \leq \frac{\varepsilon}{4 \operatorname{meas} R} \tag{43}$$

for all $\mathbf{y} \in R$ with $|\mathbf{x} - y| < \delta_{\mathbf{x}}$. Let $J_{\mathbf{x}}$ be the open cube centered at $\mathbf{x}$ and of diameter $\delta_{\mathbf{x}}$. Then

$$\overline{R} \subseteq \bigcup_n R_n \cup \bigcup_{\mathbf{x} \in R \setminus \bigcup_n I_n} J_{\mathbf{x}}.$$

Since $\overline{R}$ is closed and bounded, it is compact, and so there exist $m \in \mathbb{N}$ and $\mathbf{x}_1, \ldots, \mathbf{x}_\ell \in R \setminus \bigcup_n R_n$ such that

$$\overline{R} \subseteq \bigcup_{n=1}^{m} R_n \cup \bigcup_{k=1}^{\ell} J_{\mathbf{x}_k}.$$

Let $\mathcal{P}^\varepsilon$ be the partition of R consisting of all rectangles of the form $R_l \cap R_n$, $R_n \cap J_{\mathbf{x}_k}$, $J_{\mathbf{x}_k} \cap J_{\mathbf{x}_i}$ and let $\mathcal{P} = \{Q_1, \ldots, Q_p\}$ be any refinement of $\mathcal{P}^\varepsilon$. Let

$$F = \{i = 1, \ldots, p : \, Q_i \subseteq R_n \text{ for some } n\}.$$

Then

$$\sum_{i \in F} \operatorname{meas} Q_i \left(\sup_{Q_i} f - \inf_{Q_i} f \right) \leq 2M \sum_{i \in F} \operatorname{meas} Q_i \leq \frac{\varepsilon}{2}. \tag{44}$$

On the other hand, if $i \in \{1, \ldots, p\} \setminus F$, then $Q_i \subseteq J_{\mathbf{x}_k}$ for some $k = 1, \ldots, \ell$, and so by (43), if $\mathbf{x}, \mathbf{y} \in Q_i$, then

$$\begin{aligned} |f(\mathbf{y}) - f(\mathbf{x})| &= |f(\mathbf{y}) \pm f(\mathbf{x}_k) - f(\mathbf{x})| \\ &\leq |f(\mathbf{y}) - f(\mathbf{x}_k)| + |f(\mathbf{x}_k) - f(\mathbf{x})| \leq \frac{\varepsilon}{4 \operatorname{meas} R} + \frac{\varepsilon}{4 \operatorname{meas} R} \end{aligned}$$

and so $\sup_{Q_i} f - \inf_{Q_i} f \leq \frac{\varepsilon}{2 \operatorname{meas} R}$. In turn,

$$\begin{aligned} \sum_{i \notin F} \operatorname{meas} Q_i \left(\sup_{Q_i} f - \inf_{Q_i} f \right) &\leq \sum_{i \notin F} \operatorname{meas} Q_i \frac{\varepsilon}{2 \operatorname{meas} R} \\ &\leq \frac{\varepsilon}{2}. \end{aligned} \tag{45}$$

Summing (44) and (45), we obtain that

$$0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) \leq \varepsilon,$$

which implies that f is Riemann integrable by Theorem 319. ■

Exercise 327 *Let $R \subset \mathbb{R}^N$ be rectangle and let $f : R \to \mathbb{R}$ be a nonnegative Riemann integrable function. Assume that*

$$\int_R f(\mathbf{x}) \, d\mathbf{x} = 0$$

and prove that the set

$$E := \{\mathbf{x} \in R : \, f(\mathbf{x}) > 0\}$$

has Lebesgue measure zero.

In view of the previous theorem, we have the following.

Corollary 328 *Given a rectangle R, a bounded continuous function $f : R \to \mathbb{R}$ is Riemann integrable.*

Remark 329 *If R is a closed rectangle and $f : R \to \mathbb{R}$ is continuous, then we do not need to assume that f is bounded. Indeed, since R is closed and bounded, it is sequentially compact, and so by the Weierstrass theorem f is bounded.*

Corollary 330 *Given $f : [a, b] \to \mathbb{R}$,*

(i) if f is continuous, then f is Riemann integrable,

(ii) if f is monotone, then f is Riemann integrable.

Proof. If f is continuous, then by the Weierstrass theorem it is bounded, and thus by the previous theorem it is Riemann integrable.

If f is monotone, then it is bounded from below by $\min \{f(a), f(b)\}$ and from above by $\max \{f(a), f(b)\}$. Moreover, by Theorem 205 its set of discontinuity points is at most countable. Since a countable set has Lebesgue measure zero, it follows from the previous theorem that f is Riemann integrable. ■

Example 331 *Some examples of functions that are Riemann integrable and others that are not.*

(i) The function

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

is Riemann integrable over $[0, 1]$, since it is bounded and discontinuous only at $x = 0$.

(ii) The function

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases}$$

is Riemann integrable over $[0, 1]$, since it is bounded and its set of discontinuity points is

$$E = \left\{ \frac{1}{n} \right\}_n \cup \{0\},$$

which is countable.

(iii) The function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise}, \end{cases}$$

is bounded but not Riemann integrable over $[0, 1]$, since it is bounded and its set of discontinuity points is $[0, 1]$.

Exercise 332 *Consider the function $g : [0,1] \to [0,1]$ defined by*

$$g(x) := \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{p} & \text{if } x = \frac{p}{q} \text{ with } p, q \in \mathbb{N} \text{ relatively prime, } 0 < p < q, \\ 1 & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

(i) Prove that g is discontinuous at every rational point of $[0,1]$.

(ii) Prove that g is continuous at every irrational point of $[0,1]$.

(iii) Prove that g is Riemann integrable.

Next we discuss some properties of Riemann integration.

Proposition 333 *Given a rectangle R, let $f, g : R \to \mathbb{R}$ be Riemann integrable.*

(i) If $\lambda \in \mathbb{R}$, then λf is Riemann integrable and

$$\int_R \lambda f(\mathbf{x})\, d\mathbf{x} = \lambda \int_R f(\mathbf{x})\, d\mathbf{x}. \tag{46}$$

(ii) The functions $f + g$ and fg are Riemann integrable and

$$\int_R (f(\mathbf{x}) + g(\mathbf{x}))\, d\mathbf{x} = \int_R f(\mathbf{x})\, d\mathbf{x} + \int_R g(\mathbf{x})\, d\mathbf{x}. \tag{47}$$

(iii) If $f \le g$, then

$$\int_R f(\mathbf{x})\, d\mathbf{x} \le \int_R g(\mathbf{x})\, d\mathbf{x}.$$

(iv) The function $|f|$ is Riemann integrable and

$$\left| \int_R f(\mathbf{x})\, d\mathbf{x} \right| \le \int_R |f(\mathbf{x})|\, d\mathbf{x}.$$

Exercise 334 *Give an example of a bounded function $f : R \to \mathbb{R}$ such that $|f|$ is Riemann integrable over R, but f is not.*

Exercise 335 *Prove the previous proposition.*

Exercise 336 *Consider a bounded function $f : [a,b] \to \mathbb{R}$. Prove that for every constant $C \in \mathbb{R}$,*

$$\underline{\int_a^b} (f(x) + C)\, dx = \underline{\int_a^b} f(x)\, dx + C(b-a),$$
$$\overline{\int_a^b} (f(x) + C)\, dx = \overline{\int_a^b} f(x)\, dx + C(b-a).$$

Proposition 337 *Consider a bounded function $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then*

$$\underline{\int_a^b} f(x)\, dx = \underline{\int_a^c} f(x)\, dx + \underline{\int_c^b} f(x)\, dx, \tag{48}$$

$$\overline{\int_a^b} f(x)\, dx = \overline{\int_a^c} f(x)\, dx + \overline{\int_c^b} f(x)\, dx. \tag{49}$$

Proof. To highlight the dependence of the interval I where the lower and upper sums are taken, we write $L(f, P, I)$ and $U(f, P, I)$. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Consider the new partition $P' = P \cup \{c\}$ (note that if c is already in P, then $P' = P$). Let $P_1 := P' \cap [a, c]$ and $P_2 := P' \cap [c, b]$. Then P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$. Hence,

$$\begin{aligned}\underline{\int_a^c} f(x)\, dx + \underline{\int_c^b} f(x)\, dx &\geq L(f, P_1, [a, c]) + L(f, P_2, [c, b]) \\ &= L(f, P', [a, b]) \geq L(f, P, [a, b]),\end{aligned}$$

where in the last inequality we have used (39). Taking the supremum over all partitions P of $[a, b]$, we get

$$\underline{\int_a^c} f(x)\, dx + \underline{\int_c^b} f(x)\, dx \geq \underline{\int_a^b} f(x)\, dx = \sup_{P \text{ partition of } [a,b]} L(f, P, [a, b]). \tag{50}$$

To prove the opposite inequality, fix $\varepsilon > 0$. Using the definition of supremum, we may find a partition P_1^ε of $[a, c]$ and a partition P_2^ε of $[c, b]$ such that

$$\begin{aligned}L(f, P_1^\varepsilon, [a, c]) &\geq \underline{\int_a^c} f(x)\, dx - \varepsilon, \\ L(f, P_2^\varepsilon, [c, b]) &\geq \underline{\int_c^b} f(x)\, dx - \varepsilon.\end{aligned}$$

Then $P^\varepsilon := P_1^\varepsilon \cup P_2^\varepsilon$ is a partition of $[a, b]$ and so P_2 is a partition of $[c, b]$. Hence,

$$\begin{aligned}\underline{\int_a^b} f(x)\, dx \geq L(f, P^\varepsilon, [a, b]) &= L(f, P_1^\varepsilon, [a, c]) + L(f, P_2^\varepsilon, [c, b]) \\ &\geq \underline{\int_a^c} f(x)\, dx + \underline{\int_c^b} f(x)\, dx - 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0^+$ and using also (50), we obtain (48). The proof of (49) is similar and we omit it. ■

The next theorem will be used to prove the fundamental theorem of calculus.

Theorem 338 *Consider a bounded function $f : [a, b] \to \mathbb{R}$ and let*

$$F(x) := \begin{cases} \underline{\int_a^x} f(y)\, dy & \text{if } a < x \le b, \\ 0 & \text{if } x = a, \end{cases} \qquad G(x) := \begin{cases} \overline{\int_a^x} f(y)\, dy & \text{if } a < x \le b, \\ 0 & \text{if } x = a. \end{cases}$$

Then F and G are Lipschitz continuous and

$$F'(x_0) = G'(x_0) = f(x_0)$$

at every point $x_0 \in [a, b]$ at which f is continuous.

Proof. Step 1: We only study the function F. To prove that F is Lipschitz continuous, let $M > 0$ be such that $|f(x)| \le M$ for all $x \in [a, b]$. Fix $x, y \in [a, b]$. Without loss of generality, we may assume $x < y$. By (48) with x in place of c and $[a, y]$ in place of $[a, b]$, we have

$$F(y) = \underline{\int_a^x} f(y)\, dy + \underline{\int_x^y} f(y)\, dy = F(x) + \underline{\int_x^y} f(y)\, dy.$$

Hence,

$$|F(y) - F(x)| \le \left| \underline{\int_x^y} f(y)\, dy \right|.$$

Using (38), we have that

$$-M(y - x) \le (y - x) \inf_{[x,y]} f \le \underline{\int_x^y} f(y)\, dy \le (y - x) \sup_{[x,y]} f \le M(y - x).$$

Hence,

$$|F(y) - F(x)| \le \left| \underline{\int_x^y} f(y)\, dy \right| \le M(y - x),$$

which shows that F is Lipschitz. ■

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Proof. Step 2: Assume that f is continuous at $x_0 \in [a, b]$. We consider the case $x_0 \in (a, b)$ (the cases $x_0 = a$ and $x_0 = b$) are simpler. We want to prove that

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

For $h \ne 0$ consider the different quotient

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \begin{cases} \frac{1}{x - x_0} \underline{\int_{x_0}^x} (f(y) - f(x_0))\, dy & \text{if } x > x_0, \\ -\frac{1}{x - x_0} \underline{\int_x^{x_0}} (f(y) - f(x_0))\, dy & \text{if } x < x_0, \end{cases}$$

where we have used (48) and Exercise 336. Since f is continuous at x_0, given $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$|f(x) - f(x_0)| \le \varepsilon$$

for all $x \in [a, b]$ with $|x - x_0| \le \delta$. Take $|x - x_0| \le \delta$. Then for $x > x_0$ (the case $x < x_0$ is similar), by (38), we have

$$\begin{aligned}
-\varepsilon &\le \frac{x - x_0}{x - x_0} \inf_{y \in [x_0, x]} (f(y) - f(x_0)) \le \frac{1}{x - x_0} \underline{\int_{x_0}^{x}} (f(y) - f(x_0))\, dy \\
&\le \frac{x - x_0}{x - x_0} \sup_{y \in [x_0, x]} (f(y) - f(x_0)) \le \varepsilon,
\end{aligned}$$

which shows that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \le \varepsilon.$$

This concludes the proof. ■

As a corollary of the previous theorem, we have the mean value theorem for integrals.

Corollary 339 (Mean Value Theorem for Integrals) *Let $f : [a, b] \to \mathbb{R}$ be bounded. Assume that f is continuous in (a, b). Then there exists $c \in (a, b)$ such that*

$$\frac{1}{b - a} \underline{\int_{a}^{b}} f(x)\, dx = f(c).$$

A similar result holds for the upper integral.

Proof. Consider the function F defined in the previous theorem. Since f is continuous in (a, b), by the previous theorem, F is differentiable in (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$. Moreover, F is Lipschitz continuous in $[a, b]$ and so it is continuous in $[a, b]$. By the mean value theorem applied to the function F there exists $c \in (a, b)$ such that

$$\underline{\int_{a}^{b}} f(x)\, dx - 0 = F(b) - F(a) = F'(c)(b - a) = f(c)(b - a).$$

■

Exercise 340 *Give an example of a bounded function $f : [a, b] \to \mathbb{R}$ such that $|f|$ is Riemann integrable over $[a, b]$, but f is not.*

Remark 341 *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, we define*

$$\int_{b}^{a} f(x)\, dx := -\int_{a}^{b} f(x)\, dx. \tag{51}$$

Then property (v) should be replaced by

$$\left| \int_{b}^{a} f(x)\, dx \right| \le \left| \int_{b}^{a} |f(x)|\, dx \right|.$$

Corollary 342 *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable then for every $\varepsilon > 0$ there exists a partition P^ε of $[a, b]$ such that for every partition $P = \{x_0, \ldots, x_n\}$ that contains P^ε and for every $y_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$,*

$$\left| \int_a^b f(x)\, dx - \sum_{i=1}^{n} (x_i - x_{i-1}) f(y_i) \right| \leq \varepsilon. \tag{52}$$

Proof. Let P^ε be as in Theorem 319 and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ that contains P^ε. Then by Proposition 333, for every $y_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$,

$$\begin{aligned}\left| \int_a^b f(x)\, dx - \sum_{i=1}^{n} (x_i - x_{i-1}) f(y_i) \right| &= \left| \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\, dx - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(y_i)\, dx \right| \\ &= \left| \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (f(x) - f(y_i))\, dx \right| \leq \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(x) - f(y_i)\, dx \right| \\ &\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(x) - f(y_i)|\, dx.\end{aligned}$$

For every $x \in [x_{i-1}, x_i]$, we have that

$$|f(x) - f(y_i)| \leq \sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f,$$

and so the right-hand side of the previous inequality can be bound from above by

$$\begin{aligned}\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) dx &= \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f - \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f \\ &= U(f, P^\varepsilon) - L(f, P^\varepsilon) \leq \varepsilon.\end{aligned}$$

This concludes the proof. ■

Exercise 343 *Prove that the converse of the previous corollary holds, namely, prove that if there is a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a partition P^ε of $[a, b]$ such that for every partition $P = \{x_0, \ldots, x_n\}$ that contains P^ε and for every $y_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$,*

$$\left| L - \sum_{i=1}^{n} (x_i - x_{i-1}) f(y_i) \right| \leq \varepsilon,$$

then f is Riemann integrable and $L = \int_a^b f(x)\, dx$.

Next we discuss the composition of Riemann integrable functions. The next example shows that the composition of Riemann integrable functions may not be Riemann integrable.

Example 344 *Let g be the function defined in Exercise 332 and let $f : [0,1] \to [0,1]$ be the function*

$$f(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then f and g are both Riemann integrable, but their composition $f \circ g$ is not. Indeed, of $x \in [0,1]$

$$f(g(x)) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational,} \end{cases}$$

and we have seen that this function is not Riemann integrable.

The next proposition shows that if f is continuous, then the composition is Riemann integrable.

Proposition 345 *Let $g : [a,b] \to \mathbb{R}$ be Riemann integrable, let $f : [c,d] \to \mathbb{R}$ be continuous, and let $g([a,b]) \subseteq [c,d]$. Then $f \circ g : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof. We begin by observing that if g is continuous at some point $x \in [a,b]$, then since f is continuous, the function f will be continuous at $g(x)$ and so $f \circ g$ will be continuous at x. This shows that set of discontinuities points of $f \circ g$ is contained in the set of discontinuities points of g, and since g is Riemann integrable, this set has Lebesgue measure zero. In turn, $f \circ g$ is Riemann integrable. ■

If g is continuous and f is Riemann integrable, then $f \circ g$ may not be Riemann integrable.

23 Integration and Differentiation

You hear often that integration and differentiation are one the inverse operation of the other. Is this true? Let's see what happens to a function if we first integrate it and then differentiate. Consider a Riemann integrable function $f : [a,b] \to \mathbb{R}$ and define

$$F(x) := \int_a^x f(y)\, dy$$

for $a < x \leq b$ and $F(a) := 0$. If we now differentiate F, do we recover f? In general no.

Example 346 *Let $E = \{x_1, \ldots, x_n\} \subseteq [0,1]$ and consider the function*

$$f(x) := \begin{cases} 1 & \text{if } x = x_i \text{ for } i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

The function f is Riemann integrable, since its set of discontinuity points is E, which is finite. Define

$$F(x) := \int_0^x f(y)\, dy = 0.$$

Since $F = 0$, when we differentiate we get $F' = 0$. Hence, we lost f.

What remains true is that $F' = f$ at points in which f is continuous.

Theorem 347 *Consider a Riemann integrable function $f : [a, b] \to \mathbb{R}$ and let*

$$F(x) := \begin{cases} \int_a^x f(y)\, dy & \text{if } a < x \leq b, \\ 0 & \text{if } x = a, \end{cases}$$

Then F is well-defined, Lipschitz continuous and

$$F'(x_0) = f(x_0)$$

at every point $x_0 \in [a, b]$ at which f is continuous. In particular, $F'(x_0) = f(x_0)$ for all $x_0 \in [a, b]$ except at most a set of Lebesgue measure zero.

Proof. By Proposition 333(i), f is Riemann integrable over $[a, x]$ for every $x \in (a, b]$. Hence, F is well-defined. The result now follows from Theorem 338. ■

As a corollary of the previous theorem, we have the mean value theorem for integrals.

Corollary 348 (Mean Value Theorem for Integrals) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists $c \in (a, b)$ such that*

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(c).$$

Proof. This follows from Corollary 339. ■

Now let's do the opposite of what we did at the beginning of the section, that is, let's differentiate first and then integrate. Let $F : [a, b] \to \mathbb{R}$ be a differentiable function. If we consider F' and integrate it, do we recover F? Again the answer is no, in general.

Example 349 *Consider the function*

$$F(x) := \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function F is differentiable. The only problem could be at $x = 0$, but

$$\frac{F(x) - F(0)}{x - 0} = \frac{x^2 \sin \frac{1}{x^2} - 0}{x - 0} = x \sin \frac{1}{x^2} \to 0$$

as $x \to 0$, since $0 \leq \left| x \sin \frac{1}{x^2} \right| \leq |x| \to 0$. Hence,

$$F'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

But we cannot integrate F', since it is not bounded.

Remark 350 *There are examples of differentiable functions $F : \mathbb{R} \to \mathbb{R}$ such that F' is bounded, but F' is not Riemann integrable over any interval. They are difficult.*

What remains true is that if F' is Riemann integrable, then it is possible to integrate it to find F.

Theorem 351 (Fundamental Theorem of Calculus) *Let $F : [a,b] \to \mathbb{R}$ be a differentiable function. Assume that F' is Riemann integrable over $[a,b]$. Then*

$$F(b) - F(a) = \int_a^b F'(x)\, dx. \tag{53}$$

Proof. Fix $\varepsilon > 0$ and apply Theorem 319 to find a partition $P^\varepsilon = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that for *every* $y_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$,

$$\left| \int_a^b F'(x)\, dx - \sum_{i=1}^n (x_i - x_{i-1}) F'(y_i) \right| \leq \varepsilon. \tag{54}$$

By the mean value theorem, for every $i = 1, \ldots, n$, there exists $y_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1}) F'(y_i).$$

Hence,

$$F(b) - F(a) = F(x_n) - F(x_0) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = \sum_{i=1}^n (x_i - x_{i-1}) F'(y_i).$$

By substituting this expression in (54), we find that

$$\left| \int_a^b F'(x)\, dx - (F(b) - F(a)) \right| \leq \varepsilon.$$

By letting $\varepsilon \to 0^+$, we get (53). ■

As a corollary of the fundamental theorem of calculus, we have the formula for integration by parts.

Corollary 352 (Integration by Parts) *Let $F, G : [a,b] \to \mathbb{R}$ be a differentiable functions. Assume that F' and G' are Riemann integrable over $[a,b]$. Then*

$$\int_a^b F(x) G'(x)\, dx = F(b) G(b) - F(a) G(a) = \int_a^b F'(x) G(x)\, dx.$$

Proof. G and F are differentiable, so they are continuous. Hence, they are Riemann integrable. By Proposition 333, the functions FG' and $F'G$ are also

Riemann integrable. Consider the function $H = FG$. Then $H' = F'G + FG'$. By the fundamental theorem of calculus applied to H, we have

$$\begin{aligned} F(b)G(b) - F(a)G(a) = H(b) - H(a) &= \int_a^b H'(x)\ dx \\ &= \int_a^b (F'(x)G(x) + F(x)G'(x))\ dx, \end{aligned}$$

which concludes the proof. ■

Example 353 *Let's calculate*

$$\int_1^2 x \log^2 x\, dx.$$

Take $F(x) = \log^2 x$ *and* $G'(x) = x$*. Then* $F'(x) = \frac{2}{x}\log x$ *while* $G(x) = \frac{x^2}{2}$*. Hence,*

$$\begin{aligned} \int_1^2 x \log^2 x\, dx &= \left[\frac{x^2}{2}\log^2 x\right]_{x=1}^{x=2} - \int_1^2 \frac{x^2}{2}\frac{2}{x}\log x\, dx \\ &= 2\log^2 2 - 0 - \int_1^2 x \log x\, dx. \end{aligned}$$

We integrate by parts once more, taking $F(x) = \log x$ *and* $G'(x) = x$*. Then* $F'(x) = \frac{1}{x}$ *while* $G(x) = \frac{x^2}{2}$*. Hence,*

$$\begin{aligned} \int_1^2 x \log x\, dx &= \left[\frac{x^2}{2}\log x\right]_{x=1}^{x=2} - \int_1^2 \frac{x^2}{2}\frac{1}{x}\, dx \\ &= 2\log 2 - 0 - \frac{1}{2}\int_1^2 x\, dx = 2\log 2 - \frac{3}{4}. \end{aligned}$$

In conclusion,

$$\int_1^2 x \log^2 x\, dx = 2\log^2 2 - 2\log 2 + \frac{3}{4}.$$

Thursday, December 08, 2011

Make-up class.

Next we discuss integration by substitution. The classical formula that you see in calculus is given by

$$\int_a^b f(G(x))\, G'(x)\ dx = \int_{G(a)}^{G(b)} f(y)\ dy.$$

Theorem 354 (Change of Variables, I) *Let* $G : [a, b] \to \mathbb{R}$ *be differentiable with continuous derivative and let* $f : G([a, b]) \to \mathbb{R}$ *be continuous. Then*

$(f \circ G)\, g$ is Riemann integrable over $[a,b]$ and the following change of variables formula holds

$$\int_a^b f(G(x))\, G'(x)\, dx = \int_{G(a)}^{G(b)} f(y)\, dy,$$

where, if $G(a) \geq G(b)$, we are using the notation (51).

Proof. We will give the proof in the very special case in which both f and g are continuous.
Case 1: Assume first that $G(a) \leq G(b)$ and define the function

$$F(t) := \begin{cases} \int_{G(a)}^{t} f(y)\, dy & \text{if } G(a) < t \leq G(a)\,, \\ 0 & \text{if } t = G(a)\,. \end{cases}$$

Since f and g are continuous, by Theorem **??**, F and G are differentiable, with $F' = f$ and $G' = g$. Hence, by Theorem 276, the composite function $H := F \circ G$ is differentiable, with

$$H'(x) = F'(G(x))\, G'(x) = f(G(x))\, g(x)$$

for all $x \in [a,b]$. By the fundamental theorem of calculus applied to the function $H := F \circ G$, we have that

$$\begin{aligned} \int_{G(a)}^{G(b)} f(y)\, dy - 0 = H(b) - H(a) &= \int_a^b H'(x)\, dx \\ &= \int_a^b f(G(x))\, g(x)\, dx. \end{aligned}$$

Case 2: If $G(a) > G(b)$, we define the function

$$F(t) := \begin{cases} \int_{t}^{G(a)} f(y)\, dy & \text{if } G(b) \leq t < G(a)\,, \\ 0 & \text{if } t = G(a)\,. \end{cases}$$

By Exercise **??**, $F' = -f$. Hence, by Theorem 276, the composite function $H := F \circ G$ is differentiable, with

$$H'(x) = F'(G(x))\, G'(x) = -f(G(x))\, g(x)$$

for all $x \in [a,b]$. By the fundamental theorem of calculus applied to the function $H := F \circ G$, we have that

$$\begin{aligned} \int_{G(b)}^{G(a)} f(y)\, dy - 0 = H(b) - H(a) &= \int_a^b H'(x)\, dx \\ &= -\int_a^b f(G(x))\, g(x)\, dx. \end{aligned}$$

Using (51), we obtain

$$\int_{G(a)}^{G(b)} f(y)\ dy = -\int_{G(b)}^{G(a)} f(y)\ dy = \int_a^b f(G(x))\, g(x)\ dx.$$

■

Note that although the function $(f \circ G)\, g$ is Riemann integrable, the function $f \circ G$ need not be.

Example 355 *Since the set E of rational numbers in $[0,1]$ is countable, we can write $E = \{x_n\}_n$. For each n consider the interval $I_n := \left(x_n - \frac{\varepsilon}{2^n}, x_n + \frac{\varepsilon}{2^n}\right)$ and let*

$$U := \bigcup_{n=1}^{\infty} I_n.$$

Note that the total length of each interval is $\frac{2\varepsilon}{2^n}$, and so the

$$\sum_{n=1}^{\infty} \operatorname{meas} I_n = 2\varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = 2\varepsilon < 1$$

for $\varepsilon < \frac{1}{2}$. Hence, the set U cannot cover the whole interval $[0,1]$. Write

$$U := \bigcup_{m=1}^{\infty} (a_m, b_m),$$

where the intervals (a_m, b_m) are disjoint. For each m consider the function

$$g_m(x) := \frac{1}{2^m} \sin\left(2\pi \frac{x - a_m}{b - a_m}\right)$$

if $x \in (a_m, b_m)$ and $g_m(x) = 0$ otherwise. Define

$$g(x) := \sum_{m=1}^{\infty} g_m(x).$$

Then g is continuous. Set

$$G(x) := \int_0^x g(t)\ dt, \quad 0 \le x \le 1.$$

It can be shown that

$$G(x) = \sum_{m=1}^{\infty} \int_0^x g_m(t)\ dt.$$

Moreover, $\int_0^x g_m(t)\ dt > 0$ if $x \in (a_m, b_m)$ and is zero otherwise. Thus, $G(x) = 0$ if and only if $x \in [0,1] \setminus U$. Define

$$f(x) := \begin{cases} 1 & \textit{if } \frac{1}{x} \textit{ is an integer,} \\ 0 & \textit{otherwise.} \end{cases}$$

Then f is Riemann integrable over every interval. Let's prove that $f \circ G$ is not Riemann integrable. Consider $x \in (0,1) \backslash U$. Then $G(x) = 0$ and so $f(G(x)) = f(0) = 0$. On the other hand, if $h > 0$ in the interval $(x, x+h) \cap [0,1]$ there exists rational numbers, and so there is $y \in U$, so that $G(y) > 0$. Since $G(x) = 0$ and $G(y) > 0$ and G is continuous, there exists $z \in (0, y)$ such that $G(z) = \frac{1}{n}$ for some $n \in \mathbb{N}$. Hence, $f(G(z)) = 1$. Since we can take h arbitrarily small, we have proved that $f \circ G$ is discontinuous at every $x \in (0,1) \setminus U$. But this set does not have Lebesgue measure zero, and so $f \circ G$ cannot be Riemann integrable.

Friday, December 09, 2011

We will prove a change of variables formula that requires weaker hypotheses.

Theorem 356 (Change of Variables, II) *Let $g : [a,b] \to \mathbb{R}$ be Riemann integrable and let*

$$G(x) := G(a) + \int_a^x g(t)\ dt, \quad x \in [a,b].$$

Assume that $f : G([a,b]) \to \mathbb{R}$ is Riemann integrable. Then $(f \circ G) g$ is Riemann integrable over $[a,b]$ and the following change of variables formula holds

$$\int_a^b f(G(x))\, g(x)\ dx = \int_{G(a)}^{G(b)} f(y)\ dy,$$

where, if $G(a) \geq G(b)$, we are using the notation (51) *and* (**??**).

Note that we are not assuming that G is differentiable in $[a,b]$.

Proof. Let $M > 0$ be such that $|f(y)| \leq M$ for all $y \in G([a,b])$ and $|g(x)| \leq M$ for all $x \in [a,b]$. Since g is integrable, given $\varepsilon > 0$ there exists a partition $\mathcal{P} = \{I_1, \ldots, I_n\}$ of $[a,b]$, such that the total length of the intervals I_i such that $\operatorname{osc}_{I_i} g > \varepsilon$ is less than ε. We call these intervals, intervals of type 1. In the remaining intervals we have that $\operatorname{osc}_{I_i} g \leq \varepsilon$. We divide them in two types:

The interval I_i is type 2 if there is $x \in I_i$ such that $|g(x)| \leq \varepsilon$. In this case, we have that $|g| \leq 2\varepsilon$ in I_i. On the the hand, the interval I_i is type 3 if $|g(x)| > \varepsilon$ for all $x \in I_i$. Since $\operatorname{osc}_{I_i} g \leq \varepsilon$, we must have that either $g(x) > \varepsilon$ for all $x \in I_i$ or $g(x) < -\varepsilon$ for all $x \in I_i$. Fix one of the intervals $I_i = [y_{i-i}, y_i]$ of type 3 and assume that $g(x) > \varepsilon$ for all $x \in I_i$ (the other case is analogous). Note that if $x, y \in I_i$ with $x < y$, then

$$G(y) - G(x) = \int_x^y g(t)\ dt > \varepsilon (y - x) > 0. \tag{55}$$

Hence, G is strictly increasing in I_i. Since f is integrable over $G(I_i)$, there exists a partition $\mathcal{Q} = \{J_1, \ldots, J_m\}$ of $G(I_i)$, such that the total length of the intervals J_l such that $\operatorname{osc}_{J_l} f > \varepsilon$ is less than $\frac{\varepsilon^2}{N}$, where N is the number of intervals of type 3. Since G is strictly increasing in I_i, the intervals of the partition $\mathcal{Q} = \{J_1, \ldots, J_m\}$ of $G(I_i)$ are of the form $\{G(T_1), \ldots, G(T_m)\}$, where $\{T_1, \ldots, T_m\}$ is a partition of I_i. Moreover, $\operatorname{osc}_{J_l} f = \operatorname{osc}_{T_l} f \circ G$.

We divide the intervals T_l in two types:

The interval T_l is type 3.1 if $\operatorname{osc}_{T_l} f \circ G = \operatorname{osc}_{J_l} f > \varepsilon$. Note that by (55),

$$\sum_{T_l \text{ of type } 3.1} \operatorname{length} T_l < \frac{1}{\varepsilon} \sum_{T_l \text{ of type } 3.1} \operatorname{length} G(T_l) < \frac{1}{\varepsilon} \frac{\varepsilon^2}{N} = \frac{\varepsilon}{N}.$$

The interval T_l is type 3.2 if $\operatorname{osc}_{T_l} f \circ G = \operatorname{osc}_{J_l} f \leq \varepsilon$.

Subdivide each interval of type 3 in the above way. We obtain a partition of $[a, b]$ into intervals of type 1, 2, 3.1, and 3.2. Note that if we consider a refinement, then the intervals do not change type. Consider a refinement, say,

$$a = t_0 < t_1 < \cdots < t_\ell = b.$$

Note that the total length of the intervals of type 1 and 3.1 is less than $\varepsilon + N \frac{\varepsilon}{N} = 2\varepsilon$. Then

$$\begin{aligned} \int_{G(a)}^{G(b)} f(y)\, dy &= \sum_{i=1}^{\ell} \int_{G(t_{i-1})}^{G(t_i)} f(y)\, dy \\ &= \sum_{i=1}^{\ell} \left(G(t_i) - G(t_{i-1})\right) \lambda_i \end{aligned}$$

where $\lambda_i = 0$ if $G(t_i) = G(t_{i-1})$, and

$$\lambda_i := \frac{1}{G(t_i) - G(t_{i-1})} \int_{G(t_{i-1})}^{G(t_i)} f(y)\, dy.$$

Note that

$$\inf_{G([t_{i-1}, t_i])} f \leq \lambda_i \leq \sup_{G([t_{i-1}, t_i])} f, \tag{56}$$

or equivalently,

$$\inf_{[t_{i-1}, t_i]} f \circ G \leq \lambda_i \leq \sup_{[t_{i-1}, t_i]} f \circ G.$$

Similarly,

$$\left(G(t_i) - G(t_{i-1})\right) = \frac{\left(G(t_i) - G(t_{i-1})\right)}{t_i - t_{i-1}} (t_i - t_{i-1}) = \mu_i (t_i - t_{i-1}),$$

where

$$\inf_{[t_{i-1}, t_i]} g \leq \mu_i \leq \sup_{[t_{i-1}, t_i]} g. \tag{57}$$

Hence,

$$\int_{G(a)}^{G(b)} f(y)\, dy = \sum_{i=1}^{\ell} \lambda_i \mu_i (t_i - t_{i-1}),$$

and in turn, for any $t_{i-1} \leq z_i \leq t_i$,

$$\begin{aligned}\left|\int_{G(a)}^{G(b)} f(y)\ dy - \sum_{i=1}^{\ell} f(G(z_i))\, g(z_i)(t_i - t_{i-1})\right| &= \left|\sum_{i=1}^{\ell} [\lambda_i \mu_i - f(G(z_i))\, g(z_i)](t_i - t_{i-1})\right| \\ &\leq \sum_{i=1}^{\ell} |\lambda_i \mu_i - f(G(z_i))\, g(z_i)|(t_i - t_{i-1}).\end{aligned}$$

Now

$$\begin{aligned}\sum_{[t_{i-1},t_i] \text{ of type 1 and 3.1}} |\lambda_i \mu_i - f(G(z_i))\, g(z_i)|(t_i - t_{i-1}) &\leq M^2 \sum_{[t_{i-1},t_i] \text{ of type 1 and 3.1}} (t_i - t_{i-1}) \\ &\leq 2\varepsilon M^2.\end{aligned}$$

If $[t_{i-1}, t_i]$ is of type 2, then $|g| \leq 2\varepsilon$, and so by (57),

$$\begin{aligned}\sum_{[t_{i-1},t_i] \text{ of type 2}} |\lambda_i \mu_i - f(G(z_i))\, g(z_i)|(t_i - t_{i-1}) &\leq M \sum_{[t_{i-1},t_i] \text{ of type 2}} (|\mu_i| + |g(z_i)|)(t_i - t_{i-1}) \\ &\leq 4\varepsilon M(b-a).\end{aligned}$$

Finally, if $[t_{i-1}, t_i]$ is of type 3.2, then $\operatorname{osc}_{[t_{i-1},t_i]} g \leq \varepsilon$ and $\operatorname{osc}_{[t_{i-1},t_i]} f \circ G \leq \varepsilon$, and so by (56) and (57),

$$\begin{aligned}|\lambda_i \mu_i - f(G(z_i))\, g(z_i)| &= |\lambda_i \mu_i + \lambda_i g(z_i) - \lambda_i g(z_i) - f(G(z_i))\, g(z_i)| \\ &\leq |\lambda_i (\mu_i - g(z_i))| + |(\lambda_i - f(G(z_i)))\, g(z_i)| \\ &\leq \varepsilon M + \varepsilon M.\end{aligned}$$

Hence,

$$\sum_{[t_{i-1},t_i] \text{ of type 3.2}} |\lambda_i \mu_i - f(G(z_i))\, g(z_i)|(t_i - t_{i-1}) \leq 2\varepsilon M(b-a).$$

In conclusion,

$$\left|\int_{G(a)}^{G(b)} f(y)\ dy - \sum_{i=1}^{\ell} f(G(z_i))\, g(z_i)(t_i - t_{i-1})\right| \leq 2\varepsilon M^2 + 6\varepsilon M(b-a).$$

It follows that $(f \circ G)\, g$ is Riemann integrable over $[a, b]$ and

$$\int_{G(a)}^{G(b)} f(y)\ dy = \int_a^b f(G(x))\, g(x)\ dx.$$

■