Monday, August 29, 2011

1 Outer Measures

How do we introduce the area of a bounded set $E \subset \mathbb{R}^2$? If the set is nice, say, a rectangle $(a,b) \times (c,d)$, then

$$\operatorname{area}((a,b) \times (c,d)) = (b-a)(d-c).$$

But if E is an arbitrary set? Then the idea is to try to approximate E as closely as possible with unions of sets whose area we know, for example rectangles. So now we have two possibilities. We can define

$$\operatorname{area}_1(E) := \inf \left\{ \sum_{n=1}^{m} \operatorname{area}(R_n) : \, R_n \text{ open rectangle, } E \subset \bigcup_{n=1}^{m} R_n, \, m \in \mathbb{N} \right\}$$

or

$$\operatorname{area}_2(E) := \inf \left\{ \sum_{n=1}^{\infty} \operatorname{area}(R_n) : \, R_n \text{ open rectangle, } E \subset \bigcup_{n=1}^{\infty} R_n \right\}.$$

Does it make a difference? Consider the set $E = (0,1)^2 \cap \mathbb{Q}^2$ of all points in the unit square $(0,1)^2$ whose coordinates are rational. By the density of the rationals if a finite number of rectangles covers E it also covers $(0,1)^2$, so that $\operatorname{area}_1(E) = 1$. On the other hand, if we are allowed countably many rectangles, then $\operatorname{area}_2(E) = 0$. To see this, fix a small number $\varepsilon > 0$. Since the rational numbers are countable, we may write $E = \{(x_n, y_n)\}_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$ consider the rectangle

$$R_n := \left(x_n - \frac{\varepsilon}{4^n}, x_n + \frac{\varepsilon}{4^n} \right) \times \left(y_n - \frac{\varepsilon}{4^n}, y_n + \frac{\varepsilon}{4^n} \right),$$

whose area is $\operatorname{area}(R_n) = \frac{2\varepsilon}{2^n} \frac{2\varepsilon}{2^n} = \frac{\varepsilon^2}{2^{2n-2}}$. Since $E \subset \bigcup\limits_{n=1}^{\infty} R_n$, we have that

$$\operatorname{area}_2(E) \le \sum_{n=1}^{\infty} \operatorname{area}(R_n) \le \sum_{n=1}^{\infty} \frac{\varepsilon^2}{2^{2n-2}} \le \varepsilon^2.$$

Letting $\varepsilon \to 0^+$ proves that $\operatorname{area}_2(E) = 0$.

We will see that the first choice leads to a finitely additive measure, which is the Jordano content. This finitely additive measure is behind the Riemann integration.

The second choice leads to a countably additive measure, which is the Lebesgue measure. This measure will be behind Lebesgue integration. In this course will focus mainly on this second type of measures. The reason is that the set E in our example is a small set, and so it is more natural for it to have area zero.

Now let's try to abstract the procedure we did in the plane to arbitrary spaces. Consider a set X, which plays the role of $\mathbb{R}^2$ in the previous example. We want to measure an arbitrary set $E \subset X$.

Our starting point in the example was a family of "nice" sets that we used to cover every set. So let's take a family $\mathcal{G} \subset \mathcal{P}(X)$. An element of $\mathcal{G}$ will be called an *elementary set*. What are the properties that we need on the family $\mathcal{G}$? We want to be able to cover every set of X. This is possible if we can cover X. Thus, let us assume that there exists $\{X_n\} \subset \mathcal{G}$ with $X = \bigcup_{n=1}^{\infty} X_n$, and let's throw in $\mathcal{G}$ also the empty set.

Then we need a way to measure our elementary sets. So let's consider a function $\rho : \mathcal{G} \to [0, \infty]$ such that $\rho(\emptyset) = 0$. As we did in $\mathbb{R}^2$, for every set $E \subset X$ we can now define

$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : \{E_n\} \subset \mathcal{G},\, E \subset \bigcup_{n=1}^{\infty} E_n \right\}. \tag{1}$$

What are the properties of μ^*?

Wednesday, August 31, 2011

Definition 1 *Let X be a nonempty set. A map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an* outer measure *if*

(i) $\mu^*(\emptyset) = 0$*;*

(ii) $\mu^*(E) \le \mu^*(F)$ *for all* $E \subset F \subset X$*;*

(iii) $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \mu^*(E_n)$ *for every countable collection* $\{E_n\} \subset \mathcal{P}(X)$ *(countable subadditivity).*

Remark 2 *In several books, outer measures are called measures.*

Let's prove that the function μ^* defined in (1) is an outer measure.

Proposition 3 *Let X be a nonempty set and let $\mathcal{G} \subset \mathcal{P}(X)$ be such that $\emptyset \in \mathcal{G}$ and there exists $\{X_n\} \subset \mathcal{G}$ with $X = \bigcup_{n=1}^{\infty} X_n$. Let $\rho : \mathcal{G} \to [0, \infty]$ be such that $\rho(\emptyset) = 0$. Then the map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ defined in (1) is an outer measure.*

Proof. Since $\emptyset \in \mathcal{G}$ we have that $\mu^*(\emptyset) = 0$. If $E \subset F \subset X$ then any sequence $\{E_n\} \subset \mathcal{G}$ admissible for F in (1) is also admissible for E, and so $\mu^*(E) \le \mu^*(F)$. Finally, let $\{F_k\} \subset \mathcal{P}(X)$. Fix $\varepsilon > 0$ and for each k find a sequence $\left\{E_n^{(k)}\right\} \subset \mathcal{G}$ admissible for F_k in (1) and such that

$$\sum_{n=1}^{\infty} \rho\left(E_n^{(k)}\right) \le \mu^*(F_k) + \frac{\varepsilon}{2^k}.$$

Since $\mathbb{N}\times\mathbb{N}$ is countable, we may write $\left\{E_n^{(k)}\right\}_{k,n\in\mathbb{N}} = \{R_j\}_{j\in\mathbb{N}}$. Note that

$$\bigcup_{k=1}^{\infty} F_k \subset \bigcup_{j=1}^{\infty} R_j = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} E_n^{(k)},$$

and so (see Exercise 5 below)

$$\mu^*\left(\bigcup_{k=1}^{\infty} F_k\right) \le \sum_{j=1}^{\infty}\rho(R_j) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\rho\left(E_n^{(k)}\right) \le \sum_{k=1}^{\infty}\mu^*(F_k) + \varepsilon.$$

By letting $\varepsilon \to 0^+$ we conclude the proof. ■

Remark 4 *Note that if $E \in \mathcal{G}$, then taking $E_1 := E$, $E_n := \emptyset$ for all $n \ge 2$, it follows from the definition of μ^* that $\mu^*(E) \le \rho(E)$, with the strict inequality possible. Indeed, $\mu^* = \rho$ on $\mathcal{G}$ if and only if ρ is countably subadditive, that is,*

$$\rho(E) \le \sum_{n=1}^{\infty}\rho(E_n)$$

for all $E \subset \bigcup_{n=1}^{\infty} E_n$ with $E \in \mathcal{G}$, $\{E_n\} \subset \mathcal{G}$.

Exercise 5 *Double series.*

(i) Let $a_{n,k} \ge 0$, for $k,n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{n,k}.$$

(ii) Let $a_{nk} \ge 0$, for $k,n \in \mathbb{N}$ and define $c_m := \sum_{n+k=m+1} a_{n,k} = a_{1,m} + \cdots + a_{m,1}$. Prove that

$$\sum_{m=1}^{\infty} c_m = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{n,k}.$$

(iii) Let

$$a_{nk} := \begin{cases} 1 & \text{if } k = n, \\ -1 & \text{if } k = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{nk} \ne \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{nk}.$$

Definition 6 *Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be an outer measure. A set $E \subset X$ has σ-finite μ^** outer measure *if it can be written as a countable union of sets of finite outer measure; μ^* is said to be σ-*finite *if X has σ-finite μ^* outer measure; μ^* is said to be* finite *if $\mu^*(X) < \infty$.*

Let's see some examples. The most important example is given by the Lebesgue outer measure.

Example 7 (The Lebesgue Outer Measure) *In the Euclidean space* $\mathbb{R}^N$ *consider the family of elementary sets*

$$\mathcal{G} := \left\{ Q(x, r) : \ x \in \mathbb{R}^N, \ 0 \le r < \infty \right\}$$

and define $\rho(Q(x,r)) := r^N$, *where*

$$Q(x, r) := x + \left(-\frac{r}{2}, \frac{r}{2} \right)^N.$$

For each set $E \subset \mathbb{R}^N$ *define*

$$\mathcal{L}_o^N(E) := \inf \left\{ \sum_{n=1}^{\infty} (r_n)^N : \{Q(x_n, r_n)\} \subset \mathcal{G}, \ E \subset \bigcup_{n=1}^{\infty} Q(x_n, r_n) \right\}.$$

By Proposition 3, $\mathcal{L}_o^N$ *is an outer measure, called the* N*–dimensional* Lebesgue outer measure*. Using Remark 4 it can be shown that*

$$\mathcal{L}_o^N(Q(x,r)) = \rho(Q(x,r)) = r^N \tag{2}$$

and that $\mathcal{L}_o^N$ *is translation-invariant, i.e.,*

$$\mathcal{L}_o^N(x + E) = \mathcal{L}_o^N(E) \tag{3}$$

for all $x \in \mathbb{R}^N$ *and all* $E \subset \mathbb{R}^N$*. Moreover, for every* $\lambda > 0$,

$$\mathcal{L}_o^N(\lambda E) = \lambda^N \mathcal{L}_o^N(E). \tag{4}$$

Since $\mathcal{L}_o^N\left(\mathbb{R}^N\right) \ge \mathcal{L}_o^N(Q(x,r)) = r^N$, *by sending* $r \to \infty$, *we conclude that* $\mathcal{L}_o^N$ *is not a finite outer measure. However, it is* σ*-finite, since*

$$\mathbb{R}^N = \bigcup_{n=1}^{\infty} Q(0, n)$$

and $\mathcal{L}_o^N(Q(0,n)) = n^N < \infty$.

Friday, September 02, 2011

Another important example is given by the Hausdorff outer measure in $\mathbb{R}^N$. Loosely speaking the Hausdorff outer measure is a measure that is adapted to measure sets of lower dimensions in $\mathbb{R}^N$, say a curve in the plane or a surface in $\mathbb{R}^3$. It is also used to measure fractals.

Example 8 (The Hausdorff Outer Measure) *For* $0 \le s < \infty$ *define*

$$\alpha_s := \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2} + 1\right)},$$

where $\Gamma(t)$ *is the Euler Gamma function*

$$\Gamma(t) := \int_0^\infty e^{-x} x^{t-1} dx, \quad 0 < t < \infty.$$

Note that

$$\Gamma(n) = (n-1)!$$

for all $n \in \mathbb{N}$.

We will see later on that when $N \in \mathbb{N}$, *then* α_N *is the Lebesgue outer measure of the unit ball in* $\mathbb{R}^N$, *so that* $\mathcal{L}^N(B(x,r)) = \alpha_N r^N$ *for every ball* $B(x,r) \subset \mathbb{R}^N$.

For $0 < \delta \leq \infty$ *consider the family of elementary sets*

$$\mathcal{G}_\delta := \left\{F \subset \mathbb{R}^N : \mathrm{diam} F < \delta\right\}$$

and for every $F \in \mathcal{G}_\delta$ *define the elementary measure*

$$\rho_s(F) := \alpha_s \left(\frac{\mathrm{diam}\, F}{2}\right)^s.$$

For each set $E \subset \mathbb{R}^N$ *we define*

$$\mathcal{H}_\delta^s(E) := \inf\left\{\sum_{n=1}^\infty \alpha_s \left(\frac{\mathrm{diam} E_n}{2}\right)^s : E \subset \bigcup_{n=1}^\infty E_n, \, \mathrm{diam} E_n < \delta\right\}, \tag{5}$$

where, when $s = 0$, *we only sum only over those* $E_n \neq \emptyset$.

By Proposition 3 $\mathcal{H}_\delta^s$ *is an outer measure. We define next the Hausdorff outer measure. Since for each set* $E \subset \mathbb{R}^N$ *the function* $\delta \mapsto \mathcal{H}_\delta^s(E)$ *is decreasing, there exists*

$$\mathcal{H}_o^s(E) := \lim_{\delta \to 0^+} \mathcal{H}_\delta^s(E) = \sup_{\delta > 0} \mathcal{H}_\delta^s(E). \tag{6}$$

$\mathcal{H}^s$ *is called the* s-dimensional Hausdorff outer measure *of* E.

Let's prove that $\mathcal{H}_o^s$ is actually an outer measure.

Proposition 9 *Let* $0 \leq s < \infty$. *Then* $\mathcal{H}_o^s$ *is an outer measure.*

Proof. We use the fact that $\mathcal{H}_\delta^s$ is an outer measure. Since $\mathcal{H}_\delta^s(\emptyset) = 0$ for every $\delta > 0$, letting $\delta \to 0^+$ gives $\mathcal{H}_0^s(\emptyset) = 0$.

If $E \subset F$, then $\mathcal{H}_\delta^s(E) \leq \mathcal{H}_\delta^s(F)$, and so letting $\delta \to 0^+$ gives $\mathcal{H}_o^s(E) \leq \mathcal{H}_o^s(F)$.

To prove countable subadditivity, let $\{E_n\} \subset \mathbb{R}^N$. Since $\mathcal{H}_\delta^s$ is an outer measure, we have that

$$\mathcal{H}_\delta^s\left(\bigcup_{n=1}^\infty E_n\right) \leq \sum_{n=1}^\infty \mathcal{H}_\delta^s(E_n) \leq \sum_{n=1}^\infty \mathcal{H}_o^s(E_n),$$

where in the last inequality we have used (6). Letting $\delta \to 0^+$ and using (6) once more gives the desired inequality. ■

Exercise 10 *Prove that in the definition (6) it is possible to restrict the class of admissible sets in the covers $\{E_n\}$ to closed and convex sets (open and convex, respectively), and that the condition* $\operatorname{diam} E_n < \delta$ *can be replaced by* $\operatorname{diam} E_n \leq \delta$, *without changing the value of* $\mathcal{H}_o^s(E)$.

The outer measure $\mathcal{H}_o^s$ satisfies the following properties.

Proposition 11 *Let* $0 \leq s < \infty$. *Then*

(i) $\mathcal{H}_o^0$ *is the counting measure;*

(ii) $\mathcal{H}_o^s \equiv 0$ *if* $s > N$;

(iii) for all $E \subset \mathbb{R}^N$, $x \in \mathbb{R}^N$ *and* $\lambda > 0$ *we have*

$$\mathcal{H}_o^s(x+E) = \mathcal{H}_o^s(E), \quad \mathcal{H}_o^s(\lambda E) = \lambda^s \mathcal{H}_o^s(E).$$

(iv) $\mathcal{H}_o^N = \mathcal{L}_o^N$.

Proof. (ii) Subdivide the unit cube $Q := \left[-\frac{1}{2}, \frac{1}{2}\right]^N$ into m^N cubes of side–length $\frac{1}{m}$ and diameter $\frac{\sqrt{N}}{m}$. Let $\delta := \frac{\sqrt{N}}{m}$. Then

$$\mathcal{H}^s_{\frac{\sqrt{N}}{m}}(Q) \leq \sum_{n=1}^{m^N} \alpha_s \left(\frac{\sqrt{N}}{2m}\right)^s = \alpha_s \left(\frac{\sqrt{N}}{2}\right)^s \frac{1}{m^{s-N}} \to 0$$

as $m \to \infty$, and so $\mathcal{H}_o^s(Q) = 0$ in view of (6). Since $\mathbb{R}^N$ can be written as a countable union of unit cubes, we have that $\mathcal{H}_o^s\left(\mathbb{R}^N\right) = 0$, and so $\mathcal{H}_o^s \equiv 0$. ■

Exercise 12 *Prove properties (i) and (iii) in the previous proposition.*

We will prove property (iv) later in the semester.

Monday, September 5, 2011

Labor day, no classes.

Wednesday, September 7, 2011

Proposition 13 *Let* $E \subset \mathbb{R}^N$ *and let* $0 \leq s < t < \infty$.

(i) If $\mathcal{H}_\delta^s(E) = 0$ *for some* $0 < \delta \leq \infty$, *then* $\mathcal{H}_o^s(E) = 0$.

(ii) If $\mathcal{H}_o^s(E) < \infty$ *then* $\mathcal{H}_o^t(E) = 0$.

(iii) If $\mathcal{H}_o^t(E) > 0$ *then* $\mathcal{H}_o^s(E) = \infty$.

Proof. (i) For $s = 0$ there is nothing to prove, so assume $s > 0$. Since $\mathcal{H}_\delta^s(E) = 0$ for any $\varepsilon > 0$ we may find a sequence $\{E_n\} \subset \mathbb{R}^N$ such that $E \subset \bigcup_{n=1}^\infty E_n$, $\operatorname{diam} E_n < \delta$ for all $n \in \mathbb{N}$, and

$$\sum_{n=1}^\infty \alpha_s \left(\frac{\operatorname{diam} E_n}{2}\right)^s \leq \varepsilon.$$

It follows that $\operatorname{diam} E_n \le 2\left(\frac{\varepsilon}{\alpha_s}\right)^{1/s} =: \delta_\varepsilon$, and so $\mathcal{H}^s_{\delta_\varepsilon}(E) \le \varepsilon$. Since $\delta_\varepsilon \to 0$ as $\varepsilon \to 0^+$ it follows from (6) that $\mathcal{H}^s_o(E) = 0$.

(ii) Assume that $\mathcal{H}^s_o(E) < \infty$, fix $\delta > 0$, and find a sequence $\{E_n\} \subset \mathbb{R}^N$ such that $E \subset \bigcup_{n=1}^\infty E_n$, $\operatorname{diam} E_n < \delta$ for all $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} \alpha_s \left(\frac{\operatorname{diam} E_n}{2}\right)^s \le \mathcal{H}^s_\delta(E) + 1 \le \mathcal{H}^s_o(E) + 1.$$

Then

$$\begin{aligned} \mathcal{H}^t_\delta(E) &\le \sum_{n=1}^{\infty} \alpha_t \left(\frac{\operatorname{diam} E_n}{2}\right)^t \le \frac{\alpha_t}{\alpha_s}\left(\frac{\delta}{2}\right)^{t-s} \sum_{n=1}^{\infty} \alpha_s \left(\frac{\operatorname{diam} E_n}{2}\right)^s \\ &\le \frac{\alpha_t}{\alpha_s}\left(\frac{\delta}{2}\right)^{t-s} \left(\mathcal{H}^s_o(E) + 1\right). \end{aligned}$$

Letting $\delta \to 0$ it follows from (6) that $\mathcal{H}^t_o(E) = 0$. ■

Exercise 14 *Let $0 < s < N$. Prove that the Hausdorff measure $\mathcal{H}^s$ is not σ-finite.*

In view of the Proposition 13 the following definition makes sense:

Definition 15 *The* Hausdorff dimension *of a set $E \subset \mathbb{R}^N$ is defined by*

$$\dim_{\mathcal{H}}(E) := \inf\{0 \le s < \infty : \mathcal{H}^s_o(E) = 0\}.$$

Note that $\dim_{\mathcal{H}}$ may be any number in $[0, \infty]$, not necessarily an integer number. We remark that part (ii), together with Proposition 11(iii), implies in particular that $\dim_{\mathcal{H}}(E) \le N$ for any set $E \subset \mathbb{R}^N$. Moreover, if $t := \dim_{\mathcal{H}}(E)$, then by the previous proposition we have that $\mathcal{H}^s_o(E) = 0$ for all $s > t$, while if $t > 0$, then $\mathcal{H}^s_o(E) = \infty$ for all $0 < s < t$.

Exercise 16 ***Dimension of a set.***

(i) Consider a sequence $\{E_n\}$ of sets in $\mathbb{R}^N$. Prove that if $E_n \subset \mathbb{R}^N$ has Hausdorff dimension p_n, then

$$E := \bigcup_{n=1}^{\infty} E_n$$

has Hausdorff dimension $p := \sup_n p_n$.

(ii) Given $0 < t < N$, construct a set $E \subset \mathbb{R}^N$ of dimension t such that $\mathcal{H}^t_o(E) = 0$.[1]

[1] You can use the fact that for every $0 < s < N$, there exists a set $F \subset \mathbb{R}^N$ such that

$$0 < \mathcal{H}^s_o(F) < \infty.$$

This can be done using generalized Cantor sets.

Exercise 17 *Let $E \subset \mathbb{R}^N$ and let $f : E \to \mathbb{R}^M$ be* Hölder continuous of exponent *$\alpha \in (0,1]$, that is, there exists a constant $L > 0$ such that*

$$\|f(x) - f(y)\| \leq L \|x - y\|^{\alpha}$$

for all $x, y \in E$. Prove that for every $s > 0$,

$$\mathcal{H}_o^{s/\alpha}(f(E)) \leq \frac{\alpha_{s/\alpha} 2^s}{\alpha_s 2^{s/\alpha}} L^{s/\alpha} \mathcal{H}_o^s(E).$$

Note that when $\alpha = 1$, the function f is called *Lipschitz continuous.*

Given a continuous *curve* γ parametrized by $f : [a,b] \to \mathbb{R}^N$, the *length* of γ is defined as

$$\operatorname{length}\gamma := \sup\left\{ \sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\| : \right. \\ \left. a = t_0 < t_1 < \cdots < t_n = b,\, n \in \mathbb{N} \right\}.$$

The curve γ is *rectifiable* if $\operatorname{length}\gamma < \infty$.

Exercise 18 ***Length of a rectifiable curve.*** *Let γ be an injective rectifiable curve.*

(i) Prove that if $x,\, y \in \mathbb{R}^N$ and $S := \{tx + (1-t)y : t \in [0,1]\}$, then

$$\mathcal{H}_o^1(S) = |x - y|.$$

(ii) Prove that if $x,\, y \in \mathbb{R}^N$ and $\gamma_1 : [c,d] \to \mathbb{R}^N$ is a rectifiable curve such that $\gamma_1(c) = x$ and $\gamma_1(d) = y$, then

$$\mathcal{H}_o^1(\gamma_1([c,d])) \geq |x - y|.$$

(iii) Prove that[2]

$$\mathcal{H}_o^1(\gamma([a,b])) \geq \operatorname{length}\gamma.$$

(iv) Assume that[3]

$$|\gamma(t) - \gamma(s)| \leq |t - s|$$

for all $t, s \in [a,b]$ and prove that

$$\mathcal{H}_o^1(\gamma([a,b])) = \operatorname{length}\gamma$$

and that $\dim_{\mathcal{H}}(\gamma) = 1$.

Exercise 19 ***Length of a smooth curve.***

[2] Here you can use the fact that if $C_1, C_2 \subset \mathbb{R}^N$ are disjoint closed sets, then $\mathcal{H}_o^1(C_1 \cup C_2) = \mathcal{H}_o^1(C_1) + \mathcal{H}_o^1(C_2)$.

[3] This "assumption" actually follows by reparametrizing the curve using arclength.

(i) *Let* $g : [\alpha, \beta] \to \mathbb{R}^N$ *be a continuous function. Prove that if* $t_0 \in [\alpha, \beta]$ *then*

$$\left| \int_\alpha^\beta g(t)\, dt \right| \geq \int_\alpha^\beta |g(t)|\, dt - 2 \int_\alpha^\beta |g(t) - g(t_0)|\, dt;$$

(ii) *Given a curve* γ *of class* C^1 *parametrized by* $f : [a, b] \to \mathbb{R}^N$*, prove that*

$$\operatorname{length} \gamma = \int_a^b |f'(t)|\, dt.$$

Exercise 20 *Let* $A \subset \mathbb{R}^N$ *be an open set and let* $f : A \to \mathbb{R}$ *be a function of class* C^1*. Prove that the graph of* f*, that is,*

$$\operatorname{Gr} f := \{(x, f(x)) : x \in A\} \subset \mathbb{R}^{N+1},$$

has Hausdorff dimension N *and that*

$$\mathcal{H}^N(\operatorname{Gr} f) = \textit{surface area of } \operatorname{Gr} f = \int_A \sqrt{1 + |\nabla f(x)|^2}\, dx.$$

Exercise 21 *Let* $M \subset \mathbb{R}^N$ *be a* k*-dimensional manifold of class* C^1*,* $1 \leq k \leq N$*.*

(i) *Let* φ *be a local chart, that is,* $\varphi : A \to M$ *is a function of class* C^1 *for some open set* $A \subset \mathbb{R}^k$ *such that* $\nabla\varphi$ *has maximum rank* k *in* A*. Define* $g_{ij} := \frac{\partial \varphi}{\partial y_i} \cdot \frac{\partial \varphi}{\partial y_j}$*, where* $\cdot$ *is the inner product in* $\mathbb{R}^N$*. Prove that* $\varphi(A)$ *has Hausdorff dimension* k *and that*

$$\mathcal{H}^k(\varphi(A)) = \textit{surface area of the manifold } \varphi(A) = \int_A \sqrt{\det g_{ij}(y)}\, dy.$$

(ii) *Prove that* M *has Hausdorff dimension* k *and that* $\mathcal{H}^k(M)$ *is the standard surface measure of* M*.*

Friday, September 9, 2011

The third example is given by the Lebesgue–Stieltjes outer measure generated by an increasing function f.

In what follows an *interval* $I \subset \mathbb{R}$ is any set of $\mathbb{R}$ such that if $x, y \in I$ and $x < y$, then $[x, y] \subset I$.

Theorem 22 *Let* $I \subset \mathbb{R}$ *be an interval and let* $f : I \to \mathbb{R}$ *be a monotone function. Then* f *has at most countably many discontinuity points.*

Proof. It's in the real analysis notes. ■

Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be an increasing function. Take $\mathcal{G}$ to be the family of all intervals (a, b), where $a, b \in I$, with $a \leq b$, and define $\rho : \mathcal{G} \to [0, \infty]$ by

$$\rho((a, b)) := f(b) - f(a).$$

By Proposition 3, the application $\mu_f^* : \mathcal{P}(I) \to [0, \infty]$ given by

$$\mu_f^*(E) := \inf\left\{\sum_{n=1}^{\infty} \rho((a_n, b_n)) : \, a_n, b_n \in I, \, a_n \le b_n, \, E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\right\}, \quad E \subset I, \tag{7}$$

is an outer measure, called the *Lebesgue-Stieltjes outer measure* generated by f. It follows from the definition of μ_f^* that for every interval $(a, b) \subset I$, where $a, b \in I$, with $a \le b$,

$$\mu_f^*((a, b)) \le \rho((a, b)). \tag{8}$$

In the next theorem we will show that strict inequality may occur.

Note that if $[a', b'] \subset I$, then, since I is open, f is bounded on a interval $[a, b] \subset I$ such that $[a', b'] \subset (a, b)$, and so (8) implies that μ_f is finite on closed intervals $[a', b'] \subset I$.

Theorem 23 *Let $I \subset \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be increasing. Then for all $a, b \in I$, with $a \le b$,*

$$\mu_f^*([a, b]) = f^+(b) - f^-(a), \tag{9}$$

$$\mu_f^*((a, b)) = f^-(b) - f^+(a). \tag{10}$$

Note that under the hypotheses of the previous theorem, taking $a = b \in I$ in (9) gives

$$\mu_f^*(\{a\}) = f^+(a) - f^-(a). \tag{11}$$

Proof. We prove that if $a, b \in I$, with $a \le b$, then

$$\mu_f^*([a, b]) \le f^+(b) - f^-(a).$$

Since I is open, there exists an integer $m_0 \in \mathbb{N}$ such that $\left(a - \frac{1}{m_0}, b + \frac{1}{m_0}\right) \subset I$. Hence, the interval $\left(a - \frac{1}{m}, b + \frac{1}{m}\right)$ is an admissible cover of $[a, b]$ for $m > m_0$. By (7) we have

$$\mu_f^*([a, b]) \le f\left(b + \frac{1}{m}\right) - f\left(a - \frac{1}{m}\right).$$

Letting $m \to \infty$ gives

$$\mu_f^*([a, b]) \le f^+(b) - f^-(a).$$

To prove the opposite inequality, consider any sequence of intervals $\{(a_n, b_n)\}$ such that $[a, b] \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$. Since $\bigcup_{n=1}^{\infty} (a_n, b_n)$ is open, we may find $\varepsilon > 0$ such that $[a - \varepsilon, b + \varepsilon] \subset \bigcup_{n=1}^{\infty} (a_n, b_n)$. We claim that the set

$$\bigcup_{n=1}^{\infty} [f(a_n), f(b_n)]$$

covers $(f(a-\varepsilon), f(b+\varepsilon))$. Indeed, if $t \in (f(a-\varepsilon), f(b+\varepsilon))$, then there exists $x \in [a-\varepsilon, b+\varepsilon]$ such that $t \in [f^-(x), f^+(x)]$. Since $[a-\varepsilon, b+\varepsilon] \subset \bigcup_{n=1}^\infty (a_n, b_n)$ there exists $n \in \mathbb{N}$ such that $x \in (a_n, b_n)$. Hence

$$f(a_n) \le f^-(x) \le t \le f^+(x) \le f(b_n).$$

Thus, the claim is proved, and so

$$\begin{aligned}
\sum_{n=1}^\infty (f(b_n) - f(a_n)) &\ge \sum_{n=1}^\infty \mathcal{L}_o^1([f(a_n), f(b_n)]) \\
&\ge \mathcal{L}_o^1\left(\bigcup_{n=1}^\infty [f(a_n), f(b_n)]\right) \\
&\ge \mathcal{L}_o^1((f(a-\varepsilon), f(b+\varepsilon))) \\
&= f(b+\varepsilon) - f(a-\varepsilon) \ge f^+(b) - f^-(a).
\end{aligned}$$

Taking the supremum over all admissible covers $\{(a_n, b_n)\}$ and using (7), we obtain that

$$\mu_f^*([a,b]) \ge f^+(b) - f^-(a).$$

Thus, (9) holds. ■

Exercise 24 *Let $I \subset \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be increasing.*

(i) Prove that for all $a, b \in I$, with $a \le b$,

$$\mu_f^*((a,b]) = f^+(b) - f^+(a), \tag{12}$$

$$\mu_f^*((a,b)) = f^-(b) - f^+(a). \tag{13}$$

Exercise 25 *Let $I \subset \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be increasing.*

(i) Prove that in general μ_f^* is not translation invariant, that is,

$$\mu_f^*(x+E) \ne \mu_f^*(E).$$

(ii) Find all increasing functions $f : I \to \mathbb{R}$ for which $\mu_f^*(x+E) = \mu_f^*(E)$ for all $E \subset I$.

Exercise 26 *Let $I \subset \mathbb{R}$ be an open interval, let*

$$\begin{aligned}
\mathcal{Y} &:= \{f : I \to \mathbb{R} : \ f \text{ is increasing}\} \\
\mathcal{Z} &:= \{\mu^* : \mathcal{P}(I) \to [0,\infty] : \ \mu^* \text{ is an outer measure}\},
\end{aligned}$$

and consider the operator

$$\begin{aligned}
T : \mathcal{Y} &\to \mathcal{Z} \\
f &\mapsto \mu_f^*
\end{aligned}$$

(i) Prove that in general T is not one-to-one, that is, find $f, g \in \mathcal{Y}$ such that $f \ne g$ but $\mu_f^* = \mu_g^*$.

(ii) Given $f \in \mathcal{Y}$, find all functions $g \in \mathcal{Y}$ such that $\mu_f^* = \mu_g^*$.

Monday, September 12, 2011

2 σ-Algebras and Measures

In the previous section we have given the definition of outer measures and provided a general method for constructing outer measures. The next question is what to do with an outer measure. If we want to measure sets, an important property that is desirable is that if we take two disjoint sets, then the measure of the union should be the sum of the measures.

Unfortunately, in general an outer measure does not have this property (see your homework). To circumvent this problem Carathéodory proposed to restrict an outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$ to a smaller class of subsets for which additivity of disjoint sets holds. The class that we chose is the following:

Definition 27 *Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure. A set $E \subset X$ is said to be* μ^*-measurable *if*

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \setminus E)$$

for all sets $F \subset X$.

Remark 28 *By the subadditivity of μ^* the inequality*

$$\mu^*(F) \le \mu^*(F \cap E) + \mu^*(F \setminus E)$$

holds for all sets $F \subset X$. Hence, to prove that a set $E \subset X$ is μ^*-measurable, it suffices to show that

$$\mu^*(F) \ge \mu^*(F \cap E) + \mu^*(F \setminus E) \tag{14}$$

for all sets $F \subset X$. Moreover, it is enough to consider sets $F \subset X$ such that $\mu^*(F) < \infty$, since otherwise the inequality (14) is automatically satisfied.

We will see below in Theorem 35 that the restriction of μ^* to the class

$$\mathfrak{M}^* := \{E \subset X : \ E \text{ is } \mu^*\text{-measurable}\}$$

is additive, actually countably additive and that the class $\mathfrak{M}^*$ has some important properties, precisely it is a σ*-algebra*.

Definition 29 *Let X be a nonempty set. A collection $\mathfrak{M} \subset \mathcal{P}(X)$ is an* algebra *if*

(i) $\emptyset \in \mathfrak{M}$;

(ii) if $E \in \mathfrak{M}$ then $X \setminus E \in \mathfrak{M}$;

(iii) if E_1, $E_2 \in \mathfrak{M}$ then $E_1 \cup E_2 \in \mathfrak{M}$.

$\mathfrak{M}$ *is said to be a* σ-algebra *if it satisfies (i)–(ii) and*

(iii)′ if $\{E_n\} \subset \mathfrak{M}$ then $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}$.

To highlight the dependence of the σ-algebra $\mathfrak{M}$ on X we will sometimes use the notation $\mathfrak{M}(X)$. If $\mathfrak{M}$ is a σ-algebra then the pair $(X, \mathfrak{M})$ is called a *measurable space*. For simplicity we will often apply the term measurable space only to X.

Using De Morgan's laws and (ii) and (iii)$'$, it follows that a σ-algebra is closed under countable intersection.

Example 30 *(i) In view of (i) and (ii), every algebra contains X. Hence the smallest algebra (respectively σ-algebra) is $\{\emptyset, X\}$ and the largest is the collection $\mathcal{P}(X)$ of all subsets of X.*

(ii) If $X = [0,1)$, the family $\mathfrak{M}$ of all finite unions of intervals of the type $[a,b) \subset [0,1)$ is an algebra but not a σ-algebra. Indeed,

$$E := \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right) = \{0\} \notin \mathfrak{M}.$$

*(iii) **(Peano–Jordan measure)** For every bounded set $E \subset \mathbb{R}^N$ define*

$$m_o(E) := \inf \left\{ \sum_{n=1}^{M} (r_n)^N : E \subset \bigcup_{n=1}^{M} Q(x_n, r_n),\, M \in \mathbb{N} \right\},$$

$$m_i(E) := \sup \left\{ \sum_{n=1}^{M} (r_n)^N : \bigcup_{n=1}^{M} Q(x_n, r_n) \subset E,\, Q(x_n, r_n) \text{ pairwise disjoint, } M \in \mathbb{N} \right\},$$

where we set $m_i(E) := 0$ if there are no cubes contained in E. The maps m_o and m_i are called the Peano–Jordan outer *and* inner measures*, respectively. A bounded set $E \subset \mathbb{R}^N$ is said to be* Peano–Jordan measurable *if $m_o(E) = m_i(E)$ and the common value is called the* Peano–Jordan measure *of E and is denoted $m(E)$. Let $X = [0,1]^N$. The family*

$$\mathfrak{M} := \left\{ E \subset [0,1]^N : E \text{ is Peano–Jordan measurable} \right\}$$

is an algebra (I will not prove this), but not a σ-algebra. Indeed, if $x \in \mathbb{R}^N$, then $m_i(\{x\}) = m_o(\{x\}) = 0$, and so $\{x\}$ is Peano–Jordan measurable. However, the set $E = [0,1]^N \cap \mathbb{Q}^N$ is not Peano–Jordan measurable, since by the density of the rationals and of the irrationals, $m_i(E) = 0$, while $m_o(E) = 1$. On the other hand, since E is countable, we may write

$$E = \bigcup_{n=1}^{\infty} \{x_n\},$$

and so $\mathfrak{M}$ is not a σ-algebra.

Let X be a nonempty set. Given any subset $\mathcal{F} \subset \mathcal{P}(X)$ the smallest (in the sense of inclusion) σ-algebra that contains $\mathcal{F}$ is given by the intersection of all σ-algebras on X that contain $\mathcal{F}$.

If X is a topological space, then the *Borel σ-algebra* $\mathcal{B}(X)$ is the smallest σ-algebra containing all open subsets of X.

Definition 31 *Let X be a nonempty set and let $\mathfrak{M} \subset \mathcal{P}(X)$ be an algebra. A map $\mu : \mathfrak{M} \to [0, \infty]$ is called a* (positive) finitely additive measure *if*

$$\mu(\emptyset) = 0, \quad \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$$

for all E_1, $E_2 \in \mathfrak{M}$ with $E_1 \cap E_2 = \emptyset$.

Definition 32 *Let X be a nonempty set, let $\mathfrak{M} \subset \mathcal{P}(X)$ be a σ-algebra. A map $\mu : \mathfrak{M} \to [0, \infty]$ is called a* (positive) measure *if*

$$\mu(\emptyset) = 0, \quad \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

for every countable collection $\{E_n\} \subset \mathfrak{M}$ of pairwise disjoint sets. The triple $(X, \mathfrak{M}, \mu)$ is said to be a measure space.

Example 33 *Let*

$$\mathfrak{M} := \left\{E \subset [0,1]^N : E \text{ is Peano–Jordan measurable}\right\}.$$

It can be shown that the Peano-Jordan measure $m : \mathfrak{M} \to [0, \infty)$ is a finitely additive measure.

Definition 34 *Given a measure space $(X, \mathfrak{M}, \mu)$, the measure μ is said to be* complete *if for every $E \in \mathfrak{M}$ with $\mu(E) = 0$ it follows that every $F \subset E$ belongs to $\mathfrak{M}$.*

Theorem 35 (Carathéodory) *Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure. Then*

$$\mathfrak{M}^* := \{E \subset X : \ E \text{ is } \mu^*\text{-measurable}\} \tag{15}$$

is a σ-algebra and $\mu^* : \mathfrak{M}^* \to [0, \infty]$ is a complete measure.

Proof. Step 1: Since $\mu^*(\emptyset) = 0$, for any $F \subset X$,

$$\mu^*(F) = \mu^*(F \cap \emptyset) + \mu^*(F \setminus \emptyset),$$

thus $\emptyset \in \mathfrak{M}^*$.
Step 2: To prove that if $E \in \mathfrak{M}^*$, then $X \setminus E \in \mathfrak{M}^*$, it suffices to observe that

$$F \cap (X \setminus E) = F \setminus E, \qquad F \setminus (X \setminus E) = F \cap E.$$

Step 3: We show that if $E_1, E_2 \in \mathfrak{M}^*$, then $E_1 \cup E_2 \in \mathfrak{M}^*$. Fix a set $F \subset X$ with $\mu^*(F) < \infty$. Using the fact that E_1, $E_2 \in \mathfrak{M}^*$ we have that

$$\infty > \mu^*(F) \geq \mu^*(F \cap E_1) + \mu^*(F \setminus E_1),$$
$$\mu^*(F \setminus E_1) \geq \mu^*((F \setminus E_1) \cap E_2) + \mu^*((F \setminus E_1) \setminus E_2).$$

We now add these two inequalities and cancel $\mu^*(F \setminus E_1) < \infty$ from both sides. We get

$$\begin{aligned}\mu^*(F) &\geq \mu^*(F \cap E_1) + \mu^*((F \setminus E_1) \cap E_2) + \mu^*((F \setminus E_1) \setminus E_2) \\ &\geq \mu^*((F \cap E_1) \cup (F \setminus E_1) \cap E_2) + \mu^*((F \setminus E_1) \setminus E_2) \\ &= \mu^*(F \cap (E_1 \cup E_2)) + \mu^*(F \setminus (E_1 \cup E_2)),\end{aligned}$$

where in the second inequality we have used the subadditivity of μ^*.

Thus $\mathfrak{M}^*$ is an algebra. ■

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Proof. Step 4: To prove that $\mu^* : \mathfrak{M}^* \to [0, \infty]$ is a finitely additive measure, let E_1, $E_2 \in \mathfrak{M}^*$ be disjoint sets and let $F \subset X$. Since $E_1 \in \mathfrak{M}^*$ and E_1, E_2 are sets, we obtain

$$\begin{aligned}\mu^*(F \cap (E_1 \cup E_2)) &= \mu^*((F \cap (E_1 \cup E_2)) \cap E_1) + \mu^*((F \cap (E_1 \cup E_2)) \setminus E_1) \\ &= \mu^*(F \cap E_1) + \mu^*(F \cap E_2),\end{aligned}$$

which implies finite additivity (take $F := X$).

Using an induction argument we have that if $E_1, \ldots, E_m \in \mathfrak{M}^*$, $m \in \mathbb{N}$, are pairwise disjoint and $F \subset X$, then $\bigcup_{n=1}^m E_n \in \mathfrak{M}^*$ and

$$\mu^*\left(F \cap \bigcup_{n=1}^m E_n\right) = \sum_{n=1}^m \mu^*(F \cap E_n). \tag{16}$$

Step 5: We are now ready to prove that $\mu^* : \mathfrak{M}^* \to [0, \infty]$ is a countably additive measure. Let $\{E_n\} \subset \mathfrak{M}^*$ be any sequence of pairwise disjoint sets and let $F \subset X$. Since $\bigcup_{n=1}^m E_n \in \mathfrak{M}^*$ for any $m \in \mathbb{N}$, we have that

$$\begin{aligned}\mu^*(F) &= \mu^*\left(F \cap \bigcup_{n=1}^m E_n\right) + \mu^*\left(F \setminus \left(\bigcup_{n=1}^m E_n\right)\right) \\ &= \sum_{n=1}^m \mu^*(F \cap E_n) + \mu^*\left(F \setminus \left(\bigcup_{n=1}^m E_n\right)\right) \\ &\geq \sum_{n=1}^m \mu^*(F \cap E_n) + \mu^*\left(F \setminus \left(\bigcup_{n=1}^\infty E_n\right)\right),\end{aligned}$$

by (16) and the subadditivity of μ^*. Letting $m \to \infty$ in the previous inequality yields

$$\mu^*(F) \geq \sum_{n=1}^\infty \mu^*(F \cap E_n) + \mu^*\left(F \setminus \left(\bigcup_{n=1}^\infty E_n\right)\right). \tag{17}$$

By the properties of outer measures, the right-hand side of the previous inequality is greater than or equal to

$$\mu^*\left(F \cap \left(\bigcup_{n=1}^\infty E_n\right)\right) + \mu^*\left(F \setminus \left(\bigcup_{n=1}^\infty E_n\right)\right),$$

and so

$$\mu^*(F) \geq \mu^*\left(F \cap \left(\bigcup_{n=1}^{\infty} E_n\right)\right) + \mu^*\left(F \setminus \left(\bigcup_{n=1}^{\infty} E_n\right)\right),$$

which implies that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}^*$. On the other hand, Taking $F := \bigcup_{n=1}^{\infty} E_n$ in (17) gives

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \geq \sum_{n=1}^{\infty} \mu^*(E_n),$$

and so $\mu^* : \mathfrak{M}^* \to [0, \infty]$ is a countably additive measure.

Step 6: To prove that $\mathfrak{M}^*$ is a σ-algebra, let $\{E_n\} \subset \mathfrak{M}^*$. Then the sets

$$F_1 := E_1, \quad F_{n+1} := E_{n+1} \setminus \bigcup_{k=1}^{n} E_k$$

belong to $\mathfrak{M}^*$ and are pairwise disjoint. Hence,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n \in \mathfrak{M}^*.$$

Step 6: Finally, if $\mu^*(E) = 0$, then by the monotonicity of the outer measure, $\mu^*(F \cap E) = 0$ for all sets $F \subset X$. Hence E is μ^*-measurable and $\mu^* : \mathfrak{M}^* \to [0, \infty]$ is a complete measure. ■

Example 36

(i) *The class of all $\mathcal{L}_o^N$-measurable subsets of $\mathbb{R}^N$ is called the* σ-algebra of Lebesgue measurable sets, *and by Carathéodory's theorem, $\mathcal{L}_o^N$ restricted to this σ-algebra is a complete measure, called the N–dimensional* Lebesgue measure *and denoted by $\mathcal{L}^N$. Given a Lebesgue measurable set $E \subset \mathbb{R}^N$, we will write indifferently*

$$\mathcal{L}^N(E) \ \text{ or } \ |E|$$

for the Lebesgue measure of E.

(ii) *By Carathéodory's theorem, $\mathcal{H}_o^s$ restricted to the σ-algebra of all $\mathcal{H}_o^s$-measurable subsets of $\mathbb{R}^N$ is a complete measure denoted $\mathcal{H}^s$ and called* s-dimensional Hausdorff measure.

(iii) *Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be an increasing function. By Carathéodory's theorem, μ_f^* restricted to the σ-algebra of all μ_f^*-measurable subsets of I is a complete measure denoted μ_f and called the* Lebesgue-Stieltjes measure *generated by f.*

Using Carathéodory's theorem, we have created a large class of complete measures. The next problem is to understand the class $\mathfrak{M}^*$ of the μ^*-measurable sets. For instance, in the case of the Lebesgue measure $\mathcal{L}^N$, it is important to determine if a ball, or a cube, or an open set is Lebesgue measurable. To do that we consider a special class of outer measures.

Definition 37 *Let X be a metric space and let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be an outer measure. Then μ^* is said to be a* metric outer measure *if*

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$$

for all sets E, $F \subset X$, with

$$\operatorname{dist}(E,F) := \inf\{d(x,y) : x \in E,\ y \in F\} > 0.$$

Proposition 38 *Let X be a metric space and let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be a metric outer measure. Then every Borel set is μ^*-measurable.*

Proof. Since closed sets generate the Borel σ-algebra $\mathcal{B}(X)$, to show that $\mathfrak{M}^*$ contains $\mathcal{B}(X)$, it is enough to prove that $\mathfrak{M}^*$ contains all closed sets. Thus let $C \subset X$ be a closed set and let $F \subset X$ be such that $\mu^*(F) < \infty$. For $n \in \mathbb{N}$ define

$$E_0 := \{x \in F \setminus C : \operatorname{dist}(x,C) \geq 1\}$$
$$E_n := \left\{x \in F \setminus C : \frac{1}{n+1} \leq \operatorname{dist}(x,C) < \frac{1}{n}\right\}.$$

Note that the sets E_n are disjoint. Moreover, since C is closed we have that

$$\bigcup_{n=0}^{\infty} E_n = F \setminus C.$$

Indeed, if $x \in F \setminus C$, then $\operatorname{dist}(x,C) > 0$, and so we may find $n \in \mathbb{N}$ such that $x \in E_n$. ■

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Proof. If $x \in E_{2k}$ and $y \in E_{2k+2}$, then

$$\frac{1}{2k+1} \leq \operatorname{dist}(x,C) \leq d(x,y) + \operatorname{dist}(y,C) < d(x,y) + \frac{1}{2k+2}$$

and so $d(x,y) \geq \frac{1}{2k+1} - \frac{1}{2k+2} = \frac{1}{2(2k^2+3k+1)} > 0$, which implies that $\operatorname{dist}(E_{2k}, E_{2k+2}) > 0$ for all $k \geq 0$. By the fact that μ^* is a metric outer measure, for all $k \in \mathbb{N}$,

$$\sum_{j=0}^{k} \mu^*(E_{2j}) = \mu^*\left(\bigcup_{j=0}^{k} E_{2j}\right) \leq \mu^*(F) < \infty$$

Similarly

$$\sum_{j=1}^{k} \mu^*(E_{2j-1}) \leq \mu^*(F) < \infty.$$

Thus the series $\sum_{j=0}^{\infty} \mu^*(E_{2j})$ and $\sum_{j=1}^{\infty} \mu^*(E_{2j-1})$ are convergent. In turn, the series $\sum_{n=0}^{\infty} \mu^*(E_n)$ is convergent.

Next observe that the sets $F \cap C$ and $\bigcup_{j=0}^{n} E_j$ have positive distance, since if $x \in F \cap C$ and $y \in \bigcup_{j=0}^{n} E_j$, then

$$\frac{1}{n+1} \leq \operatorname{dist}(y, C) \leq d(x, y).$$

Hence, using again the fact that μ^* is a metric outer measure, we have that

$$\begin{aligned}
\mu^*(F \cap C) + \mu^*(F \setminus C) &= \mu^*(F \cap C) + \mu^*\left(\bigcup_{j=0}^{\infty} E_j\right) = \mu^*(F \cap C) + \mu^*\left(\bigcup_{j=0}^{n-1} E_j \cup \bigcup_{j=n}^{\infty} E_j\right) \\
&\leq \mu^*(F \cap C) + \mu^*\left(\bigcup_{j=0}^{n-1} E_j\right) + \mu^*\left(\bigcup_{j=n}^{\infty} E_j\right) \\
&\leq \mu^*(F \cap C) + \mu^*\left(\bigcup_{j=0}^{n-1} E_j\right) + \sum_{j=n}^{\infty} \mu^*(E_j) \\
&= \mu^*\left((F \cap C) \cup \bigcup_{j=0}^{n-1} E_j\right) + \sum_{j=n}^{\infty} \mu^*(E_j) \\
&\leq \mu^*(F) + \sum_{j=n}^{\infty} \mu^*(E_j).
\end{aligned}$$

Letting $n \to \infty$, we conclude that $\mu^*(F \cap C) + \mu^*(F \setminus C) \leq \mu^*(F)$ and the proof is complete. ■

Using Proposition 38 we have that:

Proposition 39 *The outer measures $\mathcal{H}_o^s$,$0 \leq s < \infty$, and $\mathcal{L}_o^N$ are metric outer measures, so that every Borel subset of $\mathbb{R}^N$ is $\mathcal{H}_o^s$-measurable and $\mathcal{L}_o^N$-measurable.*

Proof. See your homework for $\mathcal{H}_o^s$. The case $\mathcal{L}_o^N$ is left as an exercise. ■

It follows from the previous proposition and Proposition 38 that every Borel subset of $\mathbb{R}^N$ is Lebesgue measurable.It follows from the previous proposition and Proposition 38 that every Borel subset of $\mathbb{R}^N$ is Lebesgue measurable.

Example 40 ***A set that is not Lebesgue measurable.*** *On the real line we consider the equivalence relation $x \sim y$ if $x - y \in \mathbb{Q}$. By the axiom of choice we may construct a set $E \subset (0,1)$ that contains exactly one element from each equivalence class. The following properties are satisfied:*

(i) If $x \in (0,1)$, then $x \in r + E$ for some $r \in (-1,1) \cap \mathbb{Q}$. To see this, observe that by construction of E, for any $x \in (0,1)$ there exists $y \in E$ such that that $x \sim y$, that is, $x - y = r \in (-1,1) \cap \mathbb{Q}$.

(ii) If $r,\, q\in\mathbb{Q}$, with $r\neq q$, then $(r+E)\cap(q+E)=\emptyset$. Indeed, if not, then we may write $r+x=q+y$ for some $x,\, y\in E$. But then $x-y=q-r\in\mathbb{Q}\setminus\{0\}$, which implies that $x\sim y$. By the construction of E this is possible only if $x=y$, which is impossible.

We claim that E is not Lebesgue measurable. Indeed, assume by contradiction that E is Lebesgue measurable. Since $\mathcal{L}_0^1$ is translation-invariant, it follows that $r+E$ is Lebesgue measurable for all $r\in(-1,1)\cap\mathbb{Q}$, and so is the set

$$F:=\bigcup_{r\in(-1,1)\cap\mathbb{Q}}(r+E)\subset(-1,2)\,.$$

Thus by property (b) and the translation-invariance of $\mathcal{L}^1$,

$$3\geq\mathcal{L}^1(F)=\sum_{r\in(-1,1)\cap\mathbb{Q}}\mathcal{L}^1(r+E)=\sum_{r\in(-1,1)\cap\mathbb{Q}}\mathcal{L}^1(E)\,,$$

which implies that $\mathcal{L}^1(E)=0$, and, in turn, that $\mathcal{L}^1(F)=0$. This is a contradiction, since $F\supset(0,1)$ by property (a).

Monday, September 19, 2011

Exercise 41 *Let $E\subset\mathbb{R}$ be a Lebesgue measurable set with $\mathcal{L}^1(E)>0$. Prove that E contains a set, which is not Lebesgue measurable.*

Exercise 42 (Expansion in base b) *Let $b\in\mathbb{N}$ be such that $b\geq 2$ and let $x\in[0,1]$.*

(i) Prove that there exists a sequence $\{a_n\}$ of nonnegative integers such that $0\leq a_n<b$ for all $n\in\mathbb{N}$ and

$$x=\sum_{n=1}^{\infty}\frac{a_n}{b^n}.$$

Prove that the sequence $\{a_n\}$ is uniquely determined by x, unless x is of the form $x=\frac{k}{b^m}$ for some $k,m\in\mathbb{N}$, in which case there are exactly two such sequences.

(ii) Conversely, given any sequence $\{a_n\}$ of nonnegative integers such that $0\leq a_n<b$ for all $n\in\mathbb{N}$, prove that the series $\sum_{n=1}^{\infty}\frac{a_n}{b^n}$ converges to a number $x\in[0,1]$.

Example 43 ***An uncountable set with Lebesgue measure zero.*** *Divide $[0,1]$ into three equal subintervals and remove the middle one $I_{11}:=\left(\frac{1}{3},\frac{2}{3}\right)$. Divide each of the two remaining closed intervals $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$ into three equal subintervals and remove the middle ones $I_{12}:=\left(\frac{1}{3^2},\frac{2}{3^2}\right)$ and $I_{22}:=\left(\frac{7}{3^2},\frac{8}{3^2}\right)$.*

Continue in this fashion at each step n we remove 2^{n-1} middle intervals I_{1n}, ..., $I_{2^{n-1}n}$, each of length $\frac{1}{3^n}$. The Cantor set $\mathbb{D}$ *is defined as*

$$\mathbb{D} := [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{kn}.$$

The set $\mathbb{D}$ is closed (since its complement is given by a family of open intervals) and

$$\mathcal{L}^1(\mathbb{D}) = 1 - \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \operatorname{length}(I_{kn}) = 1 - \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} \frac{1}{3^n} = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 0.$$

Thus $\mathbb{D}$ has Lebesgue measure zero. To prove that $\mathbb{D}$ is uncountable observe that $x \in \mathbb{D}$ if and only if

$$x = \sum_{n=1}^{\infty} \frac{c_n}{3^n} \tag{18}$$

where each $c_n \in \{0, 2\}$. For any $x \in \mathbb{D}$ of the form (18) *define*

$$f(x) := \sum_{n=1}^{\infty} \frac{d_n}{2^n}$$

where

$$d_n := \begin{cases} 1 & \text{if } c_n = 2, \\ 0 & \text{if } c_n = 0. \end{cases}$$

The function $f : \mathbb{D} \to [0,1]$ is increasing and has the same values at the endpoints of each removed interval I_{kn} and thus f extends to a continuous function on $[0,1]$. This function is called the Cantor function*. Since $f(\mathbb{D}) = [0,1]$ it follows in particular that $\mathbb{D}$ is uncountable. Note also that $f'(x) = 0$ for all $x \in I_{kn}$, k, $n \in \mathbb{N}$ so that $f' = 0$ except on a set of Lebesgue measure zero.*

It may also be proved that there are Lebesgue measurable sets that are not Borel sets. Hence $\mathcal{L}^N : \mathcal{B}(\mathbb{R}^N) \to [0, \infty]$ is not a complete measure.

Example 44 ***A Lebesgue measurable set that is not a Borel set.*** *Consider the function $v : [0,1] \to \mathbb{R}$ defined by*

$$g(x) := x + f(x), \quad x \in [0,1].$$

Then v is continuous, strictly increasing, and $g([0,1]) = [0,2]$. Hence its inverse function $g^{-1} : [0,2] \to [0,1]$ is still continuous. Since f is constant on each of the intervals not in $\mathbb{D}$, the function g maps those intervals into intervals of the same length. Hence

$$\mathcal{L}^1(g([0,1] \setminus \mathbb{D})) = \mathcal{L}^1([0,1] \setminus \mathbb{D}) = 1.$$

In turn,

$$2 = \mathcal{L}^1 \left(g \left([0,1] \setminus \mathbb{D} \right) \right) + \mathcal{L}^1 \left(g \left(\mathbb{D} \right) \right) = 1 + \mathcal{L}^1 \left(g \left(\mathbb{D} \right) \right),$$

and so $\mathcal{L}^1 \left(g \left(\mathbb{D} \right) \right) = 1$*. Let* $E \subset g(\mathbb{D})$ *be a set that is not Lebesgue measurable (see part (i)) and consider the set* $F := g^{-1}(E) \subset \mathbb{D}$*. By the completeness of the Lebesgue measure, it follows that every subset of* $\mathbb{D}$ *(and in particular the set* F*) is Lebesgue measurable and has Lebesgue measure zero. We claim that* F *is not a Borel set. Indeed, since* $w = g^{-1}$ *is a continuous function, if* F *were a Borel set, then* $w^{-1}(F) = E$ *would be a Borel set (we will prove this fact later on). It follows that* F *cannot be a Borel set.*

Wednesday, September 21, 2011

Exercise 45 *Let* $E \subset \mathbb{R}$ *be a Lebesgue measurable set with* $\mathcal{L}^1(E) > 0$*.*

(i) Prove that for every $\theta \in (0,1)$ *there exists an open interval* $I_\theta \subset \mathbb{R}$ *such that* $\mathcal{L}^1(E \cap I_\theta) \geq \theta \mathcal{L}^1(I_\theta)$*.*

(ii) Let $\theta = \frac{3}{4}$*. Prove that for every* $x \in \left(-\frac{1}{2}\mathcal{L}^1\left(I_{3/4}\right), \frac{1}{2}\mathcal{L}^1\left(I_{3/4}\right) \right)$*, the sets* $E \cap I_{3/4}$ *and* $x + \left(E \cap I_{3/4} \right)$ *are not disjoint.*

(iii) Prove that the interval $\left(-\frac{1}{2}\mathcal{L}^1\left(I_{3/4}\right), \frac{1}{2}\mathcal{L}^1\left(I_{3/4}\right) \right)$ *is contained in* $E - E = \{x - y : x, y \in E\}$*.*

Example 46 ***A Lebesgue measurable set*** E ***such that*** $E + E$ ***is not Lebesgue measurable.*** *For every* $x \in [0,1]$*, write* $\frac{x}{2} = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$*, where* $a_n \in \{0,1,2\}$ *and consider the numbers* $y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ *and* $z = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$*, where*

$$b_n := \begin{cases} 0 & \text{if } a_n = 0, \\ 2 & \text{if } a_n = 1, \\ 2 & \text{if } a_n = 2, \end{cases} \qquad c_n := \begin{cases} 0 & \text{if } a_n = 0, \\ 0 & \text{if } a_n = 1, \\ 2 & \text{if } a_n = 2. \end{cases}$$

Then $\frac{x}{2} = \frac{y+z}{2}$*, which shows that* $[0,1] \subset \mathbb{D} + \mathbb{D}$*. In turn,* $\pm\mathbb{D} \pm \mathbb{D} \pm \cdots = \mathbb{R}$*. Consider* $\mathbb{R}$ *as a vector space over the rationals. Then*

$$\operatorname{span}_{\mathbb{Q}} \mathbb{D} = \mathbb{R}.$$

It follows from the axiom of choice that $\mathbb{D}$ *contains a Hamel basis* H *over the rationals. This means that for every* $x \in \mathbb{R}$*, with* $x \neq 0$*, we may uniquely write* $x = \sum_{i=1}^{n} r_i e_i$*, where* $r_i \in \mathbb{Q}$*,* $r_i \neq 0$*, and* $e_i \in H$ *,* $i = 1, \ldots, n$*,* $n \in \mathbb{N}$*.*

Define

$$E_0 := H \cup (-H) \cup \{0\}, \quad E_{n+1} := E_n + E_n, \quad n \in \mathbb{N} \cup \{0\}.$$

We claim that

$$\mathbb{R} = \bigcup_{n=0}^{\infty} \bigcup_{m=1}^{\infty} \frac{1}{m} E_n.$$

Indeed, if $x \in \mathbb{R}$, then by the property of H, we may write $x = \sum_{i=1}^{n} r_i e_i$, where $r_i \in \mathbb{Q}$ and $e_i \in H$, $i = 1, \ldots, n$. Let m be the least common multiple of the all the positive denominators of the rational numbers r_i, $i = 1, \ldots, n$. Then

$$x = \frac{1}{m} \sum_{i=1}^{n} (mr_i) \, e_i$$

and $mr_i \in \mathbb{Z}$. Hence, $x \in \frac{1}{m} E_n$ for n such that

$$2^n \geq \sum_{i=1}^{n} |mr_i| \, .$$

This proves the claim.

Since

$$\infty = \mathcal{L}_o^1 (\mathbb{R}) \leq \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \mathcal{L}_o^1 \left(\frac{1}{m} E_n \right),$$

it follows that at least one E_n must have positive outer measure. Let n_0 be the first nonnegative integer such that $\mathcal{L}_o^1 (E_{n_0}) > 0$. Note that, since the Cantor set has Lebesgue measure zero, then so does E_0. Hence, $n_0 \geq 1$. In particular, $\mathcal{L}_o^1 \left(E_{n_{0-1}} \right) = 0$, so that $E_{n_{0-1}}$ is Lebesgue measurable, by the completeness of the Lebesgue measure.

We claim that E_{n_0} is not Lebesgue measurable. Since $E_{n_0} = -E_{n_0}$, we have that $E_{n_0+1} = E_{n_0} - E_{n_0}$. By an exercise in your homework, E_{n_0+1} contains an open interval I containing the origin, say $I = (-\delta, \delta)$, for some $\delta > 0$. Let $e \in H$ and find $j \in \mathbb{N}$ with $j \geq 2$ such that

$$\frac{e}{j} \in I \subset E_{n_0+1} = E_{n_0} - E_{n_0}.$$

Since every element of E_{n_0} is a linear combination of elements of E_0 (and so of H) with integer coefficients, we have that

$$\frac{e}{j} = \sum_{i=1}^{n} r_i e_i,$$

where $r_i \in \mathbb{Z}$ and $e_i \in H$, $i = 1, \ldots, n$. But by the properties of the Hamel basis, we must have that $r_i = 0$ if $e_i \neq e$ and $r_i - \frac{1}{j} = 0$ if $e_i = e$, which is a contradiction, since r_i is an integer and $\frac{1}{j}$ is not. Hence, E_{n_0} is not Lebesgue measurable. The set $E := E_{n_{0-1}}$ has all the desired properties.

Proposition 47 *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

(i) If $\{E_n\}$ is an increasing sequence of subsets of $\mathfrak{M}$ then

$$\mu \left(\bigcup_{n=1}^{\infty} E_n \right) = \lim_{n \to \infty} \mu (E_n) \, .$$

(ii) If $\{E_n\}$ is a decreasing sequence of subsets of $\mathfrak{M}$ and $\mu(E_1) < \infty$ then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. (i) Define

$$F_n := E_n \setminus E_{n-1},$$

where $E_0 := \emptyset$. Since $\{E_n\}$ is an increasing sequence, it follows that the sets F_n are pairwise disjoint with $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n$, and so by the properties of measures we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) = \lim_{l\to\infty} \sum_{n=1}^{l} \mu(F_n)$$
$$= \lim_{l\to\infty} \mu\left(\bigcup_{n=1}^{l} F_n\right) = \lim_{l\to\infty} \mu(E_l).$$

(ii) Apply part (i) to the increasing sequence $\{E_1 \setminus E_n\}$ to get

$$\lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) = \lim_{n\to\infty} \mu(E_1 \setminus E_n) = \mu\left(\bigcup_{n=1}^{\infty} (E_1 \setminus E_n)\right)$$
$$= \mu\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) = \mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Since $\mu(E_1) < \infty$, we get

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

■

Example 48 *Without the hypothesis $\mu(E_1) < \infty$, property (ii) may be false. Indeed, let $E_n := [n, \infty)$. Then $\{E_n\}$ is a decreasing sequence, $\mathcal{L}^1(E_n) = \infty$ for all $n \in \mathbb{N}$, but*

$$\mathcal{L}^1\left(\bigcap_{n=1}^{\infty} E_n\right) = \mathcal{L}^1(\emptyset) = 0 \neq \lim_{n\to\infty} \mathcal{L}^1(E_n) = \infty.$$

Thursday, September 22, 2011

Make-up class: 2 hours

3 Measurable Functions

In this section we introduce the notions of measurable and integrable functions.

Definition 49 *Let X and Y be nonempty sets, and let $\mathfrak{M}$ and $\mathfrak{N}$ be algebras on X and Y, respectively. A function $f : X \to Y$ is said to be* measurable *if $f^{-1}(F) \in \mathfrak{M}$ for every set $F \in \mathfrak{N}$. If $X \subset \mathbb{R}^N$ is a Lesbesgue measurable set and $\mathfrak{M}$ is the σ-algebra of all Lebesgue measurable subsets of X, then a measurable function $f : X \to Y$ is called* Lebesgue measurable *.*

If X and Y are topological spaces, $\mathfrak{M} := \mathcal{B}(X)$ and $\mathfrak{N} := \mathcal{B}(Y)$, then a measurable function $f : X \to Y$ will be called a *Borel function*.

Proposition 50 *If $\mathfrak{M}$ is a σ-algebra on a set X and $\mathfrak{N}$ is the smallest σ-algebra that contains a given family $\mathcal{G}$ of subsets of a set Y, then $f : X \to Y$ is measurable if and only if $f^{-1}(F) \in \mathfrak{M}$ for every set $F \in \mathcal{G}$.*

Proof. The family of sets

$$\left\{F \in \mathfrak{N} : \, f^{-1}(F) \in \mathfrak{M}\right\}$$

is a σ-algebra that contains $\mathcal{G}$, and so it must coincide with $\mathfrak{N}$. ■

Remark 51 *In particular, if in the previous proposition Y is a topological space and $\mathfrak{N} = \mathcal{B}(Y)$, then it suffices to verify that $f^{-1}(A) \in \mathfrak{M}$ for every open set $A \subset Y$. Even more, if $Y = \mathbb{R}$ (respectively $Y = [-\infty, \infty]$) then it suffices to check that $f^{-1}((a, \infty)) \in \mathfrak{M}$ (respectively $f^{-1}((a, \infty]) \in \mathfrak{M}$) for every $a \in \mathbb{R}$ (why?).*

Remark 52 *If X and Y are topological spaces, then, in view of the previous remark, any continuous function $f : X \to Y$ is a Borel function. We used this fact in Example 44.*

Proposition 53 *Let $(X, \mathfrak{M})$, $(Y, \mathfrak{N})$, $(Z, \mathfrak{O})$ be measurable spaces and let $f : X \to Y$ and $g : Y \to Z$ be two measurable functions. Then $g \circ f = X \to Z$ is measurable.*

Proof. For any set $F \in \mathfrak{O}$ we have that $g^{-1}(F) \in \mathfrak{N}$ and so

$$(g \circ f)^{-1}(F) = f^{-1}\left(g^{-1}(F)\right) \in \mathfrak{M}.$$

■

Remark 54 *If X is a topological space, $(Z, \mathfrak{O})$ a measurable space, $f : X \to \mathbb{R}^N$ a continuous or Borel function and $g : \mathbb{R}^N \to Z$ a Lebesgue measurable function, then $g \circ f = X \to Z$ may not be measurable, since for any set $F \in \mathfrak{O}$ the set $g^{-1}(F)$ is not a Borel set but only a Lebesgue measurable set. See Example 44.*

On the other hand, if $(X, \mathfrak{M})$ is a measurable space, $Y = \mathbb{R}^N$ (in this case it does not matter if we take in $\mathbb{R}^N$ the Borel σ-algebra or the σ-algebra of Lebesgue measurable sets), Z is a topological space, $f : X \to \mathbb{R}^N$ is a measurable function, and $g : \mathbb{R}^N \to Z$ is a Borel function (in particular, a continuous function), then $g \circ f = X \to Z$ is measurable. Indeed,for every open set $A \subset Z$ we have that $g^{-1}(A)$ is a Borel set, and so $(g \circ f)^{-1}(F) = f^{-1}\left(g^{-1}(F)\right) \in \mathfrak{M}$.

Corollary 55 *Let $(X, \mathfrak{M})$ be a measurable space and let $f : X \to \mathbb{R}$ (respectively $f : X \to [-\infty, \infty]$) be a measurable function. Then f^2, $|f|$, , f^+, f^-, cf, where $c \in \mathbb{R}$, are measurable.*

Proof. It is enough to consider the Borel functions $g_1(t) = t^2$, $g_2(t) = |t|$, and so on, and to apply the previous result. ∎

Remark 56 *If $c = 0$ and $f : X \to [-\infty, \infty]$ the function cf is defined to be identically equal to zero. Note that by the previous remark, in this case it does not matter if in $\mathbb{R}$ we take the Borel σ-algebra or the σ-algebra of Lebesgue measurable sets.*

Given two measurable spaces $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ we denote by $\mathfrak{M} \otimes \mathfrak{N} \subset \mathcal{P}(X \times Y)$ the smallest σ-algebra that contains all sets of the form $E \times F$, where $E \in \mathfrak{M}$, $F \in \mathfrak{N}$. Then $\mathfrak{M} \otimes \mathfrak{N}$ is called the *product σ-algebra* of $\mathfrak{M}$ and $\mathfrak{N}$.

Proposition 57 *Let $(X, \mathfrak{M})$, $(Y_1, \mathfrak{N}_1), \dots, (Y_n, \mathfrak{N}_n)$ be measurable spaces and consider*

$$(Y_1 \times \ldots \times Y_n, \mathfrak{N}_1 \otimes \ldots \otimes \mathfrak{N}_n).$$

Then the vector-valued function $f : X \to Y_1 \times \ldots \times Y_n$ is measurable if and only if its components $f_i : X \to Y_i$ are measurable functions for all $i = 1, \dots, n$.

Proof. Observe that every $i = 1, \dots, n$ the projection function

$$\begin{aligned} \Pi_i : Y_1 \times \ldots \times Y_n &\to Y_i \\ (y_1, \dots, y_n) &\mapsto y_i \end{aligned}$$

is measurable. Indeed, if $F \in \mathfrak{N}_i$ then

$$\Pi_i^{-1}(F) = Y_1 \times \ldots \times Y_{i-1} \times F \times Y_{i+1} \times \ldots \times Y_n \in \mathfrak{N}_1 \otimes \ldots \otimes \mathfrak{N}_n.$$

Hence, if $f : X \to Y_1 \times \ldots \times Y_n$ is measurable, then $f_i = \Pi_i \circ f$ is also measurable.

Conversely, assume that each $f_i : X \to Y_i$ is measurable, $i = 1, \dots, n$. If $F_i \in \mathfrak{N}_i$, $i = 1, \dots, n$, then

$$f^{-1}(F_1 \times \ldots \times F_n) = f_1^{-1}(F_1) \cap \ldots \cap f_n^{-1}(F_n) \in \mathfrak{M}.$$

The conclusion follows from Proposition 50. ∎

Remark 58 *In the proof we have shown that if $(Y_1, \mathfrak{N}_1), \dots, (Y_n, \mathfrak{N}_n)$ are measurable spaces and we consider*

$$(Y_1 \times \ldots \times Y_n, \mathfrak{N}_1 \otimes \ldots \otimes \mathfrak{N}_n),$$

then for each $i = 1, \dots, n$, the projection function

$$\begin{aligned} \Pi_i : Y_1 \times \ldots \times Y_n &\to Y_i \\ (y_1, \dots, y_n) &\mapsto y_i \end{aligned}$$

is measurable. Hence for every set $F \in \mathfrak{N}_i$*, the inverse image* $\Pi_i^{-1}(F) \in \mathfrak{N}_1 \otimes \ldots \otimes \mathfrak{N}_n$*. However, if* $E \in \mathfrak{N}_1 \otimes \ldots \otimes \mathfrak{N}_n$*, then in general it is* ***not true*** *that* $\Pi_i(E) \in \mathfrak{N}_i$*. In particular, if* $E \subset \mathbb{R}^2$ *is a Lebesgue measurable set, then its projection on one of the axes may not be Lebesgue measurable. Similarly, if* $E \subset \mathbb{R}^2$ *is a Borel set, then its projection on one of the axes may not be a Borel set, but it can be shown that it is Lebesgue measurable (using the theory of analytic and of Souslin sets).*

More generally, if $f : \mathbb{R}^m \to \mathbb{R}$ *is a continuous function or a Borel function, then for every Borel set* $F \in \mathcal{B}(\mathbb{R})$ *we have that the inverse image* $f^{-1}(F)$ *is a Borel set. On the other hand, if* $E \subset \mathbb{R}^m$ *is a Lebesgue measurable set, then its image* $f(E)$ *may not be Lebesgue measurable. Similarly, if* $E \subset \mathbb{R}^m$ *is a Borel set, then its image* $f(E)$ *may not be a Borel set, but it can be shown that it is Lebesgue measurable (using the theory of analytic and of Souslin sets).*

Theorem 59 *Let* X *be a complete, separable metric space and let* $f : X \to \mathbb{R}$ *be a Borel function (or a continuous function). Then for every Borel set* $B \subset X$*, the set* $f(B)$ *is Lebesgue meaurable (but not necessarily a Borel set).*

Proposition 60 *Let* $(X, \mathfrak{M})$ *be a measurable space and let* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *be two measurable functions. Then* $f + g$*,* fg*,* $\min\{f, g\}$*,* $\max\{f, g\}$ *are measurable.*

Proof. By the previous proposition the function

$$\begin{aligned} U : X &\to \mathbb{R}^2 \\ x &\mapsto (f(x), g(x)) \end{aligned}$$

is measurable. It suffices to compose it with the continuous functions $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, \ldots, 4$, defined by

$$g_1(s,t) := s+t, \quad g_2(s,t) := st, \quad g_3(s,t) := \min\{s,t\}, \quad g_3(s,t) := \max\{s,t\},$$

and apply Proposition 53. ■

Remark 61 *The previous proposition continues to hold if* $\mathbb{R}$ *is replaced by* $[-\infty, \infty]$*, provided* $f + g$ *are well-defined, i.e.,* $(f(x), g(x)) \notin \{\pm(\infty, -\infty)\}$ *for all* $x \in X$*. Concerning* fg*, we define* $(fg)(x) := 0$ *whenever* $f(x)$ *or* $g(x)$ *is zero.*

Proposition 62 *Let* $(X, \mathfrak{M})$ *be a measurable space and let* $f_n : X \to [-\infty, \infty]$*,* $n \in \mathbb{N}$*, be measurable functions. Then* $\sup_n f_n$*,* $\inf_n f_n$*,* $\liminf_{n\to\infty} f_n$*, and* $\limsup_{n\to\infty} f_n$ *are measurable.*

Proof. For every $a \in \mathbb{R}$,

$$\left(\sup_n f_n\right)^{-1}((a, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((a, \infty]) \in \mathfrak{M},$$

and so $\sup_n f_n$ is measurable. Similarly

$$\left(\inf_n f_n\right)^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f_n^{-1}([-\infty, a)) \in \mathfrak{M},$$

which implies that $\inf_n f_n$ is measurable. Since

$$\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \sup_{k>n} f_k, \quad \liminf_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} \inf_{k>n} f_k,$$

it follows by the first part that the functions $\liminf_{n\to\infty} f_n$, and $\limsup_{n\to\infty} f_n$ are measurable. ■

Remark 63 *The previous proposition uses in a crucial way the fact that $\mathfrak{M}$ is a σ-algebra.*

Let $(X, \mathfrak{M}, \mu)$ is a measure space. We will see later on that the Lebesgue integration does not "see" sets of measure μ zero. Also, several properties (existence of pointwise limit of a sequence of functions, convergence of a series of functions) do not hold at every point $x \in X$. Thus, it is imporatnt to work with functions f that are defined only on $X \setminus E$ with $\mu(E) = 0$. For this reason, we extend Definition 49 to read as follows.

Definition 64 *Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be two measurable spaces, and let $\mu : \mathfrak{M} \to [0, \infty]$ be a measure. Given a function $f : X \setminus E \to Y$ where $\mu(E) = 0$, f is said to be* measurable over X *if $f^{-1}(F) \in \mathfrak{M}$ for every set $F \in \mathfrak{N}$.*

In general, measurability of f on $X \setminus E$ does not entail the measurability of an arbitrary extension of f to X unless μ is complete. To see this, let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be two measurable spaces, and let $\mu : \mathfrak{M} \to [0, \infty]$ be a measure. Assume that μ is not complete. Then there exists a set $E \in \mathfrak{M}$ such that $\mu(E) = 0$ but E contains a subset E_1 that does not belong to $\mathfrak{M}$. Assume that $f : X \setminus E \to Y$ is measurable (according to the previous definition) and define

$$g(x) := \begin{cases} f(x) & \text{if } x \in X \setminus E, \\ y_1 & \text{if } x \in E_1, \\ y_2 & \text{if } x \in E \setminus E_1, \end{cases}$$

where $y_1, y_2 \in Y$. Consider a set $F \in \mathfrak{N}$ that contains y_1 but not y_2 (if possible). Then

$$\begin{aligned} g^{-1}(F) &= \{x \in X \setminus E : g(x) \in F\} \cup \{x \in E : g(x) \in F\} \\ &= f^{-1}(F) \cup E_1. \end{aligned}$$

Note that $f^{-1}(F) \subset X \setminus E$. Since $f^{-1}(F) \in \mathfrak{M}$, if $f^{-1}(F) \cup E_1 \in \mathfrak{M}$, then by the properties of a σ-algebra we would have that $E_1 = \left(f^{-1}(F) \cup E_1\right) \setminus f^{-1}(F) \in \mathfrak{M}$, which is a contradiction. Thus, $f^{-1}(F) \cup E_1$ does not belong to $\mathfrak{M}$, and so the extension g is not measurable.

On the other hand, if we define

$$w(x) := \begin{cases} f(x) & \text{if } x \in X \setminus E, \\ y_1 & \text{if } x \in E, \end{cases}$$

where $y_1 \in Y$, then w is measurable. Indeed, consider a set $F \in \mathfrak{N}$. Then F contains y_1, then

$$\begin{aligned} w^{-1}(F) &= \{x \in X \setminus E : g(x) \in F\} \cup \{x \in E : g(x) \in F\} \\ &= f^{-1}(F) \cup E \in \mathfrak{M}, \end{aligned}$$

while if F does not contain y_1, then $w^{-1}(F) = f^{-1}(F) \in \mathfrak{M}$. Hence there are extensions of f that are measurable and other that are not. The next result shows that if the measure μ is complete, then this cannot happen.

Proposition 65 *Let $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ be two measurable spaces and let $f : X \to Y$ be a measurable function. Let $\mu : \mathfrak{M} \to [0, \infty]$ be a complete measure. If $g : X \to Y$ is a function such that $f(x) = g(x)$ for μ a.e. $x \in X$, then g is measurable.*

Proof. Let $E = \{x \in X : f(x) \neq g(x)\}$. By hypothesis $\mu(E) = 0$. Hence for any $F \in \mathfrak{N}$

$$\begin{aligned} g^{-1}(F) &= \{x \in X \setminus E : g(x) \in F\} \cup \{x \in E : g(x) \in F\} \\ &= \{x \in X \setminus E : f(x) \in F\} \cup \{x \in E : g(x) \in F\} \\ &= \left(f^{-1}(F) \setminus E\right) \cup \left(g^{-1}(F) \cap E\right) \in \mathfrak{M}, \end{aligned}$$

since $f^{-1}(F)$, $E \in \mathfrak{M}$ by hypothesis, while $\left(g^{-1}(F) \cap E\right) \in \mathfrak{M}$ by the fact that $\mu(E) = 0$ and μ is a complete measure. ■

Going back to the setting in which $f : X \setminus E \to Y$ with $\mu(E) = 0$, since Lebesgue integration does not take into account sets of measure zero, we will see that integration of f depends mostly on its measurability on $X \setminus E$.

Corollary 66 *Let $(X, \mathfrak{M})$ be a measurable space and let $f_n : X \to [-\infty, \infty]$, $n \in \mathbb{N}$, be measurable functions. Let $\mu : \mathfrak{M} \to [0, \infty]$ be a complete measure. If there exists* $\lim_{n\to\infty} f_n(x)$ *for μ a.e. $x \in X$, then* $\lim_{n\to\infty} f_n$ *is measurable.*

Proof. The proof follows from Propositions 62 and 65. ■

Friday, September 23, 2011

4 Lebesgue Integration of Nonnegative Functions

We are now in a position to introduce the notion of integral.

Definition 67 *Let X be a nonempty set and let $\mathfrak{M}$ be an algebra on X. A* simple function *is a measurable function $s : X \to \mathbb{R}$ whose range consists of finitely many points.*

If $c_1, \dots, c_\ell$ are the distinct values of s, then we write

$$s = \sum_{n=1}^{\ell} c_n \chi_{E_n},$$

where χ_{E_n} is the *characteristic function* of the set $E_n := \{x \in X : s(x) = c_n\}$, i.e.,

$$\chi_{E_n}(x) := \begin{cases} 1 & \text{if } x \in E_n, \\ 0 & \text{otherwise.} \end{cases}$$

If μ is a finitely additive (positive) measure on X and $s \geq 0$, then for every measurable set $E \in \mathfrak{M}$ we define the *Lebesgue integral* of s over E as

$$\int_E s \, d\mu := \sum_{n=1}^{\ell} c_n \mu(E_n \cap E), \tag{19}$$

where if $c_n = 0$ and $\mu(E_n \cap E) = \infty$, then we use the convention

$$c_n \mu(E_n \cap E) := 0.$$

Exercise 68 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $s, t : X \to [0, \infty]$ be simple functions.*

(i) Prove that the set function

$$\nu(E) := \int_E s \, d\mu, \quad E \in \mathfrak{M},$$

is a measure.

(ii) Prove that for every $E \in \mathfrak{M}$,

$$\int_E (s + t) \, d\mu = \int_E s \, d\mu + \int_E t \, d\mu.$$

Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function. We define its *(Lebesgue) integral* over a measurable set E as

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu : s \text{ simple}, 0 \leq s \leq f \right\}.$$

Theorem 69 *Let X be a nonempty set, let $\mathfrak{M}$ be an algebra on X, and let $f : X \to [0, \infty]$ be a measurable function. Then there exists a sequence $\{s_n\}$ of simple functions such that*

$$0 \leq s_1(x) \leq s_2(x) \leq \ldots \leq s_n(x) \to f(x)$$

for every $x \in X$. The convergence is uniform on any set on which f is bounded from above.

Proof. For $n \in \mathbb{N}_0$ and $0 \leq k \leq 2^{2n} - 1$ define

$$E_{n,k} := \left\{ x \in X : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\}, \quad E_n := \{x \in X : f(x) \geq 2^n\},$$

and let

$$s_n := \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{E_{n,k}} + 2^n \chi_{E_n}.$$

■

Monday, September 26, 2011

No class, Oxford

Wednesday, September 28, 2011

No class, Oxford.

Friday, September 30, 2011

Proof. If $f(x) = 0$ then $x \in E_{n,0}$, and so $s_n(x) = 0$ for all $n \in \mathbb{N}$. If $0 < f(x) < \infty$ then $x \in E_{n,\lfloor 2^n f(x) \rfloor}$ for all $n \in \mathbb{N}$ sufficiently large, where $\lfloor \cdot \rfloor$ is the integer part. Hence

$$0 \leq s_n(x) = \frac{\lfloor 2^n f(x) \rfloor}{2^n} \leq \frac{\lfloor 2^{n+1} f(x) \rfloor}{2^{n+1}} = s_{n+1}(x) \to f(x)$$

as $n \to \infty$. If $f(x) = \infty$ then $x \in E_n$ for all n, and so $s_n(x) = 2^n \nearrow \infty$.

Note that by the definition of the sets $E_{n,k}$ we have that $0 \leq f - s_n \leq \frac{1}{2^n}$ on the set where $f < 2^n$. Hence we have uniform convergence on each set on which f is bounded from above. ■

In the remainder of this subsection, $\mathfrak{M}$ is a σ-algebra and μ a (countably additive) measure. In view of the previous theorem, if $f : X \to [0, \infty]$ is a measurable function, then we define its *(Lebesgue) integral* over a measurable set E as

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu : s \text{ simple}, 0 \leq s \leq f \right\}.$$

We list below some basic properties of Lebesgue integration for nonnegative functions.

Proposition 70 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f, g : X \to [-\infty, \infty]$ be two measurable functions.*

(i) If $0 \leq f \leq g$, then $\int_E f \, d\mu \leq \int_E g \, d\mu$ for any measurable set E.

(ii) If $c \in [0, \infty]$, then $\int_E cf \, d\mu = c \int_E f \, d\mu$ (here we set $0\infty := 0$).

(iii) If $E \in \mathfrak{M}$ and $f(x) = 0$ for μ a.e. $x \in E$, then $\int_E f \, d\mu = 0$, even if $\mu(E) = \infty$.

(iv) If $E \in \mathfrak{M}$ and $\mu(E) = 0$, then $\int_E f \, d\mu = 0$, even if $f \equiv \infty$ in E.

(v) $\int_E f \, d\mu = \int_X \chi_E f \, d\mu$ for any measurable set E.

Exercise 71 *Prove the previous proposition.*

The next two results are central in the theory of integration of nonnegative functions.

Theorem 72 (Lebesgue monotone convergence theorem) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $f_n : X \to [0, \infty]$ *be a sequence of measurable functions such that*

$$0 \le f_1(x) \le f_2(x) \le \ldots \le f_n(x) \to f(x)$$

for every $x \in X$*. Then* f *is measurable and*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof. By Proposition 62 the function f is measurable, and since $f_n \le f_{n+1} \le f$ we have that $\int_X f_n \, d\mu \le \int_X f_{n+1} \, d\mu \le \int_X f \, d\mu$. In particular there exists

$$\lim_{n\to\infty} \int_X f_n \, d\mu =: \ell \in [0, \infty]$$

and $\ell \le \int_X f \, d\mu$. To prove the opposite inequality, let s be a simple function, with $0 \le s \le f$. Fix $0 < c < 1$ and for $n \in \mathbb{N}$ define

$$E_n := \{x \in X : \, f_n(x) \ge cs(x)\}.$$

Since f_n and s are measurable and $f_n \le f_{n+1}$, it follows that E_n is measurable and $E_n \subset E_{n+1}$. We claim that

$$X = \bigcup_{n=1}^{\infty} E_n.$$

To see this, fix $x \in X$. If $f(x) = 0$, then $f_n(x) = 0$ for all $n \in \mathbb{N}$ and $s(x) = 0$, and so $x \in E_n$ for all $n \in \mathbb{N}$. If $f(x) > 0$, then $f(x) > cs(x)$ and since $f_n(x) \to f(x)$, we may find $n \in \mathbb{N}$ so large that $f_n(x) > cs(x)$. Thus $x \in E_n$ and the claim is proved.

Using the fact that $f_n \ge 0$ and the definition of E_n and we have that

$$\int_X f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge \int_{E_n} cs \, d\mu = c \int_{E_n} s \, d\mu.$$

By Exercise 68, the set function

$$\nu(E) := \int_E s \, d\mu, \quad E \in \mathfrak{M},$$

is a measure, and so by Proposition 47,

$$\int_X f_n \, d\mu \ge c \int_{E_n} s \, d\mu = c\nu(E_n) \to c\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = c\nu(X).$$

Thus

$$\ell \geq c\nu(X) = c\int_X s\, d\mu.$$

Letting $c \nearrow 1$ we obtain that

$$\ell \geq \int_X s\, d\mu,$$

and given the arbitrariness of the simple function s below f, taking the supremum over all such admissible functions s yields

$$\ell \geq \int_X f\, d\mu.$$

This concludes the proof. ■

Monday, October 3, 2011

Remark 73 *The previous theorem continues to hold if we assume that $f_n(x) \to f(x)$ for μ a.e. $x \in X$. Indeed, in view of Proposition 70(iv), it suffices to re-define f_n and f to be zero in the set of measure zero in which there is no pointwise convergence.*

Example 74 *The Lebesgue monotone convergence theorem does not hold in general for decreasing sequences. Indeed, consider $X = \mathbb{R}$ and let μ be the Lebesgue measure $\mathcal{L}^1$. Define*

$$f_n := \frac{1}{n}\chi_{[n,\infty)}.$$

Then $f_n \geq f_{n+1}$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n\, dx = \infty \neq 0 = \int_{\mathbb{R}} \lim_{n\to\infty} f_n\, dx.$$

Corollary 75 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f,\, g : X \to [0,\infty]$ be two measurable functions. Then*

$$\int_X (f+g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu.$$

Proof. By Theorem 69 there exist two sequences $\{s_n\}$ and $\{t_n\}$of simple functions such that

$$0 \leq s_1(x) \leq s_2(x) \leq \ldots \leq s_n(x) \to f(x),$$
$$0 \leq t_1(x) \leq t_2(x) \leq \ldots \leq t_n(x) \to g(x)$$

for every $x \in X$. By Exercise 68,

$$\int_X (s_n + t_n)\, d\mu = \int_X s_n\, d\mu + \int_X t_n\, d\mu.$$

The conclusion follows from Lebesgue's monotone convergence theorem. ■

Corollary 76 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f_n : X \to [0, \infty]$ be a sequence of measurable functions. Then*

$$\sum_{n=1}^{\infty} \int_X f_n \, d\mu = \int_X \sum_{n=1}^{\infty} f_n \, d\mu.$$

Proof. Apply the Lebesgue monotone convergence theorem to the increasing sequence of partial sums and use linearity of the integral. ■

Example 77 *Given a doubly indexed sequence $\{a_{n,k}\}$, with $a_{n,k} \geq 0$ for all n, $k \in \mathbb{N}$, we have*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k}.$$

To see this, it suffices to consider $X = \mathbb{N}$ with counting measure and to define $f_n : \mathbb{N} \to [0, \infty]$ by $f_n(k) := a_{n,k}$. Then

$$\int_X f_n \, d\mu = \sum_{k=1}^{\infty} a_{n,k},$$

and the result now follows from the previous corollary.

Lemma 78 (Fatou lemma) *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

(i) If $f_n : X \to [0, \infty]$ is a sequence of measurable functions, then

$$f := \liminf_{n \to \infty} f_n$$

is a measurable function and

$$\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu;$$

(ii) if $f_n : X \to [-\infty, \infty]$ is a sequence of measurable functions such that

$$f_n \leq g$$

for some measurable function $g : X \to [0, \infty]$ with $\int_X g \, d\mu < \infty$, then

$$f := \limsup_{n \to \infty} f_n$$

is a measurable function and

$$\int_X f \, d\mu \geq \limsup_{n \to \infty} \int_X f_n \, d\mu.$$

Proof. (i) For $n \in \mathbb{N}$ define

$$g_n := \inf_{k \geq n} f_k.$$

Then $g_n \leq f_n$, and so

$$\int_X g_n \, d\mu \leq \int_X f_n \, d\mu.$$

Since $g_n \leq g_{n+1}$, by Lebesgue's monotone convergence theorem

$$\begin{aligned} \int_X \liminf_{n \to \infty} f_n \, d\mu &= \int_X \lim_{n \to \infty} g_n \, d\mu = \lim_{n \to \infty} \int_X g_n \, d\mu \\ &\leq \liminf_{n \to \infty} \int_X f_n \, d\mu. \end{aligned}$$

(ii) Apply the first part to the sequence $g - f_n$. ■

Example 79 *Fatou's lemma (i) fails for real-valued functions. Indeed, consider* $X = \mathbb{R}$ *and let* μ *be the Lebesgue measure* $\mathcal{L}^1$*. Define*

$$f_n := -\frac{1}{n} \chi_{[0,n]}.$$

Then

$$\liminf_{n \to \infty} \int_{\mathbb{R}} f_n \, dx = -1 < 0 = \int_{\mathbb{R}} \lim_{n \to \infty} f_n \, dx.$$

Corollary 80 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $f : X \to [0, \infty]$ *be a measurable function. Then*

$$\int_X f \, d\mu = 0$$

if and only if $f(x) = 0$ *for* μ *a.e.* $x \in X$.

Proof. Assume that $\int_X f \, d\mu = 0$ and for $n \in \mathbb{N}$ define

$$E_n := \left\{ x \in X : \, f(x) \geq \frac{1}{n} \right\}.$$

Then $f \geq \frac{1}{n} \chi_{E_n}$, and so

$$0 = \int_X f \, d\mu \geq \int_X \frac{1}{n} \chi_{E_n} \, d\mu = \frac{1}{n} \mu(E_n).$$

Since

$$E := \{x \in X : \, f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n,$$

it follows that $\mu(E) = 0$. Thus $f(x) = 0$ for μ a.e. $x \in X$.

Conversely, assume that $f(x) = 0$ for μ a.e. $x \in X$ and define $E := \{x \in X : f(x) > 0\}$. Let $f_n := n\chi_E$. Then for all $x \in X$,

$$f(x) \leq \liminf_{n\to\infty} n\chi_E(x) = \begin{cases} \infty & \text{if } x \in E, \\ 0 & \text{otherwise,} \end{cases}$$

and so by Fatou's lemma

$$\begin{aligned} 0 \leq \int_X f\,d\mu &\leq \int_X \liminf_{n\to\infty} f_n\,d\mu \leq \liminf_{n\to\infty} \int_X f_n\,d\mu \\ &= \liminf_{n\to\infty} n\mu(E) = 0. \end{aligned}$$

Hence $\int_X f\,d\mu = 0$ and the proof is concluded. ■

5 Lebesgue Integration of Functions of Arbitrary Sign

In order to extend the notion of integral to functions of arbitrary sign, consider $f : X \to [-\infty, \infty]$ and set

$$f^+ := \max\{f, 0\}, \quad f^- := \max\{-f, 0\}.$$

Note that $f = f^+ - f^-$, $|f| = f^+ + f^-$, and f is measurable if and only if f^+ and f^- are measurable. Also, if f is bounded, then so are f^+ and f^-, and in view of Theorem 69, f is then the uniform limit of a sequence of simple functions.

Definition 81 *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f : X \to [-\infty, \infty]$ be a measurable function. Given a measurable set $E \in \mathfrak{M}$, if at least one of the two integrals $\int_E f^+\,d\mu$ and $\int_E f^-\,d\mu$ is finite, then we define the* (Lebesgue) integral of f *over the measurable set E by*

$$\int_E f\,d\mu := \int_E f^+\,d\mu - \int_E f^-\,d\mu.$$

If both $\int_E f^+\,d\mu$ and $\int_E f^-\,d\mu$ are finite, then f is said to be (Lebesgue) integrable *over the measurable set E.*

In the special case that μ is the Lebesgue measure, we denote $\int_E f\,d\mathcal{L}^N$ simply by

$$\int_E f\,dx.$$

If $(X, \mathfrak{M}, \mu)$ is a measure space, with X a topological space, and if $\mathfrak{M}$ contains $\mathcal{B}(X)$, then $f : X \to [-\infty, \infty]$ is said to be *locally integrable* if it is Lebesgue integrable over every compact set.

A measurable function $f : X \to [-\infty, \infty]$ is Lebesgue integrable over the measurable set E if and only if

$$\int_E |f|\,d\mu < \infty.$$

This property is one of the main differences between the Lebesgue integral and the improper Riemann integral.

Example 82 *Consider the function*

$$f(x) := \frac{\sin x}{x}, \quad x \geq \pi.$$

We show there exists the limit

$$\lim_{\ell\to\infty} \int_\pi^\ell \frac{\sin x}{x}\,dx \in \mathbb{R},$$

so that f is Riemann-integrable in $[\pi,\infty)$, while

$$\int_\pi^\infty \left(\frac{\sin x}{x}\right)^+ dx = \int_\pi^\infty \left(\frac{\sin x}{x}\right)^- dx = \infty,$$

so that the Lebesgue integral of f is not defined.

To see the first, integrate by parts

$$\begin{aligned}\int_\pi^\ell \frac{\sin x}{x}\,dx &= \left[\frac{-\cos x}{x}\right]_{x=\pi}^{x=\ell} dx - \int_\pi^\ell \frac{\cos x}{x^2}\,dx \\ &= -\frac{\cos \ell}{\ell} + \frac{1}{\pi} - \int_\pi^\ell \frac{\cos x}{x^2}\,dx.\end{aligned}$$

Now

$$\left|\int_t^s \frac{\cos x}{x^2}\,dx\right| \leq \int_t^s \frac{1}{x^2}\,dx = -\frac{1}{s} + \frac{1}{t} \to 0$$

as $s,t\to\infty$, and so there exists $\lim_{\ell\to\infty}\int_\pi^\ell \frac{\cos x}{x^2}\,dx$. In turn, there exists

$$\begin{aligned}\lim_{\ell\to\infty}\int_\pi^\ell \frac{\sin x}{x}\,dx &= \left[\frac{-\cos x}{x}\right]_{x=\pi}^{x=\ell} dx - \int_\pi^\ell \frac{\cos x}{x^2}\,dx \\ &= \lim_{\ell\to\infty} -\frac{\cos \ell}{\ell} + \frac{1}{\pi} - \lim_{\ell\to\infty}\int_\pi^\ell \frac{\cos x}{x^2}\,dx \\ &= 0 + \frac{1}{\pi} - \int_\pi^\infty \frac{\cos x}{x^2}\,dx.\end{aligned}$$

On the other hand, by periodicity

$$\begin{aligned}\int_\pi^\infty \left|\frac{\sin x}{x}\right| dx &= \sum_{k=1}^\infty \int_{(2k-1)\pi}^{2k\pi} \left|\frac{\sin x}{x}\right| dx \\ &\geq \sum_{k=1}^\infty \frac{1}{(2k-1)\pi}\int_{(2k-1)\pi}^{2k\pi} |\sin x|\,dx \\ &= \sum_{k=1}^\infty \frac{1}{(2k-1)\pi}\int_\pi^{2\pi} |\sin x|\,dx = \frac{2}{\pi}\sum_{k=1}^\infty \frac{1}{(2k-1)\pi} = \infty,\end{aligned}$$

and so the Lebesgue integral of f is not defined.

Wednesday, October 5, 2011

Proposition 83 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let f, $g : X \to [-\infty, \infty]$ be two measurable functions.*

(i) If f and g are integrable and α, $\beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable and

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu.$$

(ii) $\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu$.

(iii) If f is Lebesgue integrable, then the set $\{x \in X : |f(x)| = \infty\}$ has measure zero.

(iv) If $f(x) = g(x)$ for μ a.e. $x \in X$, then $\int_X f^{\pm} \, d\mu = \int_X g^{\pm} \, d\mu$, so that $\int_X f \, d\mu$ is well-defined if and only if $\int_X g \, d\mu$ is well-defined, and in this case we have

$$\int_X f \, d\mu = \int_X g \, d\mu. \tag{20}$$

Exercise 84 *Prove the previous proposition.*

Property (20) shows that the Lebesgue integral does not distinguish functions that coincide μ a.e. in X. This motivates the next definition.

Finally, if $F \in \mathfrak{M}$ is such that $\mu(F) = 0$ and $f : X \setminus F \to [-\infty, \infty]$ is a measurable function in the sense of Definition 64, then we define the *(Lebesgue) integral of* f over the measurable set E as the Lebesgue integral of the function

$$g(x) := \begin{cases} f(x) & \text{if } x \in X \setminus F, \\ 0 & \text{otherwise,} \end{cases}$$

provided $\int_E g \, d\mu$ is well-defined. Note that in this case

$$\int_E g \, d\mu = \int_E \tilde{v} \, d\mu,$$

where

$$\tilde{v}(x) := \begin{cases} f(x) & \text{if } x \in X \setminus F, \\ w(x) & \text{otherwise,} \end{cases}$$

and w is an arbitrary measurable function defined on F. If the measure μ is complete, then $\int_E g \, d\mu$ is well-defined if and only if $\int_{E \setminus F} f \, d\mu$ is well-defined.

For functions of arbitrary sign we have the following convergence result.

Theorem 85 (Lebesgue dominated convergence theorem) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f_n : X \to [-\infty, \infty]$ be a sequence of measurable functions such that*

$$\lim_{n \to \infty} f_n(x) = f(x)$$

for μ a.e. $x \in X$. If there exists a Lebesgue integrable function g such that

$$|f_n(x)| \leq g(x)$$

for μ a.e. $x \in X$ and all $n \in \mathbb{N}$, then f is Lebesgue integrable and

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0.$$

In particular,

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Proof. By modifying each f_n and g on a set of measure zero and using (20) we can assume that $\lim_{n \to \infty} f_n(x) = f(x)$ exists for all $x \in X$ and that

$$|f_n(x)| \leq g(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Hence f is measurable by Proposition 62, and by Fatou's lemma

$$\int_X |f| \, d\mu \leq \liminf_{n \to \infty} \int_X |f_n| \, d\mu \leq \int_X |g| \, d\mu < \infty.$$

Thus f is integrable. Since $g \pm f_n \geq 0$, again by Fatou's lemma we have

$$\begin{aligned} \int_X g \, d\mu \pm \int_X f \, d\mu = \int_X (g \pm f) \, d\mu &\leq \liminf_{n \to \infty} \int_X (g \pm f_n) \, d\mu \\ &= \int_X g \, d\mu + \liminf_{n \to \infty} \int_X (\pm f_n) \, d\mu. \end{aligned}$$

Using the fact that $\int_X g \, d\mu \in \mathbb{R}$, we can rewrite the previous two inequalities as

$$\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu \leq \limsup_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu,$$

and so the theorem holds. ■

Example 86 *If g is not integrable then the theorem fails in general. Indeed, consider $X = [0,1]$ and let μ be the Lebesgue measure $\mathcal{L}^1$. Define*

$$f_n := n \chi_{[0, \frac{1}{n}]}.$$

Then

$$\lim_{n \to \infty} \int_0^1 f_n \, dx = 1 \neq 0 = \int_0^1 \lim_{n \to \infty} f_n \, dx.$$

Example 87 *We calculate the limit*

$$\lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{a-1} \, dx,$$

where $a > 0$. Define the sequence of functions

$$f_n(x) = \chi_{(0,n)}(x)\left(1 - \frac{x}{n}\right)^n x^{a-1}, \quad x > 0.$$

Note that f_n is Borel measurable, since it is the product of the Borel measurable function $\chi_{(0,n)}$ and the continuous (and hence Borel measurable) function $\left(1 - \frac{x}{n}\right)^n x^{a-1}$. Moreover, for all $x > 0$ and $n \in \mathbb{N}$,

$$\left|\chi_{(0,n)}(x)\left(1 - \frac{x}{n}\right)^n x^{a-1}\right| \leq e^{-x}x^{a-1}.$$

Hence, to apply the Lebesgue dominated convergence theorem, we need to show that the function $g(x) := e^{-x}x^{a-1}$ is integrable. If $x \in (0,1)$, then

$$0 \leq e^{-x}x^{a-1} \leq x^{a-1}$$

and $\int_0^1 x^{a-1}\,dx = \frac{1}{a} < \infty$, while to see what happens at infinity we write

$$e^{-x}x^{a-1} = e^{-\frac{x}{2}}x^{a-1}e^{-\frac{x}{2}}.$$

Since $\lim_{x\to\infty} e^{-\frac{x}{2}}x^{a-1} = 0$, taking $\varepsilon = 1$ we can find $M > 0$ such that

$$0 \leq e^{-\frac{x}{2}}x^{a-1} \leq 1$$

for all $x > M$. Hence $e^{-x}x^{a-1} = e^{-\frac{x}{2}}x^{a-1}e^{-\frac{x}{2}} \leq e^{-\frac{x}{2}}$ for all $x > M$ and $\int_M^\infty e^{-\frac{x}{2}}\,dx = 2e^{-\frac{1}{2}M} < \infty$. Finally, if $x \in [1, M]$ the function g is continuous on a compact set, and so it is bounded. Hence it is integrable in $[1, M]$. Thus we have shown that g is integrable. Hence, by Lebesgue dominated convergence theorem

$$\begin{aligned}
\lim_{n\to\infty}\int_0^n \left(1 - \frac{x}{n}\right)^n x^{a-1}\,dx &= \lim_{n\to\infty}\int_0^\infty \chi_{(0,n)}(x)\left(1 - \frac{x}{n}\right)^n x^{a-1}\,dx \\
&= \int_0^\infty \lim_{n\to\infty}\chi_{(0,n)}(x)\left(1 - \frac{x}{n}\right)^n x^{a-1}\,dx \\
&= \int_0^\infty e^{-x}x^{a-1}\,dx.
\end{aligned}$$

Corollary 88 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f_n : X \to [-\infty, \infty]$ be a sequence of measurable functions. If*

$$\sum_{n=1}^\infty \int_X |f_n|\,d\mu < \infty,$$

then the series $\sum_{n=1}^\infty f_n(x)$ converges for μ a.e. $x \in X$, the function

$$f(x) := \sum_{n=1}^\infty f_n(x),$$

defined for μ a.e. $x \in X$, is integrable, and

$$\sum_{n=1}^\infty \int_X f_n\,d\mu = \int_X \sum_{n=1}^\infty f_n\,d\mu.$$

Proof. Let $g(x) := \sum_{n=1}^{\infty} |f_n(x)|$. By Corollary 76

$$\int_X g\, d\mu = \sum_{n=1}^{\infty} \int_X |f_n|\, d\mu < \infty,$$

and so g is integrable, and, in turn, by Remark 83, the function g is finite μ a.e. in X. Hence the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent for μ a.e. $x \in X$, and, in particular, it converges μ a.e. $x \in X$. Moreover, if we define

$$s_n(x) := \sum_{k=1}^{n} f_k(x)$$

for μ a.e. $x \in X$, then

$$|s_n(x)| \leq \sum_{k=1}^{n} |f_k(x)| \leq g(x)$$

for μ a.e. $x \in X$ and $s_n(x) \to f(x)$ for μ a.e. $x \in X$. Thus, we are in a position to apply Lebesgue's dominated convergence theorem to conclude that

$$\sum_{n=1}^{\infty} \int_X f_n\, d\mu = \int_X \sum_{n=1}^{\infty} f_n\, d\mu,$$

which gives the desired result. ■

Friday, October 7, 2011

6 Modes of Convergence

In this subsection we study different modes of convergence and their relation to one another.

Definition 89 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f_n,\ f : X \to \mathbb{R}$ be measurable functions.*

(i) *$\{f_n\}$ is said to* converge to f pointwise μ a.e. *if there exists a set $E \in \mathfrak{M}$ such that $\mu(E) = 0$ and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X \setminus E$;

(ii) *$\{f_n\}$ is said to* converge to f almost uniformly *if for every $\varepsilon > 0$ there exists a set $E \in \mathfrak{M}$ such that $\mu(E) < \varepsilon$ and $\{f_n\}$ converges to f uniformly in $X \setminus E$, that is,*

$$\lim_{n\to\infty} \sup_{x\in X\setminus E} |f_n(x) - f(x)| = 0;$$

(iii) $\{f_n\}$ *is said to* converge to f in measure *if for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0,$$

(iv) $\{f_n\}$ *is said to* converge to f in $L^1(X)$ *if* f_n, $n \in \mathbb{N}$, *and* f *are integrable and*

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

Example 90 *The sequence of functions* $f_n(x) = x^n$ *converges to* $f(x) = 0$ *almost uniformly in* $[0,1]$ *but not uniformly. Indeed,*

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } 0 \le x < 1, \end{cases}$$

and since the limit is discontinuous, there cannot be uniform convergence in $[0,1]$*. However, given* $\varepsilon \in (0,1)$*, let* $E = [1-\varepsilon, 1]$*. Then* $\mathcal{L}^1(E) = \varepsilon$ *and*

$$\sup_{x\in[0,1]\setminus E} |f_n(x) - 0| = \sup_{x\in[0,1-\varepsilon)} |x^n| = (1-\varepsilon)^n \to 0$$

and thus we have almost uniform convergence.

The next theorem relates the types of convergence introduced in Definition 89.

Theorem 91 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* f_n, $f : X \to \mathbb{R}$ *be measurable functions.*

(i) If $\{f_n\}$ *converges to* f *almost uniformly, then it converges to* f *in measure and pointwise* μ *almost everywhere;*

(ii) if $\{f_n\}$ *converges to* f *in measure, then there exists a subsequence* $\{f_{n_k}\}$ *such that* $\{f_{n_k}\}$ *converges to* f *almost uniformly (and hence pointwise* μ *almost everywhere);*

(iii) if $\{f_n\}$ *converges to* f *in* $L^1(X)$*, then it converges to* f *in measure and there exist a subsequence* $\{f_{n_k}\}$ *and an integrable function* g *such that* $\{f_{n_k}\}$ *converges to* f *almost uniformly (and hence pointwise* μ *almost everywhere) and* $|f_{n_k}(x)| \le g(x)$ *for* μ *a.e.* $x \in X$ *and for all* $k \in \mathbb{N}$*.*

Proof. (i) Assume that $\{f_n\}$ converges to f almost uniformly. Then for every ε, $\delta > 0$ there exists a set $E_\delta \in \mathfrak{M}$ such that $\mu(E_\delta) < \delta$ and $\{f_n\}$ converges to f uniformly in $X \setminus E_\delta$. Hence we may find an integer $\bar{n} \in \mathbb{N}$ such that

$$\sup_{x\in X\setminus E_\delta} |f_n(x) - f(x)| \le \varepsilon$$

for all $n \ge \bar{n}$. In turn,

$$\mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) \le \mu(E_\delta) < \delta$$

for all $n \geq \bar{n}$, which shows that $\{f_n\}$ converges to f in measure.

To see that it converges to f pointwise μ almost everywhere, for every $k \in \mathbb{N}$ find a set $E_k \in \mathfrak{M}$ such that $\mu(E_k) < \frac{1}{k}$ and $\{f_n\}$ converges to f uniformly in $X \setminus E_k$. Without loss of generality, we may assume that that $E_{k+1} \subset E_k$ for all $k \in \mathbb{N}$. Let

$$E := \bigcap_{k=1}^{\infty} E_k.$$

By Proposition 47

$$\mu(E) = \lim_{k \to \infty} \mu(E_k) = 0.$$

If $x \in X \setminus E$, then $x \notin E_k$ for some k, and so by the uniform convergence in $X \setminus E_k$, $\lim_{n \to \infty} f_n(x) = f(x)$. This shows that $\{f_n\}$ converges to f pointwise μ almost everywhere. ■

Example 92 *Pointwise convergence μ almost everywhere does not imply convergence in measure or in $L^1(X)$ or almost uniform convergence. Let $X = \mathbb{R}$, $\mathfrak{M} = \mathcal{B}(\mathbb{R})$, and let μ be the Lebesgue measure. Define $f_n := \chi_{[n,\infty)}$. Then $\{f_n\}$ converges to zero pointwise, since for every $x \in \mathbb{R}$ we have that*

$$\chi_{[n,\infty)}(x) = 0$$

for all $n > x$. Next we show that $\{f_n\}$ does not converge to zero in measure (and hence nor in $L^1(\mathbb{R})$). Indeed, if $0 < \varepsilon < 1$, then

$$\mathcal{L}^1(\{x \in \mathbb{R} : |f_n(x) - 0| > \varepsilon\}) = \mathcal{L}^1([n,\infty)) = \infty.$$

Finally, we show that $\{f_n\}$ does not converge to zero almost uniformly. Indeed for every set $E \subset \mathcal{B}(\mathbb{R})$, with $\mathcal{L}^1(E) \leq \varepsilon$, we have that

$$\mathcal{L}^1([n,\infty) \setminus E) = \mathcal{L}^1([n,\infty)) - \mathcal{L}^1(E) = \infty,$$

and so for every n there exists $x_n \in [n,\infty) \setminus E$. Thus,

$$|f_n(x_n) - 0| = 1$$

and so

$$\sup_{x \in [n,\infty) \setminus E} |f_n(x) - 0| = 1 \nrightarrow 0.$$

Proof of Theorem 91 continued.
(ii) Assume that $\{f_n\}$ converges to f in measure. For every $k \in \mathbb{N}$ find an integer $n_k \in \mathbb{N}$ such that

$$\mu\left(\left\{x \in X : |f_n(x) - f(x)| > \frac{1}{2^k}\right\}\right) \leq \frac{1}{2^{k+1}}$$

for all $n \geq n_k$. The sequence $\{n_k\}_{k \in \mathbb{N}}$ can be chosen to be strictly increasing. We claim that $\{f_{n_k}\}$ converges to f almost uniformly. To see this, for every $k \in \mathbb{N}$ define

$$E_k := \left\{x \in X : |f_{n_k}(x) - f(x)| > \frac{1}{2^k}\right\}$$

and

$$F_k := \bigcup_{j=k}^{\infty} E_j.$$

Then

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) \leq \sum_{j=k}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^k}.$$

Hence for any fixed $\varepsilon > 0$ we may find k be so large that $\frac{1}{2^k} \leq \varepsilon$.

We prove that $\{f_{n_j}\}$ converges uniformly to f in $X \setminus F_k$. To see this, fix $\delta > 0$ and let $m \geq k$ be so large that $\frac{1}{2^m} \leq \delta$. If $x \in X \setminus F_k$, then $x \notin E_j$ for all $j \geq m$, and so

$$\left|f_{n_j}(x) - f(x)\right| \leq \frac{1}{2^j} \leq \frac{1}{2^m} \leq \delta.$$

Hence

$$\sup_{x \in X \setminus F_k} \left|f_{n_j}(x) - f(x)\right| \leq \delta$$

for all $j \geq m$. ■

Example 93 *Convergence in measure or in $L^1(X)$ does not imply pointwise convergence μ almost everywhere. Let $X = [0,1)$, $\mathfrak{M} = \mathcal{B}([0,1))$, and let μ be the Lebesgue measure. Consider the sequence of intervals $\left[0, \frac{1}{2}\right)$, $\left[\frac{1}{2}, 1\right)$, $\left[0, \frac{1}{3}\right)$, $\left[\frac{1}{3}, \frac{2}{3}\right)$, $\left[\frac{2}{3}, 1\right)$, $\left[0, \frac{1}{4}\right)$, $\left[\frac{1}{4}, \frac{2}{4}\right)$, $\left[\frac{2}{4}, \frac{3}{4}\right)$, $\left[\frac{3}{4}, 1\right)$, ... Let f_n be the characteristic function of the nth interval I_n of this sequence. Then $\{f_n\}$ converges to zero in measure and in $L^1(X)$, since*

$$\int_{[0,1)} |f_n - f| \, dx = \int_{[0,1)} |\chi_{I_n}| \, dx = \mathcal{L}^1(I_n) \to 0,$$

$$\mathcal{L}^1(\{x \in [0,1) : |f_n(x) - f(x)| > \varepsilon\}) = \mathcal{L}^1(I_n) \to 0$$

but the limit

$$\lim_{n \to \infty} f_n(x)$$

does not exist for every $x \in [0,1)$, since for every fixed x we can find infinitely many n such that x belongs to I_n, so that $f_n(x) = 1$ for those n, and infinitely many k such that x does not belong to I_k, so that $f_k(x) = 0$ for those k.

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Proof. (iii) Assume that $\{f_n\}$ converges to f in $L^1(X)$ and fix $\varepsilon > 0$. Let

$$E_{n,\varepsilon} := \{x \in X : |f_n(x) - f(x)| > \varepsilon\}.$$

Then

$$\mu(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon} \int_{E_{n,\varepsilon}} |f_n - f| \, d\mu \leq \frac{1}{\varepsilon} \int_X |f_n - f| \, d\mu \to 0$$

as $n \to \infty$. Thus $\{f_n\}$ converges to f in measure. To prove the last part of the statement, for every $k \in \mathbb{N}$ find an integer $n_k \in \mathbb{N}$ such that

$$\int_X |f_{n_k} - f| \, d\mu \le \frac{1}{2^k}.$$

The sequence $\{n_k\}_{k\in\mathbb{N}}$ can be chosen to be strictly increasing. For $x \in X$ define

$$w(x) := \sum_{k=1}^{\infty} |f_{n_k}(x) - f(x)|.$$

By Corollary 76,

$$\int_X w \, d\mu \le 1.$$

Hence by Remark 83, $w(x) < \infty$ for μ a.e. $x \in X$. At any such point $x \in X$ we have that

$$\lim_{k\to\infty} |f_{n_k}(x) - f(x)| = 0.$$

Finally, observe that

$$|f_{n_k}(x)| \le |f_{n_k}(x) - f(x)| + |f(x)| \le w(x) + |f(x)| =: g(x)$$

for all $x \in X$ and $k \in \mathbb{N}$. This concludes the proof. ■

Concerning Example 92 above, we remark that when the measure is finite, then convergence μ almost everywhere implies almost uniform convergence. This follows from the next theorem.

Theorem 94 (Egoroff) *Let $(X, \mathfrak{M}, \mu)$ be a measure space with μ finite and let $f_n : X \to \mathbb{R}$ be measurable functions converging pointwise μ almost everywhere. Then $\{f_n\}$ converges almost uniformly (and hence in measure).*

Proof. By modifying each f_n on a set of measure zero we can assume that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for all $x \in X$. For every n, $k \in \mathbb{N}$ define

$$E_{n,k} := \bigcup_{j=n}^{\infty} \left\{ x \in X : |f_j(x) - f(x)| \ge \frac{1}{k} \right\}.$$

Note that $E_{n+1,k} \subset E_{n,k}$. Moreover, since $\lim_{n\to\infty} f_n(x) = f(x)$ exists for all $x \in X$, for every fixed $k \in \mathbb{N}$,

$$\bigcap_{n=1}^{\infty} E_{n,k} = \emptyset.$$

Since $\mu(X) < \infty$, by Proposition 47 for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \mu(E_{n,k}) = \mu\left(\bigcap_{n=1}^{\infty} E_{n,k} \right) = 0.$$

Hence for any fixed $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists and integer $n_k \in \mathbb{N}$ such that

$$\mu\left(E_{n_k,k}\right) < \frac{\varepsilon}{2^k}.$$

Let

$$E := \bigcup_{k=1}^{\infty} E_{n_k,k}.$$

Then $\mu(E) < \varepsilon$ and if $x \in X \setminus E$, then $x \notin E_{n_k,k}$ for all $k \in \mathbb{N}$, which implies that

$$|f_j(x) - f(x)| < \frac{1}{k}$$

for all $j \geq n_k$. Hence

$$\sup_{x \in X \setminus E} |f_j(x) - f(x)| \leq \frac{1}{k}$$

for all $j \geq n_k$. This implies uniform convergence in $X \setminus E$. ■

In order to characterize convergence in L^1 we need to introduce the notion of equi-integrability.

Definition 95 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. A family $\mathcal{F}$ of measurable functions $f : X \to [-\infty, \infty]$ is said to be* equi-integrable *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E |f| \, d\mu \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for every measurable set $E \subset X$ with $\mu(E) \leq \delta$.

Exercise 96 *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

(i) Prove that if $f : X \to \mathbb{R}$ is integrable, then $\{f\}$ is equi-integrable.

(ii) Prove that any finite family of integrable functions is equi-integrable.

(iii) Prove that any family $\mathcal{F}$ of measurable functions for which there exists and integrable function g such that

$$|f(x)| \leq g(x)$$

for μ a.e. $x \in X$ and for all $f \in \mathcal{F}$ is equi-integrable.

Example 97 *The converse of (i) does not hold. Indeed, let $X = \mathbb{R}$, $\mathfrak{M} = \mathcal{B}(\mathbb{R})$, and let μ be the Lebesgue measure. Then for any bounded measurable function f, we have that $\{f\}$ is equi-integrable. Indeed, if $|f(x)| \leq M$ for all $x \in \mathbb{R}$, then*

$$\int_E |f| \, dx \leq M\mathcal{L}^1(E) \leq \varepsilon$$

for every Borel set $E \subset \mathbb{R}$ with $\mathcal{L}^1(E) \leq \delta$, provided we take $\delta = \frac{\varepsilon}{M+1}$. However f may not be integrable.

Theorem 98 (Vitali convergence theorem) *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f_n, f : X \to \mathbb{R}$ be integrable functions. Then $\{f_n\}$ converges to f in $L^1(X)$ if and only if*

(i) $\{f_n\}$ converges to f in measure;

(ii) $\{f_n\}$ is equi-integrable;

(iii) for every $\varepsilon > 0$ there exists $E \subset X$ with $E \in \mathfrak{M}$ such that $\mu(E) < \infty$ and

$$\int_{X \setminus E} |f_n| \, d\mu \leq \varepsilon$$

for all n.

Proof. We prove that conditions (i)–(iii) imply convergence in $L^1(X)$. By the previous exercise the function f satisfies conditions (i) and (ii).

Fix $\varepsilon > 0$ and let $\delta > 0$ be as in Definition 95 and $E_\varepsilon \in \mathfrak{M}$ be as in (iii). Since f satisfies property (i), by taking δ smaller, if necessary, we may assume that for every measurable set $E \subset X$ with $\mu(E) \leq \delta$,

$$\int_E |f| \, d\mu \leq \varepsilon \quad \text{and} \quad \int_E |f_n| \, d\mu \leq \varepsilon$$

for all $n \in \mathbb{N}$, while by taking E_ε larger, if necessary, we may assume that $X \setminus E_\varepsilon$,

$$\int_{X \setminus E_\varepsilon} |f| \, d\mu \leq \varepsilon \quad \text{and} \quad \int_{X \setminus E_\varepsilon} |f_n| \, d\mu \leq \varepsilon$$

for all $n \in \mathbb{N}$. Note that if $\mu(E_\varepsilon) = 0$, then since integration does not see sets of measure zero, we have that

$$\int_X |f - f_n| \, d\mu = \int_{X \setminus E_\varepsilon} |f - f_n| \, d\mu \leq \int_{X \setminus E_\varepsilon} |f| \, d\mu + \int_{X \setminus E_\varepsilon} |f_n| \, d\mu \leq \varepsilon + \varepsilon$$

for all $n \in \mathbb{N}$, which is what we wanted to prove.

On the other hand, if $\mu(E_\varepsilon) > 0$, then using convergence in measure, we may find an integer $\bar{n} \in \mathbb{N}$ such that

$$\mu\left(\left\{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{\mu(E_\varepsilon)}\right\}\right) < \delta_\varepsilon \tag{21}$$

for all $n \geq \bar{n}$. In turn, for all $n \geq \bar{n}$,

$$\begin{aligned}\int_X |f_n - f| \, d\mu &= \int_{X \setminus E_\varepsilon} |f_n - f| \, d\mu + \int_{E_\varepsilon} |f_n - f| \, d\mu \\ &\leq 2\varepsilon + \int_{E_\varepsilon \cap \left\{|f_n - f| \leq \frac{\varepsilon}{\mu(E_\varepsilon)}\right\}} |f_n - f| \, d\mu \\ &\quad + \int_{E_\varepsilon \cap \left\{|f_n - f| > \frac{\varepsilon}{\mu(E_\varepsilon)}\right\}} |f_n - f| \, d\mu \leq 5\varepsilon,\end{aligned}$$

where in the first inequality we have used (iii) for $\{f_n\}$ and for f, while in the last inequality we have used (21) and (ii) for $\{f_n\}$ and for f. ■

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Proof. We prove that if $\{f_n\}$ converges to f in $L^1(X)$, then $\{f_n\}$ satisfies conditions (ii) and (iii). Fix $\varepsilon > 0$ and find an integer $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$,

$$\int_X |f_n - f| \, d\mu \leq \varepsilon. \tag{22}$$

By the previous exercise, the finite family $\{f_1, \ldots, f_{\bar{n}}, f\}$ satisfies conditions (ii) and (iii). Thus, there exists $\delta > 0$ such that for every measurable set $E \subset X$ with $\mu(E) \leq \delta$,

$$\int_E |f| \, d\mu \leq \varepsilon \quad \text{and} \quad \int_E |f_n| \, d\mu \leq \varepsilon \tag{23}$$

for all $n = 1, \ldots, \bar{n}$, and there exists a set $E_\varepsilon \in \mathfrak{M}$ with finite measure such that

$$\int_{X \setminus E_\varepsilon} |f| \, d\mu \leq \varepsilon \quad \text{and} \quad \int_{X \setminus E_\varepsilon} |f_n| \, d\mu \leq \varepsilon \tag{24}$$

for all $n = 1, \ldots, \bar{n}$.

Consider a measurable set $E \subset X$ with $\mu(E) \leq \delta$. Then by (22) and (23), for all $n \geq \bar{n}$,

$$\begin{aligned} \int_E |f_n| \, d\mu &= \int_E |f_n \pm f| \, d\mu \leq \int_E |f_n - f| \, d\mu + \int_E |f| \, d\mu \\ &\leq \int_X |f_n - f| \, d\mu + \int_E |f| \, d\mu \leq \varepsilon + \varepsilon. \end{aligned}$$

Similarly, by (22) and (24), for all $n \geq \bar{n}$,

$$\begin{aligned} \int_{X \setminus E_\varepsilon} |f_n| \, d\mu &= \int_{X \setminus E_\varepsilon} |f_n \pm f| \, d\mu \leq \int_{X \setminus E_\varepsilon} |f_n - f| \, d\mu + \int_{X \setminus E_\varepsilon} |f| \, d\mu \\ &\leq \int_X |f_n - f| \, d\mu + \int_{X \setminus E_\varepsilon} |f| \, d\mu \leq \varepsilon + \varepsilon. \end{aligned}$$

This shows that $\{f_n\}$ satisfies conditions (ii) and (iii). We have already seen in Theorem 91 that convergence in $L^1(X)$ implies convergence in measure, so (i) is also satisfied. ■

Remark 99 *Note that condition (iii) is automatically satisfied when X has finite measure.*

Example 100 *Let $X = \mathbb{R}$, $\mathfrak{M} = \mathcal{B}(\mathbb{R})$, and let μ be the Lebesgue measure. The sequence $f_n = n\chi_{\left[0,\frac{1}{n}\right]}$ converges in measure to zero, satisfies (iii), but is not equi-integrable, while the sequence $f_n = \frac{1}{n}\chi_{[n,2n]}$ converges in measure to zero, is equi-integrable, but does not satisfy (iii).*

Exercise 101 *Let* $X = [0,1]$, $\mathfrak{M} = \mathcal{B}([0,1])$, *and let* μ *be the Lebesgue measure. Prove that the sequence of functions defined by* $f_n(x) := \sin nx$, $x \in [0,1]$, *satisfies conditions (ii) and (iii) but does not converges in measure.*

In view of Vitali's theorem it becomes important to understand equi-integrability.

Theorem 102 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $\mathcal{F}$ *be a family of integrable functions* $f : X \to [-\infty, \infty]$. *Consider the following conditions.*

(i) $\mathcal{F}$ *is equi-integrable.*

(ii)
$$\lim_{t\to\infty} \sup_{f\in\mathcal{F}} \int_{\{x\in X : |f|>t\}} |f| \, d\mu = 0. \tag{25}$$

(iii) ***(De la Vallée Poussin)*** *There exists an increasing function* $\gamma : [0,\infty) \to [0,\infty]$, *with*
$$\lim_{t\to\infty} \frac{\gamma(t)}{t} = \infty \tag{26}$$
such that
$$\sup_{f\in\mathcal{F}} \int_X \gamma(|f|) \, d\mu < \infty. \tag{27}$$

Then (ii) and (iii) are equivalent and either one implies (i). If in addition we assume that
$$\sup_{f\in\mathcal{F}} \int_X |f| \, d\mu < \infty,$$
then (i) implies (ii) (and so all three conditions are equivalent in this case).

Proof. Step 1: We prove that (ii) implies (i). Fix $\varepsilon > 0$, and choose $t_\varepsilon > 0$ such that
$$\sup_{f\in\mathcal{F}} \int_{\{x\in X : |f|>t_\varepsilon\}} |f| \, d\mu \le \frac{\varepsilon}{2}.$$
Then for every measurable set $F \subset X$ with $\mu(F) \le \frac{\varepsilon}{2t_\varepsilon}$ and for all $f \in \mathcal{F}$ we have
$$\begin{aligned}\int_F |f| \, d\mu &= \int_{\{x\in F : |f|>t_\varepsilon\}} |f| \, d\mu + \int_{\{x\in F : |f|\le t_\varepsilon\}} |f| \, d\mu \\ &\le \frac{\varepsilon}{2} + t_\varepsilon \mu(F) \le \varepsilon.\end{aligned}$$
Hence $\mathcal{F}$ is equi-integrable. ■

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Proof. Step 2: We prove that (ii) and (iii) are equivalent. Assume that (ii) holds and construct an increasing sequence of nonnegative integers $\{k_i\}$ such that
$$\sup_{f\in\mathcal{F}} \int_{\{x\in X : |f|>k_i\}} |f| \, d\mu \le \frac{1}{2^{i+1}}.$$

For each $l \in \mathbb{N}_0$ let b_l be the number of nonnegative integers i such that $k_i < l$. Note that $b_l \nearrow \infty$ as $l \to \infty$. Define

$$\gamma(t) := tb_l \quad \text{if } t \in [l, l+1).$$

Then

$$\frac{\gamma(t)}{t} \geq b_{\lfloor t \rfloor} \to \infty$$

as $t \to \infty$, where $\lfloor t \rfloor$ is the integer part of t. Moreover, for all $f \in \mathcal{F}$, by Example 77,

$$\begin{aligned}
\int_X \gamma(|f|)\, d\mu &= \sum_{l=0}^{\infty} \int_{\{l \leq |f| < l+1\}} \gamma(|f|)\, d\mu \\
&= \sum_{l=0}^{\infty} b_l \int_{\{l \leq |f| < l+1\}} |f|\, d\mu = \sum_{l=0}^{\infty} \sum_{i:\, k_i < l} \int_{\{l \leq |f| < l+1\}} |f|\, d\mu \\
&= \sum_{i=0}^{\infty} \sum_{l > k_i} \int_{\{l \leq |f| < l+1\}} |f|\, d\mu \leq \sum_{i=0}^{\infty} \int_{\{|f| > k_i\}} |f|\, d\mu \leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}}.
\end{aligned}$$

Conversely, assume that (iii) holds and let

$$M := \sup_{f \in \mathcal{F}} \int_X \gamma(|f|)\, d\mu < \infty.$$

Fix $\varepsilon > 0$, and find $t_\varepsilon > 0$ such that

$$\gamma(t) \geq \frac{tM}{\varepsilon} \quad \text{for all } t \geq t_\varepsilon.$$

Then for all $t \geq t_\varepsilon$ and for all $f \in \mathcal{F}$ we have

$$\int_{\{x \in X:\, |f| > t\}} |f|\, d\mu \leq \frac{\varepsilon}{M} \int_X \gamma(|f|)\, d\mu \leq \varepsilon.$$

Hence (ii) is satisfied.
Step 3: Finally assume that

$$C := \sup_{f \in \mathcal{F}} \int_X |f|\, d\mu < \infty \tag{28}$$

and that $\mathcal{F}$ is equi-integrable. Given $\varepsilon > 0$ let $\delta > 0$ be such that

$$\sup_{f \in \mathcal{F}} \int_F |f|\, d\mu \leq \varepsilon$$

for every measurable set $F \subset X$ with $\mu(F) \leq \delta$. Choose $t_\varepsilon > 0$ such that $\frac{C}{t_\varepsilon} \leq \delta$. Then, also by (28), for every $f \in \mathcal{F}$ and for all $t \geq t_\varepsilon$ we have

$$\mu(\{x \in X : |f| > t\}) \leq \frac{1}{t} \int_X |f|\, d\mu \leq \frac{C}{t} \leq \delta,$$

and so

$$\sup_{f\in\mathcal{F}}\int_{\{x\in X:\,|f|>t\}}|f|\;d\mu\leq\varepsilon,$$

and this validates (ii). ■

Remark 103 *If the measure μ is finite and nonatomic, the range of μ is $[0,\mu(X)]$ and so we may partition X into a finite number of measurable sets $E_1,\ldots,E_n$ of measure less than δ. Then*

$$\int_X|f|\;d\mu=\sum_{i=1}^{n}\int_{E_i}|f|\;d\mu\leq n\varepsilon$$

and so

$$\sup_{f\in\mathcal{F}}\int_X|f|\;d\mu\leq n\varepsilon<\infty$$

so all three conditions are equivalent.

7 L^p Spaces

Definition 104 *A* normed space *is a pair $(V,\|\cdot\|)$, where V is a vector space and $\|\cdot\|:V\to[0,\infty)$ is a* norm*, that is,*

(i) $\|v\|=0$ *if and only if* $v=0$*.*

(ii) $\|tv\|=|t|\,\|v\|$ *for all* $t\in\mathbb{R}$ *and* $v\in V$*.*

(iii) ***(Triangle inequality)*** $\|v+w\|\leq\|v\|+\|w\|$ *for all* $v,\,w\in V$*.*

Given a normed space $(V,\|\cdot\|)$, we can define the distance function $d:V\times V\to[0,\infty)$, defined by

$$d(v,w):=\|v-w\|,\quad v,w\in V.$$

We say that V is a *Banach* space if it is a complete metric space, that is, if every Cauchy sequence $\{v_n\}\subset V$ converges to some element in V.

Let $(X,\mathfrak{M},\mu)$ be a measure space. For $1\leq p<\infty$, we define the space

$$M^p(X):=\Big\{[f]:\;f:X\to[-\infty,\infty]\text{ measurable and }\|f\|_{M^p(X)}<\infty\Big\},$$

where

$$\|f\|_{M^p(X)}:=\left(\int_X|f|^p\;d\mu\right)^{1/p}.$$

For $p=\infty$, we define

$$M^\infty(X):=\{[f]:\;f:X\to\mathbb{R}\text{ measurable and bounded}\},$$

where

$$\|f\|_{M^\infty(X)} := \sup_{x\in X} |f(x)|.$$

Note that property (ii) of the previous definition is satisfied. Indeed, for $1 \le p < \infty$ and for $t \in \mathbb{R}$,

$$\|tf\|_{M^p(X)} = \left(\int_X |tf|^p \, d\mu\right)^{1/p} = \left(|t|^p \int_X |f|^p \, d\mu\right)^{1/p} = |t| \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

Next we study the triangle inequality.

Let q be the *Hölder conjugate exponent* of p, i.e.,

$$q := \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ \infty & \text{if } p = 1, \\ 1 & \text{if } p = \infty. \end{cases}$$

Note that, with an abuse of notation, we have

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In the sequel, the Hölder conjugate exponent of p will often be denoted by p'.

Theorem 105 (Hölder's inequality) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $1 \le p \le \infty$, and let q be its Hölder conjugate exponent. If f, $g : X \to [-\infty, \infty]$ are measurable functions then*

$$\int_X |fg| \, d\mu \le \left(\int_X |f|^p \, d\mu\right)^{1/p} \left(\int_X |g|^q \, d\mu\right)^{1/q} \tag{29}$$

if $1 < p < \infty$,

$$\int_X |fg| \, d\mu \le \sup_{x\in X} |g(x)| \int_X |f| \, d\mu \tag{30}$$

if $p = 1$, and

$$\int_X |fg| \, d\mu \le \sup_{x\in X} |f(x)| \int_X |g| \, d\mu \tag{31}$$

if $p = \infty$. In particular, if $f \in M^p(X)$ and $g \in M^p(X)$ then $fg \in M^1(X)$.

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Proof. If $\|f\|_{M^p(X)} = 0$ or $\|g\|_{M^q(X)} = 0$, then $f(x)\, g(x) = 0$ for μ a.e. $x \in X$ and so there is nothing to prove. Thus assume that $\|f\|_{M^p(X)}$, $\|g\|_{M^q(X)} > 0$. If $\|f\|_{M^p(X)} = \infty$ or $\|g\|_{M^q(X)} = \infty$ then the right-hand side is ∞ and so the inequality (29) holds. Hence in what follows we consider the case $\|f\|_{M^p(X)}$, $\|g\|_{M^q(X)} \in (0, \infty)$.

Assume that $1 < p < \infty$. Since the function $t \in [0, \infty) \mapsto \ln t$ is concave and $\frac{1}{p} + \frac{1}{q} = 1$, for any a, $b > 0$, we have

$$\ln\left(\frac{1}{p} a^p + \frac{1}{q} b^q\right) \ge \frac{1}{p} \ln a^p + \frac{1}{q} \ln b^q = \ln ab,$$

that is

$$\frac{1}{p}a^p + \frac{1}{q}b^q \geq ab,$$

which is known as *Young's inequality.*

If we take $a = |f(x)|$ and $b = |g(x)|$, we get

$$|f(x)\, g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q .$$

Upon integration, we obtain

$$\begin{aligned}\int_X |fg|\, d\mu &\leq \frac{1}{p}\int_X |f|^p\, d\mu + \frac{1}{q}\int_X |g|^q\, d\mu \\ &= \frac{1}{p}\|f\|_{M^p(X)}^p + \frac{1}{q}\|g\|_{M^q(X)}^q .\end{aligned}$$

To obtain the desired result, it suffices to replace f with tf, where $t > 0$, to obtain

$$\int_X |fg|\, d\mu \leq \frac{t^{p-1}}{p}\|f\|_{M^p(X)}^p + \frac{1}{tq}\|g\|_{M^q(X)}^q =: g(t).$$

By minimizing the function g, we find that for

$$t = \frac{\|g\|_{M^q(X)}^{q/p}}{\|f\|_{M^p(X)}}$$

the inequality (29) holds.

If $p = 1$ and $q = \infty$, then

$$\begin{aligned}\int_X |fg|\, d\mu &\leq \int_X |f| \sup_{x\in X}|g(x)|\, d\mu \\ &= \sup_{x\in X}|g(x)|\int_X |f|\, d\mu.\end{aligned}$$

The case $p = \infty$ is similar. ■

Exercise 106 *Prove that if $f \neq 0$ and the right-hand side of* (29) *is finite, then the equality in* (29) *holds if and only if there exists $c \geq 0$ such that*

1. $|g| = c|f|^{p-1}$ *if* $1 < p < \infty$;
2. $|g| \leq c$ *and* $|g(x)| = c$ *whenever* $f(x) \neq 0$ *if* $p = 1$;
3. $|f| \leq c$ *and* $|f(x)| = c$ *whenever* $g(x) \neq 0$ *if* $p = \infty$.

Theorem 107 (Minkowski's inequality) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $1 \leq p \leq \infty$, and let $f,\ g : X \to [-\infty, \infty]$ be measurable functions. Then,*

$$\|f + g\|_{M^p(X)} \leq \|f\|_{M^p(X)} + \|g\|_{M^p(X)} \tag{32}$$

whenever $\|f + g\|_{M^p(X)}$ is well-defined. In particular, if $f, g \in M^p(X)$, then $f + g \in M^p(X)$ and (32) *holds.*

Proof. If $\|f\|_{M^p(X)} = \infty$ or $\|g\|_{M^p(X)} = \infty$ then the right-hand side of Minkowski's inequality is ∞, and so there is nothing to prove. Thus assume that $\|f\|_{M^p(X)}$, $\|g\|_{M^p(X)} < \infty$.

We consider first the case $1 < p < \infty$. By the convexity of the function $t \in [0, \infty) \mapsto t^p$, for any a, $b > 0$, we have

$$(a+b)^p = 2^p \left(\frac{a+b}{2}\right)^p \leq \frac{2^p}{2} a^p + \frac{2^p}{2} b^p = 2^{p-1} (a^p + b^p),$$

and so

$$\int_X |f+g|^p \, d\mu \leq \int_X (|f| + |g|)^p \, d\mu \leq 2^{p-1} \left(\int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu \right),$$

which shows that $f + g \in L^p(X)$. To prove Minkowski's inequality, we observe that

$$\begin{aligned} \|f+g\|_{L^p}^p &= \int_X |f+g|^p \, d\mu = \int_X |f+g| \cdot |f+g|^{p-1} \, d\mu \\ &\leq \int_X |f| \cdot |f+g|^{p-1} \, d\mu + \int_X |g| \cdot |f+g|^{p-1} \, d\mu. \end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned} \|f+g\|_{M^p(X)}^p &\leq \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |f+g|^{(p-1)p'} \, d\mu \right)^{\frac{1}{p'}} \\ &\quad + \left(\int_X |g|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |f+g|^{(p-1)p'} \, d\mu \right)^{\frac{1}{p'}} \\ &\leq \left(\|f\|_{M^p(X)} + \|g\|_{M^p(X)} \right) \|f+g\|_{M^p(X)}^{\frac{p}{p'}}, \end{aligned}$$

where we have used the fact that $(p-1)p' = p$. If $\|f+g\|_{M^p(X)} = 0$, then there is nothing to prove, thus assume that $\|f+g\|_{M^p(X)} \in (0, \infty)$. Hence, we may divide both sides of the previous inequality by $\|f+g\|_{M^p(X)}^{\frac{p}{p'}}$ to obtain

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p},$$

where we have used the fact that $p - \frac{p}{p'} = 1$.

The cases $p = 1$ and $p = \infty$ are straightforwad. ■

In view of the previous theorem we now have that for $1 \leq p < \infty$, properties (ii) and (iii) of Definition 104 are satisfied. The problem is property (i). Indeed, if

$$\|f\|_{L^p} = \left(\int_X |f|^p \, d\mu \right)^{1/p} = 0,$$

then by Corollary 80 there exists a set $E \in \mathfrak{M}$ with $\mu(E) = 0$ such that $f(x) = 0$ for all $x \in X \setminus E$. This does not imply that the function f is zero. For example, the Dirichlet function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

has exactly this property.

To circumvent this proplem, given two measurable functions f, $g : X \to [-\infty, \infty]$, we say that f is *equivalent* to g, and we write

$$f \sim g \text{ if } f(x) = g(x) \text{ for } \mu \text{ a.e. } x \in X. \tag{33}$$

Note that $\sim$ is an equivalence relation in the class of measurable functions. Moreover, if $f(x) = 0$ for μ a.e. $x \in X$, then $f \sim 0$, or, equivalently, f belongs to equivalence class $[0]$.

Definition 108 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $1 \leq p < \infty$. We define*

$$L^p(X) := M^p(X) \diagup \sim = \left\{[f] : f : X \to [-\infty, \infty] \text{ measurable and } \|f\|_{M^p(X)} < \infty\right\}.$$

In the space $L^p(X)$ we define the norm

$$\|[f]\|_{L^p(X)} := \|f\|_{M^p(X)}.$$

Note that $\|[f]\|_{L^p}$ does not depend on the choice of the representative. We now have that $(L^p(X), \|\cdot\|_{L^p})$ is a normed space, since properties (i)-(ii) are satisfied.

Indeed, if $f \in L^p([0,1])$ (with the Lebsgue measure), then after the identification f is actually an equivalence class. Hence, for example, talking about the value $f(1)$ or $f\left(\frac{1}{2}\right)$ make no sense. Indeed, given any *number* $y \in \mathbb{R}$, in the equivalence class $[f]$ there is always a function g such that $g(1) = y$. Just define

$$g(x) := \begin{cases} y & \text{if } x = 1, \\ f(x) & \text{otherwise.} \end{cases}$$

Then f and g differ only at the point 1, and so $f \sim g$.

Wednesday, October 19, 2011

University of Puerto Rico, no class.

Friday, October 21, 2011

Midsemester break, no class.

Monday, October 24, 2011

Let's now consider the case $p = \infty$. Unlike the case $1 \leq p < \infty$, the supremum of a function changes if we change the function even at one point. Thus, we cannot take as a norm $\|[f]\|_{L^\infty(X)} := \sup_{x \in X} |f(x)|$. What we need is a notion of supremum that does not change if we modify a function on a set of measure zero.

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Given a measurable function $f : X \to [-\infty, \infty]$ we define the *essential supremum* $\operatorname{esssup} f$ of the function f as

$$\operatorname{esssup} f := \inf\{t \in \mathbb{R} : f(x) \leq t \text{ for } \mu \text{ a.e. } x \in X\}.$$

Note that if $M := \operatorname{esssup} f < \infty$, then by taking $t_n := M + \frac{1}{n}$ we can find $E_n \in \mathfrak{M}$ with $\mu(E_n) = 0$ such that

$$f(x) \leq M + \frac{1}{n} \text{ for all } x \in X \setminus E_n.$$

Take

$$E_\infty := \bigcup_{n=1}^{\infty} E_n.$$

Then $\mu(E_\infty) \le \sum_{n=1}^{\infty} \mu(E_n) = 0$, and if $x \in X \setminus E_\infty$, then

$$f(x) \le M + \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we get that $f(x) \le M$ for all $x \in X \setminus E_\infty$. Conversely, if there are $t \in \mathbb{R}$ and $E \in \mathfrak{M}$ with $\mu(E) = 0$ such that $f(x) \le t$ for all $x \in X \setminus E$, then by definition of $\operatorname{esssup} f$, we have that $\operatorname{esssup} f \le t < \infty$. This shows that $\operatorname{esssup} f < \infty$ if and only if the function f is bounded from above except on a set of measure zero.

Moreover, if $f \sim g$ then $\operatorname{esssup} f = \operatorname{esssup} g$. This leads us to the following definition.

Definition 109 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. We define*

$$L^\infty(X) := \{[f] : f : X \to [-\infty, \infty] \text{ measurable and } \operatorname{esssup} |f| < \infty\}.$$

In the space $L^\infty(X)$ we define the norm

$$\|[f]\|_{L^\infty} := \operatorname{esssup} |f|.$$

Indeed, properties (i) and (ii) are satisfied. To prove property (iii), note that if $[f]$ and $[g]$ belong to $L^\infty(X)$, then there exist $E, F \in \mathfrak{M}$ with $\mu(E) = \mu(F) = 0$ such that $|f(x)| \le \operatorname{esssup} |f|$ for all $x \in X \setminus E$ and $|g(x)| \le \operatorname{esssup} |g|$ for all $x \in X \setminus F$. Hence,

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \operatorname{esssup} |f| + \operatorname{esssup} |g|$$

for all $x \in X \setminus (E \cup F)$, which implies that $\operatorname{esssup} |f + g| \le \operatorname{esssup} |f| + \operatorname{esssup} |g|$. Thus, the triangle inequality holds.

Remark 110 *Note that in Hölder's inequality one can replace* (30) *and* (31) *with*

$$\int_X |fg| \, d\mu \le \operatorname{esssup} |g| \int_X |f| \, d\mu$$

and

$$\int_X |fg| \, d\mu \le \operatorname{esssup} |f| \int_X |g| \, d\mu,$$

respectively. Indeed, in the first case, since $|g(x)| \le \operatorname{esssup} |g|$ for all $x \in X \setminus E$, where $E \in \mathfrak{M}$ with $\mu(E) = 0$, we have that

$$\begin{aligned} \int_X |fg| \, d\mu &= \int_{X \setminus E} |f| \, |g| \, d\mu \le \int_{X \setminus E} |f| \operatorname{esssup} |g| \, d\mu \\ &= \operatorname{esssup} |g| \int_{X \setminus E} |f| \, d\mu \le \operatorname{esssup} |g| \int_X |f| \, d\mu. \end{aligned}$$

With an abuse of notation, from now on we identify a measurable function $f : X \to [-\infty, \infty]$ with its equivalence class $[f]$. Note that this is very dangerous.

We now turn to the relation between different L^p spaces.

Theorem 111 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $1 \le p < q < \infty$. Then*

(i) $L^p(X)$ is not contained in $L^q(X)$ if and only if X contains measurable sets of arbitrarily small positive measure[4]*;*

(ii) $L^q(X)$ is not contained in $L^p(X)$ if and only if X contains measurable sets of arbitrarily large finite measure[5]*.*

Proof. (i) Assume that $L^p(X)$ is not contained in $L^q(X)$. Then there exists $f \in L^p(X)$ such that

$$\int_X |f|^q \, d\mu = \infty. \tag{34}$$

For each $n \in \mathbb{N}$ let

$$E_n := \{x \in X : |f(x)| > n\}.$$

Then

$$\mu(E_n) \le \frac{1}{n^p} \int_X |f|^p \, d\mu \to 0$$

as $n \to \infty$. Thus, it suffices to show that $\mu(E_n) > 0$ for all n sufficiently large. If to the contrary, $\mu(E_n) = 0$ for infinitely many n, we have that

$$\int_X |f|^q \, d\mu = \int_{\{|f| \le n\}} |f|^q \, d\mu \le n^{q-p} \int_{\{|f| \le n\}} |f|^p \, d\mu < \infty,$$

which is a contradiction with (34). Hence, X contains measurable sets of arbitrarily small positive measure.

Conversely, assume that X contains measurable sets of arbitrarily small positive measure. Then it is possible to construct a sequence of pairwise disjoint

[4] By this we mean that for every $\varepsilon > 0$ there is a measurable set of positive measure less than ε.

[5] By this we mean that for every $M > 0$ there is a measurable set of finite positive measure gretaer than M.

sets $\{E_n\} \subset \mathfrak{M}$ such that $\mu(E_n) > 0$ for all $n \in \mathbb{N}$ and[6]

$$\mu(E_n) \searrow 0.$$

Let

$$f := \sum_{n=1}^{\infty} c_n \chi_{E_n},$$

where $c_n \nearrow \infty$ are chosen such that

$$\sum_{n=1}^{\infty} c_n^q \mu(E_n) = \infty, \quad \sum_{n=1}^{\infty} c_n^p \mu(E_n) < \infty. \tag{35}$$

Then $f \in L^p(X) \setminus L^q(X)$. ■

Wednesday, October 26, 2011

Proof. (ii) Assume that $L^q(X)$ is not contained in $L^p(X)$. Then there exists $f \in L^q(X)$ such that

$$\int_X |f|^p \, d\mu = \infty. \tag{36}$$

For each $n \in \mathbb{N}$ let

$$F_n := \left\{ x \in X : \frac{1}{n+1} < |f(x)| \le \frac{1}{n} \right\}$$

and let

$$F_\infty := \{x \in X : 0 < |f(x)| \le 1\} = \bigcup_{n=1}^{\infty} F_n.$$

If $\mu(F_\infty) < \infty$, then

$$\begin{aligned}\int_X |f|^p \, d\mu &= \int_{\{|f| \le 1\}} |f|^p \, d\mu + \int_{\{|f|>1\}} |f|^p \, d\mu \\ &\le \mu(F_\infty) + \int_{\{|f|>1\}} |f|^q \, d\mu < \infty,\end{aligned}$$

[6] For example, one could first take a set $F_0 \in \mathfrak{M}$ with $0 < \mu(F_0) \le 1$ and then by induction construct $F_n \in \mathfrak{M}$ with $0 < \mu(F_n) \le \frac{1}{3}\mu(F_{n-1})$. Note that for $i \ge n$, we have that $\mu(F_i) \le \frac{1}{3^{i-(n-1)}}\mu(F_{n-1})$. Then the sets $E_n := F_{n-1} \setminus \bigcup_{i=n}^{\infty} F_i$ have the desired properties since

$$\mu(E_n) \le \mu(F_n) \le \frac{1}{3^{n-1}}$$

and

$$\begin{aligned}\mu(E_n) &= \mu(F_{n-1}) - \sum_{i=n}^{\infty} \mu(F_i) \ge \mu(F_{n-1}) - \mu(F_{n-1}) \sum_{i=n}^{\infty} \frac{1}{3^{i-(n-1)}} \\ &= \frac{1}{2}\mu(F_{n-1}) > 0.\end{aligned}$$

which contradicts (36). Hence, $\mu(F_\infty) = \infty$. On the other hand, since for every $n \in \mathbb{N}$,

$$\infty > \int_X |f|^q \, d\mu \geq \int_{\left\{\frac{1}{n+1} < |f| \leq \frac{1}{n}\right\}} |f|^q \, d\mu \geq \frac{1}{(n+1)^q} \mu(F_n),$$

it follows that X contains measurable sets of arbitrarily large finite measure. Indeed, setting

$$G_n := \bigcup_{k=1}^{n} F_k,$$

we have that $\mu(G_n) < \infty$, while by Proposition 47(i),

$$\mu(G_n) \to \mu(F_\infty) = \infty.$$

Conversely, assume that X contains measurable sets of arbitrarily large finite measure. Then it is possible to construct a sequence of pairwise disjoint sets $\{E_n\} \subset \mathfrak{M}$ of finite measure such that[7]

$$\mu(E_n) \nearrow \infty.$$

Let

$$f := \sum_{n=1}^{\infty} c_n \chi_{E_n},$$

where $c_n \searrow 0$ are chosen such that

$$\sum_{n=1}^{\infty} c_n^q \mu(E_n) < \infty, \quad \sum_{n=1}^{\infty} c_n^p \mu(E_n) = \infty. \tag{37}$$

Then $f \in L^q(X) \setminus L^p(X)$. ■

Remark 112 *Note that the previous proof works also for p, $q > 0$. What about $q = \infty$?*

[7] For example, one could first take a set $F_0 \in \mathfrak{M}$ with $\infty > \mu(F_0) > 1$ and then by induction construct $F_n \in \mathfrak{M}$ with $\infty > \mu(F_n) > 3\mu(F_{n-1})$. It follows that $\mu(F_n) > 3^{n-i}\mu(F_i)$ for $i = 0, \ldots, n-1$. Then the sets $E_n := F_n \setminus \bigcup_{i=0}^{n-1} F_i$ have the desired properties since

$$\mu(E_n) \leq \mu(F_n) < \infty$$

and

$$\begin{aligned}
\mu(E_n) &= \mu(F_n) - \mu\left(\bigcup_{i=0}^{n-1} F_i\right) \\
&\geq \mu(F_n) - \mu(F_n) \sum_{i=0}^{n-1} \frac{1}{3^{n-i}} \\
&= \mu(F_n)\left(1 - \frac{1}{2} + \frac{1}{2}\frac{1}{3^n}\right) > \frac{(n-1)}{2}.
\end{aligned}$$

Exercise 113 *(i) Let $X = [0,1]$ and let μ be the Lebesgue measure. Show that for every $1 \leq p < \infty$ the function*

$$f(x) = \frac{1}{x^{1/p} \log^{2/p}\left(\frac{2}{x}\right)}$$

is in $L^p([0,1])$ but not in $L^q([0,1])$ for all $q > p$.

(ii) Construct sequences $c_n \nearrow \infty$ and $c_n \searrow 0$ for which conditions (35) and (37) hold, respectively.

Corollary 114 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Suppose that $1 \leq p < q \leq \infty$. If $\mu(X) < \infty$, then*

$$L^q(X) \subset L^p(X).$$

Proof. When $1 \leq q < \infty$, this follows from the previous theorem. There's also a direct proof. By Hölder's inequality (with $\frac{q}{p}$ in place of p and $|f|^p$ and 1 in place of f and g)

$$\begin{aligned} \int_X |f|^p \, d\mu &\leq \left\| |f|^p \right\|_{L^{\frac{q}{p}}} \|1\|_{L^{\left(\frac{q}{p}\right)'}} = \left(\int_X |f|^q \, d\mu \right)^{\frac{p}{q}} \|1\|_{L^{\left(\frac{q}{p}\right)'}} \\ &= \left(\int_X |f|^q \, d\mu \right)^{\frac{p}{q}} (\mu(X))^{\frac{q-p}{p}}. \end{aligned}$$

■

By identifying functions with their equivalence classes $[f]$, it follows from Minkowski's inequality that $\|\cdot\|_{L^p}$ is a norm on $L^p(X)$.

Theorem 115 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then $L^p(X)$ is a Banach space for $1 \leq p \leq \infty$.*

The proof of the previous theorem is based on the following result, which is of independent interest.

Lemma 116 *A normed space V is complete if and only if for any sequence $\{v_n\} \subset V$ such that*

$$\sum_{n=1}^{\infty} \|v_n\| < \infty$$

there exists $v \in V$ such that

$$\lim_{\ell \to \infty} \left\| \sum_{n=1}^{\ell} v_n - v \right\| = 0.$$

Exercise 117 *Prove the previous lemma.*

We turn to the proof of Theorem 115

Proof of Theorem 115. Assume that $1 \leq p < \infty$, and let $\{f_n\} \subset L^p(X)$ be such that

$$\sum_{n=1}^{\infty} \|f_n\|_{L^p} < \infty.$$

Define

$$g(x) := \left(\sum_{n=1}^{\infty} |f_n(x)|\right)^p, \quad x \in X.$$

By Minkowski's inequality, for each $\ell \in \mathbb{N}$,

$$\left(\int_X \left(\sum_{n=1}^{\ell} |f_n|\right)^p d\mu\right)^{\frac{1}{p}} = \left\|\sum_{n=1}^{\ell} |f_n|\right\|_{L^p} \leq \sum_{n=1}^{\ell} \|f_n\|_{L^p},$$

and so by the Lebesgue monotone convergence theorem,

$$\int_X g\, d\mu = \lim_{\ell \to \infty} \int_X \left(\sum_{n=1}^{\ell} |f_n|\right)^p d\mu \leq \left(\sum_{n=1}^{\infty} \|f_n\|_{L^p}\right)^p < \infty.$$

Hence by Remark 83, $g(x) < \infty$ for μ a.e. $x \in X$. At any such point $x \in X$ we have that the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent. Define the function

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} f_n(x) & \text{if } g(x) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is measurable and $|f(x)|^p \leq g(x)$ for all $x \in X$. Hence $f \in L^p(X)$. Since

$$\lim_{\ell \to \infty} \sum_{n=1}^{\ell} f_n(x) = f(x)$$

for μ a.e. $x \in X$ and

$$\left|\sum_{n=1}^{\ell} f_n(x) - f(x)\right|^p \leq g(x)$$

for μ a.e. $x \in X$ and for all $\ell \in \mathbb{N}$, it follows by the Lebesgue dominated convergence theorem that

$$\lim_{\ell \to \infty} \int_X \left|\sum_{n=1}^{\ell} f_n - f\right|^p d\mu = 0.$$

This concludes the proof. ■

Exercise 118 *Prove the case $p = \infty$.*

Next we study some density results for $L^p(X)$ spaces.

Theorem 119 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then the family of all simple functions in $L^p(X)$ is dense in $L^p(X)$ for $1 \leq p \leq \infty$.*

Proof. Assume first that $1 \leq p < \infty$. Since $(f^+)^p$, $(f^-)^p \in L^1(X)$, by Theorem 69 there exist increasing sequences $\{s_n\}$ and $\{t_n\}$ of simple functions such that $\{(s_n(x))^p\}$ converges monotonically to $(f^+)^p(x)$ for μ a.e. $x \in X$ and $\{(t_n(x))^p\}$ converges monotonically to $(f^-)^p(x)$ for μ a.e. $x \in X$. Then for each $n \in \mathbb{N}$ the function $S_n := s_n - t_n$ is still simple, belongs to $L^p(X)$, and

$$\begin{aligned}|f(x) - S_n(x)|^p &= \left|f^+(x) - s_n(x) - \left(f^-(x) - t_n(x)\right)\right|^p \\ &\leq 2^{p-1}\left(f^+(x) - s_n(x)\right)^p + 2^{p-1}\left(f^-(x) - t_n(x)\right)^p \\ &\leq 2^{p-1}\left(f^+(x)\right)^p + 2^{p-1}\left(f^-(x)\right)^p\end{aligned}$$

for μ a.e. $x \in X$. Since $f(x) - S_n(x) \to 0$ as $n \to \infty$ for μ a.e. $x \in X$, we may apply the Lebesgue dominated convergence theorem to conclude that $S_n \to f$ in $L^p(X)$.

For $p = \infty$ the result follows by the second part of Theorem 69 applied to the bounded functions f^+, f^-. ■

Friday, October 28, 2011

Next we discuss the density of continuous functions in L^p spaces.

A topological space X is a *normal space* if for every pair of disjoint closed sets $C_1, C_2 \subset X$ we may find two disjoint open set A_1, A_2 such that $A_1 \subset C_1$ and $A_2 \subset C_2$.

Remark 120 *A metric space X is a normal space. Indeed, if $C_1, C_2 \subset X$ are disjoint and closed, then the open sets*

$$\begin{aligned}A_1 &:= \{x \in X : \operatorname{dist}(x, C_1) < \operatorname{dist}(x, C_2)\}, \\ A_2 &:= \{x \in X : \operatorname{dist}(x, C_1) > \operatorname{dist}(x, C_2)\}\end{aligned}$$

are two disjoint open sets containing C_1 and C_2, respectively.

Theorem 121 (Urysohn) *A topological space X is normal if and only if for any two disjoint closed sets C_1, $C_2 \subset X$ there exists a continuous function $g : X \to [0, 1]$ such that $g \equiv 1$ in C_1 and $g \equiv 0$ in C_2.*

The next result gives conditions on X and μ that ensure the density of continuous functions in $L^p(X)$.

Theorem 122 *Let $(X, \mathfrak{M}, \mu)$ be a measure space, with X a normal space and $\mathfrak{M} \supset \mathcal{B}(X)$. Assume that*

$$\mu(E) = \sup\{\mu(C) : C \text{ closed, } C \subset E\} = \inf\{\mu(A) : A \text{ open, } A \supset E\} \quad (38)$$

for every set $E \in \mathfrak{M}$ with finite measure. Then $L^p(X) \cap C_b(X)$ is dense in $L^p(X)$ for $1 \leq p < \infty$.

Proof. Since by Theorem 119 simple functions in $L^p(X)$ are dense in $L^p(X)$, it suffices to approximate in $L^p(X)$ functions χ_E, with $E \in \mathfrak{M}$ and $\mu(E) < \infty$, by functions in $L^p(X) \cap C_b(X)$. Thus, fix $E \in \mathfrak{M}$ with $\mu(E) < \infty$, and for any $\varepsilon > 0$ find an open set $A \supset E$ and a closed set $C \subset E$ such that

$$\mu(A \setminus C) \leq \varepsilon^p.$$

By Urysohn's lemma there exists a continuous function $g : X \to [0,1]$ such that $g \equiv 1$ in C and $g \equiv 0$ in $X \setminus A$. Since $\operatorname{supp} g \subset A$ and $\mu(A) < \infty$, it follows that $g \in L^p(X) \cap C_b(X)$. Moreover,

$$\int_X |\chi_E - g|^p \, d\mu = \int_{A \setminus C} |\chi_E - g|^p \, d\mu \leq \mu(A \setminus C) \leq \varepsilon^p,$$

and the result follows. ■

Remark 123 *In a metric space we can actually consider continuous functions with support contained in a closed ball. To see this, fix* $x_0 \in X$. *Then* $\overline{B(x_0, n)} \subset B(x_0, n+1)$ *and so by Urysohn's lemma there exists a continuous function* $g_n : X \to [0,1]$ *such that* $g_n \equiv 1$ *in* C *and* $g_n \equiv 0$ *in* $X \setminus B(x_0, n+1)$. *Now given* $f \in L^p(X) \cap C_b(X)$ *(the previous theorem shows that we can always reduce to this case), consider the sequence* $f_n := f g_n$. *Then for any* $x \in X$ *find a natural number* $n_x > d(x, x_0)$. *Then* $f_n(x) = f(x) g_n(x) = f(x)$ *for all* $n \geq n_x$ *and so* $f_n(x) \to f(x)$ *for every* $x \in X$. *Moreover, since* $0 \leq g_n \leq 1$

$$\begin{aligned} |f_n - f|^p &\leq 2^{p-1} |f_n|^p + 2^{p-1} |f|^p = 2^{p-1} |f g_n|^p + 2^{p-1} |f|^p \\ &\leq 2^{p-1} |f|^p + 2^{p-1} |f|^p \end{aligned}$$

and so by the Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \int_X |f_n - f|^p \, d\mu = 0.$$

In particular, if $X = \mathbb{R}^N$ *then* $L^p(\mathbb{R}^N) \cap C_c(\mathbb{R}^N)$ *is dense in* $L^p(\mathbb{R}^N)$.

Exercise 124 *The case* $p = \infty$. *Consider the Lebesgue measure in* $\mathbb{R}^N$.

(i) Let $g : \mathbb{R}^N \to \mathbb{R}$ *be a continuous bounded function. Prove that*

$$\operatorname{esssup} |g| = \sup |g|.$$

(ii) Prove that $L^\infty(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ *is not dense in* $L^\infty(\mathbb{R}^N)$.

Saturday, October 29, 2011

Make-up class, 1:30 hours

Theorem 125 (Besicovitch covering theorem) *There exists a constant ℓ, depending only on the dimension N of $\mathbb{R}^N$, such that for any collection $\mathcal{F}$ of (nondegenerate) closed balls with*

$$\sup\left\{\operatorname{diam}\overline{B} : \overline{B} \in \mathcal{F}\right\} < \infty \tag{39}$$

there exist $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \subset \mathcal{F}$ such that each $\mathcal{F}_n$, $n = 1, \ldots, \ell$, is a countable family of disjoint balls in $\mathcal{F}$ and

$$E \subset \bigcup_{n=1}^{\ell} \bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B},$$

where E is the set of centers of balls in $\mathcal{F}$.

Lemma 126 *Let $\overline{B(x_j, r_j)} \subset \mathbb{R}^N$, $j = 0, \ldots, n$, be such that*

(i) $\overline{B(x_i, r_i)} \cap \overline{B(x_0, r_0)} \neq \emptyset$ *for all* $1 \le i \le n$*;*

(ii) $r_i > \frac{3}{4} r_0$ *for all* $1 \le i \le n$*;*

(iii) $|x_0 - x_i| > \frac{3}{4} \max\{r_i, r_0\}$ *for all* $1 \le i \le n$*;*

(iv) if $1 \le i < j \le n$ *then either* $|x_i - x_j| \ge r_i > \frac{3}{4} r_j$ *or* $|x_i - x_j| \ge r_j > \frac{3}{4} r_i$*.*

Then there exists a positive integer $c = c(N)$ such that $n \le c$.

Proof. Without loss of generality we may assume that $x_0 = 0$ and $r_0 = 1$, so that by (ii) $r_i > \frac{3}{4}$ for all $1 \le i \le n$. For $0 \le i \le n$ define

$$y_i := \begin{cases} x_i & \text{if } |x_i| \le 2, \\ 2\frac{x_i}{|x_i|} & \text{if } |x_i| > 2. \end{cases}$$

We claim that

$$|y_i - y_j| > \frac{1}{9} \tag{40}$$

for all $i, j = 0, \ldots, n$ with $i \neq j$. Note that if the claim holds, then $n + 1$ is less than or equal to the cardinality of the maximum number of points on the ball $\overline{B(0, 2)}$ with distance between distinct points greater than $\frac{1}{9}$ and with one of them being the origin.

To prove the claim, assume that $i < j$. We distinguish three cases.

Step 1: If $|x_i| \le 2$ and $|x_j| \le 2$, then

$$|y_i - y_j| > \frac{3}{4}. \tag{41}$$

Indeed, if $i = 0$, then by (iii)

$$\frac{4}{3}|y_i| = \frac{4}{3}|x_i| > \max\{r_i, 1\} \ge 1,$$

while if $i = 1, \ldots, n$, then by (ii) and (iv), $|y_i - y_j| = |x_i - x_j| > 0 \geq \frac{3}{4}$.

Step 2: If $|x_i| \leq 2$ and $|x_j| > 2$, then it is enough to consider the case $i = 1, \ldots, n$, since if $i = 0$, we have that $|y_0 - y_j| = |y_j| = 2$. Since by (i) $\overline{B(x_j, r_j)} \cap \overline{B(0,1)} \neq \emptyset$, we have that

$$r_j > |x_j - y_j| + 1, \tag{42}$$

and so $\overline{B(y_j, 1)} \subset \overline{B(x_j, r_j)}$. Indeed, for any $x \in \overline{B(y_j, 1)}$, $|x - x_j| \leq |x - y_j| + |x_j - y_j| \leq r_j$.

Hence, if $|x_i - x_j| \geq r_j$, then $x_i \notin B(x_j, r_j)$ and in turn $|y_i - y_j| = |x_i - y_j| \geq 1$. On the other hand, if $|x_i - x_j| < r_j$, then by (iv) $|x_i - x_j| \geq r_i > \frac{3}{4} r_j$, while by (iii)

$$\frac{3}{2} \geq |x_i| \frac{3}{4} > \max\{r_i, 1\} \geq r_i \geq \frac{3}{4} r_j,$$

which shows that $r_j \leq \frac{9}{8}$. Hence, also by (42),

$$\begin{aligned} |y_i - y_j| &= |x_i - y_j| \geq |x_i - x_j| - |x_j - y_j| \\ &> \frac{3}{4} r_j - (r_j - 1) = 1 - \frac{1}{4} r_j \geq 1 - \frac{9}{32}. \end{aligned}$$

Step 3: Finally, if $|x_i| \geq |x_j| > 2$, let

$$x = |x_j| \frac{x_i}{|x_i|}.$$

Then

$$|y_i - y_j| = \left| \frac{2}{|x_j|} x - \frac{2}{|x_j|} x_j \right| = \frac{2}{|x_j|} |x - x_j|, \tag{43}$$

and thus in the remaining of this step we need to estimate $|x - x_j|$.

Since

$$|x_i - x_j| \leq |x_i - x| + |x - x_j| = \left| x_i - |x_j| \frac{x_i}{|x_i|} \right| + |x - x_j| = |x_i| - |x_j| + |x - x_j|,$$

we have that

$$|x - x_j| \geq (|x_i - x_j| + |x_j|) - |x_i|.$$

By condition (i), $|x_i| \leq r_i + 1$, so

$$|x - x_j| \geq (|x_i - x_j| + |x_j|) - |x_i| \geq |x_j| - 1 + |x_i - x_j| - r_i.$$

Hence by (43),

$$\begin{aligned} |y_i - y_j| &\geq \frac{2}{|x_j|} [|x_j| - 1 + |x_i - x_j| - r_i] \\ &= 2 - \frac{2}{|x_j|} + \frac{2}{|x_j|} [|x_i - x_j| - r_i] \\ &> 1 + \frac{2}{|x_j|} [|x_i - x_j| - r_i], \end{aligned} \tag{44}$$

where we have used the fact that $|x_j| > 2$. If $|x_i - x_j| - r_i \geq 0$, then $|y_i - y_j| > 1$, while if $|x_i - x_j| - r_i < 0$, then by (iv) and (iii),

$$r_i - |x_i - x_j| \leq r_i - r_j \leq \frac{1}{3} r_j \leq \frac{4}{9} |x_j| .$$

Since $|x_j| > 2$, by (44),

$$\begin{aligned} |y_i - y_j| &> 1 + \frac{2}{|x_j|} [|x_i - x_j| - r_i] \\ &\geq 1 + \frac{2}{|x_j|} \left[-\frac{4}{9} |x_j| \right] = 1 - \frac{8}{9} = \frac{1}{9}. \end{aligned}$$

■

We are now in a position to prove the Besicovitch covering theorem

Proof of Theorem 125. For every $x \in E$ select $\overline{B(x, r_x)} \in \mathcal{F}$. For simplicity, we write $\overline{B_x} := \overline{B(x, r_x)}$. We divide the proof into three steps.

Step 1: Assume first that E is bounded. We construct a countable subfamily of $\mathcal{F}$ that still covers E. By (39) we may choose $x_1 \in E$ such that

$$r_{x_1} > \frac{3}{4} \sup_{x \in E} r_x$$

and set $E_2 := E \backslash \overline{B_{x_1}}$. By induction, assuming that $x_1, \ldots, x_n$ have been chosen, define

$$E_{n+1} := E \setminus \bigcup_{i=1}^{n} \overline{B_{x_i}}. \tag{45}$$

If E_{n+1} is empty then set $J := \{1, \ldots, n\}$ and observe that

$$E \subset \bigcup_{i=1}^{n} \overline{B_{x_i}}. \tag{46}$$

Otherwise, again by (39), select $x_{n+1} \in E_{n+1}$ such that

$$r_{x_{n+1}} > \frac{3}{4} \sup_{x \in E_{n+1}} r_x. \tag{47}$$

If $E_n \neq \emptyset$ for every n then we set $J := \mathbb{N}$, and we claim that

$$E \subset \bigcup_{n=1}^{\infty} \overline{B_{x_n}}. \tag{48}$$

This follows easily from the fact that

$$r_{x_n} \to 0. \tag{49}$$

Indeed, if (49) holds and if $x \in E$, then find n large enough such that

$$r_{x_{n+1}} < \frac{3}{4} r_x.$$

By (47) it follows that $x \notin E_{n+1}$, and so , in view of (45), we conclude that

$$x \in \bigcup_{i=1}^{n} \overline{B_{x_i}}.$$

By (39),

$$\sup_n r_n \leq \sup_{x \in E} r_x =: R < \infty. \tag{50}$$

Note that by construction if i, $j \in J$ and $i < j$ then

$$x_j \notin \overline{B_{x_i}} \text{ and } r_{x_i} > \frac{3}{4} r_{x_j}. \tag{51}$$

Moreover, the balls $B\left(x_i, \frac{r_i}{3}\right)$ and $B\left(x_j, \frac{r_j}{3}\right)$ are disjoint. Indeed, since $x_j \notin \overline{B_{x_i}}$, then

$$|x_i - x_j| > r_i = \frac{r_i}{3} + \frac{2r_i}{3} \geq \frac{r_i}{3} + \frac{r_j}{2} \geq \frac{r_i}{3} + \frac{r_j}{3}.$$

Since

$$\bigcup_{n=1}^{\infty} B\left(x_n, \frac{r_n}{3}\right) \subset \left\{x \in \mathbb{R}^N : \operatorname{dist}(x, E) < \frac{R}{3}\right\},$$

we conclude that

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{3}\right)^N < \infty,$$

where we have used the fact that E is bounded. Hence (49) holds.

Step 2: Let $\left\{\overline{B_i}\right\}_{i \in J}$ be the countable subfamily of $\mathcal{F}$ constructed in Step 1 and that still covers the bounded set E. If $J = \{1\}$ then there is nothing left to prove. Otherwise, for any fixed integer $k > 1$ let

$$I_k := \left\{i \in J : 1 \leq i < k, \overline{B_i} \cap \overline{B_k} \neq \emptyset\right\}.$$

By Lemma 126 and in view of (51) there exists a positive integer $c(N)$ such that the cardinality of I_k is less than $c(N)$.

Construct a function $f : J \to \{1, \ldots, c(N)\}$ in the following way. If $j \leq c(N)$ set $f(j) := j$. For $j > c(N)$ we define $f(j)$ recursively: assuming that $f(1), \ldots, f(j-1)$ have been assigned, set $f(j) := l$, where $l \in \{1, \ldots, c(N)\}$ satisfies the property

$$\overline{B_j} \cap \overline{B_i} = \emptyset \tag{52}$$

whenever $i \in \{1, \ldots, j-1\}$ is such that $f(i) = l$. Note that there is at least one such number l. Indeed, if not, then for every $l \in \{1, \ldots, c(N)\}$ there would exist $i_l \in \{1, \ldots, j-1\}$ such that $f(i_l) = l$ and $\overline{B_j} \cap \overline{B_{i_l}} \neq \emptyset$. In particular,

$i_l \in I_{j-1}$. Since $l \neq l'$ implies that $i_l \neq i_{l'}$, this would entail that the cardinality of I_{j-1} would be at least $c(N)$, and this is a contradiction.

For $n \in \{1, \ldots, c(N)\}$ let

$$\mathcal{F}_n := \left\{\overline{B_i} :\ f(i) = n\right\}.$$

Since if $i \in J$ then $\overline{B_i} \in \mathcal{F}_{f(i)}$, by (46) and (48) we have

$$E \subset \bigcup_{n=1}^{c(N)} \bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B}.$$

It remains to show that distinct elements of $\mathcal{F}_n$ are mutually disjoint. Indeed, if $i < j$ and $f(i) = f(j)$, then j is strictly bigger than $c(N)$, and so $\overline{B_j} \cap \overline{B_i} = \emptyset$ by (52).

Step 3: If the set E is unbounded, then for each $k \in \mathbb{N}$ apply the previous steps to the set

$$E_k := E \cap \left(B(0, kR') \setminus B(0, (k-1)R')\right),$$

where $R' >> R$ and R is defined in (50), to find

$$E_k \subset \bigcup_{n=1}^{c(N)} \bigcup_{\overline{B} \in \mathcal{F}_n^{(k)}} \overline{B},$$

where the elements of $\mathcal{F}_n^{(k)}$ are mutually disjoint. Without loss of generality, we may remove from each family $\mathcal{F}_n^{(k)}$ those elements that do not intersect E_k. Hence by (50), if $\overline{B} \in \mathcal{F}_n^{(k)}$ and $\overline{B'} \in \mathcal{F}_n^{(k+2)}$ then $\overline{B} \cap \overline{B'} = \emptyset$. For $n \in \{1, \ldots, c(N)\}$ define

$$\mathcal{F}_n := \left\{\overline{B} \in \mathcal{F}_n^{(2k)} :\ k \in \mathbb{N}\right\},$$

and if $n \in \{c(N) + 1, \ldots, 2c(N)\}$ set

$$\mathcal{F}_n := \left\{\overline{B} \in \mathcal{F}_{n-c(N)}^{(2k-1)} :\ k \in \mathbb{N}\right\}.$$

Then

$$E \subset \bigcup_{n=1}^{2c(N)} \bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B},$$

and this cover satisfies the desired properties. ■

Corollary 127 *Let $\mathcal{F}$ be a collection of (nondegenerate) closed balls. Assume that the set E of centers of balls in $\mathcal{F}$ is bounded. Then there exist $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \subset \mathcal{F}$ such that each $\mathcal{F}_n$, $n = 1, \ldots, \ell$, is a countable family of disjoint balls in $\mathcal{F}$ and*

$$E \subset \bigcup_{n=1}^{\ell} \bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B},$$

where ℓ is the number given in the previous theorem.

Proof. Since E is bounded, there exists $R > 0$ such that $E \subset B(0, R/2)$. If one of the balls in the family $\mathcal{F}$, say $\overline{B(x_0, r_0)}$ has radius $r_0 \geq R$, then for every $x \in E$,

$$|x - x_0| \leq |x| + |x_0| < \frac{R}{2} + \frac{R}{2} = R,$$

which shows that $E \subset \overline{B(x_0, r_0)}$. Thus we can take $\mathcal{F}_1 := \left\{\overline{B(x_0, r_0)}\right\}$.

Otherwise, we can assume that all balls in $\mathcal{F}$ have less than R. Hence, (39) holds and we can apply the previous theorem. ■

Monday, October 31, 2011

Exercise 128 *The case $p = \infty$. Consider the Lebesgue measure in $\mathbb{R}^N$.*

(i) Let $g : \mathbb{R}^N \to \mathbb{R}$ be a continuous bounded function. Prove that

$$\operatorname{esssup} |g| = \sup |g| .$$

(ii) Prove that $L^\infty(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ is not dense in $L^\infty(\mathbb{R}^N)$.

Exercise 129 *Consider the Lebesgue measure in $\mathbb{R}^N$. Prove that for every $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^N)$, and $h \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} |f(x+h)|^p \, dx = \int_{\mathbb{R}^N} |f(x)|^p \, dx.$$

Corollary 130 *Consider the Lebesgue measure in $\mathbb{R}^N$. Then for every $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^N)$,*

$$\lim_{h \to 0} \int_{\mathbb{R}^N} |f(x+h) - f(x)|^p \, dx = 0.$$

Proof. Step 1: Assume that $f \in C_c(\mathbb{R}^N)$. Then f is uniformly continuous, and so, given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) \in (0, 1)$ such that

$$|f(x) - f(y)| \leq \varepsilon$$

for all $x, y \in \mathbb{R}^N$ with $|x - y| \leq \delta$. Since f has compact support, there exists $B(0, R)$ such that $f = 0$ in $\mathbb{R}^N \setminus B(0, R)$. Take $|h| \leq \delta$. Then $f(x+h) = 0$ for all $x \in \mathbb{R}^N \setminus B(0, R+1)$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p \, dx &= \int_{B(0,R+1)} |f(x+h) - f(x)|^p \, dx \\ &\leq \varepsilon^p \mathcal{L}^N(B(0, R+1)),\end{aligned}$$

which proves the result in this case.

Step 2: Given $f \in L^p(\mathbb{R}^N)$ and $\varepsilon > 0$ find $g \in C_c(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |g(x) - f(x)|^p \, dx \leq \varepsilon.$$

By the previous exercise,

$$\int_{\mathbb{R}^N} |g(x+h) - f(x+h)|^p \, dx \le \varepsilon$$

for every $h \in \mathbb{R}^N$. By the previous step, there exists $\delta > 0$ such that if $|h| \le \delta$, then

$$\int_{\mathbb{R}^N} |g(x+h) - g(x)|^p \, dx \le \varepsilon.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p \, dx &= \int_{\mathbb{R}^N} |f(x+h) \pm g(x+h) \pm g(x) - f(x)|^p \, dx \\ &\le 3^{p-1} \int_{\mathbb{R}^N} |f(x+h) - g(x+h)|^p \, dx + 3^{p-1} \int_{\mathbb{R}^N} |g(x+h) - g(x)|^p \, dx \\ &\quad + 3^{p-1} \int_{\mathbb{R}^N} |f(x) - g(x)|^p \, dx \le 3^p \varepsilon.\end{aligned}$$

This concludes the proof. ■

In view of Theorem 122, it is important to understand which measures satisfy (38).

Theorem 131 *Let $\left(\mathbb{R}^N, \mathcal{B}(X), \mu\right)$ be a measure space, with μ finite on compact sets. Then*

$$\mu(E) = \sup\{\mu(K) : K \text{ compact, } K \subset E\} = \inf\{\mu(A) : A \text{ open, } A \supset E\} \tag{53}$$

for every $E \in \mathcal{B}(X)$.

To prove the previous theorem, we introduce the notion of *Dynkin class.* We will do this later.

Let $\mu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0, \infty]$ be a measure finite on compact sets. Let

$$M_1 := \left\{x \in \mathbb{R}^N : \mu(B(x,r)) = 0 \text{ for some } r > 0\right\}.$$

Note that the set M_1 is open. Using the density of the rationals in the reals it is possible to cover M_1 with countably many balls of measure zero. Therefore $\mu(M_1) = 0$.

Given a locally integrable function $f : \mathbb{R}^N \to \mathbb{R}$, the *(Hardy–Littlewood) maximal function* of f is defined by

$$\mathrm{M}(f)(x) := \sup_{r>0} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f| \, d\mu$$

for all $x \in \mathbb{R}^N \setminus M_1$.

Wednesday, November 02, 2011

Exercise 132 *Let $\mu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0, \infty]$ be a measure finite on compact sets.*

(i) Prove that for every $r > 0$ the set $\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f)(x) > t\}$ is Borel.

(ii) Prove that $\mathrm{M}(f)$ is a Borel function.

Theorem 133 *Let $\mu : \mathcal{B}(\mathbb{R}^N) \to [0, \infty]$ be a measure finite on compact sets and let $f \in L^p(\mathbb{R}^N)$, $1 \le p \le \infty$. Then*

(i) if $p = 1$ then for any $t > 0$,

$$\mu\left(\left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f)(x) > t\right\}\right) \le \frac{C(N)}{t} \int_{\mathbb{R}^N} |f| \, d\mu; \tag{54}$$

(ii) if $1 < p \le \infty$ then $\mathrm{M}(f) \in L^p(\mathbb{R}^N)$ and

$$\|\mathrm{M}(f)\|_{L^p} \le C(N,p) \|f\|_{L^p}.$$

Proof. We begin by proving (i). Let

$$E_t := \left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f)(x) > t\right\}.$$

By the definition of $\mathrm{M}(f)$, for every $x \in E_t$ we can find a ball $\overline{B(x, r_x)}$, with $r_x > 0$, such that

$$\frac{1}{\mu\left(\overline{B(x, r_x)}\right)} \int_{\overline{B(x,r_x)}} |f| \, d\mu > t. \tag{55}$$

Let $B(0, R)$ be a fixed ball and consider the family $\mathcal{F} := \left\{\overline{B(x, r_x)}\right\}_{x \in E_t \cap B(0,R)}$. By the Besicovitch covering theorem, there exist $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \subset \mathcal{F}$ such that each $\mathcal{F}_n$, $n = 1, \ldots, \ell$, is a countable family of disjoint balls in $\mathcal{F}$ and

$$E_t \cap B(0, R) \subset \bigcup_{n=1}^{\ell} \bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B}.$$

Hence, by (55),

$$\begin{aligned}
\mu(E_t \cap B(0, R)) &\le \sum_{n=1}^{\ell} \sum_{\overline{B} \in \mathcal{F}_n} \mu(\overline{B}) \le \sum_{n=1}^{\ell} \sum_{\overline{B} \in \mathcal{F}_n} \frac{1}{t} \int_{\overline{B}} |f| \, d\mu \\
&= \sum_{n=1}^{\ell} \frac{1}{t} \int_{\bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B}} |f| \, d\mu \le \frac{\ell}{t} \int_{\mathbb{R}^N} |f| \, d\mu.
\end{aligned}$$

Taking $R = k$ and letting $k \to \infty$, by Proposition 47, we obtain that

$$\mu\left(\left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f)(x) > t\right\}\right) \le \frac{\ell}{t} \int_{\mathbb{R}^N} |f| \, d\mu.$$

Note that this implies, in particular, that $\mathrm{M}(f)(x) < \infty$ for μ a.e. $x \in \mathbb{R}^N$.

In order to prove (ii) it suffices to consider $1 < p < \infty$, since the case $p = \infty$ is immediate from the definition of $\mathrm{M}(f)$. For $t > 0$ define

$$f_t(x) := \begin{cases} f(x) & \text{if } |f(x)| > \frac{t}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $f_t \in L^1\left(\mathbb{R}^N\right)$. Indeed,

$$\begin{aligned} \int_{\mathbb{R}^N} |f_t| \, d\mu &= \int_{\left\{x \in \mathbb{R}^N : |f(x)| > \frac{t}{2}\right\}} |f| \, d\mu \\ &\le \left(\frac{2}{t}\right)^{p-1} \int_{\left\{x \in \mathbb{R}^N : |f(x)| > \frac{t}{2}\right\}} |f|^p \, d\mu < \infty. \end{aligned}$$

Moreover, since $|f| \le |f_t| + \frac{t}{2}$ we have that $\mathrm{M}(f) \le \mathrm{M}(f_t) + \frac{t}{2}$, and so

$$\left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f)(x) > t\right\} \subset \left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f_t)(x) > \frac{t}{2}\right\}.$$

Part (i) applied to $f_t \in L^1\left(\mathbb{R}^N\right)$ now yields

$$\begin{aligned} \mu\left(\left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f)(x) > t\right\}\right) &\le \frac{2\ell}{t} \int_{\mathbb{R}^N} |f_t| \, d\mu \\ &= \frac{2\ell}{t} \int_{\left\{x \in \mathbb{R}^N : |f(x)| > \frac{t}{2}\right\}} |f| \, d\mu. \end{aligned} \tag{56}$$

Hence, using Theorem **??**, Fubini's theorem, and the fact that $\mu(M_1) = 0$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (\mathrm{M}(f))^p \, d\mu &= \int_{\mathbb{R}^N \setminus M_1} (\mathrm{M}(f))^p \, d\mu \\ &= \int_0^\infty \mu\left(\left\{x \in \mathbb{R}^N \setminus M_1 : (\mathrm{M}(f)(x))^p > s\right\}\right) ds \\ &= p \int_0^\infty t^{p-1} \mu\left(\left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f)(x) > t\right\}\right) dt \\ &\le 2\ell p \int_0^\infty t^{p-2} \int_{\left\{x \in \mathbb{R}^N : |f(x)| > \frac{t}{2}\right\}} |f(y)| \, d\mu(y) \, dt \\ &= 2\ell p \int_{\mathbb{R}^N} |f(y)| \left(\int_0^{2|f(y)|} t^{p-2} \, dt \right) d\mu(y) \\ &= \frac{2\ell p 2^p}{p-1} \int_{\mathbb{R}^N} |f(y)|^p \, d\mu(y). \end{aligned}$$

This proves (ii). ■

Property (ii) in the previous theorem is usually called the *weak L^1 inequality* because, as opposed to the case $p > 1$ (see (iii)), for $p = 1$ the operator

$$f \in L^1\left(\mathbb{R}^N\right) \mapsto \mathrm{M}(f)$$

is not bounded in L^1. Actually, if f is not identically zero then $\mathrm{M}(f) \notin L^1(\mathbb{R}^N)$. To see this, it suffices to assume, without loss of generality, that

$$\int_{B(0,1)} |f| \, dx > 0,$$

Then if $|x| \geq 1$ it follows that $B(0,1) \subset B(x, 2|x|)$, and thus for some $c > 0$,

$$\mathrm{M}(f)(x) \geq \frac{1}{|B(x, 2|x|)|} \int_{B(x,2|x|)} |f(y)| \, dy \geq \frac{c}{|x|^N} \int_{B(0,1)} |f| \, dx,$$

with $|x|^{-N} \notin L^1(\mathbb{R}^N \setminus B(0,1))$.

Friday, November 04, 2011

Theorem 134 (Lebesgue differentiation theorem) *Let $\mu : \mathcal{B}(\mathbb{R}^N) \to [0, \infty]$ be a measure finite on compact sets and let $f : \mathbb{R}^N \to \mathbb{R}$ be locally integrable. Then there exists a Borel set $E \subset \mathbb{R}^N$, with $\mu(E) = 0$, such that $E \supset M_1$ and for every $x \in \mathbb{R}^N \setminus E$,*

$$\lim_{r \to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)| \, d\mu(y) = 0. \tag{57}$$

Proof. Step 1: Assume that f is integrable. Given $\varepsilon > 0$, by density, we may find a function $g_\varepsilon \in C_c(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |f - g_\varepsilon| \, d\mu \leq \varepsilon.$$

Since g is uniformly continuous, given $\eta > 0$ there exists $\delta = \delta(\eta) \in (0,1)$ such that

$$|g_\varepsilon(x) - g_\varepsilon(y)| \leq \varepsilon$$

for all $x, y \in \mathbb{R}^N$ with $|x - y| \leq \delta$. Hence, for $0 < r \leq \delta$ and $x \in \mathbb{R}^N \setminus M_1$,

$$\frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |g_\varepsilon(y) - g_\varepsilon(x)| \, d\mu(y) \leq \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} \eta \, d\mu(y) = \eta.$$

This shows that

$$\lim_{r \to 0^+} \int_{\overline{B(x,r)}} |g_\varepsilon(y) - g_\varepsilon(x)| \, d\mu(y) = 0$$

for every $x \in \mathbb{R}^N \setminus M_1$. Hence,

$$\begin{aligned}
&\limsup_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)|\, d\mu(y) \\
&= \limsup_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) \pm g_\varepsilon(y) - f(x) \pm g_\varepsilon(x)|\, d\mu(y) \\
&\leq \limsup_{r\to 0^+} \left[\frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f - g_\varepsilon|\, d\mu(y) + \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |g_\varepsilon - g_\varepsilon(x)|\, d\mu(y)\right] + |g_\varepsilon(x) - f(x)| \\
&\leq \mathrm{M}(f - g_\varepsilon)(x) + 0 + |g_\varepsilon(x) - f(x)|.
\end{aligned}$$

For every $t > 0$, define

$$\begin{aligned}
G_t &:= \left\{x \in \mathbb{R}^N \setminus M_1 : \limsup_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)|\, d\mu(y) > t\right\}, \\
E_{t,\varepsilon} &:= \left\{x \in \mathbb{R}^N \setminus M_1 : \mathrm{M}(f - g_\varepsilon)(x) > t\right\}, \\
F_{t,\varepsilon} &:= \left\{x \in \mathbb{R}^N : |g_\varepsilon(x) - f(x)| > t\right\}.
\end{aligned}$$

Then by the previous inequality, we have $G_{2t} \subset E_{t,\varepsilon} \cup F_{t,\varepsilon}$. By (54),

$$\mu(E_{t,\varepsilon}) \leq \frac{C(N)}{t} \int_{\mathbb{R}^N} |f - g_\varepsilon|\, d\mu \leq \frac{C(N)\,\varepsilon}{t},$$

while

$$\mu(F_{t,\varepsilon}) \leq \frac{1}{t} \int_{\mathbb{R}^N} |f - g_\varepsilon|\, d\mu \leq \frac{\varepsilon}{t}.$$

Hence,

$$\mu(G_{2t}) \leq \mu(E_{t,\varepsilon}) + \mu(F_{t,\varepsilon}) \leq \frac{(C(N) + 1)\,\varepsilon}{t}.$$

Since G_{2t} does not depend on $\varepsilon > 0$, we can let $\varepsilon \to 0^+$ in the previous inequality to conclude that $\mu(G_{2t}) = 0$ for all $t > 0$. Let

$$E := \bigcup_{n=1}^{\infty} G_{\frac{1}{n}}.$$

Then $\mu(E) = 0$ and if $x \in \mathbb{R}^N \setminus E$, then

$$\limsup_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)|\, d\mu(y) \leq \frac{1}{n}$$

for every n, that is,

$$\limsup_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)|\, d\mu(y) = 0,$$

which implies that (57) holds.

Step 2: To remove the hypothesis that f is integrable, for every $k \in \mathbb{N}$ consider the function $f_k := f\chi_{B(0,k)}$. Then f_k is integrable, and so by Step 1 there exists a set $E_k \subset \mathbb{R}^N$, with $\mu(E_k) = 0$, such that $E_k \supset M_1$ and for every $x \in \mathbb{R}^N \setminus E_k$,

$$\lim_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f_k(y) - f_k(x)|\, d\mu(y) = 0. \tag{58}$$

Let

$$E := \bigcup_{k=1}^{\infty} E_k.$$

If $x \in \mathbb{R}^N \setminus E$, find k so large that $|x| < k-1$. Then for all $0 < r < 1$, we have that $f_k(y) = f(y)\chi_{B(0,k)}(y) = f(y)1$ for all $y \in \overline{B(x,r)}$, and so by (58), we have that (57) holds at the point x. ■

A point $x \in \mathbb{R}^N$ for which (**??**) holds is called *a Lebesgue point* of f.

Remark 135 *If $x \in \mathbb{R}^N \setminus E$, then for any family of Borel subsets $\{E_{x,r}\}_{r>0}$ such that $E_{x,r} \subset \overline{B(x,r)}$ and*

$$\mu(E_{x,r}) > \alpha\mu\left(\overline{B(x,r)}\right)$$

for some constant $\alpha > 0$ independent of $r > 0$, we have that

$$\begin{aligned}
&\limsup_{r\to 0^+} \frac{1}{\mu(E_{x,r})} \int_{E_{x,r}} |f(y) - f(x)|\, d\mu(y) \\
&\quad \le \limsup_{r\to 0^+} \frac{1}{\mu(E_{x,r})} \int_{\overline{B(x,r)}} |f(y) - f(x)|\, d\mu(y) \\
&\quad \le \frac{1}{\alpha} \lim_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)|\, d\mu(y) = 0.
\end{aligned}$$

Note that the sets $E_{x,r}$ need not contain x.

Monday, November 07, 2011

8 Radon-Nikodym's Theorem

Definition 136 *Let $(X, \mathfrak{M})$ be a measurable space and let $\mu, \nu : \mathfrak{M} \to [0,\infty]$ be two measures. The measure ν is said to be* absolutely continuous with respect to μ, *and we write $\nu \ll \mu$, if for every $E \in \mathfrak{M}$ with $\mu(E) = 0$ we have $\nu(E) = 0$.*

Example 137 *Given a measure space $(X, \mathfrak{M}, \mu)$ and a measurable function $f : X \to [0,\infty]$, we define*

$$\nu(E) := \int_E f\, d\mu, \quad E \in \mathfrak{M}. \tag{59}$$

Then ν is a measure and $\nu \ll \mu$. The Radon-Nikodym's theorem shows that when μ is σ-finite, then the opposite is also true, namely that if $\nu \ll \mu$, then v is given by (59) for some function f.

Exercise 138 *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing continuous function. Prove that*

$$\mathcal{L}_o^1 (f (E)) = \mu_f^* (E)$$

for every set $E \subset I$.

Example 139 *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing function and consider the Lebesgue-Stieltjes measure μ_f generated by f. When is $\mu_f \ll \mathcal{L}^1$? We need f to be continuous. Indeed, if f is discontinuous at some $x \in I$, then we proved that (see (11)) $\mu_f^* (\{x\}) = f^+ (x) - f^- (x) > 0$, while $\mathcal{L}^1 (\{x\}) = 0$. If f is continuous, is $\mu_f \ll \mathcal{L}^1$? By the previous exercise, if f is continuous, then*

$$\mathcal{L}_o^1 (f (E)) = \mu_f^* (E)$$

for every set $E \subset I$. Hence, $\mu_f \ll \mathcal{L}^1$ if and only if $\mathcal{L}_o^1 (f (E)) = 0$ for all Lebesgue measurable $E \subset \mathbb{R}$ with $\mathcal{L}^1 (E) = 0$. Thus, f maps sets of measure zero into sets of measure zero. Note that the Cantor function does not have this property, since $f (\mathbb{D}) = [0, 1]$. Thus, there are continuous increasing functions such that μ_f is not absolutely continuous with respect to $\mathcal{L}^1$.

Given an increasing function $f : I \to \mathbb{R}$, it can be shown that $\mu_f \ll \mathcal{L}^1$ if and only if $f : I \to \mathbb{R}$ is absolutely continuous*, that is, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \dots, \ell$, with $[a_k, b_k] \subset I$ and*

$$\sum_{k=1}^{\ell} (b_k - a_k) \leq \delta,$$

there holds

$$\sum_{k=1}^{\ell} |f (b_k) - f (a_k)| \leq \varepsilon. \tag{60}$$

Theorem 140 (Radon–Nikodym) *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0, \infty]$ be two measures, with μ σ-finite and ν absolutely continuous with respect to μ. Then there exists a unique (up to sets of measure μ zero) measurable function $f : X \to [0, \infty]$ such that*

$$\nu (E) = \int_E f \, d\mu$$

for every $E \in \mathfrak{M}$.

The function f is called the Radon-Nikodym derivative of ν with respect to μ and is denoted by $\frac{d\nu}{d\mu}$. Note that by the Lebesgue differentiation theorem if $X = \mathbb{R}^N$ and $\mathfrak{M}$ is the Borel σ-algebra, then for σ-finite measures, we have that

$$\frac{d\nu}{d\mu}(x) = \lim_{r\to 0^+} \frac{\nu\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)}$$

for μ a.e. $x \in X$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is increasing and absolutely continuous, then for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}$,

$$\frac{d\mu_f}{d\mathcal{L}^1}(x) = \lim_{r\to 0^+} \frac{\mu_f([x-r, x+r])}{\mathcal{L}^1([x-r, x+r])} = \lim_{r\to 0^+} \frac{f(x+r) - f(x-r)}{2r}.$$

In particular, if f is differentiable at x, then

$$\begin{aligned}
\lim_{r\to 0^+} \frac{f(x+r) - f(x-r)}{2r} &= \lim_{r\to 0^+} \frac{f(x+r) \pm f(x) - f(x-r)}{2r} \\
&= \frac{1}{2}\lim_{r\to 0^+} \frac{f(x+r) - f(x)}{r-0} + \frac{1}{2}\lim_{r\to 0^+} \frac{f(x) - f(x-r)}{0-r} \\
&= \frac{1}{2} f'(x) + \frac{1}{2} f'(x) = f'(x).
\end{aligned}$$

To prove the Radon-Nikodym theorem we need two lemmas.

Lemma 141 *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0, \infty]$ be two measures. For every $E \in \mathfrak{M}$ define*

$$\nu_{ac}(E) := \sup\left\{\int_E f\,d\mu : \ f : X \to [0,\infty] \ \textit{measurable}, \right. \\ \left. \int_{E'} f\,d\mu \le \nu(E') \ \textit{for all } E' \subset E, E' \in \mathfrak{M}\right\}. \tag{61}$$

Then ν_{ac} is a measure, with $\nu_{ac} \ll \mu$, and for each $E \in \mathfrak{M}$ the supremum in the definition of ν_{ac} is actually attained by a function f admissible for $\nu_{ac}(E)$. Moreover, if ν_{ac} is σ-finite, then f may be chosen independently of the set E.

Proof. Step 1: We prove that ν_{ac} is a measure, with $\nu_{ac} \ll \mu$. It follows from (61) that $\nu_{ac}(\emptyset) = 0$.

Note that if $E_1 \subset E_2$, with E_1, $E_2 \in \mathfrak{M}$, then $\nu_{ac}(E_1) \le \nu_{ac}(E_2)$. Indeed, is $f : X \to [0, \infty]$ is a measurable function such that $\int_{E'} f\,d\mu \le \nu(E')$ for all $E' \subset E_1$, $E' \in \mathfrak{M}$, then we can define

$$g(x) := \begin{cases} f(x) & \text{if } x \in E_1, \\ 0 & \text{if } x \in X \setminus E_1. \end{cases}$$

Then $\int_{E'} g\,d\mu \le \nu(E')$ for all $E' \in \mathfrak{M}$. In particular, g is admissible in the definition of $\nu_{ac}(E_2)$. Hence,

$$\int_{E_1} f\,d\mu + 0 = \int_{E_1} f\,d\mu + \int_{E_2\setminus E_1} 0\,d\mu = \int_{E_2} g\,d\mu \le \nu_{ac}(E_2)$$

and taking the supremum over all such f gives $\nu_{ac}(E_1) \leq \nu_{ac}(E_2)$.

Let $\{E_n\} \subset \mathfrak{M}$ be a countable collection of pairwise disjoint sets and let $E := \bigcup_{n=1}^{\infty} E_n$. Let $f : X \to [0,\infty]$ be a measurable function such that $\int_{E'} f \, d\mu \leq \nu(E')$ for all $E' \subset E$, $E' \in \mathfrak{M}$. Then for every $n \in \mathbb{N}$,

$$\int_{E'} f \, d\mu \leq \nu(E')$$

for all $E' \subset E_n$, $E' \in \mathfrak{M}$, and so

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu \leq \sum_{n=1}^{\infty} \nu_{ac}(E_n).$$

Hence taking the supremum over all such f, we obtain

$$\nu_{ac}(E) \leq \sum_{n=1}^{\infty} \nu_{ac}(E_n). \tag{62}$$

To prove the opposite inequality, it suffices to assume that $\nu_{ac}(E) < \infty$ (otherwise there is nothing to prove). In turn, $\nu_{ac}(E_n) \leq \nu_{ac}(E) < \infty$, and so given $\varepsilon > 0$ and $n \in \mathbb{N}$ we can find a measurable function $f_n : X \to [0,\infty]$ such that $\int_{E'} f_n \, d\mu \leq \nu(E')$ for all $E' \subset E_n$, $E' \in \mathfrak{M}$, and

$$\nu_{ac}(E_n) \leq \int_{E_n} f_n \, d\mu + \frac{\varepsilon}{2^n}.$$

Define $f := \sum_{n=1}^{\infty} \chi_{E_n} f_n$. Then for all $E' \subset E$, with $E' \in \mathfrak{M}$, we have

$$\int_{E'} f \, d\mu = \sum_{n=1}^{\infty} \int_{E' \cap E_n} f_n \, d\mu \leq \sum_{n=1}^{\infty} \nu(E' \cap E_n) = \nu(E'),$$

and so,

$$\sum_{n=1}^{\infty} \nu_{ac}(E_n) \leq \sum_{n=1}^{\infty} \left(\int_{E_n} f_n \, d\mu + \frac{\varepsilon}{2^n} \right) \leq \int_E f \, d\mu + \varepsilon \leq \nu_{ac}(E) + \varepsilon.$$

Letting $\varepsilon \to 0^+$ and using (62), we conclude that ν_{ac} is a measure.

From the definition of ν_{ac} it follows that if $E \in \mathfrak{M}$ and $\mu(E) = 0$, then $\nu_{ac}(E) = 0$, which shows that $\nu_{ac} \ll \mu$.

Step 2: We claim that for each $E \in \mathfrak{M}$ the supremum in the definition of ν_{ac} is actually reached by a measurable function f. Indeed, note first that if f, $g : X \to [0,\infty]$ are measurable functions such that

$$\int_{E'} f \, d\mu \leq \nu(E'), \quad \int_{E'} g \, d\mu \leq \nu(E')$$

for all $E' \subset E$, $E' \in \mathfrak{M}$, then $\max\{f,g\}$ satisfies the same property, since

$$\begin{aligned}\int_{E'} \max\{f,g\}\, d\mu &= \int_{E'\cap\{f\geq g\}} f\, d\mu + \int_{E'\cap\{f<g\}} g\, d\mu \qquad (63)\\ &\leq \nu\left(E'\cap\{f\geq g\}\right) + \nu\left(E'\cap\{f<g\}\right)\\ &= \nu\left(E'\right).\end{aligned}$$

Hence we can find an increasing sequence of measurable functions $f_n : X \to [0,\infty]$ such that

$$\lim_{n\to\infty} \int_E f_n\, d\mu = \nu_{ac}(E)$$

and $\int_{E'} f_n\, d\mu \leq \nu(E')$ for all $E' \subset E$, $E' \in \mathfrak{M}$. Define $f := \lim_{n\to\infty} f_n$. By the Lebesgue monotone convergence theorem we have

$$\int_E f\, d\mu = \nu_{ac}(E).$$

Step 3: Suppose first that ν_{ac} is finite, and let f be a function obtained in Step 2 for the set X. Then

$$\nu_{ac}(X) = \int_X f\, d\mu.$$

We claim that

$$\nu_{ac}(E) = \int_E f\, d\mu \qquad (64)$$

for every set $E \in \mathfrak{M}$. Since f is admissible for X, we have that $\int_{E'} f\, d\mu \leq \nu(E')$ for all $E' \in \mathfrak{M}$. Hence

$$\nu_{ac}(E) \geq \int_E f\, d\mu.$$

If this inequality were strict for some $E \in \mathfrak{M}$, then we could find a function g admissible for $\nu_{ac}(E)$ such that

$$\int_E g\, d\mu > \int_E f\, d\mu. \qquad (65)$$

Define

$$\overline{f}(x) := \begin{cases} g(x) & \text{for } x \in E, \\ f(x) & \text{elsewhere.}\end{cases}$$

Then $\overline{f}$ is admissible for $\nu_{ac}(X)$, and so

$$\nu_{ac}(X) \geq \int_X \overline{f}\, d\mu = \int_X f\, d\mu + \int_E g\, d\mu > \int_X f\, d\mu = \nu_{ac}(X),$$

where we have used (65) and the fact that $\int_X f\, d\mu < \infty$, since ν_{ac} is finite. We have reached a contradiction.

If ν_{ac} is σ-finite then we may decompose X as

$$X = \bigcup_{n=1}^{\infty} X_n,$$

where $\nu_{ac}(X_n)$ and the sets X_n are disjoint. For each $n \in \mathbb{N}$ choose a function f_n admissible for $\nu_{ac}(X_n)$ for which (64) holds with f_n in place of f and for every measurable subset of X_n. The function

$$f(x) := \sum_{n=1}^{\infty} \chi_{X_n} f_n(x)$$

has the desired property. ■

Wednesday, November 09, 2011

Definition 142 *Let $(X, \mathfrak{M})$ be a measurable space. A* signed measure *is a function $\lambda : \mathfrak{M} \to [-\infty, \infty]$ such that*

(i) $\lambda(\emptyset) = 0$;

(ii) λ takes at most one of the two values ∞ and $-\infty$, that is, either $\lambda : \mathfrak{M} \to (-\infty, \infty]$ or $\lambda : \mathfrak{M} \to [-\infty, \infty)$;

(iii) for every countable collection $\{E_i\} \subset \mathfrak{M}$ of pairwise disjoint sets we have

$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n).$$

Definition 143 *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda : \mathfrak{M} \to [-\infty, \infty]$ be a signed measure. A set $E \in \mathfrak{M}$ is said to be* positive *(respectively* negative*) if $\lambda(F) \geq 0$ (respectively $\lambda(F) \leq 0$) for all $F \subset E$ with $F \in \mathfrak{M}$.*

Example 144 *Consider the signed measure,*

$$\lambda(E) := \int_E \frac{x}{1+x^2}\,dx, \quad E \in \mathcal{B}(\mathbb{R}).$$

Then the set $[-a, b]$, $0 < a < b$, has positive λ measure without being a positive set.

Lemma 145 *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda : \mathfrak{M} \to [-\infty, \infty]$ be a signed measure. For every $E \in \mathfrak{M}$ define*

$$\lambda^+(E) := \sup\{\lambda(F) : F \subset E, F \in \mathfrak{M}\}, \tag{66}$$

$$\begin{aligned}\lambda^-(E) &:= -\inf\{\lambda(F) : F \subset E, F \in \mathfrak{M}\} \\ &= \sup\{-\lambda(F) : F \subset E, F \in \mathfrak{M}\}.\end{aligned} \tag{67}$$

Then λ^+ and λ^- are measures. Moreover, if $\lambda : \mathfrak{M} \to [-\infty, \infty)$, then for every $E \in \mathfrak{M}$ we have

$$\lambda^+(E) = \sup\left\{\lambda(F) : F \subset E, F \in \mathfrak{M}, \lambda^-(F) = 0\right\}, \tag{68}$$

λ^+ is finite, and $\lambda = \lambda^+ - \lambda^-$.

Proof. Step 1: We begin by showing that λ^+ is a measure. Note that if $E_1, E_1 \in \mathfrak{M}$, with $E_1 \subset E_2$, then $\lambda^+(E_1) \le \lambda^+(E_2)$. Since

$$\lambda^+(\emptyset) = \lambda(\emptyset) = 0$$

we have $\lambda^+(E) \ge 0$ for all $E \in \mathfrak{M}$.

Let $\{E_n\} \subset \mathfrak{M}$ be a countable collection of pairwise disjoint sets and let $E := \bigcup_{n=1}^\infty E_n$. If $\lambda^+(E_n) = \infty$ for some n, then $\lambda^+(E) = \infty$. Thus, assume that $\lambda^+(E_n) < \infty$ for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and let $F_n \subset E$, $F_n \in \mathfrak{M}$, be such that

$$\lambda(F_n) \ge \lambda^+(E_n) - \frac{\varepsilon}{2^n}.$$

Then by (66),

$$\begin{aligned}\lambda^+(E) &\ge \lambda\left(\bigcup_{n=1}^\infty F_n\right)\\ &= \sum_{n=1}^\infty \lambda(F_n) \ge \sum_{n=1}^\infty \lambda^+(E_n) - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0^+$ we conclude that

$$\lambda^+(E) \ge \sum_{n=1}^\infty \lambda^+(E_n).$$

Conversely, for any $F \subset E,\ F \in \mathfrak{M}$, let $F_n := F \cap E_n$. Then

$$\lambda(F) = \sum_{n=1}^\infty \lambda(F_n) \le \sum_{n=1}^\infty \lambda^+(E_n),$$

and taking the supremum over all such F we obtain

$$\lambda^+(E) \le \sum_{n=1}^\infty \lambda^+(E_n),$$

and this establishes that λ^+ is a measure.

Since $-\lambda$ is also a signed measure, it follows from the previous part (applied to $-\lambda$) that λ^- is a measure.

Step 2: To verify (68) it suffices to show that for $F \in \mathfrak{M}$ with $\lambda^-(F) > 0$ there exists $G \subset F$, $G \in \mathfrak{M}$, such that $\lambda^-(G) = 0$ and

$$\lambda(G) \ge \lambda(F).$$

If $\lambda(F) \le 0$, then it is enough to take $G = \emptyset$. Thus, assume that $\lambda(F) > 0$. In turn, $\lambda^-(F) < \infty$ (indeed, if $\lambda^-(F) = \infty$, then we would be able to construct an incresing sequence of sets H_k with $-\lambda(H_k) \ge k$ and so the union $H := \bigcup\limits_{k=1}^\infty H_k$ would have $-\lambda(H) = \infty$. Then $\lambda(F) = \lambda(F \setminus H) + \lambda(H) = -\infty$).

We proceed by induction. By (66) and the fact that $\lambda^{-}(F) > 0$ there exists $G_1 \subset F$, $G_1 \in \mathfrak{M}$, such that

$$-\lambda(G_1) \geq 0, \quad -\lambda(G_1) \geq \lambda^{-}(F) - 1. \tag{69}$$

Setting $F_1 := F \setminus G_1$, by $(69)_1$ we obtain

$$\lambda(F_1) = \lambda(F) - \lambda(G_1) \geq \lambda(F),$$

and since by Step 1 λ^{-} is a measure, by definition of λ^{-} we have

$$\begin{aligned}\lambda^{-}(F_1) &= \lambda^{-}(F) - \lambda^{-}(G_1) \\ &\leq \lambda^{-}(F) - (-\lambda(G_1)) \leq 1,\end{aligned}$$

where in the last inequality we have used $(69)_2$. Recursively, suppose now that $F_{n-1} \subset F$ has been selected such that $F_{n-1} \in \mathfrak{M}$, and

$$\lambda(F_{n-1}) \geq \lambda(F), \quad \lambda^{-}(F_{n-1}) \leq \frac{1}{n-1}.$$

If $\lambda^{-}(F_{n-1}) = 0$, then we may take $G := F_{n-1}$. If $\lambda^{-}(F_{n-1}) > 0$, then by (66) there exists $G_n \subset F_{n-1}$, $G_n \in \mathfrak{M}$, such that

$$-\lambda(G_n) \geq 0, \quad -\lambda(G_n) \geq \lambda^{-}(F_{n-1}) - \frac{1}{n}.$$

Setting $F_n := F_{n-1} \setminus G_n$ one can show exactly as before that

$$\infty > \lambda(F_n) \geq \lambda(F), \quad \lambda^{-}(F_n) \leq \frac{1}{n}.$$

Define $G := \bigcap_{n=1}^{\infty} F_n$. Since $\lambda(F) \in (0, \infty)$, by Proposition 47(ii) (which continues to hold for λ, since $\lambda(F_n) \in [0, \infty)$ for all n) we may let $n \to \infty$ in the previous inequalities to obtain

$$\lambda(G) \geq \lambda(F), \quad \lambda^{-}(G) = 0.$$

This concludes the proof of (68).
Step 3: Next we show that λ^{+} is finite. If $\lambda^{+}(X) = \infty$, then we would be able to find an increasing sequence of sets $X_n \in \mathfrak{M}$ such that $\lambda(X_n) \to \infty$. Let

$$X_\infty := \bigcup_{n=1}^{\infty} X_n.$$

Then

$$\lambda(X_\infty) = \lim_n \lambda(X_n) = \infty,$$

which contradicts the fact that $\lambda < \infty$. Hence, $\lambda^{+}(X) < \infty$.

Step 4: Finally, we show that $\lambda = \lambda^+ - \lambda^-$. Let $E \in \mathfrak{M}$. If $|\lambda(E)| < \infty$, then

$$\begin{aligned}\lambda^+(E) - \lambda(E) &= \sup\{\lambda(F) : F \subset E,\ F \in \mathfrak{M}\} - \lambda(E) \\ &= \sup\{\lambda(F) - \lambda(E) : F \subset E,\ F \in \mathfrak{M}\} \\ &= \sup\{-\lambda(E \setminus F) : F \subset E,\ F \in \mathfrak{M}\} \\ &= -\inf\{\lambda(G) : G \subset E,\ F \in \mathfrak{M}\} = \lambda^-(E).\end{aligned}$$

If $\lambda(E) = -\infty$, then

$$\lambda^-(E) = -\inf\{\lambda(F) : F \subset E, F \in \mathfrak{M}\} = -\lambda(E) = \infty,$$

and since $\lambda^+(E) < \infty$, it follows that

$$\infty = \lambda^-(E) = \lambda^+(E) - \lambda(E) = \infty.$$

■

Remark 146 *Note that a set E is positive if and only if $\lambda^-(E) = 0$ and negative if and only if $\lambda^+(E) = 0$.*

Friday, November 11, 2011

We are now ready to prove the Radon–Nikodym theorem.

Proof of the Radon–Nikodym theorem I. The proof is divided into several steps.

Step 1: Assume that μ and ν are finite measures. In view of Lemma 141, let f be a measurable function that realizes $\nu_{ac}(X)$, that is,

$$\nu_{ac}(X) = \int_X f\, d\mu.$$

Since ν is finite, again by Lemma 141, we have that

$$\nu_{ac}(E) = \int_E f\, d\mu$$

for all $E \in \mathfrak{M}$.

By the definition of $\nu_{ac}(X)$ for all $E \in \mathfrak{M}$ we obtain that

$$\nu'(E) := \nu(E) - \int_E f\, d\mu \geq 0. \tag{70}$$

Then ν' is a measure and $\nu' \ll \mu$. We claim that $\nu' \equiv 0$. Indeed, if this is not the case, then there exists $E_0 \in \mathfrak{M}$ with such that

$$\nu'(E_0) = \nu(E_0) - \int_{E_0} f\, d\mu > 0.$$

Since $\nu' \ll \mu$ it follows that $\mu(E_0) > 0$, and so we can find $\varepsilon > 0$ such that $\nu'(E_0) > \varepsilon\mu(E_0)$. Hence $(\nu' - \varepsilon\mu)^+(E_0) > 0$, and by Lemma 145 there exists $E_0' \subset E_0$, $E_0' \in \mathfrak{M}$, such that

$$\nu'(E_0') > \varepsilon\mu(E_0'), \quad (\nu' - \varepsilon\mu)^-(E_0') = 0.$$

Using again the fact that $\nu' \ll \mu$, we have that $\mu(E_0') > 0$. Since

$$(\varepsilon\mu - \nu')^+(E_0') = 0,$$

it follows that for any $E'' \subset E_0'$, with $E'' \in \mathfrak{M}$, we have that

$$\varepsilon\mu(E'') \leq \nu'(E'') = \nu(E'') - \int_{E''} f\, d\mu,$$

i.e.,

$$\int_{E''} \left(f + \varepsilon\chi_{E_0'}\right) d\mu \leq \nu(E''). \tag{71}$$

Therefore, by (70) and (71), for any $E \in \mathfrak{M}$,

$$\begin{aligned}\int_E \left(f + \varepsilon\chi_{E_0'}\right) d\mu &= \int_{E\setminus E_0'} f\, d\mu + \int_{E\cap E_0'} \left(f + \varepsilon\chi_{E_0'}\right) d\mu \\ &\leq \nu(E \setminus E_0') + \nu(E \cap E_0') = \nu(E).\end{aligned}$$

This implies that $f + \varepsilon\chi_{E_0'}$ is an admissible function for $\nu_{ac}(X)$, and thus

$$\begin{aligned}\nu_{ac}(X) &\geq \int_X \left(f + \varepsilon\chi_{E_0'}\right) d\mu = \int_X f\, d\mu + \varepsilon\mu(E_0') \\ &= \nu_{ac}(X) + \varepsilon\mu(E_0').\end{aligned}$$

Since $\nu_{ac}(X) < \infty$ and $\mu(E_0') > 0$, we have reached a contradiction, and the claim that $\nu' \equiv 0$ is proved.

To prove uniqueness, assume that there exists another measurable function $g : X \to [0, \infty]$ such that

$$\nu(E) = \int_E g\, d\mu$$

for every $E \in \mathfrak{M}$. Note that both f and g have finite integrals since ν is finite. Then

$$\int_E g\, d\mu = \int_E f\, d\mu$$

for every $E \in \mathfrak{M}$, and so

$$\int_E (g - f)\, d\mu = 0$$

Taking first $E := \{x \in X : g(x) \geq f(x)\}$ and then $E := \{x \in X : g(x) \leq f(x)\}$, we conclude from Corollary 80 (applied to $(g - f)\chi_E$) that $f(x) = g(x)$ for μ a.e. $x \in X$.

Step 2: Assume next that μ is finite and ν σ-finite. Consider a sequence of disjoint measurable sets X_n such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\nu(X_n) < \infty$. Applying Step 1 to the measures $\mu : \mathfrak{M}\lfloor X_n \to [0,\infty]$ and $\nu : \mathfrak{M}\lfloor X_n \to [0,\infty]$, we can find a unique sequence of measurable functions $f_n : X \to [0,\infty]$ such that for all $E \in \mathfrak{M}$,

$$\nu(E \cap X_n) = \int_{E\cap X_n} f_n \, d\mu.$$

Let

$$f(x) := \sum_{n=1}^{\infty} \chi_{X_n} f_n(x).$$

Then for all $E \in \mathfrak{M}$,

$$\nu(E) = \sum_{n=1}^{\infty} \nu(E \cap X_n) = \sum_{n=1}^{\infty} \int_{E\cap X_n} f_n \, d\mu = \sum_{n=1}^{\infty} \int_{E\cap X_n} f \, d\mu = \int_E f \, d\mu.$$

The uniqueness of f follows from the uniqueness of f_n.

Step 3: Assume that μ is finite and ν arbitrary. Without loss of generality we may assume that $\nu(X) = \infty$. Let

$$T := \sup\{\mu(E) : \ E \in \mathfrak{M}, \ \nu(E) < \infty\}.$$

Note that $T < \infty$ since $\mu(X) < \infty$. Find a sequence of increasing sets E_n, $E_n \in \mathfrak{M}$, with $\nu(E_n) < \infty$, such that

$$\lim_{n\to\infty} \mu(E_n) = T.$$

Define

$$E_\sigma := \bigcup_{n=1}^{\infty} E_n.$$

Then

$$\mu(E_\sigma) = \lim_{n\to\infty} \mu(E_n) = T.$$

Note that $\nu : \mathfrak{M}\lfloor E_\sigma \to [0,\infty]$ is σ-finite by construction, and so by the previous step there exists a unique measurable function $f_\sigma : E_\sigma \to [0,\infty]$ such that

$$\nu(E) = \int_E f_\sigma \, d\mu \tag{72}$$

for every $E \in \mathfrak{M}\lfloor E_\sigma$. We claim that $\nu : \mathfrak{M}\lfloor X \setminus E_\sigma \to [0,\infty]$ takes values only in $\{0,\infty\}$. Indeed, if there exists $F \subset X \setminus E_\sigma$, $F \in \mathfrak{M}$, such that

$$0 < \nu(F) < \infty,$$

then $\mu(F) > 0$ (since $\nu \ll \mu$), and so, since $E_n \cup F$ is admissible in the definition of T,

$$T = \mu(E_\sigma) \geq \mu(E_n \cup F) = \mu(E_n) + \mu(F). \tag{73}$$

Letting $n \to \infty$ we obtain

$$T = \mu(E_\sigma) \geq \mu(E_\sigma) + \mu(F) = T + \mu(F),$$

which is a contradiction. Note also that if $\mu(F) > 0$ for some $F \subset X \setminus E_\sigma$, $F \in \mathfrak{M}$, then $\nu(F) = \infty$. Indeed, if $\nu(F) < \infty$, then again $E_n \cup F$ is admissible in the definition of T, and the same argument as in (73) leads to a contradiction.

Define

$$f(x) := \begin{cases} f_\sigma(x) & \text{if } x \in E_\sigma, \\ \infty & \text{if } x \in X \setminus E_\sigma. \end{cases}$$

We claim that the function f has the desired properties. Let $E \in \mathfrak{M}$, $E \subset X \setminus E_\sigma$. If $\mu(E) > 0$, then, as we just showed, $\nu(E) = \infty$, and so, since $f = \infty$ on E, we get

$$\infty = \nu(E) = \int_E f \, d\mu.$$

If $\mu(E) = 0$, then $\nu(E) = 0$, since $\nu \ll \mu$.

Finally, to prove uniqueness let $g : X \to [0, \infty]$ be another measurable function such that

$$\nu(E) = \int_E g \, d\mu$$

for every $E \in \mathfrak{M}$. By uniqueness in the set E_σ (see Step 2) we have $g(x) = f_\sigma(x)$ for μ a.e. $x \in E_\sigma$. Thus it suffices to show that $g(x) = \infty$ for μ a.e. $x \in X \setminus E_\sigma$.

Assume that there exists a set $F \subset X \setminus E_\sigma$, $F \in \mathfrak{M}$, such that $\mu(F) > 0$ and $g < \infty$ on F. Then, as shown before, $\nu(F) = \infty$. Let

$$F_n := \{x \in F : \, g(x) \leq n\}.$$

Since $F_n \subset F_{n+1}$ and $F = \bigcup_{n=1}^\infty F_n$, we must have

$$\lim_{n \to \infty} \mu(F_n) = \mu(F) > 0,$$

and so $\mu(F_n) > 0$ for all n sufficiently large, say $n \geq n_0$. But then $\nu(F_{n_0}) = \infty$, which is a contradiction since

$$\infty = \nu(F_{n_0}) = \int_{F_{n_0}} g \, d\mu \leq n_0 \mu(F_{n_0}) < \infty.$$

This completes the proof of this step.

Step 4: In the general case in which μ is σ-finite and ν arbitrary, consider a sequence of disjoint measurable sets X_n such that

$$X = \bigcup_{n=1}^\infty X_n$$

and $\mu(X_n) < \infty$. We now proceed exactly as in Step 2, with the only difference that we apply Step 3 in place of Step 1. ■

Exercise 147 *Note that the Radon–Nikodym theorem fails in general without some hypotheses on μ. This is illustrated in the next two exercises.*

(i) Let X be an uncountable set and let $\mathfrak{M}$ be the family of all sets $E \subset X$ such that either E or its complement is countable. For every $E \in \mathfrak{M}$ define

$$\mu(E) := \begin{cases} \operatorname{card} E & \text{if } E \text{ is finite,} \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\nu(E) := \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{otherwise.} \end{cases}$$

Show that $\nu \ll \mu$ but the Radon–Nikodym theorem fails.

(ii) Let $X = [0,1]$, let $\mathfrak{M} := \mathcal{B}([0,1])$, and let μ, ν be respectively the counting measure and the Lebesgue measure $\mathcal{L}^1$. Prove that ν is finite, $\nu \ll \mu$, but the Radon–Nikodym theorem fails.

Monday, November 14, 2011

No class, SIAM San diego

Wednesday, November 16, 2011

No class, SIAM San diego

Friday, November 18, 2011

9 Lebesgue's Decomposition Theorem

Definition 148 *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0,\infty]$ be two measures. μ, ν are said to be* mutually singular*, and we write $\nu \perp \mu$, if there exist two disjoint sets X_μ, $X_\nu \in \mathfrak{M}$ such that $X = X_\mu \cup X_\nu$ and for every $E \in \mathfrak{M}$ we have*

$$\mu(E) = \mu(E \cap X_\mu), \quad \nu(E) = \nu(E \cap X_\nu).$$

Lemma 149 *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0,\infty]$ be two measures. For every $E \in \mathfrak{M}$ define*

$$\nu_s(E) := \sup\{\nu(F) : F \subset E, F \in \mathfrak{M}, \mu(F) = 0\}. \tag{74}$$

Then ν_s is a measure and for each $E \in \mathfrak{M}$ the supremum in the definition of ν_s is actually attained by a measurable set.

Moreover, if ν_s is σ-finite, then $\nu_s \perp \mu$.

Proof. The fact that ν_s is a measure is left as an exercise. To prove that the supremum in the definition of ν_s is attained, we consider two cases. If $\nu_s(E) = 0$, then it suffices to take $F = \emptyset$. If $\nu_s(E) > 0$, consider an increasing sequence $0 < t_n < \nu_s(E)$ such that $t_n \to \nu_s(E)$. By the definition of supremum we can find $F_n \subset E$, $F_n \in \mathfrak{M}$, with $\mu(F_n) = 0$, such that

$$t_n < \nu(F_n) \le \nu_s(E).$$

Let

$$E_n := \bigcup_{k=1}^{n} F_k.$$

Since $E_n \subset E$ and $\mu(E_n) = 0$, the set E_n is admissible in the definition of $\nu_s(E)$, and so

$$\nu_s(E) \geq \nu(E_n) \geq \nu(F_n) > t_n.$$

Letting $n \to \infty$ we conclude that

$$\nu_s(E) = \lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{k=1}^{\infty} E_k\right).$$

Define

$$E_\infty := \bigcup_{k=1}^{\infty} E_k.$$

Since $E_\infty \subset E$ and $\mu(E_\infty) = 0$, the set E_∞ is admissible in the definition of $\nu_s(E)$, and so we have proved that the supremum is reached.

To address the singularity of ν_s with respect to μ, we assume first that ν_s is finite. Choose $X_s \in \mathfrak{M}$ such that

$$\nu_s(X) = \nu(X_s) \tag{75}$$

and $\mu(X_s) = 0$. Given a set $E \in \mathfrak{M}$, let $E_s \in \mathfrak{M}$, $E_s \subset E$, be such that $\nu_s(E) = \nu(E_s)$ and $\mu(E_s) = 0$. We observe that $\nu(E_s \setminus X_s) = 0$, or else, since $E_s \cup X_s$ is admissible for $\nu_s(X)$ and ν_s is finite, we would have

$$\nu_s(X) \geq \nu(E_s \cup X_s) = \nu(X_s) + \nu(E_s \setminus X_s) > \nu(X_s),$$

and this contradicts (75). Therefore using (74) and the fact that $\mu(X_s) = 0$, we have

$$\nu_s(E) = \nu(E_s \cap X_s) \leq \nu(E \cap X_s) \leq \nu_s(E \cap X_s) \leq \nu_s(E). \tag{76}$$

The case that ν_s is σ-finite is straightforward. ∎

Theorem 150 (Lebesgue decomposition theorem) *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0,\infty]$ be two measures, with μ σ-finite. Then*

$$\nu = \nu_{ac} + \nu_s \tag{77}$$

with $\nu_{ac} \ll \mu$. Moreover, if ν is σ-finite, then $\nu_s \perp \mu$ and the decomposition (77) is unique, that is, if

$$\nu = \overline{\nu}_{ac} + \overline{\nu}_s,$$

for some measures $\overline{\nu}_{ac}$, $\overline{\nu}_s$, with $\overline{\nu}_{ac} \ll \mu$ and $\overline{\nu}_s \perp \mu$, then

$$\nu_{ac} = \overline{\nu}_{ac} \quad \textit{and} \quad \nu_s = \overline{\nu}_s.$$

Proof. In view of Lemmas 141 and 149, ν_{ac} and ν_s are measures.

Step 1: We claim that $\nu = \nu_{ac} + \nu_s$. Fix $E \in \mathfrak{M}$ and let $E_s \in \mathfrak{M}$, $E_s \subset E$, be such that $\mu(E_s) = 0$ and $\nu_s(E) = \nu(E_s)$ (see Lemma 149). If $\nu(E_s) = \infty$ then (77) is satisfied when evaluated at E.

Suppose now that $\nu(E_s) < \infty$. We claim that $\nu : \mathfrak{M}\lfloor E \setminus E_s \to [0, \infty]$ is absolutely continuous with respect to $\mu : \mathfrak{M}\lfloor E \setminus E_s \to [0, \infty]$. Indeed, let $F \in \mathfrak{M}$ be such that $F \subset E \setminus E_s$ and $\mu(F) = 0$. If $\nu(F) > 0$, then $E_s \cup F$ is admissible in the definition of $\nu_s(E)$, and so

$$\infty > \nu_s(E) = \nu(E_s) \geq \nu(E_s \cup F) = \nu(E_s) + \nu(F) > \nu(E_s),$$

and we have reached a contradiction. Therefore $\nu(F) = 0$, and the claim is proved. Hence, by the Radon-Nikodym theorem we have that

$$\nu(E \setminus E_s) = \nu_{ac}(E \setminus E_s) = \nu_{ac}(E),$$

where in the last equality we have used the fact that $\mu(E_s) = 0$ and $\nu_{ac} \ll \mu$. Hence

$$\nu(E) = \nu(E \setminus E_s) + \nu(E_s) = \nu_{ac}(E) + \nu_s(E).$$

■

Wednesday, November 10, 2011

Proof. Step 2: Suppose that ν is σ-finite. Then by Lemma 149 we have $\nu_s \perp \mu$.To prove uniqueness of the decomposition, assume that

$$\nu = \nu_{ac} + \nu_s = \overline{\nu}_{ac} + \overline{\nu}_s, \tag{78}$$

with $\overline{\nu}_{ac} \ll \mu$ and $\overline{\nu}_s \perp \mu$. Let $X_{\overline{\nu}_s} \in \mathfrak{M}$ be such that

$$\mu(X_{\overline{\nu}_s}) = 0 \quad \text{and} \quad \overline{\nu}_s(E) = \overline{\nu}_s(E \cap X_{\overline{\nu}_s}) \tag{79}$$

for every $E \in \mathfrak{M}$. Then by (78), for every $E \subset X \setminus X_{\overline{\nu}_s}$, $E \in \mathfrak{M}$, we have that $\overline{\nu}_{ac}(E) = \nu(E)$. This shows that $\nu\lfloor X \setminus X_{\overline{\nu}_s} = \overline{\nu}_{ac}\lfloor X \setminus X_{\overline{\nu}_s}$, and since $\overline{\nu}_{ac}\lfloor X \setminus X_{\overline{\nu}_s}$ is absolutely continuous with respect to $\mu\lfloor X \setminus X_{\overline{\nu}_s}$, so by the Radon-Nikodym theorem and $(79)_1$, for every $E \in \mathfrak{M}$ we have

$$\overline{\nu}_{ac}(E) = \overline{\nu}_{ac}(E \setminus X_{\overline{\nu}_s}) = \nu(E \setminus X_{\overline{\nu}_s}) = \nu_{ac}(E \setminus X_{\overline{\nu}_s}) = \nu_{ac}(E).$$

Hence $\overline{\nu}_{ac} = \nu_{ac}$, and so in the case that ν is finite, it follows from (78) that $\nu_s = \overline{\nu}_s$. If ν is σ-finite, then by restricting ν to X_n, where

$$X = \bigcup_{n=1}^{\infty} X_n, \quad \nu(X_n) < \infty,$$

we conclude that $\nu_s\lfloor X_n = \overline{\nu}_s\lfloor X_n$ for every n, which implies that $\nu_s = \overline{\nu}_s$. ■

Next we extend the previous results to signed measures.

Theorem 151 (Hahn decomposition theorem) *Let* $(X, \mathfrak{M})$ *be a measurable space and let* $\lambda : \mathfrak{M} \to [-\infty, \infty]$ *be a signed measure. Then* λ^+ *and* λ^- *are mutually singular.*

Proof. Without loss of generality we may assume that $\lambda : \mathfrak{M} \to [-\infty, \infty)$ (the case $\lambda : \mathfrak{M} \to (-\infty, \infty]$ being analogous). Let

$$\lambda^+ (X) = \sup \left\{ \lambda (E) : \ E \in \mathfrak{M}, \ \lambda^- (E) = 0 \right\}.$$

Find a sequence of increasing positive sets E_n, $E_n \in \mathfrak{M}$, such that

$$\lim_{n \to \infty} \lambda (E_n) = \lambda^+ (X).$$

Define

$$X^+ := \bigcup_{n=1}^{\infty} E_n.$$

Then X^+ is positive and

$$\lambda \left(X^+ \right) = \lim_{n \to \infty} \lambda (E_n) = \lambda^+ (X).$$

We claim that $X^- := X \setminus X^+$ is negative. Indeed, if not, then there exists $F \subset X^-$, $F \in \mathfrak{M}$, such that

$$0 < \lambda (F) < \infty.$$

But then

$$\lambda^+ (X) = \lambda \left(X^+ \right) < \lambda \left(X^+ \right) + \lambda (F) = \lambda \left(X^+ \cup F \right) < \infty,$$

and since $X^- \cup F$ is admissible in the definition of $\lambda^+ (X)$ we arrive at a contradiction. ■

We now extend the Lebesgue decomposition theorem to signed measures.

Definition 152 *Let $(X, \mathfrak{M})$ be a measurable space, let $\mu : \mathfrak{M} \to [0, \infty]$ a measure and $\lambda : \mathfrak{M} \to [-\infty, \infty]$ be a signed measure.*

(i) λ is said to be absolutely continuous with respect to μ, *and we write $\lambda \ll \mu$, if $\lambda (E) = 0$ whenever $E \in \mathfrak{M}$ and $\mu (E) = 0$.*

(ii) λ and μ are said to be mutually singular, *and we write $\lambda \perp \mu$, if there exist two disjoint sets X_μ, $X_\lambda \in \mathfrak{M}$ such that $X = X_\mu \cup X_\lambda$ and for every $E \in \mathfrak{M}$ we have*

$$\mu (E) = \mu (E \cap X_\mu), \quad \lambda (E) = \lambda (E \cap X_\lambda).$$

Note that if $\lambda \ll \mu$, then $\lambda^+ \ll \mu$ and $\lambda^- \ll \mu$.

If $(X, \mathfrak{M})$ is a measurable space, $\lambda : \mathfrak{M} \to [-\infty, \infty]$ is a signed measure, and $\mu : \mathfrak{M} \to [0, \infty]$ is a σ-finite (positive) measure, then by applying the Lebesgue decomposition theorem to λ^+ and μ (respectively to λ^- and μ) we can write

$$\lambda^+ = \left(\lambda^+ \right)_{ac} + \left(\lambda^+ \right)_s, \quad \lambda^- = \left(\lambda^- \right)_{ac} + \left(\lambda^- \right)_s,$$

where the measures $(\lambda^+)_{ac}$ and $(\lambda^+)_s$ are defined in (61) and (74), and $(\lambda^+)_{ac}$, $(\lambda^-)_{ac} \ll \mu$. Hence we can apply the Radon–Nikodym theorem to find two measurable functions f^+, $f^- : X \to [0, \infty]$ such that

$$\left(\lambda^+\right)_{ac}(E) = \int_E f^+ \, d\mu, \quad \left(\lambda^-\right)_{ac}(E) = \int_E f^- \, d\mu$$

for every $E \in \mathfrak{M}$. The functions f^+ and f^- are unique up to a set of μ measure zero.

Since either λ^+ or λ^- is finite we may define

$$\lambda_{ac} := \left(\lambda^+\right)_{ac} - \left(\lambda^-\right)_{ac}, \quad \lambda_s := \left(\lambda^+\right)_s - \left(\lambda^-\right)_s, \quad f := f^+ - f^-.$$

Then λ_{ac} is a signed measure with $\lambda_{ac} \ll \mu$. Note that if λ is positive then so are λ_{ac} and λ_s.

By the Radon–Nikodym theorem, the Lebesgue decomposition theorem, and the Jordan decomposition theorem we have proved the following result.

Theorem 153 (Lebesgue decomposition theorem) *Let $(X, \mathfrak{M})$ be a measurable space, let $\lambda : \mathfrak{M} \to [-\infty, \infty]$ be a signed measure, and let $\mu : \mathfrak{M} \to [0, \infty]$ be a σ-finite (positive) measure. Then*

$$\lambda = \lambda_{ac} + \lambda_s$$

with $\lambda_{ac} \ll \mu$, and

$$\lambda_{ac}(E) = \int_E f \, d\mu$$

for all $E \in \mathfrak{M}$. Moreover, if λ is σ-finite then $\lambda_s \perp \mu$ and the decomposition is unique, that is, if

$$\lambda = \overline{\lambda}_{ac} + \overline{\lambda}_s,$$

for some signed measures $\overline{\lambda}_{ac}$, $\overline{\lambda}_s$, with $\overline{\lambda}_{ac} \ll \mu$ and $\overline{\lambda}_s \perp \mu$, then

$$\lambda_{ac} = \overline{\lambda}_{ac} \quad \textit{and} \quad \lambda_s = \overline{\lambda}_s.$$

We call λ_{ac} and λ_s, respectively, the *absolutely continuous part* and the *singular part of* λ with respect to μ, and often we write

$$f = \frac{d\lambda_{ac}}{d\mu}.$$

Saturday, November 19, 2011

Make-up class 1:30 hours

10 Change of Variables

Theorem 154 *Let $L : \mathbb{R}^N \to \mathbb{R}^N$ be linear. Then for every Lebesgue measurable set $E \subset \mathbb{R}^N$, $L(E)$ is Lebesgue measurable and*

$$\mathcal{L}^N(L(E)) = |\det L| \, \mathcal{L}^N(E). \tag{80}$$

Proof. If $\det L = 0$ then $L(\mathbb{R}^N)$ is a subspace of $\mathbb{R}^N$ of dimension less than N. In this case $\mathcal{L}^N(L(\mathbb{R}^N)) = 0$ and so by the completeness of $\mathcal{L}^N$ for every Lebesgue measurable set $E \subset \mathbb{R}^N$, $L(E)$ is Lebesgue measurable and

$$0 = \mathcal{L}^N(L(E)) = 0 = |\det L| \, \mathcal{L}^N(E).$$

If $\det L \neq 0$, then L is invertible with continuous inverse. In particular, $L(E)$ is the inverse image of the set E through the continuous function L^{-1} and so if E is Borel, then so is $L(E)$. Thus, we can define the measure

$$\mu(E) := \mathcal{L}^N(L(E)), \quad E \in \mathcal{B}(\mathbb{R}^N).$$

Then μ is translation invariant, since by the linearity of L,

$$\mu(x+E) = \mathcal{L}^N(L(x+E)) = \mathcal{L}^N(L(x)+L(E)) = \mathcal{L}^N(L(E)) = \mu(E)$$

for every $E \in \mathcal{B}(\mathbb{R}^N)$. It follows by your homework that there exists a constant $c \geq 0$ such that

$$\mu(E) = c\mathcal{L}^N(E)$$

for every $E \in \mathcal{B}(\mathbb{R}^N)$.

If $E \subset \mathbb{R}^N$ is just Lebesgue measurable, then we can find two Borel sets $F \subset E \subset G$ such that $\mathcal{L}^N(G \setminus F) = 0$. Then $\mathcal{L}^N(L(G \setminus F)) = 0$ and so, since $E \setminus F \subset G \setminus F$, we have that $L(E \setminus F) \subset L(G \setminus F)$, which, again by completeness, implies that $L(E \setminus F)$ is measurable. In turn, $L(E) = L(F) \cup L(E \setminus F)$ is Lebesgue measurable, since $L(F)$ is a Borel set. Moreover, $\mu(E) = c\mathcal{L}^N(E)$.

Note that if L is a rotation, then $L(B(0,1)) = B(0,1)$ and so

$$\mathcal{L}^N(B(0,1)) = \mathcal{L}^N(L(B(0,1))) = c\mathcal{L}^N(B(0,1)),$$

which implies that $c = 1$. Since $|\det L| = 1$, formula (80) holds in this case.

Any invertible linear transformation can be written as a composition of linear invertible transformations of three basic types:

$$\begin{aligned} T_s(x) &:= (sx_1, x_2, \ldots, x_N), \\ A(x) &:= (x_1, x_1 + x_2, \ldots, x_N), \\ S_{ij}(x) = S_{ij}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) &:= (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N) \end{aligned}$$

for $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and where $s \in \mathbb{R} \setminus \{0\}$. Since the determinant of a composition of two invertible linear transformations is the product of their determinants, it suffices to verify (80) for these three special types of transformations.

Note that $|\det T_s| = |s|$, while $T_s\left([0,1]^N\right) = [0,s]\times[0,1]^{N-1}$ if $s>0$ and $T_s\left([0,1]^N\right) = [s,0]\times[0,1]^{N-1}$ if $s<0$. In both cases

$$|s| = \mathcal{L}^N\left(T_s\left([0,1]^N\right)\right) = c\mathcal{L}^N\left([0,1]^N\right) = c,$$

so that $c = |s| = |\det T_s|$.

We have, $\det A = 1$ and

$$A\left([0,1)^N\right) = \left\{y\in\mathbb{R}^N : y_1 \le y_2 < y_1+1,\, y_i\in[0,1) \text{ for all } i\neq 2\right\}.$$

Let

$$F_1 := \left\{x \in A\left([0,1)^N\right) : x_2 < 1\right\} = \left\{y\in\mathbb{R}^N : y_1 \le y_2 < y_1+1,\, y_i\in[0,1) \text{ for all } i\neq 2,\, y_2<1\right\},$$

$$F_2 := A\left([0,1)^N\right)\setminus F_1 = \left\{y\in\mathbb{R}^N : y_1 \le y_2 < y_1+1,\, y_i\in[0,1) \text{ for all } i\neq 2,\, y_2\ge 1\right\}.$$

Then

$$-e_2 + F_2 = \left\{z\in\mathbb{R}^N : z_1 - 1 \le z_2 < z_1,\, z_i\in[0,1) \text{ for all } i\neq 2,\, z_2\ge 0\right\},$$

and so

$$F_1\cup(-e_2+F_2) = [0,1)^N$$

and $F_1\cap(-e_2+F_2)=\emptyset$. Hence,

$$\begin{aligned} c\mathcal{L}^N\left([0,1)^N\right) &= \mathcal{L}^N\left(T_s\left([0,1]^N\right)\right) = \mathcal{L}^N\left(F_1\cup F_2\right) \\ &= \mathcal{L}^N\left(F_1\right) + \mathcal{L}^N\left(F_2\right) \\ &= \mathcal{L}^N\left(F_1\right) + \mathcal{L}^N\left(-e_2+F_2\right) \\ &= \mathcal{L}^N\left(F_1\cup(-e_2+F_2)\right) = \mathcal{L}^N\left([0,1)^N\right), \end{aligned}$$

so that $c = 1 = \det A$.

Finally, $|\det S_{ij}| = 1$ and $S_{ij}\left([0,1)^N\right) = [0,1)^N$, so that again $c=1$. ■

Exercise 155 *Let $U\subset\mathbb{R}^N$ be an open set and let $\Psi : U\to\mathbb{R}^N$ be differentiable.*

(i) Prove that $E\subset U$ is a set of Lebesgue measure zero, then $\Psi(E)$ is Lebesgue measurable and has measure zero.

(ii) Prove that if $E\subset U$ is Lebesgue measurable, then $\Psi(E)$ is Lebesgue measurable.

(iii) Prove that part (i) and (ii) continue to hold if in place of the differentiability of Ψ, we assume that

$$\lim_{y\to x}\frac{|\Psi(y)-\Psi(x)|}{|y-x|}<\infty$$

for all $x\in U$.

Theorem 156 (Change of variables for multiple integrals, I) *Let $U, V \subset \mathbb{R}^N$ be open sets and let $\Psi : U \to V$ be invertible with Ψ and Ψ^{-1} differentiable. Then the change of variables formula*

$$\int_V f(y)\, dy = \int_U f(\Psi(x)) \left|\det \nabla\Psi(x)\right|\, dx$$

holds for every Lebesgue measurable function $f : V \to [-\infty, \infty]$, which is either Lebesgue integrable or has a sign.

Proof. Step 1: For every Lebesgue measurable set $E \subset \mathbb{R}^N$ define

$$\mu(E) := \mathcal{L}^N(\Psi(E)).$$

Since $\mathcal{L}^N$ is countably additive and Ψ is one-to-one on U, it follows that μ is a Borel measure. Moreover, since Ψ is continuous, it maps compact sets into compact sets, and so μ is finite on compact sets. Moreover, μ is absolutely continuous with respect to the Lebesgue measure. Indeed, if $E \subset \mathbb{R}^N$ is such that $\mathcal{L}^N(E) = 0$, then by the previous exercise the set $\Psi(E)$ has Lebesgue measure zero. Thus, by the Radon–Nikodym theorem (see Theorem 140) there exists a nonnegative locally integrable function $\frac{d\mu}{d\mathcal{L}^N} : \mathbb{R}^N \to [0, \infty]$ such that

$$\mu(E) = \int_E \frac{d\mu}{d\mathcal{L}^N}(x)\, dx$$

for all Lebesgue measurable sets $E \subset \mathbb{R}^N$. In particular, if $E \subset U$ is a Lebesgue measurable set, then

$$\mathcal{L}^N(\Psi(E)) = \mu(E) = \int_E \frac{d\mu}{d\mathcal{L}^N}(x)\, dx. \tag{81}$$

Let now $H \subset V$ be a Lebesgue measurable set. By the previous exercise (with Ψ replaced by Ψ^{-1}), we have that the set $E := \Psi^{-1}(H)$ is Lebesgue measurable, and so by (81),

$$\begin{aligned} \int_{\Psi(\Omega)} \chi_H(y)\, dy &= \mathcal{L}^N(H) = \mathcal{L}^N(\Psi(E)) \qquad (82) \\ &= \int_E \frac{d\mu}{d\mathcal{L}^N}(x)\, dx = \int_\Omega \chi_H(\Psi(x)) \frac{d\mu}{d\mathcal{L}^N}(x)\, dx, \end{aligned}$$

where we have used the fact that $\chi_E(x) = \chi_H(\Psi(x))$ for all $x \in U$.

In view of (82), for every Lebesgue measurable simple function $s : V \to [0, \infty)$ vanishing outside a set of finite measure, we have

$$\int_V s(y)\, dy = \int_U s(\Psi(x)) \frac{d\mu}{d\mathcal{L}^N}(x)\, dx,$$

and Corollary **??**, together with the Lebesgue monotone convergence theorem, allows us to conclude that

$$\int_V f(y)\, dy = \int_U f(\Psi(x)) \frac{d\mu}{d\mathcal{L}^N}(x)\, dx$$

for every Lebesgue measurable function $f : V \to [0, \infty]$. The latter identity implies in particular that if f is integrable over V, then so is $(f \circ \Psi) \dfrac{d\mu}{d\mathcal{L}^N}$ over U. Using this fact, we conclude that

$$\int_V f(y)\, dy = \int_U f(\Psi(x)) \frac{d\mu}{d\mathcal{L}^N}(x)\, dx \tag{83}$$

for every integrable function $f : V \to \mathbb{R}$.

Step 2: Finally, to conclude the proof, it remains to show that

$$\frac{d\mu}{d\mathcal{L}^N}(x) = |\det \nabla\Psi(x)|$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$. By the Besicovitch derivation theorem (see Theorem **??**) there exists a Borel set $M \subset \Omega$, with $\mathcal{L}^N(M) = 0$, such that for every $x_0 \in \Omega \backslash M$,

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{r \to 0^+} \frac{\mu\left(\overline{Q(x_0, r)}\right)}{\mathcal{L}^N\left(\overline{Q(x_0, r)}\right)} = \lim_{r \to 0^+} \frac{\mathcal{L}^N\left(\Psi\left(\overline{Q(x_0, r)}\right)\right)}{\mathcal{L}^N\left(\overline{Q(x_0, r)}\right)} \in \mathbb{R}, \tag{84}$$

where in the second equality we have used (81). We claim that

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^N\left(\Psi\left(\overline{Q(x_0, r)}\right)\right)}{\mathcal{L}^N\left(\overline{Q(x_0, r)}\right)} \le |\det \nabla\Psi(x_0)|$$

for all $x_0 \in U$. Since Ψ is differentiable at x_0, for all $r > 0$ small and for all $x \in \overline{Q(x_0, r)}$,

$$\begin{aligned}
\Psi(x) - \Psi(x_0) &= \nabla\Psi(x_0)(x - x_0) + o(x - x_0) \\
&= \nabla\Psi(x_0)(x - x_0) + \nabla\Psi(x_0)\nabla\Psi^{-1}(\Psi(x_0))(o(x - x_0)) \\
&= \nabla\Psi(x_0)(x - x_0 + o(x - x_0)).
\end{aligned}$$

Hence, given $\varepsilon > 0$, for all $r > 0$ small, we have that

$$-\Psi(x_0) + \Psi\left(\overline{Q(x_0, r)}\right) \subset \nabla\Psi(x_0)\left(\overline{Q(0, r(1 + \varepsilon))}\right).$$

Hence, by the previous theorem,

$$\begin{aligned}
\mathcal{L}^N\left(\Psi\left(\overline{Q(x_0, r)}\right)\right) &= \mathcal{L}^N\left(-\Psi(x_0) + \Psi\left(\overline{Q(x_0, r)}\right)\right) \\
&\le \mathcal{L}^N\left(\nabla\Psi(x_0)\left(\overline{Q(0, r(1 + \varepsilon))}\right)\right) \\
&= |\det \nabla\Psi(x_0)|\, \mathcal{L}^N\left(\overline{Q(0, r(1 + \varepsilon))}\right) \\
&\le |\det \nabla\Psi(x_0)|\,(1 + \varepsilon)^N \mathcal{L}^N\left(\overline{Q(x_0, r)}\right).
\end{aligned} \tag{85}$$

It follows that

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^N\left(\Psi\left(\overline{Q(x_0,r)}\right)\right)}{\mathcal{L}^N\left(\overline{Q(x_0,r)}\right)} \le |\det \nabla\Psi(x_0)|\,(1+\varepsilon)^N$$

and letting $\varepsilon \to 0^+$, we have that

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^N\left(\Psi\left(\overline{Q(x_0,r)}\right)\right)}{\mathcal{L}^N\left(\overline{Q(x_0,r)}\right)} \le |\det \nabla\Psi(x_0)|\,.$$

It follows that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \le |\det \nabla\Psi(x_0)|$$

for every $x_0 \in \Omega \setminus M$, and so by Step 1,

$$\int_V f(y)\,dy = \int_U f(\Psi(x))\,\frac{d\mu}{d\mathcal{L}^N}(x)\,dx \le \int_U f(\Psi(x))\,|\det \nabla\Psi(x)|\,dx.$$

By interchanging U and V and Ψ and Ψ^{-1}, it follows that

$$\int_U g(x)\,dx \le \int_V g\left(\Psi^{-1}(y)\right)\left|\det \nabla\Psi^{-1}(y)\right|\,dy$$

for every integrable function $g : U \to \mathbb{R}$. Taking $g(x) := f(\Psi(x))\,|\det \nabla\Psi(x)|$, we have that

$$\begin{aligned}\int_U f(\Psi(x))\,|\det \nabla\Psi(x)|\,dx &\le \int_V f\left(\Psi\left(\Psi^{-1}(y)\right)\right)\left|\det \nabla\Psi\left(\Psi^{-1}(y)\right)\right|\left|\det \nabla\Psi^{-1}(y)\right|\,dy \\ &= \int_V f(y)\,dy,\end{aligned}$$

which concludes the proof. ■

With a lot more work, one can prove the following more general theorem.

Theorem 157 (Change of variables for multiple integrals, II) *Let $U \subset \mathbb{R}^N$ be an open set and let $\Psi : U \to \mathbb{R}^N$ be continuous. Assume that there exist a Lebesgue measurable set $F \subset \Omega$ on which Ψ is differentiable and one-to-one. If*

(i) $\mathcal{L}^N(U \setminus F) = 0$,

(ii) $\mathcal{L}^N(\Psi(U \setminus F)) = 0$,

then the change of variables formula

$$\int_{\Psi(E)} f(y)\,dy = \int_E f(\Psi(x))\,|\det \nabla\Psi(x)|\,dx$$

holds for every Lebesgue measurable set $E \subset U$ and for every Lebesgue measurable function $f : \Psi(E) \to [-\infty, \infty]$, which is either Lebesgue integrable or has a sign.

Remark 158 *A one-to-one Lipschitz function $\Psi : U \to \mathbb{R}^N$ satisfies all the hypotheses of the previous theorem. Indeed, by a theorem due to Rademacher, a Lipschitz function is differentiable for $\mathcal{L}^N$ a.e. $x \in U$. Moreover, by Exercise 155(iii), it satisfies property (ii).*

Monday, November 21, 2011

Theorem 159 (Besicovitch derivation theorem) *Let $\mu, \nu : \mathcal{B}(\mathbb{R}^N) \to [0, \infty]$ be measures finite on compact sets. Then there exists a Borel set $M \subset \mathbb{R}^N$, with $\mu(M) = 0$, such that for any $x \in \mathbb{R}^N \setminus M$,*

$$\frac{d\nu_{ac}}{d\mu}(x) = \lim_{r \to 0^+} \frac{\nu\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} \in \mathbb{R} \tag{86}$$

and

$$\lim_{r \to 0^+} \frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} = 0, \tag{87}$$

where

$$\nu = \nu_{ac} + \nu_s, \quad \nu_{ac} \ll \mu, \quad \nu_s \perp \mu. \tag{88}$$

Proof. In view of Lebesgue's differentiation theorem it remains to show (87). Since $\nu_s \perp \mu$ there exists $E_\mu \in \mathcal{B}(\mathbb{R}^N)$ such that $\mu(\mathbb{R}^N \setminus E_\mu) = \nu_s(E_\mu) = 0$. Since $\nu_s(E_\mu) = 0$, by Theorem 131, given $\varepsilon > 0$, we may find an open set U_ε containing E_μ such that $\nu_s(U_\varepsilon) < \varepsilon$. For $t > 0$, let

$$E_t := \left\{ x \in E_\mu \setminus M_1 : \limsup_{r \to 0^+} \frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} > t \right\}.$$

Then for every $x \in E_t$ we can find a ball $\overline{B(x, r_x)} \subset U_\varepsilon$, with $0 < r_x < 1$, such that

$$\frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} > t. \tag{89}$$

Consider the family $\mathcal{F} := \left\{\overline{B(x, r_x)}\right\}_{x \in E_t}$. By the Besicovitch covering theorem, there exist $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \subset \mathcal{F}$ such that each $\mathcal{F}_n$, $n = 1, \ldots, \ell$, is a countable family of disjoint balls in $\mathcal{F}$ and

$$E_t \subset \bigcup_{n=1}^{\ell} \bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B},$$

where E is the set of centers of balls in $\mathcal{F}$.

Hence, by (89),

$$\mu(E_t) \le \sum_{n=1}^{\ell} \sum_{\overline{B} \in \mathcal{F}_n} \mu(\overline{B}) \le \sum_{n=1}^{\ell} \sum_{\overline{B} \in \mathcal{F}_n} \frac{1}{t} \nu_s(\overline{B}) = \sum_{n=1}^{\ell} \frac{1}{t} \nu_s\left(\bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B}\right) \le \frac{\ell}{t} \nu_s(U_\varepsilon) \le \frac{\ell}{t} \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain that $\mu(E_t) = 0$ for all $t > 0$. Let

$$E := \bigcup_{n=1}^{\infty} E_{\frac{1}{n}}.$$

Then $\mu(E) = 0$ and if $x \in \mathbb{R}^N \setminus E$, then

$$\limsup_{r \to 0^+} \frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} \le \frac{1}{n}$$

for every n, that is,

$$\limsup_{r \to 0^+} \frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} = 0,$$

which implies (87). ■

Remark 160 *If $x \in \mathbb{R}^N \setminus E$, then for any family of Borel subsets $\{E_{x,r}\}_{r>0}$ such that $E_{x,r} \subset \overline{B(x,r)}$ and*

$$\mu(E_{x,r}) > \alpha \mu\left(\overline{B(x,r)}\right)$$

for some constant $\alpha > 0$ independent of $r > 0$, we have that

$$\begin{aligned} \limsup_{r \to 0^+} \frac{\nu_s(E_{x,r})}{\mu(E_{x,r})} &\le \limsup_{r \to 0^+} \frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu(E_{x,r})} \\ &\le \frac{1}{\alpha} \lim_{r \to 0^+} \frac{\nu_s\left(\overline{B(x,r)}\right)}{\mu\left(\overline{B(x,r)}\right)} = 0. \end{aligned}$$

Hence, also by Remark 135,

$$\frac{d\nu_{ac}}{d\mu}(x) = \lim_{r \to 0^+} \frac{\nu(E_{x,r})}{\mu(E_{x,r})} \in \mathbb{R}.$$

Corollary 161 (Lebesgue differentiation theorem) *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be monotone. Then f is differentiable for $\mathcal{L}^1$ a.e. $x \in I$.*

First proof. Without loss of generality, we can assume that f is increasing.
Step 1: Assume that f is right-continuous. Let $\mu_f : \mathcal{B}(I) \to [0,\infty]$ be the the Lebesgue-Stieltjes measure generated by f. By the previous theorem there exists a Borel set $M \subset I$, with $\mathcal{L}^1(M) = 0$, such that for any $x \in I \setminus M$, there exists

$$\frac{d(\mu_f)_{ac}}{d\mathcal{L}^1}(x) = \lim_{r\to 0^+} \frac{\mu_f\left(\overline{B(x,r)}\right)}{\mathcal{L}^1\left(\overline{B(x,r)}\right)} \in \mathbb{R}.$$

For every $x \in I^\circ$ consider the family of sets $E_{x,r} := (x, x+r]$ and $F_{x,r} := (x-r, x]$. Note that $\mathcal{L}^1(E_{x,r}) = \mathcal{L}^1(F_{x,r}) = \frac{1}{2}\mathcal{L}^1([x-r, x+r])$. Hence, by the previous remark,

$$\frac{d(\mu_f)_{ac}}{d\mathcal{L}^1}(x) = \lim_{r\to 0^+} \frac{\mu_f((x,x+r])}{\mathcal{L}^1((x,x+r])} = \lim_{r\to 0^+} \frac{\mu_f((x-r,x])}{\mathcal{L}^1((x-r,x])}.$$

But by Theorem **??**, we have that

$$\begin{aligned} f(x+r) - f(x) &= \mu_f((x, x+r]), \\ f(x) - f(x-r) &= \mu_f((x-r, x]), \end{aligned}$$

and so

$$\lim_{r\to 0^+} \frac{f(x+r) - f(x)}{r} = \lim_{r\to 0^+} \frac{f(x) - f(x-r)}{r} = \frac{d(\mu_f)_{ac}}{d\mathcal{L}^1}(x) \in \mathbb{R},$$

which shows that f is differentiable for $\mathcal{L}^1$ a.e. $x \in I$.
Step 2: Consider the function $g : I \to \mathbb{R}$ defined by $g(x) := f^+(x)$. Then g is increasing and right-continuous. Moreover, $g(x) = f(x)$ for all but countably many points (the discontinuity points of f). Moreover, $f^-(x) = g^-(x)$ for all $x \in I$. By the first step, g is differentiable for $\mathcal{L}^1$ a.e. $x \in I$.

Let $h := g - f \geq 0$. Let $\{x_n\}$ be the family of discontinuity points of f. Then $h(x) = 0$ for all $x \neq x_n$. Moreover, for every $[a,b] \subset I$ we have that

$$\sum_{x_n\in[a,b]} h(x_n) = \sum_{x_n\in[a,b]} \left(f^+(x_n) - f(x_n)\right) \leq f(b) - f(a).$$

Thus, the measure

$$\nu := \sum_n h(x_n)\,\delta_{x_n}$$

is finite on compact sets. Since $\nu \perp \mathcal{L}^1$, by the previous theorem we have that there exists a Borel set $M_1 \subset I$, with $\mathcal{L}^1(M_1) = 0$, such that for any $x \in I \setminus M$,

$$\lim_{r\to 0^+} \frac{\nu\left(\overline{B(x,r)}\right)}{\mathcal{L}^1\left(\overline{B(x,r)}\right)} = 0.$$

But

$$\left|\frac{h(x+t)-h(t)}{t}\right| \leq \frac{|h(x+t)|+|h(t)|}{|t|} \leq 4\frac{\nu\left(\overline{B(x,2|t|)}\right)}{\mathcal{L}^1\left(\overline{B(x,2|t|)}\right)} \to 0$$

as $t \to 0$. Hence, $h'(x) = 0$ for $\mathcal{L}^1$ a.e. $x \in I$. Together with the previous part, this implies that there exists in $\mathbb{R}$, $f'(x) = g'(x) - h'(x)$ for $\mathcal{L}^1$ a.e. $x \in I$. ■

Wednesday, November 23, 2011

Thanksgiving, no classes

Friday, November 25, 2011

Thanksgiving, no classes

Monday, November 28, 2011

11 Dynkin Classes

Definition 162 *Let X be a nonempty set. A collection $\mathfrak{D} \subset \mathcal{P}(X)$ is called a* d-system *or a* Dynkin class *on X if*

(i) $X \in \mathfrak{D}$;

(ii) if E, $F \in \mathfrak{D}$ and $E \subset F$, then $F \setminus E \in \mathfrak{D}$;

(iii) if $\{E_n\} \subset \mathfrak{D}$ is an increasing sequence, then $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{D}$.

Note that the intersection of an arbitrary family of Dynkin classes is still a Dynkin class. Hence, given an arbitrary subset $\mathcal{F} \subset \mathcal{P}(X)$, the smallest (in the sense of inclusion) Dynkin class $\mathfrak{D}$ that contains $\mathcal{F}$ is given by the intersection of all Dynkin classes on X that contain $\mathcal{F}$. We say that $\mathfrak{D}$ is the Dynkin class generated by $\mathcal{F}$.

Theorem 163 *Let X be a nonempty set and let $\mathcal{F} \subset \mathcal{P}(X)$ be a family closed under the formation of finite intersections. Then the σ-algebra $\mathfrak{M}$ generated by $\mathcal{F}$ coincides with the Dynkin class $\mathfrak{D}$ generated by $\mathcal{F}$.*

Proof. Since a σ-algebra satisfies properties (i)–(iii) of the previous definition, we have that $\mathfrak{M}$ is a Dynkin class containing $\mathcal{F}$ and so $\mathfrak{D} \subset \mathfrak{M}$. It remains to prove the opposite inclusion.

Step 1: We prove that $\mathfrak{D}$ is closed under finite intersections. To see this, define the family

$$\mathfrak{D}_1 := \{E \in \mathfrak{D} : E \cap F \in \mathfrak{D} \text{ for all } F \in \mathcal{F}\}.$$

We claim that $\mathfrak{D}_1$ is a Dynkin class.

(i) Note that $X \in \mathfrak{D}$ and if $F \in \mathcal{F}$, then $X \cap F = F \in \mathcal{F} \subset \mathfrak{D}$, which implies that $X \in \mathfrak{D}_1$.

(ii) Let $E_1,\ E_2 \in \mathfrak{D}_1$ with $E_1 \subset E_2$. Then for all $F \in \mathcal{F}$, we have that $E_2 \cap F$ and $E_1 \cap F$ belong to $\mathfrak{D}$ and $E_1 \cap F \subset E_2 \cap F$. It follows from (i) that $(E_2 \cap F) \setminus (E_1 \cap F) \in \mathfrak{D}$. Thus,

$$(E_2 \setminus E_1) \cap F = (E_2 \cap F) \setminus (E_1 \cap F) \in \mathfrak{D},$$

which implies that $E_2 \setminus E_1 \in \mathfrak{D}_1$.

(iii) Let $\{E_n\} \subset \mathfrak{D}_1$ be an increasing sequence. Then for all $F \in \mathcal{F}$, we have that $E_n \cap F$ and $E_{n+1} \cap F$ belong to $\mathfrak{D}$ and $E_n \cap F \subset E_{n+1} \cap F$. It follows from (iii) that $\bigcup_{n=1}^{\infty} (E_n \cap F) \in \mathfrak{D}$, but

$$\left(\bigcup_{n=1}^{\infty} E_n \right) \cap F = \bigcup_{n=1}^{\infty} (E_n \cap F) \in \mathfrak{D},$$

which implies that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{D}_1$.

Thus we have proved the claim. Next observe that since $\mathcal{F}$ is closed under the formation of finite intersections and $\mathcal{F} \subset \mathfrak{D}$, if $E \in \mathcal{F}$, then $E \cap F \in \mathcal{F} \subset \mathfrak{D}$ for all $F \in \mathcal{F}$, whihc shows that $E \in \mathfrak{D}_1$, or, equivalently that $\mathcal{F}$ is contained in $\mathfrak{D}_1$. Since $\mathfrak{D}$ was the smallest Dynkin class containing $\mathcal{F}$, we have that $\mathfrak{D}_1 \supset \mathfrak{D}$. On othe other hand, $\mathfrak{D}_1 \subset \mathfrak{D}$, and so $\mathfrak{D}_1 = \mathfrak{D}$.

Next define

$$\mathfrak{D}_2 := \{E \in \mathfrak{D} : \ E \cap F \in \mathfrak{D} \text{ for all } F \in \mathfrak{D}\}.$$

Reasoning exactly as above, we can prove that $\mathfrak{D}_2$ is a Dynkin class. Moreover, since $\mathfrak{D}_1 = \mathfrak{D}$, if $E \in \mathcal{F}$ we have that $E \cap F \in \mathfrak{D}$ for all $F \in \mathfrak{D}$, and so $E \in \mathfrak{D}_2$. Thus, $\mathfrak{D}_2$ is a Dynkin class containing $\mathcal{F}$. As before we conclude that $\mathfrak{D}_2 = \mathfrak{D}$. This, together with an induction argument, proves that $\mathfrak{D}$ is closed under finite intersections.

Step 2: We prove that $\mathfrak{D}$ is a σ-algebra. Since $X \in \mathfrak{D}$ it follows by property (ii) that $\emptyset \in \mathfrak{D}$. Again by (ii), if $E \in \mathfrak{D}$, then since $E \subset X \in \mathfrak{D}$, we have that $X \setminus E \in \mathfrak{D}$.

Hence, if $E_1,\ E_2 \in \mathfrak{D}$ then $X \setminus E_1$ and $X \setminus E_2$ belong to $\mathfrak{D}$, but then again by Step 1 and (ii),

$$\begin{aligned} E_1 \cup E_2 &= (X \setminus (X \setminus E_1)) \cup (X \setminus (X \setminus E_2)) \\ &= X \setminus ((X \setminus E_1) \cap (X \setminus E_2)) \in \mathfrak{D}. \end{aligned}$$

This proves that $\mathfrak{D}$ is an algebra. Finally, if $\{E_n\} \subset \mathfrak{D}$ then by what we just proved and induction,

$$F_n := \bigcup_{k=1}^{n} E_k \in \mathfrak{D}.$$

Since $F_n \subset F_{n+1}$, it follows by (iii) that $\bigcup_{n=1}^{\infty} F_n = \bigcup_{k=1}^{\infty} E_k \in \mathfrak{D}$. This shows that $\mathfrak{D}$ is a σ-algebra containg $\mathcal{F}$. In turn, $\mathfrak{D} \supset \mathfrak{M}$ and the proof is complete. ■

We now turn to the proof of Theorem 131.

Exercise 164 (Proof of Theorem 131) *Let* $\left(\mathbb{R}^N, \mathcal{B}(X), \mu\right)$ *be a measure space, with* μ *finite on compact sets.*

(i) Assume first that μ *is finite and let* $\mathfrak{D}$ *be the family of sets* $E \in \mathcal{B}(X)$ *for which (53) holds. Prove that* $\mathfrak{D}$ *is a Dynkin class containing all the open sets and conclude that* $\mathfrak{D} = \mathcal{B}(X)$.

(ii) Prove that

$$\mu(E) = \sup\{\mu(K) : K \text{ compact, } K \subset E\} = \inf\{\mu(A) : A \text{ open, } A \supset E\}$$

for every $E \in \mathcal{B}(X)$.

12 Product Spaces

We recall that, given two measurable spaces $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ we denote by $\mathfrak{M} \otimes \mathfrak{N} \subset \mathcal{P}(X \times Y)$ the smallest σ-algebra that contains all sets of the form $E \times F$, where $E \in \mathfrak{M}$, $F \in \mathfrak{N}$. Then $\mathfrak{M} \otimes \mathfrak{N}$ is called the *product σ-algebra* of $\mathfrak{M}$ and $\mathfrak{N}$.

Exercise 165 *Let* X *and* Y *be topological spaces and let* $\mathcal{B}(X)$ *and* $\mathcal{B}(Y)$ *be their respective Borel σ-algebras. Prove that*

$$\mathcal{B}(X) \otimes \mathcal{B}(Y) \subset \mathcal{B}(X \times Y).$$

Show also that if X *and* Y *are separable metric spaces, then*

$$\mathcal{B}(X) \otimes \mathcal{B}(Y) = \mathcal{B}(X \times Y),$$

so that in particular, $\mathcal{B}\left(\mathbb{R}^N\right) = \mathcal{B}(\mathbb{R}) \otimes \ldots \otimes \mathcal{B}(\mathbb{R})$.

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. In $X \times Y$ we consider the class of elementary sets

$$\mathcal{G} := \{F \times G : F \in \mathfrak{M}, G \in \mathfrak{N}\}$$

and we define $\rho : \mathcal{G} \to [0, \infty]$ by

$$\rho(F \times G) := \mu(F)\,\nu(G).$$

For every $E \in X \times Y$ define

$$(\mu \times \nu)^*(E) := \inf\left\{\sum_{n=1}^{\infty} \mu(F_n)\,\nu(G_n) : \{F_n\} \subset \mathfrak{M}, \{G_n\} \subset \mathfrak{N}, \right. \tag{90}$$
$$\left. E \subset \bigcup_{n=1}^{\infty}(F_n \times G_n)\right\}.$$

By Proposition 3, $(\mu \times \nu)^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure, and it is called the *product outer measure* of μ and ν. By Carathéodory's theorem, the restriction of $(\mu \times \nu)^*$ to the σ-algebra $\mathfrak{M} \times \mathfrak{N}$ of $(\mu \times \nu)^*$-measurable sets is a complete measure, denoted by $\mu \times \nu$ and called the *product measure* of μ and ν.

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Note that $\mathfrak{M} \times \mathfrak{N}$ is, in general, larger than the product σ-algebra $\mathfrak{M} \otimes \mathfrak{N}$. Indeed, we have the following result.

Theorem 166 *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. If $F \in \mathfrak{M}$ and $G \in \mathfrak{N}$, then $F \times G$ is $(\mu \times \nu)^*$-measurable and*

$$(\mu \times \nu)(F \times G) = \mu(F)\,\nu(G). \tag{91}$$

In particular, $\mathfrak{M} \otimes \mathfrak{N} \subset \mathfrak{M} \times \mathfrak{N}$.

Proof. We begin by observing that if F_1, $F_2 \in \mathfrak{M}$ and $G_1, G_2 \in \mathfrak{N}$, then

$$(F_1 \times G_1) \cap (F_2 \times G_2) = (F_1 \cap F_2) \times (G_1 \cap G_2), \tag{92}$$

$$(F_1 \times G_1) \setminus (F_2 \times G_2) = ((F_1 \setminus F_2) \times G_1) \cup ((F_1 \cap F_2) \times (G_1 \setminus G_2)). \tag{93}$$

Step 1: We claim that if $F \in \mathfrak{M}$ and $G \in \mathfrak{N}$, then $F \times G$ is $(\mu \times \nu)^*$-measurable. Fix $E \subset X \times Y$ and let $\{F_n\} \subset \mathfrak{M}$, $\{G_n\} \subset \mathfrak{N}$ be such that $E \subset \bigcup_{n=1}^{\infty} (F_n \times G_n)$. Then by (92) and (93),

$$E \setminus (F \times G) \subset \bigcup_{n=1}^{\infty} ((F_n \setminus F) \times G_n) \cup ((F_n \cap F) \times (G_n \setminus G)),$$

$$E \cap (F \times G) \subset \bigcup_{n=1}^{\infty} (F_n \cap F) \times (G_n \cap G)$$

and so

$$\begin{aligned}(\mu \times \nu)^* (E \setminus (F \times G)) + (\mu \times \nu)^* (E \cap (F \times G)) &\le \sum_{n=1}^{\infty} \mu(F_n \setminus F)\,\nu(G_n) \\ &\quad + \sum_{n=1}^{\infty} \mu(F_n \cap F)\,\nu(G_n \setminus G) + \sum_{n=1}^{\infty} \mu(F_n \cap F)\,\nu(G_n \cap G) \\ &= \sum_{n=1}^{\infty} \mu(F_n \setminus F)\,\nu(G_n) + \sum_{n=1}^{\infty} \mu(F_n \cap F)\,(\nu(G_n \setminus G) + \nu(G_n \cap G)) \\ &= \sum_{n=1}^{\infty} \mu(F_n \setminus F)\,\nu(G_n) + \sum_{n=1}^{\infty} \mu(F_n \cap F)\,\nu(G_n) = \sum_{n=1}^{\infty} \mu(F_n)\,\nu(G_n).\end{aligned}$$

Taking the supremum over all such $\{F_n\} \subset \mathfrak{M}$, $\{G_n\} \subset \mathfrak{N}$, we get

$$(\mu \times \nu)^* (E \setminus (F \times G)) + (\mu \times \nu)^* (E \cap (F \times G)) \le (\mu \times \nu)^* (E),$$

which proves the claim.
Step 2: In view of your homework, to prove (91), it remains to show that ρ is subadditive, that is, that

$$\mu(F)\,\nu(G) \leq \sum_{n=1}^{\infty} \mu(F_n)\,\nu(G_n)$$

whenever F, $F_n \in \mathfrak{M}$, G, $G_n \in \mathfrak{N}$ and $F \times G \subset \bigcup_{n=1}^{\infty} (F_n \times G_n)$. Note that

$$F \times G = \bigcup_n ((F_n \cap F) \times (G_n \cap G)), \tag{94}$$

and so

$$\chi_F(x)\,\chi_G(y) = \chi_{F\times G}(x,y) \leq \sum_n \chi_{(F_n\cap F)\times(G_n\cap G)}(x,y) = \sum_n \chi_{F_n\cap F}(x)\,\chi_{G_n\cap G}(y). \tag{95}$$

Since for each fixed $y \in Y$, the function

$$\begin{aligned} X &\to [0,\infty] \\ x &\mapsto \chi_{F_n\cap F}(x)\,\chi_{G_n\cap G}(y) \end{aligned}$$

is measurable, by Corollary 76 the function

$$\begin{aligned} X &\to [0,\infty] \\ x &\mapsto \sum_n \chi_{F_n\cap F}(x)\,\chi_{G_n\cap G}(y) \end{aligned}$$

is measurable and so, integrating (95) with respect to x, we obtain

$$\begin{aligned} \mu(F)\,\chi_G(y) &= \int_X \chi_G(y)\,\chi_F(x)\,d\mu(x) \leq \sum_n \int_X \chi_{G_n\cap G}(y)\,\chi_{F_n\cap F}(x)\,d\mu(x) \\ &= \sum_n \mu(F_n\cap F)\,\chi_{G_n\cap G}(y). \end{aligned}$$

Using Corollary 76 once more, we get that the function

$$\begin{aligned} Y &\to [0,\infty] \\ y &\mapsto \sum_n \mu(F_n\cap F)\,\chi_{G_n\cap G}(y) \end{aligned}$$

is measurable and so integrationg the previous inequality in y, we obtain

$$\begin{aligned} \mu(F)\,\nu(G) = \mu(F)\int_Y \chi_G(y)\,d\nu(y) &\leq \sum_n \mu(F_n\cap F)\int_Y \chi_{G_n\cap G}(y)\,d\nu(y) \\ &= \sum_n \mu(F_n\cap F)\,\nu(G_n\cap G) \leq \sum_n \mu(F_n)\,\nu(G_n). \end{aligned}$$

■

Corollary 167 *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. If $E \subset X \times Y$, then there exists a set $R \in \mathfrak{M} \otimes \mathfrak{N}$ containing E such that*

$$(\mu \times \nu)^* (E) = (\mu \times \nu) (R) .$$

Proof. If $(\mu \times \nu)^* (E) = \infty$ it suffices to take $R := X \times Y$. If $(\mu \times \nu)^* (E) < \infty$, then by (90), for every $k \in \mathbb{N}$ we may find $\{F_{n,k}\} \subset \mathfrak{M}$, $\{G_{n,k}\} \subset \mathfrak{N}$ be such that $E \subset \bigcup\limits_{n=1}^{\infty} (F_{n,k} \times G_{n,k})$ and

$$\sum_n \mu (F_{n,k}) \nu (G_{n,k}) \leq (\mu \times \nu)^* (E) + \frac{1}{k}. \tag{96}$$

Set

$$R_k := \bigcup_{n=1}^{\infty} (F_{n,k} \times G_{n,k}) .$$

Then $R_k \in \mathfrak{M} \otimes \mathfrak{N}$, $R_k \supset E$ and

$$(\mu \times \nu)^* (E) \leq (\mu \times \nu)^* (R_k) = (\mu \times \nu) (R_k) \leq \sum_n \mu (F_{n,k}) \nu (G_{n,k}) \leq (\mu \times \nu)^* (E) + \frac{1}{k}.$$

By replacing, R_k with $R_k \cap R_{k+1}$, without loss of generality, we may assume that $\{R_k\}$ is decreasing. Define

$$R := \bigcap_{k=1}^{\infty} R_k \in \mathfrak{M} \otimes \mathfrak{N}.$$

Then by Proposition 47, letting $k \to \infty$ in the previous inequality gives

$$(\mu \times \nu)^* (E) = \lim_{k \to \infty} (\mu \times \nu) (R_k) = (\mu \times \nu) (R) .$$

■

The next theorem gives a formula for integrating sets that are not necessarily rectangles.

Theorem 168 *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. Assume that μ and ν are complete and $E \in \mathfrak{M} \times \mathfrak{N}$ has σ-finite $\mu \times \nu$ measure. Then for μ a.e. $x \in X$ the* section

$$E_x := \{y \in Y : (x, y) \in E\}$$

belongs to the σ-algebra $\mathfrak{N}$ and for ν a.e. $y \in Y$ the section

$$E_y := \{x \in X : (x, y) \in E\}$$

belongs to the σ-algebra $\mathfrak{M}$. Moreover, the functions $y \mapsto \mu (E_y)$ and $x \mapsto \nu (E_x)$ are measurable and

$$(\mu \times \nu) (E) = \int_Y \mu (E_y) \, d\nu (y) = \int_X \nu (E_x) \, d\mu (x) .$$

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Proof. Step 1: Assume that $\mu \times \nu$ is finite. In this step we do not use the fact that μ and ν are complete. Let $\mathfrak{D}$ be the family of all sets $D \in \mathfrak{M} \times \mathfrak{N}$ such that for each fixed $y \in Y$, the function

$$\begin{aligned} (X, \mathfrak{M}) &\to [0, \infty] \\ x &\mapsto \chi_D(x, y) \end{aligned}$$

is measurable, the mapping

$$\begin{aligned} (Y, \mathfrak{N}) &\to [0, \infty] \\ y &\mapsto \int_X \chi_D(x, y) \, d\mu(x) \end{aligned}$$

is measurable, and

$$(\mu \times \nu)(D) = \int_Y \mu(D_y) \, d\nu(y).$$

By the previous theorem, $\mathfrak{D}$ contains $\mathcal{G}$. Note that by (92), $\mathcal{G}$ is closed under finite intersections. We claim that $\mathfrak{D}$ is a Dynkin class. Indeed, $X \times Y \in \mathcal{G} \subset \mathfrak{D}$. If E_1, $E_2 \in \mathfrak{D}$ with $E_1 \subset E_2$, then since a difference of two measurable functions is measurable and

$$\begin{aligned} \infty > (\mu \times \nu)(E_2) &= \int_Y \mu\left((E_2)_y\right) \, d\nu(y), \\ \infty > (\mu \times \nu)(E_1) &= \int_Y \mu\left((E_1)_y\right) \, d\nu(y), \end{aligned}$$

we have that

$$\begin{aligned} (\mu \times \nu)(E_2 \setminus E_1) &= (\mu \times \nu)(E_2) - (\mu \times \nu)(E_1) \\ &= \int_Y \mu\left((E_2)_y\right) \, d\nu(y) - \int_Y \mu\left((E_1)_y\right) \, d\nu(y) \\ &= \int_Y \left(\mu\left((E_2)_y\right) - \mu\left((E_1)_y\right)\right) \, d\nu(y) \\ &= \int_Y \mu\left((E_2 \setminus E_1)_y\right) \, d\nu(y). \end{aligned}$$

This shows that $E_2 \setminus E_1 \in \mathfrak{D}$.

Finally, if $\{E_n\} \subset \mathfrak{D}$ is an increasing sequence, the corresponding characteristic functions give an increasing sequence. Since the limit of measurable function is measurable and

$$(\mu \times \nu)(E_n) = \int_Y \mu\left((E_n)_y\right) \, d\nu(y)$$

using Proposition 47 and Lebesgue's monotone convergence theorem on the right, we have that

$$(\mu \times \nu)\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} (\mu \times \nu)(E_n) = \lim_{n\to\infty} \int_Y \mu\left((E_n)_y\right) d\nu(y) = \int_Y \lim_{n\to\infty} \mu\left((E_n)_y\right) d\nu(y)$$
$$= \int_Y \mu\left(\bigcup_{n=1}^{\infty} (E_n)_y\right) d\nu(y) = \int_Y \mu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)_y\right) d\nu(y),$$

which shows that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{D}$. This shows that $\mathfrak{D}$ is a Dynkin class and so by Theorem 163, we have that $\mathfrak{D}$ contains $\mathfrak{M} \otimes \mathfrak{N}$.

Step 2: Assume that μ and ν are complete and that $\mu \times \nu$ is finite. By the previous corollary there exists a set $R \in \mathfrak{M} \otimes \mathfrak{N}$ containing E such that

$$(\mu \times \nu)(E) = (\mu \times \nu)(R).$$

Since $(\mu \times \nu)(E) < \infty$, it follows that $(\mu \times \nu)(R \setminus E) = 0$, and so, using the previous corollary again, we may find a set $S \in \mathfrak{M} \otimes \mathfrak{N}$ containing $R \setminus E$ such that

$$\begin{aligned} 0 = (\mu \times \nu)(R \setminus E) &= (\mu \times \nu)(S) \\ &= \int_Y \left(\int_X \chi_S(x, y)\, d\mu(x)\right) d\nu(y). \end{aligned}$$

By Corollary 80, it follows that for ν a.e. $y \in Y$

$$\mu(S_y) = \int_X \chi_S(x, y)\, d\mu(x) = 0.$$

Since S contains $R \setminus E$ and the measure μ is complete, we have that for ν a.e. $y \in Y$ the set $R_y \setminus E_y$ belongs to $\mathfrak{M}$ and $\mu(R_y \setminus E_y) = 0$, or, equivalently,

$$\int_X \chi_R(x, y)\, d\mu(x) = \int_X \chi_E(x, y)\, d\mu(x).$$

In turn, the function

$$\begin{aligned} Y &\to [0, \infty] \\ y &\mapsto \int_X \chi_E(x, y)\, d\mu(x) \end{aligned}$$

is measurable. Hence upon integration

$$\begin{aligned} (\mu \times \nu)(E) &= \int_Y \left(\int_X \chi_R(x, y)\, d\mu(x)\right) d\nu(y) \\ &= \int_Y \left(\int_X \chi_E(x, y)\, d\mu(x)\right) d\nu(y). \end{aligned}$$

Step 3: Finally, assume that μ and ν are complete and that $E \in \mathfrak{M} \times \mathfrak{N}$ has σ-finite $\mu \times \nu$ measure. Without loss of generality, we may assume that $(\mu \times \nu)(E) < \infty$. By (90), we may find $\{F_n\} \subset \mathfrak{M}$, $\{G_n\} \subset \mathfrak{N}$ be such that $E \subset \bigcup_{n=1}^{\infty} (F_n \times G_n)$ and

$$\sum_n \mu(F_n)\, \nu(G_n) \le (\mu \times \nu)^*(E) + 1.$$

Using (93), without loss of generality, we may assume that the sets $F_n \times G_n$ are pairwise disjoint. Considering the σ-algebra

$$(\mathfrak{M} \times \mathfrak{N})_n := \{D \cap (F_n \times G_n) :\ D \in \mathfrak{M} \times \mathfrak{N}\},$$

we may apply Steps 1 and 2 to the finite measure $\mu \times \nu : (\mathfrak{M} \times \mathfrak{N})_n \to [0, \infty)$. In particular, we get

$$(\mu \times \nu)(E \cap (F_n \times G_n)) = \int_Y \left(\int_X \chi_{E \cap (F_n \times G_n)}(x, y)\ d\mu(x) \right) d\nu(y).$$

Summing over n gives

$$\begin{aligned}
(\mu \times \nu)(E) &= \sum_n (\mu \times \nu)(E \cap (F_n \times G_n)) = \sum_n \int_Y \left(\int_X \chi_{E \cap (F_n \times G_n)}(x, y)\ d\mu(x) \right) d\nu(y) \\
&= \int_Y \left(\int_X \sum_n \chi_{E \cap (F_n \times G_n)}(x, y)\ d\mu(x) \right) d\nu(y) \\
&= \int_Y \left(\int_X \chi_E(x, y)\ d\mu(x) \right) d\nu(y).
\end{aligned}$$

By repeating the previous proof with the role of X and Y interchanged, we conclude the proof. ■

Exercise 169 *(i) This exercise shows that without some condition on the measures μ and ν, Fubini's and Tonelli's theorems may fail in general. Let $X = Y = [0, 1]$, let $\mathfrak{M} = \mathfrak{N} = \mathcal{B}([0, 1])$, let μ be the Lebesgue measure, and let ν be the counting measure. Note that ν is not σ-finite. Show that the diagonal*

$$D := \{(x, x) :\ x \in [0, 1]\}$$

belongs to $\mathfrak{M} \otimes \mathfrak{N}$ but

$$\int_Y \mu(D_y)\ d\nu(y) \ne \int_X \nu(D_x)\ d\mu(x).$$

(ii) Finally, without the hypothesis that $E \in \mathfrak{M} \times \mathfrak{N}$ measurable function, the previous theorem fails. Let $X = Y$ be the set of all ordinals less

than or equal to the first uncountable ordinal ω_1, let $\mathfrak{M} = \mathfrak{N}$ be the σ-algebra consisting of all countable sets and their complements, and for every $F \in \mathfrak{M}$ define

$$\mu(F) = \nu(F) := \begin{cases} 1 & \text{if } F \text{ is countable,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $E = \{(x, y) \in X \times Y : x < y\}$. Prove that the sections E_x and E_y are measurable, but

$$\int_Y \mu(E_y)\, d\nu(y) \neq \int_X \nu(E_x)\, d\mu(x).$$

Remark 170 *If $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ are two measure spaces and μ and ν are complete, then $\mu \times \nu : \mathfrak{M} \times \mathfrak{N} \to [0, \infty]$ is complete. On the other hand, $\mu \times \nu : \mathfrak{M} \otimes \mathfrak{N} \to [0, \infty]$ is not complete in general. Indeed, if there exists a nonempty set $F \in \mathfrak{M}$ such that $\mu(F) = 0$ and a set $G \in \mathcal{P}(Y) \setminus \mathfrak{N}$, then the set $F \times G$ belongs to $\mathfrak{M} \times \mathfrak{N}$ since $F \times G \subset F \times Y$ and $(\mu \times \nu)(F \times Y) = \mu(F)\nu(Y) = 0$. On the other hand, by the previous exercise we have that $F \times G$ does not belong $\mathfrak{M} \otimes \mathfrak{N}$, since for every $x \in F$ the section*

$$(F \times G)_x = G$$

does not belong to $\mathfrak{N}$. In particular this can be applied to $\mathcal{L}^1 \times \mathcal{L}^1$ since we have shown that there exist sets that are not Lebesgue measurable.

Exercise 171 *Let $N = m + n$, where N, n, $m \in \mathbb{N}$. Prove that $(\mathcal{L}^n \times \mathcal{L}^m)^* = \mathcal{L}_o^N$.*

Theorem 172 (Tonelli) *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. Assume that μ and ν are complete and σ-finite, and let $f : X \times Y \to [0, \infty]$ be an $\mathfrak{M} \times \mathfrak{N}$ measurable function. Then for μ a.e. $x \in X$ the function $f(x, \cdot)$ is measurable and the function $\int_Y f(\cdot, y)\, d\nu(y)$ is measurable. Similarly, for ν a.e. $y \in Y$ the function $f(\cdot, y)$ is measurable and the function $\int_X f(x, \cdot)\, d\mu(x)$ is measurable. Moreover,*

$$\begin{aligned} \int_{X \times Y} f(x, y)\, d(\mu \times \nu)(x, y) &= \int_X \left(\int_Y f(x, y)\, d\nu(y) \right) d\mu(x) \\ &= \int_Y \left(\int_X f(x, y)\, d\mu(x) \right) d\nu(y). \end{aligned}$$

Proof. If $f = \chi_E$ or, more generally, if

$$f = \sum_{n=1}^{\ell} c_n \chi_{E_n},$$

then the result follows from the previous theorem. If $f : X \times Y \to [0, \infty]$ is an arbitrary $\mathfrak{M} \times \mathfrak{N}$ measurable function, then by Theorem 69 there exists a sequence $\{s_n\}$ of simple functions $s_n : X \times Y \to [0, \infty)$ such that

$$0 \leq s_1(x, y) \leq s_2(x, y) \leq \ldots \leq s_n(x, y) \to f(x, y)$$

for every $(x,y) \in X \times Y$. By the Lebesgue monotone convergence theorem (applied twice times) we have

$$\begin{aligned}
\int_{X\times Y} f(x,y)\ d(\mu\times\nu)(x,y) &= \lim_{n\to\infty}\int_{X\times Y} s_n(x,y)\ d(\mu\times\nu)(x,y) \\
&= \lim_{n\to\infty}\int_X\left(\int_Y s_n(x,y)\ d\nu(y)\right)d\mu(x) \\
&= \int_X\left(\lim_{n\to\infty}\int_Y s_n(x,y)\ d\nu(y)\right)d\mu(x).
\end{aligned}$$

Since by the previous theorem for all $n\in\mathbb{N}$ and for μ a.e. $x \in X$ the functions

$$y \in Y \mapsto s_n(x,y)$$

are measurable, we may apply again Lebesgue monotone convergence theorem to conclude that for μ a.e. $x \in X$,

$$\lim_{n\to\infty}\int_Y s_n(x,y)\ d\nu(y) = \int_Y f(x,y)\ d\nu(y),$$

and so

$$\int_{X\times Y} f(x,y)\ d(\mu\times\nu)(x,y) = \int_X\left(\int_Y f(x,y)\ d\nu(y)\right)d\mu(x).$$

Similarly, we have

$$\int_{X\times Y} f(x,y)\ d(\mu\times\nu)(x,y) = \int_Y\left(\int_X f(x,y)\ \ d\mu(x)\right)d\nu(y).$$

■

Exercise 173 *Prove that in the case that $f : X\times Y \to [0,\infty]$ is $\mathfrak{M}\otimes\mathfrak{N}$ measurable, then Tonelli's theorem still holds even if the measures μ and ν are not complete, and the statements are satisfied for every $x \in X$ and $y \in Y$ (as opposed to for μ a.e. $x \in X$ and for ν a.e. $y \in Y$).*

The version of Tonelli's theorem for integrable functions of arbitrary sign is the well–known Fubini's theorem:

Theorem 174 (Fubini) *Let $(X,\mathfrak{M},\mu)$ and $(Y,\mathfrak{N},\nu)$ be two measure spaces. Assume that μ and ν are complete, and let $f : X\times Y \to [-\infty,\infty]$ be $\mu\times\nu$-integrable. Then for μ a.e. $x \in X$ the function $f(x,\cdot)$ is ν-integrable, and the function $\int_Y f(\cdot,y)\ d\nu(y)$ is μ-integrable.*

Similarly, for ν a.e. $y \in Y$ the function $f(\cdot,y)$ is μ-integrable, and the function $\int_X f(x,\cdot)\ d\mu(x)$ is ν-integrable. Moreover,

$$\begin{aligned}
\int_{X\times Y} f(x,y)\ d(\mu\times\nu)(x,y) &= \int_X\left(\int_Y f(x,y)\ d\nu(y)\right)d\mu(x) \\
&= \int_Y\left(\int_X f(x,y)\ \ d\mu(x)\right)d\nu(y).
\end{aligned}$$

Proof. The proof is very similar to that of Tonelli's theorem. We consider first the case in which f is a characteristic function, then a simple function, then an nonnegative integrable function, and finally use the fact that $f = f^+ - f^-$. Note that, since f is $\mu \times \nu$-integrable, by Remark 83 the set

$$E := \{(x, y) \in E : |f(x, y)| > 0\}$$

has σ-finite $\mu \times \nu$ measure. Thus, we are in a position to apply Theorem 168(ii). ■

Exercise 175 *Prove that in the case that $f : X \times Y \to [-\infty, \infty]$ is $\mathfrak{M} \otimes \mathfrak{N}$ measurable, then Fubini's theorem still holds even if the measures μ and ν are not complete.*

Example 176 *The next example shows that Fubini's theorem fails without assuming the integrability of the function f. Consider the function*

$$f(x, y) := \frac{x^2 - y^2}{(x^2 + y^2)^2}, \quad (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Since f is continuous, it is a Borel function. Let $E = [0, 1] \times [0, 1]$. Then

$$\int_0^1 \left(\int_0^1 \int \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) dy = -\int_0^1 \frac{1}{y^2 + 1} \, dy = -\frac{1}{4}\pi,$$

while

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) dx = \int_0^1 \frac{1}{x^2 + 1} \, dx = \frac{1}{4}\pi.$$

Since, by Tonelli's theorem

$$\begin{aligned} \int_{[0,1] \times [0,1]} \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right)^+ dx dy &= \int_0^1 \left(\int_0^1 \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right)^+ dx \right) dy \\ &= \int_0^1 \left(\int_0^1 \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right)^+ dy \right) dx \end{aligned}$$

and

$$\begin{aligned} \int_{[0,1] \times [0,1]} \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right)^- dx dy &= \int_0^1 \left(\int_0^1 \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right)^- dx \right) dy \\ &= \int_0^1 \left(\int_0^1 \left(\frac{x^2 - y^2}{(x^2 + y^2)^2} \right)^- dy \right) dx, \end{aligned}$$

this implies that

$$\int_{[0,1]\times[0,1]}\left(\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right)^+ dxdy=\int_{[0,1]\times[0,1]}\left(\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right)^- dxdy=\infty,$$

so that the Lebesgue integral of f is not defined.

Example 177 *Consider the function*

$$f(x,y):=\frac{\sin^3 x}{x^4+y^2},\quad (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}.$$

Since f is continuous, it is a Borel function. Let $E=\left\{(x,y)\in\mathbb{R}^2: y>x^2,\, x>0\right\}$. Then by the Tonelli theorem

$$\begin{aligned}\int_E\left|\frac{\sin^3 x}{x^4+y^2}\right| dxdy &= \int_0^\infty\left(\int_{x^2}^\infty \frac{\left|\sin^3 x\right|}{x^4+y^2}\,dy\right)dx\\ &=\int_0^\infty\left(\frac{\left|\sin^3 x\right|}{x^2}\left[\arctan\left(\frac{y}{x^2}\right)\right]_{y=x^2}^{y=\infty}\right)dx=\frac{\pi}{4}\int_0^\infty\frac{\left|\sin^3 x\right|}{x^2}\,dx<\infty,\end{aligned}$$

and so we can apply Fubini's theorem to conclude that

$$\begin{aligned}\int_E\frac{\sin^3 x}{x^4+y^2}\,dxdy &= \int_0^\infty\left(\int_{x^2}^\infty\frac{\sin^3 x}{x^4+y^2}\,dy\right)dx\\ &=\int_0^\infty\left(\frac{\sin^3 x}{x^2}\left[\arctan\left(\frac{y}{x^2}\right)\right]_{y=x^2}^{y=\infty}\right)dx\\ &=\frac{\pi}{4}\int_0^\infty\frac{\sin^3 x}{x^2}\,dx.\end{aligned}$$

Integrating by parts,

$$\int\frac{\sin^3 x}{x^2}\,dx=-\frac{\sin^3 x}{x}-\int\left(-\frac{3\sin^2 x\cos x}{x}\right)dx.$$

Now

$$\sin^2 x\cos x=\frac{(1-\cos 2x)\cos x}{2}=\frac{\cos x-\cos x\cos 2x}{2}.$$

By Werner's formula $\cos x\cos 2x=\frac{\cos 3x+\cos x}{2}$, and so

$$\sin^2 x\cos x=\frac{\cos x-\cos 3x}{4}.$$

Hence

$$\int_0^\infty\frac{\sin^3 x}{x^2}\,dx=\int_0^\infty\frac{\cos x-\cos 3x}{4}\,dx=\frac{3\log 3}{4},$$

since we have the Frullani's integral

$$\int_0^\infty\frac{\cos x-\cos 3x}{x}\,dx=\ln 3.$$

Exercise 178 *This exercise shows that the σ-finiteness of μ and ν is not a necessary condition. Let $X = \mathbb{N}$ and let μ be the counting measure. Let $(Y, \mathfrak{N}, \nu)$ be any measure space with ν complete but not necessarily σ-finite. Prove that Fubini's and Tonelli's theorems continue to hold.*

Monday, December 05, 2011

13 Isodiametric Inequality

We leave the following preliminary result as an exercise.

Exercise 179 *Let $E, F \subset \mathbb{R}^N$ be two compact sets.*

(i) Construct a decreasing sequence of sets $\{E_n\}$ and $\{F_n\}$ such that

$$E = \bigcap_{n=1}^{\infty} E_n, \quad F = \bigcap_{n=1}^{\infty} F_n,$$

and all E_n and F_n consist of finite unions of rectangular parallelepipeds with sides parallel to the axes.

(ii) Prove that

$$\mathcal{L}^N (E_n + F_n) \to \mathcal{L}^N (E + F)$$

as $n \to \infty$.

Theorem 180 (Brunn–Minkowski inequality) *Let $E, F \subset \mathbb{R}^N$ be two Lebesgue measurable sets such that*

$$E + F := \{x + y : x \in E, y \in F\}$$

is also Lebesgue measurable. Then

$$\left(\mathcal{L}^N (E)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N (F)\right)^{\frac{1}{N}} \le \left(\mathcal{L}^N (E + F)\right)^{\frac{1}{N}}.$$

Proof. Step 1: Assume that E and F are rectangular parallelepipeds whose sides are parallel to the axes and let x_i and y_i, $i = 1, \dots, N$, be their respective side-lengths. Then

$$\mathcal{L}^N (E) = \prod_{i=1}^{N} x_i, \quad \mathcal{L}^N (F) = \prod_{i=1}^{N} y_i, \quad \mathcal{L}^N (E + F) = \prod_{i=1}^{N} (x_i + y_i).$$

By the arithmetic-geometric mean inequality

$$\left(\prod_{i=1}^{N} \frac{x_i}{x_i + y_i}\right)^{\frac{1}{N}} + \left(\prod_{i=1}^{N} \frac{y_i}{x_i + y_i}\right)^{\frac{1}{N}} \le \frac{1}{N} \sum_{i=1}^{N} \frac{x_i}{x_i + y_i} + \frac{1}{N} \sum_{i=1}^{N} \frac{y_i}{x_i + y_i} = 1,$$

which gives the Brunn-Minkowski inequality for rectangular parallelepipeds.
Step 2: We now suppose that E and F are finite unions of rectangular parallelepipeds as in Step 1. The proof is by induction on the sum of the numbers of parallelepipeds in E and in F. By interchanging E with F, if necessary, we may assume that E has at least two parallelepipeds. By translating E if necessary, we may also assume that a coordinate hyperplane, say $\{x_N = 0\}$, separates two parallelepipeds in E. Let E^+ and E^- denote the union of the parallelepipeds formed by intersecting the parallelepipeds in E with the half spaces $\{x_N \geq 0\}$ and $\{x_N \leq 0\}$ respectively. Translate F so that

$$\frac{\mathcal{L}^N\left(E^{\pm}\right)}{\mathcal{L}^N\left(E\right)} = \frac{\mathcal{L}^N\left(F^{\pm}\right)}{\mathcal{L}^N\left(F\right)},$$

where F^+ and F^- are defined analogously to E^+ and E^-. Then $E^+ + F^+ \subset \{x_N \geq 0\}$ and $E^- + F^- \subset \{x_N \leq 0\}$, and the numbers of parallelepipeds in $E^+ \cup F^+$ and in $E^- \cup F^-$ are both smaller than the number of parallelepipeds in $E \cup F$. By induction and Step 1 we obtain

$$\begin{aligned}
\mathcal{L}^N\left(E+F\right) &\geq \mathcal{L}^N\left(E^+ + F^+\right) + \mathcal{L}^N\left(E^- + F^-\right) \\
&\geq \left(\left(\mathcal{L}^N\left(E^+\right)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N\left(F^+\right)\right)^{\frac{1}{N}}\right)^N \\
&+ \left(\left(\mathcal{L}^N\left(E^-\right)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N\left(F^-\right)\right)^{\frac{1}{N}}\right)^N \\
&= \mathcal{L}^N\left(E^+\right)\left(1 + \frac{\left(\mathcal{L}^N\left(F\right)\right)^{\frac{1}{N}}}{\left(\mathcal{L}^N\left(E\right)\right)^{\frac{1}{N}}}\right)^N + \mathcal{L}^N\left(E^-\right)\left(1 + \frac{\left(\mathcal{L}^N\left(F\right)\right)^{\frac{1}{N}}}{\left(\mathcal{L}^N\left(E\right)\right)^{\frac{1}{N}}}\right)^N \\
&= \mathcal{L}^N\left(E\right)\left(1 + \frac{\left(\mathcal{L}^N\left(F\right)\right)^{\frac{1}{N}}}{\left(\mathcal{L}^N\left(E\right)\right)^{\frac{1}{N}}}\right)^N = \left(\left(\mathcal{L}^N\left(E\right)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N\left(F\right)\right)^{\frac{1}{N}}\right)^N.
\end{aligned}$$

Step 3: Assume next that E and F are compact sets. Let $\{E_n\}$ and $\{F_n\}$ be as in the previous exercise. Then for all $n \in \mathbb{N}$

$$\left(\mathcal{L}^N\left(E_n\right)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N\left(F_n\right)\right)^{\frac{1}{N}} \leq \left(\mathcal{L}^N\left(E_n + F_n\right)\right)^{\frac{1}{N}}.$$

It now suffices to let $n \to \infty$ and use Proposition 47 below to obtain

$$\left(\mathcal{L}^N\left(E\right)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N\left(F\right)\right)^{\frac{1}{N}} \leq \left(\mathcal{L}^N\left(E+F\right)\right)^{\frac{1}{N}}.$$

Step 4: Assume next that E and F are bounded Lebesgue measurable sets, with $E+F$ Lebesgue measurable and fix $\varepsilon > 0$. By your homework we may two compact sets $K_1 \subset E$, $K_2 \subset F$ such that

$$\mathcal{L}^N\left(E\right) \leq \mathcal{L}^N\left(K_1\right) + \varepsilon, \quad \mathcal{L}^N\left(F\right) \leq \mathcal{L}^N\left(K_2\right) + \varepsilon.$$

Then $K_1 + K_2 \subset E + F$, and so by the previous step

$$\left(\mathcal{L}^N\left(K_1\right)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N\left(K_2\right)\right)^{\frac{1}{N}} \leq \left(\mathcal{L}^N\left(K_1 + K_2\right)\right)^{\frac{1}{N}} \leq \left(\mathcal{L}^N\left(E+F\right)\right)^{\frac{1}{N}}.$$

Using the inequality $a^{\frac{1}{N}} + b^{\frac{1}{N}} \leq (a+b)^{\frac{1}{N}}$, $a, b \geq 0$, we now get

$$\begin{aligned}\left(\mathcal{L}^N(E)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N(F)\right)^{\frac{1}{N}} &\leq \left(\mathcal{L}^N(K_1)+\varepsilon\right)^{\frac{1}{N}} + \left(\mathcal{L}^N(K_2)+\varepsilon\right)^{\frac{1}{N}} \\ &\leq \left(\mathcal{L}^N(K_1)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N(K_2)\right)^{\frac{1}{N}} + 2\varepsilon^{\frac{1}{N}} \\ &\leq \left(\mathcal{L}^N(E+F)\right)^{\frac{1}{N}} + 2\varepsilon^{\frac{1}{N}}.\end{aligned}$$

Letting $\varepsilon \to 0$ we get Brunn-Minkowski inequality.
Step 5: Finally, in the general case, apply the previous step to $E \cap B(0,n)$ and $F \cap B(0,n)$ to obtain

$$\left(\mathcal{L}^N(E \cap B(0,n))\right)^{\frac{1}{N}} + \left(\mathcal{L}^N(F \cap B(0,n))\right)^{\frac{1}{N}} \leq \left(\mathcal{L}^N(E \cap B(0,n) + F \cap B(0,n))\right)^{\frac{1}{N}}$$

and then let $n \to \infty$ and use Proposition 47. ■

Remark 181 *(i) Note that the hypothesis that $E+F$ is measurable cannot be omitted. Indeed Sierpinsky constructed an example of two measurable sets whose sum is not measurable. However if E and F are Borel sets, then $E+F$ is an analytic set, and so it is measurable.*

(ii) Fix $\theta \in (0,1)$. By replacing E with θE and F with $(1-\theta)F$ and using the N-homogeneity of the Lebesgue measure we obtain that

$$\begin{aligned}\theta\left(\mathcal{L}^N(E)\right)^{\frac{1}{N}} + (1-\theta)\left(\mathcal{L}^N(F)\right)^{\frac{1}{N}} &= \left(\mathcal{L}^N(\theta E)\right)^{\frac{1}{N}} + \left(\mathcal{L}^N((1-\theta)F)\right)^{\frac{1}{N}} \\ &\leq \left(\mathcal{L}^N(\theta E + (1-\theta)F)\right)^{\frac{1}{N}}.\end{aligned}$$

Thus the function $f(t) := \left(\mathcal{L}^N(tE + (1-t)F)\right)^{\frac{1}{N}}$ is concave in $[0,1]$.

Using the Minkowski inequality we can prove (a weaker version of) the isodiametric inequality.

Theorem 182 (Isodiametric Inequality) *Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set. Then*

$$\mathcal{L}^N(E) \leq \alpha_N \left(\frac{\operatorname{diam} E}{2}\right)^N.$$

Proof. It is enough to prove the previous inequality for bounded sets, since otherwise the right-hand side is infinite. If $\lambda > 0$, we have that $\mathcal{L}^N(\lambda E) = \lambda^N \mathcal{L}^N(E)$ and $(\operatorname{diam}(\lambda E))^N = \lambda^N \operatorname{diam} E$, so without loss of generality we may assume that $\operatorname{diam} E \leq 1$. Let

$$F := \{-x : x \in E\}.$$

By the previous remark the function $f(t) := \left(\mathcal{L}^N(tE + (1-t)F)\right)^{\frac{1}{N}}$ is concave in $[0,1]$, and so

$$\frac{1}{2}\left(\mathcal{L}^N(E)\right)^{\frac{1}{N}} + \frac{1}{2}\left(\mathcal{L}^N(F)\right)^{\frac{1}{N}} \leq \left(\mathcal{L}^N\left(\frac{1}{2}E + \frac{1}{2}F\right)\right)^{\frac{1}{N}}.$$

But $\mathcal{L}^N(F) = \mathcal{L}^N(E)$, and so the previous inequality becomes

$$\mathcal{L}^N(E) = \mathcal{L}^N(F) \le \mathcal{L}^N\left(\frac{1}{2}E + \frac{1}{2}F\right).$$

If $x, y \in \frac{1}{2}E + \frac{1}{2}F$, then $x = \frac{x'-x''}{2}$ and $y = \frac{y'-y''}{2}$, where $x', x'', y', y'' \in E$. Hence

$$2|x-y| = |x'-x''-(y'-y'')| \le |x'-x''| + |y'-y''| \le 1+1,$$

since $\operatorname{diam} E \le 1$. This shows that $\operatorname{diam}\left(\frac{1}{2}E + \frac{1}{2}F\right) \le 1$.

Since $\frac{1}{2}E + \frac{1}{2}F = \left\{\frac{x}{2} - \frac{y}{2} : x, y \in E\right\}$ is symmetric with respect to the origin, it follows that $\frac{1}{2}E + \frac{1}{2}F \subset \overline{B\left(0, \frac{1}{2}\right)}$. Hence,

$$\mathcal{L}^N(E) \le \mathcal{L}^N\left(\frac{1}{2}E + \frac{1}{2}F\right) \le \mathcal{L}^N\left(\overline{B\left(0, \frac{1}{2}\right)}\right) = \frac{\alpha_N}{2^N}.$$

■

Wednesday, December 07, 2011

Using the isodiametric inequality and a covering theorem it is possible to prove that.

Corollary 183 $\mathcal{H}^N_o = \mathcal{L}^N_o$.

Proof. Step 1: We prove that $\mathcal{L}^N_o \le \mathcal{H}^N_o$. Fix $\delta > 0$ and consider any sequence $\{E_n\} \subset \mathbb{R}^N$ such that $E \subset \bigcup_{n=1}^\infty E_n$ and $\operatorname{diam} E_n < \delta$ for all $n \in \mathbb{N}$. Since $\operatorname{diam} E_n = \operatorname{diam} \overline{E_n}$, without loss of generality, we may assume that the sets E_n are closed and thus Lebesgue measurable.

Using the monotonicity and subadditivity of $\mathcal{L}^N_o$ together with the isodiametric inequality, we have

$$\mathcal{L}^N_o(E) \le \mathcal{L}^N_o\left(\bigcup_{n=1}^\infty E_n\right) \le \sum_{n=1}^\infty \mathcal{L}^N(E_n) \le \sum_{n=1}^\infty \alpha_N \left(\frac{\operatorname{diam} E_n}{2}\right)^N.$$

Taking the infimum over all admissible sequences $\{E_n\}$ gives

$$\mathcal{L}^N_o(E) \le \mathcal{H}^N_o(E). \tag{97}$$

Step 2: We claim that

$$\mathcal{H}^N_o \le C_N \mathcal{L}^N_o.$$

To see this, gix $\delta > 0$. Given a cube $Q(x,r)$, we have that

$$\alpha_N \left(\frac{\operatorname{diam} Q(x,r)}{2}\right)^N = \alpha_N \left(\frac{\sqrt{N} r}{2}\right)^N =: C_N r^N,$$

and so for each set $E \subset \mathbb{R}^N$, by (5) and Exercise **??**,

$$\begin{aligned}
\mathcal{H}_\delta^N(E) &:= \inf\left\{\sum_{n=1}^\infty \alpha_N\left(\frac{\operatorname{diam} E_n}{2}\right)^N : E \subset \bigcup_{n=1}^\infty E_n,\ \operatorname{diam} E_n < \delta\right\} \\
&\le \inf\left\{\sum_{n=1}^\infty \alpha_N\left(\frac{\operatorname{diam} Q(x_n, r_n)}{2}\right)^N : E \subset \bigcup_{n=1}^\infty Q(x_n, r_n),\ \sqrt{N} r < \delta\right\} \\
&= C_N \mathcal{L}_o^N(E).
\end{aligned}$$

The claim follows by letting $\delta \to 0^+$.

Step 3: We prove that $\mathcal{L}_o^N = \mathcal{H}_o^N$, fix $\varepsilon > 0$ and $\delta > 0$ and by Exercise **??** find a sequence of cubes $\{Q(x_n, r_n)\}$ with diameter less than δ such that

$$\sum_{n=1}^\infty \mathcal{L}^N(Q(x_n, r_n)) \le \mathcal{L}_o^N(E) + \varepsilon.$$

For every $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the family of all closed balls contained in $Q(x_n, r_n)$. By the Vitali–Besicovitch covering theorem there exists a countable family $\left\{\overline{B\left(x_i^{(n)}, r_i^{(n)}\right)}\right\} \subset \mathcal{F}_n$ of disjoint closed balls such that

$$\mathcal{L}_o^N\left(Q(x_n, r_n) \setminus \bigcup_i \overline{B\left(x_i^{(n)}, r_i^{(n)}\right)}\right) = 0.$$

By the previous step,

$$\begin{aligned}
&\mathcal{H}_\delta^N\left(Q(x_n, r_n) \setminus \bigcup_i \overline{B\left(x_i^{(n)}, r_i^{(n)}\right)}\right) \\
&\qquad = \mathcal{H}_o^N\left(Q(x_n, r_n) \setminus \bigcup_i \overline{B\left(x_i^{(n)}, r_i^{(n)}\right)}\right) = 0,
\end{aligned}$$

and so, since $\mathcal{H}_\delta^N$ is an outer measure,

$$\begin{aligned}
\mathcal{H}_\delta^N(E) &\le \sum_{n=1}^\infty \mathcal{H}_\delta^N(Q(x_n, r_n)) = \sum_{n=1}^\infty \mathcal{H}_\delta^N\left(\bigcup_i \overline{B\left(x_i^{(n)}, r_i^{(n)}\right)}\right) \\
&\le \sum_{n=1}^\infty \sum_i \mathcal{H}_\delta^N\left(\overline{B\left(x_i^{(n)}, r_i^{(n)}\right)}\right) \le \sum_{n=1}^\infty \sum_i \alpha_N r_i^{(n)} \\
&= \sum_{n=1}^\infty \sum_i \mathcal{L}_o^N\left(\overline{B\left(x_i^{(n)}, r_i^{(n)}\right)}\right) = \sum_{n=1}^\infty \mathcal{L}_o^N(Q(x_n, r_n)) \le \mathcal{L}_o^N(E) + \varepsilon.
\end{aligned}$$

Letting $\delta \to 0^+$ and then $\varepsilon \to 0^+$ gives the desired result, provided we can prove that $\alpha_N = \mathcal{L}^N(B(0,1))$. ■

Definition 184 *Given a set $E \subset \mathbb{R}^N$, a family $\mathcal{F}$ of nonempty subsets of $\mathbb{R}^N$ is said to be a*

(i) cover *for E if*

$$E \subset \bigcup_{F \in \mathcal{F}} F;$$

(ii) fine cover *for E if for every $x \in E$ there exists a subfamily $\mathcal{F}_x \subset \mathcal{F}$ of sets containing x such that*

$$\inf \{\operatorname{diam} F : F \in \mathcal{F}_x\} = 0. \tag{98}$$

An important consequence of the Besicovitch covering theorem is the following result.

Theorem 185 (Vitali–Besicovitch covering theorem) *Let $E \subset \mathbb{R}^N$ be a Borel set and let $\mathcal{F}$ be a family of closed balls such that each point of E is the center of arbitrarily small balls, that is,*

$$\inf \left\{ r : \overline{B(x, r)} \in \mathcal{F} \right\} = 0 \text{ for every } x \in E.$$

Let $\mu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0, \infty]$ be a measure finite on compact sets. Then there exists a countable family $\left\{\overline{B(x_n, r_n)}\right\} \subset \mathcal{F}$ of disjoint closed balls such that

$$\mu\left(E \setminus \bigcup_n \overline{B(x_n, r_n)}\right) = 0.$$

Proof. Let $\ell = \ell(N)$ be the number given in the Besicovitch covering theorem and choose $1 - \frac{1}{\ell} < \theta < 1$.

Step 1: Assume that E is bounded. We show that there exists a finite subfamily $\left\{\overline{B}_1, \ldots, \overline{B}_{m_1}\right\} \subset \mathcal{F}$ of mutually disjoint subsets of A such that

$$\mu\left(E \setminus \bigcup_{i=1}^{m_1} \overline{B}_i\right) \le \theta \mu(E). \tag{99}$$

Indeed, let

$$\mathcal{F}^{(1)} := \left\{\overline{B} \in \mathcal{F} : \operatorname{diam} F \le 1\right\}.$$

Since $\mathcal{F}$ is a fine cover of E, it follows that $\mathcal{F}^{(1)}$ satisfies the hypotheses of the Besicovitch covering theorem, and so there exist $\mathcal{F}_1^{(1)}, \ldots, \mathcal{F}_\ell^{(1)} \subset \mathcal{F}^{(1)}$ such that each $\mathcal{F}_n^{(1)}$ is a countable family of disjoint sets in $\mathcal{F}^{(1)}$ and

$$E \cap A \subset \bigcup_{n=1}^{\ell} \bigcup_{\overline{B} \in \mathcal{F}_n^{(1)}} \overline{B}.$$

Hence

$$\mu(E) \leq \sum_{n=1}^{\ell} \mu\left(E \cap \left(\bigcup_{\overline{B} \in \mathcal{F}_n^{(1)}} \overline{B}\right)\right),$$

and thus there exists $j \in \{1, \ldots, \ell\}$ such that[8]

$$\mu\left(E \cap \bigcup_{\overline{B} \in \mathcal{F}_j^{(1)}} \overline{B}\right) \geq \frac{1}{\ell} \mu(E).$$

Writing

$$\mathcal{F}_j^{(1)} = \left\{\overline{B}_{i,j}^{(1)}\right\},$$

in view of Proposition **??** we have that

$$\begin{aligned} \lim_{m \to \infty} \mu\left(E \cap \bigcup_{i=1}^{m} \overline{B}_{i,j}^{(1)}\right) = \mu\left(E \cap \bigcup_{\overline{B} \in \mathcal{F}_j^{(1)}} \overline{B}\right) &\geq \frac{1}{\ell} \mu(E) \\ &> (1-\theta) \mu(E), \end{aligned}$$

where we have used the fact that $1 - \theta < \frac{1}{\ell}$. Hence we find m_1 so large that

$$\mu\left(E \cap \bigcup_{i=1}^{m_1} \overline{B}_{i,j}^{(1)}\right) \geq (1-\theta) \mu(E). \tag{100}$$

Since $\bigcup_{i=1}^{m_1} \overline{B}_{i,j}^{(1)}$ is closed, therefore μ-measurable, we have

$$\mu(E) = \mu\left(E \cap \bigcup_{i=1}^{m_1} \overline{B}_{i,j}^{(1)}\right) + \mu\left(E \setminus \bigcup_{i=1}^{m_1} \overline{B}_{i,j}^{(1)}\right),$$

which, together with (100), establishes (99) with $B_i := B_{i,j}^{(1)}$, $i = 1, \ldots, m_1$. Note that here we have used the fact that E is bounded and μ is finite on compact sets, so that $\mu(E) < \infty$.

Set

$$A_2 := \mathbb{R}^N \setminus \bigcup_{i=1}^{m_1} \overline{B}_i$$

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[8] Here we are using the fact that if

$$a \leq \sum_{n=1}^{\ell} b_n,$$

then there exists at least one $b_n \geq \frac{1}{\ell} a$. Indeed, if not, then $b_n < \frac{1}{\ell} a$ for all $n = 1, \ldots, \ell$, and so

$$a \leq \sum_{n=1}^{\ell} b_n < \sum_{n=1}^{\ell} \frac{1}{\ell} a = a,$$

which is a contradiction.

and

$$\mathcal{F}^{(2)} := \left\{ \overline{B} \in \mathcal{F} : \operatorname{diam} \overline{B} \leq 1 \text{ and } \overline{B} \subset A_2 \right\}.$$

Just as before, we may find a finite subfamily $\left\{ \overline{B}_{m_1+1}, \ldots, \overline{B}_{m_2} \right\} \subset \mathcal{F}^{(2)}$ of mutually disjoint closed balls contained in A_2 such that

$$\begin{aligned} \mu\left(E \setminus \bigcup_{i=1}^{m_2} F_i\right) &= \mu\left((E \cap A_2) \setminus \bigcup_{i=m_1+1}^{m_2} F_i\right) \\ &\leq \theta \mu\left(E \cap A_2\right) \leq \theta^2 \mu\left(E\right). \end{aligned}$$

By induction, we construct a countable family $\mathcal{F}_0 = \{F_i\} \subset \mathcal{F}$ of pairwise disjoint subsets such that for every $k \in \mathbb{N}$,

$$\mu\left(E \setminus \bigcup_{i} \overline{B}_i\right) \leq \mu\left(E \setminus \bigcup_{i=1}^{m_k} \overline{B}_i\right) \leq \theta^k \mu\left(E\right).$$

Since $\mu\left(E\right) < \infty$ we conclude the proof in this case by letting $k \to \infty$. ■

Friday, December 09, 2011

To prove the case in which E is ubounded, we need the following proposition.

Proposition 186 *Let $(X, \mathfrak{M}, \mu)$ be a measure space with μ finite and let $\{E_j\}_{j\in J} \subset \mathfrak{M}$ be an arbitrary family of pairwise disjoint subsets of X. Then $\mu\left(E_j\right) = 0$ for all but at most countably many $j \in J$.*

Proof. Fix $k \in \mathbb{N}$ and let

$$J_k := \left\{ j \in J : \mu\left(E_j\right) > \frac{1}{k} \right\}.$$

We claim that the set J_k is finite. Indeed, if I is any finite subset of J_k, then

$$\infty > \mu\left(X\right) \geq \mu\left(\bigcup_{j\in I} E_j\right) = \sum_{j\in I} \mu\left(E_j\right) \geq \frac{1}{k} \sum_{j\in I} 1 = \frac{1}{k} \operatorname{card} I,$$

which implies that J_k cannot have more than $\lceil k\mu\left(X\right) \rceil$ elements, where $\lceil k\mu\left(X\right) \rceil$ is the integer part of $k\mu\left(X\right)$. Thus

$$\{j \in J : \mu\left(E_j\right) > 0\} = \bigcup_{k=1}^{\infty} J_k$$

is at most countable. ■

Proof of Theorem ??, continued.

Step 2: If E is unbounded, then applying Proposition 186 to the measure $\mu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0, \infty]$, choose an increasing sequence of radii $\{r_n\}$ such that $r_n \to \infty$ and

$$\mu\left(\partial B\left(0, r_n\right)\right) = 0. \tag{101}$$

Applying Step 1 to

$$E_n := E \cap \left(B\left(0, r_{n+1}\right) \setminus \overline{B\left(0, r_n\right)} \right)$$

and with the family of balls

$$\mathcal{F}_n := \left\{ \overline{B} \in \mathcal{F} : \ \ \overline{B} \subset B\left(0, r_{n+1}\right) \setminus \overline{B\left(0, r_n\right)} \right\},$$

we find a countable family of mutually disjoint balls $\left\{ \overline{B}_i^{(n)} \right\}_{i \in \mathbb{N}}$ contained in $\mathcal{F}_n$ and that cover E_n up to a set of μ outer measure zero. Therefore the collection of

$$\left\{ \overline{B}_i^{(n)} \right\}_{i,n}$$

is a family of mutually disjoint balls and

$$\begin{aligned}
\mu\left(E \setminus \bigcup_{i,n} \overline{B}_i^{(n)} \right) &= \mu\left(\left(\bigcup_{j=1}^{\infty} E \cap \left(\overline{B\left(0, r_{j+1}\right)} \setminus \overline{B\left(0, r_j\right)} \right) \right) \setminus \bigcup_{i,n} \overline{B}_i^{(n)} \right) \\
&\overset{\text{why?}}{=} \mu\left(\left(\bigcup_{j=1}^{\infty} E \cap \left(B\left(0, r_{j+1}\right) \setminus \overline{B\left(0, r_j\right)} \right) \right) \setminus \bigcup_{i,n} \overline{B}_i^{(n)} \right) \\
&= \mu\left(\left(\bigcup_{j=1}^{\infty} E_j \right) \setminus \bigcup_{i,n} \overline{B}_i^{(n)} \right) \leq \mu\left(\bigcup_{j=1}^{\infty} \left(E_j \setminus \bigcup_{i} \overline{B}_i^{(j)} \right) \right) \\
&\leq \sum_{j=1}^{\infty} \mu\left(E_j \setminus \bigcup_{i} \overline{B}_i^{(j)} \right) = 0.
\end{aligned}$$

This concludes the proof. ■

Recall that the Gamma function is defined by

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\, dx, \quad t > 0.$$

Theorem 187 *Let $N \geq 1$. Then the volume of $B\left(x_0, r\right)$ is*

$$\alpha_N = \frac{\pi^{N/2}}{\Gamma\left(1 + N/2\right)} r^N.$$

Proof. Since $B\left(x_0, r\right) = x_0 + rB\left(0, 1\right)$, by homogeneity, we have that the volume of $B\left(x_0, r\right)$ is given by r^N times the volume of the unit ball $B\left(0, 1\right)$.

Write points of $\mathbb{R}^{N+1}$ as (x, y) with $x \in \mathbb{R}^N$ and $y \in \mathbb{R}$ and consider the set

$$D := \left\{ (x, y) \in \mathbb{R}^{N+1} : \ |x|^2 < y \right\}.$$

We now condier the integral

$$\int_D e^{-y}\, dxdy.$$

We now integrate into two different ways.

$$\begin{aligned}
\int_D e^{-y}\,dxdy &= \int_0^\infty \left(\int_{B\left(0;y^{1/2}\right)} e^{-y}\,dx\right) dy \\
&= \int_0^\infty e^{-y}\left(\int_{B\left(0;y^{1/2}\right)} 1\,dx\right) dy \\
&= \int_0^\infty e^{-y}\mathcal{L}^N\left(B\left(0;y^{1/2}\right)\right) dy \\
&= \int_0^\infty e^{-y}y^{N/2}\mathcal{L}^N\left(B\left(0;1\right)\right) dy = \mathcal{L}^N\left(B\left(0;1\right)\right)\int_0^\infty e^{-y}y^{N/2}\,dy \\
&= \Gamma\left(1+N/2\right)\mathcal{L}^N\left(B\left(0;1\right)\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_D e^{-y}\,dxdy &= \int_{\mathbb{R}^N}\left(\int_{\|x\|^2}^\infty e^{-y}\,dy\right) dx \\
&= \int_{\mathbb{R}^N}\left[-e^{-y}\right]_{y=\|x\|^2}^{y=\infty} dx \\
&= \int_{\mathbb{R}^N} e^{-\|x\|^2}\,dx = \int_{\mathbb{R}^N} e^{-x_1^2-x_2^2-\cdots-x_N^2}\,dx_1dx_2\cdots dx_N \\
&= \int_{\mathbb{R}^N} e^{-x_1^2}e^{-x_2^2}\cdots e^{-x_N^2}\,dx_1dx_2\cdots dx_N \\
&= \left(\int_{\mathbb{R}} e^{-x_1^2}\,dx_1\right)\left(\int_{\mathbb{R}} e^{-x_2^2}\,dx_2\right)\cdots\left(\int_{\mathbb{R}} e^{-x_N^2}\,dx_N\right) = I^N,
\end{aligned}$$

where

$$I = \int_{\mathbb{R}} e^{-t^2}\,dt.$$

This shows that

$$\Gamma\left(1+N/2\right)\mathcal{L}^N\left(B\left(0;1\right)\right) = I^N.$$

Now for $N=2$, we know that $\mathcal{L}^2\left(B\left(0;1\right)\right)=\pi$, while $\Gamma\left(1+2/2\right)=\Gamma\left(2\right)=1$. Hence,

$$\pi = I^2,$$

which shows that $I=\pi^{1/2}$. This completes the proof. ■