21-484 Notes

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<u>Tutte's Thm:</u>

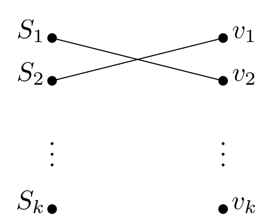
$k_o(G-S)$, G contains a perfect matching iff $k_o(G-S) \le |S|$ $\forall S \subseteq V(G)$. $*$

<u>Proof:</u>

- Induction on number of vertices in G.
- Assume that G has $*$.

$\rightarrow$ Let S be a maximal set of verties having $k_o(G-S) = |S|$.

- $\rightarrow$ S is not empty.
- $\rightarrow$ Let $G_1, \ldots, G_k$ be the components of $G-S$, then $|G_i|$ is odd $\forall 1 \le i \le k$.
- $\rightarrow$ Let S_i be the set of neighbors of G_i in S.
- $\rightarrow$ S_i is not empty. (G_i is odd and all the connected components of G are even).



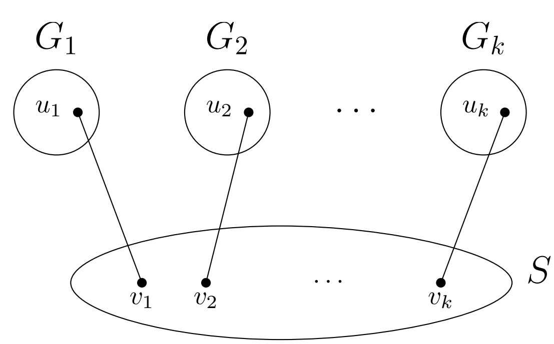
- $\rightarrow$ $**$ For every $1 \le t \le k$ and every t of the S_i's, the union of these S_i's is of size at least t.
- $\rightarrow$ Otherwise, let T be the union of the S_i's.

$$k_o(G-T) \ge t > |T| \lightning *$$

$\rightarrow$ Consider the bipartite graph with sides $\{S_1, \ldots, S_k\}$ and $S = \{v_1, \ldots, v_k\}$. There is an edge in H between S_j and v_i iff $\forall v_i \in S_j$

$\rightarrow$ by $**$+Hall's Theorem, there is a perfect matching in H.

$\rightarrow$ Let u_i be the neighbor of v_i in G_i (assuming without loss of generality that the perfect matching matched v_i to S_i).



→ Need to show that $\forall 1 \le i \le k$ and $\forall W \subseteq V(G_i - u_i)$

$$k_o(G_i - u_i - W) \le |W|$$

→ Assume otherwise: $|W| < k_o(G_i - u_i - W)$ for some i and W.

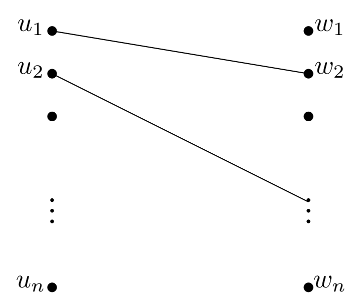
→ Since $G_i - u_i$ is eve, the parity of $|W|$ and $k_o(G_i - u_i - W)$ is the same.

→ Consider $S' = S \cup W \cup \{u_i\}$

$|S'| \ge k_o(G-S') = k_o(G-S) + k_o(G_i - u_i - W) - 1 \ge |S| + |W| + 2 - 1 = |S'|$ ↯maximality of S.

by $*$

H



→ By the induction hypothesis, there is a perfect matching M_i in G_i.

$$M = \left(\bigcup_{i=1}^{k} M_i\right) \cup \{v_i u_i\} \text{ is a perfect matching in } G$$

Theorem(Tutte-Berge formula): For every graph G, the size of a maximum matching is

$$\min_{S \subseteq V(G)} \frac{(|S| - k_o(G-S) + |V|)}{2}$$

Def: Let k be a positive integer. A k-factor in a graph G is a spanning subgraph which is k-regular.

Example: A perfect matching is a 1-factor.

Theorem (Petersen): A graph G can be decomposed into 2-factors $F_1, \dots, F_k$ if and only if G is $2k$-regular. $\left(G = \dot{\bigcup}_{i=1}^{k} F_i\right)$

Proof's idea: one direction is easy (decomposition $\implies$ $2k$-regular).

→ Assume that G is $2k$-regular. By Euler's Theorem, there is an Eulerian circuit C.

→ (Def. H)

→ H is k-regular

→ By Hall's Theorem and counting every regular bipartite graph contains a perfect matching.

→ Every perfect matching in H corresponds to a 2-factor of G.

→ repeat. ∎