Assignment 4

15-359 Probability and Computing
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Section: B
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Problem 2: Shipping error

Let G be the event that you are shipped a working server, and let B be the event that you are shipped a faulty server. Then, if T is a random variable denoting the time in days until the server crashes, $(T|G) \sim \text{Geometric}\left(\frac{1}{100}\right)$, and $(T|B) \sim \text{Geometric}\left(\frac{5}{100}\right)$. Thus, as shown in class,

$$E[T^2|G] = \frac{2 - \frac{1}{100}}{\left(\frac{1}{100}\right)^2} = 19900, \; E[T|G] = \frac{1}{1/100} = 100,$$

$$E[T^2|B] = \frac{2 - \frac{5}{100}}{\left(\frac{5}{100}\right)^2} = 780, \; E[T|B] = \frac{1}{1/20} = 20.$$

Then, since $E[T^2] = E[T^2|G]P(G) + E[T^2|B]P(B)$ and $E[T] = E[T|G]P(G) + E[T|B]P(B)$, $\text{Var}(T) = E[T^2] - E[T]^2 = \boxed{\frac{73340}{9}.}$

Problem 3: Skew and skewer

For purposes of this question, abbreviate Skewer as S. Note that, for all random variables X,

$$S(X) = E[(X - E[X])^3] = E[X^3] - 3E[X^2]E[X] + 2E[X]^3.$$

Let X and Y be independent random variables, noting that this implies that $E[X]E[Y] = E[XY]$. Thus,

$$\begin{aligned}
S(X+Y) &= E[(X+Y)^3] - 3E[X^2]E[X] + 2E[X]^3 \\
&= E[X^3] + 3E[X^2Y] + 3E[XY^2] + E[Y^3] \\
&- 3(E[X^2]E[X] - 2E[XY]E[X] + E[XY^2]) \\
&- 3(E[X^2Y] - 2E[XY][EY] + E[Y^2]E[Y]) \\
&+ 2E[X+Y]^3 \\
&= E[X^3] + E[Y^3] \\
&- 3(E[X^2]E[X] - 2E[XY]E[X]) \\
&- 3(-2E[XY][EY] + E[Y^2]E[Y]) \\
&+ 2E[X+Y]^3 \\
&= E[X^3] - 3E[X^2]E[X] + 2E[X]^3 \\
&+ E[Y^3] - 3E[Y^2]E[Y] + 2E[Y]^3
\end{aligned}$$

$$= \quad S(X) + S(Y)$$

Thus, $S(X+Y) = S(X) + S(Y)$. $\blacksquare$

Problem 4: Tails

Let X be a non-negative, discrete, integer-valued random variable.
Let $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} | 1 \leq j \leq i\}$. Then,

$$\begin{aligned}
\sum_{j=1}^{\infty} jP(X \geq j) &= \sum_{j=1}^{\infty} j \sum_{i=j}^{\infty} P(X = i) \\
&= \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} jP(X = i) \\
&= \sum_{(i,j) \in S} jP(X = i) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{i} jP(X = i) \\
&= \frac{\sum_{i=1}^{\infty} (i + i^2) P(X = i)}{2} \\
&= \boxed{\frac{E[X^2] + E[X]}{2}}
\end{aligned}$$

Problem 5: Coffee-theorem revisited

Let W be the event that the student starts off working, let C be the event that the student starts off at the coffee shop, and let X be a random variable denoting the time at which the student returns home. Note that, as computed in the solution to Problem 6 on Assignment 2, $E[X|W] = 9$ and $E[X|C] = 12$. Furthermore, conditioning on what the student does at the end of any given hour, $P(X = i|W) = \frac{2}{3}P(X = i-1|C)$, and $P(X = i|C) = \frac{2}{3}P(X = i-1|C) + \frac{1}{3}P(X = i-1|W)$. Thus, by definition of expected value, since $\sum_{i=1}^{\infty} P(X = i|C) = \sum_{i=1}^{\infty} P(X = i|W) = 1$,

$$\begin{aligned}
E[X^2|C] &= \sum_{i=2}^{\infty} i^2 P(X = i|C) \\
&= \frac{1}{3}\left(\sum_{i=2}^{\infty} i^2 2P(X = i-1|C) + P(X = i-1|W)\right) \\
&= \frac{1}{3}\left(\sum_{i=1}^{\infty} (i+1)^2 2P(X = i|C) + P(X = i|W)\right) \\
&= \frac{1}{3}\big(2(E[X^2|C] + 2E[X|C] + 1) + (E[X^2|W] + 2E[X|W] + 1)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}\left(2(E[X^2|C]+48+1)+(E[X^2|W]+18+1)\right)\\
&= \frac{1}{3}\left(2E[X^2|C]+E[X^2|W]+69\right),
\end{aligned}
$$

giving $E[X^2|C] = E[X^2|W] + 69$. A similar derivation gives $3E[X^2|W] = 51 + 2E[X^2|C]$. Solving this system of two linear equations in two variables gives $E[X^2|W] = 189$, $E[X^2|C] = 258$. Thus, since the quantity σ^2 in question is the variance of X given W, $\sigma^2 = E[X^2|W] - E[X|W]^2 = 189 - 81 = \boxed{108.}$

Problem 6: Counterexamples

A. Let X be a random variable such that $\forall$ positive integers i, $P(X = i) = \frac{6}{(\pi i)^2}$. Then,

$$\sum_{i=1}^{\infty} P(X=i) = \frac{6}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i^2} = 1,$$

but

$$E[X] = \sum_{i=1}^{\infty} i \cdot P(X=i) = \frac{6}{\pi^2}\sum_{i=1}^{\infty}\frac{1}{i} = \infty. \qquad \blacksquare$$

B. Let $X, Y : \{0,1\} \to [0,1]$ be random variables with the joint distribution below:

$P(X=i, Y=j)$	$i=0$	$i=1$
$j=0$	$1/6$	$1/2$
$j=1$	$1/12$	$1/4$

Clearly, X and Y are not independent. However, $E[XY] = \frac{1}{4} = \left(\frac{1}{12}+\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{4}\right) = E[X]E[Y]$.

Problem 7: Some Chebyshev on the side

A. Suppose that, for some $a, b \in \mathbb{R}$, a random variable X takes values a and b so that, for some $p \in [0,1]$, $P(X = a) = p$ and $P(X = b) = 1 - p$. Then, for $k = \frac{1}{\sqrt{1-p}}$, $\frac{1}{1+k^2} = p$, so that, for $\mu = E[X]$, $\sigma^2 = \mathrm{Var}(X)$, $P(X - \mu \geq k\sigma) = P(X \leq \mu) = p = \frac{1}{1+k^2}$, so that equality holds in the one-sided Chebyshev Inequality. $\blacksquare$

Problem 8: See what I mean

Let $X_1, X_2, \ldots$ be an infinite sequence of independent, identically distributed random variables with finite mean μ and finite variance σ^2. $\forall n \in \mathbb{N}$, let

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{n},$$

and let $\epsilon > 0$. Since $X_1, X_2, \ldots$ are independent, and expectation is linear, $\forall n \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{Var}\left(S_n\right) &= E\left[\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)^2\right] - E\left[\frac{X_1 + X_2 + \ldots + X_n}{n}\right]^2 \\
&= E\left[\frac{(X_1 + X_2 + \ldots + X_n)^2}{n^2}\right] - E\left[\frac{X_1 + X_2 + \ldots + X_n}{n}\right]^2 \\
&= \frac{1}{n^2}\left(E\left[(X_1 + X_2 + \ldots + X_n)^2\right] - E\left[X_1 + X_2 + \ldots + X_n\right]^2\right) \\
&= \mathrm{Var}\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) \\
&= \frac{1}{n^2}\left(\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_n)\right) \\
&= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}.
\end{aligned}$$

Also, by Linearity of Expectation, $E[S_n] = \mu$. Thus, by Chebyshev's Inequality, $\forall \epsilon_2 > 0$, for

$$n = \left(\frac{\epsilon\sigma}{\epsilon_2}\right)^2 + 1, \; k = \frac{\epsilon\sqrt{n}}{\sigma},$$

$$\begin{aligned}
P(|S_n - \mu| > \epsilon) &\leq P(|S_n - \mu| \geq \epsilon) \\
&= P(|S_n - \mu| \geq k\frac{\sigma}{\sqrt{n}}) \\
&\leq \frac{1}{k} = \frac{\sigma}{\epsilon\sqrt{n}} < \epsilon_2.
\end{aligned}$$

Thus, $\lim_{n\to\infty} P(|S_n - \mu| > \epsilon) = 0$. ■