Lecture Notes for Week 13 (First Draft)

Schwartz Space

In order to validate the computations involving Fourier transforms and derivatives, we want a function space that is invariant under differentiation and multiplication by polynomials, and has the property that its members are integrable.

Definition 13.1: Let

$$\mathcal{S}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \forall \alpha, \beta \in M_n, \ \sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha \phi(x)| < \infty\}.$$

$\mathcal{S}(\mathbb{R}^n)$ is known as *Schwartz space* or the space of *rapidly decreasing functions.*

Observe that if $\phi \in C^\infty(\mathbb{R}^n)$ then $\phi \in \mathcal{S}(\mathbb{R}^n)$ if and only if

$$P_\beta D^\alpha \phi \in L^\infty(\mathbb{R}^n) \ \text{ for all } \alpha, \beta \in M_n.$$

Moreover, it is clear that

$$\forall \phi \in \mathcal{S}(\mathbb{R}^n), \ \alpha, \beta \in M_n, \quad P_\beta D^\alpha \phi \in \mathcal{S}(\mathbb{R}^n).$$

Observe also that for every $p \in [1, \infty]$ we have

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n),$$

where $C_c^\infty(\mathbb{R}^n)$ is the space of C^∞-functions with compact support. A simple example of a function ψ that belongs to $\mathcal{S}(\mathbb{R}^n)$, but not to $C_c^\infty(\mathbb{R}^n)$ is provided by

$$\psi(x) = e^{-|x|^2} \ \text{ for all } x \in \mathbb{R}^n.$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$ we see that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for each $p \in [1, \infty)$. We shall topologize $\mathcal{S}(\mathbb{R}^n)$ a bit later on.

Lemma 13.2: Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in M_n$ be given. Then we have

(a) $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^n)$,

(b) $D^\alpha \widehat{\phi} = (-i)^{|\alpha|} (P_\alpha \phi)\widehat{\ }$,

(c) $(D^\alpha \phi)\widehat{\ } = i^{|\alpha|} P_\alpha \widehat{\phi}$.

Lemma 13.3 (Riemann-Lebesgue): Let $f \in L^1(\mathbb{R}^n)$ be given. Then $\widehat{f}$ vanishes at infinity, i.e.

$$\widehat{f}(\xi) \to 0 \quad \text{as } |\xi| \to \infty.$$

Proof: Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ we may choose a sequence $\{\phi_k\}_{k=1}^{\infty}$ in $\mathcal{S}(\mathbb{R}^n)$ such that $\|\phi_k - f\|_1 \to 0$ as $k \to \infty$. By (14) from Week 12, we have

$$\|\widehat{\phi_k} - \widehat{f}\|_\infty \to 0 \quad \text{as } k \to \infty.$$

The conclusion thus follows from the facts that $\widehat{\phi_k} \in \mathcal{S}(\mathbb{R}^n)$ and functions in $\mathcal{S}(\mathbb{R}^n)$ vanish at infinity. $\square$

Lemma 13.4: Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ be given. Then

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\phi}(\xi)\psi(\xi)\,d\xi = \int_{\mathbb{R}^n} \phi(x+y)\widehat{\psi}(y)\,dy.$$

Proof: Using the definition of Fourier transform we find that

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\phi}(\xi)\psi(\xi)\,d\xi &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-iz\cdot\xi}\phi(z)\psi(\xi)\,dz\,d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(z-x)\cdot\xi}\phi(z)\psi(\xi)\,d\xi\,dz \\
&= \int_{\mathbb{R}^n} \widehat{\psi}(x-z)\phi(z)\,dz \\
&= \int_{\mathbb{R}^n} \phi(x+y)\widehat{\psi}(y)\,dy. \quad \square
\end{aligned}$$

Theorem 13.5 (Fourier Inversion): Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then

$$\phi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\phi}(\xi)\,d\xi \quad \text{for all } x \in \mathbb{R}^n.$$

Proof: Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be given. For every $\epsilon > 0$ define $\psi_\epsilon : \mathbb{R}^n \to \mathbb{C}$ by

$$\psi_\epsilon(x) = \psi(\epsilon x) \quad \text{for all } x \in \mathbb{R}^n.$$

Let us compute $\widehat{\psi_\epsilon}$:

$$\begin{aligned}
\widehat{\psi_\epsilon}(y) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot y}\psi(\epsilon x)\,dx \\
&= \frac{\epsilon^{-n}}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i\frac{z}{\epsilon}\cdot y}\psi(z)\,dz \qquad (1) \\
&= \epsilon^{-n}\widehat{\psi}\left(\frac{y}{\epsilon}\right) \quad \text{for all } y \in \mathbb{R}^n.
\end{aligned}$$

Let $x \in \mathbb{R}^n$ be given. Using Lemma 13.4 and (1) we find that

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\phi}(\xi)\psi(\epsilon\xi)\,d\xi &= \epsilon^{-n}\int_{\mathbb{R}^n}\phi(x+y)\widehat{\psi}\left(\frac{y}{\epsilon}\right)\,dy \\
&= \int_{\mathbb{R}^n}\phi(x+\epsilon z)\widehat{\psi}(z)\,dz.
\end{aligned} \tag{2}$$

Letting $\epsilon \downarrow 0$ in (2), we obtain

$$\psi(0)\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\phi}(\xi)\,d\xi = \phi(x)\int_{\mathbb{R}^n}\widehat{\psi}(z)\,dz. \tag{3}$$

In order to complete the proof, we simply need to substitute one cleverly chosen function $\psi \in \mathcal{S}(\mathbb{R}^n)$ into (3). We shall make use of the following standard facts that are not too difficult to verify. (A proof of the claim below will be a homework exercise.)

Claim: Let

$$\psi(x) = e^{-\frac{1}{2}|x|^2} \quad \text{for all } x \in \mathbb{R}^n.$$

Then

$$\widehat{\psi} = \psi \quad \text{and} \quad \int_{\mathbb{R}^n}\widehat{\psi} = (2\pi)^{\frac{n}{2}}.$$

Substituting the function ψ from the claim (and observing that $\psi(0) = 1$) into (3) we obtain

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\phi}(\xi)\,d\xi = (2\pi)^{\frac{n}{2}}\phi(x). \quad \square$$

Taking the Fourier transform of a function twice almost returns the original function, but not quite. The result is a “reflection” of the original function. For this reason, it is helpful to introduce a reflection operator $\check{}$.

Definition 13.6 (Reflection): Given $\phi \in \mathcal{S}(\mathbb{R}^n)$, define $\check{\phi} \in \mathcal{S}(\mathbb{R}^n)$ by

$$\check{\phi}(x) = \phi(-x) \quad \text{for all } x \in \mathbb{R}^n.$$

Remark 13.7:

(a) The Fourier inversion formula can be rewritten as

$$\check{\phi} = \widehat{\widehat{\phi}} \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

(b) It is also useful to observe that

$$\check{\widehat{\phi}} = \widehat{\check{\phi}} \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Lemma 13.8: Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then

$$\int_{\mathbb{R}^n} \phi\widehat{\psi} = \int_{\mathbb{R}^n} \widehat{\phi}\psi.$$

Proof: Put $x = 0$ in Lemma 13.4. $\square$

Proposition 13.9: Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then

$$\overline{\psi} = \widehat{\widehat{\overline{\psi}}}.$$

Lemma 13.10 (Parseval's Relation): Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then

$$\int_{\mathbb{R}^n} \phi\overline{\psi} = \int_{\mathbb{R}^n} \widehat{\phi}\overline{\widehat{\psi}}.$$

Corollary 13.11: Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then

$$\|\phi\|_2 = \|\widehat{\phi}\|_2,$$

where $\|\cdot\|_2$ is the norm on $L^2(\mathbb{R}^n)$.

Example 13.12: Let $f \in \mathcal{S}(\mathbb{R}^n)$ be given. Consider the equation

$$-\Delta u(x) + u(x) = f(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{4}$$

Here Δu denotes the Laplacian of u, i.e.

$$\Delta u(x) = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}(x).$$

Let us look for solutions $u \in \mathcal{S}(\mathbb{R}^n)$ of (4). Observe that for $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(\Delta u)\widehat{\ }(\xi) = -|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

It is convenient to put

$$P(\xi) = 1 + |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Then $u \in \mathcal{S}(\mathbb{R}^n)$ satisfies (4) if and only if

$$P\widehat{u} = \widehat{f},$$

or

$$\widehat{u} = \frac{\widehat{f}}{P}.$$

It is easy to see that

$$\frac{\widehat{f}}{P} \in \mathcal{S}(\mathbb{R}^n).$$

We conclude that (4) has exactly one solution $u \in \mathcal{S}(\mathbb{R}^n)$. Moreover the solution is given by

$$\breve{u} = \left(\frac{\widehat{f}}{p}\right)^{\widehat{}}.$$

Tempered Distributions

We now endow $\mathcal{S}(\mathbb{R}^n)$ with a natural topology. For each $N \in \mathbb{N} \cup \{0\}$ put

$$|||\phi|||_N = \sum_{|\alpha|,|\beta| \le N} \|P_\beta D^\alpha \phi\|_\infty \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Then $(|||\cdot|||_N | N \in \{0\} \cup \mathbb{N})$ is a separating family of seminorms on $\mathcal{S}(\mathbb{R}^n)$. (In fact each $|||\cdot|||_N$ is actually a norm.) This family of seminorms induces a topology that turns $\mathcal{S}(\mathbb{R}^n)$ into a locally convex topological vector space. The topology is generated by the translation invariant metric

$$\rho(\phi, \psi) = \sum_{k=0}^{\infty} \frac{2^{-k} |||\phi - \psi|||_k}{1 + |||\phi - \psi|||_k} \text{ for all } \phi, \psi \in \mathcal{S}(\mathbb{R}^n).$$

Remark 13.13: Let a sequence $\{\phi_k\}_{k=1}^{\infty}$ in $\mathcal{S}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then $\phi_k \to \phi$ as $k \to \infty$ in $\mathcal{S}(\mathbb{R}^n)$ if and only if

$$\forall \alpha, \beta \in M_n, \quad P_\beta D^\alpha \phi_k \to P_\beta D^\alpha \phi \text{ uniformly on } \mathbb{R}^n \text{ as } k \to \infty.$$

It is straightforward to show that $(\mathcal{S}(\mathbb{R}^n), \rho)$ is complete.

Lemma 13.14: Let X be a Banach space and assume that $L : \mathcal{S}(\mathbb{R}^n) \to X$ is linear. Then L is continuous if and only if there exist $N \in \mathbb{N} \cup \{0\}$ and $K \in \mathbb{R}$ such that

$$\|L\phi\|_X \le K |||\phi|||_N \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Definition 13.15: By a *tempered distribution*, we mean a continuous linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The set of all tempered distributions will be denoted by $\mathcal{S}'(\mathbb{R}^n)$.

In the context of tempered distributions, the elements of $\mathcal{S}(\mathbb{R}^n)$ are often referred to as *test functions*. We write $\langle \cdot, \cdot \rangle : \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ for the duality pairing. Given $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ we write either $u(\phi)$ or $\langle u, \phi \rangle$ for the value of u at ϕ.

Example 13.16: A nonnegative Borel measure μ on $\mathbb{R}^n$ is said to be of *slow growth* provided that there exists $N \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n} \frac{d\mu(x)}{(1+|x|^2)^N} < \infty.$$

Every such measure induces a tempered distribution l_μ through the formula

$$\langle l_\mu, \phi \rangle = \int_{\mathbb{R}^n} \phi d\mu \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

When there is no danger of confusion, it is customary to use the same symbol "μ" for both the measure and the associated tempered distribution.

In order to define operations on tempered distributions, we are going to need continuity of basic operations on functions in $\mathcal{S}(\mathbb{R}^n)$. It is useful to observe that $\mathcal{S}(\mathbb{R}^n)$ is invariant under multiplication by by a class of functions that is somewhat larger than the class of polynomials. The following definition is convenient.

Definition 13.17: Let

$$\mathcal{PGD}(\mathbb{R}^n) = \{\psi \in C^\infty(\mathbb{R}^n) : \forall \alpha \in M_n,\ \exists N \in \mathbb{N},\ \lim_{|x|\to\infty} |x|^{-N} |D^\alpha \psi(x)| = 0\}.$$

Functions in $\mathcal{PGD}(\mathbb{R}^n)$ are said to be of *slow growth at infinity.*

It is immediate that $P_\alpha \in \mathcal{PGD}(\mathbb{R}^n)$ for every $\alpha \in M_n$. There does not seem to be a "standard notation" for this space of functions. I chose $\mathcal{PGD}$ with *Polynomial Growth of Derivatives* in mind. Using Leibniz product rule it is easy to see that if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{PGD}(\mathbb{R}^n)$ then $\psi\phi \in \mathcal{S}(\mathbb{R}^n)$.

Proposition 13.18: Let $\alpha \in M_n$ and $\psi \in \mathcal{PGD}(\mathbb{R}^n)$ be given. Then the mappings

- $\phi \to \widehat{\phi}$
- $\phi \to \check{\phi}$
- $\phi \to \psi\phi$
- $\phi \to D^\alpha \phi$

are continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Let $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^n)$ be given. Then the linear functional $L_f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ defined by

$$L_f(\phi) = \int_{\mathbb{R}^n} f\phi \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n),$$

is a tempered distribution. Moreover the mapping $f \to L_f$ is linear and injective from $L^p(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. [Of course, $f \to L_f$ is linear and injective from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.]

We want to extend the notions of differentiation, Fourier transform, etc. to tempered distributions. The philosophy behind taking an operation that is defined for smooth functions and extending the definition of this operation to tempered distributions is that operation should commute with the natural injection $f \to L_f$. For example to define the Fourier transform of a tempered distribution, we observe that if $f \in \mathcal{S}(\mathbb{R}^n)$ then (using Lemma 13.8) we have

$$\begin{aligned}\langle L_{\widehat{f}}, \phi\rangle &= \int_{\mathbb{R}^n} \widehat{f}\phi = \int_{\mathbb{R}^n} f\widehat{\phi} \\ &= \langle L_f, \widehat{\phi}\rangle \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$

It is therefore natural to define the Fourier transform $\widehat{u} \in \mathcal{S}'(\mathbb{R}^n)$ of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \widehat{u}, \phi\rangle = \langle u, \widehat{\phi}\rangle \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

To define the reflection $\check{u}$ of a tempered distribution u, we observe that for $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\langle L_{\check{f}}, \phi\rangle = \int_{\mathbb{R}^n} \check{f}\phi = \int_{\mathbb{R}^n} f\check{\phi} \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

(Notice that there is no minus sign associated with changing variables in the integrals above because when putting $y_k = -x_k$ we get $dy_k = -dx_k$ but then we also must interchange the upper and lower limits of integration for the k^{th} variable.) This makes it natural to define

$$\langle \check{u}, \phi\rangle = \langle u, \check{\phi}\rangle \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Let $\alpha \in M_n$, and $\psi \in \mathcal{S}(\mathbb{R}^n)$ be given In order to define $D^\alpha u$, $P_\alpha u$ and ψu, we observe that

$$\langle D^\alpha f, \phi\rangle = \int_{\mathbb{R}^n} (D^\alpha \phi)\phi = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(D^\alpha \phi) \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n). \tag{5}$$

$$\int_{\mathbb{R}^n} (P_\alpha f)\phi = \int_{\mathbb{R}^n} f(P_\alpha \phi) \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n),$$

$$\int_{\mathbb{R}^n} (\psi f)\phi = \int_{\mathbb{R}^n} f(\psi \phi) \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Integration by parts was used to obtain (5). We summarize these ideas in the following definition.

Definition 13.19: Let $\alpha \in M_n$, $\psi \in \mathcal{PGD}(\mathbb{R}^n)$, and $u \in \mathcal{S}'(\mathbb{R}^n)$ be given. We define $\psi u, D^\alpha u, \widehat{u}, \check{u} \in \mathcal{S}'(\mathbb{R}^n)$ by

(a) $\langle \psi u, \phi\rangle = \langle u, \psi\phi\rangle$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

(b) $\langle D^\alpha u, \phi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha \phi\rangle$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

(c) $\langle \widehat{u}, \phi \rangle = \langle u, \widehat{\phi} \rangle$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

(d) $\langle \check{u}, \phi \rangle = \langle u, \check{\phi} \rangle$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

We refer to $D^\alpha u$ as a *distributional derivative* (or *weak derivative*) of u and to $\widehat{u}$ as the *Fourier transform* of u.

Theorem 13.20: Let $\alpha \in M_n$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ be given. Then

(a) $(\widehat{u})\widehat{\ } = \check{u}$,

(b) $(\widehat{u})\check{\ } = (\check{u})\widehat{\ }$,

(c) $D^\alpha \widehat{u} = (-i)^{|\alpha|}(P_\alpha u)\widehat{\ }$,

(d) $(D^\alpha u)\widehat{\ } = i^{|\alpha|} P_\alpha \widehat{u}$.

Example 13.21 (Dirac Delta Function): Consider the tempered distribution δ_0 defined by

$$\langle \delta_0, \phi \rangle = \phi(0) \;\text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Let's find the Fourier transform of δ_0. Using Definitions 13.19 and 12.14 we see that

$$\langle \widehat{\delta_0}, \phi \rangle = \langle \delta_0, \widehat{\phi} \rangle = \widehat{\phi}(0) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi \;\text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

It is customary to write this formula as

$$\widehat{\delta_0} = \frac{1}{(2\pi)^{\frac{n}{2}}}.$$

Observe that

$$\check{\delta_0} = \delta_0,$$

so using Theorem 13.20, we see that

$$\delta_0 = (\widehat{\delta_0})\widehat{\ } = \frac{1}{(2\pi)^{\frac{n}{2}}} \widehat{1},$$

which can be rewritten as

$$\widehat{1} = (2\pi)^{\frac{n}{2}} \delta_0.$$

[Here I have made a slight (and very common) abuse of notation by using the same symbol "1" to indicate the number 1 and the constant function whose only value is 1.]

Example 13.22 (Heaviside Function): Let $n = 1$ and define $H : \mathbb{R} \to \mathbb{C}$ by

$$H(x) = \begin{cases} 0 \text{ for } x < 0, \\ 1 \text{ for } x \geq 0. \end{cases}$$

Let us compute the distributional derivative H' of H. For $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}\langle H', \phi\rangle &= -\langle H, \phi'\rangle \\ &= -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx = -\int_{0}^{\infty} \phi'(x)\,dx \\ &= \phi(0) = \langle \delta_0, \phi\rangle,\end{aligned}$$

where δ_0 is the one-dimensional version of the Dirac Delta from Example 13.21. In other words, the distributional derivative of the Heaviside step function is the Dirac delta, i.e. $H' = \delta_0$.

Example 13.23 (Fourier Transform on $L^2(\mathbb{R}^n)$.) Let $f \in L^2(\mathbb{R}^n)$ be given and consider the associated tempered distribution $L_f$defined by

$$\langle L_f, \phi\rangle = \int_{\mathbb{R}^n} f\phi \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Then we have

$$\langle \widehat{L_f}, \phi\rangle = \langle L_f, \widehat{\phi}\rangle = \int_{\mathbb{R}^n} f\widehat{\phi} \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Using Holder's Inequality and Corollary 13.11, we find that

$$|\langle \widehat{L_f}, \phi\rangle| \leq \|f\|_2\|\widehat{\phi}\|_2 = \|f\|_2\|\phi\|_2 \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

By the Riesz Representation Theorem (and the fact that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$), there is exactly one function $g \in L^2(\mathbb{R}^n)$ such that

$$\langle \widehat{L_f}, \phi\rangle = \int_{\mathbb{R}^n} g\phi \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Moreover this function g satisfies $\|g\|_2 = \|f\|_2$. (It is customary to write $\widehat{f} = g$.) In other words, the Fourier transform can be defined in a natural way as an isometric linear mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. This mapping is surjective and consequently it is unitary. The content of this example is known as Plancherel's Theorem.

Remark 13.24: Since (for every $p \in [1, \infty]$) $L^p(\mathbb{R}^n)$ can be identified with a subset of $\mathcal{S}'(\mathbb{R}^n)$ we can talk about the behavior of the Fourier transform on $L^p(\mathbb{R}^n)$. As before, let us write $\mathcal{F}(f) = \widehat{f}$. It is a classical result of Titchmarsh that for

$$p \in [1, 2], \quad q = \frac{p}{p-1}$$

the Fourier transform $\mathcal{F}$ maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ continuously and injectively. Moreover this mapping is surjective if and only if $p = q = 2$.

Convolution

Definition 13.25: Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be given. The convolution of ϕ with ψ is the function $\phi * \psi : \mathbb{R}^n \to \mathbb{C}$ defined by

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n} \phi(y)\psi(x-y)\,dy \quad \text{for all } x \in \mathbb{R}^n.$$

Lemma 13.26: Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in M_N$ be given. Then $\phi * \psi = \psi * \phi$, $\phi * \psi \in \mathcal{S}(\mathbb{R}^n)$, and

$$D^\alpha(\phi * \psi) = (D^\alpha \phi) * \psi = \phi * (D^\alpha \psi). \tag{6}$$

Proof: Observe first that

$$(\phi * \psi)(x) = \int_{\mathbb{R}^n} \phi(x-z)\psi(z)\,dz = (\psi * \phi)(x) \quad \text{for all } x \in \mathbb{R}^n.$$

It follows easily that $\phi * \psi \in C^\infty(\mathbb{R}^n)$ and that (6) holds. We need to verify that $\phi * \psi \in \mathcal{S}(\mathbb{R}^n)$. The definition of $\mathcal{S}(\mathbb{R}^n)$ implies that

$$\forall \alpha \in M_n, \quad P_\alpha \psi \in L^1(\mathbb{R}^n).$$

Let us put

$$C_\alpha(\psi) = \|P_\alpha \psi\|_1 \quad \text{for all } \alpha \in M_n$$

and let $\beta \in M_n$ and $x \in \mathbb{R}^n$ be given. Then we have

$$\begin{aligned}
|x^\beta(\phi * \psi)(x)| &\leq \int_{\mathbb{R}^n} |x^\beta D^\alpha \phi(x-y)\psi(y)|\,dy \\
&\leq \int_{\mathbb{R}^n} |((x-y)+y)^\beta D^\alpha \phi(x-y)\psi(y)|\,dy \\
&\leq \sum_{\gamma \leq \beta} \int_{\mathbb{R}^n} \binom{\beta}{\gamma} |(x-y)^\gamma D\alpha\phi(x-y)\psi(y)|\,dy \\
&\leq \sum_{\gamma \leq \beta} \|P_\gamma D^\alpha\|_\infty C_{\beta-\gamma}(\psi)
\end{aligned} \tag{7}$$

It follows from (7) that

$$\sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha(\phi * \psi)(x)| < \infty,$$

and consequently $\phi * \psi \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Lemma 13.27: Let $\phi, \psi, \chi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then we have

$$\phi * (\psi * \chi) = (\phi * \psi) * \chi.$$

Proof: For every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
((\phi * \psi) * \chi)(x) &= \int_{\mathbb{R}^n} (\phi * \psi)(y)\chi(x-y)\,dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(z)\psi(y-z)\chi(x-y)\,dz\,dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(z)\psi(y-z)\chi(x-y)\,dy\,dz \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(z)\psi(s)\chi(x-s-z)\,ds\,dz \\
&= \int_{\mathbb{R}^n} \phi(z)(\phi * \chi)(x-z)\,dz \\
&= (\phi * (\psi * \chi))(x). \quad \square
\end{aligned}$$

Lemma 13.28: Let $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be given. Then we have

(a) $(\phi * \psi)\widehat{\ } = (2\pi)^{\frac{n}{2}}\widehat{\phi}\widehat{\psi}$,

(b) $\widehat{\phi} * \widehat{\psi} = (2\pi)^{\frac{n}{2}}\widehat{\phi\psi}$.

Proof: (a) Let $\xi \in \mathbb{R}^n$ be given and observe that

$$\begin{aligned}
(\phi * \psi)\widehat{\ }(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\left(\int_{\mathbb{R}^n}\phi(y)\psi(x-y)\,dy\right)dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i(x-y)\cdot\xi}e^{-iy\cdot\xi}\phi(y)\psi(x-y)\,dx\,dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iz\cdot\xi}e^{-iy\cdot\xi}\phi(y)\psi(z)\,dz\,dy \\
&= (2\pi)^{\frac{n}{2}}\widehat{\phi}\widehat{\psi}.
\end{aligned}$$

For part (b), we apply (a) replacing ϕ with $\widehat{\phi}$ and ψ with $\widehat{\psi}$. This gives

$$\begin{aligned}
(\widehat{\phi} * \widehat{\psi})^{\wedge} &= (2\pi)^{\frac{n}{2}} \widehat{\widehat{\phi}}\widehat{\widehat{\psi}} \\
&= (2\pi)^{\frac{n}{2}} \check{\phi}\check{\psi} \\
&= (2\pi)^{\frac{n}{2}} (\phi\psi)^{\vee} \\
&= \widehat{\widehat{\phi\psi}}.
\end{aligned}$$

The conclusion now follows by applying the inverse transform. $\square$

Translation

In order to extend the definition of convolution of two functions in $\mathcal{S}(\mathbb{R}^n)$ to convolution of a tempered distribution with a function in $\mathcal{S}(\mathbb{R}^n)$ it is convenient to introduce a translation operator.

Definition 13.29: Given $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ define $\tau_h\phi \in \mathcal{S}(\mathbb{R}^n)$ by

$$(\tau_h\phi)(x) = \phi(x-h) \;\text{ for all } x \in \mathbb{R}^n.$$

Since we have

$$\int_{\mathbb{R}^n} \phi(x-h)\psi(x)\,dx = \int_{\mathbb{R}^n} \phi(y)\psi(y+h)\,dy$$

for all $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and all $h \in \mathbb{R}^n$, it is appropriate to make the following definition.

Definition 13.30: Given $u \in \mathcal{S}'(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ define $\tau_h u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \tau_h u, \phi\rangle = \langle u, \tau_{-h}\phi\rangle \;\text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

For the sake of completeness, we record below some simple results concerning the interaction of translations with Fourier transforms. We begin with a simple definition.

Definition 13.31: For each $z \in \mathbb{R}$ define $e_z : \mathbb{R}^n \to \mathbb{C}$ by

$$e_z(x) = e^{iz\cdot x} \;\text{ for all } x \in \mathbb{R}^n. \tag{8}$$

Observe that for each $z \in \mathbb{R}^n$ we have $e_z \in \mathcal{PGD}(\mathbb{R}^n)$ so we can talk about the product $e_z u$ when u is a tempered distribution.

Proposition 13.32: Let $h \in \mathbb{R}^n$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}'(\mathbb{R}^n)$ be given. Then we have

(a) $\tau_h \widehat{\phi} = (e_h f)\widehat{\ }$

(b) $(\tau_h \phi)\widehat{\ } = e_{-h} \widehat{\phi}$

(c) $\tau_h \widehat{u} = (e_h f)\widehat{\ }$

(d) $(\tau_h u)\widehat{\ } = e_{-h} \widehat{u}$.

The proof of Proposition 13.32 is left as an elementary exercise.

Convolution of a Tempered Distribution with a Test Function

Observe that for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} (\phi * \psi)(x) &= \int_{\mathbb{R}^n} \phi(y)\psi(x-y)\,dy \\ &= \int_{\mathbb{R}^n} \phi(y)\check{\phi}(y-x)\,dy \\ &= \int_{\mathbb{R}^n} \phi(y)(\tau_x \check{\psi})(y)\,dy \quad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

It is therefore natural to make the following definition:

Definition 13.33: Given $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ define $u * \phi : \mathbb{R}^n \to \mathbb{C}$ by

$$(u * \phi)(x) = \langle u, \tau_x \check{\phi} \rangle \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 13.34: Let $u \in \mathcal{S}(\mathbb{R}^n)$, $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$, and $\alpha \in M_n$ be given. Then we have

(a) $u * \phi \in \mathcal{PGD}(\mathbb{R}^n)$,

(b) $D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$,

(c) $(u * \phi)\widehat{\ } = (2\pi)^{\frac{n}{2}} \widehat{\phi}\widehat{u}$,

(d) $\widehat{u} * \widehat{\phi} = (2\pi)^{\frac{n}{2}} \phi u$,

(e) $u * (\phi * \psi) = (u * \phi) * \psi$.

Sobolev Spaces

Definition 13.35 Let $m \in \mathbb{N} \cup \{0\}$, $p \in [1, \infty]$ be given and put

$$W^{m,p}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n),\ |\alpha| \leq m\}.$$

We equip $W^{m,p}(\mathbb{R}^n)$ with the norm given by

$$|||u|||_{m,p} = \begin{cases} \left(\sum_{|\alpha| leqm} \|u\|_p^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \sum_{|\alpha| \leq m} \|D^\alpha u\|_\infty & \text{if } p = \infty. \end{cases}$$

Here $\| \cdot \|_p$ is the norm on $L^p(\mathbb{R}^n)$.

Remark 13.36: Using the fact the all norms on a finite-dimensional linear space are equivalent, it is easy to construct equivalent norms on $W^{m,p}(\mathbb{R}^n)$ simply by applying different norms to the finite list

$$(\|D^\alpha u\|_p | \alpha \in M_n,\ |\alpha| \leq m)$$

of real numbers. However, one must be aware that changing to one of these equivalent norms can destroy geometric properties such as uniform convexity. With the norm $|||\cdot|||_{m,p}$ as above, $W^{m,p}(\mathbb{R}^n$ is uniformly convex when $1 < p < \infty$.

Proposition 13.37: Let $m \in \mathbb{N} \cup \{0\}$ and $p\ [1, \infty]$ be given. Then $W^{m,p}(\mathbb{R}^n)$ is a Banach space.

Proof: That $W^{m,p}(\mathbb{R}^n)$ is a linear space and that $|||\cdot|||_{m,p}$ is a norm is clear. We need to check completeness. Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in $W^{m,p}(\mathbb{R}^n)$. Since $L^p(\mathbb{R}^n)$ is complete, we may choose functions $v_\alpha \in L^p(\mathbb{R}^n)$, $\alpha \in M_n$, $|\alpha| \leq m$ such that

$$\forall \alpha \in M_n,\ |\alpha| \leq m,\quad \|D^\alpha u_k - v^\alpha\|_p \to 0 \text{ as } k \to \infty.$$

Let us put $u = v_0$. We need to show that $D^\alpha u = v_\alpha$ for all multi-indices α with $|\alpha| \leq m$.

Let $\alpha \in M_n$ with $|\alpha| \leq m$ be given. Since strong convergence in $L^p(\mathbb{R}^n)$ implies weak convergence (when $1 \leq p < \infty$) and weak* convergence (when $p = \infty$) and because $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, we know that for every $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle D^\alpha u_k, \phi \rangle = \int_{\mathbb{R}^n} (D^\alpha u_k)\phi \to \int_{\mathbb{R}^n} v_\alpha \phi = \langle v_\alpha, \phi \rangle \quad \text{as } k \to \infty.$$

We also know that

$$\begin{aligned} \langle D^\alpha u_k, \phi \rangle &= (-1)^{|\alpha|} \langle u_k, D^\alpha \phi \rangle \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} u_k D^\alpha \phi \\ &\to (-1)^{|\alpha|} \int_{\mathbb{R}^n} u D^\alpha \phi = \langle D^\alpha u, \phi \rangle \quad \text{as } k \to \infty. \end{aligned}$$

It follows that $v_\alpha = D^\alpha u$. $\square$

Using the fact that $W^{m,p}(\mathbb{R}^n)$ can be identified with a closed subspace of a finite product of $L^p(\mathbb{R}^n)$ spaces, we can infer that

- $W^{m,p}(\mathbb{R}^n)$ is separable if $1 \le p < \infty$, and
- $W^{m,p}(\mathbb{R}^n)$ is reflexive if $1 < p < \infty$.

Notice that $W^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and that $W^{m,2}(\mathbb{R}^n)$ is a Hilbert space with inner product

$$(\!(u,v)\!)_m = \sum_{|\alpha|\le m} \int_{\mathbb{R}^n} D^\alpha u \overline{D^\alpha v} \quad \text{for all } u, v \in W^{m,2}(\mathbb{R}^n).$$

The spaces $W^{m,p}(\mathbb{R}^n)$ are special instances of a class of spaces known as *Sobolev spaces*. Such spaces are extremely useful in the study of partial differential equations and calculus of variations. There are entire books (e.g. Adams) devoted to Sobolev spaces and we can only begin to scratch the surface here.

We shall prove a couple of representative results concerning Sobolev spaces in the special case $p = 2$ using Fourier transforms. For each $s \in \mathbb{R}$ put

$$Q_s(\xi) = (1 + |\xi|^2)^s \quad \text{for all } \xi \in \mathbb{R}^n. \tag{9}$$

Let $m \in \mathbb{N} \cup \{0\}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ be given. Observe that (i) through (iv) below are equivalent

(i) $u \in W^{m,2}(\mathbb{R}^n)$

(ii) $D^\alpha u \in L^2(\mathbb{R}^n)$ for all multi-indices α with $|\alpha| \le m$.

(iii) $(D^\alpha u)\widehat{\ } = i^{|\alpha|} P_\alpha \widehat{u} \in L^2(\mathbb{R}^n)$ for all multi-indices α with $|\alpha| \le m$.

By elementary algebra, (iii) holds if and only if (iv) below holds:

(iv) $Q_{\frac{m}{2}} \widehat{u} \in L^2(\mathbb{R}^n)$.

The idea is that all powers ξ^α for $|\alpha| \le m$ can be controlled by $Q_{\frac{m}{2}}(\xi)$. It is extremely interesting to observe that (iv) makes sense for arbitrary real numbers m (positive or negative) and not just integers.

Definition 13.38: For each $s \in \mathbb{R}$ let

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : Q_{\frac{s}{2}} \widehat{u} \in L^2(\mathbb{R}^n\},$$

equipped with the inner product defined by

$$(u,v)_s = \int_{\mathbb{R}^n} Q_s \widehat{u}\overline{\widehat{v}} \text{ for all } u,v \in H^s(\mathbb{R}^n),$$

and the associated norm

$$\|u\|_{s,2} = \sqrt{(u,u)_s} \text{ for all } u \in H^s(\mathbb{R}^n).$$

Remark 13.39: Let $m \in \mathbb{N} \cup \{0\}$. Then $H^m(\mathbb{R}^n) = W^{m,2}(\mathbb{R}^n)$ and the norm $\|\cdot\|_{m,2}$ is equivalent to $|||\cdot|||_{m.2}$.

When $s \geq 0$ the elements of $H^s(\mathbb{R}^n)$ can be identified with functions. The functions become more regular as s increases. For $s < 0$, the elements of $H^s(\mathbb{R}^n)$ are tempered distributions, but they need not be associated with functions. It is instructive to look at an example.

Example 13.40 (Dirac Delta): Consider the tempered distribution δ_0 defined by

$$\langle \delta_0, \phi \rangle = \phi(0) \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

In Example x.x we saw that

$$\widehat{\delta_0} = \frac{1}{(2\pi)^{\frac{n}{2}}}.$$

Let $s \in \mathbb{R}$ be given. In order to have $\delta_0 \in H^s(\mathbb{R}^n)$ it is necessary and sufficient that $Q_{\frac{s}{2}} \in L^2(\mathbb{R}^n)$, i.e. that

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s \, d\xi < \infty.$$

Switching to polar coordinates, we find that the integral above is equal to

$$C_n \int_0^\infty (1+r^2)^s r^{n-1} \, dr, \tag{10}$$

where C_n is the "surface area" of the unit sphere in $\mathbb{R}^n$. The integral in (10) converges if and only if $s < -\frac{n}{2}$. We conclude that

$$\delta_0 \in H^s(\mathbb{R}^n) \iff s < -\frac{n}{2}.$$

Theorem 13.41 (Sobolev Embedding Theorem – Special Case): Let $s > \frac{n}{2}$ be given. Then

$$H^s(\mathbb{R}^n) \hookrightarrow C_0(\mathbb{R}^n). \tag{11}$$

Proof: Let $u \in H^s(\mathbb{R}^n)$ be given and observe that

$$Q_{-\frac{s}{2}} \widehat{u} \in L^2(\mathbb{R}^n). \tag{12}$$

Observe further that

$$\widehat{u} = (Q_{\frac{s}{2}}\widehat{u})Q_{-\frac{s}{2}}. \tag{13}$$

We want to show that $Q_{-\frac{s}{2}} \in L^2(\mathbb{R}^n)$, because then it will follow from (21), (13) that $\widehat{u} \in L^1(\mathbb{R}^n)$ and we will also get a useful bound for $\|\widehat{u}\|_1$. Using polar coordinates, we have

$$\int_{\mathbb{R}^n} Q_{-s} = C_n \int_0^\infty (1+r^2)^{-s} r^{n-1} \tag{14}$$

where C_n is the "surface area" of the unit sphere in $\mathbb{R}^n$. [The integrand is behaving like r^{n-2s-1} as $r \to \infty$.] Since $s > \dfrac{n}{2}$ the integral in (14) is finite. Let us put

$$K = \|Q_{-\frac{s}{2}}\|_2.$$

(Here $\|\cdot\|_2$ is the L^2-norm.) Then, by (13) and Holder's inequality, we have

$$\|\widehat{u}\|_1 \le K\|u\|_{s,2}. \tag{15}$$

By Theorem 13.20 we have

$$\breve{u} = \widehat{\widehat{u}} \tag{16}$$

and consequently we have $\breve{u} \in C_0(\mathbb{R}^n)$ and

$$\|\breve{u}\|_\infty \le \frac{1}{(2\pi)^{\frac{n}{2}}}\|\widehat{\widehat{u}}\|_1 \ \text{ for all } u \in H^S(\mathbb{R}^n). \tag{17}$$

It follows that $u \in C_0$ and

$$\|u\|_\infty \le \frac{K}{(2\pi)^{\frac{n}{2}}}\|u\|_{s,2} \ \text{ for all } u \in H^S(\mathbb{R}^n)$$

and the proof is complete. $\square$

Theorem 13.42 (Interpolation Inequality): Let $s, t \in \mathbb{R}$ with $0 < s < t$ be given. Then for every $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\|u\|_{s,2}^2 \le \epsilon\|u\|_{t,2}^2 + C(\epsilon)\|u\|_{0,2}^2 \ \text{ for all } u \in H^t(\mathbb{R}^n). \tag{18}$$

Proof: Recall Young's Inequality which says that for

$$p \in (1,\infty), \quad q = \frac{p}{p-1},$$

we have

$$AB \le \frac{A^p}{p} + \frac{B^q}{q} \ \text{ for all } A, B \ge 0. \tag{19}$$

Put

$$p = \frac{t}{s}, \quad q = \frac{t}{t-s}. \tag{20}$$

Let $\epsilon > 0$, $a \geq 0$ be given. Then using (19) with p, q as in (20) we have

$$a = \epsilon^{\frac{s}{t}} a \cdot \epsilon^{-\frac{s}{t}} 1 \leq \frac{s}{t} \epsilon a^{\frac{t}{s}} + \left(\frac{t-s}{t} \right) \epsilon^{-\frac{s}{t-s}}. \tag{21}$$

If we put

$$C(\epsilon) = \frac{t-s}{t} \epsilon^{-\frac{s}{t-s}},$$

and notice that $\dfrac{s}{t} < 1$ we infer from (**??**) that

$$a \leq \epsilon^{\frac{t}{s}} + C(\epsilon) \quad \text{for all } a \geq 0. \tag{22}$$

It follows from (22) that

$$Q_s(\xi) \leq \epsilon Q_t(\xi) + C(\epsilon) \quad \text{for all } \xi \in \mathbb{R}^n. \tag{23}$$

Given $u \in H^t(\mathbb{R}^n)$, we multiply (23) by $|\widehat{u}(\xi)|^2$ to obtain

$$Q_s(\xi) |\widehat{u}(\xi)|^2 \leq \epsilon Q_t(\xi) |\widehat{u}(\xi)|^2 + C(\epsilon) |\widehat{u}(\xi)|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{24}$$

Integration of (24) over $\mathbb{R}^n$ produces the desired conclusion. $\square$

Example 13.43: Let $f \in L^2(\mathbb{R}^n)$ be given. We seek $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$-\Delta u + u = f. \tag{25}$$

For $u \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$(-\Delta u + u)\widehat{\ } = Q_1 \widehat{u}.$$

Consequently u satisfies (25) if and only if

$$\widehat{u} = \frac{\widehat{f}}{Q_1} = Q_{-1} \widehat{f}. \tag{26}$$

It is easy to see that $Q_{-1} \in \mathcal{PGD}(\mathbb{R}^n)$ which implies that $Q_{-1}\widehat{f} \in \mathcal{S}'(\mathbb{R}^n)$. It follows that (25) has exactly one solution $u \in \mathcal{S}'(\mathbb{R}^n)$ and this solution is given by

$$\check{u} = (Q_{-1}\widehat{f})\widehat{\ }.$$

Since $f \in L^2(\mathbb{R}^n)$ we can conclude from (26) that the solution u of (25) actually satisfies $u \in H^2(\mathbb{R}^n)$.

The Wave Equation

Let us consider that initial-value problem

$$\begin{cases} u_{tt}(t,x) = \Delta u(t,x), \quad t \geq 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{27}$$

where $u_0, v_0 : \mathbb{R}^n \to \mathbb{C}$ are given functions. Here Δ is the Laplacian with respect to the spatial variables x. The partial differential equation $u_{tt} = \Delta u$ is called the *wave equation.*

Before choosing function spaces for this problem, we shall derive a formal "energy equation". Such energy equations can reveal a great deal information. Before proceeding, we observe that if $J \subset \mathbb{R}$ is an interval and $z : J \to \mathbb{C}$ is differentiable then we have

$$\begin{aligned} \frac{d}{dt}|z(t)|^2 &= 2z(t)\dot{\overline{z}}(t) \\ &= 2\overline{z}(t)\dot{z}(t). \end{aligned}$$

(The derivative of the function $t \to |z(t)|^2$ is real-valued and $\mathrm{Re}(\alpha\overline{\beta}) = \mathrm{Re}(\overline{\alpha}\beta)$ for all complex numbers α, β.)

Assuming that (27) has a sufficiently regular solution u, we multiply the equation by $\overline{u}_t$ and integrate (with respect to x) over $\mathbb{R}^n$, using integration by parts to obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |u_t(t,x)|^2\,dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u(t,x)|^2\,dx.$$

Integrating the above expression with respect to time and using the initial conditions, we find that

$$\frac{1}{2}\int_{\mathbb{R}^n} (|u_t(t,x)|^2 + |\nabla u(t,x)|^2)\,dx = \frac{1}{2}\int_{\mathbb{R}^n} (|v_0(x)|^2 + |\nabla u_0(x)|^2)\,dx \quad \text{for all } t \geq 0. \tag{28}$$

It is natural to employ function spaces such $u_t(t,\cdot)$ lives in $L^2(\mathbb{R}^n)$ and $\nabla u(t,\cdot)$ lives in $L^2(\mathbb{R}^n;\mathbb{C}^n)$ (vector-valued L^2.) It is reasonable (but not essential) to require $u(t,\cdot)$ to live in $L^2(\mathbb{R}^n)$ also (and this can be ensured by assuming that $u_0 \in L^2(\mathbb{R}^n)$.)

We shall rewrite the wave equation as a first-order system by letting $v = \dot{u}$:

$$(\dot{u}(t), \dot{v}(t)) = (v(t), \Delta u(t)).$$

Let

$$X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n),$$

equipped with the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ defined by

$$\langle\!\langle (\phi_1, \psi_1), (\phi_2, \psi_2) \rangle\!\rangle = \int_{\mathbb{R}^n} [\phi_1\overline{\phi_2} + \nabla\phi_1 \cdot \overline{\nabla\phi_2} + \psi_1\overline{\psi_2}].$$

Formally, the action of the operator A will be given by

$$A(\phi, \psi) = (\psi, \Delta\phi).$$

The relationship between the energy integral and the inner product will allow us to show that A is quasidissipative, so we will be able to use the Lumer-Phillips Theorem for Hilbert spaces. Before choosing a domain for A we look at the surjectivity of $I - A$. Let $(f, g) \in X$ be given. We need to find $(\phi, \psi) \in \mathcal{D}(A)$ such that

$$\begin{aligned} \phi - \psi &= f \\ \psi - \Delta\phi &= g. \end{aligned} \tag{29}$$

Adding the equations in (29) we obtain

$$\phi - \Delta\phi = f + g. \tag{30}$$

Since $f + g \in L^2(\mathbb{R}^n)$ it follows from Example 13.43 that (30) has a solution $\phi \in H^2(\mathbb{R}^n)$. The corresponding ψ is given by

$$\psi = \phi - f \in H^1(\mathbb{R}^n). \tag{31}$$

Consequently, a natural choice is

$$\mathcal{D}(A) = H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \tag{32}$$

and we define $A : \mathcal{D}(A) \to X$ by

$$A(\phi, \psi) = (\psi, \Delta\phi) \text{ for all } (\phi, \psi) \in \mathcal{D}(A). \tag{33}$$

Let $(\phi, \psi) \in \mathcal{D}(A)$ be given. Then we have

$$\begin{aligned} \mathrm{Re}\langle\!\langle A(\phi,\psi), (\phi,\psi)\rangle\!\rangle &= \mathrm{Re}\int_{\mathbb{R}^n} [\psi\overline{\phi} + \nabla\phi \cdot \nabla\overline{\psi} + (\Delta\phi)\overline{\psi}] \\ &= \mathrm{Re}\int_{\mathbb{R}^n} \psi\overline{\phi} \\ &\le \frac{1}{2}\int_{\mathbb{R}^n} [|\psi|^2 + |\phi|^2] \le \frac{1}{2}\|(\phi,\psi)\|_X^2. \end{aligned}$$

We conclude that A generates a linear C_0-semigroup $T : [0, \infty) \to \mathcal{L}(X; X)$ satisfying

$$\|T(t)\| \le e^{\frac{1}{2}t} \text{ for all } t \ge 0. \tag{34}$$

The growth estimate (34) is not optimal. In view of (28) one should expect

$$\int_{\mathbb{R}^n} (|u_t(t,x)|^2 + |\nabla u(t,x)|^2)\, dx$$

to remain bounded in t; however, in general, there is no reason to believe that

$$\int_{\mathbb{R}^n} |u(t,x)|^2\, dx \tag{35}$$

will be bounded in t. The growth of the integral in (35) is at most algebraic in t and depends on the spatial dimension. See, for example, for a discussion of such results.

The wave equation is reversible in time, so we cannot expect any smoothing of the initial data. Moreover, Littman has shown that if $n \geq 2$ then the solution operator for the wave equation does not form a linear C_0-semigroup on $W^{1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ unless $p = 2$. This means that the result of Problem 5 on Assignment 3 gives a rather sharp regularity result for the initial value problem (27): For each $m \in \mathbb{N}$ and $u_0 \in H^{m+1}(\mathbb{R}^n)$, $v_0 \in H^m(\mathbb{R}^n)$ the solution of (27) satisfies

$$u \in C^k([0,\infty); H^{m+1-k}(\mathbb{R}^n) \quad \text{for all } k = 0, 1, 2, \cdots m+1.$$