**21-373, Algebraic Structures**, Department of Mathematical Sciences, Carnegie Mellon University
**Fall 2011**: (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.
Luc TARTAR, University Professor of Mathematics, Wean Hall 6212, tartar@cmu.edu

Assignment 5 - Saturday October 8, 2011. Due Friday October 14

**Exercise 29**: Let $G$ be any *simple* group of order 168; let $n_p$ be the number of Sylow $p$-subgroups of $G$ (for $p = 2, 3, 7$).

a) Show that $n_2 \in \{7, 21\}$, $n_3 \in \{7, 28\}$, and $n_7 = 8$.

b) If $H$ is a Sylow-7 subgroup of $G$, show that its normalizer $N = N_G(H)$ contains seven subgroups of order 3, and that $n_3 = 28$.

**Exercise 30**: Notation of Exercise 29.

For $K$ a Sylow-2 subgroup of $G$, let $P = N_G(P)$ be its normalizer. Find how many Sylow-3 subgroups $P$ has, and show that $n_2 = 21$, and $P = K$.

**Exercise 31**: Show that a ring $R$, not necessarily unital, which satisfies $r^2 = r$ for all $r \in R$ is necessarily commutative.

**Exercise 32**: An element $r$ of a ring $R$ is said to be nilpotent if $r^n = 0$ for some $n \geq 1$.

i) If $R$ is a *commutative* ring, show that if $a$ and $b$ are nilpotent then $a + b$ is nilpotent. Show that this result may be false if $R$ is not commutative.

ii) If $R$ is a *commutative* ring, show that if $a$ is nilpotent and $r \in R$ then $a\,r$ is nilpotent. Show that this result may be false if $R$ is not commutative.

**Exercise 33**: Show that the subset $J$ of $\mathbb{Z}[x]$ of all polynomials $a_0 + a_1 x + \ldots$ (with integer coefficients) such that $a_0 = 0 \pmod 6$ and $a_1 = 0 \pmod 3$ is an ideal, and deduce that $\mathbb{Z}[x]$ is not a Principal Ideal Domain.

**Exercise 34**: Let $R$ be a (non necessarily commutative) ring and for $n \geq 1$ let $\mathcal{M}_n(R)$ be the ring of $n \times n$ matrices with entries in $R$. Let $\mathcal{J}$ be a two-sided ideal of $\mathcal{M}_n(R)$, and let $J$ be the subset of elements of $R$ which appear as the entry in row 1 – column 1 of some matrix belonging to $\mathcal{J}$; show that $J$ is a two-sided ideal of $R$ and that $\mathcal{J}$ is the set of all $n \times n$ matrices with entries in $J$.

**Exercise 35**: For $J$ an ideal in a *commutative* ring $R$, one defines $Rad(J) = \{r \in R \mid r^n \in J$ for some $n \geq 1\}$.

i) Show that $Rad(J)$ is an ideal, and that $Rad\big(Rad(J)\big) = Rad(J)$.

ii) If $J_1, \ldots, J_m$ are ideals of $R$ (and $m \geq 2$), show that $Rad(J_1 \cdots J_m) = Rad(\cap_{i=1}^m J_i) = \cap_{i=1}^m Rad(J_i)$, where $J_1 \cdots J_m$ denotes the ideal generated by products of the form $a_1 \cdots a_m$ with $a_i \in J_i$ for $i = 1, \ldots, m$, i.e. finite sums of such products.