Homework 5

Name: Shashank Singh[1]
36-705 Intermediate Statistics
Due: Thursday, October 9, 2014

1. Define $\boxed{\hat{\theta} := (1 + X)/3.}$ Then, for all $\theta \in \{1/3, 2/3\}$, the risk of $\hat{\theta}$ is $R(\theta, \hat{\theta}) = 1/3$. If any estimator maps 0 to a values besides 1/3 or 1 to a value besides 2/3, then the risk for the corresponding value of θ is at least 2/3, and hence $\hat{\theta}$ is minimax optimal. ■

2. Since $X_1, \ldots, X_n \sim$Gamma(α, β), $\mathbb{E}[X_i] = \alpha\beta$ and $\mathbb{E}[X_i^2] = \mathbb{V}[X_i^2] + \mathbb{E}^2[X_i] = \alpha\beta^2 + \alpha^2\beta^2$. Solving this system of equations gives

$$\alpha = \frac{\mathbb{E}^2[X_i]}{\mathbb{E}[X_i^2] - E^2[X_i]} \quad \text{and} \quad \beta = \frac{\mathbb{E}[X_i^2] - E^2[X_i]}{\mathbb{E}[X_i]}.$$

Hence, the method of moment estimators for α and β are

$$\boxed{\hat{\alpha} = \frac{\overline{X}^2}{\overline{X^2} - \overline{X}^2} \quad \text{and} \quad \hat{\beta} = \frac{\overline{X^2} - \overline{X}^2}{\overline{X}}.}$$

3. Since $\mathbb{E}[X_i] = \lambda$, the method of moments estimator is $\boxed{\hat{\lambda}_{MOM} = \overline{X}.}$ The log-likelihood of λ is

$$\begin{aligned} \ell(\lambda) = \log L(\lambda) = \log \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{X_i}}{X_i!} \\ &= \sum_{i=1}^n -\lambda + X_i \log(\lambda) - \log(X_i!) \\ &= -n\lambda + n\overline{X}\log(\lambda) - \sum_{i=1}^n \log(X_i!), \end{aligned}$$

and hence

$$\ell'(\lambda) = -n + n\overline{X}/\lambda,$$

so that $\ell'(\hat{\lambda}_{MLE}) = 0$ implies the maximum likelihood estimator of λ is $\boxed{\hat{\lambda}_{MLE} = \overline{X}.}$ Also, the Fisher information is

$$I(\lambda) = -\mathbb{E}\left[\ell''(\lambda)\right] = \mathbb{E}\left[\frac{d}{d\lambda}(n - n\overline{X}/\lambda)\right] = \mathbb{E}\left[n\overline{X}/\lambda^2\right] = n\lambda/\lambda^2 = \boxed{n/\lambda.}$$

4. Problem removed.

[1] sss1@andrew.cmu.edu

5. Note that
$$\hat{\beta} = \frac{\sum_{i=1}^n Y_i X_i}{\sum_{i=1}^n X_i^2} = \frac{\beta \sum_{i=1}^n X_i^2 + \sum_{i=1}^n \varepsilon_i X_i}{\sum_{i=1}^n X_i^2} = \beta + \frac{\sum_{i=1}^n \varepsilon_i X_i}{\sum_{i=1}^n X_i^2},$$
and hence it suffices to show $\frac{\sum_{i=1}^n \varepsilon_i X_i}{\sum_{i=1}^n X_i^2} \to 0$ in probability. To see this, it suffices to observe that
$$\frac{\sum_{i=1}^n \varepsilon_i X_i}{\sum_{i=1}^n X_i^2} = \frac{\frac{1}{n}\sum_{i=1}^n \varepsilon_i X_i}{\frac{1}{n}\sum_{i=1}^n X_i^2},$$
the numerator of which is approaches $\mathbb{E}[\varepsilon_i X_i] = \mathbb{E}[\varepsilon_i]\mathbb{E}[X_i] = 0$ and the denominator of which approaches $\mathbb{E}[X_i^2] = \mathbb{V}[X_i] + \mathbb{E}^2[X_i]$ (noting that the function $(x_1, x_2) \mapsto x_1/x_2$ is continuous for $x_2 \neq 0$. I didn't have time to finish this, but, presumably, $\sqrt{(}\hat{\beta} - \beta) \to \mathcal{N}(0, \sigma_2^2)$ in distribution, for some $\sigma_2 > 0$, and this can be shown using the multivariate delta method with $g(x_1, x_2) = x_1/x_2$ and the sequences $\{\varepsilon_i X_i\}_{i=1}^\infty$ and $\{X_i^2\}_{i=1}^\infty$.