Chapter 2

A Fundamental Necessary Condition for an Extremum

A very important result from basic calculus asserts that if a real-valued function F of a single real variable attains a local extremum (i.e. a local maximum or local minimum) at an interior point x_* of its domain, and if F is differentiable at x_*, then $F'(x_*) = 0$. The condition $F'(x_*) = 0$ can be used to identify points that are candidates for local extrema.

This result is often called Fermat's theorem on local extrema. It was discovered by Fermat in approximately 1635 (30 years before the discovery of calculus). Even though derivatives had not yet been invented, Fermat realized that if F attains a local extremum at x_* then the difference quotient

$$\frac{F(x_* + h) - F(x_*)}{h}, \tag{2.1}$$

although not defined at $h = 0$, must change sign as h passes through 0. Therefore the quantity in (2.1) must be very close to 0 when $|h|$ is very small. Fermat used this idea to find local extrema of certain polynomials.

The aim of this chapter is to generalize the result that "the derivative must vanish at an extremum" to a much more general framework that can be used to attack problems from the calculus of variations. The framework that we will use is that of functionals $J : \mathscr{Y} \to \mathbb{R}$, where $\mathscr{Y}$ is a subset of a real linear space (or vector space) $\mathfrak{X}$. We do not assume that $\mathfrak{X}$ is finite dimensional. Before discussing the situation in a general linear space, we review some results concerning optimization of functions $F : \mathbb{R}^n \to \mathbb{R}$. In introductory calculus courses, such functions are often optimized by freezing all but one of the variables and applying results from one-variable calculus. This is equivalent to studying rates of change of F when the input is varied in the coordinate directions. It will be our approach here to study the rate of change of F in arbitrary directions. This will make it much easier to generalize to an abstract linear space $\mathfrak{X}$. In our applications to calculus of variations problems, $\mathfrak{X}$ will be infinite dimensional and it will not be possible to introduce coordinates in a useful way.

Throughout the chapter, we will focus on global extrema, rather than local extrema, because we do not want to engage in a discussion of neighborhoods in (possibly) infinite-dimensional spaces and because in most of our applications to problems from the calculus of variations, it is global extrema that are of interest. Moreover, results will generally be stated for minima. This involves no loss of generality since the maximum of J can be found by minimizing $-J$. We begin the chapter with a discussion of notation for partial derivatives.

2.1 Notational Conventions

Let $n \in \mathbb{N}$ be given. As usual, we denote by $\mathbb{R}^n$ the set of all n-tuples (i.e., lists of length n) of real numbers. The elements of $\mathbb{R}^n$ may be thought of as "points" or as "vectors", depending the context. Typical elements of $\mathbb{R}^n$ will be denoted by $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, etc. When n has been prescribed, and the value of n is not too large (say, $n \leq 4$), we sometimes use a different letter for each component, rather than use the same letter with different subscripts. For example an element of $\mathbb{R}^2$ may be denoted by (x, y), an element of $\mathbb{R}^3$ by (x, y, z), and an element of $\mathbb{R}^4$ by (x, y, z, w), etc.

Addition and *scalar multiplication* on $\mathbb{R}^n$ are defined componentwise, i.e.

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \quad \text{for all } x, y \in \mathbb{R}^n,$$

$$cx = (cx_1, cx_2, \cdots, cx_n) \quad \text{for all } c \in \mathbb{R}, x \in \mathbb{R}^n.$$

The *zero element* of $\mathbb{R}^n$ will be denoted by

$$0 \;\text{ or }\; (0, 0, \cdots, 0).$$

With addition, scalar multiplication, and zero element as described above, $\mathbb{R}^n$ becomes a *real linear space* (i.e., a vector space over the field $\mathbb{R}$). The *dot product* (or *inner product*) of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Notice that $x \cdot x > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$.

To explain our notation for partial derivatives, let $F : \mathbb{R}^n \to \mathbb{R}$ be given:

$$F(x) = F(x_1, x_2, \cdots, x_n) \quad \text{for all } x \in \mathbb{R}^n.$$

We use two types of notation to denote the partial derivative of F with respect to its k-th argument. With $1 \leq k \leq n$, we write

$$F_{,k}(x) = F_{,k}(x_1, x_2, \cdots, x_n) = \frac{\partial}{\partial x_k} F(x) = \frac{\partial}{\partial x_k} F(x_1, x_2, \cdots, x_n).$$

The notation $\frac{\partial}{\partial x_k}$ is standard. We introduce the notation $F_{,k}$ because it makes no explicit reference to a variable name. In particular, this notation emphasizes that the derivative of F is to be taken with respect to its k-th argument – without regard to any particular symbol that is being used to denote the argument in question. This is very useful for situations involving composite functions. Assuming that the partial derivatives $F_{,1}, F_{,2}, \cdots, F_{,n}$ are continuous we define the gradient of F, $\nabla F : \mathbb{R}^n \to \mathbb{R}^n$, by

$$\begin{aligned}\nabla F(x) = \nabla F(x_1, x_2, \cdots, x_n) &= (F_{,1}(x), F_{,2}(x), \cdots, F_{,n}(x)) \\ &= \left(\tfrac{\partial}{\partial x_1}F(x), \tfrac{\partial}{\partial x_2}F(x), \cdots, \tfrac{\partial}{\partial x_n}F(x)\right).\end{aligned}$$

As a simple illustration of the notation, suppose that $F : \mathbb{R}^4 \to \mathbb{R}$ is given by

$$F(x) = x_1 + 4x_1^6x_2^3 - x_3^2 + e^{5x_4} \quad \text{for all } x \in \mathbb{R}^4.$$

Then

$$F_{,1}(\mathbf{x}) = 1{+}24x_1^5x_2^3;\ F_{,2}(\mathbf{x}) = 12x_1^6x_2^2;\ F_{,3}(\mathbf{x}) = -2x_3;\ F_{,4}(\mathbf{x}) = 5e^{5x_4} \quad \text{for all } \mathbf{x} \in \mathbb{R}^4$$

and

$$\nabla F(\mathbf{x}) = (1 + 24x_1^5x_2^3, 12x_1^6x_2^2, -2x_3, 5e^{5x_4}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^4.$$

We shall also need to consider partial derivatives of functions defined on suitable subsets of $\mathbb{R}^n$ (such as products of intervals). We use the same notation in such cases.

We conclude this section with an example that illustrates a significant advantage of the comma-notation for partial derivatives. Consider the function $f : [-1, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y, z) = (1 - x^2)^{3/2}y^3z^4 \quad \text{for all } (x, y, z) \in [-1, 1] \times \mathbb{R} \times \mathbb{R}.$$

The partial derivatives of f are given by

$$\begin{aligned}&f_{,1}(x, y, z) = -3x(1 - x^2)^{1/2}y^3z^4;\ f_{,2}(x, y, z) = 3(1 - x^2)^{3/2}y^2z^4; \\ &f_{,3}(x, y, z) = 4(1 - x^2)^{3/2}y^3z^3 \quad \text{for all } (x, y, z) \in [-1, 1] \times \mathbb{R} \times \mathbb{R}.\end{aligned}$$

(Here we are making the convention that at the endpoints $x = -1, 1$, the partial derivative with respect to the first argument of f is a one-sided derivative.) Suppose, in addition, that we are given a function $y \in \mathscr{C}^1[-1, 1]$. Then we can write the following composite function in an unambiguous way:

$$f_{,1}(x, y(x), y'(x)) = -3x(1 - x^2)^{1/2}y(x)^3y'(x)^4 \quad \text{for all } x \in [-1, 1].$$

A formula such as

$$\frac{\partial}{\partial x}f(x, y(x), y'(x))$$

could be genuinely confusing. Similarly, we write

$$\begin{aligned}f_{,2}(x, y(x), y'(x)) &= 3(1 - x^2)^{3/2}y(x)^2y'(x)^4 \\ f_{,3}(x, y(x), y'(x)) &= 4(1 - x^2)^{3/2}y(x)^3y'(x)^3 \quad \text{for all } x \in [-1, 1].\end{aligned}$$

Composites of the type above will play a central role in calculus of variations problems.

2.2 The Chain Rule

The chain rule for differentiating composite functions of one variable is one of the most important results from basic calculus. Here we record a chain rule for the composition of a function $F : \mathbb{R}^n \to \mathbb{R}$ with a function $u : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval.

Let $u : I \to \mathbb{R}^n$ be given:

$$u(t) = (u_1(t), u_2(t), \cdots, u_n(t)) \quad \text{for all } t \in I.$$

Recall that u is differentiable provided that each component of u is differentiable; in this case we write

$$u'(t) = (u_1'(t), u_2'(t), \cdots, u_n'(t)) \quad \text{for all } t \in I.$$

Theorem 2.1 (Chain Rule) *Let $I \subset \mathbb{R}$ be an interval and $u : I \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}$ be a given. Assume that u is differentiable on I and that F has continuous first-order partial derivatives on $\mathbb{R}^n$. Define $g : I \to \mathbb{R}$ by*

$$g(t) = F(u(t)) \quad \text{for all } t \in I.$$

Then g is differentiable on I and

$$\begin{aligned} g'(t) &= \nabla F(u(t)) \cdot u'(t) \\ &= \sum_{i=1}^{n} F_{,i}(u(t))u_i'(t) \quad \text{for all } t \in I. \end{aligned}$$

Moreover, if u' is continuous on I then g' is continuous on I. (In this theorem we make the convention that if I is not open, then derivatives at endpoints of I should be interpreted as appropriate one-sided derivatives.)

Since partial derivatives are defined as ordinary derivatives with respect to one variable while holding the other variables constant, we can apply the version of the chain rule given above to compute certain partial derivatives as well. We record below the relevant result below for the case when u is a vector-valued function of two variables. Here, for

$$u(t_1, t_2) = (u_1(t_1, t_2), u_2(t_1, t_2), \cdots, u_n(t_1, t_2))$$

and each $i \in \{1, 2\}$, we write

$$u_{,i}(t_1, t_2) = (u_{1,i}(t_1, t_2), u_{2,i}(t_1, t_2), \cdots, u_{n,i}(t_1, t_2)).$$

Remark 2.1 *Let $I_1, I_2 \subset \mathbb{R}$ be intervals and $u : I_1 \times I_2 \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}$ be given. Let $i \in \{1, 2\}$ be given and assume that $u_{,i}$ exists on $I_1 \times I_2$. Assume further that F has continuous first-order partial derivatives on $\mathbb{R}^n$. Define $g : I_1 \times I_2 \to \mathbb{R}$ by*

$$g(t_1, t_2) = F(u(t_1, t_2)) \quad \text{for all } (t_1, t_2) \in I_1 \times I_2.$$

Then, the partial derivative $g_{,i}$ exists on $I_1 \times I_2$ and is given by

$$\begin{aligned} g_{,i}(t_1, t_2) &= \nabla F(u(t_1, t_2)) \cdot u_{,i}(t_1, t_2) \\ &= \sum_{j=1}^{n} F_{,j}(u(t_1, t_2)u_{j,i}(t_1, t_2) \quad \text{for all } (t_1, t_2) \in I_1 \times I_2. \end{aligned}$$

2.3 Minimization Problems in $\mathbb{R}^n$

We will now review a method for determining the possible minimizers of a function $F : \mathbb{R}^n \to \mathbb{R}$. We assume that F has continuous first-order partial derivatives. Suppose that F attains a minimum at the point $y_* \in \mathbb{R}^n$. We want to establish some conditions that y_* must satisfy.

The basic idea is to move away from y_* in every direction possible and use the hypothesis that y_* minimizes F. Since y_* is a minimizer for F, if we move from y_* in any direction, the value of F cannot decrease..

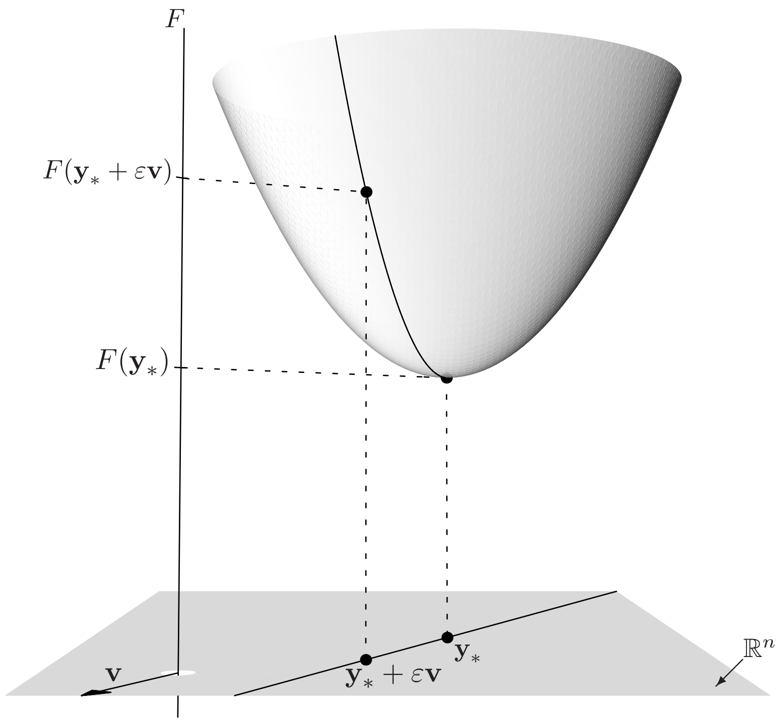

Figure 2.1: Comparison of $F(y_*)$ and $F(y_* + \varepsilon v)$

Let $v \in \mathbb{R}^n$ with $v \neq 0$ be given. We will use v as a direction in which to move from y_*. The line in the direction of v that passes through y_* is the set

$$\{y_* + \varepsilon v \,:\, \varepsilon \in \mathbb{R}\}.$$

Since F is minimized at y_*, we must have

$$F(y) \geq F(y_*) \quad \text{for all } y \in \mathbb{R}^n.$$

In particular, we have

$$F(y_* + \varepsilon v) \geq F(y_*) \quad \text{for all } \varepsilon \in \mathbb{R}.$$

Hence

$$F(y_* + \varepsilon v) - F(y_*) \geq 0 \quad \text{for all } \varepsilon \in \mathbb{R},$$

and therefore

$$\frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon} \geq 0 \quad \text{for all } \varepsilon > 0 \tag{2.2}$$

and

$$\frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon} \leq 0 \quad \text{for all } \varepsilon < 0. \tag{2.3}$$

We want to take the limit in (2.2) and (2.3) as ε tends to 0 from the right and left respectively.

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\varepsilon) := F(y_* + \varepsilon v) \quad \text{for all } \varepsilon \in \mathbb{R},$$

so that $\varphi(\varepsilon) = F(u(\varepsilon))$ with $u(\varepsilon) := y_* + \varepsilon v$. Obviously, the vector-valued function u is continuously differentiable; in fact, its derivative is just v. Since F is assumed to have continuous first-order partial derivatives, we can apply the chain rule to φ to obtain

$$\varphi'(\varepsilon) = \nabla F(u(\varepsilon)) \cdot u'(\varepsilon) = \nabla F(y_* + \varepsilon v) \cdot v \quad \text{for all } \varepsilon \in \mathbb{R}.$$

Evaluating this derivative at $\varepsilon = 0$, we find that

$$\varphi'(0) = \nabla F(y_*) \cdot \mathbf{v}. \tag{2.4}$$

Notice that by definition $\varphi'(0) = \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon}$ so that

$$\varphi'(0) = \lim_{\varepsilon \to 0} \frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon}. \tag{2.5}$$

Since $\varphi'(0)$ exists, the limit in (2.5) exists, and we may take the appropriate one-sided limits in (2.2) and (2.3) to obtain

$$\varphi'(0) = \lim_{\varepsilon \to 0^+} \frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon} \geq 0 \tag{2.6}$$

and

$$\varphi'(0) = \lim_{\varepsilon \to 0^-} \frac{F(y_* + \varepsilon \mathbf{v}) - F(y_*)}{\varepsilon} \leq 0. \tag{2.7}$$

Consequently

$$\varphi'(0) = 0,$$

by (2.6) and (2.7). Using (2.4), we have

$$\nabla F(y_*) \cdot v = \mathbf{0} \quad \text{for all } v \in \mathbb{R}^n \tag{2.8}$$

since $v \in \mathbb{R}^n$ was an arbitrary non-zero vector. (Equality in (2.8) obviously holds if $v = 0$.)

We now wish to show that (2.8) implies $\nabla F(y_*) = 0$. This will follow from

Lemma 2.1 *Let $w \in \mathbb{R}^n$ be given. Suppose that $w \cdot v = 0$ for all $v \in \mathbb{R}^n$. Then $w = 0$.*

Proof. Notice that the hypothesis of the lemma implies $w \cdot w = 0$. Thus

$$w \cdot = w_1 w^2 + w_2^2 + \cdots + w_n^2 = 0.$$

It follows that

$$w_1 = w_2 = \cdots = w_n = 0$$

and $w = 0$, as claimed. □

In view of the lemma above, we have shown that if $y_* \in \mathbb{R}^n$ minimizes the function F on $\mathbb{R}^n$, then we necessarily have

$$\nabla F(y_*) = 0.$$

If F attains only a local minimum at y_* then the method described above can still be used to show that $\nabla F(y_*) = \mathbf{0}$. Indeed, in this case, the inequality in (2.2) holds for all ε in some interval $(0, \delta)$ and the inequality in (2.3) holds for all ε in some interval $(-\delta, 0)$. We can still take the limit as $\varepsilon \to 0$ from both sides. If $\mathscr{Y}$ is a proper subset of $\mathbb{R}^n$ and $F : \mathscr{Y} \to \mathbb{R}$ attains a minimum at $y_* \in \mathscr{Y}$ then we may only be able to move away from y_* in certain directions v. This issue will be addressed in Definition 2.1.

2.4 Minimization Problems in Real Linear Spaces

We want to use a similar argument to determine the possible minimizers of a real-valued functional defined on a subset of a real linear space $\mathfrak{X}$. With $\mathfrak{X}$ a real linear space, let $\mathscr{Y} \subset \mathfrak{X}$ and $J : \mathscr{Y} \to \mathfrak{X}$ be given. Suppose that J attains a minimum at $y_* \in \mathscr{Y}$. We are going to find some conditions that y_* must satisfy.

In order to use the basic ideas from the previous section, we need to know in which directions we can move from y_*. That is, we must know those v's in $\mathfrak{X}$ such that $[y_* + \varepsilon v] \in \mathscr{Y}$ for all ε with $|\varepsilon|$ sufficiently small. Note that in general $[y_* + \varepsilon v]$ may not be in $\mathscr{Y}$ for arbitrary $v \in \mathfrak{X}$ and $\varepsilon \in \mathbb{R}$ (see Figure 2.2). To distinguish those v's that may be used to vary y_*, we make a definition.

Definition 2.1 (Admissible Variations) *Let $v \in \mathfrak{X}$ and $y \in \mathscr{Y}$ be given. We say that v is a $\mathscr{Y}$-*admissible variation *at y if there exists an open interval $I \subset \mathbb{R}$ with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for every $\varepsilon \in I$. The set of all $\mathscr{Y}$-admissible variations at y is denoted by $\mathscr{V}_y$. If the set $\mathscr{Y}$ is clear from context, we will simply use the term* admissible variation *in place of $\mathscr{Y}$-admissible variation.*

Thus, if $v \in \mathscr{V}_{y_*}$, then for all $\varepsilon \in \mathbb{R}$ such that $|\varepsilon|$ is sufficiently small, we find $[y_* + \varepsilon v] \in \mathscr{Y}$.

Let $v \in \mathscr{V}_{y_*}$ be a given admissible variation. By definition there exists an open interval $I \subset \mathbb{R}$ with $0 \in I$ and such that $[y_* + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$. Since y_* minimizes J, we must have

$$J(y_* + \varepsilon v) \geq J(y_*) \quad \text{for all } \varepsilon \in I.$$

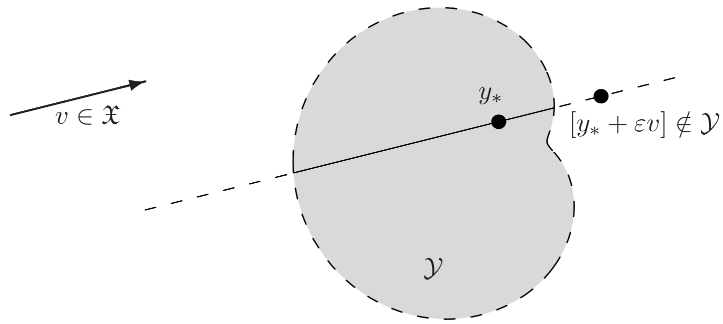

Figure 2.2: Situation where $[y_* + \varepsilon v]$ is not in $\mathscr{Y}$

Hence

$$J(y_* + \varepsilon v) - J(y_*) \geq 0 \quad \text{for all } \varepsilon \in I,$$

and therefore

$$\frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} \geq 0 \quad \text{for all } \varepsilon \in I \cap (0, +\infty) \tag{2.9}$$

and

$$\frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} \leq 0 \quad \text{for all } \varepsilon \in I \cap (-\infty, 0). \tag{2.10}$$

Since I is open and $0 \in I$, we know that neither $I \cap (0, +\infty)$ nor $I \cap (-\infty, 0)$ is empty. Suppose that $\lim_{\varepsilon \to 0} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon}$ exists. Then by (2.9) and (2.10)

$$0 \leq \lim_{\varepsilon \to 0^-} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} \leq 0.$$

Therefore, if $\lim_{\varepsilon \to 0} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon}$ exists it must be equal to zero.

2.5 Gâteaux Variations

2.5.1 Definition and Relevance to Optimization Problems

In order to state the result obtained above in a convenient form, we make the following

Definition 2.2 (Gâteaux Variation) *Let $y \in \mathscr{Y}$ and $v \in \mathscr{V}_y$ be given. The* Gâteaux variation of J at y in the direction v *is defined by*

$$\delta J(y; v) := \lim_{\varepsilon \to 0} \frac{J(y + \varepsilon v) - J(y)}{\varepsilon},$$

provided the limit exists.

In terms of this definition, our argument above proves the following:

Theorem 2.2 *Let $\mathfrak{X}$ be a real linear space. Also let $\mathscr{Y} \subset \mathfrak{X}$ and $J : \mathscr{Y} \to \mathbb{R}$ be given. Suppose that $y_* \in \mathscr{Y}$ is a minimizer (or a maximizer) for J over $\mathscr{Y}$. Let $v \in \mathscr{V}_{y_*}$ be given. Then, either $\delta J(y_*; v) = 0$ or $\delta J(y_*; v)$ does not exist.*

The theorem gives a procedure to find possible minimizers (or maximizers) of J. In our applications to problems from the Calculus of Variations, the Gâteaux variations will always exist. Therefore, to describe the procedure for identifying possible minimizers, let us assume for now that $\delta J(y; v)$ exists for all $y \in \mathscr{Y}$ and all $v \in \mathscr{V}_y$. To apply the theorem, we shall carry out the following steps:

(1) Identify the class of admissible variations $\mathscr{V}_y$ at each $y \in \mathscr{Y}$.

(2) Compute the Gâteaux Variations $\delta J(y; v)$.

(3) Analyze the condition

$$\delta J(y; v) = 0 \quad \text{for all } v \in \mathscr{V}_y. \tag{2.11}$$

In order for this approach to work, there must be sufficiently many admissible variations. If there are "too few" admissible variations, we will not be able to deduce much from (2.11). In such cases, a different approach will be employed.

2.5.2 Example 2.5.2.

For our first example involving Gâteaux variations, we look at the special case of a function from $\mathbb{R}^n$ to $\mathbb{R}$. Let $\mathfrak{X} = \mathscr{Y} = \mathbb{R}^n$ and assume that $F : \mathbb{R}^n \to \mathbb{R}$ has continuous first-order partial derivatives. It is easy to see that for each $y \in \mathbb{R}^n$, the class of admissible variations is all of $\mathbb{R}^n$, i.e.

$$\mathscr{V}_y = \mathbb{R}^n \quad \text{for all } y \in \mathbb{R}^n.$$

Given $y, v \in \mathbb{R}^n$, the arguments used in Section 2.3 show that

$$\delta F(y; v) = \nabla F(y) \cdot v.$$

Notice that if v is a unit vector then $\delta F(y; v)$ is simply the directional derivative of F at y in the direction of v.

Remark 2.2 *If $\mathscr{Y}$ is a proper subset of $\mathbb{R}^n$ and $y \in \mathscr{Y}$ is not an interior point of $\mathscr{Y}$, then $\mathscr{V}_y$ will be a proper subset of $\mathbb{R}^n$. In particular, if y is a boundary point of $\mathscr{Y}$ then $\mathscr{V}_y$ will be a proper subset of $\mathbb{R}^n$. If a function $F : \mathbb{R}^n \to \mathbb{R}$ attains a minimum (or maximum) at a boundary point y_* of $\mathscr{Y}$, we cannot conclude that $\nabla F(y_*) = 0$.*

2.5.3 Example 2.5.3.

Let $\mathfrak{X} = C[1,2]$ and put

$$\mathscr{Y} = \{y \in C[1,2] \,:\, \int_1^2 y(x)\,dx = 1\}.$$

Consider the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) = \int_1^2 x^3 y(x)^2\,dx \quad \text{for all } y \in \mathscr{Y}.$$

We wish to find all possible minimizers for J on $\mathscr{Y}$.

The first step is to determine the admissible variations. Let $y \in \mathscr{Y}$ and $v \in \mathfrak{X}$ be given. Notice that for each $\varepsilon \in \mathbb{R}$, we have $[y+\varepsilon v] \in C[1,2]$. Consequently, in order that $v \in \mathscr{V}_y$ it is necessary and sufficient that there exist an open interval I containing 0 such that

$$\int_1^2 [y(x) + \varepsilon v(x)]\,dx = 1 \quad \text{for all } \varepsilon \in I. \tag{2.12}$$

Since $y \in \mathscr{Y}$ and

$$\int_1^2 [y(x) + \varepsilon v(x)]\,dx = \int_1^2 y(x)\,dx \;+ \varepsilon \int_1^2 v(x)\,dx,$$

we see that (2.12) holds if and only if

$$\begin{aligned} \int_1^2 [y(x) + \varepsilon v(x)]\,dx &= \int_1^2 y(x)\,dx \;+ \varepsilon \int_1^2 v(x)\,dx \\ &= 1 + \varepsilon \int_1^2 v(x)\,dx \\ &= 1 \quad \text{for all } \varepsilon \in I. \end{aligned}$$

In other words (2.12) holds if and only if

$$\varepsilon \int_1^2 v(x)\,dx = 0 \quad \text{for all } \varepsilon \in I.$$

Since I must contain nonzero elements, we conclude that $v \in \mathscr{V}_y$ if and only if

$$\int_1^2 v(x)\,dx = 0.$$

(Notice that the interval I in the definition of admissible variation can be taken to be $\mathbb{R}$.) Since $\mathscr{V}_y$ is the same for each $y \in \mathscr{Y}$, we drop the subscript on $\mathscr{V}_y$ and write

$$\mathscr{V} = \{v \in \mathscr{C}[1,2] \mid \int_1^2 v(x)\,dx = 0\}.$$

The next step is to try to compute the Gâteaux variations. To this end, let $y \in \mathscr{Y}$, $v \in \mathscr{V}$, and $\varepsilon \in \mathbb{R}$ be given, and observe that

$$\begin{aligned} J(y+\varepsilon v) - J(y) &= \int_1^2 x^3(y(x)+\varepsilon v(x))^2\,dx - \int_1^2 x^3 y(x)^2\,dx \\ &= \int_1^2 x^3[y(x)^2 + 2\varepsilon y(x)v(x) + \varepsilon^2 v(x)^2]\,dx - \int_1^2 x^3 y(x)^2\,dx \\ &= 2\varepsilon \int_1^2 x^3 y(x)v(x)\,dx + \varepsilon^2 \int_1^2 v(x)^2\,dx, \end{aligned}$$

which yields

$$\frac{J(y+\varepsilon v) - J(y)}{\varepsilon} = 2\int_1^2 x^3 y(x)v(x)\,dx + \varepsilon \int_1^2 x^3 v(x)^2\,dx \tag{2.13}$$

for $\varepsilon \neq 0$. Taking the limit as $\varepsilon \to 0$ in (2.13) we find that $\delta J(y;v)$ exists and is given by

$$\delta J(y;v) = 2\int_1^2 x^3 y(x)v(x)\,dx.$$

Consequently, if y minimizes J on $\mathscr{Y}$ we must have

$$\int_1^2 x^3 y(x)v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}. \tag{2.14}$$

In order understand the implications of (2.14), we shall give a somewhat general result about continuous functions having the property that the product of the function with every continuous function having zero average also has zero average.

Lemma 2.2 *Let $a, b \in \mathbb{R}$ with $a < b$ and $w \in \mathscr{C}[a,b]$ be given and put*

$$\mathscr{U} = \{v \in C[a,b] \mid \int_a^b v(x)\,dx = 0\}.$$

Assume that

$$\int_a^b w(x)v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{U}.$$

Then there is a constant $c \in \mathbb{R}$ such that $w(x) = c$ for all $x \in \mathbb{R}$.

Proof. Observe first that for each $c \in \mathbb{R}$ and $v \in \mathscr{U}$ we have

$$\int_a^b (w(x) - c)v(x)\,dx = 0. \tag{2.15}$$

Let us put

$$c_* = \frac{1}{b-a}\int_a^b w(x)\,dx,$$

so that

$$\int_a^b (w(x) - c_*)\,dx = 0.$$

It follows that $(w - c_*) \in \mathscr{U}$. We put $c = c_*$ and $v = w - c_*$ in (2.15) to obtain

$$\int_a^b (w(x) - c_*)^2\,dx = 0. \tag{2.16}$$

Since the integrand in (2.16) is nonnegative and continuous, it must vanish identically on $[a, b]$, i.e.

$$w(x) = c_* \quad \text{for all } x \in [a, b].$$

$\square$

Let $y \in \mathscr{Y}$ be given and assume that y minimizes J on $\mathscr{Y}$. Then, by virtue of (2.14) and Lemma 2.2 with $a = 1$, $b = 2$, and $w(x) = x^3 y(x)$, we conclude that there is a constant c such that

$$x^3 y(x) = c \quad \text{for all } x \in [1, 2], \text{ i.e.}$$

$$y(x) = cx^{-3} \quad \text{for all } x \in [1, 2].$$

In order to have $y \in \mathscr{Y}$, we must have

$$1 = \int_1^2 y(x)\,dx = c\int_1^2 x^{-3}\,dx = c(\frac{1}{2} - \frac{1}{8}) = \frac{3c}{8}.$$

It follows that $c = 8/3$ and

$$y(x) = \frac{8}{3}x^{-3} \quad \text{for all } x \in [1, 2]. \tag{2.17}$$

If J has a minimum on $\mathscr{Y}$ then the minimum must be attained at the function y given by (2.17). It can be shown directly that this function y does indeed minimize J on $\mathscr{Y}$. (See Problem .)

2.5.4 How to Compute a Gâteaux Variation

With $\mathfrak{X}$ a real linear space, $\mathscr{Y} \subset \mathfrak{X}$ and $J : \mathscr{Y} \to \mathbb{R}$ given, we want to compute the Gâteaux variations of J. Fix $y \in \mathscr{Y}$ and $v \in \mathscr{V}_y$. Then we may choose an open interval $I \subset \mathbb{R}$ with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$.

As in Section 2.3, we compute the Gâteaux variation of J at y in the direction v as the derivative at 0 of a real-valued function. Define $\varphi : I \to \mathbb{R}$ by

$$\varphi(\varepsilon) := J(y + \varepsilon v) \quad \text{for all } \varepsilon \in I.$$

Notice that φ is defined on an open neighborhood of 0 and that $\varphi(0) = J(y)$. Hence

$$\frac{J(y + \varepsilon v) - J(y)}{\varepsilon} = \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} \quad \text{for all } \varepsilon \in I,\ \varepsilon \neq 0,$$

and by definition

$$\delta J(y; v) = \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} = \varphi'(0),$$

provided the limit exists. The derivative of φ at 0 exists if and only if $\delta J(y; v)$ exists; moreover, in this case we have $\varphi'(0) = \delta J(y; v)$. Another way of stating this is that

$$\delta J(y; v) = \left. \frac{d}{d\varepsilon} [J(y + \varepsilon v)] \right|_{\varepsilon = 0}$$

if the Gâteaux variation of J at y in the direction v exists. This gives a convenient way to compute the Gâteaux variation $\delta J(y; v)$ in practice – simply perturb y by εv, differentiate the expression $J(y + \varepsilon v)$ with respect to ε, and then evaluate at $\varepsilon = 0$ (assuming that the derivative exists for *vep* in some open interval about 0).

2.6 Admissible Classes Characterized by Linear Constraints

In many calculus of variations problems, the class of admissible functions will consist of all those y in one of the standard function spaces (such as $C^1[a, b]$) satisfying a list of linear constraints:

$$L_1(y) = w_1, L_2(y) = w_2, \cdots, L_k(y) = w_k,$$

where the L_i are given linear mappings from $\mathfrak{X}$ to a real linear space $\mathfrak{W}$ and the w_i are given elements of $\mathcal{W}$. When this is the case, it turns out that the class of admissible variations is the same at each $y \in \mathscr{Y}$ and is actually a subspace of $\mathfrak{X}$, namely the intersection of the null spaces of the L_i. Moreover, the interval I in the definition of admissible variation can be taken to be the entire real line. In order to avoid "re-inventing the wheel" each time this situation arises, we shall prove a very useful lemma.

Let $\mathfrak{X}$ and $\mathcal{W}$ be real linear spaces. Recall that a mapping $L : \mathfrak{X} \to$ is said to be *linear* provided that

$$L(z_1 + z_2) = L(z_1) + L(z_2) \quad \text{for all } z_1, z_2 \in \mathfrak{X},$$

and

$$L(cz) = cL(z) \quad \text{for all } c \in \mathbb{R}, z \in \mathfrak{X}.$$

The *null space* of a linear mapping L is defined by

$$\mathcal{N}(L) = \{z \in \mathfrak{X} : L(z) = 0\}.$$

It is straightforward to verify that $\mathcal{N}(L)$ is a subspace if L is linear.

Lemma 2.3 *Let $k \in \mathbb{N}$, $w_1, w_2, \cdots, w_k \in \mathcal{W}$ be given and assume that $L_1, L_2, \cdots, L_k : \mathfrak{X} \to \mathcal{W}$ are linear mappings. Define $\mathscr{Y} \subset \mathfrak{X}$ by*

$$\mathscr{Y} = \{y \in \mathfrak{X} \mid L_1(y) = w_1, L_2(y) = w_2, \cdots, L_k(y) = w_k\}.$$

Then for each $y \in \mathscr{Y}$ the class of $\mathscr{Y}$-admissible variations at y is given by

$$\mathcal{Z} = \{v \in \mathfrak{X} : L_1(v) = L_2(v) = \cdots = L_k(v) = 0\}.$$

Moreover, for all $y \in \mathscr{Y}, v \in \mathcal{Z}$, we have $y + \varepsilon v \in \mathscr{Y}$ for all $\varepsilon \in \mathbb{R}$.

Proof. Let $y \in \mathscr{Y}$ be given. We shall show that

$$\mathcal{Z} \subset \mathscr{V}_y \text{ and } \mathscr{V}_y \subset \mathcal{Z}.$$

To this end, let $v \in \mathcal{Z}$ and $\varepsilon \in \mathbb{R}$ be given. It is immediate that $[y + \varepsilon v] \in \mathfrak{X}$. Moreover, for each $i \in \{1, 2, \cdots, k\}$ we have

$$L_i(y + \varepsilon v) = L_i(y) + L_i(\varepsilon v) = L_i(y) + \varepsilon L_i(v) = L_i(y) + 0 = L_i(y) = w_i,$$

so that $L_i(y + \varepsilon v) \in \mathscr{Y}$. It follows that $v \in \mathscr{V}_y$. (Notice that the interval I in the definition of admissible variation may be taken to be $\mathbb{R}$.)

To establish the reverse inclusion, let $v \in \mathscr{V}_y$ be given and choose an open interval I with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$. Choose $\varepsilon_* \in I \backslash \{0\}$. Then, for each $i \in \{1, 2, \cdots, k\}$ we have

$$L_i(y + \varepsilon_* v) = L_i(y) + \varepsilon_* L_i(v) = w_i.$$

Since $L_i(y) = w_i$ for all $i \in \{1, 2, \cdots, k\}$ and $\varepsilon_* \neq 0$, we conclude that

$$L_i(v) = 0 \quad \text{for all } i \in \{1, 2, \cdots, k\}$$

and $v \in \mathcal{Z}$. $\square$

2.6.1 Example 2.6.1.

For our first application of the lemma, we consider the situation of Example 2.5.3:

$$\mathfrak{X} = C[1,2], \quad \mathscr{Y} = \{y \in C[1,2] \mid \int\limits_1^2 y(x)\,dx = 1\}.$$

We take $\mathcal{W} = \mathbb{R}$, $k = 1$, $w_1 = 1$ and define the linear mapping $L_1 : \mathfrak{X} \to \mathcal{W}$ by

$$L_1(z) = \int\limits_1^2 z(x)\,dx \quad \text{for all } z \in \mathfrak{X}.$$

Notice that

$$\mathscr{Y} = \{y \in \mathfrak{X} \,:\, L_1(y) = w_1\}.$$

Employing Lemma 2.3, we find that for each $y \in \mathscr{Y}$ we have

$$\mathscr{V}_y = \{v \in \mathfrak{X} \mid L_1(v) = 0\} = \{v \in C[1,2] \,:\, \int\limits_1^2 v(x)\,dx = 0\}.$$

2.6.2 Example 2.6.2.

Let $\mathfrak{X} = C^1[-1,4]$ and put

$$\mathscr{Y} = \{y \in C^1[-1,4] \,:\, y(-1) = 8,\ y(4) = 5,\ \int\limits_{-1}^4 x^6 y(x) = 2\}.$$

To apply Lemma 2.3, we put $k = 3$, $\mathcal{W} = \mathbb{R}$, $w_1 = 8$, $w_2 = 5$, $w_3 = 2$ and define the linear mappings $L_1, L_2, L_3 : \mathfrak{X} \to \mathcal{W}$ by

$$L_1(z) = z(-1),\ L_2(z) = z(4),\ L_3(z) = \int\limits_{-1}^4 x^6 z(x)\,dx \quad \text{for all } z \in \mathfrak{X}.$$

With these definitions, we have

$$\mathscr{Y} = \{y \in \mathfrak{X} \,:\, L_1(y) = w_1,\ L_2(y) = w_2,\ L_3(y) = w_3\}$$

so that the lemma tells us

$$\begin{aligned} \mathscr{V}_y &= \{v \in \mathfrak{X} \,:\, L_1(v) = L_2(v) = l_3(v) = 0\} \\ &= \{v \in C^1[-1,4] \mid v(-1) = 0,\ v(4) = 0,\ \int\limits_{-1}^4 x^6 v(x)\,dx = 0\}. \end{aligned}$$

2.7 Gâteaux Variations of Integral Functionals

It is useful to obtain a general formula for the Gâteaux variations of functionals of the type appearing in the basic problem of the calculus of variations (and variants of this problem). In order to obtain such a formula by the procedure of Section 2.5.4, we shall need to differentiate an integral with respect to a parameter appearing under the integral sign. We record below an important result that will allow us to carry out the required differentiation.

Theorem 2.3 *Let $a, b \in \mathbb{R}$ with $a < b$, an open interval I, and $g : [a, b] \times I \to \mathbb{R}$ be given. Assume that g and $g_{,2}$ are continuous on $[a, b] \times \mathbb{R}$. Define $G : I \to \mathbb{R}$ by*

$$G(\varepsilon) := \int_a^b g(x, \varepsilon)\, dx \quad \text{for all } \varepsilon \in I.$$

Then G is differentiable on $\mathbb{R}$ and

$$G'(\varepsilon) = \int_a^b g_{,2}(x, \varepsilon)\, dx \quad \text{for all } \varepsilon \in I.$$

The theorem says that

$$\frac{d}{d\varepsilon} \int_a^b g(x, \varepsilon)\, dx = \int_a^b \frac{\partial}{\partial \varepsilon} g(x, \varepsilon)\, dx;$$

in other words, we may "pass the derivative under the integral sign".

Theorem 2.4 *Let $a, b \in \mathbb{R}$ and $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given and assume that f has continuous first-order partial derivatives. Let $\mathfrak{X}$ be a subspace of $\mathscr{C}^1[a, b]$ and let $\mathscr{Y} \subset \mathfrak{X}$ be given. Define $J : \mathscr{Y} \to \mathbb{R}$ by*

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Then for each $y \in \mathscr{Y}$, $v \in \mathscr{V}_y$, the Gâteaux variation $\delta J(y; v)$ exists and is given by

$$\delta J(y; v) = \int_a^b [f_{,2}(x, y(x), y'(x))v(x) + f_{,3}(x, y(x), y'(x))v'(x)]\, dx. \tag{2.18}$$

Proof. We choose an open interval I with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$, and define $\varphi : I \to \mathbb{R}$ by

$$\varphi(\varepsilon) = J(y + \varepsilon v) = \int_a^b f(x, y(x) + \varepsilon v(x), y'(x) + \varepsilon v'(x))\, dx \quad \text{for all } \varepsilon \in I.$$

Using Theorem 2.3 and the chain rule, we find that

$$\begin{aligned}
\varphi'(\varepsilon) &= \int_a^b \frac{\partial}{\partial \varepsilon}\left[f(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x))\right]\, dx \\
&= \int_a^b \Big\{ f_{,1}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x)) \frac{\partial}{\partial \varepsilon}[x] \\
&\qquad + f_{,2}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x)) \frac{\partial}{\partial \varepsilon}[y(x)+\varepsilon v(x)] \\
&\qquad + f_{,3}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x)) \frac{\partial}{\partial \varepsilon}[y'(x)+\varepsilon v'(x)] \Big\}\, dx \\
&= \int_a^b \Big\{ f_{,2}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x)) v(x) \\
&\qquad + f_{,3}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x)) v'(x) \Big\}\, dx.
\end{aligned}$$

Evaluating this expression at $\varepsilon = 0$ gives us

$$\varphi'(0) = \int_a^b \Big\{ f_{,2}(x, y(x), y'(x)) v(x) + f_{,3}(x, y(x), y'(x)) v'(x) \Big\}\, dx.$$

Therefore, the Gâteaux variation of J at y in the direction v exists and is given by

$$\delta J(y; v) = \int_a^b \Big\{ f_{,2}(x, y(x), y'(x)) v(x) + f_{,3}(x, y(x), y'(x)) v'(x) \Big\}\, dx.$$

$\square$