Midterm 2 Study Guide
21-236 Mathematical Studies Analysis II
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Lagrange Multipliers

1. **Theorem 85 (Lagrange Multipliers)**: Let $U \subseteq \mathbb{R}^N$ be an open set, let $f : U \to \mathbb{R}$, and $\mathbf{g} : U \to \mathbb{R}^M$ with $M < N$ be C^1, and

$$F := \{\mathbf{x} \in U : \mathbf{g}(\mathbf{x}) = \mathbf{0}\}.$$

Let $\mathbf{x}_0 \in F$, and assume f attains a local extremum at $\mathbf{x}_0$. Then, if $G_{\mathbf{g}}(\mathbf{x}_0)$ has maximum rank M, there exist $\lambda_1, \lambda_2, \dots, \lambda_M \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \lambda_2 \nabla g_2(\mathbf{x}_0) + \ldots + \lambda_M \nabla g_M(\mathbf{x}_0).$$

- Proof is ≈ 2 pages, so probably too long.
- Know how to use them to find extrema.

Curves

1. **Theorem 94 (Peano Curve)** There exists a continuous function $\boldsymbol{\varphi} : [0,1] \to \mathbb{R}^2$ such that $\boldsymbol{\varphi}([0,1]) = [0,1]^2$.

2. **Theorem 101 (Length of a Smooth Curve)** Let γ be a C^1 curve with parametrization $\boldsymbol{\varphi} : I \to \mathbb{R}^N$. Recall that

$$\mathrm{Var}_I\, \boldsymbol{\varphi} := \sup \left\{ \sum_{i=1}^{n} \|\boldsymbol{\varphi}(t_i) - \boldsymbol{\varphi}(t_{i-1})\} \right\},$$

and that $L(\gamma) := \mathrm{Var}_I\, \boldsymbol{\varphi}$. Then,

$$L(\gamma) = \int_a^b \|\boldsymbol{\varphi}'(t)\| \, dt.$$

- Proof requires Lemmas 103 and 104.
- Likely just need to know how to calculate length.

3. **Lemma 103 (Triangle Inequality for Integrals)** If $\mathbf{f} : [c,d] \to \mathbb{R}^N$ is Riemann Integrable, then $\|\mathbf{f}\| : [c,d] \to \mathbb{R}$ is Riemann Integrable and

$$\left\| \int_c^d \mathbf{f}(t) \, dt \right\| \leq \int_c^d \|\mathbf{f}(t)\| \, dt.$$

4. **Lemma 104 (Another Integral Inequality)** If $\mathbf{f} : [c,d] \to \mathbb{R}^N$ is Riemann Integrable, then, for $t_0 \in [c,d]$,

$$\left\| \int_c^d \mathbf{f}(t)\ dt \right\| \geq \int_c^d \|\mathbf{f}(t)\|\ dt - 2\int_c^d \|\mathbf{f}(t) - \mathbf{f}(t_0)\|\ dt.$$

5. **Proposition 116 (Absolute Continuity implies Finite Variation)** If $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ is absolutely continuous, then $\boldsymbol{\varphi}$ has finite variation.

   - Proof relies on previous exercise.

6. **Theorem 119 (Regular curves can be Arc-Length Parametrized)** If γ is regular (i.e., it is piecewise C^1 and admits a parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ with nonzero left and right derivatives on $[c,d]$), then γ can be parametrized by arclength.

   Proof: Define a *length* function $s : [a,b] \to [0, L(\gamma)]$. Show that s is invertible. Then, since the inverse of a continuous, one-to-one function on a compact set is continuous, s^{-1} is continuous. Therefore, since, by the FTC, $s' > 0$ a.e., so that s is piecewise C^1, s^{-1} is C^1 with the usual derivative, so that $\boldsymbol{\psi}(t) := \boldsymbol{\varphi}(s^{-1}(t)), t \in [0, L(\gamma)]$ is equivalent to $\boldsymbol{\varphi}$, and, furthermore,

$$\boldsymbol{\psi}'(t) = \frac{\boldsymbol{\varphi}'(s^{-1}(t))}{\|\boldsymbol{\varphi}'(s^{-1}(t))\|},$$

   so that $\|\boldsymbol{\psi}'(t)\| = 1$. ■

Curve Integrals

1. **Theorem 129 (Fundamental Theorem of Calculus for Curves)** Let $U \subseteq \mathbb{R}^N$ be open, let $f \in C^1(U)$, let $\mathbf{x}, \mathbf{y} \in U$, and let γ be piecewise C^1 in U with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}(a) = \mathbf{y}$ and $\boldsymbol{\varphi}(b) = \mathbf{x}$. Then,

$$\int_\gamma \nabla f = f(\mathbf{x}) - f(\mathbf{y}).$$

   Proof: Define $p(t) = f(\boldsymbol{\varphi}(t))$ and note that, by the Chain Rule, p is piecewise C^1 with (a.e)

$$p'(t) = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(\boldsymbol{\varphi}(t))\varphi_i'(t).$$

   Thus, by the FTC,

$$\int_\gamma \nabla f = \int_a^b \sum_{i=1}^N \frac{\partial f}{partial x_i}(\boldsymbol{\varphi}(t))\varphi_i'(t)\ dt = \int_a^b p'(t)\ dt = p(b) - p(a) = f(\mathbf{y}) - f(\mathbf{x}). \quad ■$$

2. **Theorem 130 (Conservative Field Equivalents)**

3. **Theorem 132 (Conservative Fields are Irrotational)**

4. **Theorem 135 (Poincaré's Lemma I (Starshaped Domains))**
5. **Theorem 143 (Homotopic Curves Integrals in Irrotational Fields)**
6. **Theorem 144 (Poincaré's Lemma II (Simply Connected Domains))**
7. **Lemma 145 (Lebesgue Number)**
8. **Theorem 150 (Simple Connectivity Equivalents)**
9. **Theorem 154 (Integral Definitions of the Winding Number)**
10. **Theorem 155 (Winding Number Constant on Connected Components)**