Homework 7

21-640 Introduction to Functional Analysis
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Due: Wednesday, May 7, 2013

Problem 1

Suppose $x \in (C+K)^C$. Then, $\forall k \in K$, $x \notin C+k$, and so, since $C+k$ is closed (by continuity of $+$), there exist neighborhoods U_k and V_k with $x \in U_k$, $k \in V_k$, and $U_k \cap (C+V_k) = \emptyset$ (take neighborhoods U of x and V of 0 with $U \cap C = \emptyset$ and $V - V \subseteq U - x$, and choose $U_k := V + x, V_k := V + k$).

Since K is compact, it has a finite cover $\{V_{k_1}, \dots, V_{k_n}\} \subseteq \{V_k : k \in K\}$. Thus, for $U := \bigcap_{i=1} U_{k_i}$,

$$(C+K) \cap U \subseteq \left(C + \bigcup_{i=1}^{n} V_{k_i} \right) \cap U = \emptyset, .$$

$x \in U$, and U is open. Therefore, $(C+K)^C$ is open, and so $C+K$ is closed. ■

Problem 2

Since X is locally convex, it has a separating family of seminorms; in particular, there is a seminorm $p : X \to \mathbb{R}$ with $p(x) \neq 0$. We can then (uniquely) define a linear functional $f_0 : \operatorname{span}(x) \to \mathbb{R}$ by $f_0(x) = p(x) \neq 0$, noting $|f(y)| \leq p(y)$, $\forall y \in \operatorname{span}(x)$. Then, by the Hahn-Banach Theorem, there is an extension $f : X \to \mathbb{R}$ of f_0 with $|f| \leq p$. Thus, since p is continuous on X and $p(0) = 0$, it is immediate that f is continuous at 0, and so, by Proposition 13.3, f is continuous on X. ■

Problem 3

(a) Suppose $E \subseteq X$ is τ bounded, so that, by Theorem 14.2, $\forall x \in X$, the function $f \mapsto |f(x)|$ is bounded on E. Then, we can define $b : [0,1] \to \mathbb{R}$ by

$$b(x) := \sup_{f \in E} |f(x)|, \quad \forall x \in [0,1].$$

Note that since each f is continuous, $\forall \alpha \in \mathbb{R}$ the set

$$\{x \in [0,1] : b(x) > \alpha\} = \bigcup_{f \in E} \{x \in [0,1] : |f(x)| > \alpha\}$$

is a union of open sets and is therefore open, so that b is lower semi-continuous (thanks to Jimmy Murphy for pointing this out to me). Since the function $g := x \mapsto \frac{|x|}{1+|x|}$ is continuous and non-decreasing, $g \circ b$ is also lower semi-continuous.

Semi-continuous functions are clearly Borel measurable and hence Lebesgue measurable. Also, $g \leq 1$ on $\mathbb{R}$, so, by the Lebesgue's Dominated Convergence Theorem, $g \circ b$ is integrable on $[0,1]$.

By construction of ρ and the fact that ρ-balls give a local base of σ at 0, E is σ-bounded if and only if, $\forall \varepsilon > 0$, $\exists t_\varepsilon > 0$ such that $\forall t > t_\varepsilon, f \in E$, $\rho(f/t, 0) < \varepsilon$. Since

$$\rho(f/t, 0) = \int_0^1 \frac{|f(x)/t|}{1 + |f(x)/t|}\, dx = \int_0^1 \frac{|f(x)|}{t + |f(x)|}\, dx \leq \int_0^1 \frac{|b(x)|}{t + |b(x)|}\, dx,$$

and, again by the Dominated Convergence Theorem, the last integral approaches 0 as $t \to \infty$, for every $\varepsilon > 0$, such a t_ε must exist. ■

(b) By Theorem 14.2 and the definition of p_x, the family

$$\{\{f \in X : p_x(f) = |f(x)| < 1/n_x, \forall x \in S\} : S \subseteq [0,1] \text{ finite}, n_x \in \mathbb{N} \text{ for each } x \in S\}$$

of sets is a local base of (X, τ) at 0. Thus, letting $\varepsilon = 1/4$, to show that I is not continuous, it suffices, for an arbitrary finite set $S \subseteq [0,1]$ to construct a function f with $f = 0$ on S and

$$\rho(f, 0) = \int_0^1 \frac{|f(x)|}{1 + |f(x)|}\, dx \geq \varepsilon.$$

Although we do not explicitly give the construction, on can construct such a function f by fixing $f = 0$ on S and $f = 1$ on a set of measure at least $1/2$ (and positive distance from S), and then interpolating to make f continuous. ■

Problem 4

Suppose E is topologically bounded, and let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{K}$ and X, respectively, with $\alpha_n \to 0$ as $n \to \infty$. Suppose $U \subseteq X$ is a neighborhood of 0. Since E is topologically bounded, $\exists t_0 > 0$ such that $E \subseteq tU, \forall t > t_0$. Since $\alpha_n \to 0$, $\exists N \in \mathbb{N}$ with $\alpha_n < 1/t, \forall n > N$. since each $x_n \in tU$, $\forall n > N$, $\alpha_n x_n \in U$. Therefore, $\alpha_n x_n \to 0$ as $n \to \infty$.

Suppose, on the other hand, that E is not topologically bounded, so that there is a neighborhood V of 0 such that, $\forall n \in \mathbb{N}, \exists t_n > n$ with $E \not\subseteq t_n V$. Then, $\forall n \in \mathbb{N}$, for $\alpha_n = \frac{1}{t_n}$, $\exists x_n \in E$ with $\alpha_n x_n \notin V$, and hence $\alpha_n x_n \not\to 0$ as $n \to \infty$. ■

Problem 5

(a) $\forall x, y, z \in X$, $\rho(x + z, y + z) = F((x + z) - (y + z)) = F(x - y) = \rho(x, y)$.

(b) Let $r > 0, B := \{x \in X : \rho(x, 0) < r\}$, $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, and $x \in B$.

By part (d) of Lemma 5.32, since each V_n is convex, absorbing, and balanced,

$$\begin{aligned}\rho(\alpha x, 0) = F(\alpha x) &= \max\left\{\frac{1}{n}\min\{1, p_n(\alpha x)\} : n \in \mathbb{N}\right\}\\ &= \max\left\{\frac{1}{n}\min\{1, |\alpha| p_n(x)\} : n \in \mathbb{N}\right\}\\ &\leq \max\left\{\frac{1}{n}\min\{1, p_n(x)\} : n \in \mathbb{N}\right\} = F(x) = \rho(x, 0) < r.\end{aligned}$$

Thus, $\alpha x \in B$, and so B is balanced. ■

(c) Let $r > 0, B := \{x \in X : \rho(x, 0) < r\}$, $x, y \in B$, and $t \in (0, 1)$. Then, $\forall n \in \mathbb{N}$,

$$p_n(tx + (1 - t)y) \leq p_n(tx) + p_n((1 - t)y) = tp_n(x) + (1 - t)p_n(y),$$

by Lemma 5.32, since each V_n is convex and absorbing. Since $tp_n(x), (1 - t)p_n(y) \geq 0$,

$$\min\{1, tp_n(x) + (1 - t)p_n(y)\} \leq \min\{1, tp_n(x)\} + \min\{1, (1 - t)p_n(y)\}.$$

Thus,

$$\begin{aligned}\rho(tx + (1 - t)y, 0) &= \max\left\{\frac{1}{n}\min\{1, p_n(tx + (1 - t)y)\} : n \in \mathbb{N}\right\}\\ &\leq \max\left\{\frac{1}{n}\min\{1, tp_n(x) + (1 - t)p_n(y)\} : n \in \mathbb{N}\right\}\\ &\leq t\max\left\{\frac{1}{n}\min\{1, p_n(x)\} : n \in \mathbb{N}\right\}\\ &+ (1 - t)\max\left\{\frac{1}{n}\min\{1, p_n(y)\} : n \in \mathbb{N}\right\} = t\rho(x, 0) + (1 - t)\rho(y, 0) < r.\end{aligned}$$

Thus, $tx + (1 - t)y \in B$, and so B is convex. ■

(d) Let τ denote the topology of X. If $r \in (0, 1]$, then the ρ-ball of radius R centered at 0 is

$$\begin{aligned}
\{x \in X : F(x) < r\} &= \{x \in X : \max\{\min\{1, p_n(x)\}/n : n \in \mathbb{N}\} < r\} \\
&= \bigcap_{n \in \mathbb{N}} \{x \in X : \min\{1, p_n(x)\} < nr\} \\
&= \bigcap_{n=1}^{\lceil r \rceil} \{x \in X : p_n(x) < nr\} = \bigcap_{n=1}^{\lceil r \rceil} V(p_n, nr).
\end{aligned}$$

By Theorems 14.1 and 14.2, then, the topology induced on X by ρ is no finer than τ.

I didn't have time to figure out how to show the converse.