Homework 5

21-470 Calculus of Variations
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Problem 1

For $f : [0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y, z) = z_1^4 + z_2^2 - z_1 y_2 + y_1^3 - 2xy_2$,

$$f_{,2}(x, y, z) = \begin{bmatrix} 3y_1^2 \\ -z_1 - 2x \end{bmatrix} \quad \text{and} \quad f_{,3}(x, y, z) = \begin{bmatrix} 4z_1^3 - y_2 \\ 2z_2 \end{bmatrix}.$$

Hence, if y_* minimizes J over $\mathscr{Y}$, then the 1^{st} Euler-Lagrange Equation gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = f_{2,}(x, y(x), y'(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x)) = \begin{bmatrix} 3y_1(x)^2 - \frac{d}{dx}(4y_1'(x)^3 - y_2(x)) \\ -y_1'(x) - 2x - \frac{d}{dx} 2y_2'(x) \end{bmatrix}.$$

We also have the natural boundary conditions

$$0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot f_{,3}(x, y(x), y'(x)) \Big|_{x=0} = 4y_1'(0)^3 - y_2(0) + 2y_2'(0)$$

and

$$0 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \cdot f_{,3}(x, y(x), y'(x)) \Big|_{x=1} = 4y_1'(1)^3 - y_2(1) - y_2'(1).$$

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Problem 2

For $f : [0, \pi/2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y, z, w) = w^2 - z^2 - 2y$, if y_* minimizes J over $\mathscr{Y}$, then the 1^{st} Euler-Lagrange Equation gives that $y'' \in C^2[0, \pi/2]$ and, $\forall x \in [0, \pi/2]$,

$$\begin{aligned} 0 &= f_{2,}(x, y(x), y'(x), y''(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x), y''(x)) + \frac{d^2}{dx^2} f_{,4}(x, y(x), y'(x), y''(x)) \\ &= -2 + 2y''(x) + 2y^{(4)}(x). \end{aligned} \tag{1}$$

Solutions to $0 = y''(x) + y^{(4)}(x)$ are cos, sin, constant functions, and linear functions, a particular solution to (1) is the function $x \mapsto x^2/2$, so that the general solution to (1) is

$$y(x) = c_1 \cos(x) + c_2 \sin(x) + c_3 x + c_4 + x^2/2, \quad \forall x \in [0, \pi/2].$$

Thus, we have the boundary conditions

$$0 = f_{,4}(x, y(x), y'(x), y''(x))\big|_{x=0} = 2y''(0) = -2c_1 + 2,$$

$$\begin{aligned} 0 = y(0) = c_1 + c_4, \qquad & 0 = y(\pi/2) = c_2 + c_3\pi/2 + c_4 + \pi^2/8, \\ \text{and} \qquad & 0 = y'(\pi/2) = -c_1 + c_3 + \pi/2, \end{aligned}$$

giving $c_1 = 1, c_2 = \pi^2/8 - 1 - \pi/2, c_3 = 1 - \pi/2$, and $c_4 = -1$.

Problem 3

For $f : [1, 2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y, z, w) = x^3 w^2 - 24xy$, if y_* minimizes J over $\mathscr{Y}$, then the 1^{st} Euler-Lagrange Equation gives, $\forall x \in [0, \pi/2]$,

$$\begin{aligned} 0 &= f_{2,}(x, y(x), y'(x), y''(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x), y''(x)) + \frac{d^2}{dx^2} f_{,4}(x, y(x), y'(x), y''(x)) \\ &= -24x + \frac{d^2}{dx^2} 2x^3 y''(x), \end{aligned}$$

and hence, integrating twice gives $x^3 y''(x) = 2x^3 + c_1 x + c_2$, for some $c_1, c_2 \in \mathbb{R}$. Dividing both sides by x^3 and integrating twice more gives, for some $c_3, c_4 \in \mathbb{R}$,

$$y(x) = x^2 + c_3 x + c_4 - c_1 \ln x + \frac{c_2}{2x}, \quad \forall x \in [1, 2].$$

The given boundary conditions give

$$\begin{aligned} 6 &= y(1) = 1 + c_3 + c_4 + c_2/2 \\ -1 &= y'(1) = 2 + c_3 - c_1 - c_2/2 \\ 8 &= y(2) = 4 + 2c_3 + c_4 - c_1 \ln 2 + c_2/4 \\ 4 &= y'(2) = 4 + c_3 - c_1/2 - c_2/8. \end{aligned}$$

Solving this (nonsingular) linear system gives $c_1 = c_4 = 0, c_2 = 8, c_3 = 1$, and so

$$y(x) = x^2 + x + 4/x, \quad \forall x \in [1, 2].$$

Problem 4

Let $f : [0,3] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,y,z) = z^4 - 8z^2$.

(a) The 1^{st} Weierstrass-Erdmann Corner Condition gives that

$$4\alpha^3 - 16\alpha = f_{,3}(c, y(c), \alpha) = f_{,3}(c, y(c), \beta) = 4\beta^3 - 16\beta. \tag{2}$$

The 2^{nd} Weierstrass-Erdmann Corner Condition gives that

$$-3\alpha^4 + 8\alpha^2 = f(c, y(c), \alpha) - \alpha f_{,3}(c, y(c), \alpha) = f(c, y(c), \beta) - \alpha f_{,3}(c, y(c), \beta) = -3\beta^4 + 8\beta^2. \tag{3}$$

Equation (2) can be written as

$$(\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) = \alpha^3 - \beta^3 = 4(\alpha - \beta),$$

so that, since $c \in S(y)$ and hence $\alpha \neq \beta$,

$$\alpha^2 + \alpha\beta + \beta^2 = 4 \tag{4}$$

Equation (3) can be written as

$$3(\alpha - \beta)(\alpha + \beta)(\alpha^2 + \beta^2) = 3(\alpha^4 - \beta^4) = 8(\alpha^2 - \beta^2) = 8(\alpha - \beta)(\alpha + \beta)$$

so that (again, since $c \in S(y)$ and hence $\alpha \neq \beta$),

$$3(\alpha + \beta)(\alpha^2 + \beta^2) = 8(\alpha + \beta).$$

Thus, we have two cases: either (C1) $\alpha^2 + \beta^2 = 8/3$ or (C2) $\alpha = -\beta$.

Case (C1): ($\alpha^2 + \beta^2 = 8/3$) Using Equation (4), we have

$$\alpha\beta = 4 - 8/3 = 4/3,$$

so that $\operatorname{sign}(\alpha) = \operatorname{sign}(\beta)$ and $\beta = \frac{4}{3\alpha}$. Plugging this back into (C1) and rearranging gives

$$9\alpha^4 - 24\alpha^2 + 16 = 0,$$

so that the quadratic formula gives $\alpha^2 = 4/3$. But then (C1) gives $\beta^2 = 4/3$, and so $\alpha = \beta$ (since $\operatorname{sign}(\alpha) = \operatorname{sign}(\beta)$), contradicting the fact that $c \in S(y)$.

Case 2 (C2): ($\alpha = -\beta$) Using Equation (4) gives $\alpha^2 = 4$, so that $(\alpha, \beta) \in \{(2, -2), (-2, 2)\}$.

This is consistent with the observation that $f(x, y, z) = (z^2 - 4) - 16$, which is clearly minimized when $z = \pm 2$.

(b) Minimizers with exactly 1 corner must be piecewise linear, with slope 2 on $(0, a)$ and slope -2 on $(a, 3)$ or slope -2 on $(0, 3 - a)$ and slope 2 on $(3 - a, 3)$, for some $a \in [0, 3]$. Since $y(0) = 0$ and $y(3) = 2$, $a = 2$, and the two possible minimizers with exactly one corner are

$$y_1(x) = \begin{cases} 2x & \text{if } x \in [0,2] \\ 8 - 2x & \text{if } x \in [2,3] \end{cases} \quad \text{and} \quad y_2(x) = \begin{cases} -2x & \text{if } x \in [0,1] \\ 2x - 4 & \text{if } x \in [1,3] \end{cases}.$$