21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
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Remark 13.1: Since a Lie group G is a differentiable manifold, one can define the notion of a tangent space $T_g G$ at $g \in G$. In a general differentiable manifold, one cannot compare the tangent spaces at different points, but in a Lie group, one can relate the tangent spaces at different points by using multiplication: if R_h is the mapping of multiplication by h on the right, i.e. $g \mapsto g\,h$ for $g \in G$, then the differential DR_h maps the tangent space $T_g G$ into $T_{g\,h}G$ for every $g \in G$, and since R_h and $R_{h^{-1}}$ are inverses, one deduces that DR_h provides an isomorphism from $T_g G$ onto $T_{g\,h}G$, with inverse $DR_{h^{-1}}$. Of course, the same remark applies with L_h, the mapping of multiplication by h on the left, i.e. $g \mapsto h\,g$, and DL_h provides an isomorphism from $T_g G$ onto $T_{h\,g}G$, with inverse $DL_{h^{-1}}$. Hence $T_g G$ is isomorphic to $T_e G$ by either DL_g of DR_g.

This permits to define the *exponential map*. For a tangent vector $v \in T_e G$, one has $DL_g v \in T_g G$, and one may solve a differential equation corresponding to this (tangent) vector field, i.e. find $X(t) \in G$ for t in an open interval $I \subset \mathbb{R}$ containing 0, such that $DX(t) = DL_{X(t)}v$ for $t \in I$ and $X(0) = g_0 \in G$; the unique solution (defined in a maximal interval depending upon v and g_0) gives the exponential map $X(t) = exp(t; v)X(0)$, and with natural restrictions on the domains of definition, one has $exp(s; v) \circ exp(t; v) = exp(s + t; v)$ and $exp(\lambda\,t; v) = exp(t; \lambda\,v)$.

Although it is true in particular cases, like for $S\mathbb{O}(n)$, it is not always the case that every point in the connected component of e in G can be written as $exp(t; v)\,e$ for some $t \in \mathbb{R}$ and $v \in T_e G$. However, every point in the connected component of e in G can be written as $exp(t_m; v_m) \circ \cdots \circ exp(t_1; v_1)\,e$ for some $m \geq 1$ and $t_1, \ldots, t_m \in \mathbb{R}$ and $v_1, \ldots, v_m \in T_e G$.

Remark 13.2: An E-vector space V is called an *algebra* if it has a multiplication $(v, w) \mapsto v \cdot w \in V$ which is a bilinear mapping (so that $(\lambda\,v) \cdot w = v \cdot (\lambda\,w) = \lambda\,(v \cdot w)$ for all $v, w \in V, \lambda \in E$); usually, multiplication is asked to be associative. For a Lie group G, the tangent space $\mathcal{G} = T_e G$ is endowed with a bilinear mapping of a different kind, for which one prefers the notation $[v, w]$, called a *Lie bracket*, which is a skew-symmetric bilinear mapping (i.e. $[w, v] = -[v, w]$ for all $v, w \in \mathcal{G}$) satisfying the *Jacobi identity*, i.e. $\big[a, [b, c]\big] + \big[b, [c, a]\big] + \big[c, [a, b]\big] = 0$ for all $a, b, c \in \mathcal{G}$, and such a structure is called a *Lie algebra*.

In the case of $G = S\mathbb{O}(n)$, $\mathcal{G}$ is the space of skew-symmetric $n \times n$ matrices with entries in $\mathbb{R}$, and $[v, w] = v\,w - w\,v$, and Jacobi identity actually holds for elements of $L(V, V)$ (with $[v, w] = v\,w - w\,v$) for any E-vector space, by developing the expression, which gives twelve terms and each of the six permutations of a, b, c occur twice, one time with a $+$ sign, and one time with a $-$ sign.

For defining the Lie bracket on $\mathcal{G}$, one considers the mappings $exp(s; v)$ and $exp(t; w)$ for $v, w \in \mathcal{G}$ (which map G into itself) and one considers the commutator $exp(-t; w) \circ exp(-s; v) \circ exp(t; w) \circ exp(s; v)$: for $s = t = 0$, it maps e into e, and for s and t small, it looks like $exp(s\,t; z)$ for a vector $z \in \mathcal{G}$, which depends upon v, w in the correct bilinear way, so that one defines $[v, w] = z$.

Remark 13.3: Since a Lie group is a manifold, there are neighbourhoods of a point which look like balls in $\mathbb{R}^d$, and the connected component of any point g is the set of points h which can be reached by a (continuous) path,[1] i.e. a continuous mapping ψ from $[0, 1]$ into G such that $\psi(0) = g$ and $\psi(1) = h$. If G_0 is the connected component of e in a Lie group G, then G_0 is itself a Lie group, since the product maps $G_0 \times G_0$ into G_0.[2]

It can be shown that the Lie algebra structure of $\mathcal{G}$ permits to reconstruct what G_0 is. For any $g \in G \backslash G_0$, the connected component of g is $g\,G_0$, but if H is any discrete group, then $H \times G_0$ is a Lie group, so that the knowledge of G_0 cannot give information about what the rest of the Lie group is.

Remark 13.4: That the group of rotations $S\mathbb{O}(3)$ is connected does not tell much about its topology, and since $S\mathbb{O}(3)$ is a three-dimensional manifold embedded in $\mathbb{R}^9$, it is not so easy to "see" some of its properties.

[1] In a general topological space X, the *connected component* of $a \in X$ is the smallest subset which is both open and closed and contains a; X is said to be *connected* if the only subsets of X which are both open and closed are $\emptyset$ and X.

[2] If ψ_1 is a path from e to g_1 and ψ_2 is a path from e to g_2, then $\psi_1\psi_2$ is a path from e to $g_1 g_2$, because multiplication is continuous.

In a (connected) topological space X, a path ψ_0 from a to b is said to be *homotopic* to a path ψ_1 from a to b if there exists a *homotopy* from ψ_0 to ψ_1, i.e. a continuous mapping Ψ from $[0,1]\times[0,1]$ such that $\Psi(x,0)=\psi_0(x), \Psi(x,1)=\psi_1(x)$ for all $x\in[0,1]$, $\Psi(0,y)=a, \Psi(1,y)=b$ for all $y\in[0,1]$. X is said to be *simply connected* if for all $a,b\in X$ any two paths from a to b are homotopic; said otherwise, any path from a to a (called a *loop*) can be deformed to a constant path (i.e. $\psi(t)=a$ for all $t\in[0,1]$) by a *homotopy*.

A (non-empty) subset C of an R-vector space is *convex* if for all $c_1,c_2\in C$ the segment $[c_1c_2]$ belongs to C, i.e. $(1-t)\,c_1=t\,c_2\in C$ for all $t\in[0,1]$; any convex set of $\mathbb{R}^d$ is simply connected.

The sphere $S^{n-1}\subset\mathbb{R}^n$ is simply connected for all $n\geq 3$, but the circle $S^1\subset\mathbb{R}^2$, the torus $\mathbb{T}^2\subset\mathbb{R}^3$ (isomorphic to $S^1\times S^1$), and $S\mathbb{O}(3)$ are not simply connected, in different ways.

Remark 13.5: If ψ_1 and ψ_2 are loops from a to a, one creates a new loop $\psi_3=\psi_1\star\psi_2$ by *concatenation* i.e. going through the first loop and then through the second, by considering (for example) $\psi_3(t)=\psi_1(2t)$ for $t\in\left[0,\frac{1}{2}\right]$ and $\psi_3(t)=\psi_2(2t-1)$ for $t\in\left[\frac{1}{2},1\right]$. One then observes that for three loops ℓ_1,ℓ_2,ℓ_3 from a to a, the loop $\ell_1\star(\ell_2\star\ell_3)$ is homotopic to the loop $(\ell_1\star\ell_2)\star\ell_3$, and since being homotopic is an equivalence relation for loops from a to a, which is compatible with concatenation, one deduces that there is an associative operation on equivalence classes of loop, which has an identity, the constant loop at a. One then observes that each loop ψ_+ from a to a has an inverse ψ_- obtained by going through the loop backwards, i.e. $\psi_-(t)=\psi_+(1-t)$ for $t\in[0,1]$, since both $\psi_+\star\psi_-$ and $\psi_-\star\psi_+$ are easily seen to be homotopic to the constant loop at a. One then has defined a structure of group, the *first homotopy group* of X, denoted $\pi_1(X,a)$; if X is connected it is easy to see that the construction with another base point b gives $\pi_1(X,b)$ isomorphic to $\pi_1(X,a)$. X is simply connected if and only if $\pi_1(X,a)$ is the trivial group $\{e\}$.

Remark 13.6: A *covering map* of a topological space X is a continuous surjective mapping p from a topological space C onto X such that each $a\in X$ has a neighbourhood U which is evenly covered by p, i.e. $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is homeomorphic to U; C is called a *covering space* of X. If moreover C is simply connected, it is called a *universal cover* of X. If X is a manifold, it has a universal cover which is a manifold, and if ψ is a loop from a to a in X, and $c\in p^{-1}(a)$, then ψ lifts into a uniquely defined path from c to a point $d\in p^{-1}(a)$ which only depends upon the equivalence class of ψ in $\pi_1(X,a)$.

For $X=S^1$, one may take $C=\mathbb{R}$, and $p(t)=(\cos t,\sin t)$, so that $\pi_1(S^1,a)\simeq\mathbb{Z}$. For $X=\mathbb{T}^2$, one may take $C=\mathbb{R}^2$, and $\pi_1(\mathbb{T}^2,a)\simeq\mathbb{Z}\times\mathbb{Z}$.[3]

Remark 13.7: For $X=S\mathbb{O}(3)$, one considers for C the unit sphere $S^3\subset\mathbb{R}^4$ (which is simply connected), and the projection p from C onto $S\mathbb{O}(3)$ is defined as follows. One identifies C with the set of quaternions $U=r+x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ having unit norm, i.e. $r^2+x^2+y^2+z^2=1$, so that $U^{-1}=r-x\,\mathbf{i}-y\,\mathbf{j}-z\,\mathbf{k}$; one identifies a vector $v\in\mathbb{R}^3$ with a quaternion with real part 0, namely $q_v=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$, and one may consider that $U=\cos\theta+\sin\theta\,q_u$ for $\theta\in[0,\pi]$ and a unit vector $u\in\mathbb{R}^3$. Then one checks that the conjugation $U\,q_vU^{-1}$ corresponds to a vector q_w and that one goes from v to w by a rotation of axis u and angle 2θ,[4] and $p(U)$ is this rotation. Indeed, using the definition of the product of quaternions (i.e. $(\alpha+q_v)\,(\beta+q_w)=\gamma+q_z$ with $\gamma=\alpha\,\beta-(v,w)$, and $z=\alpha\,w+\beta\,v+v\times w$) one has $q_vU^{-1}=q_v(\cos\theta-\sin\theta\,q_u)=\sin\theta\,(v,u)+(\cos\theta\,v-\sin\theta\,v\times u)$, and $U\,q_vU^{-1}=(\cos\theta+\sin\theta\,q_u)\big(\sin\theta\,(v,u)+(\cos\theta\,v-\sin\theta\,v\times u)\big)=\gamma+q_z$ with $\gamma=\cos\theta\,\sin\theta\,(v,u)-\sin\theta\,\big(u,(\cos\theta\,v-\sin\theta\,v\times u)\big)=0$, and $z=\cos\theta\,(\cos\theta\,v-\sin\theta\,v\times u)\big)+\sin\theta\,(v,u)\,\sin\theta\,u+\sin\theta\,u\times(\cos\theta\,v-\sin\theta\,v\times u)=M\,v$ for some $M\in L(\mathbb{R}^3,\mathbb{R}^3)$; one checks easily that $M\,u=u$, so that it remains to see that when v is orthogonal to u its image $M\,v$ is obtained by a rotation: without loss of generality, one may then choose an orthonormal basis (while keeping the orientation, so that the cross product of two vectors keeps the same form) such that $u=e_3$, and check only the case $v=e_1$; noticing that $v\times u=e_1\times e_3=-e_2$ and $u\times(v\times u)=-e_3\times e_2=e_1$, one deduces that $M\,e_1=\cos^2\theta\,e_1+\sin\theta\,\cos\theta\,e_2+\sin\theta\,\cos\theta\,e_2-\sin^2\theta\,e_1=\cos 2\theta\,e_1+\sin 2\theta\,e_2$.

One has $p(-U)=p(U)$, so that p is two-to-one, hence $\pi_1(S\mathbb{O}(3),a)\simeq\mathbb{Z}_2$. It shows that on $S\mathbb{O}(3)$ there is a (non-contractible) loop at a quite different than the ones existing on S^1 (where one can count the number of turns) or the torus $\mathbb{T}^2$ (where one can count two number of turns), since if one follows it twice it becomes contractible (i.e. homotopic to the constant loop at a).

[3] If $X=X_1\times X_2$ and $a=(a_1,a_2)\in X$, one has $\pi_1(X,a)\simeq\pi_1(X_1,a_1)\times\pi_1(X_2,a_2)$.

[4] Of course, u is not defined if $\theta=0$ or $\theta=\pi$ (i.e. $U=\pm1$), but in this case the rotation is the identity.