21-484 Notes

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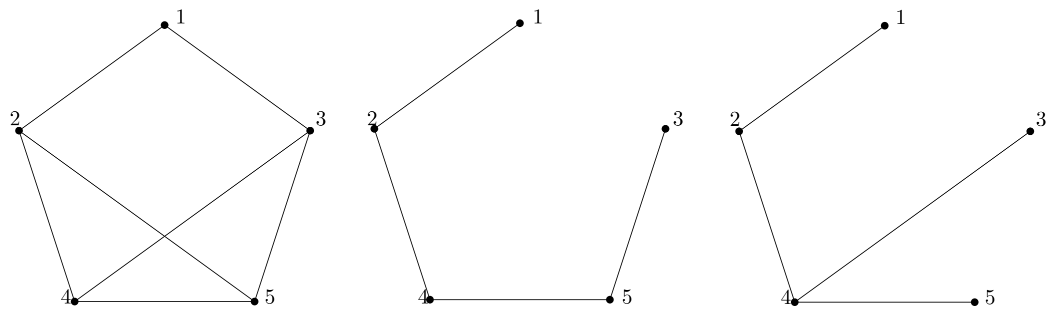
→ Recall: A subgraph H of a graph G is spanning if $V(H) = V(G)$.

Def: (p. 95) A spanning tree of a connected graph G is a spanning subgraph which is a tree.

Example: (Fig 4.7)



claim (Thm 4.10): Every connected graph contains a spanning tree.

Proof: Let H be a minimal (by number of edges) connected spanning subgraph of G.

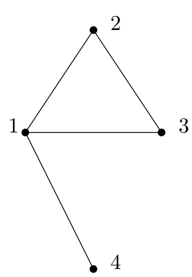
→ If H is not a tree, it contains a cycle. Removing a non-bridge from H results in a smaller connected spanning subgraph ↯.

Recall: The number of spanning trees contained in K_n is n^{n-2} (Cayley's formula).

Def: (page 48) The adjacency matrix of a graph G with n vertices is the matrix $A = A_G = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1 & ij \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Example:

$G =$ 

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Def: The Laplacian matrix of a graph G with n vertices is the $n \times n$ matrix $L = L_G = (\ell_{i,j})$ where

$$\ell_{i,j} = \begin{cases} \deg(i) & i = j \\ -1 & i \neq j \text{ and } ij \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

$$L_G = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Theorem (Thm 4.16, the matrix tree theorem, Kirchoff's theorem)

Let G be a graph and let $\lambda_1, \dots, \lambda_{n-1}$ be the non-sero eigenvalues of L_G. Then the number of spanning trees of G is

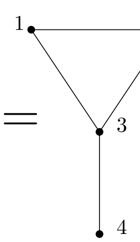
$$\frac{1}{n}\lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}$$

Equivalently: the number of spanning trees is the absolute value of any cofactor of L_G.

Def: The (i, j)-minor of a matrix A is the determinant of the matrix you get by removing the i^{th} row and the j^{th} column of A. Denote it by $M_{i,j}$.

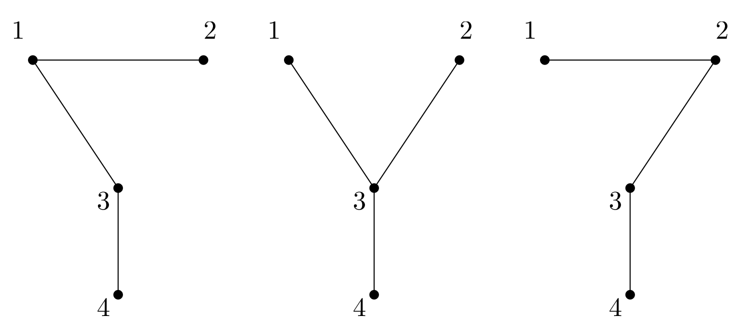
- The (i, j) cofactor of A is $C_{ij} = (-1)^{i+j} M_{i,j}$.

Example:

$G =$ 

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad L_G = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{3,3} = (-1)^6 \cdot \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

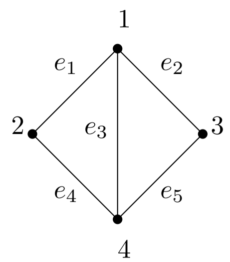


Elements from the proof:

Def (p. 48): The incidence matrix of a graph G with n vertices and m edges is the $n \times m$ matrix $B = B_G = (b_{i,j})$ where

$$b_{ij} = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ vertext belongs to the } j^{\text{th}} \text{ edge} \\ 0 & \text{otherwise} \end{cases}$$

Example:



$$B_G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Def: An oriented incedence matrix is an incedent matrix that has one 1 and one -1 in every column. Denoted by $\overrightarrow{B_G}$

When saying "the" oriented incedence matrix we mean that upper nonzero element in every column is 1.

notice: $L_G = \overrightarrow{B_G} \cdot \overrightarrow{B_G}^T$

$\rightarrow$ Also, $M_{1,1} = \overrightarrow{F} \cdot \overrightarrow{F}^T$ where $\overrightarrow{F} = \overrightarrow{B_G}$ without the first row.

$\rightarrow$ Apply the Cauchy-Binet Theorem

$$\det(M_{1,1}) = \sum_S \det(F_s) \cdot \det(F_s^T) = \sum_S \mathrm{Det}(F_s)^2$$

where s goes over all subsets of size $n-1$ of $2 \ldots, m$.

$\rightarrow$ notice that the $\det(F_s) = \pm 1$ when s spans a tree.