21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
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Remark 14.1: A *conformal transformation* is a differentiable mapping which conserves angles and orientation. Since one cannot talk about angles without an Euclidean structure, the concept cannot be applied to general (differentiable) manifolds: a *Riemannian manifold* is a manifold M such that every tangent space $T_m M$ has an Euclidean structure (varying smoothly with $m \in M$), so that one can compute the length of a (smooth) curve on M, and define *geodesics*, which locally offer the shortest path between two points, and also measure the angle between the tangents to two intersecting curves.

The initial reason for being interested in conformal mappings may have been the need to map pieces of the earth onto a flat sheet of parchment, in order to see in which direction to sail from one point to another (separated by a large body of water): once the invention of the compass had been learned from the Chinese, the direction of the (magnetic) north pole was known all the time, and sailing was done by keeping a direction on the compass, so that if a map did not conserve angles it could not be used to measure in which direction to sail.

Remark 14.2: Using $\mathbb{R}^n$ with its usual Euclidean structure, an affine mapping $x \mapsto A\,x + b$ from $\mathbb{R}^n$ to itself is conformal if and only if $det(A) > 0$ and $A^T A = c\,I$ for some $c > 0$. Indeed, if e_1 and e_2 are orthogonal unit vectors, then $A\,e_1$ and $A\,e_2$ are orthogonal, and since $\cos\theta\,e_1 + \sin\theta\,e_2$ makes an angle θ with e_1, one must have $(A\,(\cos\theta\,e_1 + \sin\theta\,e_2), A\,e_1) = \cos\theta\,||A\,(\cos\theta\,e_1 + \sin\theta\,e_2)||\,||A\,e_1||$, i.e. $\cos\theta\,||A\,e_1||^2 = \cos\theta\,\big(\cos^2\theta\,||A\,e_1||^2 + \sin^2\theta\,||A\,e_2||^2\big)^{1/2}\,||A\,e_1||$, or $||A\,e_1|| = ||A\,e_2||$, so that for an orthonormal basis $e_1,\ldots,e_n$ one has $(A\,e_i, A\,e_j) = c\,\delta_{i,j}$ for $i,j = 1,\ldots,n$ for some $c > 0$, i.e. $A^T A = c\,I$.

Of course the set of $A \in L(\mathbb{R}^n,\mathbb{R}^n)$ satisfying $A^T A = c\,I$ for some $c > 0$ is a Lie group, and its associated Lie algebra, i.e. the tangent space at I, is the set of $M \in L(\mathbb{R}^n,\mathbb{R}^n)$ such that $M^T + M = \lambda\,I$ for some $\lambda \in \mathbb{R}$. If $det(A) > 0$ and $A^T A = c\,I$ for some $c > 0$, then $A = \sqrt{c}\,A_0$ with $A_0 \in S\mathbb{O}(n)$, hence $A_0 = e^{M_0}$ for a skew-symmetric M_0, and $A = e^M$ with $M = \frac{\log(c)}{2}\,I + M_0$.

For $n = 2$, $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ gives $\alpha^2 + \gamma^2 = \beta^2 + \delta^2 = c$ and $\alpha\,\beta + \gamma\,\delta = 0$, so that $\beta = \mp\gamma, \delta = \pm\alpha$, and since $\alpha\,\delta - \beta\,\gamma > 0$, one has $A = \begin{pmatrix} \alpha & -\gamma \\ \gamma & \alpha \end{pmatrix}$, so that a smooth mapping $(x,y) \mapsto \big(P(x,y), Q(x,y)\big)$ between two open sets of the plane $\mathbb{R}^2$ is conformal if and only if $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ and $\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$, which is the Cauchy–Riemann system, expressing that $f(z) = P(x,y) + i\,Q(x,y)$ is an holomorphic function of $z = x + i\,y$, hence there are infinitely many such conformal mappings from the whole of $\mathbb{R}^2$ into itself, by considering entire functions, i.e. power series in z with an infinite radius of convergence.

Remark 14.3: However, in the approximation where the earth is considered perfectly spherical (with radius R_0), it comes with two families of circles, the meridians (great circles going through the north and south poles) and the parallels (circles at a given latitude, parallel to the equatorial plane), and one would also like that their images in the planar map be reasonably simple: there are three quite natural answers, the *stereographic projection*, the *Mercator projection*, and the *Lambert projections*.

The stereographic projection consists in taking the tangent plane T_N at the north pole N, and mapping every point M of the sphere different from the south pole S into the intersection of the line SM with P_N.

The Mercator projection consists in using the tangent cylinder at the equator and mapping each meridian of longitude θ onto the line $x = R_0\cos\theta, y = R_0\sin\theta$ with z being a function f of the latitude, and writing that the transformation is conformal gives a differential equation for f.

The Lambert projection consists for a given latitude λ_0 in using the tangent cone at latitude λ_0 (so that the Mercator projection corresponds to $\lambda_0 = 0$) and mapping each meridian of longitude θ onto a generatrix of the cone, and a differential equation must be solved for finding where to send the circle of latitude λ. The maps used nowadays by sailors are pieces of Lambert projections.

Remark 14.4: Independently of it being conformal, the stereographic projection is a simple way to show that the sphere $S^2 \subset \mathbb{R}^3$ is homeomorphic to the Aleksandrov one-point compactification of $\mathbb{R}^2$ (i.e. one adds a unique point at infinity to $\mathbb{R}^2$), and it helps showing that S^2 is simply connected: if a loop ψ from a to a is not a Peano curve (which fills the entire S^2),[1] one picks a point S not in the image of ψ and one considers the stereographic projection from S, which gives a map F from $S^2\setminus\{S\}$ onto an affine plane, and one constructs the desired homotopy Ψ by $F\big(\Psi(x,y)\big)=(1-y)\,F\big(\psi(x)\big)+y\,F(a)$ for $x,y\in[0,1]$.

If ψ is a Peano curve, and the north pole N is attained as $\psi(t_1)$, there is a largest open set $I=(t_-,t_+)$ containing t_1 such that $\psi(t)$ belongs to the (open) northern hemisphere for $t\in I$, so that $\psi(t_\pm)$ belongs to the equator, and using the stereographic projection one can transform by homotopy the path from $\psi(t_-)$ to $\psi(t_+)$ by a path which connects these two points and stays on the equator; one repeats the operation if the new loop still goes through N, and by uniform continuity of ψ each interval I has a minimum length so that after finitely many of these operations one has a loop which does not go through N, hence it is no longer a Peano curve, and the first argument applies.

Remark 14.5: If F is a smooth bijection of an open set $\Omega\subset\mathbb{R}^n$ onto Ω', f a smooth scalar function on Ω, and u is a (smooth) solution of $\Delta\,u=0$ in Ω',[2] when is it that $v(x)=f(x)\,u\big(F(x)\big)$ automatically satisfies $\Delta\,v=0$ in Ω?

One has $\frac{\partial v}{\partial x_i}=\frac{\partial f}{\partial x_i}\,u(F)+f\sum_j\frac{\partial u}{\partial y_j}\,\frac{\partial F_j}{\partial x_i}$, and $\frac{\partial^2 v}{\partial x_i^2}=\frac{\partial^2 f}{\partial x_i^2}\,u(F)+2\frac{\partial f}{\partial x_i}\sum_j\frac{\partial u}{\partial y_j}\,\frac{\partial F_j}{\partial x_i}+f\sum_j\frac{\partial u}{\partial y_j}\,\frac{\partial^2 F_j}{\partial x_i^2}+f\sum_{j,k}\frac{\partial^2 u}{\partial y_j\partial y_k}\,\frac{\partial F_j}{\partial x_i}\,\frac{\partial F_k}{\partial x_i}$, so that $\Delta\,v=X\,u+\sum_j Y_j\frac{\partial u}{\partial y_j}+\sum_{j,k}Z_{j,k}\frac{\partial^2 u}{\partial y_j\partial y_k}$, with $X=\Delta\,f$, $Y_j=f\,\Delta\,F_j+2\sum_i\frac{\partial f}{\partial x_i}\,\frac{\partial F_j}{\partial x_i}$ for all j, and $Z_{j,k}=f\,\sum_i\frac{\partial F_j}{\partial x_i}\,\frac{\partial F_k}{\partial x_i}$ for all j,k, hence it is necessary that $X=0$, $Y_j=0$ for all j, and there exists a scalar function c such that $Z_{j,k}=c\,\delta_{j,k}$ for all j,k.[3]

At points where $f\neq 0$, the condition $\sum_i\frac{\partial F_j}{\partial x_i}\,\frac{\partial F_k}{\partial x_i}=\frac{c}{f}\,\delta_{j,k}$ for all j,k means that F is either a conformal mapping or (if the Jacobian $det(\nabla\,F)$ is <0) an anti-conformal mapping, which conserves angles but changes orientation; then f should be such that $\Delta\,f=0$ and $0=f\,Y_j=\sum_i\frac{\partial}{\partial x_i}\big(f^2\frac{\partial F_j}{\partial x_i}\big)=0$. In the case where F is affine, i.e. $F(x)=A\,x+b$ with $A^TA=c\,I$, one has $\Delta\,F_j=0$ for all j and the conditions on f are $\Delta\,f=0$ and $\sum_i\frac{\partial f}{\partial x_i}\,A_{j,i}=0$ for all j, i.e. $A\,grad(f)=0$, but since A is invertible it means $grad(f)=0$, hence f is constant.

Remark 14.6: The inversion (of center 0 and power $\kappa\neq 0$) $x\mapsto\kappa\,\frac{x}{r^2}$ is an anti-conformal mapping:[4] $F_j=\frac{\kappa\,x_j}{r^2}$ gives $\frac{\partial F_j}{\partial x_i}=\frac{\kappa\,\delta_{i,j}}{r^2}-2\frac{\kappa\,x_j}{r^3}\,\frac{x_i}{r}$, i.e. $\nabla\,F=\frac{\kappa}{r^2}\,\big(I-2\frac{x}{r}\otimes\frac{x}{r}\big)$, and $I-2\frac{x}{r}\otimes\frac{x}{r}$ is a mirror symmetry (which depends upon x). Then, $0=\sum_i\frac{\partial}{\partial x_i}\big(f^2\frac{\partial F_j}{\partial x_i}\big)=\frac{\partial}{\partial x_j}\big(\frac{\kappa\,f^2}{r^2}\big)-2\sum_i\frac{\partial}{\partial x_i}\big(\frac{\kappa\,x_ix_jf^2}{r^4}\big)$, and since $\frac{\partial}{\partial x_j}\big(\frac{1}{r^2}\big)-2\sum_i\frac{\partial}{\partial x_i}\big(\frac{x_ix_j}{r^4}\big)=(4-2n)\,\frac{x_j}{r^4}$, the equation is $\frac{1}{r^2}\,\big(I-2\frac{x}{r}\otimes\frac{x}{r}\big)\,grad(f^2)=2(n-2)\,\frac{x\,f^2}{r^4}$, which gives f constant if $n=2$ and $grad(f^2)=-2(n-2)\,\frac{x\,f^2}{r^2}$ if $n\geq 3$, i.e. $f=\frac{\gamma}{r^{n-2}}$ for some constant $\gamma\neq 0$, which does satisfy $\Delta\,f=0$.

LIOUVILLE proved in 1850 a rigidity theorem about (smooth) conformal mappings in $\mathbb{R}^n$ for $n\geq 3$, that they are the so-called Möbius transformations, obtained by composition of translations, similarities, orthogonal transformations and inversions.

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[1] Giuseppe PEANO, Italian mathematician, 1858–1932. He worked in Torino (Turin), Italy.

[2] In analysis, the Laplacian Δ is the operator $\sum_j\frac{\partial^2}{\partial x_j^2}$, but one must notice that differential geometers use a different sign convention.

[3] By taking for u any polynomial of degree ≤ 2 which is *harmonic* (i.e. has zero Laplacian), i.e. $u=a+(b,x)+(M\,x,x)$ with $M\in L_s(\mathbb{R}^n,\mathbb{R}^n)$ with $trace(M)=0$, one must have $X=0$ for killing the term in a, $Y=0$ for killing the term in b, and then $\sum_{j,k}Z_{j,k}M_{j,k}=0$ for all zero-trace M means the existence of c such that $\sum_{j,k}Z_{j,k}M_{j,k}=c\,trace(M)$ for all $M\in L_s(\mathbb{R}^n,\mathbb{R}^n)$.

[4] If $F=x\,g(r)$ then $\nabla\,F=g\,I+r\,g'\frac{x}{r}\otimes\frac{x}{r}$, which has $n-1$ eigenvalues equal to g, and one eigenvalue equal to $g+r\,g'$: if F is conformal, then g is constant, and if F is anti-conformal, then $g+r\,g'=-g$, i.e. $g=\frac{\kappa}{r^2}$ for some $\kappa\neq 0$.