21-484 Notes

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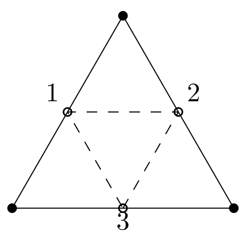
Edge coloring

$$f_k : E \to \{1, \dots, k\}$$

$\forall u, v_1, v_2 \ f(uv_1) \neq f(uv_2)$

k-edge colorable $\exists f_k$

k-edge chromatic, $\chi_1(G)$ k-edge colorable and not $k-1$ edge colorable



3-edge colorable
3-edge chromatic
4-edge colorable
not 4-edge chromatic

<ins>Vizing's Theorem:</ins> (10.12) All G $\chi_1(X) = \Delta(G)$ or $\chi_1(G) = \Delta(G) + 1$

<ins>Pr:</ins> $\chi_1(G) \geq \Delta(G)$

Take v of max degree. v has $\Delta(G)$ edges; each needs a color.

$\chi_1(G) \leq \Delta(G) + 1$

Induction on m (number of edges)

Take xy to be arbitrary.

$IH : \chi_1(G - xy) \leq \Delta(G - xy) + 1 \leq \Delta(G) + 1$

fix φ, color xy somehow

$\varphi(uv)$ is the color of uv

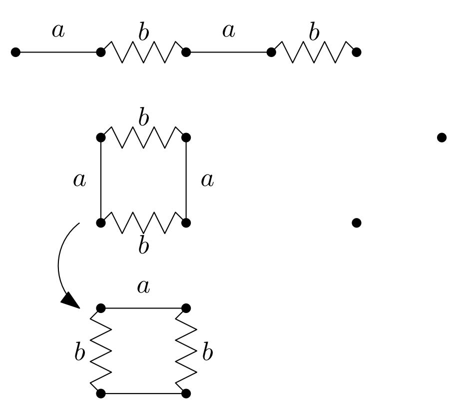
$\varphi(u)$ is the set of colors incdient with u

$\overline{\varphi}(u)$ is the set of colors missing at u

$\forall u, \overline{\varphi}(u) \neq \varnothing$

Kempe Chain $H(a, b)$

Subgraph ~~induced~~ by taking edges of colors a and b (only)



deg ≤ 2
≤ 1 color a
≤ 1 color b
(edge)

$y_0, y_1, \ldots$ vertices

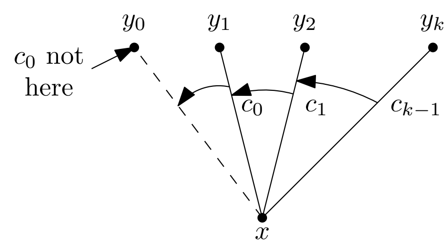
$c_0, c_1, \ldots$ colors

Set $y_0 = y$

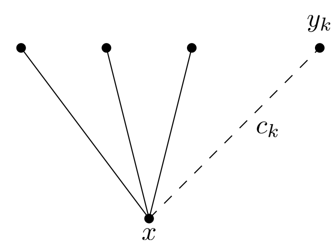
$c_i :=$ a color missing at y_i

$c_i \in \overline{\varphi}(y_i)$

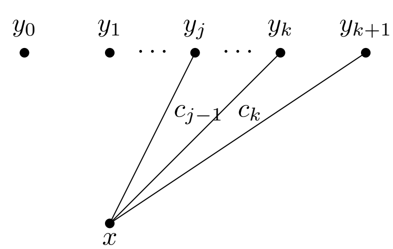
$y_{i+1} :=$ vertex such that $\varphi(xy_{i+1}) = c_i$



1 $c_k \in \overline{\varphi}(x)$ color xy_i with c_i $\forall 0 \le i \le k$.



2 y's and c's infinite



$c_k = \varphi(xy_j)$

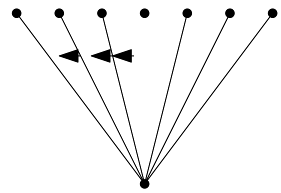
$\overline{\varphi}(x) \neq \varnothing$ Let $a \in \overline{\varphi}(x)$

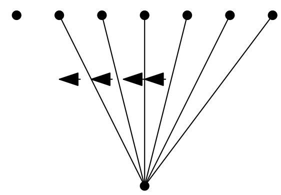
2a $a \in \overline{\varphi}(y_j)$ $\forall 0 \le i < j$ color xy_i with c_i. Color xy_j with a.

2b $a \in \overline{\varphi}(y_k)$ $\forall 0 \le i < k$ color xy_i with c_i. Color xy_k with a.

$c_k \in \varphi(x)$	$a \in \varphi(y_j)$
$c_k \in \overline{\varphi}(y_k)$	$a \in \varphi(y_k)$
$c_k \in \overline{\varphi}(y_j)$	$a \in \overline{\varphi}(x)$

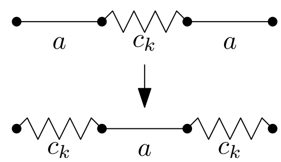
color xy_i with c_i $\forall 0 \le i < j$ uncolor xy_j





$H(C_k, a)$ each of x, y_j, y_k has degre 1. One of them is in its own compoent.

Without loss of generality



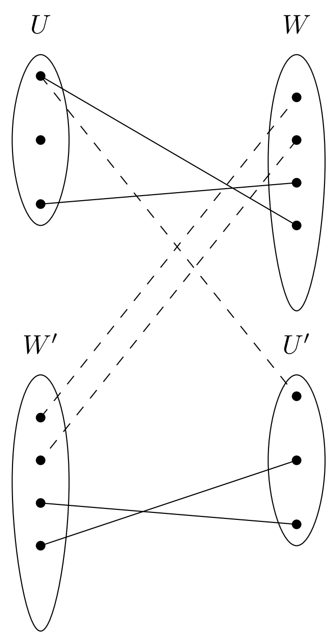
By 1, 2a, or 2b ■

<u>König's Theorem</u>: (10.17) G Bipartite, $\chi_1(G) = \Delta(G)$ [Class 1]

<u>Sketch Pf:</u> H bipartite, $\Delta(G)$-regular, $G \subseteq H$.

Given H, from exam 2 we know it has a perfect matching. Take one such matching, color some color, delete it. We now have a $\Delta(G) - 1$-regular graph, take another perfect matching. Continue this process. This gives a $\Delta(G)$ coloring of H. Restrict to G's edges.

$G = H_0$



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Copy and swap partitions. Connect corresponding minimum degree vertices. Repeat.

$H_0, H_1, \ldots, H_{\Delta(G)-\delta(G)}$ H_i bipartite

$\delta(H_{i+1}) = \delta(H_i) + 1$

$\Delta(H) = \delta(H) \Rightarrow$ Regular