Homework 3

21-484A Graph Theory
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Problem 2

Note that $V(\overline{G - \{v\}}) = V(G)\backslash\{v\} = V(\overline{G} - \{v\})$. Suppose $e \in E(\overline{G - \{v\}})$. Then, e is not incident to v in G (since $v \notin V(G - \{v\})$), and $e \notin E(G)$, so that $e \in E(\overline{G} - \{v\})$. Therefore, $E(\overline{G - \{v\}}) \subseteq E(\overline{G} - \{v\})$.

Since, by the result of a previous homework problem, either $\overline{G - \{v\}}$ or $G - \{v\}$ must be connected, and, by definition of cut-vertex, $G - \{v\}$ is disconnected, $\overline{G - \{v\}}$ must be connected. Then, since adding edges to a connected graph cannot disconnect it, $\overline{G} - \{v\}$ is connected, so that v is not a cut-vertex or $\overline{G}$. ■

Problem 3

Let G be a connected graph with at least 2 vertices.

Suppose G is non-separable, and let $e_1 = \{t, u\}, e_2 = \{u, v\}$ be two adjacent edges in G. Since G is non-separable, $G - \{u\}$, the graph produced by removing u and its incident edges from G, is connected. Thus, there exists a path from t to v in $G - \{u\}$, so that there is a path from t to v in G that does not use u. Adding u, t to this path gives a cycle containing e_1 and e_2. ■

Suppose, on the other hand, that any two adjacent edges in G lie on a common cycle. Let $t, u, v \in V(G)$, with $t \neq u \neq v$. Since G is connected, there exists a path from t to v in G. If that path does not contain u, then that path is also in $G - \{u\}$, so t and v are connected in $G - \{u\}$. Otherwise, let u_- and u_+ be the vertices before and after u, respectively, in the path from t to v. Since $\{u_-, u\}$ and $\{u, u_+\}$ are adjacent edges in G, they lie in a common cycle, so that u_-, u, and u_+ lie in a common cycle, and there is a path P from u_- to u_+ that does not include u. By choice of u_- and u_+ there exist paths P_1 and P_2 from t to u_- and from u_+ to v, respectively, neither of which includes u. Therefore, if P' is P without its first and last vertices, $P_1P'P_2$, the walk W created by concatenating P_1, P', and P_2 (in that order), is a walk from t to v that does not contain u. Therefore, W is also in $G - \{u\}$, and, since any walk from t to v contains a path from t to v, there is a path from t to v in $G - \{u\}$, so that t and v are connected in $G - \{u\}$. Therefore, since t and v are arbitrary, $G - \{u\}$ is connected, so that, since u is arbitrary, G is non-separable. ■

Problem 7

Let G be a nontrivial, connected graph.

Suppose G is Eulerian, let $m = |E(G)|$, and let $C = e_0, e_1, \dots, e_{n-1}$ be an Eulerian circuit in G. Let $v \in V(G)$. For every edge e_k incident to v, exactly one of $e_{k-1 \pmod n}$ or $e_{k+1 \pmod n}$ is also incident to v. Furthermore, since C is an Eulerian circuit, each edge incident to v appears exactly once in C. Therefore, each edge in G incident to v can be paired with exactly one other edge incident to v, so that there are an even number of edges incident to v, so that v has even degree. Thus, every vertex in G has even degree. ■

Suppose, on the other hand, that every vertex of G has even degree. Since G is connected and is not a tree (because it is non-trivial and cannot have any leaves), it must contain a cycle. We proceed by induction on the number of edges in G. The number of edges in G must be at least $n = |V(G)|$, the number of edges in a cycle on n vertices. Clearly, there is an Eulerian circuit in a graph that is just a cycle. Suppose, as an inductive hypothesis, that all G with $k \geq 1$ cycles, G is Eulerian. Suppose G contains at least 2 cycles. Let H be a graph produced by removing all edges in some cycle of G. Each connected component of H must have at least one vertex that was in that cycle (since G was connected), and each connected component has strictly fewer cycles than G. Thus, by the inductive hypothesis, each connected component of H must be Eulerian, so that we can construct an Eulerian circuit in G by connecting the Eulerian circuits in the connected components of H by their vertices on the cycle removed from G. Thus, by induction, all G with the above properties are Eulerian. ■

Problem 8

Suppose for sake of contradiction, that the Petersen graph G were Hamiltonian. Let U be the set of vertices forming the pentagonal exterior of G, and let V be the set of vertices forming star-shaped interior of G. Let C be a Hamiltonian cycle in G, and suppose we traverse C starting from a vertex in U. Since we begin and end in U, we must traverse an even number of edges spanning between U and V. Since there are 5 such edges and we must visit all the vertices in V, we must traverse either 2 or 4 such edges. Thus, we break into two cases:

Case 1: We traverse exactly 2 edges spanning between U and V. Let $u_1 \in U$ be an endpoint of some such edge, and let $v_1 \in V$ be its other endpoint. Then, in traversing C, we walk from u_1 to v_1, traverse the star until we have visited all 5 vertices, and end up on a vertex in V adjacent to v_1. Then, we exit the star, so that we are at a vertex $u_2 \in U$ that is not adjacent to u_1. Then, however, since we cannot enter the star, if we walk clockwise, we cannot visit any vertices counterclockwise from u_2 and clockwise from u_1 (of which at least one must exist because u_1 and u_2 cannot be adjacent), and similarly, if we walk counterclockwise, we fail to vist vertices clockwise from u_2 and counterclockwise from u_1. Therefore, no such Hamiltonian cycle can exist, concluding this case.

Case 2: We traverse exactly 4 edges spanning between U and V. Let $u_1 \in U$ be an endpoint of some such edge, and let $v_1 \in V$ be its other endpoint. Since, in traversing C, we enter V twice and cannot use edges more than once, we must either vist 3 vertices in V the first time we enter V, and 2 vertices the second time, or we must visit 2 vertices the first time we enter V and 3 vertices the second time. However, without loss of generality, we have the first case, because, in the second case, we can traverse C backwards to give the first case. Thus, we enter V from u_1, traverse 3 vertices in V, and then move to a vertex in U adjacent to u_1. Then, we must traverse either 1 or 3 other vertices in U before re-entering V, but, in the first case, it is not possible to visit all vertices in U and return to u_1, and in the second case, it is not possible to visit all vertices in V and return to u_1, so that there is no such Hamiltonian cycle.

Therefore, the Petersen graph is not Hamiltonian. ■