Homework 10

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36-705 Intermediate Statistics
Due: Thursday, December 4, 2014

1. Since $R(g) = \mathbb{E}[|Y - g(X)|] = \mathbb{E}[\mathbb{E}[|Y - g(X)| \mid X]]$, it suffices to show that, for any X, $\mathbb{E}[|Y - g(X)| \mid X] \geq \mathbb{E}[|Y - m(X)| \mid X]$. If $m(X) \geq g(X)$, then

$$\begin{aligned}&\mathbb{E}[|Y - g(X)| \mid X] - \mathbb{E}[|Y - m(x)| \mid X] \\ &\geq \int_{-\infty}^{m(X)} |g(X) - m(X)| p(y|X)\, dy - \int_{m(X)}^{\infty} |g(X) - m(X)| p(y|X)\, dy = 0.\end{aligned}$$

where the equality is by definition of m. The case $g(X) \leq m(X)$ is identical up to signs. ■

2. For $i \in \{n+1, \dots, 2n\}$, let $Z_i := (Y_i - \hat{g}(X_i))^2$, and let $Z_i' = Z_i|\mathcal{D}_1$. Then, for all $n \in \mathbb{N}$, $\overline{Z'} = \overline{Z}|\mathcal{D}_1 = \hat{R}|\mathcal{D}_1$. Also, by the triangle inequality, $|Z'| \leq (|Y| + |\hat{g}(X)|)^2 \leq (B + C)^2$, and so all moments of Z' exist. Since $R = \mathbb{E}[Z']$, by the Weak Law of Large Numbers,

$$(\hat{R} - R)|\mathcal{D}_1 = \left(\overline{Z'} - \mathbb{E}[Z']\right) \to 0$$

in probability. ■

3. Let $\alpha := \mathbb{E}[Y|X = 1] - \mathbb{E}[Y|X = 0]$. Then,

$$\begin{aligned}\mathbb{E}[2X_iY_i] &= \mathbb{E}[2X_iY_i|X_i = 0]\mathbb{P}[X_i = 0] + \mathbb{E}[2X_iY_i|X_i = 1]\mathbb{P}[X_i = 1] \\ &= 0 \cdot \frac{1}{2} + 2\mathbb{E}[Y_i|X_i = 1] \cdot \frac{1}{2} = \mathbb{E}[Y_i|X_i = 1],\end{aligned}$$

and, similarly, $\mathbb{E}[2(1 - X_i)Y_i] = \mathbb{E}[Y_i|X = 0]$. Thus, by the Weak Law of Large Numbers, (assuming the necessary moments of Y are finite), $\hat{\alpha}$ is a consistent estimator of α.

Note that, given X, $Y = Y(X)$, and, since X is randomly assigned, $\mathbb{E}[Y(1)|X = 1] = \mathbb{E}[Y(1)]$ and $\mathbb{E}[Y(0)|X = 0] = \mathbb{E}[Y(0)]$. Hence,

$$\begin{aligned}\alpha &= \mathbb{E}[Y|X = 1] - \mathbb{E}[Y|X = 0] \\ &= \mathbb{E}[Y(1)|X = 1] - \mathbb{E}[Y(0)|X = 0] = \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)] = \theta. \quad ■\end{aligned}$$

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