Lecture Notes for Week 11 (First Draft)

Dissipative Operators (Continued)

Theorem 11.1 (Lumer, Philips, 1961): Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to X$ is linear.

(a) If A is dissipative and there exists $\lambda_0 > 0$ such that $\lambda_0 I - A$ is surjective then A generates a linear C_0-contraction semigroup.

(b) If A generates a linear C_0-contraction semigroup then $\lambda I - A$ is surjective for every $\lambda > 0$ and for every semi-inner product $[\cdot, \cdot]$ compatible with the norm on X we have
$$\mathrm{Re}[Ax, x] \leq 0 \ \text{ for all } x \in \mathcal{D}(A).$$
In particular, A is dissipative.

Proof: Let $\lambda_0 > 0$ be given. Assume that A is dissipative and that $\lambda_0 I - A$ is surjective. Then, by Lemma 10.11, A is closed, $\rho(A) \supset (0, \infty)$, and
$$\|R(\lambda; A)\| \leq \frac{1}{\lambda} \ \text{ for all } \lambda > 0.$$
Consequently, for all $n \in \mathbb{N}$ we have
$$\|R(\lambda; A)^n\| \leq \|R(\lambda; A)\|^n \leq \frac{1}{\lambda^n} \ \text{ for all } \lambda > 0.$$
It follows from the Hille-Yosida Theorem that A generates a linear C_0-semigroup satisfying $\|T(t)\| \leq 1$ for all $t \geq 0$ (i.e., a contraction semigroup).

Assume now that A generates a linear C_0-contraction semigroup. Them, by the Hille-Yosida Theorem, A is closed and $\rho(A) \supset (0, \infty)$. It follows that $\lambda I - A$ is surjective for all $\lambda > 0$. Let $[\cdot, \cdot]$ be a semmi-inner product compatible with the norm on X. Let $x \in \mathcal{D}(A)$ and $h > 0$ be given. then we have
$$\begin{aligned} \mathrm{Re}[T(h)x - x, x] &= \mathrm{Re}[T(h)x, x] - \|x\|^2 \\ &\leq \|T(h)\| \cdot \|x\| - \|x\|^2 \\ &\leq 0. \end{aligned} \tag{1}$$
Using (1) we see that
$$\mathrm{Re}[Ax, x] = \lim_{h \downarrow 0} \mathrm{Re}\left[\frac{T(h)x - x}{h}, x\right] \leq 0. \quad \square$$

Corollary 11.2: Let X be a Banach space and $\mathcal{D}(B) \subset X$. Assume that $\mathcal{D}(B)$ is dense and that $B : \mathcal{D}(B) \to X$ is linear. Let $\omega, \lambda_0 \in \mathbb{R}$ be given with $\lambda_0 > \omega$. Assume that $\lambda_0 I - A$ is surjective and that there exists a semi-inner product compatible with the norm on X such that

$$\mathrm{Re}[Bx, x] \leq \omega \|x\|^2 \text{ for all } x \in \mathcal{D}(B).$$

Then B generates a linear C_0-semigroup satisfying

$$\|T(t)\| \leq e^{\omega t} \text{ for all } t \geq 0.$$

Proof: Put $\mathcal{D}(A) = \mathcal{D}(B)$, $A = B - \omega I$, and use Theorem 11.1. $\square$

Lemma 11.3: Let X be a reflexive Banach space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear and dissipative. Let $\lambda_0 > 0$ be given and assume that $\lambda_0 I - A$ is surjective. Then $\mathcal{D}(A)$ is dense.

Remark 11.4: Let X be a Banach space (not necessarily reflexive) and $Z \subset X$ be a linear manifold. Then Z is dense if and only if for every $y \in X$ there is a sequence $\{x_n\}_{n=1}^{\infty}$ in Z such that $x_n \rightharpoonup y$ (weakly) as $n \to \infty$. [Indeed, if Z is dense, then for every $y \in Z$ we can find a sequence of elements of Z that converges strongly to y. To see that the converse is true, if $y \notin \mathrm{cl}(Z)$ then $\mathrm{dist}(Z, y) > 0$ and by the Hahn-Banach Theorem we may choose a linear functional $x^* \in X^*$ such that $x^*(y) \neq 0$ and and $x^*(x) = 0$ for all $x \in Z$.]

Proof of Lemma 11.13: Let $y \in X$ be given. We shall construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in \mathcal{D}(A)$ for all $n \in \mathbb{N}$ and $x_n \rightharpoonup y$ (weakly) as $n \to \infty$.

By Lemma 10.11, we know that A is closed and $\rho(A) \supset (0, \infty)$. For all $n \in \mathbb{N}$, put

$$x_n = \left(I - \frac{1}{n}A\right)^{-1} y = nR(n; A)y \in \mathcal{D}(A). \tag{2}$$

A simple computation shows that

$$A\left(\frac{x_n}{n}\right) = x_n - y \text{ for all } n \in \mathbb{N}. \tag{3}$$

Lemma 10.11 also ensures that $\|R(n; A)\| \leq n^{-1}$ for all $n \in \mathbb{N}$, and consequently we have

$$\|x_n\| \leq n\|R(n; A)\| \cdot \|y\| \leq \|y\| \text{ for all } n \in \mathbb{N}.$$

Since X is reflexive and $\{x_n\}_{n=1}^{\infty}$ is bounded, we may choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $z \in X$ such that

$$x_{n_k} \rightharpoonup z \text{ (weakly) as } k \to \infty.$$

We want to show that $z = y$. Using (3) we see that

$$A\left(\frac{x_{n_k}}{n_k}\right) = x_{n_k} - y \rightharpoonup z - y \text{ (weakly) as } k \to \infty.$$

We also know that
$$\frac{x_{n_k}}{n_k} \rightharpoonup 0 \text{ (weakly) as } k \to \infty$$
(in fact; it converges strongly to 0). Since $\mathrm{Gr}(A)$ is closed and convex, it is weakly closed and we deduce that $(0, z-y) \in \mathrm{Gr}(A)$. This implies that $z = y$. $\square$

Theorem 11.5 (Lumer-Philips Theorem for Hilbert Spaces): Let X be a Hilbert space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear. Assume further that $\mathrm{Re}(Ax, x) \leq 0$ for all $x \in \mathcal{D}(A)$ and that there exists $\lambda_0 > 0$ such that $\lambda_0 I - A$ is surjective. Then A generates a linear C_0-contraction semigroup.

Example 11.6 (Heat Equation): Let $X = L^2[0,1]$ and put
$$\mathcal{D}(A) = \{u \in AC[0,1] : u' \in AC[0,1],\ u'' \in L^2[0,1],\ u(0) = u(1) = 0\},$$
$$Au = u'' \text{ for all } u \in \mathcal{D}(A).$$

We shall apply Theorem 11.5 to show that A generates a linear C_0-contraction semigroup. Let $u \in \mathcal{D}(A)$ be given. Using integration by parts, we find that
$$\begin{aligned}(Au, u) &= \int_0^1 u''(x)\overline{u(x)}\,dx \\ &= u'(x)\overline{u(x)}\Big|_{x=0}^1 - \int_0^1 u'(x)\overline{u'(x)}\,dx \\ &= -\int_0^1 |u'(x)|^2\,dx \leq 0.\end{aligned}$$
We shall show that $I - A$ is surjective. Let $g \in L^2[0,1]$ be given. We want to find $v \in \mathcal{D}(A)$ such that
$$\begin{cases} v(x) - v''(x) = g(x) & \text{a.e. } x \in [0,1] \\ v(0) = v(1) = 0. \end{cases} \tag{4}$$
It is possible to appeal to (or to prove) general theorems that ensure the existence of a suitable solution to (4); however using techniques from elementary differential equations, we can simply exhibit a suitable solution of (4), namely
$$v(x) = k \sinh x + \int_0^x \sinh(y - x) g(y)\,dy,$$
where the constant k is given by
$$k = \frac{1}{\sinh 1} \int_0^1 \sinh(1 - y)\,dy.$$
(This solution is obtained by using variation of parameters to find the general solution of $v - v'' = g$ and then choosing the constants so that $v(0) = v(1) = 0$.)

It follows from Theorem 11.5 that A generates a linear C_0 contraction semigroup $T : [0, \infty) \to \mathcal{L}(X; X)$. For $u_0 \in \mathcal{D}(A)$ let us put

$$u(t, x) = (T(t)u_0)(x) \quad \text{for all } x \in [0, 1],\ t \geq 0. \tag{5}$$

Then u is a solution of the initial-boundary value problem

$$\begin{cases} u_t(t, x) = u_{xx}(t, x) \quad x \in [0, 1],\ t \geq 0 \\ u(t, 0) = u(t, 1) = 0 \quad t \geq 0 \\ u(0, x) = u_0(x) \quad x \in [0, 1] \end{cases}$$

for the *heat equation* $u_t = u_{xx}$. Here u_t and u_x indicate partial derivatives of u with respect to the first and second argument, respectively.

The semigroup T of this example has important regularizing properties that will be addressed later. Even if $u_0 \notin \mathcal{D}(A)$, the function u produced by (5) is very smooth on $(0, \infty) \times [0, 1]$.

Nonhomogeneous Differential Equations

Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to X$ is linear and closed. Let $\tau > 0$, $f \in C([0, \tau]; X)$ and $u_0 \in X$ be given.

Consider the nonhomogeneous initial value problem

$$\begin{cases} \dot{u}(t) = Au(t) + f(t), \quad t \in (0, \tau] \\ u(0) = u_0. \end{cases} \tag{NHIVP}$$

Suppose that A generates a linear C_0-semigroup $T : [0, \infty) \to \mathcal{L}(X; X)$. We "know" that in this case, the solution to (NHIVP) should be given by

$$u(t) = T(t)u_0 + \int_0^t T(t - s)f(s)\, ds. \tag{6}$$

Equation (6) is known as the *variation of parameters formula.* We have seen that if $u_0 \in \mathcal{D}(A)$ then the mapping $t \to T(t)u_0$ is differentiable on $[0, \tau]$ and takes values in $\mathcal{D}(A)$. What about the integral term in (6)? Let us put

$$v(t) = \int_0^t T(t - s)f(s)\, ds \quad \text{for all } t \in [0, \tau].$$

It can easily happen that

$$\forall t \in (0, \tau],\ v(t) \notin \mathcal{D}(A),$$

and

$$\forall t \in (0, \tau],\ v \text{ is not differentiable at } t.$$

We shall illustrate how things can go wrong with a simple example.

Example 11.7: (See Example 8.5) Let $X = BUC(\mathbb{R})$ and define $T : [0, \infty) \to \mathcal{L}(X; X)$ by

$$(T(t)w)(x) = w(x+t) \ \text{ for all } w \in X, \ x \in \mathbb{R}, \ t \geq 0.$$

The infinitesimal generator A is given by

$$\mathcal{D}(A) = BUC^1(\mathbb{R}), \quad Aw = w' \ \text{ for all } w \in \mathcal{D}(A).$$

Let $z \in X \backslash \mathcal{D}(A)$ be given and put

$$f(t) = T(t)z \ \text{ for all } t \in [0, \tau],$$

so that

$$(f(t))(x) = z(x+t) \ \text{ for all } x \in \mathbb{R}, \ t \geq 0.$$

Observe that for all $t \in [0, \tau]$ we have $f(t) \notin \mathcal{D}(A)$. [This is because a function in X belongs to $\mathcal{D}(A)$ if and only if all of its translates belong to $\mathcal{D}(A)$.] Observe further that

$$\begin{aligned} v(t) &= \int_0^t T(t-s) f(s)\, ds = \int_0^t T(t-s) T(s) z \, ds \\ &= \int_0^t T(t) z \, ds = t T(t) z. \end{aligned}$$

It follows immediately that for all $t \in (0, \tau]$, $v(t) \notin \mathcal{D}(A)$. We also see that v is not differentiable.

The following lemma (whose proof will be a homework exercise) gives simple conditions which ensure that the integral term from the variation of parameters formula is differentiable and takes values in the domain of A.

Lemma 11.8: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup with infinitesimal generator A. Let $X_A = \mathcal{D}(A)$ equipped with the graph norm

$$\|x\|_A = \|x\| + \|Ax\| \ \text{ for all } x \in \mathcal{D}(A).$$

Let $\tau > 0$ and

$$F \in C^1([0, \tau]; X), \quad G \in C([0, \tau]; X_A)$$

be given. Put

$$f(t) = F(t) + G(t), \quad v(t) = \int_0^t T(t-s) f(s)\, ds \ \text{ for all } t \in [0, \tau].$$

Then

$$v \in C^1([0, \tau]; X) \cap C([0, \tau]; X_A)$$

and

$$\dot{v}(t) = Av(t) + f(t) \quad \text{for all } t \in [0, \tau].$$

Weak Solutions

Many authors define a "mild solution" of (NHIVP) via the variation of parameters formula (6). This approach is convenient, but not completely satisfactory, because one needs to know in advance that A generates a linear C_0-semigroup. It is desirable to have a notion of weak solution of (NHIVP) that makes no appeal to any semigroup, and then prove that if A generates a linear C_0-semigroup T the initial-value problem (NHIVP) has a unique weak solution and this solution is given by (6). The definition given here, as well as the theorem, is due to John Ball. Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to X$ is linear and closed. We shall employ the adjoint A^* of A. Let X^* denote the dual space of X and $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{K}$ denote the duality pairing.

Suppose that $u : [0, \tau] \to X$ is differentiable on $(0, \tau]$, and that $u(t) \in \mathcal{D}(A)$ for all $t \in (0, \tau]$ and that u satisfies

$$\dot{u}(t) = Au(t) + f(t), \quad t \in (0, \tau]. \tag{NHODE}$$

Then for all $x^* \in \mathcal{D}(A^*)$ we have

$$\langle x^*, \dot{u}(t) \rangle = \langle x^*, Au(t) \rangle + \langle x^*, f(t) \rangle \quad \text{for all } t \in (0, \tau].$$

We can rewrite this equation as

$$\frac{d}{dt} \langle x^*, u(t) \rangle = \langle A^* x^*, u(t) \rangle + \langle x^*, f(t) \rangle, \quad t \in (0, \tau]. \tag{7}$$

Equation (7) makes sense for a much broader class of functions u. Moreover, if u is differentiable on $(0, \tau]$, $u(t) \in \mathcal{D}(A)$ for all $t \in (0, \tau]$, and u satisfies (7) for all $x^* \in \mathcal{D}(A^*)$ then u also satisfies (NHODE). This motivates the following definition.

Definition 11.9: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Let $\tau > 0$ and $f \in C([0, \tau]; X)$ be given. Assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to X$ is linear and closed. By a weak solution of (NHODE) we mean a function $u \in C([0, \tau]; X)$ such that for every $x^* \in \mathcal{D}(A^*)$ the function $t \to \langle x^*, u(t) \rangle$ is absolutely continuous on $[0, \tau]$ and satisfies

$$\frac{d}{dt} \langle x^*, u(t) \rangle = \langle A^* x^*, u(t) \rangle + \langle x^*, f(t) \rangle \quad \text{a.e. } t \in [0, \tau].$$

Remark 11.10: Notice that with $f \in C([0, \tau]; X)$, if u is a weak solution of (NHODE) then for every $x^* \in \mathcal{D}(A^*)$ the mapping $t \to \langle x^*, u(t) \rangle$ will actually belong to $C^1[0, \tau]$ (rather than just to $AC[0, 1]$). Definition 11.9 is still appropriate under the weaker assumption that $f \in L^1([0,\tau];X)$. Moreover, Theorem 11.11 below remains valid for $f \in L^1([0,\tau];X)$. [Ball gave the definition and proved the theorem for $f \in L^1([0,\tau];X)$.]

Theorem 11.11: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to X$ is linear and closed. Let $\tau > 0$ and $f \in C([0,\tau];X)$ be given. Then (i) and (ii) below are equivalent.

(i) For every $u_0 \in X$, (NHODE) has exactly one weak solution $u \in C([0,\tau];X)$ such that $u(0) = u_0$.

(ii) A generates a linear C_0-semigroup $T : [0,\infty) \to \mathcal{L}(X;X)$.

Moreover, if (ii) [and hence also (i)] holds, then for each $u_0 \in X$, the unique weak solution u of (NHODE) satisfying $u(0) = u_0$ is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds \quad \text{for all } t \in [0,\tau].$$

In order to prove Theorem 11.11, we shall make use of the following lemma, which is of interest in its own right.

Lemma 11.12: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to X$ is linear and closed. Let $x, z \in X$ be given and assume that

$$\langle v^*, z\rangle = \langle A^*v^*, x\rangle \quad \text{for all } v^* \in \mathcal{D}(A^*). \tag{8}$$

Then $x \in \mathcal{D}(A)$ and $Ax = z$.

Proof: Suppose that $(x,z) \notin \mathrm{Gr}(A)$. Since $\mathrm{Gr}(A)$ is a closed subspace of $X \times X$ the Hahn-Banach Theorem implies that we may choose $x^*, y^* \in X^*$ satisfying

$$\langle x^*, x\rangle + \langle y^*, z\rangle \neq 0 \tag{9}$$

and

$$\langle x^*, y\rangle + \langle y^*, Ay\rangle = 0 \quad \text{for all } y \in \mathcal{D}(A). \tag{10}$$

It follows from (10) that

$$y^* \in \mathcal{D}(A^*), \quad \text{and} \quad A^*y^* = -x^*. \tag{11}$$

Using (8) and (11) we obtain

$$\langle x^*, x\rangle = -\langle y^*, z\rangle,$$

which contradicts (9). $\square$

Proof of Theorem 11.11: Assume first that A generates a linear C_0-semigroup $T : [0, \infty) \to \mathcal{L}(X; X)$. Let $u_0 \in X$ be given and put

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)\,ds, \quad \text{for all } t \in [0,\tau]. \tag{12}$$

Let $x^* \in \mathcal{D}(A^*)$ be given.

Claim: Given $x \in X$ let us put $w(t) = \langle x^*, T(t)x\rangle$ for all $t \in [0,\tau]$. Then $w \in C^1[0,\tau]$ and $\dot{w}(t) = \langle A^*x^*, T(t)x\rangle$ for all $t \in [0,\tau]$.

The claim is immediate if $x \in \mathcal{D}(A)$. If $x \notin \mathcal{D}(A)$, we may choose a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{D}(A)$ such that $x_n \to x$ as $n \to \infty$ and put $w_n(t) = \langle A^*x^*, T(t)x_n\rangle$. Then $w_n \to w$ uniformly on $[0,\tau]$ and the sequence $\{\dot{w}_n\}_{n=1}^{\infty}$ of derivatives also converges uniformly on $[0,\tau]$. If follows that $w \in C^1[0,\tau]$ and $\dot{w}(t) = \langle A^*x^*, T(t)x\rangle$ for all $t \in [0,\tau]$.

To handle the integral term in (12), observe that by the claim

$$\frac{\partial}{\partial t}\langle x^*, T(t-s)f(s)\rangle = \langle A^*x^*, T(t-s)f(s)\rangle. \tag{13}$$

The right-hand side of (13) is jointly continuous in s and t and consequently we have

$$\frac{d}{dt}\int_0^t \langle x^*, T(t-s)f(s)\rangle\,ds = \langle x^*, f(t)\rangle + \int_0^t \langle A^*x^*, T(t-s)f(s)\rangle\,ds. \tag{14}$$

It follows that the mapping $t \to \langle x^*, u(t)\rangle$ is continuously differentiable and

$$\begin{aligned}\frac{d}{dt}\langle x^*, u(t)\rangle &= \langle A^*x^*, T(t)u_0\rangle + \int_0^t \langle A^*x^*, T(t-s)f(s)\rangle\,ds + \langle x^*, f(t)\rangle \\ &= \langle A^*x^*, u(t)\rangle + \langle x^*, f(t)\rangle \quad \text{for all } t \in [0,t].\end{aligned}$$

Suppose that $\tilde{u}$ is another weak solution with $\tilde{u}(0) = u_0$. Put

$$w(t) = u(t) - \tilde{u}(t) \quad \text{for all } t \in [0,\tau],$$

and

$$W(t) = \int_0^t w(s)\,ds \quad \text{for all } t \in [0,\tau].$$

Observe that $W \in C^1([0,\tau]; X)$. Using the definition of weak solution and integrating with respect to time, we find that for all $x^* \in \mathcal{D}(A^*)$ and $t \in [0,\tau]$

$$\langle x^*, w(t)\rangle = \langle A^*x^*, \int_0^t w(s)\,ds\rangle,$$

and consequently

$$\langle x^*, \dot{W}(t)\rangle = \langle A^*x^*, W(t)\rangle.$$

It follows from Lemma 11.12 that

$$W(t) \in \mathcal{D}(A) \quad \text{and} \quad AW(t) = \dot{W}(t) \quad \text{for all } t \in [0,\tau].$$

Now fix $t \in (0,\tau]$ and define $z : [0,t] \to X$ by

$$z(s) = T(t-s)W(s) \quad \text{for all } s \in [0,t].$$

Then z is differentiable, $z(0) = 0$ (since $W(0) = 0$), and

$$\dot{z}(s) = T(t-s)AW(s) - T(t-s)AW(s) = 0 \quad \text{for all } s \in [0,t].$$

We conclude that z is constant on $[0,t]$; in particular

$$0 = z(0) = z(t) = W(t).$$

Since $W(t) = 0$ for all $t \in [0,\tau]$, we have $w(t) = 0$ for all $t \in [0,\tau]$ and $u(t) = \tilde{u}(t)$ for all $t \in [0,\tau]$.

Assume now that for every $u_0 \in X$, (NHODE) has exactly one weak solution u satisfying $u(0) = u_0$. Let us denote the value of this solution at time t by $u(t;u_0)$. Now for each $u_0 \in X$ we can define

$$T(t)u_0 = u(t;t_0) - u(t;0) \quad \text{for all } t \in [0,\tau]. \tag{15}$$

Observe that for all $x^* \in \mathcal{D}(A)$ we have

$$\frac{d}{dt}\langle x^*, T(t)u_0\rangle = \langle A^*x^*, T(t)u_0\rangle \quad \text{for all } t \in [0,\tau].$$

(In other words, $t \to T(t)u_0$ is a weak solution of $\dot{u} = Au$.) For $t > \tau$, we choose $n \in \mathbb{N}$ and $s \in [0,\tau)$ such that $t = n\tau + s$ and we define

$$T(t)u_0 = T(s)T(\tau)^n u_0.$$

It is not too difficult to show that T is a linear C_0-semigroup. This is left as an exercise. [Notice that the only weak solution of $\dot{w} = Aw$ on $[0,\tau]$ satisfying $w(0) = 0$ is identically 0 on $[0,\tau]$; otherwise there would be multiple weak solutions of (NHODE) on $[0,\tau]$ satisfying the same initial condition. This observation is useful for establishing the semigroup property. To establish continuity of T in the strong operator topology, consider the graph of the mapping $x \to T(\cdot)x$ from X to $C([0,\tau];X)$.]

Let $\hat{A}$ be the infinitesimal generator of T. We need to show that $\hat{A} = A$. To accomplish this we shall first show that A is an extension of $\hat{A}$. Let $x \in \mathcal{D}(\hat{A})$ and $x^* \in \mathcal{D}(A^*)$ be given. Then we have

$$\frac{d}{dt}\langle x^*, T(t)x\rangle = \langle A^*x^*, T(t)x\rangle,$$

and evaluating this expression at $t = 0$ we find that

$$\frac{d}{dt}\langle x^*, T(t)x\rangle\Big|_{t=0} = \langle A^*x^*, x\rangle. \tag{16}$$

Since $\hat{A}$ is the infinitesimal generator of T we also have

$$\frac{d}{dt}\langle x^*, T(t)x\rangle\Big|_{t=0} = \langle x^*, \hat{A}x\rangle. \tag{17}$$

It follows from (16) and (17) that

$$\langle x^*, \hat{A}x\rangle = \langle A^*x^*, x\rangle \ \text{ for all } x^* \in \mathcal{D}(A^*).$$

Lemma 11.12 implies that $x \in \mathcal{D}(A)$ and $Ax = \hat{A}x$.

Now let $x \in X$ and $x^* \in \mathcal{D}(A^*)$ be given. By the definition of weak solution and the construction of T we have

$$\frac{d}{dt}\langle x^*, T(t)x\rangle = \langle A^*x^*, T(t)x\rangle \ \text{ for all } t \in [0, \tau]. \tag{18}$$

Integration of (18) gives

$$\langle x^*, T(t)x\rangle - \langle x^*, x\rangle = \langle A^*x^*, \int_0^t T(s)x\,ds\rangle \ \text{ for all } x \in X,\ x^* \in \mathcal{D}(A^*). \tag{19}$$

Consequently we also have

$$\langle x^*, T(t)Ax\rangle - \langle x^*, Ax\rangle = \langle A^*x^*, \int_0^t T(s)Ax\,ds\rangle \ \text{ for all } x \in \mathcal{D}(A),\ x^* \in \mathcal{D}(A^*). \tag{20}$$

Let $x \in \mathcal{D}(A)$ and $t \in [0, \tau]$ be given. Applying Lemma 11.12 to (19) and (20) we conclude that

$$\int_0^t T(s)x\,ds \in \mathcal{D}(A), \quad \int_0^t T(s)Ax\,ds \in \mathcal{D}(A),$$

and

$$T(t)x = x + A\int_0^t T(s)x\,ds, \tag{21}$$

$$T(t)Ax = Ax + A\int_0^t T(s)Ax\,ds. \tag{22}$$

Now put

$$V(t) = \int_0^t T(s)Ax\,ds - A\int_0^t T(s)x\,ds \ \text{ for all } t \in [0, \tau],$$

and observe that $V \in C([0, \tau]; X)$ by virtue of (21). Clearly $V(0) = 0$. Let $x^* \in \mathcal{D}(A)$ be given. Using (21) and (22) and some straightforward computations we find that

$$\frac{d}{dt}\langle x^*, V(t)\rangle = \langle A^*x^*, V(t)\rangle, \quad t \in [0, \tau].$$

As noted above, the only weak solution of $\dot{w} = Aw$, $w(0) = 0$ on $[0, \tau]$ is the zero solution so we can conclude that $V(t) = 0$ for all $t \in [0, \tau]$ which yields

$$\int_0^t T(s)Ax\,ds = A\int_0^t T(s)x\,ds \quad \text{for all } t \in [0, \tau]. \tag{23}$$

Using (21) and (23) we find that

$$T(h)x - x = \int_0^h T(s)Ax\,ds \quad \text{for all } h \in (0, \tau].$$

We can conclude that

$$\lim_{h\downarrow 0} \frac{T(h)x - x}{h} = \lim_{h\downarrow 0} \frac{1}{h}\int_0^h T(s)Ax\,ds = Ax.$$

It follows that $x \in \mathcal{D}(\hat{A})$ and the proof is complete. $\square$

Compact Semigroups

Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. Let $t_0 > 0$ be given. If $T(t_0)$ is compact and $t > t_0$ then $T(t)$ is also compact because

$$T(t) = (T(t - t_0))T(t_0)$$

and the product of a bounded operator with a compact one is compact.

It follows that the set of all $t \in [0, \infty)$ such that $T(t)$ is compact is an interval of the form $[\tau, \infty)$ or (τ, ∞) with $\tau \geq 0$. Recall that the identity operator is compact if and only if X is finite-dimensional, so we want to be careful about making any assumptions that might imply $T(0)$ is compact. Semigroups having the property that $\{t \in [0, \infty) : T(t) \in \mathcal{C}(X; X)\}$ is a nonempty proper subset of $(0, \infty)$ are referred to as *eventually compact*. In 1953, Phillips gave an example which showed that the class of eventually compact semigroups is not stable under bounded perturbations of the infinitesimal generator. We shall focus here on linear C_0-semigroups having the property that $T(t)$ is compact for every $t > 0$. (This class of semigroups is, in fact, stable under bounded perturbations of the infinitesimal generator.)

Definition 11.13: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. We say that T is compact on $(0, \infty)$ provided that $T(t) \in \mathcal{C}(X; X)$ for every $t > 0$.

Our first result says that semigroups that are compact on $(0, \infty)$ are continuous in the uniform operator topology on $(0, \infty)$.

Lemma 11.14: Let X be a Banach space and $T : [0, \infty) \to \infty)$ be a linear C_0-semigroup. Assume that T is compact on $(0, \infty)$. Then T is continuous in the uniform operator topology on $(0, \infty)$.

Proof: Put

$$M = \sup\{\|T(s)\| : s \in [0,1]\}, \tag{24}$$

and notice that $M \geq 1$. Let $t, \epsilon > 0$ be given and put

$$U_t = \{T(t)x : x \in X,\ \|x\| \leq 1\},$$

$$\eta = \frac{\epsilon}{2(M+1)}. \tag{25}$$

The set U_t is totally bounded since its closure is compact. Therefore we may choose $x_1, x_2, \cdots, x_N \in \{x \in X : \|x\| \leq 1\}$ such that

$$U_t \subset \bigcup_{k=1}^{N} B_\eta(T(t)x_k) \tag{26}$$

Since each of the mappings $s \to T(s)x_k,\ k = 1, 2, \cdots, N$ is continuous at t we may choose $\delta \in (0,1]$ such that

$$\|T(t+h)x_k - T(t)x_k\| < \frac{\epsilon}{2} \text{ for all } h \in (0,\delta),\ k = 1, 2, \cdots, N. \tag{27}$$

Let $x \in X$ with $\|x\| \leq 1$ be given. In view of (26) we may choose $k \in \{1, 2, \cdots, N\}$ such that

$$\|T(t)x - T(t)x_k\| < \eta. \tag{28}$$

For $h \geq 0$ we have

$$\begin{aligned} T(t+h)x - T(t)x &= T(t+h)x - T(t+h)x_k + T(t+h)x_k - T(t)x_k \\ &\quad + T(t)x_k - T(t)x \\ &= T(h)[T(t)x - T(t)x_k] + (T(t+h)x_k - T(t)x_k) \\ &\quad + (T(t)x_k - T(t)x). \end{aligned} \tag{29}$$

Taking norms in (29) and using (24), (25), (27), (28) we find that for all $h \in [0, \delta)$

$$\|T(t+h)x - T(t)x\| < M\eta + \frac{\epsilon}{2} + \eta \leq \epsilon.$$

This establishes right continuity in the uniform operator topology at t.

Left continuity in the uniform operator topology at t follows easily from right continuity at $\frac{t}{2}$ and the observation

$$\|T(t-h) - T(t)\| \leq \left\|T\left(\frac{t}{2} - h\right)\right\| \cdot \left\|T\left(\frac{t}{2}\right) - T\left(\frac{t}{2} + h\right)\right\|. \quad \square$$

Before stating our next result about compact semigroups we make a simple, but useful, observation concerning compactness of resolvents of closed operators.

Proposition 11.15: Let X be a Banach space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A) \to X$ is linear and closed. Let $\lambda, \mu \in \rho(A)$ be given and assume that $R(\mu; A) \in \mathcal{C}(X; X)$. Then $R(\lambda; A) \in \mathcal{C}(X; X)$.

Proof: By Proposition 7.26, we have

$$R(\lambda; A) = R(\mu; A) + (\mu - \lambda)R(\lambda; A)R(\mu; A).$$

The conclusion now follows from the facts the product of a bounded operator with a compact one is compact and linear combinations of compact operators are compact. □

Theorem 11.16: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup with infinitesimal generator A. Assume that T is continuous in the uniform operator topology on $(0, \infty)$. The following three statements are equivalent.

(i) T is compact on $(0, \infty)$.

(ii) There exists $\lambda_0 \in \rho(A)$ such that $R(\lambda_0; A)$ is compact.

(iii) $R(\lambda; A)$ is compact for every $\lambda \in \rho(A)$.

Proof: In view of Proposition 11.15, we already know that (ii) $\Leftrightarrow$ (iii). Choose $M, \omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

Assume first that (i) holds. Let $\lambda > \omega$ be given. By Lemma 9.6, we know that $\lambda \in \rho(A)$ and

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt \text{ for all } x \in X.$$

Since T is continuous in the uniform operator topology on $(0, \infty)$ (and because of our bound for $\|T(t)\|$), we know that the integral

$$\int_0^\infty e^{-\lambda t}T(t)\,dt \tag{30}$$

converges in the uniform operator topology. (The lack of continuity at the left end-point does not cause any difficulties.) It follows that

$$R(\lambda; A) = \int_0^\infty e^{-\lambda t}T(t)\,dt.$$

For each $\epsilon > 0$, put

$$R_\epsilon(\lambda; A) = \int_\epsilon^\infty e^{-\lambda t}T(t)\,dt \tag{31}$$

and observe that the integral converges in the uniform operator topology. Since $T(t)$ is compact for every $t > 0$ we can conclude that $R_\epsilon(\lambda; A)$ is compact for every $\epsilon > 0$. On the other hand, for every $x \in X$ with $\|x\| \leq 1$ and every $\epsilon > 0$ we have

$$\|R(\lambda; A)x - R_\epsilon(\lambda; A)x\| \leq \|\int_0^\epsilon e^{-\lambda t} T(t)x \, dt\| \leq \epsilon M. \tag{32}$$

It follows that $R(\lambda; A)$ is the uniform limit of compact operators and is therefore compact.

Assume now that (iii) holds. We have $\rho(A) \supset (\omega, \infty)$ and

$$R(\lambda; A) = \int_0^\infty e^{-\lambda s} T(s) \, ds \quad \text{for all } \lambda > \omega. \tag{33}$$

The integral in (33) converges in the uniform operator topology.

Let $t > 0$ and $\lambda > \omega$ be given. It follows from (33) that

$$\lambda R(\lambda; A)T(t) = \lambda \int_0^\infty e^{-\lambda s} T(t+s) \, ds,$$

and consequently

$$\lambda R(\lambda; A)T(t) - T(t) = \lambda \int_0^\infty e^{-\lambda s} [T(t+s) - T(t)] \, ds. \tag{34}$$

We therefore find that for every $\delta > 0$

$$\begin{aligned} \|\lambda R(\lambda; A)T(t) - T(t)\| &\leq \int_0^\delta \lambda e^{-\lambda s} \|T(t+s) - T(t)\| \, ds \\ &\quad + \int_\delta^\infty \lambda e^{-\lambda s} \|T(t+s) - T(t)\| \, ds \\ &\leq \sup_{0 \leq s \leq \delta} \|T(t+s) - T(t)\| \\ &\quad + \frac{2\lambda M e^{\omega(t+\delta)} e^{-\lambda\delta}}{\lambda - \omega}. \end{aligned} \tag{35}$$

Let $\epsilon > 0$ be given. We may choose $\delta > 0$ such that

$$\sup_{0 \leq s \leq \delta} \|T(t+s) - T(t)\| < \frac{\epsilon}{2} \tag{36}$$

Then we choose Λ such that

$$\frac{2\lambda M e^{\omega(t+\delta)} e^{-\lambda\delta}}{\lambda - \omega} < \frac{\epsilon}{2} \quad \text{for all } \lambda > \Lambda. \tag{37}$$

It follows that (35), (36), and (37) that

$$\|\lambda R(\lambda; A)T(t) - T(t)\| \to 0 \quad \text{as } \lambda \to \infty.$$

Since $\lambda R(\lambda; A)T(t)$ is compact for every $\lambda > \omega$ we conclude that $T(t)$ is compact. $\square$

Remark 11.17: Theorem 11.16 is not completely satisfactory because it does not characterize semigroups that are compact on $(0, \infty)$ solely in terms of properties of the infinitesimal generators A. The difficulty is in ensuring continuity in the uniform operator topology on $(0, \infty)$. I do not know of a nice characterization of semigroups that are continuous in the uniform operator topology on $(0, \infty)$ solely in terms of the generators of such semigroups.

Differentiable Semigroups

Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup with infinitesimal generator A. We know that for $x \in \mathcal{D}(A)$, the mapping $t \to T(t)X$ is differentiable on $[0, \infty)$. We are interested here in semigroups having the property that for every $x \in X$, the mapping $t \to T(t)x$ is differentiable on $(0, \infty)$. Such semigroups will be called *differentiable semigroups*. There are two (equivalent) basic approaches to making a definition of differentiable semigroup – one can require differentiability on $(0, \infty)$ of the mapping $t \to T(t)x$ for every $x \in X$ or one can require that $T(t) : X \to \mathcal{D}(A)$ for all $t > 0$. (It is also possible to study semigroups that are *eventually differentiable*, i.e. semigroups for which there exists $t_0 \geq 0$ such that for every $x \in X$, the mapping $t \to T(t)x$ is differentiable on (t_0, ∞). We shall not do so here. The interested reader is referred to Section 2.4 of Pazy.)

Definition 11.18: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. We say that T is differentiable on $(0, \infty)$ provided that for all $x \in X$, the mapping $t \to T(t)x$ is differentiable on $(0, \infty)$.

Proposition 11.19: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup with infinitesimal generator A. The following two statements are equivalent.

(i) T is differentiable on $(0, \infty)$.

(ii) $T(t)[X] \subset \mathcal{D}(A)$ for all $t > 0$.

Moreover if (i) [and hence also (ii)] holds we have $T'(t)x = AT(t)x$ for every $x \in X$ and $t > 0$.

Proof: Assume that (i) holds and let $x \in X$ and $t > 0$ be given. Then we have

$$\begin{aligned} T'(t)x &= \lim_{h \to 0} \frac{T(t+h)x - T(t)x}{h} \\ &= \lim_{h \downarrow 0} \frac{T(t+h)x - T(t)x}{h} \\ &= \lim_{h \downarrow 0} \left(\frac{T(h) - I}{h} \right) T(t)x. \end{aligned}$$

It follows that $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T'(t)x$.

Assume now that (ii) holds and let $t > 0$ be given. Then we know that $T(\cdot)x$ is right differentiable at t with right derivative equal to $AT(t)x$. We need to show that the left derivative exists and also equals $AT(t)x$. Let $h \in (0, t)$ be given and observe

$$T(t)x - T(t-h)x = T\left(\frac{t}{2}\right)[T(h) - I]T\left(\frac{t}{2}\right)x.$$

It follows that

$$\lim_{h \downarrow 0} \frac{T(t-h)x - T(t)x}{-h} = T\left(\frac{t}{2}\right) AT\left(\frac{t}{2}\right) x = AT(t)x. \quad \square$$

Theorem 11.20: Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. Assume that T is differentiable on $(0, \infty)$. Then

(a) The mapping $t \to T(t)$ is of class C^∞ in the uniform operator topology on $(0, \infty)$.

(b) For every $t > 0$, $T(t) : X \to \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$.

(c) For every $t > 0$ and every $n \in \mathbb{N}$, $A^n T(t) \in \mathcal{L}(X; X)$ and

$$T^{(n)}(t) = A^n T(t) = \left(AT\left(\frac{t}{n}\right)\right)^n.$$

Before proving the theorem we observe that if an operator-valued function V is differentiable in the strong operator topology and the derivative is continuous in the uniform operator topology then V is differentiable in the uniform operator topology.

Lemma 11.21: Let X be a Banach space and $J \subset \mathbb{R}$ be an open interval. Assume that $V : J \to \mathcal{L}(X; X)$ is differentiable on J in the strong operator topology with derivative V', i.e.

$$\forall t \in J,\ x \in X, \quad V'(t)x = \lim_{h \to 0} \frac{V(t+h)x - V(t)x}{h}.$$

Assume further that the mapping $t \to V'(t)$ is continuous in the uniform operator topology. Then V is differentiable on J in the uniform operator topology, i.e.

$$\forall t \in J, \quad \lim_{h \to 0} \left\| \frac{V(t+h) - V(t)}{h} - V'(t) \right\| = 0. \tag{38}$$

Proof: For every $x \in X$ and every $t_1, t_2 \in J$ we have

$$V(t_2)x - V(t_1)x = \int_{t_1}^{t_2} V'(s)x\, ds.$$

Since V' is continuous in the uniform operator topology, the integral

$$\int_{t_1}^{t_2} V'(s)\, ds$$

exists in the uniform operator topology, so we must have

$$V(t_1) - V(t_2) = \int_{t-1}^{t_2} V'(s)\, ds \quad \text{for all } t_1, t_2 \in J. \tag{39}$$

It follows from (39) (and continuity of V') that (38) holds. $\square$

Proof of Theorem 11.20: Let $x \in X$ be given. For each $n \in \mathbb{N}$, consider the statement

(S$_n$) For every $t > 0$, $T(t)x \in \mathcal{D}(A^n)$, $T(\cdot)x$ is n-times differentiable at t,

$$T^{(n)}(t)x = A^n T(t)x = \left(AT\left(\frac{t}{n}\right)\right)^n x,$$

and $A^n T(t) \in \mathcal{L}(X; X)$.

We use induction to show that (S$_n$) holds for all $n \in \mathbb{N}$.

Base Case: Let $t > 0$ be given. By the definition of differentiable semigroup, we know that $T(\cdot)x$ is differentiable at t, and by Proposition 11.19, we know that $T(t)x \in \mathcal{D}(A)$. We also have

$$T'(t)x = \lim_{h \to 0} \frac{T(t+h)x - T(t)x}{h} = \lim_{h \downarrow 0} \left(\frac{T(h) - I}{h}\right) T(t)x = AT(t)x.$$

Since A is closed and $T(t) \in \mathcal{L}(X; X)$ we conclude that $AT(t)$ is closed. By the Closed Graph Theorem, we have $AT(t) \in \mathcal{L}(X; X)$.

Inductive Step: Let $n \in \mathbb{N}$ be given and assume that (S$_n$) holds. Let $t > 0$ and $h \in \mathbb{R}$ with $|h| < \dfrac{t}{n+1}$ be given. Then we have

$$\begin{aligned} T^{(n)}(t+h)x - T^{(n)}x &= A^n[T(t+h)x - T(t)x] \\ &= A^n T\left(\frac{nt}{n+1}\right)\left[T\left(\frac{t}{n+1} + h\right)x - T\left(\frac{t}{n+1}\right)x\right] \end{aligned} \tag{40}$$

Dividing by h, letting $h \to 0$, and using the fact A commutes with $T(\frac{t}{2})$ on $\mathcal{D}(A)$ we find that

$$
\begin{aligned}
T^{(n+1)}(t)x &= A^n T\left(\frac{nt}{n+1}\right) T'\left(\frac{t}{n+1}\right) x \\
&= A^n T\left(\frac{nt}{n+1}\right) AT\left(\frac{t}{n+1}\right) x
\end{aligned}
\tag{41}
$$

Using the semigroup property and the fact that A commutes with $T(\cdot)$ on the domain of A we infer from (41) that

$$T^{(n+1)}(t)x = A^{n+1}T(t)x.$$

On the other hand, we know that

$$A^n T\left(\frac{nt}{n+1}\right) = \left(AT\left(\frac{t}{n+1}\right)\right)^n,$$

so we also conclude from (41) that

$$T^{(n+1)}(t) = \left(AT\left(\frac{t}{n+1}\right)\right)^{n+1}.$$

The Closed Graph Theorem implies that $A^{n+1}T(t) \in \mathcal{L}(X;X)$.

We conclude that (S_n) holds for all $n \in \mathbb{N}$.

We have shown that $T(\cdot)$ is of class C^∞ in the strong operator topology. It remains to establish infinite differentiability in the uniform operator topology. In view of Lemma 11.21, it suffices to show that for every $n \in \mathbb{N}$, $T^{(n)}$ is continuous in the uniform operator topology on $(0,\infty)$.

We choose $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for all } t \geq 0.$$

Let $n \in \{0\} \cup \mathbb{N}$, $x \in X$ with $\|x\| \leq 1$, and $t_1, t_2 \in (0,\infty)$ with $t_1 \leq t_2$ be given. Then we have

$$
\begin{aligned}
T^{(n)}(t_2)x - T^{(n)}(t_1)x &= \int_{t_1}^{t_2} A^{n+1}T(s)x\,ds \\
&= \int_{t_1}^{t_2} A^{n+1}T(t_1)T(s-t_1)x\,ds.
\end{aligned}
\tag{42}
$$

Taking norms in (42) and then taking the supremum over all $x \in X$ with $\|x\| \leq 1$ we find that

$$\|T^{(n)}(t_2) - T^{(n)}(t_1)\| \leq (t_2 - t_1)Me^{\omega(t_2-t_1)}\|A^{(n+1)}T(t_1)\|.$$

It follows that $T^{(n)}$ is continuous in the uniform operator topology for every $n \in \{0\} \cup \mathbb{N}$.

This completes the proof $\Box$.

If the generator A of a differentiable semigroup T is unbounded then $\|AT(t)\|$ blows up as $t \downarrow 0$. We shall show that if A is unbounded then

$$\limsup_{t \downarrow 0} t^{-1} \|AT(t)\| \geq e^{-1}.$$

There is no upper limit to how fast $\|AT(t)\|$ can blow up as $t \downarrow 0$.

In 1995, Renardy gave an example (in Hilbert space) of a linear operator A generating a linear C_0-semigroup that is differentiable on $(0, \infty)$ and an everywhere-defined bounded linear operator L such that the semigroup generated by $A + L$ fails to be differentiable. (In fact, it is not even eventually differentiable.) Subsequently, Doytchinov, Hrusa, and Watson gave a sharp growth condition on $\|AT(t)\|$ as $t \downarrow 0$ for a differentiable semigroup T with infinitesimal generator A which guarantees that the semigroup generated by $A + L$ will be differentiable on $(0, \infty)$ for every $L \in \mathcal{L}(X; X)$.

We close this section by stating a theorem of Pazy (and a corollary) that characterizes generators of differentiable semigroups in terms of spectral properties. In order to state these results, it is convenient to introduce a family of subsets of $\mathbb{C}$. Given $a \in \mathbb{R}$ and $b > 0$, put

$$\Sigma_{b,a} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > a - b \log |\mathrm{Im}(\lambda)|\}. \tag{43}$$

Theorem 11.22: Let X be a complex Banach space and let $M, \omega \in \mathbb{R}$ be given. Let $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup satisfying $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$ and having infinitesimal generator A. Then T is differentiable on $(0, \infty)$ if and only if for every $b > 0$ there exist constants $a \in \mathbb{R}$ and $C > 0$ such that $\rho(A) \supset \Sigma_{b,a}$ and

$$\|R(\lambda; A)\| \leq C |\mathrm{Im}(\lambda)| \quad \text{for all } \lambda \in \Sigma_{b,a} \text{ with } \mathrm{Re}(\lambda) \leq \omega.$$

Corollary 11.23: Let X be a complex Banach space and let $M, \omega \in \mathbb{R}$ be given. Let $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup satisfying $\|T(t)\| \leq M e^{\omega t}$ for all $t \geq 0$ and having infinitesimal generator A. Let $\mu > \omega$ be given and assume

$$\limsup_{|\tau| \to \infty} \log |\tau| \cdot \|R(\mu + i\tau; A)\| = 0.$$

Then T is differentiable on $(0, \infty)$. [The variable τ in the limit above is real.]

Proofs of Theorem 11.22 and Corollary 11.23 are given in Section 2.4 of Pazy.