21-373, Algebraic Structures, Department of Mathematical Sciences, Carnegie Mellon University
Fall 2011: (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.
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Assignment 1 - Wednesday September 7, 2011. Due Monday September 12

Exercise 1: Let G be a group such that $g^2 = e$ for all $g \in G$. Show that G is Abelian.

Exercise 2: i) Let G be a group of order $2n$. Show that G contains an odd number of elements of order 2.
ii) Assume that n is odd. Show that if G is Abelian there is exactly one element of order 2, but that it is not always true if G is non-Abelian.

Exercise 3: i) Show that a group G cannot be the union of two proper subgroups.
ii) Give an example of a group G which is the union of three proper subgroups.

Exercise 4: Show that a group which only has a finite number of subgroups must be finite.

Exercise 5: i) Let G be an Abelian group containing elements a and b of orders m and n respectively. Show that G contains an element whose order is the least common multiple of m and n (one may start by the case where $(m,n) = 1$).
ii) Is it true if G is not Abelian?

Exercise 6: i) Show that in an Abelian group G, the set H of all elements of G with finite order is a subgroup of G.
ii) In the group $G = GL(2;\mathbb{Q})$ (the multiplicative group of non-singular 2×2 matrices with rational entries), compute the orders of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, and $A\,B$.
iii) Find in $\mathbb{Z}_2 \times \mathbb{Z}$ two elements a, b of infinite order such that $a + b$ has order 2.

Exercise 7: If G is the multiplicative group of odd integers modulo 2^{k+2}, and $k \geq 1$, show that G is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_2$ with $m = 2^k$.