21-484 Notes

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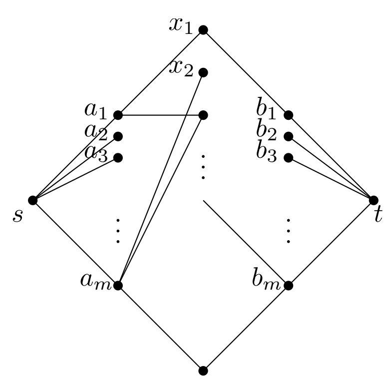
- X a set of people
  - $\boldsymbol{A} = \{A_1, \dots, A_m\}$ are subsets of X
  - we want to choose m elements $x_1, \dots, x_m$ such that $x_i \in A_i$. Such a set is called an <u>SDR</u> (system of distinct representatives)
    → Using Hall's theorem: $\exists$ SDR iff $\left|\bigcup_{i\in I} A_i\right| \geq |I|, \forall I \subseteq [m]$
  → $\boldsymbol{B} = \{B_1, \dots, B_m\}$ are subsets of X
  → A <u>CSDR</u> is a set of m x_i's such that its an SDR for $\boldsymbol{A}$ and $\boldsymbol{B}$

<u>Theorem (Ford-Fulkerson):</u> The families $\boldsymbol{A} = \{A_1, \dots, A_m\}$ and $\boldsymbol{B} = \{B_1, \dots, B_m\}$ have a CSDR iff

$$(*) \quad \left|\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right)\right| \geq |I| + |J| - m \quad \forall I, J \subseteq [m]$$

<u>Proof:</u> Define a graph G.

$$V(G) = \{s, a_1, \dots, a_m, x_1, \dots, x_{|X|}, b_1, \dots, b_m, t\}$$
$$E = \{sa_i | 1 \leq i \leq m\} \cup \{a_i x_k | x_k \in A_i\} \cup \{x_k b_j | x_k \in B_j\} \cup \{b_j t | 1 \leq j \leq m\}$$



→ An s–t path represents a common element of some A_i and B_j.

→ every s–t path has the form

$$s a_i x b_j t$$

→ $\exists$ a CSDR iff there are m internally disjoint s–t paths.
  → all the paths in such a set of paths are of length 5

→ The existence of a set of m internally disjoint s–t paths is equivalent to saying that there is no s–t cut pf size $< m$. (Menger's thm).

→ need to show that $(*) \iff$ no s–t cut of size $< m$.

→ Let $R \subseteq V(G) \setminus \{s, t\}$. Define $I = \{i \in [m] | a_i \notin R\}, J = \{j \in [m] | b_j \notin R\}$

$\to$ If R is a cut then

$$\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right) \subseteq R$$

because a path from s to t must visit some a_i then an x then b_j, this means that if a_i and b_j are in $G \setminus R$ then $x \in R$.

$\to$ for every cut R, $|R| \geq \left|\left(\bigcup_I A_i\right) \cap \left(\bigcup_J B_j\right)\right| + m - |I| + m - |J| \geq m$

$\to$ requiring that the RHS will be $\geq m$, we get $*$.

$\to$ If $*$ is false, $\exists I, J \subseteq [m]$ such that $(\bigcup A_i) \cap (\bigcup B_j) < |I| + |J| - m$.

- $\to$ for these I and J $(\bigcup A_i) \cap (\bigcup B_j) + m - |I| + m - |J| < m$
- $\to$Define R to be $(\bigcup A_i) \cap (\bigcup B_j) \cup [m] \setminus I \cup [m] \setminus J$.
- $\to$ $|R| < m$.
- $\to$ R is an s–t cut

■

Defs p. 134

$\to$ A circuit (a closed trail) in a graph G is called an Eulerian Circuit if it contains every edge of G.

$\to$ A trail is called an Eulerian trail if it visits every edge.

$\to$ A graph is Eulerian if it contains an Eulerian circuit.

$\to$ Thm (Euler 1736, Thm 6.1): A connected graph is Eulerian iff all the degrees are even.