Assignment 9

15-359 Probability and Computing
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Section: B
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Problem 4.1

Sample two random variables $X, Y \sim \text{Uniform}(0,1)$. Note that $c := \max_{x \in [0,1]} f(x) = f\left(\frac{1}{2}\right) = 15/8$ and that, $\forall x \in [0,1]$, $f(x) \geq 0$. Thus, $\forall x \in [0,1]$, $f(x)/c \in [0,1]$, so that we can use the accept/reject method (accepting if $f(X) > Y$ and rejecting otherwise) to generate the desired probability distribution. $\blacksquare$

Problem 4.3

Consider the following Java program `Simulation`:

```
public static void main(String[] args)
{
    double mean = 0;
    for(int i = 0; i < 200; i++)
        mean += trial(Double.parseDouble(args[0]));
    System.out.println(mean/200);
}
private static double trial(Double lambda) //one trial of 2000 runs
{
    //qLoad is the total work in the queue immediately after adding the
    //ith element
    double qLoad = 0;
    for(int i = 0; i < 2001; i++)
        qLoad = expDist(1) + Math.max(qLoad - expDist(lambda), 0);
    return qLoad;
}
private static double expDist(double lambda)
{
    double u = new Random().nextDouble();
    return -((Math.log(1 - u))/lambda);
}
```

`Simulation 0.5` outputs ≈ 1.6.

`Simulation 0.7` outputs ≈ 2.6.

`Simulation 0.9` outputs ≈ 6.5.

Problem 8.1

The stationary equations are:

$$\begin{bmatrix} \pi_C & \pi_M & \pi_U \\ \pi_C & \pi_M & \pi_U \\ \pi_C & \pi_M & \pi_U \end{bmatrix} = \begin{bmatrix} \pi_C & \pi_M & \pi_U \\ \pi_C & \pi_M & \pi_U \\ \pi_C & \pi_M & \pi_U \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.9 & 0.1 & 0 \end{bmatrix}$$

and $\pi_C + \pi_M + \pi_U = 1$. The solution to this system of equations is

$$\vec{\pi} = (\pi_C, \pi_M, \pi_U) = \boxed{\left(\frac{89}{121}, \frac{21}{121}, \frac{1}{11}\right)}.$$

Problem 8.2

Let P be an $m \times m$ transition matrix, and let $n \in \mathbb{N}$. $\forall i, j \in \{1, 2, \ldots, m\}$, let $p_{i,j}$ be the entry in the i^{th} row of the j^{th} column of P^n. Let $i \in \{1, 2, \ldots, m\}$, and, $\forall j \in \{1, 2, \ldots, n\}$, let E_j be the event that we end at state j after n transitions given that we are initially at state i, so that $E_j = p_{i,j}$. Thus,

$$\sum_{j=1}^{m} p_{i,j} = \sum_{j=1}^{m} E_j = 1,$$

since $E_1, E_2, \ldots, E_n$ partition the probability space of end states given that we start at state i. Therefore, $\forall n \in \mathbb{N}$, the sum of the elements in each row of P^n is 1. $\blacksquare$

Problem 8.3

Suppose A be an $m \times m$ transition matrix with the given properties, and let

$$P := \lim_{n\to\infty} (A^n) = \begin{bmatrix} \vec{\pi} \\ \vec{\pi} \\ \vdots \\ \vec{\pi} \end{bmatrix}$$

be its limiting distribution. Since, $\forall k \in \mathbb{N}$, $\left(A^k\right)^T = \left(A^T\right)^k$ (where ^{T} denotes the transpose operator),

$$P^T = \lim_{n\to\infty} \left(\left(A^T\right)^n\right) = (\vec{\pi}^T, \vec{\pi}^T, \ldots, \vec{\pi}^T),$$

so P^T is the limiting distribution for the Markov chain with transition matrix A^T. Thus, since the rows of any limiting distribution are identical,

$$\pi_1 = \pi_2 = \ldots = \pi_m. \quad \blacksquare$$

Problem 8.4

Consider a Markov chain with one state for each $i \in \mathbb{N}$, where the state representing i transitions to the state representing 0 with probability b, the state representing $i+1$ with probability p, and the state representing i (itself) with probability s. Let $\vec{pi}$ be the limiting distribution of for this Markov chain. Then, $\pi_0 = s\pi_0 + b$, and, $\forall i \geq 1$, $p_i = p\pi_{i-1} + s\pi_i$, yielding the recurrence

$$\pi_0 = \frac{b}{1-s}, \pi_i = \frac{p\pi_{i-1}}{1-s}, \forall i \geq 1.$$

The solution to this recurrence is

$$\pi_i = \boxed{b\frac{p^i}{(1-s)^{i+1}}}.$$

When $s = 0$, this reduces to $\pi_i = bp^i = b(1-b)^i$, so that the limiting distribution is $\mathrm{Geo}(b)$.