Metric Spaces

We are going to generalize many of the concepts from basic real analysis to situations involving functions whose inputs (and possibly also outputs) are functions rather than real numbers. In fact we shall study functions $f : X \to Y$, where X and Y are abstract sets endowed with some structure that will allow us to do analysis. In order to define concepts such as continuity, it will be useful to have a precise notion of "distance" between two elements of a set. The framework of a "metric space" (to be defined below) is very natural for studying mathematical concepts based on distance.

The material that follows should be regarded as a review of (or very rapid introduction to) the essentials of metric spaces. Very few proofs or examples will be given here. Students who have not previously studied metric spaces are advised to do some outside reading. Some recommended references are given below.

- W.A. Sutherland, *Introduction to Metric and Topological Spaces*
- Section I.6 of Dunford & Schwartz
- Section 1.7 and Chapter 3 of Friedman
- Chapter 1 of Goffman & Pedrick
- Chapter 1 of Kreyszig
- Chapter 7 of Royden

Definition of a Metric Space

Def: A *metric space* is a pair (X, ρ), where X is a set and $\rho : X \times X \to \mathbb{R}$ satisfies

(i) $\forall x, y \in X \quad \rho(x, y) \geq 0$,

(ii) $\forall x, y \in X, \quad \rho(x, y) = 0 \iff x = y$,

(iii) $\forall x, y \in X, \quad \rho(x, y) = \rho(y, x)$,

(iv) (triangle inequality) $\forall x, y, z \in X, \quad \rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

The function ρ is called a *metric* or *distance* on X and the elements of X are usually called *points*.

Remark: It is useful to observe that the triangle inequality can be reformulated as (iv$'$) $\forall x, y, z \in X, \quad |\rho(x, z) - \rho(y, z)| \leq \rho(x, y)$.

Indeed, it is immediate that (iv$'$) implies (iv); it is also straightforward to show that (iv$'$) follows from (iii) and (iv). We note also that (i) could be omitted from the definition of a metric because it follows from (ii), (iii), and (iv); in particular, for all x, y in X we have

$$2\rho(x, y) = \rho(x, y) + \rho(y, x) \geq \rho(x, x) = 0.$$

It is common practice to talk about "a metric space X" without any explicit reference to the metric. Some caution is required because the same set X can be equipped with more than one metric, and a change of metric can lead to dramatic changes in the metric space structure. However, in situations where there is no danger of ambiguity concerning which metric is intended, we sometimes refer to a metric space X, where X is simply the underlying set of points.

Examples:

(a) $X = \mathbb{R}$, $\rho(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. This metric is called the *standard metric* on $\mathbb{R}$. There are, of course, numerous other metrics on $\mathbb{R}$.

(b) Let X be any set. The metric ρ defined by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x = y \\ \\ 1 & \text{if } x \neq y \end{cases}$$

is called the *discrete metric* on X. The metric space of part (a) is dramatically different from the metric space obtained by equipping $\mathbb{R}$ with the discrete metric. Many useful examples and counterexamples can be obtained by considering these two spaces.

(c) $X = C[0, 1]$, the set of all continuous real-valued functions on $[0, 1]$. We shall give two different metrics on X:

$$\rho_1(f, g) = \max\{|f(x) - g(x)| : x \in [0, 1]\},$$

$$\rho_2(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

Sets in Metric Spaces

Throughout this section (X, ρ) is a given metric space.

Def: Given $\delta > 0$ and $x \in X$ we define the *ball of radius δ centered at x* by

$$B_\delta(x) = \{y \in X : \rho(y, x) < \delta\}.$$

Def: Let $S \subset X$ and $x_0 \in X$ be given.

(a) We say that x_0 is an *interior point* of S provided there exists $\delta > 0$ such that $B_\delta(x_0) \subset S$. The set of all interior points is called the *interior* of S and is denoted by $\mathrm{int}(S)$.

(b) We say that x_0 is a *point of closure* of S provided that

$$\forall \delta > 0, \quad B_\delta(x_0) \cap S \neq \emptyset.$$

The set of all points of closure of S is called the *closure* of S and is denoted by $\mathrm{cl}(S)$ or $\overline{S}$.

Intuitively, to say that x_0 belongs to the interior of S means that x_0 and all points that are sufficiently close to x_0 belong to S. To say that x_0 belongs to the closure of S means that there are points in S that are arbitrarily close to x_0. The following remark is an easy consequence of the definitions of interior and closure.

Remark: Let $A, B \subset X$.

(a) $\mathrm{int}(A) \subset A \subset \mathrm{cl}(A)$.

(b) If $A \subset B$ then $\mathrm{int}(A) \subset \mathrm{int}(B)$ and $\mathrm{cl}(A) \subset \mathrm{cl}(B)$.

Def: Let $S \subset X$. We say that S is

(a) *open* if $S = \mathrm{int}(S)$.

(b) *closed* if $S = \mathrm{cl}(S)$.

To show that S is open, it suffices to show that $S \subset \mathrm{int}(S)$. To show that S is closed, it suffices to show that $\mathrm{cl}(S) \subset S$. These observations follow immediately from the definitions and part (a) of the preceding remark.

Prop. M.1: For every $x \in X$ and every $\delta > 0$, $B_\delta(x)$ is an open set.

Proof: Let $x \in X, \delta > 0$, and $x_0 \in B_\delta(x)$ be given. We need to show that $x_0 \in \mathrm{int}(B_\delta(x))$, i.e. we must produce $\eta > 0$ such that $B_\eta(x_0) \subset B_\delta(x)$. Since $\rho(x_0, x) < \delta$, we may choose η satisfying

$$0 < \eta < \delta - \rho(x_0, x),$$

and hence also

$$\eta + \rho(x_0, x) < \delta.$$

Then we have $B_\eta(x_0) \subset B_\delta(x)$, by virtue of the triangle inequality. Indeed, if $\rho(y, x_0) < \eta$, then

$$\rho(y, x) \leq \rho(y, x_0) + \rho(x_0, x) < \delta. \ \square$$

Prop. M.2: Let $S \subset X$. Then $\text{int}(S)$ is an open set and $\text{cl}(S)$ is a closed set.

This result is not quite as trivial as it might appear. What Prop. M.2 is really saying is that $\text{int}(\text{int}(S)) = \text{int}(S)$ and $\text{cl}(\text{cl}(S)) = \text{cl}(S)$.

Proof: Put $U = \text{int}(S)$ and $C = \text{cl}(S)$. To show that U is open, let $x \in U$ be given. We want to show that $x \in \text{int}(U)$. Since $x \in \text{int}(S)$ we may choose $\delta > 0$ such that $B_\delta(x) \subset S$. We want to show that $B_\delta(x) \subset U$. Let $y \in B_\delta(x)$ be given. By Prop. M.1, $B_\delta(x)$ is open, so we may choose $\eta > 0$ such that $B_\eta(y) \subset B_\delta(x) \subset S$. This implies that $y \in \text{int}(S) = U$. We conclude that $B_\delta(x) \subset U$, so that $x \in \text{int}(U)$ and U is open.

To prove that C is closed, let $l \in \text{cl}(C)$ be given. We need to show that $l \in C = \text{cl}(S)$. Let $\delta > 0$ be given. Since $l \in \text{cl}(C)$ we may choose $z \in C$ such that $\rho(z, l) < \frac{\delta}{2}$. Since $z \in \text{cl}(S)$ we may choose $w \in S$ such that $\rho(z, w) < \frac{\delta}{2}$. By the triangle inequality, we have $\rho(l, w) < \delta$ so that $B_\delta(x) \cap S \neq \emptyset$. It follows that $l \in \text{cl}(S) = C$. $\square$

Given $S \subset X$, let $S^c = X \backslash S = \{x \in X : x \notin S\}$ denote the complement of S (relative to X). The following elementary result is very useful; it describes how interiors and closures interact with complements.

Prop. M.3: Let $S \subset X$.

(a) $\text{int}(S^c) = (\text{cl}(S))^c$

(b) $\text{cl}(S^c) = (\text{int}(S))^c$

Proof: We shall make use of the simple fact that $A \subset S^c$ if and only if $A \cap S = \emptyset$. To prove (a), let $x \in X$ be given. Then we have

$$\begin{aligned} x \in \text{int}(S^c) &\Leftrightarrow \exists \delta > 0, \quad B_\delta(x) \subset S^c \\ &\Leftrightarrow \exists \delta > 0, \quad B_\delta(x) \cap S = \emptyset \\ &\Leftrightarrow \text{not } (\forall \delta > 0, \quad B_\delta(x) \cap S \neq \emptyset) \\ &\Leftrightarrow x \in (\text{cl}(S))^c. \end{aligned}$$

Part (b) follows from (a), upon replacing S by S^c and using the fact that $(S^c)^c = S$. $\square$

Cor: Let $S \subset X$.

(a) S is open if and only if S^c is closed.

(b) S is closed if and only if S^c is open.

The empty set and the entire space X are open sets and also closed sets. In certain metric spaces, these are the only two open sets that are also closed. (Such metric spaces are said to be *connected*.) However, it is easy to give examples of metric spaces having additional sets that are both open and closed. In particular, with the discrete metric, every subset of X is both open and closed.

Prop. M.4: Let $A, B \subset X$ and $\mathcal{C}$ be a collection of subsets of X. Then

(a) $\mathrm{cl}(A \cup B) = (\mathrm{cl}(A)) \cup (\mathrm{cl}(B))$,

(b) $\mathrm{cl}(\bigcup_{S \in \mathcal{C}} S) \supset \bigcup_{S \in \mathcal{C}} \mathrm{cl}(S)$,

(c) $\mathrm{cl}(\bigcap_{S \in \mathcal{C}} S) \subset \bigcap_{S \in \mathcal{C}} \mathrm{cl}(S)$,

(d) $\mathrm{int}(A \cap B) = (\mathrm{int}(A)) \cap (\mathrm{int}(B))$,

(e) $\mathrm{int}(\bigcap_{S \in \mathcal{C}} S) \subset \bigcap_{S \in \mathcal{C}} \mathrm{int}(S)$,

(f) $\mathrm{int}(\bigcup_{S \in \mathcal{C}} S) \supset \bigcup_{S \in \mathcal{C}} \mathrm{int}(S)$

Remark: Regarding Prop. M.4, in general:

- Part(a) does not extend to unions of infinite collections of sets, i.e. the inclusion in part (b) can be strict if $\mathcal{C}$ is an infinite collection of sets.
- Strict inclusion can occur in (c) even when $\mathcal{C}$ is a finite collection of sets.
- Part (d) does not extend to intersections of infinite collections of sets, i.e. the inclusion in part (e) can be strict if $\mathcal{C}$ is an infinite collection of sets.
- Strict inclusion can occur in (f) even when $\mathcal{C}$ is a finite collection of sets.

Proof of Prop. M.4: We shall prove (d), (e), and (f). Then (a), (b), and (c) will follow from DeMorgan's Laws and Prop. M.3. We begin with (e). Let $x \in \mathrm{int}(\bigcap_{S \in \mathcal{C}} S)$ be given. Then we may choose $\delta > 0$ such that $B_\delta(x) \subset \bigcap_{S \in \mathcal{C}} S$, i.e. $B_\delta(x) \subset S$ for all $S \in \mathcal{C}$. This implies that $x \in \mathrm{int}(S)$ for all $S \in \mathcal{C}$, i.e. $x \in \bigcap_{S \in \mathcal{C}} \mathrm{int}(S)$.

To prove (d), we observe that $\mathrm{int}(A \cap B) \subset (\mathrm{int}(A)) \cap (\mathrm{int}(A)) \cap (\mathrm{int}(B))$. Let $x \in (\mathrm{int}(A)) \cap (\mathrm{int}(B))$ be given. Then we may choose $\delta_1, \delta_2 > 0$ such that $B_{\delta_1}(x) \subset A$

and $B_{\delta_2}(x) \subset B$. Put $\delta = \min\{\delta_1, \delta_2\}$ and notice that $\delta > 0$. Then $B_\delta(x) \subset A \cap B$ and consequently $x \in \mathrm{int}(A \cap B)$.

To prove (f), let $x \in \underset{S\in\mathcal{C}}{\cup} S$ be given. Then we may choose $\hat{S} \in \mathcal{C}$ such that $x \in \mathrm{int}(\hat{S})$. Consequently we may choose $\delta > 0$ such that $B_\delta(x) \subset \hat{S}$, from which we conclude that $B_\delta(x) \subset \underset{S\in\mathcal{C}}{\cup} S$, i.e. $x \in \underset{S\in\mathcal{C}}{\cup} \mathrm{int}(S)$. $\square$

Prop. M.5:

(a) The union of any collection of open sets is open.

(b) The intersection of any finite collection of open sets is open.

(c) The intersection of any collection of closed sets is closed.

(d) The union of any finite collection of closed sets is closed.

Proof: We shall prove (a) and (b). Then (c) and (d) will follow from DeMorgan's Laws and the corollary to Prop. M.3.

To prove (a), let O be a collection of open subsets of X and put $U = \underset{S\in\mathcal{O}}{\cup} S$. Since $S = \mathrm{int}(S)$ for every $S \in \mathcal{O}$, we have

$$U = \bigcup_{S\in\mathcal{O}} \mathrm{int}(S) \subset \mathrm{int}(\underset{S\in\mathcal{O}}{S}) = U,$$

and U is open.

To prove (b), let $\mathcal{F}$ be a finite collection of open subsets of X. If $\mathcal{F}$ is empty, then the intersection of $\mathcal{F}$ is X and we are done. Suppose $\mathcal{F} = \{S_1, S_2, \cdots, S_n\}$ where each S_i is open. Then, by part (d) of Prop. M.4 (and a straightforward induction argument) we have

$$\mathrm{int}(\bigcup_{i=1}^{N} S_i) = \bigcup_{i=1}^{N} \mathrm{int}(S_i) = \bigcup_{i=1}^{N} S_i. \quad \square$$

The following proposition is of interest in its own right and it will help motivate the definitions of interior and closure in a topological space.

Prop. M.6: Let $S \subset X$.

(a) The interior of S is equal to the union of all open subsets of S.

(b) The closure of S is equal to the intersection of all closed sets C such that $S \subset C$.

Proof: To prove (a), let $\mathcal{O}$ denote the collection of all open subsets of S. Let $U \in \mathcal{O}$ be given and observe that $U \subset \text{int}(S)$. (Indeed, given $x \in U$, we may choose $\delta > 0$ such that $B_\delta(x) \subset U \subset S$, which implies that $x \in \text{int}(S)$.) It follows that

$$\bigcup_{U \in \mathcal{O}} U \subset \text{int}(S).$$

On the other hand, $\text{int}(S) \in \mathcal{O}$, so that

$$\text{int}(S) \subset \bigcup_{U \in \mathcal{O}} U.$$

We conclude that

$$\text{int}(S) = \bigcup_{U \in \mathcal{O}} U.$$

To prove (b), we apply the result of part (a) to S^c and use Prop. M.3 and DeMorgan's laws. $\square$

Def: Let $S \subset X$. We say that S is *dense* (in X) provided $\text{cl}(S) = X$.

Notice that S is dense in X if and only if $B_\delta(x) \cap S \neq \emptyset$ for every $x \in X, \delta > 0$. In other words, S is dense in X if and only if every ball (no matter how small and no matter where the center is) contains points that belong to S.

Def: We say that X is *separable* provided that X has a countable dense subset.

Separable metric spaces are structurally simpler than nonseparable ones. In particular, for a separable metric space there is a countable collection of open sets that can be used to generate every open set.

Prop. M.7: X is separable if and only if there is a countable collection $\mathcal{C}$ of open subsets of X such that every open subset of X can be expressed as a union of members of $\mathcal{C}$.

Proof: Assume that X is separable and choose a countable dense set $S \subset X$. Let

$$\mathcal{C} = \{B_\delta(x) : x \in S, \delta \in \mathbb{Q}, \delta > 0\},$$

and observe that $\mathcal{C}$ is a countable collection of open subsets of X. Let U be an open subset of X. To show that U is a union of members of $\mathcal{C}$ it suffices to show that

$$\forall y \in U,\ \exists V \in \mathcal{C},\ y \in V \subset U. \tag{1}$$

To establish (1), let $y \in U$ be given. Since U is open, we may choose $\eta > 0$ such that $B_\eta(y) \subset U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we may choose $\delta \in \mathbb{Q} \cap (0, \eta]$. Notice that $B_\delta(y) \subset U$. Since $y \in \text{cl}(S)$, we may choose $x \in S$ such that $\rho(x, y) < \frac{\delta}{2}$. Observe that $B_{\frac{\delta}{2}}(x) \in \mathcal{C}$ and that

$$B_{\frac{\delta}{2}}(x) \subset B_\delta(y) \subset U,$$

and consequently (1) is satisfied.

Assume now that $\mathcal{C}$ is a countable collection of open subsets of X having the property that every open subset of X is a union of members of $\mathcal{C}$. We may assume that $\emptyset \notin \mathcal{C}$. (Indeed, if $\emptyset \in \mathcal{C}$, we may remove it from the collection, and we still have a countable collection of open sets having the property that every open set can be expressed as a union of members of $\mathcal{C}$.) For each $U \in \mathcal{C}$, we may choose $x_U \in U$. Put

$$S = \{x_U : U \in \mathcal{C}\},$$

and observe that S is countable. To see that S is dense, let $x \in X$ and $\delta > 0$ be given. Since $B_\delta(x)$ is open, it can be expressed as a union of members of $\mathcal{C}$ and consequently we may choose $U \in \mathcal{C}$ such that $x \in U \subset B_\delta(x)$. It follows that $x_U \in B_\delta(x)$ and consequently $x \in \text{cl}(S)$. $\square$

Def: Let $\mathcal{U}$ be a collection of open subsets of X. We say that $\mathcal{U}$ covers S provided that

$$S \subset \bigcup_{U \in \mathcal{U}} U.$$

Notice that if a collection of open sets covers the entire space X then the union of this collection must equal X. A simple, but very useful, characterization of separable metric spaces is contained in the following proposition.

Prop. M.8: X is separable if and only if for every $\epsilon > 0$, there is a countable collection of balls of radius ϵ that covers X.

Proof: Assume that X is separable and let $S \subset X$ be a countable dense set. Let $\epsilon > 0$ be given. Let $\mathcal{O} = \{B_\epsilon(z) : z \in S\}$ and observe that $\mathcal{O}$ is countable. We need to show that every $x \in X$ belongs to some member of $\mathcal{O}$. Let $x \in X$ be given. Since S is dense in X, we may choose $z_x \in S$ such that $\rho(x, z_x) < \epsilon$. This implies that $x \in B_\epsilon(z_x) \in \mathcal{O}$.

Assume now that for every $\epsilon > 0$ there is a countable collection of balls of radius ϵ that covers X. Then, for every $n \in \mathbb{N}$ we may choose a sequence $\{x_{n,k}\}_{k=1}^\infty$ such that the collection $\{B_{\frac{1}{n}}(x_{n,k}) : k \in \mathbb{N}\}$ of balls covers X. Let

$$S = \{x_{n,k} : n, k \in \mathbb{N}\},$$

and observe that S is countable. To prove that S is dense, let $x \in X, \delta > 0$ be given. We need to show that there exist $n, k \in \mathbb{N}$ such that $\rho(x, x_{n,k}) < \delta$. Choose $N > \frac{1}{\delta}$. Then we may choose $k \in \mathbb{N}$ such $x \in B_{\frac{1}{N}}(x_{N,k})$ and consequently we have $\rho(x, x_{N,k}) < \frac{1}{N} < \delta$. $\square$

Def: Let $S \subset X$. We say that S is *compact* provided that every collection of open sets that covers S has a finite subcollection that also covers S.

The definition of compactness may seem a bit strange if you have not encountered this concept before. However, it is one of the most important notions in basic analysis. Suppose that $K \subset X$ is compact. Then the collection of open sets

$$\{B_{\delta_x}(x) : x \in K\}$$

covers K, no matter how the radii δ_x are chosen. In particular, the radii can be chosen sufficiently small so that quantities of interest are controlled on each ball. By compactness, one can choose a finite set $F \subset K$ such that the collection of balls

$$\{B_{\delta_x}(x) : x \in F\}$$

covers K. In particular, K is covered by a *finite* number of balls of conveniently chosen radii.

Prop. M.9: Let K be a compact subset of X. Then K is closed.

Proof: We may assume that K is nonempty. (If $K = \emptyset$ there is nothing to prove.) By the corollary to Prop. M.3, it suffices to show that K^c is open. Let $z \in K^c$ be given. For each $x \in K$, we put

$$\delta_x = \frac{1}{3}\rho(z, x), \quad W_x = B_{\delta_x}(x), \quad V_x = B_{\delta_x}(z).$$

Observe that $\{W_x : x \in K\}$ is a collection of open sets that covers K. We may choose $x_1, x_2, \cdots, x_N \in X$ such that

$$K \subset \bigcup_{i=1}^{N} W_{x_i}.$$

Now let

$$\delta^* = \min\{\delta_{x_i} : i = 1, 2, \cdots, N\}.$$

By construction $B_{\delta^*}(z) \subset K^c$. It follows that $z \in \text{int}(K^c)$. Since $z \in K^c$ was arbitrary, we conclude that K^c is open. $\square$

We note for future reference that in a general topological space, compact sets need not be closed.

Prop M.10: Let K be a compact subset of X and $S \subset K$. If S is closed, then S is compact.

Proof: Assume that S is closed. Let $\mathcal{O}$ be a collection of open sets that covers S and put $\mathcal{O}' = \mathcal{O} \cup \{S^c\}$. By the corollary to Prop. M.3, S^c is open, so $\mathcal{O}'$ is a collection of open sets that covers K. (In fact, it covers X.) Since K is compact, we may choose a finite subcollection $\mathcal{F}'$ of $\mathcal{O}'$ that also covers K. (This subcollection also covers S, but it may not be a subcollection of $\mathcal{O}$ because S^c may well belong to $\mathcal{F}'$, but not to $\mathcal{O}$.) Now, put $\mathcal{F} = \mathcal{F}' \backslash \{S^c\}$ and observe that $\mathcal{F}$ is a finite subcollection of $\mathcal{O}$ that covers S. $\square$

Let (X, ρ), (Y, σ), and (Z, λ) be metric spaces.

Def: Let $f : X \to Y$ and $x \in X$ be given. We say that f is *continuous at* x provided that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sigma(f(y), f(x)) < \epsilon$ for all $y \in X$ with $\rho(y, x) < \delta$. We say that f is *continuous* (or *continuous on* X) provided that f is continuous at each point $x \in X$.

Def: We say that $f : X \to Y$ is *uniformly continuous* provided that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sigma(f(y), f(x)) < \epsilon$ for all $x, y \in X$ with $\rho(x, y) < \delta$.

Clearly, uniform continuity implies continuity; the converse implication is false in general. The difference between these two concepts is that in the definition of continuity, δ can depend on the point x as well as on ϵ, whereas for uniform continuity, δ can depend only on ϵ.

Prop. M.11: $f : X \to Y$ is continuous if and only if for every open set V in (Y, σ) the set

$$\{x \in X : f(x) \in V\}$$

is open in (X, ρ).

Cor: If $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f : X \to Z$ is continuous.

Prop. M.12: Assume that $f : X \to Y$ is continuous and $K \subset X$ is compact in (X, ρ). Then $\{f(x) : x \in K\}$ is compact in (Y, σ).

Prop. M.13: Assume that $f : X \to Y$ is continuous and that X is compact. Then f is uniformly continuous.

We denote the natural numbers by $\mathbb{N} = \{1, 2, 3, \cdots\}$. (In particular, our convention is that 1 is the smallest natural number, and therefore 0 is not a natural number. This will not be an important issue for anything, but is explicitly mentioned here because many authors consider 0 to be a natural number.)

A sequence in (X, ρ) is a mapping $x : \mathbb{N} \to X$. It is customary to write x_n in place of $x(n)$ and to denote the sequence by $\{x_n\}_{n=1}^{\infty}$.

Def: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in (X, ρ) and $l \in X$ be given. We say that l is a *limit* of $\{x_n\}_{n=1}^{\infty}$ provided that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho(x_n, l) < \epsilon$ for all $n \geq N$. A sequence in (X, ρ) is said to be *convergent* if it has a limit $l \in X$.

Remark: In a metric space, a sequence can have at most one limit. (We note for future reference that in certain topological spaces, sequences can have more than one limit.)

Proof: To prove the remark, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $l, L \in X$ be given.

Suppose that $x_n \to l$ and $x_n \to L$ as $n \to \infty$. We shall show that $l = L$. Let $\epsilon > 0$ be given. We may choose $N_1, N_2 \in \mathbb{N}$ such that

$$\rho(x_n, l) < \epsilon \text{ for all } n \geq N_1 \text{ and } \rho(x_n, L) < \epsilon \text{ for all } n \geq N_2.$$

Let us put $N = \max(N_1, N_2)$. Then, by the triangle inequality, we have

$$\rho(l, L) \leq \rho(l, x_N) + \rho(x_N, L) < \epsilon + \epsilon = 2\epsilon.$$

It follows that

$$\rho(l, L) < 2\epsilon \text{ for all } \epsilon > 0.$$

Since ρ is nonnegative, we conclude that $\rho(l, L) = 0$, and this implies that $l = L$. $\square$

Remark: If l is the limit of a sequence $\{x_n\}_{n=1}^{\infty}$ we write $x_n \to l$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = l$.

Def: A sequence $\{x_n\}_{n=1}^{\infty}$ in (X, ρ) is said to be a *Cauchy sequence* (or *fundamental sequence*) provided that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

Every convergent sequence in a metric space is a Cauchy sequence. (You should prove this as an exercise for yourself.) In some metric spaces, every Cauchy sequence is convergent. However, there are metric spaces in which some Cauchy sequences fail to be convergent.

Def: A metric space is said to be *complete* if every Cauchy sequence is convergent.

In order to use the definition to prove that a sequence is convergent, one needs to know the limit in advance. In a complete metric space, one can prove that a sequence is convergent, by showing that it is a Cauchy sequence. One does not need to know anything about possible values for the limit to show that a sequence is a Cauchy sequence.

In some sense, Cauchy sequences are "trying to converge". In an incomplete metric space, a Cauchy sequence might be trying to converge to something "outside of the metric space". In fact, it is always possible to make an incomplete metric space into a complete one by adding enough additional elements. This process is called *completion* and will be discussed later. It is some sort of "metatheorem" that if you have a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in an incomplete metric space X and there is a "natural candidate" $l \in X$ for the limit of this sequence, then indeed $x_n \to l$ as $n \to \infty$.

Def: By a *subsequence* of $\{x_n\}_{n=1}^{\infty}$ we mean a sequence of the form $\{x_{n_k}\}_{k=1}^{\infty}$ where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers.

Clearly, every subsequence of a convergent sequence is convergent to the limit of the original sequence. Moreover, every subsequence of a Cauchy sequence is a Cauchy sequence. The following elementary result is sometimes useful.

Prop. M.14: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $l \in X$ be given. If every subsequence of $\{x_n\}_{n=1}^{\infty}$ has in turn a subsequence that converges to l then $x_n \to l$ as $n \to \infty$.

As an illustration of the "metatheorem" referred to above, we have the following result.

Prop. M.15: Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X, $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$, and $l \in X$ be given. Assume that

$$x_{n_k} \to l \ \text{ as } n \to \infty.$$

Then $x_n \to l$ as $n \to \infty$.

Proof: Let $\epsilon > 0$ be given. Choose $N_1, N_2 \in \mathbb{N}$ such that

$$\rho(x_n, x_m) < \frac{\epsilon}{2} \ \text{ for all } m, n \geq N_1,$$

$$\rho(x_{n_k}, l) < \frac{\epsilon}{2} \ \text{ for all } k \geq N_2.$$

Observe that since $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers, we have

$$n_k \geq k \ \text{ for all } k \in \mathbb{N}.$$

Put $N = \max(N_1, N_2)$ and observe that $n_N \geq N_1$ and $N \geq N_2$. Then, for all $n \geq N$ we have

$$\rho(x_n, l) \leq \rho(x_n, x_{n_N}) + \rho(x_{n_N}, l) < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

We conclude that $x_n \to l$ as $n \to \infty$. $\square$

In metric spaces, sequences can be used to characterize closedness, compactness, and continuity. (However, it is important to note that, in topological spaces, sequences cannot always be used to characterize these properties.)

Prop. M.16: Let $S \subset X$ and $l \in X$ be given. Then $l \in \text{cl}(S)$ if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ such $x_n \in S$ for every $n \in \mathbb{N}$ and $x_n \to l$ as $n \to \infty$.

Prop. 3.17: Let $S \subset X$ be given. Then S is closed if and only if for every convergent sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$ for every $n \in \mathbb{N}$ we have $\lim\limits_{n\to\infty} x_n \in S$.

Prop. M.18: Let $K \subset X$ be given. Then K is compact if and only if every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in K$ for all $n \in \mathbb{N}$ has a subsequence that converges to an element of K.

Remark: You may encounter the term "sequentially compact" in the literature. The meaning is not completely standard. Some authors say that a set K is sequentially compact provided every sequence of points from K has a convergent subsequence, while other authors say that K is sequentially compact provided that every sequence of points from K has a subsequence that converges to an element of K. For subsets of metric spaces, the latter concept is equivalent to compactness, while the former is not. (Of course, when applied to an entire metric space, the two notions of sequential compactness are the same.)

Prop. M.19: Let $f : X \to Y$ and $l \in X$ be given. Then f is continuous at l if and only if $f(x_n) \to f(l)$ in (Y, σ) as $n \to \infty$ for every sequence $\{x_n\}_{n=1}^{\infty}$ in (X, ρ) such that $x_n \to l$ as $n \to \infty$.

Although continuous functions map convergent sequences to convergent sequences, they need not map Cauchy sequences to Cauchy sequences. However, uniformly continuous functions do map Cauchy sequences to Cauchy sequences.

Prop. M.20: Assume that $f : X \to Y$ is uniformly continuous and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ). Then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in (Y, σ).

Baire Category

Many of the deepest and most important consequences of completeness can be obtained from the following result of Baire.

Baire's Theorem: Let (X, ρ) be a complete metric space and $\{U_n\}_{n=1}^{\infty}$ be a sequence of subsets of X such that for every $n \in \mathbb{N}$, U_n is open and dense in X. Then

$$\bigcap_{n=1}^{\infty} U_n \quad \text{is dense in } X.$$

For the sake of completeness (no pun intended) we give a proof of Baire's Theorem. The following simple comment may make it easier to follow the proof. Let S be a subset of X. If $z \in \text{int}(S)$ then we may choose $\eta > 0$ such that $\text{cl}(B_\eta(z)) \subset S$. The validity of the comment follows immediately from the observation that

$$\text{cl}(B_\eta(z)) \subset \{y \in X : \rho(y, z) \leq \eta\} \subset B_{2\eta}(z).$$

Proof of Baire's Theorem: Let $x \in X$ and $\delta > 0$ be given. It suffices to show that $B_\delta(x)$ contains a point that belongs to each of the sets U_n. We shall accomplish this by constructing a "shrinking" sequence $\{B_{\delta_n}(x_n)\}_{n=1}^{\infty}$ of balls. We show that the sequence $\{x_n\}_{n=1}^{\infty}$ of centers converges to a point l that belongs to $B_\delta(x)$ and each of the U_n. For convenience, we choose that radii to satisfy $\delta_n \leq \frac{1}{n}$.

Since U_1 is dense in X, we may choose a point $x_1 \in U_1 \cap B_\delta(x)$. Since $U_1 \cap B_\delta(x)$ is open and $x_1 \in U_1 \cap B_\delta(x)$, we may choose $\delta_1 \in (0, 1]$ such that

$$B_{\delta_1}(x_1) \subset U_1 \cap B_\delta(x).$$

Since U_2 is dense, we may choose a point $x_2 \in U_2 \cap B_{\delta_1}(x_1)$. Moreover, since $U_2 \cap B_{\delta_1}(x_1)$ is open, we may choose $\delta_2 \in (0, \frac{1}{2}]$ such that

$$\mathrm{cl}(B_{\delta_2}(x_2)) \subset U_2 \cap B_{\delta_1}(x_1).$$

Proceeding by induction, we obtain, for each $n \in \mathbb{N}$ a point $x_n \in X$ and $\delta_n > 0$ such that

- $B_{\delta_n}(x_n) \subset U_n$,
- $\mathrm{cl}(B_{\delta_{n+1}}(x_{n+1})) \subset B_{\delta_n}(x_n)$,
- $\delta_n \leq \frac{1}{n}$.

Let $N \in \mathbb{N}, m, n \in \mathbb{N}$ with $m, n \geq N$ be given. Then $x_m, x_n \in B_{\delta_N}(x_n)$, so that

$$\rho(x_m, x_n) < 2\delta_N \leq \frac{2}{N},$$

which implies that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, we may choose $l \in X$ such that $x_n \to l$ as $n \to \infty$. Since $x_n \in B_{\delta_{N+1}}(x_{N+1})$ for all $n \geq N+1$, it follows that

$$l \in \mathrm{cl}(B_{\delta_{N+1}}(x_{N+1})) \subset B_{\delta_N}(x_N) \subset U_N.$$

Since $N \in \mathbb{N}$ was arbitrary, we conclude that

$$l \in \bigcap_{n=1}^{\infty} U_n.$$

Since $B_{\delta_1}(x_1) \subset B_\delta(x)$ and $l \in B_{\delta_1}(x_1)$, we conclude that $l \in B_\delta(x)$. $\square$

Def: Let (X, ρ) be a metric space and $S \subset X$. We say that S is

(i) *nowhere dense* provided $\mathrm{int}(cl(S)) = \emptyset$.

(ii) *meager* if S can be expressed as a countable union of nowhere dense sets.

(iii) *residual* if S^c is meager.

Remark: It is traditional to say that meager sets are of the *first category* and non-meager sets are of the *second category*.

The following result is an easy consequence of Baire's Theorem.

Baire Category Theorem: Assume that $X \neq \emptyset$ and that (X, ρ) is complete. Then X cannot be expressed as a countable union of nowhere dense sets.

The way that we shall typically use this theorem in practice is as follows. We express

$$X = \bigcup_{n=1}^{\infty} S_n,$$

where each set S_n is closed. If $X \neq \emptyset$ and (X, ρ) is complete, then we can conclude that $\text{int}(S_N) \neq \emptyset$ for some $N \in \mathbb{N}$.

Isometries, Homeomorphisms, and Completions

Let (X, ρ) and (Y, σ) be metric spaces.

Def: A mapping $f : X \to Y$ is said to be an *isometry* provided that $\sigma(f(x), f(y)) = \rho(x, y)$ for all $x, y \in X$.

In other words, isometries are mappings that preserve the distance between each pair of points.

Remark: Every isometry is injective and continuous. If an isometry is surjective (and hence bijective), then the inverse is also an isometry.

Def: We say that (X, ρ) and (Y, σ) are

(i) *homeomorphic* provided that there is a bijection $f : X \to Y$ such that f and f^{-1} are continuous. (Such a mapping f is called a *homeomorphism.*)

(ii) *uniformly homeomorphic* provided that there is a bijection $f : X \to Y$ such that f and f^{-1} are uniformly continuous. (Such a mapping f is called a *uniform homeomorphism.*)

(iii) *isometric* provided that there is a surjective isometry $f : X \to Y$.

Properties (such as compactness and separability) that are preserved under every homeomorphism are called *topological properties.* Some important properties, such as completeness, are not necessarily preserved under homeomorphisms. Properties that are preserved under every uniform homeomorphism are called *uniform properties.* It follows from Prop. M.20 that completeness is a uniform property. Metric spaces that are isometric are essentially indistinguishable from the point of view of metric space theory.

Theorem M.21: Let (X, ρ) be a metric space. Then there is a complete metric space $(\hat{X}, \hat{\rho})$ and a set $\hat{W} \subset \hat{X}$ such that $\hat{W}$ is dense in $(\hat{X}, \hat{\rho})$ and (X, ρ) is isometric to $(\hat{W}, \hat{\rho})$.

Remark: It is standard practice to call the metric space $(\hat{W}, \hat{\rho})$ *the completion* of (X, ρ). Of course, there may be more than one completion of a given metric space, but it is clear that all completions of a given metric space are isomorphic.

Subspaces

Let (X, ρ) be a metric space. If $A \subset X$ then, of course, (A, ρ) is also a metric space. We say that (A, ρ) is a *subspace* of (X, ρ). We need to be a bit careful when talking about topological properties in such situations because the interior and closure of S in (X, ρ) may be different from the interior and closure of S in (A, ρ). A very simple example is given below. If $S \subset A \subset X$, we write $\mathrm{int}^A(S)$ and $\mathrm{cl}^A(S)$ for the interior and closure of S as a subset of the metric space (A, ρ). We refer to $\mathrm{int}^A(S)$ and $\mathrm{cl}^A(S)$ as the interior and closure of S *relative* to A. We say that S is *open relative to* A provided that $S = \mathrm{int}^A(S)$ (i.e., S is an open subset of (A, ρ)) and that S is *closed relative to* A provided that $S = \mathrm{cl}^A(S)$ (i.e., S is a closed subset of (A, ρ)).

Example: Let $X = \mathbb{R}$, equipped with the standard metric and put $A = \mathbb{Q}$. Then we have

$$\mathrm{int}^{\mathbb{R}}(\mathbb{Q}) = \emptyset, \quad \mathrm{cl}^{\mathbb{R}}(\mathbb{Q}) = \mathbb{R}, \quad \mathrm{int}^{\mathbb{Q}}(\mathbb{Q}) = \mathrm{cl}^{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q}.$$

Prop. M.22: Let $S \subset A \subset X$. Then

(a) S is open relative to A if and only if there is a set $U \subset X$ such that U is open in (X, ρ) and $S = U \cap A$,

(b) S is closed relative to A if and only if there is a set $C \subset X$ such that C is closed in (X, ρ) and $S = C \cap A$,

(c) $\mathrm{cl}^A(S) = A \cap \mathrm{cl}^X(S)$.

Prop. M.23: Let $A \subset X$. If (A, ρ) is complete then A is closed in (X, ρ).

Prop. M.24: Assume that (X, ρ) is complete and that $A \subset X$ is closed in (X, ρ). Then (A, ρ) is complete.

We have seen in the simple example above that if $S \subset A \subset X$ then S may be open in (A, ρ) but fail to be open in (X, ρ). A similar comment applies to sets that are closed in (A, ρ). Compactness, however, is a more robust property as the proposition below indicates.

Prop. M.25: Let $S \subset A \subset X$. Then S is compact in (X, ρ) if and only if S is compact in (A, ρ).

Remark: Many authors define compactness first for entire metric spaces (X, ρ) and then define a set $S \subset X$ to be compact provided that the metric space (S, ρ) is compact. It follows from Prop. 4.5 that this definition of compact set is equivalent to the one given earlier.

Prop. M.26: Let $A \subset X$. If (X, ρ) is separable, then so is (A, ρ).

Prop. M.27: Let $A \subset X$ and put $\overline{A} = \mathrm{cl}^X(A)$. Then (A, ρ) is sepparable if and only if $(\overline{A}, \rho)$ is separable.

It is convenient to define separability for subsets of (X, ρ).

Def: Let $S \subset X$. We say that S is *separable* provided that the metric space (S, ρ) is separable. (Notice that this is equivalent to the requirement that there is a countable set $D \subset S$ such that $\mathrm{cl}^S(D) = S$.)

Since $\mathrm{cl}^A(S) = A \cap \mathrm{cl}^X(S)$ when $S \subset A \subset X$ we have the following simple proposition.

Prop. M.28: Let $S \subset A \subset X$. Then S is separable as a subset of (X, ρ) if and only if it is separable as a subset of (A, ρ).

Totally Bounded Sets and Compactness

Let (X, ρ) be a metric space.

Def: A set $S \subset X$ is said to be *bounded* provided that there exists $M \in \mathbb{R}$

$$\rho(x, y) \leq M \quad \text{for all } x, y \in S.$$

It is easy to see that a set S is bounded if and only if there exist $x \in X, \delta > 0$ such that $S \subset B_\delta(x)$. We have already seen that compact sets are closed. It is straightforward to show that every compact set is bounded. If $\mathbb{R}$ is equipped with the standard metric, then every closed bounded set is compact. In general, however, even if (X, ρ) is assumed to be complete, there may be closed bounded sets that fail to be compact. There is a stronger property, known as total boundedness, such that in a complete metric space, a set is compact if and only if it is closed and totally bounded.

Def: A set $S \subset X$ is said to be *totally bounded* provided that for every $\epsilon > 0$, S can be covered by a finite number of balls of radius ϵ.

It is clear that every subset of a totally bounded set is totally bounded. If a set S is totally bounded, then $\mathrm{cl}(S)$ is also totally bounded because the closure of a finite union is the union of the closures and $\mathrm{cl}(B_\eta)(x) \subset B_{2\eta}(x)$. Since this observation will be used later, we state it as a proposition.

Prop. M.29: Let (X, ρ) be a metric space and $S \subset X$. Then S is totally bounded if and only if $\mathrm{cl}(S)$ is totally bounded.

Like compactness and separability, the property of being totally bounded does not depend on which larger space the set lives in, provided that the larger metric spaces under consideration are all subspaces of a single metric space. More precisely we have the following analogue of Propositions M.25 and M.28.

Prop. M.30: Let (X, ρ) be a metric space and $S \subset A \subset X$. Then S is totally bounded in (X, ρ) if and only if it is totally bounded in (A, ρ).

Remark: Many authors define total boundedness first for entire metric spaces (X, ρ) and then define a set $S \subset X$ to be totally bounded provided that the metric space (S, ρ) is totally bounded. In view of Prop. M.30, this definition of totally bounded set is equivalent to the one given here.

Every compact metric space is complete and separable.

Prop. M.31: If (X, ρ) is compact, then it is complete.

Proof: Assume that (X, ρ) is compact and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. Then, by Prop. 3.6 there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. By Prop. M.15 the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent. $\square$

Prop. M.32: If (X, ρ) is compact then it is separable.

Proof: Assume that (X, ρ) is compact and let $\epsilon > 0$ be given. Then, since the collection $\{B_\delta(x) : x \in X\}$ of open sets covers X, we may choose a finite subcollection that covers X. Separability of X follows from Prop. M.8. $\square$

It is easy to see that every compact set is totally bounded. Indeed, suppose that $K \subset X$ is compact and let $\epsilon > 0$ be given. Then the collection $\{B_\epsilon(x) : x \in K\}$ of open sets covers K and consequently has a finite subcollection that also covers K. It is also straightforward to show that every totally bounded set is bounded. The following characterization of compact subsets of complete metric spaces is of fundamental importance.

Theorem M.33: Assume that (X, ρ) is complete and let $S \subset X$. Then S is compact if and only if S is closed and totally bounded.