Lecture Notes for Week 9 (Preliminary Draft)

The Dual of l^p, $1 \le p < \infty$

Let $p \in [1, \infty)$, $q \in (1, \infty]$ be given and assume that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Given $y \in l^q$, put

$$L_y(x) = \sum_{k=1}^{\infty} x_k y_k \quad \text{for all } x \in l^p. \tag{1}$$

By Holder's inequality, we have

$$|L_y(x)| \le \|y\|_q \|x\|_p \quad \text{for all } x \in l^p. \tag{2}$$

It follows that $L_y \in (l^p)^*$. Define $T : l^q \to (l^p)^*$ by

$$(Ty)(x) = L_y(x) \quad \text{for all } y \in l^q, x \in l^p. \tag{3}$$

Theorem 9.1: For each $p \in [1, \infty)$, the mapping T defined above is an isometric isomorphism of l^q onto $(l^p)^*$.

Proof: Let $p \in [1, \infty)$ be given and put $q = p(p-1)^{-1}$. We shall employ the standard Scahuder basis $\{e^{(n)}\}_{n=1}^{\infty}$ for l^p, where

$$e_k^{(n)} = \begin{cases} 1 & \text{for } k = n \\ 0 & \text{for } k \ne n. \end{cases}$$

Let $z \in l^q$ be given. We know that

$$\|Tz\|_* \le \|z\|_q, \tag{4}$$

by virtue of (2) and (3), and consequently

$$\|T\| \le 1. \tag{5}$$

Observe that T is injective because if $Tz = 0$ then, for each $n \in \mathbb{N}$, we have

$$0 = (Tz)e^{(n)} = \sum_{k=1}^{\infty} e_k^{(n)} z_k = z_n.$$

It remains to show that T is surjective and that $\|Ty\|_* \ge \|y\|_q$ for all $y \in l^q$.

Let $f \in (l^p)^*$ be given. We want to find $y \in l^q$ such $Ty = f$. To this end put

$$y_k = f(e^{(k)}) \text{ for all } k \in \mathbb{N}. \tag{6}$$

Since $\{e^{(k)}\}_{k=1}^{\infty}$ is a Schauder basis for l^p, we have

$$x = \sum_{k=1}^{\infty} x_k e^{(k)} \text{ for all } x \in l^p. \tag{7}$$

Since f is continuous and the series in (7) converges strongly to x we conclude that

$$f(x) = \sum_{k=1}^{\infty} x_k L(e^{(k)}) = \sum_{k=1}^{\infty} x_k y_k \text{ for all } x \in l^p.$$

To complete the proof, it is sufficient to show that $y \in l^q$ and $\|y\|_q \leq \|f\|_*$. This will imply that T is surjective (hence bijective) and that $\|Ty\|_* \geq \|y\|_q$.

Suppose first that $p = 1$ (so that $q = \infty$). It follows from (2) that

$$|y_k| \leq \|f\|_* \|e^{(k)}\| = \|f\|_* \text{ for all } k \in \mathbb{N},$$

and this tells us that $y \in l^\infty$ and $\|y\|_\infty \leq \|f\|_*$.

Suppose now that $p > 1$. If $y = 0$ we are done, so we may assume that $y \neq 0$. For every $n \in \mathbb{N}$ define $x^{(n)}$ by

$$x_k^{(n)} = \begin{cases} |y_k|^{q-1}\mathrm{sgn}(y_k) & \text{for } k \leq n \\ 0 & \text{for } k > 0. \end{cases} \tag{8}$$

Here, for $\lambda \in \mathbb{K}$,

$$sgn(\lambda) = \begin{cases} \frac{|\lambda|}{\lambda} & \text{for } \lambda \neq 0 \\ 0 & \text{for } \lambda = 0. \end{cases}$$

Observe that

$$\sum_{k=1}^{n} |y_k|^q = f(x^{(n)}) \leq \|f\|_* \|x^{(n)}\|_p \text{ for all } n \in \mathbb{N}. \tag{9}$$

Observe further that

$$\|x^{(n)}\|_p = \left(\sum_{k=1}^{n} |y_k|^q \right)^{1/p} \text{ for all } n \in \mathbb{N}, \tag{10}$$

since $(q-1)p = q$.

For n sufficiently large, using (9), (10), and the fact that $1 - \frac{1}{p} = \frac{1}{q}$, we obtain

$$\left(\sum_{k=1}^{n} |y_k|^q \right)^{1/q} \leq \|f\|_*. \tag{11}$$

Letting $n \to \infty$ in (11) we find that $y \in l^q$ and that

$$\|Ty\|_* \geq \|y\|_q. \quad \square$$

The Dual of c_0

Given $y \in l^1$, put

$$L_y(x) = \sum_{k=1}^{\infty} x_y y_k \text{ for all } x \in c_0.$$

Then we have

$$|L_y(x)| \leq \|y\|_1 \|x\|_\infty \text{ for all} x \in c_0,$$

so that $L_y \in (c_0)^*$. Now define $T : l^1 \to (c_0)^*$ by

$$Ty = L_y \text{ for all } y \in l^1.$$

Theorem 9.2: The mapping T defined above is an isometric isomorphism of l^1 onto $(c_0)^*$.

The proof is very similar to (and simpler than) the proof of Theorem 9.1 and is left as an exercise.

Remark 9.3: It follows from Theorems 9.1 and 9.2 that l^p is reflexive for $1 < p < \infty$ and that the spaces l^1, l^∞, c_0, and c are all nonreflexive. (We haven't said anything yet about the dual of c. However, if c were reflexive then c_0 would have to be reflexive since it is a closed subspace of c_0.)

L^P − Spaces

Let Ω be a nonempty open subset of $\mathbb{R}^n$. For $1 \leq p < \infty$ we denote by $L^p(\Omega)$ the set of all (equivalence classes of) measurable functions $f : \Omega \to \mathbb{K}$ such that

$$\int_\Omega |f|^p < \infty.$$

Here, by *measurable* we mean *Lebesgue measurable* and two measurable functions $f_1, f_2 : \Omega \to \mathbb{K}$ belong to the same equivalence class provided that $\mu(\{x \in \Omega : f_1(x) \neq$

$f_2(x)\}) = 0$. Here, $\mu(A)$ denotes the Lebesgue measure of a set A. We follow the standard practice of using the same symbol to denote a measurable function and its equivalence class.

We define $\|\cdot\|_p : L^p(\Omega) \to [0, \infty)$ by

$$\|f\|_p = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}} \quad \text{for all } f \in L^p(\Omega).$$

It is evident that $\|f\|_p = 0$ implies that $f = 0$ *a.e.* and that $\|\alpha f\|_p = |\alpha| \|f\|_p$ for all $\alpha \in \mathbb{K},\ f \in L^p(\Omega)$. The triangle inequality can be verified easily when $p = 1$. It also holds for $p \in (1, \infty)$ as well, but the proof is not completely trivial.

We denote by $L^\infty(\Omega)$ the set of (equivalence classes of) essentially bounded measurable functions $f : \Omega \to \mathbb{K}$. We define $\|\cdot\|_\infty : L^\infty(\Omega) \to [0, \infty)$ by

$$\|f\|_\infty = \operatorname{ess} - \sup_\Omega(|f|) = \inf\{M \in \mathbb{R} : \mu(\{x \in \Omega : |f(x)| > m\}) = 0\}.$$

We make the same conventions concerning equivalence classes of measurable functions as we do for $L^p(\Omega)$ with $p \in [1, \infty)$.

Proposition 9.4 (Minkowski's Inequality): Let $p \in [1, \infty]$ be given. If $f, g \in L^p(\Omega)$ then so is $f + g$ and

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

It follows that $(L^p(\Omega), \|\cdot\|_p)$ is a normed linear space.

Proposition 9.5 (Holder's Inequality): Let $p, q \in [1, \infty]$ with

$$\frac{1}{p} + \frac{1}{q} = 1$$

and $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ be given. Then $fg \in L^1(\Omega)$ and

$$\int_\Omega |fg| \le \|f\|_p \|g\|_q.$$

Remark 9.6: Assume that $\mu(\Omega) < \infty$ and that $1 \le p_1 \le p_2 \le \infty$. Then $L^{p_2}(\Omega) \subset L^{p_2}(\Omega)$ and there exists $K \in \mathbb{R}$ such that

$$\|u\|_{p_1} \le \|u\|_{p_2} \quad \text{for all } u \in L^{p_2}(\Omega).$$

Although the idea is very simple (and almost certainly familiar to everyone in the class), we shall give a proof of Remark 9.6 using a simple "trick" involving Holder's inequality. Assume first that $1 \le p_1 \le p_2 < \infty$ and let $u \in L^{p_2}(\Omega)$ be given. Then we have

$$\int_\Omega |u|^{p_1} = \int_\Omega (|u|^{p_2})^{\frac{p_1}{p_2}} \cdot 1.$$

Using Holder' Inequality with $p = \frac{p_2}{p_1}$, $q = \frac{p_2}{p_2 - p_1}$, $f = |u|^{p_1}$, and $g = 1$, we find that

$$\|u\|_{p_1} \leq (\mu(\Omega))^{\frac{1}{p_1} - \frac{1}{p_2}} \|u\|_{p_2}.$$

If $u \in L^\infty(\Omega)$ and $p_1 < p_2$, then

$$\int_\Omega |u|^{p_1} \leq \int_\Omega \|u\|_\infty^{p_1} \leq \|u\|_\infty^{p_1} \mu(\Omega),$$

and consequently

$$\|u\|_{p_1} \leq (\mu(\Omega))^{\frac{1}{p_1}}.$$

Theorem 9.7 (Riesz-Fisher): Assume that $1 \leq p \leq \infty$¿ Then $L^p(\Omega)$ is complete.

Theorem 9.8 (Riesz-Representation): Let $p \in [1, \infty)$, $q \in (1, \infty]$ be given and assume that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $u^* \in (L^p(\Omega))^*$ be given. Then there is exactly one $g \in L^q(\Omega)$ such that

$$u^*(f) = \int_\Omega fg;$$

moreover, $\|u^*\| = \|g\|_q$.

Corollary 9.9: $L^p(\Omega)$ is reflexive for $1 < p < \infty$.

Proposition 9.10: $L^p(\Omega)$ is separable for $1 \leq p < \infty$. $L^\infty(\Omega)$ is not separable.

Remark 9.11: It follows from Theorem and Proposition that neither $L^1(\Omega)$ nor $L^\infty(\Omega)$ is reflexive.

Let us denote by $C_c^\infty(\Omega)$ the set of all functions $u : \Omega \to \mathbb{K}$ having continuous derivatives of all orders and such that $\mathrm{spt}(u)$ is a compact subset of Ω. (Here $\mathrm{spt}(u)$ is the closure of $\{x \in \Omega : u(x) \neq 0\}$.)

Lemma 9.12: $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

Lemma 9.13 (Clarkson's Inequalities): Let $p, q \in (0, \infty)$ with

$$\frac{1}{p} + \frac{1}{q} = 1$$

and $f, g \in L^p(\Omega)$ be given. Then

(a) for $1 < p \leq 2$ we have

$$\|\frac{f+g}{2}\|_p^q + \|\frac{f-g}{2}\|_p^q \leq \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{q-1},$$

$$\|\frac{f+g}{2}\|_p^p + \|\frac{f-g}{2}\|_p^p \geq \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_P^P.$$

(b) for $2 \le p < \infty$ we have

$$\|\frac{f+g}{2}\|_p^p + \|\frac{f-g}{2}\|_p^p \le \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_P^P,$$

$$\|\frac{f+g}{2}\|_p^q + \|\frac{f-g}{2}\|_p^q \ge \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{q-1}.$$

Remark 9.14: An extremely important consequence of Clarkson's Inequalities is that they imply that $L^p(\Omega)$ is *uniformly convex* when $1 < p < \infty$. (We shall study uniformly convex spaces later. They have the useful property that weak convergence, together with convergence of the sequence of norms to the norm of the weak limit, imply strong convergence. They have other important properties as well.)

Theorem 9.15 (Riesz-Kolmogorov): Assume that $1 \le p < \infty$ and let K be a bounded subset of $L^p(\Omega)$. Then K is precompact (i.e. has compact closure) if and only if for every $\epsilon > 0$ there exists a compact set $G \subset \Omega$ and $\delta > 0$ such that (i) and (ii) below hold:

(ii)
$$\int_{\Omega\backslash G} |u|p < \epsilon^p \quad \text{for all } u \in K,$$

(ii)
$$\int_{\Omega} |\tilde{u}(x+h) - \tilde{u}(x)|^p \, dx < \epsilon^p$$

for all $h \in \mathbb{R}^n$ with $|h| < \delta$ and all $u \in K$, where

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^n\backslash\Omega. \end{cases}$$

Adjoints

Let X and Y be NLS and $T \in \mathcal{L}(X;Y)$ be given. We shall construct a bounded linear operator $T^* \in \mathcal{L}(Y^*;X^*)$ that is associated with T in a natural way. The operator T^* is called the adjoint of T. Much important information about the operator T can be obtained by studying the adjoint T^*.

If X and Y are finite-dimensional, then any linear operator $T : X \to Y$ can be represented by a matrix. (Any linear operator from Y^* to X^* can also be represented by a matrix). The matrices representing the linear transformations will, of course, depend on a choice of bases for the linear spaces. If the bases are chosen suitably, then the matrix representing T^* will be the transpose of the matrix representing T. Many of the important relations between matrices and their transposes can be generalized to linear operators and their adjoints. In situations involving infinite-dimensional spaces, we will need to make certain adjustments and additional assumptions that are not necessary in finite-dimensional spaces. We begin with a theorem that facilitates the definition of an adjoint. Since certain formulas will involve duality pairings for different spaces, we sometimes include subscripts on the pairings. More precisely, for any NLS Z, the duality pairing from $Z^* \times Z$ to $\mathbb{K}$ will be denoted by $\langle \cdot, \cdot \rangle_{Z^* \times Z}$ in situations where it may help to keep track of where certain object live.

Theorem 9.16: Let X and Y be NLS and let $T \in \mathcal{L}(X;Y)$ be given. Then there is exactly one $T^* \in \mathcal{L}(Y^*;X^*)$ such that

$$\langle T^*y^*, x \rangle_{X^* \times X} = \langle y^*, Tx \rangle_{Y^* \times Y} \quad \text{for all } y^* \in Y^*, x \in X. \tag{12}$$

Moreover $\|T^*\| = \|T\|$.

Definition 9.17: The linear operator T^* in Theorem 9.16 is called the *adjoint* of T.

Remark 9.18:

(a) It is clear that the mapping $T \to T^*$ is a linear mapping from $\mathcal{L}(X;Y)$ to $\mathcal{L}(Y^*;X^*)$, i.e. the mapping that carries an operator to its adjoint is linear.

(b) When there is no danger of confusion, we shall drop the subscripts on the duality pairings in (12).

Proof of Theorem 9.16: For each $y^* \in Y^*$, define

$$T^*y^* = y^* \circ T, \tag{13}$$

where $\circ$ indicates composition of mappings (so that (13) means $(T^*y^*)(x) = y^*(Tx)$ for all $x \in X$). For fixed y^*, T^*y^* is a mapping from X to $\mathbb{K}$. Since the composite of two continuous linear mappings is continuous and linear, it follows that $T^*y^* \in X^*$ for every $y^* \in Y^*$.

Equation (13) clearly implies (12). Let $y^* \in Y^*$ and $x^* \in X^*$ be given. Assume that (12) holds for all $x \in X$ and that

$$\langle x^*, x \rangle_{X^* \times X} = \langle y^*, Tx \rangle_{Y^* \times Y} \quad \text{for all } x \in X.$$

Then

$$\langle x^* - T^*y^*, x \rangle_{X^* \times X} = 0 \quad \text{for all } x \in X,$$

and we conclude that $x^* = T^*y^*$, which proves the required uniqueness.

It is clear that the mapping $y^* \to T^*y^*$ is linear. It remains to show that

$$\sup\{\|T^*y^*\| : y^* \in Y^*, \|y^*\| \leq 1\} = \|T\|,$$

which will establish that T^* is bounded and that $\|T^*\| = \|T\|$. Observe that

$$\begin{aligned}
\|T\| &= \sup\{\|Tx\| : x \in X, \|x\| \le 1\} \\
&= \sup\{|\langle y^*, Tx\rangle_{Y^* \times Y}| : y^* \in Y^*, x \in X, \|y^*\| \le 1, \|x\| \le 1\} \\
&= \sup\{|\langle T^* y^*, x\rangle_{X^* \times X}| : y^* \in Y^*, x \in X, \|y^*\| \le 1, \|x\| \le 1\} \\
&= \sup\{\|T^* y^*\| : y^* \in Y^*, \|y^*\| \le 1\}. \;\; \square
\end{aligned}$$

Remark 9.19: Let X be a NLS and let I_X and I_{X^*} denote the identity mappings on X and X^*, respectively. Observe that for all $x \in X, x^* \in X^*$ we have

$$\langle I_{X^*}(x^*), x\rangle = \langle x^*, x\rangle = \langle x^*, I_X(x)\rangle,$$

and consequently

$$(I_X)^* = I_{X^*}.$$

Example 9.20 (The Finite-Dimensional Case) TO BE FILLED IN.

Theorem 9.21: Let X, Y, Z be NLS and let $T \in \mathcal{L}(X; Y)$ and $S \in \mathcal{L}(Y; Z)$ be given. Then $(ST)^* = T^* S^*$.

Proof: Let $x \in X, z^* \in Z^*$ be given. Then we have

$$\begin{aligned}
\langle (ST)^* z^*, x\rangle_{X^* \times X} &= \langle z^*, (ST)x\rangle_{Z^* \times Z} = \langle z^*, S(Tx)\rangle_{Z^* \times Z} \\
&= \langle S^* z^*, Tx\rangle_{Y^* \times Y} = \langle T^*(S^* z^*), x\rangle_{X^* \times X} \\
&= \langle (T^* S^*) z^*, x\rangle_{X^* \times X}.
\end{aligned}$$

We conclude that $(ST)^* z^* = T^* S^* z^*$ for all $z^* \in Z^*$, i.e. $(ST)^* = T^* S^*$. $\square$

Second Adjoints

Let X, Y be NLS and let $T \in \mathcal{L}(X; Y)$ be given. Since $T^* \in \mathcal{L}(Y^*; X^*)$, we can talk about the adjoint T^{**} of T^*. Observe that $T^{**} \in \mathcal{L}(X^{**}; Y^{**})$. For matrices, the transpose of the transpose is the same as the original matrix. We want to understand if there is an analogue of this result for adjoints in the general setting. Of course, T^{**} and T are different kinds of objects – the inputs for T^{**} are elements of the second dual of X and the inputs for T are elements of X. We can (and will) identify X with a linear manifold $\hat{X}$ in X^{**}. If X is not reflexive, then $\hat{X} \subsetneq X^{**}$, so that T^{**} cannot be identified with T; however T^{**} can always be identified with an extension of T.

Let

$$\hat{X} = J_X[X], \quad \hat{Y} = J_Y[Y],$$

and define $\hat{T} \in \mathcal{L}(\hat{X}; \hat{Y})$ by

$$\hat{T}\hat{x} = J_Y(T((J_X)^{-1}\hat{x})) \quad \text{for all } \hat{x} \in \hat{X}.$$

Given a linear manifold $\mathcal{D}(S) \subset X^{**}$, and a linear mapping $S : \mathcal{D}(S) \to Y^{**}$ we say that S is an *extension* of T provided that

$$\mathcal{D}(S) \supset \hat{X} \quad \text{and} \quad S\hat{x} = \hat{T}\hat{x} \quad \text{for all } \hat{x} \in \hat{X}.$$

If S is an extension of T and $\mathcal{D}(S) = \hat{X}$ then we write $S = T$. (What we are really doing here is identifying T with $\hat{T}$.)