# Homework 3

21-260 Differential Equations
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Due: Friday, July 13, 2012

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**Section 7.1, Problem 15**

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Suppose $x = x_1(t), y = y_1(t)$ and $x = x_2(t), y = y_2(t)$ are both solutions to the linear homogeneous system

$$\begin{aligned} x' &= p_{1,1}(t)x + p_{1,2}(t)y & (1) \\ y' &= p_{2,1}(t)x + p_{2,2}(t)y. & (2) \end{aligned}$$

Suppose that, for some $c_1, c_2 \in \mathbb{R}$, $x = c_1x_1(t)+c_2x_2(t)$ and $y = c_1y_1(t)+c_2y_2(t)$. Then, by linearity of the derivative,

$$x' = c_1x_1'(t) + c_2x_2'(t).$$

Since $x_1$ and $x_2$ satisfy equation (1),

$$\begin{aligned} x' &= c_1(p_{1,1}(t)x_1 + p_{1,2}(t)y_1) + c_2(p_{1,1}(t)x_2 + p_{1,2}(t)y_2) \\ &= p_{1,1}(c_1x_1 + c_2x_2) + p_{1,2}(c_1y_1 + c_2y_2) \\ &= p_{1,1}x + p_{1,2}y, \end{aligned}$$

so that equation (1) is satisfied.

The proof that such choices of $x$ and $y$ also satisfy equation (2) is essentially identical, up to the naming of some terms. Thus, the superposition principle holds for this system. $\blacksquare$

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**Section 7.1, Problem 20**

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By the given relation between voltage across and current through an inductor, if $V_4$ denotes the voltage across the inductor, then

$$\frac{dI}{dt} = \frac{V_4}{1\,H}.$$

By Kirchhoff's Voltage Law, if $V_3$ denotes the voltage across the $1\,\Omega$ resistor, then

$$\frac{V_4}{1\,H} = \frac{-V_3 - V}{1\,H},$$

so that, by the given relation between voltage across and current through a resistor,

$$\frac{-V_3 - V}{1\,H} = \frac{-I(1\,\Omega) - V}{1\,H}.$$

Removing units of measurement gives the first desired result:

$$\frac{dI}{dt} = -I - V. \quad \blacksquare$$

By the given relationship between voltage across and current through a capacitor, if $I_1$ is the current through the capacitor, then

$$\frac{dV}{dt} = \frac{I_1}{0.5\,F}.$$

By Kirchhoff's Current Law, if $I_2$ denotes the current through the $2\,\Omega$ resistor, then

$$\frac{I_1}{0.5\,F} = \frac{I - I_2}{0.5\,F},$$

so that, by the given relation between voltage across and current through a resistor (and noting that the voltage across this resistor is the same as that over the capacitor),

$$\frac{I - I_2}{0.5\,F} = \frac{I - V/(2\,\Omega)}{0.5\,F}.$$

Removing units of measurement gives the second desired result:

$$\frac{dV}{dt} = 2I - V. \quad \blacksquare$$

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**Section 7.2, Problem 12**

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Gauss-Jordan elimination of the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

gives the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right],$$

so that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}}.$$

## Section 7.2, Problem 20

By definition of the inverse and identity matrices and by associativity of matrix multiplication,

$$B = BI = B(AC) = (BA)C = IC = C. \quad \blacksquare$$

## Section 7.3, Problem 4

Suppose the following hold:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \quad (3) \\ 2x_1 + x_2 + x_3 &= 0 \quad (4) \\ x_1 - x_2 + 2_3 &= 0 \quad (5) \end{aligned}$$

Noting that the sum of equations (3) and (5) is equation (4) allows us to ignore equation (4), as it does not additionally constrain the set of solutions. Eliminating the $x_1$ first term from equation (4) using equation (3) gives:

$$-3x_2 + 3x_3 = 0, \text{ or, more simply, } x_2 = x_3,$$

and using this equation to eliminate the $x_3$ term from equation (3) gives:

$$x_1 + x_2 = 0.$$

This simplified system of linear equations has the obvious solution:

$$\mathbf{x} = \boxed{\left\{ c \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}.}$$