Department of Mathematical Sciences
Carnegie Mellon University

21-740 **Functional Analysis II** **Fall 2013**

Assignment 3

Due on Friday, November 8

1. Prove or Disprove: Let X be a Banach space, $\mathcal{D}(A)$, $\mathcal{D}(B) \subset X$, and $A : \mathcal{D}(A) \to X$, $B : \mathcal{D}(B) \to X$ be closed linear operators such that $\mathcal{R}(B) \subset \mathcal{D}(A)$. Then $AB : \mathcal{D}(B) \to X$ is closed.

2. Let X, Y be Banach spaces, $\mathcal{D}(A) \subset X$ and $A : \mathcal{D}(A) \to Y$ be a linear operator. Show that A is closable if and only if
$$\forall (x, y) \in \mathrm{cl}(Gr(A)), \quad x = 0 \Rightarrow y = 0.$$

3. Let $p \in [1, \infty)$ be given, put $\mathcal{D}(A) = \mathbb{K}^{(\mathbb{N})}$ and define $A : \mathcal{D}(A) \to X$ by
$$Ax = \left(\sum_{n=1}^{\infty} n x_n, x_2, x_3, x_4, \dots \right) \quad \text{for all } x \in \mathcal{D}(A).$$
   (a) Is A closed?
   (b) Is A closable?

4. Prove or Disprove: Let X be a Banach space and let $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup. Then
$$\forall t \in [0, \infty),\ T(t) \neq 0.$$

5. Let X be a Banach space and assume that $T : [0, \infty) \to \mathcal{L}(X; X)$ is a linear C_0-semigroup with infinitesimal generator A. Let $N \in \mathbb{N}$ be given and construct Banach spaces
$$X_N \hookrightarrow X_{N-1} \hookrightarrow \cdots \hookrightarrow X_1 \hookrightarrow X_0$$
by
$$X_0 = X, \quad \|x\|_0 = \|x\| \ \text{ for all } x \in X_0,$$
$$X_1 = \mathcal{D}(A), \quad \|x\|_1 = \|x\|_0 + \|Ax\|_0 \ \text{ for all } x \in X_1,$$
and for all $n \in \{1, 2, \cdots, N\}$
$$X_N = \{x \in X_{n-1} : Ax \in X_{n-1}\}, \quad \|x\|_n = \|x\|_{n-1} + \|Ax\|_{n-1} \ \text{ for all } x \in X_n.$$
   (a) Let $n \in \{0, 1, 2, \cdots, N\}$ and $x \in X_n$ be given. Show that $T(t)x \in X_n$ for all $t \geq 0$.

(b) Let $x \in X_N$ be given and put

$$u(t) = T(t)x \quad \text{for all } t \geq 0.$$

Show that for each $k \in \{0, 1, 2, \cdots, N\}$ we have

$$u \in C^{N-k}([0, \infty); X_k).$$

6. Let X be a Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear C_0-semigroup with infinitesimal generator A. Let $L \in \mathcal{L}(X; X)$ be given. Show that $A + L$ (with domain $\mathcal{D}(A)$) is the infinitesimal generator of a linear C_0-semigroup.

7. Let X be a Banach space and assume that $T : [0, \infty)$ satisfies

   (i) $T(0) = I$,

   (ii) $T(s + t) = T(s)T(t)$ for all $s, t \in [0, \infty)$,

   (iii) For all $x \in X$ and $x^* \in X^*$ we have $x^*(T(t)x) \to x^*(x)$ as $t \to 0^+$.

   Show that T is a linear C_0-semigroup.

8. Let $X = L^2(\mathbb{R})$. Put $T(0) = I$ and for $t > 0$ put

$$(T(t)u)(x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{t}} u(x - y)\, dy, \quad \text{a.e. } x \in \mathbb{R}.$$

   Show that T is a linear C_0-semigroup and find the infinitesimal generator.