Department of Mathematical Sciences
Carnegie Mellon University

21-640 **Intro to Functional Analysis** **Spring 2013**

Assignment 1

Due on Wednesday January 23

Solutions to problems marked with an astersik should be written up and handed in.

1. Let $X = \mathbb{R}$ and put

$$\rho(x, y) = |\arctan x - \arctan y| \qquad \text{for all } x, y \in X.$$

   (a) Determine whether or not ρ is a metric on X.

   (b) If ρ is a metric determine whether or not (X, ρ) is complete?

   (c) If ρ is a metric determine whether or not (X, ρ) is separable?

2*. Prove or Disprove: Let (X, ρ) be a complete metric space and let $x \in X$ and $\delta > 0$ be given. Then

$$cl(B_\delta(x)) = \{y \in X : \rho(y, x) \leq \delta\}.$$

3. Let (X, ρ) be a metric space and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of subsets of X.

   (a) Show that $cl(S_1 \cup S_2) = [cl(S_1)] \cup [cl(S_2)]$.

   (b) What can you conclude about

$$\bigcup_{n=1}^{\infty} S_n \quad ?$$

4. Prove or Disprove: Let (X, ρ) be a metric space and $S \subset X$. Then S is nowhere dense if and only if $\text{int}(U \backslash S) \neq \emptyset$ for every nonempty open set U.

5*. Prove or Disprove: Let (X, ρ) be a complete metric space. If $S \subset X$ is meager then S^c is dense.

6*. Two metric spaces (X, ρ) and (Y, σ) are said to be

   (i) *homeomeorphic* provided there exists a bijection $f : X \to Y$ such that f and f^{-1} are continuous.

   (ii) *uniformly homeomorphic* provided there exists a bijection $f : X \to Y$ such that f and f^{-1} are uniformly continuous.

   (a) Give an example of two metric spaces that are not homeomorphic.

   (b) Give an example of two metric spaces that are homeomorphic, but not uniformly homeomorphic, or prove that no such example exists.

(c) Give an example of two metric spaces that are uniformly homeomorphic, but not isometric, or prove that no such example exists.

7. Let $X = C(\mathbb{R})$, the set of all continuous real-valued functions on $\mathbb{R}$. For every $n \in \mathbb{N}, f, g \in X$, define

$$P_n(f,g) = \max\{|f(x) - g(x)| : x \in [-n,n]\}.$$

Consider the function $\rho : X \times X \to \mathbb{R}$ defined by

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} P_n(f,g)}{1 + P_n(f,g)} \qquad \text{for all } f, g \in X.$$

(a) Verify that ρ is a metric on X.

(b) Is (X, ρ) complete?

(c) Is (X, ρ) separable?

8*. (Contraction Mapping Principle (also called Banach's Fixed-Point Theorem)) Let (X, ρ) be a complete metric space with $X \neq \emptyset$. Let $f : X \to X$ be given and assume that there exists $\alpha \in (0,1)$ such that

$$\rho(f(x), f(y)) \leq \alpha \rho(x,y) \quad \text{for all } x, y \in X.$$

(Such a mapping is said to be *strictly contractive*.) Show that f has exactly one *fixed point*, i.e. show that there is exactly one $x^* \in X$ such that $x^* = f(x^*)$.

Suggestion: Choose any point $x_0 \in X$ and for each $n = 0, 1, 2, \cdots$ put $x_{n+1} = f(x_n)$. Show that

$$\rho(x_{m+1}, x_m) \leq \alpha^m \rho(x_1, x_0).$$

Use this estimate together with the triangle inequality to show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Finally, show that the limit of this sequence has the required properties.

9. Let $f : [0,1] \to \mathbb{R}$ and $k : [0,1] \times [0,1] \to \mathbb{R}$ be given continuous functions and assume that $|k(x,y)| < 1$ for all $x, y \in [0,1]$. Show that there is exactly one continuous function $u : [0,1] \to \mathbb{R}$ satisfying

$$u(x) + \int_0^1 k(x,y) u(y) dy = f(x) \qquad \text{for all } x \in [0,1].$$

10*. Let $X = C[0,1]$ the set of all continuous real-valued functions on $[0,1]$ and define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\} \quad \text{for all } f, g \in X.$$

You may take it for granted that (X, ρ) is a complete metric space. Let S be the set of all functions in X having at least one point of differentiability in $(0,1)$. Show that $S^c \neq \emptyset$.