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Remark 14.1: $G = S_5$ has only one subgroup of order 60, which is A_5: if there was $H \leq G$ with $H \neq A_5$, one would choose $a \in A_5 \setminus H$, so that $G = H \cup (a\,H)$ (because $|H| = 60$ and $a \not\in H$) and taking the intersection with A_5, the union of $H \cap A_5$ and $(a\,H) \cap A_5$ would be A_5, but $(a\,H) \cap A_5 = a\,(H \cap A_5)$ since $a \in A_5$,[1] so that $K = H \cap A_5$ would be a subgroup of A_5 and $a\,K$ would have the same number of elements as K, which would imply that K has size 30, a contradiction since A_5 is simple hence has no proper subgroup of index ≤ 4 (because $4! < |A_5|$).

Remark 14.2: 5-cycles belong to A_5, and (by putting 1 as the first element of a cycle) there are 4! of them, which makes 24 elements of order 5. 4-cycles are odd permutations, which do not belong to A_5. 3-cycles belong to A_5, and there are $\binom{5}{3} = 10$ subsets of 3 elements, and each subset $\{a, b, c\}$ corresponds to two 3-cycles ($(a\,b\,c)$ and its square $(a\,c\,b)$), which makes 20 elements of order 3. A permutation having one 3-cycle and one 2-cycle is an odd permutation, as well as a 2-cycle, so that they do not belong to A_5. A permutation with two disjoint 2-cycles belongs to A_5, and for those fixing one element in $\{1, 2, 3, 4, 5\}$ there are 3, which makes 15 elements of order 2.

Remark 14.3: Each 5-cycle $\sigma \in A_5$ generates a cyclic subgroup H of order 5, $\{e, \sigma, \sigma^2, \sigma^3, \sigma^4\}$, so that there are 6 such Sylow 5-subgroups. Interpreting a cycle like $(1\,2\,3\,4\,5)$ as a rotation of $\frac{2\pi}{5}$ of a regular pentagon with vertices named 1, 2, 3, 4, 5 (in this order), one can then interpret the five mirror symmetries as $(2\,5)\,(3\,4)$, $(1\,3)\,(4\,5)$, $(1\,5)\,(2\,4)$, $(1\,2)\,(3\,4)$, and $(1\,4)\,(2\,3)$, which with H form a subgroup K isomorphic to D_5. Since each of these six isomorphic copies of D_5 use up five elements of order 2 and there are only 15 of them, one expects that each element of order 2 is associated with two unrelated 5-cycles (i.e. generating different cyclic subgroups): for example $(2\,5)\,(3\,4)$ is associated with $\sigma = (1\,2\,3\,4\,5)$ (and its powers $\sigma^2 = (1\,3\,5\,2\,4)$, $\sigma^3 = (1\,4\,2\,5\,3)$, and $\sigma^4 = (1\,5\,4\,3\,2)$) but also with $\pi = (1\,2\,4\,3\,5)$ (and its powers $\pi^2 = (1\,4\,5\,2\,3)$, $\pi^3 = (1\,3\,2\,5\,4)$, and $\pi^4 = (1\,5\,3\,4\,2)$).

Each K is the normalizer of the cyclic subgroup H since H is a normal subgroup of K and the only subgroup containing K is A_5 (because A_5 has no subgroups of order 20 or 30), but A_5 has no normal proper non-trivial subgroup.

Actually, any subgroup L of G of order 10 should contain a subgroup of order 5, i.e. one of the H_j, and since H_j is automatically a normal subgroup of L, L must be equal to $K_j = N_G(H_j)$.

Remark 14.4: Each 3-cycle $\sigma \in A_5$ generates a cyclic subgroup H of order 3, $\{e, \sigma, \sigma^2\}$, so that there are 10 such Sylow 3-subgroups. Considering a cycle like $(1\,2\,3)$, one can add to H the three elements (of order 2) $\tau\,(4\,5)$ where τ is a transposition on $\{1, 2, 3\}$, and obtain a subgroup K isomorphic to S_3.[2] Since each of these ten isomorphic copies of S_3 use up three elements of order 2 and there are only 15 of them, one expects that each element of order 2 is associated with two unrelated 3-cycles (i.e. generating different cyclic subgroups): for example $(1\,2)\,(3\,4)$ is associated with $\sigma = (1\,2\,5)$ (and its square $\sigma^2 = (1\,5\,2)$) but also with $\pi = (3\,4\,5)$ (and its square $\pi^2 = (3\,5\,4)$).

Each H is a normal subgroup of the corresponding K, but since A_5 has subgroups of order 12, it is simpler to invoke Sylow's theorem for being sure that K is the normalizer of H (since the orbit of H by conjugation has size 10, hence the normalizer $N_G(H)$ has order 6), and then since H is a Sylow 3-subgroup and $K = N_G(H)$, one has $N_G(K) = K$, so that if $K \leq L \leq G$ with $K \neq L$, one must have $L = G$, since Lagrange's theorem implies that the order of L is a strict multiple of 6 and a divisor of 60, so that it could only be 12 or 30 or 60, but there is no subgroup of A_5 of order 30, and the subgroups of order 12 cannot contain K, since K would automatically be a normal subgroup of such a subgroup of order 12.

Actually, any subgroup M of G of order 6 should contain a subgroup of order 3, i.e. one of the H_j, and since H_j is automatically a normal subgroup of M, M must be equal to $K_j = N_G(H_j)$. However,

[1] If $a\,h = b \in A_5$ with $h \in H$, then $h = a^{-1}b$ belongs to A_5, so that $h \in H \cap A_5$, hence $b = a\,h \in a\,(H \cap A_5)$.

[2] If $h \in H = \{e, (1\,2\,3), (1\,3\,2)\}$, then $h\,\big(\tau\,(4\,5)\big) = (h\,\tau)\,(4\,5)$, and $h\,\tau$ is a transposition on $\{1, 2, 3\}$, while the product of $\tau_1(4\,5)$ by $\tau_2(4\,5)$ is $\tau_1\tau_2$, which belongs to H.

there are (at least) two subgroups of G of order 12 containing H, since there are two distinct elements $a, b \in \{1,2,3,4,5\}$ left invariant by the 3-cycles σ and σ^2 of H_j, so that H_j is included in the isomorphic copy of A_4 leaving a fixed, and in the isomorphic copy of A_4 leaving b fixed, and the intersection of these two subgroups of order 12 leave a and b fixed, so that it is H.

Remark 14.5: Let K_1, K_2, K_3, K_4, K_5 be the five subgroups of A_5 of order 12 and isomorphic to A_4, where K_j are the permutations in A_5 which leave j fixed. K_j has a normal subgroup N_j isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, containing 3 elements of order 2, which are not repeated since for $j \neq i$ the intersection $K_i \cap K_j$ are the permutations in A_5 which leave both i and j fixed, which is one of the Sylow 3-subgroup (containing no element of order 2), so that the five Sylow 2-subgroups of A_5 are the N_j, and one has $N_G(N_j) = K_j$, since it contains K_j but cannot be larger (because a subgroup containing strictly a subgroup of order 12 must have order 60).

Remark 14.6: All the subgroups of order 3, 4, or 5 of A_5 have been accounted for, since they are the Sylow p-subgoups (10 subgroups of order 3, 5 subgroups of order 4, 6 subgroups of order 5), and their normalizers have been identified (10 subgroups of order 6 isomorphic to S_3, 5 subgroups of order 12 isomorphic to A_4, 6 subgroups of order 10 isomorphic to D_5).

Any subgroup K of order 6 contains a subgroup H of order 3, which is automatically normal in K, so that K is $N_G(H)$ for a Sylow 3-subgroup H, hence it is isomorphic to S_3, and there are 10 of them.

Any subgroup K of order 10 contains a subgroup H of order 5, which is automatically normal in K, so that K is $N_G(H)$ for a Sylow 5-subgroup H, hence it is isomorphic to D_5, and there are 6 of them.

Let K be a subgroup of order 12, which contains a 3-cycle $(x\,y\,z)$ and an element $(a\,b)\,(c\,d)$ of order 2. If the element $\in \{1,2,3,4,5\}$ fixed by $(a\,b)\,(c\,d)$ is also fixed by $(x\,y\,z)$, they belong to one K_j isomorphic to A_4, and the subgroup generated by $(x\,y\,z)$ and $(a\,b)\,(c\,d)$ must be K_j, or it would be a subgroup of order 6, automatically normal in K, but any subgroup of order 6 has been identified to be $N_G(H)$ for a Sylow 3-subgroup H, hence is its own normalizer. If the element $\in \{1,2,3,4,5\}$ fixed by $(a\,b)\,(c\,d)$ belongs to $\{x,y,z\}$, say it is x, one arrives at a contradiction: either the element of order 2 sends y onto z, and both elements belong to the normalizer $N_G(H)$ of the Sylow 3-subgroup generated by $(x\,y\,z)$, which is its own normalizer and cannot belong to a subgroup of order 12, or the element of order 2 sends y onto an element different from x and z, and the situation is like having $(1\,2\,3)$ and $(2\,4)\,(3\,5)$, but the product $(1\,2\,3)\,(2\,4)\,(3\,5)$ is $(1\,2\,4\,3\,5)$, which has order 5. The subgroups of order 12 are then the 5 subgroups isomorphic to A_4.

There are 15 subgroups of order 2, of the form $\{e,\sigma\}$ for an element $\sigma = (a\,b)\,(c\,d)$ of order 2, but what is the normalizer K of $\{e,\sigma\}$? It is the centralizer of σ, i.e. the subgroup of elements of A_5 which commute with σ, and it contains the Sylow 2-subgroup H containing σ, since H is Abelian, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, so that its order is a multiple of 4 which divides 60, i.e. it is 4 or 12, or 60, but it must then be 4, so that $K = H$, since if it was 12 or 60 K would contain an isomorphic copy of A_4 containing σ, but in A_4 an element of order 2 does not commute with an element of order 3.[3]

Lemma 14.7: Let G be any *simple* group of order 60, and for $p = 2,3,5$, let n_p be the number of Sylow p-subgroups of G. Then, one has $n_2 = 5$, $n_3 = 10$, and $n_5 = 6$. Each Sylow-2 subgroup H_i is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and two distinct Sylow 2-subgroups only intersect at $\{e\}$, so that the five Sylow 2-subgroups make 15 elements of order 2. Each Sylow-3 subgroup K_j is isomorphic to $\mathbb{Z}_3$, its normalizer $N_G(K_j)$ has order 6 and is isomorphic to S_3, and the ten Sylow 3-subgroups make 20 elements of order 3. Each Sylow-5 subgroup L_k is isomorphic to $\mathbb{Z}_5$, its normalizer $N_G(L_k)$ has order 10 and is isomorphic to D_5, and the six Sylow 5-subgroups make 24 elements of order 5.

Proof: Since G is simple with $4! = 24 < |G| < 5! = 120$, each n_p is ≥ 5. By the Sylow's theorem, $n_2 = 1 \pmod 2$ and divides 15, so that $n_2 \in \{5, 15\}$, $n_3 = 1 \pmod 3$ and divides 20, so that $n_3 = 10$, and $n_5 = 1 \pmod 5$ and divides 12, so that $n_5 = 6$. The ten Sylow 3-subgroups contain 20 elements of order 3, and the six Sylow 5-subgroups contain 24 elements of order 5, so that at most 15 elements can have order $\notin \{1,3,5\}$, and the last element is e. One wants to show that $n_2 = 5$ and that two distinct Sylow 2-subgroups only intersect at $\{e\}$, so that the five Sylow 2-subgroups use up the 15 elements. If it was not true, either $n_2 = 5$ and two distinct Sylow 2-subgroups H and H' would contain $g \neq e$, or $n_2 = 15$, and by the pigeon-hole

[3] Without loss of generality, one may take the element of order 3 to be $(1\,2\,3)$ and the element of order 2 to be $(1\,2)\,(3\,4)$, and $(1\,2\,3)\,(1\,2)\,(3\,4) = (1\,3\,4)$, while $(1\,2)\,(3\,4)\,(1\,2\,3) = (2\,4\,3)$.

principle there would exist two distinct Sylow 2-subgroups intersecting at more than $\{e\}$ (or there would be 45 elements of order 2 or 4), and one shows that it leads to a contradiction.

Since g must have order 2 (because $H \neq H'$), let $L = N_G(\langle g \rangle)$ be the normalizer of the subgroup $\langle g \rangle = \{e, g\}$ generated by g; since H and H' are Abelian (isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$), H and H' are subgroups of L, hence by Lagrange's theorem the order of L is a multiple of 4 which divides 60, so that the only possibilities are 4, 12, 20, or 60: 4 is excluded because it implies $H = H'$, 20 is excluded because the index of a subgroup must be ≥ 5, 60 is excluded because it implies that $\langle z \rangle$ is a normal subgroup of G, hence $|L| = 12$. Since L has index 5, there is an injective homomorphism from G into S_5, so that G is isomorphic to a subgroup of S_5 of order 60, hence $L = A_5$, but in A_5 two distinct Sylow 2-subgroups only intersect at $\{e\}$.

If H is a Sylow-3 subgroup, its orbit by conjugation has size 10, which is the index of its normalizer $N_G(H)$ so that $N_G(H)$ has order 6, and a group of order 6 is either isomorphic to $\mathbb{Z}_6$ or to S_3, but $\mathbb{Z}_6$ is excluded since G contains no element of order 6. If H is a Sylow-5 subgroup, its orbit by conjugation has size 6, which is the index of its normalizer $N_G(H)$ so that $N_G(H)$ has order 10, and a group of order 10 is either isomorphic to $\mathbb{Z}_{10}$ or to D_5, but $\mathbb{Z}_{10}$ is excluded since G contains no element of order 10.

If the Sylow 2-subgroup H_j is isomorphic to $\mathbb{Z}_4$, then it contains exactly one subgroup K_j of order 2, with $K_j = \{e, a_j\}$ where a_j is the only element of order 2 in H_j, so that the (two) automorphisms of H_j maps a_j onto itself, i.e. K_j is a characteristic subgroup of H_j, and since H_j is a normal subgroup of its normalizer $N_G(H_j)$, one deduces that K_j is a normal subgroup of $N_G(H_j)$, and $N_G(H_j)$ is a subgroup of G of order 12 (since the orbit of H_j under conjugation by G has size 5). Since $N_G(H_j)/K_j$ has order 6, it is either isomorphic to $\mathbb{Z}_6$ or to S_3; if π is the projection of $N_G(H_j)$ onto $N_G(H_j)/K_j$ and L is a subgroup of order 2 of $N_G(H_j)/K_j$, then $\pi^{-1}(L)$ is a subgroup of order 4 of $N_G(H_j)$, i.e. a Sylow 2-subgroup of $N_G(H_j)$, and H_j is the only one since it is a normal subgroup of $N_G(H_j)$, and because $L = \pi\big(\pi^{-1}(L)\big)$, there is only one subgroup of order 2 of $N_G(H_j)/K_j$, which is then $\simeq \mathbb{Z}_6$ (since S_3 has three subgroups of order 2). There is then an element $b \in N_G(H_j)$ such that $\pi(b)$ has order 6 in $N_G(H_j)/K_j$, and this means that $b, b^2, b^3 \notin K_j$ but $b^6 \in K_j$, hence b must have order 6 or 12 in G, and there is no such element, hence $H_j \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, so that it has three elements of order 2, hence G has fifteen elements of order 2.

Remark 14.8: If H is a Sylow 2-subgroup, its normalizer $N_G(H)$ is isomorphic to a semi-direct product $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times_\psi \mathbb{Z}_3$: one knows that H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and that its normalizer $N_G(H)$ has order 12, and contains eight elements of order 3 besides e and the three elements of order 2 in H (since H is the only Sylow 2-subgroup of $N_G(H)$), so that $N_G(H)$ has four Sylow-3 subgroups, hence it is not Abelian, and it is then a semi-direct product $H \times_\psi K$ where K is a Sylow-3 subgroup.