Homework 4
21-640 Introduction to Functional Analysis
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Collaborators: None

Problem 2

Since $cl(Y)$ is a linear manifold in X, we can take a Hamel basis $(x_i : i \in J)$ for $cl(Y)$ and extend it to a Hamel basis $(x_i : i \in I)$ with for X with $J \subseteq I$. $\exists y \in cl(Y)$ with $\|y - x_0\| = d$. Letting $z := x_0 - y$, we note that $\|z\| = d$ and that

$$z = \sum_{i \in I \setminus J} \alpha_i(z) x_i$$

(with at most finitely many $a_i(z) \neq 0$), since, if any $\alpha_i(z) \neq 0$ for some $i \in J$, $y' := x_0 - (z - \alpha_i(z)x_i) \in cl(Y)$, and $\|y' - x_0\| < \|y - x_0\| = d$, contradicting the definition of d.

Thus, letting $K := \{i \in I : \alpha(z) \neq 0\}$, $Z := \operatorname{span}((x_i : i \in K) \cup cl(Y))$, we can define $f : Z \to \mathbb{K}$ as the continuous linear functional

$$f(v) = \frac{\sum_{i \in K} \alpha_i(v)\|x_i\|}{\sum_{i \in K} \alpha_i(z)\|x_i\|}.$$

Then, $f(z) = 1$, $\sup\{|f(v)| : v \in Y, \|v\| \leq 1\} = f\left(\frac{z}{\|z\|}\right) = \frac{1}{d}$, and $f(v) = 0$, $\forall v \in cl(Y)$, so that, by Theorem 5.3, there is an extension $x^* \in X^*$ of f, with $\langle x^*, x_0 \rangle = 1$, $\|x^*\| = \frac{1}{d}$, and $\langle x^*, y \rangle = 0$, $\forall v \in Y \subseteq cl(Y)$. ■

Problem 3

We modify the proof of Theorem 6.1 (Hahn-Banach Theorem, Separation Form) in two ways.

First, we choose $x_1 \in K_1$ to be an interior point (any interior point is internal, so the original proof still holds), and we observe that 0 is then an interior point of K.

Second, we observe that, if 0 is an internal point of K (say $B_\delta(0) := \{x \in X : \|x\| \leq \delta\} \subseteq K$), then, $\forall x \in B_1(0)$, $\delta x \in K$, so that the Minkowski Functional p^K is bounded on $B_1(0)$ (by δ^{-1}). Then, since $F \leq p^K$ on X, $\|F(x)\| \leq \|\delta^{-1}x\|, \forall x \in B_1(0)$ and thus F is continuous.

We also note that the generalization of F to the case $\mathbb{K} = \mathbb{C}$ preserves continuity. ■

Problem 5

(a) If $x, y \in K_1$, then, $\forall t \in (0,1)$,

$$(tx + (1-t)y)_{m(tx+(1-t)y)} = tx_{\max\{m(x),m(y)\}} + (1-t)y_{\max\{m(x),m(y)\}} > 0,$$

so that $(tx + (1-t)y) \in K_1$, and thus K_1 is convex.

Suppose now that $x \in K_1$ and let $z \in X$ defined by $z_{m(x)+1} = -1$ and $z_i = 0$, $\forall i \neq m(x)+1$. Then, $\forall \varepsilon > 0$, for $y := x + \frac{\varepsilon}{2}z$, $y_{m(y)} = -\frac{\varepsilon}{2} < 0$, so that $y \notin K_1$. Thus, x is not an internal point of K_1, and so K_1 has no internal points. ■

(b) Let $K_2 := \{0\}$, so that K_2 is clearly convex and $K_1 \cap K_2 = \emptyset$. Suppose, for sake of contradiction, that some nontrivial linear $F : X \to \mathbb{R}$ separates K_1, K_2. Since $F(0) = 0$, either $F[K_1] \subseteq [0, \infty)$ or $F[K_1] \subseteq (-\infty, 0]$; we assume the former, as the argument in the other case is symmetric.

F is non-trivial, so $\exists x \in X$ with $F(x) \neq 0$. Let $z \in X$ defined by $z_{m(x)+1} = 1$ and $z_i = 0$, $\forall i \neq m(x)+1$. Then, let $y := z - (|F(z)| + 1)\frac{x}{F(x)}$, so that $y_{m(y)} = z_{m(z)} = 1$, so $y \in K_1$. But

$$F(y) = F(z) - (|F(z)| + 1)\frac{F(x)}{F(x)} = F(z) - (|F(z)| + 1) < 0,$$

which is a contradiction. ■

Problem 6

The function $T : c_0 \to c$ defined by

$$T\left(\{x_n\}_{n=0}^{\infty}\right) = \{x_0 + x_{n+1}\}_{n=0}^{\infty}, \quad \forall \{x_n\}_{n=0}^{\infty} \in c_0$$

is a continuous linear bijection.

Linearity is clear, since each coordinate of $T(x)$ is a linear combination of coordinates of x.

If, for some $x \in c_0$, $\|x\|_\infty \leq 1$, then, $\forall i \in \mathbb{N}$, $|x_i| \leq 1$, so that $|(T(x))_i| \leq |x_0| + |x_i| \leq 2$, and thus $\|T(x)\|_\infty \leq 2$. Therefore, T is bounded on $B_1(0)$ and is thus continuous.

$\forall y \in c$, for $L := \lim_{n\to\infty} y_n$, $(y_n - L) \to 0$ as $n \to \infty$, and $T(L, y_0 - L, y_1 - L, \ldots) = y$, so that T is surjective. T is also injective, since, if $x, y \in c_0$ with $T(x) = T(y)$, then

$$\begin{aligned} x_0 &= \lim_{n\to\infty} (T(x))_i = \lim_{n\to\infty} (T(y))_i = y_0, \\ x_i &= (T(x))_{i-1} - x_0 = (T(y))_{i-1} - y_0 = y_i, \quad \forall i \geq 1. \quad ■ \end{aligned}$$

Problem 8

Since F is non-trivial and $F(0) = 0$, $S \neq \mathbb{K}$. If S is closed, then $cl(S) = S \neq \mathbb{K}$, so S is not dense.

Since S is the pre-image of the closed set $\{\alpha\}$, if S is not closed, F is not continuous. Then, we claim, $\forall \varepsilon > 0$, $s \in \mathbb{K}$, $\exists x \in B_\varepsilon(0)$ with $F(x) = s$ (the case $s = 0$ is trivial; since F is unbounded on $B_\varepsilon(0)$, $\forall s \in \mathbb{K}\backslash\{0\}$, $\exists y \in B_\varepsilon(0)$ with $|F(y)| \geq |s| > 0$, so that $x := \frac{sy}{F(y)} \in B_\varepsilon(0)$ with $F(x) = \frac{sF(y)}{F(y)} = s$).

Then, $\forall \varepsilon > 0$, $y \in X$, $\exists x \in B_\varepsilon(0)$ with $F(x) = \alpha - F(y)$, so that $x + y \in B_\varepsilon(y)$ and $x + y \in S$, since and $F(x + y) = \alpha - F(y) + F(y) = \alpha$. Thus, S is dense. ■

It's worth noting that S is closed precisely when F is continuous, and S is dense otherwise.

Problem 9

Define $K := \left\{ f \in X : f(0) = 0 \text{ and } \int_0^1 f(x)\,dx \geq 1 \right\}$, and, $\forall n \in \mathbb{N}$, define $f_n \in X$ by

$$f_n(x) = \begin{cases} (n+1)x & \forall x \in [0, 1/n] \\ \frac{(n+1)}{n} & \forall x \in (1/n, 1] \end{cases}.$$

Clearly, each $f_n(0) = 0$ and it can be checked that

$$\int_0^1 f(x)\,dx = \frac{n+1}{2n^2} + \left(1 - \frac{1}{n}\right)\left(\frac{n+1}{n}\right) = \frac{2n^2 + n + 1}{2n^2} \geq 1,$$

so that each $f_n \in K$. Then, since $\|f_n\|_\infty = \frac{n+1}{n} \to 1$ as $n \to \infty$, $\inf\{\|f\|_\infty : f \in K\} \leq 1$.

Suppose $f \in K$. Since f is continuous and $f(0) = 0$, $\exists \delta \in (0, 1)$ such that $f < 1$ on $[0, \delta)$. Thus,

$$1 \leq \int_0^1 f(x)\,dx < \int_0^\delta 1 + \int_\delta^1 \|f\|_\infty\,dx = \delta + (1 - \delta)\|f\|_\infty,$$

so that $1 = \frac{1-\delta}{1-\delta} < \|f\|_\infty$, and so $\nexists g \in K$ with $\|g\|_\infty = \inf\{\|f\|_\infty : f \in K\} \leq 1$.

If $f^{(n)} \in K$ converge to $f \in X$, then $f(0) = \lim_{n\to\infty} f^{(n)}(0) = 0$ and (since convergence in $\|\cdot\|_\infty$ is uniform) $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f^{(n)}(x)\,dx \geq 1$, so that $f \in K$, and thus K is closed.

Finally, if $f, g \in K$, then, $\forall t \in (0, 1)$, $tf(0) + (1 - t)g(0) = 0$ and

$$\int_0^1 tf(x) + (1 - t)g(x)\,dx = t\int_0^1 f(x)\,dx + (1 - t)\int_0^1 g(x)\,dx \geq t + (1 - t) = 1,$$

so that $tf + (1 - d)g \in K$, and thus K is convex. ■

Problem 12

Let X be any infinite dimensional Banach space over $\mathbb{R}$, let $(x_i : i \in I)$ be a Hamel basis for X, and let $J \subseteq I$ be countably infinite, with $\sigma : \mathbb{N} \to J$ a bijection. Define $T : X \to X$ by

$$\alpha_i(T(x)) = \begin{cases} \alpha_i(x) & \text{if } i \in I \backslash J \\ (2n+1)^{-1}\alpha_{\sigma(2n+1)}(x) & \text{if } : i = \sigma(2n) \\ (2n+1)\alpha_{\sigma(2n)}(x) & \text{if } : i = \sigma(2n+1) \end{cases} \quad , \quad \forall x \in X, i \in I,$$

where, $\alpha_i(x)$ is the projection of x onto x_i. Clearly, T is linear and injective (T is its own inverse). Since, $\forall n \in \mathbb{N}$, $\|x_{\sigma(2n+1)}\|_\infty = \max\{|\alpha_i(x)| : i \in I\} = 1$, and $\|T(x_{\sigma(2n+1)})\|_\infty = 2n+1$, T is unbounded and hence discontinuous, but, since T is its own inverse, T^2 continuous. ■

Problem 15

Suppose $X = Y = (l^\infty, \|\cdot\|_\infty)$, and let $T \in \mathcal{L}(X;Y)$ be defined $\forall x \in X, i \in \mathbb{N}$ by $(T(x))_i = \frac{x_i}{i^2}$. We claim that $T[B]$ is closed, but that $\mathcal{R}(T)$ is not closed.

Suppose there is a sequence $x^{(n)} \in B$ with $T(x^{(n)}) \to y$, for some $y \in l^\infty$. Then, $\forall i \in \mathbb{N}, \varepsilon > 0$,

$$|y_i| \leq |(T(x^{(n)})_i| + \varepsilon = \frac{|x_i|}{i^2} + \varepsilon \leq \frac{1}{i^2} + \varepsilon,$$

so that $x := (y_1, 2^2y_2, 3^2y_3, \dots) \in B$, and hence, since $T(x) = y$, $y \in T[B]$, and so $T[B]$ is closed.

Now define, $\forall n, i \in \mathbb{N}$, $x_i^{(n)} := \begin{cases} i & \text{if } i \leq n \\ 0 & \text{else} \end{cases}$, so that $(T(x^{(n)}))_i = \begin{cases} 1/i & \text{if } i \leq n \\ 0 & \text{else} \end{cases}$. Then, for $y = (1, 1/2, 1/3, \dots)$, $\forall n \in \mathbb{N}$, $\|T(x^{(n)})i - y\| < 1/n$, so that $T(x^{(n)}) \to y$ as $n \to \infty$, but $y \notin \mathcal{R}(T)$, since $(1, 2, 3, \dots) \notin l^\infty$, and thus $\mathcal{R}(T)$ is not closed. ■