Homework 1

21-260 Differential Equations
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Section 1.1, Problem 24

Let m denote the mass of the drug in the patient's body.

(a) The drug enters the patient's bloodstream at a rate of $(5\,mg/cm^3)(100\,cm^3/h) = 500\,mg/h$ and leaves the patient's bloodstream at a rate of $(0.4\,h^{-1})m$, so that

$$\boxed{\frac{dm}{dt} = 500\,mg/h - (0.4\,h^{-1})m.}$$

(b) Quantity of the drug in the patient's bloodstream reaches equilibrium when $\frac{dm}{dt} = 0\,mg/h$, so that, by the differential equation found in part (a), $0 = 500\,mg/h - (0.4\,h^{-1})m$. This happens when $\boxed{m = 1250\,mg.}$

Section 1.3, Problem 6

The differential equation is of order 3 and is linear.

Section 1.3, Problem 18

Suppose $y = e^{rt}$, for some $r \in \mathbb{R}$. Then, $\forall i \in \mathbb{R}$, the i^{th} derivative of y with respect to t is $y^{(i)} = r^i e^{rt}$. Thus, if $y^{(3)} - 3y^{(2)} + 2y^{(1)} = 0$, then,

$$r^3e^{rt} - 3r^2e^{rt} + 2re^{rt} = 0, \quad \text{so that} \quad r^3 - 3r^2 + 2r = 0$$

(since $e^{rt} \neq 0$). Thus, $\boxed{r \in \{0, 1, 3\}.}$

Section 2.1, Problem 16

Multiplication by $e^{\int 2/t\,dt} = t^2$ gives

$$t^2y + 2ty = \cos t.$$

Thus,

$$t^2y = \int (t^2y + 2ty)\,dt = \int \cos t\,dt = \sin t + C,$$

for some $C \in \mathbb{R}$. Since $y(\pi) = 0$, $0 = (0 + C)/\pi^2$, so that $C = 0$, and thus

$$y = \boxed{\frac{\sin t}{t^2}.}$$

Section 2.2, Problem 6

Since, for $x \neq 0, y \neq \pm 1$, the equation is equivalent to the equation

$$\frac{1}{\sqrt{1 - y^2}} y' = \frac{1}{x},$$

the differential equation is seperable, so that, as shown in class,

$$\int \frac{1}{\sqrt{1 - y^2}}\, dy = \int \frac{1}{x} = \ln(|x|) + C,$$

for some $C \in \mathbb{R}$. Letting $t := \sin^{-1}(y)$ gives

$$\int \frac{1}{\sqrt{1 - y^2}}\, dy = \int \frac{\cos t}{\sqrt{1 - \sin^2 t}}\, dt = \int \frac{\cos t}{\cos t}\, dt = \int 1\, dt = t = \sin^{-1}(y).$$

Thus,

$$y = \boxed{\sin(\ln(|x|) + C).}$$

Section 2.2, Problem 16

(a) Since, for $y \neq 0$, the equation is equivalent to the equation

$$4y^3 y' = x(x^2 + 1),$$

the differential equation is separable, so that, as shown in class,

$$y^4 = \int 4y^3\, dy = \int x(x^2 + 1)\, dx = \frac{x^4}{4} + \frac{x^2}{2} + C,$$

for some constant $C \in \mathbb{R}$. Thus, for $y \neq 0$,

$$y = \left(\frac{x^4}{4} + \frac{x^2}{2} + C \right)^{1/4}.$$

Since, $y(0) = -1/\sqrt{2}$, $C^{1/4} = -1/\sqrt{2}$, so that $C = \frac{1}{4}$, and

$$y = \boxed{\left(\frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{4} \right)^{1/4}.}$$