1 Mastery set [20 points] (Yifei)

A [5 points] Let $\|\cdot\|_1$ be a norm, let $\|\cdot\|_2$ be the dual norm of $\|\cdot\|_1$, and let $\|\cdot\|_3$ be the dual norm of $\|\cdot\|_2$. The conjugate of $\|\cdot\|_1$ is the indicator function $I_{\{z:\|z\|_2\leq 1\}}$ of the unit ball under $\|\cdot\|_2$. The conjugate of $I_{\{z:\|z\|_2\leq 1\}}$ is

$$\left(x \mapsto \max_{\|z\|_2\leq 1} z^T x\right) = \|\cdot\|_3.$$

Since every norm is convex and continuous (in finite dimensions), $\|\cdot\|_3 = \|\cdot\|_1^{**} = \|\cdot\|_1$. ■

B [5 points]

The KKT conditions give

$$\begin{aligned} 0 &\in \partial\left(\frac{1}{2}\|x-y\|_2^2 + \lambda\|x\|_1\right) \\ &\in x - y + \lambda \left\{ \begin{array}{ll} [-1,1] & : \text{ if } x_i = 0 \\ \{\text{sign}(x_i)\} & : \text{ if } x_i \neq 0 \end{array} \right. . \end{aligned}$$

C [5 points] The dual program is

$$\begin{aligned} \min_{u\in\mathbb{R}^4} \quad & u_1 + u_2 \\ \text{such that} \quad & u_1 + u_2/3 - u_3 \geq 1, \\ & u_1/2 + u_2 - u_4 \geq 1, \\ \text{and} \quad & u_1, u_2 u_3, u_4 \geq 0. \end{aligned}$$

We have strong duality

Since we have strong duality, we change the optimization problem into an equivalent feasibility problem by finding point where the dual is equal to the primal; i.e., we add the constraint

$$u_1 + u_2 \leq x_1 + x_2$$

and find a feasible point over all 6 primal and dual variables.

D [5 points] The Lagrangian of the dual is

$$L(u,v,x) = -f^*(-A^T u - C^T v) - b^T u - d^T v + x^T u.$$

Thus, the dual of the dual is

$$\min_{x\in\mathbb{R}^n, x\leq 0} \max_{u\in\mathbb{R}^m, v\in\mathbb{R}^n} - f^*(-A^T u - C^T v) - b^T u - d^T v + x^T u$$