Homework 1

21-236 Mathematical Studies Analysis II
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Problem 1

(a) Let $U \subseteq \mathbb{R}^N$ be open and convex, and let $f : U \to \mathbb{R}$ be differentiable in U.

Suppose f is Lipschitz continuous in U, with Lipschitz constant $L \geq 0$. Let $x \in U$, and let

$$i = \arg \max_{k=1,\ldots,n} \left(\frac{\partial f}{\partial e_i} \right)$$

(so that $\frac{\partial f}{\partial e_i}(\mathbf{x})$ is the partial derivative of f at $\mathbf{x}$ of greatest magnitude). Then,

$$\begin{aligned} ||\nabla f(\mathbf{x})|| &= \sqrt{\sum_{k=1}^{n} \left(\frac{\partial f}{\partial e_k}(\mathbf{x}) \right)^2} \\ &\leq \sqrt{\sum_{k=1}^{n} \left(\frac{\partial f}{\partial e_i}(\mathbf{x}) \right)^2} \\ &= \sqrt{N \left(\frac{\partial f}{\partial e_i}(\mathbf{x}) \right)^2} \end{aligned}$$

Letting $\mathbf{y} = \mathbf{x} - t\mathbf{e}_i$, since f is Lipschitz continuous with Lipschitz constant L,

$$L \geq \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{||\mathbf{x} - \mathbf{y}||} = \frac{|f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})|}{||t\mathbf{e}_i||},$$

so that

$$L \geq \lim_{t \to 0} \frac{|f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})|}{||t\mathbf{e}_i||} = \frac{\partial f}{\partial e_i}(\mathbf{x}).$$

Therefore, for

$$M = \sqrt{NL^2} = L\sqrt{N} > 0,$$

we have $||\nabla f(\mathbf{x})|| \leq M$. ■

Suppose, on the other hand, that, for some $M > 0$, $\forall x \in U$, $\nabla f(\mathbf{x}) \leq M$. Since f is differentiable in U, by the Cauchy-Schwarz inequality, $\forall \mathbf{x} \in U$,

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v} \leq ||\nabla f(\mathbf{x})|| \cdot ||\mathbf{v}|| = ||\nabla f(\mathbf{x})|| \leq M.$$

Let $\mathbf{x}, \mathbf{y} \in U$. Let $\mathbf{v} = \frac{\mathbf{y}-\mathbf{x}}{||\mathbf{y}-\mathbf{x}||}$. Since f is differentiable on U, by the Mean Value Theorem, for some $\Theta \in (0,1)$ (noting that, since U is convex, so that $(\Theta\mathbf{x} + (1-\Theta)\mathbf{y}) \in U$),

$$\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{||\mathbf{x} - \mathbf{y}||} = \frac{\partial f}{\partial \mathbf{v}}(\Theta\mathbf{x} + (1-\Theta)\mathbf{y}) \leq M.$$

Thus, f is Lipschitz continuous on U. ■

(b) f has continuous partial derivatives on U and is thus differentiable on U. The partial derivatives of f are bounded above and below by 2 and -2, respectively. Thus, for $M = 2\sqrt{2}$, $||\nabla f(x,y)|| \leq M$, $\forall (x,y) \in U$. For $x = 1$, for $y > 0$, $f(x,y) > 1$, whereas, for $y < 0$, $f(x,y) < -1$, so that, $\forall L > 0$, $\exists (x_1, y_1), (x_2, y_2) \in U$ such that

$$|f(x_1,y_1) - f(x_2,y_2)| > L||fx_1,y_1) - fx_2,y_2).||$$

Therefore, f is not Lipschitz continuous on U. ■

Problem 2

Let $f : B(\mathbf{x}_0, r) \to \mathbb{R}$ be Lipschitz continuous, with Lipschitz constant L.

(a) This follows immediately from part (b).

(b) Since $\partial B(\mathbf{0}$ is a closed and bounded set in R^N, E is compact in R^N. Thus, since $\forall k \in \mathbb{N}$, $C := \{B(\mathbf{v}, \frac{1}{2k})|\mathbf{v} \in E\}$ is an open cover of E, there exists a finite open subcover $S \subseteq C$ of E. Since E is dense, $\forall B \in S$, $\exists \mathbf{v} \in E \cap B$. Thus, $\forall \mathbf{x} \in B(\mathbf{x}_0, r)$, for some $\mathbf{v} \in E$,

$$\frac{f(\mathbf{x} - \mathbf{x}_0 - \nabla f(\mathbf{x})(\mathbf{x} - \mathbf{x}_0)}{||\mathbf{x} - \mathbf{x}_0||} = \frac{f(\mathbf{x} - \mathbf{x}_0 - \frac{\partial f}{\partial \mathbf{v}}(mathbf{x} - \mathbf{x}_0)}{||\mathbf{x} - \mathbf{x}_0||} \to 0$$

as $\mathbf{x} \to \mathbf{x}_0$. Thus, since $\mathbf{x}_0$ is an interior point of $B(\mathbf{x}_0, r)$, f is differentiable at $\mathbf{x}_0$. ■

Problem 3

Let $f : \mathbb{R}^2 \backslash \{(x,y)\mathbb{R}^2 | xy = 0, x^2 + y^2 \neq 0\} \to \mathbb{R}$ such that, $\forall (x,y) \in \mathbb{R}$,

$$f(x,y) = \frac{|xy|}{xy}(x^2 + y^2).$$

Then,

$$\lim_{(x,y)\to\mathbf{0}} \frac{f(x,y) - f(\mathbf{0}) - 0}{||(x,y) - \mathbf{0}||} = \lim_{(x,y)\to\mathbf{0}} \frac{|xy|(x^2+y^2)}{xy\sqrt{x^2+y^2}} = 0,$$

since $\frac{|xy|}{xy}$ is bounded and $\sqrt{x^2+y^2} \to 0$ as $(x,y) \to \mathbf{0}$, so that f is differentiable and thus continuous at $\mathbf{0}$. However, since f is undefined on the x- and y-axes (except at $\mathbf{0}$), f has no partial derivatives at $\mathbf{0}$. ■

Problem 4

(a) f has continuous partial derivatives and is thus differentiable and continuous everywhere except $\mathbf{0}$. At $\mathbf{0}$, f is continuous if and only if $m + n \geq 2$, and differentiable if and only if $m + n \geq 3$.

(b) f has continuous partial derivatives and is thus differentiable and continuous everywhere except $\mathbf{0}$. At $\mathbf{0}$, f is continuous if and only if $m \geq 2$, $n \geq 4$, or $m = 1$ and $n = 3$, and f is differentiable if and only if $m \geq 3$, $n \geq 5$, $m = 1$ and $n = 4$, or $m = n = 2$.

(c) f has continuous partial derivatives and is thus differentiable and continuous wherever $x^2 \neq y^2$. f is discontinuous and thus not differentiable wherever $x^2 = y^2$. Wherever $x^2 = y^2$, the directional derivatives of f exist only in those directions pointing towards and away from the origin.

(d) Let $g, h : \mathbb{R}^2 \to \mathbb{R}$ such that, $\forall (x, y) \in \mathbb{R}^2$, $g(x, y) = x^2 \sin\left(\frac{1}{x}\right)$ and $h(x, y) = y^2 \sin\left(\frac{1}{y}\right)$. Then, g and h are everywhere differentiable, so that, since $f = g + h$, f is everywhere differentiable. Therefore, f is everywhere continuous, and all directional derivatives of f exist at all points.