Homework 5

21-260 Differential Equations
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
Due: Thursday, July 19, 2012

Section 7.5, Problem 4

(a) If A is the matrix of constant coefficients of the given system, then, if λ is an eigenvalue of A, then,

$$0 = (1-\lambda)(-2-\lambda) - 4 = (\lambda+3)(\lambda-2),$$

so that $\lambda \in \{-3, 2\}$. Thus, the eigenvectors of A associated with eigenvalues $\lambda = -3$ and $\lambda = 2$ are, respectively,

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus, solutions to the system of differential equations are of the form

$$\mathbf{x} = \boxed{c_1\boldsymbol{\xi}^{(1)}e^{-3t} + c_2\boldsymbol{\xi}^{(2)}e^{2t}.}$$

Since the eigenvalues of A have different signs, the solution will approach multiples a basis vector of the general solution (in particular, $\boldsymbol{\xi}^{(2)}$) as $t \to \infty$.

(b) Figure 1 below shows a direction field as well as some sample trajectories for $\mathbf{x}$.

Section 7.6, Problem 6

(a) If A is the matrix of constant coefficients of the given system of differential equations, then, if λ is an eigenvalue of A, then $\lambda = \pm 3i$. Thus, the eigenvectors of A associated with eigenvalues $\lambda = 3i$ and $\lambda = -3i$, respectively, are

$$\boldsymbol{\xi}^{(1)} = \boxed{\begin{bmatrix} 2 \\ -1+3i \end{bmatrix},} \quad \boldsymbol{\xi}^{(2)} = \boxed{\begin{bmatrix} 2 \\ -1-3i \end{bmatrix}.}$$

For $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, since $\boldsymbol{\xi} = \mathbf{a} + i\mathbf{b}$, for

$$\begin{aligned} \mathbf{x}^{(1)} &= e^{0 \cdot t}(\mathbf{a}\cos(3t) - \mathbf{b}\sin(3t)) = \mathbf{a}\cos(3t) - \mathbf{b}\sin(3t) \\ \mathbf{x}^{(2)} &= e^{0 \cdot t}(\mathbf{a}\sin(3t) + \mathbf{b}\cos(3t)) = \mathbf{a}\sin(3t) + \mathbf{b}\cos(3t), \end{aligned}$$

$\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$ is a fundamental set of real-valued solutions, so that solutions to the given system of differential equations are of the form

$$\mathbf{x} = \boxed{c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)},}$$

for some $c_1, c_2 \in \mathbb{R}$.

(b) Since the eigenvalues of A have no real part, motion of $\mathbf{x}$ over time is purely elliptical and there is no limit in $\mathbf{x}$ as $t \to \infty$. Figure 2 below shows a direction field as well as some sample trajectories for $\mathbf{x}$.

Section 7.8, Problem 4

(a) Figure 3 below shows a direction field as well as some sample trajectories for $\mathbf{x}$.

(b) If A is the matrix of constant coefficients of the given system of differential equations, then, since (as shown in part (c)), the only eigenvalue of A is negative, all solutions to the system of differential equations approach $\mathbf{0}$ as $t \to \infty$.

(c) If λ is an eigenvalue of A, then

$$0 = (-3 - \lambda)(2 - \lambda) + 25/4 = \left(\lambda + \frac{1}{2}\right)^2,$$

so that $\lambda = -\frac{1}{2}$. The only eigenvector of A is

$$\boldsymbol{\xi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus,

$$\mathbf{x}^{(1)} = \boldsymbol{\xi} e^{-\frac{1}{2}t}$$

is one solution to the system of differential equations. For $\boldsymbol{\eta}$ satisfying $(A + \frac{1}{2}I)\boldsymbol{\eta} = \boldsymbol{\xi}$, another linearly independent solution to the system of differential equations is

$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{-\frac{1}{2}t} + \boldsymbol{\eta} e^{-\frac{1}{2}t}.$$

One such value for $\boldsymbol{\eta}$ is $\begin{bmatrix} 2/5 \\ 0 \end{bmatrix}$.

Thus, solutions to the given system of differential equations are of the form

$$\mathbf{x} = \boxed{c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{1}{2}t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-\frac{1}{2}t} + \begin{bmatrix} 2/5 \\ 0 \end{bmatrix} e^{-\frac{1}{2}t} \right).}$$

Section 9.1, Problem 8

(a) If A is the matrix of constant coefficients of the given system of differential equations, then, if λ is an eigenvalue of A, then

$$0 = (-1 - \lambda)(-0.25 - \lambda)$$

so that $\boxed{\lambda \in \{-1, -0.25\}.}$ Thus, the eigenvectors of A associated with eigenvalues $\lambda = -3$ and $\lambda = 2$, respectively, are

$$\boldsymbol{\xi}^{(1)} = \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \quad \boldsymbol{\xi}^{(2)} = \boxed{\begin{bmatrix} -0.75 \\ 1 \end{bmatrix}}.$$

(b) Since both eigenvalues of A are negative, the origin is an $\boxed{\text{asymptotically stable}}$ equilibrium point.

(c) Figure 4 below shows several sample trajectories of $\mathbf{x}$, as well as some typical graphs of x_1 versus t.

Section 9.1, Problem 10

(a) As found in the solution to part (a) of Problem 6 in chapter 7.6, if A is the matrix of constant coefficients of the given system of differential equations, then, if λ is an eigenvalue of A, then $\boxed{\lambda = \pm 3i,}$ and the eigenvectors of A associated with eigenvalues $\lambda = 3i$ and $\lambda = -3i$, respectively, are

$$\boldsymbol{\xi}^{(1)} = \boxed{\begin{bmatrix} 2 \\ -1+3i \end{bmatrix}}, \quad \boldsymbol{\xi}^{(2)} = \boxed{\begin{bmatrix} 2 \\ -1-3i \end{bmatrix}}.$$

(b) As explained in the solution to part (a) of Problem 6 in chapter 7.6, all solutions to the given system of differential equations are elliptical, so that the origin is a $\boxed{\text{stable (but not asymptotically stable)}}$ equilibrium point.

(c) Figure 5 below shows several sample trajectories of $\mathbf{x}$, as well as some typical graphs of x_1 versus t.