Exam 2

21-621 Introduction to Lebesgue Integration
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Page 93, Problem 19

By Fubini's Theorem,

$$\int_0^\infty m(E_\alpha)\,d\alpha = \int_0^\infty \int_{\mathbb{R}^d} \chi_{E_\alpha}\,dx\,d\alpha = \int_{\mathbb{R}^d} \int_0^\infty \chi_{E_\alpha}\,d\alpha\,dx = \int_{\mathbb{R}^d} \int_0^{|f(x)|} 1\,d\alpha\,dx = \int_{\mathbb{R}^d} |f(x)|\,dx. \quad \blacksquare$$

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By Translation Invariance of the Lebesgue Integral over $\mathbb{R}^d$,

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^d} f\left(x - \frac{\xi}{2\|\xi\|^2}\right) e^{-2\pi i\left(x - \frac{\xi}{2\|\xi\|^2}\right)\cdot\xi}\,dx \\
&= \int_{\mathbb{R}^d} f\left(x - \frac{\xi}{2\|\xi\|^2}\right) e^{-2\pi i x\cdot\xi} e^{-2\pi i \frac{\xi\cdot\xi}{2\|\xi\|^2}}\,dx \\
&= \int_{\mathbb{R}^d} f\left(x - \frac{\xi}{2\|\xi\|^2}\right) e^{-2\pi i x\cdot\xi} e^{-\pi i}\,dx \\
&= -\int_{\mathbb{R}^d} f\left(x - \frac{\xi}{2\|\xi\|^2}\right) e^{-2\pi i x\cdot\xi}\,dx. && \text{(since } e^{-\pi i} = -1\text{)}
\end{aligned}$$

Adding $\hat{f}(\xi)$, dividing by 2, and taking absolute values on both sides gives, by the Triangle Inequality and then by Monotonicity of the Lebesgue Integral,

$$\begin{aligned}
|\hat{f}(\xi)| &= \frac{1}{2}\left|\int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\,dx - \int_{\mathbb{R}^d} f\left(x - \frac{\xi}{2\|\xi\|^2}\right) e^{-2\pi i x\cdot\xi}\,dx\right| \\
&= \frac{1}{2}\left|\int_{\mathbb{R}^d} \left(f(x) - f\left(x - \frac{\xi}{2\|\xi\|^2}\right)\right) e^{-2\pi i x\cdot\xi}\,dx\right| && \text{Linearity} \\
&\leq \frac{1}{2}\int_{\mathbb{R}^d} \left|\left(f(x) - f\left(x - \frac{\xi}{2\|\xi\|^2}\right)\right) e^{-2\pi i x\cdot\xi}\right|\,dx && \text{Triangle Inequality} \\
&\leq \frac{1}{2}\int_{\mathbb{R}^d} \left|f(x) - f\left(x - \frac{\xi}{2\|\xi\|^2}\right)\right|\,dx && \text{Monotonicity} \\
&= \frac{1}{2}\left\|f(x) - f\left(x - \frac{\xi}{2\|\xi\|^2}\right)\right\|_{L^2(\mathbb{R}^d)} \to 0
\end{aligned}$$

as $\xi \to \infty$, by Propositon 2.5 (since $\frac{\xi}{2\|\xi\|^2} \to 0$). $\blacksquare$

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(a) Define $E_\alpha := |f|^{-1}((\frac{\alpha}{2}, \infty))$, $F_\alpha := (f^*)^{-1}((\alpha, \infty))$. By regularity of the Lebesgue Measure, it suffices to show that, for any compact $K \subseteq F_\alpha$, $m(K) \leq \frac{2A}{\alpha} \int_{E_\alpha} |f|$.

By definition of f^*, $\forall x \in F_\alpha$, $\exists$ a ball B_x with $x \in B_x$ and $m(B_x) \leq \frac{1}{\alpha} \int_{B_x} |f|$.

Since K is compact, there exists a finite subcover $\mathcal{B}$ of K of such balls. Then, by the version of the Vitali Covering Lemma shown in class, there is a subset $\mathcal{B}' \subseteq \mathcal{B}$ with

$$m(K) \leq m\left(\bigcup_{B \in \mathcal{B}} B\right) \leq A \sum_{B \in \mathcal{B}'} m(B).$$

Then, by additivity of the Lebesgue Integral over sets and the definition of E_α, $\forall B \in \mathcal{B}'$,

$$m(B) \leq \frac{1}{\alpha}\left(\int_{B \setminus E_\alpha} |f| + \int_{B \cap E_\alpha} |f|\right) \leq \frac{1}{\alpha}\left(\frac{\alpha}{2} m(B) + \int_{B \cap E_\alpha} |f|\right).$$

Isolating $m(B)$ then gives $m(B) \leq \frac{2}{\alpha} \int_{B \cap E_\alpha} |f|$. Thus, since the balls in $\mathcal{B}'$ are disjoint,

$$m(K) \leq A \sum_{B \in \mathcal{B}'} m(B) \leq A \sum_{B \in \mathcal{B}'} \frac{2}{\alpha} \int_{B \cap E_\alpha} |f| \leq \frac{2A}{\alpha} \int_{E_\alpha} |f|. \quad \blacksquare$$

(b) By Fubini's Theorem,

$$\begin{aligned}
\int_{\mathbb{R}^d} (f^*)^2 = \int_{\mathbb{R}^d} \int_{(f^*)^2}^{\infty} 1 &= \int_{\mathbb{R}^d} \int_0^\infty \chi_{((f^*)^2)^{-1}((y,\infty))}\, dy \\
&= \int_0^\infty \int_{\mathbb{R}^d} \chi_{((f^*)^2)^{-1}((y,\infty))}\, dy \\
&= \int_0^\infty m(E_{\sqrt{y}})\, dy = 2 \int_0^\infty \alpha m(E_\alpha)\, d\alpha,
\end{aligned}$$

where the last equality follows from changing variables to $\alpha = \sqrt{y}$. $\blacksquare$

(c) Combining the results of parts (a) and (b) gives

$$\|f^*\|_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^d} (f^*)^2 \leq 2 \int_0^\infty 2A \int_{|f|^{-1}(\frac{\alpha}{2}, \infty)} |f|\, d\alpha \leq 4A \int_{\mathbb{R}} f^2 = C\|f\|_{L^2(\mathbb{R}^d)},$$

where $C = 4A$. Since $f \in L^2$, it follows immediately that $f^* \in L^2$. $\blacksquare$