Assignment 3: Assigned Wed 09/19. Due Wed 09/26

1. Let X be a topological space, and μ be a regular Borel measure on X. Show that X has a *maximal* open set of measure 0. Namely, show that there exists $U \subseteq X$, such that U open set, $\mu(U) = 0$ and further for any open set $V \subseteq X$ with $\mu(V) = 0$, we must have $V \subseteq U$. [The complement of U is defined to be the *support* of the measure μ, and denoted by $\operatorname{supp}(\mu)$.]
2. Let $\Sigma \supseteq \mathcal{B}(\mathbb{R}^d)$, and μ be a regular measure on $(\mathbb{R}^d, \Sigma)$. Suppose $A \in \Sigma$ is σ-finite (i.e. $A = \cup_1^\infty A_n$, and $\mu(A_n) < \infty$). Show that $\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ is compact}\}$. [This remains true if we replace $\mathbb{R}^d$ with any Hausdorff space.]
3. Let μ, ν be two measures on (X, Σ). Suppose $\mathcal{C} \subseteq \Sigma$ is a π-system such that $\mu = \nu$ on $\mathcal{C}$.
   (a) Suppose $\exists C_i \in \mathcal{C}$ such that $\bigcup_1^\infty C_i = X$ and $\mu(C_i) = \nu(C_i) < \infty$. Show that $\mu = \nu$ on $\sigma(\mathcal{C})$.
   (b) If we drop the finiteness condition $\mu(C_i) < \infty$ is the previous subpart still true? Prove or find a counter example.
4. Let $\kappa \in (0,1)$. Does there exist $E \in \mathcal{L}(\mathbb{R})$ such that for all $a < b \in \mathbb{R}$, we have $\kappa(b-a) \leqslant \lambda(I \cap (a,b)) \leqslant (1-\kappa)(b-a)$? Prove or find a counter example. [I'm aware that this looks suspiciously like a homework problem you already did. Also, this problem has a short, elegant solution using only what we've seen in class so far.]
5. For $i \in \{1,2\}$, let (X_i, Σ_i, μ_i) be two measure spaces with $\mu_i(X_i) < \infty$. Define $\Sigma_1 \otimes \Sigma_2 = \sigma\{A_1 \times A_2 \mid A_i \in \Sigma_i\}$.
   (a) Let $x_1 \in X_1$ and $A \in \Sigma_1 \otimes \Sigma_2$. Let $S_{x_1}(A) = \{x_2 \in X_2 \mid (x_1, x_2) \in A\}$, and $T_{x_2}(A) = \{x_1 \in X_1 \mid (x_1, x_2) \in A\}$. Show that $S_{x_1}(A) \in \Sigma_2$ and $T_{x_2}(A) \in \Sigma_1$.
   (b) If $A \in \mathcal{P}(X_1 \times X_2)$ is such that for all $x_i \in X_i$, $S_{x_1}(A) \in \Sigma_2$ and $S_{x_2}(A) \in \Sigma_1$. Must $A \in \Sigma_1 \otimes \Sigma_2$?
   (c) Show that there exists a measure ν on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ such that for all $A_i \in \Sigma_i$ we have $\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$.
6. *(An alternate approach to λ-systems.)* Let $\mathcal{M} \subseteq P(X)$. We say $\mathcal{M}$ is a *Monotone Class*, if whenever $A_i, B_i \in \mathcal{M}$ with $A_i \subseteq A_{i+1}$ and $B_i \supseteq B_{i+1}$ then $\bigcup_1^\infty A_i \in \mathcal{M}$ and $\bigcap_1^\infty B_i \in \mathcal{M}$. If $\mathcal{A} \subseteq P(X)$ is an algebra, then show that the *smallest* monotone class containing $\mathcal{A}$ is exactly $\sigma(A)$. [You should also address existence of a smallest monotone class containing $\mathcal{A}$.]

Optional problems, and details in class I left for you to check.

* Let X be a second countable locally compact Hausdorff space, and μ be a Borel measure on X that is finite on compact sets. Show that μ is regular.
* Is any σ-finite Borel measure on $\mathbb{R}^d$ regular?
* Show that any λ-system that is also a π-system is a σ-algebra.
* If Π is a π-system, then $\lambda(\Pi) = \sigma(\Pi)$. (We only proved $\lambda(\Pi) \subseteq \sigma(\Pi)$.)

Assignment 4: Assigned Wed 09/26. Due Wed 10/03

1. Let $f : X \to \mathbb{R}$ be measurable, and $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. True or false: $g \circ f : X \to \mathbb{R}$ is measurable? Prove or find a counter example.
2. Let (X, Σ) be a measure space, and $f, g : X \to [-\infty, \infty]$ be measurable. Suppose whenever $g = 0$, $f \neq 0$, and whenever $f = \pm\infty$, $g \in (-\infty, \infty)$. Show that $\frac{f}{g} : X \to [-\infty, \infty]$ is measurable. [Note that by the given data you will never get a 'meaningless' quotient of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. The remainder of the quotients (e.g. $\frac{1}{\infty}$) can be defined in the natural manner.]
3. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions such that $(f_n) \to f$ almost everywhere (a.e.). Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function.
   (a) If for a.e. $x \in X$, g is continuous at $f(x)$, then show $(g \circ f_n) \to g \circ f$ a.e.
   (b) Is the previous part true without the continuity assumption on g?
4. Let $C \subseteq \mathbb{R}^d$ be convex. Must C be Lebesgue measurable? Must C be Borel measurable? Prove or find counter examples. [The cases $d = 1$ and $d > 1$ are different.]
5. Let (X, Σ, μ) be a measure space, and $(X, \Sigma_\mu, \bar{\mu})$ it's completion. Show that $g : X \to [-\infty, \infty]$ is Σ_μ-measurable if and only if there exists two Σ-measurable functions $f, h : X \to [-\infty, \infty]$ such that $f = h$ μ-almost everywhere, and $f \leqslant g \leqslant h$ everywhere.
6. Let X be a metric space, $\Sigma \supseteq \mathcal{B}(X)$ a σ-algebra on X, and μ a regular finite measure on (X, Σ). Let $f : X \to \mathbb{R}$ be measurable.
   (a) For any $\varepsilon > 0$ and $i \in \mathbb{N}$, show that there exists finitely many disjoint compact sets $\{K_{i,j} \mid |j| \leqslant N_i\}$ such that
   $$\mu\Big(X - \bigcup_{j=-N_i}^{N_i} K_{i,j}\Big) < \frac{\varepsilon}{2^i}, \quad \text{and} \quad f(K_{i,j}) \subseteq \big[\frac{j}{2^i}, \frac{j+1}{2^i}\big)$$
   (b) *(Lusin's Theorem)* For any $\varepsilon > 0$ show that there exists $K_\varepsilon \subseteq X$ compact such that $f : K_\varepsilon \to \mathbb{R}$ is *continuous*, and $\mu(X - K_\varepsilon) < \varepsilon$. [HINT: Let $K_\varepsilon = \bigcap_{i=1}^\infty \bigcup_{|j| \leqslant N_i} K_{i,j}$. Define $g_i : K_\varepsilon \to \mathbb{R}$ by $g_i(x) = j/2^i$ if $x \in K_{i,j}$ and $|j| \leqslant N_i$. Show $g_i : K \to \mathbb{R}$ is continuous and $(g_i) \to f$ uniformly on K_ε.]

   A standard extension theorem now shows that for any $f : X \to \mathbb{R}$ measurable and $\varepsilon > 0$, there exists $g_\varepsilon : X \to \mathbb{R}$ *continuous* such that $\mu\{f \neq g_\varepsilon\} < \varepsilon$.

Optional problems, and details in class I left for you to check.

* Show that $f : X \to [-\infty, \infty]$ is measurable if and only if any of the following conditions hold
  (a) $\{f < a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (b) $\{f > a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (c) $\{f \leqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (d) $\{f \geqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
* Let $f : [0,1] \to [0,1]$ be the Cantor function, and $g(x) = \inf\{f = x\}$. Show that f is continuous, and the range of g is the Cantor set. Are f, g Hölder continuous? If yes, what are the largest exponents α, β for which f, g are respectively Hölder-α and Hölder-β continuous.