Homework 6

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36-705 Intermediate Statistics
Due: Thursday, October 16, 2014

1. (a) We showed on a previous assignment that the MLE for λ is $\hat{\lambda}_{MLE} = \overline{X}$. Hence, since $\mathbb{P}(X_i = 0) = e^{-\lambda}$, the MLE for θ is $\boxed{\hat{\theta}_{MLE} = e^{-\overline{X}}.}$

   (b) Note that

$$I(\lambda) = -\mathbb{E}\left[\frac{\partial^2}{\partial \lambda^2} \log L(\lambda)\right] = \mathbb{E}\left[\frac{\partial^2}{\partial \lambda^2} \lambda - x \log \lambda + \log X!\right] = \mathbb{E}\left[\frac{\partial}{\partial \lambda} 1 - x/\lambda\right] = 1/\lambda.$$

   Hence, since the asymptotic variance of the MLE is

$$v(\lambda) = \frac{\left(\frac{\partial}{\partial \lambda} e^{-\lambda}\right)^2}{I(\lambda)} = \lambda e^{-2\lambda}$$

   by asymptotic normality of the MLE, $\boxed{\sqrt{n}(\hat{\theta}_{MLE} - \theta) \to \mathcal{N}(0, \lambda e^{-2\lambda}).}$

   (c) It follows immediately from asymptotic normality that $\hat{\theta}_{MLE}$ is consistent. ■

2. (a) Since $\mathbb{E}[X_i] = \theta/2$, the method of moments estimator is $\boxed{\hat{\theta}_{MOM} = 2\overline{X}.}$ The maximum likelihood estimator is $\boxed{\hat{\theta}_{MLE} = X_{(n)}.}$

   (b) Since $\overline{X}$ is a consistent estimator for $\mathbb{E}[X_i] = \theta/2$, $2\overline{X}$ is a consistent estimator for $2\mathbb{E}[X_i] = \theta$. For $\varepsilon > 0$,

$$\mathbb{P}[|X_{(n)} - \theta| > \varepsilon] = \mathbb{P}[X_{(n)} < \theta - \varepsilon] = \mathbb{P}^n[X_i < \theta - \varepsilon] = \left(\frac{\varepsilon}{\theta}\right)^n \to 0$$

   as $n \to \infty$. Hence, $\hat{\theta}_{MLE}$ is consistent. ■

   (c) Since $\mathbb{V}[X_i] = \theta^2/12$, by the Central Limit Theorem, $\sqrt{n}(\overline{X} - \theta/2) \to \mathcal{N}(0, \theta^2/12)$. Hence,

$$\sqrt{n}(\hat{\theta}_{MOM} - \theta) = \sqrt{n}(2\overline{X} - \theta) \to \boxed{\mathcal{N}(0, \theta^2/3)}$$

   in distribution. For $t > 0$,

$$\begin{aligned}\mathbb{P}[n(\theta - \hat{\theta}_{MLE}) < t] = \mathbb{P}[X_{(n)} > \theta - t/n] &= 1 - \mathbb{P}[X_{(n)} \leq \theta - t/n] \\ &= 1 - \left(\frac{\theta - t/n}{\theta}\right)^n \to 1 - e^{-t/\theta}\end{aligned}$$

   as $n \to \infty$. Hence, $\boxed{n(\theta - \hat{\theta}_{MLE}) \to \exp(1/\theta)}$ in distribution.

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(d) By the previous part and the Gaussian tail inequality, for $\sigma = \theta/\sqrt{3}$,

$$\lim_{n\to\infty} \mathbb{P}\left[\sqrt{n}|(\hat{\theta}_{MOM} - \theta)| > \varepsilon\right] \leq \frac{2\sigma e^{-\varepsilon^2/(2\sigma^2)}}{\varepsilon} \to 0$$

as $\varepsilon \to \infty$, and hence, $\hat{\theta}_{MOM} - \theta \in O_P(n^{-1/2})$. Also by the previous part,

$$\lim_{n\to\infty} \mathbb{P}\left[n|\theta - \hat{\theta}_{MLE}| > \varepsilon\right] = e^{-\varepsilon/\theta} \to 0$$

as $\varepsilon \to \infty$, and hence $\hat{\theta}_{MLE} - \theta \in O_P(n^{-1})$. ■

3. The MLE is $\hat{\theta}_{MLE} = \overline{X}$ and $\mathsf{se}\left(\hat{\theta}_{MLE}\right) = 1/\sqrt{n}$. Hence, we reject if and and only if the Wald test statistic $\boxed{W = \sqrt{n}\left(\overline{X} - \mu_0\right),}$ satisfies $|W| > z_{\alpha/2}$.

4. (a) For $\hat{\sigma} = \sqrt{n^{-1}\sum_{i=1}^n (X_i - \overline{X})^2}$, the Wald test statistic is

$$\boxed{W = \frac{\overline{X} - \mu_0}{\hat{\sigma}/n},}$$

and we reject if and and only if $|W| > z_{\alpha/2}$.

(b) Note that, for any $\mu', \sigma \in \mathbb{R}$ the likelihood

$$L(\mu') \propto \prod_{i=1}^n \exp\left(-\frac{(X_i - \mu')^2}{2\sigma^2}\right) = \exp\left(-\frac{\sum_{i=1}^n (X_i - \mu')^2}{2\sigma^2}\right).$$

Since the MLE of μ is $\overline{X}$ and the MLE of σ is $\hat{\sigma} = \sqrt{n^{-1}\sum_{i=1}^n (X_i - \overline{X})^2}$, the LRT statistic is

$$L = -2\log\frac{L(\mu_0)}{L(\overline{X})} = \frac{1}{\hat{\sigma}^2}\sum_{i=1}^n (X_i - \mu_0)^2 - (X_i - \overline{X})^2 = \boxed{\frac{n}{\hat{\sigma}^2}(\mu_0 - \overline{X})^2,}$$

and we reject if and only if $L > \chi^2_{1,\alpha}$.

(c) For $\hat{\sigma} = \sqrt{n^{-1}\sum_{i=1}^n (X_i - \overline{X})^2}$, the Wald test statistic is

$$W = \frac{\hat{\sigma}^2 - \sigma_0^2}{\mathsf{se}(\hat{\sigma}^2)},$$

where $\mathsf{se}(\hat{\sigma}^2) =$?? and and we reject if and and only if $|W| > z_{\alpha/2}$.

(d) For $\hat{\mu} = \overline{X}$ and any $\sigma \in \mathbb{R}$, the likelihood

$$L(\sigma^2) \propto (\sigma^2)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \overline{X})^2\right). \propto (\sigma^2)^{-n/2}\exp\left(-\frac{n\hat{\sigma}^2}{2\sigma^2}\right).$$

where $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^n (X_i - \overline{X})^2$ is the MLE for σ. Hence, the LRT statistic is

$$\boxed{L = -2\log\frac{L(\sigma_0^2)}{L(\hat{\sigma}^2)} = n\left(\log(\sigma_0^2/\hat{\sigma}^2) + \frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right),}$$

and we reject if and only if $L > \chi^2_{1,\alpha}$.