Homework 5

21-651 General Topology
Name: Shashank Singh
Email: sss1@andrew.cmu.edu
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The following characterization of lower semi-continuity is used in the proofs of problems 4. and 6.

Lemma 1: $f : X \to \mathbb{R}$ is lower semi-continuous if and only if, $\forall x_0 \in X$, $\forall \epsilon > 0$, $\forall x$ is some neighborhood U of x_0 in X, $f(x) \geq f(x_0) - \epsilon$.

Proof of Lemma 1: ($\Rightarrow$) Suppose that f is lower semi-continuous, and let $x_0 \in X, \epsilon > 0$. Let $U = (f(x_0) - \epsilon, \infty)$, so that, since f is lower semi-continuous, $V := f^{-1}(U)$ is open. Then, V is a neighborhood of x_0 in X such that $f(V) \subseteq (f(x_0) - \epsilon, \infty)$, so that, $\forall x \in V$, $f(x) \geq f(x_0) - \epsilon$. ■

($\Leftarrow$) Suppose that, $\forall x_0 \in X, \epsilon > 0$, there is a neighborhood W of x_0 in X such that, $\forall x \in W$, $f(x) \geq f(x_0) - \epsilon$. Let $a \in \mathbb{R}$, and, for $V := (a, \infty)$, let $x_0 \in U := f^{-1}(V)$. Since, V is open, for some $\epsilon > 0$, $(f(x_0) - \epsilon, \infty) \subseteq (a, \infty)$. Thus, there is a neighborhood W of x_0 in X such that, $\forall x \in W$, $f(x) \geq f(x_0) - \epsilon > a$, so that $f(W) \subseteq (a, \infty)$, and thus $W \subseteq U$. Since every point in U is contained in a neighborhood in U, U is open, so that f is lower semi-continuous. ■

Problem 1

Let $I := C(X, [0, 1])$, and let e be the function on X defined by

$$e(x)(f) := f(x), \forall f \in I$$

(e is, in some sense, a restriction of the evaluation function used in Stone-Cech compatification). Note that, since, $\forall x \in X$, $e(x) : I \to [0, 1]$, $e(X) \subseteq [0, 1]^I$. e is a homeomorphism between X and $e(X)$. Since (X, τ) is Tikhonov, the proof is identical to the proof that the evaluation function used in Stone-Cech compactification is homeomorphism (as used in the proof of Theorem 142). ■

Problem 2

(i) ($\Rightarrow$) Suppose, $f : X \to [0, 1]$ is continuous, with $f = 0$ on C, and $f > 0$ on $X \backslash C$. $\forall i \in \mathbb{N}$, define $U_i := f^{-1}([0, 1/i))$. Since, $\forall i \in \mathbb{N}$, $(-1, 1/i)$ is open in the standard topology on $\mathbb{R}$, $[0, 1/i)$ is open in the topology induced on $[0, 1]$, so that, since f is continuous, $U_i \in \tau$. Then,

$$\bigcap_{i=1}^{\infty} U_i = \{x \in X : \forall i \in \mathbb{N}, f(x) \in [0, 1/i)\} = f^{-1}(\{0\}) = C,$$

so that C is a G_δ set. ■

($\Leftarrow$) Suppose C is a G_δ, so that $C = \bigcap_{i=1}^{\infty} U_i$ for $U_i \in \tau$. Note that, $\forall i \in \mathbb{N}$, $X \backslash U_i$ is closed and disjoint from C, so that, since (X, τ) is normal, by Urysohn's Lemma, there exists a continuous function $f_i : X \to [0, 1]$ with $f_i = 0$ on C and $f_i = 1$ on $X \backslash U_i$.

Define $f : X \to [0,1]$ by $f(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i}$. (Note that, since, $\forall x \in X, i \in \mathbb{N}$, $f_i(x) \in [0,1]$,

$$0 \leq f(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \leq 1,$$

so that f has range in $[0,1]$.) If $x \in C$, then, $\forall i \in \mathbb{N}$, $f_i(x) = 0$, so that $f(x) = 0$. If $x \in X \backslash C$, then, since $C = \bigcap_{i=1}^{\infty} U_i$, for some $i \in \mathbb{N}$, $x \in X \backslash U_i$. Thus, $f_i(x)/2^i > 0$, so that $f(x) \geq f_i(x)/2^i > 0$. Finally, since f is the limit of the uniformly convergent sequence $\{\sum_{i=1}^{n} \frac{f_i}{2^i}\}_{n=1}^{\infty}$ of continuous functions, f is continuous, and thus f has the desired properties. ■

(ii) $(\Rightarrow)$ Suppose $\exists f : X \to [0,1]$ continuous such that $f = 1$ on C_1, $f = 0$ on C_2, and $0 < f < 1$ on $X \backslash (C_1 \cup C_2)$. If follows from the result of part (i) that C_2 is a G_δ set. Since $g := (1 - f) : X \to [0,1]$ is continuous with $g = 0$ on C_1 and $g > 0$ on $X \backslash C_1$, by the result of part (i), C_1 is a G_δ set. ■

Problem 3

Suppose, for sake of contradiction, that there exist two distinct such extensions $g_1, g_2 : \overline{E} \to Z$. Then, $\exists x \in \overline{E}$ with $g_1(x)$ distinct from $g_2(x)$. Then, since Z is Hausdorff, $\exists U_1, U_2 \in \tau_Z$ such that $g_1(x) \in U_1, g_2(x) \in U_2$ and $U_1 \cap U_2 = \emptyset$. Since g_1 and g_2 are continuous, $V_1 := g_1^{-1}(U_1)$, $V_2 := g_2^{-1}(U_2) \in \tau_Y$, so $V_1 \cap V_2 \in \tau_Y$. Thus, $V_1 \cap V_2$ is a neighborhood of x, so that, since $x \in \overline{E}$, $\exists y \in V_1 \cap V_2 \cap E$. Then, however, $g_1(y) = f(y) = g_2(y)$, contradicting the fact that $U_1 \cap U_2 = \emptyset$.

Problem 4

Suppose f is lower semi-continuous, suppose that some convergent sequence $x_n \to a$ in X as $n \to \infty$, and let $\epsilon > 0$. Since f is lower semi-continuous, there is a neighborhood U of a such that, $\forall x \in U$, $f(x) \geq f(a) - \epsilon$. Since $x_n \to a$ as $n \to \infty$, $\exists n_0 \in \mathbb{N}$ such that, $\forall n \in \mathbb{N}$ with $n \geq n_0$, $x_n \in U$, so that $\inf_{n \geq n_0} f(x_n) \geq f(a)$. Since this is the case every $\epsilon > 0$, $\liminf_{n \to \infty} f(x_n) \geq f(a)$, and thus, f is sequentially lower semi-continuous at every $a \in X$. ■

Let τ_{coco} be the co-countable topology on $\mathbb{R}$, and let f be the identity function on $\mathbb{R}$ (mapping into the standard topology). It is easy to see that a function is sequentially continuous if and only if it is both upper and lower semi-continuous (since a sequence $x_n \to a$ as $n \to \infty$ if and only if $\liminf_{n \to \infty} x_n = a = \limsup_{n \to \infty} x_n$. Since f is sequentially continuous (as sequences converge in τ_{coco} only if they are eventually constant), f is sequentially lower and upper semi-continuous. However, since f is not continuous ($(0,1)$ is not the complement of a countable set), either f or $-f$ must not be lower semi-continuous (in particular, f is not lower semi-continuous).

Problem 5

Suppose, for sake of contradiction, that f is sequentially lower semi-continuous but not lower semi-continuous. Since f is not lower semi-continuous, for some $a \in \mathbb{R}$, $S := f^{-1}(a, \infty)$ is not open, so that some $x \in S$ is not an interior point of S. Let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a local base of X at a, with $B_{i+1} \subseteq B_i$ (we can construct such a decreasing local base of X at a by taking a countable base $\{C_i : i \in \mathbb{N}\}$ and replacing each C_i with $B_i := \bigcap_{j=1}^{i} C_i$, which is a finite intersection of open sets and is thus open). Since x is not an interior point of S, $\forall n \in \mathbb{N}$, $\exists x_n \in B_n \backslash U$, so that $x_n \to x$ as $n \to \infty$. However, since no $f(x_n)$ is in (a, ∞), $f(x_n) \leq a < f(x)$, so that $\liminf_{n\to\infty} f(x_n) \leq a < f(x)$, contradicting the fact that f is sequentially lower semi-continuous. $\blacksquare$

Problem 6

Let τ_S denote the standard topology on $\mathbb{R}$, and let $U := (X \times \mathbb{R})\backslash(\operatorname{epi} f)$. It suffices, then, to show that f is lower semi-continuous if and only if U is open in the product topology $\tau \times \tau_S$ on $X \times \mathbb{R}$.

($\Rightarrow$) Suppose f is lower semi-continuous, and suppose $(x_0, y_0) \in U$. Let $\epsilon > 0$, so that, by lower semi-continuity of f, $\exists U_1 \in \tau$ with $x \in U_1$ such that, $\forall x \in U_1$, $f(x) \geq f(x_0) - \epsilon$. Thus, for $U_2 := (-\infty, f(x_0) - \epsilon)$, $U_1 \times U_2 \subseteq U$. Furthermore, since $U_1 \in \tau$ and $U_2 \in \tau_S$, $U_1 \times U_2 \in \tau \times \tau_S$. Since, $\forall (x, y) \in U$, (x, y) has a neighborhood contained in U, U is open. $\blacksquare$

($\Leftarrow$) Suppose $U \in \tau \times \tau_S$, let $x_0 \in X$, and let $\epsilon > 0$. Since $X \times \mathbb{R}$ is the product of finitely many topological spaces, $\{U_1 \times U_2 : U_1 \in \tau, U_2 \in \tau_S\}$ gives a base for $\tau \times \tau_S$. Thus, since $(x_0, f(x_0) - \epsilon) \in U$, $\exists U_1 \in \tau, U_2 \in \tau_S$ such that $(x_0, f(x_0) - \epsilon) \in U_1 \times U_2 \subseteq U$. Then, by definition of U, $\forall x \in U_1$, $f(x) \geq f(x_0) - \epsilon$. Therefore, f is lower semi-continuous at all $x_0 \in X$. $\blacksquare$