15-359: Probability and Computing

Assignment 4 | Due: February 17, 2012

Problem 1: Genius is 99% perspiration and 1% not exploding things (10 pts.)

A persistent but inept cook is trying to make pancakes. Each pancake the cook makes turns out alright with probability p; however, with probability $1-p$, the pancake explodes! This causes no damage to the kitchen (or the cook), but the entire stack of pancakes made so far is wiped out.

What is the expected number of pancakes the cook has to try to make (one at a time) to obtain a full serving of 10 pancakes?

Problem 2: Shipping error (10 pts.)

A computer server crashes with a probability of 1% every day that it is running. However, in 1/3 of all cases, you are shipped a faulty server that crashes with a probability of 5%. Let the random variable T represent the time until a crash. What is $\mathrm{Var}(T)$?

Problem 3: Skew and skewer (10 pts.)

Let's define $\mathrm{Skewer}(X) = \mathrm{E}((X - \mathrm{E}(X))^3)$ for a random variable X. Given two independent random variables X and Y, is it true that $\mathrm{Skewer}(X+Y) = \mathrm{Skewer}(X) + \mathrm{Skewer}(Y)$?

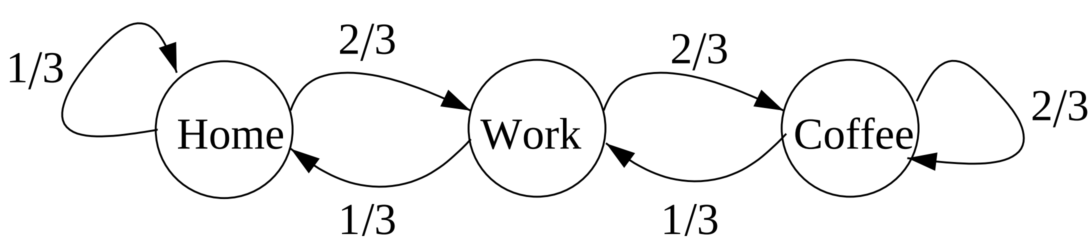
Problem 4: Tails (10 pts.)

Let X be a nonnegative, discrete, integer-valued random variable. Express in terms of $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$ the quantity

$$\sum_{x=1}^{\infty} x \, \mathrm{P}(X \geq x).$$

Problem 5: Coffee-theorem revisited (10 pts.)

Recall the poor student from Assignment 2, trapped in an endless cycle between Home, Work, and Coffee as shown in the diagram below.



Assuming the student is at work, let T be the time until she goes home. What is $\mathrm{Var}(T)$?

You may use the solution to Problem 6 on Assignment 2 without proof.

Problem 6: Counterexamples (20 pts.)

Give examples (simple ones would be appreciated) of the following:

A. A random variable X with an infinite mean.

B. A random variable X with finite mean, but infinite variance.

C. Two random variables X and Y that are not independent, yet $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$.

D. A sequence of nonnegative random variables $X_1, X_2, \ldots$ such that $\lim_{n\to\infty} \mathrm{E}(X_n) = \infty$ and yet $\lim_{n\to\infty} \mathrm{P}(X_n > 0) = 0$.

Problem 7: Some Chebyshev on the side (15 pts.)

The one-sided version of Chebyshev's inequality states: if a random variable X has mean μ and standard deviation σ, then for all $k \geq 0$,

$$\mathrm{P}(X - \mu \geq k\sigma) \leq \frac{1}{1+k^2}.$$

A. What distribution should X have, in order for equality to hold in the one-sided version of Chebyshev?

B. Suppose X is a nonnegative integer-valued random variable. Prove that

$$\mathrm{P}(X = 0) \leq \frac{\mathrm{Var}(X)}{\mathrm{E}(X^2)}.$$

Note: in recitation, we used Chebyshev's inequality in a very straightforward manner to get a similar bound with $\mathrm{E}(X)^2$ *instead. The bound you are asked to prove here is, in general, stronger.*

C. **Surprise extra credit** (15 pts.) Prove the one-sided Chebyshev's inequality.

Problem 8: See what I mean? (15 pts.)

Suppose $X_1, X_2, \ldots$ is an infinite sequence of independent, identically distributed random variables with mean μ and variance σ^2, both finite. We define S_n, for each integer n, to be the average

$$\frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Show that for any $\varepsilon > 0$, the probability $\mathrm{P}(|S_n - \mu| > \varepsilon)$ approaches 0 as $n \to \infty$.