Lecture Notes for Week 4 (First Draft)

Simultaneous Diagonalization of Compact Self-Adjoint Operators

Proposition 4.1: Let X be a (real or complex) Hilbert space and let $A, B \in \mathcal{C}(X;X)$ with $A^* = A$ and $B^* = B$ be given. Then $AB = BA$ if and only if there is an orthonormal basis $(e_j | j \in J)$ for X such that for every $j \in J$, e_j is an eigenvector for A and an eigenvector for B.

Proof: Assume first that $AB = BA$. For every $\lambda \in \sigma_p(A)$, put

$$M_\lambda = \mathcal{N}(\lambda I - B).$$

Let $\lambda \in \sigma_p(A)$ and $x \in M_\lambda$ be given. Notice that

$$\begin{aligned} B(Ax) &= A(Bx) = A(\lambda x) \\ &= \lambda(Ax), \end{aligned}$$

and consequently $Ax \in M_\lambda$. In other words, M_λ is invariant under A. Since

$$A\Big|_{M_\lambda}$$

is compact and self-adjoint, we may choose an orthonormal basis $\mathcal{O}_\lambda$ for M_λ such that every $e \in \mathcal{O}_\lambda$ is an eigenvalue for

$$A\Big|_{M_\lambda}$$

and hence also an eigenvalue for A. (Each such e is, of course, an eigenvalue for B. The desired orthonormal basis for X is given by

$$\mathcal{U} = \bigcup_{\lambda \in \sigma_p(A)} \mathcal{O}_\lambda.$$

The converse implication is immediate. $\square$

Spectral Decomposition of Compact Normal Operators

A compact normal operator need not have any eigenvalues if $\mathbb{K} = \mathbb{R}$. A simple example in $\mathbb{R}^2$ is given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Fact: If $X \neq \{0\}$, $\mathbb{K} = \mathbb{C}$, and $A \in \mathcal{C}(X;X)$ is normal, then A has an eigenvalue λ with $|\lambda| = \|A\|$.

Using this fact (which I plan to prove later), we could repeat the proof of Theorem 3.20 and obtain an analogous spectral decomposition theorem for compact normal operators on Complex Hilbert spaces. We can also obtain the desired spectral theorem for compact normal operators by applying Proposition 4.1 to ensure that a compact normal operator on a nontrivial complex Hilbert space has an eigenvalue. I shall adopt the latter approach.

Let X be a complex Hilbert space and let $A \in \mathcal{L}(X;X)$ be given. Assume that A is normal. Then we have

$$A = \frac{1}{2}(A + A^*) + i\left(\frac{1}{2i}(A - A^*)\right).$$

Let us put

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2i}(A - A^*).$$

It is immediate that

$$A = B + iC, \quad B^* = B, \quad C^* = C.$$

Moreover, since A is normal, we have

$$BC = CB.$$

Assume now that A is compact. Then A^* is compact, so B and C are both compact. Thus we can choose an orthormal basis $(e_j | j \in J)$ for X such that for every $j \in J$, e_j is an eigenvector for B and for C, i.e. for every $j \in J$ we may choose $\lambda_j, \mu_j \in \mathbb{R}$ such that

$$Ae_j = \lambda_j e_j + i\mu_j e_j.$$

Making minor adjustments to the proof of Theorem 3.20, we obtain

Theorem 4.2: Let X be a complex Hilbert space and let $A \in \mathcal{C}(X;X)$ be given and assume that A is normal. Then

(i) $\sigma(A) \subset \sigma_p(A) \cup \{0\}$

(ii) $|\lambda| \leq \|A\|$ for all $\lambda \in \sigma_p(A)$

(iii) $\sigma_p(A)$ is countable and the only possible accumulation point is 0

(iv) A has finite rank if and only if $\sigma_p(A)$ is finite

(v) $\forall \lambda, \mu \in \sigma_p(A)$ with $\lambda \neq \mu$ we have $\mathcal{N}(\lambda I - A) \perp \mathcal{N}(\mu I - A)$

(vi) For all $\lambda \in \sigma_p(A) \backslash \{0\}$, $\mathcal{N}(\lambda I - A)$ is finite dimensional and $\mathcal{R}(\lambda I - A)$ is closed

(vii) There is an orthonormal basis $(e_j|j \in J)$ for X such that for every $j \in J$, e_j is an eigenvector for A.

Spectral Resolution of Bounded Self-Adjoint Operators

Theorem 4.3: Let X be a (real or complex) Hilbert space and assume that $X \neq \{0\}$. Let $A \in \mathcal{L}(X;X)$ be given and assume that $A^* = A$. Put

$$m = \inf\{(Ax,x) : x \in X, \|x\| = 1\}, \quad M = \sup\{(Ax,x) : x \in X, \|x\| = 1\}.$$

Then there is a family $(E(\lambda)|\lambda \in \mathbb{R})$ of orthogonal projections [i.e., for every $\lambda \in \mathbb{R}$ we have $E(\lambda) \in \mathcal{L}(X;X)$, $E(\lambda)^2 = E(\lambda)$ and $E(\lambda)^* = E(\lambda)$] having the following properties.

(i) $\mathcal{R}(\lambda_1) \subset \mathcal{R}(\lambda_2)$ for $\lambda_1 \leq \lambda_2$,

(ii) For all $\lambda_0 \in \mathbb{R}$ and all $x \in X$ we have $E(\lambda)x \to E(\lambda_0)$ as $\lambda \downarrow \lambda_0$

(iii) $E(\lambda) = 0$ for all $\lambda < m$ and $E(\mu) = I$ for all $\mu \geq M$

(iv) $AE(\lambda) = E(\lambda)A$ for all $\lambda \in \mathbb{R}$

(v) For all $a, b \in \mathbb{R}$ with $a < m$ and $b \geq M$ we have

$$A = \int_a^b \lambda \, dE(\lambda)$$

Before proving Theorem 4.3, we need to develop some preliminary material concerning order on self-adjoint operators and decomposition of a self-adjoint operator into a difference of two positive self-adjoint operators.

Order on Self-adjoint Operators

Definition 4.4: Let X be a Hilbert space and $A, B \in \mathcal{L}(X;X)$ with $A^* = A$ and $B^* = B$ be given. We write $A \geq B$ [or, equivalently $B \leq A$] provided that

$$\forall x \in X, \text{ we have } (Ax,x) \geq (Bx,x).$$

[Recall that $(Ax,x), (Bx,x) \in \mathbb{R}$ for all $x \in X$ since A and B are self-adjoint.] We say that A is *positive* provived that $A \geq 0$.

You should verify the following two remarks for yourself as a simple exercise.

Remark 4.5: Let $A, B \in \mathcal{L}(X;X)$ with $A^* = A$ and $B^* = B$ be given. Assume that $AB = BA$ and $A \geq 0$, $B \geq 0$. Then we have $AB \geq 0$.

Remark 4.6: Let W and Z be closed subspaces of X. Then we have

$$W \subset Z \;\Leftrightarrow\; P_W \leq P_Z.$$

Lemma 4.7: Let X be a Hilbert space and $A \in \mathcal{L}(X;X)$ be given. Assume that $A^* = A$ and $A \geq 0$. Then there is exactly one $B \in \mathcal{L}(X;X)$ with $B^* = B$ and $B \geq 0$ such that $B^2 = A$. We call B the *square root* of A and we write $B = \sqrt{A}$. Moreover, if $C \in \mathcal{L}(X;X)$ and satisfies $AC = CA$ then C also satisfies $\sqrt{A}C = C\sqrt{A}$.

The proof of Lemma 4.7 will be a homework exercise. The idea is to use a suitable iteration scheme.

Decomposition of Self-Adjoint Operators

Let $A \in \mathcal{L}(X;X)$ with $A^* = A$ be given. Observe that

$$(A^2x, x) = (Ax, Ax) \geq 0 \;\text{ for all } x \in X,$$

so that $A^2 \geq 0$ (and, of course, A^2 is self-adjoint). We put

$$|A| = \sqrt{A^2},$$

$$A^+ = \frac{1}{2}(|A| + A),$$

$$A^- = \frac{1}{2}(|A| - A).$$

Observe that

$$A = A^+ - A^-, \quad |A| = A^+ + A^-.$$

The following remark summarizes some basic properties of the above decomposition that will be used in the proof of Theorem 4.3.

Remark 4.8: Let X be a Hilbert space and let $A, B \in \mathcal{L}(X;X)$ with $A^* = A$ be given.

(i) $A^+ \geq 0$, $A^- \geq 0$, $A^+ \geq A$, $A^- \geq -A$

(ii) $A^+A^- = A^-A^+ = 0$

(iii) If $A \geq 0$ then $A = |A| = A^+$

(iv) If $A \leq 0$ then $A = -|A| = -A^-$

(v) If $AB = BA$ then $A^+B = BA^+$ and $A^-B = BA^-$.

(vi) If $B^* = B$ and $B \geq A$ then $B \geq A^+$.

You should prove this remark for yourself as a simple exercise. (All parts are reasonably straightforward.)

Proof of Theorem 4.3: Assume that $X \neq \{0\}$, $A \in \mathcal{L}(X;X)$, and $A^* = A$. Put

$$m = \inf\{(Ax,x) : x \in X, \|x\| = 1\}, \quad M = \sup\{(Ax,x) : x \in X, \|x\| = 1\}. \tag{1}$$

For all $\lambda \in \mathbb{R}$, let

$$L(\lambda) = A - \lambda I,$$

and observe that

$$L(\lambda)L(\mu) = L(\mu)L(\lambda) \text{ for all } \lambda, \mu \in \mathbb{R}.$$

Since $\lambda \in \mathbb{R}$ and $A^* = A$ we know that $L(\lambda)$ is self-adjoint for all $\lambda \in \mathbb{R}$. A simple computation shows that

$$L(\lambda_1) \geq L(\lambda_2) \text{ for } \lambda_1 \leq \lambda_2. \tag{2}$$

It follows from part (i) of Remark 4.8 and equation (2) that

$$L(\lambda_1)^+ \geq L(\lambda_1) \geq L(\lambda_2) \text{ for } \lambda_1 \leq \lambda_2. \tag{3}$$

Using part (v) of Remark 4.8 we conclude that

$$L(\lambda_1)^+ \geq L(\lambda_2)^+ \text{ for } \lambda_1 \leq \lambda_2. \tag{4}$$

It follows immediately from (1) and parts (iii) and (iv) of Remark 4.8 that

$$L(\lambda) = |L(\lambda)| = L(\lambda)^+ \text{ for } \lambda \leq m \tag{5}$$

and

$$L(\lambda) = -|L(\lambda)| = -L(\lambda)^- \text{ for } \lambda \geq M. \tag{6}$$

It follows from (2) and Remark4.5 that

$$L(\lambda_2)^+[L(\lambda_1)^+ - L(\lambda_2)^+] \geq 0. \tag{7}$$

Rearranging (7) we find that

$$L(\lambda_2)^+ L(\lambda_1)^+ \geq (L(\lambda_2)^+)^2 \text{ for} \lambda_1 \leq \lambda_2. \tag{8}$$

It follows from (8) that

$$\mathcal{N}(L(\lambda_1)^+) \subset \mathcal{N}(L(\lambda_2)^+) \text{ for } \lambda_1 \leq \lambda_2. \tag{9}$$

To see why (9) follows from (8), let $x \in \mathcal{N}(L(\lambda_1)^+)$ be given. Using (8) we find that

$$\|L(\lambda_2)^+ x\|^2 = (L(\lambda_2)^+ x, L(\lambda_2)^+ x) = ((L(\lambda_2)^+)^2 x, x) = 0.$$

Now, for all $\lambda \in \mathbb{R}$, let $E(\lambda)$ denote the orthogonal projection onto $\mathcal{N}(L(\lambda)^+)$. It follows easily from Remark 4.6, equation (9) and the definition of $E(\lambda)$ that

$$E(\lambda_1) \leq E(\lambda_2) \text{ for } \lambda_1 \leq \lambda_2. \tag{10}$$

A simple computation gives

$$((A-\lambda I)x,x) \geq (m-\lambda)\|x\|^2 \;\; \text{for all } \lambda \in \mathbb{R}, x \in X. \tag{11}$$

Let $\lambda < m$ be given. It follows from (11) that $L(\lambda) \geq 0$ and $\mathcal{N}(L(\lambda)) = \{0\}$. We conclude that

$$\mathcal{N}(L(\lambda)^+) = \{0\} \text{ for } \lambda < m, \tag{12}$$

and consequently

$$E(\lambda) = 0 \;\; \text{for } \lambda < m. \tag{13}$$

Observe also that

$$((A-\lambda I)x,x) \leq 0 \;\; \text{for } \lambda \geq M,$$

and consequently

$$L(\lambda)^+ = 0, \; \text{i.e., } \mathcal{N}(L(\lambda))^+ = X \text{ for } \lambda \geq M. \tag{14}$$

It follows from (14) that

$$E(\lambda) = I \;\; \text{for} \lambda \geq M. \tag{15}$$