21-373, Algebraic Structures, Department of Mathematical Sciences, Carnegie Mellon University
Fall 2011: (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.
Luc TARTAR, University Professor of Mathematics, Wean Hall 6212, tartar@cmu.edu

Assignment 8 - Monday November 14, 2011. Due Monday November 21

Exercise 50: For a field E, show that every element of $E(x_1, \ldots, x_n)$ which is not in E is transcendental over E.

Exercise 51: Let E be a field, and F a field extension of E. Assume that $a, b \in F$ are algebraic over E, of degrees m and n respectively, with $(m, n) = 1$. Show that $[E(a, b) : E] = m\,n$.

Exercise 52: Let E be a field, and F a field extension of E.

i) Show that if $u \in F$ is algebraic over E then u^2 is algebraic over E.

ii) Show that if $v \in F$ is algebraic of odd degree over E, then the same is true of v^2 and one has $E(v^2) = E(v)$.

iii) If $w \in F$ is algebraic of even degree over E, can one have $E(w^2) = E(w)$?

Exercise 53: Let E be a field, and F a field extension of E. Assume that $u, v \in F$ are such that v is algebraic over $E(u)$, and that v is transcendental over E. Show that u is algebraic over $E(v)$.

Exercise 54: Let E be a field and let $F = E(x)$. Let $u = \frac{x^3}{x+1} \in F$ and let $K = E(u)$ (which is an intermediate field between E and F). Show that there exists $v \in F$ such that $F = K(v)$, and compute $[F : K]$.

Exercise 55: Let E be a field, and F a field extension of E. Let K_1, K_2 be two intermediate fields between E and F. One defines the composite field K_1K_2 as the smallest subfield of F containing $K_1 \cup K_2$.

i) Show that $[K_1K_2 : E]$ is finite if and only if $[K_1 : E]$ and $[K_2 : E]$ are finite.

ii) If $[K_1K_2 : E]$ is finite, show that $[K_1 : E]$ and $[K_2 : E]$ divide $[K_1K_2 : E]$, and that $[K_1K_2 : E] \leq [K_1 : E]\,[K_2 : E]$, with equality in the case where $[K_1 : E]$ and $[K_2 : E]$ are relatively prime.

iii) Show that if K_1 and K_2 are algebraic over E, then K_1K_2 is algebraic over E.

Exercise 56: Let E be a field, $P \in E[x]$ of degree $n \geq 1$ and let F be a splitting field extension for P over E. Show that $[F : E]$ divides $n!$.