Shashank Singh[1]

2 PSD Matrices (Adona) [25 points]

Part A [15 points]: Basic properties of PSD matrices

1. Suppose $v \in \mathbb{R}^{|I|}$. Let $k_1 < \cdots < k_{|I|}$ denote the elements of I. Define $v_I \in \mathbb{R}^n$ such that each $(v_I)_{k_i} = v_i$ and $(v_I)_k = 0$, $\forall k \notin I$. Since $X \succeq 0$,

$$v^T X_I v = v_I^T X v_I \geq 0,$$

and hence $X_I \succeq 0$. $\blacksquare$

2. If $I = \{i, j\}$, then, since $X_I \succeq 0$,

$$0 \leq \det X_I = X_{ii} X_{jj} - X_{ij}^2,$$

and hence $X_{ij}^2 \leq X_{ii} X_{jj}$.

If $X_{ii} = 0$, then each $0 \leq X_{ji}^2, X_{ij}^2 \leq X_{ii} X_{jj} = 0$, and so $X_{ji} = X_{ij} = 0$. $\blacksquare$

3. If $X = Q \Lambda Q^{-1}$, where Λ is a diagonal matrix of eigenvalues of X, then

$$\det X = \det(Q \Lambda Q^{-1}) = \det Q \det \Lambda \det(Q^{-1}) = \det Q \det \Lambda \det(Q)^{-1} = \det \Lambda.$$

Since the determinant of a diagonal matrix is trivially the product of its diagonal elements, $\det X$ is the product of the eigenvalues of X. $\blacksquare$

4. **Lemma** If $X = Y^T Z Y$ for some invertible matrix Y, then $X \succeq 0$ if and only if $Z \succeq 0$.

Proof: If $X \succeq 0$, then, $\forall v \in \mathbb{R}^n$,

$$0 \leq (Y^{-1} v)^T X Y^{-1} v = (Y^{-1} v)^T Y^T Z Y Y^{-1} v = v^T Z v,$$

and so $Z \succeq 0$. The converse follows, since $Z = (Y^{-1})^T X Y$. $\square$

It is easily checked that $X = Y^T Z Y$, where

$$Y = \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}.$$

Since Y is triangular without zeros on its diagonal, Y is invertible. Thus, by the lemma, it suffices to show that $Z \succeq 0$ if and only if $A \succeq 0$ and $C - B^T A^{-1} B \succeq 0$. One implication is immediate from Part 1. On the other hand, it is clear that any eigenvalue of either A or $C - B^T A^{-1} B$ is an eigenvalue of Z, and hence, if $A, C - B^T A^{-1} B \succeq 0$, then Z has non-negative eigenvalues and so $Z \succeq 0$. $\blacksquare$

[1] sss1@andrew.cmu.edu

Part B [10 points]: Formulating problems as SDPs

1. Let $X \in \mathbb{R}^{n \times n}$ such that each $X_{uv} = x_u^T x_v$. Since

   $$\|x_u - x_v\|^2 = x_u^T x_u - 2x_u^T x_v + x_v^T x_v,$$

   the objective function is just
   $$\frac{1}{4} X \cdot W$$
   where W_{uv} is the weight of (u, v) if $(u, v) \in \mathcal{E}$ and $W_{uv} = 0$ otherwise.

   Each constraint $\|x_u\|^2 = 1$ is simply

   $$X \cdot A_u = 1,$$

   where $(A_u)_{uu} = 1$ and all other coordinates of A_u are zero.

   Since each $\|x_u\|^2 = 1$, the constraint $\|x_{s_i} - x_{t_i}\|^2 = 4$ for $(s_i, t_i) \in T$ is equivalent to $x_{s_i} = -x_{t_i}$, and so it can be written
   $$X \cdot A_{(s_i, t_i)} = 0,$$
   where $A_{(s_i,t_i)}$ has 1 in indices (s_i, s_i) and (s_i, t_i).

   Didn't have time to finish this part.

2. Didn't have time to finish this part.