Homework
21-721 Probability
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EG.1

Consider the points as pair (P_1, P_2) distributed uniformly over the unit square $[0,1]^2$. It is easy to see that the three segments can form a triangle if and only if the length of the longest is at most the sum of the shorter two, i.e., all three segments have length at most $1/2$. Thus, if $P_1 \leq P_2$ (resp., $P_1 > P_2$), we have a triangle if and only if

1. $P_1 < 1/2$ (resp., $P_1 > 1/2$)
2. $P_2 > 1/2$ (resp., $P_2 < 1/2$)
3. and $P_2 - P_1 < 1/2$ (resp., $P_1 - P_2 < 1/2$).

Drawing these areas in a square, it becomes clear that satisfying area is $\boxed{1/4}$ the area of the square.

EG.2

Let E denote the event that the spaceships can communicate. Then,

$$\mathbb{P}[E] = \mathbb{P}[E|A \text{ both}]\mathbb{P}[A \text{ both}] + \mathbb{P}[E|A \text{ one}]\mathbb{P}[A \text{ one}] = (1)(1/4) + \mathbb{P}[E|A \text{ one}]\mathbb{P}[A \text{ one}].$$

Thus, it remains to show that

$$\mathbb{P}[E|A \text{ one}]\mathbb{P}[A \text{ one}] = \frac{1}{2\pi}.$$

It is easy to see that $\mathbb{P}[A \text{ one}] = 1/2$ and, by symmetry,

$$\mathbb{P}[E|A \text{ one}] = 2\mathbb{P}[B \text{ both}|A \text{ one}],$$

and so a simple illustration shows that

$$\mathbb{P}[E|A \text{ one}]\mathbb{P}[A \text{ one}] = \int_0^{\pi/2} \frac{\text{area}(\theta \text{ lune})}{4\pi} \frac{2\pi \sin\theta}{2\pi}\, d\theta = \int_0^{\pi/2} \frac{2\theta}{4\pi} \sin\theta \, d\theta = \boxed{\frac{1}{2\pi}},$$

where the last line follows from a simple integration by parts.

EG.3

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EG.4

E1.1: 'Probability' for subsets of $\mathbb{N}$

E4.1

E4.2

E4.3

E4.4

E4.5

E4.6: Converse to SLLN

E4.7: What's Fair about a Fair Game

Each $\mathbb{E}[X_n] = n^{-2}(n^2 - 1) - (1 - n^{-2}) = 0$. However, notice that, if

$$\mathbb{P}[X_n \neq -1 \text{ ev.}] = \mathbb{P}[X_n = n^2 - 1 \text{ i.o.}] = 0,$$

then $S_n/n \to -1$ a.s. The 1^{st} Borel-Cantelli Lemma confirms that this is the case, since

$$\sum_{n=1}^{\infty} \mathbb{P}[X_n = n^2 - 1] = \sum_{n=1}^{\infty} n^{-2} < +\infty. \quad \blacksquare$$

E4.8: Blackwell's Test of Imagination

E4.9: Tail σ-algebras

E5.1

Trivially, $\forall s \in S$, $f_n(s) \to 0$ as $n \to \infty$, but each $\mu(f_n) = n\left(\frac{1}{n}\right) = 1$. Notice that, $\forall x \in [0,1]$,

$$g(x) = \left\lfloor \frac{1}{x} \right\rfloor \geq \frac{1}{x} - 1.$$

Hence,

$$\int_S |g(x)|\, d\mu \geq \int_S \left|\frac{1}{x}\right| + 1\, d\mu = +\infty.$$

E5.2: Inclusion-Exclusion Formulae

E7.1: Inverting Laplace Transforms

E7.2: The Uniform Distribution on the Sphere $S^{n-1} \subseteq \mathbb{R}^n$

E9.1

E9.2

E10.1: Pólya's Urn

E10.2: Martingale Formulation of Bellman's Optimality Principle

Note first that, if, $y \in (0, x)$,

$$0 = \frac{d}{dy} p \log(x+y) + q \log(x-y) = (p-q)x - y,$$

then $y = (p-q)x$, and, since the appropriate second derivative is negative, this is a global maximum.

$$\begin{aligned}
\mathbb{E}_n[\log Z_{n+1} - (n+1)\alpha] &= p \log\left(Z_n + C_{n+1}\right) + q \log\left(Z_n - C_{n+1}\right) - (n+1)\alpha \\
&= p \log\left((1+p-q)Z_n\right) + q \log\left((1-(p-q))Z_n\right) - (n+1)\alpha \\
&= \log Z_n + p \log(2p) + q \log(2q) - (n+1)\alpha = \log Z_n - n\alpha
\end{aligned}$$

for the optimal strategy $C_{n+1} = (p-q)Z_n$. Hence, $n\alpha \geq \mathbb{E}[\log Z_n - \log Z_0] = \mathbb{E}[\log Z_n / Z_0]$, with equality for this strategy. $\blacksquare$

E10.3: Stopping Times

Since

$$\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\} \in \mathcal{F}_n,$$
$$\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n,$$

and

$$\{S + T \leq n\} = \bigcup_{k=1}^{n} \{S \leq k\} \cap \{T \leq n-k\} \in \mathcal{F}_n,$$

$S \wedge T$, $S \vee T$, and $S + T$ are stopping times.

E10.4

E10.5

E10.6: ABRACADABRA

For $n \in \mathbb{N}$, let S_n denote the total amount of money possessed by the first n gamblers at time n. Notice that, necessarily, $S_T = 26^{11} + 26^4 + 26$ (examine the suffixes of 'ABRACADABRA'). Also, $\forall n \in \mathbb{N}$, each of the first n gamblers expects to have \$1 at time n, so that $\mathbb{E}[S_n] = n$. Hence,

$$\mathbb{E}[T] = \mathbb{E}[S_T] = 26^{11} + 26^4 + 26.$$

Since the probability the monkey spells out 'ABRACADABRA' in any particular sequence of 11 letters is $26^{-11} > 0$, by a previous result, $\mathbb{E}[T] < \infty$. Let $C_n = 1$ for all $n \in \mathbb{N}$, and note that each

$$|S_n - n - (S_{n-1} - (n-1))| = 26^{11} + 26^4 + 26 + 1.$$

Then, by the result 10.10(c),

$$\mathbb{E}[S_n - n] = \mathbb{E}\left[\sum_{k=1}^{n}(S_n - n) - (S_{n-1} - (n-1))\right] = \mathbb{E}[C \cdot M]_T = 0,$$

so that $\mathbb{E}[S_n] = n$. $\blacksquare$

E10.7

At any time, a sequence of b consecutive cases of $X = +1$ will result in stopping. Hence, $\forall n \in \mathbb{N}$,

$$\mathbb{P}(T \leq n + b | \mathcal{F}_n) > p^b > 0,$$

so that T satisfies the conditions in Questions E10.5.

$$\mathbb{E}_n[M_{n+1}] = M_n \mathbb{E}_n \left(\frac{q}{p}\right)^{X_{n+1}} = M_n \left(p\frac{q}{p} + q\frac{p}{q}\right) = M_n(p+q) = M_n$$

$$\text{and} \quad \mathbb{E}_n[N_{n+1}] = N_n + \mathbb{E}[X_{n+1}] - (p-q) = N_n + (p-q) - (p-q) = N_n$$

so that M_n and N_n are martingales. Define $P := \mathbb{P}(S_T = 0), Q := \mathbb{P}(S_T = b)$. Then,

$$P + Q = 1$$

$$\text{and} \quad \left(\frac{q}{p}\right)^a = M_0 = \mathbb{E}[M_T] = P + Q\left(\frac{q}{p}\right)^b = 1 + Q\left(\left(\frac{q}{p}\right)^b - 1\right),$$

giving

$$Q = \frac{\left(\frac{q}{p}\right)^a - 1}{\left(\frac{q}{p}\right)^b - 1} = p^{b-a}\frac{q^a - p^a}{q^b - p^b} \quad \text{and} \quad P = 1 - Q.$$

Hence,

$$\mathbb{E}[S_T] = Qb = p^{b-a}\frac{q^a - p^a}{q^b - p^b}b. \quad \blacksquare$$

E10.8

$\forall k \in [n]$, since $\Theta \sim \text{Unif}(0,1)$, integrating by parts k times gives

$$P(B_n = k) = \int_0^1 P(B_n = k | \theta = t)\,dt = \binom{n}{k}\int_0^1 t^k(1-t)^{n-k}\,dt = \frac{1}{n+1}.$$

Bayes Rule gives

$$\begin{aligned} f_\Theta(\theta | B_1, \ldots, B_n) &= \frac{\mathbb{P}(B_1, \ldots, B_n | \Theta = \theta) f_\Theta(\theta)}{\mathbb{P}(B_1, \ldots, B_n)} \\ &= \frac{\binom{n}{B_n}\theta^{B_n}(1-\theta)^{n-B_n}}{\frac{1}{n+1}} = \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}. \end{aligned}$$

E10.9

$$\begin{aligned}
\mathbb{E}[X_T; T < \infty] = \mathbb{E}[\liminf_n X_{n\wedge T}; T < \infty] &\le \liminf_n \mathbb{E}[X_{n\wedge T}; T < \infty] && \text{(Fatou's Lemma)} \\
&\le \mathbb{E}[X_0; T < \infty] && \text{(E10.4)} \\
&\le \mathbb{E}[X_0]. && (X_0 \ge 0)
\end{aligned}$$

For $\varepsilon \in (0, c)$, define a stopping time $T_\varepsilon := \inf\{n : X_n > c - \varepsilon\}$ (with $T_\varepsilon = \infty$ if all $X_n \le c - \varepsilon$).

$$(c-\varepsilon)\mathbb{P}\left(\sup_n X_n \ge c\right) \le (c-\varepsilon)\mathbb{P}(T_\varepsilon < \infty) \le \mathbb{E}[X_{T_\varepsilon}; T_\varepsilon < \infty] \le \mathbb{E}[X_0]. \quad \blacksquare$$

E10.10

E10.11

E12.1: Branching Process

$$\mathbb{E}_n\left[\frac{Z_{n+1}}{\mu^{n+1}}\right] = \mathbb{E}_n\left[\sum_{k=1}^{Z_n} \frac{X_k^{(n+1)}}{\mu^{n+1}}\right] = \sum_{k=1}^{Z_n} \frac{\mathbb{E}_n[X_k^{(n+1)}]}{\mu^{n+1}} = \sum_{k=1}^{Z_n} \frac{1}{\mu^n} = \frac{Z_n}{\mu^n}.$$

Since (X_n) is independent,

$$\begin{aligned}
\mathbb{E}_n\left[Z_{n+1}^2\right] &= \mathbb{E}_n\left[\sum_{k=1}^{Z_n}\left(X_k^{(n+1)}\right)^2 + 2\sum_{1\le i<j\le Z_n} X_i^{(n+1)} X_j^{(n+1)}\right] \\
&= \sum_{k=1}^{Z_n} \mathbb{E}_n\left(X_k^{(n+1)}\right)^2 + 2\sum_{1\le i<j\le Z_n} \mathbb{E}_n[X_i^{(n+1)}]\mathbb{E}_n[X_j^{(n+1)}] \\
&= \left(\sigma^2+\mu^2\right)Z_n + \mu^2\left(Z_n^2 - Z_n\right) = \mu^2 Z_n^2 + \sigma^2 Z_n.
\end{aligned}$$

Note that, since (M_n) is a martingale, each $\mathbb{E}[M_n] = \mathbb{E}[M_0] = 1$. Inducting on n, we see that

$$\begin{aligned}
\mathbb{E}[M_n^2] = \mathbb{E}\left[\frac{\mu^2 Z_{n-1}^2 + \sigma^2 Z_{n-1}}{\mu^{2n}}\right] &= \mathbb{E}\left[M_{n-1}^2 + \frac{\sigma^2 M_{n-1}}{\mu^{n+1}}\right] \\
&= \mathbb{E}[M_{n-1}^2] + \frac{\sigma^2}{\mu^{n+1}} \\
&= \mathbb{E}[M_0^2] + \frac{\sigma^2}{\mu^2}\sum_{k=0}^{n-1}\mu^{-k} \uparrow 1 + \frac{\sigma^2}{\mu^2}\left(\frac{1}{1-1/\mu}\right) = 1 + \frac{\sigma^2}{\mu(\mu-1)}.
\end{aligned}$$

for all $n \in \mathbb{N}$ if and only if $\mu > 1$. Since (M_n) is bounded in $\mathcal{L}_2$, $\lim_{n\to\infty}\mathbb{E}[M_n^2] = \mathbb{E}[M_\infty^2]$, and so

$$\mathbb{V}[M_\infty] = \mathbb{E}[M_\infty^2] - \mathbb{E}[M_\infty]^2 = 1 + \frac{\sigma^2}{\mu(\mu-1)} - 1^2 = \frac{\sigma^2}{\mu(\mu-1)}. \quad \blacksquare$$

E12.2: Use of Kronecker's Lemma

Define $X_k := \frac{Y_k - 1/k}{\log k}$ for $k \geq 2$. Note that each

$$\mathbb{E}[X_k] = \frac{\mathbb{E}[Y_k] - 1/k}{\log k} = \frac{1/k - 1/k}{\log k} = 0,$$

and

$$\mathbb{V}[X_k] = \mathbb{E}[X_k^2] = \frac{(1/k)(1-1/k)^2 - (1-1/k)(1/k)^2}{\log^2 k} \in O\left(\frac{1}{k\log^2 k}\right).$$

Noting that

$$\int_2^n \frac{1}{x\log^2 x}\,dx = -\frac{1}{\log x}\bigg|_{x=2}^{x=n} \to \frac{1}{\log 2}$$

as $n \to \infty$, we see that $\sum_k \mathbb{V}[X_k]$ converges, by the integral test. Hence, $\sum_k \frac{Y_k - 1/k}{\log k} = \sum_k X_n$ converges a.s. Recalling that the sum of the first n harmonic numbers approaches $\gamma + \log n$,

$$\lim_{n\to\infty}\frac{N_n}{\log n} - 1 = \lim_{n\to\infty}\frac{N_n - \log n}{\log n} = \lim_{n\to\infty}\frac{N_n - (\log n + \gamma)}{\log n} = \lim_{n\to\infty}\frac{\sum_{k=1}^n Y_k - 1/k}{\log n} = 0,$$

by Kronecker's Lemma. $\blacksquare$

E12.3

EA13.1: Modes of Convergence

(a) Let $\varepsilon > 0$. Then,

$$\mathbb{P}[|X_n - X| > \varepsilon] \leq \mathbb{P}\left[\bigcup_{m \geq n} \{|X_m - X| > \varepsilon\}\right]$$
$$\downarrow \mathbb{P}\left[\bigcap_{n=1}^{\infty} \bigcup_{m \geq n} \{|X_m - X| > \varepsilon\}\right] = \mathbb{P}[|X_m - X| > \varepsilon \text{ i.o.}] = 0,$$

since $X_n \to X$ a.s. ■

(b) Suppose (X_n) are independent Bernoulli RV's with $\mathbb{P}[X_n = 1] = 1/n$. Then, $\forall \varepsilon > 0$,

$$\mathbb{P}[|X_n| > \varepsilon] \leq 1/n \to 0$$

as $n \to \infty$, but, by the 2^{nd} Borel-Cantelli Lemma, $\mathbb{P}[X_n = 1 \text{ i.o.}] = 1$, and so $X_n \not\to 0$ a.s. ■

(c) By the 1^{st} Borel-Cantelli Lemma,

$$\mathbb{P}[X_n \not\to X] = \mathbb{P}\left[\bigcup_{k \in \mathbb{N}} \{|X_n - X| \geq 1/k \text{ i.o.}\}\right] = 0. \quad ■$$

(d) Since, $\forall \varepsilon > 0$, $\mathbb{P}[|X_n - X| > \varepsilon] \to 0$ as $n \to \infty$, there is a subsequence (X_{n_k}) of (X_n) such that each $\mathbb{P}[|X_{n_k} - X| > 1/k] \leq 1/k^2$. Hence, by part (c), $X_{n_k} \to X$ a.s. as $k \to \infty$. ■

(e) $\Rightarrow$ follows immediately from part (d). If $X_n \not\to X$ in $\mathbb{P}$, then there exists $\varepsilon > 0$ and a subsequence (X_{n_k}) such that each $\mathbb{P}[|X_{n_k} - X| > \varepsilon] > \varepsilon$. Clearly, no subsequence of this can converge almost surely to X. ■

EA13.2

Recall the Law of the Iterated Logarithm we proved: almost surely,

$$\limsup_n \frac{S_n}{\sqrt{2n \log \log n}} = 1.$$

Notice that

$$\mathbb{P}[X_n \not\to X] = \mathbb{P}\left[\bigcup_{k \in \mathbb{N}} \{X_n > 1/k \text{ i.o.}\}\right] = \mathbb{P}\left[\bigcup_{k \in \mathbb{N}} \{aS_n > bn - \log k \text{ i.o.}\}\right]$$

Hence, if $b > 0$, then $\mathbb{P}[X_n \not\to X] = 0$, whereas, if $b \leq 0$, then $\mathbb{P}[X_n \not\to X] = 1$. On the other hand,

$$\mathbb{E}[X_n^r] = \mathbb{E}\left[\exp\left(ra \sum_{k=1}^n \xi_k - rbn\right)\right] = e^{-rbn}\mathbb{E}\left[\left(\prod_{k=1}^n e^{ra\xi_k}\right)\right]$$
$$= e^{-rbn} \prod_{k=1}^n \mathbb{E}[e^{ra\xi_k}] = e^{-rbn} \prod_{k=1}^n e^{(ra)^2/2} = e^{((ra)^2/2 - rb)n}.$$

Hence,

$$\lim_{n\to\infty} \mathbb{E}[X_n^r] \to 0 \quad \Leftrightarrow \quad (ra)^2/2 - rb < 0 \quad \Leftrightarrow \quad r < 2b/a^2. \quad \blacksquare$$

E13.1

($\Rightarrow$) If $k \in \mathbb{N}$ such that

$$\sup_{X\in\mathcal{C}} \mathbb{E}\left[|X|1_{|X|>k}\right] \leq 1,$$

then, $\forall X \in \mathcal{C}$,

$$\mathbb{E}[|X|] = \mathbb{E}[|X|1_{\{|X|>k\}}] + \mathbb{E}[|X|1_{\{|X|\leq k\}}] \leq 1 + kE[1_{\{|X|\leq k\}}] \leq 1 + k,$$

and so $\mathcal{C}$ is bounded in $\mathcal{L}_1$. If $\varepsilon > 0$, for $k \in \mathbb{N}$ such that

$$\sup_{X\in\mathcal{C}} \mathbb{E}\left[|X|1_{|X|>k}\right] \leq \varepsilon/2,$$

and $\delta = \frac{\varepsilon}{2k}$, if $\forall F \in \mathcal{F}$ with $\mathbb{P}[F] < \delta$, then

$$\mathbb{E}[X1_F] = \mathbb{E}[|X|1_{F\cap\{|X|>k\}}] + \mathbb{E}[|X|1_{F\cap\{|X|\leq k\}}] \leq \varepsilon/2 + k\mathbb{P}[F] < \varepsilon. \quad \blacksquare$$

($\Leftarrow$) Let $\varepsilon > 0$. Pick $\delta > 0$ such that, $\forall F \in \mathcal{F}$ with $\mathbb{P}[F] < \delta$,

$$\sup_{X\in\mathcal{C}} \mathbb{E}[|X|1_F] < \varepsilon.$$

By Markov's Inequality, for $k := 2A/\delta$,

$$\mathbb{P}[|X| > k] \leq \mathbb{E}[|X|]/k \leq A/k = \delta/2 \quad \Rightarrow \quad \sup_{X\in\mathcal{C}} \mathbb{E}[|X|1_{\{|X|>k\}}] < \varepsilon. \quad \blacksquare$$

E13.2

By the triangle inequality, if $A, B \in \mathbb{R}$ such that $\mathbb{E}[|X|] < A$ and $\mathbb{E}[|Y|] < B$ for all $X \in \mathcal{C}, Y \in \mathcal{D}$, then $\mathbb{E}[X + Y] < A + B$ for all $X + Y \in \mathcal{C} + \mathcal{D}$.

Let $\varepsilon > 0$. If $\exists\delta_1, \delta_2 > 0$ such that, $\forall F \in \mathcal{F}$ with $\mathbb{P}[F] < \delta_1$, $\sup_{X\in\mathcal{C}} \mathbb{E}|X|1_F < \varepsilon/2$ and $\forall F \in \mathcal{F}$ with $\mathbb{P}[F] < \delta_1$, $\sup_{Y\in\mathcal{D}} \mathbb{E}|Y|1_F < \varepsilon/2$, then, for $\delta := \min\{\delta_1, \delta_2\}$, $\forall F \in \mathcal{F}$ with $\mathbb{P}[F] < \delta$, $\sup_{X+Y\in\mathcal{C}+\mathcal{D}} \mathbb{E}|Y|1_F < \varepsilon$. Hence, by the result of Exercise 13.1, $\mathcal{C} + \mathcal{D}$ is UI. $\blacksquare$

E13.3

E14.1

E14.2

(a) Since the function $x \mapsto e^{\theta x}$ is convex, as secant bound gives

$$e^{\theta Y} \leq \frac{c-Y}{2c} e^{-\theta c} + \frac{c+Y}{2c} e^{\theta c} = \cosh(\theta c) + Y \sinh(\theta c).$$

Hence,

$$\mathbb{E}\left[e^{\theta Y}\right] \leq \mathbb{E}\left[\cosh(\theta c) + Y \sinh(\theta c)\right] = \cosh(\theta c),$$

since $Y \in [-c, c]$ and $\mathbb{E}[Y] = 0$. Also note that, $\forall x \in \mathbb{R}$,

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \leq \sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} = e^{x^2/2}. \quad \blacksquare$$

(b) By Doob's Maximal Inequality, $\forall \theta > 0$,

$$\begin{aligned}
\mathbb{P}\left[\sup_{k \leq n} M_k \geq x\right] = \mathbb{P}\left[\sup_{k \leq n} e^{\theta M_k} \geq e^{\theta x}\right] &\leq \mathbb{E}\left[e^{\theta M_n}\right] e^{-\theta x} \\
&= \mathbb{E}\left[\prod_{k=1}^{n} e^{\theta(M_k - M_{k-1})}\right] e^{-\theta x} \\
&= \mathbb{E}\left[\prod_{k=1}^{n} \cosh(\theta c_k) + (M_k - M_{k-1}) \sinh(\theta c_k)\right] e^{-\theta x} \\
&= e^{-\theta x} \prod_{k=1}^{n} \cosh(\theta c_k) \\
&\leq e^{-\theta x} \prod_{k=1}^{n} e^{\frac{1}{2}\theta^2 c_k^2} = \exp\left(\frac{1}{2}\theta^2 \sum_{k=1}^{n} c_k^2 - \theta x\right).
\end{aligned}$$

Let $c := \sum_{k=1}^n c_k^2$. To minimize over $\theta > 0$, we minimize $\frac{1}{2}\theta^2 c - \theta x$ (which is clearly convex), for which we use $\theta = x/c$, giving

$$\mathbb{P}\left[\sup_{k \leq n} M_k \geq x\right] \leq \exp\left(-\frac{1}{2}x^2/c\right). \quad \blacksquare$$

E16.1

For $0 < \varepsilon < T$, integrate e^{iz}/z around the contour composed of the intervals $[-T, -\varepsilon]$ and $[\varepsilon, T]$ and the semicircles spanning them. The integrals along the intervals approach $2i \int_0^\infty \frac{\sin(x)}{x}\, dx$. The integral along the outer semicircle approaches 0. The integral along the inner semicircle approaches $-\pi i$ (using a first order Taylor approximation of e^{iz}).

E16.2

$$\phi_Z(\theta) = \frac{1}{2}\int_{-1}^{1} e^{i\theta Z}\,dz = \frac{1}{2}\int_{-1}^{1} \cos(\theta z){+}i\sin(\theta z)\,dz = \frac{1}{2}\int_{-1}^{1} \cos(\theta z)\,dz = \frac{1}{2}(\sin(\theta){-}\sin(-\theta))/\theta = \frac{\sin(\theta)}{\theta}.$$

If X and Y are IID RV's, then

$$\phi_{X-Y} = \phi_X\phi_{-Y} = \phi_X\overline{\phi_Y} = \phi_X\overline{\phi_X} = |\phi_X|^2 \geq 0,$$

and so $\phi_{X-Y} \neq \frac{\sin(\theta)}{\theta}$. ■

E16.3

E16.4

E16.5

Use the definition of ϕ to show the LHS is the expectation of a modulus.

E16.6

E18.1

E18.2

E18.3

E18.4

E18.5

E18.6

E18.7