Lecture Notes for Week 11 (First Draft)

Linear Operators, Weak and Weak Convergence (Continued)*

Last time, it was stated that bounded linear operators that are not adjoints of some other bounded linear operator need not respect weak* convergence. We begin with an example of this phenomenon.

Example 11.1: Let $X = \mathbb{K}$ and $Y = c_0$. We identify X^* with $\mathbb{K}$ and Y^* with l^1. Define $L : l^1 \to \mathbb{K}$ by

$$L(w) = \sum_{n=1}^{\infty} w_n \quad \text{for all } w \in l^1.$$

Clearly, $L : l^1 \to \mathbb{K}$ is linear and continuous. Let us put $w^{(n)} = (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$. Then we have $w^{(n)} \overset{*}{\rightharpoonup} 0$ (weakly*) as $n \to \infty$, but $L(w^{(n)}) = (-1)^n$ for all $n \in \mathbb{N}$, so that the sequence $\{L^{(n)}\}_{n=1}^{\infty}$ fails to be weakly* convergent in $\mathbb{K}$.

Theorem 11.2: Let X and Y be NLS and let $T : X \to Y$ be a linear mapping. The following 5 statements are equivalent.

(i) T is continuous.

(ii) For every x in X and every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ we have $Tx_n \rightharpoonup Tx$ (weakly) as $n \to \infty$.

(iii) For every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \rightharpoonup 0$ (weakly) as $n \to \infty$ we have $Tx_n \rightharpoonup 0$ (weakly) as $n \to \infty$.

(iv) For every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \rightharpoonup 0$ (weakly) as $n \to \infty$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

(v) For every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \to 0$ (strongly) as $n \to \infty$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Proof: We shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i).

Assume that (i) holds and let $x \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ be given. For each $y^* \in Y^*$ we have

$$\langle y^*, Tx_n \rangle = \langle T^* y^*, x_n \rangle \to \langle T^* y^*, x \rangle = \langle y^*, Tx \rangle \quad \text{as } n \to \infty,$$

i.e. $Tx_n \rightharpoonup Tx$ (weakly) as $n \to \infty$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are clear.

To prove (v) $\Rightarrow$ (i), we shall prove the contrapositive implication, i.e. (not (i)) $\Rightarrow$ (not (v)). Assume that T is not continuous. We shall construct a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \to 0$ (strongly) as $n \to \infty$ and $\{\|Tx_n\|\}_{n=1}^{\infty}$ is unbounded. Since T is unbounded, we may choose $w_n \in X$ such that $\|w_n\| = 1$ and $\|Tw_n\| \geq n^2$ for every $n \in \mathbb{N}$. Now put

$$x_n = \frac{w_n}{n} \quad \text{for all } n \in \mathbb{N}.$$

Then we have $x_n \to 0$ as $n \to \infty$ and $\|Tx_n\| \geq n$ for all $n \in \mathbb{N}$ so that the sequence $\{Tx_n\}_{n=1}^{\infty}$ is unbounded. $\square$

We now formally state the result concerning adjoint operators and weak* convergence.

Proposition 11.3: Let X and Y be NLS and let $T \in \mathcal{L}(X;Y)$ be given. Let $y^* \in Y^*$ and a sequence $\{y_n^*\}_{n=1}^{\infty}$ in Y^* be given and assume that $y_n^* \overset{*}{\rightharpoonup} y^*$ (weakly*) as $n \to \infty$. Then we have $T^*y_n^* \overset{*}{\rightharpoonup} T^*y^*$ (weakly*) as $n \to \infty$.

Proof: For every $x \in X$ we have

$$\langle T^*y_n^*, x\rangle = \langle y_n^*, Tx\rangle \to \langle y^*, Tx\rangle = \langle T^*y^*, x\rangle \quad \text{as } n \to \infty,$$

i.e. $T^*y_n^* \overset{*}{\rightharpoonup} T^*y^*$ (weakly*) as $n \to \infty$. $\square$

Compact Linear Operators

Definition 11.4: Let X and Y be normed linear spaces. A linear mapping $T : X \to Y$ is said to be *compact* provided that $\mathrm{cl}(T[B_1(0)])$ is compact. The set of all compact linear mappings from X to Y will be denoted by $\mathcal{C}(X;Y)$.

Remark 11.5: If a linear operator T is compact, then T is bounded. In other words, $\mathcal{C}(X;Y) \subset \mathcal{L}(X;Y)$.

The next result follows easily from the definitions and the fact that compactness can be characterized by sequences in metric spaces.

Proposition 11.6: Let X and Y be NLS and $T : X \to Y$ be a linear mapping. The following 3 statements are equivalent.

(i) T is compact.

(ii) For every bounded set $A \subset X$, $\mathrm{cl}(T[A])$ is compact.

(iii) For every bounded sequence $\{x_n\}_{n=1}^{\infty}$, the sequence $\{Tx_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Definition 11.7: Let X and Y be linear spaces. A linear mapping $T : X \to Y$ is said to be of *finite rank* provided that $\mathcal{R}(T)$ is finite dimensional.

Proposition 11.8: Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear mapping.

(a) If X is finite dimensional, then T is compact.

(b) If T is continuous and of finite rank, then T is compact.

Remark 11.9: Not every compact linear operator is of finite rank. However, in certain important cases, every compact linear operator can be obtained as a limit (in the operator norm) of a sequence of bounded linear operators of finite rank.

One very pleasant feature of compact linear operators is that they map weakly convergent sequences into strongly convergent ones.

Theorem 11.10: Let X and Y be normed linear spaces, $T \in \mathcal{C}(X;Y)$, $x \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ be given. Assume that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. Then $Tx_n \to Tx$ (strongly) as $n \to \infty$.

Proof: Suppose that $\{Tx_n\}_{n=1}^{\infty}$ fails to converge strongly to Tx. Then we may choose $\epsilon > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that

$$\|Tx_{n_k} - Tx\| \geq \epsilon \text{ for all } k \in \mathbb{N}. \tag{1}$$

Since $\{x_{n_k}\}_{k=1}^{\infty}$ is bounded and T is compact, we may choose a subsequence $\{x_{n_{k_j}}\}_{j=1}^{\infty}$ and $y \in Y$ such that

$$Tx_{n_{k_j}} \to y \text{ (strongly) as } j \to \infty.$$

By Theorem 11.2,

$$Tx_{n_{k_j}} \rightharpoonup Tx \text{ (weakly) as } j \to \infty.$$

We conclude that $y = Tx$, and consequently

$$Tx_{n_{k_j}} \to Tx \text{ (strongly) as } j \to \infty,$$

which contradicts (1). It follows that $Tx_n \to Tx$ (strongly) as $n \to \infty$. $\square$

It is natural to ask whether or not a linear operator that maps weakly convergent sequences to strongly convergent ones is necessarily compact. Without some additional assumptions on the spaces, the answer is no since the identity operator $I : l^1 \to l^1$ fails to be compact (because the closed unit ball in l^1 is not compact), but sequences in l^1 are weakly convergent if and only if they are strongly convergent. However, if X is reflexive, then linear operators that map weakly convergent sequences to strongly convergent ones are automatically compact.

Theorem 11.11: Let X be a reflexive Banach space, Y be a normed linear space and $T : X \to Y$ be a linear mapping. Assume that for every weakly convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X, the sequence $\{Tx_n\}_{n=1}^{\infty}$ is strongly convergent. Then T is compact.

Proof: Let a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X be given. Since X is reflexive, we may choose a weakly convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Then, by our assumption, $\{Tx_{n_k}\}_{k=1}^{\infty}$ is strongly convergent, so T is compact by Proposition 11.6. $\square$

Proposition 11.12: Let X, Y, Z be normed linear spaces and $T \in \mathcal{L}(X;Y)$, $S \in \mathcal{L}(Y;Z)$ be given. If either T or S is compact then ST is compact.

Proof: Assume that S is compact and let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in X. Then $\{Tx_n\}_{n=1}^{\infty}$ is a bounded sequence in Y, so we may choose a subsequence $\{Tx_{n_k}\}_{k=1}^{\infty}$ such that $\{STx_{n_k}\}_{k=1}^{\infty}$ is strongly convergent in Z. The case where T is compact is similar. $\square$

Proposition 11.13: Let X and Y be normed linear spaces and let $T \in \mathcal{C}(X;Y)$ be given. Then $\mathcal{R}(T)$ is separable.

Proof: Let $n \in \mathbb{N}$ be given. Since $\mathrm{cl}(T[B_n(0)])$ is compact, we may choose a finite set $D_n = \{y_{n,k} : k = 1, 2, \cdots N_n\} \subset T[B_n(0)]$ such the collection of balls

$$\{B_{\frac{1}{n}}(y_{n,k}) : k = 1, 2, \cdots, N_n\}$$

covers $T[B_n(0)]$. (Indeed the collection of open sets $\{B_{\frac{1}{n}}(y) : y \in T[B_n(0)]\}$ covers the compact set $\mathrm{cl}(T[B_n(0)])$, so we may choose a finite subcollection that covers $\mathrm{cl}(T[B_n(0)]) \supset T[B_n(0)]$.) Put

$$D = \bigcup_{n=1}^{\infty} D_n,$$

and observe that D is countable. It is clear that D is dense in $\mathcal{R}(T)$, because given $y \in \mathcal{R}(T)$ and $\delta > 0$, we may choose $N > \delta^{-1}$ such $y \in T[B_N(0)]$ so that $B_\delta(y) \cap D_N \neq \emptyset$. $\square$

Theorem 11.14: Let X and Y be normed linear spaces and $T \in \mathcal{L}(X;Y)$ be given. Assume that T is compact. Then T^* is compact.

Proof: Let $B^* = \{x^* \in X^* : \|x^*\| < 1\}$. Since X^* is complete, it suffices to show that $T^*[B^*]$ is totally bounded. Put $B = \{x \in X : \|x\| \leq 1\}$. Since T is compact, we know that $T[B]$ is totally bounded. Let $\epsilon > 0$ be given. We may choose $x_1, x_2, \cdots, x_N \in B$ such that for every $x \in B$, there exists $i \in \{1, 2, \cdots, N\}$ such that

$$\|Tx - Tx_i\| < \frac{\epsilon}{3}. \tag{2}$$

Define $L : Y^* \to \mathbb{K}^N$ by

$$Ly^* = (y^*(Tx_1), y^*(Tx_2), \cdots y^*(Tx_n)) \text{ for all } y^* \in Y^*.$$

For definiteness, we equip $\mathbb{K}^N$ with the maximum norm. Observe that L is continuous and has finite rank. It follows that $L[B^*]$ is totally bounded. Therefore we may choose $y_1^*, y_2^*, \cdots y_m^* \in B^*$ such that for every $y^* \in B^*$, there exists $j \in \{1, 2, \cdots, m\}$ satisfying

$$\|Ly^* - Ly_j^*\| < \epsilon.$$

In other words, for every $y^* \in B^*$, there exists $j \in \{1, 2, \cdots, m\}$ such that

$$|y^*(Tx_i) - y_j^*(T(x_i)| < \frac{\epsilon}{3} \text{ for all } i = 1, 2, \cdots, N. \tag{3}$$

Let $y^* \in B^*$ be given and choose $j \in \{1, 2, \cdots, m\}$ such that (3) holds. Let $x \in B$ be given and choose $i \in \{1, 2, \cdots, N\}$ such that (2) holds. Then we have

$$\begin{aligned} |y^*(Tx) - y_j^*(Tx)| &\le |y^*(Tx) - y^*(Tx_i)| + |y^*(Tx_i) - y_j^*(Tx)| \\ &\quad + |y_j^*(Tx_i) - y^*(Tx)| \\ &\le \|y^*\| \cdot \|Tx - Tx_i\| + \epsilon + \|y_j^*\| \cdot \|Tx_i - Tx\| \\ &\le \epsilon. \end{aligned}$$

Since the above chain of inequalities holds for all $x \in X$ with $\|x\| \le 1$, we conclude that for every in B^*, there exists $j \in \{1, 2, \cdots, m\}$ such that

$$\|T^*y^* - T^*y_j^*\| < \epsilon.$$

It follows that $T^*[B^*]$ is totally bounded. $\square$

Theorem 11.15 Let X be normed linear space, Y be a Banach space, and $T \in \mathcal{L}(X; Y)$. Assume that T^* is compact. Then T is compact.

Proof: Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in X. Then $\{J_X(x_n)\}_{n=1}^\infty$ is a bounded sequence in X^{**}. By Theorem 11.14, T^{**} is compact, so we may choose a subsequence $\{x_{n_k}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\{T^{**}(J_X(x_{n_k})\}_{k=1}^\infty$ is strongly convergent in Y^{**}. In particular, $\{T^{**}(J_X(x_{n_k})\}_{k=1}^\infty$ is a Cauchy sequence in Y^{**}. Since J_Y is an isometry and

$$J_Y(T(x)) = T^{**}(J_X(x)) \text{ for all } x \in X,$$

it follows that $\{Tx_{n_k}\}_{k=1}^\infty$ is a Cauchy sequence in Y. Since Y is complete, we conclude that $\{Tx_{n_k}\}_{k=1}^\infty$ is strongly convergent in Y. $\square$

Suppose that X is a reflexive Banach space, Y a normed linear space, and $T \in \mathcal{L}(X;Y)$ has a compact adjoint. Can we conclude that T is compact? Since $T^{**} = T$ in this case, it is natural to say that the answer is yes. However, some caution is advised, because $T^{**} : X^{**} \to Y^{**}$ and $T : X \to Y$. If Y fails to be reflexive, then the image under J_Y of Y will not be all of Y^{**} and the mappings T and T^{**} have different codomains. (It could happen, for example, that T is surjective but T^{**} is not.) Examination of the proof of Theorem 11.15 reveals that if we start with a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X then there will be a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\{Tx_{n_k}\}_{k=1}^{\infty}$ is a Cauchy sequence in Y. Since X is reflexive, we can obtain a "candidate" for the limit of this sequence by extracting a weakly convergent subsequence of $\{x_{n_k}\}_{k=1}^{\infty}$ and applying T to the weak limit.

For practice working with second duals, reflexivity, and adjoints, let's formalize the argument above.

Proposition 11.16: Let X be a reflexive Banach space, Y a normed linear space, and $T \in \mathcal{L}(X;Y)$ be given. Assume that T^* is compact. Then T is compact.

Proof: Since X is reflexive, it suffices to show that T maps weakly convergent sequences to strongly convergent ones. Let $\{x_n\}_{n=1}^{\infty}$ be a weakly convergent sequence in X. Choose $x \in X$ such that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. Since J_X is continuous, we know that $J_X(x_n) \rightharpoonup x$ (weakly) as $n \to \infty$. By Theorem 11.14, $T^{**} : X^{**} \to Y^{**}$ is compact, so we have

$$T^{**}(J_X(x_n)) \to T^{**}(J_X(x)) \text{ (strongly) as } n \to \infty, \text{ i.e.,}$$

$$J_Y(T(x_n)) \to J_Y(Tx) \text{ (strongly) as } n \to \infty.$$

Since J_Y is an isometry, we conclude that $Tx_n \to Tx$ (strongly) as $n \to \infty$ and consequently T is compact. $\square$

We now give two examples below of compact operators on a space of continuous functions. You will encounter compact operators on sequence spaces on Assignment 6.

Example 11.17: Let $X = C[0,1]$, the space of all continuous functions $f : [0,1] \to \mathbb{K}$ equipped with the norm given by

$$\|f\| = \max\{|f(t)| : t \in [0,1]\} \quad \text{for all } f \in X.$$

(a) Define $T : X \to X$ by

$$(Tf)(t) = \int_0^t f(s)ds \quad \text{for all } t \in [0,1].$$

To show that T is compact, let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in X and choose $M \in \mathbb{R}$ such that $\|f_n\| \leq M$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ and all $t, \tau \in [0,1]$, we have

- $\|Tf_n\| \leq M$,
- $|(Tf_n)(t) - (Tf_n)(\tau)| \leq M|t - \tau|$,

i.e. the sequence $\{Tf_n\}_{n=1}^{\infty}$ is uniformly bounded and uniformly equicontinuous. By the Ascoli-Arzela Theorem, there is a uniformly convergent subsequence $\{Tf_{n_k}\}_{k=1}^{\infty}$ and we conclude that T is compact.

(b) Assume that $k : [0,1] \times [0,1] \to \mathbb{K}$ is continuous and define $K : X \to X$ by

$$(Kf)(t) = \int_0^1 k(t,s)f(s)ds \quad \text{for all } f \in X.$$

To show that K is compact, let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in X and choose $M \in \mathbb{R}$ such that $\|f_n\| \leq M$ for all $n \in \mathbb{N}$. Put $\overline{k} = \max\{|k(t,s)| : (t,s) \in [0,1] \times [0,1]\}$. Let $\epsilon > 0$ be given. We may choose $\delta > 0$ such that

$$|k(t,s) - k(\tau,s)| < \frac{\epsilon}{M} \quad \text{for all } s, \tau, t \in [0,1] \text{ with } |t - \tau| < \delta.$$

Now for all $n \in \mathbb{N}$ and all $t, \tau \in [0,1] \times [0,1]$ with $|t - \tau| < \delta$ we have

- $\|Kf_n\| \leq M\overline{k}$,
- $|(Kf_n)(t) - (Kf_n)(\tau)| < \epsilon$,

i.e. the sequence $\{Kf_n\}_{n=1}^{\infty}$ is uniformly bounded and uniformly equicontinuous. By the Ascoli-Arzela Theorm, there is a uniformly convergent subsequence $\{Kf_{n_k}\}_{k=1}^{\infty}$ and we conclude that K is compact.

Continuous and Compact Embeddings of Normed Linear Spaces

Let X and Y be normed linear spaces over the same field. We say that X is *continuously embedded* in Y provided that $X \subset Y$ and the identity mapping $I : X \to Y$ is linear and continuous, i.e. there exists $M \in \mathbb{R}$ such that

$$\|x\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

The assumption that the identity mapping is linear ensures that the linear structures of the two spaces are compatible. We write

$$X \hookrightarrow Y$$

to indicate that X is continuously embedded in Y.

Remark 11.18: Assume that $X \hookrightarrow Y$ and let $x \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ be given.

(a) If $x_n \to x$ (strongly) in X as $n \to \infty$ then $x_n \to x$ (strongly) in Y as $n \to \infty$.

(b) If $x_n \rightharpoonup x$ (weakly) in X as $n \to \infty$ then $x_n \rightharpoonup x$ (weakly) in Y as $n \to \infty$.

We say that X is *compactly embedded* in Y provided that $X \subset Y$ and the identity mapping $I : X \to Y$ is linear and compact. We write

$$X \hookrightarrow\hookrightarrow Y$$

to indicate that X is compactly embedded in Y.

Remark 11.19: Assume that $X \hookrightarrow\hookrightarrow Y$ and let $x \in X$ and a sequence $\{x_n\}_{n=1}^{\infty}$ be given.

(a) If $\{x_n\}_{n=1}^{\infty}$ is bounded in X then there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges strongly in Y.

(b) If $x_n \rightharpoonup x$ (weakly) in X as $n \to \infty$ then $x_n \to x$ (strongly) in Y as $n \to \infty$.

Before presenting the next group of examples, we introduce a class of spaces of continuous functions on subsets of $\mathbb{R}^n$. Let Ω be a bounded open subset of $\mathbb{R}^n$. By $C(\overline{\Omega})$ we mean the set of all uniformly continuous functions $f : \Omega \to \mathbb{K}$. Unless stated otherwise, we equip this space with $\|\cdot\|_\infty$. (Notice that each function in $C(\overline{\Omega})$ is bounded and has a unique continuous extension to $\overline{\Omega}$.)