**Lecture Notes for Week 12** (First Draft)

*Analytic Semigroups*

**Definition 12.1** (Complex Sector): For each $\phi \in (0, \pi)$ and $\omega \in \mathbb{R}$, the *open sector* of angle $\phi$ at $\omega$ is the subset of $\mathbb{C}$ defined by

$$\Delta_{\phi,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \phi\}.$$

**Theorem 12.2**: Let $X$ be a complex Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear $C_0$-semigroup with infinitesimal generator $A$. Let $K > 0$ be given. Assume that $T$ is differentiable on $(0, \infty)$ and that

$$\|AT(t)\| \leq \frac{K}{t} \quad \text{for all } t \in (0, 1].$$

Then there exists $\phi \in (0, \frac{\pi}{2})$ and an analytic mapping $\widetilde{T} : \Delta_{\phi,0} \to \mathcal{L}(X; X)$ such that (a), (b), and (c) below hold:

(a) $\widetilde{T}(t) = T(t)$ for all $t \in (0, \infty)$,

(b) $\widetilde{T}(z_1 + z_2) = \widetilde{T}(z_1)\widetilde{T}(z_2)$ for all $z_1, z_2 \in \Delta_{\phi,0}$,

(c) For every $x \in X$, we have $\widetilde{T}(z)x \to x$ as $z \to 0$, $z \in \Delta_{\phi,0}$.

**Proof**: Let $t_0$ and $N \in \mathbb{N}$ be given. Then, by Taylor's Theorem, for $t > t_0$ we have

$$T(t) = \sum_{n=0}^{N-1} \frac{(t - t_0)^n}{n!} A^n T(t_0) + R_{N-1}(t_0; t),$$

where

$$R_{N-1}(t_0; t) = \frac{1}{(N-1)!} \int_{t_0}^{t} (t - s)^{N-1} A^N T(s)\, ds. \tag{1}$$

We want to estimate the remainder. To this end, observe that

$$\begin{aligned} \|A^N T(s)\| &= \|AT\left(\frac{s}{N}\right)^N\| \\ &\leq \|AT\left(\frac{s}{N}\right)\|^N \\ &\leq K^N \left(\frac{N}{s}\right) \quad \text{for } \frac{s}{N} \leq 1. \end{aligned} \tag{2}$$

Recall from calculus that

$$N!e^N \geq N^N. \tag{3}$$

It follows from (2) and (3) that

$$\begin{aligned}\|R_N(t_0;t)\| &\leq \frac{N^N K^N}{N!}\left(\frac{t-t_0}{t_0}\right)^N \\ &\leq \left(eK\left(\frac{t-t_0}{t_0}\right)\right)^N.\end{aligned}$$

We see that if

$$eK\left(\frac{t-t_0}{t_0}\right) < 1, \tag{4}$$

then

$$R_{N-1}(t_0,t) \to 0 \ \text{ as } N \to 0,$$

so that

$$T(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} A^n T(t_0). \tag{5}$$

For $z \in \mathbb{C}$ with

$$|z - t_0| < \frac{t_0}{eK} \tag{6}$$

the power series

$$\sum_{n=0}^{\infty} \frac{(z-t_0)^n}{n!} A^n T(t_0)$$

converges absolutely and we put

$$\widetilde{T}(z) = \sum_{n=0}^{\infty} \frac{(z-t_0)^n}{n!} A^n T(t_0). \tag{7}$$

Now choose $\psi \in (0, \frac{\pi}{2})$ such that

$$\psi < \frac{1}{eK}.$$

Then for every $z \in \Delta_{\psi,0}$ there exists $t_0 > 0$ such that (6) holds. Therefore we can use the series in (7) to define $\widetilde{T} : \Delta_{\psi,0} \to \mathcal{L}(X;X)$.

Notice that

$$\widetilde{T}(t) = T(t) \ \text{ for all } t > 0,$$

and

$$\begin{aligned}\frac{d}{dz}\widetilde{T}(z) &= \sum_{n=1}^{\infty} \frac{n(z-t_0)^{n-1}}{n!} A^n(t_0) = \sum_{n=0}^{\infty} \frac{(z-t_0)^n}{n!} A^{n+1} T(t_0) \\ &= A\widetilde{T}(z).\end{aligned}$$

To verify the semigroup property, we observe that for fixed $t > 0$ and $z \in \Delta_{\psi,0}$ satisfying (6) we have

$$\begin{aligned}
\widetilde{T}(t)\widetilde{T}(z) &= \sum_{n=0}^{\infty} \frac{(z-t_0)^n}{n!} A^n T(t_0+t) \\
&= \sum_{n=0} \infty \frac{[(z+t)-(t_0+t)]^n}{n!} A^n T(t_0+t) \\
&= \widetilde{T}(z+t).
\end{aligned}$$

The semigroup property on the full sector $\Delta_{\psi,0}$ follows from a standard stepping argument.

Now choose $\phi \in (0,\psi)$ and put

$$\Omega = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) \leq 1,\ |\arg(z)| \leq \phi\},$$

$$M = \sup\{|T(s)\| : s \in [0,1]\},$$

and

$$\epsilon = \frac{\phi}{\psi}.$$

Notice that

$$0 < \epsilon < 1.$$

Let $z \in \Omega$ be given and set

$$t_0 = \mathrm{Re}(z).$$

Then we have

$$\frac{|z-t_0|}{t_0} \leq \frac{\epsilon}{eK},$$

and consequently

$$\begin{aligned}
\|\widetilde{T}(z)\| &\leq \|T(t)\| + \sum_{n=1}^{\infty} \frac{|z-t_0|^n}{n!} \|A^n T(t_0)\| \\
&\leq \|T(t)\| + \sum_{n=1}^{\infty} \epsilon^n \\
&\leq M + \frac{\epsilon}{1-\epsilon} \quad \text{for all } z \in \Omega.
\end{aligned}$$

Now we are ready to establish (c). Let $x \in X$ and $t > 0$ be given. Then we have

$$\widetilde{T}(z)T(t)x = \widetilde{T}(z+t)x \to T(t)x \quad \text{as } z \to 0. \tag{8}$$

Using (8) together with the facts that $\|\widetilde{T}\|$ is bounded on $\Omega$ and $T(t)x \to x$ as $t \downarrow 0$ we find that (c) holds. This completes the proof. $\square$

**Theorem 12.2**: Let $X$ be a complex Banach space and $\phi \in (0, \frac{\pi}{2})$ be given. Assume that $\widetilde{T} : \Delta_{\phi,0} \to \mathcal{L}(X; X)$ is analytic and that (i) and (ii) below hold:

(i) $\widetilde{T}(z_1 + z_2) = \widetilde{T}(z_1)\widetilde{T}(z_2)$ for all $z_1, z_2 \in \Delta_{\phi,0}$,

(ii) For every $x \in X$, we have $\widetilde{T}(z)x \to x$ as $z \to 0$, $z \in \Delta_{\phi,0}$.

Put $T(0) = I$ and $T(t) = \widetilde{T}(t)$ for all $t > 0$. Then $T$ is a linear $C_0$ semigroup that is differentiable on $(0, \infty)$ and there is a constant $K > 0$ such that

$$\|AT(t)\| \leq \frac{K}{t} \quad \text{for all } t \in (0, 1],$$

where $A$ is the infinitesimal generator of $A$.

**Proof**: That $T$ is a linear $C_0$-semigroup and is differentiable on $(0.\infty)$ is immediate. Let $\alpha \in (0, \phi)$ be given and put

$$\Lambda_\alpha = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) \leq 2, \ |\arg(z)| \leq \alpha\}.$$

Using the Principle of Uniform Boundedness (as in the proof of Lemma x.x) we may choose $M_\alpha > 0$ such that

$$\|\widetilde{T}(z)\| \leq M \quad \text{for all } z \in \Lambda_\alpha. \tag{9}$$

Let $t \in (0, 1]$ be given and let $C$ be the circle

$$C = \{z \in \mathbb{C} : |z - t| = (\sin\alpha)t\}$$

oriented in the counterclockwise direction. Then, by Cauchy's Theorem

$$AT(t) = T'(t) = \widetilde{T}'(t) = \frac{1}{2\pi i} \int_C \frac{\widetilde{T}(z)}{(z - t)^2}\, dz. \tag{10}$$

Taking norms in (10) and using (9) we see that

$$\|AT(t)\| \leq \frac{M_\alpha 2\pi t \sin\alpha}{2\pi((\sin\alpha)t)^2} = \frac{M_\alpha(\sin\alpha)^{-1}}{t}.$$

This completes the proof. $\square$

**Definition 12.3**: Let $X$ be a real or complex Banach space. By an *analytic semigroup* we mean a linear $C_0$-semigroup $T : \mathcal{L}(X; X) \to \mathcal{L}(X; X)$ such that $T$ is differentiable on $(0, \infty)$ and there exists a constant $K$ such that

$$\|AT(t)\| \leq Kt^{-1} \quad \text{for all } t \in (0, 1].$$

**Theorem 12.4**: Let $X$ be a complex Banach space and $\mathcal{D}(A) \subset X$. Assume that $\mathcal{D}(A)$ is dense and that $A : \mathcal{D}(A) \to x$ is linear and closed.

(a) Assume that there are constants $C, \omega \in \mathbb{R}$ such that

$$\rho(A) \supset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega\} = \Delta_{\frac{\pi}{2},\omega}$$

and

$$\|R(\lambda; A)\| \leq \frac{C}{|\lambda - \omega|} \quad \text{for all } \lambda \in \Delta_{\frac{\pi}{2},\omega}.$$

Then $A$ generates an analytic semigroup.

(b) Assume that $A$ generates an analytic semigroup $T$. Then there exist constants $C', \omega' \in \mathbb{R}$ and $\delta \in (0, \frac{\pi}{2})$ such that

$$\rho(A) \supset \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega')| < \frac{\pi}{2} + \delta\} = \Delta_{\frac{\pi}{2}+\delta,\omega'}$$

and

$$\|R(\lambda; A)\| \leq \frac{C'}{|\lambda - \omega'|} \quad \text{for all } \lambda \in \Delta_{\frac{\pi}{2}+\delta,\omega'}.$$

Moreover, for $\phi \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta)$ we have

$$T(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda; A)\, d\lambda \quad \text{for all } t > 0,$$

where $\Gamma$ is any curve from $\infty e^{-i\phi}$ to $\infty e^{i\phi}$ lying entirely in $\{\lambda \in \mathbb{C} \backslash \{\omega'\} : |\arg(\lambda - \omega')| \leq \phi\}$.

**Theorem 12.5**: Let $X$ be a real or complex Banach space and $T : [0, \infty) \to \mathcal{L}(X; X)$ be a linear $C_0$-semigroup with infinitesimal generator $A$. Then $T$ is an analytic semigroup if and only if there exist constants $C > 0$ and $\Lambda \geq 0$ such that for every $n \in \mathbb{N}$ we have

$$\|AR(\lambda; A)^{n+1}\| \leq \frac{C}{n\lambda^n} \quad \text{for all } \lambda > n\Lambda.$$

*Some Additional Comments on Analytic Semigroups*

We close our discussion of analytic semigroups with a few additional remarks.

- The class of analytic semigroups is stable under bounded perturbations of the infinitesimal generator. (This is a homework exercise.) In fact, one can perturb the generator of an analytic semigroup by a certain class of unbounded operators and still have a generator of an analytic semigroup. See, for example, Theorem 2.1 in Pazy.

- When $A$ generates an analytic semigroup, there are improved regularity results for solutions of

$$\dot{u}(t) = Au(t) + f(t).$$

  There will be a homework exercise on this topic. See the monograph of Yagi for an extensive discussion of such results.

- If $A$ generates an analytic semigroup on a complex Banach space and $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$ then one can define fractional powers of $-A$. See, for example, Section 2.6 of Pazy.

- There are some simplified generation theorems for analytic semigroups involving semi-inner products. There will be a homework exercise concerning this topic. We also give a result below for the Hilbert space case.

**Proposition 12.6**: Let $X$ be a complex Hilbert space and $\mathcal{D}(A) \subset X$. Assume that $A : \mathcal{D}(A)$ is self-adjoint. Assume further that there exists $\beta \in \mathbb{R}$ such that

$$(Ax, x) \leq \beta \|x\|^2 \quad \text{for all } x \in \mathcal{D}(A).$$

Then $A$ generates an analytic semigroup.

**Proof**: Without loss of generality, we assume that $\beta = 0$. Then we have

$$\rho(A) \supset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\} = \Delta_{\frac{\pi}{2},0}.$$

Let $\mu > 0$ and $\sigma \in \mathbb{R}$ be given and put

$$\lambda = \mu + i\sigma.$$

Then we have

$$((\lambda I - A)x, x) = \mu\|x\|^2 - (Ax, x) + i\sigma\|x\|^2 \quad \text{for all } x \in \mathcal{D}(A),$$

from which we can conclude that

$$|((\lambda - A)x, x)| \geq |\lambda| \cdot \|x\|^2 \quad \text{for all } x \in \mathcal{D}(A). \tag{11}$$

It follows from (11) that

$$\|R(\lambda; A)\| \leq \frac{1}{|\lambda|} \quad \text{for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Re}(\lambda) > 0.$$

The desired conclusion follows from part (a) of Theorem 12.4. $\square$

**Example 12.7** (Heat Equation): Let $X = L^2[0,1]$ and put

$$\mathcal{D}(A) = \{u \in AC[0,1] : u' \in AC[0,1],\ u'' \in L^2[0,1],\ u(0) = u(1) = 0\},$$

and define $A : \mathcal{D}(A) \to X$ by

$$Au = u'' \quad \text{for all } u \in \mathcal{D}(A).$$

It is left as an exercise to check that $\mathcal{D}(A^*) = \mathcal{D}(A)$, where $A^*$ is the Hilbert adjoint of $A$. Since

$$\begin{aligned}
\int_0^1 u''(x)\overline{v}(x)\,dx &= u'(x)\overline{v}(x)\Big|_0^1 - \int_0^1 u'(x)\overline{v}'(x)\,dx \\
&= -u(x)\overline{v}'(x)\Big|_0^1 + \int_0^1 u(x)\overline{v}''(x)\,dx \qquad (12) \\
&= \int_0^1 u(x)\overline{v}''(x)\,dx \quad \text{for all } u, v \in \mathcal{D}(A),
\end{aligned}$$

we conclude that $A$ is self-adjoint.

Using (13) with $u = v$ we find that

$$(Au, u) = -\int_0^1 |u'(x)|^2\,dx \le 0 \quad \text{for all } u \in \mathcal{D}(A).$$

It follows from Proposition 12.6 that $A$ generates an analytic semigroup.

*Fourier Transforms*

We begin with some useful notation.

**Definition 12.8** (Multi-indices): Let $M_n = (\mathbb{N} \cup \{0\})^n$. The elements of $M_n$ are called multi-indices

**Definition 12.9**: Let $\alpha, \beta \in M_n$ be given. We say that $\alpha \le \beta$ provided that

$$\forall j \in \{1, 2, \cdots, n\}, \quad \text{we have } \alpha_j \le \beta_j.$$

**Definition 12.10**: Let $\alpha, \beta \in M_n$ and $x \in \mathbb{R}^n$ be given. We define

- $|\alpha| = \displaystyle\sum_{j=1}^{n} \alpha_j$,
- $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$,
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

**Definition 12.11**: For $\alpha, \beta \in M_n$ with $\alpha \le \beta$ we put

$$\binom{\beta}{\alpha} = \frac{\beta!}{(\beta - \alpha)!}.$$

**Definition 12.12**: Let $\alpha \in M_n$ and $f \in C^{|\alpha|}(\mathbb{R}^n)$ be given. We define

$$(D^\alpha f)(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} f(x).$$

Observe that with the above definitions, the Binomial Theorem takes the form

$$(x+y)^\beta = \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} x^\alpha y^{\beta-\alpha} \quad \text{for all } x, y \in \mathbb{R}^n,\ \beta \in M_n,$$

and, for sufficiently smooth functions $f, g : \mathbb{R}^n \to \mathbb{C}$, Leibniz's product rule takes the form

$$D^\beta(fg) = \sum_{\alpha \leq \beta} (D^\alpha f)(D^{\beta-\alpha} g).$$

In order that we can write formulas involving Fourier transforms in a clean way, it is convenient to give a name to the mapping $x \to x^\alpha$.

**Definition 12.13**: Let $\alpha \in M_n$ be given. We define $P_\alpha : \mathbb{R}^n \to \mathbb{C}$ by

$$P_\alpha(x) = x^\alpha \quad \text{for all } x \in \mathbb{R}^n.$$

[Actually $P_\alpha$ takes values in $\mathbb{R}$, but I have taken the codomain to be $\mathbb{C}$ because the scalar field will always be $\mathbb{C}$ when we use this mapping.]

**Definition 12.14**: Let $f \in L^1(\mathbb{R}^n)$ be given. We define the *Fourier transform* $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of $f$ by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx \quad \text{for all } x \in \mathbb{R}^n. \tag{13}$$

**Remark 12.15**: There are several other conventions used in defining the Fourier transform. These involve placing factors of $2\pi$ in different places and/or changing the sign in the exponential. Also some authors (e.g., Rudin) normalize Lebesgue measure to absorb the factor $(2\pi)^{-\frac{n}{2}}$ in front of the integral. The bottom line is that one must be very careful to check what convention is being employed in situations where the numerical values of constants could be important.

**Proposition 12.16**: Let $f \in L^1(\mathbb{R}^n)$ be given. Then $\widehat{f}$ is bounded and continuous on $\mathbb{R}^n$.

**Proof**: Let $\xi \in \mathbb{R}^n$ be given and let $\{\xi^{(k)}\}_{k=1}^\infty$ be a sequence in $\mathbb{R}^n$ such that $\xi^{(k)} \to \xi$ as $k \to \infty$. Since

$$|e^{-ix\cdot\xi^{(k)}} f(x)| = |e^{-ix\cdot\xi} f(x)| = |f(x)| \quad \text{for all } x \in \mathbb{R}^n$$

(and since $f \in L^1(\mathbb{R}^n)$) the Lebesgue Dominated Convergence Theorem implies that

$$\widehat{f}(\xi^{(k)}) \to \widehat{f}(\xi).$$

It follows immediately from (13) that

$$|\widehat{f}(\xi)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

and consequently

$$\|\widehat{f}\|_\infty \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_1 \quad \text{for all } f \in L^1(\mathbb{R}^n). \tag{14}$$

This completes the proof. $\square$

An extremely useful feature of the Fourier transform is that it converts differentiation into multiplication by polynomials. This will allow us to use elementary facts from basic algebra to deduce powerful results concerning differential operators. To get an idea of how this will work, we shall perform some formal computations first and then make assumptions that justify these computations later on.

Suppose that $f : \mathbb{R}^n$ is "really nice" (meaning that is very smooth and that $f$ and derivatives of $f$ go to zero rapidly at infinity). Let $\alpha \in M_n$ be given. Then we may write

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

Applying $D^\alpha$ and differentiating under the integral sign, we find that

$$(D^\alpha \widehat{f})(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (-i)^{|\alpha|} x^\alpha e^{-ix\cdot\xi} f(x)\, dx.$$

We can rewrite this expression (without using variables) as

$$D^\alpha \widehat{f} = (-i)^{|\alpha|} (P_\alpha f)\widehat{}.$$

Using integration by parts, we find that

$$\begin{aligned}
(D^\alpha f)\widehat{}(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} D^\alpha f(x)\, dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (-1)^{|\alpha|} (-i)^{|\alpha|} \xi^\alpha e^{-ix\cdot\xi} f(x)\, dx
\end{aligned}$$

We can rewrite this as

$$(D^\alpha f)\widehat{} = i^{|\alpha|} P_\alpha \widehat{f}.$$