Lecture Notes for Week 7 (First Draft)

Duaality and Reflexivity (Continued)

Theorem 7.1: Assume that X is reflexive and let Y be a closed subspace of X. Then Y is reflexive.

Proof: Define the linear mapping $A : X^* \to Y^*$ by

$$(Ax^*)(y) = x^*(y) \text{ for all } x^* \in X^*,\ y \in Y.$$

In other words, A is simply the restriction operator for bounded linear functionals on X to the smaller domain Y. Observe that

$$\|Ax^*\|_{Y^*} \leq \|x^*\|_{X^*} \text{ for all } x^* \in X^*. \tag{1}$$

Now define the linear mapping $B : Y^{**} \to X^{**}$ by

$$(By^{**})(x^*) = y^{**}(Ax^*) \text{ for all } y^{**} \in Y^{**},\ x^* \in X^*.$$

By virtue of (1), we have

$$|(By^{**})(x^*)| \leq \|y^{**}\|_{Y^{**}}\|Ax^*\|_{Y^*} \leq \|y^{**}\|_{Y^{**}}\|x^*\|_{X^*}.$$

Let J_X, J_Y denote the canonical injections of X into X^{**} and Y into Y^{**}, respectively. (It is probably helpful to draw a diagram showing the sets $X, X^*, X^{**}, Y, Y^*, Y^{**}$ and indicating the mappings A, B, J_X, J_Y.) Let $y^{**} \in Y^{**}$ be given and put

$$x = J_X^{-1}(By^{**}).$$

Let us show that $x \in Y$. (This is a key step in the argument; in particular, it is here that we make use of the fact that Y is closed.) Suppose that $x \notin Y$. Then, since Y is closed, we may choose $x^* \in X^*$ such that $x^*(x) = 1$ and

$$x^*(y) = 0 \text{ for all } y \in Y,$$

by Corollary 5.8 (a consequence of the Hahn-Banach Theorem). This implies that $Ax^* = 0$. Therefore, we have

$$\begin{aligned} 0 = y^{**}(Ax^*) &= (By^{**})(x^*) \\ &= (J_X x)(x^*) \\ &= x^*(x). \end{aligned}$$

Since $x^*(x) = 1$, we have a contradiction, from which we conclude that $x \in Y$. Therefore, we have shown that

$$J_x^{-1}[B[Y^{**}]] \subset Y.$$

Let $y_0^{**} \in Y^{**}$ be given. We want to find $y_0 \in Y$ such that $J_Y(y_0) = y_0^{**}$. Let us put

$$y_0 = J_X^{-1}(By_0^{**}),$$

and observe that $y_0 \in Y$. Given $y^* \in Y^*$, let $x^* \in X^*$ be an extension of y^*, so that

$$y^* = Ax^*.$$

(We can choose such an extension by virtue of the Hahn-Banach Theorem.) Then we have

$$\begin{aligned} y_0^{**}(y^*) &= (By_0^{**})(x^*) \\ &= (J_X y_0)(x^*) \\ &= x^*(y_0) \\ &= y^*(y_0). \end{aligned}$$

It follows that $y_0^{**} = J_Y(y_0)$ and consequently Y is reflexive. $\square$

Proposition 7.2: Let X and Y be isomorphic normed linear spaces and assume that Y is reflexive. Then X is reflexive.

Proof: Choose a linear isomorphism $T : X \to Y$, i.e. a linear mapping such that T and T^{-1} are bounded. Define $U : X^* \to Y^*$ by

$$(Ux^*)(y) = x^*(T^{-1}y) \text{ for all } x^* \in X^*,\ y \in Y. \tag{2}$$

I claim that U is a (linear) isomorphism. To see that U is injective, let $x^* \in X^*$ be given and assume that $Ux^* = 0$. Using (2) we find that $x^*(T^{-1}y) = 0$ for all $y \in Y$, which implies that $x^*(x) = 0$ for all $x \in X$ and consequently U is injective. U is bounded because

$$|(Ux^*)(y)| \le \|x^*\|\|T\|\|y\| \text{ for all } x^* \in X^*,\ y \in Y,$$

and consequently

$$\|Ux^*\| \le \|T\|\|x^*\| \text{ for all } x^* \in X^*.$$

To see that T is surjective, let $\hat{y}^* \in Y^*$ be given and put define $\hat{x}^* \in X^*$ by

$$\hat{x}^*(x) = \hat{y}^*(Tx) \text{ for all } x \in X.$$

Using (2), we see that

$$U(\hat{x}^*)(y) = \hat{x}^*(T^{-1}y) = \hat{y}^*(TT^{-1}y) = \hat{y}^*(y) \text{ for all } y \in Y.$$

Since X^* and Y^* are complete, it follows the Bounded Inverse Theorem that U^{-1} is bounded.

Now define the linear mapping $V : X^{**} \to Y^{**}$ by

$$(Vx^{**})(y^*) = x^{**}(U^{-1}y^*) \text{ for all } x^{**} \in X^{**},\ y^* \in Y^* \tag{3}$$

I claim that V is also an isomorphism. (The details are similar to those for U. To establish the surjectivity, given $\hat{y}^{**} \in Y^{**}$, put $\hat{x}^{**}(x^*) = \hat{y}^{**}(Ux^*)$ for all $x^* \in X^*$.)

Let $x_0^{**} \in X^{**}$ be given and put

$$y_0 = J_Y^{-1}(VX_0^{*}*), \tag{4}$$

$$x_0 = T^{-1}y_0. \tag{5}$$

In order to show that J_X is surjective, it suffices to show that

$$x^*(x_0) = x_0^{**}(x^*) \text{ for all } x^* \in X^*.$$

To this end, let $x^* \in X^*$ be given and put

$$y^* = Ux^*. \tag{6}$$

Then we have

$$\begin{aligned}
x_0^{**}(x^*) &= (Vx_0^{**}(y^*) \text{ using } (3),(6) \\
&= (J_Y(y_0))(y^*) \text{ using } (4) \\
&= y^*(y_0) \text{ by the definition of } J_Y \\
&= (Ux^*)(Tx_0) \text{ using } (5),(6) \\
&= x^*(x_0) \text{ using } (2).
\end{aligned}$$

Remark 7.3: If $T : X \to Y$ is an isometric isomorphism, then the mappings U and V defined by (2) and (3) are also isometric isomorphisms.

Theorem 7.4: Let X Banach space Then X is reflexive if and only if X^* is reflexive.

Remark 7.5: It is important to require that X be a Banach space in Theorem 7.3 because it can happen that an incomplete normed linear space has a reflexive dual space.

Proof of Theorem 7.4: Assume that X is reflexive. Then X^{**} is reflexive by Proposition 7.2. Let $x_0^{***} \in (X^*)^{**}$ be given. We want to produce $x_0^* \in X^*$ such that $x_0^{***}(x^{**}) = x^{**}(x_0^*)$ for all $x^{**} \in X^{**}$. Let us define $x_0^* \in X^*$ by

$$x_0^*(x) = x_0^{***}(J_X(x)) \text{ for all } x \in X.$$

Obsereve that

$$x^{**}(x_0^*) = (J_X(x))(x_0^*) = x_0^*(x) = x_0^{***}(J_X(x)) = x_0^{***}(x^{**}) \text{ for all } x^{**} \in X^{**}.$$

It follows that X^* is reflexive.

To establish the converse, assume that X^* is reflexive. Then by what we just proved above, we know that X^{**} is reflexive. Since X is complete, we see that $J_X[X]$ is complete and hence closed. Since $J_X[X]$ is a closed subspace of X^{**}, it follows from Theorem 7.1 that $J_X[X]$ is reflexive. Since X and $J_X[X]$ are (isometrically) isomorphic, we conclude that X is reflexive. $\square$

Weak and Weak Convergence*

We now introduce and study a very important type of convergence, known as "weak convergence" in a normed linear space X. We also study a related type of convergence in X^*, known as "weak* convergence". We shall refer to convergence in terms of the norm as "strong convergence" in order to minimize the possibility of confusion with weak or weak* convergence.

Let X be a normed linear space with dual space X^*.

Definition 7.6: Let $\{x_n\}_{n=1}^\infty$ be a sequence in X and $x \in X$ be given. We say that $\{x_n\}_{n=1}^\infty$ *converges weakly* to x, or that x is a *weak limit* of $\{x_n\}_{n=1}^\infty$ provided that

$$\forall x^* \in X^*, \quad \langle x^*, x_n \rangle \to \langle x^*, x \rangle \text{ as } n \to \infty.$$

In this case we write $x_n \rightharpoonup x$ (weakly) as $n \to \infty$.

Definition 7.7: A sequence $\{x_n\}_{n=1}^\infty$ in X is said to be *weakly convergent* if there exists $x \in X$ such that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$.

Remark 7.8:

(a) A sequence can have at most one weak limit. Indeed, if $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ and $x_n \rightharpoonup y$ (weakly) as $n \to \infty$ then $\langle x^*, x - y \rangle = 0$ for all $x^* \in X^*$, which implies $x - y = 0$. (Here we are making use of Corollary 5.5.)

(b) If $x_n \to x$ (strongly) as $n \to \infty$ then $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. The converse implication is false in general.

(c) If $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ and $\{x_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{x_n\}_{n=1}^\infty$ then $x_{n_k} \rightharpoonup x$ (weakly) as $k \to \infty$.

Remark 7.9: Let X be a finite-dimensional normed linear space and let $\{x_n\}_{n=1}^\infty$ be a sequence in X and $x \in X$ be given. Then $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ if and only if $x_n \to x$ (strongly) as $n \to \infty$. (We shall give a careful proof of this result later. The key observation is the fact that if X is finite dimensional, then the component mappings for any basis are continuous linear functionals.)

In order to develop interesting examples involving weak convergence, we need characterizations of the dual spaces of some infinite-dimensional normed linear spaces. We shall make use of the following result, which will be proved later.

Proposition 7.10 Let $p \in (1, \infty)$ and put

$$q = \frac{p}{p-1}.$$

(Notice that $p^{-1} + q^{-1} = 1$.) Let $x^* \in (l^p)^*$ be given. Then there is exactly one $y \in l^q$ such that

$$\langle x^*, y \rangle = \sum_{k=1}^{\infty} x_k y_k \quad \text{for all } x \in l^p; \tag{7}$$

moreover, $\|y\|_q = \|x^*\|$.

Remark 7.11:

(a) In some sense, Proposition 7.10 says that when $p \in (1, \infty)$ and $q = p(p-1)^{-1}$ then the dual of l^p is l^q. However, some caution is appropriate here, because, strictly speaking, the elements of $(l^p)^*$ and the elements of l^q are different kinds of objects. What the proposition really says is that $(l^p)^*$ is isometrically isomorphic to l^q. This situation is frequently expressed by saying thet the dual of l^p can be identified with l^q through the pairing defined by (7).

(b) It follows immediately from Proposition 7.10 that l^p is reflexive for $1 < p < \infty$.

Remark 7.12: Let $p \in (1, \infty)$ be given and put $q = p(p-1)^{-1}$. Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a sequence in l^p and let $x \in l^p$ be given. Then $x^{(n)} \rightharpoonup x$ (weakly) in l^p as $n \to \infty$ if and only if

$$\sum_{k=1}^{\infty} x_k^{(n)} y_k \to \sum_{k=1}^{\infty} x_k y_k \quad \text{for all } y \in l^q \text{ as } n \to \infty.$$

Example 7.13: Let $p \in (1, \infty)$ be given and put $q = p(p-1)^{-1}$. Consider the sequence $\{e^{(n)}\}_{n=1}^{\infty}$ in l^p, where

$$e_k^{(n)} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

Let $y \in l^q$ be given. Then we have

$$\sum_{k=1}^{\infty} e_k^{(n)} y_k = y_n \quad \text{for all } n \in \mathbb{N}$$

Since $y_n \to 0$ as $n \to \infty$, we conclude that

$$e^{(n)} \rightharpoonup 0 \text{ (weakly) in } l^p \text{ as } n \to \infty.$$

It is clear that the sequence $\{e^{(n)}\}_{n=1}^{\infty}$ does not converge strongly. (Indeed, $\|e^{(n)} - e^{(m)}\|_p = 2^{\frac{1}{p}}$ when $m \neq n$.) Since $\|e^{(n)}\| = 1$ for all $n \in \mathbb{N}$, we see that the norm of a weak limit need not equal the limit of the norms.

Remark 7.14: Although weak convergence generally does not imply strong convergence in infinite dimensions, there do exist infinite-dimensional Banach spaces having the unusual property that weak convergence of sequences is equivalent to strong convergence. An example of such a space is l^1. We shall see later that if $x \in l^1$ and $\{x^{(n)}\}_{n=1}^{\infty}$ is a sequence in l^1 then $x^{(n)} \to x$ (strongly) as $n \to \infty$ if and only if $x^{(n)} \rightharpoonup x$ (weakly) as $n \to \infty$. This fact comes in very handy for constructing interesting examples (and counterexamples).

Theorem 7.15: Let X be a normed linear space and Y be a closed subspace of X. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$ be given. Assume that $x_n \in Y$ for all $n \in \mathbb{N}$ and that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. Then

(i) $\{x_n\}_{n=1}^{\infty}$ is bounded,

(ii) $x \in Y$,

(iii) $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

Proof: To prove (i), consider the sequence $\{J(x_n)\}_{n=1}^{\infty}$ in X^{**}. For every $x^* \in X^*$ the sequence $\{(J(x_n))(x^*)\}_{n=1}^{\infty}$ is bounded. The Principle of Uniform Boundedness implies that the sequence $\{\|J(x_n)\|\}_{n=1}^{\infty}$ is bounded. (Notice that X^* is complete.) Since $\|x_n\| = \|J(x_n)\|$ for all $n \in \mathbb{N}$, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded.

To prove (ii), suppose that $x \notin Y$. Then, by Corollary 5.8, we may choose $x^* \in X^*$ such that $\langle x^*, y\rangle = 0$ for all $y \in Y$ and $\langle x^*, x\rangle = 1$. We have $\langle x^*, x_n\rangle = 0$ for all $n \in \mathbb{N}$. This is a contradiction because $\langle x^*, x_n\rangle \to \langle x^*, x\rangle = 1$ as $n \to \infty$. It follows that $x \in Y$.

To prove (iii), let $x^* \in X^*$ be given. Then we have

$$|\langle x^*, x\rangle| = \lim_{n\to\infty} |\langle x^*, x_n\rangle| \leq \liminf_{n\to\infty} \|x^*\|\|x_n\|. \tag{8}$$

Recall that

$$\|x\| = \sup\{|\langle x^*, x\rangle| : x^* \in X^*, \|x^*\| \leq 1\}. \tag{9}$$

Combining (8) and (9) we arrive at

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \quad \square$$

Remark 7.16: Using a separation result for convex sets, we shall show later (via a modification of the proof of (ii) above) that if K is closed and convex, $x_n \in K$ for all $n \in \mathbb{N}$ and $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ then $x \in K$. We shall also show that if $F : X \to \mathbb{R}$ is a continuous convex function and $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ then

$$\liminf_{n\to\infty} F(x_n) \geq F(x).$$

These observations are of crucial importance in the calculus of variations.

Since the dual space X^* is a normed linear space in its own right, the notion of weak convergence in the dual space is meaningful. However, it is more natural to use the notion of "weak* convergence" in X^*. Unfortunately, some authors refer to "weak* convergence" as "weak convergence in the dual space". Caution is required regarding this point.

Definition 7.17: Let $\{x_n^*\}_{n=1}^\infty$ be a sequence in X^* and $x^* \in X^*$ be given. We say that $\{x_n^*\}_{n=1}^\infty$ *converges weakly** to x^*, or that x^* is a *weak* limit* of $\{x_n^*\}_{n=1}^\infty$ provided that

$$\forall x \in X, \quad \langle x_n^*, x\rangle \to \langle x^*, x\rangle \text{ as } n \to \infty.$$

In this case we write $x_n^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$.

Definition 7.18: A sequence $\{x_n^*\}_{n=1}^\infty$ in X^* is said to be *weakly* convergent* if there exists $x^* \in X^*$ such that $x_n^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$.

Remark 7.19: Notice that if X is reflexive then weak* convergence is the same thing as weak convergence in X^*. However most of the time when weak* convergence is used in practice, X is not reflexive. Since $J[X] \subset X^{**}$, it follows that weak convergence in X^* impplies weak* convergence.

Remark 7.20:

(a) A sequence can have at most one weak* limit. Indeed, if $x_n^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$ and $x_n^* \overset{*}{\rightharpoonup} y^*$ (weakly*) as $n \to \infty$ then $\langle x^* - y^*, x\rangle = 0$ for all $x \in X$, which implies $x^* - y^* = 0$.

(b) If $x_n^* \to x^*$ (strongly) as $n \to \infty$ then $x_n \overset{*}{\rightharpoonup} x$ (weakly*) as $n \to \infty$. The converse implication is false in general.

(c) If $x_n^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$ and $\{x_{n_k}^*\}_{k=1}^\infty$ is a subsequence of $\{x_n\}_{n=1}^\infty$ then $x_{n_k}^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $k \to \infty$.

(d) If $x_n^* \rightharpoonup x^*$ (weakly) as $n \to \infty$ then $x_n^* \overset{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$. The converse implication is false in general. However, when working in a dual space X^* of a nonreflexive space we shall almost always use weak* convergence rather than weak convergence.

In order to construct interesting examples involving weak* convergence, we need the dual space of a nonreflexive Banach space. We shall make use of the following result which will be proved later.

Proposition 7.21: Let $x^* \in (l^1)^*$ be given. Then there is exactly one $x \in l^\infty$ such that

$$x^*(z) = \sum_{k=1}^{\infty} x_k z_k \quad \text{for all } z \in l^1; \tag{10}$$

moreover $\|x^*\| = \|x\|_\infty$.

Remark 7.22: It follows from Proposition 7.21 that the dual of l^1 can be identified with l^∞ through the isometric isomorphism given in (10). Even though l^∞ is not strictly speaking a dual space, it is standard to talk about weak* convergence in l^∞ and to say that a sequence $\{x^{(n)}\}_{n=1}^\infty$ in l^∞ converges weakly* to $x \in l^\infty$ provided that

$$\sum_{k=1}^{\infty} x_k^{(n)} z_k \to \sum_{k=1}^{\infty} x_k z_k \quad \text{for all } z \in l^1 \text{ as } n \to \infty.$$

Example 7.23:

(a) Consider the sequence $\{e^{(n)}\}_{n=1}^\infty$ in l^∞, where

$$e_k^{(n)} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

Let $z \in l^1$ be given. Then we have

$$\sum_{n=1}^{\infty} e_k^{(n)} z_k = z_n \to 0 \quad \text{as } n \to \infty.$$

We conclude that

$$e^{(n)} \overset{*}{\rightharpoonup} 0 \text{ (weakly}*) \text{ in } l^\infty \text{ as } n \to \infty.$$

(b) Define $x \in l^\infty$ by $x_k = 1$ for all $k \in \mathbb{N}$, and consider the sequence $\{x^{(n)}\}_{n=1}^\infty$ in l^∞ given by

$$x_k^{(n)} = \begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

Observe that $x^{(n)} \in c_0$ for all $n \in \mathbb{N}$. Let $z \in l^1$ be given. Then we have

$$\sum_{k=1}^{\infty} x_k^{(n)} z_k = \sum_{k=1}^{n} z_k \to \sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} x_k z_k \quad \text{as } n \to \infty.$$

We conclude that

$$x^{(n)} \stackrel{*}{\rightharpoonup} x \text{ (weakly}*\text{) in } l^\infty \text{ as } n \to \infty.$$

The convergence cannot be weak because c_0 is a closed subspace of l^∞ and $x \notin c_0$.

Theorem 7.24: Let X be a Banach space with dual space X^*. Let $\{x_n^*\}_{n=1}^\infty$ be a sequence in X^* and let $x^* \in X^*$ be given. Assume that $x_n^* \stackrel{*}{\rightharpoonup} x^*$ (weakly*) as $n \to \infty$. Then

(i) $\{x_n^*\}_{n=1}^\infty$ is bounded,

(ii) $\|x^*\| \le \liminf_{n\to\infty} \|x_n^*\|$.

Remark 7.25:

(a) Completeness of X is essential in order to obtain the boundedness of weakly* convergent sequences. (We need to use the Principle of Uniform Boundedness.)

(b) Weak* limits can "escape" from closed subspaces of X^*. More precisely, it can happen that Z is a closed subspace of X^*, $x_n^* \in Z$ for all $n \in \mathbb{N}$, $x_n^* \rightharpoonup x^*$ (weakly*) as $n \to \infty$, but $x^* \notin Z$. (See Example 7.23(b).)

Proof of Theorem 7.24: For every $x \in X$, the sequence $\{\langle x_n^*, x\rangle\}_{n=1}^\infty$ is bounded. Since X is complete, the Principle of Uniform Boundedness implies that the sequence $\{\|x_n^*\|\}_{n=1}^\infty$ is bounded.

To prove (ii), let $x \in X$ be given. Then we have

$$|\langle x^*, x\rangle| = \lim_{n\to\infty} |\langle x_n^*, x\rangle| \le \liminf_{n\to\infty} \|x_n^*\|\|x\|. \tag{11}$$

Recall that

$$\|x^*\| = \sup\{|\langle x^*, x\rangle| : x \in X, \|x\| \le 1\}. \tag{12}$$

Combining (11) and (12) we arrive at

$$\|x^*\| \le \liminf_{n\to\infty} \|x_n^*\|. \quad \square$$

Theorem 7.26: Let X be a NLS. If X^* is separable then X is separable.

Remark 7.27: The converse implication is false as can be seen from Proposition 7.21.

Proof: Assume that X^* is separable. Choose a dense sequence $\{x_n^*\}_{n=1}^\infty$ in X^*. For each $n \in \mathbb{N}$ choose $x_n \in X$ such that $\|x_n\| = 1$ and

$$|\langle x_n^*, x_n \rangle| \geq \frac{1}{2}\|x_n^*\| \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

Let S be the set of all linear combinations of the x_n with rational coefficients. (By a rational element of $\mathbb{C}$ we mean a number of the form $a + ib$ with $a, b \in \mathbb{Q}$. Let $Y = cl(S)$ and observe that Y is a subspace of X. Suppose that $Y \neq X$. Then we may choose $x^* \in X^*$ such that $\langle x^*, y \rangle = 0$ for all $y \in Y$ and $x^* \neq 0$. Since $\{x_n^*\}_{n=1}^\infty$ is dense in X^* we may choose $\tau : \mathbb{N} \to \mathbb{N}$ such that

$$x_{\tau(n)}^* \to x^* \quad \text{as } n \to \infty.$$

Then we have

$$\|x^* - x_{\tau(n)}^*\| \geq |\langle x^* - x_{\tau(n)}^*, x_{\tau(n)} \rangle| = |\langle x_{\tau(n)}^*, x_{\tau(n)} \rangle| \geq \frac{\|x_{\tau(n)}^*\|}{2} \quad \text{for all } n \in \mathbb{N}$$

by virtue of (13) and the fact that $\langle x^*, x_{\tau(n)} \rangle = 0$ since $x_{\tau(n)}^* \in Y$. Consequently $x_{\tau(n)}^* \to 0$ as $n \to \infty$. This is a contradiction because $x^* \neq 0$. We conclude that $Y = X$. $\Box$