MATH 651: PROBLEM SET 1 SOLUTIONS ARE DUE IN CLASS ON FRI. SEP. 14.

1. Let

   $$\mathcal{B} = \{(x-r, x+r) : x \in \mathbb{R}\backslash\{0\}, r \in (0, |x|)\} \cup \{(-\infty, -a) \cup (-r, r) \cup (a, \infty) : r, a > 0\}.$$

   Show that $\mathcal{B}$ is a basis of topology (call it τ) on $\mathbb{R}$. Is τ finer or coarser than the the standard topology?

   Bonus problem: Show that $(\mathbb{R}, \tau)$ is homeomorphic to a subset of $\mathbb{R}^2$ endowed with standard topology.

2. A subset $U \subset \mathbb{R}^2$ is called radially open if at every point $x \in U$, U contains an open segment through x in every direction.

   (a) Prove that the family of radially open sets, τ in $\mathbb{R}^2$ is a topology. Is there any relation with the Euclidean topology (weaker, stronger, neither)? What is the induced topology on any straight line of $\mathbb{R}^2$? What is the induced topology on an circle?

   (b) Prove that there is an uncountable discrete closed set $E \subset R^2$. Prove that $(\mathbb{R}^2, \tau)$ does not satisfy the second axiom of countability.

3. Let (X, τ) be a nonempty topological space and $E \subset X$. We define the *boundary of* E as follows:

   $$\partial E = \{x \in X \;:\; (\forall U \in \tau \text{ such that } x \in U)\; E \cap U \neq \emptyset \text{ and } U \backslash E \neq \emptyset\}.$$

   Show that

   (i) $\partial E = \overline{E} \backslash E^o$ (closure minus the interior)

   (ii) $\partial E = \partial(X \backslash E)$

   (iii) $\partial(A \cup E) \cup \partial(A \cap E) \subseteq \partial E \cup \partial A$. Also provide an example that show that equality does not hold in general.

4. (20 points) Let $\Phi : [0, \infty)^N \to [0, \infty)$ be a function such that

   (i) $\Phi(x) = 0$ if and only $x = 0$.

   (ii) Φ is nondecreasing in each of the variables.

   (iii) Φ is subadditive, that is

   $$\Phi(a + b) \leq \Phi(a) + \Phi(b) \quad \text{for all } a, b \in [0, \infty)^N$$

   Assume that (X_i, d_i) are metric spaces. Let $X = X_1 \times \cdots \times X_N$ and $d : X \times X \to [0, \infty)$ be defined as

   $$d(x, y) = \Phi(d_1(x_1, y_1), \ldots, d_N(x_N, y_N))$$

where $x = (x_1, \ldots x_N)$ and $y = (y_1, \ldots, y_N)$. Show that (X, d) is a metric space.

Show that for distances d defined by

$$d(x,y) := \sqrt{d_1(x_1,y_1)^2 + \ldots + d_N(x_N,y_N)^2}$$
$$d(x,y) := d_1(x_1,y_1) + \cdots + d_N(x_N,y_N)$$
$$d(x,y) := \max_{i=1,\ldots,N} d_N(x_i,y_i).$$

(X, d) is a metric space.

[Hint: To show that a function Φ is subadditive on $[0,\infty)^N$ it suffices to show

$$\Phi(\alpha x) + \Phi(y) - \Phi(\alpha x + y) \geq 0 \text{ for all } \alpha \geq 0.$$

Check for $\alpha = 0$ and take a derivative in α of the LHS.]

5. (10 points) Let (A, d_A) be a bounded metric space and let $X = \mathcal{P}(A)\backslash\emptyset$. For $U, V \in X$ let

$$d_H(U,V) = \sup_{y \in V} \inf_{x \in U} d_A(x,y) + \sup_{x \in U} \inf_{y \in V} d_A(x,y). \tag{1}$$

Show that (X, d_H) is a pseudometric space (Symmetry and triangle inequality hold but perhaps not the positivity.)

Give an example that shows that (X, d) sometimes indeed does not satisfy the positivity.

6. (10 points) Given sets X and Y, their *disjoint union* is the set

$$X \sqcup Y = (X \times \{1\}) \cup (Y \times \{2\}).$$

Let A be a nonempty finite (for simplicity) set. Let

$$\mathcal{M} = \{(X, d_X) \ : \ X \subseteq A \text{ and } (X, d) \text{ is a metric space}\}$$

We would like to introduce a distance that compares elements of $\mathcal{M}$. For (X, d_X) and (Y, d_Y) in $\mathcal{M}$ let

$$\begin{aligned}\mathcal{C}(\,(X,d_X),(Y,d_Y)\,) = \{d \ : (X \sqcup Y, d) &\text{ is a metric space} \\ &\text{for all } x_1, x_2 \in X \quad d_X(x_1,x_2) = d((x_1,1),(x_2,1)) \text{ and} \\ &\text{for all } y_1, y_2 \in Y \quad d_Y(y_1,y_2) = d((y_1,2),(y_2,2))\}\end{aligned}$$

be the set of "couplings" of (X, d_X) and (Y, d_Y).

Consider the following function that measures how different two metric spaces are: For (X, d_X) and (Y, d_Y) in $\mathcal{M}$ let

$$D(\,(X,d_X),(Y,d_Y)\,) = \inf_{d \in \mathcal{C}} d_H(X \times \{1\}, Y \times \{2\})$$

where d_H is a metric on $\mathcal{P}(X \sqcup Y)\backslash\{\emptyset\}$ corresponding to d as defined by (1). Show that $(\mathcal{M}, D)$ is a pseudometric space.

7. (10 points) Consider $X = C((0,1)) := \{f : (0,1) \to \mathbb{R} : f \text{ is continuous}\}$. Consider $K_n := \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. Then

$$\bigcup_{n=1}^{\infty} K_n = (0,1).$$

Define

$$d(f,g) := \max_n \frac{1}{2^n} \frac{\max_{x \in K_n} |f(x) - g(x)|}{1 + \max_{x \in K_n} |f(x) - g(x)|}. \tag{2}$$

Show that (X, d) is a metric space.