Homework 3

15-423 Digital Signal Processing for CS
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I. Square Wave
Since the square wave is anti-symmetric, $\boxed{a_n = 0, \forall n \in \mathbb{N}.}$ $\forall$ positive $n \in \mathbb{N}$,

$$\begin{aligned} b_n = \frac{2}{T}\int_0^T x(t)\sin\left(\frac{2n\pi t}{T}\right)\,dt &= \frac{4}{T}\int_0^{T/2}\sin\left(\frac{2n\pi t}{T}\right)\,dt \\ &= -\frac{2}{\pi n}\cos\left(\frac{2n\pi t}{T}\right)\Bigg|_{t=0}^{t=T/2} \\ &= \frac{2}{\pi n}\left(1-\cos\left(n\pi\right)\right) = \boxed{\left\{\begin{array}{ll} \frac{2}{\pi n} & : \text{ if } n \text{ is odd} \\ 0 & : \text{ else} \end{array}\right. .} \end{aligned}$$

II. Triangle Wave
Since the triangle wave is anti-symmetric, $\boxed{a_n = 0, \forall n \in \mathbb{N}.}$ $\forall$ positive $n \in \mathbb{N}$, since the triangle wave is the integral of the square wave, dividing the series from problem I. by $\frac{2\pi n}{T}$ gives

$$\boxed{b_n = \frac{T}{\pi^2 n^2}.}$$

III. Amplitude-Modulated Sine Wave
Since the modulated sine wave is symmetric, $\boxed{b_n = 0, \forall \text{ positive } n \in \mathbb{N}.}$ By a trigonometric identity, $\forall t \in \mathbb{R}$,

$$sin(\omega t)\sin(\omega_1 t) = \frac{1}{2}(\cos((\omega-\omega_1)t) + \cos((\omega-\omega_1)t)).$$

Thus, $\boxed{a_{\omega-\omega_n} = a_{\omega+\omega_n} = \frac{1}{2},}$ and, for all other $\boxed{n \in \mathbb{N},\, a_n = 0.}$

IV. Gaussian
By definition of the Fourier Transform, $\forall \Omega \in \mathbb{R}$,

$$\begin{aligned} F(\Omega) &:= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-t^2\right)\cdot\exp\left(-2\pi i t\Omega\right)\,dt \\ &= \frac{\exp\left((\pi i\Omega)^2\right)}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-t^2\right)\cdot\exp\left(-2\pi i t\Omega\right)\cdot\exp\left(-(\pi i\Omega)^2\right)\,dt \\ &= \frac{\exp\left(-\pi^2\Omega^2\right)}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left(-(t+\pi i\Omega)^2\right)\,dt = \boxed{\frac{\exp\left(-\pi^2\Omega^2\right)}{\sqrt{2}},} \end{aligned}$$

since $\int_{\mathbb{R}} \exp\left(-(t+\pi i\Omega)^2\right)\,dt = \int_{\mathbb{R}} \exp\left(-t^2\right)\,dt = \sqrt{\pi}.$

V. Triangle

Since the triangle is symmetric, its Fourier transform is given by

$$
\begin{aligned}
X(\Omega) &= 2\int_0^\infty x(t)\cos(2\pi\Omega t)\ dt \\
&= 2\int_0^{T/2} x(t)\cos(2\pi\Omega t)\ dt \\
&= \int_0^{T/2} T\cos(2\pi\Omega t)\ dt - 2\int_0^{T/2} t\cos(2\pi\Omega t)\ dt \\
&= \frac{t}{2\pi\Omega}\sin(2\pi\Omega t)\bigg|_{t=0}^{t=T/2} - 2\frac{2\pi\Omega t\sin(2\pi\Omega t)+\cos(2\pi\Omega t)}{4\pi^2\Omega^2}\bigg|_{t=0}^{t=T/2} \\
&= -\frac{\cos(2\pi\Omega t)}{2\pi^2\Omega^2}\bigg|_{t=0}^{t=T/2} = \frac{1-\cos(\pi\Omega T)}{2\pi^2\Omega^2} = \boxed{(T\operatorname{sinc}(\pi\Omega))^2,}
\end{aligned}
$$

where the last equality follows from a trigonometric inequality.

VI. Triangle Wave Again

Since the triangle wave $y(t)$ is the convolution over t of $x(t)$ (as defined in part V.) and the impulse train $\sum_{i\in\mathbb{Z}}\delta\left(t-iT\right)$, whose Fourier transform is $\frac{1}{T}\sum_{i\in\mathbb{Z}}\delta\left(\Omega-\frac{i}{T}\right)$, by the Convolution Theorem,

$$
\begin{aligned}
Y(\Omega) = X(\Omega)\cdot\frac{1}{T}\sum_{i\in\mathbb{Z}}\delta\left(\Omega-\frac{i}{T}\right) &= T\operatorname{sinc}^2(\pi\Omega)\cdot\sum_{i\in\mathbb{Z}}\delta\left(\Omega-\frac{i}{T}\right) \\
&= \boxed{\begin{cases} T\operatorname{sinc}^2(\pi\Omega) & : \text{ if } \Omega = iT, i\in\mathbb{Z} \\ 0 & : \text{ else} \end{cases}.}
\end{aligned}
$$