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21-720
Fall 2012

Assignment 3: Due Wed 09/26

1. Let (X,τ) be a topological space, and μ be a regular Borel measure on X. We will show that X has a maximal open set of measure 0.

   Proof. Define $V := \{U \in \tau : \mu(U) = 0\}$ and $W := \cup V$. An arbitrary union of open sets is open, so W is open. Clearly if $U \in \tau$ and $\mu(U) = 0$ then $U \subseteq W$ so all that is left to show is that $\mu(W) = 0$. Suppose for contradiction that $\mu(W) > 0$. Since μ is regular and W is open, we may choose $K \subseteq W$ compact with $\mu(K) > 0$. The sets $\{U \in \tau : \mu(U) = 0\}$ cover W so they cover K. Since K is compact we may choose $U_1, \dots, U_n \in \{U \in \tau : \mu(U) = 0\}$ which cover K. Then by subadditivity we have that $\mu(K) \le \sum_{i=1}^n \mu(U_i) = 0$, contradicting the fact that $\mu(K) > 0$. It follows that $\mu(W) = 0$. 5/5 □

2. Let $\Sigma \supseteq \mathcal{B}(\mathbb{R}^d)$, and let μ be a regular measure on $(\mathbb{R}^d, \Sigma)$. Suppose $A \in \Sigma$ is σ-finite. We will show that $\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ is compact}\}$.

   Proof. First we claim that $\mu(B) = \sup\{\mu(K) : K \subseteq B \text{ is compact}\}$ for all $B \in \Sigma$ with $\mu(B) < \infty$. Fix $\varepsilon > 0$. Since μ is regular we choose U open containing B such that $\mu(U) < \mu(B) + \varepsilon$, and we choose V open containing $U \setminus B$ with $\mu(V) < \mu(U \setminus B) + \varepsilon = \mu(U) - \mu(B) + \varepsilon$. Again by regularity of μ and since U is open, we choose K compact contained in U so that $\mu(K) > \mu(U) - \varepsilon$. Then $K \setminus V$ is compact (this is where we would need Hausdorff if we weren't in $\mathbb{R}^d$) and is contained in B. Furthermore,

   $$\mu(K \setminus V) = \mu(K) - \mu(V) > (\mu(U) - \varepsilon) - \mu(V) > (\mu(U) - \varepsilon) + (-\mu(U) + \mu(B) - \varepsilon) = \mu(B) - 2\varepsilon.$$

   Since ε was arbitrary it follows that $\mu(B) = \sup\{\mu(K) : K \subseteq B \text{ is compact}\}$.

   Now we show the result for A which we write as $A = \cup_{i=1}^\infty A_i$ for $A_i \in \Sigma$ and $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$. If $\mu(A) < \infty$ then the previous paragraph gives us the desired result by setting $B := A$. Otherwise $\mu(A) = \infty$. Let $M > 0$ be given, we will show that there is a compact subset $K \subseteq A$ with $\mu(K) > M$. Since $\lim_{n\to\infty} \mu(\cup_{i=1}^n A_i) = \mu(A) = \infty$ there is an n large enough that $\infty > \mu(\cup_{i=1}^n A_i) > M + 1$. We then apply the previous part to find $K \subseteq \cup_{i=1}^n A_i \subseteq A$ with $\mu(K) > M + 1/2 > M$. It follows that $\sup\{\mu(K) : K \subseteq A \text{ is compact}\} > M$ for all $M > 0$ so that $\sup\{\mu(K) : K \subseteq A \text{ is compact}\} = \infty = \mu(A)$. □ 5/5

3. Let μ, ν be two measures on (X, Σ). Suppose $\mathcal{C} \subseteq \Sigma$ is a π-system such that $\mu = \nu$ on $\mathcal{C}$.

   (a) Suppose that $(C_i)_i$ elements of $\mathcal{C}$ are such that $\cup_{i=1}^\infty C_i = X$ and $\mu(C_i) = \nu(C_i) < \infty$. We will show that

   might disagree on $C_1 \cup C_2$

3. Proof: (a) By Dynkin's thm, only need to show that $\mathcal{A} := \{E \in \Sigma \mid \mu(E) = \nu(E)\}$ is a λ-system, since we've known that $\mathcal{E} \subset \mathcal{A}$ is a π-system.

1) if $A_i \in \mathcal{A}$ for all $i \in \mathbb{N}$. $A_i \subset A_{i+1}$

then $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu(\bigcup_{i=1}^{\infty} A_i) \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

2) If $A, B \in \mathcal{A}$ & $A \subset B$, then $(B \setminus A) \cap A = \phi$

So $\mu(B \setminus A) + \mu(A) = \mu(B)$, $\nu(B \setminus A) + \nu(A) = \nu(B)$

$\Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B \setminus A) \Rightarrow B \setminus A \in \mathcal{A}$

3) Since $\exists C_i \in \mathcal{C} \subset \mathcal{A}$ such that $X = \bigcup_{i=1}^{\infty} C_i$ and $\mu(C_i) = \nu(C_i) < \infty$

Let $D_1 = C_1 \subset \mathcal{A}$. $D_2 = C_1 \cup C_2$

Since $C_1, C_2 \in \mathcal{C}$, $C_1 \cap C_2 \in \mathcal{C} \subset \mathcal{A}$.

so $\mu(C_1 \cup C_2) = \mu(C_1) + \mu(C_2) - \mu(C_1 \cap C_2) = \nu(C_1) + \nu(C_2) - \nu(C_1 \cap C_2) = \nu(C_1 \cup C_2)$

$\Rightarrow D_2 \subset \mathcal{A}$ and $\mu(D_2) \le \sum_{i=1}^{2} \mu(C_i) < \infty$

$\forall \bar{C} \in \mathcal{C}$, $C_1 \cap \bar{C} \in \mathcal{C}$, $C_2 \cap \bar{C} \in \mathcal{C}$, so $(C_1 \cup C_2) \cap \bar{C} = (C_1 \cap \bar{C}) \cup (C_2 \cap \bar{C}) \in \mathcal{A}$

Suppose D_n is defined, and $\forall \bar{C} \in \mathcal{C}$, $D_n \cap \bar{C} \in \mathcal{A}$.

Set $D_{n+1} = D_n \cup C_{n+1}$, hence $\mu(D_{n+1}) = \mu(D_n) + \mu(C_{n+1}) - \mu(D_n \cap C_{n+1})$

$= \nu(D_n) + \nu(C_{n+1}) - \nu(D_n \cap C_{n+1}) = \nu(D_{n+1})$

$\Rightarrow D_{n+1} \in \mathcal{A}$ and by same method, $\forall \bar{C} \in \mathcal{C}$, $D_{n+1} \cap \bar{C} \in \mathcal{A}$.

So by mathematical induction, $\{D_n\}$ is well defined.

$D_n \in \mathcal{A}$ and $D_n \subset D_{n+1}$, $\forall n \ge 1$ and $X = \bigcup_{n=1}^{\infty} D_n$

By 1), $X \in \mathcal{A}$

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Therefore, $\mathcal{A}$ is a λ-system, hence $\mathcal{A} \supset \sigma(\mathcal{C})$

(b) A counter example is defined as following:

Let $X = \mathbb{N} = \{1, 2, 3, \cdots\}$. $\Sigma = \mathcal{P}(\mathbb{N})$

$\mu(A) = \sum_{n \in A} 2^n$, $\nu(A) = \sum_{n \in A} 3^n$

Let $\mathcal{C} := \{A \in \mathbb{N} \mid \text{there exists } N \in \mathbb{N} \text{ s.t. } \forall n \ge N, n \in A, \forall n < N, n \notin A\}$

Since all elements of $\mathcal{C}$ have infinitely many elements,

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$\forall A \in \mathcal{C}$, $\mu(A) = \infty = \nu(A)$

However, $\sigma(\mathcal{C}) = \mathcal{P}(\mathbb{N})$, and $\mu(\{1\}) = 2 \ne 3 = \nu(\{1\})$ □

4. Prof: Shall prove by contradiction that there does not exist such set.

Suppose, by contradiction, $E \in \mathcal{L}(\mathbb{R})$ is such set. $\forall a < b \in \mathbb{R}$,

$k(b-a) \le \mu(E \cap (a,b)) \le (1-k)(b-a)$ where $0 < k < \frac{1}{2}$ is a constant.

Fix $\varepsilon > 0$. Consider interval $(0,1)$, there is a sequence of intervals $\{I_n\}_{n=1}^{\infty}$ such that $E \cap (0,1) \subset \bigcup_{n=1}^{\infty} I_n$ and $\mu(E \cap (0,1)) \ge \sum_{n=1}^{\infty} \mu(I_n) - \varepsilon$

Since $\mu(E \cap (0,1)) \le 1$ is finite, $\sum_{n=1}^{\infty} \mu(I_n)$ converges.

So there is $N \in \mathbb{N}$ s.t. $\sum_{n=N+1}^{\infty} \mu(I_n) < \varepsilon$.

Set $J_1 = \left(\bigcup_{n=1}^{N} I_n\right) \cap (0,1)$ $\quad J_2 = (0,1) \setminus J_1$

It is clear that $J_1 \cap J_2 = \phi$, $J_1 \cup J_2 = (0,1)$ and

J_1, J_2 are both finite union of intervals.

Also, since $E \cap (0,1) \subset J_1 \cup \left(\bigcup_{n=N+1}^{\infty} I_n\right)$ and $\mu\left(\bigcup_{n=N+1}^{\infty} I_n\right) \leq \sum_{n=N+1}^{\infty} \mu(I_n) < \varepsilon$

$$\mu(E \cap J_1) \geq \mu\left(E \cap \left(J_1 \cup \left(\bigcup_{n=N+1}^{\infty} I_n\right)\right)\right) - \varepsilon$$

$$\geq \mu(E \cap (E \cap (0,1))) - \varepsilon \geq \mu(E \cap (0,1)) - \varepsilon \qquad (*)$$

$$\geq \sum_{n=1}^{\infty} \mu(I_n) - 2\varepsilon \geq \sum_{n=1}^{N} \mu(I_n) - 2\varepsilon \geq \mu\left(\bigcup_{n=1}^{N} I_n\right) - 2\varepsilon$$

$$\geq \mu(J_1) - 2\varepsilon$$

~~What's more, J_2 is finite union of intervals,~~

~~so $\mu(E \cap J_2) \geq k\mu(J_2)$, which leads to:~~

~~$\mu(E \cap (0,1)) = \mu(E \cap J_1) + \mu(E \cap J_2) \geq \mu(J_1) + k\mu(J_2) - 2\varepsilon$~~

and $(*)$ says: $\mu(J_1) \geq \mu(E \cap J_1) \underset{(*)}{\geq} \mu(E \cap (0,1)) - \varepsilon$

$$\Rightarrow \mu(J_1) + \varepsilon \geq \mu(E \cap (0,1)) \geq \mu(J_1) + k\mu(J_2) - 2\varepsilon$$

$$\Rightarrow k\mu(J_2) \leq 3\varepsilon$$

Let $\varepsilon \to 0$ we have: $\mu(J_2) = 0 \qquad (**)$

OTOH, $\mu(J_1) \leq \sum_{n=1}^{\infty} \mu(I_n) \leq \mu(E \cap (0,1)) + \varepsilon \leq (1-k)\cdot(1-0) + \varepsilon = 1-k+\varepsilon$

Let $\varepsilon \to 0$, $\mu(J_1) \leq 1-k$,

so $\mu(J_2) = 1 - \mu(J_1) \geq k$ that contradicts $(**)$!

5. For $i \in \{1,2\}$ let (X_i, Σ_i, μ_i) be measure spaces with $\mu_i(X_i) < \infty$. Define $\Sigma_1 \otimes \Sigma_2 = \sigma(\{A_1 \times A_2 : A_i \in \Sigma_i\})$.

(a) Fix $x_1 \in X_1, x_2 \in X_2$. Let $A \in \Sigma_1 \otimes \Sigma_2$. Let $S_{x_1}(A) := \{x_2 \in X_2 : (x_1, x_2) \in A\}$ and $T_{x_2} := \{x_1 \in X_1 : (x_1, x_2) \in A\}$. We will show that $S_{x_1}(A) \in \Sigma_2$ and $T_{x_2}(A) \in \Sigma_1$.

Proof. Let

$$S := \{B \subseteq X_1 \times X_2 : S_{x_1}(B) \in \Sigma_2, T_{x_2}(B) \in \Sigma_1\}.$$

We claim that S is a σ-algebra. We have $S_{x_1}(\emptyset) = T_{x_2}(\emptyset) = \emptyset \in \Sigma_1, \Sigma_2$. If $B \in S$ then $S_{x_1}(B^c) = (S_{x_1}(B))^c \in \Sigma_2$ and similarly $T_{x_2}(B^c) = (T_{x_2}(B))^c \in \Sigma_1$. Finally, if $(B_i)_i$ is a sequence in S, then $S_{x_1}(\cup_{i=1}^{\infty} B_i) = \cup_{i=1}^{\infty} S_{x_1}(B_i) \in \Sigma_2$ and similarly $T_{x_2}(\cup_{i=1}^{\infty} B_i) = \cup_{i=1}^{\infty} T_{x_2}(B_i) \in \Sigma_1$. Hence S is a σ-algebra. We claim that $\Sigma_1 \otimes \Sigma_2 \subseteq S$. Let $A_1 \times A_2 \in \{A_1 \times A_2 : A_i \in \Sigma_i\}$ be given. Then $S_{x_1}(A_1 \times A_2) \in \{A_2, \emptyset\} \subseteq \Sigma_2$ and $T_{x_2}(A_1 \times A_2) \in \{A_1, \emptyset\} \subseteq \Sigma_1$. It follows that $\{A_1 \times A_2 : A_i \in \Sigma_i\} \subseteq S$ and since S is a σ-algebra this implies that $S \supseteq \sigma(\{A_1 \times A_2 : A_i \in \Sigma_i\}) = \Sigma_1 \otimes \Sigma_2$ which proves the desired result. □ 5/5

(b) We claim that there is a set $A \in \mathcal{P}(\mathbb{R}^2)$ such that for all $x \in \mathbb{R}$ both $S_x(A)$ and $T_x(A)$ are Borel measurable, but $A \notin \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Proof. Since $\mathcal{P}(\mathbb{R}) \supsetneq \mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$ we may choose $M \subseteq \mathbb{R}$ that is not Borel measurable. Then set $K := M \times \{0\}$. We showed in the previous homework that $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$, so using this fact and part (a) of this problem we deduce that $K \notin \mathcal{B}(\mathbb{R}^2)$, otherwise we would have the y-section $M = T_0(A) \in \mathcal{B}(\mathbb{R})$. Next we choose $L : \mathbb{R}^2 \to \mathbb{R}^2$ a rotation by $\alpha \notin \mathbb{Z}\pi$. We know L is linear, so it is continuous, so it is measurable. It is also invertible with L^{-1} linear, continuous, and measurable. Define $A := L(K)$. It cannot be that $A \in \mathcal{B}(\mathbb{R}^2)$. This is because the inverse image of a Borel set under a measurable function is measurable, in this case our σ-algebra is $\mathcal{B}(\mathbb{R}^2)$ so the inverse image of a Borel set is Borel. That is, if $A \in \mathcal{B}(\mathbb{R}^2)$ then we would have $K = L^{-1}(A) \in \mathcal{B}(\mathbb{R}^2)$ which we already ruled out. Thus $A \notin \mathcal{B}(\mathbb{R}^2)$. However, every x-section or y-section of A is either a singleton or empty, both of which are Borel measurable. It follows that every section of A is Borel measurable, but A is not Borel measurable. □

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(c) We show that there exists a measure ν on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ such that for all $A_i \in \sigma_i$ we have $\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$.

Proof. The goal is to use the Carathéodory Extension theorem that we proved in the previous homework. To start we need an algebra $\mathcal{A}$. Define

$$\mathcal{A} := \left\{ \cup_{i=1}^{k} (A_i \times B_i) : k \in \mathbb{N}, A_i \in \Sigma_1, B_i \in \Sigma_2, i \in \{1 \dots k\} \right\}.$$

We claim that $\mathcal{A}$ is an algebra. Clearly $\emptyset \in \mathcal{A}$. And we note that by definition $\mathcal{A}$ is closed under finite unions. We note the set identities

$$(A \times B)^c = (A^c \times B^c) \cup (A \times B^c) \cup (A^c \times B)$$

where the latter is a disjoint union, and

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

By the second identity it is clear $\mathcal{A}$ is closed under intersections of finitely many elements since the composite σ-algebras are. By the first identity and fact that $\mathcal{A}$ is closed under finite intersections it follows that any element $S \in A$ may be written as a disjoint union since if $(A_i \times B_i) \cap (A_j \times B_j) \neq \emptyset$ we could replace

$$\begin{aligned}(A_i \times B_i) \cap (A_j \times B_j) &= ((A_i \times B_i) \setminus (A_j \times B_j)) \cup (A_j \times B_j) \\ &= ((A_i \times B_i) \cap ((A_j^c \times B_j^c) \cup (A_j \times B_j^c) \cup (A_j^c \times B_j))) \cup (A_j \times B_j) \\ &= ((A_i \cap A_j^c) \times (B_i \cap B_j^c)) \cup ((A_i \cap A_j) \times (B_i \cap B_j)) \cup ((A_k \cap A_j^c) \times (B_i \cap B_j)) \cup (A_j \times B_j)\end{aligned}$$

which is a disjoint union in $\mathcal{A}$. Then we note that if $A \in \mathcal{A}$ and we have the disjoint union

$$A = \bigcup_{i=1}^{n} (A_i \times B_i)$$

then

$$A^c = \left(\bigcup_{i=1}^{n} (A_i \times B_i) \right)^c = \bigcap_{i=1}^{n} (A_i \times B_i)^c$$

which is in $\mathcal{A}$ since $\mathcal{A}$ is closed under finite intersections and unions and since $(A_i \times B_i)^c$ may be expressed as a union of elements of $\mathcal{A}$. Thus we have shown that $\mathcal{A}$ is an algebra, and by definition we have $\mathcal{A} \supseteq \{A_1 \times A_2 : A_i \in \Sigma_i\}$.

We have that all elements of $A \in \mathcal{A}$ may be expressed as a disjoint union

$$A = \bigcup_{i=1}^{n} (A_i \times B_i).$$

Then we must construct a pre-measure on $\mathcal{A}$ with the desired properties. Define

$$\mu_0(A) := \sum_{i=1}^{n} \mu_1(A_i)\mu_2(B_i)$$

when A has the form of disjoint union specified above. First we check that this definition does not depend on the representation of A. Suppose both

$$A = \bigcup_{i=1}^{n} (A_i \times B_i) = \bigcup_{i=1}^{m} (C_i \times D_i)$$

have the specified form, and also suppose none of the A_i, B_i, C_i, D_i are empty, otherwise we throw them out and it doesn't change the unions. We must have $\cup_{i=1}^n A_i = \cup_{i=1}^m C_i$ and $\cup_{i=1}^n B_i = \cup_{i=1}^m D_i$ otherwise the product unions couldn't be equal. Let $L = \{L_1, \dots, L_k\}$ be a common (disjoint) partition of the A_i's and C_i's, and let and $R = \{R_1, \dots, R_\ell\}$ be a common (disjoint) partition of the B_i's and D_i's. These partitions exist and are finite since they may be made intersecting the intersection of finitely many A_i's [B_i's] with the intersection of finitely many C_i's [D_i's], and there are only finitely many (certainly bounded by $2^n \cdot 2^m$) such intersections. Then form the set P of things of the form $L_i \times R_j$ which are subsets of some $A_p \times B_q$, i.e. P is a disjoint partition of A into measurable rectangles. But then by disjointness of the partition and addivitity of μ_1, μ_2 we have

$$\sum_{i=1}^n \mu_1(A_i)\mu_2(B_i) = \sum_{L_i \times R_j \in P} \mu_1(L_i)\mu_2(R_j) = \sum_{i=1}^m \mu_1(C_i)\mu_2(D_i)$$

so we find that μ_0 does not depend on the representation of A. It is clear by definition that μ_0 is finitely additive on disjoint sets.

Finally we must show that if $(S_i)_i$ is a disjoint sequence in $\mathcal{A}$ for which $\cup_{i=1}^\infty S_i \in \mathcal{A}$ then μ_0 is countably additive for this sequence. Let such a sequence $(S_i)_i$ be given. Since $\cup_{i=1}^\infty S_i \in \mathcal{A}$ may be expressed as a disjoint union of $A_i \times B_i$ as above and by finite additivity of μ_0 over disjoint sets we have that it suffices to consider the case that $\cup_{i=1}^\infty S_i = A \times B$ for some $A \in \Sigma_1, B \in \Sigma_2$. For the same reason, that μ_1, μ_2 are additive on disjoint sets and linearity of infinite series, we may suppose that each of the S_i has the form $A_i \times B_i$ for $A_i \in \Sigma_1, B_i \in \Sigma_2$.

Note. This next paragraph gives a very slick proof of the countable addativity of μ_0, but it requires definitions of nonnegative integrals and the monotone convergence theorem, it is adapted from Terence Tao's measure theory notes. My much longer and more tedious approach follows immediately afterwards. We know

$$\chi_A(x)\chi_B(y) = \sum_{i=1}^\infty \chi_{A_i}(x)\chi_{B_i}(y)$$

by disjointness of the $A_i \times B_i$ (at most one of the terms on the right side is nonzero). Integrating with respect to μ_1 then μ_2 then applying the monotone convergence theorem twice to exchange the sum with the integrals we find

$$\mu_1(A)\mu_2(B) = \int_{X_2}\int_{X_1} \sum_{i=1}^\infty \chi_{A_i}(x)\chi_{B_i}(y)\,d\mu_1(x)\,d\mu_2(y)$$

$$= \sum_{i=1}^\infty \int_{X_2}\int_{X_1} \chi_{A_i}(x)\chi_{B_i}(y)\,d\mu_1(x)\,d\mu_2(y) = \sum_{i=1}^\infty \mu_1(A_i)\mu_2(B_i)$$

which shows that μ_0 is countably additive for this sequence $(S_i)_i$.

Now for the tedious approach that only uses elementary analysis. Our aim is to show that

$$\mu_1(A)\mu_2(B) = \sum_{i=1}^\infty \mu_1(A_i)\mu_2(B_i).$$

If either $\mu_1(A) = 0$ or $\mu_2(A) = 0$ then both sides are clearly 0, so we exclude this case. Let $\mathcal{X}_n$ be a common partition ? of $(A_i)_{i=1}^n$, and let $\mathcal{Y}_n$ be a common partition of $(B_i)_{i=1}^n$. Then let $\mathcal{P}_n$ be a common partition of $(A_i \times B_i)_{i=1}^n$. We may choose these partitions to be increasing in granularity. We note that it is not necessarily the case that $\mathcal{P}_n = \{A' \times B' : A' \in A_n, B' \in B_n\}$ but it is the case that $P_n \subseteq \{A' \times B' : A' \in A_n, B' \in B_n\}$. Then we have

$$\sum_{i=1}^{\infty} \mu_1(A_i)\mu_2(B_i) = \lim_{N\to\infty} \sum_{i=1}^{N} \mu_1(A_i)\mu_2(B_i) = \lim_{N\to\infty} \sum_{i=1}^{N} \sum_{x\times y\in\mathcal{P}_i} \mu_1(x\cap A_i)\mu_2(y\cap B_i).$$

Then since we chose the partitions to be increasing, by finite additivity of μ_1, μ_2 and disjointness of partition elements we have that this equals

$$\lim_{N\to\infty} \sum_{i=1}^{N} \sum_{x\times y\in\mathcal{P}_N} \mu_1(x\cap A_i)\mu_2(y\cap B_i)$$

$$= \lim_{N\to\infty} \sum_{x\times y\in\mathcal{P}_N} \sum_{i=1}^{N} \mu_1(x\cap A_i)\mu_2(y\cap B_i).$$

Now we use the fact that A_i and B_i were used in constructing $\mathcal{P}_N$, so that the inner sum will be nonzero exactly when $x \times y$ is a partition element contained in $A_i \times B_i$. But the A_i and B_i cover $\mathcal{X}_N$ and $\mathcal{Y}_N$ so the inner sum will be exactly $\mu_1(x)\mu_2(y)$. Hence this equals

$$\lim_{N\to\infty} \sum_{x\times y\in\mathcal{P}_N} \mu_1(x)\mu_2(y)$$

$$= \lim_{N\to\infty} \sum_{x\in\mathcal{X}_N} \sum_{\substack{y\in\mathcal{Y}_N,\\ x\times y\in\mathcal{P}_N}} \mu_1(x)\mu_2(y)$$

$$= \lim_{N\to\infty} \sum_{x\in\mathcal{X}_N} \mu_1(x) \sum_{\substack{y\in\mathcal{Y}_N,\\ x\times y\in\mathcal{P}_N}} \mu_2(y).$$

From here we will show $\leq$ and $\geq$ separately.

i. $\leq$. Since we chose the $\mathcal{P}_i$'s to be increasing in granularity, we have that

$$\lim_{N\to\infty} \sum_{x\in\mathcal{X}_N} \mu_1(x) \sum_{\substack{y\in\mathcal{Y}_N,\\ x\times y\in\mathcal{P}_N}} \mu_2(y)$$

$$\leq \lim_{N\to\infty} \sum_{x\in\mathcal{X}_N} \mu_1(x) \sum_{y\in\mathcal{Y}_N} \mu_2(y) = \lim_{N\to\infty} \sum_{x\in\mathcal{X}_N} \mu_1(x)\mu_2(\cup\mathcal{Y}_N)$$

$$= \lim_{N\to\infty} \mu_2(\cup\mathcal{Y}_N) \sum_{x\in\mathcal{X}_N} \mu_1(x) = \lim_{N\to\infty} \mu_2(\cup\mathcal{Y}_N)\mu_1(\cup\mathcal{X}_N)$$

$$= \lim_{N\to\infty} \mu_2(\cup\mathcal{Y}_N) \cdot \lim_{N\to\infty} \mu_1(\cup\mathcal{X}_N)$$

$$= \lim_{N\to\infty} \mu_2(\cup_{i=1}^{N} B_i) \cdot \lim_{N\to\infty} \mu_1(\cup_{i=1}^{N} A_i)$$

$$= \mu_2(B)\mu_1(A)$$

where in the last equality we used the fact that the limit of the measure of an increasing union is the measure of the infinite union.

ii. $\geq$. Fix $k > 0$. For $N > k$, since we picked the partitions increasing in granularity, if we look at $\{x \times y : y \in \mathcal{Y}_N, x \in \mathcal{X}_k\}$ and compare it to $\{x \times y : y \in \mathcal{Y}_N, x \in \mathcal{X}_N\}$ then we find that the latter is a finer partition than the former. Accordingly, we may define $\mathcal{P}'_{N,x}$ to be those elements of $\mathcal{P}_N$ such that x can be made as a (disjoint) union of left coordinates in $\mathcal{P}_N$. Then

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{x \in \mathcal{X}_N} \mu_1(x) \sum_{\substack{y \in \mathcal{Y}_N, \\ x \times y \in \mathcal{P}_N}} \mu_2(y) \\
&\geq \lim_{N\to\infty} \sum_{x \in \mathcal{X}_k} \mu_1(x) \sum_{\substack{y \in \mathcal{Y}_N, \\ x \times y \in \mathcal{P}'_{N,x}}} \mu_2(y) \\
&= \sum_{x \in \mathcal{X}_k} \mu_1(x) \lim_{N\to\infty} \sum_{\substack{y \in \mathcal{Y}_N, \\ x \times y \in \mathcal{P}'_{N,x}}} \mu_2(y).
\end{aligned}$$

Since $\cup \mathcal{Y}_N = \cup_{i=1}^N B_i$ and $\cup_{i=1}^\infty B_i = B$ we claim that $\cup_{i=1}^\infty(\cup \{y \in \mathcal{Y}_i : x \times y \in \mathcal{P}'_{i,x}\}) = B$. We have $\subseteq$ for free since $\mathcal{Y}_i$ is a partition created from $\cup_{i=k}^N B_k \subseteq B$, but suppose for contradiction that we do not have $\supseteq$. Then there would be $b \in B$ which is not covered over this slice of x. But $A \times B$ is a rectangle, so this would force it to not be covered for any slice of x, which would contradict the fact that the $(A_i \times B_i)_i$ cover $A \times B$. It follows that $\cup_{i=1}^\infty(\cup \{y \in \mathcal{Y}_i : x \times y \in \mathcal{P}'_{i,x}\}) = B$ and so

$$\begin{aligned}
&\sum_{x \in \mathcal{X}_k} \mu_1(x) \lim_{N\to\infty} \sum_{\substack{y \in \mathcal{Y}_N, \\ x \times y \in \mathcal{P}'_{N,x}}} \mu_2(y) \\
&= \sum_{x \in \mathcal{X}_k} \mu_1(x)\mu_2(B) \\
&= \mu_2(B) \sum_{x \in \mathcal{X}_k} \mu_1(x).
\end{aligned}$$

Hence we have shown that for all $k > 0$

$$\lim_{N\to\infty} \sum_{x \in \mathcal{X}_N} \mu_1(x) \sum_{\substack{y \in \mathcal{Y}_N, \\ x \times y \in \mathcal{P}_N}} \mu_2(y) \geq \mu_2(B) \sum_{x \in \mathcal{X}_k} \mu_1(x).$$

Taking the limit at $k \to \infty$ then gives

$$\lim_{N\to\infty} \sum_{x \in \mathcal{X}_N} \mu_1(x) \sum_{\substack{y \in \mathcal{Y}_N, \\ x \times y \in \mathcal{P}_N}} \mu_2(y) \geq \mu_2(B)\mu_1(A).$$

Thus we have shown that

$$\mu_1(A)\mu_2(B) = \sum^{\infty} \mu_1(A_i)\mu_2(B_i)$$

6. First, we note that the intersection of arbitrarily many monotone classes is a monotone class. Let a family $\{\mathcal{M}_\alpha : \alpha \in I\}$ of monotone classes be given, and define $\mathcal{M} := \bigcap_{\alpha \in I} \mathcal{M}_\alpha$.

(i) Let $A_i \in \mathcal{M}$ with $A_i \subseteq A_{i+1}$ be given. Then for every $\alpha \in I$, we have $\bigcup_1^\infty A_i \in \mathcal{M}_\alpha$. Hence $\bigcup_1^\infty A_i \in \mathcal{M}$.

(ii) Let $B_i \in \mathcal{M}$ with $B_i \supseteq B_{i+1}$ be given. Then for every $\alpha \in I$, we have $\bigcap_1^\infty B_i \in \mathcal{M}_\alpha$. Hence $\bigcup_1^\infty B_i \in \mathcal{M}$.

There is at least one monotone class in X containing $\mathcal{A}$, namely $\mathcal{P}(X)$, so we can then speak of the smallest monotone class $\bigcap \{\mathcal{M} \in \mathcal{P}(X) : \mathcal{M} \subseteq \mathcal{A} \text{ a monotone class}\}$ containing $\mathcal{A}$.

Choose $\mathcal{M}$ to be the smallest monotone class containing $\mathcal{A}$. We show that $\mathcal{M}$ is a σ-algebra. Define $\mathcal{M}' = \{A \in \mathcal{M} : A^c \in \mathcal{M}\}$. Then

(i) If $A_1, A_2, \ldots$ in $\mathcal{M}'$ with $A_i \subseteq A_{i+1}$, then $A_1^c, A_2^c, \ldots$ in $\mathcal{M}$ with $A_i^c \supseteq A_{i+1}^c$, so

$$\left(\bigcup_1^\infty A_i\right)^c = \bigcap_1^\infty A_i^c \in \mathcal{M}$$

since $\mathcal{M}$ is a monotone class. Hence $\bigcup_1^\infty A_i \in \mathcal{M}'$.

(ii) If $B_1, B_2, \ldots$ in $\mathcal{M}'$ with $B_i \supseteq B_{i+1}$, then $B_1^c, B_2^c, \ldots$ in $\mathcal{M}$ with $B_i^c \subseteq B_{i+1}^c$, so

$$\left(\bigcap_1^\infty B_i\right)^c = \bigcup_1^\infty B_i^c \in \mathcal{M}$$

since $\mathcal{M}$ is a monotone class. Hence $\bigcap_1^\infty B_i \in \mathcal{M}'$.

Thus $\mathcal{M}'$ is a monotone class. $\mathcal{A}$ is an algebra and therefore closed under complements, so $\mathcal{A} \subseteq \mathcal{M}'$. Since $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$, it follows that $\mathcal{M} \subseteq \mathcal{M}'$. Hence $\mathcal{M}$ is closed under complements.

Define $\mathcal{M}'' := \{A \in \mathcal{M} : \forall B \in \mathcal{A}, A \cup B \in \mathcal{M}\}$. Then

(i) Let $A_1, A_2, \ldots \in \mathcal{M}''$ with $A_i \subseteq A_{i+1}$ be given, and let $B \in \mathcal{A}$. Since $A_i \in \mathcal{M}''$, we have $A_i \cap B \in \mathcal{M}$. Furthermore, $A_i \cap B \subseteq A_{i+1} \cap B$. Then

$$\left(\bigcup_1^\infty A_i\right) \cap B = \bigcup_1^\infty (A_i \cap B) \ \in \mathcal{M}$$

since $\mathcal{M}$ is a monotone class. Hence $\bigcup_1^\infty A_i \subseteq \mathcal{M}''$.

(ii) Let $A_1, A_2, \ldots \in \mathcal{M}''$ with $A_i \supseteq A_{i+1}$ be given, and let $B \in \mathcal{A}$. Since $A_i \in \mathcal{M}''$, we have $A_i \cap B \in \mathcal{M}$. Furthermore, $A_i \cap B \supseteq A_{i+1} \cap B$. Then

$$\left(\bigcap_1^\infty A_i\right) \cap B = \bigcap_1^\infty (A_i \cap B) \ \in \mathcal{M}$$

since $\mathcal{M}$ is a monotone class. Hence $\bigcap_1^\infty A_i \subseteq \mathcal{M}''$.

Thus $\mathcal{M}''$ is a monotone class. We have $\mathcal{A} \subseteq \mathcal{M}''$, since $\mathcal{A}$ is an algebra. Hence $\mathcal{M}'' \supseteq \mathcal{M}$.

Now, define $\mathcal{M}''' := \{B \in \mathcal{M} : \forall A \in \mathcal{M}, A \cap B \in \mathcal{M}\}$. Then

(i) Let $B_1, B_2, \ldots \in \mathcal{M}$ be given with $B_i \subseteq B_{i+1}$, and let $A \in \mathcal{M}$. Then $A \cap B_i \in \mathcal{M}$. We have $A \cap B_i \subseteq A \cap B_{i+1}$, so

$$A \cap \left(\bigcup_1^\infty B_i\right) = \bigcup_1^\infty (A \cap B_i) \ \in \mathcal{M}$$

since $\mathcal{M}$ is a monotone class. Hence $\bigcup_1^\infty B_i \in \mathcal{M}'''$.

(ii) Let $B_1, B_2, \ldots \in \mathcal{M}$ be given with $B_i \supseteq B_{i+1}$, and let $A \in \mathcal{M}$. Then $A \cap B_i \in \mathcal{M}$. We have $A \cap B_i \supseteq A \cap B_{i+1}$, so

$$A \cap \left(\bigcap_1^\infty B_i\right) = \bigcap_1^\infty (A \cap B_i) \ \in \mathcal{M}$$

since $\mathcal{M}$ is a monotone class. Hence $\bigcap_1^\infty B_i \in \mathcal{M}'''$.

Thus $\mathcal{M}'''$ is a monotone class. Since $\mathcal{M}'' \supseteq \mathcal{M}$, we have $A \cap B \in \mathcal{M}$ for all $A \in \mathcal{M}$ and $B \in \mathcal{A}$, so $\mathcal{A} \subseteq \mathcal{M}''$. Thus $\mathcal{M}'' \supseteq \mathcal{M}$. We conclude that $\mathcal{M}$ is closed under unions. Since $\mathcal{M}$ is also closed under complements, $\mathcal{M}$ is an algebra.

Finally, we show that $\mathcal{M}$ is a σ-algebra. We already know that $\mathcal{M}$ is closed under complements; it remains to show that it is closed under countable unions.

Let $A_1, A_2, \ldots \in \mathcal{M}$ be given. Define $B_i = \bigcup_1^i A_i$. Then $B_i \in \mathcal{M}$ because $\mathcal{M}$ is an algebra. Furthermore, we have $B_i \subseteq B_{i+1}$. Then since $\mathcal{M}$ is a monotone class, we have $\bigcup_1^\infty A_i = \bigcup_1^\infty B_i \in \mathcal{M}$.

Thus $\mathcal{M}$ is a σ-algebra. Since $\mathcal{M}$ contains $\mathcal{A}$, we must have $\sigma(\mathcal{A}) \subseteq \mathcal{M}$. Now, we note that any σ-algebra is a monotone class, since it is closed under any countable unions and intersections. Since $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$, it must also be that $\mathcal{M} \subseteq \sigma(\mathcal{A})$. Hence $\mathcal{M} = \sigma(\mathcal{A})$.

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