21-484 Notes

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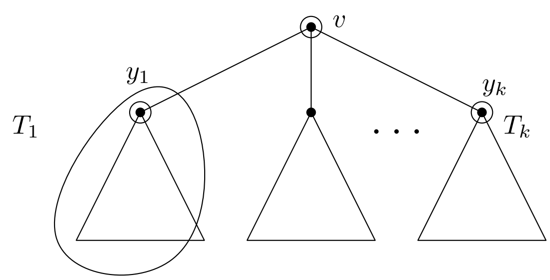
Recall: - Got the following GF for the number of rooted trees with n vertices.

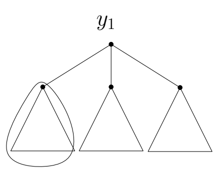
$$A(z) = \frac{z}{(1-z)^{a_1}(1-z^2)^{a_2}(1-z^3)^{a_3}\dots}$$

- Defined the weight of a vertex in a tree

- Defined a centroid of a tree.

→ Notation: T is a tree, $v \in T$ a vertex, $T_1, \dots, T_k$ are the subtrees at v, $s_1, \dots, s_k$ are the sizes of $T_1, \dots, T_k$ respectively, $y_1, \dots, y_k$ the roots of $T_1, \dots, T_k$ (the neighbor of v in T_i is y_i)





- If there is a centroid in T_1 then

$$s_1 > s_2 + s_3 + \dots + s_k$$

→ concluded: There are at worst 2 centroids in a tree. If there are 2, they are adjacent.

Claim: If $s_1 > s_2 + \dots + s_k$ then T_1 contains a centroid.

Proof: $\text{weight}(y_1) \leq \max(1 + s_2 + s_3 + \dots + s_k, s_1 - 1) \leq s_1 = \text{weight}(v)$

- The weight of all vertices in $T_2, \dots, T_k$ is at least $s_1 + 1$, hence they are not centroids.

→ T_1 must contain a centroid. (If v is a centroid, then y_1 is also a centroid, otherwise there is no centroid outside of T_1)

Corollary: v is the only centroid of T iff $\forall 1 < j \leq k = \deg(v)$

$$(*)\ s_j \leq s_1 + s_2 + \dots + s_k - s_j$$

→ If $s_j > s_1 + \dots + s_k - s_j$ for some j, then T_j contains a centroid.

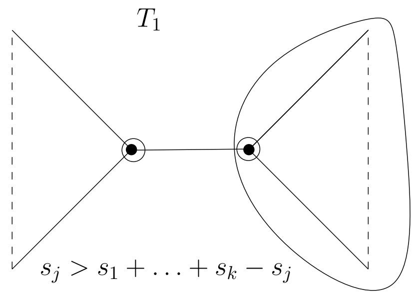
→ If $\forall 1 < j \leq k$, $s_j \leq s_1 + \dots + s_k - s_j$, then no T_j contains a centroid.

→ Strategy of counting unlabeled trees.

→ start by counting only trees with one centroid.

→ a_n is the number of rooted trees with n vertices.

→ In how many ways can we construct a rooted tree violating $\circledast$?



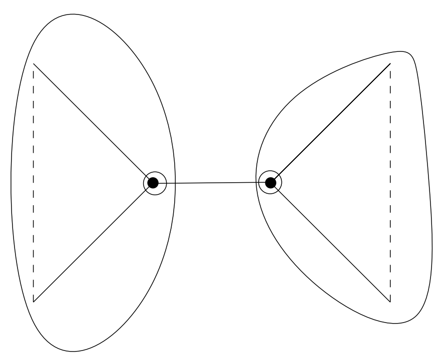
$$a_{n-1}a_1 + a_{n-2}a_2 + a_{n-3}a_3 + \ldots + a_{\lceil n/2 \rceil} a_{\lfloor \frac{n}{2} \rfloor}$$

→ the number of trees with one centroid is

$$a_n - a_{n-1}a_1 - a_{n-2}a_2 - a_{n-3}a_3 - \ldots - a_{\lceil n/2 \rceil} a_{\lfloor \frac{n}{2} \rfloor}.$$

→ move to counting trees with 2 centroids

→ If a tree has 2 centroids it must look like



→ the number of such trees is $\binom{a_{n/2}+1}{2}$ = the number of ways to choose two elements from $[a_{n/2}]$ with repetition

→ Some (easy) generating function manipulation shows

$$F(z) = A(z) - \frac{1}{2}A(z)^2 + \frac{1}{2}A(z^2) = 2 + z^2 + z^3 + 2z^4 + 3z^5 + 6z^6 + 11z^7 + \ldots$$

→ If $t(n)$ is the number of unlabeled trees with n vertices then

$$t(n) \sim c \cdot \alpha^n n^{-5/2} \text{ where } c = 0.53\ldots \text{ and } \alpha \text{ the same } \alpha \text{ for } A(z).$$

Feb 24

Mar 30

Apr 27