MATH 651: PROBLEM SET 7 SOLUTIONS ARE IN CLASS ON FRIDAY, DEC. 7.

1. (5 points) Given metric spaces X and Y let

$$C_b(X,Y) = \{f \in C(X,Y) \;:\; \sup_{x,z\in X} d_Y(f(x), f(z)) < \infty\}.$$

For $f, g \in C_b(X,Y)$ we consider

$$d_\infty(f,g) = \sup_{x\in X} d_Y(f(x), g(x)).$$

It is straightforward to check that $(C_b(X,Y), d_\infty)$ is a metric space. Show that the metric space $(C_b(X;Y), d_\infty)$ is complete if and only if (Y, d_Y) is complete.

2. (10 points) Show that every locally compact Hausdorff space is a Baire space.

3. (10 points) Consider the metric space $C([0,1])$ with the sup metric and for every $n \in \mathbb{N}$ let

$$X_n := \{f \in C([0,1]) : \text{ there is } x \in [0,1] \text{ such that } |f(x) - f(y)| \le n|x-y| \text{ for all } y \in [0,1]\}.$$

   (a) Fix $n \in \mathbb{N}$ and prove that each $f \in C([0,1])$ can be approximated by a zigzag function $g \in C([0,1])$ with very large slopes so that it does not belong to X_n and such that $d_\infty(f,g)$ is small.

   (b) Fix $n \in \mathbb{N}$ and prove that every open set $U \subset C([0,1])$ contains an open set that does not intersect X_n.

   (c) Prove that there exists a dense G_δ set in $C([0,1])$ that consists of nowhere differentiable functions.

4. (10 points) Let (X,d) be a metric space, let $E \subset X$ be a nonempty precompact set, and let $f : E \to \mathbb{R}$. Assume that there exist $L > 0$ and $\alpha \in (0,1]$ such that

$$|f(x) - f(y)| \le L(d(x,y))^\alpha$$

for all $x, y \in E$. Define

$$h(x) := \inf\{f(y) + L(d(x,y))^\alpha : y \in E\}, \quad x \in X.$$

   (a) Prove that $h(x) = f(x)$ for every $x \in E$.

   (b) Prove that

$$\inf_{x\in X} h(x) = \inf_{y\in E} f(y).$$

   (c) Prove that

$$|h(x) - h(y)| \le L(d(x,y))^\alpha$$

   for all $x, y \in X$.

5. (5 points) Show that the system of equations:

$$\frac{x^4}{3}\sin(xy) - \frac{7x-y}{8} = 0$$
$$\frac{y^2\cos y}{8} - y + \frac{e^x}{24} = 0$$

has a solution.

6. (10 points) Let $K \in C([-1,1],\mathbb{R})$. Consider the set of continuous functions $X = C([0,1],\mathbb{R})$. Given $f \in X$ let $Tf : [0,1] \to \mathbb{R}$ be defined by

$$Tf(x) := \int_0^1 K(x-y)f(y)dy.$$

Let

$$\mathcal{F} := \{Tf \ : \ f \in X \text{ and } \max_{x\in[0,1]} |f(x)| \le 1\}$$

Show that every sequence in $\mathcal{F}$ has a uniformly convergent subsequence.

7. (10 points) Let $f_0(x) = x^2$ on $[0,1]$. Let us define functions $f_n : [0,1] \to \mathbb{R}$ recursively :

$$f_{n+1}(x) = \int_0^x (f_n(s))^{1/3}\, ds \quad \text{for } n = 0,1,\ldots.$$

Show that for all n, $f_n \in C([0,1],[0,\infty))$. Then show that the sequence of function $\{f_n\}_{n=1,2,\ldots}$ has a uniformly convergent subsequence.