Assignment 1
Due on Wednesday, January 29

1. This exercise concerns the brachistochrone problem. As in lecture the y-axis is oriented downward and we take $P = (0,0)$. For simplicity we take $Q = (1,1)$. Recall that the relevant functional J is given by

$$J(y) = \int_0^1 \frac{\sqrt{1+y'(x)^2}}{\sqrt{y(x)}}\,dx.$$

Calculate (or approximate as best you can) $J(y)$ for each of the following curves joining $(0,0)$ to $(1,1)$. In particular, try to order the curves from smallest transit to largest transit time. Try to give error estimates for any approximations you make.

(a) (Line Segment) $y = x, 0 \le x \le 1$.

(b) (Quarter Circle) $y = \sqrt{1-(x-1)^2}$, $0 \le x \le 1$.

(c) (Parabola) $y = \sqrt{x}$. $0 \le x \le 1$.

(d) (Cycloid)
$$\begin{aligned} x &= \frac{c^2}{2}(\theta - \sin\theta) \\ & \qquad\qquad\qquad 0 \le \theta \le \theta_1, \\ y &= \frac{c^2}{2}(1-\cos\theta) \end{aligned}$$
where $c > 0$ and $\theta_1 \in (0, 2\pi)$ are chosen so that $x = y = 1$ when $\theta = \theta_1$.

I would like to see "rigorous" inequalities (if possible) relating the integrals that you cannot compute explicitly.

2. Let $a, b, A, B \in \mathbb{R}$ with $a < b$ be given. Let $\mathcal{Y} = \{y \in C^1[a,b] : y(a) = A,\ y(b) = B\}$ and define $J : \mathcal{Y} \to \mathbb{R}$ by

$$J(y) = \int_1^b \sqrt{1+y'(x)^2}\,dx \quad \text{for all } y \in \mathcal{Y}.$$

Prove that $J(y) \ge \sqrt{(b-a)^2 + (B-A)^2}$ for all $y \in \mathcal{Y}$.

3. This exercise concerns a special case of the minimal surface problem. Let $b, B > 0$ be given and put

$$\mathcal{Y} = \{y \in C^1[0,b] : y(0) = 0,\ y(b) = B,\ y(x) \ge 0 \text{ for all } x \in [0,b]\}.$$

Define $J : \mathcal{Y} \to \mathbb{R}$ by

$$J(y) = 2\pi \int_0^b y(x)\sqrt{1+y'(x)^2}\,dx \quad \text{for all } y \in \mathcal{Y}.$$

(a) Prove that $J(y) \geq \pi B^2$ for all $y \in \mathcal{Y}$ and give a geometric interpretation of this inequality.

(b) Does there exist a function $y_* \in \mathcal{Y}$ such that $J(y_*) = \pi B^2$?

(c) Do you think that there is a sequence $\{y_n\}_{n=1}^{\infty}$ of functions in $\mathcal{Y}$ such that $J(y_n) \to \pi B^2$ as $n \to \infty$? Give a brief explanation.