21-630 Final Exam Reference Sheet

Sunday, April 30, 2013

Existence of Solutions

Contraction Mapping Theorem: (p. 14)
$B \subset \mathbb{R}^N$ be closed, $\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_B$ a contraction. Then, $\mathcal{F}$ has a unique fixed point in $\mathcal{C}_B$.
Cauchy-Lipschitz Theorem (Part I): (p. 21)
$\varepsilon > 0$, f continuous, Lipschitz in x on $[t_0, t_0 + \delta_0] \times [x_0 - \varepsilon, x_0 + \varepsilon]$. Then, $\dot{x} = f(t, X), X(t_0) = x_0$ has a solution on $[t_0, t_0 + \delta_0]$.
Cauchy-Peano Theorem: (p. 22)
f continuous $[t_0, t_0 + \delta_0] \times [x_0 - \varepsilon, x_0 + \varepsilon]$. Then, $\exists \delta \in (0, \delta_0]$ so that $\dot{x} = f(t, X), X(t_0) = x_0$ has a solution on $[t_0, t_0 + \delta]$.
Ascola-Arzela Theorem: (p. 25)
$\{X^{(n)}\}$ pointwise-bounded, equicontinuous in $\mathcal{C}_{\mathbb{R}^N}[t_0, t_1]$. Then, $\{X^{(n)}\}$ uniformly bounded with a uniformly convergent subsequence.
Extension Theorem: (p. 35)
$D = I_t \times D_x$, I_t open interval, $D_x \subset \mathbb{R}^N$ open, $f : D \to \mathbb{R}^N$ continuous. Then, any solution of $\dot{X} = f(t, X), X(t_0) = x_0$ has a right-maximal extension, and, for any compact $S \subset D_x$, right-maximal solutions eventually leave S. Note, if f bounded, every solution can be extended to $\mathbb{R}$.

Uniqueness

Gronwall's Inequality (Simple Version): (p. 39)
$A \in \mathbb{R}, B \geq 0$, X continuous on $I = [t_0, t]$ or $I = [t_0, \infty)$,

$$X(t) \leq A + B \int_{t_0}^{t} X(s)\, ds, \quad \forall t \in I.$$

Then, $X(t) \leq Ae^{B(t-t_0)}, \forall t \in I$.
Gronwall's Inequality (Full Version): (p. 40)
$a, b \in \mathcal{C}_{\mathbb{R}}(I), b \geq 0$, X continuous on $I = [t_0, t]$ or $I = [t_0, \infty)$,

$$X(t) \leq a(t) + \int_{t_0}^{t} b(s)X(s)\, ds, \quad \forall t \in I.$$

Then, $\forall t \in I$,

$$X(t) \leq a(t) + \int_{t_0}^{t} a(s)b(s)e^{\int_s^t b(\tau)\, d\tau}\, ds.$$

Cauchy-Lipschitz Theorem (Part II): (p. 44)
Assumptions as in Cauchy-Lipschitz Theorem (Part I). $\exists \delta \in (0, \delta]$ solution unique on $[t_0, t_0 + \delta]$.

Smoothness in Initial Conditions

Continuity in Initial Conditions (p. 48)
C^1 **in Initial Conditions** (p. 51)

Linear Systems

Abel-Liouville Theorem: (p. 60)
If $\psi(t)$ is a matrix solution of $\dot{X} = A(t)X$, then $\det(\psi(t)) = \det(\psi(t_0))e^{\int_{t_0}^{t} \mathrm{tr}(A(s))\, ds}, \forall t_0, t \in I$.
Corollary 4.1 (p. 61) restates the definition of 'fundamental matrices' in terms of $\det(\psi(t))$.
Variation of Parameters: (p. 63)
If $\Phi(t)$ is a fundamental matrix solution of $\dot{X} = A(t)X$, then

$$X(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)\, ds$$

solves $\dot{X} = A(t)X + b(t), X(t_0) = x_0$. Note: if $A(t)$ constant, then $\Phi(t)\Phi^{-1}(s) = \Phi(t-s)$.
Matrix Norms and Exponentials and Jordan Canonical Form: (pp. 69-76, 83)
$|A^k| \leq |A|^k$, $|A|_\infty \leq |A| \leq N|A|_\infty$, etc.
Bounding Solutions by Spectral Radius: (p. 85)
If σ is the largest eigenvalue (by real part) of A, then $\exists C > 0$ such that $|e^{At}| \leq Ce^{\sigma t}(1 + t^{N-1})$.
As a corollary, $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ such that $|e^{At}| \leq C_\varepsilon e^{(\sigma+\varepsilon)t}$.

Stability

Homogeneous Linear Systems (p. 89)
The critical point 0 is stable for $\dot{X} = AX$ if and only if all first order eigenvalues are non-positive and higher order eigenvalues are strictly negative (in real part).
Also, 0 is asymptotically stable if and only if all eigenvalues have strictly negative real part.
Linearization (pp. 93, 97)
$f(x) = 0$, all eigenvalues of $Df(x)$ have strictly negative real part. Then x asymptotically stable.
$f(x) = 0$, some eigenvalue of $Df(x)$ has strictly positive real part. Then x is unstable.
Lyapunov Functions (pp. 108, 111, 114)
f cont., $f(t,0) = 0$, $v(t,x) \in C^1$ positive definite, $D_* v$ negative semidefinite. Then, 0 is stable.
f cont., $f(t,0) = 0$, $v(t,x) \in C^1$ positive definite, $D_* v$ negative definite, v bounded above in t near 0. Then, 0 asymptotically stable.
f cont., $f(t,0) = 0$, $v(t,x) \in C^1$, $D_* v$ negative definite, v bounded below in t near 0. Every nbhd of 0 has x with $v(0,x) > 0$. Then, 0 unstable.

Invariance Theory

Omega Limit Sets (pp. 124-125)
$\Omega(x_0)$ is positively invariant and closed. If $C^+(x_0)$ is bounded, then $\Omega(x_0)$ is nonempty, compact, and connected, and

$$\operatorname{dist}(Y(t,x_0),\Omega(x_0)) \to 0 \text{ as } t \to \infty.$$

For $S \subseteq \mathbb{R}^N$, there exists a largest positively invariant subset M_S of S.
Theorem 5.8 (p. 132)
$0 \in S \supseteq \mathbb{R}^N$ open, $w \in C^1(S)$, $w(0) = 0$, $D_* w \leq 0$ on S, $\eta \geq 0$, $0 \in H_\eta$ closed bounded connected component of $w^{-1}((-\infty,\eta])$, M largest positvely invariant subset of $H_\eta \cap (D_* w)^{-1}(\{0\})$. Then, $\forall x_0 \in H_\eta$, $\operatorname{dist}(Y(t,x_0), M) \to 0$ as $t \to \infty$. Usually, to use this, we want $M = \{0\}$.

Two-Dimensional Systems

For autonomous planar systems, $\dot{r} = \dot{X}\cos\theta + \dot{Y}\sin\theta$ and $\dot{\theta} = \dot{Y}\dfrac{\cos\theta}{r} - \dot{X}\dfrac{\sin\theta}{r}$.
Poincaré-Bendixson Theorem (p. 142)
$f : \mathbb{R}^2 \to \mathbb{R}^2$ C^1, X bounded solution of $\dot{X} = f(X)$, $\Omega(X(0))$ contains no critical point. Then, either X is periodic with $\Omega(X(0)) = C^+(X(0))$ or $\exists$ periodic solution Y with $\Omega(X(0)) = C^+(Y(0))$.
Jordan Curve Theorem (p. 146)
$C \subseteq \mathbb{R}^2$, ψ bijective, cont. mapping from unit circle into C. Then, we can partition $\mathbb{R}^2$ into 3 pathwise connected components, C, O_E, and O_I.
If C is the image of a periodic solution, then O_I contains a critical point.
Transversals (pp. 147-153)
Corollary of Divergence Theorem (p. 160)
Y nonconstant periodic solution. Then,

$$\iint_{O_I} \operatorname{div} f \, dy \, dx = 0.$$

Orbital Stability (p. 162)

Boundary Value Problems

Separate inhomogeneities due to boundary conditions and differential equation.
Green's Functions (pp. 183, 190)

Examples

Circuit Theory (p. 164)
Predator-Prey Model (p. 174)
Rigid Body Motion (p. 177)