21-484 Notes

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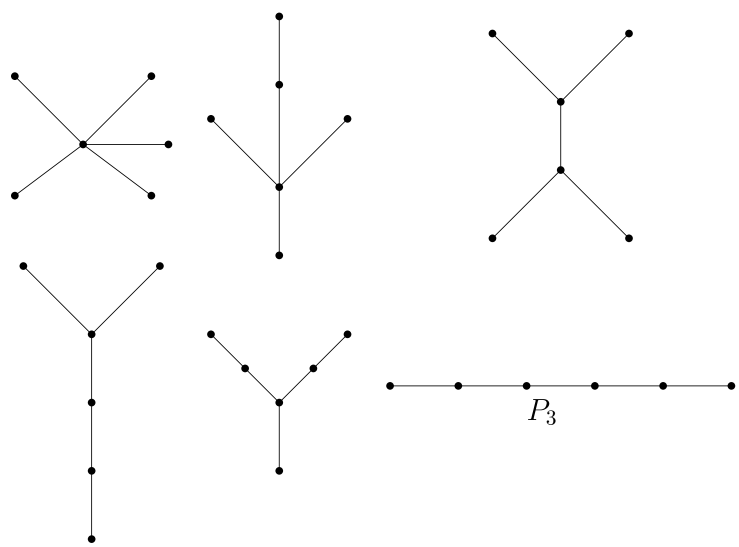
Recall:

- A bridge: $e \in G$ such that $G - e$ has more components than G.
- e is a bridge iff e lies on no cycle

Def: (p. 87-88):

- A grpah G is acyclic if it contains no cycles
- A tree is a connected acyclic graph
- Trees are usually denoted by T
- Every edge in a tree is a bridge

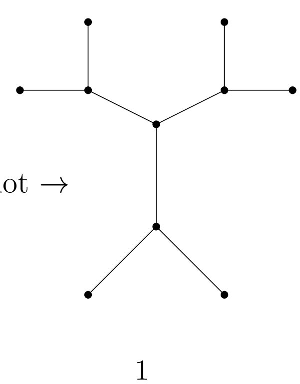
Example: (Figure 4.3): all trees with 6 verticies



Def: A caterpillar is a tree in which, after removing all the leaves, we get a path. This path is called the spine.

example: all trees with 6 vertices are caterpillars.

not $\rightarrow$



Def: A graph in which every component is a tree is called Forest.

Proposition (Thm 4.3): A graph is a tree iff every pair of veritices is connected by a unique path.

Proof:

- $\to$ If G is a tree, then it is connected, so for every $u \neq v \in V(G)$ there is a u–v path. If there are two different u–v paths, p and p' then we can form a cycle out of them.
- $\to$ If every pair $u, v \in V(G)$ are connected by a unique path, then G is connected, and G is acyclic since if we have a cycle $v_0, v_1, \ldots, v_\ell, v_0$ then $\begin{matrix} p = v_0 \ldots, v_\ell \\ p' = v_0, v_\ell \end{matrix}$ are two different v_0–v_ℓ paths.

Proposition (theorem 4.3): Every nontrivial tree has at least two end points.

Proof: Consider a path of maximal length in T. Call it P and let u and v be its enpoints. Then u and v are leaves. If, say, u has degree ≥ 2 the it has one neighbor, v_1, in the path and another, w, out of the path. Then $w, u, \ldots, v$ is a longer path in T (and by the proposition above, thats the only w–v path in T). $\lightning$ The u–v path was maximal.

- connected
- acyclic
- $|E(G)| = |V(G)| - 1$

Proposition: (Thm 4.4): In every tree with n vertices, there are $n-1$ edges.

Proof: By induction

$n = 1$, $T = \bullet$ $|V(T)| = 1, |E(T)| = 0$.

Assume that every tree with at most n vertices has $|E(T)| = |V(T)| - 1$. Given a tree with $n+1$ vertices, we know that it has a leaf u, so $T - u$ has n vertices and thus $n-1$ edges. So T has $n+1$ vertices and n edges. ✓

Corollary (Corollary 4.6): If G is a forest with k components, then it has $n - k$ edges.

Proof: count.

Theorem 4.7: In every connected graph with n vertices there are at least $n-1$ edges.

Proof: easy to verify when $n \leq 3$. Assume that G is the minimal (by number of vertices and then number of edges) graph with n vertices, at most $n-2$ edges and G is connected.

- If G is acyclic, then we have a leaf, removing the leaf will result in a graph with $n-1$ vertices, at most $n-3$ edges, which is connected. $\lightning$ contradicting minimality of G.
- If G has a cycle, then an edge on the cycle is not a bridge, so removing it we'll get a connected graph with n vertices and one less edge. $\lightning$ contradicting minimality of G.

"Proofs from the book"

n^{n-2}

$$\begin{aligned} [n] &\to [n] \\ \{1, 2, \ldots, n\} &\to \{1, 2, \ldots, n\} \end{aligned}$$

n^n

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