Lecture 11: Continuous Random Variables

Contents

1 Review of probability density function

Recall last time that we defined the probability density function for a continuous random variable as follows:

Definition 1 *The* **probability density function (p.d.f.)** *of a continuous r.v. X is a non-negative function $f_X(\cdot)$ where:*

$$\mathbf{P}\{a \le X \le b\} = \int_a^b f_X(x)dx \quad \text{and where} \quad \int_{-\infty}^{\infty} f_X(x)dx = 1$$

For example, the area shown in Figure 1 represents the probability that $5 < X < 6$.

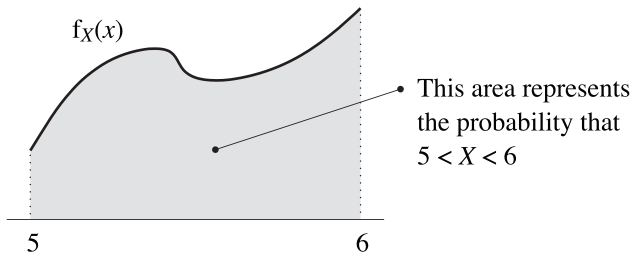

Figure 1: *Area under the curve represents the probability that r.v. X is between* 5 *and* 6*, namely* $\int_5^6 f_X(x)dx$.

The p.d.f. should be interpreted in this way:

$$f_X(x)dx \doteq \mathbf{P}\{x \leq X \leq x + dx\}$$

The *mean* or *expectation* of a continuous distribution follows immediately from it probability density function. For r.v. X, we have that:

$$\begin{aligned}
\mathbf{E}\{X\} &= \int_{-\infty}^{\infty} x \cdot f_X(x)dx \\
\mathbf{E}\left\{X^k\right\} &= \int_{-\infty}^{\infty} x^k \cdot f_X(x)dx
\end{aligned}$$

2 The Uniform distribution

Uniform(a,b), often written $U(a, b)$, models the fact that any interval of length δ between a and b is equally likely. Specifically, if $X \sim U(a, b)$, then:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Question: For $X \sim U(a, b)$, what is $F_X(x)$?

Answer:

$$F_X(x) = \int_a^x \frac{1}{b-a}dx = \frac{x-a}{b-a}$$

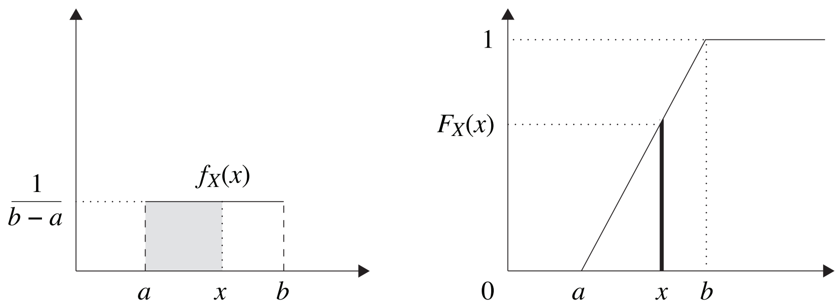

Figure 2: *The p.d.f., $f(x)$, and c.d.f., $F(x)$, functions for Uniform(a,b). The shaded region in the left graph has area equal to the height of the darkened segment in the right graph.*

Figure 2 depicts $f_X(x)$ and $F_X(x)$ graphically.

Question: What is the mean and variance of $X \sim \text{Uniform}(a, b)$?

Answer:

$$\begin{aligned}
\mathbf{E}\{X\} &= \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\frac{(b^2-a^2)}{2} = \frac{a+b}{2} \\
\mathbf{E}\left\{X^2\right\} &= \int_a^b x^2\frac{1}{b-a}dx = \frac{1}{b-a}\frac{(b^3-a^3)}{3} = \frac{a^2+ab+b^2}{3} \\
\mathbf{Var}(X) &= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}
\end{aligned}$$

3 The Exponential distribution

Exp(λ) denotes the Exponential distribution, whose probability density function drops off exponentially. We say that a random variable X is distributed exponentially with rate λ, written $X \sim \text{Exp}(\lambda)$, if:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The graph of the p.d.f. is shown in Figure 3. The c.d.f., $F_X(x) = \mathbf{P}\{X \leq x\}$, is given by

$$F_X(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

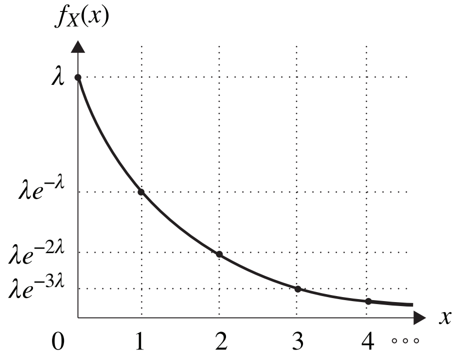

Figure 3: *Exponential probability density function.*

$$\overline{F}_X(x) = 1 - F_X(x) = e^{-\lambda x}, \ x \geq 0$$

Observe that both $f_X(x)$ and $\overline{F}(x)$ drop off by a *constant* factor, $e^{-\lambda}$, with each unit increase of x.

Question: If $X \sim \text{Exp}(\lambda)$, what is $\mathbf{E}\{X\}$?

Answer:

$$\begin{aligned}
\mathbf{E}\{X\} &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_0^{\infty} x\lambda e^{-\lambda x} dx \\
&= -xe^{-\lambda x}\Big|_0^{\infty} - \int_0^{\infty} \left(-e^{-\lambda x}\right) dx \qquad \text{(integration by parts)} \\
&= 0 + \int_0^{\infty} e^{-\lambda x} dx \\
&= \left.\frac{e^{-\lambda x}}{-\lambda}\right|_0^{\infty} \\
&= 0 - \frac{1}{-\lambda} \\
&= \frac{1}{\lambda}
\end{aligned}$$

Observe that while the λ parameter for the Poisson distribution is also its mean, for the Exponential distribution, the λ parameter is the reciprocal of the mean. For this reason, the parameter λ is referred to as the “rate” of the Exponential.

The second moment of $X \sim \text{Exp}(\lambda)$ is:

$$\mathbf{E}\left\{X^2\right\} = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{2}{\lambda^2}$$

The variance is:

$$\mathbf{Var}(X) = \mathbf{E}\left\{X^2\right\} - (\mathbf{E}\{X\})^2 = \frac{1}{\lambda^2}$$

Question: What is the squared coefficient of variation of $\text{Exp}(\lambda)$?

Answer: The squared coefficient of variation of random variable X is defined as:

$$C_X^2 = \frac{\mathbf{Var}(X)}{\mathbf{E}\{X\}^2}$$

This can be thought of as the "scaled" or "normalized" variance. When $X \sim \text{Exp}(\lambda)$, $C_X^2 = 1$.

4 Joint probabilities and independence

Definition 2 *We use* $f_{X,Y}(x,y)$ *to represent the* **joint probability density function** *between continuous r.v.'s* X *and* Y*, where:*

$$\int_c^d \int_a^b f_{X,Y}(x,y)dxdy = \mathbf{P}\{a < X < b \quad \& \quad c < Y < d\}$$

Question: What is the relationship between $f_X(x)$ and $f_{X,Y}(x,y)$?

Answer:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \qquad\qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Definition 3 *We say that continuous r.v.'s* X *and* Y *are* **independent***, written* $X \perp Y$*, if*

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y), \quad \forall x, y$$

Theorem 4 *If* $X \perp Y, \mathbf{E}\{XY\} = \mathbf{E}\{X\} \cdot \mathbf{E}\{Y\}$

Proof:

$$\begin{aligned}
\mathbf{E}\{XY\} &= \int_x \int_y xy \cdot f_{X,Y}(x,y)dxdy \\
&= \int_x \int_y xy \cdot f_X(x) \cdot f_Y(y)dxdy \quad \text{(by definition of } \perp\text{)} \\
&= \int_x x f_X(x)dx \cdot \int_y y f_Y(y)dy \\
&= \mathbf{E}\{X\}\,\mathbf{E}\{Y\}
\end{aligned}$$

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5 Conditional probabilities and expectations

For a continuous, real-valued, r.v. X, the conditional p.d.f. of X given event A is analogous to that for the discrete case, except that A is now a subset of the real line, where we define $\mathbf{P}\{X \in A\}$ to be the probability that X has value within the subset A.

Definition 5 *Let X be a continuous r.v. with p.d.f. $f_X(\cdot)$ defined over the reals. Let A be a subset of the real line with $\mathbf{P}\{X \in A\} > 0$. Then $f_{X|A}(\cdot)$ is the* **conditional p.d.f.** *of X given event A. We define:*

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{\mathbf{P}\{X \in A\}} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

As with the discrete case, the conditional p.d.f. is zero outside the conditioning set A. Within A, the conditional p.d.f. has exactly the same shape as the unconditional one, except that it is scaled by the constant factor, $\frac{1}{\mathbf{P}\{X \in A\}}$, so that $f_{X|A}(x)$ integrates to 1.

Definition 6 *Let X be a continuous r.v. with p.d.f. $f_X(\cdot)$ defined over the reals. Let A be a subset of the real line with $\mathbf{P}\{X \in A\} > 0$. The conditional expectation of X given A, written, $\mathbf{E}\{X \mid A\}$ is defined by:*

$$\mathbf{E}\{X|A\} = \int_{-\infty}^{\infty} x f_{X|A}(x)dx = \int_A x f_{X|A}(x)dx = \frac{1}{\mathbf{P}\{X \in A\}} \int_A x f_X(x)dx$$

Example: Pittsburgh Supercomputing Center

The Pittsburgh Supercomputing Center (PSC) runs large parallel jobs for scientists from all over the country. In order to charge users appropriately, jobs are grouped into different bins based on the number of CPU hours they require, each with a different price. Suppose that job durations are Exponentially-distributed with *mean* 1000 processor-hours. Further suppose that all jobs requiring less than 500 processor-hours are sent to bin 1, and all remaining jobs are sent to bin 2.

Question: Consider the following questions:

1. What is $\mathbf{P}\{\text{Job is sent to bin 1}\}$?
2. What is $\mathbf{P}\{\text{Job duration } < 200 \mid \text{job is sent to bin 1}\}$?
3. What is the conditional density of the duration X, $f_{X|Y}(t)$ where Y is the event that the job is sent to bin 1?
4. What is $\mathbf{E}\{\text{Job duration} \mid \text{job is in bin 1}\}$?

Answer: Start by recalling that for $X \sim \text{Exp}\left(\frac{1}{1000}\right)$ we have:

$$f_X(t) = \begin{cases} \frac{1}{1000}e^{-\frac{t}{1000}} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(t) = \mathbf{P}\{X \leq t\} = 1 - e^{-\frac{1}{1000}t}$$

$$\overline{F}_X(t) = \mathbf{P}\{X > t\} = e^{-\frac{1}{1000}t}$$

1. $F_X(500) = 1 - e^{-\frac{500}{1000}} = 1 - e^{-\frac{1}{2}} \approx 0.39$
2. $\frac{F_X(200)}{F_X(500)} = \frac{1-e^{-\frac{1}{5}}}{1-e^{-\frac{1}{2}}} \approx 0.46$
3. $$f_{X|Y}(t) = \begin{cases} \frac{f_X(t)}{F(500)} = \frac{\frac{1}{1000}e^{-\frac{t}{1000}}}{1-e^{-\frac{1}{2}}} & \text{if } t < 500 \\ 0 & \text{otherwise} \end{cases}$$

4.

$$
\begin{aligned}
\mathbf{E}\{\text{ Job duration } \mid \text{ job in bin 1 }\} &= \int_{-\infty}^{\infty} t\, f_{X|Y}(t)dt \\
&= \int_0^{500} t \frac{\frac{1}{1000}e^{-\frac{t}{1000}}}{1-e^{-\frac{1}{2}}} dt \\
&= \frac{1}{1-e^{-\frac{1}{2}}} \int_0^{500} t \frac{1}{1000} e^{-\frac{t}{1000}} dt \\
&= \frac{1000 - 1500e^{-\frac{1}{2}}}{1-e^{-\frac{1}{2}}} \\
&\approx 229
\end{aligned}
$$

Question: Why is the expected size of jobs in bin 1 < 250?

Answer: Consider the shape of the Exponential p.d.f. Now truncate it at 500, and scale everything by a constant needed to make it integrate to 1. There is still more weight on the smaller values, so the expected value is less than the midpoint.

Question: How would the answer to question 4 change if the job durations were distributed Uniform(0, 2000), still with mean 1000?

Answer: Logically, given that the job is in bin 1 and the distribution is uniform, we should find that the expected job duration is 250 seconds. Here's an algebraic argument:

$$
\begin{aligned}
\mathbf{E}\{\text{ Job duration } \mid \text{ job in bin 1 }\} &= \int_{-\infty}^{\infty} t\, f_{X|Y}(t)dt \\
&= \int_0^{500} t \frac{\frac{1}{2000}}{\frac{500}{2000}} dt \\
&= \int_0^{500} t \frac{1}{500} dt \\
&= \frac{1}{500} \left. \frac{t^2}{2} \right]_0^{500} \\
&= 250
\end{aligned}
$$

6 More examples

All the theorems/laws that we saw for discrete r.v.s hold for continuous r.v.s as well. This includes the Law of Total Probability, Linearity of Expectations, Linearity of Variance for independent random variables, Expectation of a product of independent random variables, etc.

In this section we include some more examples of working with continuous random variables, in particular the Exponential random variable.

Question: Suppose that the time to download file A is Exp(λ_A) while the time to download file B is Exp(λ_B). We start downloading the two files at the same time.

1. (a) What is the probability that file A finishes downloading first?
2. (b) What is the expected time until the first file finishes downloading?
3. (c) What is the distribution of the time until the first file finishes downloading?

Answer: We will answer all these questions in the next couple theorems.

Theorem 7 *Given* $X_1 \sim Exp(\lambda_1),\ X_2 \sim Exp(\lambda_2),\ X_1 \perp X_2$

$$\mathbf{P}\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Proof:

$$\begin{aligned}
\mathbf{P}\{X_1 < X_2\} &= \int_0^\infty \mathbf{P}\{X_1 < X_2 \mid X_2 = x\} \cdot f_2(x)dx \\
&= \int_0^\infty \mathbf{P}\{X_1 < x\} \cdot \lambda_2 e^{-\lambda_2 x} dx \\
&= \int_0^\infty (1 - e^{-\lambda_1 x})(\lambda_2 e^{-\lambda_2 x})dx \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 x} dx - \lambda_2 \int_0^\infty e^{-(\lambda_1 + \lambda_2)x} \\
&= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

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Theorem 8 *Given* $X_1 \sim Exp(\lambda_1),\ X_2 \sim Exp(\lambda_2),\ X_1 \perp X_2$.

Let

$$X = \min(X_1, X_2).$$

Then

$$X \sim Exp(\lambda_1 + \lambda_2)$$

Proof:

$$\begin{aligned}
\mathbf{P}\{X > t\} &= \mathbf{P}\{\min(X_1, X_2) > t\} \\
&= \mathbf{P}\{X_1 > t \text{ and } X_2 > t\} \\
&= \mathbf{P}\{X_1 > t\} \cdot \mathbf{P}\{X_2 > t\} \\
&= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \\
&= e^{-(\lambda_1 + \lambda_2)t}
\end{aligned}$$

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7 Memoryless property of the Exponential

A random variable X is said to be **memoryless** if

$$\mathbf{P}\{X > s + t \mid X > s\} = \mathbf{P}\{X > t\}\ \ \forall s, t \geq 0$$

Question: Prove that $X \sim \text{Exp}(\lambda)$ is memoryless.

Answer:

$$\begin{aligned}
\mathbf{P}\{X > s + t \mid X > s\} &= \frac{\mathbf{P}\{X > s + t\}}{\mathbf{P}\{X > s\}} \\
&= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\
&= e^{-\lambda t} \\
&= \mathbf{P}\{X > t\}
\end{aligned}$$

To understand this, think of X as being the lifetime of say a lightbulb. The above says that the probability that the lightbulb survives for at least another t seconds before burning out given that the lightbulb has survived for s seconds already, is the same as the probability that the lightbulb survives at least t seconds, *independent of* s.

Question: Does this seem realistic for a lightbulb?

Answer: Who knows.

Question: What are some real-life examples whose lifetimes can be modeled by an X such that $\mathbf{P}\{X > s + t \mid X > s\}$ goes down as s goes up?

Answer: A car's lifetime. The older a car is, the less likely that it will survive another, say, $t = 6$ years.

Distributions for which $\mathbf{P}\{X > s + t \mid X > s\}$ goes down as s goes up are said to have **increasing failure rate**. The device is more and more likely to fail as time goes on.

Question: What are some real-life examples whose lifetimes can be modeled by an X such that $\mathbf{P}\{X > s + t \mid X > s\}$ goes up as s goes up?

Answer: We will soon see that one example is UNIX job CPU lifetimes, see [9, 10]. The more CPU a job has used up so far, the more CPU it is likely to use up. Another example is computer chips. If they're going to fail, they'll do it early. That's why chip manufacturers test them for a long while.

Distributions for which $\mathbf{P}\{X > s + t \mid X > s\}$ goes up as s goes up are said to have **decreasing failure rate**. The device is less likely to fail as time goes on.

More precisely, the **failure rate function r(t)**, (a.k.a. hazard rate function), is defined as follows:

Let X be a continuous random variable with probability density function $f(t)$ and cumulative distribution function $F(t) = \mathbf{P}\{X < t\}$. Then $r(t)$ is formally defined as:

$$r(t) \equiv \frac{f(t)}{\overline{F}(t)}$$

To interpret this expression, consider the probability that a t-year old item will fail during the next dt seconds.

$$\begin{aligned} \mathbf{P}\{X \in (t, t + dt) \mid X > t\} &= \frac{\mathbf{P}\{X \in (t, t + dt)\}}{\mathbf{P}\{X > t\}} \\ &\approx \frac{f(t) \cdot dt}{\overline{F}(t)} \\ &= r(t) \cdot dt \end{aligned}$$

Thus $r(t)$ represents the instantaneous failure rate of a t-year old item.

Definition 9 *When $r(t)$ is strictly decreasing in t we say that the distribution $f(t)$ has* **decreasing failure rate***; if $r(t)$ is strictly increasing in t, we say that the distribution has* **increasing failure rate***.*

Observe that $r(t)$ is not necessarily going to always decrease with t or increase with t – it might behave differently for different t.

Question: Suppose $r(t)$ is constant. What do you know about $f(t)$?

Answer: We will prove in the next homework that $f(t)$ *must* be the Exponential p.d.f.

Bank example

Question: Suppose that the time a customer spends in a bank is Exponentially distributed with mean 10 minutes. What is $\mathbf{P}\{\text{Customer spends} > 5 \text{ min in bank}\}$

Answer: $e^{-5 \cdot 1/10} = e^{-1/2}$

Question: What is
$\mathbf{P}\{\text{Customer spends} > 15 \text{ min in bank total} \mid \text{he's there after 10 min}\}$?

Answer: Same as above answer.

The reason why the Exponential distribution is so convenient to work with is that history doesn't matter!

Question: Suppose $X \sim \text{Exp}(\lambda)$. What is $\mathbf{E}\{X|X > 10\}$?

Answer: The Exponential distribution "starts over" at 10, or any other point. Hence $\mathbf{E}\{X|X > 10\} = 10 + \mathbf{E}\{X\} = 10 + \frac{1}{\lambda}$.

Post office example

Suppose that a post office has 2 clerks. Customer B is being served by one clerk, and customer C is being served by the other clerk, when customer A walks in. All service times are Exponentially distributed with mean $\frac{1}{\lambda}$.

Question: What is $\mathbf{P}\{A \text{ is the last to leave}\}$?

Answer: $\frac{1}{2}$. Note that one of B and C will leave first. WLOG, Let us say B leaves first. Then C and A will have the same distribution on their remaining service time. It doesn't matter that C has been serving for a while.

It can be proven that the Exponential distribution is the *only* continuous-time memoryless distribution.

Question: What's the only discrete-time memoryless distribution?

Answer: The Geometric distribution.

We will later delve more deeply into the exact relationship between the Exponential and the Geometric. For now, we move to another distribution.

8 The Pareto distribution

8.1 Definition of Pareto

Pareto(α) is a distribution with a power-law tail, meaning that its density decays as a polynomial in $1/x$ rather than exponentially, as in Exp(λ). The parameter α is often referred to as the "tail parameter." It is generally assumed that $0 < \alpha < 2$. If $X \sim$ Pareto(α), then:

$$f_X(x) = \begin{cases} \alpha x^{-\alpha-1} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} F_X(x) &= 1 - x^{-\alpha} \\ \overline{F}_X(x) &= x^{-\alpha} \end{aligned}$$

While the Pareto distribution has a ski-slope shape, like that of the Exponential, its tail decreases much more slowly (compare $\overline{F}(x)$ for the two distributions). The Pareto distribution is said to have a "heavy tail," or "fat tail," where a lower α corresponds to a fatter tail because the decrease is more gradual.

8.2 Pareto in job lifetimes

A large part of my PhD research was spent on understanding the CPU requirements of UNIX jobs, see [9, 10]. I was interested in the *distribution* of the CPU requirements of UNIX jobs – this is called the CPU lifetime distribution. This is important for understanding the impact of different load balancing schemes.

Some terminology: When we talk about a job's **size** we mean its total CPU requirement. When we talk about a job's **age** we mean its total CPU usage thus far. A job's **lifetime** refers to its total CPU requirement (same thing as size). A job's **remaining lifetime** refers to its remaining CPU requirement. Observe that at any point in time, you don't know the job's remaining lifetime, just its current CPU age.

At the time that I was doing my PhD (the Dark Ages of 1994), it was believed that the CPU lifetimes of jobs were Exponentially-distributed. Refusing to believe that UNIX job lifetimes were Exponentially-distributed, I decided to measure the distribution, $F(\cdot)$, of job lifetimes and see if I could find a closed-form distribution which fit it.

I collected the CPU lifetime of millions of jobs on a wide range of different machines, including instructional, research, and administrative machines, over the course of many months. Figure 4 below shows the fraction of jobs whose size exceeds x for all jobs whose size is greater than one second.

At a first glance this plot looks like an Exponential distribution, $\overline{F}(x) = e^{-\lambda x}$. But on a closer examination you can see that it is not Exponential.

Question: How can you tell that it's not Exponential?

Answer: For an Exponential distribution, the fraction of jobs remaining should drop by a constant factor with each unit increase in x (constant failure rate). In Figure 4, we see that the faction of jobs remaining decreases by a slower and slower rate as we increase x (decreasing failure rate). In fact, looking at the graph, we see that if we start with jobs of CPU age 1 second, half of them make it to 2 seconds. Of those that make it to 2 seconds, half of those make it to 4 seconds. Of those that make it to 4 seconds, half of those make it to 8 seconds. Of those that make it to 8 seconds, half of those make it to 16 seconds, and so on.

To see the distribution more easily it helps to view it on a log-log plot as shown in Figure 5. The bumpy line shows the data and the straight line is the best curve-fit.

To see that the measured distribution is *not* an Exponential distribution, consider Figure 6 which shows the best-fit Exponential distribution in juxtaposition with the measured distribution from Figure 5.

From Figure 5 it is apparent that the tail of the distribution of jobs with lifetimes longer than 1 second decays like $\frac{1}{x}$. That is,

$$\mathbf{P}\{\text{Job size } > x \mid \text{Job size } > 1\} = \frac{1}{x}$$

At the time (mid-90's), I didn't recognize this distribution and was suspicious of

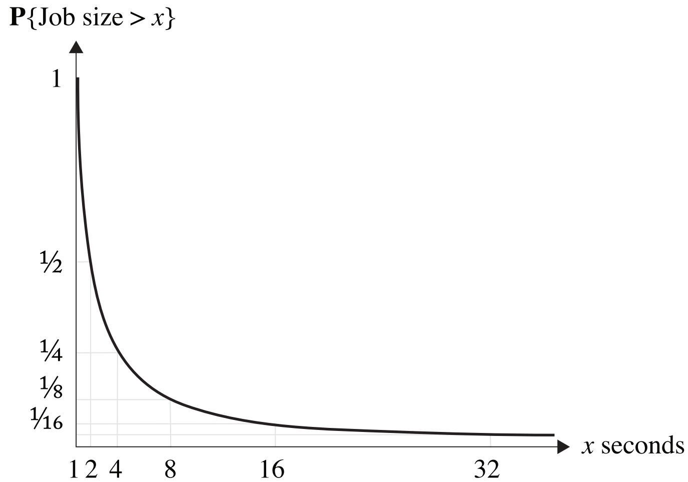

Figure 4: *Plot of measured distribution,* $\overline{F}(x) = \mathbf{P}\{Job\ size > x\}$.

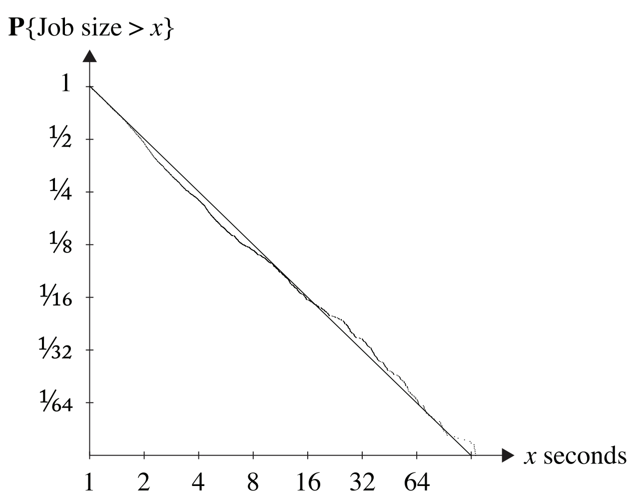

Figure 5: *Log-log plot of measured distribution,* $\overline{F}(x) = \mathbf{P}\{Job\ size > x\}$.

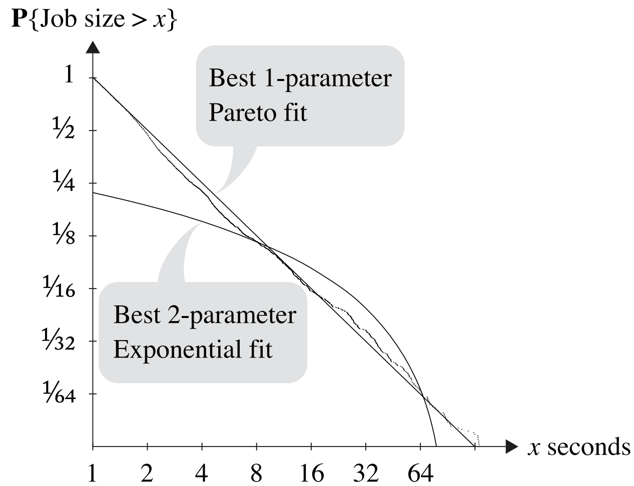

Figure 6: *Plot of measured distribution on log-log axes along with best-fit Exponential distribution.*

its simplicity, so I tried measuring different types of UNIX workloads. I also tried removing shells and daemon processes, as well as removing very short jobs. No matter what I tried, in all cases, I saw a straight line on a log-log plot, indicating the following distribution:

$$\overline{F}(x) = \frac{1}{x^{\alpha}}, \quad x \geq 1$$

where α ranged from about 0.8 to about 1.2, in my measurements, across machines. Most commonly α was very close to 1, and the regression showed goodness of fit (R^2) values of over 0.96.

8.3 Properties of the Pareto

It turns out that the distribution that I had measured has a name in economic theory. It is called the Pareto distribution, or "power-law distribution."

Definition 10 *A distribution, $\overline{F_X}(x)$, such that*

$$\overline{F}_X(x) = \mathbf{P}\{X > x\} = x^{-\alpha}, \quad \textit{for } x \geq 1$$

where $0 < \alpha < 2$ *is called a* **Pareto distribution**.

We will primarily be concentrating on the Pareto distribution with $\alpha \approx 1$.

Question: What is the failure rate of the Pareto distribution?

Answer:

$$\begin{aligned}
\overline{F}(x) &= \mathbf{P}\{X > x\} = x^{-\alpha}, \qquad x \geq 1 \\
\Rightarrow F(x) &= \mathbf{P}\{X < x\} = 1 - x^{-\alpha}, \qquad x \geq 1 \\
\Rightarrow f(x) &= \frac{dF(x)}{dx} = \alpha x^{-\alpha-1}, \qquad x \geq 1 \\
\Rightarrow r(x) &= \frac{f(x)}{\overline{F}(x)} = \frac{\alpha x^{-\alpha-1}}{x^{-\alpha}} = \frac{\alpha}{x}, \qquad x \geq 1
\end{aligned}$$

Notice that $\int_1^\infty f(x)dx = \int_1^\infty \alpha x^{-\alpha-1} dx = 1$, so $f(x)$ is a valid probability distribution.

Since $r(x) = \frac{\alpha}{x}$ decreases with x, the Pareto distribution has decreasing failure rate (DFR). Thus the older a job is (the more CPU it has used up so far), the greater its probability of using another second of CPU.

Question: What are the implications of DFR for process migration?

Answer: Older jobs (those with higher age) are likely to last longer. Thus it is often worth migrating those jobs, even though they have a lot of additional state, because they will use up more resources in the future.

Question: For a Pareto with $\alpha \leq 1$, what is the mean and variance of the distribution?

Answer: The calculations are straightforward, by integration over the density function. If $0 < \alpha \leq 1$,

$$\begin{aligned}
\mathbf{E}\{\text{Lifetime}\} &= \infty \\
\mathbf{E}\{i\text{th moment of Lifetime}\} &= \infty, \quad i = 2, 3, \ldots \\
\mathbf{E}\{\text{Remaining Lifetime} \mid \text{age } = a > 1\} &= \infty .
\end{aligned}$$

Question: How do the above answers change when α is above 1?

Answer: Both the expected lifetime and the expected remaining lifetime are now finite. Higher moments of lifetime are still infinite.

Question: Under the Pareto distribution with $\alpha = 1$, what's the probability that a job of CPU age a lives to CPU age b, where $b > a$?

Answer:

$$\mathbf{P}\{\text{Life} > b \mid \text{Life} \geq a > 1\} = \frac{1/b}{1/a} = \frac{a}{b}$$

Using the above expression, for the Pareto($\alpha = 1$) distribution, we can interpret the distribution in the following way:

- Of all the jobs currently of age 1 sec, half of those will live to age ≥ 2 sec.
- The probability that a job of age 1 sec uses $> T$ sec of CPU is $\frac{1}{T}$.
- The probability that a job of age T sec lives to be age $\geq 2T$ sec is $\frac{1}{2}$.

The following are three properties of the Pareto distribution:

Decreasing Failure Rate (DFR) — The more CPU you have used so far, the more you will continue to use.

Infinite or near-infinite Variance

"Heavy-Tail Property" — A minuscule fraction of the very largest jobs comprise half of the total system load. For example, when $\alpha = 1.1$, the largest 1% of the jobs comprise about $\frac{1}{2}$ of the load. (Note that this is stronger than the often quoted 80-20 rule.)

The last property, which we call the "heavy-tail property," comes up in many other settings. For example, in economics, when studying people's wealth, it turns out that the richest 1% of all people have more money between them than all the remaining 99% of us combined. The heavy-tailed property is often referred to as "a few big elephants (big jobs) and many, many mice (little jobs)," as illustrated in Figure 7. For comparison, in an Exponential distribution with the same mean, the largest 1% of the jobs comprise only about 5% of the total demand.

The parameter α can be interpreted as a measure of the variability of the distribution and the heavy-tailedness: $\alpha \to 0$ yields the most variable and most heavy-tailed distribution, while $\alpha \to 2$ yields the least variable, and least heavy-tailed distribution. These properties are explored in more depth in the exercises.

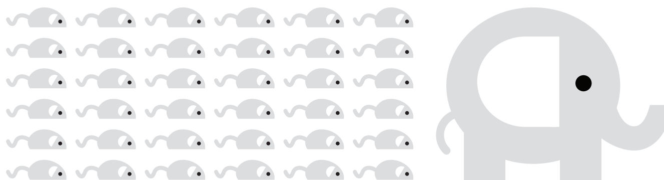

Figure 7: *Heavy-tailed property: "Elephants and mice."*

8.4 Pareto distributions are everywhere!

It's not just UNIX jobs that fit heavy-tailed Pareto distribution. Pareto job size distributions are everywhere! Here are some more practical and interesting stories:

Around 1996-98, Mark Crovella, Azer Bestavros, and Paul Barford at Boston University were measuring the sizes of files at web sites. They found that these sizes had a Pareto distribution with $\alpha \approx 1.1$. They also found similar results for the sizes of files requested from web sites. Their SURGE web workload generator is based on these findings [2, 7, 6].

Around this same time, the three Faloutsos brothers were observing a similar distribution when looking at the Internet topology. They observed, for example, that most nodes have low out-degree, but a very few nodes have very high out-degree, and the distribution of the degrees follows a Pareto distribution. This and other observations were published in their beautiful 1999 paper which won the Sigcomm Test of Time award, [8].

In 1999, Jennifer Rexford, Anees Shaikh, and Kang Shin at AT&T were working on routing IP flows to create better load balancing. They didn't want to have to re-route all flows because the overhead would be too high. Ideally, they wanted to just have to re-route only 1% of the IP flows. Would that be enough? Fortunately, their measurements showed that the number of packets in IP flows follows a heavy-tailed Pareto distribution. Consequently, the 1% largest IP flows (those with the most packets) contain about 50% of the bytes in all flows. Thus by re-routing only 1% of the flows, they were able to redistribute half the load. Their paper appeared in Sigcomm 99 [14] and generated a large group of followup papers dealing with sampling methods for how to detect which flows are large.

Around this same time, my students and I, in collaboration with colleagues at Boston University, started a project called SYNC (Scheduling Your Network Connections). The goal was to improve the performance of web servers by changing the order in which they scheduled their jobs to favor requests for small files over requests for large files. Clearly favoring requests for small files over large ones would decrease mean response time. However people hadn't tried this in the past because they were afraid that the requests for large files would "starve" or at least be treated unfairly compared to requests for small files. Using the heavy-tailed property of web file sizes, we were able to prove analytically and in implementation that this fear is unfounded for the distribution of web files. The crux of the argument is that although short requests do go ahead of long requests, all those short requests together make up very little load (more than half the load is in the top 1% of long requests) and hence don't interfere noticeably with the long requests, [1, 5, 11]. In 2004, Ernst Biersack, Idris Rai, and Guillaume Urvoy-Keller extended the SYNC results to TCP flow scheduling, by exploiting the DFR property of the Pareto distribution to discern which flows had short remaining duration [13, 12].

There are many, many more examples of the Pareto distribution in measured distributions involving jobs created by humans. Wireless session times have been shown to follow a Pareto distribution, [3]. Phone call durations have been shown to follow a distribution similar to a Pareto. Human wealth follows a Pareto distribution. Natural phenomenon too follow Pareto distributions. For example, John Doyle at Caltech has shown that the damage caused by forest fires follows a Pareto distribution, with most forest fires causing little damage, but the top few forest fires causing the majority of the damage. The same property holds for earthquakes and other natural disasters.

Given the prevalence of the Pareto distribution, there has been a large amount of research interest into **why** the Pareto distribution comes up everywhere. Ideally, we would like to prove something similar in nature to the Central Limit Theorem, which explains the ubiquity of the Normal distribution, but this time for the Pareto distribution. We do not have room to delve into the many theories proposed for the origin of the Pareto distribution, e.g., the HOT theory [4]. To date, this is still an open research problem of great practical importance.

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