ASSIGNMENT NUMBER 2, 21.630

Due Wednesday, January 30, 2013

1. Let g be a given continuous function from $[0, \infty)$ to the reals and consider $\mathcal{F}$: $\mathcal{C}\ [0, \infty) \to \mathcal{C}\ [0, \infty)$ defined by $\mathcal{F}\ [X](t) = g(t) + \int_0^t X(s)ds$.

A) For $T > 1$ is $\mathcal{F}$ a contraction from $\mathcal{C}\ [0, T]$ to itself?

B) Let $X^{(0)}(t) = 0$ and $X^{(n+1)}(t) = \mathcal{F}\ [X^{(n)}](t)$. For any $T > 0$ show that $X^{(n)}$ converges uniformly on $[0, T]$. Outline: First show that

$$|X^{(n+1)}(t) - X^{(n)}(t)| \leq ||g||\frac{t^n}{n!}$$

for any $n \geq 0$ where $||g||$ is the supremum of g over the set $[0, T]$. Then show that

$$|X^{(n+k)}(t) - X^{(n)}(t)| \leq ||g|| \sum_{l=n}^{\infty} \frac{t^l}{l!}$$

for all $n \geq 0$ and $k \geq 0$. Since $\sum_{l=0}^{\infty} \frac{t^l}{l!}$ converges, this shows that $X^{(n)}$ is uniformly Cauchy, and hence uniformly convergent, on $[0, T]$.

2. Define $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by $f(t, x) = 2t$ if $x \leq 0$, $f(t, x) = 2t - 4x/t$ if $0 < x < t^2$ and $f(t, x) = -2t$ if $t^2 \leq x$. Define $X^{(0)}(t) = 0$ and

$$X^{(n+1)}(t) = \int_0^t f(s, X^{(n)}(s))ds.$$

Compute $X^{(n)}(t)$. Does $X^{(n)}$ have any convergent subsequences? If so, do their limits satisfy

$$\frac{dX}{dt}(t) = f(t, X(t))?$$

3. Let $g \in \mathcal{C}[0, 1] = \mathcal{C}$ with $||g||_{\mathcal{C}} < 1/4$ be given.

A) Use the contraction mapping theorem to show there is $X \in \mathcal{C}$ with $||X||_{\mathcal{C}} \leq B$ and $X(t) = g(t) + \int_0^1 X^2(s)ds$ where B is chosen suitably.

B) How many solutions of $X(t) = g(t) + \int_0^1 X^2(s)ds$ are there? Find them.

4. Define $X^{(n)}(t) = \frac{t^2}{t^2 + (1-nt)^2}$ for $t \in [0, 1]$.

A) Show that $X^{(n)}$ is uniformly bounded on $[0,1]$.

B) Assume that $X^{(n)}$ is equicontinuous on $[0,1]$ and derive a contradiction from this.

5. Let $X^{(n)}$ be a sequence of continuously differentiable functions on $[0,1]$. Assume that

$$|\frac{dX^{(n)}}{dt}(t)| \leq 1000t^{-2/3}$$

for all n and all $t \in (0,1]$. For any $\epsilon > 0$ explicitly find $\delta > 0$ such that

$$|t-s| < \delta \Rightarrow |X^{(n)}(t) - X^{(n)}(s)| < \epsilon$$

for all $t, s \in [0,1]$ and all n.