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**21-355C Real Analysis, Fall 2011**
**Assignment 11**
**Due: Wednesday, Novemeber 16**

**Lemma 0.1.** Let $f_1, f_2 : X \to \mathbb{R}$, for some non-empty set $X$, be bounded.

Let $S = \sup_{x\in X} (f_1(x) + f_2(x))$. By definition of supremum, $\forall c \in X$, $f_1(c) \leq \sup_{x\in X} f_1(x)$ and $f_2(c) \leq \sup_{x\in X} f_2(x)$. Thus, $\forall c \in X$, $f_1(c) + f_2(c) \leq \sup_{x\in X} f_1(x) + \sup_{x\in X} f_2(x)$, so that $\sup_{x\in X} f_1(x) + \sup_{x\in X} f_2(x)$ is an upper bound of $A = \{f_1(x) + f_2(x) | x \in X\}$. Since $S = \sup A$ and the supremum is the *least* upper bound, $S \leq \sup_{x\in X} f_1(x) + \sup_{x\in X} f_2(x)$.

Let $I = \inf_{x\in X} (f_1(x) + f_2(x))$. By definition of infimum, $\forall c \in X$, $f_1(c) \geq \inf_{x\in X} f_1(x)$ and $f_2(c) \geq \inf_{x\in X} f_2(x)$. Thus, $\forall c \in X$, $f_1(c)+f_2(c) \geq \inf_{x\in X} f_1(x)+\inf_{x\in X} f_2(x)$, so that $\inf_{x\in X} f_1(x)+\inf_{x\in X} f_2(x)$ is a lower bound of $A = \{f_1(x)+f_2(x)|x \in X\}$. Since $I = \inf A$ and the infimum is the *greatest* lower bound, $S \geq \inf_{x\in X} f_1(x) + \inf_{x\in X} f_2(x)$. ■

**Question 0.2.** Let $f$ be a real, uniformly continuous function on a bounded domain $E \subset R$. Since $E$ is bounded, $\exists m > 0$ such that $E \subseteq (-m, m)$. Let $\epsilon = 1$. Since $f$ is uniformly continuous, $\exists \delta > 0$ such that, $\forall x, y \in E$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let $F$ be the family of sets

$$\{S_{-n}, S_{-(n-1)}, \dots, S_{-1}S_0, S_1, \dots, S_{n-1}, S_n\},$$

where $n = \lceil \frac{m}{\delta} \rceil$ and, $\forall i \in \mathbb{Z}$ with $-n \leq i \leq n$, $S_i = [i\delta - \frac{\delta}{2}, i\delta + \frac{\delta}{2}]$. It is then easily shown by induction on $|i - j|$ that, $i, j \in \mathbb{Z}$ with $-n < i, j < n$, then, $\forall x \in E \cap S_i, \forall y \in E \cap S_j$, $|f(x) - f(y)| < |i - j| + 1$. Thus, $\forall x, y \in E \cap (\cup F)$, $|f(x) - f(y)| < 2n + 1$. Furthermore, $E \subseteq \cup F$ so that $E \cap (\cup F)$. Let $x \in E$. Then, $f(E) \subseteq [-(|f(x)| + 2n + 1), |f(x)| + 2n + 1]$, so that $f$ is bounded on $E$. ■

Clearly, the condition of a bounded domain is necessary; suppose, for instance, that $f$ is the identity function on $\mathbb{R}$. Then, $\forall \epsilon > 0$, for $\delta = \epsilon$, $\forall x, y \in \mathbb{R}$ with $|x - y| < \delta$, $|f(x) - f(y)| = |x - y| < \delta = \epsilon$, so that $f$ is uniformly continuous. However, $\forall M \in \mathbb{R}$, $f(M + 1) > M$, so that $f$ is unbounded. ■

**Question 0.3.** Let $f : \mathbb{R} \to \mathbb{R}$ such that, $\forall x \in \mathbb{R}$ with $x \neq 1$, $f(x) = 0$, and $f(1) = 1$. Then, clearly, since, $\forall x, y \in \mathbb{R}$ with $x, y \neq 1$, $f(x) - f(y) = 0$, $f$ is uniformly continuous and differentiable on $(0, 1)$, with $f'(x) = 0$, $\forall x \in (0, 1)$. Then, there does not exist $c \in (0, 1)$ such that $f(1) - f(0) = (1 - 0)f'(c)$, since $1 \neq 0$. Thus, the condition that $f$ be continuous on a *closed* interval is crucial to the Mean Value Theorem (in particular, to Theorem 5.10). ■

**Question 0.5.** Let $a, b \in \mathbb{R}$, and let $f, g : (a, b) \to \mathbb{R}$ be differentiable on $(a, b)$, with $f'(x) = g'(x), \forall x \in (a, b)$. Then, $\forall x \in (a, b)$, $(f - g)'(x) = 0$. By Theorem 5.11, $(f - g)$ is a constant function. Thus, for some $c \in \mathbb{R}$, $\forall x \in (a, b)$, $f(x) = g(x) + c$. ■

**Lemma 0.6.** Taking $\alpha$ to be the identity on $\mathbb{R}$, this follows immediately from Theorem 6.12 (c). ■

**Lemma 0.7.** For some $a, b \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be continuous and non-negative on $[a, b]$, with $f(x_0) > 0$ for some $x \in [a, b]$. Since $f$ is continuous, for $\epsilon = \frac{f(x_0)}{2} > 0$, $\exists \delta > 0$ such that, $\forall x \in \mathbb{R}$ with $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$. Let $x_1 = \max\{x_0 - \delta, a\}$, and let $x_2 = \min\{x_0 + \delta, b\}$.

Thus, $\forall x \in (x_0 - x_1, x_0 + \delta)$, $f(x) > \frac{f(x_0)}{2} > 0$. Note that, since $f$ is continuous on $[a, b]$, it is integrable on $[a, b]$, by Theorem 6.8. Let $P = \{a, x_1, x_2, b\}$, so that $P$ is partition of $[a, b]$. Then, since $f \geq 0$, taking $\alpha$ to be the identity on $\mathbb{R}$, $L(P, f, \alpha) > 0$ (as $(x_2 - x_1) \sup\{f(x)|x_1 < x < x_2\} > 0$. Therefore, since $L(P, f, \alpha) \leq \int_a^b f(x)dx$, $\int_a^b f(x)dx > 0$. ■

**Question 0.8.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$, with $\int_a^b f(x)dx = 0$. By the previous lemma (Lemma 0.7), if there exists $x_0 \in [a, b]$ such that $f(x_0) > 0$, then $\int_a^b f(x)dx > 0$, contradicting the given that

$\int_a^b f(x)dx = 0$. Thus, $f \leq 0$, so that, since $f \geq 0$, $\forall x \in [a,b]$, $f(x) = 0$. $\blacksquare$

**Question 0.9.** Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : \mathbb{R} \to \mathbb{R}$ such that, $\forall x \in \mathbb{Q}$, $f(x) = 1$, and, $\forall x \in \mathbb{R}\backslash\mathbb{Q}$, $f(x) = 1$. Let $\epsilon = \frac{b-a}{2} > 0$, and, for some $n \in \mathbb{N}$, let $P = \{a, x_1, x_2, \ldots, x_n, b\}$ be a partition of $[a,b]$, with $a < x_1 < x_2 < \ldots < x_n < b$. Since $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ are both dense in $\mathbb{R}$, $\forall i \in \mathbb{N}$ with $1 \leq i \leq n$, $\exists p_i \in \mathbb{Q}$, $q_i \in \mathbb{R}\backslash\mathbb{Q}$, such that $x_{i-1} < p_i, q_i < x_i$. Thus, for $M_i = \sup\{f(x)|x \in \mathbb{R}, x_{i-1} < x < x_i\}$, $S \geq 1$ (as $f(p_i) = 1$), and, for $I = \inf\{f(x)|x \in \mathbb{R}, x_{i-1} < x < x_i\}$, $I \leq 0$ (as $f(q_i) = 0$). Thus, taking $\alpha$ to be the identity in $\mathbb{R}$, $U(P, f, \alpha) \geq (b-a)$, and $L(P, f, \alpha) \leq 0$. By Theorem 6.6, then, since for any partition $P$ of $[a,b]$, $U(P,f,\alpha) - L(P,f,\alpha) > \epsilon$ (as $b - a > 0$, so that $b - a > \frac{b-a}{2}$), by Theorem 6.6, $f \notin \mathcal{R}(\alpha)$ on $[a,b]$. $\blacksquare$

**Lemma 0.10.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be everywhere differentiable, such that $f = f'$, $g = g'$. By Theorem 5.3 (c), $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = \frac{f' - f}{g} = \frac{0}{g} = 0$. Thus, by Theorem 5.11, then $\frac{f}{g} = C_0$ for some $C_0 \in \mathbb{R}$, so that $f = C_0 g$. $\blacksquare$

**Lemma 0.11.** Let $f, g$ be as in Lemma 0.10, with the additional hypothesis that $\exists x_0 \in \mathbb{R}$ such that $f(x) = g(x)$. By the result of Lemma 0.10, $f(x_0) = C_0 g(x_0)$, so that $C_0 = 1$. Therefore, $\forall x \in \mathbb{R}$, $f(x) = g(x)$. $\blacksquare$

**Lemma 0.12.** Let $f$ be as in Lemma 0.11, with the additional hypothesis that $f(0) = 1$. Let $y \in \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{R}$ such that, $\forall x \in \mathbb{R}$, $g(x) = f(x+y)$. Then, since $\forall x \in \mathbb{R}$, $g$ is differentiable at $x$, and $g'(x) = f'(x+y)(x+y)' = f'(x+y) = f(x+y)$, so that $g = g'$. Thus, $g$ satisfies the conditions of Lemma 0.10, so that $f = C_0 g$, for some $C_0 \in \mathbb{R}$. Since $g(0) = f(y)$ and $f(0) = 1$, $f = f(y)g$. Thus, $\forall x, y \in \mathbb{R}$, $f(x+y) = f(y)g(x+y) = f(x)f(y)$. $\blacksquare$