Midterm

21-470 Calculus of Variations
Name: Shashank Singh[1]
Due: Friday, March 28, 2014

Problem 1

The 1^{st} Euler-Lagrange Equation gives,

$$0 = \frac{d}{dx} e^{x^2} e^{y'(x)}, \quad \forall x \in [0,1] \quad \Rightarrow \quad c_1 = e^{x^2} e^{y'(x)}, \quad \forall x \in [0,1]$$

for some $c_1 \in \mathbb{R}$. Solving for $y'(x)$ and integrating gives, $\forall x \in [0,1]$,

$$y'(x) = \log c_1 - x^2 \quad \Rightarrow \quad y(x) = x \log c_1 - \frac{x^3}{3} + c_2,$$

for some $c_2 \in \mathbb{R}$. Since $y(0) = 0$, $c_2 = 0$, and so, since $y(1) = 1$, $\log c_1 = 4/3$. Hence,

$$y(x) = \frac{1}{3}\left(4x - x^3\right), \quad \forall x \in [0,1].$$

Since the exponential function is convex, it is clear that J is convex. Hence this y minimizes J on $\mathscr{Y}$ and J is unbounded above on $\mathscr{Y}$.

Problem 2

The 1^{st} Euler-Lagrange Equation gives,

$$2 = \frac{d}{dx}\left(-2x^2 y'(x)\right), \quad \forall x \in [1,2] \quad \Rightarrow \quad x + c_1 = -x^2 y'(x), \quad \forall x \in [1,2]$$

for some $c_1 \in \mathbb{R}$. Solving for $y'(x)$ and integrating gives, $\forall x \in [1,2]$,

$$y'(x) = -\left(x^{-1} + c_1 x^{-2}\right) \quad \Rightarrow \quad y(x) = c_1 x^{-1} - \ln x + c_2,$$

for some $c_2 \in \mathbb{R}$. The constraint $y(1) = 0$ implies $c_1 + c_2 = 0$. The natural boundary condition is

$$-2x^2 y'(x)\Big|_{x=2} = 0 \quad \Rightarrow \quad 0 = y'(2) = \frac{c_1}{2} - \ln 2 + c_2 \quad \Rightarrow \quad c_1 = -\ln 4, c_2 = \ln 4,$$

so that

$$y(x) = \frac{-\ln 4}{x} - \ln x + \ln 4, \quad \forall x \in [1,2].$$

Since, $\forall x \in [1,2]$, the function $(y,z) \mapsto 2y - x^2 z^2$ is concave, J is concave. Hence this y maximizes J on $\mathscr{Y}$ and J is unbounded below on $\mathscr{Y}$.

[1] sss1@andrew.cmu.edu

Problem 3

Define $G : C^1[0,1] \to \mathbb{R}$ by $G(y) = \int_0^1 xy(x)\,dx$. For any $\lambda \in \mathbb{R}$, the 1^{st} Euler-Lagrange Equation for $J - \lambda G$ gives

$$-\lambda x = \frac{d}{dx} 2y'(x) \quad \Rightarrow \quad -\frac{\lambda}{4}x^2 + c_1 = y'(x),$$

for some $c_1 \in \mathbb{R}$. Integrating gives

$$y(x) = -\frac{\lambda}{12}x^3 + c_1 x + c_2, \quad \forall x \in [0,1],$$

for some $c_2 \in \mathbb{R}$. The boundary conditions $y(0) = 0$ and $y(1) = 1$ imply $c_2 = 0$ and hence $12 = -\lambda + 12c_1$. The constraint $G(y) = 1$ implies

$$60 = 60 \int_0^1 xy(x)\,dx = \int_0^1 -5\lambda x^4 + 60c_1 x^2\,dx = -\lambda x^5 + 20c_1 x^3 \Big|_{x=0}^{x=1} = -\lambda + 20c_1.$$

Hence, $c_1 = 6$ and $\lambda = 60$, so that

$$y(x) = -5x^3 + 6x, \quad \forall x \in [0,1].$$

For all $y \in \mathscr{Y}$, $(J - \lambda G)(y) = \int_0^1 y'(x)^2 - 60xy(x)\,dx$. Since, $\forall x \in [0,1]$, the function $(y, z) \mapsto z^2 - 60xy$ is convex, $J - \lambda G$ is convex. Hence, this y minimizes J on $\mathscr{Y}$.

Problem 4

J is unbounded above on $\mathscr{Y}$, since, for $n \in \mathbb{N}$, if $y_n \in \mathscr{Y}$ is defined by $y_n(x) = \sin(2nx), \forall x \in [0, \pi/2]$,

$$\begin{aligned} J(y_n) &= \int_0^{\pi/2} 2n\cos^2(2nx) - \sin^2(2nx) + 2e^x \sin(2nx)\,dx \\ &\geq \int_0^{\pi/2} 2n\cos^2(2nx) - \sin^2(2nx) - 2e^{\pi/2}\,dx = (2n-1)\frac{\pi}{4} - \pi e^{\pi/2} \to +\infty \end{aligned}$$

as $n \to \infty$. The 1^{st} Euler-Lagrange Equation gives,

$$-2y(x) + 2e^x = \frac{d}{dx} 2y'(x),$$

and so y' is continuously differentiable, with $y''(x) = e^x - y(x)$. This ODE's general solution is

$$y(x) = c_1 \cos(x) + c_2 \sin(x) + \frac{e^x}{2}.$$

The boundary conditions $y(0) = y\left(\frac{\pi}{2}\right) = 0$ imply $c_1 = -1/2$ and $c_2 = -e^{\pi/2}/2$, and so

$$y(x) = \frac{1}{2}\left(e^x - \cos(x) - e^{\pi/2}\sin(x)\right).$$

I believe, but was not able to show, that y minimizes J on $\mathscr{Y}$.

Problem 5

Let $\mathscr{V} := \{v \in C^1[a,b] : \alpha v(a) + \beta v(b) = 0\}$. If y minimizes J on $\mathscr{Y}$, then

$$0 = \delta J(y; v) = \int_a^b f_{,2}(x, y(x), y'(x))v(x) + f_{,3}(x, y(x), y'(x))v'(x)\, dx \tag{1}$$

Since $\{v \in C^1[a,b] : v(a) = v(b) = 0\} \subseteq \mathscr{V}$, Lemma 3.4 gives that the function $x \mapsto f_{,3}(x, y(x), y'(x))$ is continuously differentiable and we have the 1^{st} Euler-Lagrange Equation:

$$\frac{d}{dx} f_{,3}(x, y(x), y'(x)) = f_{,2}(x, y(x), y'(x)) \quad \forall x \in [a,b].$$

Noting that the case $\alpha = 0$ or $\beta = 0$ is already covered by Lemma 3.5, we assume $\alpha, \beta \neq 0$. Integrating Equation (1) by parts, we have, for

$$\begin{aligned} 0 &= f_{,3}(x, y(x), y'(x))v(x)\Big|_{x=a}^{x=b} + \int_a^b \left(f_{,2}(x, y(x), y'(x)) - \frac{d}{dx} f_{,3}(x, y(x), y'(x)) \right) v(x)\, dx \\ &= f_{,3}(a, y(a), y'(a))v(a) - f_{,3}(b, y(b), y'(b))v(b) \end{aligned}$$

Choosing

$$v(x) = -\left(\frac{\alpha}{\beta} + 1 \right) \frac{x-a}{b-a} + 1,$$

$v \in \mathscr{V}$ and, since $v(a) = 1, v(b) = -\frac{\alpha}{\beta}$, giving the additional constraint

$$f_{,3}(a, y(a), y'(a)) = \frac{\alpha}{\beta} f_{,3}(b, y(b), y'(b)).$$

Problem 6

Claim: g must be an affine function. That is, $\exists c_1, c_2 \in \mathbb{R}$ such that $g(x) = c_1 x + c_2, \forall x \in [0,1]$.

Proof: First note that every affine function satisfies the given condition, since

$$\int_0^1 (c_1 x + c_2)v'(x)\, dx = -c_1 \int_0^1 v(x)\, dx + c_2(v(1) - v(0)) = 0, \tag{2}$$

integrating the first term by parts and using the conditions on v. Put

$$A := \int_0^1 g(x)\, dx, \quad B := \int_0^1 \int_0^x g(t)\, dt\, dx, \quad c_1 := 6A - 12B, \quad c_2 := 6B - 2A,$$

and define $v \in C^1[0,1]$ by

$$v(x) := \int_0^x g(t) - c_1 t - c_2\, dt, \quad \forall x \in [0,1].$$

Trivially, $v(0) = 0$. Also,

$$v(1) = \int_0^1 g(t) - c_1 t - c_2 \, dt = A - \frac{c_1}{2} - c_2 = 0$$

and

$$\int_0^1 v(x)\, dx = \int_0^1 \int_0^x g(t) - c_1 t - c_2 \, dt \, dx = B - \int_0^1 \frac{c_1}{2} x^2 - c_2 x \, dx = B - \frac{c_1}{6} - \frac{c_2}{2} = 0.$$

Hence, $v \in \mathscr{V}$. Then, by Equation (2),

$$0 = \int_0^1 g(x) v'(x)\, dx = \int_0^1 (g(x) - c_1 x - c_2) v'(x)\, dx = \int_0^1 (g(x) - c_1 x - c_2)^2 \, dx,$$

and so it follows that $g(x) = c_1 x + c_2$ for all $x \in [0, 1]$.

Problem 7

Put

$$\mathscr{V} := \left\{ v \in C^1[a, b] : v(b) = v(a) - \int_a^b v(x)\, dx = 0 \right\}.$$

Then, if y minimizes J on $\mathscr{Y}$, defining $F \in C^1[a, b]$ by $F(x) = \int_a^x f_{,2}(t, y(t), y'(t))\, dt, \forall x \in [a, b]$,

$$\begin{aligned} 0 = \delta J(y; v) &= \int_a^b f_{,2}(x, y(x), y'(x)) v(x) + f_{,3}(x, y(x), y'(x)) v'(x)\, dx \\ &= \int_a^b \big(f_{,3}(x, y(x), y'(x)) - F(x) \big) v'(x)\, dx, \end{aligned}$$

using integration by parts and $F(a) = v(b) = 0$. Since

$$\left\{ v \in C^1[a, b] : v(a) = v(b) = \int_a^b v(x)\, dx = 0 \right\} \subseteq \mathscr{V},$$

by the result of Problem 6, for some $c_1, c_2 \in \mathbb{R}$,

$$f_{,3}(x, y(x), y'(x)) - F(x) = c_1 x + c_2, \quad \forall x \in [a, b].$$

Hence, the function $x \mapsto f_{,3}(x, y(x), y'(x))$ is in $C^1[a, b]$ and differentiating gives

$$\frac{d}{dx} f_{,3}(x, y(x), y'(x)) = f_{,2}(t, y(t), y'(t)) + c_1, \quad \forall x \in [a, b].$$