Assignment 3

15-359 Probability and Computing
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Section: B
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Problem 2: The randomized quicksort jumped over the lazy bubblesort

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Let $(b_1, b_2, \ldots, b_n)$ be result of sorting $\mathbf{a}$. For $i, j \in \{1, 2, \ldots, n\}$ with $i < j$, let $X_{i,j}$ be an indicator random variable which is 1 if and only if b_i and b_j are compared when $\mathbf{a}$ is quicksorted, and 0 otherwise. Then $X_{i,j} = 1$ if and only if either b_i or b_j is chosen as a pivot before any of $b_{i+1}, b_{i+2}, \ldots, b_{j-1}$, so that

$$E[X_{i,j}] = P(X = 1) = \frac{2}{j - i + 1},$$

and thus, by Linearity of Expectation,

$$E[X] = E\left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{i,j}\right] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{i,j}].$$

This summation can be re-written in terms of $k = j - i$ as

$$E[x] = \sum_{k=2}^{n} \sum_{l=0}^{n-k+1} \frac{2}{k} = \sum_{k-2}^{n} (n - k + 1)\frac{2}{k} \leq n \sum_{k-2}^{n} \frac{2}{k} \leq nH_n \in O(n \log n),$$

where H_n denotes the n^{th} harmonic number. ∎

Problem 3: This is how graduate students spend their free time

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For $i \in \{1, 2, \ldots, n\}$, let X_i be an indicator random variable which is 1 if and only if the i^{th} bin overflows and 0 otherwise. For

$$k = \frac{10 \log n}{\log \log n} + 1,$$

$E[X_i]$ is bounded above by the is bounded above by the number of ways of choosing k balls times the probability that each of those k balls is thrown into the i^{th} bin, so that

$$E[X_i] \leq \binom{n}{k} \frac{1}{n^k} = \frac{n!}{(n-k)!k!n^k} \leq \frac{n^k}{k!n^k} = \frac{1}{k!} \leq \frac{1}{k^{k/2}}.$$

Note that, since $\frac{\log \log \log n}{\log \log n}$ is bounded above by $\frac{1}{2}$,

$$k^{k/2} = \left(\frac{10 \log n}{\log \log n}\right)^{\frac{5 \log n}{\log \log n}} = n^{5\frac{\log \log n - \log \log \log n}{\log \log n}} \geq n^{5(1-0.5)} = n^{5/2}.$$

Therefore, $E[X_i] \leq n^{-5/2}$, so that, by Linearity of Expectation,

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] \leq n^{-3/2} \leq \frac{1}{n}. \qquad \blacksquare$$

The probability that no bin overflows is 1 minus the probability that at least one bin overflows. If this were greater than $\frac{1}{n}$, then

$$E[X] = \sum_{k=1}^{n} k \cdot P(k \text{ bins overflow}) > 1 \cdot P(1 \text{ bin overflows}) \geq \frac{1}{n},$$

contradicting the calculated upper bound of $E[X]$. Thus, the probability that no bin overflows is at least $1 - \frac{1}{n}$. $\blacksquare$

Problem 5: Unsorting algorithm

Call a deck of cards "randomly permuted" if and only if the deck has an equal likelihood of being in any permutation. We proceed by induction on the number of cards below the n^{th} card in the deck, (noting that each move either moves the n^{th} card up one position in the deck, or does not change the position of the n^{th} card), showing that the cards below the n^{th} card are always randomly permuted. When there are 0 no cards below the n^{th} card, the cards are vacuously randomly permuted. When there is 1 card below the n^{th} card, that card is randomly permuted, since there is only 1 permutation of 1 element. Suppose that, for some $k < n - 1$, when there are k cards below the n^{th} card, those k cards are randomly permuted, so that, for any card c and any position p below the n^{th} card, the probability that c is in position p is $\frac{1}{k}$. Then, for each position p below the n^{th} card, the probability that the $(k+1)^{st}$ card is inserted into position p is $\frac{1}{k}$. Therefore, there, for any card c below the n^{th} card (after the insertion of the $(k+1)^{st}$ card), the probability that c is in position p is $frac1k$, so that the cards below the n^{th} card are still randomly permuted. Thus, when the n^{th} card is the top card of the deck, the $(n-1)$ cards below that card are randomly permuted, so that inserting the n^{th} card into the deck produces a randomly permuted deck. $\blacksquare$

Call each repetition of steps 1 and 2 of the algorithm a "move," and use this as the unit by which to measure the algorithm's runtime. Let X be a random variable denoting the number of moves performed in unsorting a deck of n cards, and, for $i \in \{1, 2, \ldots, n-1\}$, let X_i be a random variable denoting the number of moves required to bring the n^{th} card from position $(i+1)$ in the deck to position i in the deck. Thus, $X = 1 + \sum_{i=1}^{n-1} X_i$, so that, by Linearity of Expectation,

$$E[X] = E\left[1 + \sum_{i=1}^{n-1} X_i\right] = 1 + \sum_{i=1}^{n-1} E[X_i].$$

$X_i = k$ means that $k - 1$ cards were inserted in the i positions above the n^{th} card (out of n possible positions), and then one card was inserted below the n^{th} card, so that $X_i \sim \text{Geometric}\left(\frac{n-i}{n}\right)$. Therefore, $\forall i \in \{1, 2, \ldots, n-1\}$, $E[X_i] = \frac{n}{n-i}$. Thus,

$$E[X] = 1 + \sum_{i=1}^{n-1} \frac{n}{n-i} = 1 + n \sum_{i=1}^{n-1} \frac{1}{n} = 1 + nH_{n-1} \in O(n \log n),$$

where H_k denotes the k^{th} harmonic number. ■

Problem 6: Break up the monotony (extra credit)

For notational convenience, $\forall k \in \mathbb{N}, \forall h : \{0,1\}^k \to \mathbb{R}$, we abbreviate $E[h(x_1, x_2, \ldots, x_k)]$ as $E[h]$. For $n = 0$, since f and g are constants, $E[fg] = fg = E[f]E[g]$. Suppose, as an inductive hypothesis, that, for some $n \in \mathbb{N}$, $\forall$ monotonic $h_1, h_2 : \{0,1\}^{n-1} \to \mathbb{R}$, $E[h_1 h_2] \geq E[h_1]E[h_2]$.

Let $f_0, f_1, g_0, g_1 : \{0,1\}^{n-1} \to \mathbb{R}$ such that, $\forall (x_1, x_2, \ldots, x_{n-1}) \in \{0,1\}^{n-1}$,

$$\begin{aligned} f_0(x_1, x_2, \ldots, x_{n-1}) &= f(x_1, x_2, \ldots, x_{n-1}, 0), \\ f_1(x_1, x_2, \ldots, x_{n-1}) &= f(x_1, x_2, \ldots, x_{n-1}, 1), \\ g_0(x_1, x_2, \ldots, x_{n-1}) &= g(x_1, x_2, \ldots, x_{n-1}, 0), \\ g_1(x_1, x_2, \ldots, x_{n-1}) &= g(x_1, x_2, \ldots, x_{n-1}, 1). \end{aligned}$$

Since f and g are monotonic $f_1 \geq f_0$ and $g_1 \geq g_0$, so that $E[f_1] \geq E[f_0]$ and $E[g_1] \geq E[g_0]$. Therefore, $(E[f_1] - E[f_0])\,(E[g_1] - E[g_0]) \geq 0$, and thus

$$E[f_0]E[g_0] + E[f_1]E[g_1] \geq E[f_0]E[g_1] + E[f_1]E[g_0]. \tag{1}$$

Furthermore, conditioning on the value of x_n gives

$$E[f] = \frac{1}{2}\left(E[f_0] + E[f_1]\right), E[g] = \frac{1}{2}\left(E[g_0] + E[g_1]\right). \tag{2}$$

Therefore, conditioning again on the value of x_n,

$$\begin{aligned} E[fg] &= \frac{1}{2}\left(E[f_0 g_0] + E[f_1 g_1]\right) \\ &\geq \frac{1}{2}\left(E[f_0]E[g_0] + E[f_1]E[g_1]\right) && \text{by the inductive hypothesis} \\ &\geq \frac{1}{4}\left(E[f_0]E[g_0] + E[f_0]E[g_1] + E[f_1]E[g_0] + E[f_1]E[g_1]\right) && \text{by (1)} \\ &= \left(\frac{1}{2}E[f_0] + \frac{1}{2}E[f_1]\right)\left(\frac{1}{2}E[g_0] + \frac{1}{2}E[g_1]\right) \\ &= E[f]E[g]. && \text{by (2)} \end{aligned}$$

Thus, by the Principle of Mathematical Induction, the claim in question holds $\forall n \in \mathbb{N}$. ■