21-630 Assignment # 5 Due Wed., February 20, 2013

1. Let f be continuous on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$ and satisfy

$$|f(t,x,\lambda) - f(t,y,\lambda)| \le L|x-y|$$

and

$$|f(t,x,\lambda) - f(t,x,\beta)| \le C|\lambda - \beta|^\varepsilon$$

for all t, x, y, λ and β for some $\varepsilon > 0$. Define $X(t, x_0, \lambda)$ by

$$\begin{cases} \dot{X} = f(t, X(t,x_0,\lambda), \lambda) \\ X(t_0, x_0, \lambda) = x_0. \end{cases}$$

For t_1 and $x_0 \in \mathbb{R}^n$ fixed with $t_1 > t_0$, show that

$$\lambda \mapsto X(t_1, x_0, \lambda)$$

is continuous.

2. Let $f(x,\lambda) = x + \lambda x^2 + \lambda^2 x^3$ and define $X(t, t_0, x_0, \lambda)$ by

$$\begin{cases} \dot{X}(t,t_0,x_0,\lambda) &= f(X(t,t_0,x_0,\lambda), \lambda) \\ X(t_0,t_0,x_0,\lambda) &= x_0 \end{cases}$$

A) Find $X(t,t_0,x_0,0)$ explicitly and compute $\dfrac{\partial X}{\partial x_0}(t,t_0,x_0,0)$ and $\dfrac{\partial X}{\partial t_0}(t,t_0,x_0,0)$ from it.

B) Recompute $\dfrac{\partial X}{\partial x_0}(t,t_0,x_0,0)$ and $\dfrac{\partial X}{\partial t_0}(t,t_0,x_0,0)$ by solving the linear differential equations that they satisfy.

C) Find $\dfrac{\partial X}{\partial \lambda}(t,t_0,x_0,0)$.

3. Solve $u_t(t,x) + x u_x(t,x) = u(t,x)$ with $u(0,x) = g(x)$ given.

4. Let $f(t,x)$ be continuously differentiable on $\mathbb{R} \times \mathbb{R}$ and assume that for all t, x, and y

$$\left|\frac{\partial f}{\partial x}(t,x)\right| \le L$$

and

$$\left|\frac{\partial f}{\partial x}(t,x) - \frac{\partial f}{\partial x}(t,y)\right| \le H|x-y|^p$$

for some $L > 0$, $H > 0$ and $p \in (0,1)$. Show that $\frac{\partial X}{\partial x}$ satisfies a Holder condition in x with exponent p, that is, show that given a time interval $[t_0, t_0 + T]$ there is a constant, $C > 0$, such that

$$|\frac{\partial X}{\partial x_0}(t, t_0, x_0) - \frac{\partial X}{\partial x_0}(t, t_0, x_1)| \leq C|x_0 - x_1|^p$$

for $t_0 \leq t \leq t_0 + T$. To save writing I suggest using the following notation: let

$$X^0(t) = X(t, t_0, x_0)$$

$$X^0_x(t) = \frac{\partial X}{\partial x_0}(t, t_0, x_0)$$

$$X^1(t) = X(t, t_0, x_1)$$

$$X^1_x(t) = \frac{\partial X}{\partial x_0}(t, t_0, x_1).$$