Homework 12

21-630 Ordinary Differential Equations
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Problem 1

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a) $\overline{y}$ may be a critical point, and so there need not exist a transversal T whose center is $\overline{y}$.

b) Define $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ in polar coordinates by

$$f(r, \theta) = \begin{bmatrix} -(r-1)^4 \\ (r-1)^2 + \sin^2(\theta) \end{bmatrix}.$$

There is a transversal L normal to the unit circle and centered at $(0, 1)$. We showed in Problem 2 of Assignment 11 that $\Omega((1, 1))$ is the unit circle and that the solution with $X(0) = (1, 1)$ goes through L infinitely many times, but that $\Omega((0, 1)) = \{(-1, 0)\}$, and hence, by uniqueness, the unit circle is not the orbit of a periodic solution. ■

Problem 2

Consider the Predator-Prey model discussed in lecture, with $a = b = c = d = 2$. $\forall k \in \mathbb{N}$, the solution with $X_k(0) = (2, 1/k)$ is periodic, but the solution with $X(0) = (2, 0)$ is not periodic.

Problem 3

Note that, by the chain rule,

$$\frac{d}{dt}\left((\dot{X})^2 + X^4\right) = 2\dot{X}\ddot{X} + 4X^3\dot{X} = 2\dot{X}(\ddot{X} + 2X^3) = 0,$$

and hence $(\dot{X})^2 + X^4 = C$, for some constant $C \in \mathbb{R}$. I wasn't able to finish this problem.

Problem 4

$$\begin{aligned}
\frac{d}{dt}\left(Z - X^2 - Y^2\right) &= \dot{Z} - 2X\dot{X} - 2Y\dot{Y} \\
&= 2(X^2+Y^2)^{3/2}(1-Z) + 2X^2\left(X^2+Y^2\right) \\
&- 2X\left(X\sqrt{X^2+Y^2}(1-Z) + X^3 - Y\right) \\
&- 2Y\left(Y\sqrt{X^2+Y^2}(1-Z) + X^2Y + X\right) \\
&= 2\sqrt{X^2+Y^2}(1-Z)(X^2+Y^2-(X^2+Y^2)) \\
&+ 2X^2(X^2+Y^2-(X^2+Y^2)) + 2(XY - YX) = 0,
\end{aligned}$$

Thus that $Z - X^2 - Y^2 = C$, for some constant $C \in \mathbb{R}$. Hence, converting to polar coordinates, we replace Z by $C + r^2$, giving

$$\begin{aligned}
\dot{r} &= \dot{X}\cos\theta + \dot{Y}\sin\theta \\
&= \left(r^2\cos^2\theta(1-C-r^2) + r^3\cos^4\theta - r\sin\theta\cos\theta\right) \\
&+ \left(r^2\sin^2\theta(1-C-r^2) + r^3\cos^2\theta\sin^2\theta + r\cos\theta\sin\theta\right) \\
&= r^2(1-C-r^2) + r^3\cos^2\theta = r^2(1-C-r^2+r\cos^2\theta), \\
\dot{\theta} &= \dot{Y}\frac{\cos\theta}{r} - \dot{X}\frac{\sin\theta}{r} \\
&= \left(r\sin\theta\cos\theta(1-C-r^2) + r^2\cos^3\theta\sin\theta + \cos^2\theta\right) \\
&- \left(r\cos\theta\sin\theta(1-C-r^2) + r^2\cos^3\theta\sin\theta - \sin^2\theta\right) = 1.
\end{aligned}$$

Now observe that

$$-r^2 + 1 - C \leq -r^2 + r\cos\theta + 1 - C \leq -r^2 + r + 1 - C.$$

If $C \in (0,1)$, since $-r^2 + 1 - C > 0$ for $r < \sqrt{1-C}$ and $-r^2 + r + 1 - C < 0$ for $r > s$. Thus, the annulus

$$A = \left[\sqrt{1-C}, \frac{1+\sqrt{5-4C}}{2}\right] \times \mathbb{R}$$

defined in polar coordinates is positively invariant. Since $\dot{\theta} = 1$, solutions in A are bounded away from critical points. Since A is bounded, by Poincaré-Bendixson, the planar system has a periodic solution. Since such a solution exists for each $C \in (0,1)$ and (since $Z = r^2 + C$) different values of C lead to distinct solutions, the original system has infinitely many periodic solutions. ■