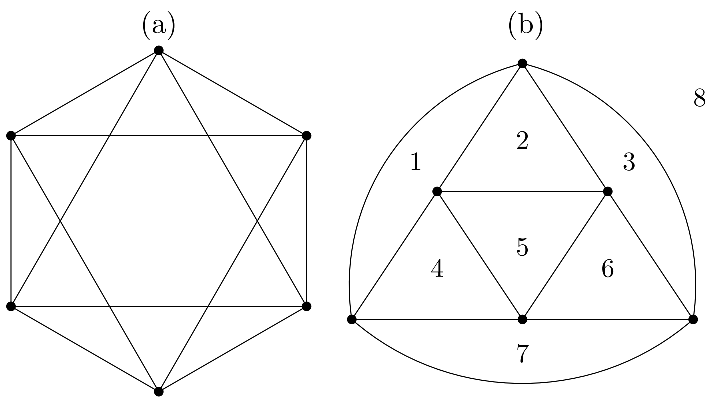
21-484 Notes

JD Nir
jnir@andrew.cmu.edu
April 13, 2012

Def: (p. 228): A graph G is called planar if it can be drawn in the plane such that no two edges intersect.

→ Such a drawing is called a plane graph.

Example: Fig 9.3

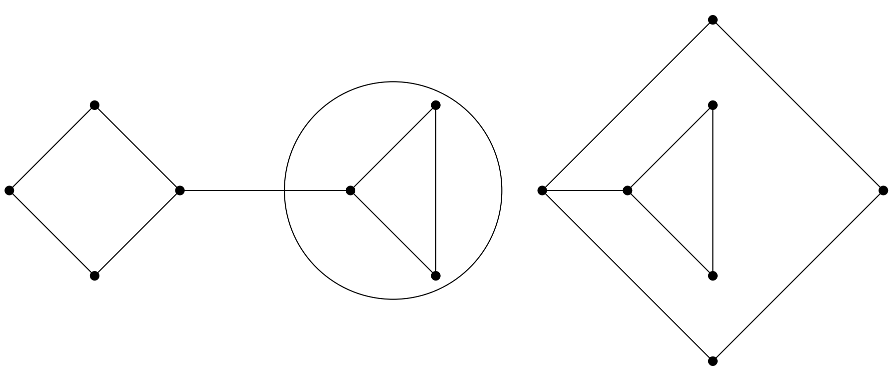


Def: (p. 230): A plane graph divides the plane into connected pieces called regions.

→ The unbounded region is called the exteriror region.

→ The subgraph of a plane graph incident with a given region R is the boundary of R.

Observations: - an edge is on the boundary of 1 region iff it is a bridge. Otherwise it is on the boundary of 2 regions.

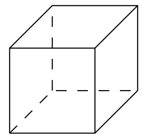


- In a connected plane graph with at least three edges every boundary contains at least three edges.

Theorem: (Thm 9.1, Euler's Identity)

If G is a connected plane graph with n vertices, m edges, and r regions, then

$$n - m + r = 2$$



Proof: If G is a tree then $r = 1$. Since $M = n - 1$ we have $n - m + r = n - (n-1) + 1 = 2$.

- Assume for the sake of contradiction that G is a plane graph, connected, not a tree, had n vertices, m edges, r regions, $n - m + r \neq 2$ and G is minimal (by number of edges) with these properties.

- G is not a tree, so there is an edge that is not a bridge. This edge lies in the boundary of two regions. Remove this edge. Now

$$\begin{aligned} n' &= n \\ m' &= m - 1 \\ r' &= r - 1 \end{aligned}$$

but $n' - m' + r' = n - m + r \neq 2$. $\lightning$ minimality G

Theorem (Thm 9.2): If G is a planar graph with $N \geq 3$ vertices and m edges then

$$m \leq 3n - 6$$

Proof: $\rightarrow$ Assume G is connected.

$\rightarrow$ Draw G as a plane graph.

$\rightarrow$ If $G \cong \bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet$, $n = 3, m = 2$ so $2 \leq 3 \cdot 3 - 6 = 2$

$\rightarrow$ Can assume that G has at least 3 edges. Hence every boundary has at least 3 edges.

$\rightarrow$ Let $m_1, m_2, \ldots, m_r$ be the number of edges in the boundaries. $m_1 \geq 3$.

$\rightarrow$ Consider $sm \geq M = \sum_{i=1}^{r} m_i \geq 3 \cdot r \Rightarrow 2m \geq 3r$

$\rightarrow$ By Euler's Identity $6 = 3n - 3m + 3r \leq 3n - 3m + 2m = 2n - m \Rightarrow m \leq 2n - 6$

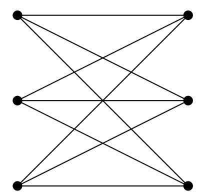
$\rightarrow$ If G is disconnected, we can add edges while maintaining planarity to get a connected planar graph. Apply this.

$\rightarrow$ Corollary: If G is planar, then $\delta(G) \leq 5$.

Proof: If G was planar with minimal degree ≥ 6 then

$$2m = \sum \deg \geq 6n$$
$$\Downarrow$$
$$m \geq 3n \ \lightning \text{ last theorem}$$

<u>Example:</u> $K_{3,3}$



$n = 6$
$m = 9$ $\quad 9 \leq 3 \cdot 6 - 6 = 12$

$\rightarrow$ $K_{3,3}$ is not planar.

<u>Proof:</u> If it was planar, we could draw it as a plane graph.

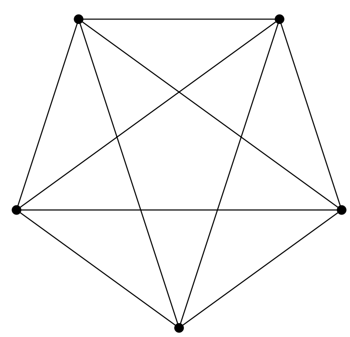
The plane graph will have $r = 2 - n + m = 2 - 6 + 9 = 5$ regions.

THe boundary of each region has at least 4 edges, since $K_{3,3}$ contains no triangles. Let m_i be the number of edges in boundaries.

$$18 = 2m = M = \sum_{i=1}^{5} m_i \geq 4r = 20 \; \lightning$$

$\rightarrow$

K_5 $\quad$ $n = 5$, $m = 10$



by the last theorem, K_5 is not planar