21-484 Notes

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Definitions: (p. 267-269)

- A proper coloring of the vertices of a graph G is a mapping $f : V(G) \to C$ such that adjacent vertices get different colors. (Also coloring of G).

$\to$ The smallest number of colors for which there is a proper coloring of G is the chromatic number of G, denoted $\chi(G)$.

$\to$ k-colorable = k-chromatic, minimum coloring

$\to$ Given a coloring of $G = (V, E)$, the set of all vertices with the same color is called color class. If V_i is a color class then $G[V_i]$ is an independent set.

$\to$ The set of all color classes is a partition of V (into independent sets).

$\to$ The independence number of G is the size of a maximum independent set. Denoted $\alpha(G)$.

$\to$ The clique number of G is the size of a maximum clique (= complete subgraph). Denoted $\omega(G)$.

Fact (Thm 10.5): Let G be a graph with n vertices. Then

$$1 \ \chi(G) \geq \omega(G) \text{ and } 2 \ \chi(G) \geq \frac{n}{\alpha(G)}$$

proof: 1 Let H be a maximum clique. Then every coloring requires at least $|V(H)|$ colors just to color H. $\chi(H) = |V(H)|$. Since $H \subseteq G$, $\chi(G) \geq \chi(H)$.

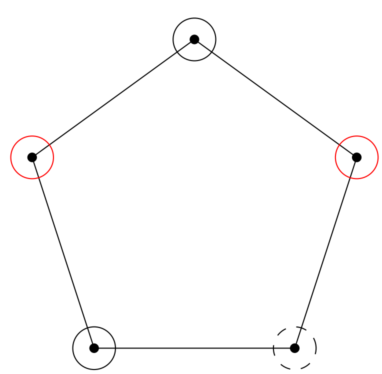
2 For a given coloring of G, let $V_1, \ldots, V_k$ be the color classes. V_i is an independent set, so $|V_i| \leq \alpha(G)$.

$$n = \sum_{i=1}^{k} |V_i| \leq k \cdot \alpha(G) \Rightarrow k \geq \frac{n}{\alpha(G)}$$

claim (Thm 10.7): For every G, $\chi(G) \leq \Delta(G) + 1$.

Proof: Color the vertices one by one. When coloring a vertex, there are at most $\delta(G)$ colors that we cannot use, so we have an available color.

Examples:

$C_5 =$ 

$\Delta(C_5) = 2$

$\chi(C_5) = 3$

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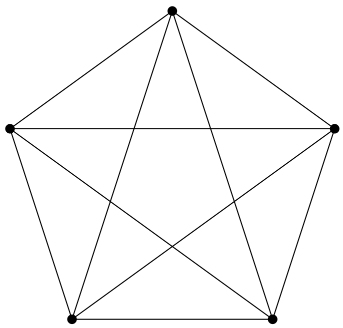
$K_5 =$ 

$\Delta(K_5) = 2$

$\chi(K_5) = 3$

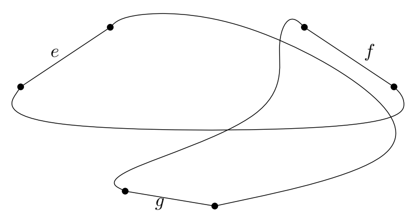
Thm (Brooks Thm, Thm 10.8): For every connected graph G other than an odd cycle or a complete graph $\chi(G) \leq \Delta(G)$.

Proof:

→ We can assume that G is connected.

→ We can assume that G is 2-connected.

- → We can decompose G into Blocks (Section 5.2), color each block separately and merge the colorings.
- → For a pair of edges e and f, let eBf iff $e = f$ or e and f lie in a common cycle.
- → B is an equivalence relation.



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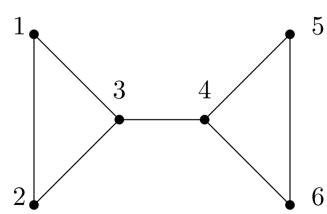
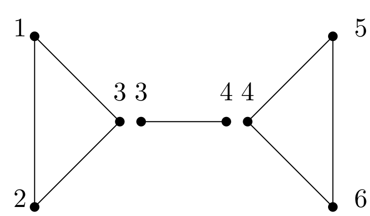
- → The equivalence classes are the blocks.
- → A block in a graph G is a maximal by inclusion nonseparable subgraph.





→ We can assume that $\Delta \geq 3$. Otherwise the graph is an even cycle, which is 2-colorable.

→ If G has a vertex v of degree less than Δ. Consider a breadth-first search tree starting from v (v is the root). Color the vertices according to distance, farthest first. At every step, the parent of the current vertex is not colored, hence there are at most $\Delta - 1$ colors that we cannot use (and we have Δ colors). In the final step we color v which has degree $< \Delta$.

→ Assumptions: G is 2-connected, Δ-regular for $\Delta \geq 3$, not complete.

→ Find a spanning tree having a root v with two neighbors of v: U, w such that u and w are leaves and $uw \notin E(G)$.