Homework 4

21-740 Introduction to Functional Analysis II
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Problem 1

Lemma 1 Suppose $\mathcal{D}(B) \subseteq X$ is dense in X, $B : \mathcal{D}(B) \to X$ is linear and closed. Then, for any integrable $g : [0, \tau] \to \mathcal{D}(B)$, $t \in [0, \tau]$, if $B \circ g$ is integrable, then

$$B \int_0^t g(s)\, ds = \int_0^t Bg(s)\, ds.$$

Proof: Claim 1: Suppose g is a simple function, i.e., for some $k \in \mathbb{N}$, $c_1, \dots, c_k \in \mathcal{D}(B)$,

$$g = \sum_{i=1}^{k} c_i \chi_{E_i},$$

where $\{E_i : i = 1, \dots, k\}$ is a partition of $[0, \tau]$ into Lebesgue-measurable sets and χ_{E_i} is the characteristic function of A_i (we refer to $\{E_i : i = 1, \dots, k\}$ as the *partition underlying* g). Then,

$$B \int_0^t g(s)\, ds = B \sum_{i=1}^{k} c_i \lambda(E_i) = \sum_{i=1}^{k} Bc_i \lambda(E_i) = \int_0^t Bg(s)\, ds,$$

where λ denotes the Lebesgue measure, proving Claim 1.

General Case: Now consider a sequences of simple functions $\{g_k\}_{k=1}^\infty$ mapping $[0, \tau]$ to $\mathcal{D}(B)$ and $\{h_k\}_{k=1}^\infty$ mapping $[0, \tau]$ to X such that

$$\sup_{t \in [0,\tau]} \|g(t) - g_k(t)\|_X \to 0 \quad \text{and} \quad \sup_{t \in [0,\tau]} \|Bg(t) - h_k(t)\|_X \to 0 \quad \text{as } k \to \infty$$

(such sequences exist because $\mathcal{D}(B)$ is dense in X and both g and $B \circ g$ are integrable.) In particular, we can take such sequences such that $g_k(s) = g(s)$ for some s in each set of the partition underlying g, and, $\forall k \in \mathbb{N}$, the partition of $[0, \tau]$ underlying g_k is a refinement of the partition underlying h_k. It follows from these constraints that

$$\sup_{t \in [0,\tau]} \|Bg_k(t) - h_k(t)\|_X \to 0 \quad \text{as } k \to \infty,$$

and so, by Claim 1,

$$\lim_{k\to\infty} B \int_0^t g_k(s)\, ds = \lim_{k\to\infty} \int_0^t Bg_k(s)\, ds = \lim_{k\to\infty} \int_0^t h_k(s)\, ds = \int_0^t Bg(s)\, ds.$$

Since B is closed, $\int g_k \to \int g$, and $B \circ \int g_k \to \int B \circ g$, it follows that $\int g : [0, \tau] \to \mathcal{D}(B)$ and $B \int g = \int B \circ g$. $\square$

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We calculate

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\left(\int_0^t T(t+h-s)f(s)\,ds-\int_0^t T(t-s)f(s)\,ds\right) &= \lim_{h\to 0}\frac{1}{h}\int_0^t (T(h)-I)T(t-s)f(s)\,ds\\
&= \lim_{h\to 0}\frac{(T(h)-I)}{h}\int_0^t T(t-s)f(s)\,ds\\
&= A\int_0^t T(t-s)f(s)\,ds = Av(t)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)\,ds &= \lim_{h\to 0} T(h)\frac{1}{h}\int_t^{t+h} T(t-s)f(s)\,ds\\
&= \lim_{h\to 0} T(h)\frac{1}{h}\int_t^{t+h} T(s)f(t-s)\,ds\\
&= If(t) = f(t).
\end{aligned}$$

Thus, $\dot{v}(t) = Av(t) + f(t), \forall t \in [0,\tau]$.

Problem 2

By a generation theorem from class, there exist constants $C, \omega \in \mathbb{R}$ and $\delta \in (0, \pi/2)$ such that

$$\left\{\lambda \in \mathbb{C} : |\arg(\lambda-\omega)| < \frac{\pi}{2}+\delta\right\} \subseteq \rho(A)$$

and

$$\|R(\lambda; A)\| \le \frac{C}{|\lambda-\omega|}, \quad \forall \lambda \in \mathbb{C} \text{ with } |\arg(\lambda-\omega)| < \frac{\pi}{2}+\delta.$$

Furthermore, by another part of the same theorem, it suffices to show that

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega + \|L\|\} \subseteq \rho(A+L) \quad \text{ and } \quad \|R(\lambda; A+L)\| \le \frac{C}{\lambda-(\omega+C\|L\|)},$$

whenever $\mathrm{Re}\,\lambda > \omega + C\|L\|$. Since L is bounded, it can shift the spectrum by at most $\|L\|$, and so

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega + \|L\|\} \subseteq \left\{\lambda \in \mathbb{C} : |\arg(\lambda-(\omega+\|L\|))| < \frac{\pi}{2}+\delta\right\} \subseteq \rho(A+L).$$

It is easily checked from the definition of the resolvent operator that

$$R(\lambda; A+L) = \sum_{k=0}^{\infty} R(\lambda; A)\left(LR(\lambda; A)\right)^k$$

(we showed this as a lemma for Problem 6 of Assignment 3). Thus, by the triangle inequality and the identity $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$, for $\operatorname{Re}\lambda - \omega > C\|L\|$,

$$\begin{aligned}
\|R(\lambda; A+L)\| \leq \sum_{k=0}^{\infty} \frac{C^{k+1}\|L\|^k}{(\lambda-\omega)^{k+1}} &= \frac{C}{\lambda-\omega}\sum_{k=0}^{\infty}\frac{C^k\|L\|^k}{(\lambda-\omega)^k} \\
&= \frac{C}{\lambda-\omega}\left(1-\frac{C\|L\|}{\lambda-\omega}\right)^{-1} \\
&= \frac{C}{\lambda-\omega}\left(\frac{\lambda-\omega}{\lambda-\omega-C\|L\|}\right) = \frac{C}{\lambda-\omega-C\|L\|}. \quad \blacksquare
\end{aligned}$$

Problem 3

We assume without loss of generality that T is quasicontractive, since there exists an equivalent norm under which T is quasicontractive, and it suffices to prove the result for an equivalent norm. In particular, suppose $\omega \in \mathbb{R}$ with $\|T(t)\| \leq e^{\omega t}, \forall t \geq 0$. Note that $\forall c \in \mathbb{R}$,

$$\|e^{cI}\| = \left\|\sum_{k=0}^{\infty}\frac{(cI)^k}{k!}\right\| = \left\|\sum_{k=0}^{\infty}\frac{c^k}{k!}I\right\| = \|e^c I\|. = e^c.$$

Since $\|T(t)\| \leq e^{\omega t}$, there exists $\varepsilon > 0$ such

$$\frac{\|T(t)\|-1}{h} \leq 1 + \frac{d}{dt}e^{\omega t}\bigg|_{t=0} = \omega + 1,$$

for all $h \in (0,\varepsilon)$. Since, $\forall t \geq 0, h \in (0,\varepsilon)$, since $T(h)$ and I commute,

$$\begin{aligned}
\|e^{t\mathcal{A}_h}\| = \left\|\exp\left(\frac{t(T(h)-I)}{h}\right)\right\| &= \left\|\exp\left(\frac{tT(h)}{h}\right)\right\| \|\exp(-tI/h)\| \\
\leq \exp(t\|T(h)\|/h)\exp(-t/h) &= \exp\left(\frac{t(\|T(h)\|-1)}{h}\right) \leq e^{(\omega+1)t}
\end{aligned}$$

By the Fundamental Theorem of Calculus, $\forall t \geq 0, h \in (0,\varepsilon), x \in X$

$$\begin{aligned}
T(t)x - e^{t\mathcal{A}_h}x &= \int_0^t \frac{d}{ds}e^{(t-s)\mathcal{A}_h}T(s)x\,ds \\
&= \int_0^t e^{(t-s)\mathcal{A}_h}(A-\mathcal{A}_h)T(s)x\,ds \\
&= \int_0^t e^{(t-s)\mathcal{A}_h}T(s)(Ax-\mathcal{A}_h x)\,ds
\end{aligned}$$

since A and each $\mathcal{A}_h$ commutes with $T(t)$ for all $t \geq 0$. Then, $\forall t \geq 0, x \in X$,

$$\begin{aligned}
\|T(t)x - e^{t\mathcal{A}_h}x\| &\leq \int_0^t \|e^{(t-s)\mathcal{A}_h}\|\|T(s)\|\|(Ax-\mathcal{A}_h x)\|\,ds \\
&\leq te^{(2\omega+1)t}\|(Ax-\mathcal{A}_h x)\| \to 0,
\end{aligned}$$

as $h \to 0$, by definition of A and $\mathcal{A}_h$. $\blacksquare$

Problem 4

If $x \in \mathcal{D}(A)$ is non-zero, then

$$\operatorname{Re}[Ax, x] = \|x\|^2 \operatorname{Re}\left[A\frac{x}{\|x\|}, \frac{x}{\|x\|}\right] \leq 0.$$

Thus, A is dissipative. As in the proof of Lemma 10.10, we have, $\forall x \in X, \lambda \in \mathbb{C}$,

$$\|(\lambda I - A)x\|\|x\| \geq |[(\lambda I - A)x, x]| = \operatorname{Re}[(\lambda I - A)x, x] = \operatorname{Re}\lambda\|x\|^2 - \operatorname{Re}[Ax, x] \geq \operatorname{Re}\lambda\|x\|^2.$$

It follows that, for $\operatorname{Re}\lambda > 0$, $\lambda I - A$ is injective and, if $\lambda I - A$ is surjective, then $R(\lambda; A) \leq \frac{1}{\operatorname{Re}\lambda}$. By a generation theorem for analytic semigroups, it suffices now to show that

$$H := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subseteq \rho(A),$$

for which we essentially follow the proof of Lemma 10.11.

Define $\Lambda := \rho(A) \cap H$. Since $\rho(A)$ is open in $\mathbb{C}$, $\Lambda \neq \emptyset$ by assumption, and H is connected, it suffices to show that Λ is closed in the relative topology on H. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence in Λ converging to $\lambda \in H$. To show $\lambda \in \Lambda$, it suffices to show $\lambda I - A$ is surjective. Let $y \in X$ and, $\forall n \in \mathbb{N}$, put

$$x_n := R(\lambda_n; A)y.$$

Noting that, since $1/\lambda_n \to 1/\lambda$ as $n \to \infty$, the sequence $\{1/\lambda_n\}_{n=1}^{\infty}$ is bounded,

$$\begin{aligned}
\|x_n - x_m\| &= \|R(\lambda_n; A)y - R(\lambda_m; A)y\| \\
&= |\lambda_n - \lambda_m|\|R(\lambda_n; A)\|\|R(\lambda_m; A)\|\|y\| \\
&\leq |\lambda_n - \lambda_m|\frac{\|y\|}{\lambda_n\lambda_m}
\end{aligned}$$

using a resolvent identity (Proposition 7.26). It follows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy, and so we may put

$$x := \lim_{n\to\infty} x_n.$$

Note that each $x_n \in \mathcal{D}(A)$ and $Ax_n \to \lambda x - y$. Since A is closed, $x \in \mathcal{D}(A)$ and $y = \lambda x - Ax$.

Problem 5

Since T is analytic, by Proposition 11.19, for $t > 0$, $T(t) : X \to \mathcal{D}(A)$. Thus, for $s, t \in [0, \tau]$ with $s < t$, $T(t-s)(f(s) - f(t)) \in \mathcal{D}(A)$, and hence $w(t) \in \mathcal{D}(A), \forall t \in [0, \tau]$.

Let $f \in C^{0,\theta}([0, \tau]; X)$ with $C > 0$ such that

$$\|f(t) - f(s)\| \leq C|t-s|^\theta, \quad \forall s, t \in [0, \tau].$$

Since T is an analytic semigroup, $\exists K > 0$ such that

$$\|AT(t)\| \leq K/t, \quad \forall t \in [0, \tau],$$

and let $s, t \in [0, \tau]$ with $s < t$. Adding forms of 0, by integral and semigroup properties,

$$\begin{aligned}
w(t) - w(s) &= \int_0^t T(t-r)(f(r) - f(t))\, dr - \int_0^s T(s-r)(f(r) - f(s))\, dr \\
&= \int_0^s T(t-r)(f(r) - f(t)) - T(s-r)(f(r) - f(s))\, dr + \int_s^t T(t-r)(f(r) - f(t))\, dr \\
&= \int_0^s (T(t-s) - I)T(s-r)(f(r) - f(s))\, dr + \int_s^t T(t-r)(f(r) - f(t))\, dr \\
&+ \int_0^s T(t-r)(f(s) - f(t))\, dr.
\end{aligned}$$

We now bound the norm of A applied to each of these three terms.

$$\begin{aligned}
\left\| A \int_0^s T(t-r)(f(s) - f(t))\, dr \right\| &\leq \int_0^s \|AT(t-r)\| \|(f(s) - f(t))\|\, dr \\
&\leq \int_0^s \frac{k}{t-r} C(t-s)^\theta\, dr \leq kC \int_0^s (t-r)^{\theta-1}\, dr \\
&= \frac{kC}{\theta}(t^\theta - (t-s)^\theta)
\end{aligned}$$

$$\begin{aligned}
\left\| A \int_s^t T(t-r)(f(r) - f(t))\, dr \right\| &\leq \int_s^t \|AT(t-r)\| \|(f(r) - f(t))\|\, dr \\
&\leq \int_s^t \frac{k}{t-r} C(t-r)^\theta\, dr \\
&\leq kC \int_0^{t-s} t^{\theta-1}\, dr = \frac{kC}{\theta}(t-s)^\theta.
\end{aligned}$$

[I didn't have time to finish writing this part up. I'm not sure I divided the pieces of Aw correctly...]

Thus, we have, $\forall s, t \in [0, \tau]$,

$$\|Aw(t) - Aw(s)\| \leq \frac{kC}{\theta}|t-s|^\theta,$$

so that $Aw \in C^{0,\theta}([0, \tau]; X)$. ■