21-484 Notes

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1. Hadwiger's conjecture: $\chi(G) = k \Rightarrow G$ contains a K_k minor.

$k = 2$ ✓ trivial

$k = 3$ $\chi(G) > 2 \iff G$ is not bipartite $\iff G$ contains an odd cycle $\Rightarrow G$ contains a K_3 minor.

$k = 4$ Proved by Hadwiger

$k = 5$ Wagner showed that this case equivalent to the 4-colors theorem.

If G is not 4 colorable $\Rightarrow G$ contains a K_5 minor $\underset{\text{Wagner's}}{\Rightarrow}$ G is not planar

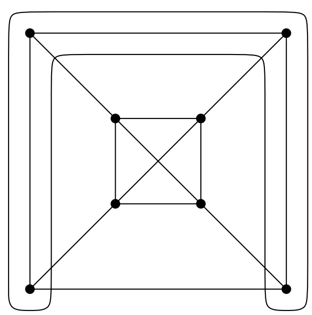
$k = 6$ Robertson, Seymour, Thomas ('93)

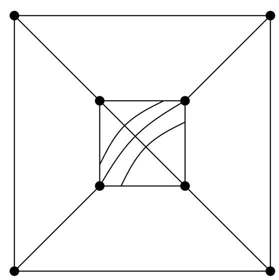
$k = 7$ known: not 6-colorable $\Rightarrow K_7$ minor or both $K_{4,4}$ and $K_{3,5}$ minors.

Hadwiger's conjecture is true for most graphs.

Theorem (Bollobás ect.): Pr[Hadwiger's conjecture is true in $G(n, 1/2)$] $\overset{n\to\infty}{\Rightarrow}$ 1

2 What's the genus of

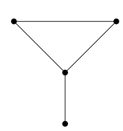




3(a):

$$k = \left(\max_{H \subseteq G} \delta(G)\right) + 1$$

$\rightarrow$ take v_n to be a vertex of degree $\delta(G)$.



$\rightarrow$ remove v_n and pick v_{n-1} in the same way.

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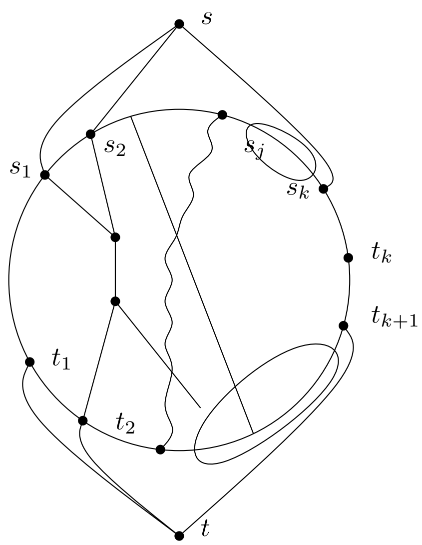
3(b):



→ add a source and a target, s, t
→ the new graph is k-connected
→ Apply Menger's Theorem
→ use planarity to argue that the paths are all s_1–t_i paths

Ramsey's theorem: $r(n,m)$ is finite.

double induction on n,m

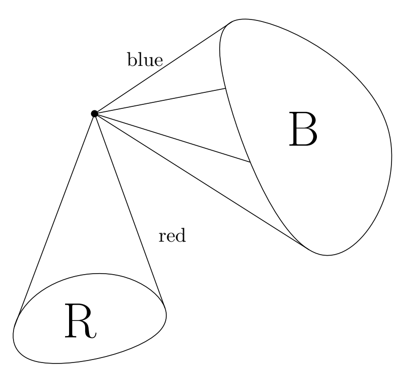
$\rightarrow r(n,1) = r(1,m) = 1$

$\rightarrow r(n,m) \le r(n-1,m) + r(n,m-1)$

proof:

Consider a two coloring of K_r where $r = r(n-1,m) + r(n,m-1)$. Pick a vertex v.

Let B be the set of vertices adjacent to v via a blue edge. Same for R.



Since $r = r(n-1,m) + r(n,m-1) = |B| + |R| + 1$ it is not the case that both $|B| < r(n-1,m)$ and $|R| < r(n,m-1)$

Assume without loss of generality $|B| \ge r(n-1,m)$. If the graph induced on B contains a red m cliquem we are done. If it contains a blue $n-1$ clique, then $|B| \cup \{v\}$ contains an n-clique.

Theorem: (11.2) $r(n_1, n_2, \ldots, n_k)$ is finite.

Proof: induct on k.

✓ $k = 2$

$\rightarrow r(n_1, n_2, n_3, \ldots, n_k)) \leq r = r(n_1, n_2, \ldots, n_{k-2}, r(n_{k-1}, n_k))$

$\rightarrow$ conider coloring of K_r with $k-1$ colors. Either we have a clique of size n_i colored i for $1 \leq i \leq k-2$ or we have a clique of size $r(n_{k-1}, n_k)$ colored in one color. Apply the induction again.

$r(F_1, F_2, \ldots, F_k)$ is finite.

$$\begin{array}{ccccc}
\mathrm{r}(s,2) & = & s & & \\
\mathrm{r}(3,3) & = & 6 & & \\
\mathrm{r}(4,3) & = & 9 & & \\
\mathrm{r}(4,4) & = & 18 & & \\
\mathrm{r}(4,5) & = & 25\ (1995) & & \\
\mathrm{r}(4,6) & \geq & 36\ (2012) & \leq & 41 \\
43 & \leq & \mathrm{r}(5,5) & \leq & 49
\end{array}$$

<u>Thm:</u> $r(n,m) \leq \binom{n+m+2}{n-1}$

$\rightarrow$ Same proof.

Erdős-Szekeres

$$r(n) \leq (1+o(1)) \frac{1}{\sqrt{\pi n}} 4^{n-1}$$

Erdős

$$r(n) \geq (1+o(1)) \frac{n}{\sqrt{2} \cdot e} \sqrt{2}^n$$

$$\underset{\text{Joel Spencer}}{(1+o(1)) \frac{\sqrt{2} n}{e} \sqrt{2}^n} \leq r(n) \leq \underset{\text{David Conlon}}{4^n \cdot n^{-\frac{c \cdot \log n}{\log \log n}}}$$