Lecture Notes for Week 14 (First Draft)

Local Convexity and Seminorms

Let X be a linear space over $\mathbb{K}$. There is a natural correspondence between topologies τ on X such that (X, τ) is a locally convex topological vector space and separating families of seminorms on X.

Theorem 14.1: Let X be a locally convex topologivcal vector space and assume that $\mathcal{B}$ is a local base such that each $V \in \mathcal{B}$ is balanced and convex. (Each $V \in \mathcal{B}$ is automatically absorbing.) Then $\{p^V : V \in \mathcal{B}\}$ is a separating family of continuous seminorms.

Proof: We know from Lemma 5.32 that each p^V is a seminorm. The Hausdorff property implies that the family $\{p^V\ V \in \mathcal{B}\}$ is separating. Let $V \in \mathcal{B}$ and $x \in V$ be given. Since V is open and scalar multiplication is continuous, we may choose $t > 1$ such that $tx \in V$. It follows that $p^V(x) < 1$. In particular, we have

$$p^V(x) < 1 \ \text{ for all } x \in V,\ V \in \mathcal{B}.$$

Let $V \in \mathcal{B}$ and $\epsilon > 0$ be given. Then for all $x, y \in X$ with $x - y \in \epsilon V$, we have

$$|p^V(x) - p^V(y)| \leq p^V(x - y) < \epsilon,$$

by virtue of Proposition 6.8(c). $\square$

Theorem 14.2: Let X be a linear space over $\mathbb{K}$ and $\mathcal{P}$ be a separating family of seminorms. For each $p \in \mathcal{P}$ and $n \in \mathbb{N}$ put

$$V(p, n) = \{x \in X : p(x) < \frac{1}{n}\}.$$

Let $\mathcal{B}$ be the collection of finite intersections of sets of the form $V(p, n)$, $p \in \mathcal{P}$, $n \in \mathbb{N}$. Then each member of $\mathcal{B}$ is balanced and convex. Moreover $\mathcal{B}$ is a local base for a topology τ such that (X, τ) is a topological vector space and each $p \in \mathcal{P}$ is continuous. Furthermore, for every $E \subset X$ we have that E is topologically bounded if and only if

$$\forall p \in \mathcal{P}, \quad p \text{ is toplogically on } E.$$

Proof: Declare $S \subset X$ to be open if and only if S can be expressed as a union of translates of members of $\mathcal{B}$. This collection of open sets is a translation invariant topology on X. To see that the Hausdorff property holds, let $x, y \in X$ with $x \neq y$ be given. Put $z = x - y$. Since $z \neq 0$ we may choose $p \in \mathcal{P}$ such that $p(z) > 0$. Put $r = p(z)$ and choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{r}{2}$. Then $x + V(p, N)$ is a neighborhood of x and $y + V(p, N)$ is a neighborhood of y such that

$$(x + V(p, N)) \cap (y + V(p, N)) = \emptyset.$$

To see that addition is continuous, let U be a neighborhood of 0. Then we may choose $p_1, p_2, \cdots, p_k \in \mathcal{P}$ and $n_1, n_2, \cdots, n_k \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{k} V(p_i, n_i) \subset U.$$

Now, put

$$W = \bigcap_{i=1}^{k} V(p_i, 2n_i).$$

Using the subadditivity of seminorms, we conclude that $W + W \subset U$, and consequently addition is continuous at $(0,0)$. Since the topology is translation invariant, we conclude that addition is continuous on $X \times X$.

To see that scalar multiplication is continuous, let $x \in X$, $\alpha \in \mathbb{K}$, and a neighborhood U of 0 be given. Define W as above and choose $s > 0$ such that $x \in sW$. (Any $s > 2n_i p_i(x)$ for all $i \in \{1, 2, \cdots, k\}$ will do.) Now choose $N \in \mathbb{N}$ with

$$N > \frac{(1 + |\alpha| s)}{s},$$

and put

$$\hat{W} = \bigcap_{i=1}^{k} V(p_i, 2Nn_i).$$

Observe that

$$\hat{W} \subset \frac{(1 + |\alpha| s)}{s} W.$$

Then for all $y \in x + \hat{W}$ and $\beta \in \mathbb{K}$ with $|\beta - \alpha| < \frac{1}{s}$ we have

$$\begin{aligned} \beta y - \alpha x &= \beta(y - x) + (\beta - \alpha)x \\ &\in |\beta| \hat{W} + |\beta - \alpha| s W \\ &\in W + W \subset U. \end{aligned}$$

It follows that scalar multiplication is continuous.

Let $p \in \mathcal{P}$ be given. From the construction of $\mathcal{B}$ we see that p is continuous at 0. It follows from Proposition 6.8 that p is continuous on X.

Let $E \subset X$ be given. Assume that E is topologically bounded. Let $p \in \mathcal{P}$ be given. Since $V(p, 1)$ is a neighborhood of zero, we may choose $k \in \mathbb{N}$ such that $E \subset kV(p, 1)$. This implies that $p(x) \leq k$ for all $x \in E$.

Assume now that p is bounded on E for each $p \in \mathcal{P}$. Let U be a neighborhood of 0. Then we may choose $p_1, p_2, \cdots, p_k \in \mathcal{P}$ and $n_1, n_2, \cdots, n_k \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{k} V(p_i, n_i) \subset U.$$

For each $i \in \{1, 2, \cdots, k\}$, we may choose $M_i \in \mathbb{R}$ such that

$$p_i(x) < M_i \text{ for all } x \in E.$$

Then for all $t > \max\{M_i n_i : i = 1, 2, \cdots, k\}$ we have

$$E \subset tU. \quad \square$$

Remark 14.3: It is not too difficult to show that if X is a locally convex topological vector space and one takes any local base $\mathcal{B}$ consisting of balanced convex sets and then applies the construction of Theorem 14.2 to the family $\{p^V : V \in \mathcal{B}\}$ of seminorms, the topology so obtained coincides with the original topology of X.

Cauchy Sequences and Completeness

Definition 14.4: Let X be a topological vector space. A sequence $\{x_n\}_{n=1}^\infty$ is said to be a *Cauchy sequence* provided that for every neighborhood U of 0 there exists $N \in \mathbb{N}$ such that

$$x_n - x_m \in U \text{ for all } m, n \in \mathbb{N} \text{ with } m, n \geq N.$$

Remark 14.5: Let X be a metrizable topological vector space and let ρ be a translation invariant metric that induces the topology. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X. Then $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in the sense of Definition 14.4 if and only if it is a Cauchy sequence in the metric space (X, ρ). In particular, all translation invariant metrics generating the topology of X have the same Cauchy sequences.

Definition 14.6: By a *Fréchet space* we mean a topological vector space whose topology is induced by a complete translation invariant metric.

Remark 14.7: It is important to keep in mind that in a Fréchet space topological boundedness of a set is not the same as boundedness in the metric sense.

Remark 14.8: Many authors require Fréchet spaces to be locally convex as part of the definition. The term *F-space* is also used in some places to indicate a topological vector space whose topology is induced by a complete translation invariant metric. Some authors require F-spaces to be locally convex, while others do not. In any event you should check carefully the convention being used with regard to local convexity whenever you encounter the term "Fréchet space" or "F-space" in the literature.

The Open Mapping Theorem and Closed Graph Theorem extend to Fréchet spaces (no local convexity needed). The statements of these results are recorded below. A good reference for the proofs is Rudin.

Theorem 14.9 (Open Mapping Theorem): Let X and Y be Fréchet spaces and assume that $T : X \to Y$ is continuous, linear, and surjective. Then T is an open mapping, i.e. $T[U]$ is open in Y for every open set $U \subset X$.

Theorem 14.10 (Closed Graph Theorem): Let X and Y be Fréchet spaces. Assume that $T : X \to Y$ is linear and that $\mathrm{Gr}(T)$ is closed. Then T is continuous.

A version of the Banach-Steinhaus Theorem (Principle of Uniform Boundedness) holds for topological vector spaces. In order to state such a result, we need to extend the definitions of sets of the first and second categories to topological vector spaces and to make a definition of equicontinuos family of linear mappings.

Definition 14.11: Let X be a topological vector space and $S \subset X$. We say that S is

(a) *nowhere dense* provided $\mathrm{int}(\mathrm{cl}(S)) = \emptyset$.

(b) of the *first category* if S can be expressed as a countable union of nowhere dense sets.

(c) of the *second category* if S it is not of the first category.

Definition 14.12: Let X and Y be topological vector spaces and $(T_i | i \in J)$ be a family of linear mappings from X to Y. We say that $(T_i | i \in J)$ is *equicontinuous* provided that for every neighborhood V of 0 in Y there is a neighborhood U of 0 in X such that $T_i[U] \subset V$ for all $i \in J$.

An good reference for the proof of Theorem 14.13 below is Rudin.

Theorem 14.13 (Banach-Steinhaus Theorem): Let X be a Fréchet space and Y be a topological vector space. Let $(T_i | i \in I)$ be a family of continuous linear mappings from X to Y and assume that

$$\forall x \in X, \quad \{T_i x : i \in J\} \text{ is topologically bounded in } Y.$$

Then $(T_i | i \in J)$ is equicontinuous.

A simple, but very useful, consequence of Theorem 14.13 is given below.

Theorem 14.14: Let X be a Fréchet space and Y be a topological vector space, and $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous linear amppings from X to Y. Assume that $\{T_n x\}_{n=1}^{\infty}$ is convergent for every $x \in X$ and put

$$Tx = \lim_{n\to\infty} T_n x \quad \text{for all } x \in X.$$

Then T is continuous.

Proof: Let W be a neighborhood of 0 in Y. It is straightforward to show that we may choose a neighborhood V of 0 in Y such that $\mathrm{cl}(V) \subset W$. Observe that $(T_n | n \in \mathbb{N})$ is equicontinuous. Therefore, we may choose a neighborhood U of 0 in X such that $T_n[U] \subset V$ for all $n \in \mathbb{N}$. It follows that $T[U] \subset \mathrm{cl}(V) \subset W$ and consequently T is continuous. $\square$

Let X be a normed linear space with dual space X^*. For each $x^* \in X^*$, put

$$p_{x^*}(x) = |x^*(x)| \text{ for all } x \in X,$$

and for each $x \in X$, put

$$q_x(x^*) = |x^*(x)| \text{ for all } x^* \in X^*.$$

Then $\{p_{x^*} : x^* \in X^*\}$ is a separting family of seminorms on X and $\{q_x : x \in X\}$ is a separating family of seminorms on X^*.

Notation: Given a normed linear space Y, we write $\overline{B}_1^Y(0)$ for the closed unit ball in Y, i.e.

$$\overline{B}_1^Y(0) = \{y \in Y : \|y\|_Y \leq 1\}.$$

Definition 14.15 (Weak Topology): The family of seminorms $\{p_{x^*} : x^* \in X^*\}$ generates a topology, denoted by $\sigma(X, X^*)$ and called the *weak topology* of X, such that $(X, \sigma(X, X^*))$ is a locally convex topological vector space and for each $x^* \in X^*$, $p_{x^*} : X \to \mathbb{R}$ is continuous. Some authors refer to $\sigma(X, X^*)$ as the X^*-topology of X.

Definition 14.16 (Weak* Topology): The family of seminorms $\{q_x : x \in X\}$ generates a topology, denoted by $\sigma(X^*, X)$ and called the *weak* topology* of X^*, such that $(X, \sigma(X^*, X))$ is a locally convex topological vector space and for each $x \in X$, $q_x : X^* \to \mathbb{R}$ is continuous. Some authors refer to $\sigma(X^*, X)$ as the X-topology of X^*

Remark 14.17: Let X be a normed linear space. Then the weak topology on X is weaker than the norm topology (i.e., every weakly open set is open).

Remark 14.18: Let X be a normed linear space. The weak* topology on X^* is weaker than the weak topology on X (i.e., $\sigma(X^*, X)$ is weaker than $\sigma(X^*, X^{**})$) and the weak topology on X^* is weaker than the norm topology on X^*. Moreover, the weak and weak* topologies on X^* coincide if and only if X is reflexive.

In situations when weak or weak* topologies are relevant, we shall refer to the norm topologies on X and X^* as the *strong topologies* on X and X^*, respectively.

Remark 14.19: Let X be a normed linear space and let $f : X \to \mathbb{K}$ be a linear functional. Then f is strongly continuous if and only if it is weakly continuous.

You should verify Remarks 14.17 – 14.19 as an exercise for yourself.

Proposition 14.20: Let X be a normed linear space and assume that $g : X^* \to \mathbb{K}$ is linear. Then g is weakly* continuous if and only if there exists $x \in X$ such that

$$g(x^*) = x^*(x) \text{ for all } x^* \in X^*. \tag{1}$$

Proof: It follows immediately from the definition of weak* topology that if g has the form (1) then g is weakly* continuous.

Suppose that g is weakly* continuous. Then the set U^* defined by

$$U^* = \{x^* \in X^* : |g(x^*)| < 1\}$$

is a weak* neighborhood of 0 in X^*. By the definition of weak* topology, we may choose $\epsilon > 0$ and $x_1, x_2, \cdots, x_N \in X$ such that

$$\{x^* \in X^* : |x^*(x_i)| < \epsilon,\ i = 1, 2, \cdots, N\} \subset U^*.$$

Observe that

$$\bigcap_{i=1}^{N} \{x^* \in X^* : x^*(x_i) = 0\} \subset \mathcal{N}(g).$$

It follows from a standard result in linear algebra that g is a linear combination of the linear functionals

$$x^* \mapsto x^*(x_i), \quad i = 1, 2, \cdots, N.$$

We conclude that there exists $x \in X$ such that (1) holds. $\square$

Theorem 14.21 (Alaoglu's Theorem): Let X be a normed linear space. Then $\overline{B}_1^{X^*}(0)$ is $\sigma(X^*, X)$-compact (i.e., the closed unit ball in X^* is weakly* compact).

Proof: Let us write B^* for $\overline{B}_1^{X^*}(0)$ and B for $\overline{B}_1^{X}(0)$. For each $x \in B$, put

$$D_x = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}.$$

$$D = \prod_{x \in B} D_x,$$

equipped with the product topology. Then D is compact by Tychonov's Theorem.

Define $\phi : B^* \to D$ by

$$(\phi(x^*))(x) = x^*(x) \ \text{ for all } x^* \in B^*,\ x \in B.$$

Let $x^*, y^* \in B^*$ be given and suppose that $\phi(x^*) = \phi(y^*)$. Then $x^*(x) = y^*(x)$ for all $x \in B$, which implies that $x^* = y^*$. It follows that ϕ is injective.

Let $x^* \in B^*$ be given and let $(x^*_\lambda)_{\lambda \in J}$ be a net in B^*. (Here, J is a directed set. If you are not familiar with nets, See pages 171 – 172 of Carothers or Appendix A in Conway for a nice quick introduction.) Then for each $x \in B$ we have

$$(\phi(x^*_\lambda))(x) = x^*_\lambda(x),$$

so that

$$\phi(x^*_\lambda) \to \phi(x^*) \Leftrightarrow x^*_\lambda \to x^*.$$

It follows that $\phi : B^* \to \phi[B^*]$ is a homeomporphism.

We shall now show that $\phi[B^*]$ is closed in D. Let $(x^*_\lambda)_{\lambda \in J}$ be a net in B^*, $z \in D$, and assume that $(\phi(x^*_\lambda))_{\lambda \in J}$ converges to z in D. Then

$$z(x) = \lim x^*_\lambda(x) \ \text{ exists for every } x \in X.$$

There is exactly one linear mapping $F : X \to \mathbb{K}$ such that

$$F(x) = z(x) \ \text{ for all } x \in B.$$

Notice that $|F(x)| \leq 1$ for all $x \in B$. We conclude that $F \in B^*$ and $\phi(F) = z$. It follows that $\phi[B^*]$ is closed. Since every closed subset of a compact set is compact, we conclude that $\phi[B^*]$ is compact. Since ϕ is a homeomorphism, we conclude that B^* is compact. $\square$

Remark 14.22: Alaglou's Theorem is one of the most important results in functional analysis. In fact, we could now talk about "Four Basic Principles of Linear Functional Analysis": Hahn Banach Theorem, Open Mapping Theorem (Closed Graph Theorem), Banach-Steinhaus Theorem, and Alaoglu's Theorem.

Remark 14.23: The use of nets is very convenient for many arguments involving weak or weak* topologies. However, one must keep in mind that arguments involving subnets can be tricky.

Theorem 14.24: Let X be a separable normed linear space and K^* be a weakly* compact subset of X^*. Then $(K^*, \sigma(X^*, X))$ is metrizable.

Proof: Choose a sequence $\{x_n\}_{n=1}^\infty$ that is dense in X and define $\rho : K^* \times K^* \to \mathbb{R}$ by

$$\rho(x^*, y^*) = \sum_{n=1}^\infty 2^{-n} \frac{|(x^* - y^*)(x_n)|}{1 + |(x^* - y^*)(x_n)|} \ \text{ for all } x^*, y^* \in K^*.$$

Let us denote the topology induced by this metric on K^* by τ. It is straightforward to verify that the identity mapping from $(K^*, \sigma(X^*, X))$ to (K^*, τ) is continuous. Since every continuous injection from a compact space to a Hausdorff space has a continuous inverse, we conclude that $(K^*, \sigma(X^*, X)) = (K^*, \tau)$. $\square$.

Lemma 14.25: Let X be a normed linear space. Then $J : (X, \sigma(X, X^*)) \to (J[X], \sigma(X^{**}, X^*))$ is a homeomorphism.

Proof: We know from previous considerations that $J : X \to J[X]$ is bijective. To show that J is continuous, it suffices to show that J is continuous at 0. Let V^{**} be a neighborhood of 0 in $(X^{**}, \sigma(X^{**}, X^*))$. Then we may choose $\epsilon > 0$ and $x^*_1, x^*_2, \cdots, x^*_k \in X^*$ such that

$$\{x^{**} \in X^{**} : |x^{**}(x^*_i)| < \epsilon,\ i = 1, 2, \cdots, k\} \subset V^{**}.$$

Let

$$W = \{x \in X : |x^*_i(x)| < \epsilon,\ i = 1, 2, \cdots, k\}.$$

Since $J(x)(x_i^*) = x_i^*(x)$ for all $x \in X$ and $i = 1, 2, \cdots, k$ we have $J[W] \subset V^{**}$ and J is continuous.

To show that $J^{-1} : (J[X], \sigma(X^{**}, X^*)) \to (X, \sigma(X, X^*))$ is continuous, let U be a neighborhood of 0 in $(X, \sigma(X, X^*)$. We may choose $\delta > 0$ and $y_1^*, y_2^*, \cdots, y_m^* \in X^*$ such that

$$\{x \in X : |y_i^*(x)| < \delta,\ i = 1, 2, \cdots, m\} \subset U.$$

Put

$$W^{**} = \{x^{**} \in X^{**} : |x^{**}(y_i^*)| < \delta,\ i = 1, 2, \cdots, m\}.$$

It is straightforward to check that if $z^{**} \in J[X] \cap W^{**}$ then $J^{-1}(z^{**}) \in U$, and consequently J^{-1} is $\sigma(X^{**}, X^*)$-continuous at 0. Since J^{-1} is linear, we conclude that $J^{-1} : (J[X], \sigma(X^{**}, X^*)) \to (X, \sigma(X, X^*))$ is continuous. $\square$

Theorem 14.26 (Goldstine's Theorem): Let X be a normed linear space. Then $J[\overline{B}_1^X(0)]$ is $\sigma(X^{**}, X^*)$-dense in $\overline{B}_1^{X^{**}}(0)$.

Proof: Put

$$B = \overline{B}_1^X(0), \quad B^{**} = \overline{B}_1^{X^{**}}(0),$$

and let $\hat{B}^{**}$ denote the $\sigma(X^{**}, X^*)$-closure of $J[B]$. By Alaoglu's Theorem, B^{**} is $\sigma(X^{**}, X^*)$-compact and consequently it is $\sigma(X^{**}, X^*)$-closed as well. Since $J[B] \subset B^{**}$, we conclude that $\hat{B}^{**} \subset B^{**}$. Since $J[B]$ is convex, it follows from that $\hat{B}^{**}$ is convex. We need to show that $\hat{B}^{**} = B^{**}$.

Suppose that $\hat{B}^{**} \neq B^{**}$. Then we may choose

$$x_0^{**} \in B^{**} \backslash \hat{B}^{**}.$$

A minor modification of the proof of Lemma 8.13 shows that we can choose a linear functional $f : X^{**} \to \mathbb{K}$, $\alpha \in \mathbb{R}$, and $\epsilon > 0$ such that f is $\sigma(X^{**}, X^*)$-continuous and

$$\mathrm{Re}(f(x^{**})) \leq \alpha < \alpha + \epsilon \leq \mathrm{Re}(f(x_0^{**})) \ \text{ for all } x^{**} \in \hat{B}^{**}.$$

By Proposition 14.20, we may choose $x_0^* \in X^*$ such that

$$f(x^{**}) = x^{**}(x_0^*) \ \text{ for all } x^{**} \in X^{**}.$$

Since $J[B] \subset \hat{B}^{**}$, we conclude that

$$\mathrm{Re}((x_0^*)(x)) \leq \alpha < \alpha + \epsilon \leq \mathrm{Re}(x^{**}(x_0^*) \ \text{ for all } x \in B.$$

Since $0 \in B$, we know that $\alpha > 0$. Let us put $y_0^* = \alpha^{-1} x_0^*$. Then we have

$$\mathrm{Re}(y_0^*(x)) \leq 1 < 1 + \frac{\epsilon}{\alpha} \leq \mathrm{Re}(x_0^{**}(y_0^*)) \ \text{ for all } x \in B. \tag{2}$$

Since $\gamma x \in B$ for all $x \in B$, $\gamma \in \mathbb{K}$ with $|\gamma\| \leq 1$, the left-most inequality in (2) tells us that

$$|y_0^*(x)| \leq 1 \ \text{ for all } x \in B,$$

and consequently $\|y_0^*\| \leq 1$ from which we conclude that

$$|x_0^{**}(y_0^*)| \leq 1. \tag{3}$$

Since $\alpha, \epsilon > 0$, the right-most inequality in (2) is inconsistent with (3). $\square$

Theorem 4.27: Let X be a normed linear space. Then X is reflexive if and only if $\overline{B}_1^X(0)$ is $\sigma(X, X^*)$-compact (i.e., if and only if the closed unit ball is weakly compact).

Proof: Assume that X is reflexive. Then, since $J : X \to X^{**}$ is a surjective isometry, we know that

$$J[\overline{B}_1^X(0)] = \overline{B}_1^{X^{**}}(0).$$

We also know that $J : (X, \sigma(X, X^*)) \to (X^{**}, \sigma(X^{**}, X^*))$ is a homeomorphism. By Alaoglu's Theorem, $\overline{B}_1^{X^{**}}(0)$ is $\sigma(X^{**}, X^*)$-compact. It follows that $\overline{B}_1^X(0)$ is $\sigma(X, X^*)$-compact.

Assume now that $\overline{B}_1^X(0)$ is $\sigma(X, X^*)$-compact. Then $J[\overline{B}_1^X(0)]$ is $\sigma(X^{**}, X^*)$-compact and hence $\sigma(X^{**}, X^*)$-closed. It follows from Goldstine's Theorem that $J[\overline{B}_1^X(0)] = \overline{B}_1^{X^{**}}(0)$. By linearity of J, we conclude that J is surjective and X is reflexive. $\square$

Theorem 4.28: Let X be a separable normed linear space and assume that $K \subset X$ is weakly compact. Then $(K, \sigma(X, X^*))$ is metrizable.

Sketch of Proof: We can construct a sequence $\{x_n^*\}_{n=1}^{\infty}$ in $X^* \backslash \{0\}$ such that

$$\bigcap_{n=1}^{\infty} \mathcal{N}(x_n^*) = \{0\}.$$

The metric $\rho : K \times K \to \mathbb{R}$ defined by

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n^*(x-y)|}{1 + |x_n^*(x-y)|} \quad \text{for all } x, y \in K,$$

does the job. $\square$

Theorem 4.29: Let X be a normed linear space and assume that $C \subset X$ is convex. Then the weak and strong closures of C coincide.

Proof: Let us denote the strong closure of C by $\overline{C}$ and the weak closure of C by $\hat{C}$. Since every weakly set is closed, we know that

$$\overline{C} \subset \hat{C}.$$

To establish the reverse inclusion, let $x_0 \in \overline{C}$ be given. Using a Hahn-Banach argument, we may choose $x^* \in X^*$, $\alpha \in \mathbb{R}$, and $\epsilon > 0$ such that

$$\mathrm{Re}((x^*)(x)) \leq \alpha \leq \alpha < \alpha + \epsilon \leq \mathrm{Re}(x^*(x_0)) \quad \text{for all } x \in \overline{C}.$$

The set $\{x \in X : \mathrm{Re}(x^*(x)) < \alpha + \frac{\epsilon}{2}\}$ is a weak neighborhood of x_0 that has empty intersection with C. It follows that

$$\hat{C} \subset \overline{C}. \quad \square$$

Proposition 4.30:

(a) Let X be a normed linear space and let $E \subset X$. Then E is topologically bounded in $(X, \sigma(X, X^*))$ (i.e., E is weakly bounded) if and only if E is norm bounded.

(b) Let X be a Banach space and let $E^* \subset X^*$. Then E^* is topologically bounded in $(X, \sigma(X^*, X))$ (i.e., E^* is weakly* bounded) if and only if E^* is norm bounded.

The proof of Proposition 4.30 is part of Assignment 8.

Inner Products Spaces

We now study normed linear spaces with additional structure due to the presence of an inner product. An inner product is a natural generalization of the dot product in Euclidean space. It gives rise to a natural notion of orthogonality that leads to a very rich geometric structure.

Definition 14.31: Let X be a linear space over $\mathbb{K}$. By an *inner product* on X we mean a function $(\cdot, \cdot) : X \times X \to \mathbb{K}$ such that

(a) $\forall x, y, z \in X, \quad (x + y, z) = (x, z) + (y, z)$,

(b) $\forall x, y \in X, \alpha \in \mathbb{K}, \quad (\alpha x, y) = \alpha(x, y)$,

(c) $\forall x, y \in \mathbb{K}, \quad (x, y) = \overline{(y, x)}$,

(d) $\forall x \in X, \quad (x, x) \geq 0$ (notice that (c) implies $(x, x) \in \mathbb{R}$),

(e) $\forall x \in X, \quad (x, x) = 0 \Leftrightarrow x = 0$.

A linear space, together with an inner product, is called an *inner-product space.*

Remark 14.32: In the above definition, the bar in (c) denotes the complex conjugate. Of course, if $\mathbb{K} = \mathbb{R}$ then $\overline{\alpha} = \alpha$ for all $\alpha \in \mathbb{K}$. It follows from (b) and (c) that $(x, \alpha y) = \overline{\alpha}(x, y)$ for all $x, y \in X, \alpha \in \mathbb{K}$. It follows form (a) and (c) that $(x, y + z) = (x, y) + (x, z)$ for all $x, y, z \in X$. In other words, an inner product is linear in the first argument and conjugate linear in the second argument. Some authors (especially in the physics literature) define inner products to be conjugate linear in the first argument and linear in the second argument. This leads to a number of small changes in the formulas that follow.

Example 14.33:

(a) $X = l^2$, $(x, y) = \sum_{k=1}^{\infty} x_k \overline{y_k}$.

(b) Let X be the set of all real or complex-valued continuous functions on $[0, 1]$. We define an inner product $(\cdot, \cdot) : X \times X \to \mathbb{K}$ by

$$(f, g) = \int_0^1 f(t)\overline{g}(t)dt.$$

Proposition 14.34 (Cauchy-Schwarz Inequality): Let X be an inner-product space and let $x, y \in X$ be given. Then

$$|(x, y)|^2 \leq (x, x)(y, y).$$

Proof: If $y = 0$ there is nothing to prove, so we assume that $y \neq 0$. Let $\lambda \in \mathbb{K}$ be given and observe that

$$0 \leq (x - \lambda y, x - \lambda y) = (x, x) - 2\mathrm{Re}[\overline{\lambda}(x, y)] + |\lambda|^2(y, y). \tag{4}$$

Rearranging (4) we obtain

$$2\mathrm{Re}[\overline{\lambda}(x, y)] \leq (x, x) + |\lambda|^2(y, y) \tag{5}$$

Let us put

$$\lambda = \frac{(x, y)}{(y, y)}. \tag{6}$$

Substituting (6) into (5) we find that

$$2\frac{|(x, y)|^2}{(y, y)} \leq (x, x) + \frac{|(x, y)|^2}{(y, y)}. \tag{7}$$

The desired conclusion follows from multiplying both sides of (7) by (y, y). $\square$

Definition 14.35: Let X be an inner product space with inner product $(\cdot, \cdot)$. We define the *norm associated with* $(\cdot, \cdot)$ by

$$\|x\| = \sqrt{(x, x)} \quad \text{for all } x \in X. \tag{8}$$

Remark 14.36: Using the norm associated with the inner product, the Cauchy-Schwarz inequality can be rewritten as

$$|(x, y)| \leq \|x\|\|y\| \quad \text{for all } x, y \in X. \tag{9}$$

To see that the function $\|\cdot\|$ defined by (8) is, in fact, a norm, we need to verify the triangle inequality. (The other properties of a norm are obviously satisfied. To this end, let $x, y \in X$ be given Observe that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(x,y). \tag{10}$$

Since $\mathrm{Re}(x,y) \leq |(x,y)|$, the Cauchy-Schwarz inequality (9) implies that

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2,$$

and we conclude that $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

When we speak of the norm in an inner-product space, we always mean the norm associated with the inner product, unless stated otherwise. Topological and metric concepts, such as continuity and distance, are understood to be defined in terms of the norm associated with the inner product and the metric associated with that norm.

Remark 14.37: Equation (10) is called the *polar identity.* Some authors use the term "polar identity" for an identity that follows from (10).

Definition 14.38: Let X be an inner-product space and let $x, y \in X$ and $A, B \subset X$ be given.

(a) We say that x is *orthogonal* to y and write $x \perp y$ provided that $(x,y) = 0$.

(b) We say that x is *orthogonal* to A and write $x \perp A$ provided that $(x,y) = 0$ for all $y \in A$.

(c) We say that A is *orthogonal* to B and write $A \perp B$ provided that $(x,y) = 0$ for all $x \in A, y \in B$.

(d) The *orthogonal complement* of A is defined by $A^{\perp} = \{y \in X : (x,y) = 0 \text{ for all } x \in A\}$.

The following well-known result is an immediate consequence of the polar identity (10).

Proposition 14.39 (Pythagorean Theorem): Let X be an inner product space and let $x, y \in X$ with $x \perp y$ be given. Then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

If we apply the polar identity to compute $\|x+y\|^2 + \|x-y\|^2$, the "cross terms" cancel and we obtain the following result.

Proposition 14.40 (Parallelogram Law): Let X be an inner product space and let $x, y \in X$ be given. Then we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{11}$$

The parallelogram law characterizes norms that come from inner products.

Proposition 14.41: Let $(X, \|\cdot\|)$ be a normed linear space and assume that (11) holds for all $x, y \in X$. Then there is an inner product $(\cdot,\cdot) : X \times X \to \mathbb{K}$ such that $\|x\| = \sqrt{(x,x)}$ for all $x \in X$.

Proof: **Case 1**: $\mathbb{K} = \mathbb{R}$

Define $(\cdot,\cdot) : X \times X \to \mathbb{R}$ by

$$(x,y) = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2\} \text{ for all } x, y \in X.$$

It is straightforward, although a bit tedious, to check that $(\cdot,\cdot)$ is an inner product.

Case 2: $\mathbb{K} = \mathbb{C}$

Define $(\cdot,\cdot) : X \times X \to \mathbb{C}$ by

$$(x,y) = \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i[\|x+iy\|^2 - \|x-iy\|^2]\} \text{ for all } x, y \in X.$$

It is straightforward, but rather tedious, to check that $(\cdot,\cdot)$ is an inner product. $\square$

Remark 14.42: Although most of the results described here are valid in both real and complex inner-product spaces, you should be aware that certain results of interest are valid in real spaces, but not complex ones, or vice versa. Two such examples are given below. Here, X is an inner-product space of dimension ≥ 2.

- If $x, y \in X$ satisfy $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, can we conclude that $x \perp y$? The answer is yes if $\mathbb{K} = \mathbb{R}$, but no if $\mathbb{K} = \mathbb{C}$.
- If $A \in \mathcal{L}(X;X)$ is such that $(Ax, x) = 0$ for all $x \in X$, can we conclude that $A = 0$? The answer is yes if $\mathbb{K} = \mathbb{C}$, but no if $\mathbb{K} = \mathbb{R}$.

Most of the deep results about inner-product spaces, require the space to be complete.

Definition 14.43: An inner-product space that is complete with respect to the norm associated with the inner product is called a *Hilbert space*.