***10.90** Refer to Exercise 10.5.

a Find the power of test 2 for each of the following alternatives: $\theta = .1, \theta = .4, \theta = .7$, and $\theta = 1$.

b Sketch a graph of the power function.

c Compare the power function in part (b) with the power function that you found in Exercise 10.89 (this is the power function for test 1, Exercise 10.5). What can you conclude about the power of test 2 compared to the power of test 1 *for all* $\theta \geq 0$?

10.91 Let $Y_1, Y_2, \ldots, Y_{20}$ be a random sample of size $n = 20$ from a normal distribution with unknown mean μ and known variance $\sigma^2 = 5$. We wish to test $H_0 : \mu = 7$ versus $H_a : \mu > 7$.

a Find the uniformly most powerful test with significance level .05.

b For the test in part (a), find the power at each of the following alternative values for μ: $\mu_a = 7.5, 8.0, 8.5$, and 9.0.

c Sketch a graph of the power function.

10.92 Consider the situation described in Exercise 10.91. What is the smallest sample size such that an $\alpha = .05$-level test has power at least .80 when $\mu = 8$?

10.93 For a normal distribution with mean μ and variance $\sigma^2 = 25$, an experimenter wishes to test $H_0 : \mu = 10$ versus $H_a : \mu = 5$. Find the sample size n for which the most powerful test will have $\alpha = \beta = .025$.

10.94 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with *known* mean μ and unknown variance σ^2. Find the most powerful α-level test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 = \sigma_1^2$, where $\sigma_1^2 > \sigma_0^2$. Show that this test is equivalent to a χ^2 test. Is the test uniformly most powerful for $H_a : \sigma^2 > \sigma_0^2$?

10.95 Suppose that we have a random sample of four observations from the density function

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{2\theta^3}\right) y^2 e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$. [*Hint:* Make use of the χ^2 distribution.]

b Is the test given in part (a) uniformly most powerful for the alternative $\theta > \theta_0$?

10.96 Suppose Y is a random sample of size 1 from a population with density function

$$f(y \mid \theta) = \begin{cases} \theta y^{\theta - 1}, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta > 0$.

a Sketch the power function of the test with rejection region: $Y > .5$.

b Based on the single observation Y, find a uniformly most powerful test of size α for testing $H_0 : \theta = 1$ versus $H_a : \theta > 1$.

***10.97** Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed random variables with discrete probability function given by

	y		
	1	2	3
$p(y \mid \theta)$	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

where $0 < \theta < 1$. Let N_i denote the number of observations equal to i for $i = 1, 2, 3$.

a Derive the likelihood function $L(\theta)$ as a function of N_1, N_2, and N_3.

b Find the most powerful test for testing $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_a$, where $\theta_a > \theta_0$. Show that your test specifies that H_0 be rejected for certain values of $2N_1 + N_2$.

c How do you determine the value of k so that the test has nominal level α? You need not do the actual computation. A clear description of how to determine k is adequate.

d Is the test derived in parts (a)–(c) uniformly most powerful for testing $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$? Why or why not?

10.98 Let $Y_1, \ldots, Y_n$ be a random sample from the probability density function given by

$$f(y \mid \theta) = \begin{cases} \left(\dfrac{1}{\theta}\right) m y^{m-1} e^{-y^m/\theta}, & y > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

with m denoting a known constant.

a Find the uniformly most powerful test for testing $H_0 : \theta = \theta_0$ against $H_a : \theta > \theta_0$.

b If the test in part (a) is to have $\theta_0 = 100$, $\alpha = .05$, and $\beta = .05$ when $\theta_a = 400$, find the appropriate sample size and critical region.

10.99 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having a Poisson distribution with mean λ.

a Find the form of the rejection region for a most powerful test of $H_0 : \lambda = \lambda_0$ against $H_a : \lambda = \lambda_a$, where $\lambda_a > \lambda_0$.

b Recall that $\sum_{i=1}^{n} Y_i$ has a Poisson distribution with mean $n\lambda$. Indicate how this information can be used to find any constants associated with the rejection region derived in part (a).

c Is the test derived in part (a) uniformly most powerful for testing $H_0 : \lambda = \lambda_0$ against $H_a : \lambda > \lambda_0$? Why?

d Find the form of the rejection region for a most powerful test of $H_0 : \lambda = \lambda_0$ against $H_a : \lambda = \lambda_a$, where $\lambda_a < \lambda_0$.

10.100 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having a Poisson distribution with mean λ_1. Let $X_1, X_2, \ldots, X_m$ denote an independent random sample from a population having a Poisson distribution with mean λ_2. Derive the most powerful test for testing $H_0 : \lambda_1 = \lambda_2 = 2$ versus $H_a : \lambda_1 = 1/2,\ \lambda_2 = 3$.

10.101 Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a population having an exponential distribution with mean θ.

a Derive the most powerful test for $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, where $\theta_a < \theta_0$.

b Is the test derived in part (a) uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_a : \theta < \theta_0$?

10.102 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Bernoulli-distributed population with parameter p. That is,

$$p(y_i \mid p) = p^{y_i}(1-p)^{1-y_i}, \qquad y_i = 0, 1.$$

a Suppose that we are interested in testing $H_0 : p = p_0$ versus $H_a : p = p_a$, where $p_0 < p_a$.

i Show that

$$\frac{L(p_0)}{L(p_a)} = \left[\frac{p_0(1-p_a)}{(1-p_0)p_a}\right]^{\sum y_i} \left(\frac{1-p_0}{1-p_a}\right)^n.$$

ii Argue that $L(p_0)/L(p_a) < k$ if and only if $\sum_{i=1}^{n} y_i > k^*$ for some constant k^*.

iii Give the rejection region for the most powerful test of H_0 versus H_a.

b Recall that $\sum_{i=1}^{n} Y_i$ has a binomial distribution with parameters n and p. Indicate how to determine the values of any constants contained in the rejection region derived in part [a(iii)].

c Is the test derived in part (a) uniformly most powerful for testing $H_0: p = p_0$ versus $H_a: p > p_0$? Why or why not?

***10.103** Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from a uniform distribution over the interval $(0, \theta)$.

a Find the most powerful α-level test for testing $H_0: \theta = \theta_0$ against $H_a: \theta = \theta_a$, where $\theta_a < \theta_0$.

b Is the test in part (a) uniformly most powerful for testing $H_0: \theta = \theta_0$ against $H_a: \theta < \theta_0$?

***10.104** Refer to the random sample of Exercise 10.103.

a Find the most powerful α-level test for testing $H_0: \theta = \theta_0$ against $H_a: \theta = \theta_a$, where $\theta_a > \theta_0$.

b Is the test in part (a) uniformly most powerful for testing $H_0: \theta = \theta_0$ against $H_a: \theta > \theta_0$?

c Is the most powerful α-level test that you found in part (a) unique?

10.11 Likelihood Ratio Tests

Theorem 10.1 provides a method of constructing most powerful tests for simple hypotheses when the distribution of the observations is known except for the value of a single unknown parameter. This method can sometimes be used to find uniformly most powerful tests for composite hypotheses that involve a single parameter. In many cases, the distribution of concern has more than one unknown parameter. In this section, we present a very general method that can be used to derive tests of hypotheses. The procedure works for simple or composite hypotheses and whether or not other parameters with unknown values are present.

Suppose that a random sample is selected from a distribution and that the likelihood function $L(y_1, y_2, \dots, y_n \mid \theta_1, \theta_2, \dots, \theta_k)$ is a function of k parameters, $\theta_1, \theta_2, \dots, \theta_k$. To simplify notation, let Θ denote the vector of all k parameters—that is, $\Theta = (\theta_1, \theta_2, \dots, \theta_k)$—and write the likelihood function as $L(\Theta)$. It may be the case that we are interested in testing hypotheses only about one of the parameters, say, θ_1. For example, if as in Example 10.24, we take a sample from a normally distributed population with unknown mean μ and unknown variance σ^2, then the likelihood function depends on the *two* parameters μ and σ^2 and $\Theta = (\mu, \sigma^2)$. If we are interested in testing hypotheses about only the mean μ, then σ^2—a parameter not of particular interest to us—is called a *nuisance parameter*. Thus, the likelihood function may be a function with both unknown nuisance parameters and a parameter of interest.

Suppose that the null hypothesis specifies that Θ (may be a vector) lies in a particular set of possible values—say, Ω_0—and that the alternative hypothesis specifies that Θ lies in another set of possible values Ω_a, which does not overlap Ω_0. For example, if we sample from a population with an exponential distribution with mean λ (in this case, λ is the only parameter of the distribution, and $\Theta = \lambda$), we might be

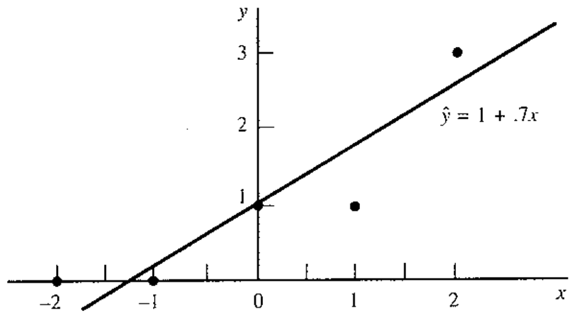

FIGURE 11.6 Plot of data points and least-squares line for Example 11.1

The five points and the fitted line are shown in Figure 11.6. ■

In this section, we have determined the least-squares estimators for the parameters β_0 and β_1 in the model $E(Y) = \beta_0 + \beta_1 x$. The simple example used here will reappear in future sections to illustrate other calculations. Exercises of a more realistic nature are presented at the ends of the sections, and two examples involving data from actual experiments are presented and analyzed in Section 11.9. In the next section, we develop the statistical properties of the least-squares estimators $\hat{\beta}_0$ and $\hat{\beta}_1$. Subsequent sections are devoted to using these estimators for a variety of inferential purposes.

Exercises

11.1 If $\hat{\beta}_0$ and $\hat{\beta}_1$ are the least-squares estimates for the intercept and slope in a simple linear regression model, show that the least-squares equation $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ always goes through the point $(\overline{x}, \overline{y})$. [*Hint:* Substitute $\overline{x}$ for x in the least-squares equation and use the fact that $\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x}$.]

11.2 **Applet Exercise** How can you improve your understanding of what the method of least-squares actually does? Access the applet *Fitting a Line Using Least Squares* (at www.thomsonedu.com/statistics/wackerly). The data that appear on the first graph is from Example 11.1.

a What are the slope and intercept of the blue horizontal line? (See the equation above the graph.) What is the sum of the squares of the vertical deviations between the points on the horizontal line and the observed values of the y's? Does the horizontal line fit the data well? Click the button "Display/Hide Error Squares." Notice that the areas of the yellow boxes are equal to the squares of the associated deviations. How does SSE compare to the sum of the areas of the yellow boxes?

b Click the button "Display/Hide Error Squares" so that the yellow boxes disappear. Place the cursor on right end of the blue line. Click and hold the mouse button and drag the line so that the slope of the blue line becomes negative. What do you notice about the lengths of the vertical red lines? Did SSE increase of decrease? Does the line with negative slope appear to fit the data well?

c Drag the line so that the slope is near 0.8. What happens as you move the slope closer to 0.7? Did SSE increase or decrease? When the blue line is moved, it is actually pivoting around a fixed point. What are the coordinates of that pivot point? Are the coordinates of the pivot point consistent with the result you derive in Exercise 11.1?