Lecture Notes for Week 2 (First Draft)

Idempotent Operators and Projections (Continued)

Proposition 2.1: Let X be a Hilbert space, $E \in \mathcal{L}(X;X)$ be given and assume that E is idempotent. Let $M = \mathcal{R}(E)$. Then $E = P_M$ if and only if

$$\mathcal{N}(E) = \mathcal{R}(E)^{\perp}.$$

Proof: Assume first that $E = P_M$. Then $I - E = P_{M^{\perp}}$, so $\mathcal{R}(I - E) = M^{\perp}$. By Remark 1.19, we know that $\mathcal{N} = \mathcal{R}(I - E) = M^{\perp}$. Since M is a closed subspace, we know that $(M^{\perp})^{\perp} = M$, and consequently $\mathcal{N}(E) = \mathcal{R}(E)^{\perp}$.

Assume now that $\mathcal{N}(E) = \mathcal{R}(E)^{\perp} = M^{\perp}$. Then, by Remark 1.19, $\mathcal{R}(I-E) = M^{\perp}$. Let $x \in X$ be given and observe that

$$x = Ex + (I - E)x.$$

Since $Ex \in M$ and $(I - E)x \in M^{\perp}$, it follows from the Projection Theorem that $Ex = P_M x$. $\square$

Remark 2.2: Let X be a normed linear space and assume that $E \in \mathcal{L}(X;X)$ is idempotent. Then

$$\|E\| = \|E^2\| \leq \|E\|^2,$$

and consequently either $E = 0$ or $\|E\| \geq 1$. Notice that in Example 1.20, the idempotent operator E has norm equal to $\sqrt{2}$.

Proposition 2.3: Let X be a Hilbert space and let $E \in \mathcal{L}(X;X)$ be given. Assume that E is idempotent and $E \neq 0$. Let $M = \mathcal{R}(E)$. Then $E = P_M$ if and only if $\|E\| = 1$.

Proof: Assume first that $E = P_M$. Let $x \in X$ be given. Then

$$x = P_M x + (I - P_M)x,$$

and $(P_M x, (I - P_M)x) = 0$. It follows from the Pythagorean Theorem that

$$\|x\|^2 = \|P_M x\|^2 + \|(I - P_M)x\|^2 \geq \|P_M x\|^2 = \|Ex\|^2.$$

Consequently, we have $\|E\| \leq 1$. By Remark 2.2, we also have $\|E\| \geq 1$.

Assume now that $\|E\| = 1$. Since $\mathcal{R}(E)$ is closed, it suffices to show that $\mathcal{R}(E) = \mathcal{N}(E)^{\perp}$. Let $x \in \mathcal{N}(E)^{\perp}$ be given. Since $\mathcal{R}(I - E) = \mathcal{N}(E)$ we have

$$\begin{aligned} 0 &= (x - Ex, x) = \|x\|^2 - (Ex, x) \\ &\geq \|x\|^2 - \|Ex\|\|x\| = \|x\|(\|x\| - \|Ex\|). \end{aligned} \tag{1}$$

If $x = 0$ then certainly $x \in \mathcal{R}(E)$, so we may assume that $x \neq 0$. It follows from (1) that $\|Ex\| \geq \|x\|$. Since $\|E\| = 1$ we know that $\|Ex\| \leq \|x\|$ and consequently $\|Ex\| = \|x\|$. Using (1) again, we see that

$$\|Ex\|^2 = \|x\|^2 = (Ex, x),$$

and consequently

$$\|x - Ex\|^2 = \|x\|^2 + \|Ex\|^2 - 2(Ex, x) = 0.$$

It follows that $x = Ex$ and $x \in \mathcal{R}(E)$.

Now let $y \in \mathcal{R}(E)$ be given. Let us write

$$y = x + z$$

with $x \in \mathcal{N}(E)$ and $z \in \mathcal{N}(E)^{\perp}$. Since $\mathcal{N}(E)^{\perp} \subset \mathcal{R}(E)$ we know that $Ez = z$. Consequently, we have

$$y = Ey = E(x + z) = Ez = z,$$

and $y \in \mathcal{N}(E)^{\perp}$. $\square$

Theorem 2.4: Let X be a Hilbert space and $E \in \mathcal{L}(X; X)$ be given. Assume that E is idempotent and that $E \neq 0$. Put $M = \mathcal{R}(E)$. The following four statements are equivalent:

(i) $E = P_M$.

(ii) $\|E\| = 1$.

(iii) E is self-adjoint.

(iv) E is normal.

Proof: We have already shown that (i) holds if and only if (ii) holds. Moreover the implication (iii) $\Rightarrow$ (iv) is immediate. It suffices to prove (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i).

Assume that (i) holds and let $x, y \in X$ be given. We write

$$x = x_1 + x_2, \quad y = y_1 + y_2, \quad x_1, y_1 \in M, \; x_2, y_2 \in M^{\perp}.$$

Then we have

$$(Ex, y) = (Ex_1 + Ex_2, y_1 + y_2) = (x_1, y_1 + y_2) = (x_1, y_1),$$

and also

$$(x, Ey) = (x_1 + x_2, Ey_1 + Ey_2) = (x_1 + x_2, y_1) = (x_1, y_1).$$

We conclude that

$$(Ex, y) = (x, Ey) \text{ for all } x, y \in X$$

and E is self-adjoint.

Assume now that (iv) holds. Then

$$\|Ex\| = \|E^*x\| \text{ for all } x \in X$$

and $\mathcal{N}(E) = \mathcal{N}(E^*)$. Since $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ for every $A \in \mathcal{L}(X;X)$ we conclude that $\mathcal{N}(E) = \mathcal{R}(E)^\perp$ and $E = P_M$ by virtue of . $\square$

Proposition 2.5: Let X be a Hilbert space and $E \in \mathcal{L}(X;X)$ be given. Assume that E is idempotent and put $M = \mathcal{R}(E)$. Then $E = P_M$ if and only if

$$(Ex, x) \geq 0 \text{ for all } x \in X. \tag{2}$$

(The condition (2) means that for every $x \in X$, the quantity (Ex, x) is real and nonnegative.)

Proof: Assume first that $E = P_M$. Let $x \in X$ be given and write

$$x = x_1 + x_2, \quad x_1 \in M, \; x_2 \in M^\perp.$$

Then we have

$$(Ex, x) = (Ex_1 + Ex_2, x_1 + x_2) = (x_1, x_1) \geq 0.$$

Conversely, assume that (2) holds. If $\mathbb{K} = \mathbb{C}$ then E is self-adjoint by Proposition 1.6 and using Thoerem 2.4, we conclude that $E = P_M$.

Assume that $\mathbb{K} = \mathbb{R}$ and let $y \in \mathcal{R}(E)$ and $z \in \mathcal{R}(I - E)$ be given. Then, since $\mathcal{R}(I - E) = \mathcal{N}(E)$, we have

$$0 \leq (Ey + Ez, y + z) = (y, y + z) = \|y\|^2 + (y, z).$$

We conclude that

$$\|y\|^2 + (y, z) \geq 0 \text{ for all } y \in \mathcal{R}(E), \; z \in \mathcal{R}(I - E). \tag{3}$$

I claim that (3) implies that $(y, z) = 0$ for all $y \in \mathcal{R}(E)$, $z \in \mathcal{R}(I - E)$. To see why the claim is true, let $y \in \mathcal{R}(E)$ and $z \in \mathcal{R}(I - E)$ be given. Then we have

$$\|y\|^2 + t(y, z) \geq 0 \text{ for all } t \in \mathbb{R}. \tag{4}$$

If $(y, z) \neq 0$, then we can choose t to have the opposite sign of (y, z) and be large in magnitude. This will contradict (4). $\square$