Homework 10

21-260 Differential Equations
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Section 10.1, Problem 6

An obvious particular solution to the given nonhomogeneous differential equation is $y(t) = \frac{1}{2}x$.

The characteristic equation of the homogeneous differential equation associated with the given nonhomogeneous equation is $r^2 + 2 = 0$, whose roots are $r \in \{-\sqrt{2}i, \sqrt{2}i\}$. Thus, the solutions to the associated homogeneous equation are of the form

$$y(t) = c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right),$$

for some $c_1, c_2 \in \mathbb{R}$, and so, since the particular and general solutions are independent, solutions to the given nonhomogeneous equation are of the form

$$y(t) = \boxed{c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right) + \frac{1}{2}x.}.$$

Since $y(0) = 0$, $\boxed{c_1 = 0.}$ Then, since $y(\pi) = 0$, $c_2 = \frac{-\pi}{\sin\left(\sqrt{2}\pi\right)} \approx \boxed{3.26.}$

Section 10.1, Problem 19

In the first case, suppose $\lambda > 0$. Then, the characteristic equation of the given differential equation is $r^2 - \mu^2 = 0$, whose roots are $r = \pm\mu$. Then, solutions to the differential equation are of the form

$$y(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x), \quad c_1, c_2 \in \mathbb{R}.$$

Since $y(0) = 0$, $c_1 = 0$. Since $0 = y'(L) = c_2\mu\cosh(\mu L)$, and since $\mu \neq 0$ and cosh is everywhere strictly positive, c_2. Thus, the given differential equation has no positive eigenvalues.

In the second case, suppose $\lambda = 0$, so that $y'' = 0$. Then, solutions to the differential equation are of the form

$$y(x) = c_1 x + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

Since $y(0) = 0$, $c_2 = 0$, and, since $0 = y'(L) = c_1$, $c_1 = 0$. Thus, the given differential equation does not have zero as an eigenvalue.

In the last case, suppose $\lambda < 0$. Then, the characteristic equation of the given differential equation is $r^2 + \mu^2 = 0$, whose roots are $r = \pm\mu i$. Then, solutions to the differential equation are of the form

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x), \quad c_1, c_2 \in \mathbb{R}.$$

Since $y(0) = 0, c_1 = 0$. Since $y'(L) = 0$, so that $c_2\mu\cos(\mu L) = 0$ and we are interested only in the case $c_2 \neq 0$ (and $\mu \neq 0$), $\cos(\mu L) = 0$, and thus $\mu L - \frac{\pi}{2}$ is an integer multiple of π. Then,

$$\mu = \frac{\pi/2 + n\pi}{L}$$

and the eigenvalues of the given differential equation are, $\forall n \in \mathbb{N}$,

$$\lambda_n = \boxed{\left(\frac{\pi/2 + n\pi}{L}\right)^2,}$$

with associated eigenfunctions

$$y_n(x) = \boxed{\sin\left(\frac{\pi/2 + n\pi}{L}x\right).}$$

Section 10.2, Problem 16

(a) Three periods of the graph of f are plotted below:

(b) Suppose a_n and b_{n+1} ($\forall n \in \mathbb{N}$) are such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi x) + b_n \sin(n\pi x)\right).$$

$a_0 = \int_{-1}^{1} f(x)\,dx = \boxed{1.}$ $\forall n \in \mathbb{N}\backslash\{0\}$, the Euler-Fourier formulas and integration by parts give that

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 f(x)\cos(n\pi x)\,dx \\ &= 2\int_0^1 (1-x)\cos(n\pi x)\,dx \\ &= \boxed{\frac{2 - 2\cos(\pi n)}{\pi^2 n^2}.} \end{aligned}$$

Since f is even, $\forall n \in \mathbb{N}\backslash\{0\}$, $b_n = \boxed{0.}$

Section 10.3, Problem 4

(a) Suppose a_n and b_{n+1} ($\forall n \in \mathbb{N}$) are such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi x) + b_n \sin(n\pi x)\right).$$

$a_0 = \int_{-1}^{1} f(x)\,dx = \boxed{4/3.}$

$\forall n \in \mathbb{N}\setminus\{0\}$, the Euler-Fourier formulas and integration by parts give that

$$\begin{aligned} a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_0^1 f(x)\cos(n\pi x)\,dx \\ &= 2\int_0^1 (1-x^2)\cos(n\pi x)\,dx \\ &= \boxed{\frac{4\sin(\pi n) - 4\pi n\cos(\pi n)}{\pi^3 n^3}.} \end{aligned}$$

Since f is even, $\forall n \in \mathbb{N}\setminus\{0\}$, $b_n = \boxed{0.}$

(b) Three periods of the graph of f are plotted below:

Section 10.4, Problem 16

(a) The Euler-Fourier formula for a sine series and integration by parts give that, for

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right),$$

$\forall n \in \mathbb{N}\backslash\{0\}$,

$$\begin{aligned}
b_n &= \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right) \\
&= \int_0^1 x\sin\left(\frac{n\pi x}{2}\right) + \int_1^2 \sin\left(\frac{n\pi x}{2}\right) \\
&= \frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right) - \frac{2}{\pi n}\cos\left(\frac{\pi n}{2}\right) + \frac{2}{\pi n}\cos\left(\frac{\pi n}{2}\right) - \frac{2}{\pi n}\cos(\pi n) \\
&= \boxed{\frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right) - \frac{2}{\pi n}\cos(\pi n).}
\end{aligned}$$

(b) Three periods of the graph of f are plotted below: