21-630 Assignment #11 Due Wed., April 17, 2013

1. Assume that $f : \mathbb{R}^N \to \mathbb{R}^N$ is continuously differentiable and that $X(t)$ is a solution of

$$\frac{dX}{dt} = f(X(t))$$

for which $\Omega(X(0))$ is bounded and nonempty. Define

$S_1 = \{x : \text{dist}(x, \Omega(X(0))) < 1\}$,

$S_2 = \{x : 1 \leq \text{dist}(x, \Omega(X(0))) \leq 2\}$,

$S_3 = \{x : 3 < \text{dist}(x, \Omega(X(0)))\}$.

Suppose that $C^+(X(0))$ is unbounded.

A) Show that there are nonnegative sequences $\{t_k\}$ and $\{s_k\}$ that tend to infinity as $k \to \infty$ and satisfy $X(t_k) \in S_1$ and $X(s_k) \in S_3$ for every k.

B) Show that there is a nonnegative sequence $\{\tau_k\}$ that tends to infinity as $k \to \infty$ and satisfies $X(\tau_k) \in S_2$ for every k.

C) Derive a contradiction and conclude that $C^+(X(0))$ must be bounded.

2. Consider the system

$$\frac{dX}{dt} = f(X, Y)$$

$$\frac{dY}{dt} = g(X, Y)$$

that takes the form

$$\frac{dr}{dt} = -(r-1)^4$$

$$\frac{d\theta}{dt} = (r-1)^2 + \sin^2(\theta)$$

in polar coordinates. If $X^2(0) + Y^2(0) > 1$, what is the omega limit set? If $X(0) = 0$ and $Y(0) = 1$, what is the omega limit set?

3. Let $f \in C^1(\mathbb{R}^2)$ and let X be a nonconstant solution of

$$\frac{dX}{dt} = f(X).$$

Assume that $C^+(X(0))$ is bounded. Prove that if $\Omega(X(0)) \cap C^+(X(0))$ is nonempty, then X is periodic.

4. Consider the system

$$\frac{dX}{dt} = X + XY^2 - (X^2+Y^2)^2 X + (X^2+Y^2)Y$$

$$\frac{dY}{dt} = Y + Y^3 - (X^2+Y^2)^2 Y - (X^2+Y^2)X.$$

Prove that there is a nonconstant periodic solution to this system. Suggestion: consider the system in polar coordinates.