Department of Mathematical Sciences
Carnegie Mellon University

21-640 **Functional Analysis** **Spring 2013**

Assignment 4

Due on Wednesday, March 6

Solutions to problems marked with an asterisk should be written up and handed in.

1. Let $\mathbb{K} = \mathbb{R}$ and $X = \mathbb{R}^2$. Give an example of nonempty convex sets K_1 and K_2, such that K_1 has an internal point, but there is no linear functional $F : \mathbb{R}^2 \to \mathbb{R}$ such that
$$F(x) - F(y) < 0 \text{ for all } x \in K_1,\ y \in K_2.$$

2.* Let X be a normed linear space and Y be a linear manifold in X. Let $x_0 \in X$ and $d > 0$ be given and assume that
$$\inf\{\|y - x_0\| : y \in Y\} = d.$$
Show that there exists $x^* \in X^*$ such that $\langle x^*, x_0\rangle = 1$, $\|x^*\| = \frac{1}{d}$ and $\langle x*, y\rangle = 0$ for all $y \in Y$.

3.* Let X be a normed linear space and K_1, K_2 be disjoint convex subsets of X such that K_1 has nonempty interior and K_2 is nonempty. Show that there is a nontrivial continuous linear functional that separates K_1 and K_2.

4. Let X be a normed linear space, F be a finite, linearly independent, subset of X, and $\alpha : F \to \mathbb{K}$ be given. Show that there exists $x^* \in X^*$ such that $x^*(x) = \alpha(x)$ for all $x \in F$.

5.* Let $X = \mathbb{R}^{(\mathbb{N})}$ For each $x \in X \backslash \{0\}$ let $m(x) = \max\{n \in \mathbb{N} : x_n \neq 0\}$. Put
$$K_1 = \{x \in X \backslash \{0\} : x_{m(x)} > 0\}.$$
   (a) Show that K_1 is convex and has no internal points.
   (b) Find a nonempty convex set $K_2 \subset X$ such $K_1 \cap K_2 = \emptyset$ and there is no nontrivial linear functional that separates K_1 and K_2.

6.* Is there a continuous linear bijection $T : c_0 \to c$? Explain.

7. Let $bv := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_{n+1} - x_n| < \infty \right\}$ equipped with the norm defined by
$$\|x\| := |x_1| + \sum_{n=1}^{\infty} |x_{n+1} - x_n| \qquad \forall x \in X.$$
Find a Schauder basis for bv or show that bv has no Schauder basis.

8.* Let X be a normed linear space and $F : \mathbb{X} \to \mathbb{K}$ be a nontrivial linear functional. Let $\alpha \in \mathbb{K}$ be given and put $S = \{x \in X : F(x) = \alpha\}$. Show that S is either closed or dense, but not both.

9.* Let $X := C[0,1]$ equipped with the norm $\|\cdot\|_\infty$ given by

$$||f||_\infty := \max\{|f(x)| : x \in [0,1]\} \qquad \forall f \in X.$$

Give an example of a nonempty closed convex set $K \subset X$ having no element of minimum norm (i.e. there is no $g \in K$ such that $||g||_\infty = \inf\{||f||_\infty : f \in K\}$).

10. Let $\mathbb{K} = \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$ be given. Let $B[a,b]$ denote the set of all bounded functions $f : [a,b] \to \mathbb{R}$. Show that there is a linear mapping $I : B[a,b] \to \mathbb{R}$ satisfying the following conditions

(i) $I(f) = \displaystyle\int_a^b f(x)dx$ for all $f \in C[a,b]$,

(ii) $I(f) \geq 0$ for all $f \in B[a,b]$ with $f(x) \geq 0$ for all $x \in [a,b]$.

11. Let X be an infinite-dimensional Banach space. Show that there exist convex sets $K_1, K_2 \subset X$ such that $K_1 \cap K_2 = \emptyset$, $K_1 \cup K_2 = X$, $\mathrm{cl}(K_1) = \mathrm{cl}(K_2) = X$.

12.* Prove or Disprove: Let X be a Banach space and $T : X \to X$ be a linear mapping such that T is injective and T^2 is continuous. Then T is continuous.

13. (Banach Limits) Let $\mathbb{K} = \mathbb{R}$ and let $X = l^\infty$ equipped with the standard norm. Define the shift operator $S : X \to X$ by

$$(Sx)_k = x_{k+1} \text{ for all } x \in X,\ k \in \mathbb{N}.$$

Show that there exists a continuous linear functional $L : X \to \mathbb{R}$ satisfying

(i) For all $x \in l^\infty$ with $x_k \geq 0$ for all $k \in \mathbb{N}$, we have $L(x) \geq 0$.

(ii) $L(x) = L(Sx)$ for all $x \in l^\infty$.

(iii) $\liminf_{k\to\infty} x_k \leq L(x) \leq \limsup_{k\to\infty} x_k$ for all $x \in l^\infty$.

(iv) $L(x) = \lim_{k\to\infty} x_k$ for all $x \in c$. (Here c is the set of all convergent sequences.)

14. Let X be a linear space and $(p_i | i \in \mathbb{N})$ be a separating family of seminorms. Define the metric $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x,y) = \sup\left\{\frac{1}{i}\left(\frac{p_i(x-y)}{1+p_i(x-y)}\right) : i \in \mathbb{N}\right\}.$$

Show that for every $x_0 \in X$ and every $r > 0$, the open ball

$$B_r(x_0) = \{x \in X : \rho(x, x_0) < r\}$$

is convex.

15.* Prove or disprove: Let X and Y be Banach spaces and let $T \in \mathcal{L}(X;Y)$ be given. Put $B = \{x \in X : \|x\| \leq 1\}$. If $T[B]$ is closed then $\mathcal{R}(T)$ is closed.