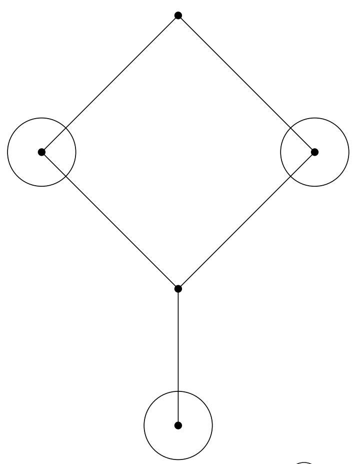
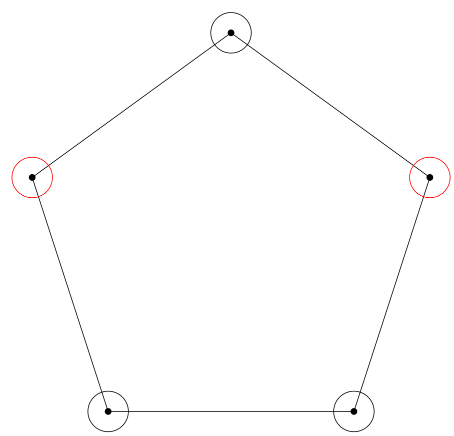
21-484 Notes

JD Nir
jnir@andrew.cmu.edu
February 1, 2012

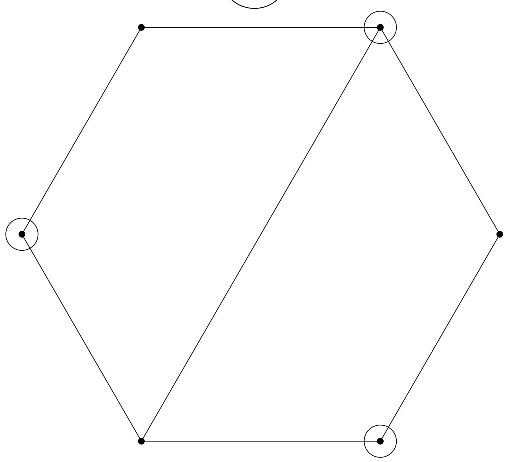
Def: (p. 19): A graph G is complete if every pair of distinct vertices is an edge.
(p. 20): A graph G is empty if every pair of distinct vertices is a non-edge.
$\rightarrow$ The complete graph on n vertices is denoted by K_n.
$\rightarrow$ $\overline{K_n}$ is empty
(p. 21): A graph G is called bipartite if $V(G)$ can be partitioned into two nonempty sets $U \dot{\cup} W = V(G)$ such that $G[U], G[W]$ are empty. U and W are called partite sets or parts.
(p. 19): A path on n vertices is denoted by P_n.
A cycle on n vertices is denoted by C_n.

Examples:





(C_5)



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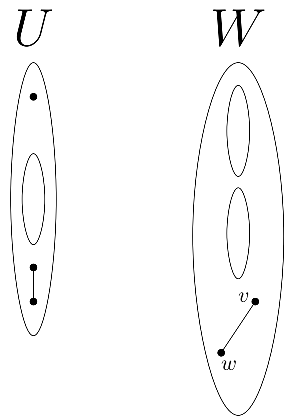
Proposition (Theorem 1.12): A non-trivial graph G is bipartite iff it contains no odd cycles.

Proof: If G contains an odd cycle, then G is not bipartite:



Assume that $v_1, v_2, \dots, v_n, v_1$ is an odd cycle in G. Assume for the sake of contradiction that $U \dot\cup W = V(G)$ is a partition of the vertex set such that $G[U]$ and $G[W]$ are empty. Without loss of generality, assume that $v_1 \in W$. Since $v_1v_2 \in E(G)$, we know $v_2 \in U$, then $v_3 \in U$.

Continuing in this way (formally, by induction) we see that $v_i \in W$ iff i is odd. n is odd, so $v_n \in W$, but then $v_nv_1 \in G[W]$. $\lightning$

$\rightarrow$ If G is not bipartite then it contains an odd cycle:

- Assume that G is connected.
- Let $u \in V(G)$. Define
$$U = \{v | d(u,v) \text{ is even}\}$$
$$W = \{v | d(u,v) \text{ is odd}\}$$
- Clearly, $U \dot\cup W = V(G)$.
- U is not empty, $u \in U$. W is not empty because G is not trivial.
- Since G is not bipartite, one of $G[U]$ or $G[W]$ is not empty.
- assume that $vw \in E(G[W])$. Let $d(u,v) = 2s+1$ and $d(u,w) = 2t+1$, also let $p' = v_0, v_1, \dots, v_{2s+1}$ be a u–v path. Let $p'' = w_0, \dots, w_{2t+1}$ be a u–w geodesic path.
- $u \in p' \cap p''$. Let x be the last common vertex between p' and p''.
- $i = d(u,x)$
- the subpath of p', $v_0, v_1, \dots, x$ is geodesic, so $x = v_i$.
- the subpath of p'', $w_0, w_1, \dots, x$ is geodesic, so $w_i = x = v_i$.
- Consider the cycle $w = w_{2t+1}, w_{2t}, \dots, w_i = v_i, v_{i+1}, \dots, v_{2s+1} = v, w$. It is of length $2t+1-i) + (2s+1-i) + 1 = 2(t+1-i+s)+1$ which is odd.

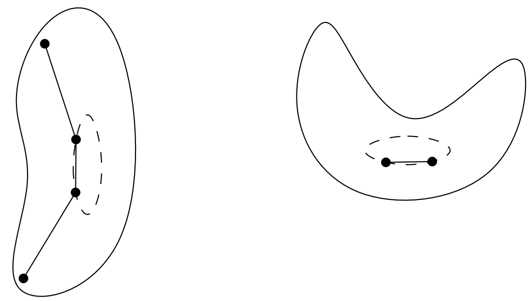
$\rightarrow$ If $vw \in E(G[U])$ then notice that $u \neq v$ and $u \neq w$. Otherwise, the other vertex $\in W$.

$\rightarrow$ Continue in the same manner.

$\rightarrow$ G is bipartite iff every connected component of G is bipartite or trivial.

Trees: Defs: (p. 86) - Let G be a connected graph, and let $e \in E(G)$. Then e is a bridge if $G - e$ is disconnected. If G is disconnected, then e is a bridge of G if it is a bridge of a component of G.

Claim: an edge is a bridge iff it lies on no cycle.



Proof: Assume $e \in G_1$, G_1 a component of G. If $e = uw$ is not a bridge is not a bridge then $G_1 - e$ is connected, so there is a u–w path in $G_1 - e$. Add e to this path to get a cycle in G_1.

If e is part of a cycle $\underset{e}{u, w,}, v_1, \dots, v_n, u$, define $p = w, v_1, \dots, v_n, u$.

$\forall x, y \in V(G_1)$, we know that there is an x–y path in G_1. If e is not on the path, then x and y are connected in $G_1 - e$.

If e is on the path, replace it by p to get an x–y walk. ■