21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
Spring 2012: Monday, Wednesday, Friday, 10:30 am, Doherty Hall 1211.
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Assignment 2 - Tuesday February 14, 2012. Due Monday February 20

Exercise 6: Let W be a non-negative continuous function on $[\alpha,\beta]\subset\mathbb{R}$ which is not identically 0, and consider on $\mathbb{R}[x]$ the Euclidean structure defined by the scalar product $(f,g)=\int_\alpha^\beta f(x)\,g(x)\,W(x)\,dx$ for all $f,g\in\mathbb{R}[x]$. Let $P_0=1,P_1,\ldots,P_n,\ldots$ be the orthogonal monic polynomials obtained by the Gram–Schmidt orthogonalization process from the basis $1,x,\ldots,x^n,\ldots$ of $\mathbb{R}[x]$.
i) Show that for $n\geq 2$ there exist $\lambda_n,\mu_n\in\mathbb{R}$ such that $P_n=(x+\lambda_n)\,P_{n-1}+\mu_n P_{n-2}$.
ii) Denoting a_m the coefficient of x^{m-1} in P_m for $m\geq 1$, and b_m the coefficient of x^{m-2} in P_m for $m\geq 2$, express λ_n,μ_n in terms of the a_js and b_ks. Deduce the induction relation satisfied by the Legendre polynomials (i.e. the case $W=1$ with $\alpha=-1,\beta=+1$).

Exercise 7: For any field E and $n\geq 2$, find an $n\times n$ matrix A with entries in E such that, whatever the field extension F of E, one cannot find an $n\times n$ matrix B with entries in F satisfying $B^2=A$.

Exercise 8: i) Let A,B be $n\times n$ matrices with entries in a field E, and such that $A^n=0$ and $det(B)\neq 0$. Show that if X is an $n\times n$ matrix (with entries in E) such that $A\,X+X\,B=0$, then $X=0$.
ii) Let M be an $n\times n$ matrix which is block diagonal (i.e. $M_{i,j}=0$ for $i\neq j$), where for $i=1,\ldots,m$, $M_{i,i}$ is $d_i\times d_i$ matrix (with $d_1+\ldots+d_m=n$) having only eigenvalue λ_i, with $\lambda_1,\ldots,\lambda_m$ distinct. Show that if Y is an $n\times n$ matrix which commutes with M, then Y is block diagonal (so that only its entries in the $d_i\times d_i$ blocks may be non-zero).

Exercise 9: If U_m is an $m\times m$ invertible real matrix, v a (column) vector in $\mathbb{R}^m$, $w=U_m^T v$, $\alpha\in\mathbb{R}$, and A the $(m+1)\times(m+1)$ (symmetric) real matrix defined by

$$A=\begin{pmatrix} U_m^T U_m & w \\ w^T & \alpha \end{pmatrix}.$$

i) show that A is positive definite (i.e. $(A\,x,x)>0$ for all non-zero $x\in\mathbb{R}^{m+1}$) if and only if $\alpha>||v||^2$, and prove that in this case one has

$$A=U_{m+1}^T U_{m+1} \text{ with } U_{m+1}=\begin{pmatrix} U_m & v \\ 0 & \beta \end{pmatrix} \text{ for some } \beta>0.$$

ii) Show that if B is an $n\times n$ positive definite symmetric matrix, then there exists a lower triangular matrix L with positive diagonal elements such that $B=L\,L^T$.
iii) Show that an $n\times n$ symmetric matrix C is positive definite if and only if its principal determinants $\Delta_1,\ldots,\Delta_n$ are >0, where Δ_j is the determinant of the $j\times j$ matrix made of the first j rows and the first j columns of C, for $j=1,\ldots,n$.

Exercise 10: Let G be an interior point of a (non-degenerate) triangle (in $\mathbb{R}^2$) with vertices A_1,A_2,A_3 and for $i<j$ let $A_{i,j}=\frac{A_i+A_j}{2}$.
i) Given a scalar $\alpha_i\in\mathbb{R}$ and a vector $\xi_i\in\mathbb{R}^2$ for each vertex A_i, a scalar value $\beta_{i,j}$ for each middle $A_{i,j}$, as well as a scalar $\gamma\in\mathbb{R}$ and a vector $\eta\in\mathbb{R}^2$ for G, show that there is a unique continuous function ψ in the triangle such that for all $i<j$ the restriction $\psi_{i,j}$ of ψ to the triangle with vertices A_i,A_j,G is a polynomial of degree ≤ 3 satisfying $\psi_{i,j}(A_i)=\alpha_i, D\psi_{i,j}(A_i)=\xi_i$, $\psi_{i,j}(A_j)=\alpha_j, D\psi_{i,j}(A_j)=\xi_j$, $\frac{\partial\psi_{i,j}(A_{i,j})}{\partial n}=\beta_{i,j}$, and $\psi_{i,j}(G)=\gamma, D\psi_{i,j}(G)=\eta$ (where D is the total derivative, and $\frac{\partial}{\partial n}$ is the normal derivative to the side).
ii) Given a scalar $\alpha_i\in\mathbb{R}$ and a vector $\xi_i\in\mathbb{R}^2$ for each vertex A_i, a scalar value $\beta_{i,j}$ for each middle $A_{i,j}$, show that there is a unique choice of $\gamma\in\mathbb{R}$ and a vector $\eta\in\mathbb{R}^2$ to use at G such that the corresponding ψ is continuously differentiable on the triangle.