Lecture Notes: Calculus of Variations

Mikil D. Foss William J. Hrusa

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Notation

$\forall$	for all
$\exists$	there exists
$A \subset B$	A is a subset of B (possibly with $A = B$)
$A \backslash B$	$\{x \in A \mid x \notin B\}$
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \cdots\}$
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^n$	with $n \in \mathbb{N}$, the set of all ordered n-tuples. In general, a point $\mathbf{x} \in \mathbb{R}^n$ can be represented by $(x_1, x_2, \cdots, x_n)$. For points in $\mathbb{R}^2$ and $\mathbb{R}^3$, however, we typically use (x, y) and (x, y, z) respectively.
$C([a, b]; \mathbb{R}^n)$	the set of all continuous mappings from the interval $[a, b]$ to $\mathbb{R}^n$
$C^k([a, b]; \mathbb{R}^n)$	with $k \in \mathbb{N}$, the set of all k-times continuously differentiable mappings from the interval $[a, b]$ to $\mathbb{R}^n$
$C[a, b]$	the set of all continuous functions from the interval $[a, b]$ to $\mathbb{R}$
$C^k[a, b]$	with $k \in \mathbb{N}$, the set of all k-times continuously differentiable functions from $[a, b]$ to $\mathbb{R}$
$f_{,k}$	with f a function from $\mathbb{R}^n$ to $\mathbb{R}$ and $1 \leq k \leq n$, we generally denote the partial derivative of f with respect to its k-th argument by $f_{,k}$ when it exists.
∇	the gradient operator. If f is a function from $\mathbb{R}^n$ to $\mathbb{R}$ and f has continuous partial derivatives, then $\nabla f = (f_{,1}, f_{,2}, \cdots, f_{,n})$.
$\delta J(y; v)$	the Gâteaux variation of J at y in the direction v
$\mathscr{V}_y$	the class of admissible variations at y
V	an alternative notation for a class of admissible variations
$\mathscr{Y}$	a class of admissible functions
Y	an alternative notation for a class of admissible functions
$\mathfrak{X}$	a real linear space

Chapter 1

Introduction

Many problems that arise in a wide variety of applications lead to mathematical formulations that require maximizing or minimizing an integral involving an unknown function and one or more of its derivatives. Typically, the unknown function is required to satisfy certain constraints, e.g. its values on the boundary of its domain may be prescribed. The branch of applied mathematics that deals with such problems is generally referred to as the *Calculus of Variations*.

The aim of this course is to provide an introduction to the calculus of variations with emphasis on basic principles rather than on recipes to solve particular problems. The main prerequisites are calculus (including functions of several variables), linear algebra, a bit of exposure to differential equations, and some mathematical maturity (i.e., the ability to follow and construct proofs).

We begin with a simple example.

1.1 Shortest Path Between Two Points

Working in the x-y plane, we seek a curve of shortest length joining the points (a, A) and (b, B), where a, b, A, B are given real numbers with $a < b$. For now, we consider only those curves that are graphs of continuously differentiable functions $y : [a, b] \to \mathbb{R}$ (see Figure 1.1). Later on we shall consider problems with parametric curves. By the arclength formula, the length of such a curve is given by

$$\int_a^b \sqrt{1 + y'(x)^2}\, dx. \tag{1.1}$$

The expression in (1.1) takes a continuously differentiable function as input and returns a real number as output. Our problem is to minimize the value of this expression over all continuously differentiable functions $y : [a, b] \to \mathbb{R}$ satisfying $y(a) = A$ and $y(b) = B$.

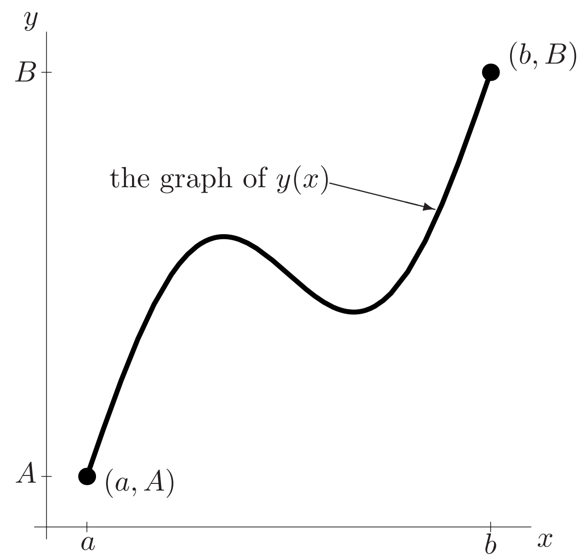

Figure 1.1: A typical curve that we consider

For this purpose it is convenient to put[1]

$$\mathscr{Y} := \left\{ y \in C^1[a,b] \,:\, y(a) = A \text{ and } y(b) = B \right\}$$

and define the functional $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b \sqrt{1 + y'(x)^2}\, dx \quad \text{for all } y \in \mathscr{Y}.$$

(Here, $C^1[a,b]$ denotes the set of all continuously differentiable functions $y : [a,b] \to \mathbb{R}$. A more detailed, and precise, definition of this set of functions is given in Appendix .) The problem can now be stated very succinctly: Minimize J over $\mathscr{Y}$.

The term *functional* is traditionally used for a real (or complex) valued function whose domain is a set of functions. Moreover, we shall refer to $\mathscr{Y}$ as the *class of admissible functions*.

As is well known, the minimum for J over $\mathscr{Y}$ is attained at the line segment described by

$$y(x) := A + \frac{B - A}{b - a}(x - a) \quad \text{for all } x \in [a,b].$$

It is possible to prove directly that

$$J(y) \geq \sqrt{(b-a)^2 + (B-A)^2} \quad \text{for all } y \in \mathscr{Y}$$

[1] We use the symbol := for an equality in which the left-hand side is defined by the right-hand side.

using a standard inequality from vector calculus. (See Exercise .)

1.2 Example 1.2

For this example, we take the class of admissible functions to be

$$\mathscr{Y} := \left\{ y \in C^1[0,1] \,:\, y(0) = 0 \text{ and } y(1) = 1 \right\}.$$

(There is nothing special about the numbers 0 and 1; they were simply chosen for convenience.) We consider the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) := \int_0^1 \left[y(x)^2 + y'(x)^2 \right] dx \quad \text{for all } y \in \mathscr{Y}. \tag{1.2}$$

Our problem is to minimize J over $\mathscr{Y}$. It is not obvious what a minimizer for J should look like. It is not even clear that a minimizer exists. (We use the term *minimizer* for a function $y \in \mathscr{Y}$ that minimizes J.)

Notice that $J(y) \geq 0$ for all $y \in \mathscr{Y}$. One might first guess that $J(y)$ can be made very small in magnitude by taking $y(x)$ to be a large power of x. Indeed, if $y(x) = x^k$ with k very large, then $y(x)$ and $y'(x)$ are nearly zero over most of the interval $[0,1]$. As x approaches 1, however, $y(x)$ must rise rapidly to 1, and consequently $y'(x)$ is very large when x is close to 1 (see Figure 1.2). It turns

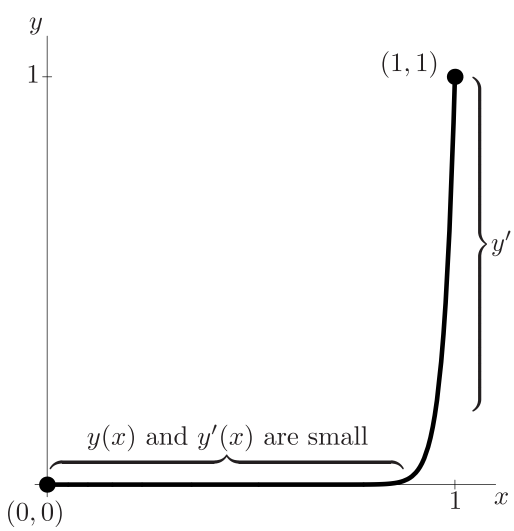

Figure 1.2: A $y \in \mathscr{Y}$ that rapidly increases to 1 as x nears 1

out that J charges a large "penalty" for this rapid rise and the value of $J(y)$ is quite large for such a function y, as we shall now see.

Let us compute the values of J for some functions in $\mathscr{Y}$ (see Figure 1.3). We look first at the function y_1 defined by $y_1(x) := x$ for all $x \in [0,1]$ and note

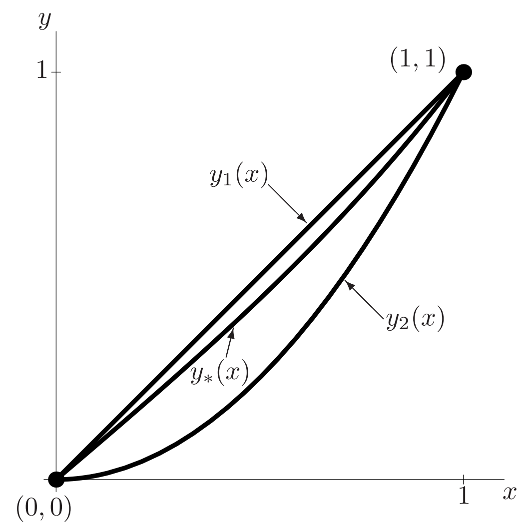

Figure 1.3: Some functions found in $\mathscr{Y}$

that $y_1'(x) = 1$ for all $x \in [0,1]$. Substituting y_1 and y_1' into (1.2), we find that

$$J(y_1) = \int_0^1 \left[y_1(x)^2 + y_1'(x)^2\right] dx = \int_0^1 \left[x^2 + 1\right] dx = \left[\frac{1}{3}x^3 + x\right]\bigg|_0^1 = \frac{4}{3}.$$

Let us now consider $y_2(x) := x^2$. Then $y_2'(x) = 2x$ and

$$\begin{aligned} J(y_2) &= \int_0^1 \left[y_2(x)^2 + y_2'(x)^2\right] dx = \int_0^1 \left[x^4 + 4x^2\right] dx = \left[\frac{1}{5}x^5 + \frac{4}{3}x^3\right]\bigg|_0^1 \\ &= \frac{1}{5} + \frac{4}{3} > \frac{4}{3} = J(y_1). \end{aligned}$$

More generally, for $y_\alpha(x) := x^\alpha$ with $\alpha \geq 1$, we have

$$\begin{aligned} J(y_\alpha) &= \int_0^1 \left[x^{2\alpha} + \alpha^2 x^{2\alpha-2}\right] dx = \left[\frac{1}{2\alpha+1}x^{2\alpha+1} + \frac{\alpha^2}{2\alpha-1}x^{2\alpha-1}\right]\bigg|_0^1 \\ &= \frac{1}{2\alpha+1} + \frac{\alpha^2}{2\alpha-1}. \end{aligned}$$

It is easy to see that $J(y_\alpha) \to \infty$ as $\alpha \to \infty$. In particular, this shows that J does not attain a maximum on $\mathscr{Y}$. It can be shown that the function y_* given by $y_*(x) := \frac{e^x - e^{-x}}{e - e^{-1}}$ for all $x \in [0,1]$ (see Figure 1.3) minimizes J on $\mathscr{Y}$. For this function, we have $y_*'(x) = \frac{e^x + e^{-x}}{e - e^{-1}}$ and

$$
\begin{aligned}
J(y_*) &= \int_0^1 \left[y_*(x)^2 + y_*'(x)^2 \right] \, dx = \frac{1}{(e - e^{-1})^2} \int_0^1 \left[(e^x - e^{-x})^2 + (e^x + e^{-x})^2 \right] \, dx \\
&= \frac{2}{(e - e^{-1})^2} \int_0^1 \left[e^{2x} + e^{-2x} \right] \, dx = \frac{1}{(e - e^{-1})^2} \left[e^{2x} - e^{-2x} \right] \Big|_0^1 = \frac{e^2 - e^{-2}}{(e - e^{-1})^2} \\
&= \frac{e^2 + 1}{e^2 - 1} < \frac{4}{3} = J(y_1).
\end{aligned}
$$

Summarizing our computations, we have found that

$$
\frac{e^2 + 1}{e^2 - 1} = J(y_*) < \frac{4}{3} = J(y_1) < \frac{23}{15} = J(y_2).
$$

In this case, there was no obvious guess for a minimizer. In "ordinary calculus" one looks for minima and maxima by considering points where the derivative vanishes (and also places where the derivative fails to exist as well as boundary points of the domain). We shall see that it is possible to define an appropriate "derivative" of functionals like the one in (1.2). Setting such a derivative equal to zero leads to a differential equation. For the specific functional in (1.2) the differential equation is $y''(x) - y(x) = 0$. Functions y in $\mathscr{Y}$ are required to satisfy $y(0) = 0$ and $y(1) = 0$, so we are led to the boundary value problem

$$
\begin{cases} y''(x) - y(x) = 0; \\ y(0) = 0 \text{ and } y(1) = 1. \end{cases} \tag{1.3}
$$

The reason why a minimizer for J on $\mathscr{Y}$ must satisfy (1.3) will be discussed later (Chapter 3). It is interesting to notice that although we only assumed the functions in $\mathscr{Y}$ have continuous first-order derivatives, the differential equation is a second-order equation. The theory to be developed will imply that for certain problems (such as this one), a minimizer must have additional smoothness properties beyond what seems "natural" from the definition the functional.

It is important to observe that although the functional has an especially simple form (and the boundary conditions are very simple), there is no "obvious" candidate for a minimizer. We need to develop some machinery that will help us to identify possible minimizers (and maximizers) and also some machinery to determine whether or not the functions that we find are actually minimizers.

1.3 The Basic Problem in the Calculus of Variations

We now formulate a much more general problem that includes the first two examples as special cases. Let $a, b, A, B \in \mathbb{R}$ with $a < b$ and $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

be given. For now, we assume that f is continuous. When we actually prove theorems, we will need to impose stronger assumptions on f. Let

$$\mathscr{Y} := \left\{ y \in C^1[a,b] \,:\, y(a) = A \text{ and } y(b) = B \right\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

The basic problem in the calculus of variations is to maximize or minimize J over $\mathscr{Y}$. Most of the time, we will talk about minimization problems. There is no loss of generality in focusing on minimization, because maximizing J is equivalent to minimizing $-J$.

There are many possible variants of the basic problem. Some of these are:

(a) Boundary conditions other than $y(a) = A$ and $y(b) = B$.

(b) Additional types of constraints on the admissible functions, e.g.

$$\int_a^b g(x, y(x), y'(x))\, dx = c,$$

where $g : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ are given. Numerous of other types of constraints are possible. For example, in certain problems the graphs of the admissible functions are required to lie above (or below) some obstacle in the $x - y$ plane.

(c) The functional takes as input a parameter as well as a function, e.g.

$$J(y, \alpha) = \int_a^b f(x, y(x), y'(x), \alpha)\, dx \quad \text{for all } y \in \mathscr{Y}, \alpha \in \mathbb{R},$$

and the problem is to find a function $y \in \mathscr{Y}$ and a real number $\alpha \in \mathbb{R}$ so that the pair (y, α) minimizes J over $\mathscr{Y} \times \mathbb{R}$;

(d) $J(y)$ may involve higher-order derivatives of y, e.g.

$$J(y) = \int_a^b f(x, y(x), y'(x), y''(x))\, dx.$$

(e) The functions $y \in \mathscr{Y}$ may be vector-valued.

(f) The functions $y \in \mathscr{Y}$ may be functions of more than one variable.

(g) Combinations of (1)–(6).

1.4 The Brachistochrone Problem (1696)

1.4.1 Description

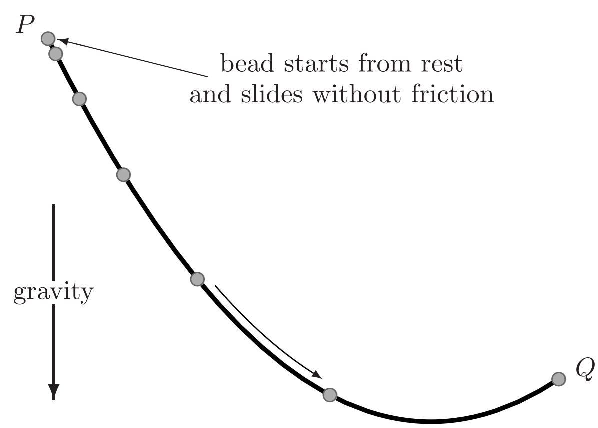

Figure 1.4: Illustration for the statement of the brachistochrone problem. The position of the bead is shown at equal time intervals.

The birth of the calculus of variations is often associated with the following mathematical challenge issued by JOHANN BERNOULLI in 1696: *Let two horizontally and vertically displaced points P and Q be given in a vertical plane with gravity acting downward. Find the smooth curve joining the two points such that a heavy particle (or bead) starting from rest at the higher point will slide without friction along the curve to reach the lower point in the shortest possible time* (Figure 1.4).

The desired curve is called a *brachistochrone*, from the Greek words $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$ (=shortest) and $\chi\rho o\nu o\varsigma$ (=time). Solutions to this problem were obtained by JOHANN BERNOULLI, his brother JAKOB BERNOULLI, L'HÔPITAL, LEIBNITZ, and NEWTON. NEWTON's solution was published anonymously. It is part of mathematical folklore that, upon reading the anonymous solution, JOHANN BERNOULLI immediately recognized it as the work of NEWTON and exclaimed: "*I can tell the lion by his claw*".

1.4.2 Mathematical Formulation of the Brachistochrone Problem

For definiteness, we assume that P lies above Q and that Q lies to the right of P. We begin by choosing a convenient coordinate system. We orient the y-axis so that the positive direction is downward, and take the x-axis to have the usual orientation. Without loss of generality, we assume that $P = (0, 0)$ and consequently $Q = (b, B)$ with $b, B > 0$ (see Figure 1.5).

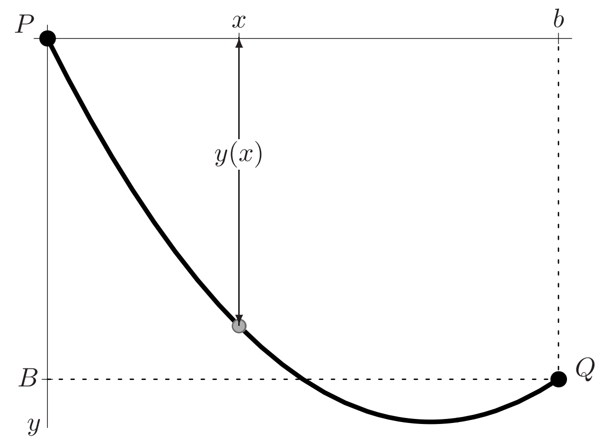

Figure 1.5: Setup for the brachistochrone problem

Next, we establish some notation. Let us denote the mass of the bead by m, the total time of transit from P to Q by T, and the speed of the bead by v. Also let t be the time variable and g be the acceleration due to gravity (which we take to be constant). We assume that the curve along which the bead travels is the graph of a smooth function y with $y(x) > 0$ for all $x \in (0, b]$.

We want to find an expression for T in terms of the function y. To this end, we let l be the total length of curve and $s : [0, b] \to \mathbb{R}$ be the arclength function for y, i.e.

$$s(x) = \int_0^x \sqrt{1 + y'(\tau)^2}\, d\tau \quad \text{for all } x \in [0, b].$$

Thus $s(0) = 0$, $s(b) = l$ and $ds = \sqrt{1 + y'(x)^2}dx$. Now the speed of the bead is $v = \frac{ds}{dt}$, or in terms of differentials, we have $dt = \frac{1}{v}ds$. Thus, the total time of transit for the bead is

$$T = \int_0^l \frac{1}{v}\, ds. \tag{1.4}$$

To express T in terms of the function y, we need to find a relationship between v and y. For this purpose we use the fact that the total energy of the bead is conserved: (Kinetic Energy)+(Potential Energy) is constant. The kinetic energy of the bead is given by $\frac{1}{2}mv^2$, while the potential energy is $-mgy$. (The minus sign in the potential energy is due to the fact that the y-axis is oriented downward.) Thus conservation of energy implies

$$\frac{1}{2}mv^2 - mgy = C \quad \text{for all } x \in [a, b],$$

where C is a constant. Since the bead starts from rest at the point $P = (0,0)$, we know that $y = v = 0$ when $x = 0$. Thus $C = 0$, and

$$\frac{1}{2}mv^2 - mgy = 0 \Rightarrow v^2 = 2gy \Rightarrow v = \sqrt{2gy(x)} \quad \text{for all } x \in [a,b]. \tag{1.5}$$

Here we have used the fact that $v \geq 0$. Substituting our expression for v and using the fact that $ds = \sqrt{1 + y'(x)^2}dx$ in (1.4) yields

$$T = \sqrt{\frac{1}{2g}} \int_0^b \frac{\sqrt{1 + y'(x)^2}}{\sqrt{y(x)}}\, dx. \tag{1.6}$$

This is our expression for the transit time of the bead in terms of the function y.

Since we are only considering curves that join $(0,0)$ to (b,B), we require the admissible functions y to satisfy $y(0) = 0$ and $y(b) = B$. We also want $y(x) > 0$ for all $x \in (0,b]$. There are some subtle issues involved with giving a precise mathematical formulation of the brachistochrone problem. In particular, since $y(0) = 0$, the integral in (1.6) is guaranteed to be singular at $x = 0$. Moreover, it turns out that the "solution" to this problem has a vertical tangent at the origin, so we cannot require the admissible functions to belong to $C^1[0,b]$. Therefore, we put[2]

$$\mathscr{Y} := \left\{ y \in C[0,b] \cap C^1(0,b] \,:\, y(0) = 0, y(b) = B \text{ and } y(x) > 0 \text{ for all } x \in (0,b] \right\}.$$

(The functions in $\mathscr{Y}$ are required to be continuous on $[0,b]$ and continuously differentiable on $(0,b]$. They are not assumed to be differentiable (from the right) at 0.) It is not difficult to show that there are functions $y \in \mathscr{Y}$ for which the integral in (1.6) diverges to $+\infty$. Consequently, we must either reduce the domain for T or allow T to take the value $+\infty$. We shall adopt the latter approach and define the functional $J : \mathscr{Y} \to \mathbb{R} \cup \{+\infty\}$ by

$$J(y) := \int_0^b \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Our problem is to minimize J over $\mathscr{Y}$. Notice that in the definition of J, we have dropped the constant factor $\sqrt{\frac{1}{2g}}$. Indeed if $c > 0$ is a constant, then minimizing J is equivalent to minimizing cJ.
(It is sometimes very convenient in minimization problems to have a simpler domain for the functional at the expense of allowing the functional to take the value $+\infty$.)

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[2]We employ a standard "abuse" of notation here. Strictly speaking, the sets $C[0,b]$ and $C^1(0,b]$ are disjoint because the functions in these sets have different domains ($[0,b]$ versus $(0,b]$). By $C[0,b] \cap C^1(0,b]$ we mean the set of all functions in $C[0,b]$ whose restrictions to the smaller domain $(0,b]$ belong to $C^1(0,b]$.

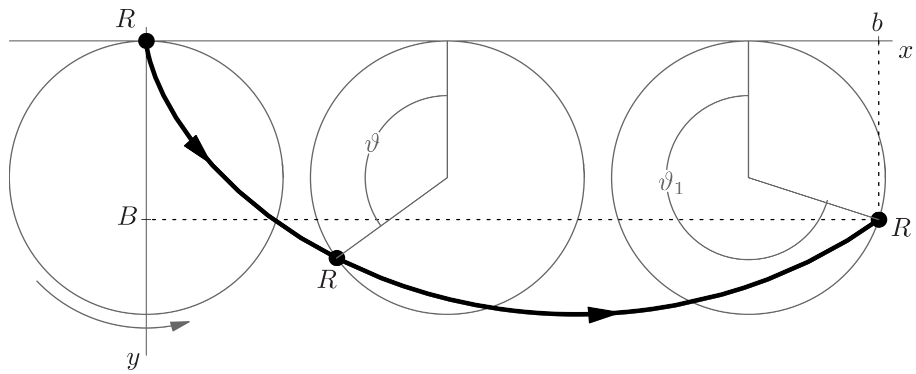

Figure 1.6: Generation of a cycloid

The minimizer for this problem is a cycloid with a cusp at the point $(0,0)$. A cycloid is the curve followed by a point R on the circumference of a circle as the circle is rolled along a horizontal line (see Figure 1.6). A useful way to represent the cycloid is by the parametric equations

$$x = \frac{c^2}{2}(\theta - \sin\theta) \text{ and } y = \frac{c^2}{2}(1 - \cos\theta) \quad \text{for all } \theta \in [0, \theta_1],$$

where $\theta_1 \in (0, 2\pi)$ and $c \in \mathbb{R}$. The quantity $\frac{c^2}{2}$ is the radius of the circle that is being rolled. The values for c and θ_1 should be chosen so that $x(\theta_1) = b$, $y(\theta_1) = B$. The condition $\theta_1 \in (0, 2\pi)$ ensures that the only cusp is at the origin, i.e. the solution involves only one "arch" of the cycloid.

1.4.3 Jakob Bernoulli's Brachistochrone Problem (1697)

In 1697, after Johann Bernoulli presented his solution to the brachistochrone problem, his brother, Jakob Bernoulli, issued a counter-challenge: *In a vertical plane with gravity acting downward, let P be a given point and let L be a given vertical line that does not contain P. Find the curve joining P and L such that a bead starting from rest at P will slide without friction along the curve and reach the line L in the least possible time.*

For definiteness, we assume that L lies to the right of P. We orient the coordinate axes in the same way as above and we assume that $P = (0,0)$. The mathematical formulation of this problem is essentially the same as that for Johann Bernoulli's brachistochrone problem, but with one important exception. The admissible class of functions is now

$$\mathscr{Y} := \left\{ y \in C[0,b] \cap C^1(0,b] \,:\, y(0) = 0 \text{ and } y(x) > 0 \text{ for all } x \in (0,b] \right\}.$$

With $J : \mathscr{Y} \to \mathbb{R} \cup \{+\infty\}$ again given by

$$J(y) := \int_0^b \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, dx \quad \text{for all } y \in \mathscr{Y},$$

the problem is to minimize J over $\mathscr{Y}$. Since the minimizer for Johann Bernoulli's problem is a cycloid, it follows immediately that the minimizer for Jakob Bernoulli's problem (if one exists) is also a cycloid. (Indeed, the solution to Jakob Bernoulli's problem is a solution to Johann Bernoulli's problem for *some* value of B.) It turns out that the there is exactly one solution which is a cycloid with a cusp at $(0, 0)$ and a horizontal tangent at $x = b$. This is an example of a problem with *one end free* because there is no boundary condition imposed on the admissible functions at $x = b$. As we shall see in Chapter 3, when we do not impose a boundary condition at an endpoint, the minimization procedure will yield a boundary condition, called a *natural boundary condition* at the free end. In this particular problem, the natural boundary condition is $y'(b) = 0$.

1.5 The Hanging Cable Problem

We wish to find the shape that an inextensible thin cable will assume hanging under its own weight when it is pinned at both ends. We assume that the cable is homogeneous and has a uniform cross-section. Let ρ be the density of the cable (mass per unit length) and let g be the acceleration due to gravity (which we assume to be constant and acting downward). The shape of the cable should minimize the potential energy. Let us assume that the endpoints of the cable are pinned at (a, A) and (b, B) with $a < b$ and that the configuration of the cable can be represented as the graph of a function $y \in C^1[a, b]$ (see Figure 1.7).

We need to express the potential energy of the cable in terms of the function y. Let l denote the (fixed) length of the cable and let s be the arclength function for the graph of y. We think of subdividing the cable into small pieces of length Δs. The mass of a portion of the cable having length Δs is $\rho\Delta s$. Thus the potential energy for this portion of the cable is approximately $gy\rho\Delta s$, and the total potential energy for the cable is approximately

$$\sum \rho g y \Delta s.$$

Taking the limit as Δs tends to zero, we can express the total energy as

$$E = \int_0^l \rho g y\, ds = \int_a^b \rho g y(x)\sqrt{1 + y'(x)^2}\, dx = \rho g \int_a^b y(x)\sqrt{1 + y'(x)^2}\, dx. \tag{1.7}$$

The problem is to minimize the potential energy over an appropriate class of admissible functions. Since the cable is inextensible and has length l, the total length of the graph of an admissible function y must be l. Hence y must satisfy

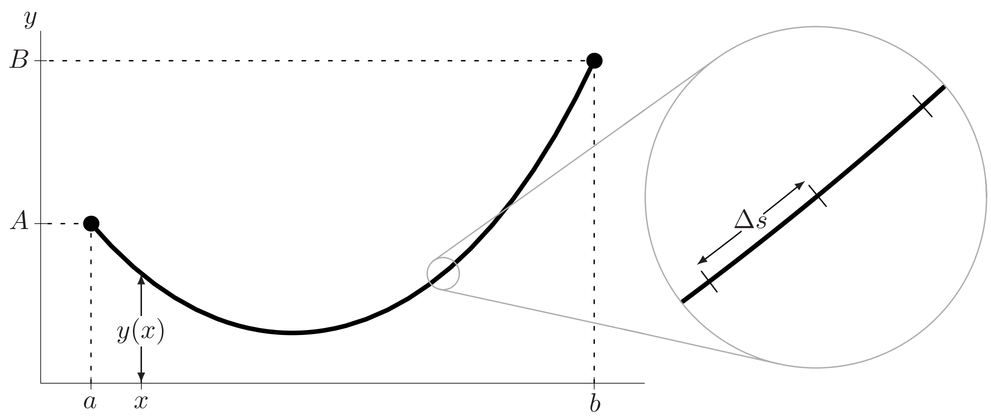

Figure 1.7: Setup for the hanging cable problem

$$\int_a^b \sqrt{1+y'(x)^2}\,dx = l. \tag{1.8}$$

The cable is also pinned at its ends, so we must have

$$y(a) = A \text{ and } y(b) = B.$$

An appropriate class of admissible functions is therefore

$$\mathscr{Y} := \left\{ y \in C^1[a,b] \,:\, y(a) = A,\ y(b) = B \text{ and } \int_a^b \sqrt{1+y'(x)^2}\,dx = l \right\}.$$

Using (1.7), we define the functional $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b y(x)\sqrt{1+y'(x)^2}\,dx \quad \text{for all } y \in \mathscr{Y}.$$

Our problem is to minimize J over $\mathscr{Y}$. (Again, we have dropped a positive multiplicative constant in the definition of J.)

1.6 Some Other Classical Problems

In this section we briefly describe several other classical problems that fall within the domain of the calculus of variations.

1.6.1 Minimal Surface of Revolution

Given $a, b, A, B \in \mathbb{R}$ with $a < b$ and $A, B > 0$, we seek a smooth curve in the $x - y$ plane joining the points (a, A) and (b, B) such that the surface generated by rotating the curve about the x-axis will have the smallest possible area. For now, we consider only curves that are graphs of smooth functions $y : [a, b] \to \mathbb{R}$. (However as we shall discuss later, it is very important in this problem to consider parametric curves as well.) In order to avoid certain technical complications, we will not allow curves that cross (or even touch) the x-axis. The surface area of such a curve is given by

$$S = 2\pi \int_a^b y(x)\sqrt{1 + y'(x)^2}\, dx.$$

Dropping the constant factor 2π, we wish to minimize over

$$\mathscr{Y} := \left\{ y \in C^1[a, b] | y(a) = A, y(b) = B \text{ and } y(x) > 0 \text{ for all } x \in [a, b] \right\}$$

the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) = \int_a^b y(x)\sqrt{1 + y'(x)^2}\, dx.$$

This problem has many very interesting features and will be discussed in much more detail in Section 3.7.

1.6.2 Geodesic Problems

It is often crucial to know the shortest path joining two given locations. If a straight-line path is feasible, then this will provide the desired route. However, if a straight-line path is not possible, then the most efficient route may not be obvious. For example, the paths may be constrained to lie on a surface, such as the surface of a sphere. The curves of shortest length on a given surface are called *geodesics* on the surface. The geodesics on a sphere are arcs of great circles. Calculus of variations plays an important role in the study of geodesics. Geodesics are discussed briefly in Section and there are several exercises concerning special types of geodesics in Chapter . However, we will not devote a great deal of attention to geodesics in this book. The interested reader should consult [ ] for more information.

1.6.3 The Isoperimetric Problem

The so-called isoperimetric problem is concerned with finding planar simple closed curves of fixed length that enclose the maximum possible area. It has been known since ancient times that such curves are circles, but a satisfactory proof of this fact is not easy. We shall discuss a version of this problem in Section 4.5.

1.7 Overview of the Text

The notes are organized as follows. In Chapter 2, we introduce our notation for partial derivatives, review some material concerning minimization problems in $\mathbb{R}^n$, and discuss a general necessary condition for extrema of functionals defined on subsets of a real linear space. This condition is based on an extension of the idea of directional derivatives. Chapter 3 is devoted to applying the ideas from Chapter 2 to problems from the calculus of variations. We begin by using ideas of LAGRANGE to develop a theory that applies to situations in which the admissible functions y and the integrand f have continuous second-order derivatives. We then employ ideas of DU-BOIS REYMOND to study situations in which the admissible functions and the integrand are assumed only to have continuous first-order derivatives. We shall see that minimizers (and maximizers) must satisfy certain differential equations called Euler-Lagrange equations.

Certain types of constraints that arise naturally (such as the isoperimetric constraint (1.8) cannot be treated adequately by the methods of Chapters 2 and 3. In Chapter 4, we introduce the method of Lagrange Multipliers which can often help in treating such constraints.

Chapter 5 is concerned with convexity and its role in establishing that the candidates for minima found by applying the necessary conditions are indeed minimizers. Chapters 6 and 7 are concerned with extending the ideas and techniques of the first five chapters to problems in which the admissible functions are vector-valued and problems with functionals involving second-order derivatives of the admissible functions. In Chapter 8 we briefly investigate sufficient conditions for minimality that are based on constructing fields of solutions to the Euler-Lagrange equations. Although this approach can be very difficult to implement in practice, it is helpful in certain problems where the approach of Chapter 5 does not apply.

Finally, in Chapter 9 we consider situations in which the admissible functions are not required to be continuously differentiable. We study the classical Weierstrass-Erdmann corner conditions as well a phenomenon discovered by LAVRENTIEV in which there can be a gap between the minimum value of J over an enlarged class of admissible functions and the values of J that can be obtained by using continuously differentiable functions.

Chapter 2

A Fundamental Necessary Condition for an Extremum

A very important result from basic calculus asserts that if a real-valued function F of a single real variable attains a local extremum (i.e. a local maximum or local minimum) at an interior point x_* of its domain, and if F is differentiable at x_*, then $F'(x_*) = 0$. The condition $F'(x_*) = 0$ can be used to identify points that are candidates for local extrema.

This result is often called FERMAT's theorem on local extrema. It was discovered by FERMAT in approximately 1635 (30 years before the discovery of calculus). Even though derivatives had not yet been invented, FERMAT realized that if F attains a local extremum at x_* then the difference quotient

$$\frac{F(x_* + h) - F(x_*)}{h}, \tag{2.1}$$

although not defined at $h = 0$, must change sign as h passes through 0. Therefore the quantity in (2.1) must be very close to 0 when $|h|$ is very small. FERMAT used this idea to find local extrema of certain polynomials.

The aim of this chapter is to generalize the result that "the derivative must vanish at an extremum" to a much more general framework that can be used to attack problems from the calculus of variations. The framework that we will use is that of functionals $J : \mathscr{Y} \to \mathbb{R}$, where $\mathscr{Y}$ is a subset of a real linear space (or vector space) $\mathfrak{X}$. We do not assume that $\mathfrak{X}$ is finite dimensional. Before discussing the situation in a general linear space, we review some results concerning optimization of functions $F : \mathbb{R}^n \to \mathbb{R}$. In introductory calculus courses, such functions are often optimized by freezing all but one of the variables and applying results from one-variable calculus. This is equivalent to studying rates of change of F when the input is varied in the coordinate directions. It will be our approach here to study the rate of change of F in arbitrary directions. This will make it much easier to generalize to an abstract linear space $\mathfrak{X}$. In our applications to calculus of variations problems, $\mathfrak{X}$ will be infinite dimensional and it will not be possible to introduce coordinates in a useful way.

Throughout the chapter, we will focus on global extrema, rather than local extrema, because we do not want to engage in a discussion of neighborhoods in (possibly) infinite-dimensional spaces and because in most of our applications to problems from the calculus of variations, it is global extrema that are of interest. Moreover, results will generally be stated for minima. This involves no loss of generality since the maximum of J can be found by minimizing $-J$. We begin the chapter with a discussion of notation for partial derivatives.

2.1 Notational Conventions

Let $n \in \mathbb{N}$ be given. As usual, we denote by $\mathbb{R}^n$ the set of all n-tuples (i.e., lists of length n) of real numbers. The elements of $\mathbb{R}^n$ may be thought of as "points" or as "vectors", depending the context. Typical elements of $\mathbb{R}^n$ will be denoted by $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, etc. When n has been prescribed, and the value of n is not too large (say, $n \leq 4$), we sometimes use a different letter for each component, rather than use the same letter with different subscripts. For example an element of $\mathbb{R}^2$ may be denoted by (x, y), an element of $\mathbb{R}^3$ by (x, y, z), and an element of $\mathbb{R}^4$ by (x, y, z, w), etc.

Addition and *scalar multiplication* on $\mathbb{R}^n$ are defined componentwise, i.e.

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \quad \text{for all } x, y \in \mathbb{R}^n,$$

$$cx = (cx_1, cx_2, \cdots, cx_n) \quad \text{for all } c \in \mathbb{R}, x \in \mathbb{R}^n.$$

The *zero element* of $\mathbb{R}^n$ will be denoted by

$$0 \text{ or } (0, 0, \cdots, 0).$$

With addition, scalar multiplication, and zero element as described above, $\mathbb{R}^n$ becomes a *real linear space* (i.e., a vector space over the field $\mathbb{R}$). The *dot product* (or *inner product*) of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Notice that $x \cdot x > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$.

To explain our notation for partial derivatives, let $F : \mathbb{R}^n \to \mathbb{R}$ be given:

$$F(x) = F(x_1, x_2, \cdots, x_n) \quad \text{for all } x \in \mathbb{R}^n.$$

We use two types of notation to denote the partial derivative of F with respect to its k-th argument. With $1 \leq k \leq n$, we write

$$F_{,k}(x) = F_{,k}(x_1, x_2, \cdots, x_n) = \frac{\partial}{\partial x_k} F(x) = \frac{\partial}{\partial x_k} F(x_1, x_2, \cdots, x_n).$$

The notation $\frac{\partial}{\partial x_k}$ is standard. We introduce the notation $F_{,k}$ because it makes no explicit reference to a variable name. In particular, this notation emphasizes that the derivative of F is to be taken with respect to its k-th argument – without regard to any particular symbol that is being used to denote the argument in question. This is very useful for situations involving composite functions. Assuming that the partial derivatives $F_{,1}, F_{,2}, \cdots, F_{,n}$ are continuous we define the gradient of F, $\nabla F : \mathbb{R}^n \to \mathbb{R}^n$, by

$$\begin{aligned}\nabla F(x) = \nabla F(x_1, x_2, \cdots, x_n) &= (F_{,1}(x), F_{,2}(x), \cdots, F_{,n}(x)) \\ &= \left(\tfrac{\partial}{\partial x_1} F(x), \tfrac{\partial}{\partial x_2} F(x), \cdots, \tfrac{\partial}{\partial x_n} F(x)\right).\end{aligned}$$

As a simple illustration of the notation, suppose that $F : \mathbb{R}^4 \to \mathbb{R}$ is given by

$$F(x) = x_1 + 4x_1^6 x_2^3 - x_3^2 + e^{5x_4} \quad \text{for all } x \in \mathbb{R}^4.$$

Then

$$F_{,1}(\mathbf{x}) = 1{+}24x_1^5 x_2^3;\ F_{,2}(\mathbf{x}) = 12x_1^6 x_2^2;\ F_{,3}(\mathbf{x}) = -2x_3;\ F_{,4}(\mathbf{x}) = 5e^{5x_4} \quad \text{for all } \mathbf{x} \in \mathbb{R}^4$$

and

$$\nabla F(\mathbf{x}) = (1 + 24x_1^5 x_2^3, 12x_1^6 x_2^2, -2x_3, 5e^{5x_4}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^4.$$

We shall also need to consider partial derivatives of functions defined on suitable subsets of $\mathbb{R}^n$ (such as products of intervals). We use the same notation in such cases.

We conclude this section with an example that illustrates a significant advantage of the comma-notation for partial derivatives. Consider the function $f : [-1, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y, z) = (1 - x^2)^{3/2} y^3 z^4 \quad \text{for all } (x, y, z) \in [-1, 1] \times \mathbb{R} \times \mathbb{R}.$$

The partial derivatives of f are given by

$$\begin{aligned}&f_{,1}(x, y, z) = -3x(1 - x^2)^{1/2} y^3 z^4;\ f_{,2}(x, y, z) = 3(1 - x^2)^{3/2} y^2 z^4; \\ &f_{,3}(x, y, z) = 4(1 - x^2)^{3/2} y^3 z^3 \quad \text{for all } (x, y, z) \in [-1, 1] \times \mathbb{R} \times \mathbb{R}.\end{aligned}$$

(Here we are making the convention that at the endpoints $x = -1, 1$, the partial derivative with respect to the first argument of f is a one-sided derivative.) Suppose, in addition, that we are given a function $y \in \mathscr{C}^1[-1, 1]$. Then we can write the following composite function in an unambiguous way:

$$f_{,1}(x, y(x), y'(x)) = -3x(1 - x^2)^{1/2} y(x)^3 y'(x)^4 \quad \text{for all } x \in [-1, 1].$$

A formula such as

$$\frac{\partial}{\partial x} f(x, y(x), y'(x))$$

could be genuinely confusing. Similarly, we write

$$f_{,2}(x, y(x), y'(x)) = 3(1 - x^2)^{3/2} y(x)^2 y'(x)^4$$

$$f_{,3}(x, y(x), y'(x)) = 4(1 - x^2)^{3/2} y(x)^3 y'(x)^3 \quad \text{for all } x \in [-1, 1].$$

Composites of the type above will play a central role in calculus of variations problems.

2.2 The Chain Rule

The chain rule for differentiating composite functions of one variable is one of the most important results from basic calculus. Here we record a chain rule for the composition of a function $F : \mathbb{R}^n \to \mathbb{R}$ with a function $u : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval.

Let $u : I \to \mathbb{R}^n$ be given:

$$u(t) = (u_1(t), u_2(t), \cdots, u_n(t)) \quad \text{for all } t \in I.$$

Recall that u is differentiable provided that each component of u is differentiable; in this case we write

$$u'(t) = (u_1'(t), u_2'(t), \cdots, u_n'(t)) \quad \text{for all } t \in I.$$

Theorem 2.1 (Chain Rule) *Let $I \subset \mathbb{R}$ be an interval and $u : I \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}$ be a given. Assume that u is differentiable on I and that F has continuous first-order partial derivatives on $\mathbb{R}^n$. Define $g : I \to \mathbb{R}$ by*

$$g(t) = F(u(t)) \quad \text{for all } t \in I.$$

Then g is differentiable on I and

$$\begin{aligned} g'(t) &= \nabla F(u(t)) \cdot u'(t) \\ &= \sum_{i=1}^{n} F_{,i}(u(t))u_i'(t) \quad \text{for all } t \in I. \end{aligned}$$

Moreover, if u' is continuous on I then g' is continuous on I. (In this theorem we make the convention that if I is not open, then derivatives at endpoints of I should be interpreted as appropriate one-sided derivatives.)

Since partial derivatives are defined as ordinary derivatives with respect to one variable while holding the other variables constant, we can apply the version of the chain rule given above to compute certain partial derivatives as well. We record below the relevant result below for the case when u is a vector-valued function of two variables. Here, for

$$u(t_1, t_2) = (u_1(t_1, t_2), u_2(t_1, t_2), \cdots, u_n(t_1, t_2))$$

and each $i \in \{1, 2\}$, we write

$$u_{,i}(t_1, t_2) = (u_{1,i}(t_1, t_2), u_{2,i}(t_1, t_2), \cdots, u_{n,i}(t_1, t_2)).$$

Remark 2.1 *Let $I_1, I_2 \subset \mathbb{R}$ be intervals and $u : I_1 \times I_2 \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}$ be given. Let $i \in \{1, 2\}$ be given and assume that $u_{,i}$ exists on $I_1 \times I_2$. Assume further that F has continuous first-order partial derivatives on $\mathbb{R}^n$. Define $g : I_1 \times I_2 \to \mathbb{R}$ by*

$$g(t_1, t_2) = F(u(t_1, t_2)) \quad \text{for all } (t_1, t_2) \in I_1 \times I_2.$$

Then, the partial derivative $g_{,i}$ exists on $I_1 \times I_2$ and is given by

$$\begin{aligned} g_{,i}(t_1, t_2) &= \nabla F(u(t_1, t_2)) \cdot u_{,i}(t_1, t_2) \\ &= \sum_{j=1}^{n} F_{,j}(u(t_1, t_2)u_{j,i}(t_1, t_2) \quad \text{for all } (t_1, t_2) \in I_1 \times I_2. \end{aligned}$$

2.3 Minimization Problems in $\mathbb{R}^n$

We will now review a method for determining the possible minimizers of a function $F : \mathbb{R}^n \to \mathbb{R}$. We assume that F has continuous first-order partial derivatives. Suppose that F attains a minimum at the point $y_* \in \mathbb{R}^n$. We want to establish some conditions that y_* must satisfy.

The basic idea is to move away from y_* in every direction possible and use the hypothesis that y_* minimizes F. Since y_* is a minimizer for F, if we move from y_* in any direction, the value of F cannot decrease..

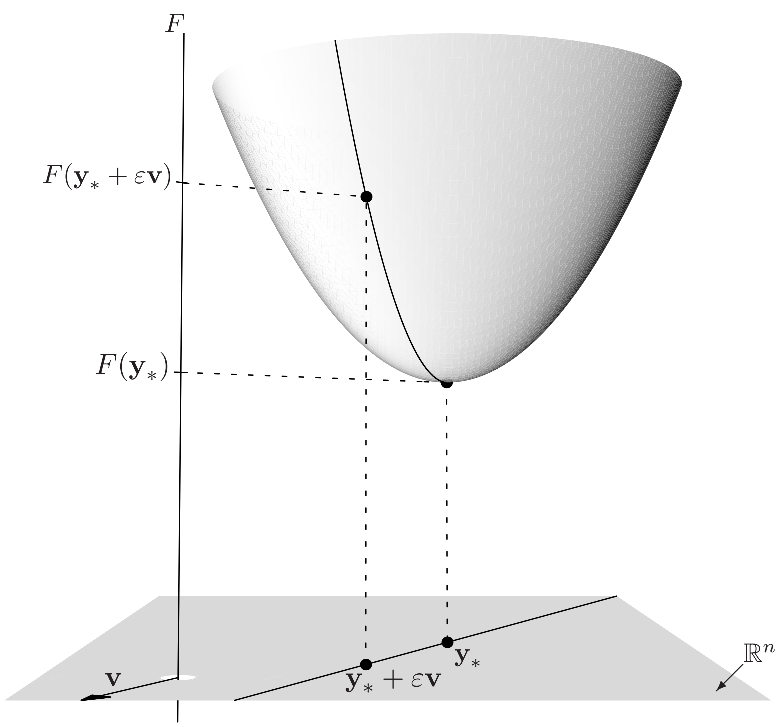

Figure 2.1: Comparison of $F(y_*)$ and $F(y_* + \varepsilon v)$

Let $v \in \mathbb{R}^n$ with $v \neq 0$ be given. We will use v as a direction in which to move from y_*. The line in the direction of v that passes through y_* is the set

$$\{y_* + \varepsilon v \, : \, \varepsilon \in \mathbb{R}\}.$$

Since F is minimized at y_*, we must have

$$F(y) \geq F(y_*) \quad \text{for all } y \in \mathbb{R}^n.$$

In particular, we have

$$F(y_* + \varepsilon v) \geq F(y_*) \quad \text{for all } \varepsilon \in \mathbb{R}.$$

Hence

$$F(y_* + \varepsilon v) - F(y_*) \geq 0 \quad \text{for all } \varepsilon \in \mathbb{R},$$

and therefore

$$\frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon} \geq 0 \quad \text{for all } \varepsilon > 0 \tag{2.2}$$

and

$$\frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon} \leq 0 \quad \text{for all } \varepsilon < 0. \tag{2.3}$$

We want to take the limit in (2.2) and (2.3) as ε tends to 0 from the right and left respectively.

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\varepsilon) := F(y_* + \varepsilon v) \quad \text{for all } \varepsilon \in \mathbb{R},$$

so that $\varphi(\varepsilon) = F(u(\varepsilon))$ with $u(\varepsilon) := y_* + \varepsilon v$. Obviously, the vector-valued function u is continuously differentiable; in fact, its derivative is just v. Since F is assumed to have continuous first-order partial derivatives, we can apply the chain rule to φ to obtain

$$\varphi'(\varepsilon) = \nabla F(u(\varepsilon)) \cdot u'(\varepsilon) = \nabla F(y_* + \varepsilon v) \cdot v \quad \text{for all } \varepsilon \in \mathbb{R}.$$

Evaluating this derivative at $\varepsilon = 0$, we find that

$$\varphi'(0) = \nabla F(y_*) \cdot \mathbf{v}. \tag{2.4}$$

Notice that by definition $\varphi'(0) = \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon}$ so that

$$\varphi'(0) = \lim_{\varepsilon \to 0} \frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon}. \tag{2.5}$$

Since $\varphi'(0)$ exists, the limit in (2.5) exists, and we may take the appropriate one-sided limits in (2.2) and (2.3) to obtain

$$\varphi'(0) = \lim_{\varepsilon \to 0^+} \frac{F(y_* + \varepsilon v) - F(y_*)}{\varepsilon} \geq 0 \tag{2.6}$$

and

$$\varphi'(0) = \lim_{\varepsilon \to 0^-} \frac{F(y_* + \varepsilon \mathbf{v}) - F(y_*)}{\varepsilon} \leq 0. \tag{2.7}$$

Consequently

$$\varphi'(0) = 0,$$

by (2.6) and (2.7). Using (2.4), we have

$$\nabla F(y_*) \cdot v = \mathbf{0} \quad \text{for all } v \in \mathbb{R}^n \tag{2.8}$$

since $v \in \mathbb{R}^n$ was an arbitrary non-zero vector. (Equality in (2.8) obviously holds if $v = 0$.)

We now wish to show that (2.8) implies $\nabla F(y_*) = 0$. This will follow from

Lemma 2.1 *Let $w \in \mathbb{R}^n$ be given. Suppose that $w \cdot v = 0$ for all $v \in \mathbb{R}^n$. Then $w = 0$.*

Proof. Notice that the hypothesis of the lemma implies $w \cdot w = 0$. Thus

$$w \cdot = w_1 w^2 + w_2^2 + \cdots + w_n^2 = 0.$$

It follows that

$$w_1 = w_2 = \cdots = w_n = 0$$

and $w = 0$, as claimed. □

In view of the lemma above, we have shown that if $y_* \in \mathbb{R}^n$ minimizes the function F on $\mathbb{R}^n$, then we necessarily have

$$\nabla F(y_*) = 0.$$

If F attains only a local minimum at y_* then the method described above can still be used to show that $\nabla F(y_*) = \mathbf{0}$. Indeed, in this case, the inequality in (2.2) holds for all ε in some interval $(0, \delta)$ and the inequality in (2.3) holds for all ε in some interval $(-\delta, 0)$. We can still take the limit as $\varepsilon \to 0$ from both sides. If $\mathscr{Y}$ is a proper subset of $\mathbb{R}^n$ and $F : \mathscr{Y} \to \mathbb{R}$ attains a minimum at $y_* \in \mathscr{Y}$ then we may only be able to move away from y_* in certain directions v. This issue will be addressed in Definition 2.1.

2.4 Minimization Problems in Real Linear Spaces

We want to use a similar argument to determine the possible minimizers of a real-valued functional defined on a subset of a real linear space $\mathfrak{X}$. With $\mathfrak{X}$ a real linear space, let $\mathscr{Y} \subset \mathfrak{X}$ and $J : \mathscr{Y} \to \mathfrak{X}$ be given. Suppose that J attains a minimum at $y_* \in \mathscr{Y}$. We are going to find some conditions that y_* must satisfy.

In order to use the basic ideas from the previous section, we need to know in which directions we can move from y_*. That is, we must know those v's in $\mathfrak{X}$ such that $[y_* + \varepsilon v] \in \mathscr{Y}$ for all ε with $|\varepsilon|$ sufficiently small. Note that in general $[y_* + \varepsilon v]$ may not be in $\mathscr{Y}$ for arbitrary $v \in \mathfrak{X}$ and $\varepsilon \in \mathbb{R}$ (see Figure 2.2). To distinguish those v's that may be used to vary y_*, we make a definition.

Definition 2.1 (Admissible Variations) *Let $v \in \mathfrak{X}$ and $y \in \mathscr{Y}$ be given. We say that v is a $\mathscr{Y}$-*admissible variation* at y if there exists an open interval $I \subset \mathbb{R}$ with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for every $\varepsilon \in I$. The set of all $\mathscr{Y}$-admissible variations at y is denoted by $\mathscr{V}_y$. If the set $\mathscr{Y}$ is clear from context, we will simply use the term* admissible variation *in place of $\mathscr{Y}$-admissible variation.*

Thus, if $v \in \mathscr{V}_{y_*}$, then for all $\varepsilon \in \mathbb{R}$ such that $|\varepsilon|$ is sufficiently small, we find $[y_* + \varepsilon v] \in \mathscr{Y}$.

Let $v \in \mathscr{V}_{y_*}$ be a given admissible variation. By definition there exists an open interval $I \subset \mathbb{R}$ with $0 \in I$ and such that $[y_* + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$. Since y_* minimizes J, we must have

$$J(y_* + \varepsilon v) \geq J(y_*) \quad \text{for all } \varepsilon \in I.$$

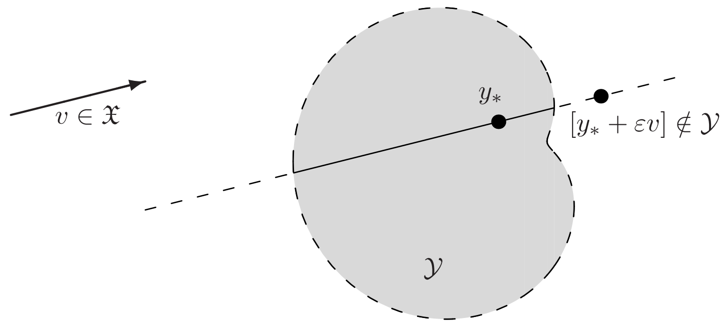

Figure 2.2: Situation where $[y_* + \varepsilon v]$ is not in $\mathscr{Y}$

Hence

$$J(y_* + \varepsilon v) - J(y_*) \geq 0 \quad \text{for all } \varepsilon \in I,$$

and therefore

$$\frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} \geq 0 \quad \text{for all } \varepsilon \in I \cap (0, +\infty) \tag{2.9}$$

and

$$\frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} \leq 0 \quad \text{for all } \varepsilon \in I \cap (-\infty, 0). \tag{2.10}$$

Since I is open and $0 \in I$, we know that neither $I \cap (0, +\infty)$ nor $I \cap (-\infty, 0)$ is empty. Suppose that $\lim_{\varepsilon \to 0} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon}$ exists. Then by (2.9) and (2.10)

$$0 \leq \lim_{\varepsilon \to 0^-} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon} \leq 0.$$

Therefore, if $\lim_{\varepsilon \to 0} \frac{J(y_* + \varepsilon v) - J(y_*)}{\varepsilon}$ exists it must be equal to zero.

2.5 Gâteaux Variations

2.5.1 Definition and Relevance to Optimization Problems

In order to state the result obtained above in a convenient form, we make the following

Definition 2.2 (Gâteaux Variation) *Let $y \in \mathscr{Y}$ and $v \in \mathscr{V}_y$ be given. The* Gâteaux variation of J at y in the direction v *is defined by*

$$\delta J(y; v) := \lim_{\varepsilon \to 0} \frac{J(y + \varepsilon v) - J(y)}{\varepsilon},$$

provided the limit exists.

In terms of this definition, our argument above proves the following:

Theorem 2.2 *Let $\mathfrak{X}$ be a real linear space. Also let $\mathscr{Y} \subset \mathfrak{X}$ and $J : \mathscr{Y} \to \mathbb{R}$ be given. Suppose that $y_* \in \mathscr{Y}$ is a minimizer (or a maximizer) for J over $\mathscr{Y}$. Let $v \in \mathscr{V}_{y_*}$ be given. Then, either $\delta J(y_*; v) = 0$ or $\delta J(y_*; v)$ does not exist.*

The theorem gives a procedure to find possible minimizers (or maximizers) of J. In our applications to problems from the Calculus of Variations, the Gâteaux variations will always exist. Therefore, to describe the procedure for identifying possible minimizers, let us assume for now that $\delta J(y; v)$ exists for all $y \in \mathscr{Y}$ and all $v \in \mathscr{V}_y$. To apply the theorem, we shall carry out the following steps:

(1) Identify the class of admissible variations $\mathscr{V}_y$ at each $y \in \mathscr{Y}$.

(2) Compute the Gâteaux Variations $\delta J(y; v)$.

(3) Analyze the condition

$$\delta J(y; v) = 0 \quad \text{for all } v \in \mathscr{V}_y. \tag{2.11}$$

In order for this approach to work, there must be sufficiently many admissible variations. If there are "too few" admissible variations, we will not be able to deduce much from (2.11). In such cases, a different approach will be employed.

2.5.2 Example 2.5.2.

For our first example involving Gâteaux variations, we look at the special case of a function from $\mathbb{R}^n$ to $\mathbb{R}$. Let $\mathfrak{X} = \mathscr{Y} = \mathbb{R}^n$ and assume that $F : \mathbb{R}^n \to \mathbb{R}$ has continuous first-order partial derivatives. It is easy to see that for each $y \in \mathbb{R}^n$, the class of admissible variations is all of $\mathbb{R}^n$, i.e.

$$\mathscr{V}_y = \mathbb{R}^n \quad \text{for all } y \in \mathbb{R}^n.$$

Given $y, v \in \mathbb{R}^n$, the arguments used in Section 2.3 show that

$$\delta F(y; v) = \nabla F(y) \cdot v.$$

Notice that if v is a unit vector then $\delta F(y; v)$ is simply the directional derivative of F at y in the direction of v.

Remark 2.2 *If $\mathscr{Y}$ is a proper subset of $\mathbb{R}^n$ and $y \in \mathscr{Y}$ is not an interior point of $\mathscr{Y}$, then $\mathscr{V}_y$ will be a proper subset of $\mathbb{R}^n$. In particular, if y is a boundary point of $\mathscr{Y}$ then $\mathscr{V}_y$ will be a proper subset of $\mathbb{R}^n$. If a function $F : \mathbb{R}^n \to \mathbb{R}$ attains a minimum (or maximum) at a boundary point y_* of $\mathscr{Y}$, we cannot conclude that $\nabla F(y_*) = 0$.*

2.5.3 Example 2.5.3.

Let $\mathfrak{X} = C[1,2]$ and put

$$\mathscr{Y} = \{y \in C[1,2] \, : \, \int_1^2 y(x)\,dx = 1\}.$$

Consider the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) = \int_1^2 x^3 y(x)^2\,dx \quad \text{for all } y \in \mathscr{Y}.$$

We wish to find all possible minimizers for J on $\mathscr{Y}$.

The first step is to determine the admissible variations. Let $y \in \mathscr{Y}$ and $v \in \mathfrak{X}$ be given. Notice that for each $\varepsilon \in \mathbb{R}$, we have $[y+\varepsilon v] \in C[1,2]$. Consequently, in order that $v \in \mathscr{V}_y$ it is necessary and sufficient that there exist an open interval I containing 0 such that

$$\int_1^2 [y(x) + \varepsilon v(x)]\,dx = 1 \quad \text{for all } \varepsilon \in I. \tag{2.12}$$

Since $y \in \mathscr{Y}$ and

$$\int_1^2 [y(x) + \varepsilon v(x)]\,dx = \int_1^2 y(x)\,dx \; + \varepsilon \int_1^2 v(x)\,dx,$$

we see that (2.12) holds if and only if

$$\begin{aligned} \int_1^2 [y(x) + \varepsilon v(x)]\,dx &= \int_1^2 y(x)\,dx \; + \varepsilon \int_1^2 v(x)\,dx \\ &= 1 + \varepsilon \int_1^2 v(x)\,dx \\ &= 1 \quad \text{for all } \varepsilon \in I. \end{aligned}$$

In other words (2.12) holds if and only if

$$\varepsilon \int_1^2 v(x)\,dx = 0 \quad \text{for all } \varepsilon \in I.$$

Since I must contain nonzero elements, we conclude that $v \in \mathscr{V}_y$ if and only if

$$\int_1^2 v(x)\,dx = 0.$$

(Notice that the interval I in the definition of admissible variation can be taken to be $\mathbb{R}$.) Since $\mathscr{V}_y$ is the same for each $y \in \mathscr{Y}$, we drop the subscript on $\mathscr{V}_y$ and write

$$\mathscr{V} = \{v \in \mathscr{C}[1,2] \mid \int_1^2 v(x)\,dx = 0\}.$$

The next step is to try to compute the Gâteaux variations. To this end, let $y \in \mathscr{Y}$, $v \in \mathscr{V}$, and $\varepsilon \in \mathbb{R}$ be given, and observe that

$$\begin{aligned} J(y+\varepsilon v) - J(y) &= \int_1^2 x^3(y(x)+\varepsilon v(x))^2\,dx - \int_1^2 x^3 y(x)^2\,dx \\ &= \int_1^2 x^3[y(x)^2 + 2\varepsilon y(x)v(x) + \varepsilon^2 v(x)^2]\,dx - \int_1^2 x^3 y(x)^2\,dx \\ &= 2\varepsilon \int_1^2 x^3 y(x)v(x)\,dx + \varepsilon^2 \int_1^2 v(x)^2\,dx, \end{aligned}$$

which yields

$$\frac{J(y+\varepsilon v) - J(y)}{\varepsilon} = 2\int_1^2 x^3 y(x)v(x)\,dx + \varepsilon \int_1^2 x^3 v(x)^2\,dx \tag{2.13}$$

for $\varepsilon \neq 0$. Taking the limit as $\varepsilon \to 0$ in (2.13) we find that $\delta J(y;v)$ exists and is given by

$$\delta J(y;v) = 2\int_1^2 x^3 y(x)v(x)\,dx.$$

Consequently, if y minimizes J on $\mathscr{Y}$ we must have

$$\int_1^2 x^3 y(x)v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}. \tag{2.14}$$

In order understand the implications of (2.14), we shall give a somewhat general result about continuous functions having the property that the product of the function with every continuous function having zero average also has zero average.

Lemma 2.2 *Let $a, b \in \mathbb{R}$ with $a < b$ and $w \in \mathscr{C}[a,b]$ be given and put*

$$\mathscr{U} = \{v \in C[a,b] \mid \int_a^b v(x)\,dx = 0\}.$$

Assume that

$$\int_a^b w(x)v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{U}.$$

Then there is a constant $c \in \mathbb{R}$ such that $w(x) = c$ for all $x \in \mathbb{R}$.

Proof. Observe first that for each $c \in \mathbb{R}$ and $v \in \mathscr{U}$ we have

$$\int_a^b (w(x) - c)v(x)\, dx = 0. \tag{2.15}$$

Let us put

$$c_* = \frac{1}{b-a} \int_a^b w(x)\, dx,$$

so that

$$\int_a^b (w(x) - c_*)\, dx = 0.$$

It follows that $(w - c_*) \in \mathscr{U}$. We put $c = c_*$ and $v = w - c_*$ in (2.15) to obtain

$$\int_a^b (w(x) - c_*)^2\, dx = 0. \tag{2.16}$$

Since the integrand in (2.16) is nonnegative and continuous, it must vanish identically on $[a, b]$, i.e.

$$w(x) = c_* \quad \text{for all } x \in [a, b].$$

□

Let $y \in \mathscr{Y}$ be given and assume that y minimizes J on $\mathscr{Y}$. Then, by virtue of (2.14) and Lemma 2.2 with $a = 1$, $b = 2$, and $w(x) = x^3 y(x)$, we conclude that there is a constant c such that

$$x^3 y(x) = c \quad \text{for all } x \in [1, 2], \text{ i.e.}$$

$$y(x) = cx^{-3} \quad \text{for all } x \in [1, 2].$$

In order to have $y \in \mathscr{Y}$, we must have

$$1 = \int_1^2 y(x)\, dx = c \int_1^2 x^{-3}\, dx = c(\frac{1}{2} - \frac{1}{8}) = \frac{3c}{8}.$$

It follows that $c = 8/3$ and

$$y(x) = \frac{8}{3} x^{-3} \quad \text{for all } x \in [1, 2]. \tag{2.17}$$

If J has a minimum on $\mathscr{Y}$ then the minimum must be attained at the function y given by (2.17). It can be shown directly that this function y does indeed minimize J on $\mathscr{Y}$. (See Problem .)

2.5.4 How to Compute a Gâteaux Variation

With $\mathfrak{X}$ a real linear space, $\mathscr{Y} \subset \mathfrak{X}$ and $J : \mathscr{Y} \to \mathbb{R}$ given, we want to compute the Gâteaux variations of J. Fix $y \in \mathscr{Y}$ and $v \in \mathscr{V}_y$. Then we may choose an open interval $I \subset \mathbb{R}$ with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$.

As in Section 2.3, we compute the Gâteaux variation of J at y in the direction v as the derivative at 0 of a real-valued function. Define $\varphi : I \to \mathbb{R}$ by

$$\varphi(\varepsilon) := J(y + \varepsilon v) \quad \text{for all } \varepsilon \in I.$$

Notice that φ is defined on an open neighborhood of 0 and that $\varphi(0) = J(y)$. Hence

$$\frac{J(y + \varepsilon v) - J(y)}{\varepsilon} = \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} \quad \text{for all } \varepsilon \in I,\ \varepsilon \neq 0,$$

and by definition

$$\delta J(y; v) = \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} = \varphi'(0),$$

provided the limit exists. The derivative of φ at 0 exists if and only if $\delta J(y; v)$ exists; moreover, in this case we have $\varphi'(0) = \delta J(y; v)$. Another way of stating this is that

$$\delta J(y; v) = \left. \frac{d}{d\varepsilon} \left[J(y + \varepsilon v) \right] \right|_{\varepsilon = 0}$$

if the Gâteaux variation of J at y in the direction v exists. This gives a convenient way to compute the Gâteaux variation $\delta J(y; v)$ in practice – simply perturb y by εv, differentiate the expression $J(y + \varepsilon v)$ with respect to ε, and then evaluate at $\varepsilon = 0$ (assuming that the derivative exists for *vep* in some open interval about 0).

2.6 Admissible Classes Characterized by Linear Constraints

In many calculus of variations problems, the class of admissible functions will consist of all those y in one of the standard function spaces (such as $C^1[a, b]$) satisfying a list of linear constraints:

$$L_1(y) = w_1, L_2(y) = w_2, \cdots, L_k(y) = w_k,$$

where the L_i are given linear mappings from $\mathfrak{X}$ to a real linear space $\mathfrak{W}$ and the w_i are given elements of $\mathcal{W}$. When this is the case, it turns out that the class of admissible variations is the same at each $y \in \mathscr{Y}$ and is actually a subspace of $\mathfrak{X}$, namely the intersection of the null spaces of the L_i. Moreover, the interval I in the definition of admissible variation can be taken to be the entire real line. In order to avoid "re-inventing the wheel" each time this situation arises, we shall prove a very useful lemma.

Let $\mathfrak{X}$ and $\mathcal{W}$ be real linear spaces. Recall that a mapping $L : \mathfrak{X} \to$ is said to be *linear* provided that

$$L(z_1 + z_2) = L(z_1) + L(z_2) \quad \text{for all } z_1, z_2 \in \mathfrak{X},$$

and

$$L(cz) = cL(z) \quad \text{for all } c \in \mathbb{R}, z \in \mathfrak{X}.$$

The *null space* of a linear mapping L is defined by

$$\mathcal{N}(L) = \{z \in \mathfrak{X} \, : \, L(z) = 0\}.$$

It is straightforward to verify that $\mathcal{N}(L)$ is a subspace if L is linear.

Lemma 2.3 *Let $k \in \mathbb{N}$, $w_1, w_2, \cdots, w_k \in \mathcal{W}$ be given and assume that $L_1, L_2, \cdots, L_k : \mathfrak{X} \to \mathcal{W}$ are linear mappings. Define $\mathscr{Y} \subset \mathfrak{X}$ by*

$$\mathscr{Y} = \{y \in \mathfrak{X} \mid L_1(y) = w_1, L_2(y) = w_2, \cdots, L_k(y) = w_k\}.$$

Then for each $y \in \mathscr{Y}$ the class of $\mathscr{Y}$-admissible variations at y is given by

$$\mathcal{Z} = \{v \in \mathfrak{X} \, : \, L_1(v) = L_2(v) = \cdots = L_k(v) = 0\}.$$

Moreover, for all $y \in \mathscr{Y}, v \in \mathcal{Z}$, we have $y + \varepsilon v \in \mathscr{Y}$ for all $\varepsilon \in \mathbb{R}$.

Proof. Let $y \in \mathscr{Y}$ be given. We shall show that

$$\mathcal{Z} \subset \mathscr{V}_y \text{ and } \mathscr{V}_y \subset \mathcal{Z}.$$

To this end, let $v \in \mathcal{Z}$ and $\varepsilon \in \mathbb{R}$ be given. It is immediate that $[y + \varepsilon v] \in \mathfrak{X}$. Moreover, for each $i \in \{1, 2, \cdots, k\}$ we have

$$L_i(y + \varepsilon v) = L_i(y) + L_i(\varepsilon v) = L_i(y) + \varepsilon L_i(v) = L_i(y) + 0 = L_i(y) = w_i,$$

so that $L_i(y + \varepsilon v) \in \mathscr{Y}$. It follows that $v \in \mathscr{V}_y$. (Notice that the interval I in the definition of admissible variation may be taken to be $\mathbb{R}$.)

To establish the reverse inclusion, let $v \in \mathscr{V}_y$ be given and choose an open interval I with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$. Choose $\varepsilon_* \in I \backslash \{0\}$. Then, for each $i \in \{1, 2, \cdots, k\}$ we have

$$L_i(y + \varepsilon_* v) = L_i(y) + \varepsilon_* L_i(v) = w_i.$$

Since $L_i(y) = w_i$ for all $i \in \{1, 2, \cdots, k\}$ and $\varepsilon_* \neq 0$, we conclude that

$$L_i(v) = 0 \quad \text{for all } i \in \{1, 2, \cdots, k\}$$

and $v \in \mathcal{Z}$. $\square$

2.6.1 Example 2.6.1.

For our first application of the lemma, we consider the situation of Example 2.5.3:

$$\mathfrak{X} = C[1,2], \quad \mathscr{Y} = \{y \in C[1,2] \mid \int\limits_1^2 y(x)\,dx = 1\}.$$

We take $\mathcal{W} = \mathbb{R}$, $k = 1$, $w_1 = 1$ and define the linear mapping $L_1 : \mathfrak{X} \to \mathcal{W}$ by

$$L_1(z) = \int\limits_1^2 z(x)\,dx \quad \text{for all } z \in \mathfrak{X}.$$

Notice that

$$\mathscr{Y} = \{y \in \mathfrak{X} : L_1(y) = w_1\}.$$

Employing Lemma 2.3, we find that for each $y \in \mathscr{Y}$ we have

$$\mathscr{V}_y = \{v \in \mathfrak{X} \mid L_1(v) = 0\} = \{v \in C[1,2] : \int\limits_1^2 v(x)\,dx = 0\}.$$

2.6.2 Example 2.6.2.

Let $\mathfrak{X} = C^1[-1,4]$ and put

$$\mathscr{Y} = \{y \in C^1[-1,4] : y(-1) = 8,\ y(4) = 5,\ \int\limits_{-1}^4 x^6 y(x) = 2\}.$$

To apply Lemma 2.3, we put $k = 3$, $\mathcal{W} = \mathbb{R}$, $w_1 = 8$, $w_2 = 5$, $w_3 = 2$ and define the linear mappings $L_1, L_2, L_3 : \mathfrak{X} \to \mathcal{W}$ by

$$L_1(z) = z(-1),\ L_2(z) = z(4),\ L_3(z) = \int\limits_{-1}^4 x^6 z(x)\,dx \quad \text{for all } z \in \mathfrak{X}.$$

With these definitions, we have

$$\mathscr{Y} = \{y \in \mathfrak{X} : L_1(y) = w_1,\, L_2(y) = w_2,\, L_3(y) = w_3\}$$

so that the lemma tells us

$$\begin{aligned} \mathscr{V}_y &= \{v \in \mathfrak{X} : L_1(v) = L_2(v) = l_3(v) = 0\} \\ &= \{v \in C^1[-1,4] \mid v(-1) = 0,\ v(4) = 0,\ \textstyle\int_{-1}^4 x^6 v(x)\,dx = 0\}. \end{aligned}$$

2.7 Gâteaux Variations of Integral Functionals

It is useful to obtain a general formula for the Gâteaux variations of functionals of the type appearing in the basic problem of the calculus of variations (and variants of this problem). In order to obtain such a formula by the procedure of Section 2.5.4, we shall need to differentiate an integral with respect to a parameter appearing under the integral sign. We record below an important result that will allow us to carry out the required differentiation.

Theorem 2.3 *Let $a, b \in \mathbb{R}$ with $a < b$, an open interval I, and $g : [a,b] \times I \to \mathbb{R}$ be given. Assume that g and $g_{,2}$ are continuous on $[a,b] \times \mathbb{R}$. Define $G : I \to \mathbb{R}$ by*

$$G(\varepsilon) := \int_a^b g(x,\varepsilon)\,dx \quad \text{for all } \varepsilon \in I.$$

Then G is differentiable on $\mathbb{R}$ and

$$G'(\varepsilon) = \int_a^b g_{,2}(x,\varepsilon)\,dx \quad \text{for all } \varepsilon \in I.$$

The theorem says that

$$\frac{d}{d\varepsilon}\int_a^b g(x,\varepsilon)\,dx = \int_a^b \frac{\partial}{\partial\varepsilon} g(x,\varepsilon)\,dx;$$

in other words, we may "pass the derivative under the integral sign".

Theorem 2.4 *Let $a, b \in \mathbb{R}$ and $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given and assume that f has continuous first-order partial derivatives. Let $\mathfrak{X}$ be a subspace of $\mathscr{C}^1[a,b]$ and let $\mathscr{Y} \subset \mathfrak{X}$ be given. Define $J : \mathscr{Y} \to \mathbb{R}$ by*

$$J(y) = \int_a^b f(x, y(x), y'(x))\,dx \quad \text{for all } y \in \mathscr{Y}.$$

Then for each $y \in \mathscr{Y}$, $v \in \mathscr{V}_y$, the Gâteaux variation $\delta J(y; v)$ exists and is given by

$$\delta J(y;v) = \int_a^b [f_{,2}(x, y(x), y'(x))v(x) + f_{,3}(x, y(x), y'(x))v'(x)]\,dx. \tag{2.18}$$

Proof. We choose an open interval I with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{Y}$ for all $\varepsilon \in I$, and define $\varphi : I \to \mathbb{R}$ by

$$\varphi(\varepsilon) = J(y + \varepsilon v) = \int_a^b f(x, y(x) + \varepsilon v(x), y'(x) + \varepsilon v'(x))\,dx \quad \text{for all } \varepsilon \in I.$$

Using Theorem 2.3 and the chain rule, we find that

$$\begin{aligned}
\varphi'(\varepsilon) &= \int_a^b \frac{\partial}{\partial\varepsilon}\left[f(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x))\right] dx \\
&= \int_a^b \Big\{ f_{,1}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x))\frac{\partial}{\partial\varepsilon}[x] \\
&\qquad + f_{,2}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x))\frac{\partial}{\partial\varepsilon}[y(x)+\varepsilon v(x)] \\
&\qquad + f_{,3}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x))\frac{\partial}{\partial\varepsilon}[y'(x)+\varepsilon v'(x)]\Big\}\, dx \\
&= \int_a^b \Big\{ f_{,2}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x))v(x) \\
&\qquad + f_{,3}(x, y(x)+\varepsilon v(x), y'(x)+\varepsilon v'(x))v'(x)\Big\}\, dx.
\end{aligned}$$

Evaluating this expression at $\varepsilon = 0$ gives us

$$\varphi'(0) = \int_a^b \Big\{ f_{,2}(x, y(x), y'(x))v(x) + f_{,3}(x, y(x), y'(x))v'(x)\Big\}\, dx.$$

Therefore, the Gâteaux variation of J at y in the direction v exists and is given by

$$\delta J(y; v) = \int_a^b \Big\{ f_{,2}(x, y(x), y'(x))v(x) + f_{,3}(x, y(x), y'(x))v'(x)\Big\}\, dx.$$

$\square$

Chapter 3

Euler-Lagrange Equations

In this chapter, we apply the results of Sections 2.4 through 2.7 to some problems from the calculus of variations. We shall see that solutions of these problems must satisfy differential equations (called Euler-Lagrange equations). We begin with the so-called C^2-theory in which it is assumed that the integrand f and the admissible functions y have continuous second-order derivatives. After developing this theory and investigating some examples, we will develop an analogous C^1-theory in which the integrand and admissible functions are assumed only to have continuous first-order derivatives.

3.1 C²-Theory

Let $a, b, A, B \in \mathbb{R}$ with $a < b$ and $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given. We assume that f has continuous second-order partial derivatives. Let $\mathfrak{X} = C^2[a, b]$ and define $\mathscr{Y} \subset \mathfrak{X}$

$$\mathscr{Y} := \left\{ y \in C^2[a, b] \, : \, y(a) = A \text{ and } y(b) = B \right\}.$$

(It is readily verified that $C^2[a, b]$ is a real linear space and is a subspace of $C^1[a, b]$.) We wish to minimize the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) := \int_a^b f(x, y(x), y'(x)) \, dx \quad \text{for all } y \in \mathscr{Y}.$$

Let $y_* \in \mathscr{Y}$ be given and suppose that J attains a minimum at $y_* \in \mathscr{Y}$. We want to carry out the following three steps:

(1) Identify the class of admissible variations $\mathscr{V}_{y_*}$.

(2) Determine $\delta J(y_*; v)$ for each $v \in \mathscr{V}_{y_*}$.

(3) Figure out what can be deduced from the condition $\delta J(y_*; v) = 0$ for all $v \in \mathscr{V}_{y_*}$.

For step (1), we apply Lemma 2.3. We take $\mathcal{W} = \mathbb{R}$, let $w_1 = A$, $w_2 = B$, and define $L_1, L_2 : \mathfrak{X} \to \mathcal{W}$ by

$$L_1(z) = z(a) \quad L_2(z) = z(b) \quad \text{for all } z \in \mathfrak{X}.$$

It is clear that L_1 and L_2 are linear. We conclude from Lemma 2.3 that for each $y \in \mathscr{Y}$ the class of admissible variations at y is (see Figure 3.1)

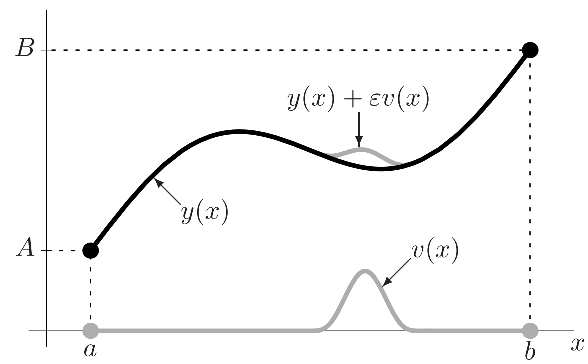

Figure 3.1: Example of a $v \in \mathscr{V}_y$ and $y + \varepsilon v$

$$\mathscr{V}_y = \left\{ v \in C^2[a,b] \,:\, v(a) = v(b) = 0 \right\},$$

and that the interval I in the definition of admissible variation can be taken to be $\mathbb{R}$. Since $\mathscr{V}_y$ does not depend upon y, we drop the subscript and simply denote the admissible variations by $\mathscr{V}$.

For step (2), we use Theorem 2.4. For each $y \in \mathscr{Y}$ and $v \in \mathscr{V}$, we know that $\delta J(y; v)$ exists and is given by

$$\delta J(y; v) = \int_a^b \left\{ f_{,2}(x, y(x), y'(x)) v(x) + f_{,3}(x, y(x), y'(x)) v'(x) \right\} dx.$$

We now carry out step (3). Since y_* is a minimizer for J over $\mathscr{Y}$, Theorem 2.2 tells us that

$$\delta J(y_*; v) = \int_a^b \left\{ f_{,2}(x, y_*(x), y_*'(x)) v(x) + f_{,3}(x, y_*(x), y_*'(x)) v'(x) \right\} dx = 0 \quad \text{for all } v \in \mathscr{V}. \tag{3.1}$$

To analyze (3.1), let us define $F, G \in C^1[a,b]$ by

$$F(x) := f_{,2}(x, y_*(x), y_*'(x)) \quad \text{and} \quad G(x) := f_{,3}(x, y_*(x), y_*'(x)) \quad \text{for all } x \in [a,b], \tag{3.2}$$

so we can rewrite (3.1) as

$$\int_a^b \Big\{F(x)v(x) + G(x)v'(x)\Big\}\,dx = 0 \quad \text{for all } v \in \mathscr{V}. \tag{3.3}$$

Since $G \in C^1[a,b]$, we may integrate the second term by parts to obtain

$$\begin{aligned}\int_a^b G(x)v'(x)\,dx &= [G(x)v(x)]\Big|_a^b - \int_a^b G'(x)v(x)\,dx \\ &= G(b)v(b) - G(a)v(a) - \int_a^b G'(x)v(x)\,dx \quad \text{for all } v \in \mathscr{V}. \end{aligned} \tag{3.4}$$

Using the fact that $v(a) = v(b) = 0$ for every $v \in \mathscr{V}$ we find that

$$\int_a^b G(x)v'(x)\,dx = -\int_a^b G'(x)v(x)\,dx \quad \text{for all } v \in \mathscr{V}.$$

Thus (3.3) becomes

$$\int_a^b \{F(x)v(x) - G'(x)v(x)\}\,dx = \int_a^b \{F(x) - G'(x)\}\,v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}.$$

Recalling the definitions (3.2), we conclude that if y_* minimizes J on $\mathscr{Y}$ then we must have

$$\int_a^b \left\{f_{,2}(x, y_*(x), y_*'(x)) - \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right]\right\} v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}. \tag{3.5}$$

The following lemma due to Lagrange will permit us to draw a very powerful conclusion from (3.5).

Lemma 3.1 (Lagrange) *Let $g \in C[a,b]$ be given, and assume that $\int_a^b g(x)v(x)\,dx = 0$ for all $v \in C^2[a,b]$ satisfying $v(a) = v(b) = 0$. Then $g(x) = 0$ for all $x \in [a,b]$.*

Applying this lemma to (3.5) yields the following condition on y_*:

$$f_{,2}(x, y_*(x), y_*'(x)) - \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] = 0 \quad \text{for all } x \in [a,b]. \tag{E-L$_1$}$$

The equation (E-L)_1 is called the first Euler-Lagrange equation or simply the Euler-Lagrange equation for J. If y_* is a minimizer (or a maximizer) for J on $\mathscr{Y}$ then it must satisfy this equation.

Before proving Lagrange's lemma, let us examine the Euler-Lagrange equation for some examples.

3.1.1 Example 3.1.1 (cf. Example 1.2)

Consider the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) := \int_0^1 \left[y(x)^2 + y'(x)^2 \right] dx \quad \text{for all } y \in \mathscr{Y},$$

where

$$\mathscr{Y} := \left\{ y \in C^2[0,1] \,:\, y(0) = 0 \text{ and } y(1) = 1 \right\}.$$

This is the same minimization problem as in Section 1.2, except that the admissible functions are assumed to belong to $C^2[0,1]$ rather than to $C^1[0,1]$. If we define $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,y,z) := y^2 + z^2 \quad \text{for all } (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R},$$

then J has the form

$$J(y) = \int_0^1 f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

To find the Euler-Lagrange equation for this functional, we need the partial derivatives of f with respect to its second and third arguments. We easily find

$$f_{,2}(x,y,z) = 2y \qquad \text{and} \qquad f_{,3}(x,y,z) = 2z \quad \text{for all } (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}.$$

Therefore

$$f_{,2}(x, y(x), y'(x)) = 2y(x) \qquad \text{and} \qquad f_{,3}(x, y(x), y'(x)) = 2y'(x) \quad \text{for all } y \in \mathscr{Y},$$

and the first Euler-Lagrange equation for J is

$$2y(x) - \frac{d}{dx}\left[2y'(x)\right] = 0 \quad \text{for all } x \in [0,1]. \tag{E-L$_1$}$$

Thus a minimizer for J on $\mathscr{Y}$ must satisfy

$$\begin{cases} y''(x) - y(x) = 0; \\ y(0) = 0 \text{ and } y(1) = 1. \end{cases} \tag{3.6}$$

Let us solve the boundary value problem (3.6). The Euler-Lagrange equation is a second-order homogeneous linear differential equation with constant coefficients. The characteristic equation is $r^2 - 1 = 0$ which has roots $r = \pm 1$. It follows that the solution to (3.6) has the form

$$y(x) = C_1 e^x + C_2 e^{-x}$$

for some constants C_1 and C_2. It only remains to find C_1 and C_2 so that the boundary conditions are satisfied. We have

$$y(0) = 0 \Rightarrow C_2 = -C_1;$$

and

$$y(1) = 1 \Rightarrow C_1 e + C_2 e^{-1} = 1 \Rightarrow C_1 = \frac{1}{e - e^{-1}} \text{ and } C_2 = \frac{-1}{e - e^{-1}}.$$

Thus the only solution to (3.6) is given by

$$y(x) = \frac{e^x - e^{-x}}{e - e^{-1}}.$$

What we know at this point is that if there is a minimizer, it is given by the formula above. It turns out that this function is indeed a minimizer. We shall establish this last claim in Section 5.7.1.

3.1.2 Example 3.1.2

For this example, we take

$$\mathscr{Y} := \left\{ y \in C^2[1, 32] \, : \, y(1) = 1 \text{ and } y(32) = 2 \right\}$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_1^{32} x^2 y'(x)^6 \, dx \quad \text{for all } y \in \mathscr{Y}.$$

The integrand for J is the function $f : [1, 32] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$f(x, y, z) = x^2 z^6 \quad \text{for all } (x, y, z) \in [1, 32] \times \mathbb{R} \times \mathbb{R}.$$

To write $(\text{E-L})_1$ for J, we observe that

$$f_{,2}(x, y, z) = 0 \quad \text{and} \quad f_{,3}(x, y, z) = 6x^2 z^5 \quad \text{for all } (x, y, z) \in [1, 32] \times \mathbb{R} \times \mathbb{R}.$$

Therefore, the first Euler-Lagrange equation for this example is

$$\frac{d}{dx} \left[x^2 y'(x)^5 \right] = 0 \quad \text{for all } x \in [1, 32], \tag{E-L$_1$}$$

so a minimizer for J on $\mathscr{Y}$ must satisfy

$$\begin{cases} \dfrac{d}{dx} \left[x^2 y'(x)^5 \right] = 0; \\ y(1) = 1 \text{ and } y(32) = 2. \end{cases} \tag{3.7}$$

To solve (3.7), observe that we can integrate the Euler-Lagrange equation. This yields

$$x^2 y'(x)^5 = C_1 \Rightarrow y'(x) = C_2 x^{-\frac{2}{5}} \Rightarrow y(x) = C_3 x^{\frac{3}{5}} + C_4.$$

(Note that we have made the substitutions C_3 for $\frac{5}{3}C_2$ and C_2 for $C_1^{\frac{1}{5}}$). We now use the boundary conditions to find values for C_3 and C_4:

$$y(1) = 1 \Rightarrow C_3 + C_4 = 1 \Rightarrow C_3 = 1 - C_4;$$

and

$$y(32) = 2 \Rightarrow 8C_3 + C_4 = 2 \Rightarrow C_4 = \frac{6}{7} \Rightarrow C_3 = \frac{1}{7}.$$

The only solution to (3.7) is therefore

$$y(x) = \frac{1}{7}x^{\frac{3}{5}} + \frac{6}{7}.$$

We shall show in Section 5.7.1 that the solution found above is, in fact, a minimizer.

3.2 Proof of Lagrange's Lemma

In this section, we provide a proof of Lagrange' lemma. For ease of reference, we restatte the lemma here in a self-contained form.

Lemma 3.1 (Lagrange) *Let $a, b \in \mathbb{R}$ with $a < b$ and $g \in C[a,b]$ be given and put*

$$\mathscr{V} := \left\{ v \in C^2[a,b] \,:\, v(a) = v(b) = 0 \right\}.$$

Assume that $\displaystyle\int_a^b g(x)v(x)\,dx = 0$ *for all* $v \in \mathscr{V}$. *Then* $g(x) = 0$ *for all* $x \in [a,b]$.

Our proof of this lemma relies on the construction of a function from $\mathscr{V}$ having special properties.

Lemma 3.2 *Let $a, b, \alpha, \beta \in \mathbb{R}$ with $a < \alpha < \beta < b$ be given. Define the function $v_* : [a,b] \to \mathbb{R}$ by (see Figure 3.2)*

$$v_*(x) := \begin{cases} 0, & a \le x < \alpha; \\ (x-\alpha)^4(x-\beta)^4, & \alpha \le x \le \beta; \\ 0, & \beta < x \le b. \end{cases}$$

Then $v_* \in C^2[a,b]$ and $v_*(a) = v_*(b) = 0$.

Proof of Lemma 3.2. It is clear that $v_* \in C[a,b]$ and $v_*(a) = v_*(b) = 0$. So we need only show that $v_*', v_*'' \in C[a,b]$.

First we show that v_*' is continuous on $[a,b]$. We easily find that v_* is differentiable at each point in $[a,\alpha) \cup (\alpha,\beta) \cup (\beta,b]$ and

$$v_*'(x) = 0 \quad \text{for all } x \in [a,\alpha) \cup (\beta,b]$$

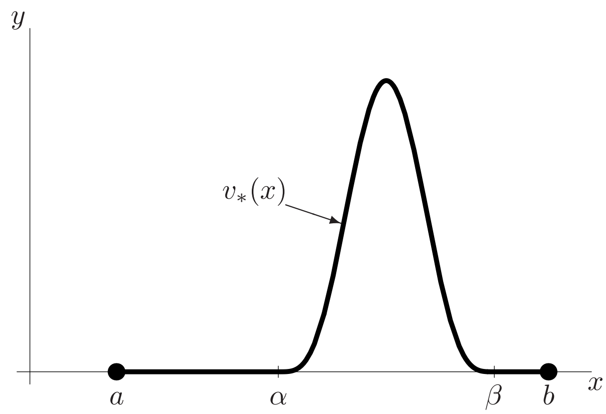

Figure 3.2: Graph of v_*

and

$$v_*'(x) = 4(x-\alpha)^4(x-\beta)^3 + 4(x-\alpha)^3(x-\beta)^4 \quad \text{for all } x \in (\alpha, \beta).$$

We see that v_*' is continuous at each point in $[a, \alpha) \cup (\alpha, \beta) \cup (\beta, b]$, and it only remains to show that v_*' is continuous at α and β. Examining the limit as x tends to α from the left and right, we find

$$\lim_{x \to \alpha^-} v_*'(x) = 0 = \lim_{x \to \alpha^+} \left[4(x-\alpha)^4(x-\beta)^3 + 4(x-\alpha)^3(x-\beta)^4\right] = \lim_{x \to \alpha^+} v_*'(x).$$

It follows that v_* is differentiable at α and v_*' is continuous at α. It is similarly established that v_*' is continuous at β. Thus $v_*' \in C[a, b]$.

Now we prove that $v_*'' \in C[a, b]$. We have

$$v_*''(x) = 0 \quad \text{for all } x \in [a, \alpha) \cup (\beta, b]$$

and

$$\begin{aligned} v_*''(x) = 12&(x-\alpha)^4(x-\beta)^2 \\ &+ 32(x-\alpha)^3(x-\beta)^3 + 12(x-\alpha)^2(x-\beta)^4 \quad \text{for all } x \in (\alpha, \beta). \end{aligned}$$

Obviously v_*'' is continuous on $[a, \alpha) \cup (\alpha, \beta) \cup (\beta, b]$. As with the first derivative of v_*, an examination of the one-sided limits of v_*'' at α and β shows that the second derivative of v_* is continuous at α and β. It follows that $v_*'' \in \mathscr{C}[a, b]$.

Therefore $v_* \in \mathscr{C}^2[a, b]$, as claimed. □

Now we turn to the proof of Lagrange's Lemma.

Proof of Lemma 3.1. We will prove the contrapositive assertion. To this end, suppose that g is not identically zero on the interval $[a, b]$. We want to show that there exists a $v \in \mathscr{V}$ such that $\int_a^b g(x)v(x)\,dx \neq 0$. (This will prove the lemma.)

Since g is not identically zero, there must exist an $x_0 \in (a,b)$ such that $g(x_0) \neq 0$. Without loss of generality, we assume that $g(x_0) > 0$. By the continuity of g, we may choose $\delta > 0$ so that $a < x_0 - \delta < x_0 < x_0 + \delta < b$ and $g(x) > 0$ for every $x \in (x_0 - \delta, x_0 + \delta)$ (see Figure 3.3). Define $v_* : [a,b] \to \mathbb{R}$ by

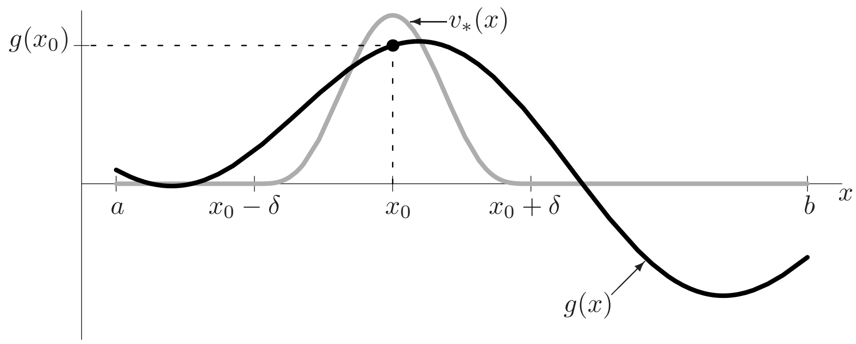

Figure 3.3: Example of a $v_* \in \mathscr{V}$

$$v_*(x) := \begin{cases} 0, & a \leq x < x_0 - \delta; \\ (x - x_0 + \delta)^4 (x - x_0 - \delta)^4, & x_0 \leq x \leq x_0 + \delta; \\ 0, & x_0 + \delta < x \leq b. \end{cases}$$

By Lemma 3.2, we know that $v_* \in \mathscr{V}$. Observe that we have $g(x)v_*(x) > 0$ for every $x \in (x_0 - \delta, x_0 + \delta)$ and $g(x)v_*(x) = 0$ elsewhere on [a,b]. It follows that $\int_a^b g(x)v(x)\,dx > 0$ and the proof is complete. $\square$

3.3 Problems with Free Endpoints

So far we have focused on problems where the values of the admissible functions are prescribed at both endpoints. For these problems, we have found that a minimizer (or maximizer) must satisfy the Euler-Lagrange equation as well as the prescribed boundary conditions. Now we wish to determine what conditions must be satisfied by a minimizer (or maximizer) when the values of the admissible functions y are prescribed at only one end, or when the values of y are not prescribed at either end. An endpoint at which the value of y is not prescribed is called a *free end*. As we shall see, in problems with free ends, the Euler-Lagrange equations still holds and at each free end a *natural boundary condition* must hold.

We begin with some notation. Let $a, b, A, B \in \mathbb{R}$ with $a < b$ be given. Put

$$\mathfrak{X} := C^2[a,b].$$

Let $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function with continuous second-order partial derivatives. Define the functional $J : \mathfrak{X} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathfrak{X}.$$

We now specify a variety of classes of admissible functions and indicate the corresponding classes of admissible variations (which can easily be determined using Lemma 2.3). Let

$$\begin{aligned}
\mathrm{Y}_{a,b} &:= \left\{ y \in C^2[a,b] \,:\, y(a) = A \text{ and } y(b) = B \right\};\\
\mathrm{Y}_a &:= \left\{ y \in C^2[a,b] \,:\, y(a) = A \right\};\\
\mathrm{Y}_b &:= \left\{ y \in C^2[a,b] \,:\, y(b) = B \right\};\\
\mathrm{Y} &:= C^2[a,b];\\
\mathrm{V}_{a,b} &:= \left\{ v \in C^2[a,b] \,:\, v(a) = v(b) = 0 \right\};\\
\mathrm{V}_a &:= \left\{ v \in C^2[a,b] \,:\, v(a) = 0 \right\};\\
\mathrm{V}_b &:= \left\{ v \in C^2[a,b] \,:\, v(b) = 0 \right\};\\
\mathrm{V} &:= C^2[a,b].
\end{aligned}$$

Notice that $\mathrm{Y}_{a,b}$ is the usual class of admissible functions when both endpoints are pinned, and earlier we found that $\mathrm{V}_{a,b}$ is the class of admissible variations for each $y \in \mathrm{Y}_{a,b}$. A function in the class Y_a only has its value at a prescribed; it is not required to have any particular value at the endpoint b, and we say that the functions in Y_a have a free endpoint at b. Since a function $y \in \mathrm{Y}_a$ has no prescribed value at b, an admissible variation v for y need not satisfy $v(b) = 0$. It is straightforward to check that V_a, as defined above, is the class of admissible variations for every $y \in \mathrm{V}_a$. Analogously Y_b is the class of admissible functions with a free endpoint at a, and V_b is the class of admissible variations for every $y \in \mathrm{Y}_b$. The set Y is the class of admissible functions for a problem with both endpoints free. The class of admissible variations for each $y \in \mathrm{Y}$ is V (which is all of $C^2[a,b]$).

For the sake of definiteness, we focus on the problem of minimizing J when both endpoints are free. We therefore suppose that J attains a minimum over Y at $y_* \in \mathrm{Y}$. We wish to find some useful conditions that y_* must satisfy.

By Theorem 2.2, for each $v \in \mathrm{V}$ either $\delta J(y_*; v) = 0$ or $\delta J(y_*; v)$ does not exist. Using Theorem 2.4 we find that for each $v \in \mathrm{V}$, the Gâteaux variation of J at y_* in the direction v exists and

$$\delta J(y_*; v) = \int_a^b \left\{ f_{,2}(x, y_*(x), y_*'(x)) v(x) + f_{,3}(x, y_*(x), y_*'(x)) v'(x) \right\} dx = 0 \quad \text{for all } v \in \mathrm{V}. \tag{3.8}$$

Since f has continuous second partial derivatives and $y_* \in C^2[a,b]$, we may define $F, G \in C^1[a,b]$ by

$$F(x) := f_{,2}(x, y_*(x), y_*'(x)) \text{ and } G(x) := f_{,3}(x, y_*(x), y_*'(x)) \quad \text{for all } x \in [a,b]. \tag{3.9}$$

Thus (3.8) becomes

$$\int_a^b \Big\{ F(x)v(x) + G(x)v'(x) \Big\}\, dx = 0 \quad \text{for all } v \in \mathrm{V}. \tag{3.10}$$

As in Section 3.1, we observe that $G \in C^1[a,b]$ and we may therefore integrate the second term in the integrand in (3.10) by parts. Doing so, we obtain

$$\begin{aligned} \int_a^b G(x)v'(x)\, dx &= [G(x)v(x)]\Big|_a^b - \int_a^b G'(x)v(x)\, dx \\ &= G(b)v(b) - G(a)v(a) - \int_a^b G'(x)v(x)\, dx \quad \text{for all } v \in \mathrm{V}. \end{aligned} \tag{3.11}$$

For problems with both ends pinned, the first two terms on the right-hand side of (3.11) were equal to 0. Now, however, since $v \in \mathrm{V}$ tells us nothing about the values of $v(a)$ or $v(b)$, we must carry these two terms along with us. Using (3.11) in (3.10), we find that

$$G(b)v(b) - G(a)v(a) + \int_a^b \{F(x) - G'(x)\}\, v(x)\, dx = 0 \quad \text{for all } v \in \mathrm{V}. \tag{3.12}$$

We now make a key observation: since $\mathrm{V}_{a,b} \subset \mathrm{V}$, if (3.12) holds for all $v \in \mathrm{V}$, then it must hold for all $v \in \mathrm{V}_{a,b}$. Therefore we deduce from (3.12) that

$$G(b)v(b) - G(a)v(a) + \int_a^b \{F(x) - G'(x)\}\, v(x)\, dx = 0 \quad \text{for all } v \in \mathrm{V}_{a,b}. \tag{3.13}$$

Now for each $v \in \mathrm{V}_{a,b}$, we have $v(a) = v(b) = 0$. Substituting this into (3.13), we find

$$\int_a^b \left\{ F(x) - \frac{d}{dx}\,[G(x)] \right\} v(x)\, dx = 0 \quad \text{for all } v \in \mathrm{V}_{a,b}.$$

This is exactly the same condition we found in Section 3.1, so Lagrange's Lemma (Lemma 3.1) allows us to conclude that

$$F(x) - G'(x) = 0 \quad \text{for all } x \in [a,b]. \tag{3.14}$$

With (3.14) in hand, we return to (3.12). We now have

$$G(b)v(b) - G(a)v(a) = 0 \quad \text{for all } v \in \mathrm{V}, \tag{3.15}$$

since (3.14) tells us that the integral term must be zero. We will argue next that (3.15) implies $G(a) = G(b) = 0$. We first choose $v_a \in \mathrm{V}$ such that $v_a(a) = 1$ and $v_a(b) = 0$. For example, we may choose the linear function

$$v_a(x) = \frac{b-x}{b-a}.$$

Since $v_a \in \mathrm{V}$, we have from (3.15) that

$$G(b)v_a(b) - G(a)v_a(a) = -G(a) = 0 \Rightarrow G(a) = 0. \tag{3.16}$$

Similarly, we choose $v_b \in \mathrm{V}$ such that $v_b(b) = 1$ and $v_b(a) = 0$, and with this choice we conclude that

$$G(b)v_b(b) - G(a)v_b(a) = 0 \Rightarrow G(b) = 0. \tag{3.17}$$

Thus $G(a) = G(b) = 0$.

So we have found that (3.10) implies the following:

(1) $F(x) - G'(x) = 0 \quad \text{for all } x \in [a, b]$;

(2) $G(a) = 0$;

(3) $G(b) = 0$.

Recalling the definitions (3.9), we conclude that if y_* minimizes the functional J over Y, then we must have

$$f_{,2}(x, y_*(x), y_*'(x)) - \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] = 0 \quad \text{for all } x \in [a, b]; \tag*{(E-L)$_1$}$$

$$f_{,3}(a, y_*(a), y_*'(a)) = 0; \tag*{(NBC)$_a$}$$

$$f_{,3}(b, y_*(b), y_*'(b)) = 0. \tag*{(NBC)$_b$}$$

The equation (E-L)_1 is the Euler-Lagrange equation we found in Section 3.1. The conditions (NBC)_a and (NBC)_b are called the natural boundary conditions at a and b respectively.

We have shown that if y_* minimizes J over the class Y, then y_* must satisfy (E-L)_1, (NBC)_a and (NBC)_b. If instead y_* were a minimizer for J over the class Y_a, then y_* is pinned at a and there is no natural boundary condition to be satisfied at a while there is a natural boundary condition (NBC)_b at b. An analogous conclusion may be made if y_* minimizes J over Y_b. To summarize this section, we list the following:

(no free ends) If y_* minimizes J over $\mathrm{Y}_{a,b}$,
then y_* must satisfy $(\text{E-L})_1$, $y_*(a) = A$ and $y_*(b) = B$.

(free end at a) If y_* minimizes J over Y_b,
then y_* must satisfy $(\text{E-L})_1$, $(\text{NBC})_a$ and $y_*(b) = B$.

(free end at b) If y_* minimizes J over Y_a,
then y_* must satisfy $(\text{E-L})_1$, $y_*(a) = A$ and $(\text{NBC})_b$.

(both ends free) If y_* minimizes J over Y,
then y_* must satisfy $(\text{E-L})_1$, $(\text{NBC})_a$ and $(\text{NBC})_b$.

3.3.1 Example 3.3.1

We illustrate our results from the previous section with an example (compare to Examples 1.2 and 3.1.1). Set

$$\mathscr{Y} := \left\{ y \in C^2[0,1] \,:\, y(1) = 1 \right\}.$$

The functions in $\mathscr{Y}$ have a free endpoint at 0. We consider the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) := \int_0^1 \left[y(x)^2 + y'(x)^2 \right] dx \quad \text{for all } y \in \mathscr{Y}.$$

If we define $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,y,z) := y^2 + z^2 \quad \text{for all } (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R},$$

then J has the form

$$J(y) = \int_0^1 f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

To write out the Euler-Lagrange equation and natural boundary condition for this problem, we compute the partial derivatives

$$f_{,2}(x,y,z) = 2y \text{ and } f_{,3}(x,y,z) = 2z \quad \text{for all } (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}.$$

The first Euler-Lagrange equation may thus be written as

$$2y(x) - 2y''(x) = 0 \quad \text{for all } x \in [0,1], \tag{E-L$_1$}$$

and the natural boundary condition at 0 is

$$f_{,3}(0, y(0), y'(0)) = 2y'(0) = 0. \tag{NBC$_0$}$$

Hence, a minimizer for J over $\mathscr{Y}$ must satisfy

$$\begin{cases} y''(x) - y(x) = 0; \\ y'(0) = 0 \text{ and } y(1) = 1. \end{cases} \tag{3.18}$$

We will now solve (3.18). In Section 3.1.1, we found that the general solution to the differential equation above has the form

$$y(x) = C_1 e^x + C_2 e^{-x}.$$

Therefore, it only remains to find C_1 and C_2 such that $y'(0) = 0$ and $y(1) = 1$. We have

$$y'(0) = 0 \Rightarrow C_1 - C_2 = 0 \Rightarrow C_1 = C_2;$$

and

$$y(1) = 1 \Rightarrow C_1 e + C_2 e^{-1} = 1 \Rightarrow C_1 = \frac{1}{e + e^{-1}} \text{ and } C_2 = \frac{1}{e + e^{-1}}.$$

Hence the only solution to (3.18) is

$$y(x) = \frac{e^x + e^{-x}}{e + e^{-1}}.$$

3.4 The Second Euler-Lagrange Equation

This section contains a very useful alternative to the first Euler-Lagrange equation.

Theorem 3.1 *Let $a, b \in \mathbb{R}$ with $a < b$ be given, and suppose that the function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has continuous second partial derivatives. Let $y \in C^2[a, b]$ be given and assume that y satisfies*

$$f_{,2}(x, y(x), y'(x)) = \frac{d}{dx}\left[f_{,3}(x, y(x), y'(x))\right] \quad \text{for all } x \in [a, b]. \tag{E-L$_1$}$$

Then there exists $c \in \mathbb{R}$ such that

$$f(x, y(x), y'(x)) - y'(x) f_{,3}(x, y(x), y'(x)) = c + \int_a^x f_{,1}(t, y(t), y'(t))\, dt \quad \text{for all } x \in [a, b]. \tag{E-L$_2$}$$

Equation (E-L)$_2$ is called the second Euler-Lagrange equation. In contrast with the first Euler-Lagrange equation, the second does not involve second-order derivatives of y. However, the second equation is not necessary simpler than the first, due to the presence of the integral term.

The proof of Theorem 3.1 is not too difficult and is left as an exercise. The idea is to show that (E-L)$_2$ holds if and only if

$$y'(x) \left\{ f_{,2}(x, y(x), y'(x)) - \frac{d}{dx}\left[f_3(x, y(x), y'(x))\right] \right\} = 0 \quad \text{for all } x \in [a, b].$$

This indicates that a function y might satisfy $(\text{E-L})_2$ without satisfying $(\text{E-L})_1$. In particular, constant functions always satisfy $(\text{E-L})_2$.

In some problems, the first Euler-Lagrange equation is more convenient than the second, while in other problems the second Euler-Lagrange equation is preferable to the first. In certain situations, it is helpful to use both equations. Therefore it is prudent to keep both $(\text{E-L})_1$ and $(\text{E-L})_2$ in mind when tackling a problem.

3.4.1 An Important Special Case of $(\text{E-L})_2$

If the integrand f has no explicit x-dependence, i.e, if $f_{,1} \equiv 0$, then the second Euler-Lagrange equation reduces to a first-order differential equation (involving an undetermined constant) because the integral term vanishes. In particular, if

$$f(x, y, z) = F(y, z) \quad \text{for all } x \in [a, b], y \in \mathbb{R}, z \in \mathbb{R},$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function with continuous second-order partial derivatives, then the second Euler-Lagrange equations takes the form

$$F(y(x), y'(x)) - y'(x) F_{,2}(y(x), y'(x)) = c, \tag{E-L$_2$}$$

for some $c \in \mathbb{R}$.

3.4.2 The Brachistochrone Problem

Let $b, B > 0$ be given. For the brachistochrone problem (Section 1.4.2), we found that the functional to be minimized was

$$J(y) = \int_0^b \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, dx.$$

The integrand for J is the function $f : [0, b] \times (0, \infty) \times \mathbb{R} \to \mathbb{R}$ given by

$$f(x, y, z) := \frac{(1 + z^2)^{\frac{1}{2}}}{y^{\frac{1}{2}}} \quad \text{for all } (x, y, z) \in [0, b] \times (0, \infty) \times \mathbb{R}.$$

We are interested in minimizing J over a class of admissible functions satisfying $y(0) = 0$ and $y(b) = B$. The theory that we have developed so far does not apply to this problem because the integrand is not well behaved when $y = 0$ and we are forcing y to vanish at the left endpoint of the interval. Nevertheless, it is instructive to ignore this technical difficulty for the moment and formally determine the Euler-Lagrange equations. To write the first Euler-Lagrange equation, we compute

$$f_{,2}(x, y, z) = -\frac{(1 + z^2)^{\frac{1}{2}}}{2y^{\frac{3}{2}}} \quad \text{for all } (x, y, z) \in [0, b] \times (0, \infty) \times \mathbb{R}$$

and

$$f_{,3}(x,y,z) = \frac{z}{y^{\frac{1}{2}}(1+z^2)^{\frac{1}{2}}} \quad \text{for all } (x,y,z) \in [0,b] \times (0,\infty) \times \mathbb{R},$$

which yields

$$-\frac{(1+y'(x)^2)^{\frac{1}{2}}}{2y(x)^{\frac{3}{2}}} = \frac{d}{dx}\left[\frac{y'(x)}{y(x)^{\frac{1}{2}}(1+y'(x)^2)^{\frac{1}{2}}}\right] \quad \text{for all } x \in [0,b]. \qquad \text{(E-L)}_1$$

It looks as though it would be very difficult to find solutions to this equation.

Since f has no explicit x-dependence, i.e., since $f_{,1}(x,y,z) = 0$ for all $(x,y,z) \in [0,b] \times (0,\infty) \times \mathbb{R}$, the second Euler-Lagrange equation becomes

$$\frac{(1+y'(x)^2)^{\frac{1}{2}}}{y(x)^{\frac{1}{2}}} - \frac{y'(x)^2}{y(x)^{\frac{1}{2}}(1+y'(x)^2)^{\frac{1}{2}}} = c \quad \text{for all } x \in [0,b], \qquad \text{(E-L)}_2$$

for some $c \in \mathbb{R}$. After some simplification, we find that y satisfies (E-L)_2 if and only if it satisfies

$$y(x)(1+y'(x)^2) = \frac{1}{c^2} \quad \text{for all } x \in [0,b], \tag{3.19}$$

for some $c \in \mathbb{R}\backslash\{0\}$. We observe that if y satisfies (3.19) for some $c \in \mathbb{R}\backslash\{0\}$, then $y'(x)^2$ must be large whenever $y(x) > 0$ is small. So if $y \in \mathscr{Y}$ satisfies (E-L)_2 and $y(0) = 0$, we should expect the graph for y to have a vertical tangent at 0.

3.5 Transit Time for a Boat

Suppose that we wish to steer a boat from a given location on the bank of a river with current to a given location on the opposite bank in such a way that the transit time is minimized. For simplicity, we assume the following:

(1) the river banks are straight and parallel;

(2) the current in the river always runs parallel to the river banks;

(3) the strength of the current depends only upon the distance from the initial river bank;

(4) the speed of our boat relative to the water is constant;

(5) the speed of the current is always less than the speed of the boat.

To set up the problem, we choose our coordinate system so that one of the river banks coincides with the y-axis, the other bank is represented by the line $x = b$ with $b > 0$, and our boat's initial position is the origin. We let $Q = (b, B)$ be the destination. We assume that the river flows in the positive y-direction. The

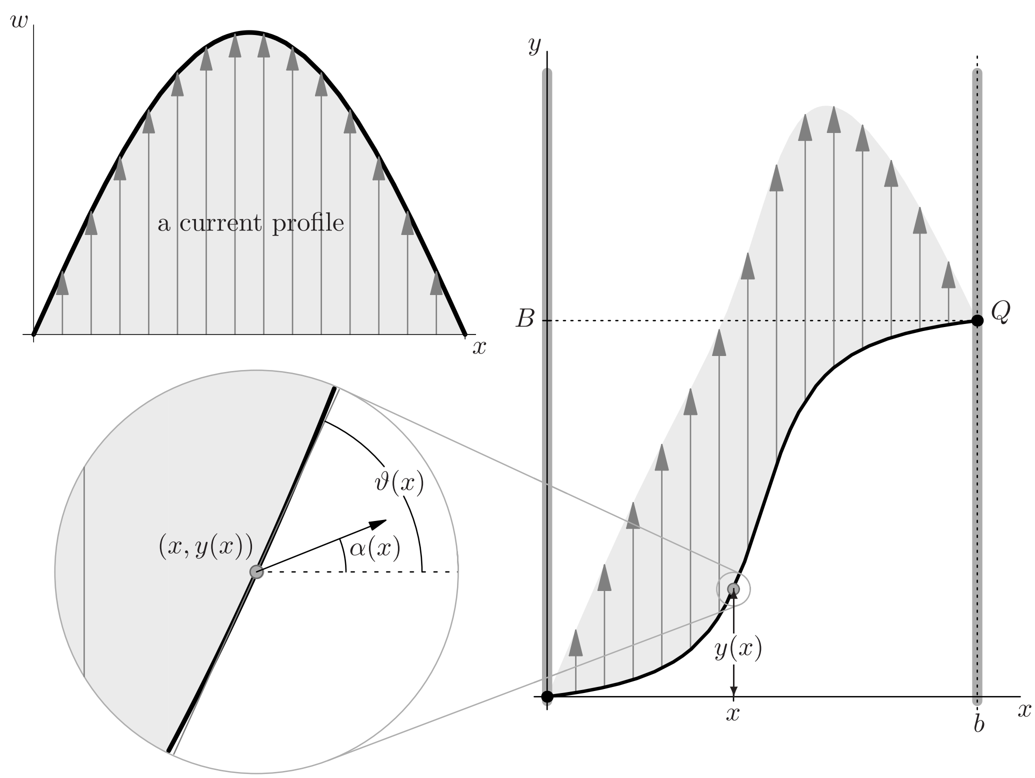

Figure 3.4: Setup for finding the transit time for a boat

speed of the boat relative to the water is denoted by ω, time is denoted by t, and T is the total time to cross the river. We assume that the path of the boat can be represented as the graph of some function $y \in C^1[0, b]$.

Let $\alpha : [0, b] \to (-\frac{\pi}{2}, \frac{\pi}{2})$ be the angle (measured off the x-axis) in which the boat is steered, let $\theta : [0, b] \to (-\frac{\pi}{2}, \frac{\pi}{2})$ be the angle at which the boat is traveling, and let $w : [0, b] \to [0, \omega)$ be the velocity of the current (we are incorporating assumption (5) here). We assume that $w \in C[0, b]$; we will impose additional smoothness assumptions on w later. If we are x-units from the initial river bank, the velocity of the current is $w(x)$, and we are steering the boat in the direction $\alpha(x)$ at the speed ω relative to the water. Though we are steering in the direction $\alpha(x)$, the boat is actually moving in the direction $\theta(x)$ because of the river's current.

If the velocity of the boat (relative to land) in the x-direction is $\frac{dx}{dt}$, then the total time required to traverse the river is

$$T = \int_0^b \frac{1}{\frac{dx}{dt}} \, dx. \tag{3.20}$$

Since the current in the river is assumed to be orthogonal to the x-axis, the current contributes nothing to the boat's velocity in the x-direction, and this velocity is given by

$$\frac{dx}{dt} = \omega \cos \alpha(x).$$

Thus (3.20) may be rewritten as

$$T = \frac{1}{\omega} \int_0^b \sec \alpha(x)\, dx. \tag{3.21}$$

We want to express $\sec \alpha$ in terms of the path of the boat y and the velocity of the current w. When the boat is at a point $(x, y(x))$, the direction in which the boat is traveling is $\theta(x)$. So the slope of the tangent line to the graph of y at x is

$$\begin{aligned} y'(x) = \tan \theta(x) &= \frac{\text{boat's speed in the } y\text{-direction}}{\text{boat's speed in the } x\text{-direction}} \\ &= \frac{\omega \sin \alpha(x) + w(x)}{\omega \cos \alpha(x)} = \frac{\sin \alpha(x) + \frac{w(x)}{\omega}}{\cos \alpha(x)} \\ &= \tan \alpha(x) + \frac{w(x)}{\omega} \sec \alpha(x). \end{aligned} \tag{3.22}$$

For convenience, we define the normalized current $e : [0, b] \to [0, 1)$ by

$$e(x) := \frac{w(x)}{\omega}.$$

Solving for $\tan \alpha$ in (3.22), we find

$$\tan \alpha(x) = y'(x) - e(x) \sec \alpha(x). \tag{3.23}$$

In order to eliminate $\tan \alpha$ from (3.23), we square both sides of (3.23) and use the trigonometric identity $\tan^2 \alpha = \sec^2 \alpha - 1$ to obtain

$$[y'(x) - e(x) \sec \alpha(x)]^2 = \tan^2 \alpha(x) = \sec^2 \alpha(x) - 1.$$

Collecting terms to one side, we find

$$[1 - e(x)^2] \sec^2 \alpha(x) + 2y'(x)e(x) \sec \alpha(x) - y'(x)^2 - 1 = 0.$$

Utilizing the quadratic formula yields

$$\sec \alpha(x) = \frac{-y'(x)e(x) \pm \sqrt{y'(x)^2 + 1 - e(x)^2}}{1 - e(x)^2}. \tag{3.24}$$

At this point, we need to understand what to do with the $\pm$ sign in (3.24). Since $\alpha(x) \in (\frac{-\pi}{2}, \frac{\pi}{2})$,we know that $\sec \alpha(x) > 0$ for every $x \in [0, b]$. Therefore, we want the positive root in (3.24). Upon examining (3.24), we see that this root is given by

$$\sec\alpha(x) = \frac{\sqrt{y'(x)^2 + 1 - e(x)^2} - y'(x)e(x)}{1 - e(x)^2}.$$

Having found an expression for $\sec\alpha$ in terms of y and w, we substitute it into (3.21) and find that

$$T = \frac{1}{\omega}\int_0^b \frac{\sqrt{y'(x)^2 + 1 - e(x)^2} - y'(x)e(x)}{1 - e(x)^2}\,dx. \tag{3.25}$$

This is the total transit time for the boat when it is steered along the graph of y across the river.

The optimal path across the river is that which minimizes the expression for T given in (3.25). Our problem, therefore, is to minimize the functional $J : \mathscr{Y} \to \mathbb{R}$ given by

$$J(y) := \int_0^b \frac{\sqrt{1 - e(x)^2 + y'(x)^2} - e(x)y'(x)}{1 - e(x)^2}\,dx \quad \text{for all } y \in \mathscr{Y},$$

where

$$\mathscr{Y} := \left\{ y \in C^2[0,b] \,:\, y(0) = 0 \text{ and } y(b) = B \right\}.$$

Notice that if the river has no current, then $w(x) = 0$ which implies $e(x) = 0$ for every $x \in [0,b]$. In such a situation, equation (3.25) collapses to

$$T = \frac{1}{\omega}\int_0^b \sqrt{1 + y'(x)^2}\,dx.$$

This formula says that the transit time is simply the length of the path (i.e., the total distance traveled) divided by the speed. The minimizer in this case is just the straight line joining the origin to the point (b,B), and the transit time for the boat is the distance between the origin and (b,B) divided by the speed of the boat. This is exactly what one should expect to occur when there is no current in the river.

Returning to the more general problem, let us try to find solutions to one of the Euler-Lagrange equations for J. The integrand $f : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for J is given by

$$f(x,y,z) := \frac{\sqrt{1 - e(x)^2 + z^2}}{1 - e(x)^2} - \frac{e(x)z}{1 - e(x)^2} \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}.$$

In order to ensure that f has continuous second-order partial derivatives, we assume that $w \in C^2[0,b]$ so that $e \in C^2[0,b]$. Computing the partial derivatives of f we find that

$$f_{,2}(x,y,z) = 0 \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}$$

and

$$f_{,3}(x,y,z) = \frac{1}{1-e(x)^2}\left(\frac{z}{\sqrt{1-e(x)^2+z^2}} - e(x)\right) \quad \text{for all } (x,y,z) \in [0,b]\times\mathbb{R}\times\mathbb{R},$$

so the first Euler-Lagrange equation becomes

$$\frac{d}{dx}\left[\frac{1}{1-e(x)^2}\left(\frac{y'(x)}{\sqrt{1-e(x)^2+y'(x)^2}} - e(x)\right)\right] = 0. \qquad \text{(E-L)}_1$$

Thus y is a solution to (E-L)_1 if and only if

$$\frac{1}{1-e(x)^2}\left(\frac{y'(x)}{\sqrt{1-e(x)^2+y'(x)^2}} - e(x)\right) = \beta$$

for some $\beta \in \mathbb{R}$. Multiplying through by $1-e(x)^2$, we obtain

$$\beta(1-e(x)^2) = \frac{y'(x)}{\sqrt{1-e(x)^2+y'(x)^2}} - e(x). \tag{3.26}$$

Let us define $\gamma : [0,b]\times\mathbb{R} \to \mathbb{R}$ by

$$\gamma(x,\beta) := e(x) + \beta(1-e(x)^2) \quad \text{for all } x \in [0,b], \beta \in \mathbb{R},$$

so that (3.26) becomes

$$\frac{y'(x)}{\sqrt{1-e(x)^2+y'(x)^2}} = \gamma(x,\beta) \quad \text{for all } x \in [0,b], \beta \in \mathbb{R}. \tag{3.27}$$

It follows from (3.27) that $y'(x)$ and $\gamma(x,\beta)$ always have the same sign. Observe also that (3.27) implies

$$|\gamma(x,\beta)| < 1 \quad \text{for all } x \in [0,b], \beta \in \mathbb{R}.$$

Squaring (3.27) and rearranging terms gives

$$y'(x)^2 = \frac{\gamma(x,\beta)^2(1-e(x)^2)}{1-\gamma(x,\beta)^2}. \tag{3.28}$$

Using the fact that $y'(x)$ and $\gamma(x,\beta)$ have the same sign, we can take square roots in (3.28) to obtain

$$y'(x) = G(x,\beta),$$

where

$$G(x,\beta) := \frac{\gamma(x,\beta)\sqrt{1-e(x)^2}}{\sqrt{1-\gamma(x,\beta)^2}}. \tag{3.29}$$

The fundamental theorem of calculus gives

$$y(x) = C + \int_0^x G(t,\beta)\,dt \quad \text{for all } x \in [0,b].$$

Now $y(0) = 0$ implies $C = 0$ and consequently

$$y(b) = B \Rightarrow \int_0^b G(t, \beta)\, dt = B.$$

The solution to $(\text{E-L})_1$ satisfying the boundary conditions is therefore given by

$$y(x) = \int_0^x G(t, \beta)\, dt \quad \text{for all } x \in [0, b],$$

where G is defined by (3.29) and $\beta \in \mathbb{R}$ is chosen to satisfy

$$\int_0^b G(t, \beta)\, dt = B.$$

3.6 C¹-Theory

We will now investigate conditions that must be satisfied by minimizers (or maximizers) under weaker smoothness assumptions on the integrand and on the admissible functions. In particular we will now assume that f and y have continuous first-order derivatives, but we do not make any assumptions regarding second-order derivatives. By examining the argument presented in Sections 3.1 and 3.3, we see that the main obstacle is that the integration by parts that was performed in (3.4) is no longer valid under the relaxed assumptions of the C^1-theory. The key idea for overcoming this obstacle is to integrate the other term in (3.1) by parts. An appropriate analogue of Lagrange's Lemma will lead us again to the first Euler-Lagrange equation. As in Section 3.3, we will deal with several types of problems simultaneously.

Let $a, b, A, B \in \mathbb{R}$ with $a < b$ be given. Put

$$\mathfrak{X} := C^1[a, b].$$

Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function with continuous first-order partial derivatives. Consider the functional $J : \mathfrak{X} \to \mathbb{R}$ defined by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathfrak{X}.$$

We define several types of classes of admissible functions and the corresponding admissible variations. Put

$$\begin{aligned}
\mathrm{Y}_{a,b} &:= \left\{ y \in C^1[a,b] \,:\, y(a) = A \text{ and } y(b) = B \right\};\\
\mathrm{Y}_{a} &:= \left\{ y \in C^1[a,b] \,:\, y(a) = A \right\};\\
\mathrm{Y}_{b} &:= \left\{ y \in C^1[a,b] \,:\, y(b) = B \right\};\\
\mathrm{Y} &:= C^1[a,b];\\
\mathrm{V}_{a,b} &:= \left\{ v \in C^1[a,b] \,:\, v(a) = v(b) = 0 \right\};\\
\mathrm{V}_{a} &:= \left\{ v \in C^1[a,b] \,:\, v(a) = 0 \right\};\\
\mathrm{V}_{b} &:= \left\{ v \in C^1[a,b] \,:\, v(b) = 0 \right\};\\
\mathrm{V} &:= C^1[a,b].
\end{aligned}$$

These are essentially the same classes given in Section 3.3 – the only difference being that they are no longer subsets of $\mathscr{C}^2[a,b]$ but subsets of $\mathscr{C}^1[a,b]$ instead.

For now, we focus on the problem of minimizing J over $\mathrm{Y}_{a,b}$ – this is the problem with both endpoints fixed. Later we will discuss problems with free endpoints. Let $y_* \in \mathrm{Y}_{a,b}$ be given and suppose that J attains a minimum at y_*

For each $v \in \mathrm{V}_{a,b}$, Theorem 2.4 tells us that the Gâteaux variation of J at y_* in the direction v exists and gives us a formula for $\delta J(y; v)$. Moreover, by Theorem 2.2, $\delta J(y_*; v)$ must be zero for each $v \in \mathrm{V}$ Therefore, we have

$$\delta J(y_*; v) = \int_a^b \left\{ f_{,2}(x, y_*(x), y_*'(x))v(x) + f_{,3}(x, y_*(x), y_*'(x))v'(x) \right\} dx = 0 \quad \text{for all } v \in \mathrm{V}_{a,b}. \tag{3.30}$$

Define $F, G : [a,b] \to \mathbb{R}$ by

$$F(x) := f_{,2}(x, y_*(x), y_*'(x)) \text{ and } G(x) := f_{,3}(x, y_*(x), y_*'(x)) \quad \text{for all } x \in [a,b]. \tag{3.31}$$

and notice that $F, G \in \mathscr{C}[a,b]$ by virtue of our smoothness assumptions on f and y. We may now rewrite (3.30) as

$$\int_a^b \left\{ F(x)v(x) + G(x)v'(x) \right\} dx = 0 \quad \text{for all } v \in \mathrm{V}_{a,b}. \tag{3.32}$$

In Section 3.1, we knew that G was continuously differentiable and we were able to integrate the second term in (3.32) by parts. Now, instead, we know only that F and G are continuous. To proceed from (3.32), we will integrate the first term by parts.

This idea is due to du Bois-Reymond. Define $H \in \mathscr{C}^1[a,b]$ by

$$H(x) := \int_a^x F(t)\,dt \quad \text{for all } x \in [a,b]. \tag{3.33}$$

Observe that $H' = F$, so (3.32) becomes

$$\int_a^b \Big\{ H'(x)v(x) + G(x)v'(x) \Big\}\, dx = 0 \quad \text{for all } v \in \mathrm{V}_{a,b}.$$

Integrating the term involving H by parts yields

$$[H(x)v(x)]\Big|_a^b - \int_a^b H(x)v'(x)\,dx + \int_a^b G(x)v'(x)\,dx = 0 \quad \text{for all } v \in \mathrm{V}_{a,b}. \tag{3.34}$$

Since $v \in \mathrm{V}_{a,b}$ implies $v(a) = v(b) = 0$, the first term in (3.34) is zero. Thus

$$\int_a^b \Big\{ G(x) - H(x) \Big\} v'(x)\,dx = 0 \quad \text{for all } v \in \mathrm{V}_{a,b}. \tag{3.35}$$

At this point, we need to use a lemma of du Bois-Reymond.

Lemma 3.3 (du Bois-Reymond) *Let $a, b \in \mathbb{R}$ with $a < b$ and $w \in C[a, b]$ be given. Put*

$$\mathscr{V} := \left\{ v \in \mathscr{C}^1[a, b] \mid v(a) = v(b) = 0 \right\}.$$

Assume that $\int_a^b w(x)v'(x)\,dx = 0$ for all $v \in \mathscr{V}$. Then there exists a $c \in \mathbb{R}$ such that $w(x) = c$ for all $x \in [a, b]$.

Proof. As in the proof of Lagrange's Lemma, the key idea involves a judicious choice of $v \in \mathscr{V}$. Notice that the only possible c that can work is the average value of w over the interval $[a, b]$.

Put

$$c_* := \frac{1}{b-a} \int_a^b w(t)\,dt.$$

We will show that $w(x) = c_*$ at each $x \in [a, b]$. It suffices to establish

$$\int_a^b [w(x) - c_*]^2\,dx = 0. \tag{3.36}$$

Since $\int_a^b w(x)v'(x)\,dx = 0$ for every $v \in \mathscr{V}$, we see that

$$\begin{aligned} \int_a^b [w(x) - c_*]v'(x)\,dx &= \int_a^b w(x)v'(x)\,dx - c_*[v(x)]\Big|_a^b \\ &= \int_a^b w(x)v'(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}. \end{aligned}$$

To obtain (3.36), we would like to choose $v \in \mathscr{V}$ such that $v'(x) = w(x) - c_*$ at each $x \in [a, b]$. Let us define $v_* \in C^1[a, b]$ by

$$v_*(x) := \int_a^x [w(t) - c_*]\, dt.$$

Notice that $v_*'(x) = w(x) - c_*$ at each $x \in [a, b]$, so our proof will be complete if we show that $v_* \in \mathscr{V}$. In other words, it only remains to show $v_*(a) = v_*(b) = 0$. By its definition, we have $v_*(a) = 0$ and

$$\begin{aligned} v_*(b) &= \int_a^b [w(t) - c_*]\, dx = \int_a^b w(t)\, dt - \int_a^b c_*\, dt = \int_a^b w(t) - (b - a)c_* \\ &= \int_a^b w(t)\, dt - \int_a^b w(t)\, dt = 0. \end{aligned}$$

Thus we find $v_* \in \mathscr{V}$ and therefore (3.36) must hold. It follows that $w(x) = c_*$ at each $x \in [a, b]$ and this proves the lemma. □

With du Bois-Reymond's Lemma in hand, we now return to (3.35). It follows that there exists some $c \in \mathbb{R}$ such that

$$G(x) - H(x) = c \Rightarrow G(x) = H(x) + c \quad \text{for all } x \in [a, b].$$

By definition (3.33), we see that $H + c$ is continuously differentiable. Hence G is continuously differentiable, and we have

$$G'(x) = H'(x) = F(x) \quad \text{for all } x \in [a, b]. \tag{3.37}$$

Now by definition (3.31), we have that the function $x \mapsto f_{,3}(x, y(x), y'(x))$ is continuously differentiable on $[a, b]$ and

$$f_{,2}(x, y_*(x), y_*'(x)) = \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] \quad \text{for all } x \in [a, b] \tag*{(E-L)$_1$}$$

For future reference, we restate our conclusion in (3.37) as a lemma.

Lemma 3.4 *Let $a, b \in \mathbb{R}$ with $a < b$ and $F, G \in C[a, b]$ be given. Put*

$$\mathscr{V} := \left\{v \in C^1[a, b] \,:\, v(a) = v(b) = 0\right\}.$$

Assume that

$$\int_a^b \left\{F(x)v(x) + G(x)v'(x)\right\} dx = 0 \quad \text{for all } v \in \mathscr{V}.$$

Then, we have $G \in C^1[a, b]$ and $G'(x) = F(x)$ at each $x \in [a, b]$.

3.6.1 Problems with Free Endpoints

Continuing our development of the $\mathscr{C}^1$-theory for problems in the calculus of variations, we state a corresponding version of Lemma 3.4 for the case when $\mathscr{V}$ is replaced by V_a, V_b or V.

Lemma 3.5 *Let $a, b \in \mathbb{R}$ with $a < b$ and $F, G \in C[a,b]$ be given.*

(i) If $\int_a^b \left\{F(x)v(x) + G(x)v'(x)\right\} = 0$ for every $v \in \mathrm{V}_a$, then $G(b) = 0$, $G \in C^1[a,b]$ and $G'(x) = F(x)$ at each $x \in [a,b]$.

(ii) If $\int_a^b \left\{F(x)v(x) + G(x)v'(x)\right\} = 0$ for every $v \in \mathrm{V}_b$, then $G(a) = 0$, $G \in \mathscr{C}^1[a,b]$ and $G'(x) = F(x)$ at each $x \in [a,b]$.

(iii) If $\int_a^b \left\{F(x)v(x) + G(x)v'(x)\right\} = 0$ for every $v \in \mathrm{V}$, then $G(a) = G(b) = 0$, $G \in C^1[a,b]$ and $G'(x) = F(x)$ at each $x \in [a,b]$.

Proof. We only prove (i), the proof for the others being similar (see Section 3.3).

Therefore, we are assuming that

$$\int_a^b \left\{F(x)v(x) + G(x)v'(x)\right\} = 0 \quad \text{for all } v \in \mathrm{V}_a. \tag{3.38}$$

Since $\mathrm{V}_{a,b} \subset \mathrm{V}_a$, we may use Lemma 3.4 to conclude that $G \in C^1[a,b]$ and $G'(x) = F(x)$ at each $x \in [a,b]$.

Once we know that $G \in \mathscr{C}^1[a,b]$, our argument can follow that given in Section 3.3. We integrate the second term in (3.38) by parts and write

$$\begin{aligned}\int_a^b \left\{F(x)v(x) + G(x)v'(x)\right\} dx &= [G(x)v(x)]\Big|_a^b + \int_a^b \left\{F(x) - \frac{d}{dx}[G(x)]\right\} v(x)\, dx \\ &= G(b)v(b) - G(a)v(a) \\ &= -G(a)v(a) = 0 \quad \text{for all } v \in \mathrm{V}_a.\end{aligned}$$

By choosing a $v \in \mathrm{V}_a$ such that $v(a) = 1$, we deduce that $G(a) = 0$.

So $G(a) = 0$ and $G \in C^1[a,b]$ with $G'(x) = F(x)$ at each $x \in [a,b]$. The lemma is proved. □

Recall that $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to have continuous first partial derivatives, and our functional was given by

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathfrak{X},$$

where $\mathfrak{X} = C^1[a,b]$. The first Euler-Lagrange equation for J is

$$f_{,2}(x, y_*(x), y_*'(x)) = \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] \quad \text{for all } x \in [a,b], \tag{E-L$_1$}$$

and the natural boundary conditions at a and b, respectively, are

$$f_{,3}(a, y_*(a), y_*'(a)) = 0 \tag{NBC$_a$}$$

and

$$f_{,3}(b, y_*(b), y_*'(b)) = 0. \tag{NBC$_b$}$$

Using Lemma 3.5, we deduce the following conclusions:

(both ends pinned) If y_* minimizes J over $\mathrm{Y}_{a,b}$,
then y_* must satisfy $(\text{E-L})_1$, $y_*(a) = A$ and $y_*(b) = B$.

(free end at a) If y_* minimizes J over Y_b,
then y_* must satisfy $(\text{E-L})_1$, $(\text{NBC})_a$ and $y_*(b) = B$.

(free end at b) If y_* minimizes J over Y_a,
then y_* must satisfy $(\text{E-L})_1$, $y_*(a) = A$ and $(\text{NBC})_b$.

(both ends free) If y_* minimizes J over Y,
then y_* must satisfy $(\text{E-L})_1$, $(\text{NBC})_a$ and $(\text{NBC})_b$.

We remark that if y_* minimizes (or maximizes) J over one of the sets $\mathrm{Y}_{a,b}$, Y_a, Y_b or Y, then there exists a $c \in \mathbb{R}$ such that

$$f(x, y_*(x), y_*'(x)) - y_*'(x) f_{,3}(x, y_*(x), y_*'(x)) = c + \int_a^x f_{,1}(t, y_*(t), y_*'(t))\, dt \quad \text{for all } x \in [a,b], \tag{E-L$_2$}$$

but the proof is completely elementary.

3.6.2 Example 3.6.2 (cf. Examples 1.2, 3.1.1 and 3.3.1)

Let us consider an example. Put

$$\mathscr{Y} := \left\{ y \in C^1[0,1] \,:\, y(0) = 0 \text{ and } y(1) = 1 \right\}.$$

We want to minimize the functional $J : \mathscr{Y} \to \mathbb{R}$ given by

$$J(y) := \int_0^1 \left[y(x)^2 + y'(x)^2 \right] dx \quad \text{for all } y \in \mathscr{Y}$$

over the class $\mathscr{Y}$. This functional was also used in Sections 1.2, 3.1.1 and 3.3.1.

By our results in the previous section, if J attains a minimum at $y \in \mathscr{Y}$, then $2y'$ must be continuously differentiable and y must satisfy

$$2y(x) = \frac{d}{dx}\left[2y'(x)\right] \quad \text{for all } x \in [0,1]. \tag{E-L$_1$}$$

Therefore, a minimizer for J over $\mathscr{Y}$ must satisfy

$$\begin{cases} y''(x) = y(x); \\ y(0) = 0 \text{ and } y(1) = 1. \end{cases} \tag{3.39}$$

Notice that from Lemma 3.4 one concludes that $y \in C^2[0,1]$. In fact, if y is a minimizer for J, then y must be in $C^\infty[0,1]$: since y satisfies (3.39) and $y \in C^2[0,1]$, we find $y'' \in C^2[0,1]$, and thus $y \in C^4[0,1]$. This in turn implies $y \in C^6[0,1]$. By induction, we find $y \in C^\infty[0,1]$. So even though our admissible class $\mathscr{Y}$ includes functions that may have only one continuous derivative, a minimizer for J must actually have continuous derivatives of all orders.

3.6.3 Minimizers Might Not be in $\mathbf{C^2}$

We remark that it may not always be possible to conclude that a C^1-minimizer for a functional has a continuous second derivative. It is true that a minimizer for a functional J with integrand f must satisfy

$$f_{,2}(x, y(x), y'(x)) = \frac{d}{dx}\left[f_{,3}(x, y(x), y'(x))\right] \quad \text{for all } x \in [a,b], \tag{E-L$_1$}$$

and by Lemma 3.4, we know that the derivative on the right hand side of (E-L)$_1$ exists and is continuous. It is the composition $x \mapsto f_{,3}(x, y(x), y'(x))$, however, that is continuously differentiable. Without more information regarding $f_{,3}$, Lemma 3.4 tells us nothing about the differentiability of y'.

Indeed, there are many examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that the composition $x \mapsto f(g(x))$ is continuously differentiable, but g itself is not differentiable.

The following example has a C^1-minimizer that is not of class C^2. Let

$$\mathscr{Y} = \left\{ y \in C^1[0,1] \,:\, y(0) = 0,\, y(0) = 1 \right\}$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^1 (4y'(x)^2 - 9x)^2 \, dx \quad \text{for all } y \in \mathscr{Y},$$

and notice that $J(y) \geq 0$ for all $y \in \mathscr{Y}$. Is is straightforward to check that $J(y_*) = 0$ for the function $y_* \in \mathscr{Y}$ given by $y_*(x) = x^{3/2}$., and consequently y_* minimizes J on $\mathscr{Y}$. Notice that $y_* \notin C^2[0,1]$.

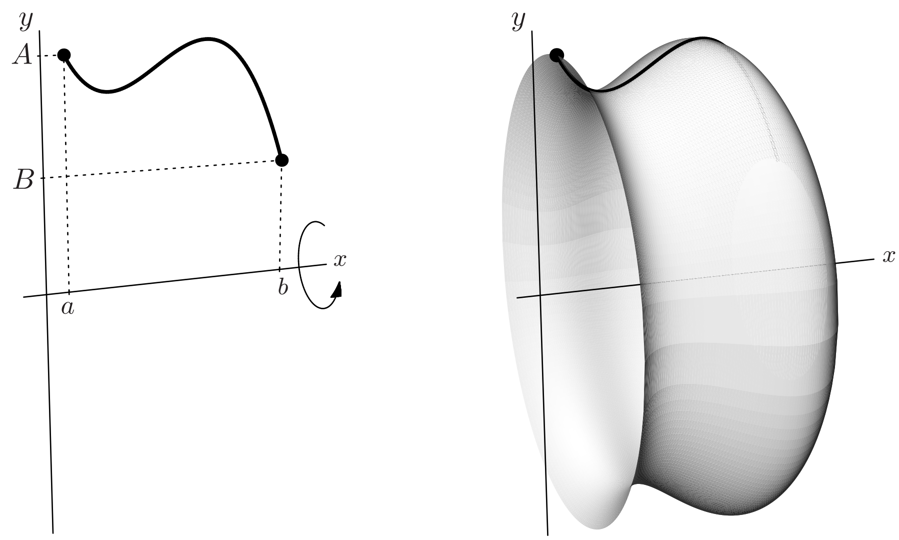

Figure 3.5: A surface of revolution

3.7 Minimal Surface of Revolution Problem

Let $a, b, A, B \in \mathbb{R}$ with $a < b$ and $A, B > 0$ be given and put

$$\mathscr{Y} := \left\{ y \in C^1[a,b] \,:\, y(a) = A\,, y(b) = B\,, y(x) > 0 \quad \text{for all } x \in [a,b] \right\}.$$

Given a function $y \in \mathscr{Y}$, we consider the surface generated by revolving the graph of y about the x-axis. The area of such a surface is

$$S = 2\pi \int_a^b y(x)\sqrt{1 + y'(x)^2}\, dx.$$

Our problem is to minimize, over $\mathscr{Y}$, the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) := \int_a^b y(x)\sqrt{1 + y'(x)^2}\, dx \quad \text{for all } y \in \mathscr{Y}.$$

We begin by observing that the theory that we have discussed so far does not apply directly to this problem because of the pointwise constraint $y(x) > 0$ in the definition of $\mathscr{Y}$. However, since $A, B > 0$, it is not difficult to show that for each $y \in \mathscr{Y}$, the class of admissible variations at y is

$$\mathscr{V}_y = \left\{ v \in C^1[a,b] \,:\, v(a) = v(b) = 0 \right\}.$$

(See Exercise) Therefore, the derivation $(\text{E-L})_1$ given in Section still applies. The integrand $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$f(x, y, z) := y(1 + z^2)^{\frac{1}{2}} \quad \text{for all } (x, y, z) \in [a, b] \times \mathbb{R} \times \mathbb{R},$$

so that

$$f_{,2}(x, y, z) = (1 + z^2)^{\frac{1}{2}} \quad \text{for all } (x, y, z) \in [a, b] \times \mathbb{R} \times \mathbb{R}$$

and

$$f_{,3}(x, y, z) = \frac{yz}{(1 + z^2)^{\frac{1}{2}}} \quad \text{for all } (x, y, z) \in [a, b] \times \mathbb{R} \times \mathbb{R}.$$

Thus the first Euler-Lagrange equation is

$$(1 + y'(x)^2)^{\frac{1}{2}} = \frac{d}{dx}\left[\frac{y(x)y'(x)}{(1 + y'(x)^2)^{\frac{1}{2}}}\right] \quad \text{for all } x \in [a, b]. \tag{E-L$_1$}$$

Although we do not know *a priori* that a minimizer will belong to $C^2[a, b]$, we do know that if y is a minimizer then the mapping

$$x \mapsto \frac{y(x)y'(x)}{(1 + y'(x)^2)^{\frac{1}{2}}}$$

is continuously differentiable on [a,b]. This fact can be used to show that if y is a minimizer, then $y \in C^2[a, b]$. (See Exercise .) Once it is known that $y \in C^2[a, b]$, $(\text{E-L})_1$ can be simplified to

$$y''(x)y(x) = 1 + y'(x)^2 \quad \text{for all } x \in [a, b]. \tag{3.40}$$

(See Exercise .)

It is also useful to look at the second Euler-Lagrange equation. Since the integrand for J has no explicit dependence on x, the second Euler-Lagrange equation reduces to

$$y(x)(1 + y'(x)^2)^{\frac{1}{2}} - \frac{y(x)y'(x)^2}{(1 + y'(x)^2)^{\frac{1}{2}}} = c_1 \quad \text{for all } x \in [a, b] \tag{E-L$_2$}$$

for some constant $c_1 \in \mathbb{R}$. Upon putting the left hand side of $(\text{E-L})_2$ over a common denominator, we find that y satisfies $(\text{E-L})_2$ if

$$\frac{y(x)}{(1 + y'(x)^2)^{\frac{1}{2}}} = c_1 \quad \text{for all } x \in [a, b]. \tag{3.41}$$

At this point we observe that the denominator in (3.41) is always strictly positive and therefore the following hold:

(1) if $c_1 = 0$, then $y(x) = 0$ at each $x \in [a, b]$;

(2) if $c_1 < 0$, then $y(x) < 0$ at each $x \in [a, b]$;

(3) if $c_1 > 0$, then $y(x) > 0$ at each $x \in [a, b]$.

Consequently, if y satisfies (E-L)$_2$, then y must have the same sign throughout the interval $[a, b]$. Since we seek a solution with $y(a), y(b) > 0$, we assume that $c_1 > 0$. This tells us that $y(x) > 0$ for all $x \in [a, b]$. Using (3.40), we conclude that $y''(x) > 0$ for all $x \in [a, b]$. Using (3.41) and the fact that $c_1 > 0$ we find that $y(x) \geq c_1$ for all $x \in [a, b]$.

It follows from (3.41) that

$$y'(x)^2 = \left(\frac{y(x)}{c_1}\right)^2 - 1 \quad \text{for all } x \in [a, b], \tag{3.42}$$

which suggests a hyperbolic-cosine substitution for y. Using the properties that we have established about a minimizer y, it is possible to show that there is a function $u \in C^1[a, b]$ such that

$$y(x) = c_1 \cosh u(x) := \frac{c_1}{2}\left(e^{u(x)} + e^{-u(x)}\right) \quad \text{for all } x \in [a, b]. \tag{3.43}$$

(See Problem .) It follows that

$$y'(x) = c_1(\sinh u(x))u'(x) = \frac{c_1}{2}\left(e^{u(x)} - e^{-u(x)}\right)u'(x) \quad \text{for all } x \in [a, b]. \tag{3.44}$$

Substituting (3.43) and (3.44) into (3.41) and using the identity $\cosh^2 z - 1 = \sinh^2 z$ we find that

$$(c_1 u'(x))^2 = 1 \quad \text{for all } x \in [a, b],$$

which gives

$$u'(x) = \pm\frac{1}{c_1} \quad \text{for all } x \in [a, b].$$

Since u' is continuous on $[a, b]$, we must have either

$$u(x) = \frac{x + c_2}{c_1} \quad \text{for all } x \in [a, b],$$

or

$$u(x) = \frac{-x + c_2}{c_1} \quad \text{for all } x \in [a, b],$$

for some constant c_2. Thus we have

$$y(x) = c_1 \cosh\left(\frac{x + c_2}{c_1}\right) \quad \text{for all } x \in [a, b],$$

or

$$y(x) = c_1 \cosh\left(\frac{-x + c_2}{c_1}\right) \quad \text{for all } x \in [a, b].$$

Since the hyperbolic cosine is an even function (and c_2 is unspecified), we can ignore the second formula for y. (Indeed, if we replace c_2 by $-c_2$ in the second formula for y, we get the first formula.)

It is straightforward to check that the functions we have just found do indeed satisfy $(\text{E-L})_1$ for our functional. Moreover, we have found all possible solutions to $(\text{E-L})_1$ with $y(x) > 0$ for all $x \in [a,b]$, since every C^2-solution of $(\text{E-L})_1$ is also a solution of $(\text{E-L})_2$. In other words, our argument above proves that a function $y \in C^1[a,b]$ with $y(x) > 0$ for all $x \in [a,b]$ is a solution of $(\text{E-L})_1$ if and only if y is of the form

$$y(x) = c_1 \cosh\left(\frac{x+c_2}{c_1}\right) \quad \text{for all } x \in [a,b] \tag{3.45}$$

for some $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

Thus to find a solution of $(\text{E-L})_1$ that is in $\mathscr{Y}$, it only remains to choose $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$ such that

$$A = c_1 \cosh\left(\frac{a+c_2}{c_1}\right) \text{ and } B = c_1 \cosh\left(\frac{b+c_2}{c_1}\right). \tag{3.46}$$

This is a system of two (nonlinear) equations with two unknowns. Whether or not it is possible to choose c_1, c_2 satisfying (3.46) depends upon the values of a, A, b, B. It turns out that in some cases there is a unique choice, but in other cases there is either no way or two ways in which to choose c_1 and c_2.

Let us now make an observation. Consider the parametric curve $(x,y) : [0,1] \to \mathbb{R}^2$ given by

$$(x(t), y(t)) := \begin{cases} \left(a, 3A\left(\frac{1}{3} - t\right)\right), & 0 \le t < \frac{1}{3}; \\ \left(a + 3(b-a)\left(t - \frac{1}{3}\right), 0\right), & \frac{1}{3} \le t < \frac{2}{3}; \\ \left(b, 3B\left(t - \frac{2}{3}\right)\right), & \frac{2}{3} \le t \le 1 \end{cases}$$

(which is not the graph of a function). The surface of revolution generated by this curve consists of a disk with a radius A, a disk with radius B and a line joining the centers of the two disks. The area of this surface is simply $\left(A^2 + B^2\right)\pi$. Although, the parametric curve given above is not in $\mathscr{Y}$, we can approximate it using functions from $\mathscr{Y}$; moreover, we can choose our approximations so that the associated surface areas approach $\left(A^2 + B^2\right)\pi$. For example, for each $n \in \mathbb{N}$ define $y_n \in \mathscr{Y}$ by

$$y_n(x) := A\frac{(b-x)^n}{(b-a)^n} + B\frac{(x-a)^n}{(b-a)^n} \quad \text{for all } x \in [a,b].$$

As n gets large, the functions y_n closely approximate the parametric curve, and the area of the surface generated by y_n becomes very close to the sum of the areas for the two disks with radii A and B. What this tells us is that the minimal surface area must be at least as small as $\left(A^2 + B^2\right)\pi$. That is, if there is a minimal value for J, then it cannot be larger than $\frac{1}{2}\left(A^2 + B^2\right)$.

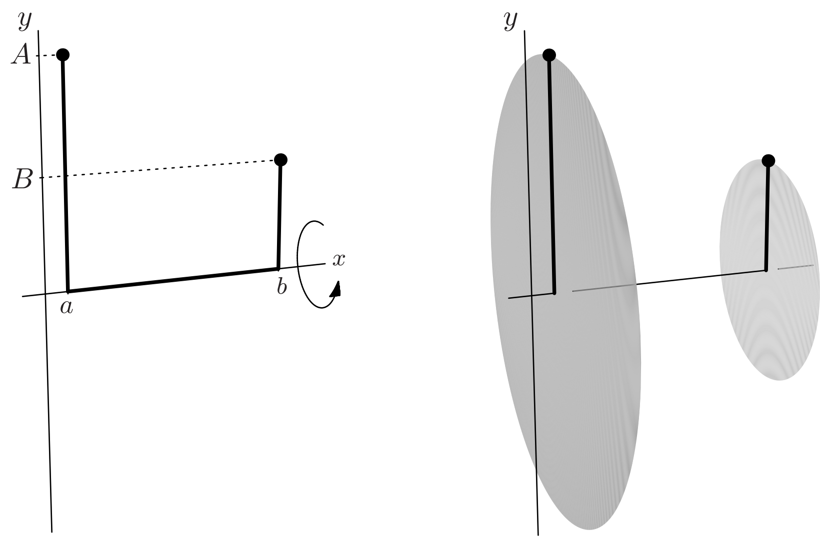

Figure 3.6: Surface of revolution generated by $(x(t), y(t))$

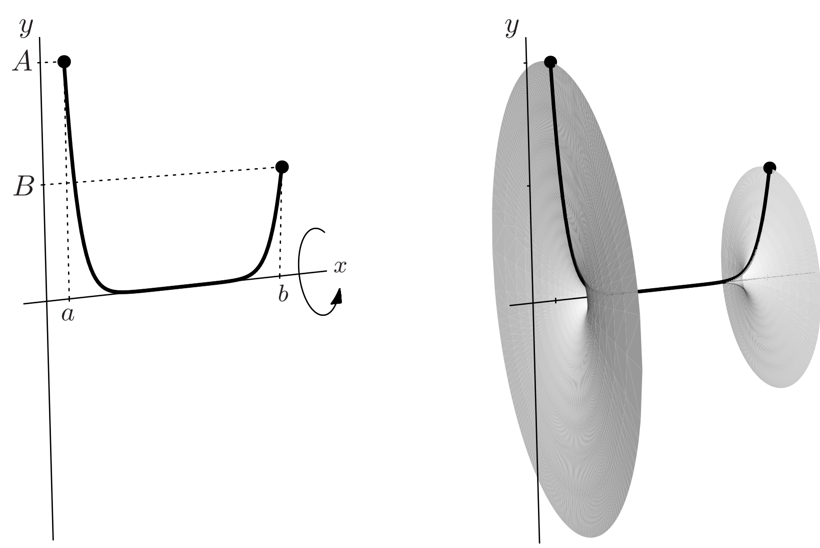

Figure 3.7: Surface of revolution generated by y_n

Chapter 4

Lagrange Multipliers

We now turn our attention to problems where the admissible functions are required to lie on a level set of a constraining functional. Unless the constraining functional is linear, the set over which we wish to optimize will not lead to classes of admissible variations that are large enough to be helpful. The method of Lagrange multipliers provides a flexible and straightforward way to deal with these types of problems. We will develop the method within a sufficiently abstract framework so that it can be applied to many different kinds of problems.

4.1 Too Few Admissible Variations

In this section we investigate a simple (and natural) situation where there are simply not enough admissible variations to provide *any* information about maximizers or minimizers. Let $\mathfrak{X} = C^1[0,\pi]$ and put

$$\mathscr{S} = \{y \in C^1[0,\pi] \,:\, y(0) = y(\pi) = 0 \;,\; \int_0^\pi y(x)^2\,dx = 1\}.$$

For definiteness, suppose that we wish to minimize the functional $J : \mathscr{S} \to \mathbb{R}$ defined by

$$J(y) = \int_0^\pi y'(x)^2\,dx \quad \text{for all } y \in \mathscr{S}.$$

In order to apply the approach of Chapter 2, we need to identify the class of $\mathscr{S}$-admissible variations for each $y \in \mathscr{S}$. To this end, let $y \in \mathscr{S}, v \in \mathfrak{X}$ be given be given. In order to have $v \in \mathscr{V}_y$, it is necessary (and sufficient) to have an open interval I with $0 \in I$ such that $[y + \varepsilon v] \in \mathscr{S}$ for all $\varepsilon \in I$. Suppose that such an interval I exists. Then we must have

$$\begin{aligned}\int_0^\pi [y(x)+\varepsilon v(x)]^2\,dx &= \int_0^\pi y(x)^2\,dx + 2\varepsilon\int_0^\pi y(x)v(x)\,dx + \varepsilon^2\int_0^\pi v(x)^2\,dx \\ &= 1 + 2\varepsilon\int_0^\pi y(x)v(x)\,dx + \varepsilon^2\int_0^\pi v(x)^2\,dx \\ &= 1.\end{aligned}$$

It follows easily that

$$2\int_0^\pi y(x)v(x)\,dx + \varepsilon\int_0^\pi v(x)^2 = 0 \quad \text{for all } \varepsilon \in I\backslash\{0\}, \tag{4.1}$$

which implies that

$$\int_0^\pi v(x)^2\,dx = 0. \tag{4.2}$$

(Indeed, if the equality in (4.1) holds for two distinct values of ε, we can immediately conclude that (4.2) holds.) It follows from (4.2) that $v(x) = 0$ for all $x \in [0, \pi]$. Therefore we have $\mathscr{V}_y = \{0\}$ for all $y \in \mathscr{S}$. Since $\delta J(y; 0) = 0$ for every $y \in \mathscr{S}$, we can conclude nothing from the fact that the Gâteaux variations must vanish for all admissible variations.

The idea will be use the larger domain $\mathscr{Y} = \{y \in C^1[0, \pi] : y(0) = y(\pi) = 0\}$, for which there is a rich class of admissible variations, and consider two functionals $J, G : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) = \int_0^\pi y'(x)^2\,dx \quad \text{and} \quad G(y) = \int_0^\pi y(x)^2\,dx \quad \text{for all } y \in \mathscr{Y}.$$

Notice that

$$\mathscr{S} = \{y \in \mathscr{Y} \mid G(y) = 1\}.$$

By studying the behavior of the pair (J, G) of functionals on $\mathscr{Y}$, we can draw some very useful conclusions about maxima and minima of J on $\mathscr{S}$.

4.2 The Lagrange Multiplier Method in Real Linear Spaces

Let $\mathfrak{X}$ be a real linear space and $\mathscr{Y} \subset \mathfrak{X}$, $J, G : \mathscr{Y} \to \mathbb{R}$, $c \in \mathbb{R}$ be given. Define

$$\mathscr{S} := \{y \in \mathscr{Y} \mid G(y) = c\}.$$

If y is a member of $\mathscr{S}$, then in addition to being in $\mathscr{Y}$, it must satisfy the constraint $G(y) = c$. Our problem is to minimize (or maximize) J on $\mathscr{S}$. If the functional G is nonlinear, then the class of $\mathscr{S}$-admissible variations may be too small to be useful, as the example in Section 4.1 shows.

Rather than focusing on the class $\mathscr{S}$ as being the domain of the functional to be minimized, we think about the pair (J, G) of functionals with domain $\mathscr{Y}$. We define $F : \mathscr{Y} \to \mathbb{R}^2$ by

$$F(y) := (J(y), G(y)) \quad \text{for all } y \in \mathscr{Y}.$$

Suppose that $y_* \in \mathscr{S}$ is a minimizer for J over $\mathscr{S}$, i.e. $J(y) \geq J(y_*)$ for every $y \in \mathscr{S}$. Let us investigate what this tells about F. For one thing, since $y_* \in \mathscr{S}$, we have

$$F(y_*) = (J(y_*), c).$$

Now if $J(y) \geq J(y_*)$ for every $y \in \mathscr{S}$, then there cannot be any $y \in \mathscr{Y}$ such that

$$F(y) = (\alpha, c)$$

with $\alpha < J(y_*)$. Another way of saying this is that for every $\alpha < J(y_*)$, the point (α, c) is not in the range of F.

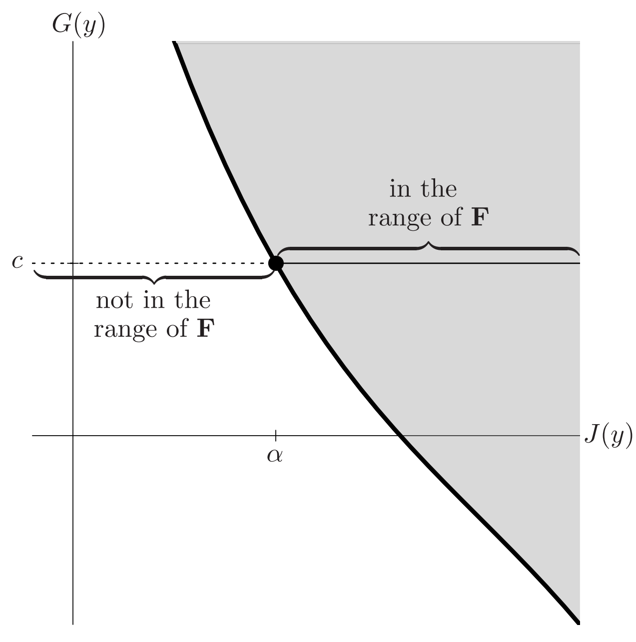

Figure 4.1: (α, c) is in the boundary of the range of F

This means that there is a point $z_* = F(y_*)$ in the range of F for which there are points arbitrarily close to z_* that are not in the range of F. (In other words, $F(y_*)$ is not an interior point of the range of F.) This property has important implications regarding F. In order to take advantage of this observation, we will make use of the following

Lemma 4.1 *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable mapping, and let $x_* \in \mathbb{R}^2$ be given. Suppose that* $\det \nabla\Phi(x_*) \neq 0$. *Then there exists $\delta > 0$ such that*

$$\left\{z \in \mathbb{R}^2 \,:\, \|z - \Phi(x_*)\| < \delta\right\} \subset \mathrm{Range}(\Phi).$$

In order to use Lemma 4.1 it will be convenient to employ a pair of admissible directions, so that we can use F to create a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. Given $y \in \mathscr{Y}$, we want $u, w \in \mathfrak{X}$ such that

$$y + \eta u + \mu w \in \mathscr{Y}$$

for all (η, μ) in some neighborhood of $(0,0)$ in $\mathbb{R}^2$. In order to highlight the important ideas, we shall treat in detail a vety important case in which the situation regarding admissible variations is rather straightforward. For simplicity we assume that $\mathscr{V}$ is a subspace of $\mathfrak{X}$ such that

$$y + v \in \mathscr{Y} \quad \text{for all } y \in \mathscr{Y} \text{ and } v \in \mathscr{V}. \tag{4.3}$$

This implies that for every $y \in \mathscr{Y}$ we have

$$y + \eta u + \mu w \in \mathscr{Y} \quad \text{for all } u, w \in \mathscr{V} \text{ and } \eta, \mu \in \mathbb{R}. \tag{4.4}$$

Observe that if $\mathscr{V}$ is the 0 subspace then (4.3) is automatically, but this is of no use. However, in situations when $\mathscr{Y}$ is characterized by linear constraints, then the set $\mathscr{V}_y$ of admissible variations is a subspace that is independent of $\mathscr{Y}$ and this common subspace $\mathscr{V}$ satisfies (4.3) (and hence also (4.4)).

To use Lemma 4.1, let $u, w \in \mathscr{V}$ be given and define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(\eta, \mu) := (J(y_* + \eta u + \mu w), G(y_* + \eta u + \mu w)) \quad \text{for all } (\eta, \mu) \in \mathbb{R}^2. \tag{4.5}$$

Notice that $\Phi(0,0) = (J(y_*), G(y_*)) = (J(y_*), c)$.

Let us assume that Φ is continuously differentiable. Then from our discussion above, there is no neighborhood of $\Phi(0,0)$ that is contained in the Range(Φ), since y_* minimizes J on $\mathscr{S}$. In other words, for every $\delta > 0$

$$\left\{(\eta, \mu) \in \mathbb{R}^2 \,:\, \Phi(\eta, \mu) - \Phi(0,0)\| < \delta\right\} \not\subset \mathrm{Range}(\Phi).$$

Hence, Lemma 4.1 implies that $\det \nabla\Phi(0,0) = 0$. So if y_* is a minimizer for J over $\mathscr{S}$, then $\det \nabla\Phi(0,0) = 0$.

We thus have an important condition on the gradient of Φ at the point $(0,0)$. Let us now determine $\nabla\Phi(0,0)$. By definition

$$\Phi(\eta, \mu) = (\Phi_1(\eta, \mu), \Phi_2(\eta, \mu)) = (J(y_*{+}\eta u{+}\mu w), G(y_*{+}\eta u{+}\mu w)) \quad \text{for all } (\eta, \mu) \in \mathbb{R}^2.$$

Computing the partial derivatives of the components of Φ with respect to η at $(0,0)$, we find that

$$\Phi_{1,1}(0,0) = \frac{\partial}{\partial \eta}\left[J(y_* + \eta u + \mu w)\right]\Big|_{(\eta,\mu)=(0,0)} = \frac{\partial}{\partial \eta}\left[J(y_* + \eta u)\right]\Big|_{\eta=0} = \delta J(y_*; u)$$

and

$$\Phi_{2,1}(0,0) = \frac{\partial}{\partial \eta}\left[G(y_* + \eta u + \mu w)\right]\Big|_{(\eta,\mu)=(0,0)} = \frac{\partial}{\partial \eta}\left[G(y_* + \eta u)\right]\Big|_{\eta=0} = \delta G(y_*; u).$$

Similarly, the partial derivatives of the components of Φ with respect to μ at $(0,0)$ are given by

$$\Phi_{1,2}(0,0) = \delta J(y_*; w)$$

and

$$\Phi_{2,2}(0,0) = \delta G(y_*; w).$$

Consequently, we have

$$\nabla \Phi(0,0) = \begin{pmatrix} \delta J(y_*; u) & \delta J(y_*; w) \\ \delta G(y_*; u) & \delta G(y_*; w) \end{pmatrix}.$$

It follows that

$$\det \nabla \Phi(0,0) = \delta J(y_*; u)\delta G(y_*; w) - \delta J(y_*; w)\delta G(y_*; u).$$

Therefore, if y_* minimizes J over $\mathscr{S}$, we must have

$$\delta J(y_*; u)\delta G(y_*; w) - \delta J(y_*; w)\delta G(y_*; u) = 0 \quad \text{for all } u, w \in \mathscr{V}. \tag{4.6}$$

There are only two ways that (4.6) can hold:

(1) either $\delta G(y_*; v) = 0$ for all $v \in \mathscr{V}$, or

(2) there exists a $\lambda \in \mathbb{R}$ such that $\delta J(y_*; v) = \lambda \delta G(y_*; v)$ for all $v \in \mathscr{V}$.

To see why this is the case, suppose that (1) does not hold. Then we may choose $w \in \mathscr{V}$ such that $\delta G(y_*; w) \neq 0$. For (4.6) to hold, we must have

$$\delta J(y_*; u) = \left[\frac{\delta J(y_*; w)}{\delta G(y_*; w)}\right] \delta G(y_*; u) \quad \text{for all } u \in \mathscr{V}.$$

So the λ in (2) may be set to the value of $\frac{\delta J(y_*; w)}{\delta G(y_*; w)}$.

4.2.1 Lagrange Multipliers in $\mathbb{R}^n$

We will apply our results from the previous section to the case when $\mathfrak{X} = \mathscr{Y} = \mathbb{R}^n$. Assume that $f, g : \mathbb{R}^n \to \mathbb{R}$ have continuous first-order partial derivatives, and let $c \in \mathbb{R}$ be given. Put

$$\mathscr{S} := \{x \in \mathbb{R}^n \mid g(x) = c\}.$$

We may take $\mathscr{V} = \mathbb{R}^n$.

Suppose that f attains a minimum over $\mathscr{S}$ at the point x_*. The Gateaux variations of f and g are given by

$$\delta f(x_*; v) = \nabla f(x_*) \cdot v,\ \delta g(x_*; v) = \nabla g(x_*) \cdot v \quad \text{for all } v \in \mathbb{R}^n.$$

It follows from the chain rule that mapping Φ will be continuously differentiable. Thus by our argument in the previous section, we conclude that

(1) either $\nabla g(x_*) = 0$, or

(2) there exists a $\lambda \in \mathbb{R}$ such that $\nabla f(x_*) = \lambda \nabla g(x_*)$.

This is the standard method of Lagrange multipliers in $\mathbb{R}^n$. The quantity λ is usually referred to as a Lagrange multiplier.

Remark 4.1 *Assume that $c \in \mathrm{Range}(g)$ and that $\mathscr{S}$ is bounded. Since g is continuous it follows that $\mathscr{S}$ is closed, and therefore compact. (It is also nonempty.) Consequently, f attains a minimum and a maximum on $\mathscr{S}$, i.e., there exist $x_*, y_* \in \mathscr{S}$ such*

$$f(x_*) \leq f(x) \leq f(y_*) \quad \text{for all } x \in \mathbb{R}^n.$$

4.2.2 Example of the Lagrange Multiplier Method in $\mathbb{R}^2$

Define $f, g : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = x_1 x_2 \text{ and } g(x_1, x_2) = x_1^2 + x_2^2 \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

Set

$$\mathscr{S} := \left\{(x_1, x_2) \in \mathbb{R}^2 \mid g(x_1, x_2) = 1\right\}.$$

We consider the problem of minimizing f over the class $\mathscr{S}$.

First, we determine those $x \in \mathscr{S}$ such that $\nabla g(x) = 0$. For the gradient of g, we have

$$\nabla g(x_1, x_2) = (2x_1, 2x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

So

$$\nabla g(x) = 0 \Leftrightarrow x = 0,$$

but $0 \notin \mathscr{S}$. Therefore 0 is not a possible minimizer for f over $\mathscr{S}$.

Let us now look for those $x \in \mathscr{S}$ for which there exists a $\lambda \in \mathbb{R}$ such that $\nabla f(x) = \lambda \nabla g(x)$. The gradient of f is given by

$$\nabla f(x_1, x_2) = (x_2, x_1) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2,$$

so we want those points in $\mathscr{S}$ where

$$(x_2, x_1) = \lambda(2x_1, 2x_2).$$

We therefore seek solutions to

$$\begin{cases} x_2 = 2\lambda x_1; \\ x_1 = 2\lambda x_2; \\ x_1^2 + x_2^2 = 1. \end{cases} \tag{4.7}$$

Notice that this is a system of three equations with three unknowns. Further note that if λ, x_1 or x_2 is zero, then $x_1 = x_2 = 0$. Since $0 \notin \mathscr{S}$, we must have

$\lambda \neq 0$, $x_1 \neq 0$ and $x_2 \neq 0$. So we may substitute the second relation in (4.7) into the first and divide by x_2. Doing so, we see that

$$x_2 = 2\lambda(2\lambda x_2) \Rightarrow \lambda = \pm\frac{1}{2}.$$

Now if $\lambda = \frac{1}{2}$, then $x_1 = x_2$; while $\lambda = -\frac{1}{2}$ implies that $x_1 = -x_2$. Using the third condition in (4.7), we find $x_1 = \pm\frac{1}{\sqrt{2}}$. Thus the possible minimizers for f over $\mathscr{S}$ are

$$\begin{array}{rl} \text{with } \lambda = \frac{1}{2} & \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right); \\ \text{with } \lambda = -\frac{1}{2} & \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right). \end{array}$$

The last two points are the minimizers for f over $\mathscr{S}$, while the first two points are actually maximizers.

4.2.3 Eigenvalues of a Real Symmetric Matrix

There are very important connections between Lagrange multipliers and eigenvalues. In this section we use the method of Lagrange multipliers to show that every real symmetric matrix has at least one (real) eigenvalue.

Let $n \in \mathbb{N}$ be given and let A be an $n \times n$ matrix with real entries. We assume that A is symmetric, i.e. that $A = A^T$, where A^T is the transpose of A. Recall that a real number λ is said to be an *eigenvalue* of A provided that there is a nonzero vector $x \in \mathbb{R}^n$ such that

$$Ax^T = \lambda x^T.$$

Since we regard the elements of $\mathbb{R}^n$ as row vectors, we have taken the transpose of x to turn it into a column vector (i.e., an $n \times 1$ matrix before multiplying it on the left by an $n \times n$ matrix.

Let us define $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = xAx^T \quad \text{for all } x \in \mathbb{R}^n,$$

$$g(x) = x \cdot x \quad \text{for all } x \in \mathbb{R}^n,$$

and put

$$\mathscr{S} = \{x \in \mathbb{R}^n \,:\, g(x) = 1\}.$$

It is straightforward to verify that f and g are continuously differentiable and

$$(\nabla f(x))^T = 2Ax^T, \;\; (\nabla g(x))^T = 2x^T \quad \text{for all } x \in \mathbb{R}^n. \tag{4.8}$$

We note that the assumption $A = A^T$ has been used in an essential way to derive the formula for ∇f. In fact this is the only part of the argument where symmetry of A is used.

It is clear that $1 \in \text{Range}(g)$ and that $\mathscr{S}$ is bounded, so by Remark 4.1, we may choose $x_* \in \mathscr{S}$ that minimizes f over $\mathscr{S}$. Since $x_* \in \mathscr{S}$, we know that

$\nabla g(x_*) \neq 0$ by virtue of our formula for ∇g. Therefore, we may choose $\lambda \in \mathbb{R}$ such that $\nabla f(x_*) = \lambda \nabla g(x_*)$. In view of (4.8), we conclude that

$$Ax_*^T = \lambda x_*^T,$$

and λ is an eigenvalue of A since $0 \notin \mathscr{S}$.

The number λ produced by the minimization procedure is the *smallest* eigenvalue of A. We know that f also attains a maximum on $\mathscr{S}$. If we maximize f on $\mathscr{S}$, the corresponding number λ will be the *largest* eigenvalue of A. (Of course, there is no guarantee that the largest and smallest eigenvalues of A will be different, as we can see by taking A to be the identity matrix.)

4.3 An Example of the Lagrange Multiplier Method in the Calculus of Variations

We now illustrate the Lagrange multiplier method for a calculus of variations problem with an integral constraint, namely the problem from Section 4.1. We put $\mathfrak{X} = \mathscr{C}^1[0,\pi]$,

$$\mathscr{Y} := \left\{ y \in C^1[0,\pi] \mid y(0) = y(\pi) = 0 \right\},$$

and define $G, J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^\pi y'(x)^2\,dx \text{ and } G(y) := \int_0^\pi y(x)^2\,dx \quad \text{for all } y \in \mathscr{Y}.$$

Our problem is to minimize J over

$$\mathscr{S} := \{ y \in \mathscr{Y} \,:\, G(y) = 1 \}.$$

The subspace $\mathscr{V}$ will be

$$\mathscr{V} := \left\{ v \in C^1[0,1] \,:\, v(0) = v(\pi) = 0 \right\}.$$

Notice that this is simply the class of $\mathscr{Y}$-admissible variations. For now, we take it for granted that the function Φ will be continuously differentiable; this technical issue will be addressed later.

We first determine the functions $y \in \mathscr{S}$ such that $\delta G(y; v) = 0$ for every $v \in \mathscr{V}$. These functions must satisfy the first Euler-Lagrange equations for G:

$$2y(x) = 0 \quad \text{for all } x \in [0,\pi]. \tag{E-L$_1$}$$

This, however, implies $y(x)$ is zero at each $x \in [0,\pi]$, but the zero function is not a member of $\mathscr{S}$, so 0 is not a possible minimizer for J over $\mathscr{S}$.

We now seek those $y \in \mathscr{S}$ where there is a $\lambda \in \mathbb{R}$ such that $\delta J(y; v) - \lambda \delta G(y; v) = 0$ for every $v \in \mathscr{V}$. It is straightforward to verify that such y must satisfy the first Euler-Lagrange equation for the functional $J - \lambda G$. Therefore, we seek those y satisfying

$$-2\lambda y(x) = \frac{d}{dx}\left[2y'(x)\right] \quad \text{for all } x \in [0, \pi] \tag{E-L$_1$}$$

and the conditions

$$y(0) = y(\pi) = 0 \text{ and } \int_0^\pi y(x)^2\, dx = 1.$$

We now solve

$$\begin{cases} y''(x) + \lambda y(x) = 0; \\ y(0) = y(\pi) = 0; \\ \int_0^\pi y(x)^2\, dx = 1. \end{cases} \tag{4.9}$$

There are three cases to consider: the first case is when $\lambda < 0$, the second is $\lambda = 0$ and the last is $\lambda > 0$.

For the first case, we may choose $\omega \in \mathbb{R}$ so that $\omega > 0$ and $\lambda = -\omega^2$. Let us first look for the general solution to

$$y''(x) - \omega^2 y(x) = 0 \quad \text{for all } x \in [0, \pi]. \tag{4.10}$$

The characteristic equation for (4.10) is $r^2 - \omega^2 = 0$. The roots of the characteristic equation are therefore $r = \pm\omega$ and the general solution to (4.10) is

$$y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}. \tag{4.11}$$

It is easily checked that $c_1 = c_2 = 0$ in order for the function y given in (4.11) to satisfy $y(0) = y(\pi) = 0$. So for this case, the first two conditions in (4.9) imply that $y(x) = 0$ at each $x \in [0, \pi]$. Clearly, the last condition in (4.9) cannot be satisfied. So there is no solutions to (4.9) when $\lambda < 0$.

We now look at the second case, when $\lambda = 0$. The first condition in (4.9) reduces to

$$y''(x) = 0 \quad \text{for all } x \in [0, \pi].$$

For this equation, the general solution is

$$y(x) = c_1 x + c_2.$$

As in the first case, the condition $y(0) = y(\pi) = 0$ is satisfied only when $c_1 = c_2 = 0$. Thus, we again have no solutions for this case.

In the last case, we have $\lambda > 0$. So we may choose $\omega \in \mathbb{R}$ so that $\omega > 0$ and $\lambda = \omega^2$. We want the general solution for

$$y''(x) + \omega^2 y(x) = 0 \quad \text{for all } x \in [0, \pi]. \tag{4.12}$$

The roots to the characteristic equation $r + \omega^2 = 0$ for (4.12) are $r = \pm i\,\omega$. The general solution to (4.12) is

$$y(x) = c_1 \sin \omega x + c_2 \cos \omega x. \tag{4.13}$$

Let us now find c_1, c_2 such that $y(0) = y(\pi) = 0$. We have

$$y(0) = 0 \Rightarrow c_2 = 0 \Rightarrow y(x) = c_1 \sin \omega x,$$

and

$$y(\pi) = 0 \Rightarrow c_1 \sin \omega\pi = 0 \Rightarrow c_1 = 0 \text{ or } \sin \omega\pi = 0.$$

Now if $c_1 = 0$, then $y(x) = 0$ at each $x \in [0, \pi]$, and we have already seen that this does not satisfy all the conditions in (4.9). So we want $\sin \omega\pi$ to be zero, or in other words, we want ω to be an integer. Since $\omega > 0$, we take ω to be a natural number $n \in \mathbb{N}$. Therefore, we have

$$y(x) = c_1 \sin nx \quad \text{for all } x \in [0, \pi],$$

for some $c_1 \in \mathbb{R} \backslash \{0\}$ and $n \in \mathbb{N}$. Now we will try to satisfy the third condition in (4.9). We need

$$\int_0^\pi c_1^2 \sin^2 nx \, dx = c_1^2 \int_0^\pi \sin^2 nx \, dx = 1.$$

Thus

$$c_1^2 \frac{\pi}{2} = 1 \Rightarrow c_1 = \pm\sqrt{\frac{2}{\pi}}.$$

So for every $n \in \mathbb{N}$, the functions

$$y(x) = \pm\sqrt{\frac{2}{\pi}} \sin nx \tag{4.14}$$

are all the possible minimizers for J over the class $\mathscr{S}$.

Let us compute the value of J for the functions given in (4.14). For y given by (4.14), we find that

$$y'(x) = \pm n\sqrt{\frac{2}{\pi}} \cos nx.$$

Thus

$$J(y) = \int_0^\pi y'(x)^2 \, dx = \frac{2n^2}{\pi} \int_0^\pi \cos^2 nx \, dx = \frac{2n^2}{\pi} \frac{\pi}{2} = n^2, \tag{4.15}$$

where $n \in \mathbb{N}$. The smallest value of J for the functions given in (4.14) is 1, and this value is provided by the functions

$$y(x) = \pm\sqrt{\frac{2}{\pi}} \sin x.$$

So these are the only two possible minimizers for J over $\mathscr{S}$. We note that the computation in (4.15) shows that there is no maximizer for J over $\mathscr{S}$, since we may take $n \in \mathbb{N}$ as large as desired.

4.4 Summary of the Lagrange Multiplier Method for Basic Problems

Let $a, b, A, B, c \in \mathbb{R}$ with $a < b$ be given, and let $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be functions with continuous first partial derivatives. In order to handle problems with fixed endpoints and free endpoints simultaneously, let $\alpha, \beta \in \{0, 1\}$ be given and put

$$\mathscr{Y} := \left\{ y \in C^1[a, b] \, : \, \alpha y(a) = \alpha A \text{ and } \beta y(b) = \beta B \right\}.$$

Notice that if $\alpha = 1$ then the value of y at a is prescribed to be A, while if $\alpha = 0$, then we have a free end at a. (Similar comments apply to β and the end at b.) Define $J, G : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x)) \, dx \text{ and } G(y) := \int_a^b g(x, y(x), y'(x)) \, dx \quad \text{for all } y \in \mathscr{Y},$$

and let

$$\mathscr{S} := \{ y \in \mathscr{Y} \, : \, G(y) = c \}.$$

We take $\mathscr{V}$ to be

$$\mathscr{V} = \{ v \in C^1[a, b] \, : \, \alpha v(a) = \beta v(b) = 0 \}.$$

In order to apply the Lagrange multiplier procedure, we need to know that the mapping Φ is continuously differentiable. The required smoothness of Φ is assured by the following lemma.

Lemma 4.2 *Assume that $M : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has continuous first partial derivatives and let $y, u, w \in C^1[a, b]$ be given. Define $\Psi : \mathbb{R}^2 \to \mathbb{R}$ by*

$$\Psi(\eta, \mu) = \int_a^b M(x, y(x) + \eta u(x) + \mu w(x), y'(x) + \eta u'(x) + \mu w'(x)) \, dx \quad \text{for all } \eta, \mu \in \mathbb{R}.$$

Then Ψ is continuously differentiable.

We define the augmented integrand $L : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$L(x, y, z, \lambda) := f(x, y, z) - \lambda g(x, y, z) \quad \text{for all } (x, y, z, \lambda) \in [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

Consider the functional $H : \mathscr{Y} \times \mathbb{R} \to \mathbb{R}$ given by

$$H(y, \lambda) := \int_a^b L(x, y(x), y'(x), \lambda) dx \quad \text{for all } y \in \mathscr{Y} \text{ and } \lambda \in \mathbb{R}.$$

There are two basic steps to finding all the possible minimizers (and maximizers) for J over the class $\mathscr{S}$:

(Step 1) find all solutions (if any exist) to (E-L)_1 for the functional G that belong to $\mathscr{S}$ and satisfy any relevant natural boundary conditions.

(Step 2) holding λ fixed, find all solutions (if any exist) to (E-L)_1 for the functional H that belong to $\mathscr{S}$ any relevant natural boundary conditions (NBC).

For (Step 2), one treats λ as a constant in order to find (E-L)_1 for H. So the first Euler-Lagrange equation for H is given by

$$L_{,2}(x, y(x), y'(x), \lambda) = \frac{d}{dx}\left[L_{,3}(x, y(x), y'(x), \lambda)\right]. \qquad \text{(E-L)}_1$$

The set of possible minimizers (and maximizers) for J over $\mathscr{S}$ consists of all the solutions found from (Step 1) and (Step 2). We emphasize that the solutions, if any, from (Step 1) must be considered as *potential* minimizers (and maximizers).

4.5 An Isoperimetric Problem

For this problem, let

$$\mathscr{Y} := \left\{y \in C^1[-1,1] \,:\, y(-1) = y(1) = 0\right\}$$

and define $J, G : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_{-1}^{1} y(x)\,dx \text{ and } G(y) := \int_{-1}^{1} \sqrt{1 + y'(x)^2}\,dx \quad \text{for all } y \in \mathscr{Y}.$$

We want to maximize the functional J over the class

$$\mathscr{S} := \{y \in \mathscr{Y} \mid G(y) = l\},$$

where $l > 2$ is some fixed constant. The integrands for J and G are given by

$$f(x,y,z) = y \text{ and } g(x,y,z) = \left(1 + z^2\right)^{\frac{1}{2}} \quad \text{for all } (x,y,z) \in [-1,1] \times \mathbb{R} \times \mathbb{R},$$

respectively, and the augmented integrand for $J - \lambda G$ is given by

$$L(x,y,z,\lambda) = y - \lambda\left(1 + z^2\right)^{\frac{1}{2}} \quad \text{for all } (x,y,z,\lambda) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

We will now try to solve our problem using the method of Lagrange multipliers.

For (Step 1), we need to find all $y \in \mathscr{S}$ that satisfy the first Euler-Lagrange equation for G, which is

$$\frac{d}{dx}\left[\frac{y'(x)}{(1 + y'(x)^2)^{\frac{1}{2}}}\right] = 0 \quad \text{for all } x \in [-1,1]. \qquad \text{(E-L)}_1$$

The general solution to this equation is easily found to be

$$y(x) = c_1 x + c_2 \quad \text{for all } x \in [-1,1].$$

For y to be a member of $\mathscr{Y}$, we need $y(-1) = y(1) = 0$ and this implies $c_1 = c_2 = 0$. Thus $y(x) = 0$ at each $x \in [-1, 1]$. For y to be in $\mathscr{S}$, we also need $G(y)$ to be equal to l. With $y(x) = 0$ at each $x \in [-1, 1]$, however, we find

$$G(y) = \int_{-1}^{1} \sqrt{1 + y'(x)^2}\, dx = \int_{-1}^{1} dx = 2 < l.$$

So there are no functions $y \in \mathscr{S}$ that satisfy the first Euler-Lagrange equation for G. This completes (Step 1).

We now proceed to (Step 2). The first Euler-Lagrange equation for $J - \lambda G$ is

$$L_{,2}(x, y(x), y'(x), \lambda) = \frac{d}{dx}\left[L(x, y(x), y'(x), \lambda)\right] \quad \text{for all } x \in [-1, 1]. \qquad \text{(E-L)}_1$$

Upon computing the necessary partial derivatives of L, we find that $y \in \mathscr{S}$ must satisfy

$$1 = \frac{d}{dx}\left[-\frac{\lambda y'(x)}{(1 + y'(x)^2)^{\frac{1}{2}}}\right] \quad \text{for all } x \in [-1, 1]. \qquad \text{(E-L)}_1$$

Clearly $\lambda \neq 0$. Letting $\mu = -\lambda$, we see that y satisfies (E-L)_1 if and only if

$$\frac{y'(x)}{(1 + y'(x)^2)^{\frac{1}{2}}} = \frac{x + c_1}{\mu} \quad \text{for all } x \in [-1, 1]. \tag{4.16}$$

(It is important to keep track of the sign of the Lagrange multiplier because this information plays a role in determining whether the candidate we find will actually provide a maximum.) At this point we observe that the denominator on the left hand side of (4.16) is always positive. So (4.16) implies that $y'(x)$ always has the same sign as $\frac{x+c_1}{\mu}$. Upon solving (4.16) for y', we have

$$y'(x) = \frac{\frac{x+c_1}{\mu}}{\sqrt{1 - \left(\frac{x+c_1}{\mu}\right)^2}} \quad \text{for all } x \in [-1, 1]. \tag{4.17}$$

Integrating (4.17) we find that

$$y(x) = -\mu\sqrt{1 - \left(\frac{x + c_1}{\mu}\right)^2} + c_2 = \lambda\sqrt{1 - \left(\frac{x + c_1}{\lambda}\right)^2} + c_2 \quad \text{for all } x \in [-1, 1].$$

This is the equation for a circle of radius $|\lambda|$ centered at the point $(-c_1, c_2)$. We need y to satisfy $y(-1) = y(1) = 0$ for it to be a member of $\mathscr{Y}$, which leads to

$$\lambda\sqrt{1 - \left(\frac{-1 + c_1}{\lambda}\right)^2} + c_2 = \lambda\sqrt{1 - \left(\frac{1 + c_1}{\lambda}\right)^2} + c_2 \Rightarrow (c_1 - 1)^2 = (c_1 + 1)^2$$
$$\Rightarrow c_1 = 0.$$

Thus

$$y(x) = \lambda\sqrt{1 - \left(\frac{x}{\lambda}\right)^2} + c_2 \quad \text{for all } x \in [-1, 1]. \tag{4.18}$$

For y to be in $\mathscr{S}$, we now need to satisfy the constraint $G(y) = l$. Computing the derivative for y given in (4.18), we find that

$$y'(x) = -\frac{\frac{x}{\lambda}}{\sqrt{1 - \left(\frac{x}{\lambda}\right)^2}} \Rightarrow 1 + y'(x)^2 = \frac{1}{1 - \left(\frac{x}{\lambda}\right)^2} \quad \text{for all } x \in [-1, 1].$$

Therefore

$$G(y) = \int_{-1}^{1} \sqrt{1 + y'(x)^2}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1 - \left(\frac{x}{\lambda}\right)^2}}\, dx = 2|\lambda| \arcsin \frac{1}{|\lambda|}.$$

Thus for y to be in $\mathscr{S}$, we must have

$$|\lambda| \arcsin \frac{1}{|\lambda|} = \frac{l}{2}. \tag{4.19}$$

It is not difficult to prove that there is a value for $|\lambda|$ satisfying (4.19) if and only if $2 < l \leq \pi$. Provided that $2 < l \leq \pi$, the condition $y(1) = 0$ implies that

$$c_2 = -\lambda\sqrt{1 - \frac{1}{\lambda^2}}.$$

Whence

$$y(x) = \lambda\left[\sqrt{1 - \frac{x^2}{\lambda^2}} - \sqrt{1 - \frac{1}{\lambda^2}}\right],$$

with $|\lambda|$ satisfying (4.19). When $l = \pi$, the function y above does not belong to $C^1[-1, 1]$ because there are vertical tangents at the endpoints. We shall show in the next chapter that when λ is positive, our solution is a maximizer, and when λ is negative, we have a minimizer.

Chapter 5

Convexity

The notion of a convex functional is a very natural and useful generalization of the idea from basic calculus of a function whose graph is "concave up". Such functionals play a fundamental role in the study of minimization problems. One important reason for this is the fact that if a convex functional J has an element y_* in its domain such $\delta J(y; v) = 0$ for all v in a suitable class of admissible variations, then J must attain a minimum at y_*. We begin by looking at convex functions from $\mathbb{R}$ to $\mathbb{R}$.

5.1 Convex functions from $\mathbb{R}$ to $\mathbb{R}$

Let $f : \mathbb{R} \to \mathbb{R}$ be given. The notion of "concavity" plays an important role in graphing functions in basic calculus. In calculus courses, concavity is typically defined in terms of monotonicty of the derivative which is related to the sign of the second derivative. In more sophisticated mathematical contexts a function from $\mathbb{R}$ to $\mathbb{R}$ whose graph is every concave up is said to be a "convex function". We want to give a definition of convexity that does not assume differentiability. Let us recall some characterizations of convex functions from basic calculus.

(1) If f is twice differentiable on $\mathbb{R}$ then f is convex if and only if $f(x) \geq 0$ for all $x \in \mathbb{R}$.

(2) If f is differentiable on $\mathbb{R}$ then f is convex provided that f' is increasing[1]

(3) If f is differentiable on $\mathbb{R}$ then f is convex if and only if the graph of f always lies above its tangent lines.

(4) Without assuming that f is differentiable: f is convex if and only if given any secant line for the graph of f, the secant line intersects the graph of f at only two points and between the points of intersection the graph of f lies below the secant line.

[1] When we say that a function $g : \mathbb{R} \to \mathbb{R}$ is *increasing* we mean that $g(x_2) \geq g(x_1)$ whenever $x_2 \geq x_1$. If $g(x_2) > g(x_1)$ whenever $x_2 > x_1$ we say that g is *strictly increasing*.

Let us formulate these last two characterizations analytically. The third condition states that the graph of f is supported by its tangent lines. Assume that f is differentiable and let $x_0 \in \mathbb{R}$ be given. The tangent line to the graph of f at x_0 is described by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

The graph of f lies everywhere above this tangent line if and only if

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \quad \text{for all } x \in \mathbb{R}. \tag{5.1}$$

Observe that (5.1) has the following very important consequence: If $f'(x_0) = 0$, then $f(x) \geq f(x_0)$ for all $x \in \mathbb{R}$. In other words, if f is differentiable and convex and $f'(x_0) = 0$, then f attains a minimum at x_0.

For item (4), let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ be given. The characterization states that at each $x \in [x_1, x_2]$, the graph of f lies below the line segment joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. It is not difficult to check that this will be the case if and only if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \quad \text{for all } t \in [0, 1]. \tag{5.2}$$

It is quite natural to use (5.2) for the official definition of *convex function*.

5.2 The Definition of Convexity in a Real Linear Space

In this section, we generalize the notion of convexity to functions defined on subsets of real linear spaces. Let $\mathfrak{X}$ be a real linear space. We will base the definition on (5.2), because this condition did not require any differentiability of the function in question. We will be interested in defining convexity of functionals whose domains are proper subsets of $\mathfrak{X}$. In order to do this we want be sure that the domain of the functional has the property that whenever two points are in the domain, then so is the line segment joining the two points. This motivates the following

Definition 5.1 *Let $\mathscr{Y} \subset \mathfrak{X}$ be given. We say that $\mathscr{Y}$* is convex *provided that*

$$ty_1 + (1-t)y_2 \in \mathscr{Y} \quad \text{for all } y_1, y_2 \in \mathscr{Y} \text{ and } t \in [0, 1].$$

Now, we define convexity for functionals.

Definition 5.2 *Let $\mathscr{Y} \subset \mathfrak{X}$ be a convex set, and let $J : \mathscr{Y} \to \mathbb{R}$ be given. We say that J* is convex *provided that*

$$J(ty_1 + (1-t)y_2) \leq tJ(y_1) + (1-t)J(y_2) \quad \text{for all } y_1, y_2 \in \mathscr{Y} \text{ and } t \in [0, 1].$$

We emphasize that Definition 5.2 makes sense only when the domain of J is a convex set. Before we explore the significance of convexity in the context of a general linear space $\mathfrak{X}$, it is instructive to examine convex functions from $\mathbb{R}^n$ to $\mathbb{R}$.

5.3 Convex Functions on $\mathbb{R}^n$

We shall now relate convexity of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ to conditions involving ∇f. Before stating the main theorem, we will extend the idea of an increasing function to mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. To motivate this extension let us reexamine what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be increasing. Observe that g is increasing if and only if: for all $x_1, x_2 \in \mathbb{R}$, we have

$$g(x_2)-g(x_1) \geq 0 \text{ when } (x_2 - x_1) \geq 0 \text{ and } g(x_2)-g(x_1) \leq 0 \text{ when } (x_2 - x_1) \leq 0.$$

An equivalent way to state this condition is

$$[g(x_2) - g(x_1)](x_2 - x_1) \geq 0 \quad \text{for all } x_1, x_2 \in \mathbb{R}.$$

We make the following definition.

Definition 5.3 *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given. We say that g is monotone provided that*

$$[g(x) - g(y)] \cdot (x - y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^n.$$

We are now ready to generalize items (2) and (3) from Section 5.1.

Theorem 5.1 *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ has continuous first-order partial derivatives. The following three statements are equivalent:*

(i) f is convex;

(ii) $f(x) \geq f(y) + \nabla f(x) \cdot (x - y)$ for all $x, y in \mathbb{R}^n$;

(iii) $[\nabla f(x) - \nabla f(y)] \cdot (x - y) \geq 0$ for all $x, y \in \mathbb{R}^n$.

Proof. To establish the theorem, we will prove that (i)⇒(ii)⇒(iii)⇒(i).

To prove that (i)⇒(ii), we assume that f is convex. Let $x, y \in \mathbb{R}^n$ be given. By Definition 5.2, we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \quad \text{for all } t \in [0, 1]. \tag{5.3}$$

Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) := f(tx + (1 - t)y) \quad \text{for all } t \in \mathbb{R}.$$

Notice that $g(0) = f(y)$ and $g(1) = f(x)$. Using the chain rule (Theorem 2.1), we find that g is differentiable and

$$g'(t) = \nabla f(tx + (1 - t)y) \cdot (x - y) \quad \text{for all } t \in \mathbb{R}. \tag{5.4}$$

We deduce from (5.3) that for each $t \in (0, 1]$

$$f(tx + (1 - t)y) - f(y) \leq t[f(x) - f(y)],$$

which gives

$$\frac{f(tx+(1-t)y)-f(y)}{t} \le f(x)-f(y),$$

i.e.

$$\frac{g(t)-g(0)}{t} \le f(x)-f(y). \tag{5.5}$$

Letting $t \to 0^+$ in (5.5) yields

$$g'(0) \le f(x)-f(y).$$

Using (5.4) with $t=0$ now gives us

$$\nabla f(x) \cdot (x-y) \le f(x)-f(y)$$

and consequently we have

$$f(x) \ge f(y) + \nabla f(y) \cdot (x-y).$$

We have thus proven that (i)⇒(ii).

Next, we prove that (ii)⇒(iii). Assume that (ii) holds and let $x, y \in \mathbb{R}^n$ be given. Then we have

$$f(x) \ge f(y) + \nabla f(y) \cdot (x-y). \tag{5.6}$$

and (by interchanging x and y) we also have

$$f(y) \ge f(x) + \nabla f(x) \cdot (y-x). \tag{5.7}$$

Adding (5.6) and (5.7) yields

$$f(x)+f(y) \ge f(y)+f(x)+\nabla f(y)\cdot(x-y)+\nabla f(x)\cdot(y-x)$$

and consequently we have

$$0 \ge \nabla f(y)\cdot(x-y)+\nabla f(x)\cdot(x-y)$$

which gives

$$0 \ge [\nabla f(y) - \nabla f(x)] \cdot (x-y). \tag{5.8}$$

Multiplying (5.8) by -1 we obtain

$$[\nabla f(y)-\nabla f(x)]\cdot(y-x) \ge 0,$$

and the implication (ii)⇒(iii) is established.

To complete the proof of the theorem, it only remains to show that (iii)⇒(i). Assume that (iii) holds and let $x, y \in \mathbb{R}^n$ be given. Define the function $G : [0,1] \to \mathbb{R}$ by

$$G(t) = f(tx+(1-t)y) - tf(x) - (1-t)f(y) \quad \text{for all } t \in [0,1].$$

Observe that $G(0) = G(1) = 0$. We want to show that $G(t) \leq 0$ at each $t \in [0,1]$. We shall accomplisf this by first showing that the derivative of G is increasing over $[0,1]$ and then use this fact to conclude that $G(t) \leq 0$ for all $t \in [0,1]$.

Let us look at the derivative of G. We have

$$G'(t) = \nabla f(tx + (1-t)y) \cdot (x - y) - f(x) + f(y) \quad \text{for all } t \in [0,1].$$

Let $s, t \in [0,1]$ be given and observe that

$$\begin{aligned} G'(t) - G'(s) &= [\nabla f(tx + (1-t)y) - \nabla f(sx + (1-s)y)] \cdot (x - y) \\ &= [\nabla f(y + t(x-y)) - \nabla f(y + s(x-y))] \cdot (x - y). \end{aligned} \tag{5.9}$$

Putting $\widetilde{x} := tx + (1-t)y$ and $\widetilde{y} := sx + (1-s)y$, we can rewrite (5.9) as

$$\begin{aligned} [G'(t) - G'(s)](t-s) &= (t-s)\,[\nabla f(\widetilde{x}) - \nabla f(\widetilde{y})] \cdot (x - y) \\ &= [\nabla f(\widetilde{x}) - \nabla f(\widetilde{y})] \cdot (\widetilde{x} - \widetilde{y}) \\ &\geq 0. \end{aligned}$$

For the last step, we used our assumption that (iii) holds. Since $t, s \in [0,1]$ were arbitrary, we have shown that

$$[G'(t) - G'(s)]\,(t-s) \geq 0 \quad \text{for all } t, s \in [0,1].$$

Consequently G' is increasing on the interval $[0,1]$. It follows that the maximum of G on $[0,1]$is 0, i.e. the maximum value of G is attained at the endpoints of $[0,1]$. Indeed, suppose that G attains a maximum at $t_0 \in (0,1)$. Since G is continuously differentiable, we must have $G'(t_0) = 0$. Moreover, we must have $G'(t) \leq 0$ at each $t \in [0, t_0]$ and $G'(t) \geq 0$ at each $t \in [t_0, 1]$, since G' is increasing $[0,1]$. This implies that G also attains a minimum at t_0. The only way that this can occur is if G is constant on $[0,1]$, and since $G(0) = G(1) = 0$, we have shown that is G attains a maximum at some $t_0 \in (0,1)$, then $G(t) = 0$ at each $t \in (0,1)$. Thus the maximum value for G is 0. In conclusion, we have shown that $G(t) \leq 0$ at each $t \in [0,1]$, and using the definition of G, we see that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Consequently f is convex and (iii)⇒(i). This completes the proof. □

Corollary 5.1 *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function with continuous first-order partial derivatives. Let $y_* \in \mathbb{R}^n$ be given and suppose that $\nabla f(y_*) = 0$. Then y_* minimizes f on $\mathbb{R}^n$.*

The proof follows immediately from (ii) and the observation that for all $y \in \mathbb{R}^n$ we have

$$f(y) \geq f(y_*) + \nabla f(y_*) \cdot (y - y_*) = f(y_*),$$

since $\nabla f(y_*) = 0$.

5.4 Convexity and Minima in Real Linear Spaces

In this section, we prove an analogue of Corollary 5.1 for functionals defined on convex subsets of real linear spaces. Let $\mathfrak{X}$ be a real linear space and let $\mathscr{Y} \subset \mathfrak{X}$ be convex. Let $\mathscr{V}$ be a subspace of $\mathfrak{X}$ and assume that

$$z - y \in \mathscr{V} \text{ and } y + v \in \mathscr{Y} \quad \text{for all } y, z \in \mathscr{Y} \text{ and } v \in \mathscr{V}.$$

We will prove the following

Theorem 5.2 *Assume that $J : \mathscr{Y} \to \mathbb{R}$ is convex and let $y_* \in \mathscr{Y}$ be given. Suppose that $\delta J(y_*; v)$ exists and $\delta J(y_*; v) = 0$ for each $v \in \mathscr{V}$. Then y_* minimizes J over $\mathscr{Y}$.*

Proof. Let $y \in \mathscr{Y}$ be given. We want to show that $J(y) \geq J(y_*)$. Let $v = y - y_* \in \mathscr{V}$, so that $y = y_* + v$. Let $t \in (0, 1]$ be given. Since J is convex, we have the following inequalities:

$$J(ty + (1-t)y_*) \leq tJ(y) + (1-t)J(y_*).$$

$$\begin{aligned} J(t(y_* + v) + (1-t)y_*) &\leq t[J(y) - J(y_*)] + J(y_*) \\ J(y_* + tv) &\leq J(y_*) + t[J(y) - J(y_*)] \\ J(y_* + tv) - J(y_*) &\leq t[J(y) - J(y_*)] \\ \frac{J(y_* + tv) - J(y_*)}{t} &\leq J(y) - J(y_*). \end{aligned}$$

Since $\delta J(y_*; v)$ exists and is equal to zero, we may let $t \to 0^+$ in the last inequality to conclude that

$$0 = \lim_{t \to 0^+} \frac{J(y_* + tv) - J(y_*)}{t} \leq J(y) - J(y_*).$$

and consequently $J(y) \geq J(y_*)$, i.e. J attains a minimum over $\mathscr{Y}$ at y_*. $\square$

5.5 Convexity of Functionals Defined by Integrals

Let $a, b \in \mathbb{R}$ with $a < b$ be given. Put $\mathfrak{X} = C^1[a, b]$ and assume that $\mathscr{Y} \subset \mathfrak{X}$ is convex. Let a continuous function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given, and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

In order to apply Theorem 5.2 to J, we need to know when J is convex.

Given $y_1, y_2 \in \mathscr{Y}$, let us look at the quantity

$$J(ty_1 + (1-t)y_2) = \int_a^b f(x, ty_1(x) + (1-t)y_2(x), ty_1'(x) + (1-t)y_2'(x))\, dx$$

with $t \in [0,1]$. Suppose that $f(x,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is convex for each $x \in [a,b]$, i.e. that f is convex with respect to its last two arguments. Then

$$\begin{aligned}
&\int_a^b f(x, ty_1(x) + (1-t)y_2(x), ty_1'(x) + (1-t)y_2'(x))\, dx \\
\leq &\int_a^b \Big\{ tf(x, y_1(x), y_1'(x)) + (1-t)f(x, y_2(x), y_2'(x)) \Big\}\, dx \\
= &t\int_a^b f(x, y_1(x), y_1'(x))\, dx + (1-t)\int_a^b f(x, y_2(x), y_2'(x))\, dx \\
= &tJ(y_1) + (1-t)J(y_2).
\end{aligned}$$

We conclude that if the mapping $(y,z) \mapsto f(x,y,z)$ is convex for each $x \in [a,b]$, then

$$J(ty_1 + (1-t)y_2) \leq tJ(y_1) + (1-t)J(y_2) \quad \text{for all } t \in [0,1]$$

and J is convex. It is important to note that although convexity of $f(x,\cdot,\cdot)$ is sufficient for convexity of J, it is not necessary. (See Exercise .)

5.6 Second Derivative Test for Convexity

The definition of convexity, as well as the characterizations established so far, may be difficult to check directly. Recall that for functions of one real variable the second derivative test is a very convenient way to test for convexity. It is possible to define second-order Gateaux variations and use them to characterize convexity of "smooth" functionals on convex subsets of a general linear space. We shall not adopt this approach here. (However, readers are encouraged to try to define second-order Gateaux variations and explore their relationship with convexity.) Instead, we will develop a second derivative test for functions on $\mathbb{R}^n$ and use this test to show that integrands have sufficient convexity to render the functional J convex.

Let us first look at the situation in one-dimension. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. and let $x_1, x_2 \in \mathbb{R}$ be given. Taylor's Theorem tells us that there exists a point c_{x_1,x_2} between x_1 and x_2 such that

$$f(x_1) = f(x_2) + f'(x_2)(x_1 - x_2) + \frac{1}{2}f''(c_{x_1,x_2})(x_1 - x_2)^2.$$

If $f''(c_{x_1,x_2}) \geq 0$ for all $x \in \mathbb{R}$, we find that

$$f(x_1) \geq f(x_2) + f'(x_2)(x_1 - x_2).$$

and by Theorem 5.1, the function f is convex. One can also show that if f is convex then $f''(x) \geq 0$ for all $x \in \mathbb{R}$. In other words, for twice continuously differentiable functions, nonnegativity of the second derivative characterizes convexity.

Now we consider the more general situation when $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives. Define the Hessian matrix $H : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by

$$H(x) := \begin{pmatrix} f_{,1,1}(x) & f_{,1,2}(x) & \cdots & f_{,1,n}(x) \\ f_{,2,1}(x) & f_{,2,2}(x) & \cdots & f_{,2,1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{,n,1}(x) & f_{,n,2}(x) & \cdots & f_{,n,n}(x) \end{pmatrix}.$$

Since f has continuous second-order partial derivatives, we see that

$$\big(H(x)\big)_{ij} = f_{,i,j}(x) = f_{,j,i}(x),$$

and the Hessian matrix is symmetric, i.e. $H(x)^{\mathsf{T}} = H(x)$ at each $x \in \mathbb{R}^n$. Using Taylor's Theorem, for each $x, y \in \mathbb{R}^n$ there exists some $c_{x,y}$ on the line segment joining x and y such that

$$f(x) = f(y) + \nabla f(y) \cdot (x - y) + \frac{1}{2}(x - y)H(c_{x,y})(x - y)^{\mathsf{T}}. \tag{5.10}$$

(The form of Taylor's theorem used above can be obtained by applying Taylor's theorem for functions of one variable to the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = f(tx + (1 - t)y)$.) We see that if the last term in (5.10) is always nonnegative, then Theorem 5.1 yields the convexity of f. Let us make a definition

Definition 5.4 *Let $A \in \mathbb{R}^{n\times n}$ be given. We say that A* is positive definite *if*

$$xAx^{\mathsf{T}} > 0 \quad \text{for all } x \in \mathbb{R}^n \backslash \{0\}.$$

We say that A is positive semidefinite *if*

$$xAx^{\mathsf{T}} \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

It follows from (5.10)that if the Hessian matrix $H(x)$ for f is positive semidefinite at every $x \in \mathbb{R}^n$, then f is convex. We now have a test for the convexity of f using the second-order partial derivatives of f.

We provide here an important characterization of positive definite and positive semidefinite symmetric matrices.

Theorem 5.3 *Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric. Then*

(i) A is positive semidefinite if and only if all the eigenvalues of A are nonnegative;

(ii) A is positive definite if and only if all the eigenvalues of A are strictly positive.

Proof.

□

We now give an alternative characterization of symmetric positive definite known as *Sylvester's Theorem*. Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric. Define $A_1 \in \mathbb{R}^{1\times 1}$, $A_2 \in \mathbb{R}^{2\times 2}$,..., $A_n \in \mathbb{R}^{n\times n}$ by

$$\begin{aligned} A_1 :=& (A_{11}), \\ A_2 :=& \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \\ A_i :=& \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1i} \\ A_{12} & A_{22} & \cdots & A_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1i} & A_{2i} & \cdots & A_{ii} \end{pmatrix} \quad \text{for } i \in \{3, 4, \cdots, n-1\}, \end{aligned}$$

and

$$A_n := A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Notice that we have used the symmetry of A for our definitions.

Theorem 5.4 *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and let $A_1, A_2, \cdots, A_n$ be as above. Then A is positive definite if and only if $\det A_i > 0$ for every $i \in \{1, 2, \cdots, n\}$.*

As an example, suppose that $n = 3$ so that

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}.$$

Then according to the theorem, the matrix A is positive definite if and only if $A_{11} > 0$, $A_{11}A_{22} - A_{12}^2 > 0$ and $\det A > 0$.

Extending the above characterization of positive definiteness to positive semidefiniteness is a bit delicate. One needs to consider additional subdeterminants of A. (See, for example,) A discussion of the general case is outside the scope of this course. However, the result for $n = 2$ is easy to state (and also to prove).

Remark 5.1 *Assume that $A \in \mathbb{R}^{2\times 2}$ is symmetric.*

(1) A is positive definite if and only if $A_{11} > 0$ and $A_{11}A_{22} - A_{12}^2 > 0$;

(2) A is positive semidefinite if and only if $A_{11} \geq 0$, $A_{22} \geq 0$ and $A_{11}A_{22} - A_{12}^2 \geq 0$.

5.7 Applying the Second Derivative Test to Calculus of Variations Problems

Let $a, b \in \mathbb{R}$ with $a < b$ be given. Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function with continuous second-order partial derivatives. Put $\mathfrak{X} := C^1[a, b]$ and assume that $\mathscr{Y} \subset \mathfrak{X}$ is convex. As usual, define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

We have the following

Theorem 5.5 *Suppose at each $(x, y, z) \in [a, b] \times \mathbb{R} \times \mathbb{R}$ that f satisfies*

(a) $f_{,2,2}(x, y, z) \geq 0$,

(b) $f_{,3,3}(x, y, z) \geq 0$,

(c) $f_{,2,2}(x, y, z) f_{,3,3}(x, y, z) - [f_{,2,3}(x, y, z)]^2 \geq 0$.

Then, the functional J is convex.

Proof. The proof for this theorem follows the results stated in the previous section and our discussion in Section 5.5. □

5.7.1 Example 5.7.1 (cf. Examples 1.2, 3.1.1, 3.3.1 and 3.6.2)

Set

$$\mathscr{Y} := \left\{ y \in C^1[0, 1] \,:\, y(0) = 0 \text{ and } y(1) = 1 \right\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^1 \left[y(x)^2 + y'(x)^2 \right] dx \quad \text{for all } y \in \mathscr{Y}.$$

The integrand $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for J is given by

$$f(x, y, z) := y^2 + z^2 \quad \text{for all } x \in [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

We find that

$$f_{,2}(x, y, z) = 2y \text{ and } f_{,3} = 2z \quad \text{for all } (x, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$$

and that

$$f_{,2,2}(x, y, z) = 2; f_{,2,3}(x, y, z) = 0 \text{ and } f_{,3,3}(x, y, z) = 2 \quad \text{for all } (x, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

By Theorem 5.5, the functional J is convex. Therefore any $y \in \mathscr{Y}$ that satisfies

$$2y(x) = 2y''(x) \quad \text{for all } x \in [0, 1] \qquad \text{(E-L)}_1$$

is a minimizer for J over the class $\mathscr{Y}$.

5.7.2 Example 5.7.2 (cf. Section 3.5)

In this section, we recall the problem of minimizing the transit time for a boat crossing a river with current. We found, in Section 3.5, that we needed to minimize the functional $T : \mathscr{Y} \to \mathbb{R}$ given by

$$T(y) := \frac{1}{\omega} \int_0^b \frac{\sqrt{1 - e(x)^2 + y'(x)^2} - e(x)y'(x)}{1 - e(x)^2}\, dx \quad y \in \mathscr{Y},$$

where

$$\mathscr{Y} := \left\{ y \in C^1[0,b] \,:\, y(0) = 0 \text{ and } y(b) = B \right\}.$$

In the definition of T, we have $\omega > 0$ and $0 \leq e(x) < 1$ at each $x \in [0,b]$. The integrand $f : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for T is given by

$$f(x,y,z) := \frac{1}{\omega\,(1 - e(x)^2)} \left[\sqrt{1 - e(x)^2 + z^2} - e(x)z \right] \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}.$$

So

$$f_{,2}(x,y,z) = 0 \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}$$

and

$$f_{,3}(x,y,z) = \frac{1}{\omega\,(1 - e(x)^2)} \left[\frac{z}{\sqrt{1 - e(x)^2 + z^2}} - e(x) \right] \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}.$$

Whence

$$f_{,2,2}(x,y,z) = f_{,2,3}(x,y,z) = 0 \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}.$$

and

$$\begin{aligned} f_{,3,3}(x,y,z) &= \frac{1}{\omega\,(1 - e(x)^2)} \left[\frac{\sqrt{1 - e(x)^2 + z^2} - \frac{z^2}{\sqrt{1 - e(x)^2 + z^2}}}{1 - e(x)^2 + z^2} \right] \\ &= \frac{1}{\omega\,(1 - e(x)^2)} \frac{1 - e(x)^2}{(1 - e(x)^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{1}{\omega} \frac{1}{(1 - e(x)^2 + z^2)^{\frac{3}{2}}} \qquad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Since $\omega > 0$, Theorem 5.5 implies that J is convex. It follows that any $y \in \mathscr{Y}$ that satisfies the first Euler-Lagrange equation for T is a minimizer for T over $\mathscr{Y}$.

5.7.3 Example 5.7.3

For this example, we put

$$\mathscr{Y} := \left\{ y \in C^1[0,\pi] \mid y(0) = y(\pi) = 0 \right\}$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^\pi \left[y'(x)^2 - y(x)^2 \right] \, dx \quad \text{for all } y \in \mathscr{Y}.$$

For each $n \in \mathbb{N}$ put $y_n(x) := \sin nx$ for $x \in [0,\pi]$. Then, we compute that

$$J(y_n) = \int_0^\pi \left[n^2 \cos^2 nx - \sin^2 nx \right] \, dx = \left(n^2 - 1 \right) \frac{\pi}{2} \quad \text{for all } n \in \mathbb{N}.$$

Thus J has no maximum value over $\mathscr{Y}$.

Let us see if we can use Theorem 5.5 to determine whether J is convex or not. The integrand $f : [0,\pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for J is

$$f(x,y,z) = z^2 - y^2 \quad \text{for all } (x,y,z) \in [0,\pi] \times \mathbb{R} \times \mathbb{R}.$$

Thus

$$f_{,2,2}(x,y,z) = -2 \quad \text{for all } (x,y,z) \in [0,\pi] \times \mathbb{R} \times \mathbb{R},$$

and we cannot use Theorem 5.5 to conclude that J is convex.

5.8 Convexity and the Lagrange Multiplier Method

Let $\mathfrak{X}$ be a real linear space and let $\mathscr{Y} \subset \mathfrak{X}$ be convex. Assume that $\mathscr{V}$ is a subspace of $\mathfrak{X}$ such that

$$y - z \in \mathscr{V} \text{ and } y + v \in \mathscr{Y} \quad \text{for all } y, z \in \mathscr{Y} \text{ and } v \in \mathscr{V}.$$

Let $J, G : \mathscr{Y} \to \mathbb{R}$ and $c \in \mathbb{R}$ be given. Put

$$\mathscr{S} := \{ y \in \mathscr{Y} \mid G(y) = c \}.$$

Theorem 5.6 *Let $y_* \in \mathscr{S}$ and $\lambda \in \mathbb{R}$ be given. Assume that the functional $(J - \lambda G)$ is convex and that $\delta J(y_*; v)$ and $\delta G(y_*; v)$ exist for all $v \in \mathscr{V}$. Assume further that*

$$\delta J(y_*; v) = \lambda G(y_*; v) \quad \text{for all } v \in \mathscr{V}.$$

Then y_* minimizes J over $\mathscr{S}$.

Proof. Define $H : \mathscr{Y} \to \mathbb{R}$ by

$$H(y) := J(y) - \lambda G(y) \quad \text{for all } y \in \mathscr{Y}.$$

By assumption, the functional H is convex over $\mathscr{Y}$ and $\delta H(y_*; v) = 0$ for every $v \in \mathscr{V}$. From our discussion in Section 5.5, we conclude that $H(w) \geq H(y_*)$ for every $w \in \mathscr{Y}$. Now let $y \in \mathscr{S}$ be given. Since $y \in \mathscr{S} \subset \mathscr{Y}$, we have $H(y) \geq H(y_*)$. Thus

$$\begin{aligned} J(y) - \lambda G(y) \geq J(y_*) - \lambda G(y_*) &\Rightarrow J(y) - \lambda c \geq J(y_*) - \lambda c \\ &\Rightarrow J(y) \geq J(y_*). \end{aligned}$$

We have shown that y_* minimizes J over $\mathscr{S}$. □

5.8.1 Example 5.8.1 (cf. Section 4.5)

Set

$$\mathscr{Y} := \left\{ y \in C^1[-1,1] \,:\, y(-1) = y(1) = 0 \right\}.$$

Define $J, G : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_{-1}^{1} y(x)\,dx \text{ and } G(y) := \int_{-1}^{1} \sqrt{1 + y'(x)^2}\,dx \quad \text{for all } y \in \mathscr{Y},$$

and put

$$\mathscr{S} := \{ y \in \mathscr{Y} \mid G(y) = l \},$$

where $l > 2$. The augmented integrand $L : (x, y, z, \lambda) : [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $(J - \lambda G)$ is

$$L(x, y, z, \lambda) := y - \lambda \sqrt{1 + z^2} \quad \text{for all } (x, y, z, \lambda) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

We find that

$$L_{,2,2}(x, y, z, \lambda) = L_{,2,3}(x, y, z, \lambda) = 0 \quad \text{for all } (x, y, z, \lambda) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

and

$$L_{3,3}(x, y, z, \lambda) = -\frac{\lambda}{(1 + z^2)^{\frac{3}{2}}} \quad \text{for all } (x, y, z, \lambda) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

So the convexity of L with respect to the y and z arguments depends on the sign of the Lagrange multiplier λ. Thus the convexity of the functional $J - \lambda G$ depends on the sign of λ. We conclude that

(1) if $\lambda < 0$, then the functional $(J - \lambda G)$ is convex and a $y \in \mathscr{S}$ satisfying $(\text{E-L})_1$ for $(J - \lambda G)$ is a minimizer for J over the class $\mathscr{S}$;

(2) if $\lambda > 0$, then the functional $-(J - \lambda G)$ is convex and a $y \in \mathscr{S}$ satisfying $(\text{E-L})_1$ for $(J - \lambda G)$ is a maximizer for J over the class $\mathscr{S}$.

Chapter 6

Vector-Valued Minimizers

In this chapter we study minimization problems in which the unknown function y takes values in $\mathbb{R}^n$. As one would expect, solutions of such problems must satisfy appropriate analogues of the first and second Euler-Lagrange equations. It is interesting to note that although the first Euler-Lagrange equation will actually be a system of n (scalar) differential equations for the n unknown components of y, the second Euler-Lagrange will be a single (scalar) equation. Consequently, unless $n = 1$, the second equation will not provide enough information to completely determine solutions to minimization problems.

The treatment of boundary conditions will be a very important issue. In addition to cases where the values of the admissible functions y are either completely prescribed or completely free at each endpoint, we can consider situations where some components of y are prescribed at an endpoint, but the other components are left free. There are other important types of boundary conditions as we shall see below.

6.1 Basic Theory

Let $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N}$ and $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given. Assume that f is continuously differentiable. We will take $\mathfrak{X} := C^1([a, b]; \mathbb{R}^n)$ as our underlying linear space. The functions in $\mathfrak{X}$ map $[a, b]$ into $\mathbb{R}^n$. We consider the problem of minimizing functionals of the form

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx$$

over a subset of $\mathfrak{X}$.

As mentioned above, in addition to problems in which the values of y are either completely prescribed or completely free at each endpoint, there are many other important possibilities. For example, with $n = 3$, we may wish to consider boundary conditions such as

$$y_1(a) + 2y_2(a) - y_3(a) = 5 \text{ and } y_1(a) + 3y_3(a) = -4. \tag{6.1}$$

Notice that (6.1) can be viewed as a pair of linear equations in the three variables $y_1(a), y_2(a), y_3(a)$. In general, we should not prescribe more than n such equations at an endpoint because otherwise the system of linear equations would either be inconsistent or redundant. If fewer than n equations are prescribed, we can always make them into exactly n equations by augmenting the system with $0 = 0$ as many times as necessary.

Therefore, we consider boundary conditions of the form

$$\mathcal{A}y(a)^\mathsf{T} = \xi^\mathsf{T} \text{ and } \mathcal{B}y(b)(b)^\mathsf{T} = \eta^\mathsf{T}, \tag{6.2}$$

with $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n\times n}$ and $\xi, \eta \in \mathbb{R}^n$. Notice that if $y(a)$ is completely prescribed then an appropriate choice for $\mathcal{A}$ would be $\mathcal{A} = I$, where I is the $n \times n$ identity matrix and we would simply take ξ to be the value prescribed for $y(a)$. (Observe that any invertible matrix $\mathcal{A}$ would also work – but with a different choice of ξ.) For a problem with a completely free end at b, an appropriate choice for $\mathcal{B}$ and η would be $\mathcal{B} = 0, \eta = 0$. Then

$$\mathcal{B}y(b)^\mathsf{T} = 0^\mathsf{T} = \eta^\mathsf{T}$$

regardless of what $y(b)$ is.

As a final illustration, suppose that $n = 3$ and we want the boundary conditions expressed in (6.1) Then we can take

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \xi^\mathsf{T} = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix}.$$

In anticipation of computing the Gateaux variations of J, we extend the notation discussed in Section 2.1. We still denote the partial derivative of f with respect to its first argument by $f_{,1}(x, y, z)$. However, since the second and third arguments of f are vectors, it is appropriate to introduce *partial gradients*. We use $f_{,2}(x, y, z)$ to denote the partial gradient of f with respect to the components of the second argument. Similarly, $f_{,3}(x, y, z)$ denotes the partial gradient of f with respect to the components of the third argument. As an example, suppose that $n = 3$ and

$$f(x, y, z) = x^2y_1^2 + x^3y_2y_3 + z_1^2 + z_3^2 + y_1z_2 \quad \text{for all } (x, y, z) \in [a, b] \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Then

$$\begin{aligned} f_{,1}(x, y, z) &= 2xy_1^2 + 3x^2y_2y_3 && \text{for all } (x, y, z) \in [a, b] \times \mathbb{R}^3 \times \mathbb{R}^3, \\ f_{,2}(x, y, z) &= (2xy_1 + z_2, x^3y_3, x^3y_2) && \text{for all } (x, y, z) \in [a, b] \times \mathbb{R}^3 \times \mathbb{R}^3 \end{aligned}$$

and

$$f_{,3}(x, y, z) = (2z_1, y_1, 2z_3) \qquad \text{for all } (x, y, z) \in [a, b] \times \mathbb{R}^3 \times \mathbb{R}^3.$$

We now summarize our problem. Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n\times n}$ and $\xi, \eta \in \mathbb{R}^n$ be given and put

$$\mathscr{Y} := \left\{ y \in C^1([a,b];\mathbb{R}^n) \,:\, \mathcal{A}y(a)^{\mathsf{T}} = \xi^{\mathsf{T}} \text{ and } \ \mathcal{B}y(b)^{\mathsf{T}} = \eta^{\mathsf{T}} \right\}.$$

We assume that ξ^{T} is in the range of $\mathcal{A}$ and that η^{T} is in the range of $\mathcal{B}$ (otherwise $\mathscr{Y}$ is empty). Define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

We wish is to minimize J over $\mathscr{Y}$.

We will now derive the analogue of $(\text{E-L})_1$ for J. The space of admissible variations for each $y \in \mathscr{Y}$ is easily seen to be

$$\mathscr{V} := \left\{ v \in C^1([a,b];\mathbb{R}^n) \,:\, \mathcal{A}v(a)^{\mathsf{T}} = \mathcal{B}v(b)^{\mathsf{T}} = 0 \right\}.$$

Let $y \in \mathscr{Y}$ and $v \in \mathscr{V}$ be given. A straightforward computation utilizing the chain rule yields

$$\delta J(y; v) = \int_a^b \Big\{ f_{,2}(x, y(x), y'(x)) \cdot v(x) + f_{,3}(x, y(x), y'(x)) \cdot v'(x) \Big\}\, dx.$$

Suppose that $y_* \in \mathscr{Y}$ is a minimizer for J over $\mathscr{Y}$. Then y_* must satisfy $\delta J(y_*; v) = 0$ for every $v \in \mathscr{V}$. To derive the Euler-Lagrange equations for J and the natural boundary conditions we need to use some clever choices for $v \in \mathscr{V}$. Put

$$\mathscr{W} := \left\{ w \in C^1[a,b] \,:\, w(a) = w(b) = 0 \right\}.$$

Observe that the elements of $\mathscr{W}$ are scalar-valued functions. Let $w \in \mathscr{W}$ and $\mu \in \mathbb{R}^n$ be given. Notice that $v := w\mu \in \mathscr{V}$. With this choice for v, we have

$$\delta J(y_*; v) = \int_a^b \Big\{ f_{,2}(x, y_*(x), y_*'(x)) \cdot \mu w(x) + f_{,3}(x, y_*(x), y_*'(x)) \cdot \mu w'(x) \Big\}\, dx.$$

If y_* minimizes J over $\mathscr{Y}$, then the above expression is zero for each $w \in \mathscr{W}$ and $\mu \in \mathbb{R}^n$. Putting $F(x) = f_{,2}(x, y_*(x), y_*'(x)) \cdot \mu$ and $G(x) = f_{,3}(x, y_*(x), y_*'(x)) \cdot \mu$ for $x \in [a,b]$, Lemma 3.4 implies that $G \in C^1[a,b]$ and

$$\frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x)) \cdot \mu \right] = f_{,2}(x, y_*(x), y_*'(x)) \cdot \mu \quad \text{for all } x \in [a,b] \text{ and } \mu \in \mathbb{R}^n.$$

Since, G is continuously differentiable for every choice of $\mu \in \mathbb{R}^n$, we can conclude that the mapping $x \mapsto f_{,3}(x, y_*(x), y_*'(x))$ is continuously differentiable and consequently

$$\left\{ f_{,2}(x, y_*(x), y_*'(x)) - \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x)) \right] \right\} \cdot \mu = 0 \quad \text{for all } x \in [a,b] \text{ and } \mu \in \mathbb{R}^n.$$

By Lemma 2.1, we have

$$f_{,2}(x, y_*(x), y_*'(x)) = \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] \quad \text{for all } x \in [a,b]. \tag{E-L$_1$}$$

Now, we derive the associated natural boundary conditions. To analyze the endpoint $x = a$, we put $w(x) = \frac{x-b}{a-b}$ for $x \in [a,b]$, so that $w(a) = 1$ and $w(b) = 0$. Suppose that $\lambda \in \mathbb{R}^n$ satisfies

$$\mathcal{A}\lambda^\mathsf{T} = 0,$$

i.e. the vector λ^T is in the null-space of $\mathcal{A}$, and put $v = w\lambda$. Since $w(b) = 0$ and λ^T is in the null-space of $\mathcal{A}$, we find $v \in \mathscr{V}$. With this choice for v, we have

$$\delta J(y_*; v) = \int_a^b \left\{ f_{,2}(x, y_*(x), y_*'(x)) \cdot \lambda w(x) + f_{,3}(x, y_*(x), y_*'(x)) \cdot \lambda w'(x) \right\} dx.$$

Upon integrating the second term by parts and using the fact that y_* satisfies (E-L)$_1$ for J, we have

$$\delta J(y_*; v) = f_{,3}(x, y_*(x), y_*'(x)) \cdot \lambda w(x)\Big|_a^b = 0 \Rightarrow f_{,3}(a, y_*(a), y_*'(a)) \cdot \lambda = 0.$$

Thus y_* must satisfy the natural boundary conditions

$$f_{,3}(a, y_*(a), y_*'(a)) \cdot \lambda = 0 \quad \text{for all } \lambda \in \mathbb{R}^n \text{ satisfying } \mathcal{A}\lambda^\mathsf{T} = 0. \tag{NBC$_a$}$$

Similarly, the natural boundary conditions at $x = b$ are

$$f_{,3}(b, y_*(b), y_*'(b)) \cdot \lambda = 0 \quad \text{for all } \lambda \in \mathbb{R}^n \text{ satisfying } \mathcal{B}\lambda^\mathsf{T} = 0. \tag{NBC$_b$}$$

We have just proved the following

Theorem 6.1 *Let $a, b \in \mathbb{R}$ with $a < b$ be given, and let $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given. Assume that f is continuously differentiable. Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n \times n}$ and $\xi, \eta \in \mathbb{R}^n$ be such that ξ^T and η^T are in the range of $\mathcal{A}$ and $\mathcal{B}$, respectively. Put*

$$\mathscr{Y} := \left\{ y \in C^1([a,b]; \mathbb{R}^n) \,:\, \mathcal{A}y(a)^\mathsf{T} = \xi^\mathsf{T} \text{ and } \mathcal{B}y(b)^\mathsf{T} = \eta^\mathsf{T} \right\},$$

and define the functional $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Let $y_* \in \mathscr{Y}$ be given and assume that y_* minimizes (or maximizes) J over $\mathscr{Y}$. Then the mapping $x \mapsto f_{,3}(x, y_*(x), y_*'(x))$ is continuously differentiable on $[a,b]$ and y_* must satisfy the system of n differential equations

$$f_{,2}(x, y_*(x), y_*'(x)) = \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] \quad \text{for all } x \in [a,b], \tag{E-L$_1$}$$

and the natural boundary conditions

$$f_{,3}(a, y_*(a), y_*'(a)) \cdot \lambda = 0 \quad \text{for all } \lambda \in \mathbb{R}^n \textit{ satisfying } \mathcal{A}\lambda^{\mathsf{T}} = 0 \qquad (\text{NBC})_a$$

and

$$f_{,3}(b, y_*(b), y_*'(b)) \cdot \lambda = 0 \quad \text{for all } \lambda \in \mathbb{R}^n \textit{ satisfying } \mathcal{B}\lambda^{\mathsf{T}} = 0. \qquad (\text{NBC})_b$$

Let us make a few remarks.

Remark 6.1 *If for each $x \in [a, b]$ the function $f(x, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is convex, then J is convex and any $y_* \in \mathscr{Y}$ that satisfies (E-L)$_1$, (NBC)$_a$ and (NBC)$_b$ must be a minimizer for J over $\mathscr{Y}$.*

Remark 6.2 *Constraints of the form*

$$\int_a^b g(x, y(x), y'(x))\, dx = c,$$

where $g : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$ are given, can be handled by the method of Lagrange multipliers.

Remark 6.3 *Minimizers for J over $\mathscr{Y}$ must also satisfy an analogue of the second Euler-Lagrange equation. If J attains a minimum over $\mathscr{Y}$ at $y \in \mathscr{Y}$, then y must satisfy the integro-differential equation*

$$f(x, y(x), y'(x)) - y'(x) \cdot f_{,3}(x, y(x), y'(x)) = c + \int_a^x f_{,1}(t, y(t), y'(t))\, dt \quad (\text{E-L})_2$$
$$\text{for all } x \in [a, b],$$

for some $c \in \mathbb{R}$. Notice that (E-L)$_2$ is a single equation with n unknown functions. The fact that a minimizer satisfies the second Euler-Lagrange equation, though useful, does not provide as much information about the minimizer as the fact that it satisfies the first Euler-Lagrange equation.

6.2 Example 6.2

Let us look at an example with $n = 2$. Put $\mathfrak{X} = C^1([0, 1]; \mathbb{R}^2)$,

$$\mathscr{Y} := \left\{ y \in \mathscr{C}^1([0, 1]; \mathbb{R}^2) \,:\, y_1(0) = y_2(0) = 0 \text{ and } y_1(1) + y_2(1) = 1 \right\},$$

and define the functional $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^1 \left[y_1'(x)^2 + y_2'(x)^2 + 2y_1(x)y_2(x) \right] dx \quad \text{for all } y \in \mathscr{Y}.$$

We wish to minimize J over $\mathscr{Y}$.

First, we write the boundary conditions in the form (6.2). Since both components of each admissible y are prescribed to be 0 at $x=0$, we take $\mathcal{A} := I$ and $\xi := 0$. To write the boundary conditions at $x = 1$ in the form of (6.2), we may take

$$\mathcal{B} := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \eta^{\mathsf{T}} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

With these definitions, we see that

$$\mathscr{Y} = \left\{ y \in C^1([0,1];\mathbb{R}^2) \,:\, \mathcal{A}y(0)^{\mathsf{T}} = \xi^{\mathsf{T}} \text{ and } \mathcal{B}y(1)^{\mathsf{T}} = \eta^{\mathsf{T}} \right\}.$$

The integrand $f : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ for J is given

$$f(x,y,z) = z_1^2 + z_2^2 + 2y_1y_2 \quad \text{for all } (x,y,z) \in [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2.$$

Consequently, we have

$$f_{,2}(x,y,z) = (2y_2, 2y_1) \quad \text{for all } (x,y,z) \in [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2, \tag{6.3}$$

and

$$f_{,3}(x,y,z) = (2z_1, 2z_2) \quad \text{for all } (x,y,z) \in [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2. \tag{6.4}$$

At $x = 0$, there are no natural boundary conditions, because the null-space for $\mathcal{A} = I$ is the singleton $\{0\}$. This is what should be expected, since the value of $y \in \mathscr{Y}$ is completely prescribed at $x = 0$. To find the natural boundary conditions at $x = 1$, we need to determine the null-space of $\mathcal{B}$, i.e.we want to find those $\lambda \in \mathbb{R}^2$ such that

$$\mathcal{B}\lambda^{\mathsf{T}} = 0.$$

Using our definition for $\mathcal{B}$, we require

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0.$$

Thus, the null-space for $\mathcal{B}$ consists of those $\lambda \in \mathbb{R}^2$ satisfying

$$\lambda_2 = -\lambda_1.$$

Using this and (6.4), the natural boundary conditions at $x = 1$ are

$$(2y_1'(1), 2y_2'(1)) \cdot (\lambda_1, -\lambda_1) = 0, \quad \text{for all } \lambda_1 \in \mathbb{R}. \tag{NBC$_1$}$$

or equivalently

$$y_1'(1) = y_2'(1). \tag{NBC$_1$}$$

We now write down the first Euler-Lagrange equations. From (6.3) and (6.4), we have

$$\begin{aligned} & f_{,2}(x, y(x), y'(x)) = \frac{d}{dx}\left[f_{,3}(x, y(x), y'(x))\right] && \text{(E-L)}_1 \\ \Rightarrow & (2y_2(x), 2y_1(x)) = \frac{d}{dx}\left[(2y_1'(x), 2y_2'(x))\right] \\ \Rightarrow & (2y_2(x), 2y_1(x)) = (2y_1''(x), 2y_2''(x)) \\ \Rightarrow & \left\{ \begin{array}{l} y_1''(x) = y_2(x) \\ y_2''(x) = y_1(x) \end{array} \right. \quad \text{for all } x \in [0,1]. \end{aligned}$$

To find the general solution to (E-L)_1, observe that since $y \in \mathscr{Y} \subset C^1([0,1];\mathbb{R}^2)$, if y satisfies the above equations, then y must have continuous derivatives of all orders. Thus

$$\left\{ \begin{array}{l} y_1''(x) = y_2(x) \\ y_2''(x) = y_1(x) \end{array} \right. \Rightarrow y_1^{(4)}(x) = y_2''(x) \Rightarrow y_1^{(4)}(x) - y_1(x) = 0 \quad \text{for all } x \in [0,1].$$

The roots for the characteristic equation $r^4 - 1 = 0$ are $r = \pm i, \pm 1$. It follows that y_1 must have the form

$$y_1(x) = c_1 \cosh x + c_2 \sinh x + c_3 \cos x + c_4 \sin x. \tag{6.5}$$

For the second component, we have

$$\begin{aligned} & y_2(x) = y_1''(x) \\ \Rightarrow & y_2(x) = c_1 \cosh x + c_2 \sinh x - c_3 \cos x - c_4 \sin x. \end{aligned} \tag{6.6}$$

Imposing the boundary condition $y_1(0) = y_2(0) = 0$ leads to

$$\left\{ \begin{array}{l} y_1(0) = c_1 + c_3 = 0 \\ y_2(0) = c_1 - c_3 = 0 \end{array} \right. \Rightarrow c_1 = 0 \text{ and } c_3 = 0.$$

Substituting these values for c_1 and c_3 into (6.5) and (6.6) yields

$$y_1(x) = c_2 \sinh x + c_4 \sin x \quad \text{and} \quad y_2(x) = c_2 \sinh x - c_4 \sin x. \tag{6.7}$$

Let us now impose the boundary condition $y_1(1) + y_2(1) = 1$. From (6.7), we need

$$c_2 \sinh 1 + c_4 \sin 1 + c_2 \sinh 1 - c_4 \sin 1 = 1.$$

Thus

$$c_2 = \frac{1}{2\sinh 1},$$

and we now have that

$$y_1(x) = \frac{\sinh x}{2\sinh 1} + c_4 \sin x \quad \text{and} \quad y_2(x) = \frac{\sinh x}{2\sinh 1} - c_4 \sin x. \tag{6.8}$$

Lastly, we impose the natural boundary condition $y_1'(1) - y_2'(1) = 0$. Using (6.8), we have

$$y_1'(1) - y_2'(1) = \frac{\cosh 1}{2\sinh 1} + c_4 \cos 1 - \frac{\cosh 1}{2\sinh 1} + c_4 \cos 1 = 2c_4 \cos 1 = 0 \Rightarrow c_4 = 0.$$

Thus, the only possible minimizer for J over $\mathscr{Y}$ is

$$y(x) = (y_1(x), y_2(x)) = \left(\frac{\sinh x}{2\sinh 1}, \frac{\sinh x}{2\sinh 1}\right) \quad \text{for all } x \in [0,1].$$

6.3 Lagrangian Constraints

Let $a, b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}$ with $n \geq 2$ be given. In this section we consider problems in which the admissible functions $y : [a,b] \to \mathbb{R}^n$ are required to satisfy the constraint

$$g(y(x)) = 0 \quad \text{for all } x \in [a,b],$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a given smooth function satisfying

$$\nabla g(z) \neq 0 \quad \text{for all } z \in \mathbb{R}^n \text{ with } g(z) = 0. \tag{6.9}$$

In other words, we are limiting our attention to admissible functions whose ranges are subsets of the 0-level surface of g.

For simplicity, we shall only treat the case where y is completely prescribed at both endpoints. Let $\xi, \eta \in \mathbb{R}^n$ be given and put

$$\mathscr{Y} = \{y \in C^1([a,b];\mathbb{R}^n) \,:\, y(a) = \xi, y(b) = \eta \text{ and } g(y(x)) = 0 \quad \text{for all } x \in [a,b]\}.$$

Let $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given. For technical reasons, we assume that f is twice continuously differentiable and that g is twice continuously differentiable. We shall also assume twice continuous differentiability of our candidate for a minimizer.

We consider the functional $J : \mathscr{Y} \to \mathbb{R}$ defined by

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Theorem 6.2 *Let $y_* \in \mathscr{Y} \cap C^2([a,b];\mathbb{R}^n)$ be given and assume that y_* minimizes J on $\mathscr{Y}$. Then there exists a function $\lambda \in C[a,b]$ such that*

$$f_{,2}(x, y_*(x), y_*(x)) - \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] = \lambda(x)\nabla g(y_*(x)) \quad \text{for all } x \in [a,b].$$

We shall prove the theorem only in the special case when the constraining function g has the form

$$g(z) = z_n - \psi(z_1, z_2, \cdots, z_{n-1}) \quad \text{for all } z \in \mathbb{R}^n, \tag{6.10}$$

for some function $\psi \in C^2(\mathbb{R}^{n-1})$. The idea is that the first $n-1$ components of y will yield a minimizer for a standard variational problem. The first Euler-Lagrange equation for the "reduced" problem will yield the desired conclusion. (DETAILS TO BE FILLED IN)

Chapter 7

Second-Order Problems

7.1 C^4-Theory

We now investigate minimization problems for functionals that involve the second derivative of the unknown function. We start with a C^4-theory which can be obtained a bit more simply than the corresponding C^2-theory. Moreover, we consider only the case in which the unknown function is scalar-valued.

Given $a, b \in \mathbb{R}$ with $a < b$, we consider boundary conditions that are any combination of the following

(1) $y(a) = A_0$ with $A_0 \in \mathbb{R}$ given;

(2) $y'(a) = A_1$ with $A_1 \in \mathbb{R}$ given;

(3) $y(b) = B_0$ with $B_0 \in \mathbb{R}$ given;

(4) $y'(b) = B_1$ with $B_1 \in \mathbb{R}$ given.

There are 16 possible sets of boundary conditions (including problems with completely free ends). We want to treat all of the various possibilities in a unified way. For this purpose let $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \{0, 1\}$ be given and put

$$\mathscr{Y} := \left\{ y \in C^4[a,b] \,:\, \alpha_0 y(a) = \alpha_0 A_0, \alpha_1 y'(a) = \alpha_1 A_1, \beta_0 y(b) = \beta_0 B_0, \beta_1 y'(b) = \beta_1 B_1 \right\}.$$

If $\alpha_0 = 0$, then the boundary value for y at a is unprescribed. On the other hand, if $\alpha_0 = 1$, then the boundary value of y at a must be A_0. Similar remarks hold for the other boundary conditions.

We now state our problem. Let $f : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function with continuous third-order partial derivatives. Define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

We wish to minimize J over $\mathscr{Y}$.

For each $y \in \mathscr{Y}$, the class of admissible variations at y is easily seen to be

$$\mathscr{V} := \left\{ v \in C^4[a,b] \,:\, \alpha_0 v(a) = \alpha_1 v'(a) = \beta_0 v(b) = \beta_1 v'(b) = 0 \right\}.$$

Define

$$\mathscr{V}_0 := \left\{ v \in C^4[a,b] \,:\, v(a) = v'(a) = v(b) = v'(b) = 0 \right\},$$

and notice that $\mathscr{V}_0 \subset \mathscr{V}$.

In order to find an appropriate analogue of the first Euler-Lagrange equation, we need to compute the Gâteaux variation of J at $y \in \mathscr{Y}$ in the direction $v \in \mathscr{V}$. Let $y \in \mathscr{Y}$ and $v \in \mathscr{V}$ be given. For each $\varepsilon \in \mathbb{R}$, we have

$$J(y + \varepsilon v) = \int_a^b f(x, y(x) + \varepsilon v(x), y'(x) + \varepsilon v'(x), y''(x) + \varepsilon v''(x))\, dx.$$

Thus

$$\begin{aligned}\frac{d}{d\varepsilon}\left[J(y + \varepsilon v)\right] = \int_a^b \Big\{ & f_{,2}(x, y(x) + \varepsilon v(x), y'(x) + \varepsilon v'(x), y''(x) + \varepsilon v''(x)) v(x) \\ & + f_{,3}(x, y(x) + \varepsilon v(x), y'(x) + \varepsilon v'(x), y''(x) + \varepsilon v''(x)) v'(x) \\ & + f_{,4}(x, y(x) + \varepsilon v(x), y'(x) + \varepsilon v'(x), y''(x) + \varepsilon v''(x)) v''(x) \Big\}\, dx.\end{aligned}$$

Evaluating the above expression at $\varepsilon = 0$ yields

$$\begin{aligned}\delta J(y; v) = \int_a^b \Big\{ & f_{,2}(x, y(x), y'(x), y''(x)) v(x) + f_{,3}(x, y(x), y'(x), y''(x)) v'(x) \\ & + f_{,4}(x, y(x), y'(x), y''(x)) v''(x) \Big\}\, dx.\end{aligned}$$

Suppose that J attains a minimum over $\mathscr{Y}$ at y_*. Then, for each $v \in \mathscr{V}$, the Gâteaux variation $\delta J(y_*; v)$ is zero. Define $F, G, H : [a,b] \to \mathbb{R}$ by

$$F(x) := f_{,2}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b],$$

$$G(x) := f_{,3}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b]$$

and

$$H(x) := f_{,4}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a,b].$$

With these definitions, we may write

$$\delta J(y_*; v) = \int_a^b \Big\{ F(x) v(x) + G(x) v'(x) + H(x) v''(x) \Big\} = 0 \quad \text{for all } v \in \mathscr{V}.$$

Notice that

$$\int_a^b H(x)v''(x)\,dx = H(x)v'(x)\Big|_a^b - \int_a^b H'(x)v'(x)\,dx.$$

Consequently, if $\delta J(y_*;v) = 0$ for every $v \in \mathscr{V}$, then

$$H(x)v'(x)\Big|_a^b + \int_a^b \Big\{F(x)v(x) + [G(x) - H'(x)]\,v'(x)\Big\} = 0 \quad \text{for all } v \in \mathscr{V}. \tag{7.1}$$

Now

$$\int_a^b [G(x) - H'(x)]\,v'(x)\,dx = [G(x) - H'(x)]\,v(x)\Big|_a^b - \int_a^b [G'(x) - H''(x)]\,dx,$$

and substitution into (7.1) yields

$$\begin{aligned}\Big\{H(x)v'(x) + [G(x) - H'(x)]\,v(x)\Big\}\Big|_a^b \\ + \int_a^b \Big\{F(x) - G'(x) + H''(x)\Big\}v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}.\end{aligned} \tag{7.2}$$

Since (7.2) must hold for each $v \in \mathscr{V}$, it must hold for each $v \in \mathscr{V}_0 \subset \mathscr{V}$. Whence

$$\int_a^b \Big\{F(x) - G'(x) + H''(x)\Big\}v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.3}$$

We state without proof the following

Lemma 7.1 *Let $g \in C[a,b]$ be given. Assume that*

$$\int_a^b g(x)v(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0.$$

Then $g(x) = 0$ at each $x \in [a,b]$.

(The proof is very similar to the proof of Lemma 3.1. One can use $(x-\alpha)^6(x-\beta)^6$ instead of $(x-\alpha)^4(x-\beta)^4$ in the construction of $v_*(x)$.) Using Lemma 7.1, and equation(7.3) we find that

$$F(x) - G'(x) + H''(x) = 0 \quad \text{for all } x \in [a,b]. \tag{7.4}$$

We conclude that if y_* minimizes J over $\mathscr{Y}$, then y_* must satisfy

$$\begin{aligned} f_{,2}(x, y_*(x), y_*'(x), y_*''(x)) - \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x), y_*''(x))\right] \\ + \frac{d^2}{dx^2}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right] = 0 \quad \text{for all } x \in [a,b]. \end{aligned} \tag{E-L$_1$}$$

We turn now to the natural boundary conditions. Since we now know that y_* must satisfy (E-L)$_1$, the condition in (7.2) reduces to

$$H(b)v'(b) + [G(b) - H'(b)]v(b) - H(a)v'(a) - [G(a) - H'(a)]v(a) = 0 \quad \text{for all } v \in \mathscr{V}. \tag{7.5}$$

First suppose that $\alpha_1 = 0$. Then we may choose $v \in \mathscr{V}$ such that $v(a) = v(b) = v'(b) = 0$ and $v'(a) = 1$. Thus (7.5) implies that $H(a) = 0$ and consequently

$$f_{,4}(a, y_*(a), y_*'(a), y_*''(a)) = 0.$$

If $\alpha_1 = 1$, then there is no associated natural boundary condition. We may express both of these situations simultaneously with the single condition

$$(1 - \alpha_1) f_{,4}(a, y_*(a), y_*'(a), y_*''(a)) = 0. \tag{7.6}$$

Now, we suppose that $\alpha_0 = 0$. Then we may choose $v \in \mathscr{V}$ such that $v'(a) = v(b) = v'(b) = 0$ and $v(a) = 1$. Thus (7.5) implies that $G(a) - H'(a) = 0$ and consequently

$$f_{,3}(a, y_*(a), y_*'(a), y_*''(a)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\Big|_{x=a} = 0.$$

As before, if $\alpha_0 = 1$, then there is no associated natural boundary condition, and we may express both possibilities together as

$$(1 - \alpha_0)\left\{f_{,3}(a, y_*(a), y_*'(a), y_*''(a)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\Big|_{x=a}\right\} = 0. \tag{7.7}$$

We conclude that the natural boundary conditions at $x = a$ are

$$\begin{cases} (1 - \alpha_0)\left\{f_{,3}(a, y_*(a), y_*'(a), y_*''(a)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\big|_{x=a}\right\} = 0; \\ (1 - \alpha_1) f_{,4}(a, y_*(a), y_*'(a), y_*''(a)) = 0. \end{cases} \tag{NBC$_a$}$$

Similarly, the natural boundary conditions at $x = b$ are

$$\begin{cases} (1 - \beta_0)\left\{f_{,3}(b, y_*(b), y_*'(b), y_*''(b)) - \frac{d}{dx}\left[f_{,4}(x, y_*(x), y_*'(x), y_*''(x))\right]\big|_{x=b}\right\} = 0; \\ (1 - \beta_1) f_{,4}(b, y_*(b), y_*'(b), y_*''(b)) = 0. \end{cases} \tag{NBC$_b$}$$

7.2 Example 7.2

For this example, we put

$$\mathscr{Y} := \left\{ y \in C^4[0,1] \,:\, y(0) = 0, y'(0) = 1 \text{ and } y'(1) = -1 \right\},$$

so that $\alpha_0 = \alpha_1 = \beta_1 = 1$ and $\beta_0 = 0$. Define $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,y,z,w) := y^2 + z^2 + w^2 \quad \text{for all } (x,y,z,w) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

Let the functional $J : \mathscr{Y} \to \mathbb{R}$ be given by

$$J(y) := \int_0^1 f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Computing the partial derivatives for f, we have at each $(x,y,z,w) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ that

$$f_{,2}(x,y,z,w) = 2y; \;\; f_{,3}(x,y,z,w) = 2z; \;\text{ and } f_{,4}(x,y,z,w) = 2w.$$

The Euler-Lagrange equation for J is thus

$$2y(x) - \frac{d}{dx}\left[2y'(x)\right] + \frac{d^2}{dx^2}\left[2y''(x)\right] = 0 \quad \text{for all } x \in [0,1]. \tag{E-L$_1$}$$

To find solutions to (E-L)$_1$, we would need to solve the fourth order equation

$$y(x) - y''(x) + y^{(4)}(x) = 0 \quad \text{for all } x \in [0,1].$$

Let us look at the natural boundary condition at $x = 1$ (there is only one since $\alpha_0 = \alpha_1 = \beta_1 = 1$). We have

$$2y'(1) - \frac{d}{dx}\left[2y''(x)\right]\bigg|_{x=1} = 0. \tag{NBC$_1$}$$

Since $y \in \mathscr{Y}$ implies $y'(1) = -1$, the natural boundary condition is

$$y^{(3)}(1) = -1. \tag{NBC$_1$}$$

So if $y \in \mathscr{Y}$ minimizes J over $\mathscr{Y}$, then y satisfies

$$\begin{cases} y(x) - y''(x) + y^{(4)}(x) = 0 \\ y(0) = 0, y'(0) = 1, y'(1) = -1, y^{(3)}(1) = -1. \end{cases}$$

7.3 C^2-Theory

We now look at what happens if the admissible functions are assumed to be of class C^2 rather than class C^4. We can still obtain the Euler-Lagrange quations under the more natural assumption that the admissible functions are only *twice* continuously differentiable, but we will need to work a little bit harder.

Let $a, b, A_0, A_1, B_0, B_1 \in \mathbb{R}$ with $a < b$ and $f : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \{0, 1\}$ be given. We assume that f has continuous first-order partial derivatives.

We put

$$\mathscr{Y} := \left\{ y \in C^2[a, b] \,:\, \alpha_0 y(a) = \alpha_0 A_0, \alpha_1 y'(a) = \alpha_1 A_1, \beta_0 y(b) = \beta_0 B_0, \beta_1 y'(b) = \beta_1 B_1 \right\}.$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

For each $y \in \mathscr{Y}$, the class of admissible variations at y is easily seen to be

$$\mathscr{V} := \left\{ v \in C^2[a, b] \,:\, \alpha_0 v(a) = \alpha_1 v'(a) = \beta_0 v(b) = \beta_1 v'(b) = 0 \right\}.$$

Define

$$\mathscr{V}_0 := \left\{ v \in C^2[a, b] \,:\, v(a) = v'(a) = v(b) = v'(b) = 0 \right\},$$

and notice that $\mathscr{V}_0 \subset \mathscr{V}$

Let $y_* \in \mathscr{Y}$ be given and assume that y_* minimizes J over $\mathscr{Y}$. Then, for each $v \in \mathscr{V}$, the Gâteaux variation $\delta J(y_*; v)$ is zero. The previously derived formula for Gâteaux variations remains valid under the weaker smoothness assumptions of this section. As before, we define $F, G, H : [a, b] \to \mathbb{R}$ by

$$F(x) := f_{,2}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a, b],$$

$$G(x) := f_{,3}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a, b],$$

and

$$H(x) := f_{,4}(x, y_*(x), y_*'(x), y_*''(x)) \quad \text{for all } x \in [a, b].$$

With these definitions, we have

$$\delta J(y_*; v) = \int_a^b \left\{ F(x)v(x) + G(x)v'(x) + H(x)v''(x) \right\} = 0 \quad \text{for all } v \in \mathscr{V}.$$

In particular, since $\mathscr{V}_0 \subset \mathscr{V}$ we have

$$\delta J(y_*; v) = \int_a^b \left\{ F(x)v(x) + G(x)v'(x) + H(x)v''(x) \right\} = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.8}$$

The idea is to integrate by parts in such a way that we get

$$\int_a^b w(x)v''(x) = 0 \quad \text{for all } v \in \mathscr{V}_0$$

for some function w.

For this purpose, it will be convenient to define $\widetilde{F}, \widetilde{G}, \widehat{F}; [a,b] \to \mathbb{R}$ by

$$\widetilde{F}(x) := \int_a^x F(t)\,dt \qquad \widetilde{G}(x) := \int_a^x G(t)\,dt \quad \text{for all } x \in [a,b],$$

$$\widehat{F}(x) := \int_a^x \widetilde{F}(t)\,dt \quad \text{for all } x \in [a,b].$$

Observe that $\widetilde{F}, \widetilde{G} \in C^1[a,b]$ and $\widehat{F} \in C^2[a,b]$. Since $\widetilde{G}'(x) = G(x)$ for all $x \in [a,b]$ (and $v'(a) = v'(b) = 0$ for all $v \in \mathscr{V}_0$) we have

$$\int_a^b G(x)v'(x)\,dx = -\int_a^b \widetilde{G}(x)v''(x)\,dx \quad \text{for all } v \in \mathscr{V}_0.$$

In addition, we have $\widehat{F}''(x) = \widetilde{F}'(x) = F(x)$ for all $x \in [a,b]$ and consequently, after integrating by parts twice and using the boundary conditiona for v, we find that

$$\int_a^b F(x)v(x)\,dx = \int_a^b \widehat{F}(x)v''(x)\,dx \quad \text{for all } v \in \mathscr{V}_0.$$

Substituting these formulas into (7.8) we obtain

$$\int_a^b \Big\{\widehat{F}(x) - \widetilde{G}(x) + H(x)\Big\}v''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.9}$$

Lemma 7.2 *Let $w \in C[a,b]$ be given and assume that*

$$\int_a^b w(x)v''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0.$$

Then there exist $c_0, c_1 \in \mathbb{R}$ such that

$$w(x) = c_0 + c_1 x \quad \text{for all } x \in [a,b].$$

Proof. We shall find $d_0, d_1 \in \mathbb{R}$ such that $w(x) = d_0 + d_1(x-a)$ for all $x \in [a,b]$. Let $d_0, d_1 \in \mathbb{R}$ be given. Observe that

$$\int_a^b v''(x)\,dx = \int_a^b xv''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0,$$

and consequently

$$\int_a^b (w(x) - d_0 - d_1(x-a))v''(x)\,dx = 0 \quad \text{for all } v \in \mathscr{V}_0. \tag{7.10}$$

We want to construct $v_* \in \mathscr{V}_0$ such that

$$v_*''(x) = w(x) - d_0 - d_1(x-a) \quad \text{for all } x \in [a,b]. \tag{7.11}$$

In order for (7.11) to hold and also have $v_*'(a) = 0$ we need to have

$$\begin{aligned} v_*'(x) &= \int_a^x \{w(t) - d_0 - d_1 t\}\,dt \\ &= -d_0(x-a) - \tfrac{d_1}{2}(x-a)^2 + \int_a^x w(t)\,dt \quad \text{for all } x \in [a,b], \end{aligned} \tag{7.12}$$

and consequently, since we also want $v(a) = 0$ we need to have

$$\begin{aligned} v_*(x) &= \int_a^x \left\{ \int_a^t w(\tau)\,d\tau - d_0(t-a) - \tfrac{d_1}{2}(t-a)^2 \right\} dt \\ &= -\tfrac{d_0}{2}(x-a)^2 - \tfrac{d_1}{6}(x-a)^3 + \int_a^x \int_a^t w(\tau)\,d\tau \quad \text{for all } x \in [a,b]. \end{aligned} \tag{7.13}$$

Whether or not the expression in (7.13) gives a function in $\mathscr{V}_0$ depends on the values of d_0 and d_1. If v_* is defined by (7.13) then $v_*(a) = v_*'(a) = 0$ automatically. We need to choose d_0, d_1 (if possible) so that

$$v_*'(b) = -d_0(b-a) - \frac{d_1}{2}(b-a)^2 + \int_a^b w(x)\,dx = 0$$

and

$$v_*(b) = -\frac{d_0}{2}(b-a)^2 - \frac{d_1}{6}(b-a)^3 + \int_a^b \int_a^x w(t)\,dt\,dx = 0.$$

In other words, we should choose d_0 and d_1 (if possible) so that the linear system

$$\begin{pmatrix} (b-a) & \frac{(b-a)^2}{2} \\ \frac{(b-a)^2}{2} & \frac{(b-a)^3}{6} \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} = \begin{pmatrix} \int_a^b w(x)\,dx \\ \int_a^b \int_a^x w(t)\,dt\,dx \end{pmatrix} \tag{7.14}$$

is satisfied. The determinant of the coefficient matrix in (7.14) is

$$\frac{1}{6}(b-a)^3(b-a) - \left(\frac{1}{2}(b-a)^2\right)^2 = -\frac{1}{12}(b-a)^4 \neq 0,$$

and consequently there is exactly one choice for the pair (d_0, d_1) such that (7.14) is satisfied. If we make this choice for (d_0, d_1) and define v_* by (7.13) then $v_* \in \mathscr{V}_0$ and (7.11) holds so we have

$$\int_a^b (w(x) - d_0 - d_1(x-a))^2\, dx = 0.$$

We conclude that $w(x) - d_0 - d_1(x-a) = 0$ for all $x \in [a,b]$ and the proof of Lemma 7.2 is complete. □

Using Lemma 7.2 and equation (7.9) we may choose $c_0, c_1 \in \mathbb{R}$ such that

$$\widehat{F}(x) - \widetilde{G}(x) + H(x) = c_0 + c_1 x \quad \text{for all } x \in [a,b].$$

Since

$$H(x) = \widetilde{G}(x) - \widehat{F}(x) + c_0 + c_1 x \quad \text{for all } x \in [a,b],$$

and $\widehat{F}, \widetilde{G} \in C^1[a,b]$, we conclude that

$$H \in C^1[a,b] \text{ and } H'(x) = \widetilde{G}'(x) - \widehat{F}'(x) + c_1 = G(x) - \widetilde{F}(x) + c_1 \quad \text{for all } x \in [a,b].$$

Since

$$H'(x) - G(x) = c_1 - \widetilde{F}(x) \quad \text{for all } x \in [a,b]$$

and $\widetilde{F} \in C^1[a,b]$ with $\widetilde{F}' = F$ we conclude that

$$H' - G \in C^1[a,b] \text{ and } (H' - G)'(x) + F(x) = 0 \quad \text{for all } x \in [a,b].$$

(Notice that if $y_* \in C^4[a,b]$ and f has continuous third-order partial derivatives then the equation $(H' - G)' + F = 0$ reduces to $(\text{E-L})_1$ from Section 7.1.) With this additional information about F, G, H, we can return to the equation

$$\int_a^b \Big\{ F(x)v(x) + G(x)v'(x) + H(x)v''(x) \Big\}\, dx = 0 \quad \text{for all } v \in \mathscr{V}$$

and integrate by parts as we did in the C^4-theory to obtain

$$\Big\{ H(x)v'(x) + (G(x) - H'(x))v(x) \Big\}\Big|_a^b = 0 \quad \text{for all } v \in \mathscr{V}. \tag{7.15}$$

The same argument as in the C^4-theory can be used and we get exactly the same natural boundary conditions.

We summarize these results in a theorem.

Theorem 7.1 *Let $a, b, A_0, A_1, B_0, B_1 \in \mathbb{R}$ with $a < b$ and $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \{0,1\}$ be given. Put*

$$\mathscr{Y} := \{y \in C^2[a,b] \,:\, \alpha_0 y(a) = \alpha_0 A_0, \alpha_1 y'(a) = \alpha_1 A_1, \beta_0 y(b) = \beta_0 B_0, \beta_1 y'(b) = \beta_1 B_1\}.$$

Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with continuous first partial derivatives be given and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Lrt $y_* \in \mathscr{Y}$ be given and assume that y_* minimizes J on $\mathscr{Y}$. Define $F, G, H : [a, b] \to \mathbb{R}$ by $F(x) = f_{,2}(x, y_*(x), y_*'(x), y_*''(x))$, $G(x) = f_{,3}(x, y_*(x), y_*'(x), y_*''(x))$, and $H(x) = f_{,4}(x, y_*(x), y_*'(x), y_*''(x))$ for all $x \in [a, b]$. Then, $H \in C^1[a, b]$, $(H' - G) \in C^1[a, b]$ and

$$F(x) + (H' - G)'(x) = 0 \quad \text{for all } x \in [a, b],$$

$$(1 - \alpha_0)\{G(a) - H'(a)\} = 0 \quad (1 - \alpha_1)H(a) = 0,$$

$$(1 - \beta_0)\{G(b) - H'(b)\} = 0 \quad (1 - \beta_1)H(b) = 0.$$

7.4 Example 7.4

Set

$$\mathscr{Y} := \left\{ y \in C^2[0, 1] \,:\, y(0) = y(1) = 0 \text{ and } y'(1) = 1 \right\}.$$

We take $\alpha_0 = \beta_0 = \beta_1 = 1$ and $\alpha_1 = 0$. Define $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, y, z, w) := 4y + z^2 + w^2 \quad \text{for all } x \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

We want to minimize the functional $J : \mathscr{Y} \to \mathbb{R}$ given by

$$J(y) := \int_0^1 f(x, y(x), y'(x), y''(x))\, dx \quad \text{for all } y \in \mathscr{Y}$$

over the class $\mathscr{Y}$.

We find that

$$f_{,2}(x, y, z, w) = 4; \;\; f_{,3}(x, y, z, w) = 2z; \;\; \text{and } f_{,4}(x, y, z, w) = 2w$$

for all $(x, y, z, w) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. If $y \in \mathscr{Y}$ is a minimizer, then $2y'' \in C^1[0, 1]$ and $2y''' - 2y' \in C^1[0, 1]$, from which we conclude that $y \in C^4[0, 1]$. The Euler-Lagrange equation for J is

$$4 - \frac{d}{dx}[2y'(x)] + \frac{d^2}{dx^2}[2y''(x)] = 0 \tag{E-L$_1$}$$

The only natural boundary condition is

$$2y''(0) = 0. \tag{NBC$_0$}$$

So, we seek those functions satisfying

$$\begin{cases} y^{(4)}(x) - y''(x) = -2 \\ y(0) = y''(0) = y(1) = 0 \text{ and } y'(1) = 1. \end{cases}$$

We first find general solutions to the homogeneous equation

$$y^{(4)}(x) - y''(x) = 0 \tag{H}$$

The roots to the characteristic equation $r^4 - r^2 = 0$ are $r = \pm 1, 0$, with zero being a root with multiplicity 2. So the general solution to (H) is

$$y_H(x) = c_1 \cosh x + c_2 \sinh x + c_3 + c_4 x.$$

We see that a particular solution to

$$y^{(4)}(x) - y''(x) = 2 \tag{7.16}$$

is

$$y_P(x) = x^2.$$

By summing the general solution to (H) and the particular solution y_P, we have

$$y(x) = c_1 \cosh x + c_2 \sinh x + c_3 + c_4 x + x^2,$$

which is the general solution to (7.16).

We now impose the condition $y(0) = 0$. We have

$$y(0) = c_1 + c_3 = 0 \Rightarrow c_3 = -c_1.$$

So

$$y(x) = c_1 \cosh x + c_2 \sinh x - c_1 + c_4 x + x^2.$$

For the condition $y''(0) = 0$, we have

$$y''(0) = c_1 + 2 = 0 \Rightarrow c_1 = -2.$$

Thus

$$y(x) = -2 \cosh x + c_2 \sinh x + c_4 x + 2 + x^2.$$

The condition $y(1) = 0$ implies

$$y(1) = -2 \cosh 1 + c_2 \sinh 1 + c_4 + 3 = 0,$$

while the condition $y'(1) = 1$ implies

$$y'(1) = -2 \sinh 1 + c_2 \cosh 1 + c_4 + 4 = 0.$$

Thus

$$2(\cosh 1 - \sinh 1) + c_2(\cosh 1 - \sinh 1) = -1 \Rightarrow c_2 = \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1},$$

and

$$c_4 = 2\cosh 1 - 3 - c_2 \sinh 1 = 2\cosh 1 - 3 - \sinh 1 \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1}.$$

The only possible minimizer for J over $\mathscr{Y}$ is

$$\begin{aligned} y(x) &= -2\cosh x + \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1} \sinh x \\ &+ \left[2\cosh 1 - 3 - \sinh 1 \frac{2(\sinh 1 - \cosh 1) - 1}{\cosh 1 - \sinh 1}\right] x + 2 + x^2 \quad \text{for all } x \in [0,1]. \end{aligned}$$

7.5 Two Remarks Regarding Second-Order Problems

Remark 7.1 *If at each $x \in [a,b]$, the function $f(x,\cdot,\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is convex and $\mathscr{Y}$ is a convex set, then J is convex over $\mathscr{Y}$. In such a case, a solution to $(E\text{-}L)_1$ satisfying all boundary conditions is a minimizer for J over $\mathscr{Y}$.*

Remark 7.2 *Lagrange multiplier techniques can be used to handle constraints of the form*

$$\int_a^b g(x, y(x), y'(x), y''(x))\, dx = c,$$

where $g : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ are given.

Chapter 8

Sufficiency via Field Theory

Let $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N}$, $A, B \in \mathbb{R}^n$, and $f \in C^1([a,b] \times \mathbb{R}^n \times \mathbb{R}^n)$ be given. Put

$$\mathscr{Y} = \{y \in C^1([a,b];\mathbb{R}^n) \,:\, y(a) = A \text{ and } y(b) = B\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Recall that if $y_* \in \mathscr{Y}$ minimizes J over $\mathscr{Y}$ then y_* satisfies the first Euler-Lagrange system

$$f_{,2}(x, y_*(x), y_*'(x)) = \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] \quad \text{for all } x \in [a,b]. \qquad \text{(E-L)}_1$$

We also know that if $f(x,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is convex for every $x \in [a,b]$, then any function $y_* \in \mathscr{Y}$ that satisies (E-L)_1 does actually minimize J over $\mathscr{Y}$. In this chapter, we develop conditions that are weaker than convexity of $f(x,\cdot,\cdot)$ and still ensure that a solution y_* of (E-L)_1 does, in fact, minimize J over $\mathscr{Y}$. Roughly speaking these conditions will involve convexity of the mapping $z \mapsto f(x,y,z)$ for fixed values of $x \in [a,b]$, $y \in \mathbb{R}^n$, together with existence of an suitable collection of stationary functions.

8.1 The Method of Weierstrass

In his lectures of 1879, Weierstrass presented an approach for comparing the value of $J(\widetilde{y})$ for a given (but arbitrary) $\widetilde{y} \in \mathscr{Y}$ to the value of $J(y_*)$, where $y_* \in \mathscr{Y}$ satisfies (E-L)_1. This approach requires that for each $\xi \in (a,b]$ there is exactly one solution $\psi(\cdot,\xi)$ of (E-L)_1 on $[a,\xi]$ satisfying

$$\psi(a,\xi) = A \text{ and } \psi(\xi,\xi) = \widetilde{y}(\xi).$$

Assuming the existence of such a family of solutions of $(\text{E-L})_1$ (and some additional "regularity" of ψ), there is an extremely elegant formula which ensures that if f is convex in its third argument, then $J(y) \geq J(y_*)$. The "fly in the ointment" here is that it is generally quite difficult to verify that a family of solutions of $(\text{E-L})_1$ having the requisite properties exists (even in the scalar case $n = 1$).

8.1.1 Example 8.1.1

Before discussing the method of Weierstrass in generality, it seems instructive to look at an example where $f(x, \cdot, \cdot)$ fails to be convex, but Weierstrass' approach works and there are closed-form expressions for the important ingredients needed for this approach.

Here, we assume that $n = 1$. Let $L \in (0, \pi)$ be given, put

$$\mathscr{Y} = \{y \in C^1[0, L] \,:\, y(0) = y(L) = 0\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) = \int_0^L [y'(x)^2 - y(x)^2]\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Solutions of the first Euler-Lagrange equation for this integrand are automatically twice continuously differentiable and obey the differential equation

$$y'' + y = 0. \tag{8.1}$$

Solutions of (8.1) satisfying $y(0) = 0$ are given by

$$y(x) = k \sin x,$$

where k is a constant.

Let $\widetilde{y} \in \mathscr{Y}$ be given. For each $\xi \in (a, b]$, there is exactly one solution $\psi(\cdot, \xi)$ of (8.1) satisfying $\psi(0, \xi) = 0$ and $\psi(\xi, \xi) = \widetilde{y}(\xi)$. This solution is given by

$$\psi(x, \xi) = \frac{\widetilde{y}(\xi) \sin x}{\sin \xi} \quad \text{for all } x \in [0, \xi]. \tag{8.2}$$

Notice that a difficulty occurs if $L \geq \pi$ because the denominator in (8.2) vanishes when $\xi = \pi$. (For $L = \pi$, this difficulty can be overcome by a limiting arguemnt. However, if $L > \pi$ then J is unbounded below and consequently the difficulty cannot possibly be overcome in this case.)

The unique solution y_* of (8.2) on $[0, L]$ satisfying $y_*(0) = y_*(L) = 0$ is given by $y_*(x) = 0$ for all $x \in [0, L]$. Observe that $y_*(x) = \psi(L, x)$ for all $x \in [0, l]$.

Using L'Hôpital's rule, we find that

$$\lim_{\xi \to 0^+} \frac{\widetilde{y}(\xi)}{\sin \xi} = \widetilde{y}'(0).$$

It is therefore natural to put $\psi(0,0)=0$.

With ψ given by (8.2) we have

$$\psi_{,1}(x,\xi)=\frac{\widetilde{y}(\xi)\cos x}{\sin\xi} \tag{8.3}$$

and

$$\psi_{,2}(x,\xi)=\left(\frac{\widetilde{y}'(\xi)\sin\xi-\widetilde{y}(\xi)\cos\xi}{\sin^2\xi}\right)\sin x. \tag{8.4}$$

Remark 8.1 *For future reference, we note that*

(i) For fixed ξ the function $x\mapsto\psi_{,1}(x,\xi)$ is differentiable and we have

$$\psi_{,1,1}(x,\xi)=-\frac{\widetilde{y}(\xi)\sin x}{\sin\xi}.$$

(ii) For fixed x the function $\xi\mapsto\psi_{,1}(x,\xi)$ is differentiable and we have

$$\psi_{,1,2}(x,\xi)=\left(\frac{\widetilde{y}'(\xi)\sin-\widetilde{y}(\xi)\cos\xi}{\sin^2\xi}\right)\cos x.$$

(iii) For fixed ξ the function $x\mapsto\psi_{,2}(x,\xi)$ is differentiable and we have

$$\psi_{,2,1}(x,\xi)=\left(\frac{\widetilde{y}'(\xi)\sin\xi-\widetilde{y}'(\xi)\cos\xi}{\sin^2\xi}\right)\cos x.$$

(iv) The partial derivative $\psi_{,2,2}(x,\xi)$ does not exist unless $\widetilde{y}$ is twice differentiable at ξ. Nevertheless we have equality of mixed partials, namely $\psi_{,1,2}=\psi_{,2,1}$.

(v) There is a type of uniform convergence of $\psi(\cdot,\xi)$ to y_* and $\psi_{,1}(\cdot,\xi)$ to y_*' as $\xi\to L^-$. More precisely, for every $\epsilon>0$ there exists $\delta>0$ such that

$|\psi(x,\xi)-y_*(x)|<\epsilon$ *and* $|\psi_{,1}(x,\xi)-y_*'(x)|<\epsilon$ whenever $\xi>L-\delta$ and $0<x<\xi$.

Define $\sigma:[a,b]\to\mathbb{R}$ by

$$\sigma(\xi)=-\int_0^\xi[\psi_{,1}(x,\xi)^2-\psi(x,\xi)^2]\,dx-\int_\xi^L[\widetilde{y}'(x)^2-\widetilde{y}(x)^2]\,dx. \tag{8.5}$$

Ignorning the minus signs in (8.5), what we are doing in this formula is evaluating J along a path that rides the graph of a solution of the Euler-Lagrange equation from $(0,0)$ to $(\xi,\widetilde{y}(\xi))$ and then follows the graph of $\widetilde{y}$ from $(\xi,\widetilde{y}(\xi))$ to $(L,0)$.

Observe that

$$\sigma(L)=-\int_0^L[\psi_{,1}(x,L)^2-\psi(x,L)^2]\,dx=-\int_0^L[y_*(x)^2-y_*(x)^2]\,dx=-J(y_*)$$

and

$$\sigma(0) = -\int_0^L [\widetilde{y}'(x)^2 - \widetilde{y}(x)^2]\, dx = -J(\widetilde{y}).$$

It follows that

$$J(\widetilde{y}) - J(y_*) = \sigma(L) - \sigma(0).$$

If we can show that σ is continuous on $[0, L]$, differentiable on $(0, L)$, and $\sigma'(\xi) \geq 0$ for all $\xi \in (0, L)$ then we will know that $\sigma(L) \geq \sigma(0)$, which will ensure that $J(\widetilde{y}) \geq J(y_*)$.

Substitution of (8.2) and (8.3) into (8.5) gives

$$\sigma(\xi) = -\frac{\widetilde{y}(\xi)^2}{\sin^2 \xi} \int_0^\xi [\cos^2 x - \sin^2 x]\, dx - \int_\xi^L [\widetilde{y}'(x)^2 - \widetilde{y}(\xi)^2]\, dx. \tag{8.6}$$

The first integral can be computed explicitly using some trig identities. (We shall do so shortly.) The second integral in (8.6) cannot be computed explicitly; however, we can easily find the derivative (with respect to ξ) of this integral simply by applying the fundamental theorem of calculus.

To differentiate the term involving the first integral on the right-hand side of (8.6), let us put

$$F(\xi) = \frac{\widetilde{y}(\xi)^2}{\sin^2 \xi} \int_0^\xi [\cos^2 x - \sin^2 x]\, dx.$$

Using the double angle formulas

$$\cos^2 x - \sin^2 x = \cos 2x \text{ and } 2 \sin x \cos x = \sin 2x,$$

we see that

$$\int_0^\xi [\cos^2 x - \sin^2 x]\, dx = \sin \xi \cos \xi,$$

and we conclude that

$$F(\xi) = \widetilde{y}(\xi)^2 \cot \xi \quad \text{for all } \xi \in (0, L].$$

It follows that

$$\sigma(\xi) = -\widetilde{y}(\xi)^2 \cot \xi - \int_\xi^L [\widetilde{y}'(x)^2 - \widetilde{y}(x)^2]\, dx \quad \text{for all } \xi \in (0, L]. \tag{8.7}$$

Using L'Hôpital's rule, we find that

$$\lim_{\xi \to 0^+} \widetilde{y}(\xi)^2 \cot \xi = 0,$$

so that σ is continuous at 0. Continuity of σ at L is evident.

Employing the chain rule, product rule, fundamental theorem of calculus, and the identity $\csc^2 x - 1 = \cot^2 x$, we can differentiate the expression in (8.6) to obtain

$$\begin{aligned}
\sigma'(\xi) &= -2\widetilde{y}(\xi)\widetilde{y}'(\xi)\cot\xi + \widetilde{y}(\xi)^2\csc^2\xi + \widetilde{y}'(\xi)^2 - \widetilde{y}(\xi)^2 \\
&= \widetilde{y}(\xi)^2\cot^2\xi - 2\widetilde{y}(\xi)\widetilde{y}'(\xi)\cot\xi + \widetilde{y}'(x)^2 \\
&= (\widetilde{y}(\xi)\cot\xi - \widetilde{y}'(\xi))^2 \\
&\geq 0 \quad \text{for all } \xi \in (0, L).
\end{aligned}$$

We conclude that

$$J(\widetilde{y}) \geq J(y_*) = 0 \quad \text{for all } \widetilde{y} \in \mathscr{Y}.$$

In other words, the zero function minimizes J over $\mathscr{Y}$.

8.1.2 General Formula and the Weierstrass Excess Function

We now return to the general setting described at the beginning of the chapter. (The admissible functions take values in $\mathbb{R}^n$, where $n \in \mathbb{N}$ is a given natural number, not necessarily 1, and $a, b \in \mathbb{R}$, $A, B \in \mathbb{R}^n$, and $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are unspecified.)

Let $\widetilde{y}, y_* \in \mathscr{Y}$ be given and assume that y_* satisfies $(\text{E-L})_1$. We require that Assumption (WM) below holds. [(WM) stands for Weierstrass Method here.]

ASSUMPTION (WM): For every $\xi \in (a, b]$ there is exactly one solution $\psi(\cdot, \xi)$ of $(\text{E-L})_1$ satisfying

$$\psi(a, \xi) = A \text{ and } \psi(\xi, \xi) = \widetilde{y}(\xi). \tag{8.8}$$

We put

$$\psi(a, a) = A.$$

As in Section 8.1.1, we define $\sigma : [a, b] \to \mathbb{R}$ by

$$\sigma(\xi) = -\int_a^\xi f(x, \psi(x, \xi), \psi_{,1}(x, \xi))\,dx - \int_\xi^b f(x, \widetilde{y}(x), \widetilde{y}'(x))\,dx, \tag{8.9}$$

so that

$$\sigma(a) = -J(\widetilde{y}) \text{ and } \sigma(b) = -J(y_*).$$

We want to know that σ is continuous on $[a, b]$, differentiable on (a, b) and satisfies $\sigma'(\xi) \geq 0$ for all $\xi \in (a, b)$.

Continuity of σ at 0 will be ensured if we know that

$$\begin{aligned} \exists M \in \mathbb{R}, \delta > 0 \quad &\text{such that } |f(x, \psi(x,\xi), \psi_{,1}(x,\xi))| \leq M \\ &\text{for all } x, \xi \text{ with } a < x < \xi. \end{aligned} \tag{8.10}$$

(The condition in (8.10) guarantees that the first integral on the right-hand side of (8.9) remains tends to 0 as $\xi \to a^+$.) Continuity of σ at b will be ensured if we have

$$\begin{aligned} &\forall \epsilon > 0, \quad \exists \delta > 0 \text{ such that} \\ &\qquad |\psi(x,\xi) - y_*(x)| < \epsilon \text{ and} \\ &\qquad |\psi_{,1}(x,\xi) - y_*'(x)| < \epsilon \text{ whenever } \xi > L - \delta \text{ and } 0 < x < \xi. \end{aligned} \tag{8.11}$$

The second integral in (8.9) is automatically differentiable and its derivative is given by the fundamental theorem of calculus. In order to differentiate the first integral we shall make use of a result known as Leibniz' rule (for differentiating integrals). We record below a special case of this result that is sufficient for the present purpose.

Remark 8.2 (Special Case of Leibniz' Rule) *Let*

$$S := \{(x,\xi) \in \mathbb{R}^2 : a \leq x \leq \xi,\ a \leq \xi \leq b\} \tag{8.12}$$

and let $\rho : S \to \mathbb{R}$ be given.Assume that ρ and $\rho_{,2}$ are continuous on S and put

$$R(\xi) = \int_a^\xi \rho(x,\xi)\, dx \quad \text{for all } \xi \in [a,b].$$

Then R is differentiable and

$$R'(\xi) = \rho(\xi,\xi) + \int_a^\xi \rho_{,2}(x,\xi)\, dx \quad \text{for all } \xi \in [a,b].$$

In other words

$$\frac{d}{d\xi} \int_a^\xi \rho(x,\xi)\, dx = \rho(\xi,\xi) + \int_a^\xi \frac{\partial}{\partial \xi} \rho(x,\xi)\, dx.$$

Let us put

$$F(\xi) = \int_a^\xi f(x, \psi(x,\xi), \psi_{,1}(x,\xi))\, dx \tag{8.13}$$

and

$$G(\xi) = \int_{\xi}^{b} f(x, \widetilde{y}(x), \widetilde{y}'(x))\, dx. \tag{8.14}$$

We want to apply Remark 8.2 with

$$\rho(x, \xi) = f(x, \psi(x, \xi), \psi_{,1}(x, \xi)).$$

In order to do so, we need to make some assumptions concerning $\psi_{,2}$ and $\psi_{,1,2}$. (Looking ahead, in order to make use of the fact that $\psi(\cdot, \xi)$ satisfies the first Euler Lagrange equation, it will useful to make some assumptions concerning $\psi_{,1,1}$ and $\psi_{,2,1}$.) We assume that

$$\psi_{,2}, \psi_{1,1}, \psi_{1,2}, \psi_{2,1} \text{ exist and are continuous on } S, \tag{8.15}$$

and

$$\psi_{1,2}(x, \xi) = \psi_{2,1}(x, \xi) \quad \text{for all } (x, \xi) \in S, \tag{8.16}$$

where S is given by (8.12). [These are assumptions are at least plausible in veiew of Remark 8.1.]

Using Remark 8.2 and the fundamental theorem of calculus we find that

$$F'(\xi) = f(\xi, \psi(\xi, \xi), \psi_{,1}(\xi, \xi)) + \int_{a}^{\xi} \frac{\partial}{\partial \xi} f(x, \psi(x, \xi), \psi_{,1}(x, \xi))\, dx, \tag{8.17}$$

$$G'(\xi) = -f(\xi, \widetilde{y}(\xi), \widetilde{y}'(\xi)). \tag{8.18}$$

Since (for fixed ξ) $\psi(\cdot, \xi)$ satisfies $(\text{E-L})_1$ we know that

$$f_{,2}(x, \psi(x, \xi), \psi_{,1}(x, \xi) = \frac{d}{dx} f_{,3}(x, \psi(x, \xi), \psi_{,1}(x, \xi)) \quad \text{for all } x \in [a, \xi]. \tag{8.19}$$

Using the chain rule, (8.16), and (8.19) we see that

$$\begin{aligned}
\tfrac{\partial}{\partial \xi} f(x, \psi(x, \xi), \psi_{,1}(x, \xi)) \quad &= \quad f_{,2}(x, \psi(x, \xi), \psi_{,1}(x, \xi)) \cdot \psi_{,2}(x, \xi) \\
&\quad\; + f_{,3}(x, \psi(x, \xi), \psi_{,1}(x, \xi)) \cdot \psi_{,1,2}(x, \xi) \\
&= \quad \left(\tfrac{d}{dx} f_{,3}(x, \psi(x, \xi), \psi_{,1}(x, \xi))\right) \cdot \psi_{,2}(x, \xi) \\
&\quad\; + f_{,3}(x, \psi(x, \xi), \psi_{,1}(x, \xi)) \cdot \psi_{.2.1}(x, \xi) \\
&= \quad \tfrac{\partial}{\partial x}[f_{,3}(x, \psi(x, \xi), \psi_{,1}(x, \xi)) \cdot \psi_{,2}(x, \psi(x, \xi))].
\end{aligned}$$

It follows that

$$\begin{aligned}\int_a^\xi \frac{\partial}{\partial \xi} f(x,\psi(x,\xi),\psi_{,1}(x,\xi))\,dx &= \int_a^\xi \frac{\partial}{\partial x}[f_{,3}(x,\psi(x,\xi),\psi_{,1}(x,\xi))\cdot\psi_{,2}(x,\xi)]\,dx \\ &= f_{,3}(\xi,\psi(\xi,\xi),\psi_{,1}(\xi,\xi))\cdot\psi_{,2}(\xi,\xi) \\ &\quad -f_{,3}(a,\psi(a,\xi),\psi_{,1}(a,\xi))\cdot\psi_{,2}(a,\xi).\end{aligned} \tag{8.20}$$

Since $\psi(a,\xi)=A$ for all $\xi\in[a,b]$ we conclude that

$$\psi_{,2}(a,\xi)=0 \quad \text{for all } \xi\in[a,b]. \tag{8.21}$$

Furthermore, since $\psi(\xi,\xi)=\widetilde{y}(\xi)$ for all $\xi\in[a,b]$ we conclude that

$$\psi_{,1}(\xi,\xi)+\psi_{,2}(\xi,\xi)=\widetilde{y}'(\xi) \quad \text{for all } \xi\in[a,b]. \tag{8.22}$$

Substituting (8.21) and (8.22) into (8.20) (and recalling that $\psi(\xi,\xi)=\widetilde{y}(\xi)$) we find that

$$\int_a^\xi \frac{\partial}{\partial\xi} f(x,\psi(x,\xi),\psi_{,1}(x,\xi))\,dx = f_{,3}(\xi,\widetilde{y}(\xi),\psi_{,1}(\xi,\xi))\cdot(\widetilde{y}'(\xi)-\psi_{,1}(\xi,\xi)). \tag{8.23}$$

Combining (8.17) and (8.23) we find that

$$F'(\xi)=f(\xi,\widetilde{y}(\xi),\psi_{,1}(\xi,\xi)+f_{,3}(\xi,\widetilde{y}(\xi),\psi_{,1}(\xi,\xi))\cdot(\widetilde{y}'(\xi)-\psi_{,1}(\xi,\xi)). \tag{8.24}$$

Recalling that $\sigma(\xi)=-F(\xi)-G(\xi)$, and using (8.18), (8.24) we obtain

$$\begin{aligned}\sigma'(\xi)=f(\xi,\widetilde{y}(\xi),\widetilde{y}'(\xi)) \quad &-f(\xi,\widetilde{y}(\xi),\psi_{,1}(\xi,\xi)) \\ &-f_{,3}(\xi,\widetilde{y}(\xi),\psi_{,1}(\xi,\xi))\cdot(\widetilde{y}'(\xi)-\psi_{,1}(\xi,\xi)).\end{aligned} \tag{8.25}$$

The following defininition is quite natural in view of (8.25).

Definition 8.1 (Weierstrass Excess Function) *Given $f\in C^1([a,b]\times\mathbb{R}^n\times\mathbb{R}^n)$, the* Weierstrass excess function *associated with f is the function $\mathcal{E}:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ defined by*

$$\begin{aligned}\mathcal{E}(x,y,z,w)= \quad & f(x,y,w)-f(x,y,z)-f_{,3}(x,y,z)\cdot(w-z) \\ & \text{for all } x\in[a,b],\ y,z,w\in\mathbb{R}^n.\end{aligned} \tag{8.26}$$

The next is result an immediate consequence of Theorem 5.1 and Definition 8.1.

Proposition 8.1 *Let $f\in C^1([a,b]\times\mathbb{R}^n\times\mathbb{R}^n)$ be given. The following two statements are equivalent.*

(i) $\mathcal{E}(x,y,z.w)\geq 0$ *for all* $x\in[a,b]$, $y,z,w\in\mathbb{R}^n$.

(ii) For every $x \in [a,b]$, $y \in \mathbb{R}^n$ the mapping $z \to f(x,y,z)$ is convex on $\mathbb{R}^n$.

Putting all of the pieces together we see that

$$J(\widetilde{y}) - J(y_*) = \int_a^b \mathcal{E}(x, \widetilde{y}(x), \psi_{,1}(x,x), \widetilde{y}'(x))\, dx. \tag{8.27}$$

Consequently, if $y_* \in \mathscr{Y}$ satisfies (E-L$_1$ and there is a family ψ having all of the properties above and for every $x \in [a,b]$, $y \in \mathbb{R}^n$ the function $z \mapsto f(x,y,z)$ is convex on $\mathbb{R}^n$, then we can be sure that $J(\widetilde{y}) \geq J(y_*)$. Unfortunately, it seems to be extremely difficult to impose reasonable coonditions directly on f that will the existence of such a family ψ.

8.2 Fields

A powerful and beautiful alternative approach to obtaining a conclusion similar in spirit to Weierstrass' formula (8.27) is due to Hilbert and is based on the notion of *fields* of stationary functions.

8.2.1 Basic Definitions

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ be given and put

$$D = (\alpha, \beta) \times \mathbb{R}^n.$$

It is convenient to have the integrand f defined on an open set that "matches up" nicely with D. We therefore consider integrands $f : (\alpha,\beta) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Definition 8.2 *By a* field[1] *on D we mean a function $\Phi \in C^1(D;\mathbb{R}^n)$. The differential equation*

$$y'(x) = \Phi(x, y(x)) \tag{8.28}$$

is called the field equation *associated with Φ.*

Before proceeding, we pause to make a few comments about differential equations of the form (8.28). By a solution of (8.28) we mean a function $y_* \in C^1(I;\mathbb{R}^n)$, where $I \subset (\alpha,\beta)$ is an interval (with nonempty interior) such that $y_*'(x) = \Phi(x, y_*(x))$ for all $x \in I$. It is a standard result from the theory of differential equations that every solution $y_* : I \to \mathbb{R}^n$ of (8.28) can be extended to be defined on a maximal interval of existence $I_{max} \supset I$ in such a way that the extended function is a solution of (8.28); the maximal interval of existence is necessarily open. (To say that I_{max} is the maximal interval of existence of a solution means that the solution cannot be extended to an interval $I' \supsetneq I_{max}$ in such a way that it remains a solution.) Unless stated otherwise, when we speak of a solution of (8.28) it is understood that this solution has been extended to be defined on its maximal interval of existence.

[1] The term "field" is used with different meanings in other branches of mathematics.

Definition 8.3 *Solutions of the field equation* (8.28) *are called* field trajectories.

Remark 8.3 *Let Φ be a field on D. It is a consequence of the basic existence-uniqueness theory for first-order differential equations that through each point $(\xi, \eta) \in D$ passes exactly one field trajectory. In other words, given $(\xi, \eta) \in D$, the field equation (8.28) has exactly one (maximally extended) solution y_* satisfying $y_*(\xi) = \eta$.*

Fields having the property that all field trajectories satisfy the first Euler-Lagrange equation for a given $f : (\alpha, \beta) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are of special importance in the calculus of variations.

Definition 8.4 *Let $f \in C^1((\alpha, \beta) \times \mathbb{R}^n \times \mathbb{R}^n)$ be given and let Φ be a field on $D = (\alpha, \beta) \times \mathbb{R}^n$. We say that Φ is a* stationary field *for f on D provided that every field trajectory satisfies the first Euler-Lagrange equation for f on its maximal interval of existence; in other words, if y_* is a field trajectory with maximal interval of existence I_{max} then the mapping $x \mapsto f_{,3}(x, y_*(x), y_*'(x))$ is continuously differentiable on I_{max} and*

$$f_{,2}(x, y_*(x), y_*'(x)) = \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] \quad \text{for all } x \in I_{max} \qquad \text{(E-L)}_1$$

Around 1900, Hilbert realized the relevance to variational problems of certain line integrals of the form

$$\int_\Gamma P(x, y)\, dx + Q(x, y) \cdot dy$$

where $P : D \to \mathbb{R}$ and $Q : D \to \mathbb{R}^n$ are constructed in a special way from a variational integrand f and a field Φ.

Definition 8.5 *Given $f \in C^1((\alpha, \beta) \times \mathbb{R}^n \times \mathbb{R}^n)$ and a field Φ on $D = (\alpha, \beta) \times \mathbb{R}^n$, define $P : D \to \mathbb{R}$ and $Q :\to \mathbb{R}^n$ by*

$$Q(x, y) = f_{,3}(x, y, \Phi(x, y)) \quad \text{for all } (x, y) \in D, \tag{8.29}$$

$$P(x, y) = f(x, y, \Phi(x, y)) - \Phi(x, y) \cdot Q(x, y) \quad \text{for all } (x, y) \in D. \tag{8.30}$$

We say that Φ is an exact field *for f provided that there is a function $U \in C^1(D)$ such that*

$$P(x, y) = U_{,1}(x, y), \quad Q(x, y) = U_{,2}(x, y) \quad \text{for all } (x, y) \in D. \tag{8.31}$$

U is called a potential *for the pair (P, Q).*

We recall an important result from calculus concerning exact differentials. (Note that $(\alpha, \beta) \times \mathbb{R}^n$ is simply connected.)

Remark 8.4 *Assume that $f \in C^1((\alpha, \beta)) \times \mathbb{R}^n \times \mathbb{R}^n)$ and that the functions P and Q defined by (8.30) and (8.29) are continuously differentiable on $D = (\alpha, \beta) \times \mathbb{R}^n$. Then Φ is exact if and only if*

$$(Q_{,2}(x, y))^{\mathsf{T}} = Q_{,2}(x, y) \text{ and } Q_{,1}(x, y) = P_{,2}(x, y) \quad \text{for all } (x, y) \in D.$$

8.2.2 Some Formulas Involving Gradients

Before discussing the significance of exact fields, we make a few conventions and observations concerning gradients of vector-valued functions.

Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. We identify the gradient of F evaluated at $y \in \mathbb{R}^n$, denoted by $\nabla F(y)$, with an $n \times n$ matrix. The i^{th} row of $\nabla F(y)$ is simply the gradient of the the i^{th} component of F evaluated at y; in other words

$$(\nabla F(y))_{ij} = F_{i,j}(y).$$

Remark 8.5 *[A Chain Rule]: If $g : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable and we define $h : \mathbb{R}^n \to \mathbb{R}$ by*

$$h(y) = g(F(y)) \quad \text{for all } y \in \mathbb{R}^n,$$

then h is continuously differentiable and

$$\nabla h(y) = \nabla g(F(y))\nabla F(y). \tag{8.32}$$

(Notice that the right-hand side of (8.32) is a row vector (with n components) times an $n \times n$ matrix and is therefore a row vector.)

Remark 8.6 *[Another Chain Rule] Let $I \subset \mathbb{R}$ be an interval. If $G : \mathbb{R}^n \to \mathbb{R}^n$ and $y_* : I \to \mathbb{R}^n$ are continuously differentiable and*

$$H(x) = G(y_*(x)) \quad \text{for all } x \in [a, b]$$

then H is continuously differentiable and

$$H'(x) = y_*'(x)(\nabla G(y_*(x)))^{\mathsf{T}} \quad \text{for all } x \in [a, b]. \tag{8.33}$$

(Notice that the right-hand side of (8.33) is a row vector (with n components) times an $n \times n$ matrix and is therefore a row vector.)

Remark 8.7 *[A Product Rule] If $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable then*

$$\nabla(F \cdot G)(y) = F(y)\nabla G(y) + G(y)\nabla F(y) \quad \text{for all } y \in \mathbb{R}^n. \tag{8.34}$$

(Notice that each summand on the right-hand side of (8.34) is a row vector (with n components) times an $n \times n$ matrix and is therefore a row vector.)

8.2.3 Exact Fields are Stationary

We now show that every exact field is stationary.

Theorem 8.1 *Let $f \in C^1((\alpha, \beta) \times \mathbb{R}^n \times \mathbb{R}^n)$ be given and assume that $f_{,3}$ is continuously differentiable. Put $D = (\alpha, \beta) \times \mathbb{R}^n$ and define $P : D \to \mathbb{R}$, $Q : D \to \mathbb{R}^n$ by (8.29) and (8.30), and let Φ be a field on D. If Φ is exact then Φ is stationary.*

Proof. Assume that Φ is exact and let y_* be a field trajectory with maximal interval of existence I_{max}. We need to show that y_* satisfies (E-L)_1 on I_{max}.

Recall that

$$\begin{aligned} Q(x,y) &= f_{,3}(x,y,\Phi(x,y)), \\ P(x,y) &= f(x,y,\Phi(x,y)) - \Phi(x,y)\cdot Q(x,y) \quad \text{for all } (x,y)\in D. \end{aligned}$$

Since $f_{,3}$ and Φ are assumed to be continuously differentiable, we know that P and Q are continuously differentiable. Using Remarks 8.5 and 8.7 we find that

$$\begin{aligned} P_{,2}(x,y) &= f_{,2}(x,y,\Phi(x,y)) + f_{,3}(x,y,\Phi(x,y))\Phi_{,2}(x,y) \\ &\quad -\Phi(x,y)Q_{,2}(x,y) - Q(x,y)\Phi_{,2}(x,y) \quad \text{for all } (x,y)\in D. \end{aligned} \tag{8.35}$$

Using (8.29), (8.35), and the fact that $P_{,2} = Q_{,1}$, we see that

$$Q_{,1}(x,y) = f_{,2}(x,y,\Phi(x,y)) - \Phi(x,y)Q_{,2}(x,y) \quad \text{for all } (x,y)\in D,$$

so that

$$f_{,2}(x,y,\Phi(x,y)) = Q_{,1}(x,y) + \Phi(x,y)Q_{,2}(x,y) \quad \text{for all } (x,y)\in D. \tag{8.36}$$

Using (8.36) and the fact that y_* is a field trajectory, we obtain

$$f_{,2}(x,y_*(x),y_*'(x)) = Q_{,1}(x,y_*(x)) + y_*'(x)Q_{,2}(x,y_*(x)) \quad \text{for all } x\in I_{max}. \tag{8.37}$$

On the other hand, since

$$f_{,3}(x,y_*(x),y_*'(x)) = Q(x,y_*(x)) \quad \text{for all } x\in I_{max},$$

we have

$$\begin{aligned} \tfrac{d}{dx}f_{,3}(x,y_*(x)y_*'(x)) &= \tfrac{d}{dx}Q(x,y_*(x)) \\ &= Q_{,1}(x,y_*(x)) + y_*'(x)(Q_{,2}(x,y_*(x)))^{\mathsf{T}} \\ &\qquad \text{for all } x\in I_{max}. \end{aligned} \tag{8.38}$$

Observe that Remark 8.6 was used in obtaining (8.38). Using the fact that $Q_{,2}(x,y) = (Q_{,2}(x,y))^{\mathsf{T}}$ (see Remark 8.4), it follows from (8.37) and (8.38) that

$$f_{,2}(x,y_*(x),y_*'(x)) = \frac{d}{dx}f_{,3}(x,y_*(x),y_*'(x)) \quad \text{for all } x\in I_{max},$$

and the proof is complete. $\square$

The converse of Theorem 8.1 is not true in general, but *is* true when $n = 1$.

Theorem 8.2 *Assume that $n=1$. Let $f\in C^1((\alpha,\beta)\times\mathbb{R}\times\mathbb{R})$ be given and assume that $f_{,3}$ is continuously differentiable. Put $D=(\alpha,\beta)\times\mathbb{R}$ and define $P: D\to\mathbb{R}$, $Q: D\to\mathbb{R}^n$ by (8.29) and (8.30), and let Φ be a field on D. If Φ is stationary then Φ is exact.*

The proof of Theorem 8.2 is an exercise. (See Problem) It is important that the set D in Theorem 8.2 is simply connected.

8.3 Exact Fields and Hilbert's Invariant Integral

Assume that Φ is an exact field for f and that P and Q are given by (8.30) and (8.29). Then (using "traditional" notation) the line integral

$$\int_{\Gamma} P(x,y)\,dx + Q(x,y)\cdot dy \tag{8.39}$$

is path independent, meaning that given $(\xi_1,\eta_1),(\xi_2,\eta_2) \in D$ the value of the line integral in (8.39) is the same for all piecewise smooth directed paths Γ in D having initial point (ξ_1,η_1) and terminal point (ξ_2,η_2). (In fact, the value of the integral is $U(\xi_2,\eta_2) - U(\xi_1,\eta_1)$, where U is any potential for the pair (P,Q).) When P and Q are given by (8.30) and (8.29), the integral in (8.39) is called *Hilbert's Invariant Integral* for f associated with the field Φ. The idea will be to consider directed paths generated by functions in $\widetilde{y} \in C^1([a,b];\mathbb{R}^n)$ and exploit the fact that the value of the integral depends only on $(a,\widetilde{y}(a))$ and $(b,\widetilde{y}(b))$.

Formally, if $\widetilde{\Gamma}$ is the graph of a function $\widetilde{y} \in C^1([a,b];\mathbb{R}^n)$, oriented so that $(a,\widetilde{y}(a))$ is the initial point and $(b,\widetilde{y}(b))$ is the terminal point then

$$\int_{\widetilde{\Gamma}} P(x,y)\,dx + Q(x,y)\cdot dy = \int_a^b \left\{P(x,\widetilde{y}(x)) + Q(x,\widetilde{y}(x))\cdot \widetilde{y}'(x)\right\} dx. \tag{8.40}$$

Rather than attemp to give a precise meaning to line integrals of the type (8.39), we shall work directly with integrals of the type appearing on the right-hand side of (8.40) and express the values of these integrals in terms of a potential U for the pair (P,Q) and $(a,\widetilde{y}(a))$, $(b,\widetilde{y}(b))$.

Choose a potential $U \in C^1(D)$ for the pair (P,Q) and let $a,b \in \mathbb{R}$ be given with

$$\alpha < a < b < \beta.$$

Then, for each $\widetilde{y} \in C^1([a,b];\mathbb{R}^n)$, we have

$$\int_a^b \left\{P(x,\widetilde{y}(x)) + Q(x,\widetilde{y}(x))\cdot \widetilde{y}'(x)\right\} dx = U(b,\widetilde{y}(b)) - U(a,\widetilde{y}(a)). \tag{8.41}$$

If y_* is a field trajectory with maximal interval of existence $I_{max} \supset [a,b]$ then, since $y_*'(x) = \Phi(x,y_*(x))$ for all $x \in [a,b]$, we have

$$P(x,y_*(x)) + Q(x,y_*(x))\cdot y_*'(x) = f(x,y_*(x),y_*'(x)) \quad \text{for all } x \in [a,b]$$

and consequently

$$\int_a^b \left\{P(x,y_*(x)) + Q(x,y_*(x))\cdot y_*'(x)\right\} dx = \int_a^b f(x,y_*(x),y_*'(x))\,dx; \tag{8.42}$$

in other words, we have

$$\int_a^b f(x, y_*(x), y_*'(x))\, dx = U(b, y_*(b)) - U(a, y_*(a)). \tag{8.43}$$

Let $\widetilde{y} \in C^1([a,b];\mathbb{R}^n)$ be given. By adding and subtracting terms, and using the definitions of P and Q, we see that

$$\begin{aligned} f(x,\widetilde{y}(x),\widetilde{y}'(x)) &= f(x,\widetilde{y}(x),\widetilde{y}'(x)) - f(x,\widetilde{y}(x),\Phi(x,\widetilde{y}(x)) \\ &\quad +f(x,\widetilde{y}(x),\Phi(x,\widetilde{y}(x)) \\ &= f(x,\widetilde{y}(x),\widetilde{y}'(x)) - f(x,\widetilde{y}(x),\Phi(x,\widetilde{y}(x)) \\ &\quad +P(x,\widetilde{y}(x)) + Q(x,\widetilde{y}(x))\cdot\Phi(x,\widetilde{y}(x)) \\ &= f(x,\widetilde{y}(x),\widetilde{y}'(x)) - f(x,\widetilde{y}(x),\Phi(x.\widetilde{y}(x))) \\ &\quad +Q(x,\widetilde{y}'(x))\cdot(\Phi(x,\widetilde{y}(x)) - \widetilde{y}'(x)) \\ &\quad +\Big\{P(x,\widetilde{y}(x)) + Q(x,\widetilde{y}(x))\cdot\widetilde{y}'(x)\Big\} \\ &\quad \text{for all } x\in[a,b]. \end{aligned} \tag{8.44}$$

Using the fact that $Q(x,\widetilde{y}(x)) = f_{,3}(x,\widetilde{y}(x),\Phi(x,\widetilde{y}(x)))$, and recalling the definition of the Weierstrass excess function $\mathcal{E}$, we see that

$$\begin{aligned} f(x,\widetilde{y}(x)\widetilde{y}'(x)) &= \Big\{P(x,\widetilde{y}(x)) + Q(x,\widetilde{y}(x))\Big\} + \mathcal{E}(x,\widetilde{y}(x),\Phi(x,\widetilde{y}(x)),\widetilde{y}'(x)) \\ &\quad \text{for all } x\in[a,b]. \end{aligned} \tag{8.45}$$

Integrating (8.45) from a to b and using (8.41) we obtain

$$\begin{aligned} \int_a^b f(x,\widetilde{y}(x),\widetilde{y}'(x))\,dx &= U(b,\widetilde{y}(b)) - U(a,\widetilde{y}(a)) \\ &\quad + \int_a^b \mathcal{E}(x,\widetilde{y}(x),\Phi(x,\widetilde{y}(x)),\widetilde{y}'(x))\,dx \end{aligned} \tag{8.46}$$

Remark 8.8 *Notice that since $\mathcal{E}(x, y_*(x), y_*'(x), y_*'(x)) = 0$ for all $x \in [a,b]$, it follows from (8.46) that if y_* is a field trajectory then (8.43) holds.*

We summarize the result of the computations above in a theorem.

Theorem 8.3 *Let $f \in C^1((\alpha,\beta)\times\mathbb{R}^n\times\mathbb{R}^n)$ be given, define P and Q by (8.30) and (8.29) and assume that Φ is an exact field for f on $D = (\alpha,\beta)\times\mathbb{R}^n$. Let U be a potential for the pair (P,Q) and let $a,b\in\mathbb{R}$ be given with $\alpha < a < b < \beta$. Then (8.46) holds for every $\widetilde{y} \in C^1([a,b];\mathbb{R}^n)$.*

8.3.1 Fixed Endpoints

We now apply Theorem 8.3 to problems with fixed endpoints.

Theorem 8.4 *Let $a, b, \alpha, \beta \in \mathbb{R}$ with $\alpha < a < b < \beta$, $A, B \in \mathbb{R}^n$, and $f \in C^1([a,b] \times \mathbb{R}^n \times \mathbb{R}^n)$ be given. Put*

$$\mathscr{Y} = \{y \in C^1([a,b];\mathbb{R}^n) \,:\, y(a) = A \text{ and } y(b) = B\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

Assume that Φ is an exact field for f on $D = (\alpha, \beta) \times \mathbb{R}^n$. Let $y_*, \widetilde{y} \in \mathscr{Y}$ be given and assume that y_* is (the restriction to $[a,b]$ of) a field trajectory. Then we have

$$J(\widetilde{y}) - J(y_*) = \int_a^b \mathcal{E}(x, \widetilde{y}(x), \Phi(x, \widetilde{y}(x)), \widetilde{y}'(x))\, dx. \tag{8.47}$$

Proof. Choose a potential U for the pair (P, Q), where P and Q are given by (8.30) and (8.29). By (8.43) we have

$$J(y_*) = U(b, B) - U(a, A), \tag{8.48}$$

and by Theorem 8.3 we have

$$J(\widetilde{y}) = U(b, B) - U(a, A) + \int_a^b \mathcal{E}(x, \widetilde{y}(x), \Phi(x, \widetilde{y}(x)), \widetilde{y}'(x))\, dx. \tag{8.49}$$

Combining (8.48) and (8.49) we arrive at (8.47). $\square$

Remark 8.9 *There is, of course, a connection between (8.47) and the formula (8.14) from Section 8.1.2. Suppose that for each fixed ξ, $\psi(\cdot, \xi)$ is a stationary function and also that $\psi(x, x) = \widetilde{y}(x)$ for all x. If there were a field Φ such that $\psi_{,1}(x, \xi) = \Phi(x, \psi(x, \xi))$ then we would have $\psi_{,1}(x, x) = \Phi(x, \widetilde{y}(x))$ and, at least formally, (**??**) would look identical to (8.47). However, for the family ψ whose existence is postulated in Assumption (WM), we have $\psi(a, \xi) = A$ for all ξ (multiple stationary functions passing through the same point) and consequently (a, A) cannot be a point of smoothness of Φ because this would violate the condition that through each point in D passes exactly one field trajectory.*

Remark 8.10 *If the assumptions of Theorem 8.4 hold and the mapping $z \mapsto f(x, y, z)$ is convex for all $x \in [a,b]$, $y \in \mathbb{R}^n$ then y_* minimizes J on $\mathscr{Y}$.*

8.3.2 Example 8.3.2

Let $n = 1$, $\alpha = 0$, and $\beta = \pi$, so that $D = (0, \pi)\times\mathbb{R}$. Define $f : (0, \pi)\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$f(x, y, z) = z^2 - y^2 \quad \text{for all } x \in (0, \pi), y, z \in \mathbb{R}.$$

Solutions of the first Euler-Lagrange equation for this integrand are of class C^2 and satisfy

$$y'' + y = 0.$$

In order to construct a suitable field Φ for f on D we want a one-parameter family of solutions of the Euler-Lagrange equation having the property that exactly one solution from this family passes through each point in D. Such a family of solutions is given by

$$y(x) = k \sin x,$$

where k is a constant. To find a formula for Φ, we observe that since this family of solutions is characterized by

$$\frac{y(x)}{\sin x} \text{ is constant,}$$

we know that

$$\frac{d}{dx}\left(\frac{y(x)}{\sin x}\right) = 0,$$

which gives

$$\frac{y'(x)\sin x - y(x)\cos x}{\sin^2 x} = 0,$$

so that

$$y'(x) = y(x)\cot x \quad \text{for all } x \in (0, \pi). \tag{8.50}$$

We put

$$\Phi(x, y) = y \cot x \quad \text{for all } (x, y) \in D. \tag{8.51}$$

This field was constructed in such a way that we know that it is stationary for f. Therefore, since $n = 1$, we know it is also exact by Theorem 8.2. For purposes of illustration, we shall verify directly that this field is exact by explicitly exhibiting a potential U. Using (8.30) and (8.29), the formula for f, and the fact that $f_{,3}(x, y, z) = 2z$, we find that

$$Q(x, y) = 2y \cot x,$$

$$P(x, y) = y^2 \cot^2 x - y^2 - 2y^2 \cot^2 x = -y^2(1 + \cot^2 x) = -y^2 \csc^2 x.$$

We seek $U : D \to \mathbb{R}$ such that

$$U_{,1}(x, y) = -y^2 \csc^2 x, \quad U_{,2}(x, y) = 2y \cot x \quad \text{for all } (x, y) \in D. \tag{8.52}$$

It is easy to see that the function $U : D \to \mathbb{R}$ defined by

$$U(x, y) = y^2 \cot x \quad \text{for all } (x, y) \in D$$

satisfies 8.52.

Let $a, b \in (0, \pi)$ with $a < b$ be given and put

$$\mathscr{Y} = \{y \in C^1[a,b] \,:\, y(a) = 0,\ y(b) = 0\},$$

$$J(y) = \int_a^b [y'(x)^2 - y(x)^2]\,dx \quad \text{for all } y \in \mathscr{Y}.$$

Put $y_*(x) = 0$ for all $x \in (0, \pi)$, and observe that y_* is a field trajectory. The Weierstrass excess function for this integrand is given by

$$\begin{aligned} \mathcal{E}(x,y,z,w) &= w^2 - z^2 - (z^2 - y^2) - 2z(w - z) \\ &= w^2 - z^2 - 2zw + 2z^2 = w^2 - 2zw + z^2 = (w - z)^2. \end{aligned}$$

It follows that y_* minimizes J on $\mathscr{Y}$.

Remark 8.11 *(i) Using a limiting argument, it can be shown that*

$$luint0\pi[y'(x)^2 - y(x)^2]\,dx \geq 0$$

For all $y \in C^1[0,\pi]$ with $y(0) = y(\pi) = 0$.

(ii) Let $\mu \in \mathbb{R}$ be given and define $\Phi_\mu : (\mu, \pi + \mu) \times \mathbb{R}\mathbb{R}$ by

$$\Phi_\mu(x,y) = y\cot(x - \mu) \quad \text{for all } (x,y) \in (\mu, \pi + \mu).$$

Then Φ_μ is an exact field for the integrand $f(x,y,z) = z^2 - y^2$ on $(\mu, \pi + \mu) \times \mathbb{R}$.

8.3.3 Free Endpoints

8.3.4 Example

8.3.5 The Hamilton Jacobi Equation

There is an extremely important connection between exact fields and a partial differential equation know as the *Hamilton Jacobi equation.* We shall merely touch the surface here.

For fixed $(x,y) \in D$ and $q \in \mathbb{R}^n$, we consider the equation

$$f_{,3}(x,y,z) = q \tag{8.53}$$

and assume that there is exactly one solution z that can be expressed as a smooth function of (x,y,q). More specifically, we assume that there is a smooth function $g : (\alpha,\beta) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f_{,3}(x,y,g(x,y,q)) = q \quad \text{for all } x \in (\alpha,\beta), y, q \in \mathbb{R}^n. \tag{8.54}$$

We define the *Hamiltonian* $H : (\alpha, \beta) \times \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$ by

$$-H(x, y, q) = f(x, y, g(x, y, q)) - q \cdot g(x, y, q). \tag{8.55}$$

Given a field $Phi \in C^1(D; \mathbb{R}^n)$, we put

$$\textbf{??} \begin{cases} Q(x, y) = f_{,3}(x, y, \Phi(x, y)) \\ P(x, y) = f(x, y, \Phi(x, y)) - Q(x, y) \cdot \Phi(x, y) \quad \text{for all } (x, y) \in D. \end{cases} \tag{8.56}$$

Notice that

$$\Phi(x, y) = g(x, y, Q(x, y)) \quad \text{for all } (x, y) \in D. \tag{8.57}$$

Using the definitions of H and Q and (8.57), we find that

$$\begin{aligned} -H(x, y, Q(x, y)) &= f(x, y, g(x, y, Q(x, y))) - Q(x, y) \cdot \Phi(x, y) \\ &= P(x, y) \quad \text{for all } (x, y) \in D. \end{aligned} \tag{8.58}$$

Suppose now that Φ is exact. Then we may choose a potential $U \in C^1(D)$ such that

$$P(x, y) = U_{,1}(x, y), \quad Q(x, y) = U_{,2}(x, y) \quad \text{for all } (x, y) \in D. \tag{8.59}$$

Combining (8.58) and (8.59) we arrive at

$$W_{,1}(x, y) + H(x, y, W_{,2}(x, y)) = 0 \tag{8.60}$$

Equation (8.60) is called the *Hamilton-Jacobi equation*. It is a first-order non-linear partial differential equation.

If one has a solution U of (8.60) on an open set $\widetilde{D} \subset D$, it can be used to construct an exact field Φ on $\widetilde{D}$.

8.3.6 Example

Let $n = 1$ and put $D = (0, \pi) \times \mathbb{R}$ and define $f : (0, \pi) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, y, z) = z^2 - y^2 \quad \text{for all } x \in (0, \pi) \times \mathbb{R} \times \mathbb{R}.$$

For this example, equation (8.53) becomes

$$2z = q,$$

which is equivalent to $z = \frac{1}{2}q$, so we can put

$$g(x, y, q) = \frac{1}{2}q.$$

We find that

$$\begin{aligned} -H(x, y, q) &= -y^2 + \left(\tfrac{1}{2}q\right)^2 - q\left(\tfrac{1}{2}q\right) \\ &= -y^2 - \tfrac{1}{4}q^2. \end{aligned}$$

The Hamilton-Jacobi equation becomes

$$U_{,1}(x,y) + y^2 + \frac{1}{4}U_{,2}(x,y)^2 = 0. \tag{8.61}$$

In Example we found that the function $U : (0,\pi) \times \mathbb{R}$ defined by

$$U(x,y) = y^2 \cot x \quad \text{for all } (x,y) \in (0,\pi) \times \mathbb{R} \tag{8.62}$$

is the potential for an exact field. Let us verify that U satisfies the Hamilton Jacobi equation. We have

$$U_{,1}(x,y) = -y^2 \csc^2 x, \quad U_2(x,y) = 2y \cot x,$$

and consequently

$$\begin{aligned} U_{,1}(x,y) + y^2 + \tfrac{1}{4}U_{,2}(x,y)^2 &= -y^2 \csc^2 x + y^2 + \tfrac{1}{4}(4y^2)\cot^2 x \\ &= y^2(-\csc^2 x + 1 + \cot^2 x) \\ &= 0, \end{aligned}$$

which verifies that U is indeed a solution of the Hamilton Jacobi equation.

Chapter 9

Relaxing the Smoothness Requirements for the Admissible Functions

In this chapter, we consider the possibility of minimizing a functional of the form

$$J(y) = \int_a^b f(x, y(x), y'(x))\, dx,$$

over classes of admissible functions that need not have continuous derivatives. Recall that when we first minimized such functionals, we assumed that the admissible functions were of class C^2; we then developed a theory in which the admissible functions were only required to be of class C^1. It is natural to see if we can further relax the smoothness of the admissible functions. An important reason for this is that if we are trying to find the "true minimum" of J, then we need to consider the largest possible appropriate class of admissible functions. However, we shall always require our admissible functions y to satisfy the fundamental theorem of calculus in the sense that y' must be (Riemann) integrable (possibly in the improper sense) and

$$y(x) = y(a) + \int_a^x y'(t)\, dt \quad \text{for all } x \in [a, b]. \tag{9.1}$$

Indeed, if the admissible functions are not required to satisfy this important property, then the entire character of the minimization problem changes dramatically. There are important problems in which it is not appropriate to require the admissible functions to satisfy (9.1), but in such problems the functional J often includes additional terms to penalize jumps in y, and the treatment of such problems is beyond the scope of these notes. Notice that (9.1) implies that

y is continuous on $[a, b]$. For simplicity, we consider only problems in which the values of the admissible functions are prescribed at both endpoints.

9.1 Example 9.1

Suppose that we wish to minimize

$$J(y) := \int_0^1 \left(y'(x)^2 - 1\right)^2 \, dx$$

subject to $y(0) = y(1) = 0$. If we allow we allow admissible functions to have jump discontinuities in their derivatives, then it is clear that the minimizers for J will those functions y such that $y'(x) = \pm 1$ at each $x \in [0, 1]$ and the minimum value of J will be zero. If we require y to be in $C^1[0, 1]$, however, then we always find that $J(y) > 0$.

We can construct "approximate minimizers" for J using C^1 functions. More precisely, for each $n \in \mathbb{N}$, there exists $y_n \in C^1[0, 1]$ with $y_n(0) = y_n(1) = 0$ such that

$$0 < J(y_n) \leq \frac{1}{n}.$$

9.2 Example 9.2 (Heinricher & Mizel, 1986)

Suppose that we want to minimize

$$J(y) := \int_0^1 \left(y(x)^2 - x\right)^2 y'(x)^6 \, dx$$

subject to $y(0) = 0$ and $y(1) = 1$. Notice that $J(y)$ is always nonnegative. We see that if $y_*(x) = \sqrt{x}$ for all $x \in [0, 1]$, then $J(y_*) = 0$, but $y_* \notin C^1[0, 1]$, since its derivative $y_*'(x)$ blows up as x approaches zero.

One might guess that, as in the previous example, we can approximate the "true minimum for" J using some $y_n \in C^1[0, 1]$ such that $y_n(0) = 0$ and $y_n(1) = 1$ and letting $n \to \infty$. For this example, such an approximation is impossible. Heinricher and Mizel showed that if $y \in C^1[0, 1]$ and satisfies $y(0) = 0$ and $y(1) = 1$, then

$$J(y) \geq \frac{1}{6} \left(\frac{3}{5}\right)^6.$$

Consequently, if one restricts the class of admissible functions to those with continuous derivatives over $[0, 1]$, then the "true minimum" for J cannot even be approached. This pathology, originally discovered in 1926 by Lavrentiev, is known as Lavrentiev's phenomenon.

9.3 Continuous Piecewise $\mathbf{C^1}$ Functions

In this section, we introduce a class of functions that is appropriate for minimizing functionals such as the one in Example 9.1 Let $a, b \in \mathbb{R}$ with $a < b$ be given.

Definition 9.1 *Let $y : [a,b] \to \mathbb{R}$ be given. We define the* singular set for y *by*

$$S(y) := \{x \in (a,b) \mid y \text{ is not differentiable at } x\}.$$

Definition 9.2 *Let $y : [a,b] \to \mathbb{R}$ be given. We say that y* is a continuous piecewise C^1 function *provided the following conditions are satisfied:*

(1) $y \in C[a,b]$;

(2) $S(y)$ is a finite set;

(3) y' is continuous on the set $(a,b)\backslash S(y) := \{x \in (a,b) \mid x \notin S(y)\}$;

(4) for each $c \in S(y)$, we have $\lim_{x\to c^-} y'(x)$ and $\lim_{x\to c^+} y'(x)$ exist in $\mathbb{R}$;

(5) $\lim_{x\to a^+} y'(x)$ and $\lim_{x\to b^-} y'(x)$ exist in $\mathbb{R}$.

The class of all functions satisfying items (1)–(5) will be denoted by $\widehat{C}^1[a,b]$.

If a function y belongs to $\widehat{C}^1[a,b]$, then y is continuous, has no vertical tangents and has at most a finite number of "corners"; the corners in the graph of y are at the points $(c, y(c))$ with $c \in S(y)$. For notational convenience, if $y \in \widehat{C}^1[a,b]$ and $c \in (a,b)$, we write $y'(c^-)$ for $\lim_{x\to c^-} y'(x)$ and $y'(c^+)$ for $\lim_{x\to c^+} y'(x)$.

9.4 Minimizers with "Corners"

We first establish a version of the fundamental theorem of calculus and integration by parts for functions in $\widehat{C}[a,b]$. Before ststing the theorem we make a remark concerning integrability of derivatives of $\widehat{C}^1$-functions.

Remark 9.1 *Let $y \in \widehat{C}^1[a,b]$ be given. Then, for every $\gamma \in (a,b]$, the restriction of y to $[a,\gamma]$ belongs $\widehat{C}^1[a,\gamma]$ and y' is Riemann integrable on $[a,\gamma]$. The fact that there is a (possibly nonempty) finite subset of $[a,\gamma]$ on which y' is not well-defined does not cause any problems. If you feel more comfortable having an integrand that is everywhere defined on $[a,\gamma]$, you can assign values to $y'(c)$ for $c \in S(y)$ in any manner you wish. Changing the values of y' on a finite set does not alter the value of its integral.*

Theorem 9.1 (Fundamental Theorem of Calculus) *Let $y \in \widehat{C}^1[a,b]$ be given. Then, for every $\gamma \in (a,b]$ we have*

$$\int_a^\gamma y'(x)\,dx = y(\gamma) - y(a).$$

Proof. Let $\gamma \in (a, b]$ be given and put $P := \{a, \gamma\} \cup (S(y) \cap (a, \gamma))$. Notice that P is a non-empty and finite set with $a, \gamma \in P$ and $P \subset [a, \gamma]$. Therefore, we may write $P = \{x_0, x_1, \cdots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = \gamma$. Using the standard fundamental theorem of calculus, we find that

$$\begin{aligned}
\int_a^b y'(x)\,dx &= \int_{x_0}^{x_n} y'(x)\,dx \\
&= \int_{x_0}^{x_1} y'(x)\,dx + \int_{x_1}^{x_2} y'(x)\,dx + \cdot + \int_{x_{n-1}}^{x_n} y'(x)\,dx \\
&= y(x_1) - y(x_0) + y(x_2) - y(x_1) + \cdots + y(x_n) - y(x_{n-1}) \\
&= y(x_n) - y(x_0) = y(b) - y(a).
\end{aligned} \tag{9.2}$$

□

Theorem 9.2 (Integration by Parts) *Let $u, v \in \widehat{C}^1[a, b]$ be given. Then*

$$\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx.$$

Proof. Set $P := \{a, b\} \cup S(u) \cup S(v)$. We may write $P = \{x_0, x_1, \cdots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$. Using the standard integration by parts formula on each subinterval $[x_{i-1}, x_i]$, we find that

$$\begin{aligned}
\int_a^b u(x)v'(x)\,dx &= \int_{x_0}^{x_n} u(x)v'(x)\,dx \\
&= \int_{x_0}^{x_1} u(x)v'(x)\,dx + \int_{x_1}^{x_2} u(x)v'(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} u(x)v'(x)\,dx \\
&= u(x)v(x)\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u'(x)v(x)\,dx + u(x)v(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} u'(x)v(x)\,dx \\
&\qquad + \cdots + u(x)v(x)\Big|_{x_{n-1}}^{x_n} - \int_{x_{n-1}}^{x_n} u'(x)v(x)\,dx \\
&= u(x_1)v(x_1) - u(x_0)v(x_0) + u(x_2)v(x_2) - u(x_1)v(x_1) \\
&\qquad + \cdots + u(x_n)v(x_n) - u(x_{n-1})v(x_{n-1}) - \int_{x_0}^{x_n} u'(x)v(x)\,dx \\
&= u(x_n)v(x_n) - u(x_0)v(x_0) - \int_{x_0}^{x_n} u'(x)v(x)\,dx \\
&= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx.
\end{aligned}$$

□

We now pose our minimization problem. Let $A, B \in \mathbb{R}$ be given and put

$$\mathscr{Y} := \left\{ y \in \widehat{C}^1[a,b] \,:\, y(a) = A \text{ and } y(b) = B \right\}.$$

Let $f : [a,b] \times \mathbb{R} \times \mathbb{R}$ be a given function with continuous first-order partial derivatives and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\,dx \quad \text{for all } y \in \mathscr{Y}.$$

As usual, we wish to minimize J over $\mathscr{Y}$. (The fact that there might be some finite set on which y' is not defined does not cause any difficulties with defining the integral.)

For each $y \in \mathscr{Y}$, the class of admissible variations is easily seen to be

$$\mathscr{V} := \left\{ v \in \widehat{C}^1[a,b] \,:\, v(a) = v(b) = 0 \right\}.$$

Given $y \in \mathscr{Y}$ and $v \in \mathscr{V}$, it is straightforward to verify that the Gâteaux variation at y in the direction v exists and is given by the usual formula

$$\delta J(y; v) = \int_a^b \Big\{ f_{,2}(x, y(x), y'(x))v(x) + f_{,3}(x, y(x), y'(x))v'(x) \Big\}\, dx.$$

(Indeed, we can take a partition for [a,b] consisting of a, b together with all corner points for y and all corner points for v, break the integral for $J(y + \varepsilon v$ into a sum of integrals, differentiate each integral separately, and re-assemble the pieces.)

Suppose that J attains a minimum over $\mathscr{Y}$ at $y_* \in \mathscr{Y}$. Then $\delta J(y_*; v) = 0$ for every $v \in \mathscr{V}$. Define $G, H : [a, b] \to \mathbb{R}$ by

$$G(x) := f_{,3}(x, y_*(x), y_*'(x)) \text{ and } H(x) := \int_a^x f_{,2}(t, y_*(t), y_*'(t))\, dt \quad \text{for all } x \in [a, b].$$

(For $x \in S(y_*)$ we can take $G(x)$ to be any convenient real number.) Notice that $H \in \widehat{C}^1[a, b]$. Using our integration by parts theorem and our definition of $\mathscr{V}$, we may write

$$\begin{aligned}
\delta J(y_*; v) &= \int_a^b \Big\{ H'(x)v(x) + G(x)v'(x) \Big\}\, dx \\
&= H(b)v(b) - H(a)v(a) + \int_a^b \Big\{ G(x) - H(x) \Big\} v'(x)\, dx \\
&= \int_a^b \Big\{ G(x) - H(x) \Big\} v'(x)\, dx \quad \text{for all } v \in \mathscr{V}.
\end{aligned}$$

Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = G(x) - H(x) \quad \text{for all } x \in [a, b],$$

so that we have that

$$\int_a^b g(x)v'(x)\, dx = 0 \quad \text{for all } v \in \mathscr{V}.$$

We now proceed along the lines of the proof of the du Bois-Reymond Lemma (Lemma 3.3). Let k be a constant to be specified later. Since $\int_a^b g(x)v'(x)\, dx = 0$ for every $v \in \mathscr{V}$, we have

$$\int_a^b [g(x) - k]\, v'(x)\, dx = 0 \quad \text{for all } v \in \mathscr{V}.$$

We want to find a $v_* \in \mathscr{V}$ such that $v_*'(x) = g(x) - k$ at each $x \in (a,b)\backslash S(v_*)$. Since we want $v_*(a) = 0$ and $v_*'(x) = g(x) - k$ for all $x \in (a,b)\backslash S(v_*)$, we put

$$v_*(x) := \int_a^x [g(t) - k]\, dt \quad \text{for all } x \in [a,b].$$

Observe that with this definition we have $v_* \in \widehat{C}^1[a,b]$, $v_*(a) = 0$ and $v_*'(x) = g(x) - k$ for all $x \in (a,b)\backslash S(v_*)$. We need to choose $k \in \mathbb{R}$ so that $v_*(b) = 0$. From our definition for v_*, we see that

$$v_*(b) = \int_a^b [g(t) - k]\, dt = \int_a^b g(t)\, dt - (b-a)k,$$

so clearly, we should choose

$$k = \frac{1}{b-a} \int_a^b g(t)\, dt.$$

With this choice of k, we find that $v_* \in \mathscr{V}$ and

$$\int_a^b [g(x) - k] v_*'(x)\, dx = 0 \Rightarrow \int_a^b [g(x) - k]^2\, dx = 0.$$

We conclude that $g(x) = k$ for all $x \in [a,b]$ at which g is continuous. It follows that there exists a $k \in \mathbb{R}$ such that

$$f_{,3}(x, y_*(x), y_*'(x)) - \int_a^x f_{,2}(t, y_*(t), y_*'(t))\, dt = k \quad \text{for all } x \in (a,b)\backslash S(y_*). \tag{IE-L$_1$}$$

The above equation is often referred to as the integrated first Euler-Lagrange equation.

We have shown that if $y_* \in \mathscr{Y}$ is a minimizer for J over $\mathscr{Y}$, then y_* must satisfy (IE-L)$_1$ for all $x \in (a,b)\backslash S(y_*)$. The points $c \in S(y_*)$ are called corner points. For $x \notin S(y_*)$, we can differentiate both sides of (IE-L)$_1$ to obtain

$$f_{,2}(x, y_*(x), y_*'(x)) = \frac{d}{dx}\left[f_{,3}(x, y_*(x), y_*'(x))\right] \quad \text{for all } x \in (a,b)\backslash S(y). \tag{E-L$_1$}$$

Set $P := \{a,b\} \cup S(y_*) = \{x_0, x_1, \cdots, x_n\}$, with $x_0, x_1, \cdots < x_n$ and suppose that (E-L)$_1$ holds at each $x \in (a,b)\backslash S(y_*) = (x_0,x_1) \cup (x_1,x_2) \cup \cdots \cup (x_{n-1}, x_n)$. Then, there exist $k_1, k_2, \cdots, k_n \in \mathbb{R}$ such that

$$f_{,3}(x, y(x), y'(x)) - \int_a^x f_{,2}(t, y(t), y'(t))\, dt = k_i \quad \text{for all } x \in (x_{i-1}, x_i).$$

If the integrated first Euler-Lagrange equation holds, then all the k_i's must be the same.

We now state our first corner condition.

Theorem 9.3 (1^{st} Weierstrass-Erdmann Corner Condition) *Let $y \in \widehat{C}^1[a,b]$ be given, and suppose that y satisfies (IE-L)$_1$. Then, for each $c \in S(y)$ we find that*

$$f_{,3}(c, y(c), y'(c^+)) = f_{,3}(c, y(c), y'(c^-)).$$

Proof. Notice that $x \mapsto \int\limits_a^x f_{,2}(t, y(t), y'(t))\,dt$ is continuous on $[a,b]$. In particular

$$\lim_{x \to c^-} \int_a^x f_{,2}(t, y(t), y'(t))\,dt = \lim_{x \to c^+} \int_a^x f_{,2}(t, y(t), y'(t)) \quad \text{for all } c \in S(y).$$

Thus (IE-L)$_1$ implies

$$\lim_{x \to c^-} f_{,3}(x, y(x), y'(x)) = \lim_{x \to c^+} f_{,3}(x, y(x), y'(x)) \quad \text{for all } c \in S(y),$$

and therefore

$$f_{,3}(c, y(c), y'(c^-)) = f_{,3}(c, y(c), y'(c^+)) \quad \text{for all } c \in S(y),$$

since $f_{,3}$ is continuous and $y \in \widehat{C}^1[a,b]$. □

Corollary 9.1 *Suppose that $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has continuous second-order partial derivatives. Suppose further that $y_* \in \mathscr{Y}$ minimizes J over $\mathscr{Y}$. Let $c \in S(y_*)$ be given, and put $\alpha := y_*'(c^-)$ and $\beta := y_*'(c^+)$. Then, there exists a λ between α and β such that*

$$F'(\lambda) = f_{,3,3}(c, y_*(c), \lambda) = 0.$$

Proof. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(z) := f_{,3}(c, y_*(c), z) \quad \text{for all } z \in \mathbb{R}.$$

Since y_* minimizes J over $\mathscr{Y}$, it must satisfy (IE-L)$_1$. Therefore it satisfies the 1^{st} Weierstrass-Erdmann corner condition. That is

$$F(\alpha) = f_{,3}(c, y_*(c), \alpha) = f_{,3}(c, y_*(c), \beta) = F(\beta).$$

By Rolle's theorem, there exists a λ between α and β such that $F'(\lambda) = 0$. From the definition of F, we have

$$f_{,3,3}(c, y_*(c), \lambda) = 0.$$

□

Notice, in particular, that the corollary above tells us that when $f_{,3,3}(x,y,z) > 0$ at each $(x,y,z) \in [a,b]\times\mathbb{R}\times\mathbb{R}$, then no minimizer for J can have corners.

Before formulating our second corner condition, we state the following fact: if $y_* \in \mathscr{Y}$ minimizes J over $\mathscr{Y}$, then there exists a $K \in \mathbb{R}$ such that y_* satisfies

$$f(x, y_*(x), y_*'(x)) - y_*'(x) f_{,3}(x, y_*(x), y_*'(x)) = K + \int_a^x f_{,1}(t, y_*(t), y_*'(t))\, dt \quad \text{for all } x \in (a,b)\backslash S(y_*). \tag{E-L$_2$}$$

In other words, the second Euler-Lagrange Equation holds. Before proving this fact, we shall record an important consequence of (E-L)$_2$, namely another corner condition, and look at a couple of examples. The proof that (E-L)$_2$ holds will be given in Section 9.5

Theorem 9.4 (2$^{\text{nd}}$ Weierstrass-Erdmann Corner Condition) *Let $y \in \widehat{C}^1[a,b]$ be given, and assume that y satisfies (E-L)$_2$. Then, for each $c \in S(y)$, we have*

$$\begin{aligned} f(c, y(c), y'(c^-)) - y'(c^-) f_{,3}(c, y(c), y'(c^-)) \\ = f(c, y(c), y'(c^+)) - y'(c^+) f_{,3}(c, y(c), y'(c^+)). \end{aligned}$$

Proof. The result follows from the continuity of $x \mapsto \int_a^x f_{,1}(t, y(t), y'(t))\, dt$. $\square$

9.4.1 Example 9.4.1

Let

$$\mathscr{Y} := \left\{ y \in \widehat{C}^1[a,b] \mid y(a) = A \text{ and } y(b) = B \right\}.$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b y(x)^2 \left(y'(x) - 1\right)^2\, dx \quad \text{for all } \in \mathscr{Y}.$$

The integrand $f : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ for J is given by

$$f(x,y,z) := y^2(z-1)^2 \quad \text{for all } (x,y,z) \in [a,b]\times\mathbb{R}\times\mathbb{R}.$$

Thus for each $(x,y,z) \in [a,b]\times\mathbb{R}\times\mathbb{R}$, we find that

$$f_{,1}(x,y,z) = 0; \; f_{,2}(x,y,z) = 2y(z-1)^2;$$

$$f_{,3}(x,y,z) = 2y^2(z-1); \text{ and } f_{,3,3}(x,y,z) = 2y^2.$$

Notice that

$$f_{,3,3}(c, y(c), \lambda) = 0 \Rightarrow y(c) = 0,$$

so if c is a corner point for a minimizer y, then $y(c)$ must be zero.

Suppose that y minimizes J over $\mathscr{Y}$ and that $c \in S(y)$ is a corner point. Put $\alpha := y'(c^-)$ and $\beta := y'(c^+)$. The 1st corner condition becomes

$$2y(c)^2(\alpha - 1) = 2y(c)^2(\beta - 1).$$

This, however, provides no new information, since at a corner point $y(c) = 0$. The 2nd corner condition becomes

$$y(c)^2(\alpha - 1)^2 - 2\alpha y(c)^2(\alpha - 1) = y(c)^2(\beta - 1)^2 - 2\beta y(c)^2(\beta - 1).$$

Again since we know that $y(c) = 0$, we get no new information from the above equation.

Let us now look at $(\text{IE-L})_1$ and $(\text{E-L})_2$ for J. We have

$$2y(x)^2(y'(x) - 1) = k + \int_a^x 2y(t)(y'(t) - 1)^2\, dt \quad \text{for all } x \in (a, b)\backslash S(y) \quad (\text{IE-L})_1$$

and

$$y(x)^2(y'(x) - 1)^2 - 2y(x)^2 y'(x)(y'(x) - 1) = K \quad \text{for all } x \in (a, b)\backslash S(y), \quad (\text{E-L})_2$$

for some $k, K \in \mathbb{R}$. If there is a corner point c for y, then $y(c) = 0$, and this tells us that $K = 0$. So if y has a corner point, then $(\text{E-L})_2$ reduces to

$$y(x)^2(y'(x) - 1)^2 - 2y(x)^2 y'(x)(y'(x) - 1) = 0 \quad \text{for all } x \in (a, b)\backslash S(y).$$

For those $x \in (a, b)\backslash S(y)$ with $y(x) \neq 0$, the above equation implies that $y'(x) = \pm 1$. We conclude that if a minimizer y has at least one corner point then

(1) for those $x \in (a, b)\S(y)$ where $y(x) \neq 0$, we have $y'(x) = \pm 1$;

(2) if c is a corner point for y, then $y(c) = 0$.

Thus either $y'(x) = \pm 1$ or $y(x) = 0$. We note, for example, that if $a = 0, b = 1, A = 0, B = 2$, there can be no corner points for a minimizer, since minimizers with corners can only have graphs with slopes of 0 or ± 1. More generally, a necessary condition for there to be a minimizer with at least one corner point is

$$\frac{|B - A|}{|b - a|} < 1.$$

9.4.2 Example 9.4.2

Put

$$\mathscr{Y} := \left\{ y \in \widehat{C}^1[a, b] \mid y(a) = A \,:\, y(b) = B \right\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b \left[y'(x)^4 - y'(x)^2 \right] dx.$$

We wish to find out what happens at a corner point of a minimizer for J over $\mathscr{Y}$.

Remark 9.2 *We could use the fact that*

$$y'(x)^4 - y'(x)^2 = \left(y'(x)^2 - \frac{1}{2}\right)^2 - \frac{1}{4}$$

to analyze J directly and see that $\pm\frac{1}{\sqrt{2}}$ are possible for values of $y'(c^-), y'(c^+)$ at a corner point c. For purposes of illustartion we shall use the Weierstrass-Erdmann corner conditions and see how the conclusions compare with what we just observed by above completing the square in the integrand.

The integrand $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for J is given by

$$f(x,y,z) := z^4 - z^2 \quad \text{for all } (x,y,z) \in [a,b] \times \mathbb{R} \times \mathbb{R},$$

so that

$$f_{,1}(x,y,z) = 0;\ f_{,2}(x,y,z) = 0;\ f_{,3}(x,y,z) = 4z^3 - 2z$$

and

$$f_{,3,3}(x,y,z) = 12z^2 - 2 \quad \text{for all } (x,y,z) \in [a,b] \times \mathbb{R} \times \mathbb{R}.$$

We notice that if c is a corner point for a minimizer $y \in \mathscr{Y}$, then

$$f_{,3,3}(c, y(c), \lambda) = 12\lambda^2 - 2 = 0 \Rightarrow \lambda = \pm\sqrt{\frac{1}{6}}.$$

By Corollary 9.1, if c is a corner point for y, then at least one of $\pm\sqrt{\frac{1}{6}}$ must lie between $y'(c^-)$ and $y'(c^+)$.

Now let us look at the 1st and 2nd corner conditions. Suppose that c is a corner point for a minimizer y, and put $\alpha := y'(c^-)$ and $\beta := y'(c^+)$. For the 1st corner condition, we have

$$4\alpha^3 - 2\alpha = 4\beta^3 - 2\beta. \tag{9.3}$$

The 2nd corner condition gives

$$\alpha^4 - \alpha^2 - \alpha(4\alpha^3 - 2\alpha) = \beta^4 - \beta^2 - \beta(4\beta^3 - 2\beta). \tag{9.4}$$

Since we are supposing that c is a corner point for y, we seek solutions to (9.3) and (9.4) such that $\alpha \neq \beta$. Using this, we may rewrite (9.3) as

$$\begin{aligned} 4\alpha^3 - 4\beta^3 = 2\alpha - 2\beta \Rightarrow & 4(\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) = 2(\alpha - \beta) \\ \Rightarrow & \alpha^2 + \alpha\beta + \beta^2 = \frac{1}{2}. \end{aligned} \tag{9.5}$$

We also rewrite (9.4) as

$$\begin{aligned} 3\alpha^4 - \alpha^2 = 3\beta^4 - \beta^2 \Rightarrow & 3\alpha^4 - 3\beta^4 = \alpha^2 - \beta^2 \\ \Rightarrow & 3(\alpha - \beta)(\alpha + \beta)(\alpha^2 + \beta^2) = (\alpha - \beta)(\alpha + \beta) \\ \Rightarrow & 3(\alpha + \beta)(\alpha^2 + \beta^2) = \alpha + \beta. \end{aligned} \tag{9.6}$$

From this last expression, we deduce that either $\alpha = -\beta$ or $\alpha^2 + \beta^2 = \frac{1}{3}$.

First, we consider the case where $\alpha^2 + \beta^2 = \frac{1}{3}$. Substituting this back into (9.5) yields

$$\alpha\beta = \frac{1}{6} \Rightarrow \beta = \frac{1}{6\alpha}. \tag{9.7}$$

Now, using $\alpha^2 + \beta^2 = \frac{1}{3}$ gives us

$$\begin{aligned} \alpha^2 + \frac{1}{36\alpha^2} = \frac{1}{3} &\Rightarrow \alpha^4 + \frac{1}{36} = \frac{1}{3}\alpha^2 \\ &\Rightarrow \left(\alpha^2 - \frac{1}{6}\right)^2 = 0 \\ &\Rightarrow \alpha = \pm\sqrt{\frac{1}{6}}. \end{aligned}$$

If $\alpha = \sqrt{\frac{1}{6}}$, then (9.7) implies that $\beta = \sqrt{\frac{1}{6}} = \alpha$; also, if $\alpha = -\sqrt{\frac{1}{6}}$, we find that $\beta = -\sqrt{\frac{1}{6}} = \alpha$. In either case, our solutions violate the condition that $\alpha \neq \beta$, so these are not valid solutions.

We now turn to the cases where $\alpha = -\beta$. Substituting this back into (9.5) yields

$$\beta^2 = \frac{1}{2} \Rightarrow \beta = \pm\sqrt{\frac{1}{2}}.$$

If $\alpha = \sqrt{\frac{1}{2}}$, then $\beta = -\sqrt{\frac{1}{2}}$; and if $\alpha = -\sqrt{\frac{1}{2}}$, then $\beta = \sqrt{\frac{1}{2}}$.

In conclusion, if c is a corner point for a minimizer $y \in \mathscr{Y}$ for J, then

$$y'(c^-) = \pm\sqrt{\frac{1}{2}} \text{ and } y'(c^+) = -y'(c^-).$$

These conclusions are completely consistent with Remark 9.2.

9.5 The Second Euler Lagrange Equation

(TO BE FILLED IN)

9.6 The Cauchy-Schwarz Inequality

We now wish to prove our statement regarding the Heinricher & Mizel Example. Namely, we will show that the "true minimum" (namely 0) for the functional J defined in Section 9.2 cannot be approached using functions in $C^1[0,1]$ satisfying $y(0) = 0$ and $y(1) = 1$. To do this, we need to establish an inequality for integrals. The proof of this inequality makes use of a preliminary result (Hölder's inequality) which we develop in this and the next section.

Hölder's inequality is a generalization of the Cauchy-Schwarz inequality. The Cauchy-Schwarz inequality can be obtained easily from the following elementary algebraic inequality.

Proposition 9.1 *Let $\alpha, \beta \in \mathbb{R}$ be given. We have*

$$|\alpha\beta| \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2.$$

Proof. Observe that

$$\begin{aligned}(\alpha+\beta)^2 \geq 0 \Rightarrow & \alpha^2 + 2\alpha\beta + \beta^2 \geq 0 \\ \Rightarrow & -\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2.\end{aligned}$$

Similarly

$$\begin{aligned}(\alpha-\beta)^2 \geq 0 \Rightarrow & \alpha^2 - 2\alpha\beta + \beta^2 \geq 0 \\ \Rightarrow & \alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2.\end{aligned}$$

It follows that

$$|\alpha\beta| \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2.$$

□

Using this simple inequality, we can prove a very useful inequality for integrals.

Definition 9.3 *For functions $f \in C[a,b]$, put*

$$\|f\|_2 := \left(\int_a^b f(x)^2 \, dx \right)^{\frac{1}{2}}$$

Notice that for every $c \in \mathbb{R}$, we have $\|cf\|_2 = |c| \cdot \|f\|_2$. Furthermore for $f \in C[a,b]$ we have $\|f\|_2 = 0$ if and only if $f(x) = 0$ for all $x \in [a,b]$.

We now state the Cauchy-Schwarz inequality for integrals.

Theorem 9.5 *Cauchy-Schwarz Inequality Let $f, g \in C[a,b]$ be given. Then*

$$\int_a^b |f(x)g(x)| \, dx \leq \|f\|_2 \|g\|_2.$$

Proof. If either $\|f\|_2 = 0$ or $\|g\|_2 = 0$, then the inequality is trivially satisfied because both sides vanish. We therefore may assume that $\|f\|_2 \neq 0$ and $\|g\|_2 \neq 0$. Let us define $F, G \in C[a,b]$ by

$$F(x) := \frac{f(x)}{\|f\|_2} \text{ and } G(x) := \frac{g(x)}{\|g\|_2} \quad \text{for all } x \in [a,b].$$

Using Proposition 9.1 and our definitions of F and G, we have

$$\begin{aligned}\int_a^b |F(x)G(x)|\,dx \leq& \frac{1}{2}\int_a^b |F(x)|^2\,dx + \frac{1}{2}\int_a^b |G(x)|^2\,dx \\ =& \frac{1}{2\|f\|_2^2}\int_a^b |f(x)|^2\,dx + \frac{1}{2\|g\|_2^2}\int_a^b |g(x)|^2\,dx \\ =& \frac{1}{2} + \frac{1}{2} = 1.\end{aligned}$$

Thus

$$\begin{aligned}\int_a^b |F(x)G(x)|\,dx \leq 1 \Rightarrow& \frac{1}{\|f\|_2\|g\|_2}\int_a^b |f(x)g(x)\,dx \leq 1 \\ \Rightarrow& \int_a^b |f(x)g(x)|\,dx \leq \|f\|_2\|g\|_2\end{aligned}$$

$\square$

Remark 9.3 (Triangle Inequality) *Let $f, g \in C[a,b]$ be given. Then we have*

$$\begin{aligned}\|f+g\|_2^2 &= \int_a^b \left\{ f(x)^2 + 2f(x)g(x) + g(x)^2 \right\} dx \\ &= \|f\|_2^2 + \|g\|_2^2 + 2\int_a^b f(x)g(x)\,dx \leq \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2\|g\|_2.\end{aligned}$$

Since $\|f\|_2, \|g\|_2 \geq 0$ we conclude that

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2 \quad \text{for all } f, g \in C[a,b].$$

It follows that $\|\cdot\|_2$ is a norm on $C[a,b]$.

9.7 Hölder's Inequality

In this section, we will prove Hölder's inequality, which is a generalization of Theorem 9.5. In the next section, we use Hölder's inequality to prove a special case of Jensen's inequality. Once we have Jensen's inequality in hand, we will prove our statement regarding the Heinricher & Mizel example.

First, we give a geometrical explanation of Proposition 9.1. Suppose that $\alpha, \beta > 0$. The proposition states that

$$\alpha\beta \leq \frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2. \tag{9.8}$$

The left-hand side of (9.8) is the area of a rectangle with sides of length α and β. The right hand side of (9.8) is sum of the area of two triangles: one with base and height α, the other with base and height β. From Figure 9.1, we see

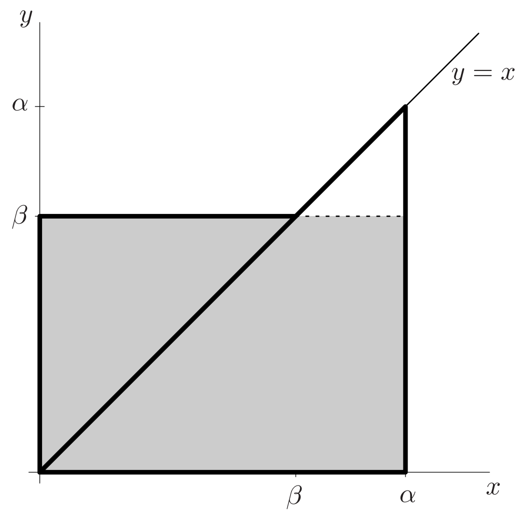

Figure 9.1: A geometrical interpretation of Proposition 9.1

that the area of the rectangle is indeed at least as small as the sum of the areas for the triangles. (The figure was drawn for the case $\alpha > \beta$. If $\beta > \alpha$, we can simply interchange α and β.) Observe also from the figure that equality holds in (9.8) if and only if $\alpha = \beta$.

For the Cauchy-Schwarz inequality, the areas of the two triangles are separated by the line $y = x$. We will generalize (9.8) by using regions separated by the curve $y = x^\gamma$, with $\gamma > 0$, rather than by a straight line (see Figure 9.2). From the figure, it is apparent that the area of the rectangle with sides of length α and β is not larger than the sum of the areas A_1 and A_2. That is

$$\alpha\beta \leq A_1 + A_2. \tag{9.9}$$

(Figure 9.2 was drawn for the case $\alpha > \beta$. You should convince yourself by drawing an appropriate figure that (9.9) remains valid when $\beta \geq \alpha$.) We find that

$$A_1 = \int_0^\alpha x^\gamma \, dx = \frac{1}{\gamma+1}\alpha^{\gamma+1}$$

and

$$A_2 = \int_0^\beta y^{\frac{1}{\gamma}} \, dy = \frac{\gamma}{\gamma+1}\beta^{\frac{\gamma+1}{\gamma}}.$$

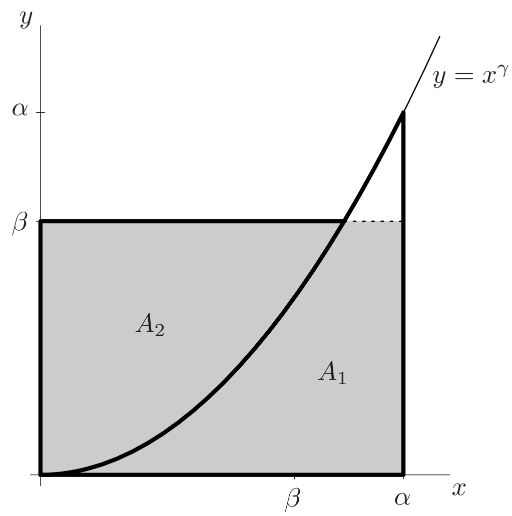

Figure 9.2: A geometrical interpretation of Young's inequality

Thus

$$\alpha\beta \le \frac{1}{\gamma+1}\alpha^{\gamma+1} + \frac{\gamma}{\gamma+1}\beta^{\frac{\gamma+1}{\gamma}}.$$

Put $p = \gamma + 1$ and $q = \frac{\gamma+1}{\gamma}$. Then we have

$$\alpha\beta \le \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q. \tag{9.10}$$

Observe that $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The exponents p and q are often called conjugate exponents. Now, we have shown (9.10) for $\alpha, \beta > 0$, but it is straightforward to extend the inequality to arbitrary $\alpha, \beta \in \mathbb{R}$, with appropriate absolute values inserted. Observe also that starting with $p > 1$ we can define $\gamma = p - 1$ and obtain (9.10).

Theorem 9.6 (Young's Inequality) *Let $\alpha, \beta, p \in \mathbb{R}$ with $p > 1$ be given. Choose $q \in \mathbb{R}$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$|\alpha\beta| \le \frac{1}{p}|\alpha|^p + \frac{1}{q}|\beta|^q.$$

Notice that when $p = 2$, we have $q = 2$ and Young's inequality reduces to (9.8). Observe also that given $p > 1$ there is exactly one $q \in \mathbb{R}$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, namely

$$q = \frac{p}{p-1}$$

and this value of q is strictly graeter than 1.

Before generalizing Theorem 9.5, we need a definition.

Definition 9.4 *For each $p > 1$ and function $f \in C[a,b]$, put*

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

The quantity $\|f\|_p$ is usually referred to as the L^p-norm of f. We point out that for every $c \in \mathbb{R}$, we have $\|cf\|_p = |c| \cdot \|f\|_p$. Furthermore, for $f \in C[a,b]$ we have $\|f\|_p = 0$ if and only if $f(x) = 0$ for all $x \in [a,b]$.

We will now state and prove

Theorem 9.7 (Hölder's Inequality) *Let $p, q \in \mathbb{R}$ be such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g \in C[a,b]$ be given. Then we have*

$$\int_a^b |f(x)g(x)| \leq \|f\|_p \|g\|_q.$$

Proof. The proof proceeds along the lines of the proof for Theorem 9.5.

If either $\|f\|_p = 0$ or $\|g\|_q = 0$, then there is nothing to prove. So we assume that $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

Define $F, G : [a,b] \to \mathbb{R}$ by

$$F(x) := \frac{f(x)}{\|f\|_p} \text{ and } G(x) := \frac{g(x)}{\|g\|_q} \quad \text{for all } x \in [a,b].$$

As before, we find that

$$\int_a^b |F(x)G(x)| \, dx \leq 1,$$

since $\|F\|_p = \|G\|_q = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. It follows that

$$\int_a^b |f(x)g(x)| \, dx \leq \|f\|_p \|g\|_q.$$

□

Remark 9.4 (Triangle Inequality) *It can be shown that for each $p > 1$ we have*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \quad \text{for all } f, g \in C[a,b].$$

Consequently $\| \cdot \|_p$ is a norm on $C[a,b]$.

9.8 Jensen's Inequality (A Special Case)

Using Hölder's inequality, we can easily prove

Theorem 9.8 (Jensen's Inequality (a special case)) *Let $p \in \mathbb{R}$ with $p > 1$ and $f \in C[a,b]$ be given. Then we have*

$$\frac{1}{(b-a)^{p-1}} \left(\int_a^b |f(x)|\, dx \right)^p \leq \int_a^b |f(x)|^p\, dx.$$

Proof. Put $q := \frac{p}{p-1}$, so $\frac{1}{p} + \frac{1}{q} = 1$. Using Hölder's inequality, we may write

$$\begin{aligned} \int_a^b |f(x)|\, dx &= \int_a^b |f(x)|^p \cdot 1\, dx \\ &\leq \left(\int_a^b |f(x)|^p\, dx \right)^{\frac{1}{p}} \left(\int_a^b 1^q\, dx \right)^{\frac{1}{q}} \\ &= \left(\int_a^b |f(x)|^p\, dx \right)^{\frac{1}{p}} (b-a)^{\frac{1}{q}}. \end{aligned}$$

Thus

$$\begin{aligned} \left(\int_a^b |f(x)|\, dx \right)^p &\leq (b-a)^{\frac{p}{q}} \int_a^b |f(x)|^p\, dx \\ \Rightarrow \frac{1}{(b-a)^{\frac{p}{q}}} \left(\int_a^b |f(x)|\, dx \right)^p &\leq \int_a^b |f(x)|^p\, dx \\ \Rightarrow \frac{1}{(b-a)^{p-1}} \left(\int_a^b |f(x)|\, dx \right)^p &\leq \int_a^b |f(x)|^p\, dx. \end{aligned}$$

$\square$

9.9 Example 9.9 (cf. Example 9.2; Heinricher & Mizel, 1986)

Let us recall the example given in Section 9.2. We set

$$\mathscr{Y} \in \{y \in C1[0,1] \,:\, y(0) = 0 \text{ and } y(1) = 1\}$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^1 \left(y(x)^2 - x\right)^2 y'(x)^6 \, dx \quad \text{for all } y \in \mathscr{Y}.$$

As pointed out in Section 9.2, if $y_*(x) := \sqrt{x}$ for all $x \in [0,1]$, then $J(y_*) = 0$ but $y_* \notin \mathscr{Y}$ since $y_* \notin C^1[0,1]$. In this section, we will prove

$$J(y) \geq \frac{1}{32}\left(\frac{3}{5}\right)^6 \quad \text{for all } y \in \mathscr{Y}.$$

In other words, we will show that no matter which y from $\mathscr{Y}$ is used, the value for J can never be smaller than $\frac{1}{32}\left(\frac{3}{5}\right)^6$. This phenomenon is called Lavrentiev's phenomenon and is quite remarkable since we know $y_*(x) = \sqrt{x}$ makes $J(y_*)$ zero.

To prove our statement, let $y \in \mathscr{Y}$ be given. Since $y \in C[0,1]$, we may choose $M > 0$ such that

$$|y(x_2) - y(x_1)| \leq M|x_2 - x_1| \quad \text{for all } x_1, x_2 \in [0,1].$$

In particular, we have

$$|y(x) - y(0)| = |y(x)| \leq M|x - 0| = Mx \quad \text{for all } x \in [0,1].$$

Thus

$$y(x)^2 \leq M^2x^2 \leq \frac{1}{2}x \quad \text{for all } x \in [0, \tfrac{1}{2M^2}].$$

Therefore

$$-\sqrt{\frac{x}{2}} \leq y(x) \leq \sqrt{\frac{x}{2}} \quad \text{for all } x \in [0, \tfrac{1}{2M^2}].$$

This inequality tells us that near $x = 0$, the graph of y must be between the graphs of the functions $x \mapsto -\sqrt{\frac{x}{2}}$ and $x \mapsto \sqrt{\frac{x}{2}}$. Since $y(1) = 1$, it follows that there must be at least one point where the graph of y crosses the graph of $x \mapsto \sqrt{\frac{x}{2}}$. We may therefore choose $\beta \in (0,1)$ to be the smallest strictly positive x-value at which the graph of y crosses the graph of one of the functions $x \mapsto -\sqrt{\frac{x}{2}}$ or $x \mapsto \sqrt{\frac{x}{2}}$. That is, we choose $\beta \in (0,1)$ such that for every $x \in (0,\beta)$

$$-\sqrt{\frac{x}{2}} \leq y(x) \leq \sqrt{\frac{x}{2}}$$

and

$$|y(\beta)| = \sqrt{\frac{\beta}{2}}.$$

With β so chosen, we deduce that

$$\begin{aligned} x \geq 2y(x)^2 &\Rightarrow x - y(x)^2 \geq y(x)^2 \\ &\Rightarrow \left(y(x)^2 - x\right)^2 \geq y(x)^4 \quad \text{for all } x \in [0,\beta]. \end{aligned}$$

We now prove a positive lower bound for the value of $J(y)$. Using the above inequality and the fact that $0 < \beta < 1$, we have

$$\begin{aligned} J(y) &= \int_0^1 \left(y(x)^2 - x\right)^2 y'(x)^6 \, dx \\ &\geq \int_0^\beta \left(y(x)^2 - x\right)^2 y'(x)^6 \, dx \\ &\geq \int_0^\beta y(x)^4 y'(x)^6 \, dx \\ &\geq \int_0^\beta \left|y(x)^{\frac{2}{3}} y'(x)\right|^6 \, dx \end{aligned}$$

Now we use Jensen's inequality, with $p = 6$, we write

$$\begin{aligned} J(y) &\geq \frac{1}{(\beta - 0)^5} \left[\int_0^\beta |y(x)^{\frac{2}{3}} y'(x)| \, dx \right]^6 \\ &\geq \frac{1}{\beta^5} \left[\frac{3}{5} y(x)^{\frac{5}{3}} \Big|_0^\beta \right]^6 \\ &\geq \frac{1}{\beta^5} \left(\frac{3}{5}\right)^6 \left[y(\beta)^{\frac{5}{3}} - y(0)^{\frac{5}{3}} \right]^6 \\ &\geq \frac{1}{\beta^5} \left(\frac{3}{5}\right)^6 \left(\frac{1}{2}\right)^5 \beta^5 \\ &\geq \frac{1}{32} \left(\frac{3}{5}\right)^6 . \end{aligned}$$

We have shown that for any $y \in \mathscr{Y}$, the value of J is no smaller than $\frac{1}{32}\left(\frac{3}{5}\right)^5$. With more advanced techniques, one can actually show that the value of J is no smaller than $\frac{1}{6}\left(\frac{3}{5}\right)^6$. In any case, there is no way that the value for J can be driven to zero using functions from $\mathscr{Y}$ despite the fact that $J(y_*)$ is zero.