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21-355C Real Analysis, Fall 2011
Assignment 9
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Question 0.1. Let g be a differentiable real function of $\mathbb{R}$ such that, $\forall x \in \mathbb{R}$, $|g'(x)| \leq M$, for some $M \in \mathbb{R}$. Let $\epsilon = \frac{1}{M+1} > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ such that, $\forall x \in \mathbb{R}$, $f(x) = x + \epsilon g(x)$. Then, by Theorem 5.3, since the identity and g are differentiable on $\mathbb{R}$, f is differentiable on $\mathbb{R}$ and, $\forall x \in \mathbb{R}$, $f'(x) = 1 + \epsilon g'(x)$. $\forall x \in \mathbb{R}$, since $-(M+1) < g'(X)$, $-1 < \epsilon g'(x)$, so $f'(x) > 0$. Suppose, for sake of contradiction, that f were not injective; in particular, suppose $\exists x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Then, by the Mean Value Theorem (in particular, by Theorem 5.10), $\exists x \in \mathbb{R}$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$. Since $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, this implies that $f'(x) = 0$, contradicting the above proof that $f'(x) > 0$. Thus, f is injective. ■

Question 0.2. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable its domain, such that $\lim_{x\to\infty} f'(x) = 0$. Let $g : \mathbb{R} \to \mathbb{R}$ such that, $\forall x \in \mathbb{R}$, $g(x) = f(x+1) - f(x)$. Let $\epsilon \in \mathbb{R}$ with $\epsilon > 0$. Since $\lim_{x\to\infty} f'(x) = 0$, $\exists x_0 \in (0, \infty)$ such that, $\forall x \in \mathbb{R}$ with $x > x_0$, $|f'(x)| < \epsilon$. Let $x \in \mathbb{R}$ with $x > x_0$. Suppose, for sake of contradiction that $g(x) > \epsilon$. Then, by the Mean Value Theorem (in particular, by Theorem 5.10), $\exists c \in (x, x+1)$ such that $|g(x)| = |f(x+1) - f(x)| = |((x+1) - x)f'(c)| = |f'(c)|$. However, since $c > x_0$, this contradicts the result that $|f'(c)| < \epsilon$. Therefore, $\forall x \in \mathbb{R}$ with $x > x_0$, $|g(x)| < \epsilon$, so that $\lim_{x\to\infty} g(x) = \infty$. ■

Question 0.3.
Question 0.4.
Lemma 0.5.
Question 0.6.
Question 0.7.
Question 0.8.
Question 0.9.
Question 0.10.