5/5

4.1. Suppose X is some measure space, and $f : X \to \mathbb{R}$ is a measurable function, $g : \mathbb{R} \to \mathbb{R}$ a Lebesgue measurable function. In this case, $g \circ f(x)$ need not be measurable. For example, consider the case when X is $(\mathbb{R}, \mathcal{B}, \lambda_{\mathcal{B}})$, the real line with the Borel σ-algebra, and the Lebesgue measure restricted to Borel. Then let A be some subset of $\mathbb{R}$ which is Lebesgue measurable, but not Borel measurable, and let $g = \chi_A$. Then the pre-image of any set is either A, A^C or $\mathbb{R}$ which are all necessarily Lebesgue measurable, so g is Lebesgue measurable. Then let f be the identity, which is certainly measurable. Now certainly $\{1\}$ is a Borel set, but the pre-image under $g \circ f$ is A, which we assumed was not Borel measurable, so $g \circ f$ is not measurable.

~~except you change σ-algebras~~ $f^{-1}(A)=$ is not Borel measur

2. Consider the function $h : ((-\infty, \infty) \setminus \{0\}) \to$ ... partial derivatives of h

$$\frac{\partial h}{\partial x} = \frac{1}{y} \qquad\qquad \frac{\partial h}{\partial y} = -\frac{x}{y^2}$$

exist and are continuous for all $x, y \in (-\infty, \infty) \setminus 0$. Hence h is continuous. Then for every $U \subseteq (-\infty, \infty) \setminus 0$ open we have $h^{-1}(U)$ relatively open in $(-\infty, \infty) \setminus \{0\}$; since $(-\infty, \infty) \setminus \{0\}$ is open, h^{-1} is open in $\mathbb{R}$. Now, consider the extension $h' : [-\infty, \infty]^2 \setminus \{(0, 0), (\pm\infty, \pm\infty)\} \to [-\infty, \infty]$ defined by $h'(x, y) = x/y$. For any $A \subseteq (-\infty, \infty) \setminus \{0\}$, we have $h'^{-1}(A) = h^{-1}(A)$, since $h'(x, y) \in (-\infty, \infty) \setminus \{0\}$ implies that $x, y \notin \{-\infty, 0, \infty\}$. We note that

$$\begin{aligned} h'^{-1}(\{-\infty\}) &= \{(x, 0) : x < 0\} \\ h'^{-1}(\{\infty\}) &= \{(x, 0) : x > 0\} \\ h'^{-1}(\{0\}) &= \{(0, x) : x \neq 0\} \cup \{(x, -\infty) : x \in (-\infty, \infty)\} \cup \{(x, \infty) : x \in (-\infty, \infty)\} \end{aligned}$$

Each of these is a Borel set. For any open set U, we have

$$h'^{-1}(U) = h^{-1}(U \cap (-\infty, 0) \cap (0, \infty)) \cup h^{-1}(U \cap \{-\infty\}) \cup h^{-1}(U \cap \{0\}) \cup h^{-1}(U \cap \{\infty\})$$

Since $U \cap (-\infty, 0) \cap (0, \infty)$ is open and $h'^{-1}(V)$ is open for $V \subseteq (-\infty, 0) \cap (0, \infty)$ open, $h'^{-1}(U \cap (-\infty, 0) \cap (0, \infty))$ is open. Thus, $h'^{-1}(U)$ is a Borel set.

Now, we have that h' is Borel measurable. Since f and g are measurable, we have that $k : [-\infty, \infty]^2 \to [-\infty, \infty] \setminus \{(0, 0), (\pm\infty, \pm\infty)\}$ defined by $k(x) = (f(x), g(x))$ is measurable. Then since h' is Borel, $h' \circ k = \frac{f}{g}$ is measurable.

5/5

3. (a) Let $Y \subseteq X$ be the set of points on which $(f_n) \to f$, and let $Z \subseteq X$ be the set of points x for which g is continuous at $f(x)$. Let $x \in Y \cap Z$ be given, and let $\epsilon > 0$ be given. Since $x \in Z$, there exists some $\delta > 0$ such that $|g(y) - g(f(x))| < \epsilon$ for all $y \in \mathbb{R}$ with $|y - f(x)| < \delta$. Since $x \in Y$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \delta$ for all $n > N$. Then for all $n > N$, we have $|g(f_n(x)) - g(f(x))| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $(g \circ f_n)(x) \to (g \circ f)(x)$ as $n \to \infty$ for all $x \in Y \cap Z$. Since $X \setminus Y$ and $X \setminus Z$ have measure zero, $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$ has measure zero. Thus $(g \circ f_n) \to g \circ f$ almost everywhere.

5/5

(b) We give a counter-example. Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{1}{n}$ and $g : \mathbb{R} \to \mathbb{R}$ by

$$g(y) = \begin{cases} 1, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

3/3

f_n is clearly measurable for all $n \in \mathbb{N}$, and g is Borel measurable since $f^{-1}(A) \in \{\emptyset, (-\infty, 0], (0, \infty), \mathbb{R}\}$ for all $A \in \mathcal{P}(\mathbb{R})$. For $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$, we have $(f_n) \to f$ everywhere. However, for any $x \in \mathbb{R}$, we have $(g \circ f_n)(x) = g\left(\frac{1}{n}\right) = 1$ for all $n \in \mathbb{N}$, while $(g \circ f)(x) = g(0) = 0$. Thus $(g \circ f_n)(x) \not\to (g \circ f)(x)$ for all $x \in \mathbb{R}$.

4. For $d = 1$, any convex $C \in \mathbb{R}$ is an interval. All intervals are Borel sets, so any such C is Borel and therefore Lebesgue measurable.

For $\mathbb{R}^d$ with $d > 1$, there exist convex sets which are not Borel measurable. Let $d > 1$ be given, and define $f : \mathbb{R} \to \mathbb{R}^d$ by

$$f(t) = (\cos t, \sin t, 0, \ldots, 0)$$

which is a continuous function. Choose some non-Borel measurable set $E \subseteq [0, 2\pi)$. Take

$$A := \{(x, y, 0, \ldots, 0) : |x| < 1, |y| < 1\} \cup f(E)$$

Note that $f(E) \subseteq \partial(\{(x, y, 0, \ldots, 0) : |x| < 1, |y| < 1\})$. A consists of an open disk and certain portions of its boundary. This set is convex: any line connecting two points in A is a subset of a chord in the disk, and thus touches the boundary at most at its endpoints.

Assume for sake of contradiction that A is a Borel set. Since f is continuous, the inverse image of any Borel set is a Borel set, so $f^{-1}(A) = \bigcup_{n \in \mathbb{Z}} (2\pi n + E)$ is Borel. But this would imply that $E = f^{-1}(A) \cap [0, 2\pi)$ is a Borel set, which is false. Thus, A is not Borel measurable.

Convex sets in $\mathbb{R}^d$ should always be Lebesgue measurable, but I have no proof of this. Intuitively, the boundary of a convex set is simple enough that it will always be a null set. Since the interior of any set is open and therefore Lebesgue measurable, the measurability of all convex sets follows from the completeness of the Lebesgue measure. 5/5

5. First, assume that there exist two Σ-measurable functions $f, h : X \to \infty$ such that $f = h$ μ-almost everywhere and $f \leq g \leq h$; we show that g is Σ_μ-measurable. Let $a \in [-\infty, \infty]$ be given. Since f and h are Σ-measurable, $f^{-1}([-\infty, a])$ and $h^{-1}([-\infty, a])$ are measurable. Since $f \leq g \leq h$, we have $h^{-1}([-\infty, a]) \subseteq g^{-1}([-\infty, a]) \subseteq f^{-1}([-\infty, a])$. Finally, since f and h differ only on a null set N, we have $f^{-1}([-\infty, a]) \setminus h^{-1}([-\infty, a]) \subseteq N$, so $\bar{\mu}(f^{-1}([-\infty, a]) \setminus h^{-1}([-\infty, a])) = 0$. Hence $\mu(f^{-1}([-\infty, a])) = \mu(h^{-1}([-\infty, a]))$.

Thus we have two Σ-measurable sets $h^{-1}([-\infty,a])$ and $f^{-1}([-\infty,a])$ with $h^{-1}([-\infty,a]) \subseteq g^{-1}([-\infty,a]) \subseteq f^{-1}([-\infty,a])$ and $\mu(h^{-1}([-\infty,a])) = \mu(f^{-1}([-\infty,a]))$, so $g^{-1}([-\infty,a])$ is Σ_μ-measurable. Then the inverse image of $[-\infty,a]$ under g is Σ_μ-measurable for all $a \in [-\infty,\infty]$, so g is Σ_μ-measurable.

Now, assume instead that g is Σ_μ-measurable; we find two Σ-measurable functions f and h with $f \le g \le h$ which are equal μ-almost everywhere.

Since g is Σ_μ-measurable, $g^{-1}(\{-\infty\})$ and $g^{-1}(\{\infty\})$ are Σ_μ-measurable, so there exist $G_{-\infty} \subseteq g^{-1}(\{-\infty\})$ and $G_\infty \subseteq g^{-1}(\{\infty\})$ which are Σ-measurable such that $\overline{\mu}(g^{-1}(\{-\infty\}) - G_{-\infty}) = 0$ and $\overline{\mu}(g^{-1}(\{\infty\}) - G_\infty) = 0$. Let $i \in \mathbb{N}$ be given. For $j \in \mathbb{Z}$, we have $g^{-1}\left(\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right)\right)$ Σ_μ-measurable, so there exists a Σ-measurable set $G_{i,j} \subseteq g^{-1}\left(\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right)\right)$ such that $\overline{\mu}\left(g^{-1}\left(\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right)\right) - G_{i,j}\right) = 0$. Set $E_i := \left(G_{-\infty} \cup G_\infty \cup \bigcup_{j\in\mathbb{Z}} G_{i,j}\right)^c$, which is Σ-measurable since it is the complement of a countable union of Σ-measurable sets. Furthermore, we have

$$
\begin{aligned}
\overline{\mu}(E_i) &= \overline{\mu}(E_i \cap g^{-1}(\{-\infty\})) + \overline{\mu}(E_i \cap g^{-1}(\{\infty\})) + \sum_{j\in\mathbb{Z}} \overline{\mu}\left(E_i \cap g^{-1}\left(\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right)\right)\right) \\
&= \overline{\mu}(g^{-1}(\{-\infty\}) - G_{-\infty}) + \overline{\mu}(g^{-1}(\{\infty\}) - G_\infty) + \sum_{j\in\mathbb{Z}} \overline{\mu}\left(g^{-1}\left(\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right)\right) - G_{i,j}\right) \\
&= 0
\end{aligned}
$$

We now define $f_i, h_i : X \to \mathbb{R}$ by

$$
f_i(x) = \begin{cases} \frac{j}{2^i}, & \text{if } x \in G_{i,j} \\ -\infty, & \text{if } x \in G_{-\infty} \\ \infty, & \text{if } x \in G_\infty \\ -\infty, & \text{if } x \in E_i \end{cases}
\qquad
h_i(x) = \begin{cases} \frac{j+1}{2^i}, & \text{if } x \in G_{i,j} \\ -\infty, & \text{if } x \in G_{-\infty} \\ \infty, & \text{if } x \in G_\infty \\ \infty, & \text{if } x \in E_i \end{cases}
$$

For any set $A \subseteq \mathbb{R}$, $f_i^{-1}(A)$ and $h_i^{-1}(A)$ are the union of a subset of the countable set of Σ-measurable sets $\{G_{i,j} : i \in \mathbb{N}, j \in \mathbb{Z}\} \cup \{G_{-\infty}, G_\infty, E_i\}$ and are therefore Σ-measurable. Hence f_i and h_i are Σ-measurable functions. In comparing $f_i(x)$ and $h_i(x)$ with $g(x)$ for $x \in X$, we have the following cases:

- [Case: $x \in G_{-\infty}$] Then $x \in g^{-1}(\{-\infty\})$, so $g(x) = -\infty$. We have $f_i(x) = g_i(x) = -\infty$, so $f_i(x) = g_i(x) = h_i(x)$.
- [Case: $x \in G_\infty$] Then $x \in g^{-1}(\{\infty\})$, so $g(x) = \infty$. We have $f_i(x) = g_i(x) = \infty$, so $f_i(x) = g_i(x) = h_i(x)$.
- [Case: $x \in G_{i,j}$] Then $x \in g^{-1}\left(\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right)\right)$, so $\frac{j}{2^i} \le g(x) < \frac{j+1}{2^i}$. We have $f_i(x) = \frac{j}{2^i}$ and $h_i(x) = \frac{j+1}{2^i}$, so $g(x) - f_i(x) < \frac{1}{2^i}$, $h_i(x) - g(x) < \frac{1}{2^i}$, and $f_i(x) \le g_i(x) \le h_i(x)$.
- [Case: $x \in E_i$] Then $f_i(x) = -\infty$ and $h_i(x) = \infty$, so $f_i(x) \le g_i(x) \le h_i(x)$.

We thus have $f_i \le g \le h_i$, and $g(x) - f_i(x) < \frac{1}{2^i}$ and $h_i(x) - g(x) < \frac{1}{2^i}$ for all $x \notin E_i$. Take $E = \bigcup_1^\infty E_i$. Since each E_i is μ-null, E is μ-null as well. Define $f = \sup_{i\in\mathbb{N}} f_i$ and $h = \inf_{i\in\mathbb{N}} h_i$. Since f and h are the infimum and supremum of sequences of Σ-measurable functions, they are themselves Σ-measurable. For all $i \in \mathbb{N}$, we have $0 \leq g(x) - f_i(x) < \frac{1}{2^i}$ and $0 \leq h_i(x) - g(x) < \frac{1}{2^i}$ for all $x \notin E$. Thus

$$g(x) - \frac{1}{2^i} \leq f(x) \leq g(x) \qquad\qquad g(x) \leq h(x) \leq g(x) + \frac{1}{2^i}$$

for all $i \in \mathbb{N}$ and $x \notin E$, so we conclude that $f(x) = g(x) = h(x)$ for all $x \notin E$. For $x \in E$, we still have that $f(x) \leq g(x) \leq h(x)$, since $f_i(x) \leq g(x) \leq h_i(x)$ for all $i \in \mathbb{N}$. Thus, we have two Σ-measurable functions f and h such that $f \leq g \leq h$ and $f = g = h$ μ-almost everywhere (since E is μ-null).

6. Let X be a metric space, and $\Sigma \supseteq \mathcal{B}(X)$ a σ-algebra on X, and μ a regular finite measure on (X, Σ). Let $f : X \to \mathbb{R}$ be measurable.

(a) For any $\varepsilon > 0$ and $i \in \mathbb{N}$, we claim that there exist finitely many disjoint compact sets $\{K_{i,j} : |j| \leq N_i\}$ such that

$$\mu\left(X - \bigcup_{j=-N_i}^{N_i} K_{i,j}\right) < \frac{\varepsilon}{2^i}, \quad \text{and} \quad f(K_{i,j}) \subseteq [\frac{j}{2^i}, \frac{j+1}{2^i}).$$

Proof. Fix $\varepsilon > 0$ and $i \in \mathbb{N}$. Let $F_{i,j} := f^{-1}([\frac{j}{2^i}, \frac{j+1}{2^i})) \in \Sigma$ for all j. Note that the $F_{i,j}$ are disjoint and that

$$X = \bigcup_{j \in \mathbb{Z}} F_{i,j}$$

so we have

$$\mu(X) = \sum_{j \in \mathbb{Z}} \mu(F_{i,j})$$

Fix $\delta > 0$ with $\delta < \frac{\varepsilon}{2^i}$. It follows that we may choose N_i large enough that

$$\mu(X) < \sum_{|j| \leq N_i} \mu(F_{i,j}) + \frac{\varepsilon}{2^i} - \delta.$$

Since any metric space is Hausdorff, and $\Sigma \supseteq \mathcal{B}(X)$, and μ is regular and finite (hence all sets are σ-finite), we may apply problem 2 of assignment 3 to show that for all $A \in \Sigma$ we have $\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ is compact}\}$. In particular, since all the $F_{i,j} \in \Sigma$, for each $|j| < N_i$ we may choose a compact set $K_{i,j} \subseteq F_{i,j}$ such that $\mu(K_{i,j}) + \frac{\delta}{2^{j+2}} > \mu(F_{i,j})$. The $K_{i,j}$ are all disjoint because they are subsets of the $F_{i,j}$ which are disjoint. And furthermore we have

$$\begin{aligned}
\mu(X) < \sum_{|j| \leq N_i} \mu(F_{i,j}) + \frac{\varepsilon}{2^i} - \delta &< \sum_{|j| \leq N_i} (\mu(K_{i,j}) + \frac{\delta}{2^{j+2}}) + \frac{\varepsilon}{2^i} - \delta \\
&= \sum_{|j| \leq N_i} \mu(K_{i,j}) + \sum_{|j| \leq N_i} (\frac{\delta}{2^{j+2}}) + \frac{\varepsilon}{2^i} - \delta \\
&< \sum_{|j| \leq N_i} \mu(K_{i,j}) + 2\sum_{j=0}^{\infty} (\frac{\delta}{2^{j+2}}) + \frac{\varepsilon}{2^i} - \delta \\
&= \sum_{|j| \leq N_i} \mu(K_{i,j}) + \delta + \frac{\varepsilon}{2^i} - \delta \\
&= \sum_{|j| \leq N_i} \mu(K_{i,j}) + \frac{\varepsilon}{2^i}. \\
&= \mu\left(\bigcup_{|j| \leq N_i} K_{i,j}\right) + \frac{\varepsilon}{2^i}
\end{aligned}$$

Rearranging and using the fact that $\mu(X)$ is finite and $\bigcup_{|j| \leq N_i} K_{i,j} \subseteq X$ this then gives

5/5

$$\mu\left(X - \bigcup_{j=-N_i}^{N_i} K_{i,j}\right) < \frac{\varepsilon}{2^i}$$

as desired. $\square$

(b) *(Lusin's theorem)* For any $\varepsilon > 0$ we claim that there exists $K_\varepsilon \subseteq X$ compact such that $f : K_\varepsilon \to \mathbb{R}$ is *continuous*, and $\mu(X - K_\varepsilon) < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. And define $K_\varepsilon := \bigcap_{i=1}^{\infty} \bigcup_{|j| \le N_i} K_{i,j}$ for the $K_{i,j}$ defined in the previous part. We have that K_ε is compact by the two lemmas. Then we have

$$\mu(X - K_\varepsilon) = \mu\left(X - \bigcap_{i=1}^{\infty} \bigcup_{|j|<N_i} K_{i,j}\right) = \mu\left(\bigcup_{i=1}^{\infty} \bigcap_{|j|<N_i} (X - K_{i,j})\right)$$

$$\le \sum_{i=1}^{\infty} \mu\left(\bigcap_{|j|<N_i} (X - K_{i,j})\right) = \sum_{i=1}^{\infty} \mu\left(X - \bigcup_{|j|<N_i} K_{i,j}\right)$$

$$< \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon.$$

All that remains to show is that $f : K_\varepsilon \to \mathbb{R}$ is continuous. We do this indirectly. For each $i \in \mathbb{N}$ define $g_i : K_\varepsilon \to \mathbb{R}$ by

$$g_i(x) := \frac{j}{2^i}$$

if $x \in K_{i,j}$ and $|j| \le N_i$. The $K_{i,j}$'s are disjoint so the g_i's are well defined. We claim that the g_i's are continuous. Fix $x \in K_{i,j}$ for some $i \in \mathbb{N}, |j| < N_i$. Let a sequence (x_n) in K_ε converging to x be given. The each x_n is in some K_{i,j_n} for $|j_n| < N_i$. But for $j_1 \ne j_2$ we have $K_{i,j_1} \cap K_{i,j_2} = \emptyset$. We recall from real analysis that in a metric space, compact sets that do not intersect must have positive distance from each other. Since $(x_n) \to x$ it follows that for all n large $x_n \in K_{i,j}$, otherwise the sequence could not converge to x. Thus for all n large $g_i(x_n) = \frac{j}{2^i}$ and so the we find the limit

$$\lim_{n\to\infty} g_i(x_n) = \frac{j}{2^i} = g_i(x).$$

It follows that g_i is sequentially continuous for all $i \in \mathbb{N}$, and since X is a metric space, we find that g_i is continuous for all $i \in \mathbb{N}$. Finally we claim that $(g_i) \to f$ uniformly on K_ε. We note that by the condition $f(K_{i,j}) \subseteq [\frac{j}{2^i}, \frac{j+1}{2^i})$ for all $i \in \mathbb{N}, |j| < N_i$, and by construction of g_i we have that

$$\sup_{x \in K_\varepsilon} \|g_i(x) - f(x)\| \le \frac{1}{2^i}$$

for all $i \in \mathbb{N}$. It follows that $(g_i) \to f$ uniformly. This finishes the proof since the uniform limit of continuous functions from a topological space to a metric space, in our case $X \to \mathbb{R}$, is continuous. $\square$

5/5