21-630 Assignment #12 Due Wed., April 24, 2013

1. Assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and let L be a transversal. Assume that X is a solution of

$$\frac{dX}{dt} = f(X(t))$$

with $C^+(X(0))$ bounded and that X crosses L infinitely many times.

a) Identify the error in the following argument (the error that really can't be fixed): We may choose a sequence, t_k, of distinct positive times that tend to infinity for which $X(t_k) \in L$ for each k. By Lemma 6.2 $X(t_k)$ converges to some point, $\overline{x} \in L$. Let Y be the solution with $Y(0) = \overline{x}$. Now $\overline{x} \in \Omega(X(0))$ and $\Omega(X(0))$ is positively invariant, so $C^+(Y(0)) \subset \Omega(X(0))$. Furthermore, $\Omega(X(0))$ is closed so it follows that $\Omega(Y(0)) \subset \Omega(X(0))$. Choose $\overline{y} \in \Omega(Y(0))$ and let T be a transversal whose center is $\overline{y}$. By Lemma 6.3 $\Omega(X(0)) \bigcap (T$ delete endpoints) has only one element, namely $\overline{y}$. But by the corollary to Lemma 6.1, Y must cross T infinitely many times and can do so only at $\overline{y}$. Therefore Y is periodic.

b) Give an example of f that has a solution X (with $C^+(X(0))$ bounded) that crosses a transversal infinitely many times, but whose omega limit set is not the orbit of a periodic solution.

2. Give an example where there is a sequence of periodic solutions, X_k, for which $X_k(0)$ converges, but the solution whose initial value is

$$\lim_{k\to\infty} X_k(0)$$

is not periodic.

3. Let $x_0 > 0$ and define $X(t)$ by

$$\frac{d^2X}{dt^2} + 2X^3 = 0,$$

$X(0) = x_0$, and $\frac{dX}{dt}(0) = 0$. Let $T = \min\{t > 0 : X(t) = 0\}$, you don't have to show that this is well defined. Note that X is $4T$ periodic, you don't have to show this either. Show that there is a positive constant, C such that

$$T = \frac{C}{x_0}$$

for all $x_0 > 0$. Hint: use the energy equation, i.e. $(\frac{dX}{dt})^2 + X^4$ =constant, to express T as an integral. The problem may be completed without finding an antiderivative for this integral.

4. Consider the system

$$\frac{dX}{dt} = X\sqrt{X^2+Y^2}(1-Z) + X^3 - Y$$

$$\frac{dY}{dt} = Y\sqrt{X^2+Y^2}(1-Z) + X^2Y + X$$

$$\frac{dZ}{dt} = 2(X^2+Y^2)^{3/2}(1-Z) + 2X^2(X^2+Y^2).$$

Prove that this system has infinitely many periodic solutions. The idea is to show that $Z - X^2 - Y^2$ is constant for every solution. Then use this to eliminate Z. Then you have a planar system.