Assignment 5: Assigned Wed 10/03. Due Wed 10/10

1. (a) Suppose $I \subseteq \mathbb{R}^d$ is a cell, and $f : I \to \mathbb{R}$ is Riemann integrable. Show that f is measurable, Lebesgue integrable and that the Lebesgue integral of f equals the Riemann integral.

   (b) Is the previous subpart true if we only assume that an improper (Riemann) integral of f exists? Prove or find a counter example.

2. (a) Let (X, Σ, μ) be a complete measure space, $f : X \to [-\infty, \infty]$ be measurable and suppose $\int_X f \, d\mu$ is defined. If $g : X \to [-\infty, \infty]$ is such that $f = g$ a.e., then show $\int_X f \, d\mu = \int_X g \, d\mu$.

   All the convergence theorems we've seen so far hold if we replace pointwise convergence with a.e. convergence. I ask you to prove one below; you should verify the others on your own.

   (b) Suppose (f_n) is a sequence of measurable functions, $f_n \geqslant 0$ a.e., and $(f_n) \to f$ a.e. on E. Show that $\liminf \int_E f_n \, d\mu \geqslant \int_E f \, d\mu$.

3. Let $f : \mathbb{R}^d \to [-\infty, \infty]$ be an integrable function such that $\int_I f \, d\lambda = 0$ for all cells I. Must $f = 0$ a.e.? Prove or find a counter example.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a measurable function. We define the *Laplace Transform* of f to be the function $F(s) = \int_0^\infty \exp(-st) f(t) \, dt$ wherever defined.

   (a) If $\int_0^\infty |f(t)| \, dt < \infty$, show that $F : [0, \infty) \to \mathbb{R}$ is continuous.

   (b) If $\int_0^\infty t|f(t)| \, dt < \infty$, show that $F : [0, \infty) \to \mathbb{R}$ is differentiable.

   (c) If f is continuous and bounded, compute $\lim_{s \to \infty} sF(s)$.

5. (a) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be linear, and $A \in \mathcal{L}$. Show that $\lambda(T(A)) = |\det(T)| \lambda(A)$. [Hint: Check it separately for $\det(T) = 0$. For $\det(T) \neq 0$, write T as a product of elementary transformations, and check the result for cells. (This should have been on HW1, but I 'inadvertently' added the assumption that T was orthogonal.)]

   (b) *(Linear change of variable)* Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable, $T : \mathbb{R}^d \to \mathbb{R}^d$ an invertible linear transformation, and $E \in \mathcal{L}(\mathbb{R}^d)$. Show that

$$\int_{T^{-1}(E)} (f \circ T) |\det T| \, d\lambda = \int_E f \, d\lambda.$$

Optional problems, and details in class I left for you to check.

- For simple functions, check that $\int_E s$ is well defined.
- For positive functions check $f \leqslant g \implies \int_E f \leqslant \int_E g$.
- For arbitrary integrable functions, check $\int_E \alpha f \, d\mu = \alpha \int_E f \, d\mu$.
- If $\int_X f \, d\mu < \infty$, then show $f < \infty$ a.e.
- If $\int_X |f| \, d\mu = 0$, then show that $f = 0$ a.e.
- Prove the following generalization of Fatou's Lemma: If $f_n \geqslant 0$ are measurable, then $\liminf \int_E f_n \, d\mu \geqslant \int_E \liminf f \, d\mu$.
- Finish the proof of showing $\int_X g \, d\mu = \int_Y g \circ f \, d\mu_{f^{-1}}$. Use this to give show $\int_{\mathbb{R}^d} f(x+y) \, dx = \int_{\mathbb{R}^d} f(x) \, dx$. (This trick also helps with Q5(b).)