Homework 2
21-651 General Topology
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Problem 1

(a) Let $x \in \mathbb{R}^2$, and suppose, for sake of contradiction, that $\beta \subseteq \tau$ is a countable local base of τ at x (τ is the radially open set topology). $\forall \theta \in [0, 2\pi)$, let P_θ be the parabola with vertex at x, rotated counterclockwise by θ, without the point x itself. For any given $B \in \beta$, $T_B := \{\theta : P_\theta \cap B = \emptyset\}$ cannot be dense in $[0, 2\pi)$, since each B is radially open. Thus, by the Baire Category Theorem, $\bigcup_{B \in \beta} T_B$ is not dense in $[0, 2\pi)$, so that some $P_\theta \cap \bigcup_{B \in \beta} B = \emptyset$. Thus, if $U := B(x, 1) \backslash P_\theta$, U is radially open, but there is no $B \in \beta$ with $B \subseteq U$, contradicting the choice of β as a countable local base of τ at x. $\blacksquare$

(b) Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $(X \times Y, \tau)$ be the product topology. Suppose $(x, y) \in X \times Y$ ($x \in X$,$y \in Y$). Then, there exist countable local bases $\beta_x \subseteq \tau_X$ and $\beta_y \subseteq \tau_Y$ of x and y respectively. Let $f : \mathbb{N} \to \beta_X$ and $g : \mathbb{N} \to \beta_Y$ be bijections, and let $\beta_{xy} := \{\bigcap_{i=2}^{i} f(i) \times g(i) : i \in \mathbb{N}\} \subseteq \tau$.

We show that the set

$$\tau_{xy} := \{U \in \mathcal{P}(X \times Y) : x \in U \text{ implies } \exists B \in \beta_{xy}, B \subseteq U\}$$

is a topology containing $\mathcal{F} := \{U \times V : U \in \tau_X, V \in \tau_Y\}$, so that, by definition of the product topology, $\tau \subseteq \tau_{xy}$, implying that $(X \times Y, \tau)$ is first countable.

Vacuously, $\emptyset \in \tau_{xy}$, and, trivially, $X \times Y \in \tau_{xy}$.

Suppose $(x, y) \in U_1, U_2, \dots, U_k \in \tau_{xy}$. Since there exist $B_1, B_2, \dots, B_k \in \beta_{xy}$ with $B_1 \subseteq U_1, \dots B_k \subseteq U_k$, there is some $B \in \beta_{xy}$ with $B \subseteq \bigcap_{i=1}^k B_i \subseteq \bigcap_{i=1}^k U_i$, so that $\bigcap_{i=1}^k U_i \in \tau_{xy}$ and τ_{xy} is closed under finite intersections (if there is some U_i not containing x, then x is not in the intersection, so the intersection is vacuously in τ_{xy}).

Suppose $U_i \in \tau_{xy}$, $\forall i \in I$, with $x \in U_i$ for some $i \in I$. Then, since $\exists B \in \beta_{xy}$ with $B \subseteq U_i$, $B \subseteq \bigcup_{i \in I} U_i$, so that $\bigcup_{i \in I} U_i \in \tau_{xy}$, and τ_{xy} is closed under arbitrary unions (if there is no U_i with $x \in U_i$, then x is not in the union, so the intersection is vacuously in τ_{xy}).

Thus, τ_{xy} is a topology. Since $U \in \tau_X, V \in \tau_Y$ implies $\exists f(i) \times g(i) \subseteq U \times V$, $\mathcal{F} \subseteq \tau_{xy}$. $\blacksquare$

(c) Suppose (X, τ) is a first countable topological space and $A \subseteq X$ (and let τ_A be the topology induced on A by τ). Let $x \in A$. Since $x \in X$ and (X, τ) is first countable, there exists a countable local base $\beta_x \subseteq \tau$ of x. Let

$$\beta'_x := \{B \cap A : B \in \beta_x\} \subseteq \tau_A.$$

By construction of the induced topology, $\forall U_A \in \tau_A$ with $x \in U_A$, $U_A = A \cap U$, for some $U \in \tau$. By definition of a local base, $\exists B \in \beta_x$ such that $B \subseteq U$, so that $B \cap A \subseteq U_A$. Thus, β'_x is a local base of τ_A at x.

Thus, since β'_x is countable ($B \mapsto B \cap A$ is a surjection from β_x), (A, τ_A) is first countable. ■

Problem 2

Suppose (X, τ) is a second countable topological space, and let β be a countable base for (X, τ). By the Axiom of Choice, $\forall B \in \beta$, we can pick some $x_B \in B$. Then, let $E := \{x_B : B \in \beta\}$.

Since β is countable, E is countable as well ($B \mapsto x_B$ is a surjection).

Suppose C is a closed set with $E \subseteq C$. Then, $U := X \backslash C$ is open, so that, for some $A \subseteq \beta$, $U = \bigcup_{B \in A} B$. If U is nonempty, $\exists B \in A$, so that $x_B \in U$, contradicting the fact that $E \subseteq C = X \backslash U$. Therefore, $C = X$.

Thus, X is the intersection of all closed sets containing E, so that, by Proposition 26 (ii) in the notes, $\overline{E} = X$, and so E is dense. Since E is countable and dense, X is separable. ■

Problem 3

By the result of problem 2, it suffices to show that (A, d) is second countable.

Since (X, d) is separable, there exists a countable dense set $S \subseteq X$. Then, let $f : \mathbb{N} \to S$ be a bijection. Define

$$\beta := \{B(f(i), 1/n) : i, n \in \mathbb{N}\},$$

where $B(x, r)$ denotes the ball of radius $r > 0$ centered at $x \in X$. Suppose $U \subseteq X$ is open and $x \in U$. Since U is open, for some $n \in \mathbb{N}$, $B(x, 1/n) \subseteq U$. Since S is dense, $\exists i \in \mathbb{N}$ such that $d(x_i, x) < \frac{1}{2n}$, and thus, by the triangle inequality, $\forall y \in B(x_i, \frac{1}{2n})$, $d(x, y) \leq d(x, x_i) + d(x_i, y)$, implying $B(x_i, \frac{1}{2n}) \subseteq B(x, 1/n)$. Thus, for any open set $U \subseteq X$, $\forall x \in U$, $\exists B_x \in \beta$ with $B_x \subseteq U$, so that $U = \bigcup_{x \in U} B_x$, implying that β is a base for (X, d). Since there is an obvious bijection between β and $\mathbb{N}^2$, β is countable, and so (X, d) is second countable.

Clearly, second countability is heritable (the set $\beta' := \{A \cap B : B \in \beta\}$ is a countable base for (A, d)), so that (A, d) is second countable. ■

Problem 4

Since, $\forall n \in \mathbb{N}$, $C([-n, n])$ is separable with respect to the d_∞ metric, there exists a set S_n which is countable and dense in $[-n, n]$ with respect to the d_∞ metric. Let $S := \bigcup_{i=1}^{\infty} S_n \subseteq C(\mathbb{R}, \mathbb{R})$. Let $f \in C(\mathbb{R}, \mathbb{R})$, and let $\epsilon > 0$. $\forall n \in \mathbb{N}$, by choice of S_n, $\exists f_n \in S_n$ such that $d_\infty(f, f_n) < \epsilon$. Thus, $f_n \to f$ uniformly on $[-n, n]$. As given in Example 34 of the notes, this implies that $d(f, f_n) \to 0$ on $\mathbb{R}$. Thus, $\exists f_n$ such that $d(f, f_n) < \epsilon$, so that S is dense in $\mathbb{R}$.

Since S is a countable union of countable sets, S is countable, so that $(C(\mathbb{R}, \mathbb{R}), d)$ is separable. ■

Problem 5

The following Lemma is used in the proofs of parts (i) and (ii) of this problem:

Lemma: Suppose $x_0, x_1, x_2 \in X \subseteq \mathbb{R}^n$ and there exists a polygonal path with endpoints x_0, x_1 and range in X. If the line segment $S(x_1, x_2)$ between x_1 and x_2 is contained in X, then there exists a polygonal path with endpoints x_0, x_2 and range in X.

Proof of Lemma: Let $\gamma : [a, b] \to \mathbb{R}^n$ be a polygonal path with endpoints x_0, x_1 and range in X, and let $\varphi : [a, b+1] \to \mathbb{R}^n$ be the following function:

$$\varphi(t) = \begin{cases} \gamma(t) & : t \in [a, b] \\ (b+1)x_1 - bx_2 + t(x_2 - x_1) & : t \in [b, b+1] \end{cases}$$

Since $\varphi([b, b+1]) = S(x_1, x_2) \subseteq X$ and $\gamma([a, b]) \subseteq X$, φ has range in X. Thus, since $\varphi([a, b+1])$ is piecewise affine (the partition of the domain is the same as that for γ, with the additional segment $(b, b+1]$) and $\varphi(a) = \gamma(a) = x_0$ and $\varphi(b+1) = x_2$, φ is polygonal path with the desired properties, proving the lemma. ■

(i) Since clearly $x_0 \in U$, U is nonempty. Suppose $x_1 \in U$. Since O is open, there exists an open ball B of radius $r > 0$ centered at x_1 with $B \subseteq O$. Let $x_2 \in B$. Since balls in $\mathbb{R}^n$ are convex, the segment $S(x_1, x_2) \subseteq O$. Thus, by the above Lemma, $x_2 \in U$, so that $B \subseteq U$. Then, since every point in U is the center of some open ball contained in U, U is open. ■

(ii) Let $x_1 \in V$. Since O is open, there exists an open ball B of radius $r > 0$ centered at x_1 with $B \subseteq O$. Let $x_2 \in B$. Since balls in $\mathbb{R}^n$ are convex, the segments $S(x_2, x_1) \subseteq O$. If it were the case that x_2 is not in V, then, by the above lemma, x_1 is not in V, which is a contradiction. Thus, $x_2 \in V$, so that $B \subseteq U$. Then, since every point in V is the center of some open ball contained in V, V is open. ■

Problem 6

Suppose, for sake of contradiction, that Y is not connected. Then, there exists some set $U_1 \in \tau_X$ (the topology on X) such that $V_1 := U_1 \cap Y$ is clopen in the topology τ_Y on Y (and V_1 is neither Y nor $\emptyset$). Since V_1 is closed in τ_Y, there is some $U_2 \in \tau_X$ such that $V_2 := U_2 \cap Y = Y \backslash U_1$ is open in τ_X.

Since $S := \bigcap_{i \in I} Y_i$ is nonempty, let $x \in S$. Since V_1 and V_2 partition Y, $x \in V_1$ or $x \in V_2$; without loss of generality, $x \in V_1$ (the proof is identical up to variable names when $x \in V_2$). Let $y \in V_2$ (since V_2 is nonempty), so that, for some $i \in I$, $y \in Y_i$. Since $x, y \in Y_i$, $W_1 := U_1 \cap Y_i = V_1 \cap Y_i$ and $W_2 := U_2 \cap Y_i = V_2 \cap Y_i$ are nonempty. Furthermore, W_1 and W_2 are open in τ_{Y_i}, the topology induced on Y_i by τ_X. Thus, since $W_2 = Y_i \backslash W_1$, W_1 is clopen in τ_{Y_i}, contradicting the fact that Y_i is connected. ■

Problem 7

Let G be the given graph, and suppose $f : [a, b] \to G$, with $f(a) = (0, 1), f(b) = (1/\pi, 1)$. Let $c := \sup\{x : f(x) = (0, 1)\}$, and let $\delta > 0$. $\exists n \in \mathbb{N}$ such that $frac12\pi n < f_1(d + \delta)$, the first component of $f(d + \delta)$. Then, since $\sin(2\pi n) = 0$, so that by the Intermediate Value Theorem, $\exists x \in (c, c + \delta)$ such that $f(x) = (frac12\pi n, 0)$, and thus $\|f(x) - f(c)\| > 1$, despite $|c - x| < \delta$, implying f is not continuous. Thus, G is not pathwise connected.

Suppose, for sake of contradiction, that G is not connected, so that there exists a clopen set $S \in \tau_G$ (where τ_G is the topology induced on G by the standard topology τ on $\mathbb{R}^2$) that is neither $\emptyset$ nor G. Let $T := G \backslash S$. Either $(1/\pi, 1) \in S$ or $(1/\pi, 1) \in T$; without loss of generality, $(1/\pi, 1) \in S$ (the proof is identical in the case $(1/\pi, 1) \in T$, up to some variable names). It must be the case that $T \backslash \{(0, 1)\} = \emptyset$, since the alternative would imply that $G \backslash \{(0, 1)\}$ is disconnected, contradicting the fact that the function $x \mapsto (x, f(x))$ is continuous on $(0, 1]$, which is a convex and thus connected set.

Thus, since T is nonempty, $T = \{(0, 1)\}$. Since $T \in \tau_G$, $\exists U \in \tau$ such that $T = G \cap U$. Since U is open, $\exists \delta > 0$ such that the ball $B := B((0, 1), \delta) \subseteq U$, so that $B \cap G \subseteq T$. However, if we pick $x = \frac{1}{2\pi n} < \delta$, then $(x, 1) \in T$, which is a contradiction. Therefore, $T = \emptyset$, so that the only clopen sets $S \subseteq G$ are $S = \emptyset$ and $S = G$, and thus G is connected. $\blacksquare$