21-640 **Intro to Functional Analysis** **Spring 2013**

A Few Practice Problems on Hilbert Spaces

1. Let X be a Hilbert space and $A \in \mathcal{L}(X)$ be given. Show that
$$\|Ax\| = \|x\| \text{ for all } x \in X$$
if and only if
$$(Ax, Ay) = (x, y) \text{ for all } x, y \in X.$$

2. Let X be a Hilbert space and $A \in \mathcal{L}(X; X)$ be given. Assume that $A_H^* = A$ and there there exists $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in X$. What can you conclude regarding A?

3. Let X be a complex Hilbert space and let $A \in \mathcal{L}(X; X)$ be given. Show that $A_H^* = A$ if and only if $(Ax, x) \in \mathbb{R}$ for all $x \in X$.

4. Let X be a Hilbert space and $A \in \mathcal{L}(X; X))$ be given. Show that $A_H^* A = A A_H^*$ if and only if $\|Ax\| = \|A_H^* x\|$ for all $x \in X$.

5. Let X be a complex Hilbert space and $A \in \mathcal{L}(X; X)$. Show that A is compact if and only if $(Ax_n, x_n) \to 0$ as $n \to \infty$ for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \rightharpoonup 0$ (weakly) as $n \to \infty$. What happens with regard to this result in real Hilbert spaces?