21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
Spring 2012: Monday, Wednesday, Friday, 10:30 am, Doherty Hall 1211.
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Assignment 3 - Thursday February 23, 2012. Due Wednesday February 29

Exercise 11: On $V = \mathbb{C}^n$, one considers the norms $||v||_1 = \sum_{i=1}^n |v_i|$, $||v||_2 = \sqrt{\sum_{i=1}^n |v_i|^2}$, and $||v||_\infty = \max_{i=1}^n |v_i|$. Each of these norm induces on $L(\mathbb{R}^n, \mathbb{R}^n)$ a norm $|||\cdot|||_j$, defined by $|||A|||_i = \sup_{v\neq 0} \frac{||A\,v||_i}{||v||_i}$.
i) Compute $|||A|||_1$ and $|||A|||_\infty$. Using the fact that $|||A||| < 1$ implies that $I - A$ is invertible (since the series $\sum_{k\geq 0} A^k$ converges), deduce that (for $n \geq 2$) each eigenvalue of A belongs to $D_i \subset \mathbb{C}$ for some $i \in \{1, \ldots, n\}$, where D_i is the closed disc $\{z \in \mathbb{C} \mid |z - A_{i,i}| \leq R_i = \sum_{j\neq i} |A_{i,j}|$.
ii) For $n \geq 2$, show that each eigenvalue of A belongs to $\Delta_{i,j} \subset \mathbb{C}$ for some $i \neq j \in \{1, \ldots, n\}$, where $\Delta_{i,j} = \{z \in \mathbb{C} \mid |z - A_{i,i}|\,|z - A_{j,j}| \leq R_i R_j\}$.
iii) What is $|||A|||_2$?

Exercise 12: If V is an Euclidean space, one writes $M_2 \geq M_1$ for $M_1, M_2 \in L(V, V)$ to mean that $(M_2 v, v) \geq (M_1 v, v)$ for all $v \in V$.
i) Show that $\geq$ is not a (partial) order on $L(V, V)$ if the dimension of V is ≥ 2. Show that $\geq$ is a (partial) order on $L_s(V, V)$, the space of symmetric mappings from V into itself.
ii) If $dim(V) = n$, and $A, B \in L_s(V, V)$ satisfy $A \geq 0, B \geq 0$, show that $\sum_{i,j=1}^n A_{i,j} B_{i,j} \geq 0$ in any orthonormal basis. Show that if, for a given orthonormal basis, $A \in L_s(V, V)$ satisfies $\sum_{i,j=1}^n A_{i,j} B_{i,j} \geq 0$ for all $B \in L_s(V, V)$ with $B \geq 0$, then one has $A \geq 0$.
iii) For $V = \mathbb{R}^2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, what is the set $\mathcal{X}$ of $M \in L_s(V, V)$ satisfying $M \geq A$ and $M \geq B$. Show that $\mathcal{X}$ has no minimum, i.e. there is no $M_0 \in \mathcal{X}$ satisfying $M_0 \leq M$ for all $M \in \mathcal{X}$.

Exercise 13: Notation of Exercise 12.
i) For $P \in \mathbb{R}[x]$, show that $a\,I \leq M \leq b\,I$ implies $\alpha\,I \leq P(M) \leq \beta\,I$ with $\alpha = \min\{P(s) \mid a \leq s \leq b\}$ and $\beta = \max\{P(s) \mid a \leq s \leq b\}$. Deduce that $||P(M)|| \leq \max\{|\alpha|, |\beta|\}$.
ii) Show that for $A, B \in L_s(V, V)$, the operator $DP(A)\!:\!B = \lim_{0\neq\varepsilon\to 0} \frac{P(A+\varepsilon\,B)-P(A)}{\varepsilon}$ exists (notice that A and B may not commute). If $dim(V) = n$, in an orthonormal basis of eigenvectors of A with $A\,e_i = \lambda_i e_i$ and if the λ_i are distinct, show that $C = DP(A)\!:\!B$ means $C_{i,j} = B_{i,j} K_{i,j}$ for all i, j, with $K \in L_s(V, V)$ defined by $K_{i,i} = P'(\lambda_i)$ for all i and $K_{i,j} = \frac{P(\lambda_i)-P(\lambda_j)}{\lambda_i-\lambda_j}$ for all $i \neq j$.

Exercise 14: Notation of Exercise 12 and Exercise 13.
i) For $a < b$ given, show that the condition $a\,I \leq M_1 \leq M_2 \leq b\,I$ implies $P(M_1) \leq P(M_2)$ is equivalent to $K \geq 0$ for all choices $a \leq \lambda_1 < \ldots < \lambda_n \leq b$.
ii) For $a < b$ given, show that the condition $a\,I \leq M_1 \leq M_2 \leq b\,I$ implies $||P(M_2) \leq P(M_1)|| \leq \kappa\,||M_2 - M_1||$ is equivalent to the condition that for all choices $a \leq \lambda_1 < \ldots < \lambda_n \leq b$ the eigenvalues $\mu_1, \ldots, \mu_n$ of K satisfy $\max_j |\mu_j| \leq \kappa$.

Exercise 15: Notation of Exercise 12. Let V be a finite dimensional Euclidean space, and $A \in L_s(V, V)$ satisfying $0 \leq A \leq I$. One defines $B_0 = 0$, and then by induction $2B_{n+1} = A + B_n^2$ for $n \geq 0$. Show that $B_n \in L_s(V, V)$, and $0 \leq B_1 \leq B_2 \leq \ldots \leq I$. Deduce that B_n converges to a limit $B_\infty = \Phi(A)$. Show that $I - \Phi(A)$ is the unique element $C \in L_s(V, V)$ with $C \geq 0$ and $C^2 = I - A$.