21-373, Algebraic Structures, Department of Mathematical Sciences, Carnegie Mellon University
Fall 2011: (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.
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Assignment 3 - Solution

Exercise 15: *Prove that D_{12} and S_4 are not isomorphic.*
Solution: Both groups have order 24, but D_{12} has a subgroup isomorphic to $\mathbb{Z}_{12}$, so that it has (at least) $\varphi(12) = 4$ elements of order 12 (but since the 12 elements outside this subgroup have order 2, there are exactly 4 elements of order 12), so that it cannot be isomorphic to S_4, which has no element of order 12 (since the orders of elements in S_4 are 1, 2, 3, or 4).

Exercise 16: *Write the cycle decompositions of all the elements of order 4 in S_4, and of all the elements of order 2 in S_4.*
Solution: In general, any $a \in S_n$ different from e (because one does not write down the cycles of length 1) can be written as a product of disjoint cycles, and the order of a is the least common multiple of the lengths of the cycles. a has order 4 if and only if it has cycles of length 2 or 4 and at least one cycle of length 4. For $n = 4$, there is only room for one cycle, so that one makes it start with 1 and after the 1 one writes the six permutations of 2, 3, and 4, hence there are six such elements of order 4, $(1\,2\,3\,4)$, $(1\,2\,4\,3)$, $(1\,3\,2\,4)$, $(1\,3\,4\,2)$, $(1\,4\,2\,3)$, and $(1\,4\,3\,2)$. a has order 2 if and only if it only has cycles of length 2 (transpositions), and at least one such cycle, so that for $n = 4$, there is only room for one or two cycles of length 2, hence there are six elements of order 2 having only one cycle of length 2, $(1\,2)$, $(1\,3)$, $(1\,4)$, $(2\,3)$, $(2\,4)$, and $(3\,4)$, and there are three elements of order 2 having two cycles of length 2, $(1\,2)\,(3\,4)$, $(1\,3)\,(2\,4)$, and $(1\,4)\,(2\,3)$.

Exercise 17: *Let σ the 8-cycle $(1\,2\,3\,4\,5\,6\,7\,8)$, τ the 12-cycle $(1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12)$, and ω the 14-cycle $(1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\,13\,14)$. For which positive integer i is σ^i an 8-cycle? For which positive integer j is τ^j a 12-cycle? For which positive integer k is ω^k a 14-cycle?*
Solution: In a cycle of length n, the image of i is $i+1$ modulo n, and in its kth power it is $i+k$ modulo n, so that if the greatest common divisor (k,n) is $d > 1$, the kth power has cycles of length $\frac{n}{d}$; one then needs k to relatively prime with n for the kth power to be also a cycle of length n.

i must then be relatively prime with 8, so that $i = 1,3,5,7 \pmod 8$, but since one wants $i > 0$, it means that i has one of the forms $1+8m, 3+8m, 5+8m, 7+8m$ for some $m \geq 0$.

j must then be relatively prime with 12, so that $j = 1,5,7,11 \pmod 8$, but since one wants $j > 0$, it means that j has one of the forms $1+12m, 5+12m, 7+12m, 11+12m$ for some $m \geq 0$.

k must then be relatively prime with 14, so that $j = 1,3,5,7,11,13 \pmod 8$, but since one wants $k > 0$, it means that k has one of the forms $1+14m, 3+14m, 5+14m, 7+14m, 11+14m, 13+14m$ for some $m \geq 0$.

Exercise 18: *Show that in the three following cases, the centralizer of H is H, and the normalizer of H is G:*

i) $G = S_3$ *and* $H = \{e, (123), (132)\}$,
ii) $G = D_4$ *and* $H = \{e, a^2, b, a^2 b\}$,
iii) $G = D_5$ *and* $H = \{e, a, a^2, a^3, a^4\}$.

[In a group G, for any subset $X \subset G$, the centralizer of X is $C_G(X) = \bigcap_{x \in X} C_G(x)$ (where the centralizer $C_G(x)$ is the stabilizer of x for the action of conjugation, i.e. $\{g \in G \mid g\,x = x\,g\}$). In D_n, a denotes an element of order n and b an element of order 2, satisfying $b\,a = a^{-1} b$.]
Solution: In each of the three cases, H is normal subgroup of G, because H has half the number of elements of G, so that $N_G(H) = G$.

In each of the three cases G is non-Abelian, and H is Abelian, since it is isomorphic to $\mathbb{Z}_3$ in the first case, to $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the second case, and to $\mathbb{Z}_5$ in the third case, so that $C_G(H)$ contains H and since a centralizer is a subgroup, $C_G(H)$ is either H or G. Since $C_G(e) = G$ in any group, it is not just the fact that G is non-Abelian which can give the result, and again the fact that H has half the number of elements of G helps: for any $a \in G \setminus H$, one has $G = H \cup a\,H$, and a cannot belong to $C_G(H)$, for if a was commuting with all the elements of H then for any $h_1, h_2, h_3 \in H$ one would deduce that $a\,h_1$ commutes with h_2 and $a\,h_3$, and since h_1 also commutes with h_2 and $a\,h_3$, the group G would be Abelian.

Exercise 19: *For $m \geq 1$ and $q_1, \ldots, q_m \in \mathbb{Q}^*$, prove that the (finitely generated) subgroup $H = \langle q_1, \ldots, q_m \rangle$ of $\mathbb{Q}$ is a subgroup of $K = \left\langle \frac{1}{D} \right\rangle$, where D is the least common multiple of the denominators of $q_1, \ldots, q_m$. Show that H is cyclic (hence $\mathbb{Q}$ is not finitely generated).*
Solution: One writes $q_i = \frac{a_i}{b_i}$ with $a_i \in \mathbb{Z}, b_i \in \mathbb{N}^\times$ and $(a_i, b_i) = 1$, and if D is the least common multiple of $b_1, \ldots, b_m$, one has $D\, q_i = c_i \in \mathbb{Z}$, and $H = \left\{ q = \sum_{i=1}^m k_i q_i \mid k_1, \ldots, k_m \in \mathbb{Z} \right\} = \left\{ q = \frac{1}{D} \sum_{i=1}^m k_i c_i \mid k_1, \ldots, k_m \in \mathbb{Z} \right\}$, but $K = \left\{ \sum_{i=1}^m k_i c_i \mid k_1, \ldots, k_m \in \mathbb{Z} \right\}$ is a subgroup of $\mathbb{Z}$, which is not reduced to $\{0\}$, so that it is $d\,\mathbb{Z}$ for some $d \in \mathbb{N}^\times$, hence H is made of the multiples of $\frac{d}{D}$; in particular, $H \neq \mathbb{Q}$, so that $\mathbb{Q}$ is not a finitely generated (Abelian) group.

Exercise 20: *A non trivial Abelian group G is called divisible if for each $a \in G$ and each positive integer k there exists $b \in G$ with $k\, b = a$. Show that $\mathbb{Q}$ is divisible, that no finite Abelian group is divisible, and that $G_1 \times G_2$ is divisible if and only if both G_1 and G_2 are divisible.*
Solution: For $a \in \mathbb{Q}$ and $k \in \mathbb{N}^\times$, $k\, b = a$ has the (unique) solution $b = a\, \frac{1}{k}$.

If in a divisible Abelian group, an element $a \neq 0$ has a finite order d, and p is a prime divisor of d, then $c = \frac{d}{p}\, a$ has finite order p, so that $c \neq 0$. If for $m \geq 1$, one then solves $p^m b = c$, one deduces that $p^{m+1} b = p\, c = 0$, so that b has a finite order which divides p^{m+1}, but it does not divide p^m (since $c \neq 0$), hence it is p^{m+1}, and letting m vary gives infinitely many different elements (since they have different orders), so that G is infinite. To be complete, one must consider the other case, where no non-zero element has a finite order, but it only happens for an infinite group, since in a finite group G every element has an order which divides $|G|$.

If $G_1 \times G_2$ is divisible, then solving $k\,(b_1, b_2) = (g_1, 0)$ or $k\,(b_1, b_2) = (0, g_2)$ gives solutions of $k\, b = g_1 \in G_1$ and of $k\, b = g_2 \in G_2$, so that both G_1 and G_2 are divisible. Conversely, if both G_1 and G_2 are divisible, one solves $k\, b_1 = g_1$ in G_1 and $k\, b_2 = g_2$ in G_2, and one then has $k\,(b_1, b_2) = (g_1, g_2)$ in $G_1 \times G_2$.

Exercise 21: *Show that the group of rigid motion symmetries of a platonic solid (tetrahedron, cube, octahedron, dodecahedron, icosahedron) have respectively orders 12, 24, 24, 60, 60, i.e. $2E$, where E is the number of edges. Show that for the tetrahedron this group is isomorphic to a subgroup of S_4, and that for the cube or the octahedron this group is isomorphic to S_4.*
[A Platonic solid is a convex polyhedron which is regular, so that its faces all are regular polygons with k sides, and ℓ edges arrive at each vertex, so that the number of faces F, of edges E, and of vertices V satisfy $k\, F = \ell\, V = 2E$; using $k, \ell \geq 3$ (which implies $k, \ell \leq 5$) and the relation $F - E + V = 2$ (that the Euler characteristic of the sphere $\mathbb{S}^2$ is 2), one finds there are five such regular polyhedron: the tetrahedron (4 triangular faces), the hexahedron = cube (6 square faces), the octahedron (8 triangular faces), the dodecahedron (12 pentagonal faces), and the icosahedron (20 triangular faces).]
Solution: If for each polygonal face one puts a mark on each of its k edges, each edge ends up with two marks, giving $F\, k = 2E$. If for each vertex one puts a mark on each of its ℓ edges arriving there, each edge ends up with two marks, giving $V\, \ell = 2E$. Assuming that $F - E + V = 2$, one finds that the only solutions are the five Platonic polyhedra: tetrahedron ($F = 4$, $E = 6$, $V = 4$, $k = 3$, $\ell = 3$), hexahedron = cube ($F = 6$, $E = 12$, $V = 8$, $k = 4$, $\ell = 3$), octahedron ($F = 8$, $E = 12$, $V = 6$, $k = 3$, $\ell = 4$), dodecahedron ($F = 12$, $E = 30$, $V = 20$, $k = 5$, $\ell = 3$), icosahedron ($F = 20$, $E = 30$, $V = 12$, $k = 3$, $\ell = 5$).

If one selects two adjacent vertices A and B, one counts the number of rigid motions sending the regular polyhedron on itself by picking A' as one of the V vertices and putting A there, and then by picking B' as one of the ℓ neighbours of A' and putting B there (and after that one rotates around the axis $A'B'$ so that the two faces adjacent to the edge $A\,B$ end up on the two faces adjacent to the edge $A'B'$): the number of possibilities is then V times ℓ, i.e. $2E$ (12 for the tetrahedron, 24 for the cube and for the octahedron, 60 for the dodecahedron and for the icosahedron). Composition of two such rigid motions, or the inverse of such a rigid motion is also a rigid motion, so that one has a group (and e is the rigid motion $x \mapsto x$ for all x in the polyhedron).

Since the tetrahedron has 4 vertices, which are mapped to the 4 vertices, it gives an element of S_4, and the group of rigid motions is then a subgroup of S_4. For the cube, one observes that two opposite vertices are always sent to opposite vertices (since if each edge has length 1, it is the only vertex at distance $\sqrt{3}$), so that it gives a permutation of the 4 diagonals joining a vertex to its opposite vertex, hence the group of rigid motions is a subgroup of S_4, but since it has the same order, it is S_4. The same observation holds for the octahedron if one replaces opposite vertices by opposite faces.