Homework 6

21-640 Introduction to Functional Analysis
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Problem 1

Suppose $\alpha_n \to 0$ as $n \to \infty$. Consider the sequence $\{T_k\}_{k=1}^{\infty}$ of operators in $\mathcal{L}(l^2; l^2)$ defined for all $x \in l^2$ by

$$(T_k x)_n = \begin{cases} \alpha_n x_n & : \text{ if } n \leq k \\ 0 & : \text{ else} \end{cases}.$$

Suppose $\varepsilon > 0$. Then, for n_0 sufficiently large, $\forall n \geq n_0$, $\alpha_n < \epsilon$, and hence, $\forall x \in l^2$ with $\|x\|_2 = 1$,

$$\|(T_n - T)x\|_2 \leq \varepsilon \|x\|_2 = \varepsilon,$$

so $T_n \to T$ in the uniform operator topology. Thus, by Proposition 11.8 and Theorem 12.11, since each T_n has finite rank and is thus compact, T is compact. Thus, $\alpha_n \to 0$ is sufficient for T compact.

Suppose $\alpha_n \not\to 0$ as $n \to \infty$. Then, since,

$$\|Te^{(n)} - Te^{(m)}\|_2 = \alpha_n^2 + \alpha_m^2, \quad \forall n, m \in \mathbb{N},$$

$\{Te^{(n)}\}$ has no Cauchy subsequence and hence no convergent subsequence. Thus, $\alpha_n \to 0$ is also necessary for T compact. ■

Problem 2

I solved this problem with a hint from Jimmy Murphy suggesting that I design components of Tx to be 'local averages' of components of x.

To construct a counterexample T, we make use of the triangle numbers defined by

$$t_1 = 0, t_{k+1} = t_k + k, \forall k \in \mathbb{N}.$$

Define $T : l^2 \to l^2$ for all $x \in l^2, n \in \mathbb{N}$ by

$$(Tx)_n := \frac{1}{k} \sum_{n=t_k}^{t_{k+1}-1} x_n, \quad \text{for} \quad t_k \leq n \leq t_{k+1} - 1,$$

$$\left(\text{so} \quad Tx = \left(x_1, \frac{x_2 + x_3}{2}, \frac{x_2 + x_3}{2}, \frac{x_4 + x_5 + x_6}{3}, \frac{x_4 + x_5 + x_6}{3}, \frac{x_4 + x_5 + x_6}{3}, \dots \right)\right).$$

T is linear, since as each component is a linear combination of components of x. To see that T is continuous (and $\|T\| \leq 1$), it suffices observe that, by the Cauchy-Schwarz inequality, $\forall x \in l^2, k \in \mathbb{N}$,

$$\sum_{n=t_k}^{t_{k+1}-1} ((Tx)_n)^2 = k \left(\sum_{n=t_k}^{t_{k+1}-1} \frac{x_n}{k} \right)^2 \leq k \left(\sum_{n=t_k}^{t_{k+1}-1} x_n^2 \right) \left(\sum_{n=t_k}^{t_{k+1}-1} \frac{1}{k^2} \right) = \sum_{n=t_k}^{t_{k+1}-1} x_n^2.$$

Thus, $T \in \mathcal{L}(l^2; l^2)$. Also, $\forall n \in \mathbb{N}$ with $t_k \leq n \leq t_{k+1} - 1$, $\|Te^{(n)}\|_2 = \frac{1}{k^2}$, so $Te^{(n)} \to 0$ strongly. Now consider $\{x^{(n)}\}_{n=1}^{\infty}$ in l^2 defined by

$$x_i^{(n)} := \begin{cases} \frac{1}{\sqrt{n}} & : \text{ if } t_n \leq i \leq t_{n+1} - 1 \\ 0 & : \text{ else} \end{cases} .$$

$\{x^{(n)}\}_{n=1}^{\infty}$ is bounded, since the n non-zero components of $x^{(n)}$ are identically $\frac{1}{\sqrt{n}}$, giving

$$\|x^{(n)}\| = n \left(\frac{1}{\sqrt{n}} \right)^2 = 1.$$

Furthermore, $\forall n \in \mathbb{N}$, $Tx^{(n)} = x^{(n)}$, and so, for $n, m \in \mathbb{N}$,

$$\|Tx^{(n)} - Tx^{(m)}\| = n \left(\frac{1}{\sqrt{n}} \right)^2 + m \left(\frac{1}{\sqrt{m}} \right)^2 = 2,$$

and so $\{Tx^{(n)}\}_{n=1}^{\infty}$ has no convergent subsequence. Consequently, T is not compact. ■

Problem 3

Let $L, R \in \mathcal{L}(l^2; l^2)$ denote the shift operators defined by

$$Lx = (x_2, x_3, \dots), \quad Rx = (0, x_1, x_2, \dots), \quad \forall x \in l^2.$$

$\forall n \in \mathbb{N}$, put $T_n = L^n$ and $T = 0$. $\forall x \in l^2$,

$$\|T_n x\|_2 = \sum_{k=n+1}^{\infty} x_k^2 \to 0$$

as $n \to \infty$, and so $T_n \to T$ in the strong operator topology. An easy induction argument using $(L^{n-1}L)^* = L^*(L^{n-1})^*$ shows $T_n^* = R^n$. Thus, $\|T_n^* x\|_2 = \|x\|_2$, so $T_n^* \not\to 0 = T^*$ as $n \to \infty$. ■

Problem 4

Assume X is complete and Y is weakly sequentially complete (e.g., by Theorem 8.5, if Y is reflexive).

For all $x \in X$, the condition that $\{y^*(T_n x)\}_{n=1}^{\infty}$ is convergent for all $y^* \in Y^*$ is equivalent to $\{T_n x\}_{n=1}^{\infty}$ being weakly convergent. Thus, we can define $T : X \to Y$ by assigning Tx to be the weak limit of $\{T_n x\}_{n=1}^{\infty}$ (which is in Y by weak sequential completeness). Since the weak limit operator is linear, T is linear. Since X is complete, by the Principle of Uniform Boundedness, T is bounded, and hence $T \in \mathcal{L}(X; Y)$. ■

Problem 5

(a) Let $X = l^2, Y = l^1$, and suppose $T \in \mathcal{L}(X;Y)$. If $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in X, by Theorem 8.1, $\{x_n\}_{n=1}^{\infty}$ has a weakly convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Since continuous linear operators respect weak convergence, $\{Tx_{n_k}\}_{k=1}^{\infty}$ is weakly convergent, and hence, since, in l^1, weak convergence is equivalent to strong convergence, $\{Tx_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{Tx_n\}_{n=1}^{\infty}$. Thus, $T \in C(X;Y)$. Since $\mathcal{C}(X;Y) \subseteq \mathcal{L}(X;Y)$, $\mathcal{C}(X;Y) = \mathcal{L}(X;Y)$. ■

(b) I wasn't able to solve this problem.

Problem 6

(a) T is not surjective. If it were, by the Open Mapping Theorem, $T[B_1(0)]$ would be open and hence contain a non-empty ball B. Since Y is infinite dimensional, B would contain a sequence with no convergent subsequence, contradicting the compactness of T. ■

(b) Let $X = (l^\infty, \|\cdot\|_\infty)$, and let $Y = (l^\infty, \|\cdot\|)$ where $\|x\| := \sup_{n\in\mathbb{N}} x_n/n$, and let $I \in \mathcal{L}(X;Y)$ be the identity (which is continuous, since clearly $\|\cdot\|$ is bounded by $\|\cdot\|_\infty$, and note that I is surjective.

Now consider the sequence $\{I_k\}_{k=1}^{\infty}$ in $\mathcal{L}(X;Y)$ defined, $\forall x \in l^\infty, n \in \mathbb{N}$ by

$$(I_k x)_n = \begin{cases} \alpha_n x_n & : \text{ if } n \leq k \\ 0 & : \text{ else} \end{cases}.$$

$\forall x \in l^2$ with $\|x\| = 1$, $\|(I_k - I)x\|_2 \leq \frac{1}{k}\|x\| = \frac{1}{k}$, and so $I_k \to I$ in the uniform operator topology. Thus, by Proposition 11.8 and Theorem 12.11, since each I_n has finite rank and is thus compact, I is compact. ■

Problem 7

(a) Suppose $T \in \mathcal{L}(l^2; l^2)$. As shown in the solution to part (a) of Problem 5, T is compact. Then, by part (a) of Problem 6, T is not surjective. ■

(b) By the identification of $(c_0)^*$ with l^1 and $(l^2)^*$ with $(l^2)^*$, if $L \in \mathcal{L}(c_0; l^2)$, then, as shown in the solution to part (a) of Problem 5, $L^* : l^2 \to l^1$ would be compact. Then, by Theorem 11.15, T would be compact, and hence, by part (a) of Problem 6, T is not surjective.

Problem 8

If $x = 0$, then, as $n \to \infty$, $\|x_n\| \to \|x\|$ immediately implies $x_n \to x$. Thus, we assume $x \neq 0$. Then, since $\|x_n\| \to \|x\|$ as $n \to \infty$, by considering only n sufficiently large, we may assume, without loss of generality, that each $x_n \neq 0$, and we may therefore define

$$z := \frac{x}{\|x\|} \quad \text{and } z_n := \frac{x_n}{\|x_n\|}, \quad \forall n \in \mathbb{N}.$$

It suffices to show $z_n \to z$ strongly, and so, since each $\|z_n\| = \|z\| = 1$, by uniform convexity, it suffices to show that $\|z_n + z\| \to 2$. By part (iii) of Theorem 7.15, since $z_n + z \rightharpoonup 2z$ weakly,

$$2 = \|2z\| \leq \liminf_{n\to\infty} \|z_n + z\| \leq \limsup_{n\to\infty} \|z_n + z\| \leq \limsup_{n\to\infty} \|z_n\| + \|z\| = \|z\| + \|z\| = 2.$$

Problem 9

(a) Let $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$, and let $t \in (0,1)$. Since $\|tx + (1-t)y\| \leq t\|x\| + (1-t)\|y\| = 1$, it suffices to show that $\|tx + (1-t)y\| \neq 1$. If it were the case that $\|\frac{1}{2}x + \frac{1}{2}y\| = 1$, then $\|x+y\| = 2$, and hence, by uniform convexity (using the constant sequences $\{x\}_{i=1}^{\infty}, \{y\}_{i=1}^{\infty}$), $x = y$. Thus, it suffices to show that, if $\|tx+(1-t)y\| = 1$, then $\|\frac{1}{2}x + \frac{1}{2}y\| = 1$.

The case $t = 1/2$ is trivial, and the case $t \in (1/2, 1)$ follows by switching x and y, so we may assume $t \in (0, 1/2)$. Then, $tx + (1-t)y$ is a convex combination of x and $\frac{1}{2}x + \frac{1}{2}y$, and so, for some $t_2 \in (0,1)$,

$$1 = \|tx + (1-t)y\| = \|t_2 x + (1-t_2)(\frac{1}{2}x + \frac{1}{2}y)\| \leq t_2 + (1-t_2)\|(\frac{1}{2}x + \frac{1}{2}y)\|,$$

and so $1 \leq \|(\frac{1}{2}x + \frac{1}{2}y)\| \leq 1$. ■

(b) Let $X' := \prod_{i=1}^{\infty} \mathbb{R}^2$, let

$$X := \left\{ x \in X' : \sum_{i=1}^{\infty} \|(x_i, y_i)\|_k < \infty \right\},$$

(where $\|\cdot\|_k$ denotes the usual k-norm on $\mathbb{R}^2$) and, define $\|\cdot\| : X \to \mathbb{R}$ for all $x \in X$ by

$$\|x\| = \sum_{i=1}^{\infty} \|(x_i, y_i)\|_k.$$

The proof that $(X, \|\cdot\|)$ is a Banach space is essentially identical to the proof for l^1.

Suppose $x, y \in X$, $\|x\| = \|y\| = 1$ and $x \neq y$ (say $x_n \neq y_n$). Then, since each $(R^2, \|\cdot\|_k)$ is strictly convex, $\|tx_n + (1-t)y_n\| < \frac{\|x_n\|_n + \|y_n\|_n}{2}$ (and each $\|tx_k + (1-t)y_k\| \leq \frac{\|x_k\|_k + \|y_k\|_k}{2}$), and so

$$\|tx + (1-t)y\| = \sum_{k=1}^{\infty} \|tx_k + (1-t)y_k\|_k < \sum_{k=1}^{\infty} \frac{\|x_i\|_k + \|y_i\|_k}{2} = \frac{1}{2}(\|x\| + \|y\|) = 1.$$

Thus, X is strictly convex. However, suppose $\forall n \in \mathbb{N}, x^{(n)}$ and $y^{(n)}$ have $x_n^{(n)} = (1,0), y_n^{(n)} = (0,1)$, and $x_i^{(n)} = y_i^{(n)} = 0$ for $i \neq n$. Then, each $\|x^{(n)}\| = \|y^{(n)}\| = 1$ and, as $n \to \infty$,

$$\|x^{(n)} + y^{(n)}\| = 2^{\frac{1}{1+1/n}} \to 2$$

but

$$\|x^{(n)} - y^{(n)}\| = 2^{\frac{1}{1+1/n}} \to 2.$$

Therefore, X is not uniformly convex. The proof that X is separable is similar to the proof of separability for l^1. I'm not quite sure about reflexivity...