1 Mastery set [25 points]

For any $p \in [1, \infty]$, let

$$B_p := \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$$

denote the p-norm unit ball.

Q1. [2+2+3]

(a) The projection is given in each coordinate by

$$(P_{\mathbb{R}^n_+}(x))_i = \max\{0, x_i\}.$$

(b) The projection is

$$P_{B_2}(x) = \begin{cases} x & : \text{ if } \|x\|_2 \le 1 \\ \frac{x}{\|x\|} & : \text{ else} \end{cases}.$$

(c) The projection is given in each coordinate by

$$(P_{B_\infty}(x))_i = \begin{cases} -1 & : \text{ if } x_i < 1 \\ x_i & : \text{ if } -1 \le x \le 1 \\ 1 & : \text{ else} \end{cases}.$$

Q2. [3+3]

(a) Let

$$x^* := \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \frac{1}{2}\|x - z\|_2^2 - \lambda \sum_{i=1}^n \log(|x_i|).$$

Taking an appropriate derivative in each coordinate gives

$$0 = x_i^* - z_i - \frac{\lambda}{x_i^*} = (x_i^*)^2 - z_i x_i^* - \lambda.$$

The quadratic formula gives

$$\boxed{x_i^* = \frac{z_i \pm \sqrt{z_i^2 + 4\lambda}}{2}.}$$

(Since the regularization term decreases away from 0, we use the negative value if $z_i < 0$ and the positive value otherwise).

(b) Let

$$x^* := \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \frac{1}{2}\|x - z\|_2^2 + \lambda(x^T A x + b^T x + c).$$

Taking an appropriate gradient gives

$$\begin{aligned} 0 &= (x^* - z) + \lambda((A + A^T)x^* + b) \\ \text{or} \quad z - \lambda b &= (2\lambda A + I)x^*, \end{aligned}$$

since $A = A^T$. Since A is positive semidefinite and $\lambda \ge 0$, $(2\lambda A + I)$ is invertible, so

$$\boxed{x^* = (2\lambda A + I)^{-1}(z - \lambda b).}$$

Q3. [4+4+4]

(a) The dual program is

$$\begin{aligned} \min_{u \in \mathbb{R}^4} \quad & 4u_1 + 2u_2 \\ \text{such that} \quad & u_1 + u_2 \ge 2, \\ & -u_1 - u_2 \ge 1, \\ \text{and} \quad & u_1, u_2 \ge 0. \end{aligned}$$

The original problem is unbounded, and the dual is unfeasible.

(b) The dual program is

$$\begin{aligned} \min_{u \in \mathbb{R}^4} \quad & -4u_1 + 2u_2 \\ \text{such that} \quad & -u_1 + u_2 \ge 2, \\ & -u_1 + u_2 \ge 1, \\ \text{and} \quad & u_1, u_2 \ge 0. \end{aligned}$$

The original problem is infeasible, and the dual is unbounded.

(c) The dual program is

$$\begin{aligned} \min_{u \in \mathbb{R}^4} \quad & -4u_1 + 2u_2 \\ \text{such that} \quad & -u_1 + u_2 \ge 2, \\ & u_1 - u_2 \ge 1, \\ \text{and} \quad & u_1, u_2 \ge 0. \end{aligned}$$

The original and dual programs are both infeasible.