Homework 2
36-705
Due: Thursday Sept 18 by 3:00
Do not submit homework by email. Bring a paper copy to Mari-Alice.

1. Let $X_1, \ldots, X_n$ be independent random variables. We do not assume that they are identically distributed. Let $\mu_i = \mathbb{E}(X_i)$. Assume that $a_i \leq X_i \leq b_i$ for $i = 1, \ldots, n$. Prove an exponential bound for
$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i - \frac{1}{n}\sum_{i=1}^n \mu_i > t\right)$$
for $t > 0$.

2. Let X have mean μ and variance σ^2. Prove that
$$\mathbb{P}(X - \mu \geq t) \leq \frac{\sigma^2}{t^2 + \sigma^2}.$$

3. A random variable X is sub-Gaussian if there is some $c > 0$ such that
$$\mathbb{E}[e^{tX}] \leq e^{c^2 t^2/2}$$
for all t. Suppose that X is sub-Gaussian. Show that $\mathbb{E}(X) = 0$ and $\mathrm{Var}(X) \leq c^2$.
Hint: Taylor expand the moment generating function and Taylor expand $e^{c^2 t^2/2}$.

4. Let $X \sim \mathrm{Unif}(0, 1)$. Compute the moment generating function of X and show that $X - 1/2$ is sub-Gaussian.

5. Let $X_1, \ldots, X_n \sim \mathrm{Unif}(0, 1)$. Let $Y = \max_{1 \leq i \leq n} X_i$.
In class, we derived a method for bounding $\mathbb{E}[\max_{1 \leq i \leq n} X_i]$ when X_i is sub-Gaussian. Use this method to bound $\mathbb{E}(Y)$. Now find an exact expression for $\mathbb{E}(Y)$. Compare the exact result to the bound.

6. Suppose that X is sub-Gaussian. Show that there exists some $a > 0$ such that, for all $t \geq 0$, $\mathbb{P}(|X| \geq t) \leq 2e^{-at^2}$.

7. An improvement on Hoeffding's inequality is Bernstein's inequality. Let $X_1, \ldots, X_n$ be iid, with mean μ, $\mathrm{Var}(X_i) = \sigma^2$ and $|X_i| \leq c$. Then Bernstein's inequality says that
$$\mathbb{P}\left(|\overline{X}_n - \mu| > t\right) \leq 2\exp\left\{-\frac{nt^2}{2\sigma^2 + 2ct/3}\right\}.$$
(i) Show that, when σ is sufficiently small, this bound is tighter than Hoeffding's inequality for some values of t.
(ii) Suppose that $\sigma = 1/n$. Show that Hoeffding's inequality implies $\overline{X}_n - \mu = O_P\left(\sqrt{\frac{1}{n}}\right)$ but that Bernstein's inequality implies $\overline{X}_n - \mu = O_P(1/n)$.

8. Prove or disprove the following:
   (i) If $X_n = O_P(a_n)$ and $Y_n = O(b_n)$ then $X_n Y_n = O_P(a_n b_n)$.
   (ii) If $X_n = O_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n + Y_n = O_P(\max\{a_n, b_n\})$.
   (iii) If $X_n = O_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_n Y_n = o_P(a_n)$.
   (iv) If $X_n = o_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_n + Y_n = o_P(a_n + b_n)$.
   (v) If $X_n = o_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n / Y_n = o_P(a_n / b_n)$.