Lecture Notes for Week 6 (First Draft)

A General Spectral Representation Theorem

Let X be a complex Banach space and $T \in \mathcal{L}(X;X)$ be given. For $|\lambda| > r_\sigma(T)$ we have

$$R(\lambda;T) = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{T}{\lambda}\right)^n. \tag{1}$$

Let $k \in \mathbb{N}$ be given. It follows from (1) that

$$\lambda^k R(\lambda;T) = \sum_{n=0}^{\infty}\lambda^{k-n-1}T^n \quad \text{for } |\lambda| > r_\sigma(T). \tag{2}$$

Let C be a circle in the complex plane with center at 0 and radius strictly greater than $r_\sigma(T)$, oriented counterclockwise. Recall that if m is an integer, then

$$\int_C \lambda^m \, d\lambda = \begin{cases} 2\pi i & \text{if } m = -1, \\ 0 & \text{if } m \neq -1. \end{cases} \tag{3}$$

It follows from (2) and (3) that

$$\int_C \lambda^k R(\lambda;T)\, d\lambda = 2\pi i A^k.$$

We have just proved the following important result.

Theorem 6.1: Let X be a complex Banach space (with $X \neq \{0\}$) and $T \in \mathcal{L}(X;X)$ and $k \in \mathbb{N}$ be given. Let C be a circle in the complex plane with center at 0, radius strictly greater than $r_\sigma(T)$, and oriented in the counterclockwise direction. Then

$$T^k = \frac{1}{2\pi i}\int_C \lambda^k R(\lambda;T)\, d\lambda;$$

in particluar

$$T = \frac{1}{2\pi i}\int_C \lambda R(\lambda;T)\, d\lambda.$$

This gives us a way to "extend" ordinary scalar analytic functions to analytic operator-valued functions. Let $T \in \mathcal{L}(X;X)$ and $\rho > r_\sigma(T)$ be given. Assume that

$$f : \{z \in \mathbb{C} : |z| < \rho\} \to \mathbb{C}$$

is analytic and let C be a circle in $\mathbb{C}$ centered at 0, having radius γ, with $r_\sigma(T) < \gamma < \rho$, and oriented in the counterclockwise direction. Then we can define

$$f(T) = \frac{1}{2\pi i}\int_C f(\lambda)R(\lambda;T)\,d\lambda.$$

One can prove many nice properties of $f(T)$; in particular

$$\mu \in \sigma(f(T)) \;\Leftrightarrow\; \mu = f(\lambda) \text{ for some } \lambda \in \sigma(T).$$

Moreover if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } z \in \{w \in \mathbb{C} : |w| < \rho\},$$

then

$$f(T) = \sum_{n=0}^{\infty} a_n T^n.$$

Remark 6.2: In the integrals above, the circle C can be deformed to other "rectifiable" Jordan curves that lie within the domain of analyticity of f, contain $\sigma(T)$ in the interior region, and have counterclockwise orientation.

Some Simple Remarks about Spectra

Recall that an operator $B \in \mathcal{L}(X;X)$ is bijective if and only if the adjoint B^* is bijective and in this case we have

$$(B^{-1})^* = (B^*)^{-1}.$$

Moreover if $T \in \mathcal{L}(X;X)$ and $\lambda \in \mathbb{K}$ then

$$(\lambda I - T)^* = \lambda I - T^*.$$

(Here * indicates the Banach-space adjoint.)

Observe also that if $\lambda \neq 0$ and $T \in \mathcal{L}(X;X)$ is bijective then

$$\frac{1}{\lambda}I - T^{-1} \text{ is bijective } \;\Leftrightarrow\; \lambda I - T \text{ is bijective.}$$

The above observations yield the following simple, but very useful, results.

Remark 6.3: Let X be a Banach space and let $T \in \mathcal{L}(X;X)$ be given.

(a) Then $\sigma(T) = \sigma(T^*)$. Moreover, for all $\lambda \in \rho(T)$ we have $R(\lambda;T^*) = (R(\lambda;T))^*$.

(b) Assume that $0 \in \rho(T)$ and let $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then $\lambda \in \sigma(T^{-1})$ if and only if $\lambda^{-1} \in \sigma(T)$.

Applying Remark 6.3 to the special case when X is a Hilbert space we obtain the following useful observations.

Remark 6.4: Let X be a Hilbert space and let $A \in \mathcal{L}(X;X)$ be given. Let A^* denote the Hilbert space adjoint of A.

(a) Then $\sigma(A^*) = \{\lambda \in \mathbb{K} : \overline{\lambda} \in \sigma(A)\}$. Moreover, for all $\lambda \in \rho(A)$ we have $R(\overline{\lambda}; A^*) = (R(\lambda; A))^*$.

(b) Assume that $0 \in \rho(A)$ and let $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then $\lambda \in \sigma(A^{-1})$ if and only if $\lambda^{-1} \in \sigma(A)$.

(c) If A is self adjoint then $\sigma(A) \subset \mathbb{R}$.

(d) If A is unitary then $|\lambda| = 1$ for all $\lambda \in \sigma(A)$.

Spectral Theory of Compact Operators

Proposition 6.5: Let X be a Banach space and let $T \in \mathcal{C}(X;X)$ be given. Then $\sigma_p(T)$ is countable and 0 is the only possible accumulation point.

Proof: For $\epsilon > 0$ put

$$\Lambda_\epsilon = \{\lambda \in \sigma_p(T) : |\lambda| \geq \epsilon.\}.$$

It suffices to show that Λ_ϵ is a finite set for every $\epsilon > 0$. Let $\epsilon_0 > 0$ be given and suppose that Λ_{ϵ_0} is infinite. Then we may choose an injective sequence $\{\lambda_n\}_{n=1}^\infty$ in Λ_{ϵ_0} and a sequence $\{x_n\}_{n=1}^\infty$ of corresponding eigenvectors. For each $n \in \mathbb{N}$, put

$$M_n = \text{span}\{x_1, x_2, \cdots, x_n\},$$

and observe that M_n is invariant under T, i.e. $T[M_n] \subset M_n$, and that $M_n \subset M_{n+1}$. Observe also that each M_n is a closed subspace of X.

Let $n \in \mathbb{N}$ and $x \in M_n$ be given. Then we may choose $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$ such that

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n.$$

Therefore we have

$$(\lambda_n I - T)x = (\lambda_n - \lambda_1)x_1 + (\lambda_n - \lambda_2)x_2 + \cdots + (\lambda_n - \lambda_{n-1})x_{n-1} + 0.$$

It follows that

$$(\lambda_n I - T)[M_n] \subset M_{n-1}. \text{ for } n \geq 2.$$

Using the Riesz Lemma, we may choose a sequence $\{y_n\}_{n=1}^{\infty}$ such that

$$y_n \in M_n, \quad \|y_n\| = 1 \text{ for all } n \in \mathbb{N},$$

$$\forall n \geq 2, \text{ we have } \|y_n - x\| \geq \frac{1}{2} \text{ for all } x \in M_{n-1}.$$

Now let $m, n \in \mathbb{N}$ with $m < n$ be given and put

$$x = Ty_m + (\lambda_n I - T)y_n.$$

Notice that $x \in M_{n-1}$ since $Ty_m \in M_m \subset M_{n-1}$ and $(\lambda_n I - T)y_n \in M_{n-1}$. We have

$$Ty_n - Ty_m = \lambda_n y_n - (\lambda_n I - T)y_n - Ty_m = \lambda_n \left(y_n - \frac{1}{\lambda_n} x \right). \tag{4}$$

It follows from (4) that

$$\|Ty_n - Ty_m\| \geq \frac{1}{2}|\lambda_n| \geq \frac{1}{2}\epsilon_0.$$

We conclude that $\{Ty_n\}_{n=1}^{\infty}$ has no convergent sequence, which is impossible because $\{y_n\}_{n=1}^{\infty}$ is bounded and T is compact. $\square$

Proposition 6.6: Let X be a Banach space and let $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then $\mathcal{N}(\lambda I - T)$ is finite dimensional.

Proof: Let
$$K = \{x \in \mathcal{N}(\lambda I - T) : \|x\| \leq 1\},$$
i.e., the closed unit ball in $\mathcal{N}(\lambda I - T)$ equipped with the norm of X. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence in K. Then

$$Tx_n = \lambda x_n \text{ for all } n \in \mathbb{N}.$$

Since T is compact and $\{x_n\}_{n=1}^{\infty}$ is bounded, we can extract a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\{Tx_{n_k}\}_{k=1}^{\infty}$ is strongly convergent. Since $\lambda \neq 0$, we see that $\{x_{n_k}\}_{k=1}^{\infty}$ is also strongly convergent. Let
$$x = \lim_{k \to \infty} x_{n_k},$$
and observe that $x \in K$ since K is closed. We conclude that K is compact and consequently $\mathcal{N}(\lambda I - T)$ is finite dimensional. $\square$

Proposition 6.7: Let X be a Banach space and let $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then $\mathcal{R}(\lambda I - T)$ is closed.

Proof: Put $W = \mathcal{N}(\lambda I - T)$ and observe that W is finite dimensional by Proposition 6.6. Therefore, we may choose a closed subspace Z of X such that

$$X = W \oplus Z.$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and put

$$y_n = (\lambda I - T)x_n \quad \text{for all } n \in \mathbb{N}.$$

Let $y \in X$ be given and assume that $y_n \to y$ as $n \to \infty$. We need to show that $y \in \mathcal{R}(\lambda I - T)$.

We may choose sequences $\{w_n\}_{n=1}^{\infty}$ in W and $\{z_n\}_{n=1}^{\infty}$ in Z such that

$$x_n = w_n + z_n \quad \text{for all } n \in \mathbb{N}.$$

Since $(\lambda I - T)w_n = 0$ for all $n \in \mathbb{N}$ we see that

$$(\lambda I - T)z_n \to y \text{ as } n \to \infty. \tag{5}$$

I claim that $\{z_n\}_{n=1}^{\infty}$ is bounded. To verify the claim, suppose that $\{z_n\}_{n=1}^{\infty}$ is unbounded. Then we may choose a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that

$$\|z_{n_k}\| > k \quad \text{for all } k \in \mathbb{N}.$$

Put

$$v_k = \frac{z_k}{\|z_k\|} \quad \text{for all } k \in \mathbb{N},$$

and observe that $\|v_k\| = 1$ for all $k \in \mathbb{N}$ and that

$$(\lambda I - T)v_k \to 0 \text{ as } k \to \infty. \tag{6}$$

Since T is compact, we may choose a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ such that $\{Tv_{k_j}\}_{j=1}^{\infty}$ is strongly convergent. Since $\lambda \neq 0$, we conclude from (6) that $\{v_{k_j}\}_{j=1}^{\infty}$ is also strongly convergent; put

$$v = \lim_{j\to\infty} v_{k_j}.$$

We have $\|v\| = 1$ (since the convergence is strong), $v \in Z$ (since Z is closed), and $(\lambda I - T)v = 0$ (which means that $v \in W$). This is impossible, because $W \cap Z = \{0\}$. We conclude that $\{z_n\}_{n=1}^{\infty}$ is bounded.

Since $\{z_n\}_{n=1}^{\infty}$ is bounded, we may extract a subsequence $\{z_{n_j}\}_{j=1}^{\infty}$ such that $\{Tz_{n_j}\}_{j=1}^{\infty}$ is strongly convergent. Since $\lambda \neq 0$, it follows from (5) that $\{z_{n_j}\}_{j=1}^{\infty}$ is strongly convergent; put

$$z = \lim_{j\to\infty} z_{n_j}.$$

We have $(\lambda I - T)z = y$ and $y \in \mathcal{R}(\lambda I - T)$. $\square$

Corollary 6.8: Let X be a Banach space and let $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then

$$\mathcal{R}(\lambda I - T) = {}^{\perp}\mathcal{N}(\lambda I - T^*), \text{ and}$$
$$\mathcal{R}(\lambda I - T^*) = \mathcal{N}(\lambda I - T)^{\perp}.$$

In spectral analysy in finite dimensions, when there are not enough eigenvectors corresponding to an eigenvalue λ of an $N \times N$ matrix B, then we look at the null spaces of higher powers of $(\lambda I - B)$. The same idea is useful for compact operators in infinite-dimensional space.

We now want to look at powers of the form $(\lambda I - T)^n$ where T is a compact operator, $\lambda \neq 0$, and n is a nonnegative integer. For $n \geq 1$ we introduce

$$L = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \lambda^{n-k} T^k, \tag{7}$$

and observe that L is compact since T is compact.

Proposition 6.9: Let X be a Banach space, $T \in \mathcal{L}(X;X)$, $\lambda \in \mathbb{K}\backslash\{0\}$, and n be a nonnegative integer. Then $\mathcal{N}((\lambda I - T)^n)$ is finite dimensional and $\mathcal{R}((\lambda I - T)^n)$ is closed.

Proof: If $n = 0$ then $(\lambda I - T)^n = I$ and the conclusion is immediate. If $n > 0$ then

$$(\lambda I - T)^n = \mu I - L,$$

where $\mu = \lambda^n \neq 0$ and L is given by (7) and is therefore compact. The conclusion follows from Propositions 6.6 and 6.7. $\square$

Lemma 6.10: Let X be a Banach space, $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then there exists a nonnegative integer p such that $\mathcal{N}((\lambda I - T)^n)) = \mathcal{N}((\lambda I - T)^{n+1})$ for all integers $n \geq p$. Let $p_*(\lambda;T)$ be the smallest such nonnegative integer. If $p_*(\lambda;T) > 0$ then

$$\mathcal{N}((\lambda I - T)^0) \subset \mathcal{N}((\lambda I - T)^1) \subset \cdots \subset \mathcal{N}((\lambda I - T)^{p_*(\lambda;T)},$$

with all inclusions strict.

Proof: Put $N_m = \mathcal{N}((\lambda I - T)^m)$ and notice that

$$N_m \subset N_n \ \text{ for all nonnegative integers } m, n \text{ with } m \leq n.$$

Suppose that $N_n \neq N_{n+1}$ for all nonnegative integers n. Then, by the Riesz Lemma, we may choose a sequence $\{y_n\}_{n=1}^{\infty}$ such that

$$y_n \in N_{n+1}\backslash N_n, \quad \|y_n\| = 1, \quad \operatorname{dist}(y_n, N_n) \geq \frac{1}{2} \ \text{ for all } n \in \mathbb{N}. \tag{8}$$

Let $m, n \in \mathbb{N}$ with $m < n$ be given and observe that

$$(\lambda I - T)^{n+1} y_n = 0, \quad (\lambda I - T)^n y_m = 0.$$

It follows that

$$\begin{aligned}(\lambda I - T)^n[(\lambda I - T)y_n + Ty_m] &= (\lambda I - T)^{n+1} y_n + T(\lambda I - T)^n y_m \\ &= 0,\end{aligned}$$

and we conclude that

$$(\lambda I - T)y_n + Ty_m \in N_n. \tag{9}$$

Observe that

$$\begin{aligned} Ty_n - Ty_m &= \lambda y_n - (\lambda y_n - Ty_n + Ty_m) \\ &= \lambda[y_n - \lambda^{-1}(\lambda y_n - Ty_n + Ty_m)] \end{aligned} \tag{10}$$

It follows from (8), (9), and (10) that

$$\|Ty_n - Ty_m\| \geq \lambda\|y_n - y_m\| \geq \frac{|\lambda|}{2}.$$

This is impossible because $\{Ty_n\}_{n=1}^{\infty}$ must have a strongly convergent subsequence by virtute of the fact that T is compact and $\{y_n\}_{n=1}^{\infty}$ is bounded. It follows that there is some nonnegative integer k such that $N_k = N_{k+1}$.

Let a nonnegative integer n be given and assume that $N_n = N_{n+1}$. We shall show that $N_{n+1} = N_{n+2}$. It suffices to show that $N_{n+2} \subset N_{n+1}$. Let $x \in N_{n+2}$ be given. Then

$$0 = (\lambda I - T)^{n+2}x = (\lambda I - T)^{n+1}(\lambda I - T)x,$$

and consequently $(\lambda I - T)x \in N_{n+1}$. Since $N_{n+1} = N_n$, we see that $(\lambda I - T)x \in N_n$, which gives

$$(\lambda I - T)^{n+1}x = (\lambda I - T)^n(\lambda I - T)x = 0. \quad \square$$

Let $T \in \mathcal{L}(X;X)$ and $\lambda \in \backslash\{0\}$ be given and make the following observations:

- $T^* \in \mathcal{L}(X^*;X^*)$,
- $\mathcal{R}((\lambda I - T)^n) = {}^{\perp}\mathcal{N}(((\lambda I - T)^n)^*)$ for every nonnegative integer n,
- $((\lambda I - T)^n)^* = (\lambda I - T^*)^n$ for every nonnegative integer n,

In view of these observations and Lemma 6.10, we have

Lemma 6.11: Let X be a Banach space and $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then there exists a nonnegative integer q such that $\mathcal{R}((\lambda I - T)^n) = \mathcal{R}((\lambda I - T)^{n+1})$ for all integers n with $n \geq q$. Let $q_*(\lambda;T)$ be the smallest such integer. If $q_*(\lambda;T) > 0$ then

$$\mathcal{R}((\lambda I - T)^0) \supset \mathcal{R}((\lambda I - T)^1) \supset \cdots \supset \mathcal{R}((\lambda I - T)^{q_*(\lambda;T)},$$

with all inclusions strict.

Lemma 6.12: Let X be a Banach space and $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be gievn. Let $p_*(\lambda;T)$ and $q_*(\lambda;T)$ be as in Lemmas 6.10 and 6.11. Then we have $p_*(\lambda;T) = q_*(\lambda;T)$.

Proof: For each nonnegative integer n, let us put

$$N_n = \mathcal{N}((\lambda I - T)^n), \quad R_n = \mathcal{R}((\lambda I - T)^n).$$

For ease of notation let us write

$$p = p_*(\lambda; T), \quad q = q_*(\lambda; T).$$

We shall show first that $q \geq p$. Since $R_q = R_{q+1}$ we have

$$(\lambda I - T)[R_q] = R_q. \tag{11}$$

To establish the desired inequality we want to show that $N_{q+1} = N_q$. For this purpose, it is convenientshall show that

$$(\lambda I - T)\Big|_{R_q} \text{ is injective.} \tag{12}$$

Suppose that $x_1 \in R_q \backslash \{0\}$ satisfies $(\lambda I - T)x_1 = 0$. By virtue of (11), we may choose $x_2 \in R_q \backslash \{0\}$ such that $(\lambda I - T)x_2 = x_1$. Proceedingly inductively, we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ satisfying

$$(\lambda I - T)^{n-1} x_n = x_1 \neq 0, \quad (\lambda I - T)^n x_n = (\lambda I - T)x_1 = 0 \ \text{ for all } n \in \mathbb{N}.$$

In other words, we have $x_n \in N_n \backslash N_{n-1}$ for all $n \in \mathbb{N}$. This contradicts Lemma 6.10 and we conclude that (12) holds.

We now prove that $N_{q+1} = N_q$. We already know that $N_q \subset N_{q+1}$. Suppose that $N_q \neq N_{q+1}$. Then we may choose $x \in N_{q+1} \backslash N_q$. Put $y = (\lambda I - T)^q x \in R_q$ and notice that $y \neq 0$. However

$$(\lambda I - T)y = (\lambda I - T)^{q+1} x = 0,$$

which contradicts (12). We conclude that $N_q = N_{q+1}$ and consequently $q \geq p$.

To establish the reverse inequality, we shall show that $R_{p+1} = R_p$. For this purpose it is convenient to show that

$$N_q + \mathcal{R}(\lambda I - T) = X. \tag{13}$$

To establish (13), let $x \in X$ be given. Since $R_q = R_{q+1}$, we may choose $y \in X$ such that

$$(\lambda I - T)^q x = (\lambda I - T)^{q+1} y.$$

Let us put

$$x_1 = x - (\lambda I - T)y, \quad x_2 = (\lambda I - T)y.$$

It is immediate that $x_2 \in \mathcal{R}(\lambda I - T)$. Since

$$(\lambda I - T)^q x_1 = (\lambda I - T)^q x - (\lambda I - T)^{q+1} y = 0,$$

we see that $x_1 \in N_q$ and (13) is established.

Since $N_q \subset N_p$, it follows from (13) that

$$N_p + \mathcal{R}(\lambda I - T) = X. \tag{14}$$

We already know that $R_{p+1} \subset R_p$. To establish the reverse inclusion, let $x \in R_p$ be given. Then we may choose $y \in X$ such that

$$x = (\lambda I - T)^p y.$$

By (14) we may choose $y_1 \in N_p$ and $y_2 \in \mathcal{R}(\lambda I - T)$ such that $y = y_1 + y_2$. Now, we may choose $y_3 \in X$ such that $y_2 = (\lambda I - T)y_3$. It follows that

$$x = (\lambda I - T)^p y_1 + (\lambda I - T)^{p+1} y_3 = (\lambda I - T)^{p+1} y_3.$$

We conclude that $x \in R_{p+1}$. This implies that $R_p = R_{p+1}$ which implies that $p \geq q$ and the proof is complete. $\square$

Proposition 6.13: Let X be a Banach space and $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$. be given. Assume that $\lambda \in \sigma(T)$. Then $\lambda \in \sigma_p(T)$.

Proof: Suppose that $\lambda \notin \sigma_p(T)$. Then $\mathcal{N}(\lambda I - T) = \{0\}$, so $p_*(\lambda; T) = 0$. By Lemma 6.12, we also have $q_*(\lambda; T) = 0$ which implies that $\mathcal{R}(\lambda I - T) = X$ and consequently $\lambda I - T$ is bijective, which contradicts the assumption that $\lambda \in \sigma(T)$. $\square$

Theorem 6.14: Let X be a Banach space and $T \in \mathcal{C}(X;X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Let $p = p_*(\lambda; T)$ where $p_*(\lambda; T)$ is as in Lemma 6.10. Then

$$\mathcal{N}((\lambda I - T)^p) \oplus \mathcal{R}((\lambda I - T)^p) = X.$$

Proof: Let N_n and R_n be as in the proof of Lemma 6.12 and observe that $R_{2p} = R_p$. Let $x \in X$ be given. Then we may choose $z \in X$ such that $(\lambda I - T)^{2p} z = (\lambda I - T)^p x$. Now put

$$y = (\lambda I - T)^p z \in R_p,$$

and observe that

$$(\lambda I - T)^p y = (\lambda I - T)^{2p} z = (\lambda I - T)^p x.$$

It follows that $x - y \in N_p$ and we have

$$x = (x - y) + y.$$

To see that the decomposition is unique, let $\tilde{y} \in R_p$ be given with $x - \tilde{y} \in N_p$. Put

$$z = y - \tilde{y} \in R_p.$$

We want to show that $z = 0$. We may choose $w \in X$ such that

$$z = (\lambda I - T)^p w.$$

Observe that
$$z = (x - \tilde{y}) - (x - y) \in N_p,$$
and consequently
$$(\lambda I - T)^{2p} w = (\lambda I - T)^p z = 0.$$
Since $N_{2p} = N_p$, we find that $w \in N_p$, and this gives
$$0 = (\lambda I - T)^p w = z. \quad \square$$

Theorem 6.15: Let X be a Banach space and let $T \in \mathcal{C}(X; X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then
$$\dim(\mathcal{N}(\lambda I - T)) = \dim(\mathcal{N}(\lambda I - T^*)).$$

To establish Theorem 6.15, we shall make use of the following lemma, whose proof is left as an exercise.

Lemma 6.16: Let X be a Banach space and let $\{x_1^*, x_2^*, \cdots, x_n^*\}$ be a linearly independent subset of X^*. Then there exist $x_1, x_2, \cdots, x_n \in X$ such that for all $i, j \in \{1, 2, \cdots, n\}$ we have
$$x_i^*(x_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

To prove Theorem 6.15, it is convenient to show that
$$\dim(\mathcal{N}(\lambda I - T)) \geq \dim(\mathcal{N}(\lambda - T^*)) \tag{15}$$
and then look at $\dim(\mathcal{N}(\lambda I - T^{**}))$.

Lemma 6.17: Let X be a Banach space and $T \in \mathcal{C}(X, X)$ and $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then (15) holds.

Proof: We know that both of the null spaces in question have finite dimension. Let
$$n = \dim(\mathcal{N}(\lambda I - T)), \quad m = \dim(\mathcal{N}(\lambda I - T^*)).$$
Let $\{x_1, x_2, \cdots, x_n\}$ be a basis for $\mathcal{N}(\lambda I - T)$ and $\{y_1^*, y_2^*, \cdots, y_m^*\}$ be a basis for $\mathcal{N}(\lambda I - T^*)$. By a straightforward application of the Hahn-Banach Theorem we may choose $x_1^*, x_2^*, \cdots, x_n^* \in X^*$ such that
$$x_i^*(x_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$
Using Lemma 6.16, we may choose $y_1, y_2, \cdots, y_m \in X$ such that
$$y_i^*(y_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Suppose that $n < m$ and define $L \in \mathcal{L}(X;X)$ by

$$Lx = \sum_{i=1}^{n} x_i^*(x)y_i.$$

Observe that $L \in \mathcal{C}(X;X)$ since L has finite rank. Now put

$$S = T + L,$$

and observe that S is also compact.

We shall show that $\mathcal{N}(\lambda I - S) = \{0\}$. To this end, let $x \in \mathcal{N}(\lambda I - S)$ be given. Then $Sx = \lambda x$ so that

$$(\lambda I - T)x = Lx.$$

It follows that for all $j \in \{1, 2, \cdots, n\}$ we have

$$0 = (\lambda y_j^* - T^* y_j^*)(x),$$

and consequently $x_j^*(x) = 0$ for all $j \in \{1, 2, \cdots, m\}$. It follows that $(\lambda I - T)x = 0$ and therefore we may choose $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$ such that

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n. \tag{16}$$

For each $j \in \{1, 2, \cdots, n\}$, we apply x_j^* to (16) to conclude that $x_j^*(x) = \alpha_j$. It follows that $x = 0$ and consequently $\mathcal{N}(\lambda I - S) = \{0\}$.

Since $\lambda \neq 0$, S is compact and $\lambda \notin \sigma_p(S)$, it follows from Proposition 6.13 that $\lambda \in \rho(T)$ and consequently that $\mathcal{R}(\lambda I - S) = X$. Thus we may choose $z \in X$ such that

$$(\lambda I - S)z = y_{n+1}.$$

Then by the choice of $y_1, y_2, \cdots, y_m$ we see that

$$1 = y_{n+1}^*(y_{n+1}) = y_{n+1}^*(x) = y_{n+1}^*\left((\lambda I - T)x - \sum_{i=1}^{n} x_i^*(x)y_i\right) = 0.$$

This is, of course, a contradiction and consequently it is not possible to have $n < m$. □

Proof of Theorem 6.15: By Lemma 6.16 we have

$$\dim(\mathcal{N}(\lambda I - T) \geq \dim(\mathcal{N}(\lambda I - T^*)).$$

Applying Lemma 6.16 to T^* we see that

$$\dim(\mathcal{N}(\lambda I - T^*) \geq \dim(\mathcal{N}(\lambda I - T^{**})).$$

On the other hand, since $\lambda I - T^{**}$ is an extension of $\lambda I - T$ we have

$$\dim(\mathcal{N}(\lambda I - T^{**}) \geq \dim(\mathcal{N}(\lambda I - T)),$$

and we are done. $\square$

The Fredholm Alternative

There is an important principle known as the *Fredholm Alternative* which expresses some of the main conclusions concerning spectral theory of compact operators in terms of solutions of equations of the form $(\lambda I - T)x = y$.

Theorem 6.18 (The Fredholm Alternative): Let X be a Banach space and let $T \in \mathcal{C}(X;X)$, $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Then exactly one of (a) or (b) below must hold:

(a) For every $y \in X$ the equation $(\lambda I - T)x = y$ has exactly one solution $x \in X$. has only the trivial solution $x = 0$.

(b) There exists $x \in X\backslash\{0\}$ such that $(\lambda I - T)x = 0$

Moreover, if (a) holds then for every $y^* \in X^*$ the equation $(\lambda I - T^*)x^* = y^*$ has exactly one solution $x^* \in X^*$. If (b) holds then (i), (ii), and (iii) below hold.

(i) The number of linearly independent solutions of $(\lambda I - T)x = 0$ in X is finite and is the same as the number of linearly independent solutions of $(\lambda I - T^*)x^* = 0$ in X^*.

(ii) For a given $y \in X$, the equation $(\lambda I - T)x = y$ has a solution $x \in X$ if and only if $y \in {}^{\perp}\mathcal{N}(\lambda I - T)$.

(iii) For a given $y^* \in X^*$, the equation $(\lambda I - T^*)x^* = y^*$ has a solution $x^* \in X^*$ if and only if $y^* \in \mathcal{N}(\lambda I - T)^{\perp}$.

Fredholm Operators

TO BE FILLED IN

General Linear Operators Between Normed Linear Spaces

We now turn our attention to linear operators that need not be bounded.

Let X and Y be normed linear spaces over $\mathbb{K}$.

Definition 6.19: Let $\mathcal{D}(A) \subset X$. When we say that $A : \mathcal{D}(A) \to Y$ is a linear operator we mean that $\mathcal{D}(A)$ is a linear manifold and

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \ \text{ for all } x, y \in \mathcal{D}(A),\ \alpha, \beta \in \mathbb{K}.$$

When dealing with a linear operator A whose domain could be properly included in X we shall write $A : \mathcal{D}(A) \to Y$ rather than try to introduce special terminology to indicaate that the domain might be properly included in X because there does not seem to be any universally accepted terminology for this and it is more important to avoid potential confusion rather than to have a more elegant way of phrasing results.

Definition 6.20: Let $\mathcal{D}(A) \subset X$ and assume that $A : \mathcal{D}(A) \to Y$ is linear. We say that

(i) A is *bounded* provided there exists $K \in \mathbb{R}$ such that

$$\|Ax\|_Y \le K\|x\|_X \text{ forall } x \in X.$$

(ii) A is *unbounded* if it is not bounded.

(iii) A is closed provided that $\mathrm{Gr}(A)$ is closed in $X \times Y$.

Theorem 6.21 (Closed Graph Theorem): Assume that X and Y are Banach spaces that $\mathcal{D}(A) \subset X$ and that $A : \mathcal{D}(A) \to Y$ is linear. Assume further that A is closed and that $\mathcal{D}(A)$ is closed. Then A is bounded.

Remark 6.22: Let X and Y be normed linear spaces, $\mathcal{D}(A) \subset X$, and assume that $A : \mathcal{D}(A) \to Y$ is linear. Then A is closed if and only if for every $x \in X$, $y \in Y$ and every sequence $\{x_n\}_{n=1}^\infty$ in $\mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$ we have $x \in \mathcal{D}(A)$ and $Ax = y$.

Example 6.23 Let $X = Y = C[0,1]$ equipped with the supremum norm. Let us put $\mathcal{D}(A) = C^1[0,1]$, $\mathcal{D}(B) = C^\infty[0,1]$, and define

$$Au = u' \text{ for all } u \in \mathcal{D}(A), \quad Bu = u' \text{ for all } u \in \mathcal{D}(B).$$

Notice that A is an extension of B. Using Remark we see that A is closed, but B is not. [Indeed if $\{u_n\}_{n=1}^\infty$ is a sequence in $C^1[0,1]$ and $u_n \to u$ uniformly and $u_n' \to v$ uniformly as $n \to \infty$ then $v \in C^1[0,1]$ and $u' = v$. To see that B is not closed, let any function $w \in C[0,1] \backslash C^1[0,1]$ be given. Then we may choose a sequence $\{w_n\}_{n=1}^\infty$ of polynomials such that $w_n \to w$ uniformly as $n \to \infty$. Then we can define

$$u_n(x) = \int_0^x w_n(t)\,dt \text{ for all } n \in \mathbb{N},\ x \in [0,1.$$

Then we have $u_n \to u$ uniformly and $u_n' \to w$ uniformly as $n \to \infty$, but $w \notin \mathcal{D}(B)$.]