ASSIGNMENT NUMBER 4, 21.630 Spring 2013

Due Wednesday, February 13, 2013

1. Let $A > 0$ and b and X be continuous and nonnegative on $[t_0, \infty)$. Assume that

$$X(t) \leq A + \int_{t_0}^{t} b(s)\sqrt{X(s)}ds$$

for all $t \geq t_0$ and show that

$$X(t) \leq \left(\sqrt{A} + \frac{1}{2}\int_{t_0}^{t} b(s)ds\right)^2.$$

2. A) Solve

$$R(t) = 1 + \int_0^t \frac{1}{R(s)}ds$$

for $R(t)$.

B) Assume that X is continuous and positive on $[0, \infty)$ and satisfies

$$X(t) \leq 1 + \int_0^t \frac{1}{X(s)}ds$$

for $t \geq 0$. Does $X(t) \leq R(t)$ follow? Either prove that it does or give a counter example.

3. Assume that $f(t, x) = F(t, |x|)x$ and that solutions of $\dot{X} = f(t, X(t))$ are unique and exist for all t. Define $X(t, t_0, x_0)$ by

$$\frac{dX}{dt} = f\left(t, X(t, t_0, x_0)\right)$$

$$X(t_0, t_0, x_0) = x_0.$$

Show that if R is orthogonal ($|Rx| = |x|$ for all $x \in \mathbb{R}^n$, R linear) then

$$f(t, Rx) = Rf(t, x).$$

Then show that

$$X\left(t, t_0, Rx_0\right) = RX\left(t, t_0, x_0\right).$$

4. Assume that $f(t, x)$ is continuous and that solutions of $\dot{X} = f(t, X(t))$ are unique and exist for all t. Define $X(t, t_0, x_0)$ by $\dot{X} = f\left(t, X(t, t_0, x_0)\right)$ and

$X(t_0, t_0, x_0) = x_0$. Assume that for every x_0, $\lim_{t \to +\infty} X(t, t_0, x_0)$ exists (and is finite). Is $x_0 \mapsto \lim_{t \to +\infty} X(t, t_0, x_0)$ continuous? Prove this or show it to be false.