Math 720: Homework.

Do, but don't turn in optional problems. *Keep in mind that there is a firm 'no late homework' policy.*

Assignment 1: Assigned Wed 09/05. Due Wed 09/12

Following the notation of Cohn, I use λ to denote the Lebesgue measure.

1. For each of the following sets, compute the Lebesgue outer measure.
   (a) Any countable set. (b) The Cantor set. (c) $\{x \in [0,1] \mid x \notin \mathbb{Q}\}$.
2. (a) If $V \subseteq \mathbb{R}^d$ is a subspace with $\dim(V) < d$, then show that $\lambda(V) = 0$.
   (b) If $P \subseteq \mathbb{R}^2$ is a polygon show that $\operatorname{area}(P) = \lambda(P)$.
3. (a) Say μ is a *translation invariant* measure on $(\mathbb{R}^d, \mathcal{L})$ (i.e. $\mu(x + A) = \mu(A)$ for all $A \in \mathcal{L}$, $x \in \mathbb{R}^d$). Show that $\exists c \geqslant 0$ such that $\mu(A) = c\lambda(A)$.
   (b) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal linear transformation, and $A \in \mathcal{L}$. Show that $T(A) \in \mathcal{L}$ and $\lambda(T(A)) = \lambda(A)$. [Hint: Express T in terms of elementary transformations.]
4. (a) Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. For any $A \subseteq X$ define

   $$\mu^*(A) = \inf\Big\{ \sum_1^\infty \rho(E_i) \Bigm| E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_1^\infty E_j \Big\}.$$

   Show that μ^* is an outer measure.
   (b) Let (X, d) be any metric space, $\delta > 0$ and define $\mathcal{E}_\delta = \{B(x, r) \mid x \in X, r \in (0, \delta)\}$. Given $\alpha > 0$ define $\rho(B(x, r)) = c_\alpha r^\alpha$, where $c_\alpha = \pi^{\alpha/2}/\Gamma(1+\alpha/2)$ is a normalization constant. Let $H^*_{\alpha,\delta}$ be the outer measure obtained with this choice of ρ and the collection of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta \to 0} H^*_{\alpha,\delta}$. Show H^*_α is an outer measure and restricts to a measure H_α on a σ-algebra that contains all Borel sets. The measure H_α is called the *Hausdorff measure of dimension* α. [Don't reprove Caratheodory.]
   (c) If $X = \mathbb{R}^d$, and $\alpha = d$ show that H_d is the Lebesgue measure.
   (d) Let $S \in \mathcal{B}(X)$. Show that there exists (a unique) $d \in [0, \infty]$ such that $H_\alpha(S) = \infty$ for all $\alpha \in (0, d)$, and $H_\alpha(S) = 0$ for all $\alpha \in (d, \infty)$. This number is called the *Hausdorff dimension* of the set S.
   (e) Compute the Hausdorff dimension of the Cantor set.

Details in class I left for you to check. (Do it, but don't turn it in.)

- We saw in class $\ell(I) = I$ for closed cells. Show it for arbitrary cells.
- Show that $m^*(a + E) = m^*(E)$ for all $a \in \mathbb{R}^d$, $E \subseteq \mathbb{R}^d$.
- Show that the arbitrary intersection of σ-algebras on X is also a σ-algebra.
- Verify that the counting measures and delta measures are measures.
- When proving Caratheodory, we proved in class Σ is a σ-algebra, and that $\mu^*|_\Sigma$ is *finitely* additive. Show that $\mu^*|_\Sigma$ is countably additive.