21-373, Algebraic Structures, Department of Mathematical Sciences, Carnegie Mellon University
Fall 2011: (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.
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Assignment 7 - Sunday November 6, 2011. Due Friday November 11

Exercise 43: i) Prove that the ring $2\mathbb{Z}$ and the ring $3\mathbb{Z}$ are not isomorphic.
ii) Prove that the ring $\mathbb{Z}[x]$ and the ring $\mathbb{Q}[x]$ are not isomorphic.

Exercise 44: Decide which of the following are ideals of the ring $\mathbb{Z}[x]$:
i) the set of all polynomials whose constant term is a multiple of 3,
ii) the set of all polynomials whose coefficient of x^2 is a multiple of 3,
iii) the set of all polynomials whose constant term, coefficient of x, and coefficient of x^2 are zero,
iv) the set of all polynomials in which only even powers of x appear (i.e. $\mathbb{Z}[x^2]$),
v) the set of all polynomials whose sum of all coefficients is zero,
vi) the set of all polynomials whose sum of all coefficients of even powers of x is zero, and whose sum of all coefficients of odd powers of x is zero,
vii) the set of all polynomials P such that $P'(0) = 0$.

Exercise 45: Let R be a commutative unital ring, and let $P_1, \ldots, P_n$ be prime ideals.
i) Suppose that A is an ideal such that for $i = 1, \ldots, n$ there exists $a_i \in A \cap P_i$ such that $a_i \notin P_j$ for all $j \neq i$, and let $b = a_1 + (a_2 \cdots a_n)$; show that $b \in A$ but $b \notin P_1 \cup \cdots \cup P_n$.
ii) Show that if an ideal B is such that $B \subset P_1 \cup \cdots \cup P_n$, then $B \subset P_i$ for some $i \in \{1, \ldots, n\}$.

Exercise 46: Let R be a ring with at least one non-zero element, and such that for each non-zero $a \in R$ there is a *unique* $b \in R$ satisfying $a\,b\,a = a$, which one writes $b = \psi(a)$.
i) Show that multiplication is regular (i.e. for each non-zero $r \in R$, $r\,x = r\,y$ implies $x = y$ and $x\,r = y\,r$ implies $x = y$).
ii) Show that $a\,b\,a = a$ implies $b\,a\,b = b$, i.e. if $b = \psi(a)$, then $a = \psi(b)$.
iii) Show that there is an identity for multiplication, and that R is a division ring.

Exercise 47: Let p be an odd prime, and let $R \subset \mathbb{Q}$ be the set of rational numbers whose denominator in reduced form (i.e. $\frac{a}{b}$ with $b \in \mathbb{Z}^*$ and $a \in \mathbb{Z}$ satisfying $(a, b) = 1$) is not divisible by p, and let $J \subset R$ be the set of such rational numbers whose numerator in reduced form is a multiple of p.
i) Show that R is a subring of $\mathbb{Q}$ and J is an ideal of R.
ii) If $\frac{a}{b}, \frac{c}{d} \in R$ (so that $b, d \neq 0 \pmod p$), one writes that $\frac{a}{b} = \frac{c}{d} \pmod p$ if $\frac{a}{b} - \frac{c}{d} \in J$. Show that $1 + \frac{1}{2} + \ldots + \frac{1}{p-1} = 0 \pmod p$.

Exercise 48: (Putnam 1996-A5) If p is a prime greater than 3, and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.
(For example $\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} = 7 + 21 + 35 + 35 = 2.7^2$.)

Exercise 49: One considers the ring of Gaussian integers, $\mathbb{Z}[i] = \{z = a + i\,b \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$, with $V(z) = z\,\overline{z} = a^2 + b^2$.
i) If x_0 is a positive integer and $y_0 = a + b\,i \in \mathbb{Z}[i]$, show that there exists $q, r \in \mathbb{Z}[i]$ with $y_0 = q\,x_0 + r$ with either $r = 0$ or $r \neq 0$ and $V(r) \leq \frac{V(x_0)}{2}$.
ii) If $x \in \mathbb{Z}[i]$ with $x \neq 0$ and $y \in \mathbb{Z}[i]$, show by considering $x_0 = x\,\overline{x}$ that $y = q\,x + r$ with either $r = 0$ or $V(r) \leq \frac{V(x)}{2}$, so that $\mathbb{Z}[i]$ is an Euclidean domain.
iii) Show that $Z[\sqrt{-2}] = \{z = a + i\,\sqrt{2}\,b \mid a, b \in \mathbb{Z}\}$ with $V(z) = z\,\overline{z} = a^2 + 2b^2$, is an Euclidean domain.