21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
Spring 2012: Monday, Wednesday, Friday, 10:30 am, Doherty Hall 1211.
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Assignment 4 - Friday March 2, 2012. Due Wednesday March 7

Exercise 16: Let E be a field, and let $\frac{P}{Q} \in E(x)$ be a non-zero rational fraction. Assume that $Q = Q_1^{k_1} \cdots Q_m^{k_m}$, where $k_1, \ldots, k_m \geq 1$ and $Q_1, \ldots, Q_m$ are distinct monic irreducible polynomials of degree $d_1, \ldots, d_m \geq 1$.
i) Show that there is a decomposition $\frac{P}{Q} = A + \sum_{i=1}^m \frac{B_i}{Q_i^{k_i}}$ with $A, B_1, \ldots, B_m \in E[x]$ with $degree(B_i) < k_i d_i$ for $i = 1, \ldots, m$.
ii) Show that $A, B_1, \ldots, B_m$ are determined in a unique way.

Exercise 17: For $m \geq 1$, let $a, b, c_1, \ldots, c_m, \xi_1, \ldots, \xi_m \in \mathbb{R}$ with $a, c_1, \ldots, c_m \geq 0$, and let $f = a\,x + b + \sum_{i=1}^m \frac{c_i}{\xi_i - x} \in \mathbb{C}(x)$.
i) Show that f maps H into itself, where H is the "upper half plane", i.e. $\{z = \alpha + i\,\beta \mid \alpha, \beta \in \mathbb{R}, \beta > 0\}$.
ii) Let $g \in \mathbb{C}(x)$ be such that it has no poles in H and that $g(z) \in H$ for all $z \in H$. Show that if $g \in \mathbb{C}[x]$, it has the form $a\,x + b$ with $a, b \in \mathbb{R}$ and $a \geq 0$, and that if $g \in \mathbb{C}(x) \setminus \mathbb{C}[x]$, it has the above form f for some $m \geq 1$

Exercise 18: Let V be a finite-dimensional Euclidean space, let $f \in \mathbb{R}(x)$ have the form of Exercise 17, and let $[a_-, a_+] \subset \mathbb{R}$ (with $a_- < a_+$) be an interval containing no pole of f.
i) Show that for $A \in L_s(V, V)$ satisfying $a_- I \leq A \leq a_+ I$, and any decomposition of $f = \frac{P}{Q}$ with Q having no poles in $[a_-, a_+]$, then $Q(A)$ is invertible, and $P(A)\big(Q(A)\big)^{-1}$ is an element of $L_s(V, V)$ independent of the representation of f chosen, so that one denotes it $f(A)$.
ii) Show that if $A_1, A_2 \in L_s(V, V)$ satisfy $a_- I \leq A_1 \leq A_2 \leq a_+ I$, then one has $f(A_1) \leq f(A_2)$.

Exercise 19: Let V be a finite-dimensional Euclidean space, and let $M \in L(V, V)$ be such that there exists $\alpha > 0$ such that $(M\,v, v) \geq \alpha\,||v||^2$ for all $v \in V$.
i) Show that M is invertible, with $||M^{-1}|| \leq \frac{1}{\alpha}$, and that the (complex) eigenvalues λ_j of M satisfy $\Re(\lambda_j) \geq \alpha$ for all j.
ii) Show that for every $B \in L(V, V)$, $X = \int_0^\infty e^{-t\,M^T} B\, e^{-t\,M}\,dt$ defines an element $X \in L(V, V)$, and show that X is the unique solution of $X\,M + M^T X = B$.
ii) Show that if $B \in L_s(V, V)$, then $X \in L_s(V, V)$, and if moreover $\beta\,(M + M^T) \leq B \leq \gamma\,(M + M^T)$ with $0 \leq \beta \leq \gamma$, then $\beta\,I \leq X \leq \gamma\,I$.

Exercise 20: Let V be a finite-dimensional Euclidean space, and let $A \in L(V, V)$ be such that there exists $\alpha > 0$ such that $(A\,v, v) \geq \alpha\,||v||^2$ for all $v \in V$. For $C \in L_s(V, V)$ satisfying $C \geq 0$, and $D \in L_s(V, V)$ satisfying $D \geq 0$, one wants to solve $X\,A + A^T X + X\,C\,X = D$, and show that there exists a solution $X \in L_s(V, V)$ satisfying $0 \leq X \leq \gamma\,I$, with $\gamma \geq 0$ chosen so that $D \leq \gamma\,(A + A^T) + \gamma^2 C$.
i) Show that if $X_n \in L_s(V, V)$ satisfies $0 \leq X_n \leq \gamma\,I$, and $\mu \geq \gamma\,||C||$, there is a unique $X_{n+1} \in L_s(V, V)$ satisfying $X_{n+1}\,(A + C\,X_n + \mu\,I) + (A^T + X_n C + \mu\,I)\,X_{n+1} = 2\mu\,X_n + X_n C\,X_n + D$, and that $0 \leq X_{n+1} \leq \gamma\,I$.
ii) If moreover $X_n A + A^T X_n + X_n C\,X_n \leq D$, show that $X_n \leq X_{n+1}$, and that $X_{n+1} A + A^T X_{n+1} + X_{n+1} C\,X_{n+1} \leq D$.
iii) Starting from $X_0 = 0$, show that X_n converges to a solution.