# 21-484 Notes

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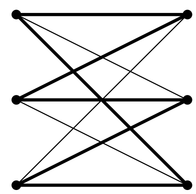
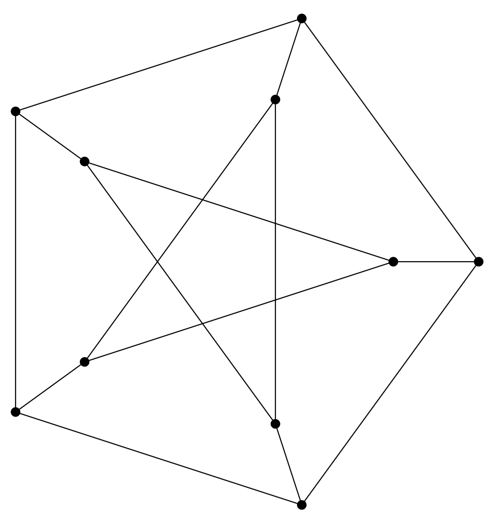
Def: (Page 141): Let $G$ be a graph.

- $\rightarrow$ A cycle $C$ containing every vertex of $G$ is called a Hamiltonian cycle.
- $\rightarrow$ A path $C$ containing every vertex of $G$ is called a Hamiltonian path.
- $\rightarrow$ If $G$ contains a Hamiltonian cycle then $G$ is Hamiltonian.

Examples: 1. $K_{3,3}$ is Hamiltonian



2. The Petersen Graph is not Hamiltonian



The Petersen Graph

Claim: (Thm 6.5) If $G$ is Hamiltonian then for every non empty set $S \subseteq V(G)$

$$k(G - S) \leq |S|$$

Proof:

- Let $G_1, \ldots, G_k$ be the components of $G - S$.
- $C$ is a Hamiltonian cycle in $G$.
- If you walk along $C$, then every time that you leave $G_i$, you encounter a vertex of $S$.
- $|S| \geq k = k(G - S)$

Theorem (Ore): Let $G$ be a graph with $n \geq 3$ vertices. If

$$\deg u + \deg v \geq n \, (*)$$

for every pair of nonadjacent vertices $u$ and $v$, then $G$ is Hamiltonian.

Proof:

- Let $G$ be a graph having $(*)$ that is not Hamiltonian.
- Add edges as long as the result is not Hamiltonian. Call the result $H$.
- $G \subseteq H$
- $H$ is not complete graph.

$\rightarrow$ $H$ has $(*)$

$\rightarrow$ adding any edge to $H$ yields a Hamiltonian

- Let $x, y$ be two non adjacent vertices of $H$.

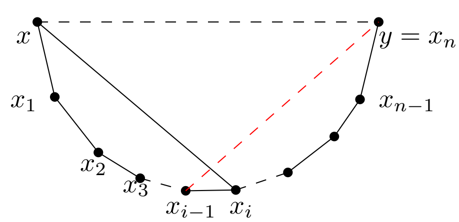
$\rightarrow$ Let $e = xy$

$\rightarrow$ $H + e$ is Hamiltonian

$\rightarrow$ every Hamiltonian cycle in $H + e$ uses $e$.

$\Rightarrow$ There is an $x$–$y$ Hamiltonian path in $H$. Let $x_0 = x, x_1, x_2, \ldots, x_n = y$ be such a Hamiltonian path.

$\rightarrow$ If $xx_i$ is an edge, then $x_{i-1}y$ is not an edge. Otherwise we get a Hamiltonian cycle $x, x_i, x_{i+1}, \ldots, x_n = y, x_{i-1}, x_{i-2}, \ldots, x_0 = x$




$\Rightarrow$ for every neightbor of $x$ in $\{x_1, \ldots, x_n\}$ there is a non neighbor of $y$ in $\{x_0, \ldots, x_{n-1}\}$

$\Rightarrow$ $\deg(y) \leq n - 1 - \deg(x)$

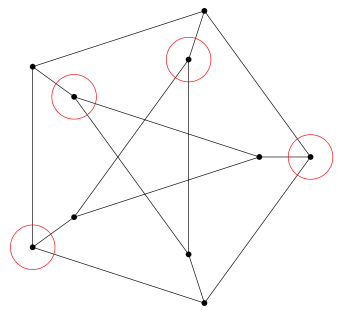
$\Rightarrow$ $\deg(x) + \deg(y) \leq n - 1$ ↯

$\rightarrow$ Corollary (Dirac's Thm): If $\delta(G) \geq n/2$ then $G$ is Hamiltonian.

Def: For a graph $G$, $\alpha(G)$ denotes the independence number of $G$ which is the size of a maximal independent set (a set of vertices spanning no edges).

Recall that $\kappa(G)$ is the vertex connectivity of $G$.

Theorem (Chvátal and Erdös): If $\alpha(G) \leq \kappa(G)$ then $G$ is Hamiltonian.



$\rightarrow \alpha(PG) = 4 \rightarrow \kappa(PG) = 3$

$\rightarrow$ Theorem (Chvátal and Erdös): If $\alpha(G) \leq \kappa(G) + 1$ then $G$ is Hamiltonian