Homework 8

21-260 Differential Equations
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Section 6.1, Problem 16

If F is the Laplace transform of f, then, by definition,

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} t \sin(at)\, dt.$$

Integration by Parts (differentiating the te^{-st} term and integrating the $\sin(at)$ term) gives

$$\begin{aligned} F(s) &= -te^{-st}\frac{\cos(at)}{a}\Big|_{t=0}^{t=\infty} + \frac{1}{a}\int_0^\infty \left(e^{-st} - \frac{t}{s}e^{-st}\right)\cos(at)\, dt \\ &= \frac{1}{a}\int_0^\infty \left(e^{-st} - \frac{t}{s}e^{-st}\right)\cos(at)\, dt. \end{aligned}$$

Integration by Parts again (differentiating the $\frac{t}{s}e^{-st}$ term, integrating the $\cos(at)$ term, and ignoring the $e^{-st}\cos(at)$ term by linearity of the integral for the time being)

$$\begin{aligned} F(s) &= \frac{1}{a}\int_0^\infty - \frac{1}{sa}\int_0^\infty te^{-st}\cos(at)\, dt \\ &= \frac{1}{a}\int_0^\infty - \frac{1}{sa}\left(\frac{F(s)}{sa} - \int_0^\infty e^{-st}\sin(at)\, dt\right). \end{aligned}$$

Solving for $F(s)$ in the above equation gives

$$F(s) = \left(1 + \frac{1}{s^2a^2}\right)^{-1}\left(\frac{1}{a}g_1(s) + \frac{1}{sa^2}g_2(s)\right) = \frac{s^2a}{s^2a^2+1}g_1(s) + \frac{s}{s^2a^2+1}g_2(s)$$

where $g_1(s) = \int_0^\infty e^{-st}\cos(at)\, dt$ and $g_2 = \int_0^\infty e^{-st}\sin(at)\, dt$.

Table 6.2.1 shows that

$$\int_0^\infty e^{-st}\cos(at)\, dt = \mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2},$$

and that

$$\int_0^\infty e^{-st}\sin(at)\, dt = \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}.$$

Thus,

$$F(s) = \boxed{\frac{s^3a}{(s^2a^2+1)(s^2+a^2)} + \frac{sa}{(s^2a^2+1)(s^2+a^2)}.}$$

Section 6.1, Problem 26abc

(a) Suppose $p > 0$. Then, Integration by Parts (differentiating x^p and integrating e^{-x}) gives that

$$\begin{aligned}
\Gamma(p+1) &= \int_0^\infty x^p e^{-x}\,dx \\
&= -x^p e^{-x}\Big|_{x=0}^{x=\infty} - \int_0^\infty p x^{p-1}\left(-e^{-x}\right)\,dx \\
&= p\int_0^\infty x^{p-1}e^{-x}\,dx = p\Gamma(p). \quad \blacksquare
\end{aligned}$$

(b)

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_{x=0}^{x=\infty} = \lim_{x\to\infty}\left(-e^{-x}\right) - \left(-e^{-0}\right) = 0 - (-1) = 1. \quad \blacksquare$$

(c) Since the factorial is defined by the recurrence relation

$$n! = n \cdot (n-1)! \text{ for } n \in \mathbb{N}\backslash\{0\}, \quad 0! = 1,$$

the desired result follows immediately by the Principle of Mathematical Induction (the result of part (b) above is the base case, while the result of part (a) above is the general case). $\blacksquare$

Section 6.2, Problem 14

As given in equation (16) is Section 2 of Chapter 6, if Y is the Laplace transform of the solution to the given initial value problem, then

$$Y(s) = \frac{(s-4)+1}{s^2-4s+4} = \frac{(s-2)-1}{s^2-4s+4} = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

Table 6.2.1 shows that $\frac{1}{s-2}$ is the Laplace transform of the function e^{2t} of t, and that $\frac{1}{(s-2)^2}$ is the Laplace transform of the function te^{2t} of t. Thus, by linearity of the inverse Laplace transform, the solution to the given initial value problem is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} = \boxed{e^{2t} - te^{2t}.}$$

Section 6.2, Problem 20

As given in equation (16) is Section 2 of Chapter 6, if Y is the Laplace transform of the solution to the given initial value problem, then (using Table 6.2.1) to compute $\mathcal{L}\{\cos(2t)\}$)

$$Y(s) = \frac{s}{s^2+\omega^2} + \frac{s}{(s^2+4)(s^2+\omega^2)} = \frac{s}{s^2+\omega^2} + \frac{s}{(\omega^2-4)(s^2+4)} - \frac{s}{(\omega^2-4)(s^2+\omega^2)}.$$

Table 6.2.1 shows that $\frac{s}{s^2+\omega^2}$ is the Laplace transform of the function $\cos(\omega t)$, and that $\frac{s}{s^2+4}$ is the Laplace transform of $\cos(2t)$. Thus, by linearity of the inverse Laplace transform (noting that the $\frac{1}{\omega^2-4}$ terms are constants in s), the solution to the given initial value problem is

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} + \frac{1}{\omega^2-4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{\omega^2-4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} \\ &= \boxed{\cos(\omega t) + \frac{\cos(2t)}{\omega^2-4} + \frac{\cos(\omega t)}{\omega^2-4}.} \end{aligned}$$

Section 6.3, Problem 12

On the interval $[0,\infty)$,

$$\begin{aligned} f(t) &= t + (2-t)u_2(t) + (7-t-2)u_5(t) - (7-t)u_7(t) \\ &= \boxed{t + (2-t)u_2(t) + (5-t)u_5(t) - (7-t)u_7(t).} \end{aligned}$$