21-484 Notes

JD Nir
jnir@andrew.cmu.edu
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→ Recall: Dirac's Fan Lemma: A graph is k-connected iff it has at least $k+1$ vertices and for every vertex x and every set $U \subset V(G) \setminus x, |U| \geq k$, there is an x, U–fan of size k.

an x, U–fan is a collection of paths from x to vertices of U such that for every two paths the only common vertex is x.

Theorem (Chvátal-Erdős): Let G be a graph with at least 3 vertices such that $\alpha(G) \leq \kappa(G)$. Then G is Hamiltonian.

Proof: → Let $k = \kappa(G)$ and let C be a longest cycle in G.

→ Denote the vertices of C cyclically by $V(C) = \{v_0, \ldots, v_{\ell-1}\}$ (think of the indices as the elements of $\mathbb{Z}_\ell$)

→ AFSOC that C is not a Hamiltonian cycle.

→ Let V be a vertex of G out of C.

→ Let $\mathcal{F}$ be a $v, V(C)$ fan of maximumal size. Denote $\mathcal{F} = \{P_i | i \in I\}$ where P_i is a v–v_i path.

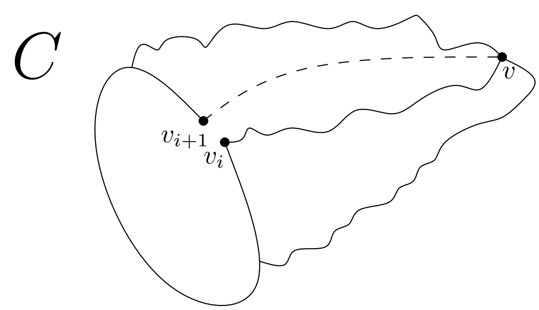
→ observe:

→ By the Fan Lemma

* $|\mathcal{F}| = |I| \geq \min(|C|, k)$ using the fact that a k-connected graph is also $k-1$ connected

→ for every $i \in I$, $v_{i+1}v \notin E(G)$. Otherwise

$$(C \cup P_i \cup P_{i+1}) - v_i v_{i+1} \text{ is a cycle longer than } C$$

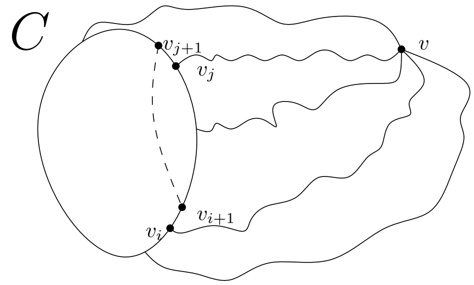


→ for every $j \notin I$, $vv_j \notin E(G)$.

⇒ if $i \in I$ then $i+1 \notin I$.

⇒ $|T| < |I|$

⇒ $|I| \geq k$ (from *)



$\rightarrow$ If $i, j \in I$ then $v_{i+1}v_{j+1} \notin E$. Otherwise the cycle

$$\underbrace{v_{j+1}, \ldots, v_i}_{C}, \overleftarrow{P}_{i+1}, P_j, \underbrace{v_{j-1}, \ldots, v_{i+1}}_{\overleftarrow{C}}, v_{j+1}$$

has length $|C| - 2 + |P_i| + |P_j| + 1 > C$

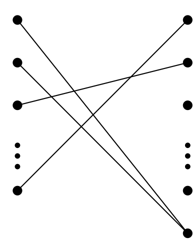
$\rightarrow$ the set $S = \{v_{i+1} | i \in I\} \cup \{v\}$ is an independent set.

$\rightarrow |S| = |I| + 1 > k$

$\rightarrow \alpha(G) \geq |S| > k = \kappa(G)$ ↯

$\rightarrow$ The Petersen Graph shows that this is tight (having $\alpha(PG) = 4$ and $\kappa(PG) = 3$ and being non-Hamiltonian.)

$\rightarrow$ Consider $K_{s,s+1}$



$K_{s,s+1}$

$\kappa(K_{s,s+1}) = s$

$\alpha(K_{s,s+1}) = s + 1$

not Hamiltonian, so the Theorem is tight.

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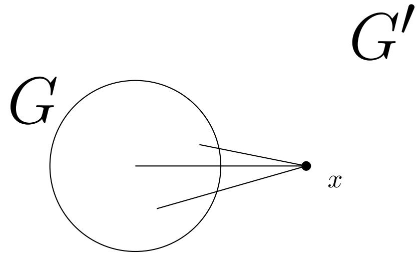
Corollary: If a graph G has $\alpha(G) \leq \kappa(G) + 1$ then G contains a Hamiltonian path.

–Proof:



$\alpha(G') = \alpha(G)$

$\kappa(G') = \kappa(G) + 1$

$\rightarrow$ By the Chvátal-Erdős theorem, G' contains a Hamiltonian cycle. Thus G contains a Hamiltonian path.