**Lecture Notes for Week 1** (First Draft)

This course will be a continuation of 21-640 (*Introduction to Functional Analysis*) from last semester. The main topics that I plan to cover are

I. More on Hilbert Spaces (including Spectral Theory)

II. Spectral Theory in Banach Spaces

III. Unbounded Linear Operators

IV. Semigroups of Linear Operators

V. Fourier Transforms & Applications

VI. Nonlinear Operators

In order to make the course accessible to students who took 21-640 from another instructor, I will begin by reviewing a few notational conventions and basic results concerning Hilbert spaces.

*Inner Products and Hilbert Spaces*

We shall use $\mathbb{K}$ to denote a field that is either $\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a linear space over $\mathbb{K}$. (Unless stated otherwise, all linear spaces are over $\mathbb{K}$.) By an *inner product* on $X$ we mean a mapping $(\cdot,\cdot) : X \times X \to \mathbb{K}$ satisfying the following 5 conditions:

(i) $\forall\ x, y, z \in X,\ (x + y, z) = (x, z) + (y, z)$,

(ii) $\forall x, y \in X,\ \alpha \in \mathbb{K},\ (\alpha x, y) = \alpha(x, y)$,

(iii) $\forall\ x, y \in X,\ (x, y) = \overline{(x, y)}$,

(iv) $\forall\ x \in X,\ (x, x) \geq 0$,

(v) $\forall x \in X,\ (x, x) = 0 \Rightarrow x = 0$.

A linear space equipped with an inner product is called an *inner product space*. If $(X, (\cdot,\cdot))$ is an inner product space, then the function $\|\cdot\| : X \to \mathbb{R}$ defined by

$$\|x\| = \sqrt{(x, x)} \text{ for all } x \in X$$

is a norm on $X$; this norm obeys the *Cauchy-Schwartz inequality*

$$|(x,y)| \leq \|x\|\|y\| \quad \text{for all } x, y \in X. \tag{1}$$

When we speak of the norm on an inner product space, we always mean the norm associated with the inner product via (1).

One of the most important features of Hilbert spaces is the notion of orthogonal projections. We record two very important results that were proved in 21-640.

**Theorem 1.1** (Projection onto Closed Convex Sets): Let $X$ be a Hilbert space and assume that $K \subset X$ is nonempty, closed, and convex. Then there is exactly one $y_0 \in K$ such that

$$\|x - y_0\| = \inf\{\|x - y_0\| : y \in K\}. \tag{2}$$

**Remark 1.2**:

(a) If $X$ is a reflexive Banach space and $K$ is a nonempty, closed, convex subset of $X$, then for every $x \in X$ there is at least one $y_o \in K$ satisfying (2).

(b) If $X$ is a uniformly convex Banach space and $K$ is a nonempty, closed, convex subset of $X$, then for every $x \in X$ there is exactly one $y_o \in K$ satisfying (2).

(c) It is not difficult to give an example of a nonempty, closed, convex subset $K$ of a nonreflexive Banach space and a point $x \in X$ such that there is no point $y_0 \in K$ satisfying (2).

Recall that if $S$ is a subset of a Hilbert space $X$, then the *orthogonal complement* $S^\perp$ of $S$ is defined by

$$S^\perp = \{y \in X : (x,y) = 0 \quad \text{for all } x \in S\}.$$

Moreover $S^\perp$ is always a closed subspace of $X$.

**Theorem 1.3** (Projection Theorem): Let $X$ be a Hilbert space and $M$ be a closed subspace of $X$. Let $x \in X$ be given. Then there is exactly one pair $(y,z) \in M \times M^\perp$ such that $x = y + z$.

Let $M$ be a closed subspace of a Hilbert space $X$. For each $x \in X$ let $P_M x$ denote the unique element of $M$ such that

$$\|x - P_m x\| \leq \|x - y\| \quad \text{for all } y \in M.$$

Then $P_M : X \to X$ is linear and

$$x - P_M x \in M^\perp \quad \text{for all } x \in X.$$

Notice that

$$x = P_M x + (x - P_M x) \text{ for all} x \in X$$

which implies

$$\|x\|^2 = \|P_M x\|^2 + \|x - P_M x\|^2 \text{ for all } x \in X$$

since $P_M x$ is orthogonal to $x - P_M x$. It follows that

$$\|P_M x\| \le \|x\| \text{ for all } x \in X,$$

and consequently $P_M$ is continuous with $\|P_M\| \le 1$. Observe that

$$P_{M^\perp} = I - P_M.$$

We refer to $P_M$ as the *orthogonal projection* onto $M$. We refer to a linear operator that is the orthogonal projection onto some closed subspace as an *orthogonal projection.*

Recall that a list $(e_j | j \in J)$ is said to be *orthonormal* provided that

$$\forall\ i, j \in J \text{ we have } (e_j, e_j) = 1 \text{ and } (e_i, e_j) = 0 \text{ whenever } i \ne j.$$

(Here $J$ can be any index set.) An orthonormal list $(e_j | j \in J)$ is said to be an *orthonormal basis* provided that it is *maximal* in the sense that

$$\forall x \in X\ (x \perp e_j \text{ for all } j \in J) \Rightarrow x = 0.$$

If $(e_j | j \in J)$ is an orthonormal basis then for every $x \in X$ we have

$$x = \sum_{j \in J} (x, e_j) e_j.$$

(Convergence of the above sum to $x$ means that for every $\epsilon > 0$ there is a finite set $F \subset J$ such that for every finite set $G$ with $F \subset G \subset J$ we have

$$\|x - \sum_{j \in G} (x, e_j) e_j\| < \epsilon.)$$

A very convenient feature of Hilbert spaces is that they can be identified with their own dual spaces via the inner product.

**Theorem 1.4** (Riesz Representation Theorem): Let $X$ be a Hilbert space and let $x^* \in X^*$ be given. Then there exists exactly one $y \in X$ such that

$$x^*(x) = (x, y) \text{ for all } x \in X;$$

moreover $\|x^*\|_* = \|y\|$.

It is convenient to introduce the *Riesz Operator* $\hat{R} : X \to X^*$ defined by

$$(\hat{R}(y))(x) = (x, y) \text{ for all } x, y \in X.$$

Recall that $\hat{R}$ is conjugate linear and isometric.

Let $X$ be a Hilbert space and $A \in \mathcal{L}(X; X)$ be given. Since $X$ is also a Banach space, $A$ has an adjoint $A_B^* \in \mathcal{L}(X^*; X^*)$ in the sense of Banach spaces defined by

$$(A_B^* x^*)(x) = x^*(Ax) \text{ for all } x^* \in X^*, \ x \in X.$$

The Hilbert space adjoint $A_H^* \in \mathcal{L}(X; X)$ of $A$ is defined by

$$(Ax, y) = (x, A_H^* y) \text{ for all } x, y \in X.$$

Notice that the Banach and Hilbert adjoints are related by the formula

$$A_H^* = \hat{R}^{-1} A_B^* \hat{R}.$$

When $X$ is a Hilbert space, the Hilbert adjoint is generally more convenient than the Banach Adjoint. Unless stated otherwise, when $X$ is a Hilbert space and $A \in \mathcal{L}(X; X)$ we use $A^*$ to denote the Hilbert adjoint.

Recall that if $X$ is a Hilbert space, then

$$(A^*)^* = A \text{ and } \mathcal{N}(A) = \mathcal{R}(A^*) \text{ for all } A \in \mathcal{L}(X; X).$$

*Self-Adjoint and Normal Operators*

**Definition 1.5**: Let $X$ be a Hilbert space and let $A \in \mathcal{L}(X; X)$ be given. We say that $A$ is

(a) *self-adjoint* if $A = A^*$.

(b) *normal* if $AA^* = A^*A$.

Clearly, every self-adjoint operator is normal, but not conversely.

**Proposition 1.6**: Assume that $X$ is a complex Hilbert space and let $A \in \mathcal{L}(X; X)$ be given. Then $A$ is self-adjoint if and only if $(Ax, x) \in \mathbb{R}$ for all $x \in X$.

**Proof**: Assume first that $A^* = A$ and let $x \in X$ be given. Then we have

$$(Ax, x) = (A^*x, x) = (x, Ax) = \overline{(Ax, x)},$$

so $(Ax, x) \in \mathbb{R}$.

Conversely, assume that $(Az, z) \in \mathbb{R}$ for all $z \in X$. Let $x, y \in X$ and $\alpha \in \mathbb{C}$ be given. Then

$$(A(x + \alpha y), x + \alpha y) \in \mathbb{R}.$$

Expanding the above expression and using the fact that $(Ax, x), (A(\alpha y), (\alpha y)) \in \mathbb{R}$ we find that

$$\overline{\alpha}(Ax, y) + \alpha(Ay, x) \in \mathbb{R}. \tag{3}$$

Putting $\alpha = 1$ and $\alpha = i$ in (3) we find that

$$(Ax, y) + (Ay, x) = (y, Ax) + (x, Ay), \tag{4}$$

$$-(Ax, y) + (Ay, x) = (y, Ax) - (x, Ay). \tag{5}$$

Adding (4) and (**??**) we find that

$$(AY, x) = (y, Ax) \quad \text{for all} x, y \in X,$$

which implies that $A = A^*$. $\square$

**Proposition 1.7**: Let $X$ be a Hilbert space and let $A \in \mathcal{L}(X; X)$ be given. Assume that $A$ is self-adjoint and that $X \neq \{0\}$. Then

$$\|A\| = \sup\{|(Ax, x)| : x \in X, \|x\| = 1\}.$$

**Proof**: Put

$$M = \sup\{|(Ax, x)| : x \in X, \|x\| = 1\}.$$

We shall show $\|A\| \leq M$ and $M \leq \|A\|$. (Notice that the definition of $M$ implies $|(Az, z)| \leq M\|z\|^2$ for all $z \in X$.)

Let $x \in X$ with $\|x\| = 1$ be given. Then we have

$$|(Ax, x)| \leq \|Ax\|\|x\| \leq \|A\|,$$

and consequently

$$M \leq \|A\|.$$

(Notice that self-adjointness of $A$ is not needed to conclude that $M \leq \|A\|$.

To establish the reverse inequality, let $x, y \in X$ with $\|x\| = \|y\| = 1$ be given. Then we have

$$\begin{aligned} (A(x + y), (x + y)) &= (Ax, x) + (Ax, y) + (Ay, x) + (Ay, y) \\ &= (Ax, x) + (Ax, y) + (y, Ax) + (Ay, y) \\ &= (Ax, x) + 2\mathrm{Re}(Ax, y) + (Ay, y). \end{aligned}$$

Replacing $y$ with $-y$ in the above, we find that

$$(A(x - y), (x - y)) = (Ax, x) - 2\mathrm{Re}(Ax, y) + (Ay, y).$$

Subtracting the expressions for $(A(x+y),(x+y))$ and $(A(x-y),(x-y))$ we find that

$$4\mathrm{Re}(Ax,y) = (A(x+y),(x+y)) - (A(x-y),(x-y)).$$

Taking the absolute value and using the fact that $|(Az,z)| \le M\|z\|^2$ we find that

$$4|\mathrm{Re}(Ax,y)| \le M(\|x+y\|^2 + \|x-y\|^2) = 2M(\|x\|^2 + \|y\|^2).$$

Since $\|x\| = \|y\| = 1$ we conclude that

$$|\mathrm{Re}(Ax,y)| \le M. \tag{6}$$

Choose $\theta \in [0,2\pi)$ such that

$$(Ax,y) = e^{i\theta}\|(Ax,y)|$$

and put $z = e^{-i\theta}x$. Observe that $\|z\| = 1$. Then we have

$$|(Ax,y)| = e^{-i\theta}(Ax,y) = (Az,y).$$

It follows that $(Az,y)$ is real and nonnegative. Since $\|z\| = \|y\| = 1$, it follows from

$$|(Ax,y)| = |\mathrm{Re}(Az,y)| \le M.$$

We conclude that

$$\|Ax\| = \sup\{(Ax,y) : y \in X, \|y\| = 1\} \le M.$$

Taking the supremum over $x \in X$ with $\|x\| = 1$ we find that

$$\|A\| \le M,$$

and we are done. $\square$

**Corollary 1.8**: Let $X$ be a Hilbert space and $A \in \mathcal{L}(X;X)$ be given and assume that $A$ is self-adjoint. Assume further that $(Ax,x) = 0$ for all $x \in X$. Then $A = 0$.

**Remark 1.9**: If $\mathbb{K} = \mathbb{C}$ then self-adjointness of $A$ is not needed in the above corollary.

**Proposition 1.10**: Let $X$ be a Hilbert space and $A \in \mathcal{L}(X;X)$ be given. Then $A$ is normal if and only if $\|Ax\| = \|A^*x\|$ for all $x \in X$.

**Proof**: Let $x \in X$ be given. Then we have

$$\begin{aligned}
\|Ax\|^2 - \|A^*x\|^2 &= (Ax,Ax) - (A^*x,A^*x) \\
&= (A^*Ax,x) - (AA^*x,x) \\
&= ((A^*A - AA^*)x,x).
\end{aligned}$$

Since $A^*A - AA^*$ is self-adjoint, the result follows easily from Corollary . $\square$

**Corollary 1.11**: Let $X$ be a Hilbert space and assume that $A \in \mathcal{L}(X;X)$ is normal. Then $\mathcal{N}(A) = \mathcal{N}(A^*)$.

*Isometric and Unitary Operators*

**Definition 1.12**: Let $X$ be a Hilbert space and $A \in \mathcal{L}(X;X)$ be given. We say that $A$ is

(a) an *isometry* provided that $\|Ax\| = \|x\|$ for all $x \in X$.

(b) *unitary* if it is a surjective isometry.

Notice that every isometry is injective and consequently every unitary operator is bijective.

**Example 1.13**: Let $X = l^2$ and define the right and left shift operators $R, L \in \mathcal{L}(L^2; l^2)$ by

$$Rx = (0, x_1, x_2, x_3, \cdots) \text{ for all } x \in L^2,$$
$$Lx = (x_2, x_3, x_4, \cdots) \text{ for all } x \in l^2.$$

Recall that $R^* = L$ and $L^* = R$. Notice that $R$ is an isometry, but fails to be surjective and $L$ is surjective, but fails to be an isometry. Notice also that $LR = I$, but

$$RLx = (0, x_2, x_3, x_4, \cdots) \text{ for all } x \in X,$$

so neither $R$ nor $L$ is normal.

**Proposition 1.14**: Let $X$ be a Hilbert space and $A \in \mathcal{L}(X;Y)$ be given. Then $A$ is an isometry if and only if

$$(Ax, Ay) = (x, y) \text{ for all } x, y \in X. \tag{7}$$

**Proof**: If (7) holds, then putting $y = x$ gives

$$\|Ax\|^2 = \|x\|^2 \text{ for all } x \in X$$

and $A$ is an isometry.

Conversely, assume that $A$ is an isometry and let $x, y \in X$ and $\alpha \in \mathbb{K}$ be given. Then we have

$$\begin{aligned} \|A(x + \alpha y)\|^2 &= (Ax + \alpha Ay, Ax + \alpha Ay) \\ &= \|Ax\|^2 + 2\mathrm{Re}[\alpha(Ay, Ax)] + |\alpha|^2\|Ay\|^2 \qquad (8) \\ &= \|x\|^2 + 2\mathrm{Re}[\alpha(Ay, Ax)] + |\alpha|^2\|y\|^2. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\|A(x+\alpha y)\|^2 &= \|x+\alpha y\|^2 \\ &= \|x\|^2 + 2\mathrm{Re}[\alpha(y,x)] + |\alpha|^2\|y\|^2.\end{aligned} \tag{9}$$

Combining (8) and (9) we obtain

$$\mathrm{Re}[\alpha(Ay,Ax) = \mathrm{Re}[\alpha(y,x)]. \tag{10}$$

If $\mathbb{K}=\mathbb{R}$ we are done. If $\mathbb{K}=\mathbb{C}$ then we put $\alpha=1$ and $\alpha=i$ in (10). $\square$

**Proposition 1.15** Let $X$ be a Hilbert space and $A \in \mathcal{L}(X;X)$ be given. Then $A$ is an isometry if and only if $A^*A = I$.

**Proof**: Observe that

$$\begin{aligned} A \text{ is isometric} &\Leftrightarrow (Ax,Ay)=(x,y) \text{ for all } x,y\in X \\ &\Leftrightarrow (A^*Ax,y)=(x,y) \text{ for all } x,y\in X \\ &\Leftrightarrow A^*Ax = x \text{ for all } x\in X.\end{aligned}$$

**Remark 1.16**: The order of the product $A^*A$ in Proposition is important. Observe that $R$ is isometric, but $RR^* \neq I$ and $LL^* = I$, but $L$ is not isometric, where $R$ and $L$ are as in Example 1.13.

**Proposition 1.17**: Let $X$ be a Hilbert space and $A \in \mathcal{L}(X;X)$ be given. Assume that $A$ is an isometry. Then $A$ is normal if and only if $A$ is surjective.

**Proof**: Assume first that $A$ is normal. Then we have

$$I = A^*A = AA^*,$$

which implies that $A$ is surjective.

Assume now that $A$ is surjective. Then $A$ is bijective and $A^{-1}$ is an isometry. Using , we see that

$$(A^{-1})^*A = I.$$

Since the adjoint of $A^{-1}$ is the inverse of $A^*$ we have

$$(A^*)^{-1}A^{-1} = I,$$

and this implies that

$$(AA^*)^{-1} = I$$

and consequently

$$AA^* = I.$$

Since $A$ is isometric, we know that $A^*A = I$ and consequently $AA^* = A^*A$. $\square$

*Idempotent Operators and Orthogonal Projections*

For the definition of idempotent operator (and the subsequent remark), we let $X$ be a normed linear space.

**Definition 1.18**: Let $X$ be a normed linear space and $E \in \mathcal{L}(X;X)$ be given. We say that $E$ is *idempotent* provided that $E^2 = E$.

Observe that for any linear operator $E$ we have

$$(I - E)^2 - (I - E) = E^2 - E.labelEq : 3.idem \tag{11}$$

Observe also that if $E$ is idempotent the $y \in \mathcal{R}(E)$ if and only if $y + Ey$.

**Remark 1.19**: Let $X$ be a normed linear space and $E \in \mathcal{L}(X;X)$ be given. Then

(a) $E$ is idempotent if and only if $I - E$ is idempotent.

(b) If $E$ is idempotent then $\mathcal{R}(E) = \mathcal{N}(I-E)$ and $\mathcal{R}(I-E) = \mathcal{N}(E)$. In particular, every idempotent operator has closed range.

If $X$ is a Hilbert space, then every orthogonal projection is idempotent, but not conversely. It is instructive to look at a simple example in $\mathbb{R}^2$.

**Example 1.20**: Let $X = \mathbb{R}^2$ and put

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is straightforward to check that $E$ is idempotent, $E$ is not normal, and $\|E\| = \sqrt{2}$.