Lecture Notes for Week 12 (First Draft)

Continuous and Compact Embeddings (Continued)

Example 12.1 Let $p_1, p_2 \in (1, \infty)$ be given. Then we have

$$l^1 \hookrightarrow l^{p_1} \hookrightarrow l^{p_2} \hookrightarrow c_0 \hookrightarrow c \hookrightarrow l^\infty.$$

None of these embeddings can be compact. Indeed the sequence $\{e^{(n)}\}_{n=1}^\infty$ is bounded in each one of the above spaces and it is also true that when $n \neq m$, we have $\|e^{(n)} - e^{(m)}\| \geq 1$ in all of the spaces (and consequently there can be no strongly convergent subsequence).

Remark 12.2: For continuous and compact embeddings, it is not necessary to have $X \subset Y$. The identity operator can be replaced with any linear injection $I : X \to Y$. Continuity of I is required for a continuous embedding and compactness of I is required for a compact embedding. If there could be any doubt about what the mapping I might be, this should be made clear when the embeddings are mentioned.

Example 12.3 Let Ω be a nonempty bounded subset of $\mathbb{R}^n$ and let $p_1, p_2 \in (1, \infty)$ be given. Then we have

$$C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^{p_2}(\Omega) \hookrightarrow L^{p_1}(\Omega) \hookrightarrow L^1(\Omega).$$

It is not too difficult to show that none of these embeddings can be compact. Remark 12.2 applies here because the elements of $C(\overline{\Omega})$ are individual functions and the elements of $L^\infty(\Omega)$ are equivalences classes of functions. In this situation the injection mapping is clear from context.

Example 12.4: Let $\gamma \in (0, 1]$ be given. By $C^{0,\gamma}[0, 1]$, we mean the set of all functions $f : [0, 1] \to \mathbb{K}$ satisfying

$$|f|_{0,\gamma} = \sup \left\{ \frac{|f(t) - f(s)|}{|t - s|^\gamma} : s, t \in [0, 1], s \neq t \right\} < \infty. \tag{1}$$

If $\gamma < 1$, these functions are said to be Holder continuous with exponent γ; if $\gamma = 1$ they are said to be Lipschitz continuous. Notice that $|\cdot|_{0,\gamma}$ is a seminorm, but not a norm, because it vanishes on constant functions. We equip $C^{0,\gamma}$ with the norm defined by

$$\|f\|_{0,\gamma} = \|f\|_\infty + |f|_{0,\gamma} \quad \text{for all } f \in C^{0,\gamma}[0, 1].$$

(An equivalent norm is obtained by replacing $\|f\|_\infty$ with $|f(0)|$ or with $|f(a)|$ for any conveniently chose $a \in [0, 1]$.)

Let $X = (C^{0,\gamma}[0, 1], \|\cdot\|_{0,\gamma})$, $Y = (C[0, 1], \|\cdot\|_\infty)$ and $\{f_n\}_{n=1}^\infty$ be a bounded sequence in X. It is easy to see that $\{f_n\}_{n=1}^\infty$ is uniformly bounded and uniformly equicontinuous. By the Ascoli-Arzela Theorem, there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ that is uniformly convergent. We conclude that $X \hookrightarrow\hookrightarrow Y$. As an exercise for yourself, you should verify that if $0 < \gamma < \alpha \leq 1$ then

$$C^{0,\alpha}[0,1] \hookrightarrow\hookrightarrow C^{0,\gamma}[0,1].$$

Operator Topologies and Sequences of Bounded Linear Operators

In applications, we frequently encounter sequences of bounded linear operators that are generated by some sort of approximation scheme. There are several different (and important) types of convergence associated with such sequences.

Definition 12.5: Let $T \in \mathcal{L}(X;Y)$ and $\{T_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}(X;Y)$. We say that

(a) $T_n \to T$ in the *uniform operator topology* as $n \to \infty$ provided that $\|T_n - T\| \to 0$ as $n \to \infty$.

(b) $T_n \to T$ in the *strong operator topology* as $n \to \infty$ provided that for every $x \in X$ we have $T_n x \to Tx$ (strongly) as $n \to \infty$.

(c) $T_n \to T$ in the *weak operator topology* as $n \to \infty$ provided that for every $x \in X$ we have $T_n x \rightharpoonup Tx$ (weakly) as $n \to \infty$.

It is clear that convergence in the uniform operator topology implies convergence in the strong operator topology implies convergence in the weak operator topology. As the examples below illustrate, the converse implications are false in general.

Example 12.6: Let $X = Y = l^2$.

(a) For every $n \in \mathbb{N}$ define $T_n \in \mathcal{L}(X;Y)$ by

$$(T_n x)_k = \begin{cases} x_k & \text{if } k \leq n \\ \\ 0 & \text{if } k > n. \end{cases}$$

It is easy to see that $T_n \to I$ in the strong operator topology and that the sequence $\{T_n\}_{n=1}^{\infty}$ fails to be convergent in the uniform topology

(b) For every $n \in \mathbb{N}$ define $L_n \in \mathcal{L}(X;Y)$ by

$$(L_n x)_k = \begin{cases} 0 & \text{if } k \leq n \\ \\ x_{k-n} & \text{if } k > n. \end{cases}$$

Let $x \in X$ be given. Observe that $\|L_n x\| = \|x\|$ for all $n \in \mathbb{N}$ and that $L_n x \to 0$ componentwise as $n \to \infty$. It follows that $L_n x \rightharpoonup 0$ (weakly) as $n \to \infty$. The sequence $\{L_n x\}_{n=1}^{\infty}$ fails to be strongly convergent unless $x = 0$; indeed if it were to converge strongly, the limit would have to be equal to the weak limit which is zero, but $\|L_n x\| = \|x\|$ for all $n \in \mathbb{N}$. Therefore $L_n \to 0$ in the weak operator topology as $n \to \infty$ but the sequence $\{L_n\}_{n=1}^{\infty}$ fails to converge in the strong operator topology.

Remark 12.7: Let X be a normed linear, Y be a Banach space, and $\{T_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}(X;Y)$. Recall that if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|T_n - T_m\| < \epsilon$ for all $m, n \geq N$ then there exists $T \in \mathcal{L}(X;Y)$ such $T_n \to T$ in the uniform operator topology as $n \to \infty$. (This is the content of Proposition 2.23.)

We now give an analogous result for Cauchy sequences in the strong operator topology.

Proposition 12.8: Let X, Y be Banach spaces and $\{T_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}(X;Y)$. Assume that for every $x \in X$ the sequence $\{T_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Then there exists $T \in \mathcal{L}(X;Y)$ such that $T_n \to T$ in the strong operator topology as $n \to \infty$.

Proof: Since Y is complete, the sequence $\{T_n x\}_{n=1}^{\infty}$ is convergent for every $x \in X$. Define $T : X \to Y$ by

$$Tx = \lim_{n\to\infty} T_n x \quad \text{for all } x \in X.$$

It is clear that T is linear. Since X is complete, the Principle of Uniform Boundedness implies that we may choose $M \in \mathbb{R}$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. It follows that

$$\|Tx\| \leq M\|x\| \quad \text{for all } x \in X,$$

and consequently T is bounded. $\square$

An analogue of Proposition 12.8 for the weak operator topology is part of Assignment 6.

It is important to understand if convergence of a sequence of bounded linear operators implies convergence of the sequence of adjoints.

Proposition 12.9: Let X, Y be normed linear spaces and let $T \in \mathcal{L}(X;Y)$ and a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{L}(X;Y)$ be given.

(a) If $T_n \to T$ in the uniform operator topology (of $\mathcal{L}(X;Y)$) as $n \to \infty$ then $T_n^* \to T^*$ in the uniform operator topology (of $\mathcal{L}(Y^*;X^*)$) as $n \to \infty$.

(b) If $T_n \to T$ in the weak operator topology (of $\mathcal{L}(X;Y)$) as $n \to \infty$ then for every $y^* \in Y^*$ we have $T_n^* y^* \overset{*}{\rightharpoonup} T^* y^*$ (weakly*) in X^* as $n \to \infty$.

(c) If X is reflexive and $T_n \to T$ in the weak operator topology (of $\mathcal{L}(X;Y)$) as $n \to \infty$ then $T_n^* \to T^*$ in the weak operator topology (of $\mathcal{L}(Y^*;X^*)$) as $n \to \infty$.

Proof: To prove (a) we simply observe that $\|T_n^* - T^*\| = \|T_n - T\|$ for all $n \in \mathbb{N}$.

To prove (b), assume that $T_n \to T$ in the weak operator topology as $n \to \infty$ and let $x \in X, y^* \in Y^*$ be given. Then, as $n \to \infty$, we have

$$(T_n^* y^*)(x) = y^*(T_n x) \to y^*(Tx) = (T^* y^*)(x).$$

Part (c) follows immediately from (b) and the observation that if X is reflexive then weak and weak* convergence in X^* are the same thing. $\square$

Remark 12.10: Convergence of a sequence of operators in the strong operator topology does not imply convergence of the sequence of adjoints in the strong operator topology. You are asked to find an example in Assignment 6.

If a sequence of compact operators converges in the uniform operator topology (and the target space is complete), then the limit is a compact operator. This result is very useful in applications.

Theorem 12.11: Let X be a normed linear space and Y be a Banach space. Let $T \in \mathcal{L}(X;Y)$ and a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{C}(X;Y)$ be given. Assume that $T_n \to T$ in the uniform operator topology as $n \to \infty$. Then $T \in \mathcal{C}(X;Y)$.

Proof: We want to show that $\mathrm{cl}(T[B_1(0)])$ is compact. Since Y is complete, it suffices to show that $T[B_1(0)]$ is totally bounded, i.e. for every $\epsilon > 0$, $T[B_1(0)]$ can be covered by a finite number of balls of radius ϵ. Let $\epsilon > 0$ be given. Since $T_n \to T$ in the uniform operator topology as $n \to \infty$ we may choose $N \in \mathbb{N}$ such that

$$\|T_N - T\| < \frac{\epsilon}{4}. \tag{2}$$

Observe that

$$\begin{aligned} \|Tx - Ty\| &\leq \|(T - T_N)x\| + \|T_N x - T_N y\| + \|(T - T_N)y\| \\ &\leq 2\|T - T_N\| + \|T_N x - T_N y\| \quad \text{for all } x, y \in B_1(0). \end{aligned} \tag{3}$$

Since T_N is compact, we may cover $\mathrm{cl}(T_N[B_1(0)])$ (and hence also $T_N[B_1(0)]$) by a finite number of balls of radius $\frac{\epsilon}{2}$; choose $x_1, x_2, \cdots, x_k \in B_1(0)$ such the centers of these balls are $T_N(x_1), T_N(x_2), \cdots, T_N(x_k)$:

$$T_N[B_1(0)] \subset \bigcup_{i=1}^{k} B_{\frac{\epsilon}{2}}(T_N(x_i)). \tag{4}$$

Then, by the triangle inequality, (2), (3), and (4), we have

$$T[B_1(0)] \subset \bigcup_{i=1}^{k} B_{\epsilon}(T(x_i)). \quad \square$$

Remark 12.12: We see from part (a) of Example 12.6 that convergence in the strong operator topology does not preserve compactness. Indeed, for each $n \in \mathbb{N}$, the operator T_n is continuous and has finite rank, so it is compact. On ther other hand $T_n \to I$ in the strong operator toplogy as $n \to \infty$ and I is not compact.

Remark 12.13 (Approximation Problem): It follows immediately from Theorem 12.11 that if X and Y are Banach spaces and $\{T_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{L}(X;Y)$ such that each T_n has finite rank and $T_n \to T$ in the uniform operator topology as $n \to \infty$ then T is compact. An important question that remained open for a long time is the following: If X and Y are Banach spaces and $T \in \mathcal{C}(X;Y)$ does there exist a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{L}(X;Y)$ such that each T_n has finite rank and $T_n \to T$ in the uniform operator topology as $n \to \infty$. The question was answered negatively by Per Enflo in 1973. We shall see that the answer is yes if Y is a Hilbert space.

Topological Vector Spaces

We now consider vector spaces having a topology that is naturally adapted to the vector space structure, but the topology need not be induced by a norm (or even by a metric). Such spaces are important because there are useful (non-metrizable) topologies (such as weak and weak* topologies) generated by classes of linear functionals on infinite-dimensional normed linear spaces. Moreover the theory of weak differentiation and distributions involves basic spaces of functions having natural topologies that are not induced by metrics. Such spaces also occur naturally in the study of holomorphic functions on open subsets of $\mathbb{C}$.

Definition 12.14: By a *topological vector space* (abbreviated *TVS*) we mean a pair (X, τ) where X is a linear space over $\mathbb{K}$ and τ is a topology on X such that

(i) (X, τ) is a Hausdorff space,

(ii) Addition is continuous from $X \times X$ to X,

(iii) Scalar Multiplication is continuous from $\mathbb{K} \times X$ to $\mathbb{X}$.

Remark 12.15: Many authors do not include item (i) (the Hausdorff property) as part of the definition of TVS. The Hausdorff property is needed for most results of interest and it is satisfied in all of the important examples of TVS that I have encountered. Rudin replaces (i) with the T1 separation axiom and then proves that T1, together with (ii) and (iii), imply the Hausdorff property.

Remark 12.16: In situations when there is no danger of confusion about what the topology could be, we refer to a "topological vector space X" and omit any specific reference to the topology τ.

Proposition 12.17: Let X be a topological vector space and $a \in X$, $\lambda \in \mathbb{K}\backslash\{0\}$ be given. Define the mappings $T_a, M_\lambda : X \to X$ by

$$T_a(x) = a + x, \quad M_\lambda(x) = \lambda x \ \text{ for all } x \in X.$$

Then T_a and M_λ are homeomorphisms of X onto X.

Proof: The vector space axioms and the definitions of T_a and M_λ imply that T_a and M_λ are bijective and that

$$(T_a)^{-1}(x) = -a + x, \quad (M_\lambda)^{-1}(x) = \lambda^{-1}x \ \text{ for all } x \in X.$$

Continuity of T_a, $(T_a)^{-1}$, M_λ and $(M_\lambda)^{-1}$ follow from continuity of addition and scalar multilpication. $\square$

A very important consequence of this proposition is *translation invariance* of the topology.

Corollary 12.18: Let X be a topological vector space and $V \subset X$ be given. The following three statements are equivalent.

(a) V is open.

(b) There exists $x \in X$ such that $x + V$ is open.

(c) $x + V$ is open for every $x \in X$.

In view of Corollary 12.18, the topology in a TVS is completely determined by a local base at any point.

Conventions on Terminology: By a *local base* for a TVS we mean a local base at 0. By a *neighborhood*, we mean an open neighborhood; in particular, a neighborhood of a point x is simply any open set U with $x \in U$.

Another simple, but very useful, consequence of Corollary 12.18 is that if $A, U \subset X$ and U is open then $A + U$ is open.

Proposition 12.9: Let X be a topological vector space and $A, U \subset X$. Assume that U is open. Then $A + U$ is open.

Proof: Observe that

$$A + U = \bigcup_{x \in A} (x + U)$$

and apply Corollary 12.18. $\square$

The notion of boundedness of a set plays an important role in the theory of topological vector spaces. Some care should be exercised with the terminology, because if the topology is induced by a metric, the sets which are metrically bounded can change when a different metric inducing the same topology is employed. We want important properties of sets to depend only on the topology and the linear structure.

Definition 12.20: Let X be a topological vector space and $E \subset X$ be given. We say that E is *topologically bounded* provided that for every neighborhood V of 0, there exists $t_0 > 0$ such that $E \subset tV$ for all $t > t_0$.

Remark 12.21: Many authors use the term "bounded" in place of "topologically bounded".

The following definition captures many important properties that a given TVS may or may not have.

Definition 12.22: Let X be a topological vector space. We say that X is

(a) *locally convex* if there is a local base $\mathcal{B}$ whose members are cionvex.

(b) *locally bounded* if there is a bounded neighborhood of 0.

(c) *locally compact* if there is a neighborhood V of 0 such that $\mathrm{cl}(V)$ is compact.

(d) *metrizable* if the topology is induced by some metric.

(e) *normable* if the topology is induced by a norm.

In order to give an overview of the big picture, the important characterizations of the properties in Definition 2.22 are given in the remark below. We will discuss these statements in detail later.

Remark 12.23: Let X be a topological vector space. Then

(i) X is locally convex if and only if the topology is generated by a separating family of seminorms.

(ii) X is locally compact if and only if X is finite dimensional.

(iii) X is metrizable if and only if there is a countable local base.

(iv) X is normable if and only if X is locally convex and locally bounded.

Lemma 12.24: Let X be a topological vector space and $A, B \subset X$. Then

$$\mathrm{cl}(A) + \mathrm{cl}(B) \subset \mathrm{cl}(A + B).$$

Proof: Consider the function $F : X \times X \to X$ defined by

$$F(x + y) = x + y \quad \text{for all } x, y \in X.$$

Since F is continuous we know that

$$F[\mathrm{cl}(A), \mathrm{cl}(B)] \subset \mathrm{cl}(F[A, B]),$$

which is precisely the desired conclusion. $\square$

Lemma 12.25: Let X be a topological vector space and K be a convex subset of X. Then $\mathrm{int}(K)$ and $\mathrm{cl}(K)$ are convex.

Proof: Let $U = \mathrm{int}(K)$ and $C = \mathrm{cl}(K)$. Since K is convex we know that

$$tK + (1-t)K \subset K \quad \text{for all } t \in (0, 1).$$

Let $t \in (0, 1)$ be given. Since $U \subset K$ we know that

$$tU + (1-t)U \subset K.$$

Since tU and $(1-t)U$ are open, it follows from Proposition 12.19 that $tU + (1-t)$ is open. Since $tU + (1-t)U$ is open and $tU + (1-t)U \subset K$ we know that

$$tU + (1-t)U \subset \mathrm{int}(K) = U,$$

and consequently U is convex.

To see that C is convex, let $t \in (0, 1)$ be given and observe that

$$tC = \mathrm{cl}(tK), \quad (1-t)C = \mathrm{cl}((1-t)K).$$

Using Lemma 2.24, and the fact that $tK + (1-t)K \subset K$ we find that

$$tC + (1-t)C \subset \mathrm{cl}(tK + (1-t)K) \subset \mathrm{cl}(K) = C,$$

and consequently C is convex. $\square$

Lemma 12.26: Let X be a topological vector space. Every neighborhood of 0 contains a balanced neighborhood of 0.

Proof: Let W be a neighborhood of 0. Since scalar multiplication is continuous at $(0, 0)$ we may choose a neighborhood V of 0 and $\delta > 0$ such that

$$\alpha V \subset W \quad \text{for all } \alpha \in \mathbb{K} \text{ with } |\alpha| < \delta.$$

Let $A = \{\alpha \in \mathbb{K} : |\alpha| < \delta\}$ and put

$$U = \bigcup_{\alpha \in A} \alpha V.$$

Since αV is open for every $\alpha \in A \backslash \{0\}$ (and since $0 \in \alpha V$ for $\alpha \neq 0$) we know that U is open. Furthermore, since $\gamma\alpha \in A$ for every $\gamma \in \mathbb{K}$ with $|\gamma| \leq 1$ we conclude that U is balanced. Finally, since $\alpha V \subset W$ for all $\alpha \in A$, we see that U is balanced.

Corollary 2.27: Let X be a topological vector space and $E \subset X$. Assume that for every neighborhood U of 0, there exists $t_0 > 0$ such that $E \subset t_0 U$. Then E is topologically bounded.

Proof: Let V be a neighborhood of 0. By Lemma 2.26, we may choose a balanced neighborhood U of 0 such that $U \subset V$. By assumption, we may choose $t_0 > 0$ such that $E \subset t_0 U$. Let $t > t_0$ be given. Then $t^{-1} t_0 U \subset U$ because U is balanced. Therefore, we have

$$E \subset t_0 U = t(t^{-1} t_0 U) \subset tU \subset tV,$$

and consequently E is bounded. $\square$

Lemma 12.28: Let X be a topological vector space. Every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

Proof: Suppose that W is a convex neighborhood of 0. Let $F = \{\alpha \in \mathbb{K} : |\alpha| = 1\}$ and put

$$K = \bigcap_{\alpha \in F} \alpha W.$$

Notice that K is convex and $K \subset W$ (since $1 \in F$). By Lemma 12.26, we may choose a balanced neighborhood U of 0 such that $U \subset W$. Since U is balanced, we know that $\alpha^{-1} U = U$ for all $\alpha \in F$ and we conclude that

$$U \subset K.$$

Let $V = \mathrm{int}(K)$ and observe that $U \subset V$, so $0 \in V$. Furthermore, $V \subset W$ because $K \subset W$. By Lemma 12.25, we know that V is convex. It remains to show that V is balanced. To show that V is balanced, we shall first show that K is balanced. To this end, let $\gamma \in \mathbb{K}$ with $|\gamma| \leq 1$ be given. If $\gamma = 0$ then obviously $\gamma K \subset K$. Assume that $\gamma \neq 0$ and put $\beta = \frac{\gamma}{|\gamma|}$. Then we have

$$\gamma K = |\gamma| \beta K = \bigcap_{\alpha \in F} |\gamma| \beta \alpha W = \bigcap_{\alpha \in F} |\gamma| \alpha W. \tag{5}$$

For every $\alpha \in F$, αW is a convex set containing 0, and consequently $|\gamma| \alpha W \subset \alpha W$. Therefore (5) tells us that $\gamma K \subset K$ and K is balanced.

To conclude that V is balanced, first observe that $0V \subset V$ since $0 \in V$. Let $\gamma \in \mathbb{K}$ with $0 < |\gamma| \leq 1$ be given. Then, by Proposition 12.17, $\mathrm{int}(\gamma K) = \gamma V$ and consequently

$$\gamma V \subset \gamma K \subset K.$$

Since γV is open, we can conclude $\gamma V \subset \mathrm{int}(K) = V$. $\square$

Lemma 2.29: Let X be a topological vector space and V be a neighborhood of 0. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of strictly positive numbers such that $r_n \to \infty$ as $n \to \infty$.

Then

$$X = \bigcup_{n=1}^{\infty} r_n V.$$

Proof: Let $x \in X$ be given and put

$$A = \{\alpha \in \mathbb{K} : \alpha x \in V\}.$$

Since the mapping $\alpha \to \alpha x$ is continuous and V is an open set containing 0 we know that A is an open subset of $\mathbb{K}$ and that $0 \in A$. Therefore we may choose $N \in \mathbb{N}$ such that $(r_N)^{-1} \in A$. It follows that $x \in r_N V$. $\square$

Lemma 2.30: Let X be a topological vector space. Suppose that V is a bounded neighborhood of 0 and let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of strictly positive real numbers such that $\delta_n \to 0$ as $n \to \infty$. Then $\{\delta_n V : n \in \mathbb{N}\}$ is a local base.

Proof: Let U be a neighborhood of 0. We need to shoe that there exists $n \in \mathbb{N}$ such that $\delta_n V \subset W$. Since V is bounded, we may choose $t_0 > 0$ such that $V \subset tU$ for all $t > t_0$. Choose $N \in \mathbb{N}$ such that $\delta_n t_0 < 1$. Then we have $\delta_N V \subset U$. $\square$

Remark 2.31: In view of Remark 12.23 (iii) and Lemma 2.30, any topological vector space having a bounded neighborhood of 0 is metrizable. In other words, if X fails to be metrizable, then X cannot have a bounded neighborhood of 0.