MATH 759: PROBLEM SET 1 **DUE IN CLASS ON FRI. SEP 13TH.**

1. Let $F : \mathbb{R}^n \to \mathbb{R}$ and $n \geq 2$. The value $\alpha \in \mathbb{R}$ is caller *regular* if for all $x \in \mathbb{R}^n$ such that $F(x) = \alpha$, $DF(x) \neq 0$.
   (i) Show that for any α regular, the set $\mathcal{M}_\alpha = \{x \in \mathbb{R}^n \ : \ F(x) = \alpha\}$ considered with topology it has as a subset of $\mathbb{R}^n$ is a differentiable manifold.
   (ii) Also show that $\mathcal{M}_\alpha$ is orientable.
2. Show that the tangent bundle, $T\mathcal{M}$, defined in the lecture is indeed a differentiable manifold. Also if $v \in T_p\mathcal{M}$ is given in the chart (U, ϕ) as $a^1 \frac{\partial}{\partial x_1} + \cdots + a^n \frac{\partial}{\partial x_n}$ and in the chart (V, ψ) as $b^1 \frac{\partial}{\partial y_1} + \cdots + b^n \frac{\partial}{\partial y_n}$, find the relationship between coefficients a_i and b_j.
3. If *df* $\in T_p\mathcal{M}^*$ is given in the chart (U, ϕ) as $\alpha_1 dx_1 + \cdots + \alpha_n dx_n$ and in the chart (V, ψ) as $\beta_1 dy_1 + \cdots + \beta_n dy_n$, find the relationship between coefficients α_i and β_j.
4. Let (U, ϕ) be a coordinate chart. Let $x_i(p)$ be the coordinates of ϕ^{-1}, that is $\phi^{-1}(p) = (x_1(p), \ldots, x_n(p))$. Show that $dx_1, \ldots dx_n$ are linearly independent and span $T_p\mathcal{M}^*$
5. Assume $\mathcal{M}$ is a connected manifold and $f : \mathcal{M} \to \mathbb{R}$ is such that $df = 0$. Show that f is constant.
6. Recall that $P^2 = S^2/\sim$ where $x \sim y$ is $x = -y$ or $x = y$. Consider $F : P^2 \to \mathbb{R}^4$ defined by
   $$F([x], [y], [z]) = (x^2 - y^2, xy, xz, yz).$$
   (Note that the mapping is well defined and does not depend on the choice of the representative of an equivalence class.) Show that F is an embedding.
7. Let $(G, \cdot)$ be a group, $\mathcal{M}$ a manifold and $G \times \mathcal{M} \to \mathcal{M}$ a properly discontinuous action.
   (i) Show that $\mathcal{M}/G$ is orientable if and only if there exists an orientation of $\mathcal{M}$ that is preserved by all $\Phi_g : \mathcal{M} \to \mathcal{M}$ for all $g \in G$.
   (ii) Show that P^2 is not orientable and that P^3 is orientable.
8. Let $\mathcal{M}$ be a compact manifold and v a smooth vector field on $\mathcal{M}$. Let $p \in \mathcal{M}$. Show that there exists a unique curve $\gamma \in C^\infty(\mathbb{R}, \mathcal{M})$ such that
   $$(\forall t \in \mathbb{R}) \ \ \gamma'(t) = v(\gamma(t)) \text{ and } \gamma(0) = p.$$
   You can use any theorem on ODE in Euclidean space.