21-484 Notes

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-	3 out of 4 exams	70%
-	few quizzes	10%
-	excercises 9	20%

→ Book: Graph Theory — Tom Bohman, John Mackey — CMU 21-484

Introduction to Graph Theory

Chartrand + Zhang

Definitions and terminology

A graph G is an ordered pair (V, E) where:

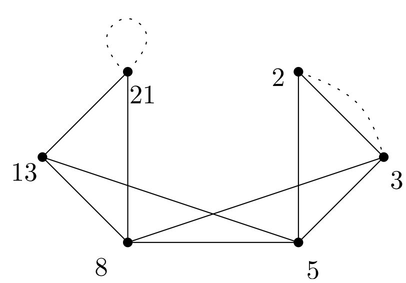
- V if a nonempty (finite) set
- E is a set of subsets of size 2 of elements of V.

V is called the vertex set and its elements are vertices.

E is called the edge set and its elements are edges.

Given a graph G, we use $V(G)$ to denote its vertes set and $E(G)$ for the edge set.

Example 1.2: $V = \{2, 3, 5, 8, 13, 21\}$ $E = \{\{2,3\}, \{2,5\}, \{3,5\}, \{3,8\}, \{5,8\}, \{5,13\}, \{8,13\}, \{8,21\}, \{13,21\}\}$



→ Remark: There are no parallel edges an no loops in "our" graphs

"our" graphs are sometimes called simple graphs

→ Hence: A graph is an irreflexive symmetric binary relation over a ground set.

→ what do we mean by $G = H$ (for graphs G and H)?

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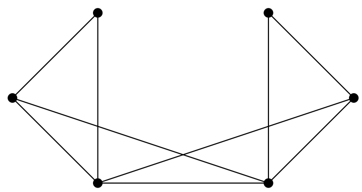
$\to$ Sometimes we only care about the structure of the graph



unlabeled
graph

- for brevity, we may denote the edge $\{u, v\}$ as uv
- If uv is an edge of G then the verticies u and v are said to be adjacent. Also, u is a neighbor of v. If $e = uv \in E$ then u and v are called the endpoints of e.
- THe set of all neighbors of a vertex u is called the neighborhood of u and is denoted by $N(u) = \{v \in V \mid uv \in E\}$
- The degree of a verte u is denoted $d(u) = |N(u)|$.

claim: For any (multi) graph $G = (V, E)$,

$$\sum_{v \in V} d(v) = 2|E|$$

Proof: double counting.

Corollary: The sum of the degrees is always even

Remark: The claim holds also for multigraphs with loops if a loop contributes 2 to the degree.

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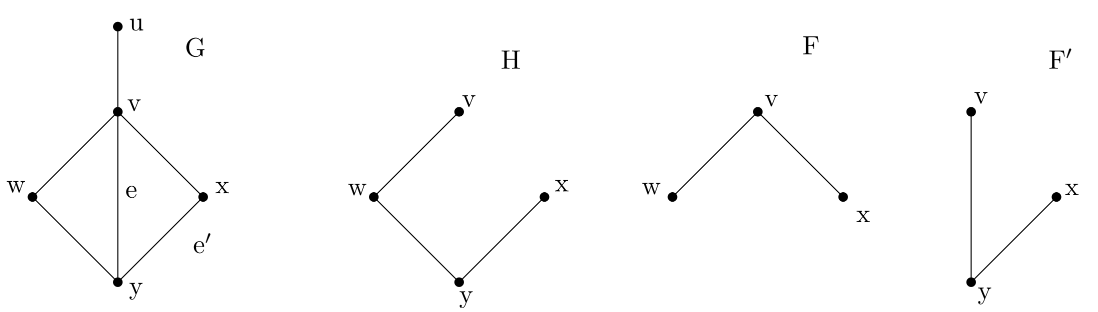
<u>recall:</u>
- $\deg(v) = d(v)$
- claim (Theorem 2.1): If G is a (multi)graph then $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$

<u>Def:</u>

- A graph H is called a <u>subgraph</u> of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \subseteq G$. Also say "G contains H as a subgraph".
- If $H \subseteq G$ and $H \neq G$ then H is a <u>proper</u> subgraph of G.
- If $H \subseteq G$ and $V(H) = V(G)$ then H is a <u>spanning</u> subgraph of G.
- If $H \subseteq G$ and $E(H) = E(G)|_{V(H)} = \{uv \in E(G) | u, v \in V(H)\}$ then H is an <u>induced</u> subgraph of G.
- Let $\varnothing \neq S \subseteq V(G)$ be a set of verticies and let $\varnothing \neq X \subseteq E(G)$ be a set of edges.
    - Then $G[S] = \langle S \rangle$ is the induced subgraph over S $G[S] = (S, E(G)|_S)$
    - And $G[X] = \langle X \rangle$ is the induced subgraph over X, $G[X] = \left(\bigcup_{e \in X} e, X \right)$

Example (1.15)

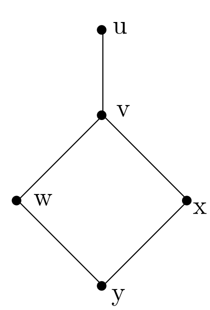


<u>here:</u> $H \subseteq G$
$F \not\subseteq H$

let $S = \{x, v, w\}$ then $F = G[S]$

let $X = \{e, e'\}$ then $F' = G[X]$

$F = G - \{y, u\}$



= G - e

Def. (page 11)

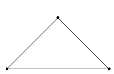
- A walk in a graph G is a sequence of verticies $v_0, v_1, \ldots, v_\ell$ such that $\forall 1 \leq i \leq \ell, v_{i-1}v_i \in E(G)$
- If G is a multigraph then a walk is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell$ s.t. $\forall 1 \leq i \leq \ell$ $e_i = \{v_{i-1}, v_i\} \in E(G)$
- also called a v_0-v_ℓ walk
- If $v_0 = v_\ell$ then the walk is closed, otherwise it's open.
- length is measured by edges, so in the definition above the length is ℓ.
- a walk is a trail if no edge is tranversed more than once.
- a walk is a path if no vertex is visited more than once.

claim (Theorem 1.6): If G contains a u-v walk of length ℓ, then it contains a path from u to v of length $\leq \ell$.

Proof:

- Assume that G has no u-v path
- let P be the shortest u-v walk, $P = v_0, v_1, \ldots, v_k$
- P is not a path, so $\exists i < j$ s.t. $v_i = v_j$
- $P' = v_0, \ldots, v_i, v_{j+1}, \ldots, v_k$ is a shorter u-v walk ■

more Def. (p. 13):

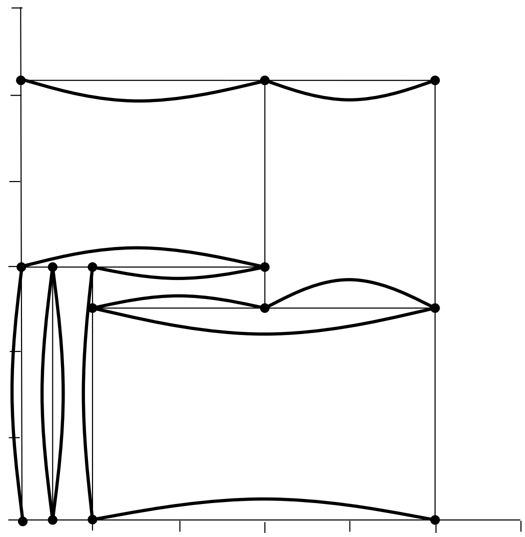
- A circuit is a closed trail of length ≥ 3.
- A cycle is closed trail in which no vertex appears twice except for the first and last.
- A k-cycle is a cycle of length k.
- similarly: even cycle, odd cycle
- 3-cycle: 

Example: You are given a rectangle divided into smaller rectangles s.t. each small rectangle has at least on side of integer length.

Show that the big rectangle also has at least one side of integer length.

standard solution:

- integrate $e^{2\pi i(x+y)}$ over the plane
- notice that the integral over a rectangle is 0 iff one side is integer



Define a graph G:

V - the corners of all rectangle

E - Pick two parallel "integer sides" from each small rectangle. Make the endpoints of the integer sides adjacent in G.

Notice:

1. the corners of the big rectangle are of degree 1.
2. the other verticies have degree 2 or 4 (because every such vertex is the corner of 2 or 4 small rectangle.

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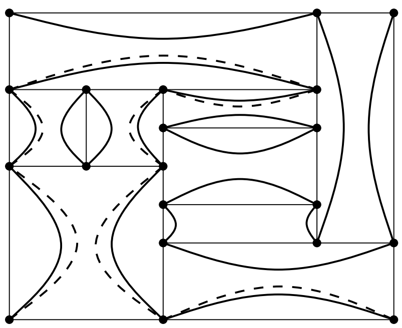
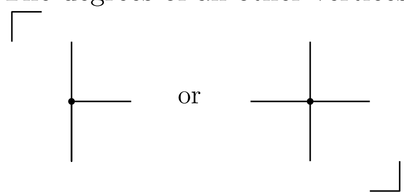


→ The degree of a vertex is the number of small rectangles containing it
→ The degrees of the corners of the big rectangle are 1
→ The degrees of all other vertices are either 2 or 4



→ So (1), start a trail from the lower left corner (of the big rectangle) and continue as long as you can
→ notice, you will not stop on a non-corner vertex.
⇒ there is a trail between two corners.
→ the trail is made entirely of integer edges.
→ there is an integer length side in the big rectangle.

→ So (2), the connected component containing the lower left corner should contain another corner (the sum of the degrees in the connected component is even)

→ ∃ path from the lower left corner to another corner.

Remark: Also works if instead of integer we have algebraic.

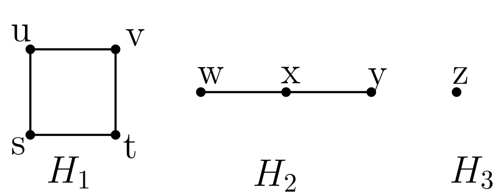
Def:(p.13-14)

- If a graph contains a path from u to v, then u and v are connected in the graph.
- If every two verticies in a graph G are connected, then G is connected.
- If G is not connected, it is disconnected.

Remark: The path $p = v_0$ shows that v_0 is connected to itself (by a path of length 0). So, the trivial graph—the simple graph with one vertex—is connected.

Def: (p.14) A connected subgraph of a graph G that is not a proper subgraph of any other connected subgraph, is called a "component" of G or a connected component.

Example: $H = H_1 \cup H_2 \cup H_3$:



Fact: (Theorem 1.7): The connectivity relation is an equivalence relation.

That is: if uRv iff there is a path from u to v, then R is an equivalence relation.

Proof:

1. R is reflexive ✓
2. R is symmetric since if $v_0, \ldots, v_\ell$ is a u-v path, then $v_\ell, \ldots, v_0$ is a v-u path.
3. R is transitive; if $v_0, \ldots, v_\ell$ is a u-v path and $v'_0, \ldots, v'_k$ is a v-w path, then $v_0, \ldots, v_\ell, v_0 1, \ldots, v'_k$ is a u-w walk. A u-w walk contains a u-w path.

claim (Thrm 2.4): If for any two vertices x, y in a greaph G with n verticies we have

$$\deg x + \deg y \geq n - 1$$

then G is connected.

Proof: If $x = y$ then x is an x-y path (len. 0)

If $xy \in E(G)$ then x, y is an x-y path (len. 1)

If $xy \notin E(G)$, then $y \notin N(x)$ and $y \notin N(y)$ and $x \notin N(y)$ and $x \notin N(x)$.

$\rightarrow N(X) \cup N(y) \subseteq |V(G) \setminus \{x, y\}\,| = n - 2$

$|N(x)| + |N(y)| \geq n - 1$

$\Rightarrow N(x) \cap N(y) \neq \varnothing$

$\Rightarrow \exists w$ such that $wx, wy \in E(G)$; x, w, y is x-y path in G.

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Def. (p. 15-16) Let G be a graph and let u, v be two vertices of G.

- The distance between u and b is the length of a shortest path connecting u and v, if such a path exists. If there is no u–v path in G, then the distance is undefinted (sometimes it is ∞).

  notation: $\mathrm{dist}_G(u, v)$ or $\mathrm{dist}(u, v)$ ($d_G(u, v)$ or $d(u, v)$)

- The maximal distance between any two verticies in G is the diameter of G, denoted $\mathrm{diam}(G)$

Example: Seen: If G has n verticies and for every $u, v \in V(G)$ we have

$$\deg(u) + \deg(v) \geq n - 1$$

then G is connected.

In fact, $\mathrm{diam}(G) \leq 2$.

Proof: Same proof: need to show: $\forall u, v \in V(G).\mathrm{dist}(u, v) \leq 2$

- $u = v$ ✓
- $uv \in E(G)$ ✓
- $uv \notin E(G)$ – we've seen that this implies $\exists w \in V(G).uw, wv \in E(G)$ ✓

Def: (p. 43): Given a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the degree sequence of G to be $\deg(v_1), \deg(v_2), \ldots, \deg(v_n)$.

(p.31): an isolated vertex is a vertex of degree zero.

an end point (or a leaf) is a vertex of degree one.

the minimal degree of G is $\min\limits_{v \in V(G)} \deg(v)$, denoted by $\delta(G)$.

the maximal degree of G is $\max\limits_{v \in V(G)} \deg(v)$, denoted by $\Delta(G)$.

Claim: The degree sequence of any nontrivial graph has repetitions.

Proof: Let G be a graph with n verticies. Then $\delta(G) \geq 0$ and $\Delta(G) \leq n - 1$.

- notice that if G has an isolated vertex, then $\Delta(G) \leq n - 2$
- If $\Delta(G) = n - 1$, then G does not contain an isolated vertex ($\delta(G) \geq 1$).

$\Rightarrow$ For any graph $\Delta(G) - \delta(G) \leq n - 2$, so the range of possible degrees is of size $\leq n - 1$.

–Pigeon-Hole Principle

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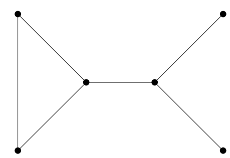
Def. (p. 43): A finite sequence of non-negative integers is called graphical if it is the degree sequence of some graph.

Example: (2.9) : Which of the following is graphical?

1. 3,3,2,2,1,1



2. 6,5,5,4,3,3,3,2,2 X sum of the degrees is odd
3. 7,6,4,4,3,3,3 X max degree $\leq n-1$
4. 3,3,3,1 X Each of the verticies of degree 3 must be connected to each other vertex, but the vertex of degree 1 can only be connected to one of them.

Lemma: (Theorem 2.10): A non-increasing sequence $S = d_1, d_2, \ldots, d_n$ $(n \geq 2)$ of non-negative integers, where $d_1 \geq 1$ is graphical if and only if the sequence

$$S_1 = d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphical.

Proof: If S_1 is graphical, then there is a graph G with degree sequence s_1. Assume that $V(G) = \{v_2, \ldots, v_n\}$ and that $\deg(v_i) = \begin{cases} d_i - 1 & 2 \leq i \leq d_1 + 1 \\ d_i & d_1 + 2 \leq i \leq n \end{cases}$

Construct G' by adding a vertex v_1 and the edges

$$v_i v_j \ 2 \leq j \leq d_1 + 1$$

Assume that s is graphical.

If G has a vertex such that $d(v_1) = d_1$ and the degrees of the neightbors of v_1 are $d_2, \ldots, d_{d_1+1}$, then removing v_1 yields a graph with degree sequene s_1.

$*$Assume that there is no G such that G has a vertex v of degree d_1 and the degrees of the neighbors of v are $d_2, \ldots, d_{d_1+1}$.

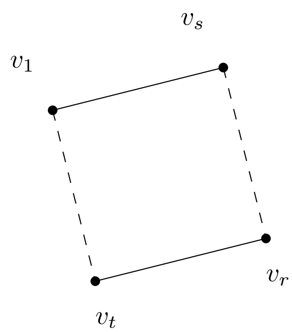
Let G be a graph such that

- the degree sequence of G is S
- the maximal sum (over verticies of degree d_1) of the degrees of neighbors of a vertex of degree d_1 is maximal (over all graphs with degree sequence S).

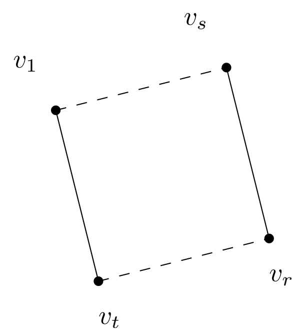
Let $V(G) = \{v_1, \ldots, v_n\}$, and assume that $\deg(v_1) = d_1$ and

$$\sum_{u \in N(v_1)} \deg(u) \text{ is maximal (over all such graphs and verticies of degree } d_1)$$

- by $\circledast$ the degrees of the neighbors of v_1 are <u>not</u> $d_2, \ldots, d_{d_1+1}$
  - $\rightarrow$ v_1 has a neighbor v_s such that there is a non neighbor of v_1, v_t, such that $\deg(v_t) > \deg(v_s)$.
  - $\rightarrow$ $\exists v_r$ such that $v_r v_t \in E(G)$ but $v_r v_s \notin E(G)$.



def G' by



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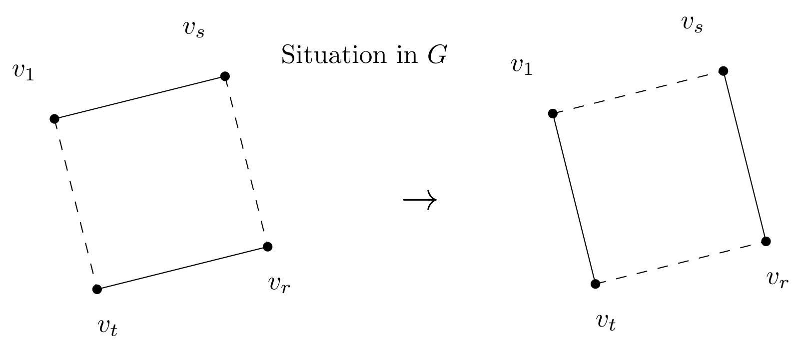
Recall: A graphical sequence.

- Want to prove: $S = d_1, d_2, \dots, d_n$, $d_1 \geq 1, n \geq 2$, s monotonically non increasing is graphical iff $s_1 = d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n$ is graphical.
- need to show that if s is graphical then there is a graph G with vertex set $\{v_1, \dots, v_n\}$ such that
  - the degree sequence of G is S
  - $\deg_G(v_1) = d_1$
  - the degrees of the neighbors of v_1 are $d_2, \dots, d_{d_1+1}$
- Assume that there is no such graph. Let G be the graph with $V(G) = \{v_1, \dots, v_n\}$ such that
  1. the degree sequence of G is s
  2. $\deg_G(v_1) = d_1$
  3. $\sum\limits_{v \in N_G(v_1)} \deg_G(v)$ is maximal (over all verticies of degree d_1 in G and over all graphs satisfying 1 and 2)

→ There is a neighbor of v_1, v_2, and a nonneighbor of v_1, v_t, such that $\deg(v_t) > \deg(v_s)$

→ $\exists v_r$. $v_r v_t \in E(G)$ and $v_r v_s \notin E(G)$.

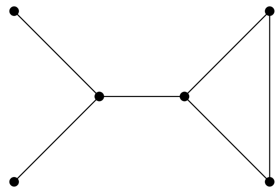


→ Define G' by removing $v_1 v_s$ and $v_t v_r$ and adding $v_s v_r, v_1 v_t$

- Notice:
  1. $V(G') = \{v_1, \dots, v_n\}$
  2. $d'_G(v_1) = d_1$
  3. The degree sequence of G' is s
  4. $\sum\limits_{v \in N_{G'}(v_1)} \deg_{G'}(v) > \sum\limits_{v \in N_G(v_1)} \deg_G(v)$ $\lightning$

Example (like 2.12 but shorter)

Is 3,3,2,2,1,1 graphical?

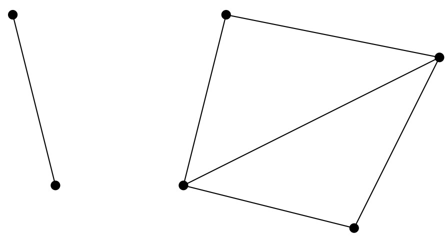


✓

2,1,1,1,1

0,0,1,1 $\to$ 1,1,0,0

0,0,0



Graphical sequences:

Do not define a graph

Do not define connectivity

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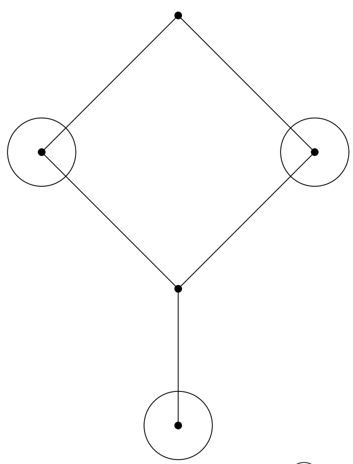
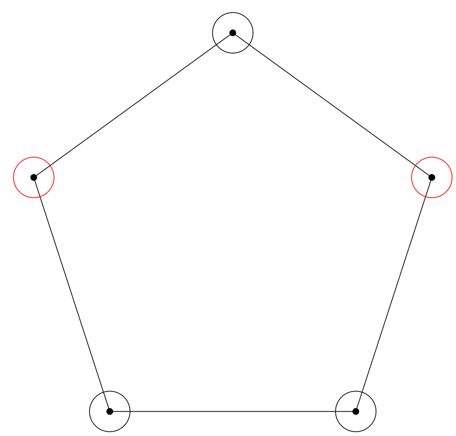
Def: (p. 19): A graph G is complete if every pair of distinct vertices is an edge.
(p. 20): A graph G is empty if every pair of distinct vertices is a non-edge.
$\rightarrow$ The complete graph on n vertices is denoted by K_n.
$\rightarrow$ $\overline{K_n}$ is empty
(p. 21): A graph G is called bipartite if $V(G)$ can be partitioned into two nonempty sets $U \dot{\cup} W = V(G)$ such that $G[U], G[W]$ are empty. U and W are called partite sets or parts.
(p. 19): A path on n vertices is denoted by P_n.
A cycle on n vertices is denoted by C_n.

Examples:





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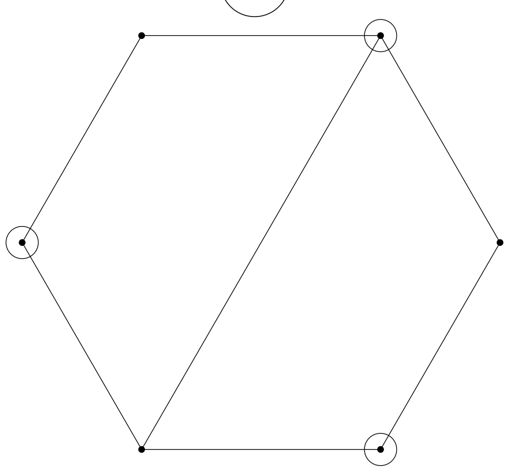
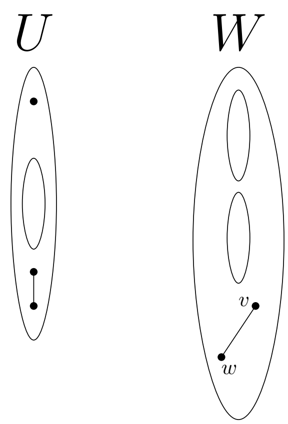


Proposition (Theorem 1.12): A non-trivial graph G is bipartite iff it contains no odd cycles.

<u>Proof:</u> If G contains an odd cycle, then G is not bipartite:



Assume that $v_1, v_2, \ldots, v_n, v_1$ is an odd cycle in G. Assume for the sake of contradiction that $U \dot{\cup} W = V(G)$ is a partition of the vertex set such that $G[U]$ and $G[W]$ are empty. Without loss of generality, assume that $v_1 \in W$. Since $v_1v_2 \in E(G)$, we know $v_2 \in U$, then $v_3 \in U$.

Continuing in this way (formally, by induction) we see that $v_i \in W$ iff i is odd. n is odd, so $v_n \in W$, but then $v_nv_1 \in G[W]$. $\lightning$

$\rightarrow$ If G is not bipartite then it contains an odd cycle:

- Assume that G is connected.
- Let $u \in V(G)$. Define
$$U = \{v | d(u,v) \text{ is even}\}$$
$$W = \{v | d(u,v) \text{ is odd}\}$$
- Clearly, $U \dot{\cup} W = V(G)$.
- U is not empty, $u \in U$. W is not empty because G is not trivial.
- Since G is not bipartite, one of $G[U]$ or $G[W]$ is not empty.
- assume that $vw \in E(G[W])$. Let $d(u,v) = 2s+1$ and $d(u,w) = 2t+1$, also let $p' = v_0, v_1, \ldots, v_{2s+1}$ be a u–v path. Let $p'' = w_0, \ldots, w_{2t+1}$ be a u–w geodesic path.
- $u \in p' \cap p''$. Let x be the last common vertex between p' and p''.
- $i = d(u,x)$
- the subpath of p', $v_0, v_1, \ldots, x$ is geodesic, so $x = v_i$.
- the subpath of p'', $w_0, w_1, \ldots, x$ is geodesic, so $w_i = x = v_i$.
- Consider the cycle $w = w_{2t+1}, w_{2t}, \ldots, w_i = v_i, v_{i+1}, \ldots, v_{2s+1} = v, w$. It is of length $2t+1-i) + (2s+1-i) + 1 = 2(t+1-i+s)+1$ which is odd.

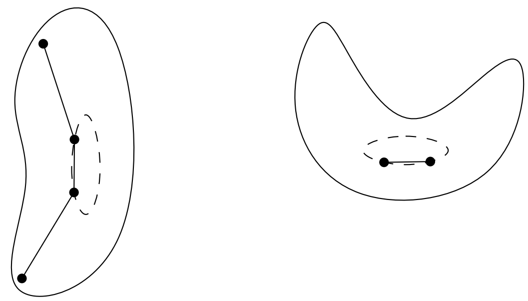
$\rightarrow$ If $vw \in E(G[U])$ then notice that $u \neq v$ and $u \neq w$. Otherwise, the other vertex $\in W$.

$\rightarrow$ Continue in the same manner.

$\rightarrow$ G is bipartite iff every connected component of G is bipartite or trivial.

Trees: Defs: (p. 86) - Let G be a connected graph, and let $e \in E(G)$. Then e is a bridge if $G - e$ is disconnected. If G is disconnected, then e is a bridge of G if it is a bridge of a component of G.

Claim: an edge is a bridge iff it lies on no cycle.



Proof: Assume $e \in G_1$, G_1 a component of G. If $e = uw$ is not a bridge is not a bridge then $G_1 - e$ is connected, so there is a u–w path in $G_1 - e$. Add e to this path to get a cycle in G_1.

If e is part of a cycle $u, \underset{e}{w},, v_1, \ldots, v_n, u$, define $p = w, v_1, \ldots, v_n, u$.

$\forall x, y \in V(G_1)$, we know that there is an x–y path in G_1. If e is not on the path, then x and y are connected in $G_1 - e$.

If e is on the path, replace it by p to get an x–y walk. ■

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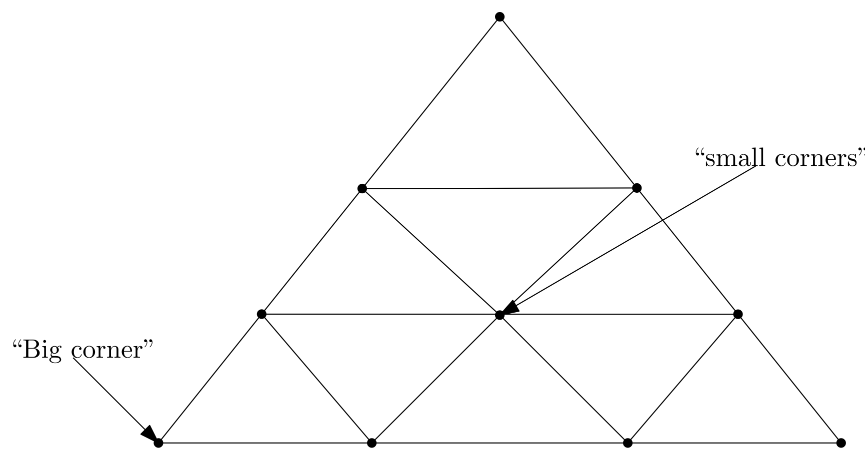
February 3, 2012

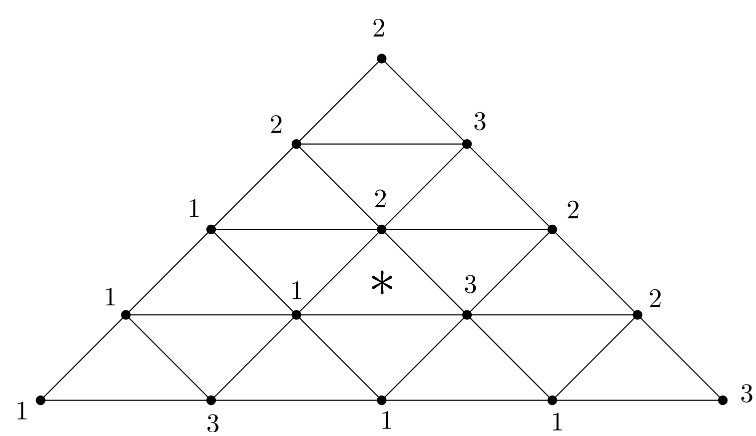
Def:

- A triangulation of a triangle is a subdivision of the triangle into smaller triangles.



- A Sperner labeling of a triangulation is a labeling of the corners by 1,2,3 such that
  - → The big corners are labeled 1,2,3
  - → A small corner lying on the line connecting two Big corners labeled i, j can only be labeled i or j.

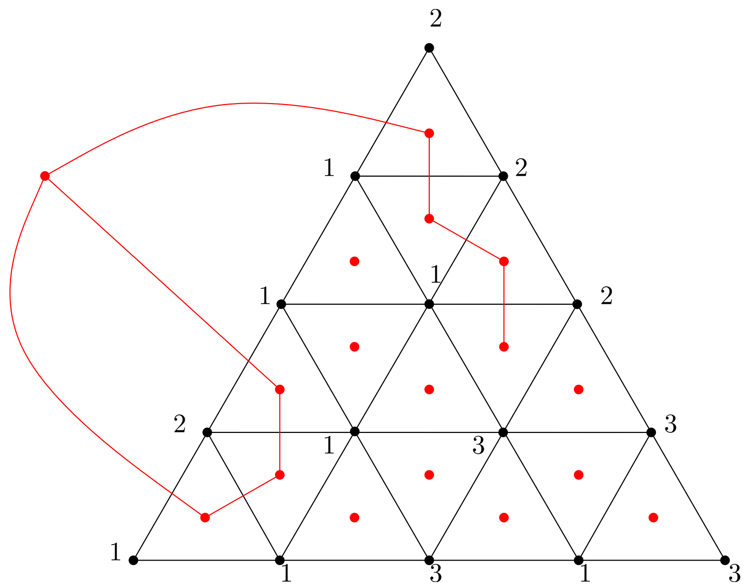
Example:



Lemma: (Sperner's lemma) In every Sperner's labeling there is a small triangle lableed 1,2,3.

Proof: Define the following Graph G.

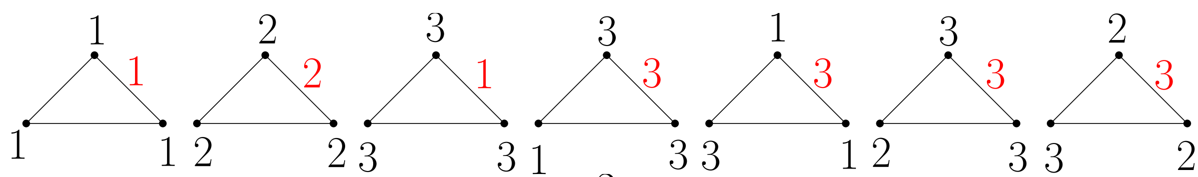
- The vertex set is the set of small triangles plus another vertex representing the outer face.
- There is an edge between two vertices if there is a side who's endpoints are labeled 1,2.



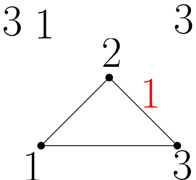
Notice:

1. the degree of an inner vertex is 0,1,or 2

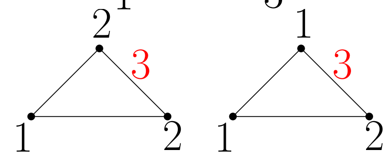
0:



1:



2:



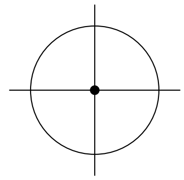
2. the degree of an inner vertex is 1 iff it is labeled 1,2,3

3. the degree of the outer vertex is odd because we start with 1 and end with 2. Let x be the number of lines moving from $1 \to 2$. Let y be the number of lines moving from $2 \to 1$. $x - y = 1$ so $x + y$ is odd.

$\to$ since the sum of degrees in a graph is even, we must have an inner vertex with odd degree. Actualy, we proved that there is an odd number of such triangles.

Application: Proving Brouwer's Fixed point Thm.

Thm: Every continuous function t from $D = \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\right\}$ to itself has a fixed point x_0 such that $f(x_0) = x_0$



Proof: - having a fixed point is a topological property

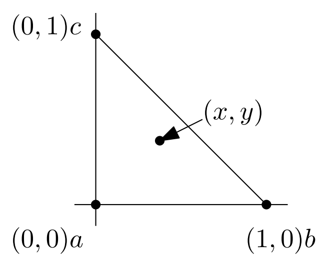
- If $f : G \to G$ is continuous and we know that the FP theorem holds in H, and there is $h : G \to H$ continuous and bijective

$$(h \circ f \circ h^{-1})(x_0) = x_0$$
$$f \circ (h^{-1}(x_0)) = h^{-1}(x_0) \quad \text{Can prove on triangles}$$

$\to$ Use Barycentric coordinates

$\to$ write (x,y) as a convex combination of a, b, c

$$(x,y) \mapsto (1 - x - y, x, y)$$



$\to$ let F be a continuous function from $\triangle abc$ to itself, assume $f(x,y,z) = (x', y', z')$ label (x,y,z)

1 if $x' < x$

2 if $x' \geq x$ bu $y' < y$

3 if $x' \geq x, y' \geq y$ but $z' < z$

Notice: $\to$ if a point can not be labeled, then $a' \geq a, b' \geq b, c' \geq c \Rightarrow a' = a, b' = b, c' = c \Rightarrow$ found a fixed point

$\to$ a is labeled 1 (or it is a fixed point)

$\to$ b is labeled 2 (or it is a fixed point)

$\to$ c is labeled 3 (or it is a fixed point)

$\to$ if (x,y) is on the a–b line, then $y = 0$, so the Barycentric coordinates $(1 - x, x, 0)$ in particular, the 3$^{\text{rd}}$ coordinate will not become smaller. So such (x,y) will be labeled 1 or 2 (or be a fixed point)

$\to$ true for all sides

$\to$ can apply Sperner's lemma

21-484 Notes

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February 6, 2012

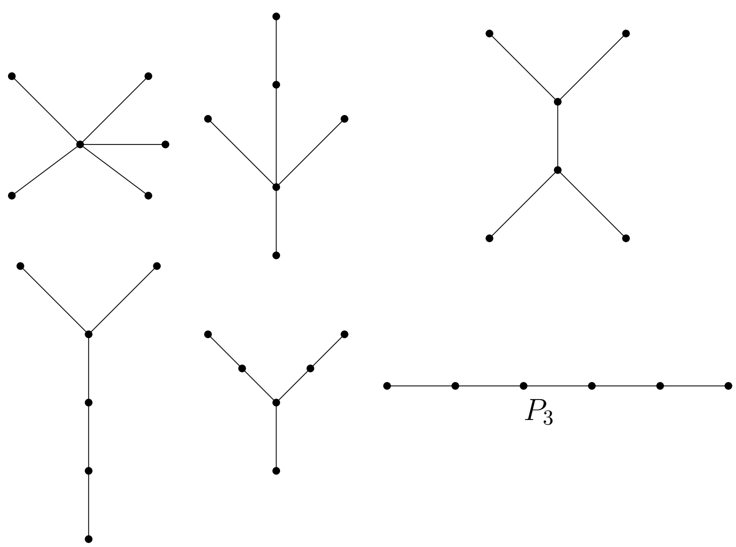
Recall:

- A bridge: $e \in G$ such that $G - e$ has more components than G.
- e is a bridge iff e lies on no cycle

Def: (p. 87-88):

- A grpah G is acyclic if it contains no cycles
- A tree is a connected acyclic graph
- Trees are usually denoted by T
- Every edge in a tree is a bridge

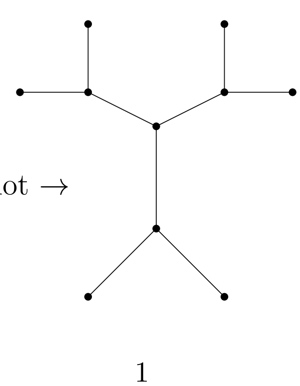
Example: (Figure 4.3): all trees with 6 verticies



Def: A caterpillar is a tree in which, after removing all the leaves, we get a path. This path is called the spine.

example: all trees with 6 vertices are caterpillars.

not $\rightarrow$



Def: A graph in which every component is a tree is called Forest.

Proposition (Thm 4.3): A graph is a tree iff every pair of veritices is connected by a unique path.

Proof:

- $\rightarrow$ If G is a tree, then it is connected, so for every $u \neq v \in V(G)$ there is a u–v path. If there are two different u–v paths, p and p' then we can form a cycle out of them.
- $\rightarrow$ If every pair $u, v \in V(G)$ are connected by a unique path, then G is connected, and G is acyclic since if we have a cycle $v_0, v_1, \ldots, v_\ell, v_0$ then $\begin{matrix} p = v_0 \ldots, v_\ell \\ p' = v_0, v_\ell \end{matrix}$ are two different v_0–v_ℓ paths.

Proposition (theorem 4.3): Every nontrivial tree has at least two end points.

Proof: Consider a path of maximal length in T. Call it P and let u and v be its enpoints. Then u and v are leaves. If, say, u has degree ≥ 2 the it has one neighbor, v_1, in the path and another, w, out of the path. Then $w, u, \ldots, v$ is a longer path in T (and by the proposition above, thats the only w–v path in T). ↯ The u–v path was maximal.

- connected
- acyclic
- $|E(G)| = |V(G)| - 1$

Proposition: (Thm 4.4): In every tree with n vertices, there are $n - 1$ edges.

Proof: By induction

$n = 1$, $T = \bullet$ $|V(T)| = 1, |E(T)| = 0$.

Assume that every tree with at most n vertices has $|E(T)| = |V(T)| - 1$. Given a tree with $n + 1$ vertices, we know that it has a leaf u, so $T - u$ has n vertices and thus $n - 1$ edges. So T has $n + 1$ vertices and n edges. ✓

Corollary (Corollary 4.6): If G is a forest with k components, then it has $n - k$ edges.

Proof: count.

Theorem 4.7: In every connected graph with n vertices there are at least $n - 1$ edges.

Proof: easy to verify when $n \leq 3$. Assume that G is the minimal (by number of vertices and then number of edges) graph with n vertices, at most $n - 2$ edges and G is connected.

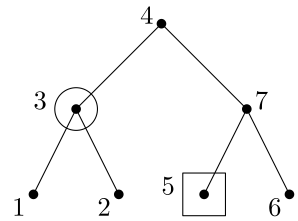
- If G is acyclic, then we have a leaf, removing the leaf will result in a graph with $n-1$ vertices, at most $n - 3$ edges, which is connected. ↯ contradicting minimality of G.
- If G has a cycle, then an edge on the cycle is not a bridge, so removing it we'll get a connected graph with n vertices and one less edge. ↯ contradicting minimality of G.

"Proofs from the book"

n^{n-2}

$$\begin{aligned} [n] &\rightarrow [n] \\ \{1, 2, \ldots, n\} &\rightarrow \{1, 2, \ldots, n\} \end{aligned}$$

n^n


4
3
7
1
2
5
6

21-484 Notes

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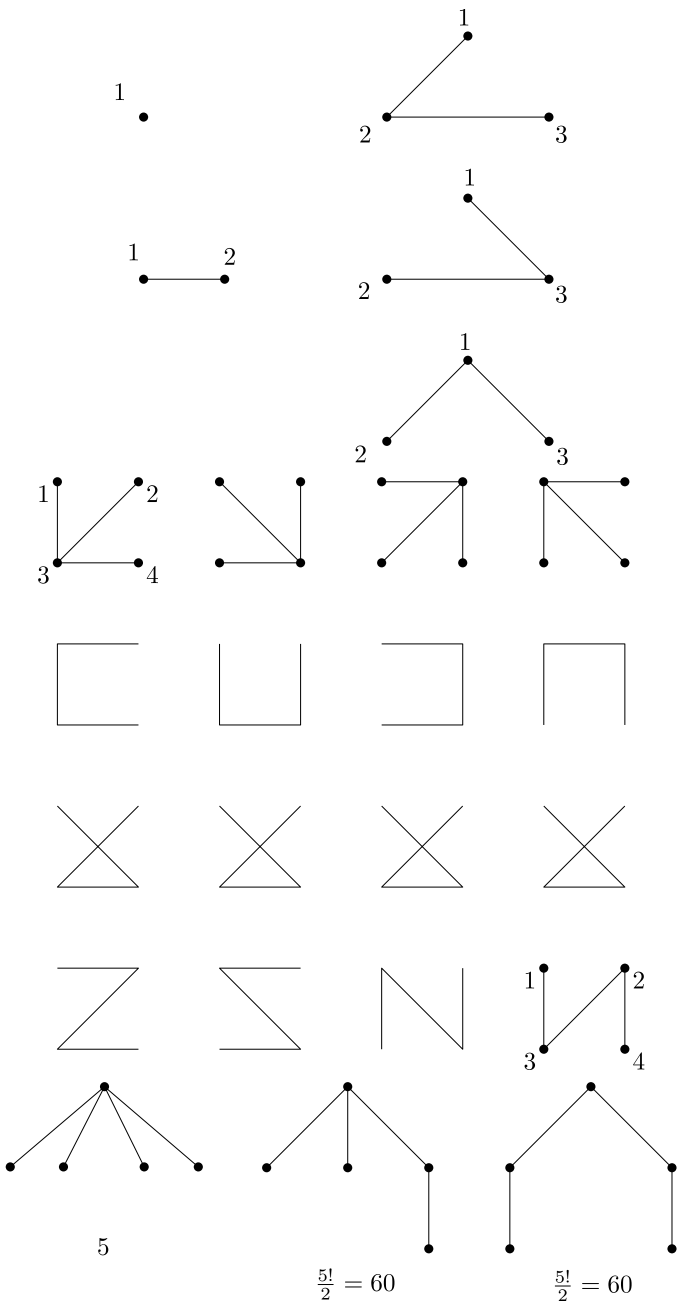
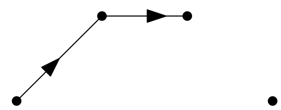
February 8, 2012



$1, 1, 3, 16, 125$

n^{n-2}

Theorem (Cayley's Formula, Thm. 4.15): The number of labeled trees with vertex set $[n] = \{1, 2, \ldots, n\}$ is n^{n-2}.



Def: A directed graph is a graph in which the set of edges is a set of order pairs (instead of 2-sets)

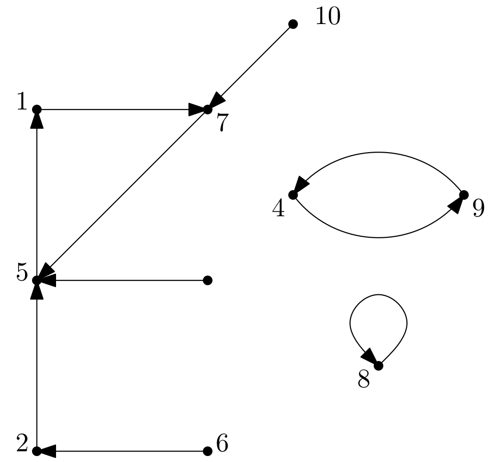
- Generally, in a directed graph we allow loops.

Proof: (Joyal) Let $\mathcal{T}_n$ be the set of all trees with vertex set $[n]$ and two marking $\bigcirc, \square$. Let T_n be the number of trees with vertex set $[n]$. Clearly $|\mathcal{T}_n| = n^2 \cdot T_n$. We are going to show that $|\mathcal{T}_n| = n^n$, by showing a bijection between $\mathcal{T}_n$ and $[n]^{[n]}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$$

→ Let $f : [n] \to [n]$ be any function from $[n]^{[n]}$.

→ Let $\vec{G}_f$ be the directed graph $([n], \{(i, f(i)) | i \in [n]\})$



→ The outdegree of every vertex (the number of edges going out of the vertex) is 1 (since f is a function)

→ In every connected component, the number of edges is the same as the number of vertices.

→ Every conponent is unicyclic (a tree + one edge)

→ This cycle is a directed cycle (otherwise, we will have a vertex with outdegree 2).

→ Let M be the set of all vertices in cycles

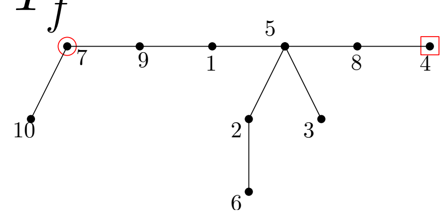
$$M = \{1, 4, 5, 7, 8, 9\}$$

→ notice: M is the largest subset of $[n]$ such that $f|_m$ is a bijection.

→ Define T_f:
- $V[t_f] = [n]$
- write $f|_m = f(m_1), f(m_2), \ldots, f(m_{|M|})$
- create a path $f(m_1), f(m_2), \ldots, f(m_{|M|})$

$$f|_m = \begin{pmatrix} 1 & 4 & 5 & 7 & 8 & 9 \\ 7 & 9 & 1 & 5 & 8 & 4 \end{pmatrix}$$

T_f



→ mark $f(m_1)$ by a $\bigcirc$
mark $f(m_{|M|})$ by a $\square$

→ for any vertex i out of m add $\{i, f(i)\}$

→ To complete the proof we need to show that the maaping $f \to T_f$ is a bijection, by describing the inverse map.

→ write the elements of the $\bigcirc \to \square$ path
write them again sorted to get $f|_m$.
any element not in M is mapped to the next vertex in the path connecting it to the $\bigcirc \to \square$ path

21-484 Notes

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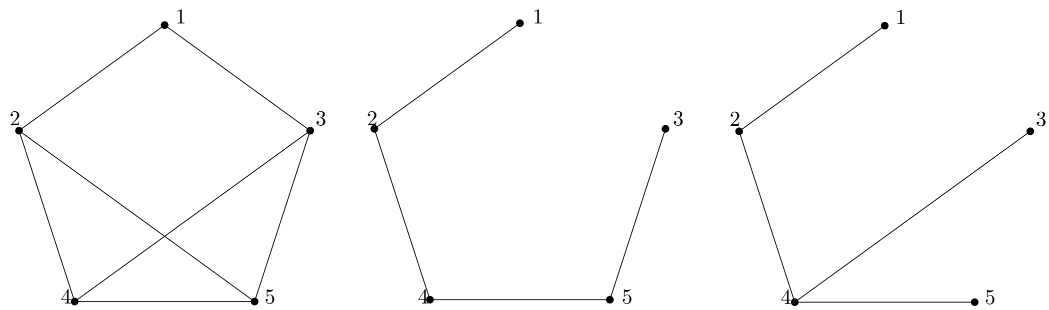
$\rightarrow$ Recall: A subgraph H of a graph G is spanning if $V(H) = V(G)$.

Def: (p. 95) A spanning tree of a connected graph G is a spanning subgraph which is a tree.

Example: (Fig 4.7)



claim (Thm 4.10): Every connected graph contains a spanning tree.

Proof: Let H be a minimal (by number of edges) connected spanning subgraph of G.

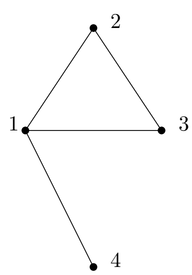
$\rightarrow$ If H is not a tree, it contains a cycle. Removing a non-bridge from H results in a smaller connected spanning subgraph $\lightning$.

Recall: The number of spanning trees contained in K_n is n^{n-2} (Cayley's formula).

Def: (page 48) The adjacency matrix of a graph G with n vertices is the matrix $A = A_G = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} 1 & ij \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Example:

$G =$ 

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Def: The Laplacian matrix of a graph G with n vertices is the $n \times n$ matrix $L = L_G = (\ell_{i,j})$ where

$$\ell_{i,j} = \begin{cases} \deg(i) & i = j \\ -1 & i \neq j \text{ and } ij \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

$$L_G = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Theorem (Thm 4.16, the matrix tree theorem, Kirchoff's theorem)

Let G be a graph and let $\lambda_1, \ldots, \lambda_{n-1}$ be the non-sero eigenvalues of L_G. Then the number of spanning trees of G is

$$\frac{1}{n}\lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}$$

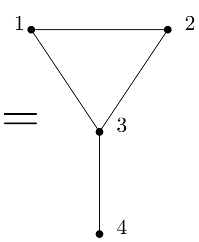
Equivalently: the number of spanning trees is the absolute value of any cofactor of L_G.

Def: The (i,j)-minor of a matrix A is the determinant of the matrix you get by removing the i^{th} row and the j^{th} column of A. Denote it by $M_{i,j}$.

- The (i,j) cofactor of A is $C_{ij} = (-1)^{i+j} M_{i,j}$.

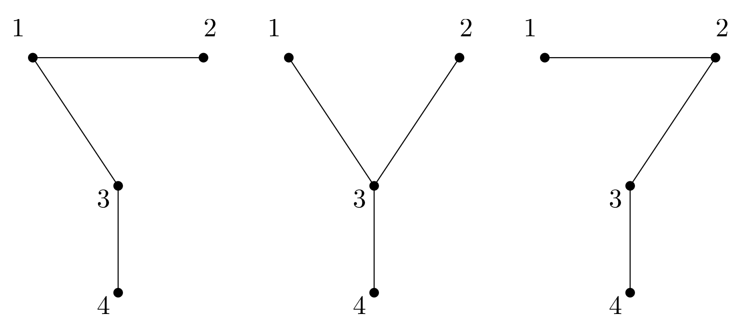
Example:

$G =$



$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad L_G = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$C_{3,3} = (-1)^6 \cdot \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

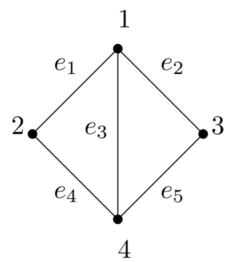


Elements from the proof:

Def (p. 48): The incidence matrix of a graph G with n vertices and m edges is the $n \times m$ matrix $B = B_G = (b_{i,j})$ where

$$b_{ij} = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ vertext belongs to the } j^{\text{th}} \text{ edge} \\ 0 & \text{otherwise} \end{cases}$$

Example:



$$B_G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Def: An oriented incedence matrix is an incedent matrix that has one 1 and one -1 in every column. Denoted by $\overrightarrow{B_G}$

When saying "the" oriented incedence matrix we mean that upper nonzero element in every column is 1.

notice: $L_G = \overrightarrow{B_G} \cdot \overrightarrow{B_G}^T$

$\rightarrow$ Also, $M_{1,1} = \overrightarrow{F} \cdot \overrightarrow{F}^T$ where $\overrightarrow{F} = \overrightarrow{B_G}$ without the first row.

$\rightarrow$ Apply the Cauchy-Binet Theorem

$$\det(M_{1,1}) = \sum_S \det(F_s) \cdot \det(F_s^T) = \sum_S \mathrm{Det}(F_s)^2$$

where s goes over all subsets of size $n-1$ of $2 \ldots, m$.

$\rightarrow$ notice that the $\det(F_s) = \pm 1$ when s spans a tree.

21-484 Notes

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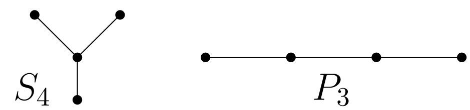
February 13, 2012

- Recall: There are 4^{4-2} labeled trees with four vertices.

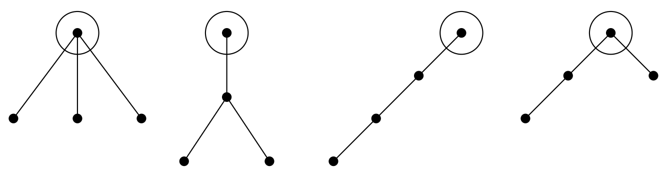
→ Notice that these are the different unlabeled trees with 4 vertices:



→ Def: (page 88) - a tree in which one of the vertices is distinguished as the root is called a rooted tree and denoted (T, v).

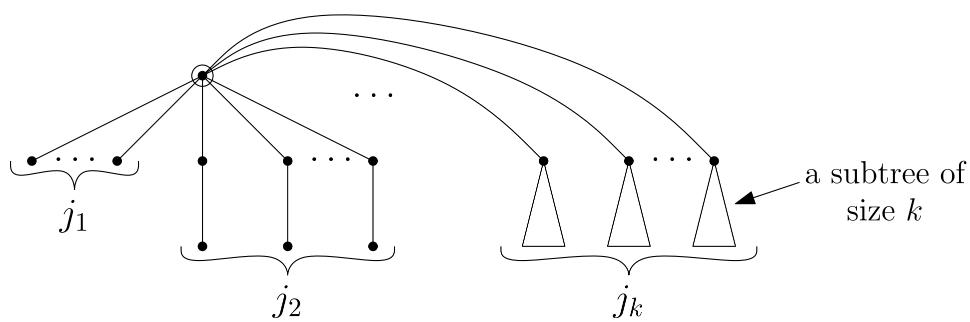
→ Remark: People also consider rooted graphs, in which we may have a set of roots: (G, R)

Example: There are 4 rooted trees with 4 vertices.


$= ((\{1,2,3,4\}, \{\{1,2\}, \{2,3\}, \{3,4\}\}), 3)$

- Let a_n be the number of rooted trees with n vertices.

- $a_1 = 1$



→ Let (T, v) be a rooted tree.

- Let $T_1, \ldots, T_d$ be the subtrees at v.

- let j_i be the number of subtrees of size i.

- In how many ways can we choose the subtrees of size k?

$\binom{a_k + j_k - 1}{j_k}$ – choosing j_k elements out of a_k elements with repetition and without order.

$$\Rightarrow^{n>1} a_n = \sum_{j_1+2j_2+3j_3+\ldots+(n-1)j_{n-1}=n-1} \binom{a_1+j_1-1}{j_1}\binom{a_2+j_2-1}{j_2}\cdots\binom{a_{n-1}+j_{n-1}-1}{j_{n-1}}$$

→ got a recursive formula

→ recall: $\dfrac{1}{(1-x)^s} = \displaystyle\sum_{k=0}^{\infty} \binom{s+k-1}{k} x^k$ (Newton's generalized Binomial theorem)

→ let $A(z)$ be the generating function for the sequence a_n.

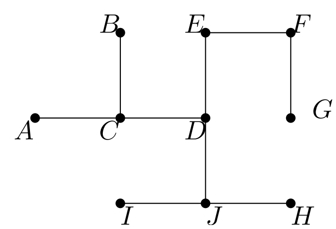
→ $A(z) = \dfrac{z}{(1-z)^{a_1}(1-z^2)^{a_2}(1-z^3)^{a_3}\cdots}$

→ (take log and some simplifying) $A(z) = z \cdot exp(A(z) + \frac{1}{2}A(z^2) + \frac{1}{3}A(z^3) + \ldots)$

→ (not trivial) $a_n = \dfrac{1}{\alpha^{n-1}\cdot n} \cdot \sqrt{\beta/2\pi n} + O(n^{-5/2}\alpha^{-n})$

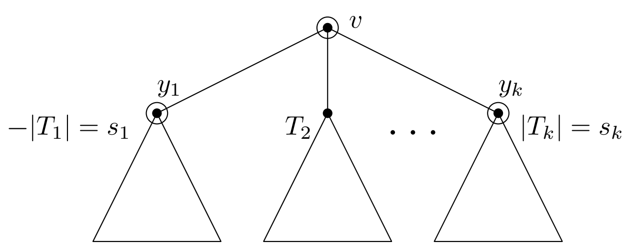
where $1/\alpha \approx 2.955765285652\ldots$ $\alpha\sqrt{\beta/2\pi n} \approx 0.439924012571\ldots$

→ Let T be a tree, v a vertex in T. The weight of v is the size of the maximal subtree at v_1.



weight$(D) = 3$ weight$(E) = \max(2,7) = 7$

- A vertex of minimal weight is called a centroid.



- $T_1, \ldots, T_k$ are the subtrees at v, their sizes are $s_i = |T_i|$, the root of T_i is the neighbor of v in T_i and it is denoted by y_i.

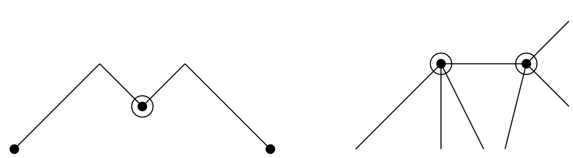
→ weight $(y_i) \geq 1 + s_2 + \ldots + s_k = n - s_1$

→ If there is a centroid of T in T_1, w, then weight$(v) = \max(s_1, \ldots, s_k) \geq$ weight$(w) \geq 1 + s_2 + s_3 + \ldots + s_k$

This is possible only if $s_1 > s_2 + \ldots + s_k$ $(*)$

→ At most one subtree of a given vertex can contian a centroid of T.

→ There are at most 2 centroids and, if there are two, they are adjacent.



→ $(*)$ is iff.

21-484 Notes
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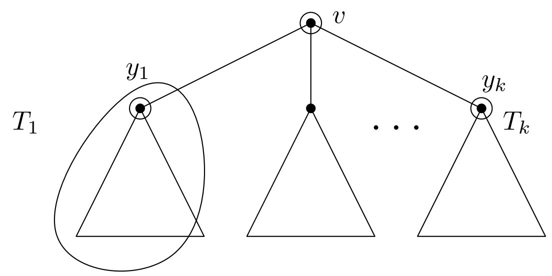
Recall: - Got the following GF for the number of rooted trees with n vertices.

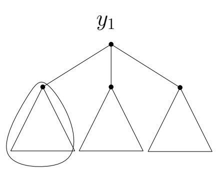
$$A(z) = \frac{z}{(1-z)^{a_1}(1-z^2)^{a_2}(1-z^3)^{a_3}\dots}$$

- Defined the weight of a vertex in a tree

- Defined a centroid of a tree.

→ Notation: T is a tree, $v \in T$ a vertex, $T_1, \dots, T_k$ are the subtrees at v, $s_1, \dots, s_k$ are the sizes of $T_1, \dots, T_k$ respectively, $y_1, \dots, y_k$ the roots of $T_1, \dots, T_k$ (the neighbor of v in T_i is y_i)





- If there is a centroid in T_1 then

$$s_1 > s_2 + s_3 + \dots + s_k$$

→ concluded: There are at worst 2 centroids in a tree. If there are 2, they are adjacent.

Claim: If $s_1 > s_2 + \dots + s_k$ then T_1 contains a centroid.

Proof: weight$(y_1) \leq \max(1 + s_2 + s_3 + \dots + s_k, s_1 - 1) \leq s_1 =$ weight(v)

- The weight of all vertices in $T_2, \dots, T_k$ is at least $s_1 + 1$, hence they are not centroids.

→ T_1 must contain a centroid. (If v is a centroid, then y_1 is also a centroid, otherwise there is no centroid outside of T_1)

Corollary: v is the only centroid of T iff $\forall 1 < j \leq k = \deg(v)$

$$* \quad s_j \leq s_1 + s_2 + \dots + s_k - s_j$$

→ If $s_j > s_1 + \dots + s_k - s_j$ for some j, then T_j contains a centroid.

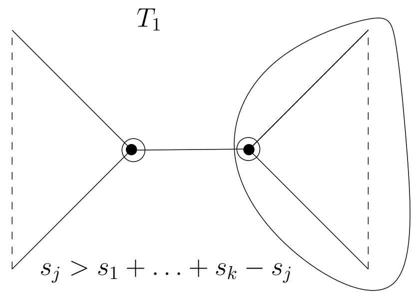
→ If $\forall 1 < j \leq k$, $s_j \leq s_1 + \dots + s_k - s_j$, then no T_j contains a centroid.

→ Strategy of counting unlabeled trees.

→ start by counting only trees with one centroid.

→ a_n is the number of rooted trees with n vertices.

→ In how many ways can we construct a rooted tree violating $*$?



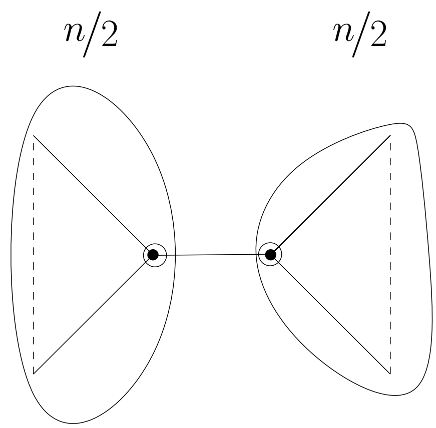
$$a_{n-1}a_1 + a_{n-2}a_2 + a_{n-3}a_3 + \ldots + a_{\lceil n/2 \rceil}a_{\lfloor \frac{n}{2} \rfloor}$$

→ the number of trees with one centroid is

$$a_n - a_{n-1}a_1 - a_{n-2}a_2 - a_{n-3}a_3 - \ldots - a_{\lceil n/2 \rceil}a_{\lfloor \frac{n}{2} \rfloor}.$$

→ move to counting trees with 2 centroids

→ If a tree has 2 centroids it must look like



→ the number of such trees is $\binom{a_{n/2}+1}{2}$ = the number of ways to choose two elements from $[a_{n/2}]$ with repetition

→ Some (easy) generating function manipulation shows

$$F(z) = A(z) - \frac{1}{2}A(z)^2 + \frac{1}{2}A(z^2) = 2 + z^2 + z^3 + 2z^4 + 3z^5 + 6z^6 + 11z^7 + \ldots$$

→ If $t(n)$ is the number of unlabeled trees with n vertices then

$$t(n) \sim c \cdot \alpha^n n^{-5/2} \text{ where } c = 0.53\ldots \text{ and } \alpha \text{ the same } \alpha \text{ for } A(z).$$

Feb 24

Mar 30

Apr 27

21-484 Notes

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Connectivity - Def: (p. 108): A vertex v in a graph G is a cut-vertex if the number of connected components in $G - \{v\}$ is greater than the number of connected components in G.

- notation: (p. 145): The number of connected components in G is denoted by $\kappa(G)$.

- Claim (Theorem 5.1): A vertex incident with a bridge is a cut-vertex if an only if its degree is at least two.

Proof: let $e = uv$ be a bridge. If v is a leaf, then $\kappa(G - \{v\}) = \kappa(G)$. If $\deg(v) \geq 2$, then let $w \neq u$ be another neighbor of v.

$\rightarrow$ In $G - \{v\}$, e is not an edge. Since e was a bridge then there is no w–u path in $G - \{v\}$ (uvw is a u–w path in G using e, so every u–w path uses e). So u and w are not connected in $G - \{v\}$, $\kappa(G - \{v\}) > \kappa(G)$.

Def (p. 111): A non-trivial connected graph with no cut-vertices is called a nonseparable graph.

Remark: $K_2 =$ •——• is nonseparable.

Proposition (Theorem 5.7): A graph with at least 3 vertices is nonseparable if and only if every two vertices lie on a common cycle.

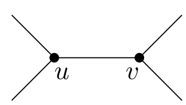
Proof: Let G be a graph with at least 3 vertices.

$\rightarrow$ Assume that every two vertices lie on a common cycle. Assume for the sake of contradiction that v is a cut vertex.
- G is connected (since there is a cycle between every two vertices). $\kappa(G) = 1$.
- $\kappa(G - \{v\}) \geq 2$, so there are u and w in separate connected components in $G - \{v\}$.
- In G, u and w lie on a common cycle, so there are $\geq$ two disjoint u–w paths.
- v can lie in at most one of these paths.

$\Rightarrow$ There is a u–v path in $G - \{v\}$ $\lightning$.

$\rightarrow$ Assume that G is a nonseparable. Assume for the sake of contradiction that not all pairs of vertices lie on a common cycle, and let u,v be two vertices such that $\text{dist}(u, v)$ is minimal.

$\rightarrow$ if $d(u, v) = 1$ then none of them are leaves since then the other vertex is a cut-vertex. Also, can't be that both are leaves.

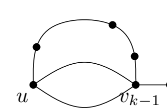


$\Rightarrow$ $\deg(u)$, $\deg(v) \geq 2$. No common cycle containing both u and v means that uv is a bridge.

$\rightarrow$ By the claim, we get that both are cut-vertices $\lightning$.

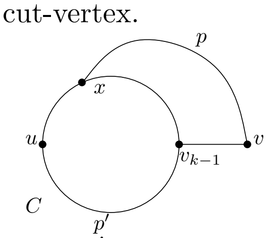
$\rightarrow$ Assume $d(u, v) \geq 2$ and let $v_0 = u, v_1, \dots, v_k = v$ be a u–v path.

$\rightarrow$ u and v_{k-1} lie on a common cycle, $C = \{u_0 = u, u_1, \dots, u_\ell = u\}$ (by minimality of u,v).



$\rightarrow$ There is a u–v path p in $G - \{v_{k-1}\}$, otherwise v_{k-1} is a cut-vertex.

$\rightarrow$ Let $x = u_i$ be the last common vertex between c and p.



$\rightarrow$ call the part of C connecting u and v_{k-1} and not containing x, p'.

We have found a cycle: u, p', go backwards on p until $x, u_{i-1}, \ldots, u_0 = u$ common to u and v. ↯ ■

21-484 Notes

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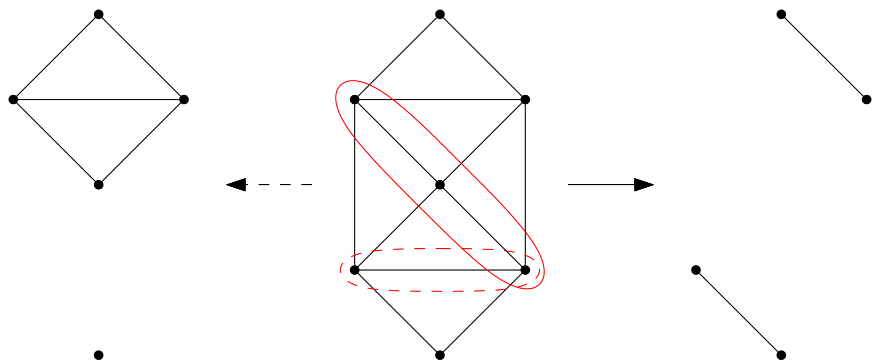
February 22, 2012

(Havel-Hakimi)

<u>Def:</u> (p. 115-116):

- A <u>vertex-cut</u> in a graph G is a set U such that $G - U$ is disconnected.
- A vertex-cut of minimal cardinality is called a <u>minimal vertex cut</u>.
- A vertex-cut U such that no proper subset of U is a vertex-cut is called a <u>minimal vertex-cut</u>.



→ Every minimum vertex-cut is minimal.

→ A graph contains a vertex-cut iff it is not complete.

→ The vertex-connectivity of a graph G, denoted by $\kappa(G)$, is the size of a smallest set U such that $G = U$ is disconnected or trivial.

→ The number of connected components in G will be denoted $k(G)$.

→ A graph is said to be k-connected if $\kappa(G) \geq k$.

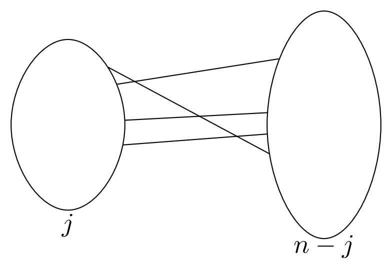
<u>Def (p. 116-117):</u>

- An <u>edge-cut</u> $X \subseteq E(G)$ is a set of edges such that $G - X$ is disconnected.
- An edge-cut with minimal size is called a <u>minimum edge-cut</u>.
- An edge-cut for which no proper subset is an edge-cut is called a <u>minimal edge-cut</u>.
- $\lambda(G)$, the <u>edge-connectivity</u> of G, is the size of a minimal $X \subseteq E(G)$ such that $G - X$ is disconnected or trivial.
- G is <u>k-edge-connected</u> if $\lambda(G) \geq k$.

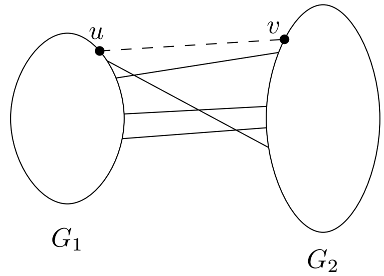
Property (theorem 5.11, Whitney): For all G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$

Proof:

- $\rightarrow$ If G is disconnected or trivial, $\kappa(G) = \lambda(G) = 0$ ✓
- $\rightarrow$ If G is complete graph then $\kappa(K_n) = n - 1 = \lambda(K_n)$
  - $\rightarrow$ removing all $n-1$ edges incident with one vertex disconnects the graph.
  - $\rightarrow$ Let X be an edge-cut and assume that $G - X$ has two components of size j and $n - j$.
  - $\rightarrow$ $|X| = j(n-j)$
  - $\rightarrow$ both components are not empty, so $j \geq 1, n - j \geq 1$
  - $\rightarrow$ $0 \leq (j-1)(n-j-1) = j(n-j) - j - n + j + 1 = j(n-j) - n + 1$
  - $\Rightarrow$ $|X| = j(n-j) \geq n-1$
  - $\Rightarrow$ $\lambda(K_n) = n - 1 = \delta(K_n)$.



- $\rightarrow$ Assume that G is non-trivial, connected, not complete and with at least three vertices.
- $\rightarrow$ $\lambda(G) \leq \delta(G)$ ✓(removing all edges incident with a vertex of minimum degree disconnects the graph).
- $\rightarrow$ Let X be a minimum edge-cut, and let G_1 and G_2 be the two components of $G - X$.



- $\rightarrow$ If all of the edges between G_1 and G_2 are in G, then $|X| = j(n-j)$ where j is the number of vertices in G_1. Then $j \geq 1$ and $n - j \geq 1 \Rightarrow j(n-j) \geq n-1$ contradicting the facts that $\delta(G \ncong K_n) < n - 1$ and $\lambda(G) \leq \delta(G)$.
- $\rightarrow$ Since not all the edges between G_1 and G_2 are in G, then we have $u \in G_1$ and $v \in G_2$ such that $uv \notin E(G)$.
- $\rightarrow$ Define U as follows. For every $e \in X$, pick a vertex incident with e as follows.
  - $\rightarrow$ if $u \in e$, pick $e \cap V(G_2)$
  - $\rightarrow$ otherwise, pick $e \cap V(G_2)$

- $|U| \leq |X|$
- $X \cap E(G - U) = \varnothing$

$\blacksquare$

21-484 Notes

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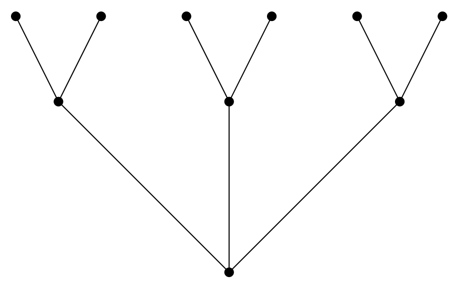
<u>Moore Bound:</u>

$$n(\delta, g) = \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i & g =: 2r + 1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i & g =: 2r \text{ is even} \end{cases}$$

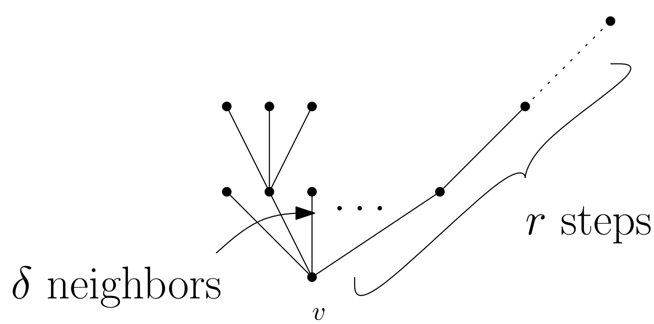
Every graph with minimal degree $\delta \geq 2$ and girth g has at least $n_0(\delta, g)$ vertices.

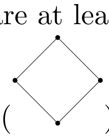
$\rightarrow$ $\delta \geq 2 \Rightarrow g$ is finite

$\rightarrow$ <u>Main idea:</u> The ball of radius $\sim r$ around a vertex/edge is a tree.



$\rightarrow$ <u>Proof:</u> assume $g = 2r + 1$ is odd. Pick a vertex v.



- $\rightarrow$ There are at least δ neighbors of v.
- $\rightarrow$ There are at least $\delta(\delta - 1)$ neighbors of neighbors of v (assuming $g > 3$), otherwise we get a 4 cycle ().
- $\rightarrow$ There are $\delta(\delta - 1)^i$ vertices in the i^{th} level (vertices of distance i from v) if $r \geq i$.

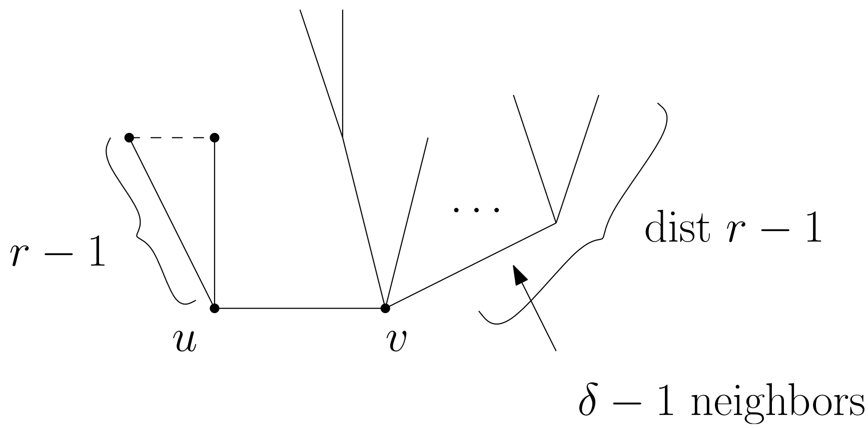
  Otherwise we get a cycle of length $2i \leq 2r < g$.

  <u>Summing up:</u> $\underset{v\uparrow}{1} + \underset{N(v)\uparrow}{\delta} + \underset{N(N(v))\uparrow}{\delta(\delta - 1)} \ldots + \underset{\underbrace{N(N(\cdots(N(v))\cdots))}_{r \text{ times}}\uparrow}{\delta(\delta - 1)^{r-1}}$

$\longrightarrow$ If g is even, we do the same around an edge:

$\rightarrow$ Pick an edge $e = uv$. The tree of depth $r-1$ around each endpoint has $\sum_{i=0}^{r-1}(\delta-1)^i$ vertices, as before (otherwise we get a cycle of length $< r-1+r-1 = 2r-2 < g$)

$\rightarrow$ The two trees are disjoint since otherwise we get a cycle of length $r-1+1+r-1 = 2r-1 < g$.



Q2: diam $G = \max_{u,v} \text{dist}(u,v)$

$\text{ecc}_G(u) = \max_{v} \text{dist}(u,v)$

$\text{radius}(G) = \min_{u} \text{ecc}_G(u)$.

$$\text{rad} \leq \text{diam} \leq 2\text{rad}$$

$\text{rad} = \min_u \max_v \text{dist}(u,v) \leq \max_u \max_v \text{dist}(u,v) = \text{diam}$

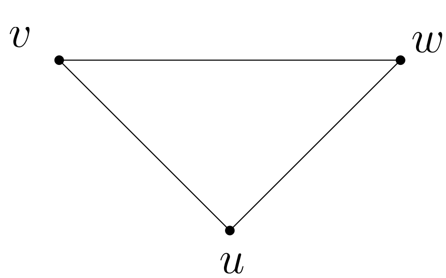
For the left inequality, let u be a vertex such that $\text{ecc}_G(u) = \text{rad}(G)$ (u is called a center of G). Let v and w be two vertices such that

$$\text{dist}(u,w) = \text{diam}(G)$$

Notice that by the triangle inequality we have

$$\text{diam} = \text{diam}(v,w) \leq \text{dist}(v,u) + \text{dist}(u,w) \leq 2\text{rad}(G)$$

To see that the triangle inequality holds, consider the greatest v–u path followed by the geometric u–w path.

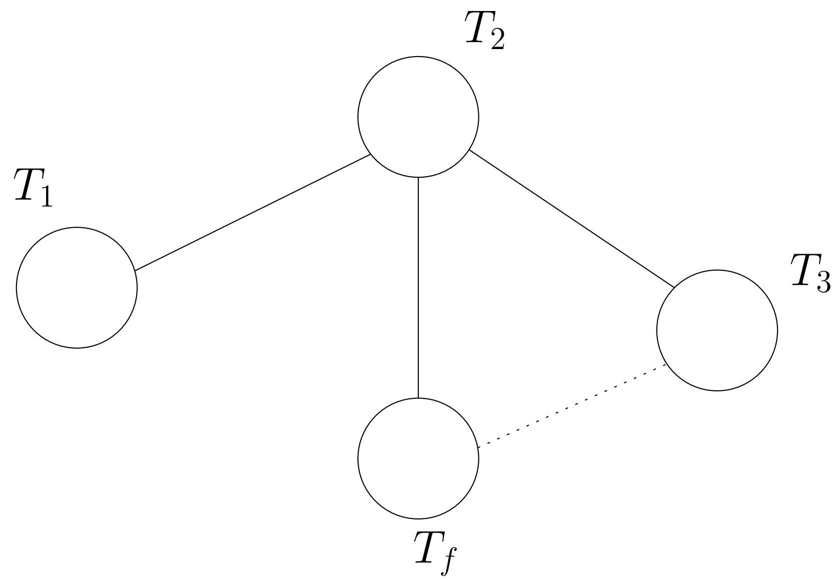


$\text{dist}(v,w) \leq \text{dist}(v,u) + \text{dist}(u,w)$

Take care of disconnected graphs.

$F = T_1 \cup \cdots \cup T_f$, $V(F) = [n]$.

→ Main idea: Think of T_i as vertices.



$f^{f-2} t_1 t_2$

21-484 Notes

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<u>Def:</u> (p. 125): Let G be a graph, u, v are vertices in G.

→ a set $S \subseteq V(G)$ is called a <u>u–v separating set</u> if $G - S$ is disconnected and u and v are in different connected components of $G - S$.

→ <u>also:</u> "S separates u and v"

→ A minimal (by size) u–v separating set is called a minimal u–v separating set.

→ <u>Notice:</u> the size of a u–v separating set is at least $\kappa(G)$.

<u>Def:</u> → Let P be a u–v path in G. A vertex of p that is not u or v is called an <u>internal vertex</u> of P.

→ A set of u–v paths, $P_1, \ldots, P_k$ is called <u>internally disjoint</u> if there is no commmon internal vertex between any two paths of the set.

→ Theorem (Thm 5.16, Menger's Theorem)

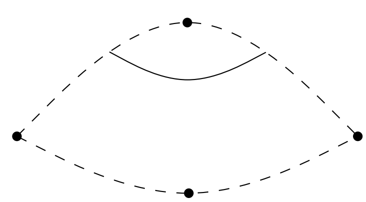
Let G be a graph, and let u and v be two nonadjacent vertices. Then the size of a minimum separating set equals the number of maximal internally disjoint u–v paths.

<u>Proof:</u> Let G be a graph and let u and v be two nonadjacent vertices.

→ Let S be a u–v separating set. Clearly every u–v path must contain a vertex from S.

→ therefore, the number of internally disjoint u–v paths is at most $|S|$.

→ Let k be the size of a minimal u–v separating set.

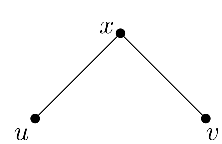


→ By induction on the number of edges in G.

→ If G is an empty graph, everything is zero. ✓

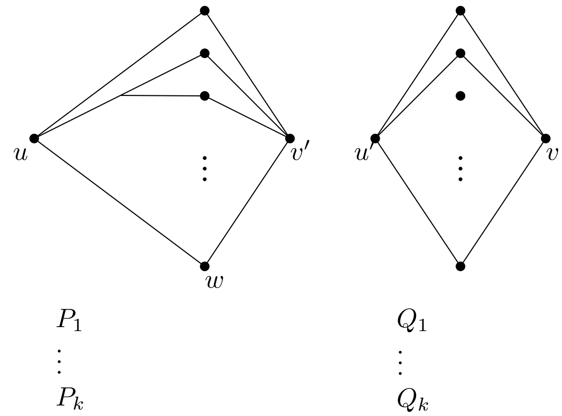
→ Assume the theorem for all graphs with $< m$ edges.

case 1: If there is a separating set S containing a vertex x adjacent to beoth u and v, let $G' = G - \{x\}$.



→ Notice that $S - \{x\}$ is a minimal u–v separating set in G' (Since $G' - (S - \{x\}) = G - S$.)
→ By the induction hypothesis we have $k - 1$ interally disjoint u–v paths in $G - \{x\}$. Adding the path uxv, we get a set of k internally disjoint u–v paths in G.

case 2: Assume there is a separating set W such that one vertex of W is not a neighbor of u and at least one vertex of W is not a neighbor of v.



→ Let V_u be the vertex set containing the component containing u in $G - W$. Let G_u be the graph spanned over $V_u \cup W$, $G_u = G[V_u \cup W]$. (G_u is a connected graph). → Define G'_u by adding another vertex v' and all the edges of the form $v'w_1, v'w_2, \ldots, v'w_k$.

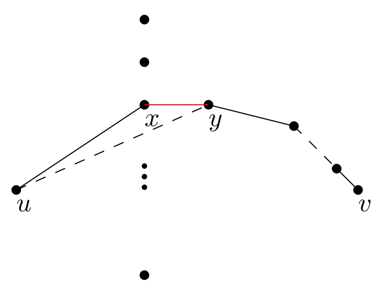
→ G'_u has fewer edges than G because in G u is not adjacent to at least to at least one member of w.

→ By the induction hypothesis, there are k internally disjoint u–v' paths $P_1, \ldots, P_k$, where $w_i \in P_i$.

⟶ Repeat the process with V_v, G_v, G'_v and u' to get k internally disjoint v–u' paths $Q_1, \ldots, Q_k$ when $w_i \in Q_i$.

→ The paths P_i without u' and Q' without v' are k internally disjoint u–v paths.

→ Assume that in every minimal u–v separating set all the vertices are adjacent to u or all of them are adjacent to v.



→ Let $P = u, x, y, \ldots, v$ be a geodesic u–v path.

→ Let $G' = G - \{e = xy\}$.

→ Let Z be a minimal u–v separating set in G'. Assume $|Z| < k$.

→ $Z \cup \{x\}$ is a minimal u–v separating set in G, because $G - (Z \cup \{x\}) = G' - Z$.

→ by our assumption, all the members of Z are adjacent to u.

→ $Z - \{y\}$ is also a minimal separating set in G.

→ y is also adjacent to u, but then there is a u–v path shorter than P. ↯

→ Therefore, $|Z| = k$, and there are k internally disjoint u–v paths in G'. ■

21-484 Notes

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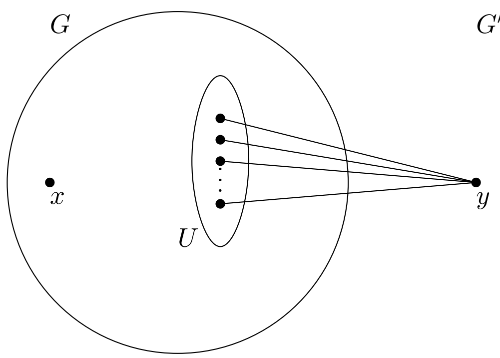
<u>Recall:</u> <u>Menger's Theorem:</u> If G is a graph and $x, y \in V(G)$, $xy \notin E(G)$ then the size of a minimal x–y separating set equals the maximum number of internally disjoint x–y paths.

<u>Theorem (Dircac):</u> Let G be a k-connected graph (with $k \geq 2$). Then for every set $S \subseteq V(G)$, $|S| = k$, there is a cycle $C \in G$ such that $S \subseteq V(C)$.

<u>Def:</u> Let G be a graph, $X \in V(G)$, $U \subseteq V(G) \setminus \{x\}$. An x, U–fan is a set of paths from x to vertices of U such that for every pair of paths the only common vertex is x.

<u>Lemma:</u> (Fan Lemma): A graph is k-connected iff it has at least $k+1$ vertices and for every vertex x and every set $U \subseteq V \setminus \{x\}$, $|U| \geq k$, there is an x, U–of size k.

<u>Proof:</u> Assume that G is k-connected. Let x be a vertex. Let U be a set of at least k other vertices in G.



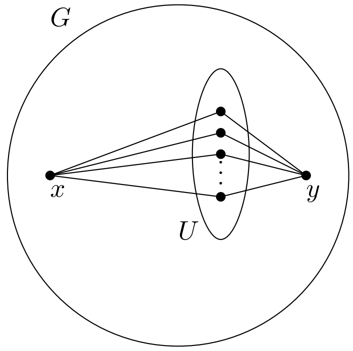
Define G' by adding another vertex y and all the edges of the form uy for $u \in U$. G' is also k-connected since removing at most $k-1$ vertices leaves y connected to at least one vertex from U and also leaves G connected.

$\rightarrow$ A minimal x–y separating set is of size at least k.

$\rightarrow$ By Menger's Theorem there exists a set of k internally disjoint x–y paths in G'.

$\rightarrow$ We get an x, U–fan of size $\geq k$.

Assume that G satisfies the fan condition.



- $\delta(G) \geq k$
- let x and y be two non-adjacent vertices in G.
- let $U = N(y)$
    - $|U| \geq k$
    - $x \notin U$

$\rightarrow$ By the assumption, there is an x, U–fan of size k.

$\rightarrow$ assing the edges between U and y we get a set of $\geq k$ internally disjoint x–y paths.

$\Rightarrow$ (Menger's) the size of any x–y separating set $\geq k$.

$\Rightarrow$ G is k-connected

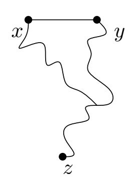
<u>Proof:</u> Induction on k.

$k = 2$. Let x, y be two vertices of a 2-connected graph G.

$\rightarrow$ If $xy \in E(G)$ consider a third vertex z.

$\rightarrow$ By 2-connectivity, $G - \{x\}$ contains a y–z path p.

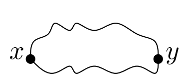
$\rightarrow$ By 2-connectivity, $G - \{y\}$ contains a x–z path p'.



$\rightarrow$ There is an x–y path (in the x–y walk pp') not using the edge xy.

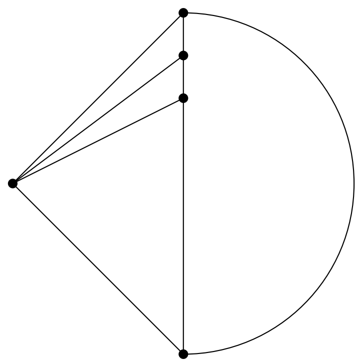
$\rightarrow$ together with xy we get a cycle.

- If $x, y \notin E(G)$, then by 2-connectivity and Menger's theorem, we get two internally disjoint x–y paths. ✓



→ $k > 2$.

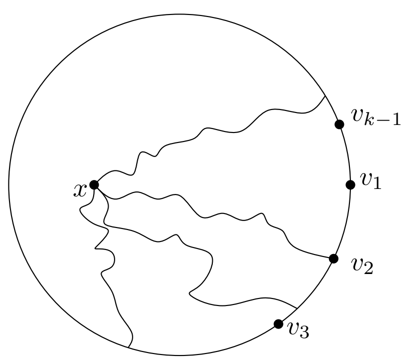
- → → G is k connected, $S \subseteq V(G)$ of size k.
- → let $x \in S$.
- → Since G is also $k-1$ connected, there is a cycle C containing all the vertices in $S \setminus \{x\}$. (Induction hypothesis)
- → If $|C| = k-1$
- → By the Fan lemma, there is an x, C–fan of size $k-1$.



- → So there are internally disjoint paths from x to every vertex of C.
- → taking two consecutive vertices y, x in X we get a new cycle

$$x(\text{path from } x \text{ to } y)\ (\text{path of } C \text{ from } y \text{ to } z)\ (\text{path from } z \text{ to } x).$$

- → Assume that $|C| \geq k$.



- → Let $v_1, v_2, \ldots, v_{k-1}$ be the vertices of $S \setminus \{x\}$ ordered according to appearance on C.
- → Let V_i be the v_i–v_{i+1} path on C. (V_{k-1} is the v_{k-1}–v_1 path on C).
- → By the fan lemma, k-connectivity og G, $|C| \geq k$, we gave j "disjoint" paths from x to C.
- → The paths have k endpoints in C, so there is a set V_i containing two such endpoints y, z. (Pigeon-hole principle)
- → The cycle (the x–y path) (the y–z segment on C out of V_i) (the z–x path) is the required cycle. ■

21-484 Notes

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March 18, 2012

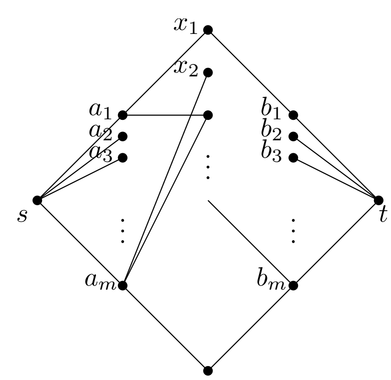
- X a set of people
  - $\boldsymbol{A} = \{A_1, \ldots, A_m\}$ are subsets of X
  - we want to choose m elements $x_1, \ldots, x_m$ such that $x_i \in A_i$. Such a set is called an <u>SDR</u> (system of distinct representatives)
    - $\rightarrow$ Using Hall's theorem: $\exists$ SDR iff $\left|\bigcup_{i \in I} A_i\right| \geq |I|, \forall I \subseteq [m]$
  - $\rightarrow$ $\boldsymbol{B} = \{B_1, \ldots, B_m\}$ are subsets of X
  - $\rightarrow$ A <u>CSDR</u> is a set of m x_i's such that its an SDR for $\boldsymbol{A}$ and $\boldsymbol{B}$

<u>Theorem (Ford-Fulkerson):</u> The families $\boldsymbol{A} = \{A_1, \ldots, A_m\}$ and $\boldsymbol{B} = \{B_1, \ldots, B_m\}$ have a CSDR iff

$$(*) \quad \left|\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right)\right| \geq |I| + |J| - m \quad \forall I, J \subseteq [m]$$

<u>Proof:</u> Define a graph G.

$$V(G) = \left\{s, a_1, \ldots, a_m, x_1, \ldots, x_{|X|}, b_1, \ldots, b_m, t\right\}$$
$$E = \{sa_i | 1 \leq i \leq m\} \cup \{a_i x_k | x_k \in A_i\} \cup \{x_k b_j | x_k \in B_j\} \cup \{b_j t | 1 \leq j \leq m\}$$



$\rightarrow$ An s–t path represents a common element of some A_i and B_j.

$\rightarrow$ every s–t path has the form

$$sa_i x b_j t$$

$\rightarrow$ $\exists$ a CSDR iff there are m internally disjoint s–t paths.
  $\rightarrow$ all the paths in such a set of paths are of length 5

$\rightarrow$ The existence of a set of m internally disjoint s–t paths is equivalent to saying that there is no s–t cut pf size $< m$. (Menger's thm).

$\rightarrow$ need to show that $(*)$ $\iff$ no s–t cut of size $< m$.

$\rightarrow$ Let $R \subseteq V(G) \setminus \{s, t\}$. Define $I = \{i \in [m] | a_i \notin R\}, J = \{j \in [m] | b_j \notin R\}$

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→ If R is a cut then

$$\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right) \subseteq R$$

because a path from s to t must visit some a_i then an x then b_j, this means that if a_i and b_j are in $G \setminus R$ then $x \in R$.

→ for every cut R, $|R| \geq \left|\left(\bigcup_I A_i\right) \cap \left(\bigcup_J B_j\right)\right| + m - |I| + m - |J| \geq m$

→ requiring that the RHS will be $\geq m$, we get $*$.

→ If $*$ is false, $\exists I, J \subseteq [m]$ such that $(\bigcup A_i) \cap (\bigcup B_j) < |I| + |J| - m$.
  → for these I and J $(\bigcup A_i) \cap (\bigcup B_j) + m - |I| + m - |J| < m$
  →Define R to be $(\bigcup A_i) \cap (\bigcup B_j) \cup [m] \setminus I \cup [m] \setminus J$.
  → $|R| < m$.
  → R is an s–t cut

■

Defs p. 134

→ A circuit (a closed trail) in a graph G is called an Eulerian Circuit if it contains every edge of G.

→ A trail is called an Eulerian trail if it visits every edge.

→ A graph is Eulerian if it contains an Eulerian circuit.

→ Thm (Euler 1736, Thm 6.1): A connected graph is Eulerian iff all the degrees are even.

21-484 Notes

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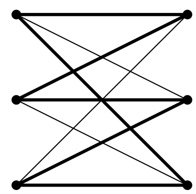
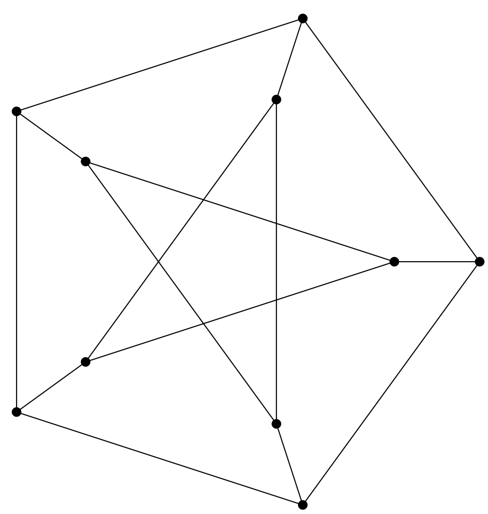
Def: (Page 141): Let G be a graph.

- $\rightarrow$ A cycle C containing every vertex of G is called a Hamiltonian cycle.
- $\rightarrow$ A path C containing every vertex of G is called a Hamiltonian path.
- $\rightarrow$ If G contains a Hamiltonian cycle then G is Hamiltonian.

Examples: 1. $K_{3,3}$ is Hamiltonian



2. The Petersen Graph is not Hamiltonian



The Petersen Graph

Claim: (Thm 6.5) If G is Hamiltonian then for every non empty set $S \subseteq V(G)$

$$k(G - S) \leq |S|$$

Proof:

- Let $G_1, \ldots, G_k$ be the components of $G - S$.
- C is a Hamiltonian cycle in G.
- If you walk along C, then every time that you leave G_i, you encounter a vertex of S.
- $|S| \geq k = k(G - S)$

Theorem (Ore): Let G be a graph with $n \geq 3$ vertices. If

$$\deg u + \deg v \geq n \, (*)$$

for every pair of nonadjacent vertices u and v, then G is Hamiltonian.

Proof:

- Let G be a graph having $(*)$ that is not Hamiltonian.
- Add edges as long as the result is not Hamiltonian. Call the result H.
- $G \subseteq H$
- H is not complete graph.

$\rightarrow$ H has $(*)$

$\rightarrow$ adding any edge to H yields a Hamiltonian

- Let x, y be two non adjacent vertices of H.

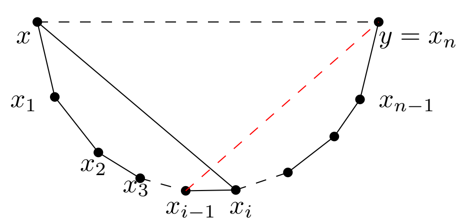
$\rightarrow$ Let $e = xy$

$\rightarrow$ $H + e$ is Hamiltonian

$\rightarrow$ every Hamiltonian cycle in $H + e$ uses e.

$\Rightarrow$ There is an x–y Hamiltonian path in H. Let $x_0 = x, x_1, x_2, \ldots, x_n = y$ be such a Hamiltonian path.

$\rightarrow$ If xx_i is an edge, then $x_{i-1}y$ is not an edge. Otherwise we get a Hamiltonian cycle $x, x_i, x_{i+1}, \ldots, x_n = y, x_{i-1}, x_{i-2}, \ldots, x_0 = x$



$\Rightarrow$ for every neightbor of x in $\{x_1, \ldots, x_n\}$ there is a non neighbor of y in $\{x_0, \ldots, x_{n-1}\}$

$\Rightarrow$ $\deg(y) \leq n - 1 -$ $\deg(x)$

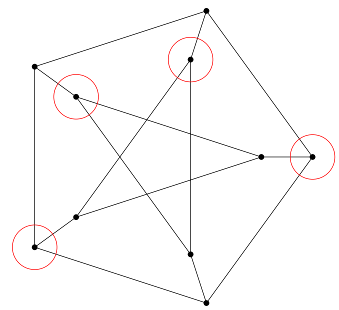
$\Rightarrow$ $\deg(x) +$ $\deg(y) \leq n - 1$ ↯

$\rightarrow$ Corollary (Dirac's Thm): If $\delta(G) \geq n/2$ then G is Hamiltonian.

<u>Def:</u> For a graph G, $\alpha(G)$ denotes the independence number of G which is the size of a maximal independent set (a set of vertices spanning no edges).

Recall that $\kappa(G)$ is the vertex connectivity of G.

Theorem (Chvátal and Erdös): If $\alpha(G) \leq \kappa(G)$ then G is Hamiltonian.



$\to \alpha(PG) = 4 \to \kappa(PG) = 3$

$\to$ Theorem (Chvátal and Erdös): If $\alpha(G) \leq \kappa(G) + 1$ then G is Hamiltonian

21-484 Notes

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$\rightarrow$ Recall: Dirac's Fan Lemma: A graph is k-connected iff it has at least $k+1$ vertices and for every vertex x and every set $U \subset V(G) \setminus x, |U| \geq k$, there is an x, U–fan of size k.

an x, U–fan is a collection of paths from x to vertices of U such that for every two paths the only common vertex is x.

Theorem (Chvátal-Erdős): Let G be a graph with at least 3 vertices such that $\alpha(G) \leq \kappa(G)$. Then G is Hamiltonian.

Proof: $\rightarrow$ Let $k = \kappa(G)$ and let C be a longest cycle in G.

$\rightarrow$ Denote the vertices of C cyclically by $V(C) = \{v_0, \ldots, v_{\ell-1}\}$ (think of the indices as the elements of $\mathbb{Z}_\ell$)

$\rightarrow$ AFSOC that C is not a Hamiltonian cycle.

$\rightarrow$ Let V be a vertex of G out of C.

$\rightarrow$ Let $\mathcal{F}$ be a $v, V(C)$ fan of maximumal size. Denote $\mathcal{F} = \{P_i | i \in I\}$ where P_i is a v–v_i path.

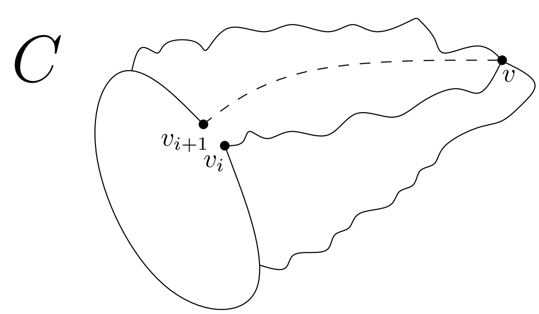
$\rightarrow$ observe:

$\rightarrow$ By the Fan Lemma

$*$ $|\mathcal{F}| = |I| \geq \min(|C|, k)$ using the fact that a k-connected graph is also $k-1$ connected

$\rightarrow$ for every $i \in I$, $v_{i+1}v \notin E(G)$. Otherwise

$$(C \cup P_i \cup P_{i+1}) - v_i v_{i+1} \text{ is a cycle longer than } C$$

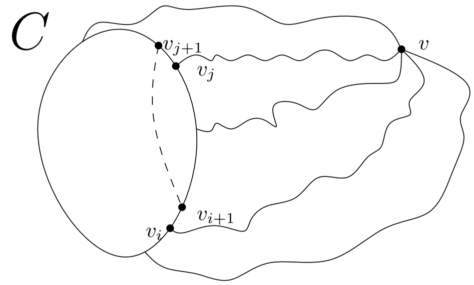


$\rightarrow$ for every $j \notin I$, $vv_j \notin E(G)$.

$\Rightarrow$ if $i \in I$ then $i+1 \notin I$.

$\Rightarrow |T| < |I|$

$\Rightarrow |I| \geq k$ (from $*$)



→ If $i, j \in I$ then $v_{i+1}v_{j+1} \notin E$. Otherwise the cycle

$$\underbrace{v_{j+1}, \dots, v_i}_{C}, \overleftarrow{P}_{i+1}, P_j, \underbrace{v_{j-1}, \dots, v_{i+1}}_{\overleftarrow{C}}, v_{j+1}$$

has length $|C| - 2 + |P_i| + |P_j| + 1 > C$

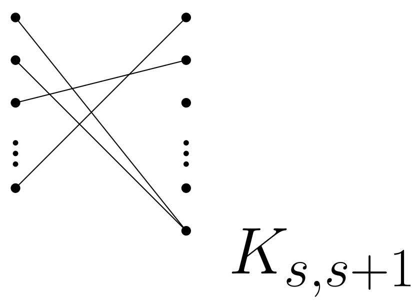
→ the set $S = \{v_{i+1} | i \in I\} \cup \{v\}$ is an independent set.

→ $|S| = |I| + 1 > k$

→ $\alpha(G) \geq |S| > k = \kappa(G)$ ↯

→ The Petersen Graph shows that this is tight (having $\alpha(PG) = 4$ and $\kappa(PG) = 3$ and being non-Hamiltonian.)

→ Consider $K_{s,s+1}$



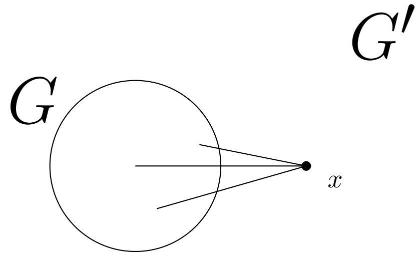
$\kappa(K_{s,s+1}) = s$

$\alpha(K_{s,s+1}) = s + 1$

not Hamiltonian, so the Theorem is tight.

Corollary: If a graph G has $\alpha(G) \leq \kappa(G) + 1$ then G contains a Hamiltonian path.

–Proof:



$\alpha(G') = \alpha(G)$

$\kappa(G') = \kappa(G) + 1$

$\rightarrow$ By the Chvátal-Erdős theorem, G' contains a Hamiltonian cycle. Thus G contains a Hamiltonian path.

21-484 Notes

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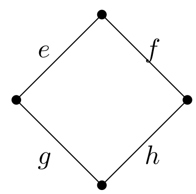
wmacrae@andrew.cmu.edu Def: (p. 184): A set of edges in a grpah G is independent or is a matching if every two edges are disjoint.

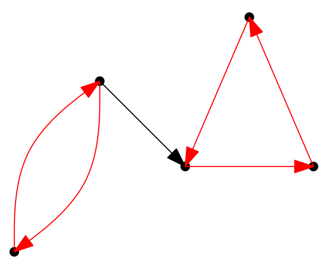
Example:

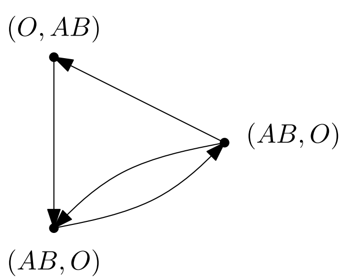


$$\varnothing, \{e\}, \{f\}, \{g\}, \{h\}, \quad \underbrace{\{e, g\}, \{f, g\}}_{\text{Perfect matchings (p.194)}}$$

Application:

(Patient, Doner)





<u>Def:</u> (p. 185): Let G be a bipartite graph, $G = (V = U \cup W, E)$. For a set $X \subseteq U$ we define the neighborhood of X, $N(X)$, to be

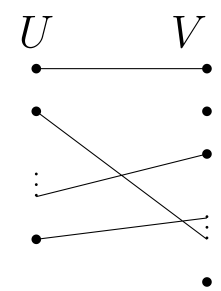
$$N(X) = \{w \in W | \exists u \in X. uw \in E\}$$

Theorem (Hall, Theorem 8.3): $G = (V = U \cup W, E)$ be a bipartite graph. Then there is a matching of size $|U|$ if and only if for every $X \subset U$,

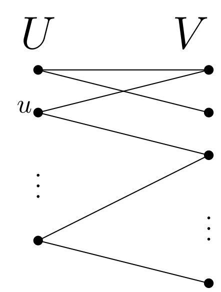
$$|N(X)| \geq |X| \; (*).$$

<u>Proof:</u> If G has a matching M of size $|U|$, then every vertex of U lies in a unique edge. For every $X \subseteq U$

$$|X| = |\{w \in W | \exists u \in X. uw \in M\}| \leq |N(X)|$$



Assume the condition $(*)$, assume for the sake of contradiction that there is no matching of size $|U|$. Pick a maximum matching and let $u \in U$ be an unmatched vertex.



An alternating path is a path in which the edges alternate between matching edges and nonmatching edges. Let S be the set of all vertices s such that there is a u–s alternating path of maximal length.

$\rightarrow S \cap W = \varnothing$. Otherwise, there is a maximal alternating path of odd length. Such a path starts and ends with a nonmatching edge. Define $M' = M \setminus$ the matching edges in the path $\cup$ the nonmatching edges in the path $|M'| > |M|$. ↯

21-484 Notes

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→ Thm: If $G = (U \cup W, E)$ is a bipartite graph, then G has a matching of size $|U|$ iff $\forall X \subset U. |N(X)| \geq |X|$. $(*)$

Proof: Saw that having a matching pf size $|U|$ implies $(*)$.

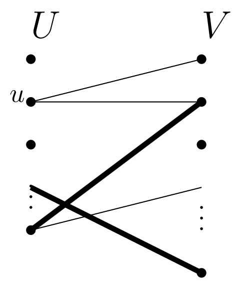
Assume that G has $(*)$ and that M is a maximal matching, $|M| < |U|$.

Then, $\exists u \in U$ that is not matched.

Define an alternating path. Consider the set S of all vertices v such that there is an alternating u–v path.

→ $u \in S$

→ If $w \in W \cap S$, then it is not an endpoint of a maximal alternating path. Otherwise, we could swap and non-matching edges in the path and get a larger matching.

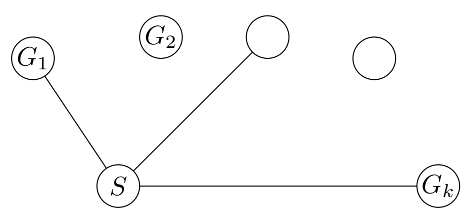


→ Let $U' = U \cap S, W' = W \cap S$.

→ There is a matching edge going from every vertex of W' to a vertex of U'.

$\Rightarrow |W'| \leq |U' \setminus \{u\}| \Rightarrow |W'| < |U'|$ ↯ $(*)$

Tutte's Theorem



A graph $G = (V, E)$ is a perfect matching iff for every set $S \subseteq V$ the number of connected components of odd size in $G[V \setminus S]$ is at most the size of S.

Proof: Assume that G has a perfect matching, and let S be a set of vertices. Then, since the perfect matching M matchs an even number of vertices in every connected component of $G[U \setminus S]$, every odd component contains at least one vertex that is not matched with another vertex from this component. Such a vertex must be matched wtih a vertex set S.

→ Let $k_o(G - S)$ be the number of odd connected components in $G[u \setminus S]$.

→ Assume that G obeys

$$k_o(G - S) \leq |S| \text{ for every } S \subseteq V. \ (*)$$

→ (∗), G has an even number of vertices.

→ By induction, $|V| = 2$ •——• ✓

→ Let $n \geq 4$.

→ Assume that (∗) implies the existence of a perfect matching in every graph with fewer than n vertices.

→ Let S be a maximal set of vertices with the property

$$k_o(G - S) = |S|$$

→ S is not empty. Every connected graph has a vertex that is not a cut vertex. A leaf of a spanning tree, ...

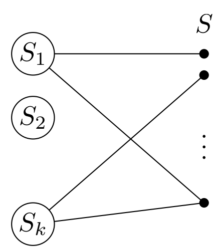
→ let u be a noncut vertex. $k_o(G - \{u\}) = 1 = |\,\{u\}\,|$

→ let $G_1, \ldots, G_k$ be the connected component in $G(V \setminus S)$.

→ All the G_i's are odd, otherwise we can add a non cut vertex from an even G_i to S.

→ Let S_i be the set of vertices in S having a neighbor in G_i.

→ S_i is not empty. (G_i was even in G, and now all the components are odd).



21-484 Notes

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Tutte's Thm:

$k_o(G-S)$, G contains a perfect matching iff $k_o(G-S) \leq |S|\ \forall S \subseteq V(G)$. $(*)$

Proof:

- Induction on number of vertices in G.
- Assume that G has $(*)$.

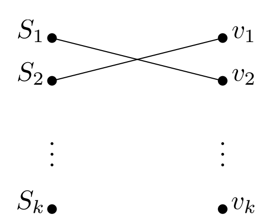
$\rightarrow$ Let S be a maximal set of verties having $k_o(G-S) = |S|$.

$\rightarrow$ S is not empty.

$\rightarrow$ Let $G_1, \ldots, G_k$ be the components of $G-S$, then $|G_i|$ is odd $\forall 1 \leq i \leq k$.

$\rightarrow$ Let S_i be the set of neighbors of G_i in S.

$\rightarrow$ S_i is not empty. (G_i is odd and all the connected components of G are even).



$\rightarrow$ $(**)$ For every $1 \leq t \leq k$ and every t of the S_i's, the union of these S_i's is of size at least t.

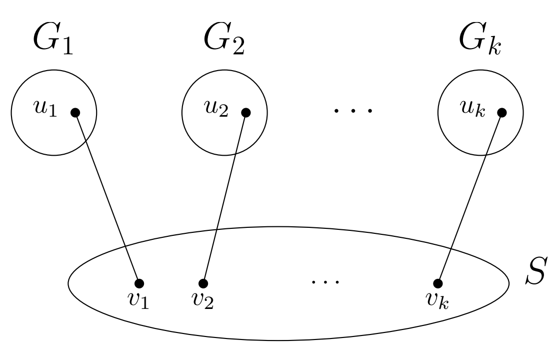
$\rightarrow$ Otherwise, let T be the union of the S_i's.

$$k_o(G-T) \geq t > |T| \ \lightning\ (*)$$

$\rightarrow$ Consider the bipartite graph with sides $\{S_1, \ldots, S_k\}$ and $S = \{v_1, \ldots, v_k\}$. There is an edge in H between S_j and v_i iff $\forall v_i \in S_j$

$\rightarrow$ by $(**)$+Hall's Theorem, there is a perfect matching in H.

$\rightarrow$ Let u_i be the neighbor of v_i in G_i (assuming without loss of generality that the perfect matching matched v_i to S_i).



→ Need to show that $\forall 1 \leq i \leq k$ and $\forall W \subseteq V(G_i - u_i)$

$$k_o(G_i - u_i - W) \leq |W|$$

→ Assume otherwise: $|W| < k_o(G_i - u_i - W)$ for some i and W.

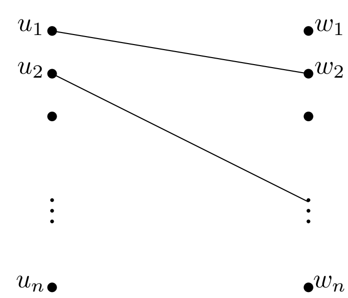
→ Since $G_i - u_i$ is eve, the parity of $|W|$ and $k_o(G_i - u_i - W)$ is the same.

→ Consider $S' = S \cup W \cup \{u_i\}$

$|S'| \geq k_o(G - S') = k_o(G-S) + k_o(G_i - u_i - W) - 1 \geq |S| + |W| + 2 - 1 = |S'|$ ↯maximality of S.

by $(*)$

H



→ By the induction hypothesis, there is a perfect matching M_i in G_i.

$$M = \left(\bigcup_{i=1}^{k} M_i \right) \cup \{v_i u_i\} \text{ is a perfect matching in } G$$

<u>Theorem(Tutte-Berge formula)</u>: For every graph G, the size of a maximum matching is

$$\min_{S \subseteq V(G)} \frac{(|S| - k_o(G - S) + |V|)}{2}$$

<u>Def:</u> Let k be a positive integer. A k-factor in a graph G is a spanning subgraph which is k-regular.
<u>Example:</u> A perfect matching is a 1-factor.
<u>Theorem (Petersen):</u> A graph G can be decomposed into 2-factors $F_1, \ldots, F_k$ if and only if G is $2k$-regular. $\left(G = \dot{\bigcup}_{i=1}^{k} F_i \right)$

<u>Proof's idea:</u> one direction is easy (decomposition $\implies$ $2k$-regular).
→ Assume that G is $2k$-regular. By Euler's Theorem, there is an Eulerian circuit C.
→ (Def. H)
→ H is k-regular
→ By Hall's Theorem and counting every regular bipartite graph contains a perfect matching.
→ Every perfect matching in H corresponds to a 2-factor of G.
→ repeat. ■

21-484 Notes

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2 Cvátal Erdős $\alpha(G) \leq \kappa(G)$

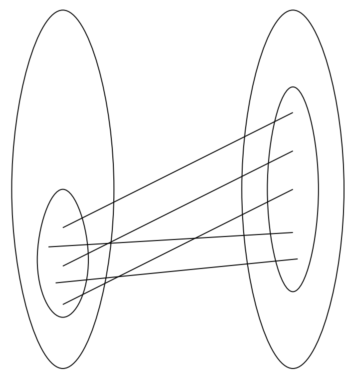
4 Take a maximum cycle

3 Find one

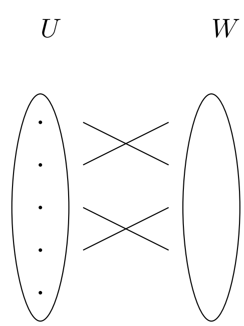
1 Ore: $\forall u, v$ non-adjacent $\deg(u) + \deg(v) \geq n$

$r = |U| \leq |W|$. G has a matching of cardinality r iff

$$\forall S \subseteq, |N(S)| \geq |S|$$



G is d-regular



$$\begin{aligned}
\sum_{u \in U} \deg(u) &= \sum_{v \in W} \deg(v) \\
\sum_{u \in U} d &= \sum_{v \in W} d \\
|U|d &= |W|d \\
|U| &= |W|
\end{aligned}$$

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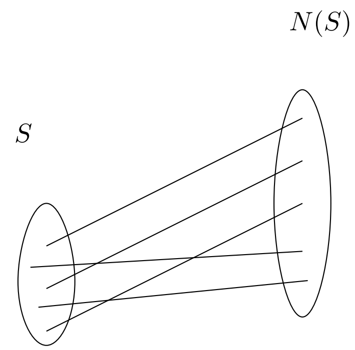
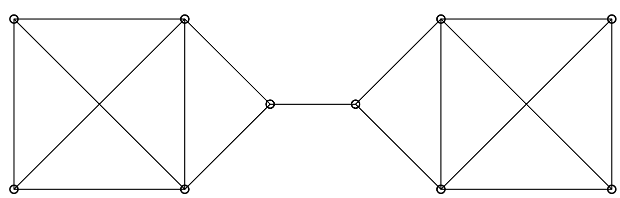
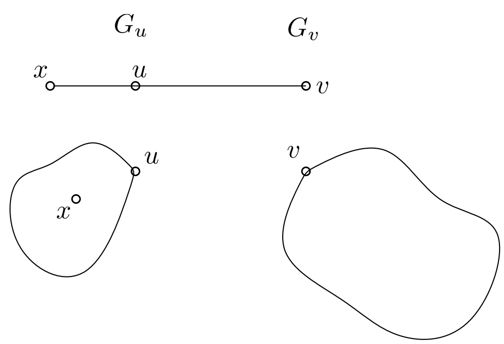
Let $S \subseteq U$

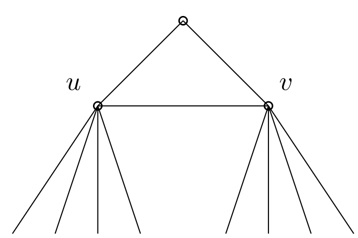


$$
\begin{array}{lcl}
\sum_{u \in S} \deg(u) & = & \sum_{v \in N(S)} \deg(v) \\
\sum_{u \in S} d & \leq & \sum_{v \in N(S)} d \\
|S| \cdot d & \leq & |N(S)| \cdot d \\
|N(S)| & \geq & |S|
\end{array}
$$

$$\delta(G) \geq \kappa(G)$$



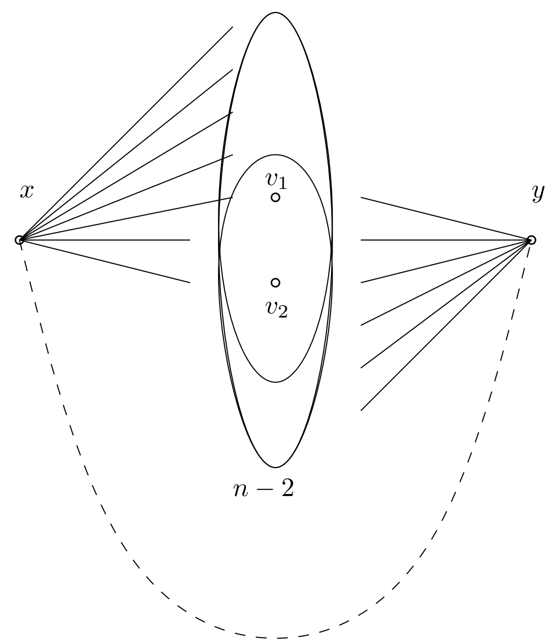




G $n \geq 3$

deg $(V) \geq {}^{n}\!/_{2}$

nonseparable



2 internally disjoint paths

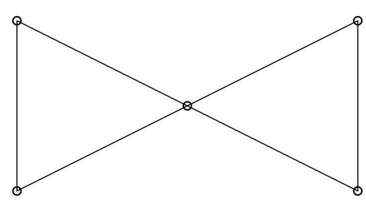
Mengers $\Rightarrow$ nonseparable.

G $n \geq 3$

$(n-1)$–Connected

G is K_n

Whitney's Theorem: $\kappa(G) \leq \lambda(G) \leq \delta(G)$



$1 + \kappa(G) = \lambda(G)$

Menger's Theorem

Fan Lemma

21-484 Notes

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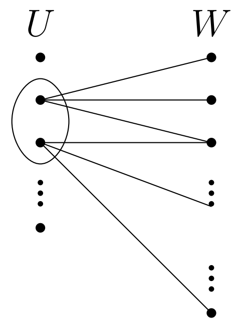
April 6, 2012

→ Show that a nonempty regular bipartite graph contains a perfect matching.

→ Matchings + bipartite ⇒ Hall's Thm.



→ Let $G = (U, W, E)$ be a $d > 0$ regular graph. Need to show

$$X \subseteq U \Rightarrow |X| \leq |N(X)|$$

→ Indeed, the sum of degrees of vertices in X is $d \cdot |X|$

→ If $|N(X)| < |X|$, then the sum of degrees of vertices in $N(X)$ is $d \cdot |N(X) < d \cdot |X|$ ↯ since all edges leaving X are going in $N(X)$.

→ The same for W

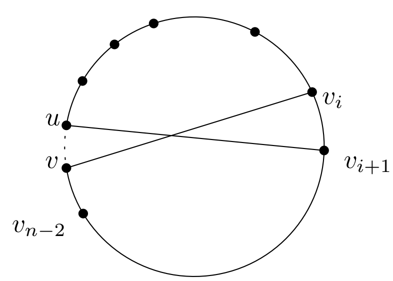
→ u and v are two nonadjacent vertices in G, such that $d(u) + d(v) \geq \underset{\substack{\uparrow \\ \text{number of vertices}}}{n}$

$$G + uv \text{ is Hamiltonian} \iff G \text{ is Hamiltonian}$$

→ if G contains a Hamiltonian cycle C, then $G + uv$ also contains C.

→ Assume that $G + uv$ contains a Hamiltonian cycle C.

→ If C does not use uv, we are done.



→ Let $C = (v_1, \ldots, v_{n-2}, v, u, v_1)$.

→ we want to find two vertices v_i, v_{i+1} such that

$$vv_i \in E(G) \text{ and } uv_{i+1} \in E(G)$$

→ Let I be the set of indices of neighbors of v.

→ Let $J = \{i+1 | i \in I, i < n-2\}$

$|J| = |I| - 1 = d(v) - 1$

→ u must have a neighbor with index in J, since there are $n-2-|J| = n-2-d(v)+1 = n-1-d(v)$ vertices out of J (and u and v) but the degree of u is $d(u) \geq n - d(v)$

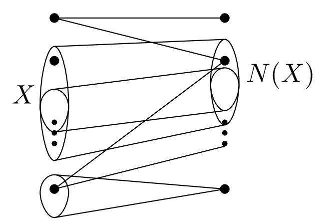
→ So there are v_i and v_{i+1} as required.

→ the cycle $u, v_1, \ldots, v_i, v, v_{n-2}, v_{n-3}, \ldots, v_{i+1}, u$ is Hamiltonian.

G is bipartite $G = (U \cup W, E)$. The size of a maximal matching is

$$|U| - \max_{X \subseteq U}(|X| - |N(X)|) \ (*)$$

→ There is no matching of size bigger than, because $(*)$, because if X is the maximal set then we can match all vertices of $U \setminus X$ plus $N(X)$ vertices from X: $|U \setminus X| + |N(X)| = |U| - |X| + |N(X)|$



→ Let G be such a graph and let x be a maximal set.

→ We want to match all the vertices in $U \setminus X$ with vertices from $W \setminus N(X)$.

→ Need to show: $\forall Y \subseteq U \setminus X$, $|Y| \leq |N(Y) \setminus N(X)|$

indeed, if $|Y| > |N(Y) \setminus N(X)|$, consider $X \cup Y$

$$|X \cup Y| - |N(X \cup Y)| = |X| + |Y| - |N(X)| - |N(Y) \setminus N(X)| = |X| - |N(X)| + |Y| - \underbrace{|N(Y) \setminus N(X)|}_{>0}$$

↯ maximality of X.

→ Have a matching that matches all vertices of $U \setminus X$ outside of $N(X)$.

→ We want to show: $Z \subseteq N(X) \Rightarrow |Z| \leq |N(Z)|$. Assume $|Z| > |N(Z)|$, consider $X \setminus N(Z)$.

$$|X \setminus N(Z)| - |N(X \setminus N(Z))| = |X| - |N(Z)| - N(X \setminus X \setminus N(Z))|$$

<u>claim:</u> $|N(X \setminus N(Z))| \leq |N(X)| - |Z|$ a vertex of Z can not be in $N(X \setminus N(Z))$.

$$|X| - |N(Z)| - |N(X)| + |Z|. \text{ ↯ maximality of } X.$$

Q_k $\kappa(Q_k) = \lambda(Q_k) = k$.

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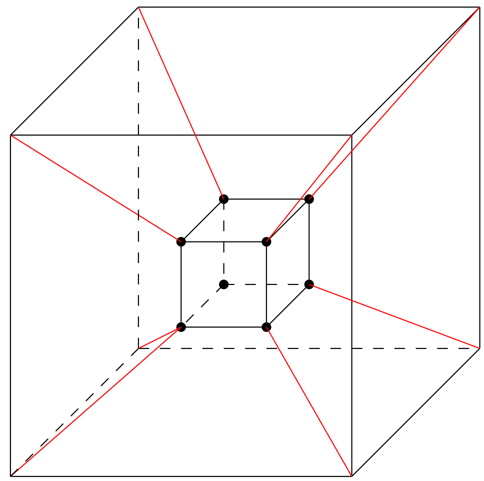

21-484 Notes

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Definitions: (p. 267-269)

- A proper coloring of the vertices of a graph G is a mapping $f : V(G) \to C$ such that adjacent vertices get different colors. (Also coloring of G).

$\to$ The smallest number of colors for which there is a proper coloring of G is the chromatic number of G, denoted $\chi(G)$.

$\to$ k-colorable = k-chromatic, minimum coloring

$\to$ Given a coloring of $G = (V, E)$, the set of all vertices with the same color is called color class. If V_i is a color class then $G[V_i]$ is an independent set.

$\to$ The set of all color classes is a partition of V (into independent sets).

$\to$ The independence number of G is the size of a maximum independent set. Denoted $\alpha(G)$.

$\to$ The clique number of G is the size of a maximum clique (= complete subgraph). Denoted $\omega(G)$.

Fact (Thm 10.5): Let G be a graph with n vertices. Then

$$1\ \chi(G) \geq \omega(G) \text{ and } 2\ \chi(G) \geq \frac{n}{\alpha(G)}$$

proof: 1 Let H be a maximum clique. Then every coloring requires at least $|V(H)|$ colors just to color H. $\chi(H) = |V(H)|$. Since $H \subseteq G$, $\chi(G) \geq \chi(H)$.

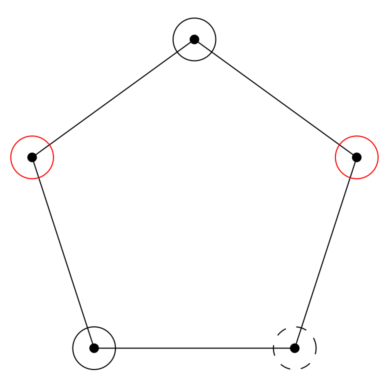
2 For a given coloring of G, let $V_1, \ldots, V_k$ be the color classes. V_i is an independent set, so $|V_i| \leq \alpha(G)$.

$$n = \sum_{i=1}^{k} |V_i| \leq k \cdot \alpha(G) \Rightarrow k \geq \frac{n}{\alpha(G)}$$

claim (Thm 10.7): For every G, $\chi(G) \leq \Delta(G) + 1$.

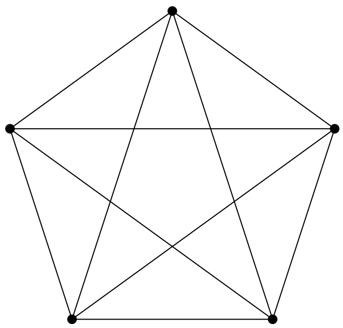
Proof: Color the vertices one by one. When coloring a vertex, there are at most $\delta(G)$ colors that we cannot use, so we have an available color.

Examples:

$C_5 =$ 

$\Delta(C_5) = 2$

$\chi(C_5) = 3$

$K_5 =$ 

$\Delta(K_5) = 2$

$\chi(K_5) = 3$

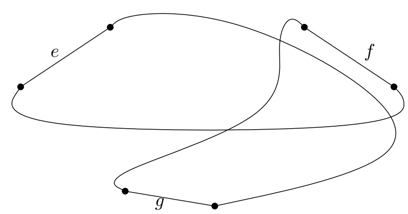
Thm (Brooks Thm, Thm 10.8): For every connected graph G other than an odd cycle or a complete graph $\chi(G) \leq \Delta(G)$.

Proof:

→ We can assume that G is connected.

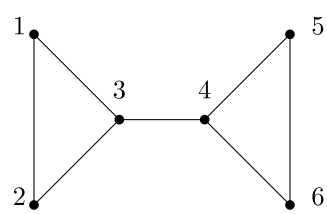
→ We can assume that G is 2-connected.

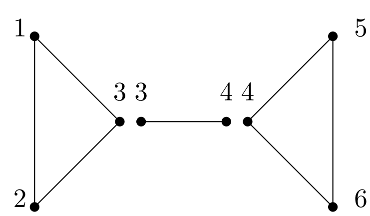
- → We can decompose G into Blocks (Section 5.2), color each block separately and merge the colorings.
- → For a pair of edges e and f, let eBf iff $e = f$ or e and f lie in a common cycle.
- → B is an equivalence relation.



check

- → The equivalence classes are the blocks.
- → A block in a graph G is a maximal by inclusion nonseparable subgraph.





$\rightarrow$ We can assume that $\Delta \geq 3$. Otherwise the graph is an even cycle, which is 2-colorable.

$\rightarrow$ If G has a vertex v of degree less than Δ. Consider a breadth-first search tree starting from v (v is the root). Color the vertices according to distance, farthest first. At every step, the parent of the current vertex is not colored, hence there are at most $\Delta - 1$ colors that we cannot use (and we have Δ colors). In the final step we color v which has degree $< \Delta$.

$\rightarrow$ Assumptions: G is 2-connected, Δ-regular for $\Delta \geq 3$, not complete.

$\rightarrow$ Find a spanning tree having a root v with two neighbors of v: U, w such that u and w are leaves and $uw \notin E(G)$.

21-484 Notes

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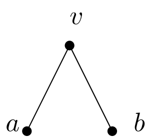
→ <u>Brooks Theorem:</u> G is connected, not complete, not odd cycle $\chi(G) \leq \Delta(G)$.

→ Assume that G is 2-connected.

→ Assume that $\Delta(G) \geq 3$

→ Assume that G is Δ-regular

→ Find 

and $G - a - b$ is connected.

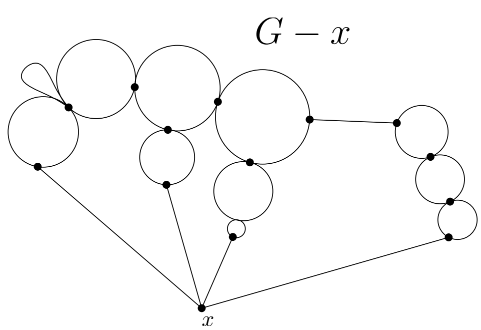
→ Once we have 3 vertices a, b, v such that $av, bv \in E(G)$, $ab \notin E(G)$ and $G - a - b$ is connected, color a and b by color 1. Find a spanning tree for $G - a - b$; root the tree at v. Color vertices according to their place in the tree from leaves towards the root. This can be done becasue every vertex has at most $\Delta - 1$ colored enighbors (its parent is not colored).

→ Whe we try to color v, it has at most $\Delta - 2$ colored neighbors besides a, b. But a and b are both colored 1.

→ Consider a vertex x that is not adjacent to all ofther vertices.

→ If $G - x$ is still 2-connected, find a vertex of distance 2 from x (call it y). Let v be a common neighbor of x and y. Letting x and y have the roles of a ad b works. Indeed, $G - x - y$ is connected because $G - x$ is 2-connected.

→ Assume that $G - x$ is not 2-connected. Consider the block decomposition of $G - x$.



→ We have a tree of blocks, there are at least two end blocks (because every tree has at least two leaves).

→ An end block B_i has a vertex j_i such that every other block B_k is either disjoint from B_i, or they have j_i as their only common vertex.

→ Let B_1 and B_2 be two end blocks. There are two vertices $b_1 \in B_2$, $b_2 \in B_2$ such that $b_1 \neq j_1$, $b_2 \neq j_2$, and $xb_1 \in E(G)$ and $xb_2 \in E(G)$.

→ otherwise, if such b_1 does not exist, then j_1 is a cut vertex of G ↯ G is 2-connected.

→ x, b_1, b_2 can have the roles of v, a, b.

→ b_1 and b_2 are neighbors of x, by the above. They are not adjacent because they are in different blocks in G.

→ $G - b_1 - b_2 - x$ is connected, because neither b_1 nor b_2 was a joint and $d_G(x) \geq 3$. ■

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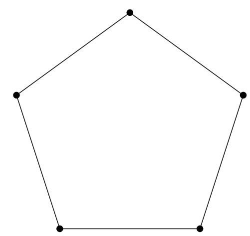

$\chi(C_5) = 3$
$\omega(C_5) = 2$

Theorem 10.10: $\forall$ integer k there is a triangle free graph with chromatic number k.

$\rightarrow$ For every forest F, $\chi(F) \leq 2$.

Theorem (Erdős): For all integers k, ℓ, $\exists G$ such that girth$(G) > \ell$ and $\chi(G) > k$.

Proof: $\rightarrow$ set $0 < \theta < \frac{1}{\ell}$ constant

$\rightarrow$ define: $p = n^{-1+\theta}$

$\rightarrow$ Consider a graph on n vertices such that every possible edge is in G with probability P, independently of all other edges.

Let X be the number of short ($\leq \ell$) cycles in G. X is a random variable.

$$\mathbb{E}[x] = \sum_{i=3}^{\ell} (\# \text{ of } i\text{-cycles in } K_n) \cdot p^i = \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{\underset{\substack{\uparrow \\ \text{starting point} \\ +\text{ direction}}}{2 \cdot i}} \leq \sum_{i=3}^{\ell} n^i p^i \leq 2 \cdot (np)^\ell = 2n^{\theta\ell} \underset{\substack{n \text{ is large} \\ \text{enough}}}{<} \frac{n}{\log n}$$

$\Pr[X \geq n/2] \leq \frac{\mathbb{E}[X]}{n/2} \leq \frac{2n}{\log n \cdot n} = \frac{2}{\log n} \overset{n \to \infty}{\to} 0$

Markov's inequality: if X is a non-negative random variable with expectation then for any positive real a

$$\Pr[X > a] \leq \frac{\mathbb{E}[X]}{a}$$

21-484 Notes

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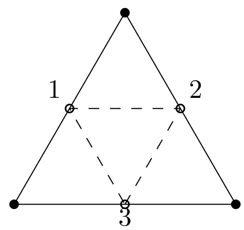
Edge coloring

$$f_k : E \to \{1, \ldots, k\}$$

$\forall u, v_1, v_2\ f(uv_1) \neq f(uv_2)$

k-edge colorable $\exists f_k$

k-edge chromatic, $\chi_1(G)$ k-edge colorable and not $k-1$ edge colorable



3-edge colorable
3-edge chromatic
4-edge colorable
not 4-edge chromatic

<u>Vizing's Theorem:</u> (10.12) All G $\chi_1(X) = \Delta(G)$ or $\chi_1(G) = \Delta(G) + 1$

<u>Pr:</u> $\chi_1(G) \geq \Delta(G)$

Take v of max degree. v has $\Delta(G)$ edges; each needs a color.

$\chi_1(G) \leq \Delta(G) + 1$

Induction on m (number of edges)

Take xy to be arbitrary.

$IH : \chi_1(G - xy) \leq \Delta(G - xy) + 1 \leq \Delta(G) + 1$

fix φ, color xy somehow

$\varphi(uv)$ is the color of uv

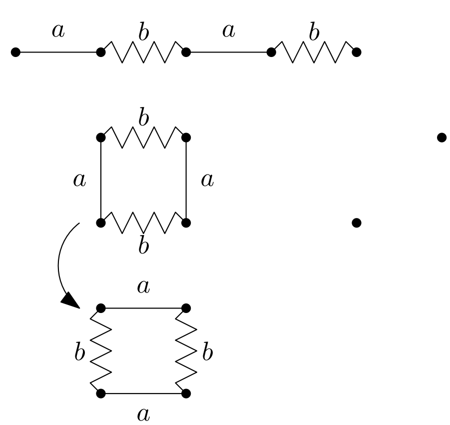
$\varphi(u)$ is the set of colors incdient with u

$\overline{\varphi}(u)$ is the set of colors missing at u

$\forall u, \overline{\varphi}(u) \neq \varnothing$

Kempe Chain $H(a, b)$

Subgraph ~~induced~~ by taking edges of colors a and b (only)



deg ≤ 2
≤ 1 color a
≤ 1 color b
(edge)

$y_0, y_1, \ldots$ vertices

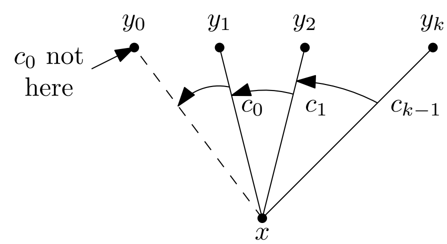
$c_0, c_1, \ldots$ colors

Set $y_0 = y$

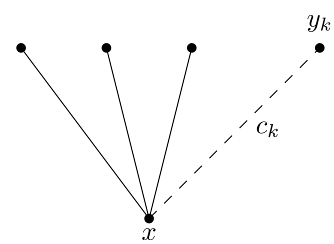
$c_i :=$ a color missing at y_i

$c_i \in \overline{\varphi}(y_i)$

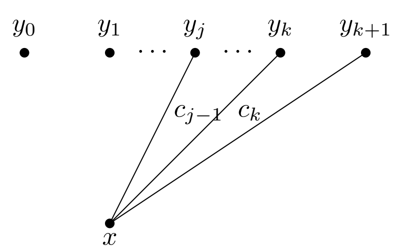
$y_{i+1} :=$ vertex such that $\varphi(xy_{i+1}) = c_i$



1 $c_k \in \overline{\varphi}(x)$ color xy_i with c_i $\forall 0 \leq i \leq k$.



2 y's and c's infinite



$c_k = \varphi(xy_j)$

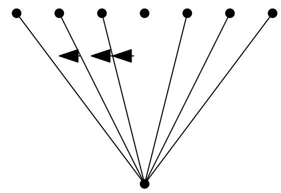
$\overline{\varphi}(x) \neq \varnothing$ Let $a \in \overline{\varphi}(x)$

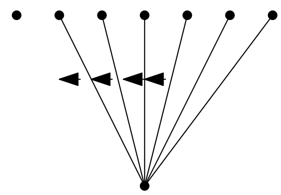
2a $a \in \overline{\varphi}(y_j)$ $\forall 0 \leq i < j$ color xy_i with c_i. Color xy_j with a.

2b $a \in \overline{\varphi}(y_k)$ $\forall 0 \leq i < k$ color xy_i with c_i. Color xy_k with a.

$c_k \in \varphi(x)$ $\quad a \in \varphi(y_j)$
$c_k \in \overline{\varphi}(y_k)$ $\quad a \in \varphi(y_k)$
$c_k \in \overline{\varphi}(y_j)$ $\quad a \in \overline{\varphi}(x)$

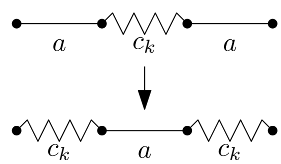
color xy_i with c_i $\forall 0 \leq i < j$ uncolor xy_j





$H(C_k, a)$ each of x, y_j, y_k has degre 1. One of them is in its own compoent.

Without loss of generality



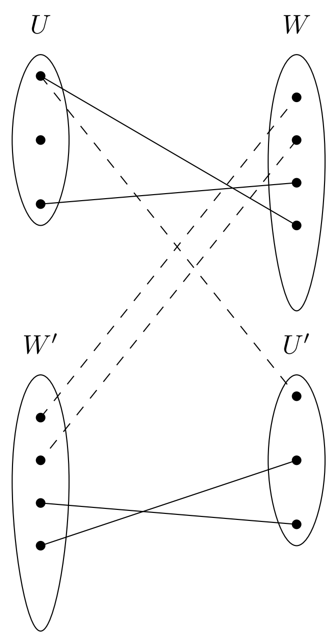
By (1), (2a), or (2b) ■

<u>König's Theorem</u>: (10.17) G Bipartite, $\chi_1(G) = \Delta(G)$ [Class 1]

<u>Sketch Pf:</u> H bipartite, $\Delta(G)$-regular, $G \subseteq H$.

Given H, from exam 2 we know it has a perfect matching. Take one such matching, color some color, delete it. We now have a $\Delta(G) - 1$-regular graph, take another perfect matching. Continue this process. This gives a $\Delta(G)$ coloring of H. Restrict to G's edges.

$G = H_0$



Copy and swap partitions. Connect corresponding minimum degree vertices. Repeat.

$H_0, H_1, \ldots, H_{\Delta(G)-\delta(G)}$ H_i bipartite

$\delta(H_{i+1}) = \delta(H_i) + 1$

$\Delta(H) = \delta(H) \Rightarrow$ Regular

21-484 Notes

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<u>Thm:</u> $\forall k, \ell \;.\; \exists G$ such that $\text{girth}(G) > \ell$ and $\chi(G) > k$.

<u>Tools:</u> 1. Markov's inequality

If X is a nonnegative random variable with expectation, then

$$\Pr[X > a] \leq \frac{\mathbb{E}[X]}{a}$$

<u>Proof:</u>

$$\begin{array}{ccccccccc} \mathbb{E}[X] & := & \sum\limits_{x=0}^{\infty} x \cdot \Pr[X = x] & = & \sum\limits_{0 \leq x \leq a} x \cdot \Pr[X = x] & + & \sum\limits_{x > a} x \cdot \Pr[X = x] & \geq & \\ & & & \geq & 0 & + & a \sum\limits_{x > a} \Pr[X = x] & = & a \cdot \Pr[X > a] \end{array}$$

2. Stirling's Approximation

$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}(1 + O(\frac{1}{n}))$

$n! \geq \left(\frac{n}{e}\right)^n$

$\rightarrow$ Set $0 < \theta < {}^1\!/\!\ell$

$\rightarrow$ Set $p = n^{-1+\theta} = \frac{n^\theta}{n}$

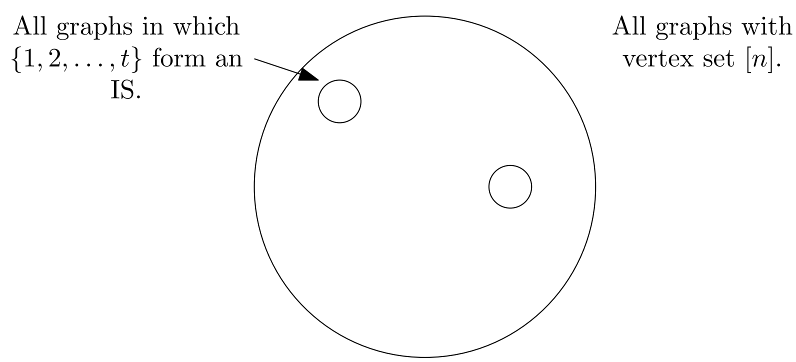
$\rightarrow$ Consider a graph G with n vertices in which every edge is in the graph with probability p.

$\rightarrow$ $X =$ number of cycles of length $\leq \ell$ in the random graph.

$\rightarrow$ $\mathbb{E}[X] = \sum\limits_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} \cdot p^i \leq \cdots \leq \frac{n}{\log n}$

$\rightarrow$ Apply Markov's inequality $\Pr[X \geq {}^n\!/\!_2] \leq \dfrac{\frac{n}{\log n}}{\frac{n}{2}} = \dfrac{2}{\log n} \overset{n\to\infty}{\longrightarrow} 0$

$\rightarrow$ <u>Def:</u> $t = \left\lceil \frac{3 \ln n}{p} \right\rceil \approx \frac{n^{1-\theta}}{3 \ln n}$



$\rightarrow$

$\Pr[\alpha(G) \geq t] \leq \binom{n}{t}(1-p)^{\binom{t}{2}} \leq \left(\frac{ne}{t}\right)^t e^{-p\binom{t}{2}} =$
$\left(\frac{en}{t}\right)^t e^{-pt(t-1)\frac{1}{2}} = \left(\frac{en}{t} e^{-\frac{1}{2}p(t-1)}\right)^t \leq ene^{-\frac{1}{2}\ 3 \ln n}\ .$
$\leq ene^{-1.4 \ln n} = enn^{-1.4} = en^{-0.4} \overset{n\to\infty}{\longrightarrow} 0$

$\rightarrow$ The probability that $X \geq n/2$ or $\alpha(G) \geq t$ is tending to 0 when $n \to \infty$.

$\rightarrow$ There is a graph G such that $X < n/2$ and $\alpha(G) < t$.

$\rightarrow$ Delete one vertex from every short cycle. Let G' be the graph spanned on the remaining vertices.

$\rightarrow$ $|V(G')| \geq n/2$

$\rightarrow$ G' contains no cycles of length $\leq \ell$.

$\rightarrow$ $\alpha(G') \leq \alpha(G) < t \Rightarrow \chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{t} = \frac{\frac{n}{2}}{\frac{3\ln n}{p}} = \frac{np}{6\ln n} = \frac{n^\theta}{6\ln n} \underset{\substack{\uparrow \\ n \text{ large} \\ \text{enough}}}{\geq} k$ ■

21-484 Notes

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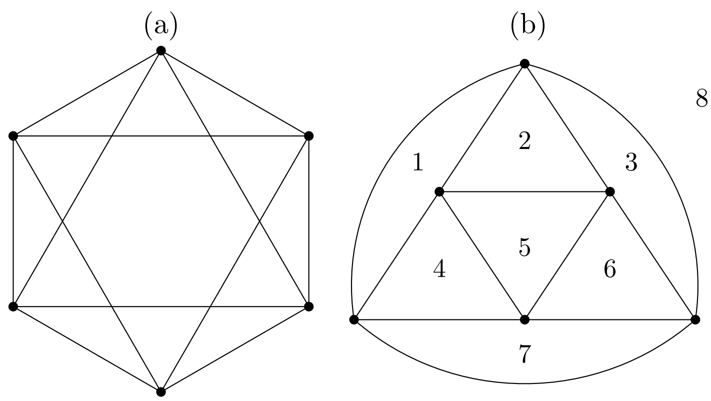
Def: (p. 228): A graph G is called planar if it can be drawn in the plane such that no two edges intersect.

→ Such a drawing is called a plane graph.

Example: Fig 9.3

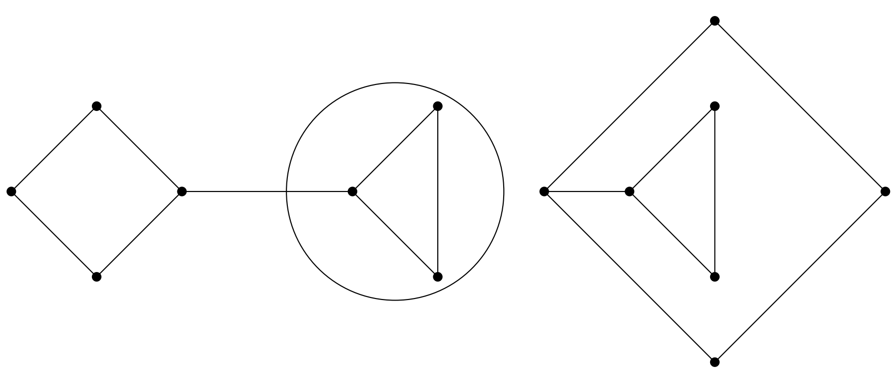


Def: (p. 230): A plane graph divides the plane into connected pieces called regions.

→ The unbounded region is called the exteriror region.

→ The subgraph of a plane graph incident with a given region R is the boundary of R.

Observations: - an edge is on the boundary of 1 region iff it is a bridge. Otherwise it is on the boundary of 2 regions.

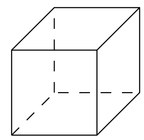


- In a connected plane graph with at least three edges every boundary contains at least three edges.

Theorem: (Thm 9.1, Euler's Identity)

If G is a connected plane graph with n vertices, m edges, and r regions, then

$$n - m + r = 2$$



Proof: If G is a tree then $r = 1$. Since $M = n - 1$ we have $n - m + r = n - (n-1) + 1 = 2$.

- Assume for the sake of contradiction that G is a plane graph, connected, not a tree, had n vertices, m edges, r regions, $n - m + r \neq 2$ and G is minimal (by number of edges) with these properties.

- G is not a tree, so there is an edge that is not a bridge. This edge lies in the boundary of two regions. Remove this edge. Now

$$\begin{aligned} n' &= n \\ m' &= m - 1 \\ r' &= r - 1 \end{aligned}$$

but $n' - m' + r' = n - m + r \neq 2$. $\lightning$ minimality G

Theorem (Thm 9.2): If G is a planar graph with $N \geq 3$ vertices and m edges then

$$m \leq 3n - 6$$

Proof: $\to$ Assume G is connected.

$\to$ Draw G as a plane graph.

$\to$ If $G \cong \bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$, $n = 3, m = 2$ so $2 \leq 3 \cdot 3 - 6 = 2$

$\to$ Can assume that G has at least 3 edges. Hence every boundary has at least 3 edges.

$\to$ Let $m_1, m_2, \ldots, m_r$ be the number of edges in the boundaries. $m_1 \geq 3$.

$\to$ Consider $sm \geq M = \sum_{i=1}^{r} m_i \geq 3 \cdot r \Rightarrow 2m \geq 3r$

$\to$ By Euler's Identity $6 = 3n - 3m + 3r \leq 3n - 3m + 2m = 2n - m \Rightarrow m \leq 2n - 6$

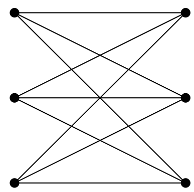
$\to$ If G is disconnected, we can add edges while maintaining planarity to get a connected planar graph. Apply this.

$\to$ Corollary: If G is planar, then $\delta(G) \leq 5$.

Proof: If G was planar with minimal degree ≥ 6 then

$$\begin{gathered} 2m = \sum \deg \geq 6n \\ \Downarrow \\ m \geq 3n \ \lightning \text{ last theorem} \end{gathered}$$

Example: $K_{3,3}$



$n = 6$
$m = 9$ $\quad 9 \leq 3 \cdot 6 - 6 = 12$

$\rightarrow$ $K_{3,3}$ is not planar.

Proof: If it was planar, we could draw it as a plane graph.

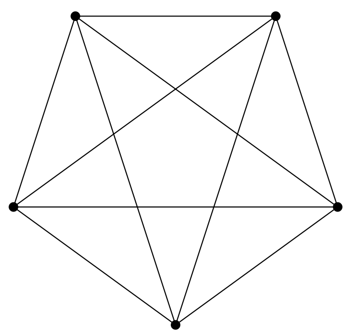
The plane graph will have $r = 2 - n + m = 2 - 6 + 9 = 5$ regions.

THe boundary of each region has at least 4 edges, since $K_{3,3}$ contains no triangles. Let m_i be the number of edges in boundaries.

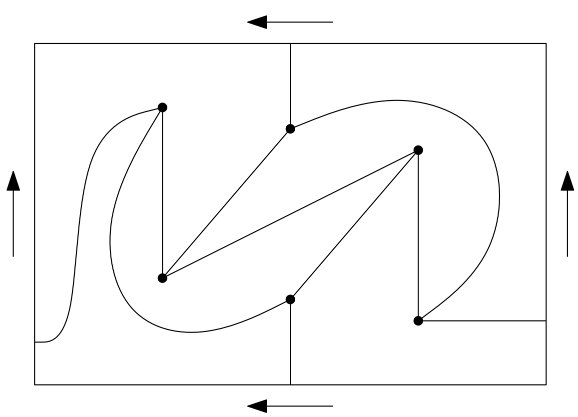
$$18 = 2m = M = \sum_{i=1}^{5} m_i \geq 4r = 20 \;\lightning$$

$\rightarrow$

K_5 $\quad n = 5$, $m = 10$



by the last theorem, K_5 is not planar



21-484 Notes

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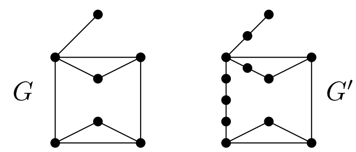
<u>Def (p. 235):</u> A graph G' is a <u>subdivision</u> of a graph G, if G' can be obtained from G by replacing edges by paths.

<u>Example:</u>



G' is a subdivision of G.

<u>Thm:</u> (9.7, Kuratowski's theorem): G is planar if and only if it does not contain a subdivision of K_5 as a subgraph or a subdivision of $K_{3,3}$ as a subgraph.

→ If G is planar, then G does not contain a K_5 subgraph or a $K_{3,3}$ subgraph because these graphs are not planar.

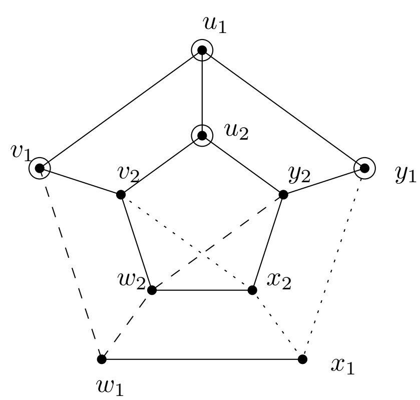
→ K_5 is not planar since it has 5 vertices and 10 edges, $10 > 3 \cdot 5 - 6$

→ a subdivision operation is replacing one edge uv by a path uwv where w is a new vertex adjacent only to u and v.

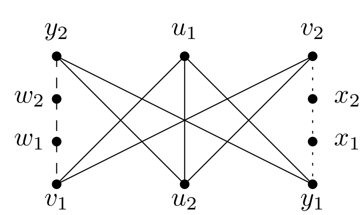
→ If we do k subdivision operations of K_5 we end with $5 + k$ vertices, $10 + k$ edges, so...

→ Should prove similarly to the proof that $K_{3,3}$ is not planar.

<u>Example (9.8):</u>

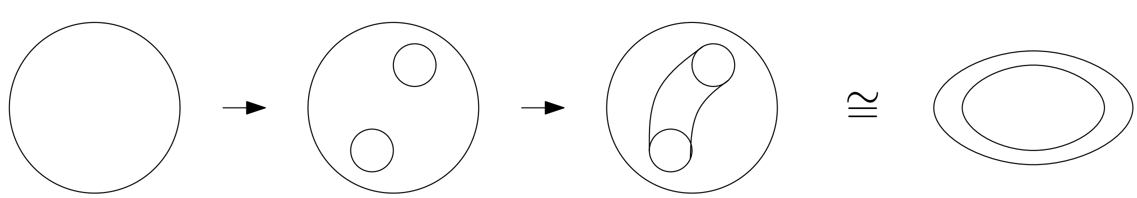


$n = 10$
$m = 17$ $\quad 3 \cdot 10 - 6 = 24$



<u>placeholder</u>

"Adding a handle to a surface"



→ If you add k handles to the sphere, you get S_k, which is a surface of genus k.

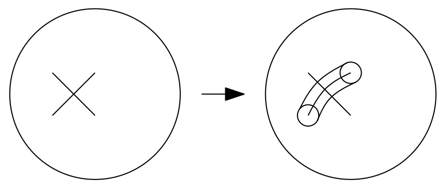
→ Def (p. 244): - A graph G is embeddable in S_k if it can be drawn on S_k such that two edges do not intersect.

- A Graph G has genus if it can be embedded in $S_{\gamma(G)}$ but can not be embedded in $S_{\gamma G-1}$.

Claim: $\delta(G)$ is finite for all graphs G.

Proof: Draw G on the sphere such that every intersection point which is not a vertex is an intersection point of at most 2 edge.

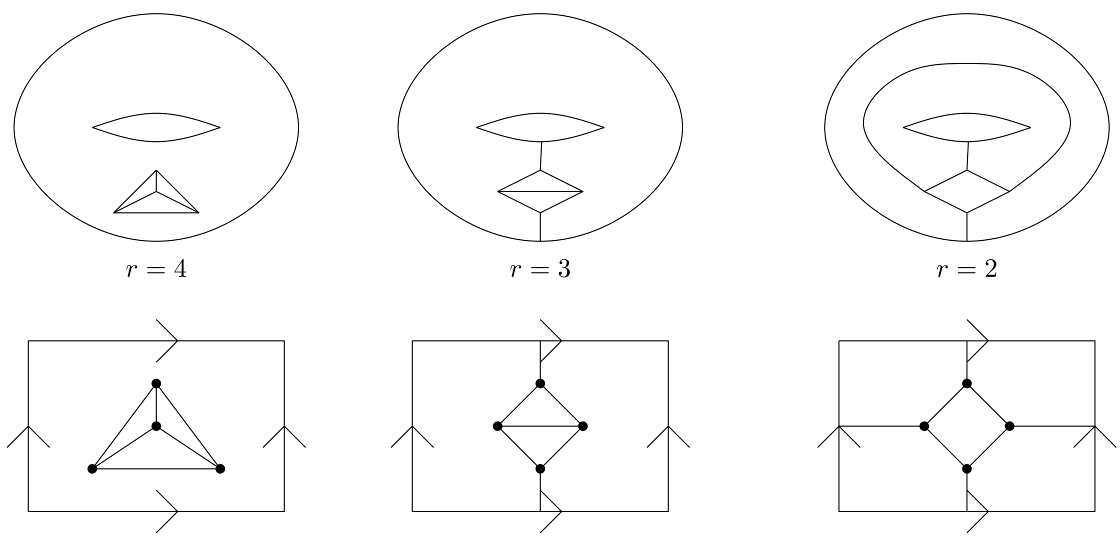
→ Add a handle for each intersection point



Def: A region of a surface is called 2-cell if any closed curve can be continuously contracted in the region to a single point.

- A 2-cell embedding of a graph is an embedding such that every region is a 2-cell region.

Example: (9.25 + 9.27)



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Recall: - 2-Cell region

- 2-Cell embedding

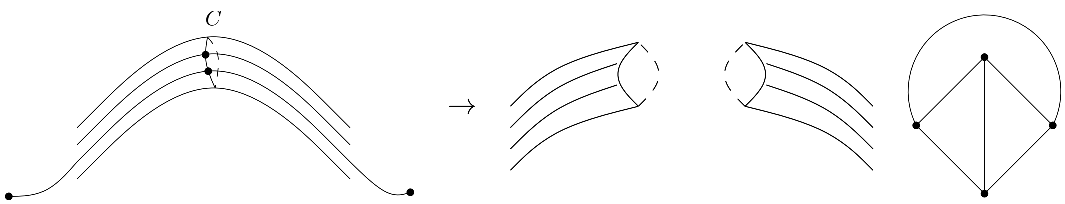
Theorem: (9.9): Let G be a connected graph, 2-cell embedded on a surface of genus k, and G has n vertices, m edges and r regions.

Then $n - m + r = 2 - 2k$

Proof ideas: Induction on k

- $k = 0$ – Euler's identity

- $k > 0$

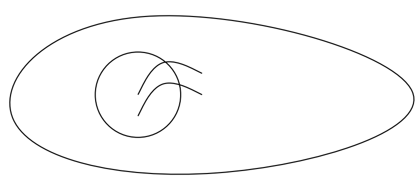


$n' = n + t + 2$ $\quad$ $n'' = n + 2t + 4$

$m = \ldots$

Claim: Let G be a connected graph, let $k = \delta(G)$. Any embedding of G on a surface of genus k is a 2-cell embedding.

Proof: Assume for the sake of contradiction that G is embedded on S_k and there is a region that is not a 2-cell region. There is a closed curve C that is not continuously contractable in the region to a point.



This region contains a handle and there are no edges or vertices on the handle. Remove the handle to ger S_{k-1} in which G is embedded. $\lightning$ $\delta(G) = k$.

Corollary: (9.10) If G is connected, embedded on a surface of genus $\delta(G)$ and n, m, r are as usual, then

$$n - m + r = 2 - 2\delta(G).$$

$\rightarrow$ following the same proof that showed $m \leq 3n - 6$ for planar graphs give. If G has n vertices and m edges then

$$m \leq 3n + 6(\delta(G) - 1)$$

$\rightarrow$ G is embeddable on S_0 (the sphere) iff it is planar.

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→ If G is planar, embed it in the plan, take a curve surrounding G and contract it to a point to get an embedding of G on S_0. For the other direction, start with a point not on an edge, and "tear" through it to get an embedding in the plane.

Def: (p.249-250)

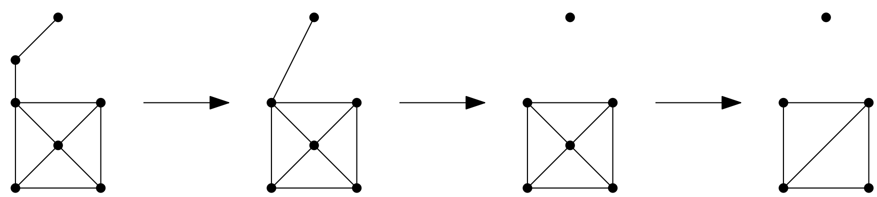
→ Let G be a graph and assume $uv \in E(G)$

contracting the edge uv measn the following:

- remove the vertices u and v
- add a new vertex w
- add edges between w and all the vertices in $N(u) \cup N(v)$.

→ A minor of a graph G is a graph that can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions.

Example:



fact: If H is a subdivision of G then G is a minor of H.

→ contract the new paths back to an edge.

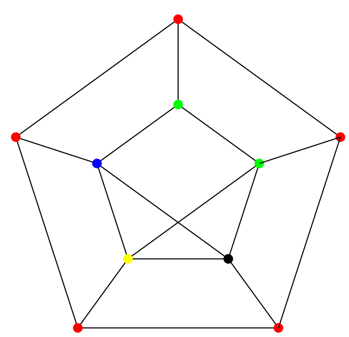
claim: If H is a minor of G then $\delta(H) \leq \delta(G)$.

→ contracting an edge does not increase the genus.

Thm (9.15, Wagner's thm)

A graph is planar iff it does not contain a K_5 or a $K_{3,3}$ minor.

Example: recall that the graph below is not planar. We showed that by finding a $K_{3,3}$ subdivision in it.



→ there is no K_5 subdivision in it.

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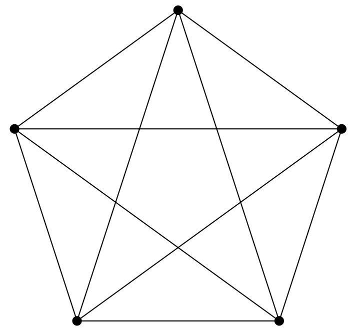

There is a K_5 minor.

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Wagner's Theorem: G is planar iff no K_5 or $K_{3,3}$ minor.

Def: (p. 252): A graph G is minimally nonembeddable in S_k if

→ G is not embeddable in S_k.

→ removing any vertex or any edge or contracting any edge results in a graph embeddable in S_k.

→ Wagner's Theorem: K_5 and $K_{3,3}$ are the only minimally nonembeddable graphs in S_0.

→ G is either not planar or has a K_5 or $K_{3,3}$ minor.

→ Theorem (Seymor and Robertson, 1983-2004, graph minor theorem)

For any infinite set of graphs, there are two graphs such that one is a minor of the other.

Corollary: Every family of graphs closed under taking minors can be defined by a finite set of forbidden minors.

→ Otherwise we have an infinite set of forbidden minors $\lightning$ the theorem above.

Corollary: (Cor 9.7) For all $k \geq 0$, the set of minimally nonembeddable graphs in S_k is finite.

Corollary 9.8 $\forall k \geq 0$ there is a finite set of graphs S such that G is embeddable in S_k iff it does not have an H minor for every H in S.

→ For S_1, the set of forbidden minors is of size at least 800.

Recall: For every planar G, $\delta(G) \leq 5$.

Corollary: If G is planar, then $\chi(G) \leq 6$.

Proof: Given a graph $G = G_n$ we can find a vertex of degree ≤ 5, call it v_n and remove it to get G_{n-1}.

→ G_{n-1} is also planar, so we can repeat this process.

→ Color the vertices in order. Notice that when coloring a vertex, it has at most 5 colored neighbors. So one of the 6 colors is available.

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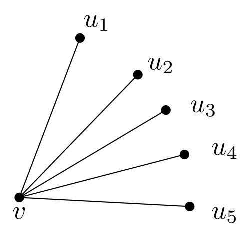
Theorem: Every planar graph is 5 colorable.

Proof: By induction on the number of vertices. If G has a vertex of degree ≤ 4, remove it, color the resulting graph with 5 colors, and add it back.

→ Since $\delta(G) \leq 5$, there is a vertex of degree 5. call it v.

→ Draw G in the plane to get a plane graph and name the neighbors of v according to the order in which they appear in the plane graph.

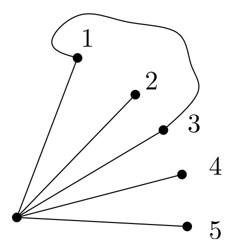


→ Color $G \setminus \{v\}$ with 5 colors.

→ If 2 of the u_i's are colored in the same color, we're done.

→ Assume u_i is colored by color i.

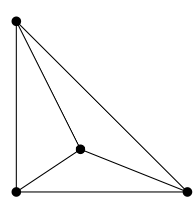
→ Let G_{13} be the graph spanned by vertices colored 1 or 3.



→ If u_1 and u_3 are not in the same component, switch colors in the component containing u_3.

→ If u_1 and u_3 are in the same component, there is a u_1–u_3 path in which all vertices are colored 1 or 3.

→ Repeat for G_{24}. If they are in the same component there is a u_2–u_4 path in which all vertices are colored 2 or 4. ↯ planarity



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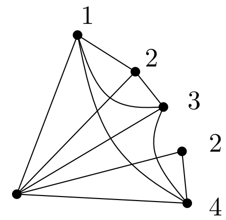
→ Kempe:

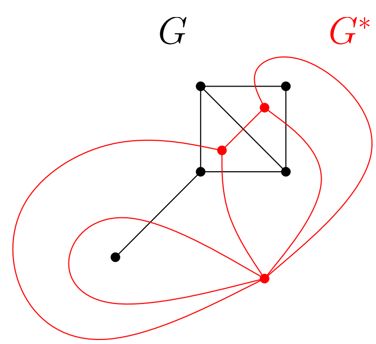


Heawood

Def: (p. 267)

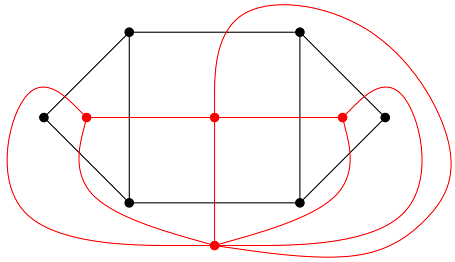
Let G be a plane graph. The dual of G, denoted G^*, is a plane multigraph. From each region of G we pick one inner point to be a vertex of G^*. For every edge of G we add a curve connecting the vertices of G^* corresponding to the regions incident with e, such that this curve intersects e once and does not intersect anything else (including itself).

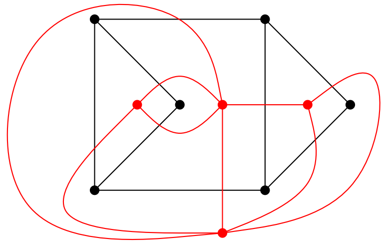
Example:



→ The name "dual" is justified.

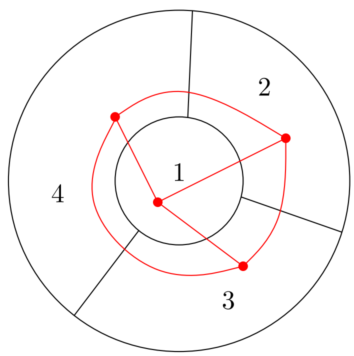
→ We also talk about the dual of a planar graph, which is not unique.





$\Rightarrow$ The dual of a planar graph depends on the embedding.

Thm: "Every map can be colored in 4 colors."



$\rightarrow$ Taking the dual of a map gives a planar graph.

$\rightarrow$ Every planar graph is 4-colorable

About the proof:

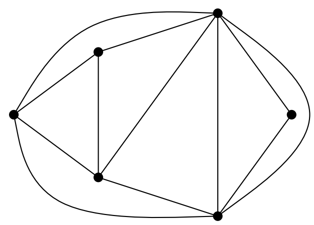
$\rightarrow$ reducible configurations.

$\rightarrow$ unavoidable set of reducible configurations.

5-color thm

$$\left.\begin{array}{c} \text{1 vertex of deg 0} \\ \text{1 vertex of deg 1} \\ \vdots \\ \text{1 vertex of deg 5} \end{array}\right\} \text{unavoidable}$$

$\rightarrow$ Triangulation:



$$2e = 3r$$

<u>Ramsey Theory</u>

<u>Def:</u> (p. 299)

The <u>ramsey number</u> $r(F_1, F_2)$ is the minimal number r such that in every red-blue coloring of the edges of K_r there is either a red copy of F_1 or a blur copy of F_2.

→ $r(n, m)$ is $r(K_n, K_m)$

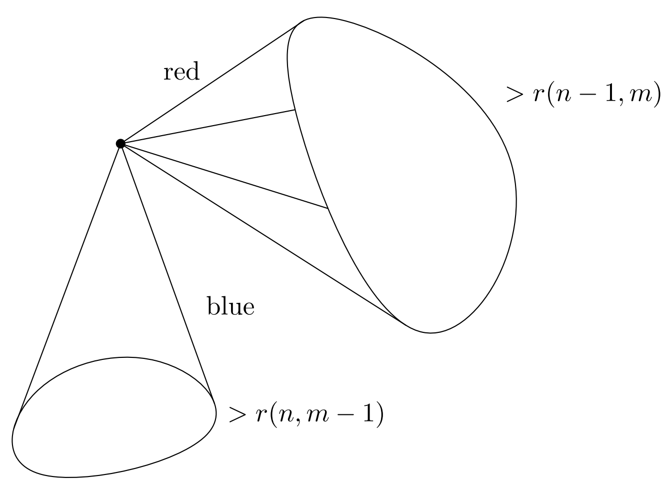
→ $r(n) = r(n, n)$

<u>Thm (Ramsey):</u> $r(n, m)$ is finite.

<u>Proof:</u> by double induction.

$r(1, n) = r(m, 1) = 1$

→ Assume that $r(n', m')$ is finite for all pairs $n', m' < n, m$.



→ Let $r = r(n-1, m) + r(n, m-1)$

→ Fix a vertex v.

→ Let N_{red} be the set of vertices adjacent to v via a red edge.

→ N_{blue}.

→ If both $|N_{\text{red}} < r(n-1, m)$ and $|N_{\text{blue}}| < r(n, m-1)$ ↯

$$r = |N_{\text{red}}| + |N_{\text{blue}}| + 1 < r(n-1, m) + r(n, m-1) + 1$$

→ One of them is large enough and we can finish

21-484 Notes

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1. Hadwiger's conjecture: $\chi(G) = k \Rightarrow G$ contains a K_k minor.

$k = 2$ ✓ trivial

$k = 3$ $\chi(G) > 2 \iff G$ is not bipartite $\iff G$ contains an odd cycle $\Rightarrow G$ contains a K_3 minor.

$k = 4$ Proved by Hadwiger

$k = 5$ Wagner showed that this case equivalent to the 4-colors theorem.

If G is not 4 colorable $\Rightarrow G$ contains a K_5 minor $\underset{\text{Wagner's}}{\Rightarrow}$ G is not planar

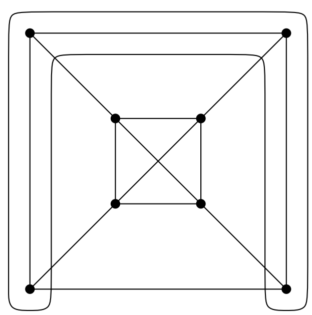
$k = 6$ Robertson, Seymour, Thomas ('93)

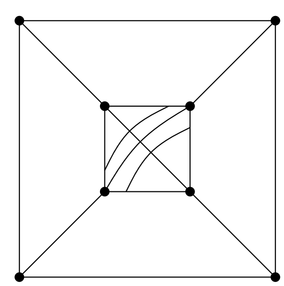
$k = 7$ <u>known:</u> not 6-colorable $\Rightarrow K_7$ minor or both $K_{4,4}$ and $K_{3,5}$ minors.

Hadwiger's conjecture is true for most graphs.

Theorem (Bollobás ect.): Pr[Hadwiger's conjecture is true in $G(n, 1/2)$] $\overset{n\to\infty}{\to}$ 1

2 What's the genus of

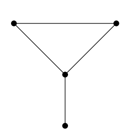




<u>3(a):</u>

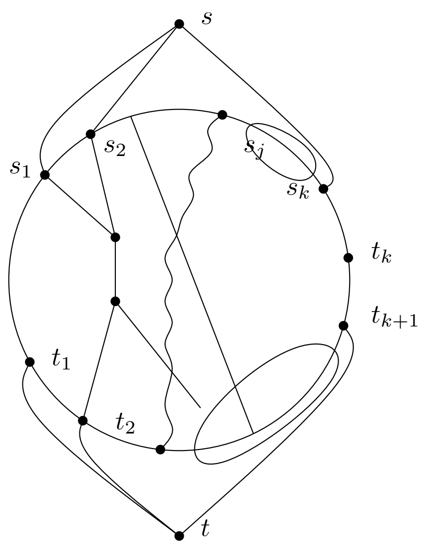
$$k = \left(\max_{H \subseteq G} \delta(G)\right) + 1$$

$\to$ take v_n to be a vertex of degree $\delta(G)$.



$\to$ remove v_n and pick v_{n-1} in the same way.

3(b):



→ add a source and a target, s, t
→ the new graph is k-connected
→ Apply Menger's Theorem
→ use planarity to argue that the paths are all s_1–t_i paths

Ramsey's theorem: $r(n, m)$ is finite.

double induction on n, m

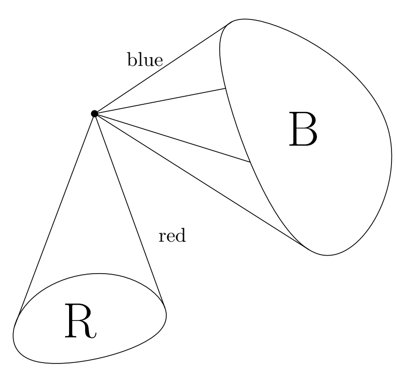
$\rightarrow r(n, 1) = r(1, m) = 1$

$\rightarrow r(n, m) \leq r(n-1, m) + r(n, m-1)$

proof:

Consider a two coloring of K_r where $r = r(n-1, m) + r(n, m-1)$. Pick a vertex v.

Let B be the set of vertices adjacent to v via a blue edge. Same for R.



Since $r = r(n-1, m) + r(n, m-1) = |B| + |R| + 1$ it is not the case that both $|B| < r(n-1, m)$ and $|R| < r(n, m-1)$

Assume without loss of generality $|B| \geq r(n-1, m)$. If the graph induced on B contains a red m cliquem we are done. If it contains a blue $n-1$ clique, then $|B| \cup \{v\}$ contains an n-clique.

Theorem: (11.2) $r(n_1, n_2, \ldots, n_k)$ is finite.

Proof: induct on k.

✓ $k = 2$

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$\rightarrow r(n_1, n_2, n_3, \ldots, n_k)) \leq r = r(n_1, n_2, \ldots, n_{k-2}, r(n_{k-1}, n_k))$

$\rightarrow$ conider coloring of K_r with $k-1$ colors. Either we have a clique of size n_i colored i for $1 \leq i \leq k-2$ or we have a clique of size $r(n_{k-1}, n_k)$ colored in one color. Apply the induction again.

$r(F_1, F_2, \ldots, F_k)$ is finite.

r(s,2)	$=$	s		
r(3,3)	$=$	6		
r(4,3)	$=$	9		
r(4,4)	$=$	18		
r(4,5)	$=$	25 (1995)		
r(4,6)	$\geq$	36 (2012)	$\leq$	41
43	$\leq$	r(5,5)	$\leq$	49

<u>Thm:</u> $r(n,m) \leq \binom{n+m+2}{n-1}$

$\rightarrow$ Same proof.

Erdős-Szekeres

$$r(n) \leq (1+o(1)) \frac{1}{\sqrt{\pi n}} 4^{n-1}$$

Erdős

$$r(n) \geq (1+o(1)) \frac{n}{\sqrt{2} \cdot e} \sqrt{2}^n$$

$$\underset{\text{Joel Spencer}}{(1+o(1))\frac{\sqrt{2}n}{e}\sqrt{2}^n} \leq r(n) \leq \underset{\text{David Conlon}}{4^n \cdot n^{-\frac{c \cdot \log n}{\log \log n}}}$$