21-640 **Intro to Functional Analysis** **Spring 2013**

Assignment 7

Due on Wednesday, May 1

Please hand in Solutions to all 5 problems.

1. Let X be a topological vector space, C be a closed subset of X, and K be a compact subset of X. Show that $C + K$ is closed.

2. Let X be a locally convex topological vector space and let $x \in X \backslash \{0\}$ be given. Show that there exists a continuous linear functional $f : X \to \mathbb{K}$ such that $f(x) \neq 0$.

3. Let $\mathbb{K} = \mathbb{C}$ and let X be the set of all continuous functions $f : [0, 1] \to \mathbb{C}$. Define the metric ρ on X by

$$\rho(f, g)) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

Let (X, σ) denote X equipped with the topology induced by ρ and let (X, τ) denote X equipped with the topology defined by the separating family $\{p_x : x \in [0, 1]\}$ of seminorms, where

$$p_x(f) = |f(x)| \text{ for all } x \in X.$$

Let I denote the identity mapping on X.

(a) Show that $I : (X, \tau) \to (X, \sigma)$ is bounded. (Notice that this is equivalent to showing that every set that is topologically τ-bounded is also topologically σ-bounded.)

(b) Show that $I : (X, \tau) \to (X\sigma)$ fails to be continuous.

(Note: This is also an example of a linear mapping between two TVS that is sequentially continuous, but not continuous.)

4. Let X be a topologival vector space and let E be a subset of X. Show that E is topologically bounded if and only if the following property holds: For every sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathbb{K}$ such that $\alpha_n \to 0$ as $n \to \infty$ and every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in E$ for all $n \in \mathbb{N}$, we have $\alpha_n x_n \to 0$ as $n \to \infty$.

5. Let X be a lovcally convex topological vector space with a countable local base. Let $\{V_n : n \in \mathbb{N}\}$ be a local base such that for each $n \in \mathbb{N}$, V_n is balanced and convex. (Each V_n is automatically absorbing.) For each $n \in \mathbb{N}$ put

$$p_n(x) = p^{V_n}(x) \text{ for all } x \in X,$$

where p^{V_n} is the Minkowski functional for V_n. Define $F : X \to \mathbb{R}$ by

$$F(x) = \max\left\{\frac{1}{n}\min\{1, p_n(x)\} : n \in \mathbb{N}\right\} \quad \text{for all } x \in X.$$

Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) = F(x - y) \quad \text{for all } x, y \in X.$$

(a) Convince yourself that ρ is a translation invariant metric on X. You do not need to hand anything in for this part, just make sure that you convince yourself, and that you could convince me if I were to ask for details.

(b) Show that each open ρ-ball centered at 0 is balanced.

(c) Show that each open ρ-ball is convex.

(d) Show that ρ induces the topology of X.

(Observe that radii $r > 1$ are irrelevant since $F(x) \leq 1$ for all $x \in X$. For $0 < r \leq 1$, there is an elegant formula for $\{x \in X : F(x) < r\}$.)