21-484 Notes

JD Nir
jnir@andrew.cmu.edu
March 5, 2012

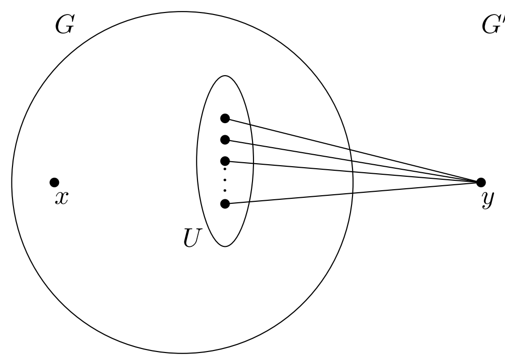
Recall: Menger's Theorem: If G is a graph and $x, y \in V(G)$, $xy \notin E(G)$ then the size of a minimal x–y separating set equals the maximum number of internally disjoint x–y paths.

Theorem (Dircac): Let G be a k-connected graph (with $k \geq 2$). Then for every set $S \subseteq V(G)$, $|S| = k$, there is a cycle $C \in G$ such that $S \subseteq V(C)$.

Def: Let G be a graph, $X \in V(G)$, $U \subseteq V(G) \setminus \{x\}$. An x, U–fan is a set of paths from x to vertices of U such that for every pair of paths the only common vertex is x.

Lemma: (Fan Lemma): A graph is k-connected iff it has at least $k+1$ vertices and for every vertex x and every set $U \subseteq V \setminus \{x\}$, $|U| \geq k$, there is an x, U–of size k.

Proof: Assume that G is k-connected. Let x be a vertex. Let U be a set of at least k other vertices in G.



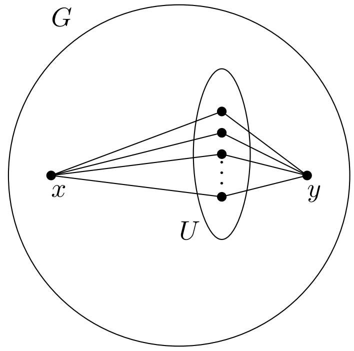
Define G' by adding another vertex y and all the edges of the form uy for $u \in U$. G' is also k-connected since removing at most $k-1$ vertices leaves y connected to at least one vertex from U and also leaves G connected.

→ A minimal x–y separating set is of size at least k.

→ By Menger's Theorem there exists a set of k internally disjoint x–y paths in G'.

→ We get an x, U–fan of size $\geq k$.

Assume that G satisfies the fan condition.



- $\delta(G) \geq k$
- let x and y be two non-adjacent vertices in G.
- let $U = N(y)$
  - $|U| \geq k$
  - $x \notin U$

$\rightarrow$ By the assumption, there is an x, U–fan of size k.

$\rightarrow$ assing the edges between U and y we get a set of $\geq k$ internally disjoint x–y paths.

$\Rightarrow$ (Menger's) the size of any x–y separating set $\geq k$.

$\Rightarrow$ G is k-connected

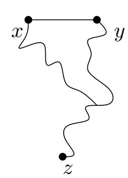
<u>Proof:</u> Induction on k.

$k = 2$. Let x, y be two vertices of a 2-connected graph G.

$\rightarrow$ If $xy \in E(G)$ consider a third vertex z.

$\rightarrow$ By 2-connectivity, $G - \{x\}$ contains a y–z path p.

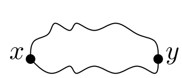
$\rightarrow$ By 2-connectivity, $G - \{y\}$ contains a x–z path p'.



$\rightarrow$ There is an x–y path (in the x–y walk pp') not using the edge xy.

$\rightarrow$ together with xy we get a cycle.

- If $x, y \notin E(G)$, then by 2-connectivity and Menger's theorem, we get two internally disjoint x–y paths. ✓



→ $k > 2$.

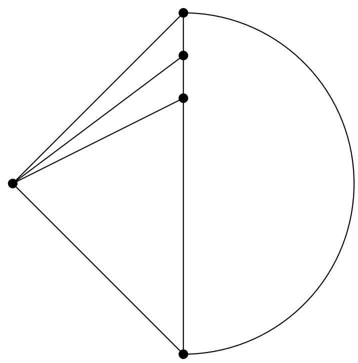
→ → G is k connected, $S \subseteq V(G)$ of size k.

→ let $x \in S$.

→ Since G is also $k-1$ connected, there is a cycle C containing all the vertices in $S \setminus \{x\}$. (Induction hypothesis)

→ If $|C| = k-1$

→ By the Fan lemma, there is an x, C–fan of size $k-1$.

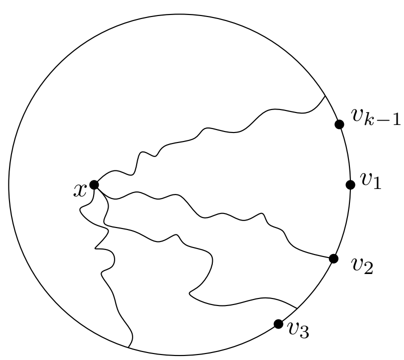


→ So there are internally disjoint paths from x to every vertex of C.

→ taking two consecutive vertices y, x in X we get a new cycle

$$x(\text{path from } x \text{ to } y)\ (\text{path of } C \text{ from } y \text{ to } z)\ (\text{path from } z \text{ to } x).$$

→ Assume that $|C| \geq k$.



→ Let $v_1, v_2, \ldots, v_{k-1}$ be the vertices of $S \setminus \{x\}$ ordered according to appearance on C.

→ Let V_i be the v_i–v_{i+1} path on C. (V_{k-1} is the v_{k-1}–v_1 path on C).

→ By the fan lemma, k-connectivity og G, $|C| \geq k$, we gave j "disjoint" paths from x to C.

→ The paths have k endpoints in C, so there is a set V_i containing two such endpoints y, z. (Pigeon-hole principle)

→ The cycle (the x–y path) (the y–z segment on C out of V_i) (the z–x path) is the required cycle. ■