Chapter 5

Convexity

The notion of a convex functional is a very natural and useful generalization of the idea from basic calculus of a function whose graph is "concave up". Such functionals play a fundamental role in the study of minimization problems. One important reason for this is the fact that if a convex functional J has an element y_* in its domain such $\delta J(y; v) = 0$ for all v in a suitable class of admissible variations, then J must attain a minimum at y_*. We begin by looking at convex functions from $\mathbb{R}$ to $\mathbb{R}$.

5.1 Convex functions from $\mathbb{R}$ to $\mathbb{R}$

Let $f : \mathbb{R} \to \mathbb{R}$ be given. The notion of "concavity" plays an important role in graphing functions in basic calculus. In calculus courses, concavity is typically defined in terms of monotonicty of the derivative which is related to the sign of the second derivative. In more sophisticated mathematical contexts a function from $\mathbb{R}$ to $\mathbb{R}$ whose graph is every concave up is said to be a "convex function". We want to give a definition of convexity that does not assume differentiability. Let us recall some characterizations of convex functions from basic calculus.

(1) If f is twice differentiable on $\mathbb{R}$ then f is convex if and only if $f(x) \geq 0$ for all $x \in \mathbb{R}$.

(2) If f is differentiable on $\mathbb{R}$ then f is convex provided that f' is increasing[1]

(3) If f is differentiable on $\mathbb{R}$ then f is convex if and only if the graph of f always lies above its tangent lines.

(4) Without assuming that f is differentiable: f is convex if and only if given any secant line for the graph of f, the secant line intersects the graph of f at only two points and between the points of intersection the graph of f lies below the secant line.

[1] When we say that a function $g : \mathbb{R} \to \mathbb{R}$ is *increasing* we mean that $g(x_2) \geq g(x_1)$ whenever $x_2 \geq x_1$. If $g(x_2) > g(x_1)$ whenever $x_2 > x_1$ we say that g is *strictly increasing*.

Let us formulate these last two characterizations analytically. The third condition states that the graph of f is supported by its tangent lines. Assume that f is differentiable and let $x_0 \in \mathbb{R}$ be given. The tangent line to the graph of f at x_0 is described by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

The graph of f lies everywhere above this tangent line if and only if

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \quad \text{for all } x \in \mathbb{R}. \tag{5.1}$$

Observe that (5.1) has the following very important consequence: If $f'(x_0) = 0$, then $f(x) \geq f(x_0)$ for all $x \in \mathbb{R}$. In other words, if f is differentiable and convex and $f'(x_0) = 0$, then f attains a minimum at x_0.

For item (4), let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ be given. The characterization states that at each $x \in [x_1, x_2]$, the graph of f lies below the line segment joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. It is not difficult to check that this will be the case if and only if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \quad \text{for all } t \in [0,1]. \tag{5.2}$$

It is quite natural to use (5.2) for the official definition of *convex function*.

5.2 The Definition of Convexity in a Real Linear Space

In this section, we generalize the notion of convexity to functions defined on subsets of real linear spaces. Let $\mathfrak{X}$ be a real linear space. We will base the definition on (5.2), because this condition did not require any differentiability of the function in question. We will be interested in defining convexity of functionals whose domains are proper subsets of $\mathfrak{X}$. In order to do this we want be sure that the domain of the functional has the property that whenever two points are in the domain, then so is the line segment joining the two points. This motivates the following

Definition 5.1 *Let $\mathscr{Y} \subset \mathfrak{X}$ be given. We say that $\mathscr{Y}$* is convex *provided that*

$$ty_1 + (1-t)y_2 \in \mathscr{Y} \quad \text{for all } y_1, y_2 \in \mathscr{Y} \text{ and } t \in [0,1].$$

Now, we define convexity for functionals.

Definition 5.2 *Let $\mathscr{Y} \subset \mathfrak{X}$ be a convex set, and let $J : \mathscr{Y} \to \mathbb{R}$ be given. We say that J* is convex *provided that*

$$J(ty_1 + (1-t)y_2) \leq tJ(y_1) + (1-t)J(y_2) \quad \text{for all } y_1, y_2 \in \mathscr{Y} \text{ and } t \in [0,1].$$

We emphasize that Definition 5.2 makes sense only when the domain of J is a convex set. Before we explore the significance of convexity in the context of a general linear space $\mathfrak{X}$, it is instructive to examine convex functions from $\mathbb{R}^n$ to $\mathbb{R}$.

5.3 Convex Functions on $\mathbb{R}^n$

We shall now relate convexity of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ to conditions involving ∇f. Before stating the main theorem, we will extend the idea of an increasing function to mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. To motivate this extension let us reexamine what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be increasing. Observe that g is increasing if and only if: for all $x_1, x_2 \in \mathbb{R}$, we have

$$g(x_2)-g(x_1) \geq 0 \text{ when } (x_2 - x_1) \geq 0 \text{ and } g(x_2)-g(x_1) \leq 0 \text{ when } (x_2 - x_1) \leq 0.$$

An equivalent way to state this condition is

$$[g(x_2) - g(x_1)](x_2 - x_1) \geq 0 \quad \text{for all } x_1, x_2 \in \mathbb{R}.$$

We make the following definition.

Definition 5.3 *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be given. We say that g is monotone provided that*

$$[g(x) - g(y)] \cdot (x - y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^n.$$

We are now ready to generalize items (2) and (3) from Section 5.1.

Theorem 5.1 *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ has continuous first-order partial derivatives. The following three statements are equivalent:*

(i) f *is convex;*

(ii) $f(x) \geq f(y) + \nabla f(x) \cdot (x - y)$ *for all* $x, y in \mathbb{R}^n$*;*

(iii) $[\nabla f(x) - \nabla f(y)] \cdot (x - y) \geq 0$ *for all* $x, y \in \mathbb{R}^n$*.*

Proof. To establish the theorem, we will prove that (i)⇒(ii)⇒(iii)⇒(i).

To prove that (i)⇒(ii), we assume that f is convex. Let $x, y \in \mathbb{R}^n$ be given. By Definition 5.2, we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \quad \text{for all } t \in [0, 1]. \tag{5.3}$$

Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) := f(tx + (1 - t)y) \quad \text{for all } t \in \mathbb{R}.$$

Notice that $g(0) = f(y)$ and $g(1) = f(x)$. Using the chain rule (Theorem 2.1), we find that g is differentiable and

$$g'(t) = \nabla f(tx + (1 - t)y) \cdot (x - y) \quad \text{for all } t \in \mathbb{R}. \tag{5.4}$$

We deduce from (5.3) that for each $t \in (0, 1]$

$$f(tx + (1 - t)y) - f(y) \leq t[f(x) - f(y)],$$

which gives

$$\frac{f(tx+(1-t)y)-f(y)}{t} \le f(x)-f(y),$$

i.e.

$$\frac{g(t)-g(0)}{t} \le f(x)-f(y). \tag{5.5}$$

Letting $t \to 0^+$ in (5.5) yields

$$g'(0) \le f(x)-f(y).$$

Using (5.4) with $t=0$ now gives us

$$\nabla f(x)\cdot(x-y) \le f(x)-f(y)$$

and consequently we have

$$f(x) \ge f(y)+\nabla f(y)\cdot(x-y).$$

We have thus proven that (i)⇒(ii).

Next, we prove that (ii)⇒(iii). Assume that (ii) holds and let $x, y \in \mathbb{R}^n$ be given. Then we have

$$f(x) \ge f(y)+\nabla f(y)\cdot(x-y). \tag{5.6}$$

and (by interchanging x and y) we also have

$$f(y) \ge f(x)+\nabla f(x)\cdot(y-x). \tag{5.7}$$

Adding (5.6) and (5.7) yields

$$f(x)+f(y) \ge f(y)+f(x)+\nabla f(y)\cdot(x-y)+\nabla f(x)\cdot(y-x)$$

and consequently we have

$$0 \ge \nabla f(y)\cdot(x-y)+\nabla f(x)\cdot(x-y)$$

which gives

$$0 \ge [\nabla f(y)-\nabla f(x)]\cdot(x-y). \tag{5.8}$$

Multiplying (5.8) by -1 we obtain

$$[\nabla f(y)-\nabla f(x)]\cdot(y-x) \ge 0,$$

and the implication (ii)⇒(iii) is established.

To complete the proof of the theorem, it only remains to show that (iii)⇒(i). Assume that (iii) holds and let $x, y \in \mathbb{R}^n$ be given. Define the function $G : [0,1] \to \mathbb{R}$ by

$$G(t) = f(tx+(1-t)y) - tf(x) - (1-t)f(y) \quad \text{for all } t \in [0,1].$$

Observe that $G(0) = G(1) = 0$. We want to show that $G(t) \leq 0$ at each $t \in [0,1]$. We shall accomplisf this by first showing that the derivative of G is increasing over $[0,1]$ and then use this fact to conclude that $G(t) \leq 0$ for all $t \in [0,1]$.

Let us look at the derivative of G. We have

$$G'(t) = \nabla f(tx + (1-t)y) \cdot (x-y) - f(x) + f(y) \quad \text{for all } t \in [0,1].$$

Let $s, t \in [0,1]$ be given and observe that

$$\begin{aligned} G'(t) - G'(s) &= [\nabla f(tx + (1-t)y) - \nabla f(sx + (1-s)y)] \cdot (x-y) \\ &= [\nabla f(y + t(x-y)) - \nabla f(y + s(x-y))] \cdot (x-y). \end{aligned} \tag{5.9}$$

Putting $\widetilde{x} := tx + (1-t)y$ and $\widetilde{y} := sx + (1-s)y$, we can rewrite (5.9) as

$$\begin{aligned} [G'(t) - G'(s)](t-s) &= (t-s)\,[\nabla f(\widetilde{x}) - \nabla f(\widetilde{y})] \cdot (x-y) \\ &= [\nabla f(\widetilde{x}) - \nabla f(\widetilde{y})] \cdot (\widetilde{x} - \widetilde{y}) \\ &\geq 0. \end{aligned}$$

For the last step, we used our assumption that (iii) holds. Since $t, s \in [0,1]$ were arbitrary, we have shown that

$$[G'(t) - G'(s)]\,(t-s) \geq 0 \quad \text{for all } t, s \in [0,1].$$

Consequently G' is increasing on the interval $[0,1]$. It follows that the maximum of G on $[0,1]$is 0, i.e. the maximum value of G is attained at the endpoints of $[0,1]$. Indeed, suppose that G attains a maximum at $t_0 \in (0,1)$. Since G is continuously differentiable, we must have $G'(t_0) = 0$. Moreover, we must have $G'(t) \leq 0$ at each $t \in [0, t_0]$ and $G'(t) \geq 0$ at each $t \in [t_0, 1]$, since G' is increasing $[0,1]$. This implies that G also attains a minimum at t_0. The only way that this can occur is if G is constant on $[0,1]$, and since $G(0) = G(1) = 0$, we have shown that is G attains a maximum at some $t_0 \in (0,1)$, then $G(t) = 0$ at each $t \in (0,1)$. Thus the maximum value for G is 0. In conclusion, we have shown that $G(t) \leq 0$ at each $t \in [0,1]$, and using the definition of G, we see that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Consequently f is convex and (iii)$\Rightarrow$(i). This completes the proof. $\square$

Corollary 5.1 *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function with continuous first-order partial derivatives. Let $y_* \in \mathbb{R}^n$ be given and suppose that $\nabla f(y_*) = 0$. Then y_* minimizes f on $\mathbb{R}^n$.*

The proof follows immediately from (ii) and the observation that for all $y \in \mathbb{R}^n$ we have

$$f(y) \geq f(y_*) + \nabla f(y_*) \cdot (y - y_*) = f(y_*),$$

since $\nabla f(y_*) = 0$.

5.4 Convexity and Minima in Real Linear Spaces

In this section, we prove an analogue of Corollary 5.1 for functionals defined on convex subsets of real linear spaces. Let $\mathfrak{X}$ be a real linear space and let $\mathscr{Y} \subset \mathfrak{X}$ be convex. Let $\mathscr{V}$ be a subspace of $\mathfrak{X}$ and assume that

$$z - y \in \mathscr{V} \text{ and } y + v \in \mathscr{Y} \quad \text{for all } y, z \in \mathscr{Y} \text{ and } v \in \mathscr{V}.$$

We will prove the following

Theorem 5.2 *Assume that $J : \mathscr{Y} \to \mathbb{R}$ is convex and let $y_* \in \mathscr{Y}$ be given. Suppose that $\delta J(y_*; v)$ exists and $\delta J(y_*; v) = 0$ for each $v \in \mathscr{V}$. Then y_* minimizes J over $\mathscr{Y}$.*

Proof. Let $y \in \mathscr{Y}$ be given. We want to show that $J(y) \geq J(y_*)$. Let $v = y - y_* \in \mathscr{V}$, so that $y = y_* + v$. Let $t \in (0, 1]$ be given. Since J is convex, we have the following inequalities:

$$J(ty + (1-t)y_*) \leq tJ(y) + (1-t)J(y_*).$$

$$\begin{aligned} J(t(y_* + v) + (1-t)y_*) &\leq t[J(y) - J(y_*)] + J(y_*) \\ J(y_* + tv) &\leq J(y_*) + t[J(y) - J(y_*)] \\ J(y_* + tv) - J(y_*) &\leq t[J(y) - J(y_*)] \\ \frac{J(y_* + tv) - J(y_*)}{t} &\leq J(y) - J(y_*). \end{aligned}$$

Since $\delta J(y_*; v)$ exists and is equal to zero, we may let $t \to 0^+$ in the last inequality to conclude that

$$0 = \lim_{t \to 0^+} \frac{J(y_* + tv) - J(y_*)}{t} \leq J(y) - J(y_*).$$

and consequently $J(y) \geq J(y_*)$, i.e. J attains a minimum over $\mathscr{Y}$ at y_*. □

5.5 Convexity of Functionals Defined by Integrals

Let $a, b \in \mathbb{R}$ with $a < b$ be given. Put $\mathfrak{X} = C^1[a, b]$ and assume that $\mathscr{Y} \subset \mathfrak{X}$ is convex. Let a continuous function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given, and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

In order to apply Theorem 5.2 to J, we need to know when J is convex.

Given $y_1, y_2 \in \mathscr{Y}$, let us look at the quantity

$$J(ty_1 + (1-t)y_2) = \int_a^b f(x, ty_1(x) + (1-t)y_2(x), ty_1'(x) + (1-t)y_2'(x))\,dx$$

with $t \in [0,1]$. Suppose that $f(x,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is convex for each $x \in [a,b]$, i.e. that f is convex with respect to its last two arguments. Then

$$\begin{aligned}
&\int_a^b f(x, ty_1(x) + (1-t)y_2(x), ty_1'(x) + (1-t)y_2'(x))\,dx \\
\leq &\int_a^b \Big\{ tf(x, y_1(x), y_1'(x)) + (1-t)f(x, y_2(x), y_2'(x)) \Big\}\,dx \\
=&t\int_a^b f(x, y_1(x), y_1'(x))\,dx + (1-t)\int_a^b f(x, y_2(x), y_2'(x))\,dx \\
=&tJ(y_1) + (1-t)J(y_2).
\end{aligned}$$

We conclude that if the mapping $(y,z) \mapsto f(x,y,z)$ is convex for each $x \in [a,b]$, then

$$J(ty_1 + (1-t)y_2) \leq tJ(y_1) + (1-t)J(y_2) \quad \text{for all } t \in [0,1]$$

and J is convex. It is important to note that although convexity of $f(x,\cdot,\cdot)$ is sufficient for convexity of J, it is not necessary. (See Exercise .)

5.6 Second Derivative Test for Convexity

The definition of convexity, as well as the characterizations established so far, may be difficult to check directly. Recall that for functions of one real variable the second derivative test is a very convenient way to test for convexity. It is possible to define second-order Gateaux variations and use them to characterize convexity of "smooth" functionals on convex subsets of a general linear space. We shall not adopt this approach here. (However, readers are encouraged to try to define second-order Gateaux variations and explore their relationship with convexity.) Instead, we will develop a second derivative test for functions on $\mathbb{R}^n$ and use this test to show that integrands have sufficient convexity to render the functional J convex.

Let us first look at the situation in one-dimension. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. and let $x_1, x_2 \in \mathbb{R}$ be given. Taylor's Theorem tells us that there exists a point c_{x_1,x_2} between x_1 and x_2 such that

$$f(x_1) = f(x_2) + f'(x_2)(x_1 - x_2) + \frac{1}{2}f''(c_{x_1,x_2})(x_1 - x_2)^2.$$

If $f''(c_{x_1,x_2}) \geq 0$ for all $x \in \mathbb{R}$, we find that

$$f(x_1) \geq f(x_2) + f'(x_2)(x_1 - x_2).$$

and by Theorem 5.1, the function f is convex. One can also show that if f is convex then $f''(x) \geq 0$ for all $x \in \mathbb{R}$. In other words, for twice continuously differentiable functions, nonnegativity of the second derivative characterizes convexity.

Now we consider the more general situation when $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives. Define the Hessian matrix $H : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by

$$H(x) := \begin{pmatrix} f_{,1,1}(x) & f_{,1,2}(x) & \cdots & f_{,1,n}(x) \\ f_{,2,1}(x) & f_{,2,2}(x) & \cdots & f_{,2,1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_{,n,1}(x) & f_{,n,2}(x) & \cdots & f_{,n,n}(x) \end{pmatrix}.$$

Since f has continuous second-order partial derivatives, we see that

$$\big(H(x)\big)_{ij} = f_{,i,j}(x) = f_{,j,i}(x),$$

and the Hessian matrix is symmetric, i.e. $H(x)^\mathsf{T} = H(x)$ at each $x \in \mathbb{R}^n$. Using Taylor's Theorem, for each $x, y \in \mathbb{R}^n$ there exists some $c_{x,y}$ on the line segment joining x and y such that

$$f(x) = f(y) + \nabla f(y) \cdot (x - y) + \frac{1}{2}(x - y)H(c_{x,y})(x - y)^\mathsf{T}. \tag{5.10}$$

(The form of Taylor's theorem used above can be obtained by applying Taylor's theorem for functions of one variable to the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = f(tx + (1 - t)y)$.) We see that if the last term in (5.10) is always nonnegative, then Theorem 5.1 yields the convexity of f. Let us make a definition

Definition 5.4 *Let $A \in \mathbb{R}^{n\times n}$ be given. We say that A is* positive definite *if*

$$xAx^\mathsf{T} > 0 \quad \text{for all } x \in \mathbb{R}^n \backslash \{0\}.$$

We say that A is positive semidefinite *if*

$$xAx^\mathsf{T} \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

It follows from (5.10)that if the Hessian matrix $H(x)$ for f is positive semidefinite at every $x \in \mathbb{R}^n$, then f is convex. We now have a test for the convexity of f using the second-order partial derivatives of f.

We provide here an important characterization of positive definite and positive semidefinite symmetric matrices.

Theorem 5.3 *Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric. Then*

(i) A is positive semidefinite if and only if all the eigenvalues of A are nonnegative;

(ii) A is positive definite if and only if all the eigenvalues of A are strictly positive.

Proof.

□

We now give an alternative characterization of symmetric positive definite known as *Sylvester's Theorem*. Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric. Define $A_1 \in \mathbb{R}^{1\times 1}$, $A_2 \in \mathbb{R}^{2\times 2}$,..., $A_n \in \mathbb{R}^{n\times n}$ by

$$\begin{aligned}
A_1 &:=(A_{11}),\\
A_2 &:= \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}\\
A_i &:= \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1i} \\ A_{12} & A_{22} & \cdots & A_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1i} & A_{2i} & \cdots & A_{ii} \end{pmatrix} \qquad \text{for } i \in \{3, 4, \cdots, n-1\},
\end{aligned}$$

and

$$A_n := A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

Notice that we have used the symmetry of A for our definitions.

Theorem 5.4 *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and let $A_1, A_2, \cdots, A_n$ be as above. Then A is positive definite if and only if $\det A_i > 0$ for every $i \in \{1, 2, \cdots, n\}$.*

As an example, suppose that $n = 3$ so that

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}.$$

Then according to the theorem, the matrix A is positive definite if and only if $A_{11} > 0$, $A_{11}A_{22} - A_{12}^2 > 0$ and $\det A > 0$.

Extending the above characterization of positive definiteness to positive semidefiniteness is a bit delicate. One needs to consider additional subdeterminants of A. (See, for example,) A discussion of the general case is outside the scope of this course. However, the result for $n = 2$ is easy to state (and also to prove).

Remark 5.1 *Assume that $A \in \mathbb{R}^{2\times 2}$ is symmetric.*

(1) A is positive definite if and only if $A_{11} > 0$ and $A_{11}A_{22} - A_{12}^2 > 0$;

(2) A is positive semidefinite if and only if $A_{11} \geq 0$, $A_{22} \geq 0$ and $A_{11}A_{22} - A_{12}^2 \geq 0$.

5.7 Applying the Second Derivative Test to Calculus of Variations Problems

Let $a, b \in \mathbb{R}$ with $a < b$ be given. Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function with continuous second-order partial derivatives. Put $\mathfrak{X} := C^1[a, b]$ and assume that $\mathscr{Y} \subset \mathfrak{X}$ is convex. As usual, define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_a^b f(x, y(x), y'(x))\, dx \quad \text{for all } y \in \mathscr{Y}.$$

We have the following

Theorem 5.5 *Suppose at each $(x, y, z) \in [a, b] \times \mathbb{R} \times \mathbb{R}$ that f satisfies*

(a) $f_{,2,2}(x, y, z) \geq 0$,

(b) $f_{,3,3}(x, y, z) \geq 0$,

(c) $f_{,2,2}(x, y, z) f_{,3,3}(x, y, z) - [f_{,2,3}(x, y, z)]^2 \geq 0$.

Then, the functional J is convex.

Proof. The proof for this theorem follows the results stated in the previous section and our discussion in Section 5.5. □

5.7.1 Example 5.7.1 (cf. Examples 1.2, 3.1.1, 3.3.1 and 3.6.2)

Set

$$\mathscr{Y} := \left\{ y \in C^1[0, 1] \,:\, y(0) = 0 \text{ and } y(1) = 1 \right\},$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^1 \left[y(x)^2 + y'(x)^2 \right] dx \quad \text{for all } y \in \mathscr{Y}.$$

The integrand $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for J is given by

$$f(x, y, z) := y^2 + z^2 \quad \text{for all } x \in [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

We find that

$$f_{,2}(x, y, z) = 2y \text{ and } f_{,3} = 2z \quad \text{for all } (x, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$$

and that

$$f_{,2,2}(x, y, z) = 2; f_{,2,3}(x, y, z) = 0 \text{ and } f_{,3,3}(x, y, z) = 2 \quad \text{for all } (x, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}.$$

By Theorem 5.5, the functional J is convex. Therefore any $y \in \mathscr{Y}$ that satisfies

$$2y(x) = 2y''(x) \quad \text{for all } x \in [0, 1] \tag{E-L$_1$}$$

is a minimizer for J over the class $\mathscr{Y}$.

5.7.2 Example 5.7.2 (cf. Section 3.5)

In this section, we recall the problem of minimizing the transit time for a boat crossing a river with current. We found, in Section 3.5, that we needed to minimize the functional $T : \mathscr{Y} \to \mathbb{R}$ given by

$$T(y) := \frac{1}{\omega} \int_0^b \frac{\sqrt{1 - e(x)^2 + y'(x)^2} - e(x)y'(x)}{1 - e(x)^2}\, dx \quad y \in \mathscr{Y},$$

where

$$\mathscr{Y} := \left\{ y \in C^1[0,b] \,:\, y(0) = 0 \text{ and } y(b) = B \right\}.$$

In the definition of T, we have $\omega > 0$ and $0 \le e(x) < 1$ at each $x \in [0,b]$. The integrand $f : [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for T is given by

$$f(x,y,z) := \frac{1}{\omega\,(1 - e(x)^2)} \left[\sqrt{1 - e(x)^2 + z^2} - e(x)z \right] \quad \text{for all } (x,y,z) \in [0,b]\times\mathbb{R}\times\mathbb{R}.$$

So

$$f_{,2}(x,y,z) = 0 \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}$$

and

$$f_{,3}(x,y,z) = \frac{1}{\omega\,(1 - e(x)^2)} \left[\frac{z}{\sqrt{1 - e(x)^2 + z^2}} - e(x) \right] \quad \text{for all } (x,y,z) \in [0,b]\times\mathbb{R}\times\mathbb{R}.$$

Whence

$$f_{,2,2}(x,y,z) = f_{,2,3}(x,y,z) = 0 \quad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}.$$

and

$$\begin{aligned} f_{,3,3}(x,y,z) &= \frac{1}{\omega\,(1 - e(x)^2)} \left[\frac{\sqrt{1 - e(x)^2 + z^2} - \frac{z^2}{\sqrt{1 - e(x)^2 + z^2}}}{1 - e(x)^2 + z^2} \right] \\ &= \frac{1}{\omega\,(1 - e(x)^2)} \frac{1 - e(x)^2}{(1 - e(x)^2 + z^2)^{\frac{3}{2}}} \\ &= \frac{1}{\omega} \frac{1}{(1 - e(x)^2 + z^2)^{\frac{3}{2}}} \qquad \text{for all } (x,y,z) \in [0,b] \times \mathbb{R} \times \mathbb{R}. \end{aligned}$$

Since $\omega > 0$, Theorem 5.5 implies that J is convex. It follows that any $y \in \mathscr{Y}$ that satisfies the first Euler-Lagrange equation for T is a minimizer for T over $\mathscr{Y}$.

5.7.3 Example 5.7.3

For this example, we put

$$\mathscr{Y} := \left\{ y \in C^1[0,\pi] \mid y(0) = y(\pi) = 0 \right\}$$

and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_0^\pi \left[y'(x)^2 - y(x)^2 \right] \, dx \quad \text{for all } y \in \mathscr{Y}.$$

For each $n \in \mathbb{N}$ put $y_n(x) := \sin nx$ for $x \in [0,\pi]$. Then, we compute that

$$J(y_n) = \int_0^\pi \left[n^2 \cos^2 nx - \sin^2 nx \right] \, dx = \left(n^2 - 1 \right) \frac{\pi}{2} \quad \text{for all } n \in \mathbb{N}.$$

Thus J has no maximum value over $\mathscr{Y}$.

Let us see if we can use Theorem 5.5 to determine whether J is convex or not. The integrand $f : [0,\pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for J is

$$f(x,y,z) = z^2 - y^2 \quad \text{for all } (x,y,z) \in [0,\pi] \times \mathbb{R} \times \mathbb{R}.$$

Thus

$$f_{,2,2}(x,y,z) = -2 \quad \text{for all } (x,y,z) \in [0,\pi] \times \mathbb{R} \times \mathbb{R},$$

and we cannot use Theorem 5.5 to conclude that J is convex.

5.8 Convexity and the Lagrange Multiplier Method

Let $\mathfrak{X}$ be a real linear space and let $\mathscr{Y} \subset \mathfrak{X}$ be convex. Assume that $\mathscr{V}$ is a subspace of $\mathfrak{X}$ such that

$$y - z \in \mathscr{V} \text{ and } y + v \in \mathscr{Y} \quad \text{for all } y, z \in \mathscr{Y} \text{ and } v \in \mathscr{V}.$$

Let $J, G : \mathscr{Y} \to \mathbb{R}$ and $c \in \mathbb{R}$ be given. Put

$$\mathscr{S} := \{ y \in \mathscr{Y} \mid G(y) = c \}.$$

Theorem 5.6 *Let $y_* \in \mathscr{S}$ and $\lambda \in \mathbb{R}$ be given. Assume that the functional $(J - \lambda G)$ is convex and that $\delta J(y_*; v)$ and $\delta G(y_*; v)$ exist for all $v \in \mathscr{V}$. Assume further that*

$$\delta J(y_*; v) = \lambda G(y_*; v) \quad \text{for all } v \in \mathscr{V}.$$

Then y_* minimizes J over $\mathscr{S}$.

Proof. Define $H : \mathscr{Y} \to \mathbb{R}$ by

$$H(y) := J(y) - \lambda G(y) \quad \text{for all } y \in \mathscr{Y}.$$

By assumption, the functional H is convex over $\mathscr{Y}$ and $\delta H(y_*; v) = 0$ for every $v \in \mathscr{V}$. From our discussion in Section 5.5, we conclude that $H(w) \geq H(y_*)$ for every $w \in \mathscr{Y}$. Now let $y \in \mathscr{S}$ be given. Since $y \in \mathscr{S} \subset \mathscr{Y}$, we have $H(y) \geq H(y_*)$. Thus

$$\begin{aligned} J(y) - \lambda G(y) \geq J(y_*) - \lambda G(y_*) &\Rightarrow J(y) - \lambda c \geq J(y_*) - \lambda c \\ &\Rightarrow J(y) \geq J(y_*). \end{aligned}$$

We have shown that y_* minimizes J over $\mathscr{S}$. □

5.8.1 Example 5.8.1 (cf. Section 4.5)

Set

$$\mathscr{Y} := \left\{ y \in C^1[-1,1] \, : \, y(-1) = y(1) = 0 \right\}.$$

Define $J, G : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) := \int_{-1}^{1} y(x)\, dx \text{ and } G(y) := \int_{-1}^{1} \sqrt{1 + y'(x)^2}\, dx \quad \text{for all } y \in \mathscr{Y},$$

and put

$$\mathscr{S} := \{ y \in \mathscr{Y} \mid G(y) = l \},$$

where $l > 2$. The augmented integrand $L : (x, y, z, \lambda) : [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $(J - \lambda G)$ is

$$L(x, y, z, \lambda) := y - \lambda \sqrt{1 + z^2} \quad \text{for all } (x, y, z, \lambda) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

We find that

$$L_{,2,2}(x, y, z, \lambda) = L_{,2,3}(x, y, z, \lambda) = 0 \quad \text{for all } (x, y, z, \lambda) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

and

$$L_{3,3}(x, y, z, \lambda) = -\frac{\lambda}{(1 + z^2)^{\frac{3}{2}}} \quad \text{for all } (x, y, z, \lambda) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

So the convexity of L with respect to the y and z arguments depends on the sign of the Lagrange multiplier λ. Thus the convexity of the functional $J - \lambda G$ depends on the sign of λ. We conclude that

(1) if $\lambda < 0$, then the functional $(J - \lambda G)$ is convex and a $y \in \mathscr{S}$ satisfying (E-L)_1 for $(J - \lambda G)$ is a minimizer for J over the class $\mathscr{S}$;

(2) if $\lambda > 0$, then the functional $-(J - \lambda G)$ is convex and a $y \in \mathscr{S}$ satisfying (E-L)_1 for $(J - \lambda G)$ is a maximizer for J over the class $\mathscr{S}$.