21-238, Math Studies Algebra 2, Department of Mathematical Sciences, Carnegie Mellon University
Spring 2012: Monday, Wednesday, Friday, 10:30 am, Doherty Hall 1211.
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Assignment 5 - Wednesday April 25, 2012. Due Wednesday May 2

Exercise 21: Let F be a finite dimensional Galois extension of E, and let K be an intermediate field. Show that there is a smallest field L such that $K \subset L \subset F$ and L is a Galois extension of E, characterized by $Aut_L(F) = \bigcap_{\sigma \in Aut_E(F)} \sigma \big(Aut_K(F)\big)\sigma^{-1}$.

Exercise 22: Let E be a field, $P \in E[x]$, and F a splitting field extension for P over E. One assumes that P splits in F as $(x-u_1)^{m_1} \cdots (x-u_k)^{m_k}$ (with $u_1, \ldots, u_k$ distinct and $m_1, \ldots, m_k \geq 1$). Let $v_0, v_1, \ldots, v_k \in F$ be the coefficients of $Q = (x-u_1) \cdots (x-u_k) \in F[x]$ (so that $v_k = 1$), and let $K = E(v_0, \ldots, v_k)$.
i) Show that F is a splitting field extension for Q over K.
ii) Show that F is a Galois extension of K.
iii) Show that $Aut_K(F) = Aut_E(F)$.

Exercise 23: Let $E \subset \mathbb{C}$ be the field generated by $\mathbb{Q}$ and $\{\sqrt{b} \mid b \in \mathbb{Q}\} \subset \mathbb{C}$, i.e. a splitting field extension for the set $S = \{x^2 + a \mid a \in \mathbb{Q}\} \subset \mathbb{Q}[x]$.
i) Show that $E = \mathbb{Q}(A)$ where $A = \{\sqrt{p} \mid p = -1 \text{ or } p \text{ is a prime integer}\}$.
ii) If $\sigma \in Aut_{\mathbb{Q}}(E)$, show that $\sigma^2 = id_E$. Show that every group G such that $g^2 = e$ for all $g \in G$ is Abelian.
iii) Show that for every subset $B \subset A$, there exists $\sigma \in Aut_{\mathbb{Q}}(E)$ such that $\sigma(\sqrt{p}) = \sqrt{p}$ for all $p \in B$ and $\sigma(\sqrt{p}) = -\sqrt{p}$ for all $p \in A \setminus B$.

Exercise 24: Notation of Exercise 23.
i) Show that $Aut_{\mathbb{Q}}(E)$ is uncountable and has an uncountable number of subgroups of index 2.
ii) Show that the set of extension fields of $\mathbb{Q}$ included in E and having dimension 2 over $\mathbb{Q}$ is countable.
iii) Show that $[E\!:\!\mathbb{Q}] \leq \aleph_0$, so that $[E\!:\!\mathbb{Q}] < |Aut_{\mathbb{Q}}(E)|$.

Exercise 25: Let F be a finite extension of E.
i) Show that there exists a finite field extension G of F such that G is a normal field extension of E, and no proper subfield of G containing F is a normal field extension of E.
ii) Show that if F is separable over E, then G is a Galois extension of E.