Midterm 1

21-630 Ordinary Differential Equations
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Problem 1

$\forall t \geq 0$, define $F_t : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$F_t \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} X_1(t,x,v) \\ X_2(t,x,v) \\ V_1(t,x,v) \\ V_2(t,x,v) \end{pmatrix} = \begin{pmatrix} x_1 + \int_0^t V_1(s,x,v)\,ds \\ x_2 + \int_0^t V_2(s,x,v)\,ds \\ v_1 + \int_0^t E_1(s,X(s,x,v))\,ds \\ v_2 + \int_0^t E_2(s,X(s,x,v))\,ds \end{pmatrix},$$

so that $J(t)$ is the Jacobian of F_t. Thus, for $t = 0$,

$$F_t \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}, \text{ so that } \det(J(t)) = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

I wasn't sure where to go from here; since E, and hence the system, are not necessarily linear, we can't use the Abel/Liouville Theorem or similar results to show that $\det(J(t))$ is constant.

Problem 2

Let $T = (4|x_0|)^{-2} > 0$, $B = [-2|x_0|, 2|x_0|]$, $\mathcal{C}_B := \{X : [0,T] \to B : X \text{ is continuous}\}$, and $\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_B$ defined by

$$\mathcal{F}(X)(t) = x_0 + \int_0^{t/2} 2\cos(t)X^3(s)\,ds + \int_0^t X^3(s)\,ds, \quad \forall X \in \mathcal{C}_B, t \in [0,T].$$

We show that the range of $\mathcal{F}$ is indeed contained in $\mathcal{C}_B$ and then that $\mathcal{F}$ is a contraction, so that, by the Contraction Mapping Theorem, $\mathcal{F}$ has a fixed point (which clearly has the desired properties).

Suppose that $X \in \mathcal{C}_B$. Then, $\forall t \in [0,T]$,

$$\begin{aligned} |\mathcal{F}(X)(t)| &\leq |x_0| + \int_0^{t/2} |2\cos(t)X^3(s)|\,ds + \int_0^t |X^3(s)|\,ds \\ &\leq |x_0| + 2\int_0^{t/2} |X^3(s)|\,ds + \int_0^t |X^3(s)|\,ds \\ &\leq |x_0| + 2\int_0^{T/2} 8|x_0|^3\,ds + \int_0^T 8|x_0|^3\,ds \\ &= |x_0| + 8\frac{|x_0|^3}{(4|x_0|)^2} + 8\frac{|x_0|^3}{(4|x_0|)^2} = |x_0| + \frac{1}{2}|x_0| + \frac{1}{2}|x_0| = 2|x_0|, \end{aligned}$$

so that $\|\mathcal{F}(X)\| \leq 2|x_0|$ and thus $\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_B$. Suppose now that $X, Y \in \mathcal{C}_B$. Then, $\forall t \in [0, T]$,

$$\begin{aligned}
|\mathcal{F}(X)(t) - \mathcal{F}(Y)(t)| &\leq \int_0^{t/2} |2\cos(t)||X^3(s) - Y^3(s)|\,ds + \int_0^t |X^3(s) - Y^3(s)|\,ds \\
&\leq 2\int_0^{t/2} |X^3(s) - Y^3(s)|\,ds + \int_0^t |X^3(s) - Y^3(s)|\,ds \\
&= 2\int_0^{t/2} |X^2(s) + XY(s) + Y^2(s)||X(s) - Y(s)|\,ds \\
&+ \int_0^t |X^2(s) + XY(s) + Y^2(s)||X(s) - Y(s)|\,ds \quad \text{(difference of cubes)} \\
&\leq 2\int_0^{T/2} 3|x_0|^2\|X - Y\|\,ds + \int_0^T 3|x_0|^2\|X - Y\|\,ds \\
&\leq 3\frac{|x_0|^2\|X - Y\|}{(4|x_0|)^2} + 3\frac{|x_0|^2\|X - Y\|}{(4|x_0|)^2} = \frac{3}{16}\|X - Y\|,
\end{aligned}$$

so that $\|\mathcal{F}(X) - \mathcal{F}(Y)\| \leq \frac{3}{16}\|X - Y\|$, and thus $\mathcal{F}$ is a contraction. ■

Problem 3

Since f is bounded, $\exists M_f > 0$ with $|f| \leq M_f$ on $\mathbb{R}$. Note first that, for all $X : [0, T] \to \mathbb{R}$ satisfying

$$X(t) = x_0 + \int_0^{\min(t+\delta,T)} f(X(s))\,ds \quad \forall t \in [0, T], \tag{1}$$

$$|X(t) - x_0| \leq \int_0^{\min(t+\delta,T)} M_f\,ds \leq \int_0^T M_f\,ds = M_f T.$$

Since f' is continuous and $I := [x_0 - M_f T, x_0 + M_f T]$ is compact, $\exists M_{f'} > 0$ with $|f'| \leq M_{f'}$ on I. Thus, $\forall t \in [0, T]$, for X, Y satisfying Equation (1), by the Mean Value Theorem, $\exists c \in I$ with

$$|f(X(t)) - f(Y(t))| = |f'(c)||X(t) - Y(t)| \leq M_{f'}|X(t) - Y(t)|.$$

Thus,

$$\begin{aligned}
|X(t) - Y(t)| &\leq \int_0^{\min(t+\delta,T)} |f(X(s)) - f(Y(s))|\,ds \\
&\leq \int_0^{\min(t+\delta,T)} M_{f'}|X(s) - Y(s)|\,ds.
\end{aligned}$$

This is close, but not quite sufficient to use Gronwall to get the desired result…

Problem 4

A) Since f is continuous and $D := [t_0, t_1] \times [-C, C]^N$ is compact, for some $M \in \mathbb{R}$, $|f| \leq M$ on D. Note that C bounds $\{X^{(k)}\}$ uniformly on $[t_0, t_1]$. Furthermore, letting $\varepsilon > 0$, for $\delta := \frac{\varepsilon}{M}$, $\forall k \in \mathbb{N}, s, t \in [t_0, t_1]$ with $|t - s| < \delta$, using the integral equation form of (ODE),

$$|X^{(k)}(t) - X^{(k)}(s)| \leq \int_s^t |f(u, X^{(k)}(u))| \, du \leq \int_s^t M \, du = M(t - s) < \varepsilon.$$

Thus, $\{X^{(k)}\}$ is equicontinuous on $[t_0, t_1]$, so, by the Ascoli-Arzela Theorem, some subsequence $\{X^{(n_k)}\}$ converges uniformly to some continuous $X : [t_0, t_1] \to \mathbb{R}^N$.

Since f is continuous, and $D := [t_0, t_1] \times [-C, C]$ is compact, f is uniformly continuous on D, so that the sequence $\{t \mapsto f(t, X^{(n_k)}(t))\}$ converges uniformly to the function $t \mapsto f(t, X(t))$, allowing us to pass the limit into the integral in the following equality:

$$\begin{aligned}
X(t) = \lim_{k \to \infty} X^{(n_k)}(t) &= \lim_{k \to \infty} \left(X^{(n_k)}(t_0) + \int_{t_0}^t f(s, X^{(n_k)}(s)) \, ds \right) \\
&= X(t_0) + \int_{t_0}^t \lim_{k \to \infty} f(s, X^{(n_k)}(s)) \, ds = X(t_0) + \int_{t_0}^t f(s, X(s)) \, ds,
\end{aligned}$$

$\forall t \in [t_0, t_1]$, so that X is a solution to (ODE). ■

B) Let $\varepsilon > 0$. Since, f is continuous on the compact set $D := [t_0, t_1] \times [-C, C]$, f is uniformly continuous on D. Thus, $\exists \delta > 0$, such that, $\forall x_1, x_2 \in [-C, C]$

$$|x_1 - x_2| < \delta \quad \text{implies} \quad |f(t, x_1) - f(t, x_2)| < \frac{\varepsilon}{2(t_1 - t_0)}.$$

From part A and the uniqueness of solutions with the same initial conditions, we have a subsequence $\{X^{(n_k)}\}$ converging uniformly to X. Thus, we can take $n \in \mathbb{N}$ such that, $\forall t \in [t_0, t_1], |X^{(n)}(t) - X(t)| < \delta$, and then, since $X^{(k)}(t_0) \to X(t_0)$ as $k \to \infty$, we can take $k_0 \in \mathbb{N}$ such that, $\forall k \geq k_0$, $|X^{(k)}(t_0) - X(t_0)| < |X^{(n)}(t_0) - X(t_0)|$.

Then, $\forall t \in [t_0, t_1]$,

$$\begin{aligned}
|X^{(k)}(t) - X(t)| &\leq |X^{(k)}(t_0) - X(t_0)| + \int_{t_0}^t |f(s, X^{(k)}(s)) - f(s, X(s))| \, ds \\
&\leq |X^{(k)}(t_0) - X(t_0)| + \int_{t_0}^t |f(s, X^{(n)}(s)) - f(s, X(s))| \, ds \\
&< \frac{\varepsilon}{2} + \int_{t_0}^t \frac{\varepsilon}{2(t_1 - t_0)} \, ds = \frac{\varepsilon}{2} + \frac{\varepsilon(t - t_0)}{2(t_1 - t_0)} \, ds = \frac{\varepsilon}{2} + \frac{\varepsilon(t - t_0)}{2(t_1 - t_0)} \, ds \leq \varepsilon,
\end{aligned}$$

so that $X^{(k)} \to X$ uniformly as $k \to \infty$. ■