Homework 5

21-236 Mathematical Studies Analysis II
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Problem 1

Let $f : [a, b] \to \mathbb{R}$.

(a) $\forall \delta > 0$, let $\mathcal{I}_\delta$ denote the set of finite sets of nonoverlapping intervals $(a_k, b_k) \subseteq [a, b]$ such that

$$\sum_{k=1}^{l} (b_k - a_k) \leq \delta.$$

Suppose that f belongs to $AC([a, b])$, and let $\epsilon > 0$. By definition of Absolute Continuity, $\exists \delta > 0$, such that, $\forall \{(a_1, b_1), \dots, (a_l, b_l)\} \in \mathcal{I}_\delta$,

$$\sum_{k=1}^{l} |f(b_k) - f(a_k)| \leq \epsilon,$$

Then, for this choice of δ, by the Triangle Inequality,

$$\left| \sum_{k=1}^{l} f(b_k) - f(a_k) \right| \leq \sum_{k=1}^{l} |f(b_k) - f(a_k)| \leq \epsilon$$

$\forall \{(a_1, b_1), \dots, (a_l, b_l)\} \in \mathcal{I}_\delta$, proving the desired condition. ■

Suppose, on the other hand, that, $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$\left| \sum_{k=1}^{l} f(b_k) - f(a_k) \right| \leq \epsilon$$

$\forall \{(a_1, b_1), \dots, (a_l, b_l)\} \in \mathcal{I}_\delta$. Let $\epsilon > 0$, and let δ be such that

$$\left| \sum_{k=1}^{l} f(b_k) - f(a_k) \right| \leq \frac{\epsilon}{2},$$

$\forall \{(a_1, b_1), \dots, (a_l, b_l)\} \in \mathcal{I}_\delta$.

Consider any set $S := \{(a_1, b_1), \dots, (a_l, b_l)\} \in \mathcal{I}_\delta$. Partition S into P and N, where P contains those intervals $(a, b) \in S$ with $f(b) - f(a) \geq 0$, and N contains those intervals $(a, b) \in S$ with

$f(b) - f(a) < 0$. Then, $P, N \in \mathcal{I}_\delta$, so that, by choice of δ,

$$\begin{aligned}
\sum_{(a,b)\in S} |f(b) - f(a)| &= \left(\sum_{(a,b)\in P} |f(b) - f(a)|\right) + \left(\sum_{(a,b)\in N} |f(b) - f(a)|\right) \\
&= \left|\sum_{(a,b)\in P} f(b) - f(a)\right| + \left|\sum_{(a,b)\in N} f(b) - f(a)\right| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

so that f belongs to $AC([a,b])$. ■

(b) Let $\epsilon = 1$, and let δ be such that

$$\left|\sum_{k=1}^{l} f(b_k) - f(a_k)\right| \epsilon$$

for every finite number of intervals (a_k, b_k), $k \in \{1, \ldots, l\}$, with $[a_k, b_k] \subseteq [a, b]$ and

$$\sum_{k=1}^{l} (b_k - a_k) \leq \delta.$$

Let $x, y \in [a, b]$ (without loss of generality, $x \leq y$.), and let $n = \lfloor \frac{y-x}{\delta} \rfloor$. Let $\forall i \in \{0, 1, \ldots, n\}$, let $a_i = x + i\delta$. Then,

$$|f(y) - f(x)| = \left|f(y) - f(a_n) + \sum_{i=2}^{n} f(a_i) - f(a_{i-1})\right| = 1 + N - 1 = N \leq \frac{1}{\delta}|y - x|,$$

so that f is Lipschitz continuous with Lipschitz constant $1/\delta$. ■

Problem 2

Define $C = \mathbb{R}^2 \backslash U$ (noting that, since U is open, C is closed), and let $f : \mathbb{R}^2 \to [0, \infty)$ be the distance function from C (as defined in Assignment 2). Since K is compact, and, as shown in Assignment 2, f is Lipschitz continuous and therefore continuous, by the Weierstrass Theorem, f achieves a minimum on K at some $\mathbf{x} \in K$. Since $K \subseteq U$ and U is open, for some $r > 0$, $B(\mathbf{x}, r) \subseteq U$, so that $f(\mathbf{x}) \geq r$. Therefore, for $t = \inf_{(\mathbf{x},\mathbf{y}) \in K \times C} \|\mathbf{x} - \mathbf{y}\|$, $t > 0$.

Let $R \subseteq \mathbb{R}^2$ be a rectangle with $K \subseteq R$ (such a rectangle exists because K is compact). Cut R into a grid of closed squares $S_1, S_2, \dots, S_k$ of side length $s := \frac{t}{2}$ (we can assume s divides the lengths of R because we can make R larger if we wish). Then, by choice of s, for each S_i, $S_i \cap K = \emptyset$ or $S_i \cap C = \emptyset$; let S be the set of S_i such that $S_i \cap K \neq \emptyset$ (so that $S_i \cap C = \emptyset$). Orient each square counterclockwise. For each $S_i \in S$, consider the four curves comprising its edges. Call these curves $\gamma_1, \gamma_2, \dots, \gamma_n$.

Note the following:

1. If $\mathbf{x} \in K$, the winding number around $\mathbf{x}$ of the union of the curves comprising the edges of the square including $\mathbf{x}$ is 1.
2. For every γ_i whose range is entirely in K, there is a corresponding γ_j which is the same curve under the opposite orientation.
3. If two curves are identical but have oppositve orientation, then the sum of their winding numbers around any point is 0.

Let G be the set of γ_i's whose range is in $U \backslash K$, and let γ be the union of the curves in G. Using induction on the number of squares surrounded by γ and the above three observations, it can be shown that, $\forall x \in K$, $\text{ind}_\gamma(\mathbf{x}) = 1$.

Furthermore, the range of γ is in $U := \bigcup_{S_i \in S} S_i$. Since U is bounded, $\mathbb{R}^2 \backslash U$ is unbounded, so that, since $C \subseteq \mathbb{R}^2 \backslash U$, by Theorem 155, since the range of γ is in U, $\text{ind}_\gamma(C) = \{0\}$. Therefore, γ is a curve with the desired properties $\blacksquare$.

Problem 3

(a) Define $E := \mathbb{R}^2 \backslash ([1, \infty) \times [-1, 1])$. Then, E is simply connected. Let γ be a continuous closed curve with range $\Gamma \subseteq E$, and let $\phi : [a, b] \to \mathbb{R}^2$ be a parametric representation of γ (with components ϕ_1, ϕ_2).

Define $\mathbf{h} : [a, b] \times [0, 1] \to \mathbb{R}^2$ such that, $\forall s \in [a, b], \forall t \in [0, 1]$,

$$\mathbf{h}(s,t) = \begin{cases} \begin{bmatrix} \phi_1(s) - 2t\phi_1(s) \\ \phi_2(s) \end{bmatrix} & \text{when } t \in [0, \frac{1}{2}] \\ \begin{bmatrix} 0 \\ \phi_2(s) - (2t-1)\phi(s) \end{bmatrix} & \text{when } t \in [\frac{1}{2}, 1] \end{cases}.$$

Then, $\forall s \in [a, b]$, $\mathbf{h}(s, 0) = \phi(s)$ and $\mathbf{h}(s, 1) = \mathbf{0}$, and, since $\phi(a) = \phi(b)$, $\forall t \in [0, 1]$, $\mathbf{h}(a, t) = \mathbf{v}(b, t)$. Note that $\mathbf{h}([a, b] \times [\frac{1}{2}, 1]) = \{(0, y) : y \in \mathbb{R}\}$. Thus, if there exists $(s, t) \in [a, b] \times [0, 1]$ such that $\mathbf{h}(s, t) \notin E$, $t \in [0, \frac{1}{2})$. However if this were the case, then, $\phi(s) = \mathbf{h}(s, 0) =\notin E$, which is impossible, since $\phi([a, b]) \subset E$. Finally, it is easily checked that $\mathbf{h}$ is linear in t when t is restricted to $[0, \frac{1}{2}]$ and when t and restricted to $[\frac{1}{2}, 1]$, so that, because the domain and range of $\mathbf{h}$ have finite dimension, and ϕ is continuous, $\mathbf{h}$ is continuous on its domain.

Therefore, $\mathbf{h}$ is a homotopy from ϕ to the origin, so that ϕ is homotopic to a point, and therefore E is simply connected. $\blacksquare$

(b) Define $E := \mathbb{R}^3 \backslash \{(0, 0, 0)\}$. Then, E is simply connected. Let γ be a continuous closed curve with range $\Gamma \subseteq E$, and let $\phi : [a, b] \to \mathbb{R}^2$ be a parametric representation of γ (with components ϕ_1, ϕ_2, ϕ_3).

Let $\mathbf{h}_1 : [a, b] \times [0, 1] \to \mathbb{R}^3$ such that, $\forall (s, t) \in [a, b] \times [0, 1]$, if (r, θ, z) is the 'cylindrical' representation of $\phi(s)$ (i.e., $\phi(s) = (r \cos\theta, r \sin\theta, z)$, where $\theta \in [0, 2\pi)$),

$$\mathbf{h}_1(s,t) = \begin{bmatrix} (r + t(1-r))\cos\theta \\ (r + t(1-r))\sin\theta \\ z + t(1-z) \end{bmatrix}.$$

Let $\mathbf{h_2} : [a, b] \times [0, 1] \to \mathbb{R}^3$ such that, $\forall (s, t) \in [a, b] \times [0, 1]$,

$$\mathbf{h}_2(s,t) = \begin{bmatrix} \phi_1(s) - t\phi_1(s) \\ \phi_2(s) - t\phi_2(s) \\ 1 \end{bmatrix}.$$

Define $\mathbf{h} : [a, b] \times [0, 1] \to \mathbb{R}^2$ such that, $\forall s \in [a, b], \forall t \in [0, 1]$,

$$\mathbf{h}(s,t) = \begin{cases} \mathbf{h}_1(s, 2t) & \text{when } t \in [0, \frac{1}{2}] \\ \mathbf{h}_2(s, 2t-1) & \text{when } t \in [\frac{1}{2}, 1] \end{cases}.$$

Then, $\forall s \in [a, b]$, $\mathbf{h}(s, 0) = \phi(s)$ and $\mathbf{h}(s, 1) = (0, 0, 1)$, and, $\forall t \in [0, 1]$, $\mathbf{h}(a, t) = \mathbf{h}(b, t)$. Since each of $\mathbf{h}_1$ and $\mathbf{h_2}$ is linear on its domain, and $\forall s \in [a, b]$, $\mathbf{h}_1(s, \frac{1}{2}) = \mathbf{h}_2(s, \frac{1}{2})$, and ϕ is continuous on $[a, b]$, $\mathbf{h}$ is continuous on its domain. Finally, since $\mathbf{h}_1([a, b] \times [0, 1]), \mathbf{h}_2([a, b] \times [0, 1]) \subseteq E$, $\mathbf{h}([a, b] \times [0, 1]) \subseteq E$. Therefore, $\mathbf{h}$ is a homotopy from ϕ to a point, so that E is simply connected. $\blacksquare$

(c) $\mathbb{R}^3 \backslash$line is not simply connected. First, consider $E := \mathbb{R}^3 \backslash \{(0,0,z) : z \in \mathbb{R}\}$. Let $\mathbf{g} : E \to \mathbb{R}^3$ such that, $\forall (x,y,z) \in E$,

$$g(x,y,z) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right).$$

As shown in Example 133,

$$\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x}.$$

Furthermore,

$$\frac{\partial g_1}{\partial z} = \frac{\partial g_2}{\partial z} = \frac{\partial g_3}{\partial x} = \frac{\partial g_3}{\partial y} = 0,$$

so that $\mathbf{g}$ is irrotational.

Let γ be the closed curve with parametric representation $\phi : [0, 2\pi] \to \mathbb{R}^3$ such that, $\forall t \in [0, 2\pi]$, $\phi(t) = (\cos t, \sin t, 0)$ (i.e., γ is the unit circle in the xy-plane at the origin), so that the range of γ is in E. Since $g_3 = 0$, $\int_\gamma \mathbf{g}$ is the same as the integral computed in Example 133, so that

$$\int_\gamma \mathbf{g} = 2\pi \neq 0,$$

and thus $\mathbf{g}$ is not conservative.

Since E is open, by Theorem 144 (Poincare's Lemma), if E were simply connected, then $\mathbf{g}$ would have to be conservative, so that E cannot be simply connected. A similar proof, using an appropriately re-oriented field $\mathbf{g}$ and unit circle γ, suffices for proving that $\mathbb{R}^3$ minus *any* line is not simply connected. $\blacksquare$.

Problem 4

Note first that, since $\log(a)$ is defined if and only if $a > 0$, we are concerned only with the domain $E := \{(x, y) \in \mathbb{R} : x, y > 0\}$.

Define $\mathbf{g} : E \to \mathbb{R}^2$ and $h : E \to \mathbb{R}$ such that, $\forall (x, y) \in E$, $\mathbf{g}(x, y) = (2\ln(x, y) + 1, x/y)$ and $h(x, y) = x$. Then, $\forall x, y \in E$,

$$\frac{\partial(hg)_1}{\partial y} = \frac{2x}{y} = \frac{\partial(hg)_2}{\partial x},$$

so that $h\mathbf{g}$ is irrotational. Since E is open and simply connected, and $h\mathbf{g}$ is irrotational, by Theorem 144 (Poincare's Lemma), $h\mathbf{g}$ is conservative, so that, for some $f : E \to \mathbb{R}$, $h\mathbf{g} = \nabla f$. By the given differential equation, then,

$$y' = -\frac{h(x, y(x))g_1(x, y(x))}{h(x, y(x))g_2(x, y(x))} = \frac{\frac{\partial f}{\partial x}(x, y(x))}{\frac{\partial f}{\partial y}(x, y(x))},$$

and, consequently,

$$\frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))y'(x) = 0.$$

Then, if $F := (x \mapsto f(x, y(x)))$, by the Chain Rule, $\forall x \in \mathbb{R}$,

$$F'(x) = \frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))y'(x) = 0,$$

so that F is constant on every connected component of its domain ($\{x \in \mathbb{R} : x > 0\}$).

Integrating f with respect to x and with respect to y gives

$$f(x, y) = \int h(x, y(x))g_1(x, y(x))\, dx = x^2 \ln(xy) + c_x$$

and

$$f(x, y) = \int h(x, y(x))g_2(x, y(x))\, dx = x^2 \ln(y) + c_y,$$

where c_x is constant with respect to x and c_y is constant with respect to y. Then, however,

$$0 = x^2 \ln(xy) + c_x - (x^2 \ln(y) + c_y) = x^2 \ln(x) + c_x - c_y,$$

so that c_x is independent of y, and therefore, $f(x, y) = x^2 \ln(xy) + c$, where c is constant with respect to both x and y. Then,

$$0 = \frac{\partial F}{\partial y}(x) = \frac{\partial}{\partial y}(x^2 \ln(xy) + c) = x^2 y'/y.$$

Since $x, y \neq 0$, $y' = 0$, so that

$$-\frac{2\ln(xy) + 1}{\frac{x}{y}} = 0,$$

and, consequently, $\ln(xy) + 1 = 0$. Solving for y in terms of x then gives

$$\boxed{y = \frac{e^{-1/2}}{x}.}$$