Homework 7b

21-260 Differential Equations
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Section 7.3, Problem 8

The vectors are linearly dependent:

$$-2\mathbf{x}^{(3)} + 5\mathbf{x}^{(2)} = \mathbf{x}^{(1)}.$$

Section 7.3, Problem 22

If A is the given matrix, then characteristic polynomial of A (computed by cofactor expansion about the first row) is

$$|A - \lambda I| = (1-\lambda)((1-\lambda)^2 + 4) = (1-\lambda)(\lambda^2 - 2\lambda + 5).$$

Aside from the obvious root $\lambda = 1$, the quadratic formula gives the two roots

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = 1 \pm 2i,$$

so that the eigenvalues of A are

$$\lambda \in \boxed{\{1, 1+2i, 1-2i\}.}$$

Solving for the eigenvector $\mathbf{x}_1$ associated with the eigenvalue $\lambda = 1$ gives the system of equations (the first line does not constrain $\mathbf{x}_1$)

$$\begin{aligned} x_1 &= x_1 \\ 2x_1 + x_2 - 2x_3 &= x_2 \\ 3x_1 + 2x_2 + x_3 &= x_3, \end{aligned}$$

whose solutions are multiples the first eigenvector,

$$\boxed{\mathbf{x}_1 = \begin{bmatrix} \frac{2}{3} \\ -1 \\ \frac{2}{3} \end{bmatrix}.}$$

Solving for the eigenvectors $\mathbf{x}_2$ and $\mathbf{x}_3$ associated with the eigenvalues $\lambda = 1 + 2i$ and $\lambda = 1 - 2_i$, respectively, gives the systems of equations

$$\begin{aligned} x_1 &= (1 \pm 2i)x_1 \\ 2x_1 + x_2 - 2x_3 &= (1 \pm 2i)x_2 \\ 3x_1 + 2x_2 + x_3 &= (1 \pm 2i)x_3, \end{aligned}$$

which have the solutions which are multiples of

$$\mathbf{x}_2 = \boxed{\begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}}, \quad \mathbf{x}_3 = \boxed{\begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}}.$$

Section 7.3, Problem 31

The cleanest proof is immediate from the fact that the determinant of a matrix is the product of its eigenvalues, but this fact doesn't appear to be at our disposal.

It suffices however to observe that, if a square matrix A with 0 as an eigenvalue were invertible, then, for some associated non-zero eigenvector $\mathbf{x}$,

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A(0 \cdot \mathbf{x}) = A\mathbf{0} = \mathbf{0},$$

which is impossible. Therefore, any matrix with 0 as an eigenvalue is not invertible, so that it has determinant 0. $\blacksquare$

Section 7.4, Problem 6

(a) By definition of the Wronskian,

$$W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right] = \left|\begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix}\right| = 2t^2 - t^2 = \boxed{t^2.}$$

(b) Since the 2×2 matrix with columns $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ as columns has determinant 0 if and only if its columns are linearly dependent and its determinant is the Wronskian computed in part (a), $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent if and only if $t^2 \neq 0$. Thus, the solutions are linearly independent on $\boxed{(-\infty, 0)}$ and on $\boxed{(0, \infty).}$

(c) Since $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent on the intevals given in part (b), by Theorem 7.4.2, all solutions of the set of homogeneous differential equations satisfied by $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are of the form

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t),$$

for some $c_1, c_2 \in \mathbb{R}$ (i.e., they are linear combinations of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$).

(d) The system of equations is

$$\mathbf{x}' = \mathbf{x}_2$$

and

Section 7.5, Problem 12

If A is the matrix such that $\mathbf{x}' = A\mathbf{x}$, then the eigenvalues of A are $\lambda \in \{-1, 8\}$, where the eigenvalue $\lambda = -1$ has algebraic multiplicity 2. The eigenvectors associated with the eigenvalues (-1), (-1), and 8, respectively, are

$$\boldsymbol{\xi}_1 = \begin{bmatrix} -0.4941 \\ -0.4720 \\ 0.7301 \end{bmatrix}, \quad \boldsymbol{\xi}_2 = \begin{bmatrix} -0.5580 \\ -0.8161 \\ 0.1500 \end{bmatrix}, \quad \boldsymbol{\xi}_3 = \begin{bmatrix} -0.6667 \\ -0.3333 \\ 0.6667 \end{bmatrix}.$$

Thus, solutions to the homogeneous system of linear, first-order differential equations are functions of the form

$$\mathbf{x}(t) = \boxed{c_1\boldsymbol{\xi}_1 e^{-t} + c_2\boldsymbol{\xi}_2 e^{-t} + c_3\boldsymbol{\xi}_3 e^{8t},}$$

with $c_1, c_2, c_3 \in \mathbb{R}$.