MATH 759: PROBLEM SET 3 — DUE IN CLASS ON MON. OCT. 21.

1. Let G be a Lie group. Show that there exists a nontrivial left-invariant Borel measure μ on G. That is show that there exists a measure μ such that for any measurable subset A of G

$$\mu(L_g(A)) = \mu(A)$$

   where $L_g(x) = gx$ for all $g, x \in G$. Show that μ is unique up to multiplication by a constant.

2. Prove the properties (iv)-(vi) of Lie derivative (Section 1.19 in the lecture notes).

3. Consider a manifold $\mathcal{M}$ and a coordinate chart (U, ϕ) such that U is open, bounded and simply connected. Let ω be a 1-form supported in $\phi(U)$ and such that $d\omega = 0$. Show that there exists a function f such that $\omega = df$.

4. A k-form ω is called *exact* if there exists a $k-1$ form α such that $\omega = d\alpha$. A form ω is *closed* if $d\omega = 0$. Note that every exact form is closed. Show that if $\mathcal{M}$ is a compact manifold and ω a exact form then $\int_{\mathcal{M}} \omega = 0$. Show that there exists a manifold $\mathcal{M}$ and a 1-form ω such that there is no function f for which $\omega = df$.

5. Consider the *catenoid* and *helicoid* surfaces in $\mathbb{R}^3$. The catenoid surface is the surface of revolution obtained by rotating around the z-axis the graph of the function $x = \cosh z$. Helicoid is the surface obtained as the union of straight lines passing through the points $(0, 0, z)$ and $(\cos z, \sin z, z)$ with $z \in \mathbb{R}$. Show that the two surfaces are locally isometric. Find an explicit local isometry and check in the local charts that it is indeed a local isometry. Are the surfaces globally isometric?

   Hint 1. Express both of the surfaces in local coordinates. Note that both surfaces have a natural angular coordinate. Try to find a natural matching of the other coordinate.

   Hint 2. Go on `www.youtube.com` and search for `catenoid, helicoid`. Google `catenoid, helicoid`.

6. Consider the quadratic form G on $\mathbb{R}^{n+1}$ defined as follows:

$$G(x, x) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$

   Let H be the hypersurface

$$H = \{x \in \mathbb{R}^{n+1} \;:\; G(x, x) = -1 \text{ and } x_0 > 0\}.$$

   Note that at any $p \in H$ the tangent space can be considered a hyperplane of $\mathbb{R}^{n+1}$ and so we define $g : T_pH \times T_pH \to \mathbb{R}$ by $g(x, x) = G(x, x)$. Show that (H, g) is a Riemannian manifold.

On H we define a mapping

$$f(x) = \Pi_n\left(s - \frac{2(x-s)}{G(x-s,x-s)}\right)$$

where we fix $s = (-1, 0, \ldots, 0)$ and Π_n is the projection to last n coordinates (i.e. $\Pi_n(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$) . Show that f is a diffeomorphism from H to unit disk $B(0,1)$ in $\mathbb{R}^n$. Furthermore show that the pull-back of g to $B(0,1)$, $\tilde{g} = f^*g$ is given as follows: at $y \in B(0,1)$ and for tangent vector $v \in \mathbb{R}^n$,

$$\tilde{g}(v,v) = \frac{1}{1-|y|^2}\, v \cdot v.$$