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4 Statistical estimation and penalty methods [25 points] (Sashank)

A [KL divergence estimation].

(a) First, note that, $\forall x \in \mathcal{X}$, if

$$0 = \left(\frac{d}{dg(x)} \log(g(x))p(x) - g(x)q(x) \right) (g^*(x)) = \frac{p(x)}{g^*(x)} - q(x),$$

then $g^*(x) = \frac{p(x)}{q(x)}$, so that $g* = \frac{p}{q}$ maximizes $\log(g^*(x))p(x) - g^*(x)q(x)$. If $\mathrm{KL}(\mathbb{P}\|\mathbb{Q}) = \infty$, then $\int_{\mathcal{X}} \log(g^*(x))p(x) - g^*(x)q(x) = \infty$. Thus, assume $\mathrm{KL}(\mathbb{P}\|\mathbb{Q}) < \infty$. Then,

$$\begin{aligned}
\mathrm{KL}(\mathbb{P}\|\mathbb{Q}) &= \int_{\mathcal{X}} p(x) \log \frac{p(x)}{q(x)}\, dx \\
&= \int_{\mathcal{X}} \left(\log \frac{p(x)}{q(x)} \right) p(x) - p(x) + p(x)\, dx \\
&= \int_{\mathcal{X}} (\log g^*(x))\, p(x) - g^*(x)q(x)\, dx + 1 \\
&= \int_{\mathcal{X}} \sup_{g>0} (\log g(x))\, p(x) - g(x)q(x)\, dx + 1 \\
&= \sup_{g>0} \int_{\mathcal{X}} (\log g(x))\, p(x) - g(x)q(x)\, dx + 1
\end{aligned}$$

(we use the Dominated Convergence Theorem switch the sup and the integral). ■

(b) Not quite clear on how to do this; don't we need some sort of smoothing to get an estimate of $\frac{p(x)}{q(x)}$ that is finite almost everywhere?

(c) If $\frac{p(x)}{q(x)} = w^T x$, then

$$KL(\mathbb{P}, \mathbb{Q}) = \sup_{\substack{w \in \mathbb{R}^d \\ w^T x = \log \frac{p(x)}{q(x)}}} \int_{\mathcal{X}} w^T x p(x)\, dx - \int \exp(w^T x) q(x)\, dx + 1.$$

This maximization problem is concave and smooth in w, so we should be able to use any simple constrained optimization algorithm. The dual problem is

$$\inf_{v \geq 0} \sup_{w \in \mathbb{R}^d} \int_{\mathcal{X}} w^T x p(x)\, dx - \int \exp(w^T x) q(x)\, dx + 1 + \int_{\mathcal{X}} v(x) \left(w^T x - \frac{p(x)}{q(x)} \right).$$

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B [Penalty methods].

(i) Since f and h convex and differentiable, the KKT conditions for (P) are

$$\begin{aligned} 0 &= \nabla f(x^*) + \sum_{i=1}^{m} u_i^* \nabla h_i(x^*) \\ u_i^* h_i(x^*) &= 0, \quad (i \in \{1, \dots, m\}) \\ h(x^*) &\leq 0, \\ \text{and} \quad u^* &\geq 0. \end{aligned}$$

$(P(c))$ is unconstrained and h^+ need not be differentiable, so the only condition for $(P(c))$ is

$$0 \in \partial \left(f(x_c^*) + cp(x_c^*) \right) = \nabla f(x_c^*) + c \sum_{i=1}^{m} \partial h_i^+(x_c^*),$$

where x_c^* is a primal solution of $(P(c))$.

(ii) We show $0 \in \nabla f(\widetilde{x}) + c \sum_{i=1}^{m} \partial h_i^+(\widetilde{x})$. Since $0 = \nabla f(\widetilde{x}) + \sum_{i=1}^{m} u_i^* \nabla h_i(\widetilde{x})$, it suffices to show

$$u_i^* \nabla h_i(\widetilde{x}) \in c \partial h_i^+(\widetilde{x}),$$

for each $i \in \{1, \dots, m\}$. Suppose $h_i(\widetilde{x}) < 0$. Then, since each $u_i^* h_i(\widetilde{x}) = 0$, $u_i^* = 0$, and, since $h_i^+(\widetilde{x}) = 0$, so that h_i^+ achieves a local minimum at $\widetilde{x}$,

$$u_i^* \nabla h_i(\widetilde{x}) = 0 \in c \partial h_i^+(\widetilde{x}).$$

Now suppose $h_i(\widetilde{x}) = 0$. Then, $h_i^+(\widetilde{x}) = h_i(\widetilde{x})$, so that, since $h_i^+ \geq h_i$ and h_i is convex,

$$ch_i^+(z) - ch_i^+(\widetilde{x}) \geq u_i^* h_i(z) - u_i^* h_i(\widetilde{x}) \geq u_i^* \nabla h_i(\widetilde{x}) \cdot (z - \widetilde{x}) \qquad \forall z \in \mathbb{R}^n$$

(using $c \geq \|u\|_\infty$), and so

$$u_i^* \nabla h_i(\widetilde{x}) \in c \partial h_i^+(\widetilde{x}).$$

Since $h(\widetilde{x}) \leq 0$, this covers all cases. ■

(iii) Since u^* is a solution to the dual, $u^* \geq 0$. Since the dual is bounded, there is a solution x^* to the primal. Then, since $c > \|u^*\|_\infty$ and $u^* \geq 0$, by strong duality,

$$\begin{aligned} f(\widetilde{x}) + c \sum_{i=1}^{m} h_i^+(\widetilde{x}) &\geq f(\widetilde{x}) + \sum_{i=1}^{m} u_i^* h_i^+(\widetilde{x}) \geq f(\widetilde{x}) + \sum_{i=1}^{m} u_i^* h_i(\widetilde{x}) \\ &\geq f(x^*) + \sum_{i=1}^{m} u_i^* h_i(x^*) = f(x^*) = f(x^*) + c \sum_{i=1}^{m} h_i^+(x^*) \end{aligned}$$

since each $u_i^* h_i(x^*) = 0 = h_i^+(x^*)$. Thus, since $\widetilde{x}$ minimizes the first expression, $f(\widetilde{x}) = f(x^*)$ and $\widetilde{x}$ is feasible, so $\widetilde{x}$ solves (P). ■