Midterm 1
15-423 Digital Signal Processing for CS
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Due: After Spring Break

1 Signals

a. In all senses of boundedness, $a^n u[n]$ is bounded if and only if $|a| \in [0, 1)$, in which case 1 bounds the signal, $\frac{1}{1-a}$ bounds the signal's integral, and $\frac{1}{1-a^2}$ bounds the integral of the signal's energy.

b. Since, $\forall t \in \mathbb{R}$, $u(t) + u(-t) = 1$,

$$\mathcal{E}v(x(t)) = \frac{1}{2}(\cos(2\pi t)u(t) + \cos(2\pi t)u(-t)) = \frac{1}{2}\cos(2\pi t),$$

so $\mathcal{E}v(x(t))$ is periodic with period 1.

$$\mathcal{O}d(x(t)) = \frac{1}{2}(\cos(2\pi t)u(t) - \cos(2\pi t)u(-t)) = \cos(2\pi t)\left(u(t) - \frac{1}{2}\right),$$

so $\mathcal{O}d(x(t))$ is not periodic (although it is periodic with period 1 if restricted to $t < 0$ or $t \geq 0$).

2 Systems

For boundedness, I wasn't sure which definition to use, so I answered in terms of 1-, 2-, and ∞-norms, corresponding to the integral, square integral (energy), and suprememum, respectively, of the input and output.

a. The system is memoryless and causal ($y(t)$ depends only on $x(t)$), non-linear (if $x_1(0) = x_2(0) = 0$, then $e^{x_1(0)+x_2(0)} = 1 < 2 = e^{x_1(0)} + e^{x_2(0)}$), time-invariant, unstable in 1- and 2-norms, and stable in the ∞-norm.

b. The system is causal but not memoryless ($y[n]$ depends on $x[n]$ and $x[n-1]$), non-linear (if $x_1[0] = x_2[1] = 0, x_1[1] = x_2[0] = 1$, then $(x_1[1] + x_2[1])(x_1[0] + x_2[0]) = 1 > 0 = x_1[1]x_1[0] + x_2[1]x_2[0]$), time-invariant, and stable in all three norms.

c. The system is neither causal nor memoryless ($y(-2)$ depends on $x(-1)$ and $y(2)$ depends on $x(1)$), but is linear (each coordinate is a linear combination of coordinates of the input), not time-invariant, and stable with repsect to all three norms.

d. The system is neither causal nor memoryless ($y[1]$ depends on $x[2]$ and $y[-1]$ depends on $x[-2]$), but is linear, not time-invariant, and stable with respect to all three norms.

e. The system is memoryless but not causal, linear, not time-invariant, and stable with respect to all three norms.

3 Discrete Time Fourier Analysis

Problem III

a. The period of the cosine term is 10. The period of the sine term is 20. Since the summed signal repeats precisely when both the sine term and the cosine term repeat, the period is $\text{lcm}(10, 20) = 20$.

b. Since $\cos(0.4n)$ is not periodic, it has no Fourier series.

Problem V

a. b must divide a.

b. Define, $\forall n \in \mathbb{N}$,

$$y[n] = \begin{cases} x\left[\frac{an}{b}\right] : & \text{if } n = k\frac{b}{a}, k \in \mathbb{N} \\ & 0 \text{ else .} \end{cases}$$

c. Let I denote the unit pulse train of minimum period $N \in \mathbb{N}$ such that $N\frac{a}{b} \in \mathbb{N}$. Then, since $y = x \cdot I$, by the Convolution Theorem, $Y = X * \mathcal{I}$, where $\mathcal{I}$ denotes the Fourier transform of I (which is itself an impulse train).

Problem VI

a.

$$\mathcal{F}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]e^{i\Omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{i(-\Omega)n} = X(-\Omega). \quad \blacksquare$$

b.

$$-x(t) = x^*(t) = \left(\mathcal{F}^{-1}\left\{X(\Omega)\right\}\right)^*\right) = \int_{-\infty}^{\infty} X^*(\Omega)e^{-j\Omega t} = \int_{-\infty}^{\infty} X^*(-\Omega)e^{j\Omega t} = \mathcal{F}^{-1}\left\{X^*(-\Omega)\right\}.$$

Thus, by linearity of the Fourier Transform, since x

$$\mathcal{F}\left\{x^*(t)\right\} = X^*(-\Omega) = -X(-\Omega). \quad \blacksquare$$

c. By part a., if x is symmetric, then

$$X(-\Omega) = \mathcal{F}\{x[-n]\} = \mathcal{F}\{x[n]\} = X(\Omega),$$

and so X is also symmetric. Also, as shown in part b. (without needing x imaginary)

$$X(-\Omega) = X(\Omega) = \mathcal{F}\left\{x[n]\right\} = \mathcal{F}\left\{x^*[n]\right\} = X^*(-\Omega),$$

and so X is real. $\blacksquare$