Homework 3

21-236 Mathematical Studies Analysis II
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Problem 1

(a) Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be convex. Let $x \in I$, and let $y, z \in I$ with $z < x < y$. Then $z < x < y$, $\exists \theta \in (0, 1)$ such that $x = z + \theta(y - z)$. Thus, since f is convex,

$$\begin{aligned}
\frac{f(x) - f(z)}{x - z} &= \frac{f(z + \theta(y - z)) - f(z)}{z + \theta(y - z) - z} \\
&\leq \frac{f(z) + \theta(f(y) - f(z)) - f(z)}{\theta(y - z)} \\
&= \frac{\theta(f(y) - f(z))}{\theta(y - z)} \\
&= \frac{f(y) - f(z)}{y - z} \\
&= \frac{(1 + \theta)(f(y) - f(z)))}{(1 + \theta)(y - z)} \\
&= \frac{f(y) - (f(z) + \theta(f(y) - f(z)))}{y - (z + \theta(y - z))} \\
&\leq \frac{f(y) - f(z + \theta(y - z))}{y - (z + \theta(y - z))} \\
&= \frac{f(y) - f(x)}{y - x}.
\end{aligned}$$

Thus, $z \mapsto \frac{f(x)-f(z)}{x-z}$ is increasing on I, so that, since x is an accumulation point of I, by Theorem 204 (as per the notes for Real Analysis I), $f'_-(x) := \lim_{z \to x-} \left(\frac{f(x)-f(z)}{x-z} \right)$ and $f'_+(x) := \lim_{z \to x+} \left(\frac{f(z)-f(x)}{z-x} \right)$ exist. ■

(b) $\forall x, y \in I$ with $x < y$, also by Theorem 204, since $x \mapsto \frac{f(y)-f(x)}{y-x}$ is also increasing on I,

$$f'_-(x) \leq \frac{f(y) - f(x)}{y - x} \leq f'_-(y) \leq f'_+(y). \qquad ■$$

(c) Let $x, y \in I$. If $x < y$, then, by the result of part (b),

$$f'_-(x) \leq \frac{f(y) - f(x)}{y - x} = \frac{f(x) - f(y)}{x - y}.$$

Thus, since $(x-y) > 0$, multiplying by $(x-y)$ and adding $f(y)$ gives $f(x) \geq f(y) + f'_-(x)(x-y)$. If $y < x$, then, by the result of part(b),

$$\frac{f(x) - f(y)}{x - y} \leq f'_-(x).$$

Thus, since $x - y < 0$, multiplying by $(x-y)$ and adding $f(y)$ gives $f(x) \geq f(y) + f'_-(x)(x-y)$. If $x = y$, then $x - y = 0$, so, trivially, $f(x) \geq f(y) + f'_-(x)(x-y)$. Thus, $\forall x, y \in I$, $f(x) \geq f(y) + f'_-(x)(x-y)$. ■

(d) Let $U \subseteq \mathbb{R}$ be convex, and let $g : U \to \mathbb{R}$ be differentiable and convex. Let $\mathbf{v} = \mathbf{y} - \mathbf{x}$. Since U is convex, if S is the line segment between $\mathbf{x}$ and $\mathbf{y}$, then $S \subseteq U$. Since U is open, $\exists \delta_1, \delta_2$ such that $B(\mathbf{x}, \delta_1), B(\mathbf{y}, \delta_2) \subseteq U$. Thus, for $I = (-\delta_1, \|\mathbf{v}\| + \delta_2) \subseteq \mathbb{R}$ we can define $h : I \to \mathbb{R}$, such that, $\forall t \in (-\delta_1, \|\mathbf{v}\| + \delta_2)$, $h(t) = g(\mathbf{x} + t\frac{\mathbf{v}}{\|\mathbf{v}\|})$. Since g is convex, h is also convex. Furthermore, I is open, so that, by the result of part (c),

$$h(0) \geq h(\|\mathbf{v}\|) + h'_-(0)(\|\mathbf{v}\|).$$

Since g is differentiable, h is differentiable (so that, $\forall t \in I$, $h'(t) = h'_-(t)$),

$$g(\mathbf{y}) + \nabla g(\mathbf{x}) \cdot \mathbf{v} = g(\mathbf{y}) + \frac{\partial g}{\partial \mathbf{v}}(\mathbf{x}) = h(\|\mathbf{v}\|) + \frac{dh}{dt}(0)\|\mathbf{v}\| = h(\|\mathbf{v}\|) + h'_-(0)\|\mathbf{v}\| \leq h(0) = g(\mathbf{x}).$$ ■

Problem 2

(a) Let $h \in C([a, b])$ such that

$$\int_a^b h(x)v'(x)\, dx = 0,$$

$\forall v \in C^1([a, b])$ such that $v(a) = v(b) = 0$. Suppose, for sake of contradiction, that h is non-constant, so that there exists some $x_1, x_2 \in [a, b]$ such that $h(x_1) \neq h(x_2)$. Without loss of generality, $h(x_1) < h(x_2)$ (since h is constant if and only if $-h$ is constant), and $0 < h(x_1)$ (since h is constant if and only if $h - h(x_1) + 1$ is constant).

Since h is continuous, $\exists \delta_1, \delta_2 > 0$ such that, $\forall x \in [x_1 - \delta_1, x_1 + \delta_1]$, $h(x) < m$, for some $m \leq \frac{h(x_1) + h(x_2)}{2}$, and $\forall x \in [x_2 - \delta_2, x_2 + \delta_2]$, $h(x) > M$, for some $M \geq \frac{h(x_1) + h(x_2)}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$, and let $c = \max\{a, x_1 - \delta_1\}$, $d = x_1 + \delta_1$, $e = x_2 - \delta_2$, and $f = \min\{b, x_2 + \delta_2\}$.

Define $v : [a, b] \to \mathbb{R}$ piecewise as follows:

$$f(x) = \begin{cases} 0 & : \quad x \in [a, c] \cup [f, b] \\ 2 & : \quad x \in [d, e] \\ (\frac{2}{d-c}(x - c))^2 & : \quad x \in (c, \frac{c+d}{2}] \\ 2 - (\frac{2}{d-c}(x - d))^2 & : \quad x \in (\frac{c+d}{2}, d) \\ 2 - (\frac{2}{f-e}(x - e))^2 & : \quad x \in (e, \frac{e+f}{2}] \\ (\frac{2}{f-e}(x - f))^2 & : \quad x \in (\frac{e+f}{2}, f) \end{cases}$$

Then, $v \in C^1([a,b])$, $\forall x \in (c,d)$. Since v is constant on $[a,c]$, $[d,e]$ and $[f,b]$, $v' = 0$ on these intervals. Since, $\forall y \in [c,d]$, $v(y) = 2 - v(e + y - c)$,

$$\int_c^d v' = -\int_e^f v' \neq< 0.$$

Thus, since $h \geq m > 0$ on $[c,d]$ and $m < M \leq h$

$$\int_c^d hv' + \int_e^f hv' < \int_c^d mv' + \int_e^f Mv' < 0.$$

Thus,

$$\begin{aligned}
\int_a^b hv' &= \int_a^c hv' + \int_c^d hv' + \int_d^e hv' + \int_e^f hv' + \int_f^b hv' \\
&= \int_c^d hv' + \int_e^f hv' < 0,
\end{aligned}$$

contradicting the given that

$$\int_a^b hv' = 0. \qquad \blacksquare$$

(b) Since $p \in C([a,b])$, p is integrable on $[a,b]$, so that is has an antiderivative $P \in C^1([a,b])$. Thus, since $v(a) = v(b) = 0$, integrating by parts gives

$$\begin{aligned}
\int_a^b [pv + qv'] &= P(b)v(b) - P(a)v(a) + \int_a^b [qv' - Pv'] \\
&= \int_a^b [qv' - Pv'] \\
&= \int_a^b [q - P]\, v'
\end{aligned}$$

By the result of part (a), $h := q - P$ is a constant function. As the sum of two functions in $C^1([a,b])$, q is differentiable, and, furthermore, $q' = P' + h' = p$. $\blacksquare$

(c) Let $\alpha, \beta \in \mathbb{R}$, and let $X = \{f \in C^1([a,b]) : f(a) = \alpha, f(b) = \beta\}$. Suppose some $f_0 \in X$ minimizes G over X. Recall that, by the result of Problem 3, part (c) of Assignment 2,

$$\int_\alpha^\beta \left[\frac{\partial g}{\partial y}(x, f_0(x), f_0'(x))v(x) + \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))v'(x)\right] = 0,$$

$\forall v \in C^1([a,b])$ with $v(\alpha) = v(\beta) = 0$. Thus, by the result of part (b), for $q(x) := \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))$, $p(x) := \frac{\partial g}{\partial y}(x, f_0(x), f_0'(x))$, $q \in C^1([a,b])$, and furthermore, $q' = p$; i.e.

$$\frac{d}{dx}\left(\frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))\right) = \frac{\partial g}{\partial y}(x, f_0(x), f_0'(x)). \qquad \blacksquare$$

(d) Suppose that, $\forall x \in [a,b]$, $h := (y,z) \mapsto g(x,y,z)$ is convex, and suppose $f_0 \in X$ satisfies (1). Then, $\forall f \in X$, since h is convex, by the result of Problem 1, part (d),

$$\begin{aligned}
G(f) - G(f_0) &= \int_a^b g(x, f(x), f'(x)) - g(x, f_0(x), f_0'(x))\, dx \\
&\geq \int_a^b \nabla g(x, f_0(x), f_0'(x)) \cdot ((x, f(x), f'(x)) - (x, f_0(x), f_0'(x)))\, dx \\
&= \int_a^b \frac{\partial g}{\partial y}(x, f_0(x), f_0'(x))(f(x) - f_0(x)) \\
&+ \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))(f'(x) - f_0'(x))\, dx
\end{aligned}$$

Since $f(a) = \alpha = f_0(a)$ and $f(b) = \beta = f_0(b)$, integration by parts gives

$$\int_a^b \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))(f'(x) - f_0'(x))\, dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))\right)(f(x) - f_0(x))\, dx,$$

so that, by equation (1),

$$\int_a^b \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))(f'(x) - f_0'(x))\, dx = -\int_a^b \frac{\partial g}{\partial y}(x, f_0(x), f_0'(x))(f(x) - f_0(x))\, dx.$$

Therefore,

$$G(f) - G(f_0) \geq \int_a^b 0\, dx = 0,$$

so that $G(f) \geq G(f_0)$, and f_0 minimizes G over X. ■

Problem 3

(a) Let $X = \{f \in C^1([0,1]) : f(0) = f(1) = 0\}$, and let $G : X \to \mathbb{R}$ such that, $\forall f \in X$, $G(f) = \int_0^1 e^{-(f'(x))^2}\, dx$. Note that, since the exponential function is strictly positive, $G > 0$. Suppose, for sake of contradiction, that some $f_0 \in X$ minimized G over X. Then, for $h = 2f$, $h \in X$, and, since $G(f) > 0$,

$$G(h) = \int_0^1 e^{-(h'(x))^2} = \int_0^1 e^{-4} e^{-(f_0'(x))^2} < \int_0^1 e^{-(f_0'(x))^2},$$

contradicting the choice of f_0 as a minimizer of G on X. Thus, G has no minimum on X. ■

(b) Let $X = \{f \in C^1([0,1]) : f(0) = f(1) = 0\}$, and let $G : X \to \mathbb{R}$ such that, $\forall f \in X$, $G(f) = \int_0^1 \left[(f'(x))^2 - 1\right]^2$. Suppose, for sake of contradiction, that some $f_0 \in X$ minimized G over X. Let $g : [0,1] \times \mathbb{R} \times \mathbb{R}$ such that, $\forall (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}$, $g(x,y,z) = (z^2 - 1)^2$, so that,

$\forall f \in X$, $G(f) = \int_0^1 g(x, f(x), f'(x))\,dx$. Note that $\frac{\partial g}{\partial y} = 0$, so that, by the result of Problem 2, part (c), $\forall x \in [0,1]$,

$$4(f_0'(x)f_0''(x) - f_0'(x)f_0''(x)) = \frac{d}{dx}\left(\frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))\right) = 0.$$

Thus, either $f_0' = (f_0')^3$, or $f_0'' = 0$; in the former case, $\forall x \in [0,1]$, $f_0'(x) \in \{-1, 0, 1\}$. Since f_0' is continuous, this means that f_0' is constant -1, 0, or 1. Thus, in any case, $f_0'' = 0$. By the Fundamental Theorem of Calculus, integration gives, $\forall x \in [0,1]$, $f_0(x) = ax + b$ for some constants a and b. However, the only such f_0 with $f_0(0) = f_0(1) = 0$ is the constant function $f_0 = 0$. Thus, if f_0 minimizes G over X, then $f_0 = 0$. However, it can be seen that, for $h : [0,1] \to \mathbb{R}$ such that, $\forall x \in [0,1]$, $h(x) = x - x^2$, $h \in X$ and $G(h) < G(0)$, which is a contradiction. Thus, G has no minimum on X. ■

(c) Let $X = \{f \in C^1([a,b]) : f(0) = 0, f(1) = 1\}$, and let $G : X \to \mathbb{R}$ such that, $\forall f \in X$, $G(f) = \int_0^1 \left[x(f'(x))^2\right]$. Suppose, for sake of contradiction, that some $f_0 \in X$ minimized G over X. Let $g : [0,1] \times \mathbb{R} \times \mathbb{R}$ such that, $\forall (x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{R}$, $g(x,y,z) = xz^2$, so that, $\forall f \in X$, $G(f) = \int_0^1 g(x, f(x), f'(x))\,dx$. Note that $\frac{\partial g}{\partial y} = 0$, so that, by the result of Problem 3, part(c) of Assignment 2,

$$\int_0^1 2xf_0'(x)v'(x)\,dx = \int_0^1 \frac{\partial g}{\partial z}(x, f_0(x), f_0'(x))v'(x)\,dx = 0,$$

$\forall v \in C^1([0,1])$ with $v(0) = v(1) = 0$. Thus, by the result of part (a), $2xf_0'(x) = c_1$ for some constant $c_1 \in \mathbb{R}$, so that $f_0' = \frac{c_1}{2x}$. By the Fundamental Theorem of Calculus, integration gives, $\forall x \in [0,1]$, $f_0(x) = \frac{c_1}{2}(\log(x) + c_2)$, for some constant $c_2 \in \mathbb{R}$. However, since $\lim_{x\to 0+}\log(0) = \infty$, either $c_2 \neq 0$ and $\lim_{x\to 0+} f_0(0) = \pm\infty$, contradicting the constraint on f_0 of $f_0(0) = 0$, or $c_2 = 0$, contradicting the constraint on f that $f(1) = 1$. Thus, G has no minimum on X. ■