21-630 Assignment # 7 Due Wed., March 20, 2013

1. Assume that

$$\left|e^{At}\right| \leq Ce^{-\lambda t}$$

for $t \geq 0$ with $\lambda > 0$ and that b is continuous on $[0, \infty)$ with

$$\lim_{t \to \infty} |b(t)| = 0.$$

Let $\dot{X}(t) = AX(t) + b(t)$ for $t \geq 0$ and show that

$$\lim_{t \to +\infty} |X(t)| = 0.$$

2. Let $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$ with $\omega > 0$.

   (A) Find complex matrices P and D with D diagonal so that $A = PDP^{-1}$. Use this to show $e^{At} = \cos(\omega t)I + \omega^{-1}\sin(\omega t)A$.

   (B) With no complex numbers compute A^k for all k and use the definition

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k$$

   to show that

$$e^{At} = \cos(\omega t)I + \omega^{-1}\sin(\omega t)A.$$

3. Consider $\dot{Y}(t) = AY(t) + F(t, Y(t))$ with

$$\left|e^{At}\right| \leq B \qquad B > 0, t \geq 0$$

and

$$|F(t, y)| \leq b(t)|y|$$

with $\displaystyle\int_0^{\infty} b(t)dt$ finite. Show that 0 is stable.

4. Show that 0 is unstable for the scalar equation

$$\dot{Y}(t) = \frac{1}{1+t} Y(t)$$

by explicitly solving it. (So $\int_0^\infty b(t)dt$ finite is needed in problem 4.)