Lecture Notes for Week 10 (First Draft)

Second Adjoints (Continued)

Theorem 10.1: Let X and Y be normed linear spaces and let $T \in \mathcal{L}(X;Y)$ be given. Then T^{**} is an extension of T. If X is reflexive then $T^{**} = T$.

Proof: What we need to show is that

$$T^{**}(J_X(x)) = J_Y(Tx) \quad \text{for all } x \in X. \tag{1}$$

Let $x \in X$ be given. Then, for all $y^* \in Y^*$, we have

$$\begin{aligned} \langle T^{**}(J_X(x)), y^* \rangle_{Y^{**} \times Y^*} &= \langle J_X(x), T^* y^* \rangle_{X^{**} \times X^*} = \langle T^* y^*, x \rangle_{X^* \times X} \\ &= \langle y^*, Tx \rangle_{Y^* \times Y} = \langle J_Y(Tx), y^* \rangle_{Y^{**} \times Y^*}. \end{aligned}$$

We conclude that (1) holds. $\square$

Adjoints, Null Spaces, Ranges, and Inverses

A square matrix is invertible if and only if the transpose is invertible. In this case the transpose of the inverse is the inverse of the transpose. We now give a generalization of this result to operators between NLS.

Theorem 10.2: Let X be a Banach space, Y be a normed linear space, and $T \in \mathcal{L}(X;Y)$ be given. Then the following two statements are equivalent:

(i) T is bijective and $T^{-1} \in \mathcal{L}(Y;X)$,

(ii) T^* is bijective and $(T^*)^{-1} \in \mathcal{L}(X^*;Y^*)$.

If (i) (and hence also (ii)) holds then $(T^*)^{-1} = (T^{-1})^*$.

Remark 10.3: Completeness of X is not needed for the implication (i) $\Rightarrow$ (ii) in Theorem 10.2.

Proof of Theorem 10.2: Assume that (i) holds. Then we have

$$TT^{-1} = I_Y, \quad T^{-1}T = I_X.$$

Applying Theorem 9.21 (concerning adjoints of products) and Remark 9.19 (concerning adjoints of identity operators), we obtain

$$(T^{-1})^* T^* = (I_Y)^* = I_{Y^*}, \tag{2}$$

$$T^*(T^{-1})^* = (I_X)^* = I_{X^*}. \tag{3}$$

We conclude that T^* is bijective and that $(T^*)^{-1} = (T^{-1})^*$.

Assume that (ii) holds. Then, by what we just proved, T^{**} is bijective and $T^{**} \in \mathcal{L}(Y^{**}; X^{**})$. Since T^{**} is an extension of T, we conclude that T is injective. It remains to show that T is surjective. This is where we use the completeness of X. Let $\hat{X} = J_X[X]$. Since X is complete, we know that $\hat{X}$ is a closed subspace of X^{**}. Since T^{**} is an isomorphism from X^{**} to Y^{**}, we conclude that $T^{**}[\hat{X}]$ is a closed subspace of Y^{**}. Since T^{**} is an extension of T, we conclude that $T[X]$ is a closed subspace of Y.

Suppose that $T[X] \neq Y$. Then, by a corollary to the Hahn-Banach Theorem, we may choose $y^* \in Y^*$ such that $y^* \neq 0$ and $y^*(y) = 0$ for all $y \in T[X]$. This means that $y^*(Tx) = 0$ for all $x \in X$. It follows that $T^*y^* = 0$. Since T^* is injective, we infer that $y^* = 0$, which is impossible. We conclude that T is surjective, and hence bijective. Boundedness of T^{-1} follows from (3) and the boundedness of $(T^{**})^{-1}$. $\square$

Another very useful feature of transposes of matrices is the "orthogonality relationship" between the range of a matrix and the null space of the transpose. This relationship can be generalized to linear operators between infinite-dimensional NLS, but the situation becomes more complicated in infinite dimensions.

Definition 10.4: Let X be a NLS and let $A \subset X$, $B \subset X^*$ be given. The *annihilator* of A is the subset $A^\perp$ of X^* defined by

$$A^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ forall } x \in A\}.$$

The *pre-annihilator* of B is the subset ${}^\perp B$ of X defined by

$${}^\perp B = \{x \in X : x^*(x) = 0 \text{ for all } x^* \in B\}.$$

Remark 10.5: Many authors call both sets "annihilators". I think that it is a good idea to use a different term because a dual space of a NLS is a NLS in its own right and the meaning of the "annihilator of a subset of X^*" could become genuinely ambiguous.

The following remark concerning annihilators and pre-annihilators is easily verified.

Remark 10.6: Let X be a normed linear space and $A \subset X$, $B \subset Y$ be given. Then

(i) $A^\perp$ is a closed subspace of X^*.

(ii) ${}^\perp B$ is a closed subspace of X.

(iii) $A \subset^\perp (A^\perp)$

(iv) $B \subset ({}^{\perp}B)^{\perp}$

It is natural to ask when we might have $A = {}^{\perp}(A^{\perp})$ for a set $A \subset X$. By item (ii) of the remark above, there is no hope for this inequality unless A is a closed subspace.

In fact, we have the following simple, but useful, result.

Proposition 10.7: Let X be a NLS and $A \subset X$. Then ${}^{\perp}(A^{\perp}) = \text{cl}(\text{span}(A))$.

The proof of this Proposition 10.7 is part of Assignment 5.

Remark 10.8: It is possible to have a closed subspace Z of X^* such that $Z \subsetneq ({}^{\perp}Z)^{\perp}$. You are asked to give an example in Assignment 6. What is actually true about this situation is that for any set $B \subset X^*$, $({}^{\perp}B)^{\perp}$ is the weak* closure of span(B).

Theorem 10.9: Let X, Y be NLS and let $T \in \mathcal{L}(X;Y)$ be given. Then we have

(i) $\mathcal{N}(T^*) = (\mathcal{R}(T))^{\perp}$, and

(ii) $\mathcal{N}(T) = {}^{\perp}(\mathcal{R}(T^*))$.

Proof: To prove (i), let $y^* \in Y^*$ be given and observe that

$$\begin{aligned} y^* \in \mathcal{N}(T^*) \quad &\Leftrightarrow \quad T^*y^* = 0 \\ &\Leftrightarrow \quad (T^*y^*)(x) = 0 \text{ for all } x \in X \\ &\Leftrightarrow \quad y^*(Tx) = 0 \text{ for all } x \in X \\ &\Leftrightarrow \quad y^* \in (\mathcal{R}(T))^{\perp}. \end{aligned}$$

To prove (ii), let $x \in X$ be given and observe that

$$\begin{aligned} x \in \mathcal{N}(T) \quad &\Leftrightarrow \quad Tx = 0 \\ &\Leftrightarrow \quad y^*(Tx) = 0 \text{ for all } y^* \in Y^* \\ &\Leftrightarrow \quad (T^*y^*)(x) = 0 \text{ for all } y^* \in Y^* \\ &\Leftrightarrow \quad x \in {}^{\perp}(\mathcal{R}(T^*)). \quad \square \end{aligned}$$

We now investigate the relationship between $\mathcal{R}(T)$ and ${}^{\perp}(\mathcal{N}(T^*))$. In view of Theorem 10.9, it might be tempting to conjecture that these two linear manifolds are the same. However, since the null space of a continuous linear operator is closed, and the range of a continuous linear operator need not be closed, we can see that the linear manifolds in question cannot be equal in general. However, we have the following result.

Theorem 10.10: Let X, Y be normed linear spaces and let $T \in \mathcal{L}(X;Y)$ be given. Then

$$\mathrm{cl}(\mathcal{R}(T)) =^{\perp} (\mathcal{N}(T^*)).$$

The proof follows immediately from Theorem 10.9 (i) and Proposition 10.7.

Finally, we consider the relationship between $\mathcal{R}(T^*)$ and $(\mathcal{N}(T))^{\perp}$. It might be tempting to conjecture that $\mathrm{cl}(\mathcal{R}(T^*))$ is equal to $(\mathcal{N}(T))^{\perp}$; however this is not true in general because, even for a closed subspace Z of X^*, it can happen that the annihilator of the pre-annihilator of Z is strictly larger than Z – and this phenomenon does occur sometimes for closures of ranges of adjoints. However, we do have the following result.

Theorem 10.11: Let X, Y be Banach spaces, let $T \in \mathcal{L}(X;Y)$ be given, and assume that $\mathcal{R}(T)$ is closed. Then $\mathcal{R}(T^*)$ is closed and

$$\mathcal{R}(T^*) = (\mathcal{N}(T))^{\perp}.$$

To prove Theorem 10.11, we shall make use of the following simple lemma, which is based on the open mapping theorem.

Lemma 10.12: Let X, Y be Banach spaces, let $T \in \mathcal{L}(X;Y)$ be given, and assume that $\mathcal{R}(T)$ is closed. Then there exists $K \in \mathbb{R}$ with the following property: For every $y \in \mathcal{R}(T)$, there exists $x \in X$ such that $y = Tx$ and $\|x\| \le K\|y\|$.

Proof: Since Y is a Banach space and $\mathcal{R}(T)$ is closed, $(\mathcal{R}, \|\cdot\|_Y)$ is a Banach space. Obviously, $T : (X, \|\cdot\|_X) \to (\mathcal{R}(T), \|\cdot\|_Y)$ is linear, continuous, and surjective. By the open mapping theorem, we may choose $\delta > 0$ such that

$$T[B_1^X(0)] \supset \{y \in \mathcal{R}(T) : \|y\| < \delta\}.$$

Let $y \in \mathcal{R}(T)$ be given. If $y = 0$, there is nothing to prove, so we assume that $y \neq 0$. Put

$$z = \frac{\delta y}{2\|y\|}$$

and notice that $z \in \mathcal{R}(T)$ and $\|z\| < \delta$. We may therefore choose $w \in B_1^X(0)$ such that $Tw = z$. Now put

$$x = \frac{2\|y\|}{\delta} w.$$

Observe that

$$Tx = y \quad \text{and} \quad \|x\| \le \frac{2}{\delta}\|y\|. \quad \square$$

Proof of Theorem 10.11: Let $x^* \in \mathcal{N}(T)^{\perp}$ be given. Notice that if $x_1, x_2 \in X$ satisfy $Tx_1 = Tx_2$ then $x^*(x_1) = x^*(x_2)$. Consequently, we may choose a linear functional $g : \mathcal{R}(T) \to \mathbb{K}$ such such that

$$g(Tx) = x^*(x) \quad \text{for all } x \in X.$$

Choose K as in Lemma 10.12. Let $y \in \mathcal{R}(T)$ be given. Then we may choose $x \in X$ such that $y = Tx$ and $\|x\| \leq K\|y\|$. It follows that

$$|g(y)| \leq K\|x^*\|\|y\|.$$

By the Hahn-Banach Theorem, we may choose $y^* \in Y^*$ such that

$$y^*(y) = g(y) \quad \text{for all } y \in \mathcal{R}(T).$$

Then, for all $x \in X$ we have

$$(T^*y^*)(x) = y^*(Tx) = x^*(x),$$

which says that $x^* = T^*y^*$, i.e. $x^* \in \mathcal{R}(T^*)$. We conclude that

$$\mathcal{N}(T)^\perp \subset \mathcal{R}(T^*).$$

Applying the annihilator to the expression in Theorem 10.9 (ii) and using the fact that $({}^\perp B)^\perp \supset B$ for every $B \in X^*$ we find that

$$\mathcal{R}(T^*) \subset \mathcal{N}(T)^\perp.$$

It follows that $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$. Since $\mathcal{N}(T)^\perp$ is closed, we conclude that $\mathcal{R}(T^*)$ is closed. $\square$

Lemma 10.13: Let X, Y be Banach spaces and $S \in \mathcal{L}(X; Y)$ be given. Assume that there exists $c > 0$ such that

$$\|S^*y^*\| \geq c\|y^*\| \quad \text{for all } y^* \in Y^*.$$

Then S is surjective.

The proof of Lemma 10.13 is part of Assignment 5.

Lemma 10.14: Let X, Y be Banach spaces and let $T \in \mathcal{L}(X; Y)$ be given. Assume that $\mathcal{R}(T^*)$ is closed. Then $\mathcal{R}(T)$ is closed.

Proof: Let $Z = \text{cl}(\mathcal{R}(T))$ and notice that $(Z, \|\cdot\|_Y)$ is complete. Define $S \in \mathcal{L}(X; Z)$ by

$$Sx = Tx \quad \text{for all } x \in X.$$

Observe that $\mathcal{R}(S)$ is dense in Z and consequently $(\mathcal{R}(S))^\perp = \{0\}$, because a continuous linear functional that vanishes on a dense set, must be the zero functional. (Here the $^\perp$ operator is the one that takes a set in Z and produces a set in Z^*.) It follows from Theorem 10.9 (i) that

$$\mathcal{N}(S^*) = (\mathcal{R}(S))^\perp = \{0\},$$

and S^* is injective.

Let $z^* \in Z^*$ be given. Choose an extension $y^* \in Y^*$ of z^*. Then, for all $x \in X$ we have

$$(T^*y^*)(x) = y^*(Tx) = z^*(Sx) = (S^*z^*)(x). \tag{4}$$

We conclude that $T^*y^* = S^*z^*$, which tells us that $\mathcal{R}(S^*) \subset \mathcal{R}(T^*)$. Similarly, if we start with $y^* \in Y^*$ and let z^* denote the restriction of y^* to Z, then (4) holds for all $x \in X$ which implies that $\mathcal{R}(T^*) \subset \mathcal{R}(S^*)$, and consequently we have

$$\mathcal{R}(S^*) = \mathcal{R}(T^*).$$

This implies that $\mathcal{R}(S^*)$ is closed. Since S^* is injective and has closed range, the Bounded Inverse Theorem implies the existence of a constant $c > 0$ such that

$$\|S^*z^*\| \geq c\|z^*\| \ \text{ for all } z^* \in Z^*.$$

Lemma 10.13 implies that S is surjective. This tells us that $\mathcal{R}(T) = Z$, and consequently $\mathcal{R}(T)$ is closed. $\square$

The following theorem summarizes several key results that we have established concerning ranges of operators and adjoints.

Theorem 10.15: Let X, Y be Banach spaces and let $T \in \mathcal{L}(X;Y)$ be given. The following four statements are equivalent

(i) $\mathcal{R}(T)$ is closed,

(ii) $\mathcal{R}(T^*)$ is closed,

(iii) $\mathcal{R}(T) = {}^{\perp}\mathcal{N}(T^*)$,

(iv) $\mathcal{R}(T^*) = \mathcal{N}(T)^{\perp}$.

Remark 10.16: The implications "(i) $\Leftrightarrow$ (iii)" and "(iv) $\Rightarrow$ (ii) in Theorem 10.15 do not require completeness.

The right and left shift operators can be used to illustrate many important features of adjoints. For simplicity, we look at these operators as acting from l^2 to l^2. (The results can easily be generalized to other l^p spaces.

Example 10.17: Let $X = Y = l^2$. We shall identify $(l^2)^*$ with l^2 as usual. Define $R, L : l^2 \to l^2$ as follows:

$$Rx = (0, x_1, x_2, x_3, \cdots), \quad Lx = (x_2, x_3, x_4, x_5, \cdots) \ \text{ for all } x \in l^2.$$

Observe that R is an isometry (and hence is injective), but is not surjective; L is surjective, but fails to be injective. Since

$$\sum_{k=1}^{\infty}(Rx)_k y_k = \sum_{j=1}^{\infty} x_j (Ly)_j = x_1 y_2 + x_2 y_3 + x_3 y_4 + \cdots \quad \text{for all } x, y \in l^2,$$

conclude that

$$R^* = L, \quad L^* = R.$$

If we think of the elements of l^2 as column vectors, we can associate R and L with infinite matrices $\mathbf{R}$ and $\mathbf{L}$:

$$\mathbf{R} = \begin{pmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Notice that $\mathbf{R}^{\mathrm{T}} = \mathbf{L}$ and $\mathbf{L}^{\mathrm{T}} = \mathbf{R}$.

Example 10.18: Let $X = Y = L^2[0,1]$ and let $k \in C([0,1] \times [0,1])$ be given. As usual, we identify $(L^2[0,1])^*$ with $L^2[0,1]$. For every $f \in L^2[0,1]$, put

$$(Kf)(x) = \int_0^1 k(x,y) f(y)\, dy, \quad x \in [0,1].$$

It is straightforward to show that $K \in \mathcal{L}(L^2[0,1]; L^2[0,1])$. It is also straightforward to show that the adjoint K^* of K is given by

$$(K^* g)(x) = \int_0^1 k(y,x) g(y)\, dy.$$

Linear Mappings, Weak and Weak Convergence*

Let X, Y be normed linear spaces and let $T \in \mathcal{L}(X;Y)$ and $L \in \mathcal{L}(Y^*;X^*)$ be given.

Question 1: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$ be given. Assume that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$. Can we conclude that $Tx_n \rightharpoonup Tx$ (weakly) in Y as $n \to \infty$?

This question can be answered affirmatively by using the adjoint T^*. Let $y^* \in Y^*$ be given. Then we have

$$y^*(Tx_n) = (T^* y^*)(x_n) \to (T^* y^*)(x) = y^*(Tx) \quad \text{as } n \to \infty.$$

We conclude that $Tx_n \rightharpoonup Tx$ (weakly) in Y as $n \to \infty$.

Question 2: Let $\{y_n^*\}_{n=1}^{\infty}$ be a sequence in in Y^* and $y^* \in Y^*$ be given. Assume that $y_n^* \overset{*}{\rightharpoonup} y^*$ (weakly*) in Y^* as $n \to \infty$. Can we conclude that $Ly_n^* \overset{*}{\rightharpoonup} Ly^*$ (weakly*) in X^* as $n \to \infty$?

If $L = U^*$ for some $U \in \mathcal{L}(X;Y)$ then we have

$$(Ly_n^*)(x) = y_n^*(Ux) \to y^*(Ux) = (Ly^*)(x) \quad \text{as } n \to \infty,$$

and consequently $Ly_n^* \overset{*}{\rightharpoonup} Ly^*$ (weakly*) in X^* as $n \to \infty$. However, if L is not the adjoint of some member of $\mathcal{L}(X;Y)$ then the argument given above cannot be used. In fact, bounded linear operators that are not adjoints need not respect weak* convergence. We will see an example at the beginning of class next time.