21-484 Notes

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$\rightarrow$ Thm: If $G = (U \cup W, E)$ is a bipartite graph, then G has a matching of size $|U|$ iff $\forall X \subset U. |N(X)| \geq |X|$. $(*)$

Proof: Saw that having a matching pf size $|U|$ implies $(*)$.

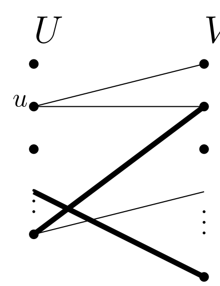
Assume that G has $(*)$ and that M is a maximal matching, $|M| < |U|$.

Then, $\exists u \in U$ that is not matched.

Define an alternating path. Consider the set S of all vertices v such that there is an alternating u–v path.

$\rightarrow u \in S$

$\rightarrow$ If $w \in W \cap S$, then it is not an endpoint of a maximal alternating path. Otherwise, we could swap and non-matching edges in the path and get a larger matching.

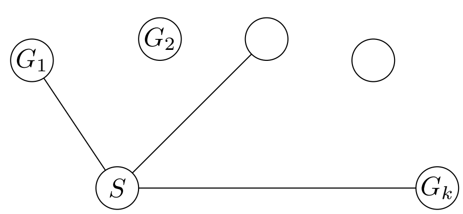


$\rightarrow$ Let $U' = U \cap S, W' = W \cap S$.

$\rightarrow$ There is a matching edge going from every vertex of W' to a vertex of U'.

$\Rightarrow |W'| \leq |U' \setminus \{u\}| \Rightarrow |W'| < |U'| \lightning (*)$

Tutte's Theorem



A graph $G = (V, E)$ is a perfect matching iff for every set $S \subseteq V$ the number of connected components of odd size in $G[V \setminus S]$ is at most the size of S.

Proof: Assume that G has a perfect matching, and let S be a set of vertices. Then, since the perfect matching M matchs an even number of vertices in every connected component of $G[U \setminus S]$, every odd component contains at least one vertex that is not matched with another vertex from this component. Such a vertex must be matched wtih a vertex set S.

$\rightarrow$ Let $k_o(G - S)$ be the number of odd connected components in $G[u \setminus S]$.

$\rightarrow$ Assume that G obeys

$$k_o(G - S) \leq |S| \text{ for every } S \subseteq V. \; (*)$$

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→ $*$, G has an even number of vertices.

→ By induction, $|V| = 2$ ✓

→ Let $n \geq 4$.

→ Assume that $*$ implies the existence of a perfect matching in every graph with fewer than n vertices.

→ Let S be a maximal set of vertices with the property

$$k_o(G - S) = |S|$$

→ S is not empty. Every connected graph has a vertex that is not a cut vertex. A leaf of a spanning tree, ...

→ let u be a noncut vertex. $k_o(G - \{u\}) = 1 = |\,\{u\}\,|$

→ let $G_1, \ldots, G_k$ be the connected component in $G(V \setminus S)$.

→ All the G_i's are odd, otherwise we can add a non cut vertex from an even G_i to S.

→ Let S_i be the set of vertices in S having a neighbor in G_i.

→ S_i is not empty. (G_i was even in G, and now all the components are odd).