Homework 2

21-236 Mathematical Studies Analysis II
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Problem 1

(a) Let $\mathbf{x} \in \mathbf{R}^N$. Let $d = f(\mathbf{X}) + 1$, and let $D = \overline{B(\mathbf{x}, d)} \cap C$. Since D is the intersection of two closed sets, D is closed, and, since $\overline{B(\mathbf{x}, d)}$ is bounded, D is bounded, so that D is compact. Let $g : D \to \mathbb{R}$ such that, $\forall \mathbf{y} \in D$, $g(\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$. Since g is a continuous function on a compact set, by the Weierstrass Theorem, g attains a minimum on D; in particular, $\exists \mathbf{y_x} \in D$ such that $g(\mathbf{y_x}) = \min g(D)$. By definition of infimum, since $g(\mathbf{y_x})$ is a lower bound of $\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{y} \in C\}$. Since $\mathbf{y_x} \in C$, $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{y_x}\|$. ■

(b) Let $\mathbf{y} \in C$. Since $\mathbf{z}$ is in the line segment with endpoints $\mathbf{x}$ and $\mathbf{y_x}$,

$$\|\mathbf{y_x} - \mathbf{z}\| = \|\mathbf{y_x} - \mathbf{x}\| - \|\mathbf{x} - \mathbf{z}\|.$$

By choice of $\mathbf{y_x}$ as $\inf\{\|\mathbf{x} - \mathbf{y} : \mathbf{y} \in C\|\}$,

$$\|\mathbf{y_x} - \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\|.$$

Thus, by the Reverse Triangle Inequality,

$$\|\mathbf{y_x} - \mathbf{z}\| \leq \|\mathbf{y} - \mathbf{x}\| - \|\mathbf{x} - \mathbf{z}\| \leq \|\mathbf{y} - \mathbf{z}\|.$$

Therefore, since $\|\mathbf{y_x} - \mathbf{z}\|$ is a lower bound of $\{\|\mathbf{z} - \mathbf{y}\| : \mathbf{y} \in C\}$ and $\mathbf{y_x} \in C$, by definition of infimum, $f(\mathbf{z}) = \|\mathbf{y_x}\|$. ■

(c) Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N$. As shown in part (a), $\exists \mathbf{y}_1, \mathbf{y}_2 \in C$ such that $f(\mathbf{x}_1) = \|\mathbf{x}_1 - \mathbf{y}_1\|, f(\mathbf{x}_2) = \|\mathbf{x}_2 - \mathbf{y}_2\|$. By the Reverse Triangle Inequality, and the choice of $\mathbf{y}_1$,

$$\|\mathbf{x}_1 - \mathbf{x}_2\| \geq \|\mathbf{x}_1 - \mathbf{y}_2\| - \|\mathbf{y}_2 - \mathbf{x}_2\| \geq \|\mathbf{x}_1 - \mathbf{y}_1\| - \|\mathbf{y}_2 - \mathbf{x}_2\| = f(\mathbf{x}_1) - f(\mathbf{x}_2).$$

An identical proof shows that $\|\mathbf{x}_1 - \mathbf{x}_2\| \geq f(\mathbf{x}_2) - f(\mathbf{x}_1)$, so that $\|\mathbf{x}_1 - \mathbf{x}_2\| \geq |f(\mathbf{x}_1) - f(\mathbf{x}_2)|$, so that f is Lipschitz continuous with Lipschitz constant at most 1. If f is differentiable at some $\mathbf{x} \in \mathbb{R}^N$, then

$$||\nabla f(\mathbf{x})|| \leq \lim_{\mathbf{y} \to \mathbf{x}} \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|} \leq 1. \quad ■$$

(d) Suppose f is differentiable in $\mathbb{R}^N \backslash C$. Since, $\forall$ directions $\mathbf{v} \in \mathbb{R}^N$, $\forall \mathbf{x} \in \mathbb{R}^N$, $\left|\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x})\right| = \|\nabla f(\mathbf{x})\|\|\mathbf{v}\|$, it follows from the equality case of the Cauchy-Schwarz Inequality that the gradient of f is in the direction in which the directional derivative of f is maximised. For $\mathbf{v} = \mathbf{y_x} - \mathbf{x}$, by the Reverse Triangle Inequality

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}) = \lim_{t \to 0} \left(\frac{f(\mathbf{x} + t\mathbf{v}) - f(\mathbf{v})}{t} \right) \leq \lim_{t \to 0} \left(\frac{\|t\mathbf{v}\|}{t} \right) = \|\mathbf{v}\| = 1.$$

Since, as shown in part (c), $\|\nabla f(\mathbf{x})\| \leq 1$, $\|\nabla f(\mathbf{x})\| = 1$. ■

(e) Let $\mathbf{x} \in \mathbb{R}^N \backslash C$, and let $\mathbf{y}, \mathbf{z} \in C$ such that $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x} - \mathbf{z}\|$. Suppose, for sake of contradiction, that f were differentiable at $\mathbf{x}$. Then, the gradient of f is well-defined, and, furthermore, as shown in the solution to part (d) above,

$$\nabla f(\mathbf{x}) = \frac{\mathbf{y} - \mathbf{x}}{\|\mathbf{y} - \mathbf{x}\|} = \frac{\mathbf{z} - \mathbf{x}}{\|\mathbf{z} - \mathbf{x}\|}.$$

Then, $(\mathbf{y} - \mathbf{x})$ and $(\mathbf{z} - \mathbf{x})$ must be parallel, so that there must exists a line segment containing $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{x}$. Since $\|\mathbf{y} - \mathbf{x}\| = \|\mathbf{z} - \mathbf{x}\|$, $\mathbf{y} = \mathbf{z}$, contradicting the given that $\mathbf{y}$ and $\mathbf{z}$ are distinct.

(f) Let $N = 1$, let $C = \{-1, 1\} \subseteq \mathbb{R}^N$. Then, by the result of part (e), f is not differentiable at 0. ■

Problem 2

Let $E \subseteq \mathbb{R}^N$. Then, E has property (a) if and only if E has property (b).

Since, if U and V are disjoint sets, then $E \cap U \cap V = \emptyset$, if E has property (a), then E has property (b).

Suppose, on the other hand, that E has propety (b), so that there exist two open sets U and V such that

$$E \subseteq U \cup V, \; E \cap U \neq \emptyset \neq E \cap V, \; \text{and } E \cap U \cap V = \emptyset.$$

Let $U' = U \cap E$, and let $V' = V \cap E$. Since $E \subseteq U \cup V$, $E \subseteq U' \cup V'$. Let $f : \mathbb{R}^N \to [0, \infty)$ be the distance function from V', as defined in problem 1. Since U is open, $\forall \mathbf{x} \in U'$, $\exists r_{\mathbf{x}} > 0$ such that $B(\mathbf{x}, r_{\mathbf{x}}) \subseteq U$. Suppose, for sake of contradiction, that, for some $\mathbf{x}_0 \in U'$, $f(x) = 0$. Then, $\exists \mathbf{y} \in V_\prime$ such that $\|\mathbf{y} - \mathbf{x}\| < r_{\mathbf{x}}$. However, this would imply that $\mathbf{y} \in U \cap V_\prime = U \cap E \cap V$, contradicting the given. Thus, $\forall \mathbf{x} \in U_\prime$, $f(\mathbf{x}) > 0$. A similar proof by contradiction shows that, letting $g : \mathbb{R}^N \to [0, \infty)$ be the distance function from U', $\forall \mathbf{y} \in V_\prime$, $g(\mathbf{y}) > 0$. Let $U^* = \cup_{\mathbf{x} \in U'} B(\mathbf{x}, f(\mathbf{x}/4)$, and let $V^* = \cup_{\mathbf{y} \in V'} B(\mathbf{y}, g(\mathbf{y}/4)$. By construction, $U^* \cap V^* = \emptyset$. Since a union of open sets is open, U^* and V^* are open. Since $U' \subseteq U^*$ and $V' \subseteq V^*$, $E \subseteq U^* \cup V^*$. Thus, E has property (a). ■

Problem 3

(a) Let $\epsilon > 0$, and let $\epsilon_2 = \frac{\epsilon}{b-a}$. Since g is continuous, $\exists \delta > 0$ such that, $\forall f, h \in C^1([a, b])$ with $\|(x, f(x), f') - (x, h(x), h'(x))\| < \delta$, $|g(x, f(x), f'(x)) - g(x, f(x), f'(x)| < \epsilon_2$.

Thus, $\forall f, h \in C^1([a, b])$ with $\|(x, f(x), f') - (x, h(x), h'(x))\| < \delta$, by the Triangle Inequality,

$$\begin{aligned}
|G(f) - G(h)| &= \left| \int_a^b g(x, f(x), f') - g(x, h(x), h'(x) \right| dx \\
&\leq \int_a^b \left| g(x, f(x), f') - g(x, h(x), h'(x) \right| dx \\
&< \int \epsilon_2 dx = \epsilon_2 (b - a) = \epsilon.
\end{aligned}$$

Thus, G is continuous. $\blacksquare$

(b)

(c) Suppose f_0 is such that

$$\min_{f\in X} G(f) = G(f_0).$$

Let $v \in C^1([a,b])$ with $v(a) = v(b) = 0$. Consider the function $h : \mathbb{R} \to \mathbb{R}$ such that, $\forall t \in \mathbb{R}$, $h(t) = G(f + tv)$. Clearly, h must have a local minimum at $t = 0$, since G has a local minimum at f. By Theorem 282 (in the notes for Real Analysis I), $\frac{\partial G}{\partial v}(f) = \frac{dh}{dt}(0) = 0$. $\blacksquare$

(d) Let $h \in C([a,b])$ such that, $\forall v \in C^1([a,b])$ with $v(a) = v(b) = 0$,

$$\int_a^b h(x)v(x)\,dx = 0.$$

Suppose, for sake of contradiction, that $h \neq 0$. Without loss of generality, for some $x_0 \in [a,b]$, $h(x_0) > 0$ (otherwise, we take $(-h)$). Since h is continuous, for some $\delta > 0$, letting $c = \max\{x_0 - \delta, a\}, d = \min\{x_0 + \delta, b\}$. $\forall x \in [c,d]$, $|f(x) - f(x_0)| < f(x_0)$, so that $f(x) > 0$. Let $w = d - c$. Define $v : [a,b] \to \mathbb{R}$ piecewise as follows:

$$f(x) = \begin{cases} 0 & : \quad x \in [a,c] \cup [d,b] \\ (w(x-c))^2 & : \quad x \in (c, c + \frac{w}{4}] \\ \frac{1}{2} - (w(x - (c+1)))^2 & : \quad x \in (c + \frac{w}{4}, c + \frac{3w}{4}] \\ (w(x-(c+2)))^2 & : \quad x \in (c + \frac{3w}{4}, d) \end{cases}$$

Then, $v \in C^2([a,b])$, $\forall x \in (c,d), v(x) > 0$, and, $\forall x \in [a,c] \cup [d,b]$, $v(x) = 0$. Thus, $\forall x \in (c,d), h(x)v(x) > 0$, and, $\forall x \in [a,c] \cup [d,b]$, $h(x)v(x) = 0$. Then, however,

$$\int_a^b h(x)v(x)\,dx > 0,$$

which is a contradiction. Therefore, $h = 0$. $\blacksquare$

Problem 4

(a) $\min_{f\in X} G(f) = \boxed{L^2(b-a).}$ In particular, G is minimized by $f_0 := \left(x \mapsto L\frac{x-a}{b-a}\right)$. For suppose $f \in X$. Then, letting $F = f - f_0$,

$$G(f) = G(F + f_0) = \int_a^b \left((F + f_0)'\right)^2 = \int_a^b \left(F'\right)^2 + 2F'f_0' + \left(f_0'\right)^2.$$

By the Fundamental Theorem of Calculus, since $f_0'(x)$ is a constant with respect to x and $f(a) = f_0(a) = 0$ and $f(b) = f_0(b) = L$, so that $F(a) = F(b) = 0$,

$$\int_a^b 2F'(x)f_0'(x)\,dx = 2f_0'(x)\int_a^b F'(x)\,dx = 2f_0'(x)\left(F(b) - F(a)\right) = 2f_0'(x)(0) = 0.$$

Thus, since $f_0'(x)$ is a constant with respect to x,

$$G(f) = \int_a^b \left(F'\right)^2 + C,$$

for some constant, so that G is minimized when $F = 0$, and thus $f = f_0$. $\blacksquare$

(b) $\forall f \in C^1([a,b])$ and $\forall$ directions $v \in C^1([a,b])$ with $v(0) = v(1) = 0$, integrating by parts gives

$$G(f) = \int_0^1 \left(f'(x)\right)^2 - x^2 f'(x)\, dx.$$

By the result of Exercise 3, part (c), if f minimizes G, then

$$\int_0^1 \left(2f'(x) - x^2\right) v'(x)\, dx = \int_0^1 \frac{\partial}{\partial f'} \left(\left(f'(x)\right)^2 - x^2 f'(x)\right)\, dx = \frac{\partial G}{\partial v}(f) = 0.$$

Since $v(0) = v(1) = 0$, integrating by parts again shows that

$$\int_0^1 \left(f'' - 2x\right) v(x)\, dx = 0,$$

so that, by the result of Exercise 3, part (d), $f''(x) = 2x$.

Integrating with respect to x shows that, for $f \in C^1([a,b])$ such that, $\forall x \in [a,b]$, $f(x) = \boxed{\frac{1}{3}\left(x^3 - x\right),}$ f is the unique function satisfying the constraints $f''(x) = 2x$ and $f(0) = f(1) = 0$. Thus, G is minimized at f. $\blacksquare$