

**21-373, Algebraic Structures**, Department of Mathematical Sciences, Carnegie Mellon University  
**Fall 2011:** (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.  
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Assignment 2 - Friday September 16, 2011. Due Wednesday September 21

**Exercise 8:** If  $G$  is a group (not necessarily finite), show that every subgroup  $H$  of index  $[G:H] = 2$  is a normal subgroup.

Is it true if  $H$  has index 3?

**Exercise 9:** If  $G$  is a group and  $H, K$  are two subgroups of  $G$ , show that  $HK$  is a subgroup of  $G$  if and only if  $HK = KH$ .

**Exercise 10:** (Putnam 1968-B2) If  $A$  is a subset of a finite group  $G$ , and  $A$  contains more than one half of the elements of  $G$ , show that each element of  $G$  is the product of two elements of  $A$ .

**Exercise 11:** (Putnam 1969-B1) Let  $n$  be a positive integer such that  $n \equiv 23 \pmod{24}$ . Show that the sum of all the divisors of  $n$  is divisible by 24.

**Exercise 12:** (Putnam 1972-A5) Show that if  $n$  is an integer  $\geq 2$ , then  $n$  does not divide  $2^n - 1$ .

**Exercise 13:** (Putnam 1972-B3) Let  $a$  and  $b$  be two elements in a group such that  $aba = ba^2b$ ,  $a^3 = e$  and  $b^{2n-1} = e$  for some positive integer  $n$ . Show that  $b = e$ .

**Exercise 14:** (Putnam 1976-B2) Suppose that  $G$  is a group generated by two elements  $a$  and  $b$ , and that  $a^4 = b^7 = ab a^{-1}b = e$ , with  $a^2 \neq e$  and  $b \neq e$ .

- i) How many element of  $G$  are of the form  $c^2$  with  $c$  in  $G$ ?
- ii) Write each such square as  $a^m b^n$  for some  $m, n \in \mathbb{Z}$ .