Homework 3

21-640 Introduction to Functional Analysis
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Due: Friday, February 15, 2013
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Problem 3

We suppose $\mathcal{R}(T)$ is not of the first category in Y, and show that $\mathcal{R}(T) = Y$. Since $\mathcal{R}(T)$ is a linear manifold, it suffices to show that $\mathcal{R}(T)$ contains some ball, say, of radius $\delta > 0$, in Y, since, then, $\mathcal{R}(T)$ contains a Hamel basis of Y (normalized to have each element's norm less than δ). Didn't have time to finish writing this one.

Problem 4

By Remark 2.14, the identity function $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is discontinuous. Then, by the Closed Graph Theorem, the graph $\text{Gr}(I) = \{(x, x) : x \in X\}$ is not closed, so that there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X with $(x_n, x_n) \to (y, z)$ as $n \to \infty$ and $y \neq z$. From the definition of the product norm on X^2, it follows that $\|x_n - y\| \to 0$ and $\|x_n - z\| \to 0$ as $n \to \infty$. ■

Problem 6

Let $X = Y = c_0$ (equipped with the norm $\|\cdot\| = \|\cdot\|_\infty$, so that X and Y are Banach spaces), and define $T : X \to Y$ such that, for all sequences $\{x_n\}_{n=1}^{\infty} \in c_0$, $T\left(\{x_n\}_{n=1}^{\infty}\right) = \left\{n^{-1}x_n\right\}_{n=1}^{\infty}$. Clearly T is linear and injective, and, since $\sup\{\|Tx\| : x \in c_0, \|x\| = 1\} = 1$, T is continuous.

$\forall k \in \mathbb{N}$, since the sequence whose first k terms are 1 and whose remaining terms are 0 is in c_0, the sequence $S_k = \{x_n\}_{n=1}^{\infty}$ with $x_n = n^{-1}$ for $n \leq k$ and $x_n = 0$ otherwise is in c_0, and furthermore $\|S_k - \{1\}_{n=1}^{\infty}\| = n^{-1} \to 0\|$ as $k \to \infty$. However, since the constant sequence $\{1\}_{n=1}^{\infty} \notin c_0$, so $\{n^{-1}\}_{n=1}^{\infty} \notin T[X]$, and thus $T[X]$ is not closed. ■

Problem 7

Since V is continuous, the graph $\text{Gr}(V)$ is closed, so that $V[Y] \times Y = \{(Vy, y) : y \in Y\}$ is closed in $Z \times Y$. Thus, by definition of the product norm, $V[Y]$ is closed in Z, and thus, since V is linear, $V[Y]$ is a Banach space with $V : Y \to V[Y]$ bijective. Then, by the Bounded Inverse Theorem, $V^{-1} : V[Y] \to Y$ is continuous. Then, since $U = V^{-1} \circ T$, U is continuous. ■

Problem 8

Since $X \subseteq Y$, $\forall x_n \in \mathbb{R}^{\mathbb{N}}$, $\|x_n\|_Y \to 0$ as $n \to \infty$ implies $\|x_n\|_X \to 0$ as $n \to \infty$. Suppose some sequence $(x_n, x_n) \to (x, y)$ in $(X, \|\cdot\|_X) \times (X, \|\cdot\|_Y)$. Then, since $\|x_n - x\| \to 0$ a $n \to \infty$, $\|x_n - x\|_X \to 0$ as $n \to \infty$. It follows that $x = y$, so that the graph $\mathrm{Gr}(I)$ of the identity $I : (X, \|\cdot\|_X) \to (X, \|\cdot\|_Y)$ is closed.

By the Closed Graph Theorem, then, I is continuous. It follows, by Remark 2.14 that $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent norms on X, implying the desired result.

Problem 9

Suppose Y is a Banach space over $\mathbb{K}$, and let $L : X \to \mathbb{K}$ be discontinuous and linear (we showed the existence of such mappings in Problem 2 of Assignment 2). Since L is linear, the graph $X := \mathrm{Gr}(L)$ is a normed linear space (under the product norm). Let $\pi : X \to \mathbb{K}$ be the projection mapping $(x, Tx) \mapsto Tx$, noting that, by definiton of the product topology, projections are continuous. Then, $f : Y \to X$ defined by $f(x) = (x, Tx)$ is discontinuous, since otherwise $L = \pi \circ f$ would be continuous.

Since f is bijective, define $T = f^{-1}$. Since f is discontinuous, it is immediate from the topological definition of continuity that T is not open. However, since T is the just projection of X into Y, T is linear, continuous, and surjective, as desired. ■

Problem 10

Since $|a_n| \to \infty$ as $n \to \infty$, we can take a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ with $a_{n_k} > 4^k$, $\forall k \in \mathbb{N}$. Define $g : \mathbb{R} \to \mathbb{R}$ to be the 2π-periodic function defined by

$$g(x) = \sum_{k=1}^{\infty} 2^{-k} \sin(n_k x), \quad \forall x \in \mathbb{R}.$$

It is easy to show that the sequence of continuous functions $\sum_{k=1}^{n} 2^{-k} \sin(n_k x) \to g$ uniformly on $\mathbb{R}$ as $n \to \infty$, so that g is continuous. Note also that, $\forall n, m \in \mathbb{N}$,

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} \pi - \frac{\sin(4\pi n)}{4n} \geq 1 & : n = m \\ 0 & : \text{otherwise} \end{cases}.$$

Therefore, $\forall k \in \mathbb{N}$,

$$a_{n_k} \int_0^{2\pi} g(x)\sin(n_k x)\,dx \geq 4^k \sum_{i=1}^{\infty} \int_0^{2\pi} 2^{-i} \sin(n_i x)\sin(n_k x)\,dx \geq 4^k \left(2^{-k}\right) = 2^k \to \infty$$

as $k \to \infty$ (where we use the Bounded Convergence Theorem move the summation outside the integral). Thus, since it has an unbounded subsequence, the desired sequence is unbounded. ■