MATH 651: PROBLEM SET 4 SOLUTIONS ARE IN CLASS ON WED. OCT 24.

1. (10 points) Prove that $[0,1]^{\mathbb{N}}$ with the box topology is not compact. Box topology on a product is the smallest topology in which any product of open sets is open, more precisely it is the topology where sets of the form $\Pi_{i=1}^{\infty} U_i$, for U_i open in $[0,1]$, form a subbase.

2. (10 points) Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of nonempty topological spaces, and let $E_\alpha \subset X_\alpha$ be nonempty for every $\alpha \in \Lambda$. Fix $g \in \prod_{\alpha\in\Lambda} E_\alpha$ and consider the set
$$E := \left\{ f \in \prod_{\alpha\in\Lambda} E_\alpha : \ f(\alpha) = g(\alpha) \text{ for all but finitely many } \alpha \in \Lambda \right\}.$$
Prove that
$$\overline{E} = \overline{\prod_{\alpha\in\Lambda} E_\alpha}.$$

3. (15 points)
   (i) Show that the product of any (nonempty) family of nonempty Hasudorff topological spaces is Hausdorff.
   (ii) Show that the product of any (nonempty) family of nonempty completely regular topological spaces is completely regular.
   (iii) Let $(\mathbb{R}, \tau)$ be the Sorgenfrey line (that is τ is the topology with basis $[a, b)$ for $a, b \in \mathbb{R}$). Recall that $(\mathbb{R}, \tau)$ is a normal space. Show that the product topology on $\mathbb{R} \times \mathbb{R}$ is not normal.

4. (10 points) Prove that in $\mathbb{R}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{R}\}$ with the box topology, the set
$$U := \{f : \mathbb{N} \to \mathbb{R} : \ f \text{ is bounded}\}$$
is both open and closed, so that $\mathbb{R}^{\mathbb{N}}$ is not connected.

5. (15 points) Show that if $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a bounded continuous injection then the closure of $\gamma(\mathbb{R})$ in $\mathbb{R}^2$ is a compactification of $\mathbb{R}$ if $\gamma((a, b))$ is open in $\gamma(\mathbb{R})$ for all $a, b \in \mathbb{R}$.

   Find a compactification X of $\mathbb{R}$ which is homeomorphic to a subset of $\mathbb{R}^2$ such that the function $\cos : \mathbb{R} \to \mathbb{R}$ has a continuous extension to X. [Hint: Find γ such that the function $\cos \circ \gamma^{-1} : \gamma(\mathbb{R}) \to \mathbb{R}$ has a continuous extension to $X = \overline{\gamma(\mathbb{R})}$.]

6. (5 points) Show that $[0,1]^{\mathbb{N}}$ is sequentially compact.