Homework 1

21-470 Calculus of Variations
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Due: Wednesday, January 29, 2014

Problem 1

[I wasn't able to get very far with the integrals in parts (b) and (c), and wasn't able to determine much about the constants in part (d). Based on numerical computation, I believe (a) offers the slowest solution, followed by (b), (c), and then (d).]

(a) Since $y'(x) = 1$ for all $x \in [0, 1]$,

$$J(y) = \int_0^1 \frac{\sqrt{1 + y'(x)^2}}{\sqrt{y(x)}}\,dx = \sqrt{2}\int_0^1 x^{-1/2}\,dx = 2\sqrt{2}x^{1/2}\Big|_{x=0}^{x=1} = 2\sqrt{2}.$$

(b) Since

$$y(x) = \sqrt{x - x^2} \quad \text{and} \quad y'(x) = \frac{1 - x}{\sqrt{2x - x^2}}$$

for all $x \in (0, 1]$,

$$J(y) = \int_0^1 \frac{\sqrt{1 + (1 - x)^2/(2x - x^2)}}{(2x - x^2)^{1/4}}\,dx = \int_0^1 \frac{\sqrt{2x - x^2 + x^2 - 2x + 1}}{(2x - x^2)^{3/4}}\,dx$$
$$= \int_0^1 (2x - x^2)^{-3/4}\,dx.$$

(c) Since $y'(x) = x^{-1/2}/2$ for all $x \in (0, 1]$, using the inequality $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$,

$$J(y) = \int_0^1 \frac{\sqrt{1 + 1/(4x)}}{x^{1/4}}\,dx = \int_0^1 \sqrt{x^{-1/2} + x^{-3/2}/4}\,dx.$$

(d) Observing that

$$\frac{dx}{d\theta} = \frac{c^2}{2}(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin\theta}{1 - \cos\theta},$$

so that

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{\sin\theta}{1 - \cos\theta}\right)^2 = \frac{1 - \cos^2\theta}{(1 - \cos\theta)^2} = \frac{1 + \cos\theta}{1 - \cos\theta},$$

and changing variables from x to t, $J(y)$ simplifies significantly:

$$J(y) = \int_0^{\theta_1} \frac{\sqrt{1 + \frac{1+\cos\theta}{1-\cos\theta}}}{\sqrt{\frac{c^2}{2}(1 - \cos\theta)}} \frac{c^2}{2}(1 - \cos\theta)\,d\theta = \frac{c}{\sqrt{2}}\int_0^{\theta_1} \sqrt{1 + \frac{1 + \cos\theta}{1 - \cos\theta}}\sqrt{1 - \cos\theta}\,d\theta$$
$$= \frac{c}{\sqrt{2}}\int_0^{\theta_1} \sqrt{1 - \cos\theta + 1 + \cos\theta}\,d\theta = \frac{c}{\sqrt{2}}\int_0^{\theta_1} \sqrt{2}\,d\theta = c\theta_1.$$

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Problem 2

Since

$$\frac{d^2}{dx^2}\sqrt{1+x^2} = \frac{d}{dx}\frac{x}{\sqrt{1+x^2}} = \left(1+x^2\right)^{-3/2} \geq 0,$$

the function $x \mapsto \sqrt{1+x^2}$ is convex. Thus, by Jensen's Inequality and a linear change of variables

$$\begin{aligned}
J(y) = \int_a^b \sqrt{1+y'(x)^2}\,dx &= (b-a)\int_0^1 \sqrt{1+y'((b-a)u+a)^2}\,du && \text{(Change of Variables)}\\
&\geq (b-a)\sqrt{1+\left(\int_0^1 y'((b-a)u+a)\,du\right)^2} && \text{(Jensen's Inequality)}\\
&= \sqrt{(b-a)^2+\left(\int_a^b y'(x)\,dx\right)^2} && \text{(Change of Variables)}\\
&= \sqrt{(b-a)^2+(B-A)^2},
\end{aligned}$$

where the last step uses that $\int_a^b y'(x) = B - A$ by the Fundamental Theorem of Calculus.

Problem 3

(a) The bound represents the 'vertical component' of the surface area:

$$\begin{aligned}
J(y) = 2\pi\int_0^b y(x)\sqrt{1+y'(x)^2}\,dx &\geq 2\pi\int_0^b y(x)\sqrt{y'(x)^2}\,dx\\
&\geq \pi\int_0^b 2y(x)y'(x)\,dx = \pi y(x)^2\Big|_{x=0}^{x=b} = \pi B^2,
\end{aligned}$$

since $y \in \mathcal{Y}$ is non-negative. The bound essentially states that the minimal surface area is at least the surface area of the disc of radius B centered at $(b,0)$, normal to the x-axis. ■

(b) If, for any $x \in [0,b]$, $y(x) > 0$, then $y(x)\sqrt{1+y'(x)} > y(x)\sqrt{y'(x)^2}$. Since y and y' are continuous, it follows that the first inequality in part (a) above can be made strict, so that no $y \in \mathcal{Y}$ achieves $J(y) = \pi B^2$. ■

(c) There is indeed such a sequence. For $n \in \mathbb{N}$, define $y_n \in \mathcal{Y}$ by $y_n(x) = B(x/b)^n, \forall x \in [0,b]$. Then, using the inequality $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for $a,b \geq 0$,

$$\begin{aligned}
J(y_n) = 2\pi\int_0^b y_n(x)\sqrt{1+y_n'(x)^n} &= 2\pi\int_0^b B(x/b)^n\sqrt{1+(Bnx^{n-1}/b^n)^2}\,dx\\
&\leq 2\pi B\int_0^b (x/b)^n(1+Bnx^{n-1}/b^n)\,dx\\
&= 2\pi B\left(\frac{b}{n+1}+\frac{B}{2}\right) \to \pi B^2
\end{aligned}$$

as $n \to \infty$. By the result of part (a), it follows that $J(y_n) \to \pi B^2$ as $n \to \infty$. ■