MATH 759 DIFFERENTIAL GEOMETRY
LECTURE NOTES

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1. Differentiable Manifolds

Lecture 1: Definition of a manifold.

Definition 1. A *manifold*, $\mathcal{M}$, of dimension d is a Hausdorff topological space with countable basis, such that its every point has a neighborhood, Ω, that is homeomorphic to an open subset, U, of $\mathbb{R}^d$. The homeomorphism $x : U \to \Omega$ is called a *coordinate chart*.

An *atlas* is a family of coordinate charts $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ such that sets $\Omega_\alpha = x_\alpha(U_\alpha)$ constitute an open covering of $\mathcal{M}$.

Definition 2. A *differentiable manifold, $\mathcal{M}$, of dimension d is a manifold which has a differentiable atlas, that is an atlas such that for all $\alpha, \beta \in I$ for which $\Omega_\alpha \cap \Omega_\beta \neq \varnothing$*

$$x_\beta^{-1} \circ x_\alpha : x_\alpha^{-1}(\Omega_\alpha \cap \Omega_\beta) \to x_\beta^{-1}(\Omega_\alpha \cap \Omega_\beta)$$

is differentiable.

A chart is *compatible* with the given differentiable atlas if the union of the chart and the atlas is again a differentiable atlas. A maximal differentiable atlas is called a *differentiable structure*.

Exercise 3. Let A be a differentiable atlas for manifold $\mathcal{M}$. Show that the union of all differentiable atlases containing A is again a differentiable atlas. Thus for any differential atlas there exists a unique differentiable structure containing it.

All manifolds and functions considered in the rest of the notes are assumed to be differentiable, unless specified otherwise.

Lecture 2.

Examples of manifolds:

Example 4.

- Graph of a function

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- Sphere S^n
- Projective space P^n
- Torus T^n

Definition 5. Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds and $f : \mathcal{M} \to \mathcal{N}$ a mapping. f is said to be *differentiable* if for every two charts (U, ϕ) of $\mathcal{M}$ and (V, ψ) of $\mathcal{N}$ the function $\psi^{-1} \circ f \circ \phi$ is a smooth function on $\phi^{-1}(f^{-1}(\psi(V)))$.

Example 6.
- If $\mathcal{M} = (a, b)$ then f is a curve.
- If $\mathcal{N} = \mathbb{R}$ then we just say that f is a differentiable function on $\mathcal{M}$.

Definition 7 (Product manifold)**.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be manifolds. The product manifold is the set $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ on which the coordinate charts are given as follows: Let (U_1, ϕ_1) and (U_2, ϕ_2) be local charts for $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Then $(U_1 \times U_2, \phi_1 \times \phi_2)$, where $\phi_1 \times \phi_2(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$ is a coordinate chart for $\mathcal{M}$

Exercise 8. Show that the definition above provides a differentiable structure on $\mathcal{M}$.

1.1. **Tangent vectors.** Let $p \in \mathcal{M}$. Two descriptions of the tangent space $T_p\mathcal{M}$.

The first description is using the (differentiable) curves γ which for some $\varepsilon > 0$ map $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ so that $\gamma(0) = p$. On this set a relation $\sim$ is defined as follows $\gamma_1 \sim \gamma_2$ if there exists a coordinate chart ϕ, that contains p and

$$(\phi^{-1} \circ \gamma_1)'(0) = (\phi^{-1} \circ \gamma_2)'(0). \tag{1}$$

It is straightforward to show that given any chart ϕ that contains p, $\gamma_1 \sim \gamma_2$ is and only if (1) holds for that particular chart. It follows that $\sim$ is an equivalence relation.

Let $T_p\mathcal{M}$ be the set of equivalence classes. For $[\gamma_1], [\gamma_2] \in T_p\mathcal{M}$ and $a_1, a_2 \in \mathbb{R}$ we define $a_1[\gamma_1] + a_2[\gamma_2] := [\phi \circ (a_1\phi^{-1} \circ \gamma_1 + a_2\phi^{-1} \circ \gamma_2)]$ where ϕ is a coordinate chart for which $\phi(0) = p$. It is straightforward to check that the definition above does not depend on the choice of the curve that represents the equivalence class or the coordinate chart. Furthermore this gives $T_p\mathcal{M}$ the structure of a vector space.

We claim that $T_p\mathcal{M}$ is isomorphic as a vector space to $\mathbb{R}^n$, that is that there exists an invertible linear mapping $L : T_p\mathcal{M} \to \mathbb{R}^n$. More precisely, given the coordinate chart ϕ as above. Consider $L : [\gamma] \mapsto (\phi^{-1} \circ \gamma)'(0)$. It is straightforward to show that L is linear and invertible.

Lecture 3.

The second description of tangent vectors is related. It is discussed in detail in the textbook, here we only mention a few key facts. Let $\mathcal{D}$ be the set of smooth real-valued functions on $\mathcal{M}$. Given a point $p \in \mathcal{M}$

- The set of tangent vectors at p, $T_p\mathcal{M}$, is a subset of the set of linear functions from $\mathcal{D}$ to $\mathbb{R}$.
- The set of tangent vectors can be thought as the set of directional derivatives at p. More precisely given a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = p$ one can define $\gamma'(0) \in T_p\mathcal{M}$ to be the directional derivative in the direction defined by γ. That is for any $f \in \mathcal{D}$
$$\gamma'(0) : f \mapsto (f \circ \gamma)'(0).$$
- $T_p\mathcal{M}$ defined as above is also isomorphic to $\mathbb{R}^n$ and in particular given a coordinate chart (U, ϕ) containing p one can represent any tangent vector v at p as $v = a^1 \frac{\partial}{\partial x_1} + \cdots + a^n \frac{\partial}{\partial x_n}$. By this we mean that for any $f \in \mathcal{D}$, the value of $v(f)$ is as follows:
$$v(f) = a^1 \frac{\partial}{\partial x_1}(f \circ \phi)(\phi^{-1}(p)) + \cdots a^n \frac{\partial}{\partial x_n}(f \circ \phi)(\phi^{-1}(p)).$$

We remark that we write the indexes for coordinates of vectors as superscripts rather than subscripts.

Lecture 4: Tangent bundle, Differential.

Consider the disconnected union $T\mathcal{M} = \bigcup_{p \in \mathcal{M}} \{p\} \times T_p\mathcal{M}$ There is a natural projection $\Pi : T\mathcal{M} \to \mathcal{M}$ given by $\Pi : (p, v) \mapsto p$. $T\mathcal{M}$ is endowed with the manifold structure in the following way. Let (U, ϕ) be a coordinate chart for $\mathcal{M}$. Then on $U \times \mathbb{R}^n$ the mapping $\Phi : (x, a) \mapsto (\phi(x), a^1 \frac{\partial}{\partial x_1} + \cdots + a^n \frac{\partial}{\partial x_n})$ provides a coordinate chart for TM. In fact it is first used to define the topology on $T\mathcal{M}$. In particular the basis of topology is $\{\Phi(V) \ : \ V \subset U, V - \text{ open }, (U, \phi) - \text{ coordinate chart for } \mathcal{M}\}$.

Exercise 9. Show that $T\mathcal{M}$ defined above is indeed a differentiable manifold. Also if $v \in T_p\mathcal{M}$ is given in the chart (U, ϕ) as $a^1 \frac{\partial}{\partial x_1} + \cdots + a^n \frac{\partial}{\partial x_n}$ and in the chart (V, ψ) as $b^1 \frac{\partial}{\partial y_1} + \cdots + b^n \frac{\partial}{\partial y_n}$, find the relationship between coefficients a_i and b_j.

Definition 10 (Differential)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds, and let $f : \mathcal{M} \to \mathcal{N}$. Let $p \in \mathcal{M}$ and $q = f(p)$. The *differential of f at p* is the mapping
$$df_p : T_p\mathcal{M} \to T_q\mathcal{N}$$
defined as follows:
$$\text{For all } v \in T_p\mathcal{M}) \qquad df_p(v)[g] = v[g \circ f] \qquad \text{for all } g \in \mathcal{D}_q$$

1.2. **Fiber bundle.** First we consider the general, topological definition of fiber bundle.

Definition 11 (Fiber bundle)**.** Let F be a vector space, E and B be topological spaces, and $\pi : E \to B$. (E, B, π, F) is a *fiber bundle* is π as a surjection and for each $x \in E$ there exists a neighborhood Ω of $\pi(x)$ in B and a homeomorphism $\phi : \pi^{-1}(\Omega) \to \Omega \times F$ such that the diagram below commutes.

$$\begin{tikzcd} \pi^{-1}(\Omega) \arrow[r, "\phi"] \arrow[d, "\pi"] & \Omega \times F \arrow[dl, "\mathrm{proj}_1"] \\ \Omega & \end{tikzcd}$$

Above proj_1 is the projection to the first coordinate of the product $\Omega \times F$.

E is called the total space, B the base space, F a fiber, π the projection and ϕ a trivialization. In the setting of differential geometry we will consider both E and B to be manifolds, F to be finite dimensional, and require ϕ to be a diffeomorphism.

A *section* of a fiber bundle is a mapping $v : B \to E$ such that $\pi \circ v = id_B$.

For us, the main example of a fiber bundle is the tangent bundle $T\mathcal{M}$. Vector field is nothing but a section of the tangent bundle. The set of vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$.

Lecture 5: Cotangent bundle, immersion, submersion, embedding, orientation.

1.3. **Cotangent space.** Consider $f : \mathcal{M} \to \mathcal{N}$, where $\mathcal{N} = \mathbb{R}$. Then there exists a natural identification of $T_q\mathcal{N}$ with $\mathbb{R}$: Consider the coordinate chart given by the identity mapping $i_d : \mathbb{R} \to \mathbb{R}$. Using this chart, any vector $w \in T_q\mathbb{R}$ can be written as $w = c\frac{d}{dx}$ (then for $g : \mathbb{R} \to \mathbb{R}$, $w[g] = c(g \circ i_d)'(i_d^{-1}(q)) = cg'(q)$). The identification between $T_q\mathbb{R}$ and $\mathbb{R}$ is then given by $w \mapsto c$. Using this identification the differential df_p becomes a linear mapping $df_p : T_p\mathcal{M} \to \mathbb{R}$. In other words $df_p \in T_p\mathcal{M}^*$ the dual space to $T_p\mathcal{M}$. The space $T_p\mathcal{M}^*$ is called the cotangent space, and its element are called covectors.

We furthermore claim that for $f : \mathcal{M} \to \mathbb{R}$ and any $v \in T_p\mathcal{M}$

$$df(v) = v[f].$$

To see this note that if $df(v)$ is identified with $c \in \mathbb{R}$ then for $g : \mathbb{R} \to \mathbb{R}$ $df(v)[g] = cg'(q)$. So we need to show that $df(v)[g] = v[f]g'(q)$. Let $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = v$, in other words the curve which represents the tangent vector v. Then, using chain rule,

$$df(v)[g] = v[g \circ f] = \frac{d}{dt}(g \circ f \circ \gamma)(0) = g'(f(\gamma(0))\frac{d}{dt}(f \circ \gamma)(0) = g'(q)v[f]$$

as desired.

We note that since $T_p\mathcal{M}$ is n dimensional so is $T_p\mathcal{M}^*$. To find coordinates for $T_p\mathcal{M}^*$ write the inverse of a coordinate chart as $\phi^{-1}r = (x_1(r), \ldots, x_n(r))$ then each x_i is a differentiable function in a neighborhood of p and this we can consider $dx_1, \ldots dx_n$.

Exercise 12. Show that $dx_1, \ldots dx_n$ are linearly independent and span $T_p\mathcal{M}^*$

Therefore any $\omega \in T_p\mathcal{M}^*$ can be written in the form $\omega = \sum_i \alpha_i dx_i$. Note also that $\sum_i \alpha_i dx_i = d(\sum_i \alpha_i x_i)$ which implies that any element of $T_p\mathcal{M}^*$ is a differential of a function.

1.4. **Cotangent bundle.** Consider the set $T\mathcal{M}^* = \bigcup_{p\in\mathcal{M}}\{p\} \times T_p\mathcal{M}^*$. $T\mathcal{M}^*$ is endowed with the manifold structure in the following way. Let (U, ϕ) be a coordinate chart for $\mathcal{M}$. Then on $U \times \mathbb{R}^n$ the mapping $(x, \alpha) \mapsto (\phi(x), \alpha_1 dx_1 + \cdots + \alpha_n dx_n)$ provides a coordinate chart for $T\mathcal{M}^*$.

Exercise 13. Show that $T\mathcal{M}^*$ defined above is indeed a differentiable manifold. Also if $df \in T_p\mathcal{M}^*$ is given in the chart (U, ϕ) as $\alpha_1 dx_1 + \cdots + \alpha_n dx_n$ and in the chart (V, ψ) as $\beta_1 dx_1 + \cdots + \beta_n dx_n$, find the relationship between coefficients α_i and β_j. Compare this relationship to one obtained for tangent vectors in Exercise 9.

1.5. **Immersions, Submersions, and Embeddings, Orientation.** Here we followed Do Carmo [2] Chapter 0, Sections 3 and 4 (up to page 22).

Only that we also defined submersions: A mapping $F : \mathcal{M} \to \mathcal{N}$ is a *submersion* is for all $p \in \mathcal{M}$ the differential dF_p is an onto mapping from $T_p\mathcal{M}$ to $T_{F(p)}\mathcal{N}$.

LECTURE 6: (PROPERLY DISCONTINUOUS) GROUP ACTIONS.

Do Carmo Chapter 0, Section 4, pages 22-25.

LECTURE 7: VECTOR FIELDS, FLOWS, BRACKET.

Do Carmo Chapter 0, Section 5, pages 25-28. The existence and smoothness of (local) flows follows directly from results on existence, uniqueness and continuous dependence (and smooth dependance) on initial data for systems of ODE. Most graduate ODE book would have the results needed, for example see the book by Chicone [1].

Furthermore a careful explanation why XY is not a vector field was given.

LECTURE 8: LIE DERIVATIVE OF A VECTOR FIELD, PARACOMPACTNESS, PARTITION OF UNITY.

Definition 14 (Lie derivative)**.** Let X and Y be vector fields, and $p \in \mathcal{M}$. Let Φ_t be the flow of the vector field X.

$$L_X Y|_p = \lim_{t\to 0} \frac{d\Phi_{-t} Y|_{\Phi_t(p)} - Y|_p}{t} = \left.\frac{d}{dt}\right|_{t=0} d\Phi_{-t} Y|_{\Phi_t(p))} \tag{2}$$

Lemma 15.

$$L_X Y = [X, Y]$$

Proof. Recall that for function $f : \mathcal{M} \to \mathbb{R}$

$$Xf(p) = \left.\frac{d}{dt}\right|_{t=0} f(\Phi_t(p)).$$

Let Ψ_s be the flow of the vector field Y. Note that

$$L_X Y|_p = \frac{d}{dt}\bigg|_{t=0} Y|_{\Phi_t(p)}(f \circ \Phi_{-t}) = \frac{\partial^2 H}{\partial t \partial s}\bigg|_{(0,0)}$$

where

$$H(t,s) = f(\Phi_{-t}(\Psi_s(\Phi_t(p))).$$

In particular since

$$\frac{\partial H}{\partial s}\bigg|_{(t,0)} = Y|_{\Phi_t(p)}(f \circ \Phi_{-t}).$$

Consider $K(t,s,r) = f(\Phi_r(\Psi_s(\Phi_t(p))))$. Then $H(t,s) = K(t,s,-t)$. Thus

$$\frac{\partial^2 H}{\partial t \partial s}\bigg|_{(0,0)} = \frac{\partial^2 K}{\partial t \partial s}\bigg|_{(0,0,0)} - \frac{\partial^2 K}{\partial r \partial s}\bigg|_{(0,0,0)}.$$

We now compute

$$\begin{aligned}
\frac{\partial K}{\partial s}\bigg|_{(t,0,0)} &= Yf(\Phi_t(p)), \\
\frac{\partial^2 K}{\partial t \partial s}\bigg|_{(0,0,0)} &= XYf(p), \\
\frac{\partial K}{\partial r}\bigg|_{(0,s,0)} &= Xf(\Psi_s(p)), \\
\frac{\partial^2 K}{\partial r \partial s}\bigg|_{(0,0,0)} &= YXf(p).
\end{aligned}$$

Hence

$$\frac{\partial^2 H}{\partial t \partial s}\bigg|_{(0,0)} = [X,Y]f(p).$$

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Lecture 9: Problem Session.

Lecture 10: Lie Groups. Lecture 11: Lie Algebras.

Definition 16 (Lie Group)**.** A *Lie group* is a group, $(G,\cdot)$ which is also a differentiable manifold and such that the mapping $\cdot : G \times G \to G$ defined by $(x,y) \mapsto x \cdot y$ is a differentiable and that the inversion mapping mapping $g \mapsto g^{-1}$ is differentiable.

Example 17. The following are Lie groups:

- $(\mathbb{R}, +)$
- $(\mathbb{C} \backslash \{0\}, \cdot)$
- $(S^1, \cdot)$ is a subgroup of $(\mathbb{C} \backslash \{0\}, \cdot)$

- $GL(n, \mathbb{R})$ general linear group is the group of invertible $n \times n$ matrices with the group operation being the product of matrices. $\mathbb{R}$ is there to indicate that matrix coefficients are real numbers.
- $SL(n, \mathbb{R})$ special linear group is the group of $n \times n$ matrices with determinant 1.
- $O(n)$ orthogonal group — orthogonal matrices with real coefficients.
- $SO(n)$ special orthogonal group — orthogonal matrices with real coefficients and determinant 1.

Definition 18 (Lie Algebra)**.** A *Lie algebra* is a vector space, $\mathfrak{g}$, (over some field F) endowed with an operation (called the Lie bracket), $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which

(i) (is bilinear) For all $a, b, c \in F$ and $x, y, z \in \mathfrak{g}$

$$[ax + by, z] = a[x, z] + b[y, z]$$

and

$$[x, by + cz] = b[x, y] + c[x, z].$$

(ii) (alternates on $\mathfrak{g}$) For all $x \in \mathfrak{g}$

$$[x, x] = 0.$$

(iii) (satisfies the Jacobi identity) For all $x, y, z \in \mathfrak{g}$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

Bilinearity and alternating on $\mathfrak{g}$ imply that Lie bracket is antycommutative: For all $x, y \in \mathfrak{g}$, $[x, y] = -[y, x]$.

Example 19. The following are lie algebras. In all examples the filed $F = \mathbb{R}$.

- The set of vector fields on a manifold, endowed with the bracket $[X, Y] = XY - YX$.
- $\mathbb{R}^3$, with $[x, y] = x \times y$ (the cross product).
- $M(n, \mathbb{R})$ the set of $n \times n$ matrices with $[A, B] = AB - BA$.
- $\mathrm{sl}(n)$ the set of $n \times n$ matrices with trace zero. The Lie bracket is the commutator.
- $\mathrm{so}(n)$ the set of $n \times n$ skew-symmetric matrices ($A^T = -A$). Again, the Lie bracket is the commutator.

1.6. **Lie algebras associated to Lie groups.** There is a natural way to associate a Lie algebra to a Lie group. In a sense the Lie algebra is an infinitesimal description of the Lie group. Given a Lie group $(G, \cdot)$, let $e \in G$ be the identity element. Then the associated Lie algebra $\mathfrak{g} = T_e\mathcal{M}$ with the bracket defined as follows. Given an element $x \in G$ let $L_x : G \to G$ be the so-called *left translation* defined by

$$L_x(y) = x \cdot y \qquad \text{for all } y \in G.$$

Note that L_x gives an action of the group G on manifold G, and that it is a diffeomorphism. Given a vector $X_e \in T_e\mathcal{M} = \mathfrak{g}$ one defines a vector field X by

$$X|_x = dL_x X_e \qquad \text{for all } x \in \mathcal{M}.$$

We note that the vector field X is *left-invatiant*, meaning that for all $x, y \in G$ $dL_x X|_y = X|_{xy}$.

The Lie bracket $[\![\cdot,\cdot]\!]$ is, for all $X_e, Y_e \in \mathfrak{g}$ given by

$$[\![X_e, Y_e]\!] = [X, Y]|_e.$$

Exercise 20. Identify the Lie Algebras associated to the Lie groups given in the Example 17.

1.7. **Partition of unity.** Let (X, τ) be a topological space. $\mathcal{U}$ is an *open cover* of X if $\mathcal{U} \subset \tau$ and $\bigcup \mathcal{U} = X$. $\mathcal{W}$ is a *subcover* of an open cover $\mathcal{U}$ if $\mathcal{W} \subset \mathcal{U}$ and $\bigcup W = X$. $\mathcal{W}$ is a *refinement* of an open cover $\mathcal{U}$ if $\mathcal{W}$ is an open cover and for all $W \in \mathcal{W}$ there exists $U \in \mathcal{U}$ such that $W \subset U$. An open cover $\mathcal{U}$ is *locally finite* if for all $x \in X$ there exists an open neighborhood Ω or x such that Ω intersects at most finitely many sets in $\mathcal{U}$.

A topological space (X, τ) is *compact* if every open cover has a finite subcover. A space is *locally compact* if every point has a neighborhood whose closure is compact. A space is *paracompact* if every open cover has a locally finite refinement.

Lemma 21. *Every Hausdorff, locally compact topological space with countable basis is paracompact. In particular every manifold (as we defined it) is paracompact.*

The proof of this lemma can be found in Warner [3][Chapter 1, Section 1.9] as well as in the notes by Brian Conrad (Lemma 2.6) `http://math.stanford.edu/~conrad/diffgeomPage/handouts/paracompact.pdf`

Lemma 22 (176 it topology notes)**.** *Every paracompact space is normal.*

Theorem 23 (Urysohn metrication theorem, 179 in topology notes)**.** *A topological space is metrizable and separable if and only if it is normal and has countable basis.*

Corollary 24. *Every manifold is metrizable.*

Given a cover $\mathcal{U}$ a collection of continuous functions $\{\phi_i : i \in I\}$ is a partition of unity subordinated to $\mathcal{U}$ if $\phi_i \in C(X, [0,1])$ and for all $i \in I$ there exists $U \in \mathcal{U}$ such that $\operatorname{supp} \phi_i \subset U$. A partition of unity is locally finite if for all $x \in X$ there exists an open neighborhood Ω, such that $\{i \in I : \operatorname{supp} \phi_i \cap \Omega \neq \emptyset\}$ is finite.

Lemma 25. *Let (X, τ) be a paracompact space and let $\mathcal{U}$ be an open cover. Then there exists a locally finite partition of unity subordinated to $\mathcal{U}$.*

The proof follows from Lemma 178 in the general topology notes (also attached).

Corollary 26. *Let $\mathcal{M}$ be a manifold and $\{(U_i, \psi_i) : i \in I\}$ an atlas. Then there exists a partition of unity $\{\phi_j : j \in J\}$ subordinated to the open cover $\{\psi_i(U_i) : i \in I\}$ and such that all functions ϕ_j are smooth.*

Proof. The fact that there exists a partition of unity $\{\tilde{\phi}_j : j \in J\}$ subordinated to $\mathcal{U}$ follows from the previous lemma. Let η be a mollifier (a smooth function from

$\mathbb{R}^n \to [0,\infty)$, supported in $B(0,1)$ and such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$ and $\eta(0) > 0$). Let $\eta_\varepsilon(x) = \eta(x/\varepsilon)/\varepsilon^n$. Consider $j \in J$. Then there exists $i \in I$, depending on j, such that $\operatorname{supp} \tilde{\phi}_j \circ \psi_i \subset U_i$. Standard properties of convolution imply that for $\varepsilon_j > 0$ small enough the function $(\tilde{\phi}_j \circ \psi_i) * \eta_{\varepsilon_j}$ is smooth and its support is contained in U_i. Furthermore note that $(\tilde{\phi}_j \circ \psi_i) * \eta_{\varepsilon_j}$ is positive where $\tilde{\phi}_j \circ \psi_i$ is positive. Let $\phi(p) = \sum_{j \in J}((\phi_j \circ \psi_i) * \eta_{\varepsilon_j}) \circ \psi_i^{-1}(p)$. Note that ϕ is strictly positive on $\mathcal{M}$. We define $\phi_j = ((\phi_j \circ \psi_i) * \eta_{\varepsilon_j}) \circ \psi_i^{-1}/\phi$. It is straightforward to check that it is the desired partition of unity. ∎

Lecture 12: Tensors.

The book by Warner [3] has a more extensive (than what we covered) introduction to tensors.

1.8. **Tensor product of Vector Spaces.** Given two vector spaces V and W over $\mathbb{R}$ the tensor product can be defined via the *universal mapping property*: $V \times W$ is the vector space over $\mathbb{R}$ for which there exists a bilinear mapping $\varphi : V \times W \to V \otimes W$ such that for any vector space U over $\mathbb{R}$ and every **bilinear** mapping $l_2 : V \times W \to U$ there exists a **linear** mapping $l : V \otimes W \to U$ such that $l \circ \varphi = l_2$, that is that the following diagram commutes.

$$\begin{tikzcd}
V \otimes W \arrow[dr, "l"] & \\
V \times W \arrow[u, "\varphi"] \arrow[r, "l_2"] & U
\end{tikzcd}$$

We will show below that the $V \otimes W$ exists. One can also shaw that such $V \otimes W$ is unique (and hence well defined), up to isomorphisms (of vector spaces).

1.9. **Construction of $V \otimes W$.** The *free vector space* $F(V,W)$ is the vector space of linear combinations of elements of $V \times W$, that is of

$$F(V,W) = \left\{ \sum_i \alpha_i(v_i, w_i) \text{ with } \alpha_i \in \mathbb{R},\ v_i \in V \text{ and } w_i \in W \right\}.$$

Let $\varphi((v,w)) = (v,w)$ for all $(v,w) \in V \times W$. Given l_2 a bilinear mapping from $V \times W \to U$ one can define $l : F(V,W) \to U$ by setting $l((v,w)) = l_2(v,w)$. In such a way l is defined for all base elements of $F(V,W)$ and is thus uniquely determined on $F(V,W)$. However $F(V,W)$ is not the desired tensor product, since the function φ is not bilinear.

Let

$$R(V,W) = \operatorname{span}\{(v_1+v_2,w)-(v_1,w)-(v_2,w),\ a(v,w)-(av,w),\\ (v,w_1+w_2)-(v,w_2)-(v,w_2),\ a(v,w)-(v,aw)\\ :\ v,v_1,v_2 \in V, w, w_1, w_2 \in W, a \in \mathbb{R}\}.$$

Note that $R(V,W)$ is always a subset of the kernel of l. We now define $V \otimes W$ to be the quotient space $F(V,W)/R(V,W)$. The denote by $v \otimes w$ the equivalence class to which (v,w) belongs. We define $\varphi((v,w)) = v \otimes w$. We note that if for l_2 as above, we define $l(v \otimes w) = l_2(v,w)$ this does not depend on the choice of the representative of the class $v \otimes w$ and determines l uniquely. Furthermore the diagram commutes, that is $l \circ \varphi = l_2$.

1.10. **Basis of $V \otimes W$.** Note that if $\{e_i : i = 1,\ldots,n\}$ is a basis of V and $\{f_j : j = 1,\ldots,m\}$ is a basis of W then for $v = a_i e_i$ (here we use the summation convention) and $w = b_j f_j$ the properties above imply than $v \otimes w = a_i b_j e_i \otimes f_j$. It is straightforward to check that the vectors $e_i \otimes f_j$ are linearly independent. Thus they form a basis of $V \otimes W$. Consequently

$$\dim V \otimes W = (\dim V)(\dim W).$$

1.11. **Properties of the tensor product.** Furthermore it is straightforward to check that

$$(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z).$$

Lemma 27. *Let $L_2(V \times W)$ be the set of all bilinear functions from $V \otimes W \to \mathbb{R}$. Then*

$$(V \otimes W)^* \cong L_2(V \times W) \cong V^* \otimes W^*.$$

Exercise 28. Prove the lemma above.

Lecture 13: Tensor Algebra, Exterior Algebra.

1.12. **Tensor algebra.** Let $V^{\otimes r} = V \otimes \cdots \otimes V$ (r copies of V) and $V^{*\otimes s} = V^* \otimes \cdots \otimes V^*$ (s copies of V^*). Let for $r,s \geqslant 0$, $V_{r,s} = V^{\otimes r} \otimes V^{*\otimes s}$ where we consider $V_{0,0} = \mathbb{R}$. The tensor algebra is the direct sum (of vector spaces) $TV = \sum_{r,s\geqslant 0} V_{r,s}$.

There are two "operations" of interest on TV.

- *Contraction* is a mapping from $V_{r,s} \to V_{r-1,s-1}$ defined by
$$v_1 \otimes \cdots \otimes v_r \otimes \alpha_1 \otimes \cdots \otimes \alpha_s \mapsto \alpha_1(v_r)\, v_1 \otimes \cdots \otimes v_{r-1} \otimes \alpha_2 \otimes \cdots \otimes \alpha_s.$$
- *Tensor product* is a mapping from $V_{r_1,s_1} \times V_{r_2,s_2} \to V_{r_1+r_2,s_1+s_2}$ defined by
$$(v_1 \otimes \cdots \otimes v_{r_1} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{s_1},\ w_1 \otimes \cdots \otimes w_{r_2} \otimes \beta_1 \otimes \cdots \otimes \beta_{s_2})\\ \mapsto v_1 \otimes \cdots \otimes v_{r_1} \otimes w_1 \otimes \cdots \otimes w_{r_2} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{s_1} \otimes \beta_1 \otimes \cdots \otimes \beta_{s_2}$$

Let $I_k(V)$ be the subspace of $V_{k,0}$ generated by the vectors of the form

$$v_1 \otimes \cdots \otimes v_k \qquad \text{where } v_i = v_j \text{ for some } i \neq j$$

We consider the quotient space

$$\Lambda^k(V) = V_{k,0}/I_k(V).$$

We denote the equivalence class to which $v_1 \otimes \cdots \otimes v_k$ belongs by $v_1 \wedge \cdots \wedge v_k$.

Let S_k be the set of permutations of the set $\{1, \ldots, k\}$. The permutation that interchanges two elements (while leaving others in place) is called an inversion. The sign of a permutation, $\mathrm{sign}(\sigma)$ is 1 if the permutation can be written as a composition of an even number of inversions. Otherwise the sign is -1.

The following properties hold:

- $v_1 \wedge v_2 = -v_2 \wedge v_1$
- If $\sigma \in S_k$ is an inversion then
$$v_1 \wedge \cdots \wedge v_k = -v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}$$
- Also
$$v_1 \wedge \cdots \wedge v_k = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}. \tag{3}$$
- Let $n = \dim V$. The dimension of $\Lambda^k(V) = \binom{n}{k}$. Furthermore if $\{e_i\}_{i=1,\ldots,n}$ is a basis of V then vectors of the form
$$e_{\theta_1} \wedge \cdots \wedge e_{\theta_k}$$
where $1 \leqslant \theta_1 < \cdots < \theta_k \leqslant n$ forms a basis of $\Lambda^k(V)$. In particular note that $\Lambda^n(V) \cong \mathbb{R}$.

The *exterior algebra*, $\Lambda(V)$ is the direct sum $\sum_{k=\infty}^n \Lambda^k(V)$.

1.13. **Universal mapping property.** Given two vector spaces V and W over $\mathbb{R}$ the multilinear (i.e. linear in each component) mapping

$$h : V^k \to W$$

is called *alternating* if for all $v_1, \ldots, v_k \in V$ permutations $\sigma \in S_k$

$$h(v_1, \ldots v_k) = \mathrm{sign}(\sigma) h(v_{\sigma(1)}, \ldots, v_{\sigma(k)}.$$

We denote the set of alternating k-multilinear mappings from V^k to W by $\mathrm{Alt}_k(V, W)$.

Note that the mapping $\varphi : V^k \to \Lambda^k(V)$ defined by $(v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k$ is an alternating multilinear mapping. Furthermore the pair $\Lambda^k(V), \varphi$ satisfies the following universality property. For and vector space W (over $\mathbb{R}$) and any alternating multilinear mapping $h : V^k \to W$ there exists a unique linear mapping $\tilde{h} : \Lambda^k(V) \to W$ such that the diagram below commutes:

$$\begin{tikzcd}
\Lambda^k(V) \arrow[dr, "\tilde{h}"] & \\
V^k \arrow[u, "\varphi"] \arrow[r, "h"] & W
\end{tikzcd}$$

Example 29. Consider $V = \mathbb{R}^3$. Then $\Lambda^2(V) \cong \mathbb{R}^3$ and in particular $e_1 \wedge e_2 \mapsto e_3$, $e_1 \wedge e_3 \mapsto -e_2$, $e_2 \wedge e_3 \mapsto e_1$ defines an isomorphism. Under such identification the vedge is the cross product:

$$v_1 \wedge v_2 = v_1 \times v_2.$$

Furthermore

$$v_1 \wedge v_2 \wedge v_3 = \det[v_1\ v_2\ v_3].$$

Exercise 30. Show that there exists a natural (canonical) isomorphism (i.e. ones that does not depend on the choice of the basis) that establish:

$$\Lambda^k(V^*) \cong \mathrm{Alt}_k(V, \mathbb{R}) \cong (\Lambda^k(V))^*.$$

Hint: Given $\alpha_i \in V^*$, define $\Phi : \Lambda^k(V^*) \to \mathrm{Alt}(V^k, \mathbb{R})$ by defining in on $\alpha_1 \wedge \cdots \wedge \alpha_k$ and then extending it linearly. For $v_i \in V$ we set

$$\Phi(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \ldots, v_k) = \sum_{\sigma \in S_p} \mathrm{sign}(\sigma) \alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)}).$$

For the other isomorphism utilize the universal mapping property.

We will often use the identifications above without mentioning that explicitly.

Lecture 14: Tensors, Differential forms.

1.14. **Exterior product.** Note that fo $\Lambda^n(V^*) \cong \mathbb{R}$ and that for $k > n$, $\Lambda^n(V^*)$ is the trivial vector space (it contains only the zero element). On it the exterior algebra one can consider the exterior product, $\wedge$, which for any $r, s \geqslant 0$ is a mapping $\Lambda^r(V^*) \times \Lambda^s(V^*) \to \Lambda^{r+s}(V^*)$ defined as follows. For $f \in \Lambda^r(V^*)$ and $g \in \Lambda^r(V^*)$, $v \in V^{r+s}$

$$(4)\quad f \wedge g(v_1, \ldots, v_{r+s}) = \frac{1}{r!\, s!} \sum_{\sigma \in S_{r+s}} \mathrm{sign}(\sigma) f(v_{\sigma_1}, \ldots, v_{\sigma(r)})\, g(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}).$$

If either r or s is 0, then the wedge product defined to be just to product with a scalar (an element of $\mathbb{R}$). Note that $f \wedge g = (-1)^{rs} g \wedge f$.

Exercise 31. Show that for $\alpha_i \in V^*$ for $i = 1, \ldots, r+s$

$$(\alpha_1 \wedge \cdots \wedge \alpha_r) \bigwedge (\alpha_{r+1} \wedge \cdots \wedge \alpha_{r+s}) = \alpha_1 \wedge \cdots \wedge \alpha_{r+s}$$

where by $\bigwedge$ we denoted the exterior product defined in (4).

1.15. **Tensor bundles.** are generalizations of the tangent and cotangent bundle. That is the (r, s) tensor bundle $T_{r,s}\mathcal{M}$ is the bundle over $\mathcal{M}$ for which a fiber over point $p \in \mathcal{M}$ is $T_p\mathcal{M}^{\otimes r} \otimes T_p\mathcal{M}^{*\otimes s}$. The coordinate charts are given as follows. Let $\phi : U \to \mathcal{M}$ be a coordinate chart for $\mathcal{M}$. Then $\Phi : U \times \mathbb{R}^{rn} \times \mathbb{R}^{sn} \to T_{r,s}\mathcal{M}$ given by

$$(x, a) \mapsto \left(\phi(x),\; a^{i_1,\dots,i_r}_{j_1,\dots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}\right)$$

where we used the summation convention and n is the dimension of the manifold. We also use the convention to index the coordinates corresponding to vectors as upper indexes, and those corresponding to covectors as lower indexes.

We also consider the *exterior k-bundle*, $\Lambda^*_k(\mathcal{M})$ where the fibers are $\Lambda_k(T_p\mathcal{M}^*)$. The *exterior bundle* is the one with fibers $\Lambda(T_p\mathcal{M}^*)$.

1.16. **Tensors and differential forms.** A tensor is a section of a tensor bundle, while a differential k-form is a section of the exterior k-bundle. The sections of the $T_{1,0}\mathcal{M}$ tensor bundle are the vector fields, which we denote by $\mathcal{X}(\mathcal{M})$. The set of k-forms is denoted by $E^k(\mathcal{M})$.

Let (U, ϕ) and (V, ψ) be two coordinate charts such that $p \in \phi(U) \cap \psi(V)$. Let $T \in T_{r,s}(\mathcal{M})$. Then in a neighborhood of p there exist smooth functions $a^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ and $b^{k_1,\dots,k_r}_{l_1,\dots,l_s}$ such that

$$\begin{aligned}
T &= a^{i_1,\dots,i_r}_{j_1,\dots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s} \\
&= b^{k_1,\dots,k_r}_{l_1,\dots,l_s} \frac{\partial}{\partial y_{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial y_{k_r}} \otimes dy_{l_1} \otimes \cdots \otimes dy_{l_s}
\end{aligned}$$

Here we used the summation convention. To find the relationship between coefficients note that

$$\begin{aligned}
& a^{i_1,\dots,i_r}_{j_1,\dots,j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}(y_{m_1}, \dots, y_{m_r}, \frac{\partial}{\partial y_{q_1}}, \dots, \frac{\partial}{\partial y_{q_s}}) \\
=\; & b^{k_1,\dots,k_r}_{l_1,\dots,l_s} \frac{\partial}{\partial y_{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial y_{k_r}} \otimes dy_{l_1} \otimes \cdots \otimes dy_{l_s}(y_{m_1}, \dots, y_{m_r}, \frac{\partial}{\partial y_{q_1}}, \dots, \frac{\partial}{\partial y_{q_s}}).
\end{aligned}$$

Thus

$$a^{i_1,\dots,i_r}_{j_1,\dots,j_s} \frac{\partial y_{m_1}}{\partial x_{i_1}} \cdots \frac{\partial y_{m_r}}{\partial x_{i_r}} \cdot \frac{\partial x_{j_1}}{\partial y_{q_1}} \cdots \frac{\partial x_{j_s}}{\partial y_{q_s}} = b^{m_1,\dots,m_r}_{q_1,\dots,q_s}. \tag{5}$$

Similarly any differential k-form can be written as

$$\begin{aligned}
\omega &= \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \alpha_{i_1,\dots,i_k}\, dx_{i_1} \wedge dx_{i_k} \\
&= \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \beta_{j_1,\dots,j_k}\, dy_{j_1} \wedge dy_{j_k}.
\end{aligned}$$

Hence

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \alpha_{i_1,\ldots,i_k}\, dx_{i_1} \wedge dx_{i_k} \left(\frac{\partial}{\partial y_{l_1}}, \ldots, \frac{\partial}{\partial y_{l_k}}\right)$$
$$= \sum_{1\leqslant j_1<\cdots<j_k\leqslant n} \beta_{j_1,\ldots,j_k}\, dy_{j_1} \wedge dy_{j_k} \left(\frac{\partial}{\partial y_{l_1}}, \ldots, \frac{\partial}{\partial y_{l_k}}\right).$$

Thus

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \alpha_{i_1,\ldots,i_k} \frac{1}{k!} \sum_{\sigma\in S_k} \operatorname{sign}(\sigma) \frac{\partial x_{i_1}}{\partial y_{l_{\sigma(1)}}} \cdots \frac{\partial x_{i_k}}{\partial y_{l_{\sigma(k)}}} = \frac{1}{k!}\beta_{l_1,\ldots,l_k}. \tag{6}$$

Lecture 15: Differential forms: Exterior product and exterior derivative.

On the level of vector spaces the exterior product was defined in Section 1.14. To define exterior product of differential forms one just uses the exterior product in each fiber, that is the product is defined pointwise. So for $k \geqslant 0$ and $l \geqslant 0$, $\wedge : E^k(\mathcal{M}) \times E^l(\mathcal{M}) \to E^{k+l}(\mathcal{M})$ is defined by

$$\omega_1 \wedge \omega_2|_p = \omega_1|_p \wedge \omega_2|_p$$

where the wedge on the right hand side, $\wedge : \Lambda^k(T_p\mathcal{M}^*) \times \Lambda^l(T_p\mathcal{M}^*) \to \Lambda^{k+l}(T_p\mathcal{M}^*)$ the one defined in Section 1.14. Note that for $k = 0$ $E^k(\mathcal{M}) = C^\infty(\mathcal{M})$, the set of smooth real-valued functions on $\mathcal{M}$. Then for $f \in E^0(\mathcal{M})$ and $\omega \in E^l(\mathcal{M})$, $f \wedge \omega = f\omega$, that is at every point the form ω is multiplied by the value of the function at the point.

Properties

(i) $\omega_1 \wedge \omega_2 = (-1)^{kl}\omega_2 \wedge \omega_1$
(ii) $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$
(iii) Exterior product is a bilinear mapping.
(iv) If k is odd then for any k-form ω: $\omega \wedge \omega = 0$.

1.17. **Pullback of covariant tensors and differential forms.** Let S be a $(0, s)$ tensor. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a differentiable mapping and S a $(0, s)$ tensor on $\mathcal{N}$. The pull back of S is the $(0, s)$ tensor Φ^*S on $\mathcal{M}$ defined as follows:

$$\Phi^*S|_p(v_1, \ldots, v_s) = S|_{\Phi(p)}(D\Phi v_1, \ldots D\Phi v_s) \quad \text{for all } v_1, \ldots, v_s \in T_p\mathcal{M}.$$

Pullback of differential forms is defined analogously. If f is a function then $\Phi^*f = \Phi \circ f$.

Properties

(i) $\Phi^*(S_1 \otimes S_2) = \Phi^*(S_1) \otimes \Phi^*(S_2)$
(ii) For any differential forms ω_1 and ω_2 on $\mathcal{M}$, $\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*(\omega_1) \wedge \Phi^*(\omega_2)$
(iii) Exterior derivative and pullback commute; that is for any from ω

$$\Phi^*(d\omega) = d\Phi^*(\omega).$$

(iv) If $\Psi : \mathcal{N} \to \mathcal{P}$ then $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$. Consequently, if Φ is a diffeomorphism then $(\Phi^*)^{-1} = (\Phi^{-1})^*$

We remark that these properties (except for (iv), which is straightforward) have been proved as part of the problem set 2.

We note that if Φ is a diffeomorphism then we can push and pull at the same time. Namely if T is an (r, s) tensor on $\mathcal{N}$ we define $\Phi^* S$ to be an (r, s) tensor on $\mathcal{M}$ by

(7)

$$\Phi^*(S)|_p(f_1, \dots, f_r, X_1, \dots, X_s) = S|_{\Phi(p)}(f_1 \circ \Phi^{-1}, \dots, f_r \circ \Phi^{-1}, D\Phi_p X_1, \dots D\Phi_p X_s)$$

We note that this "extended" pull-back still satisfies the property (i) above.

1.18. **Exterior derivative.** Exterior derivative d is a linear mapping from $E(\mathcal{M})$ to $E(\mathcal{M})$ such that d maps k-forms to $k+1$-forms and that

(i) If f is a function (that is a 0-form) then df is the differential of the function.
(ii) If f is a function then $ddf = 0$.
(iii) For k form ω_1 and l-form ω_2

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \omega_2.$$

Let us first investigate the properties of d, assuming that it exists. Let f be a smooth function and ω a k-form. Then by property (iii)

$$d(f\omega) = d(f \wedge \omega) = df \wedge \omega + f \wedge d\omega.$$

Furthermore note that if in some coordinate system $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ then, by induction on k and using the properties (ii) and (iii) one obtains that $d\omega = 0$. These two properties and the linearity of d imply that if we consider a general k form ω written in local coordinates:

$$\omega = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \alpha_{i_1, \dots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{8}$$

then

$$d\omega = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} d\alpha_{i_1, \dots, i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{9}$$

Lemma 32. *On any manifold $\mathcal{M}$ there exists a unique exterior derivative. It's form in local coordinates is given by* (9).

Proof. By the argument above the exterior derivative in local coordinates must have form (9). It thus suffices to show that d defined by (9) does not depend on the coordinate chart considered. So let (U, ϕ) and (V, ψ) be two local charts in a neighborhood of some point $p \in \mathcal{M}$. Let $\Omega = \phi(U) \cap \psi(V)$ and let $\tilde{U} = \phi^{-1}(\Omega)$ and $\tilde{V} = \psi^{-1}(\Omega)$. Note that $\tilde{U}$, $\tilde{V}$ and Ω are manifolds in their own right and that $\tilde{U}$ and $\tilde{V}$ are submanifolds of $\mathbb{R}^n$. Let us name the coordinates on $\tilde{U}$ by x and those on $\tilde{V}$ by y. In the x coordinates ω takes form (8) while in y coordinates it takes form

$$\omega = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \beta_{j_1,\ldots,j_k}\, dy_{j_1} \wedge dy_{j_k}.$$

Let ω_1 be the pullback of ω to $\tilde{U}$, that is $\omega_1 = \phi^*\omega$. Likewise let $\omega_2 = \psi^*\omega$. By property (iv) of pull-back it follows that $\omega_1 = (\psi^{-1} \circ \phi)^*\omega_2$. We note that ω_1 has the form (8), only that the coordinates are now on $\tilde{U}$ and not on Ω; likewise for ω_2.

We also note that the mapping $\omega_1 \mapsto d\omega_1$ defined by (9) (with ω replaced by ω_1) is an exterior derivative on manifold $\tilde{U}$ since it satisfies all properties of the exterior derivative. An analogous exterior differentiation exists on $\tilde{V}$:

$$d\omega_2 = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} d\beta_{j_1,\ldots,j_k} \wedge dy_{j_1} \wedge dy_{j_k}.$$

Since $\omega_1 = (\psi^{-1} \circ \phi)^*\omega_2$ by property (iii) of the pullback it follows that $d\omega_1 = (\psi^{-1} \circ \phi)^* d\omega_2$. Therefore $(\phi^{-1})^* d\omega_1 = (\phi^{-1})^* \circ (\psi^{-1} \circ \phi)^* d\omega_2 = (\psi^{-1})^* d\omega_2$ on Ω, which proves the claim. ■

Lemma 33. *For any form ω, $dd\omega = 0$.*

Proof. Form (9) and properties (iii) and (ii) follows

$$\begin{aligned} dd\omega = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} & dd\alpha_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ & + (-1)^1 d\alpha_{i_1,\ldots,i_k} \wedge ddx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_k} \\ & + \quad \cdots\cdots \\ & + (-1)^k d\alpha_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge ddx_{i_k} \\ = 0. & \end{aligned}$$

■

Lecture 16: Interior product.

Given a vector field X the interior product is a mapping from $(0, s)$ tensors to $(0, s-1)$ tensors and also from k-forms to $k-1$-forms. For tensors it is a special form of a contraction defined in Lecture 1.12. Given a $(0, s)$ tensor T we define

$$(i_X S)|_p(v_1, \ldots, v_{s-1}) = S|_p(X(p), v_1, \ldots, v_s) \tag{10}$$

Interior product for differential forms is defined analogously.

Properties

(i) $i_X(df) = df(X) = X[f]$.
(ii) $i_X \circ i_Y = -i_Y \circ i_X$ and, consequently, $i_X \circ i_X = 0$.

(iii) For k-form α and l-form β, $i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta)$.

Proofs are straightforward, and rely on the definitions and properties of the wedge product.

1.19. **Lie derivative of tensors and differential forms.** Let $\mathcal{M}$ ba a manifold and $p \in \mathcal{M}$ Given a vector field X, let Φ_t be the flow associated to the vector field. We note that there exists a neighborhood Ω of p and $\tau > 0$ such that for all $t \in [-\tau, \tau]$, Φ_t is defined on Ω and is a diffeomorphism (between Ω and $\Phi_t(\Omega)$).

Let T be a (r, s) tensor on $\mathcal{M}$. We define the *Lie derivative* of T at p to be

$$L_X T = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* T \tag{11}$$

Where the general pull-back is defined using (7).

We note that for vector fields

$$L_X Y = \left.\frac{d}{dt}\right|_{t=0} \Phi_t^* Y = \left.\frac{d}{dt}\right|_{t=0} d\Phi_{-t} Y$$

since $\Phi_t^{-1} = D\Phi_{-t}$. This confirms that the new definition of Lie derivative agrees with the old one.

Properties. Let X be a vector field on $\mathcal{M}$.

(i) If f is a smooth function on $\mathcal{M}$ then $L_X f = X[f]$.
(ii) If Y is a vector field then $L_X Y = [X, Y]$.
(iii) If S, T are tensord: $L_X(S \otimes T) = (L_X S) \otimes T + S \otimes (L_X T)$
(iv) If α and β are differential forms: $L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge L_X\beta$.
(v) If Y is a vector field and ω a differential form then: $L_X(i_Y\omega) = i_Y(L_X\omega)$.
(vi) If ω is a differential form then $L_X(d\omega) = dL_X(\omega)$.

The proofs are straightforward from the definition and properties of the pull-back, see Section 1.17. [1]

Theorem 34 (Cartan's formula)**.**

$$L_X = d \circ i_X + i_X \circ d$$

Proof. If f is a function then

$$L_X f = X[f] = i_X df = (d \circ i_X + i_X \circ d)f.$$

Furthermore

$$L_X df = dL_X f = d(X[f]) = d(i_X(df)) = (d \circ i_X + i_X \circ d)df.$$

Consider the mapping on the set of differential forms defined by

$$P_X\omega = (d \circ i_X + i_X \circ d)\omega.$$

[1]During the lecture, I mentioned in the proof of (vi) that the derivatives commute since they are with respect to different variables. That is in fact not necessary. Namely in the expression in local coordinates we looked at the derivatives would commute even with respect to the same variables.

Note that P_X is linear and maps k-forms to k-forms. A direct calculation shows that

$$P_X(\alpha \wedge \beta) = P_X\alpha \wedge \beta + \alpha \wedge P_X\beta. \tag{12}$$

If we consider an arbitrary 1-form ω then to show that $L_X\omega = P_X\omega$ it is enough to show the equality in a coordinate chart, where ω can be written as $\omega = \sum_i \alpha_i dx_i$. By linearity it is enough to consider $\alpha_i dx_i$, for which the equality follows by using the property (12) and that $L_X\alpha_i = P_X\alpha_i$ and $L_X dx_i = P_X dx_i$.

Using the property (12) and the property (iv) of Lie derivative, by induction if follows that $L_X = P_X$ for any differential form. ■

Lecture 17: Integration of n-forms on oriented manifolds.

Let $\mathcal{M}$ be an oriented manifold and ω a differential form. All of the charts we use belong to the same orientation. We want to define $\int_{\mathcal{M}} \omega$.

Let us first consider the case that there exists a coordinate chart (U, ϕ) such that $\operatorname{supp} \omega \subset \phi(U)$. Then there exists a smooth function $\alpha : \mathcal{M} \to \mathbb{R}$ supported in $\phi(U)$ such that

$$\omega = \alpha dx_1 \wedge \cdots \wedge dx_n.$$

We define

$$\int_{\mathcal{M}} \omega = \int_U \alpha \circ \phi \, dx_1 \cdots dx_n.$$

To show that this is a good definition we need to ensure that it does not depend on the chart chosen. So let (V, ψ) be another chart so that $\operatorname{supp}(\omega) \subset \psi(V)$. Then there exists β such that $\omega = \beta dy_1 \wedge \cdots \wedge dy_n$ Then using (6), the fact that the charts are from the same orientation and the change of variables in an integral we obtain

$$\begin{aligned} \int_V \beta \circ \psi dy_1 \dots dy_n &= \int_V \alpha \circ \psi \det\left(\frac{\partial x}{\partial y}\right) dy_1 \dots dy_n \\ &= \int_V \alpha \circ \psi \left|\det\left(\frac{\partial x}{\partial y}\right)\right| dy_1 \dots dy_n \\ &= \int_U \alpha \circ \phi \, dx_1 \dots dx_n. \end{aligned}$$

Now let us consider a general n-form on an oriented compact manifold $\mathcal{M}$. Let f_i, $i = 1, \dots, m$ be a partition of unity subordinate to some atlas. Let $\omega_i = f_i\omega$. Then ω_i are n-forms and $\sum_i = 1^m \omega_i = \omega$. We define

$$\int_{\mathcal{M}} \omega = \sum_{i=1}^{m} \int_{\mathcal{M}} \omega_i$$

where the integrals on the right-hand side are as defined above.

Exercise 35. Show that the definition above does not depend on the choice of atlas (from the differential structure of the given orientation) or the partition of unity.

Hint: Given two atlases $\{(U_i, \phi_i) : i = 1, \dots k\}$ and $\{(V_j, \psi_j) : j = 1, \dots l\}$ consider the open cover of $\mathcal{M}$ that consists of intersections of the form $\phi_i(U_i) \cap \psi_j(V_j)$. If f_i and g_j are partitions of unity suborinate to atlases above, note that the products $f_i g_j$ constitute a partition of unity subordinate to the cover made of intersections. Use this to "bridge" between the expressions for the integral in the two atlases given.

We remark that if the same form is integrated with respect to the other orientation of manifold then the sign of the integral changes!

Lecture 18: Stokes Theorem.

1.20. **Manifolds with boundary.** Let $H^n = \{x \in \mathbb{R}^n : x_1 \leqslant 0\}$. Let $\partial H = \{0\} \times \mathbb{R}^{d-1}$, that is let ∂H be the boundary of H, when it is considered as a subset of $\mathbb{R}^d$. [2]

Definition 36. A *manifold with boundary*, $\mathcal{M}$, of dimension n is a Hausdorff topological space with countable basis, such that its every point has a neighborhood, Ω, that is homeomorphic to an open subset, U, of H^n.

We note that if in some chart (U, ϕ) and some $p \in \Omega = \phi(U) \subset \mathcal{M}$, it holds that $\phi^{-1}(p) \in \partial H$, then for any other chart the image of p belongs to ∂H. We denote the set of such p by $\partial\mathcal{M}$. We note that $\partial\mathcal{M}$ is a closed subset of $\mathcal{M}$.

Definition 37. A *differentiable manifold with boundry, $\mathcal{M}$, of dimension n is a manifold with boundary which has a differentiable atlas, that is an atlas such that for all $\alpha, \beta \in I$ for which $\Omega_\alpha \cap \Omega_\beta \neq \varnothing$*

$$\phi_\beta^{-1} \circ \phi_\alpha : \phi_\alpha^{-1}(\Omega_\alpha \cap \Omega_\beta) \to \phi_\beta^{-1}(\Omega_\alpha \cap \Omega_\beta)$$

is differentiable.

Note that if $\mathcal{M}$ is a differentiable manifold with boundary then $\mathcal{M} \backslash \partial\mathcal{M}$ is a differential manifold (without boundary), while (by restricting the charts to ∂H) we see that ∂M is an (n-1) dimensional manifold.

Let us mention that for every manifold with boundary $\mathcal{M}$ there exists a manifold $\mathcal{M}^*$ such that $\mathcal{M}$ is embedded in $\mathcal{M}*$.

Consider now an oriented manifold with boundary $\mathcal{M}$. Recall that a manifold is oriented if the derivative of the transition maps between charts has a positive determinant, or (equivalently) if there is a volume form on $\mathcal{M}$. We note that at every $p \in \partial M$ the tangent space to boundary ∂M can be seen as a subset of $T_p\mathcal{M}$. Furthermore $T_p\mathcal{M}$ can be divided into inwards vectors, $T_p^{in}\mathcal{M}$, and outward

[2] Note that this is a different half-space than the one used in the lectures. This one makes the orientation of the half-space compatible with the natural orientation of ∂H.

vectors, $T_p^{out}\mathcal{M}$, as follows: $v \in T_p\mathcal{M}$ is a inward tangent vector if $v \in T_p$ and there exists a curve $\gamma : [0, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. We set $T_p^{out}\mathcal{M} = T_p\mathcal{M} \backslash T_p^{in}\mathcal{M}$. Note that $T_p^{in}\mathcal{M}$ is a closed set and contains $T_p\partial\mathcal{M}$. Also note that $\frac{\partial}{\partial x_1}$ is an outward tangent vector at $p \in \partial\mathcal{M}$.

We say that the orientation of ∂M is compatible with the orientation of $\mathcal{M}$ if for every oriented coordinate chart (U, ϕ) of $\partial\mathcal{M}$ and every oriented chart (V, ψ) of $\mathcal{M}$, and every $p \in \phi(U) \cap \psi(V)$ (which implies that $\psi^{-1}(p) \in \partial H$) then $\det(D((\psi|_{\partial H})^{-1} \circ \phi)) > 0$.

1.21. **Stokes theorem.**

Theorem 38. *Let $\mathcal{M}$ be an oriented n-dimensional differentiable manifold with boundary and ω an $n-1$-form. Consider an orientation of $\partial\Omega$ compatible with that of $\mathcal{M}$. Then*

$$\int_{\mathcal{M}} d\omega = \int_{\partial M} \omega. \tag{13}$$

Proof. Consider first the case that ω has compact support and is supported in a single coordinate chart (U, ϕ). Then $\omega = \sum_{i=1}^n f_i dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots dx^n$, where the hat above a term indicates that it is being omitted. Due to linearity of the objects involver it is enough to consider each of the terms separately. Thus, wolog,

$$\omega = f dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n.$$

Therefore

$$d\omega = (-1)^{i-1} \frac{\partial f}{\partial x_i} dx^1 \wedge \cdots \wedge dx^n.$$

Then

$$\int_{\mathcal{M}} d\omega = \int_U (-1)^{i-1} \frac{\partial f}{\partial x_i} dx^1 \cdots dx^n.$$

Since f is a smooth and compactly supported function in U we can extend it by zero to a smooth function on H^n.

Case 1^o. Assume $i \neq 1$. There exists $a > 0$ such that $\operatorname{supp} f \subset \{x \in H^n : -a < x_i < a\}$. Therefore

$$\begin{aligned}\int_{\mathcal{M}} d\omega &= \int_{\mathbb{R}^{n-2}} \int_{-\infty}^0 \int_{-a}^a (-1)^{i-1} \frac{\partial f}{\partial x_i} dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= \int_{\mathbb{R}^{n-2}} \int_{-\infty}^0 f|_{x_i = a} - f|_{x_i = -a} dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0.\end{aligned}$$

Case 2^o. Assume $i = 1$. Then

$$\begin{aligned}\int_{\mathcal{M}} d\omega &= \int_{\mathbb{R}^{n-1}} \int_{-a}^0 \frac{\partial f}{\partial x_1} dx^1 \cdots dx^n \\ &= \int_{\mathbb{R}^{n-1}} f(0, x_2, \ldots, x_n) dx^2 \ldots dx^n.\end{aligned}$$

To compute $\int_{\partial\mathcal{M}}\omega$ we note that $dx_1 = 0$ on $\partial M \cap \phi(U)$ and thus the integral is zero if $i \neq 1$. It remains to consider $i = 1$. Then $\int_{\partial\mathcal{M}}\omega = \int_{\mathbb{R}^{n-1}} f(0, x_2, \ldots, x_n)dx^2 \cdots dx^n$, which establishes the claim.

Let us now consider a general $n-1$-form ω. Let $\{(U_i, \phi_i) : i \in I\}$ be an atlas on $\mathcal{M}$. Let η_i be a locally finite partition of unity subordinated to the atlas. Then, applying the above case to forms $\eta_i\omega$,

$$\int_{\mathcal{M}} d\omega = \sum_{i\in I}\int_{\mathcal{M}} d(\eta_i\omega) = \sum_{i\in I}\int_{\partial\mathcal{M}} \eta_i\omega = \int_{\partial\mathcal{M}}\omega.$$

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References

[1] Carmen Chicone. *Ordinary differential equations with applications*, volume 34 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1999.

[2] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[3] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1983. Corrected reprint of the 1971 edition.