Assignment 6
Due on Friday, May 2

1. (Brachistoichrone Revisited) In order to avoid technical complications arising from the singularity at the initial point, we assume that the bead has a strictly positive initial speed, instead of starting from rest. This leads to minimizing

$$J(y) = \int_0^b \sqrt{\frac{1 + y'(x)^2}{\gamma + y(x)}}\, dx$$

subject to the boundary conditions $y(0) = 0$, $y(b) = B$. Here γ, b, and B are given strictly positive constants. We can restrict our attention to functions satisfying $y(x) > 0$ for all $x \in [0, b]$. We choose $\eta \in (0, \gamma)$. Then we can put

$$\mathscr{Y} = \{y \in C^1[0, b] : y(0) = 0,\ y(b) = B,\ y(x) > -\eta \text{ for all } x \in [0, b]\},$$

and proceed to minimize J on $\mathscr{Y}$. Make the substitution

$$u(x) = \sqrt{\gamma + y(x)} \text{ for all } x \in [0, b].$$

What nice thing happens when you make this substitution? Explain. You do not need to solve the problem. (You may, of course, hand in a complete solution if you like.)

2. Let
$$\mathscr{Y} = \{y \in C^1[0, 1] : y(0) = 0,\ y(1) = 1\},$$
and define $J : \mathscr{Y} \to \mathbb{R}$ by

$$J(y) = \int_0^1 (y(x)^3 - x)^2 |y'(x)|^{\frac{9}{2}} dx \text{ for all } y \in \mathscr{Y}.$$

Show that there exists $\alpha > 0$ such $J(y) \geq \alpha$ for all $y \in \mathscr{Y}$.

3. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and $f : (\alpha, \beta) \times \mathbb{R} \times \mathbb{R}$ be given and assume that f is of class C^2. Let $\Phi \in C^1((\alpha, \beta) \times \mathbb{R})$ be given and assume that Φ is a stationary field for f. Show that Φ is an exact field for f.

4. Define $f : (0, \pi) \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y, z) = z_1^2 + z_2^2 + 2y_1y_2 \text{ for all } x \in (0, \pi), y, z \in \mathbb{R}^2.$$

Now define $u, w, \phi, \chi : (0, \pi) \to \mathbb{R}$ by

$$u(x) = \frac{1}{2}(\coth x + \cot x), \quad w(x) = \frac{1}{2}(\coth x - \cot x),$$

$$\phi(x) = \frac{1}{2}\left(\coth^2 x + \cot^2 x\right), \quad \chi(x) = \frac{1}{2}\left(\coth^2 x - \cot^2 x\right) \quad \text{for all } x \in (0, \pi),$$

and $\Phi : (0, \pi)\mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(x, y) = (u(x)y_1 + w(x)y_2, w(x)y_1 + u(x)y_2) \quad \text{for all } x \in (0, \pi), y \in \mathbb{R}^2.$$

(a) Show that Φ is an exact field for f on $(0, \pi) \times \mathbb{R}^2$. You may take it for granted that

$$\Phi(x, y) \cdot \Phi(x, y) = -\phi(x)(y_1^2 + y_2^2) + 2\chi(x)y_1y_2 \quad \text{for all } x \in (0, \pi) \times \mathbb{R}^2.$$

(b) Let $a, L \in (0, \pi)$ with $a < L$ be given and put

$$\mathscr{Y} = \{y \in C^1([a, L]; \mathbb{R}^2) : y(a) = y(L) = (0, 0)\}.$$

Use Hilbert's invariant integral to analyze the inequality

$$\int_a^L \left\{y_1'(x)^2 + y_2'(x)^2 + 2y_1(x)y_2(x)\right\} dx \geq 0 \quad \text{for all } y \in \mathscr{Y}.$$

(c) (Optional) Can you find a simpler eaxct field that is of use for this problem?