Homework 6

21-236 Mathematical Studies Analysis II
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Problem 1

Let $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^2$ parametrize γ, and let $\mathbf{h}_1, \mathbf{h}_2 : [a, b] \times [0, 1] \to \mathbb{R}^2$ such that, $\forall (s, t) \in [a, b] \times [0, 1]$, if (ρ, θ) is the polar representation of $\boldsymbol{\varphi}(s)$ (i.e., $\boldsymbol{\varphi}(s) = (\rho \cos\theta, \rho \sin\theta)$, $\theta \in [0, 2\pi]$),

$$\mathbf{h_1}(s, t) = (\rho + t(r - \rho), \theta)$$

and

$$\mathbf{h_2}(s, t) = \left(1, \theta + t\left((2\pi\omega)\frac{(s - a)}{(b - a)} - \theta\right)\right),$$

where $\omega = \operatorname{ind}_\gamma(x_0, y_0)$. Then, let $\mathbf{h} : [a, b] \times [0, 1]$ such that, $\forall (s, t) \in [a, b] \times [0, 1]$,

$$\mathbf{h}(s, t) = \begin{cases} (x_0, y_0) + \mathbf{h}_1(s, 2t), & t \in \left[0, \frac{1}{2}\right] \\ (x_0, y_0) + \mathbf{h}_2(s, 2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Let γ_2 be the continuous closed curve parametrized by $\boldsymbol{\phi} : [a, b] \to \mathbb{R}^2$ such that, $\forall s \in [a, b]$, $\boldsymbol{\phi}(s) = (x_0, y_0) + (r \cos(\omega s), r \sin(\omega s))$. Then, $\forall s \in [a, b]$, $\mathbf{h}(s, 0) = \boldsymbol{\varphi}(s)$ and $\mathbf{h}(s, 1) = \boldsymbol{\phi}(s)$, and, because $\sin 0 = \sin(2\pi)$ and $\cos 0 = \cos(2\pi)$ $\forall t \in [0, 1]$, $\mathbf{h}(a, t) = \mathbf{h}(b, t)$. $\forall (s, t) \in [a, b] \times [0, \frac{1}{2}]$, $\mathbf{h}(s, t)$ is on the line segment from $\boldsymbol{\varphi}(s) \in U$ to (x_0, y_0), and,

$\forall (s, t) \in [a, b] \times [\frac{1}{2}, 1]$ $\mathbf{h}(s, t) \in \partial B((x_0, y_0), r)$

so that, since U is starshaped with respect to (x_0, y_0) and by choice of r, $\forall (s, t) \in [a, b] \times [0, 1]$, $h(s, t) \in U$.

Therefore, $\mathbf{h}$ is a homotopy from γ to γ_2 so that, since $\mathbf{g}$ is irrotational and of class C^1, by Theorem 143,

$$\int_\gamma \mathbf{g} = \int_{\gamma_2} \mathbf{g}.$$

However, γ_2 is the union of ω identical curves parametrized by ψ, so that

$$\int_\gamma \mathbf{g} = \int_{\gamma_2} \mathbf{g} = \sum_{i=1}^{\omega} \int_{\gamma_1} \mathbf{g} = \operatorname{ind}_\gamma \int_{\gamma_1} \mathbf{g} \quad \blacksquare.$$

Problem 2

Let $\boldsymbol{\varphi} : [a, b] \subseteq \mathbb{R} \to V$ be a parametrization of γ (with components $\varphi_1, \varphi_2, \varphi_3$), let γ_2 be the curve parametrized by $\boldsymbol{\phi} : [a, b] \to V$ such that, $\forall s \in [a, b]$,

$$\boldsymbol{\phi}(s) = (\varphi_1(s), \varphi_2(s), 0),$$

and let $\mathbf{h} : [a, b] \times [0, 1] \to V$ such that, $\forall (s, t) \in [a, b] \times [0, 1]$,

$$\mathbf{h}(s, t) = (\varphi_1(s), \varphi_2(s), \varphi_3(s) - t\varphi_3(s)).$$

Note that, $\forall s \in [a, b]$, $\mathbf{h}(s, 0) = \boldsymbol{\varphi}(s)$ and $\mathbf{h}(s, 1) = \boldsymbol{\phi}(s)$, and, $\forall t \in [0, 1]$, $\mathbf{h}(a, t) = \mathbf{h}(b, t)$. Since $\mathbf{h}$ is linear in t and $\boldsymbol{\varphi}$ is continuous (as γ is continuous), $\mathbf{h}$ is continuous. Since $h_1 = \varphi_1$ and $h_2 = \varphi_2$, and $\boldsymbol{\varphi}$ has range in V, h has range in V. Therefore, $\mathbf{h}$ is a homotopy from γ to γ_2, and γ and γ_2 are homotopic in V. By Theorem 143, then, since $\mathbf{g}$ is an irrotational field of class C^1,

$$\int_\gamma \mathbf{g} = \int_{\gamma_2} \mathbf{g}.$$

Note that γ_2 in $\mathbb{R}^3$ is the same curve as Π_γ is $\mathbb{R}^2$, so that

$$\int_{\gamma_2} \mathbf{g} = \int_{\Pi_\gamma} \mathbf{g}.$$

By the result of Problem 1 above, since $U := \mathbb{R}^2 \backslash \{(0, 0)\}$ is open and starshaped with respect to $(0, 0)$, and $\partial B((0, 0), 1) \subseteq U$,

$$\int_\gamma \mathbf{g} = \int_{\gamma_2} \mathbf{g} = \int_{\Pi_\gamma} \mathbf{g} = \operatorname{ind}_{\Pi_\gamma}(0, 0) \int_{\gamma_1} \mathbf{g},$$

as γ_1 in $\mathbb{R}^3$ is the same curve as unit circle centered at the origin in $\mathbb{R}^2$. ■

Problem 3

(a) Let $h : U \to \mathbb{R}^3$ such that, $\forall (x, y, z) \in U$,

$$h(x, y, z) = \frac{1}{2} \ln(x^2 + y^2).$$

Then, by the Quotient Rule,

$$\frac{\partial g_1}{\partial y}(x, y, z) = \frac{-xz(2y)}{(x^2 + y^2)^2} = \frac{\partial g_2}{\partial x}(x, y, z).$$

Furthermore,

$$\frac{\partial g_1}{\partial z}(x, y, z) = \frac{x}{x^2 + y^2} = \frac{1}{2} \frac{2x}{x^2 + y^2} = \frac{\partial h}{\partial x}(x, y, z) = \frac{\partial g_3}{\partial x}(x, y, z),$$

and

$$\frac{\partial g_2}{\partial z}(x, y, z) = \frac{y}{x^2 + y^2} = \frac{1}{2} \frac{2y}{x^2 + y^2} = \frac{\partial h}{\partial y}(x, y, z) = \frac{\partial g_3}{\partial y}(x, y, z).$$

Thus, $\mathbf{g}$ is irrotational. ■

(b) $\forall (x,y,z) \in \mathbb{R}^3$, if $z = 0$, then $\mathbf{g}(x,y,z) = \mathbf{0}$. Thus, $\forall t \in [0, 2\pi]$, $\mathbf{g}(\boldsymbol{\varphi}(t)) = \mathbf{0}$, so that, by definition of the curve integral,

$$\int_\gamma \mathbf{g} = \int_0^{2\pi} \sum_{i=1}^{3} 0 \cdot (\varphi_i'(t))\, dt = 0. \quad \blacksquare$$

(c) Let γ be a piecewise C^1 closed, oriented curve with range $\Gamma \subseteq U$. By the result of Problem 2 above, since $\mathbf{g}$ is irrotational and C^1,

$$\int_\gamma \mathbf{g} = \mathrm{ind}_{\Pi_\gamma}((0,0)) \int_{\gamma_1} \mathbf{g},$$

where γ_1 is the closed curve parametrized by $\boldsymbol{\varphi}(t) = (\cos t, \sin t, 0), t \in [0, 2\pi]$. Then, by the result of part (b) above,

$$\int_\gamma \mathbf{g} = \mathrm{ind}_{\Pi_\gamma}((0,0)) \cdot 0 = 0.$$

By Theorem 130, then, $\mathbf{g}$ is conservative. $\blacksquare$

(d) Let $f : U \to \mathbb{R}^3$ such that, $\forall (x,y,z) \in U$,

$$f(x,y,z) = \frac{z}{2}\ln(x^2+y^2) - \frac{\ln 2}{2}.$$

Then,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= \frac{xz}{x^2+y^2} = g_1(x,y,z) \\
\frac{\partial f}{\partial y}(x,y,z) &= \frac{yz}{x^2+y^2} = g_2(x,y,z) \\
\frac{\partial f}{\partial z}(x,y,z) &= \frac{1}{2}\log(x^2+y^2) = g_3(x,y,z)
\end{aligned}$$

(where $g_3 = h$ as defined in part (a). Furthermore, $f(1,1,1) = \frac{1}{2}\ln(1^2+1^2) - \frac{\ln 2}{2} = 0$, so that f has the desired properties. $\blacksquare$

Problem 4

(a) **Case 1:** $(N = M)$ Let L be the Lipschitz constant of $\mathbf{g}$, and R be a rectangle with $E \subseteq R$ (such a rectangle exists because E is Peano-Jordan Measurable). Since meas $E = 0$, $\forall \delta = \frac{\epsilon}{L^M}$, there exists a partition P of R such that $U(\chi_E, P) < \delta$. Let $\epsilon > 0$, and let Q be a refinement of P such that each rectangle R in Q has diagonals of length no more than ϵ, with rectangles $Q_1, Q_2, \ldots, Q_l$. Since Q is a refinement of P, $U(\chi_E, Q) < \delta$. Since $\mathbf{g}$ is Lipschitz, $\forall \mathbf{x}, \mathbf{y} \in R_i$, $\|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x})\| < L\|\mathbf{x} - \mathbf{y}\|$, so that each $\mathbf{g}(Q_i)$ can be covered by a rectangle C_i of diagonal at most $L\epsilon$; let S be a cover of $\mathbf{g}(E)$ of such rectangles. Then, $U(\chi_{\mathbf{g}(E)}, S) < L^M U(\chi_{\mathbf{g}(E)} < \epsilon$, so that $\int_S \chi_{\mathbf{g}(E)} = 0$ and thus $\operatorname{meas} \mathbf{g}(E) = 0$, concluding this case.

Case 2: $(N < M)$ Let

$$E' := \{(\mathbf{x}, \mathbf{0}) \in \mathbb{R}^M : \mathbf{x} \in E\}.$$

If χ_E is integrable over R over some rectangle R with $E \subseteq R$, then, $\forall k \in \mathbb{N}$, $\chi_{E'}$ is integrable over the rectangle $R' := R \times [0,0]^{M-N} \subseteq \mathbb{R}^M$. In particular, since $\chi_{E'}$ is bounded and $\operatorname{meas}_M R' = 0$,

$$\int_{R'} \chi_{E'} = 0.$$

Let $\mathbf{f} : E' \to \mathbb{R}^M$ such that, $\forall \mathbf{y} = (\mathbf{x}, \mathbf{0}) \in E'$ $(\mathbf{x} \in E)$, $\mathbf{f}(\mathbf{y}) = \mathbf{g}(\mathbf{x})$. Then, $\mathbf{f}$ is Lipschitz, and E' is Peano-Jordan Measurable with measure zero, so that, by the result of Case 1 above, $\mathbf{f}(E')$ is Peano-Jordan Measurable with measure zero. Therefore, since $\mathbf{f}(E') = \mathbf{g}(E)$, $\mathbf{g}(E)$ is Peano-Jordan Measurable with measure zero. ■

(b) Since $\mathbf{g}$ is of class C^1, its derivative is continuous. Thus, since $\overline{E}$ is compact, (it is clearly closed, and it is bounded because it is Peano-Jordan Measurable) and any continuous function on a compact domain is bounded, the derivative of $\mathbf{g}$ is bounded on $\overline{E}$. Therefore, $\mathbf{g}$ is Lipschitz over $\overline{E}$. By the result of part (a), then, $\mathbf{g}(E)$ is Peano-Jordan measurable with measure zero. ■

(c) i. By the Inverse Function Theorem, $\forall \mathbf{x} \in E^\circ$, $\exists r > 0$ such that $B(\mathbf{x}, r)) \subseteq E^\circ$ and $\mathbf{g}(B(\mathbf{x}, r))$ is open. Thus, if $\mathbf{x} \in E^\circ$, $\exists r_{\mathbf{x}} > 0$ such that $B(\mathbf{g}(\mathbf{x}, r_{\mathbf{x}})) \subseteq (\mathbf{g}(E))^\circ$. Therefore, $\mathbf{g}(\mathbf{x}) \in (\mathbf{g}(E))^\circ$, and so $\mathbf{g}(E^\circ) \subseteq (\mathbf{g}(E))^\circ$. Since $\mathbf{g}$ is continuous, $\overline{\mathbf{g}(E)} \subseteq \mathbf{g}(\overline{E})$. Thus,

$$\partial \mathbf{g}(E) = \overline{\mathbf{g}(E)} \backslash (\mathbf{g}(E))^\circ \subseteq \overline{\mathbf{g}(E)} \backslash \mathbf{g}(E^\circ) \subseteq \mathbf{g}(\overline{E}) \backslash \mathbf{g}(E^\circ) \subseteq \mathbf{g}(\partial E).$$

ii. Since E is Peano-Jordan measurable, by Theorem 177, ∂E is Peano-Jordan measurable with measure zero. Thus, by the result of part (b) above, $\mathbf{g}(\partial E)$ is Peano-Jordan measurable with measure zero. By the result of part i. above and Theorem 167,

$$\operatorname{meas} \partial \mathbf{g}(E) \leq \operatorname{meas} \mathbf{g}(\partial E) = 0,$$

so that $\operatorname{meas} \partial \mathbf{g}(E) = 0$. Therefore, by Theorem 177, $\mathbf{g}(E)$ is Peano-Jordan measurable. ■