Lecture Notes for Week 13 (First Draft)

Linear Mappings Between TVS

Definition 13.1: Let X and Y be topological vector spaces. A function $F : X \to Y$ is said to be *uniformly continuous* provided that for every neighborhood V of 0 in Y there is a neighborhood U of 0 in X such that

$$F(x) - F(y) \in V \quad \text{for all } x, y \in X \text{ with } x - y \in U.$$

Definition 13.2: Let X and Y be topological vector spaces and $T : X \to Y$ be a linear mapping. We say that T is bounded provided that $T[E]$ is topologically bounded for every topologically bounded set $E \subset X$.

Proposition 13.3: Let X and Y be topological vector spaces and assume that $T : X \to Y$ is linear. The following three statements are equivalent,

(i) T is uniformly continuous.

(ii) T is continuous on X.

(iii) T is continuous at 0.

Proof: It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). We need to show that (iii) $\Rightarrow$ (i).

Assume that T is continuous at 0. Let V be a neighborhood of 0 in Y. Since T is continuous at 0 and $T0 = 0$ we may choose a neighborhood U of 0 in V such that $T[U] \subset V$. Let $x, y \in X$ with $x - y \in U$ be given. Then $T(x - y) \in V$. Since $T(x - y) = Tx - Ty$, we conclude that $Tx - Ty \in V$. $\square$

Proposition 13.4: Let X and Y be topological vector spaces and assume that $T : X \to Y$ is linear. If T is continuous then T is bounded.

Remark 13.5: The converse of Proposition 13.4 is false in general, but is true if X is metrizable.

Proof of Proposition 13.4: Assume that T is continuous and let E be a topologically bounded subset of X. Let V be a neighborhood of 0 in Y. Since T is continuous we may choose a neighborhood U of 0 in X such that $T[U] \subset V$. Since E is topologically bounded, we may choose $t_0 > 0$ such that $E \subset tU$ for all $t > t_0$. Then for all $t > t_0$ we have

$$T[E] \subset T[tU] = tT[U] \subset tV,$$

and consequently $T[E]$ is topologically bounded. $\square$

Theorem 13.6: Let X be a topological vector space and assume that $f : X \to \mathbb{K}$ is linear and nontrivial (i.e. $f(x) \neq 0$ for some $x \in X$). The following four statements are equivalent.

(a) f is continuous,

(b) $\mathcal{N}(f)$ is closed,

(c) $\mathcal{N}(f)$ is not dense in X,

(d) There is a neighborhood W of 0 such that f is bounded on W.

Proof: We shall show that

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a).$$

Assume that f is continuous. Since $\{0\}$ is a closed subset of $\mathbb{K}$ and f is continuous, we conclude that $\mathcal{N}(f) = \{x \in X : f(x) \in \{0\}\}$ is closed, and consequently (a) $\Rightarrow$ (b).

Since f is not the zero functional, we know that $\mathcal{N}(f) \neq X$, and consequently (b) $\Rightarrow$ (c).

Assume that (c) holds. Then

$$\mathrm{int}((\mathcal{N}(f))^c) \neq \emptyset.$$

Therefore we may choose $x \in (\mathcal{N}(f))^c$ and a neighborhood U of 0 such that

$$(x + U) \cap \mathcal{N}(f) = \emptyset.$$

By Lemma 12.26, we may choose a balanced neighborhood W of 0 with $W \subset U$. Observe that

$$(x + W) \cap \mathcal{N}(f) = \emptyset. \tag{1}$$

Then $f[W]$ is a balanced subset of $\mathbb{K}$. This implies that either $f[W]$ is bounded or $f[W] = K$. Suppose that $f[W] = \mathbb{K}$. Then we may choose $y \in W$ such that $f(y) = -f(x)$. This implies that $x + y \in \mathcal{N}(f)$, which is impossible because of (1). We conclude that $f[W]$ is bounded and (d) holds.

Assume that there is a neighborhood W of 0 such that f is bounded on W. We shall show that f is continuous at 0. Choose $M > 0$ such that $|f(x)| < M$ for all $x \in W$. Let $\epsilon > 0$ be given and put $V = \epsilon M^{-1} W$. Then we have $|f(x)| < \epsilon$ for all $x \in V$ and f is continuous at 0. By Proposition 13.3, f is continuous. $\square$

Metrization

Remark 13.7: Let X be a metrizable topological vector space and let ρ be a metric that generates the topology. Put $V_n = \{x \in X : \rho(x, 0) < \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Then $\{V_n : n \in \mathbb{N}\}$ is a countable local base.

Theorem 13.8: Let X be a topological vector space and assume that X has a countable base. Then there is a metric $\rho : X \times X \to [0, \infty)$ such that

(a) ρ induces the topology on X,

(b) ρ is translation invariant,

(c) Each open ball centered at 0 is balanced.

If X is locally convex, then there exists a metric $\rho : X \times X \to [0, \infty)$ such that (a), (b), (c) holds and

(d) Each open ball is convex.

Lemma 13.9: Let X be a topological vector space and V be a neighborhood of 0. Then there exists a balanced neighborhood U of 0 such that $U + U \subset V$.

Proof: Let V be a neighborhood of 0. Since addition is continuous at $(0, 0)$ we may choose neighborhoods V_1 and V_2 of 0 such that $V_1 + V_2 \subset V$. By Lemma 12.26, we may choose balanced neighborhoods U_1 and U_2 of 0 such that $U_1 \subset V_1$ and $U_2 \subset V_2$. Let us put $U = U_1 \cap U_2$. Then U is a balanced neighborhood of 0 and $U + U \subset V$. □

Proof of Theorem 13.8: We shall prove the existence of a metric ρ satisfying (a), (b), and (c). It is possible to construct the metric in such a way that if X is locally convex, then (d) also holds. We shall not do so, because when X is locally convex, there is a different construction of ρ satisfying (a), (b), (c), and (d). This construction will be part of Assignment 7. Choose a local base $\{V_n : n \in \mathbb{N}\}$ such that each V_n is balanced and

$$V_{n+1} + V_{n+1} \subset V_n. \tag{2}$$

Let $\mathcal{F}$ denote the set of all nonempty finite subsets of $\mathbb{N}$. Given $F \in \mathcal{F}$ and $n \in \mathbb{N}$, we write $n < F$ to indicate that $n < k$ for all $k \in F$.

For each $F \in \mathcal{F}$, put

$$V^F = \sum_{n \in F} V_N,$$

$$q^F = \sum_{n \in F} 2^{-n}.$$

Notice that each V^F is balanced. Using an induction argument and (2), we can show that for all $F \in \mathcal{F}$ and $n \in \mathbb{N}$, we have

$$q^F < 2^{-n} \Rightarrow n < F \Rightarrow V^F \subset V_n.$$

Define the function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} \inf\{q^F : x \in V^F\} & \text{if } x \in \bigcup_{F \in \mathcal{F}} V^F \\ \\ 1 & \text{if } x \notin \bigcup_{F \in \mathcal{F}} V^F. \end{cases}$$

Notice that $f(x) \in [0,1]$ for all $x \in X$. Furthermore, since

$$\forall x \in X,\ F \in \mathcal{F}\ x \in F \Leftrightarrow -x \in F,$$

it follows that $f(-x) = f(x)$ for all $x \in X$. I claim that the function $\rho : X \times X \to [0,\infty)$ defined by

$$\rho(x,y) = f(x-y)$$

does the trick.

Clearly $0 \le \rho(x,y) = \rho(y,x) < \infty$ for all $x, y \in X$. Since $0 \in V^F$ for all $F \in \mathcal{F}$, we know that $f(0) = 0$. Since X has the Hausdorff property, if $x \in X \backslash \{0\}$, we may choose N sufficiently large so that $x \notin V^F$ whenever $N < F$ and consequently

$$f(x) > 0 \ \text{ for all } x \in X \backslash \{0\}.$$

We conclude that

$$\forall x, y \in X,\ \rho(x,y) = 0 \ \Leftrightarrow x = y.$$

To establish the triangle inequality for ρ, it suffices to show that

$$f(x+y) \le f(x) + f(y) \ \text{ for all } x, y \in X. \tag{3}$$

Let $x, y \in X$ be given. If $f(x) + f(y) \ge 1$ then we get $f(x+y) \le f(x) + f(y)$ "for free" because $f(z) \le 1$ for all $z \in X$. Assume that $f(x) + f(y) < 1$. Then we may choose $\epsilon > 0$ such that

$$f(x) + f(y) + 2\epsilon < 1.$$

We may also choose $F, G \in \mathcal{F}$ such

$$x \in V^F, \quad q^F < f(x) + \epsilon, \quad y \in V^G, \quad q^G < f(y) + \epsilon.$$

There is exactly one $H \in \mathcal{F}$ such that

$$q^H = q^F + q^G.$$

By virtue of (2) we have

$$V^F + V^G \subset V^H.$$

It follows that $x + y \in V^H$ and consequently

$$f(x+y) \le q^H = q^F + q^G < f(x) + f(y) + 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that (3) holds.

We have established that ρ is a metric. It is immediate that ρ is translation invariant because

$$\rho(x+z, y+z) = f(x+z-y-z) = f(x-y) = \rho(x,y) \quad \text{for all } x, y, z \in X.$$

For each $\delta > 0$ put

$$B_\delta = \{x \in X : \rho(x,0) < \delta\}.$$

It is straightforward to show that $\{V_\delta : \delta > 0\}$ is a local base for X. It follows that the topology of X is induced by ρ.

To see that each open ball centered at 0 is balanced, it suffices to show that

$$f(\alpha x) \leq f(x) \quad \text{for all } x \in X,\ \alpha \in \mathbb{K} \text{ with } |\alpha| \leq 1. \tag{4}$$

Since each V^F is balanced, we know that if $x \in V^F$ and $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$ then $\alpha x \in V^F$ and consequently $f(\alpha x) \leq f(x)$ and (4) is satisfied.

Normability

Theorem 13.10: Let X be a topological vector space. Then X is normable if and only if 0 has a bounded convex neighborhood.

Proof: If X is normable, then we may choose a norm $\|\cdot\|$ that induces the topology and it is straightforward to check that $\{x \in X : \|x\| < 1\}$ is a bounded convex neighborhood of 0.

Assume that X has a bounded convex neighborhood U of zero. By Lemma 12.28, we may choose a balanced convex neighborhood W of 0 such that $W \subset U$. Continuity of scalar multiplication implies that W is absorbing. Let us define the function $\|\cdot\| : X \to [0,\infty)$ by

$$\|x\| = p^U(x), \quad \text{for all } x \in X,$$

where p^U is the Minkowski functional for U. From previous considerations, we know that $\|\cdot\|$ is a seminorm. To show that it is a norm, we need to show that

$$\forall x \in X, \quad \|x\| = 0 \Rightarrow x = 0.$$