Lecture 17: Laplace Transforms and Central Limit Theorem

Contents

Recall that for a *discrete* random variable X we defined the z-transform of X, denoted by $\widehat{X}(z)$, which is a function of z that can be differentiated to get the moments of X. There is a similar definition for a *continuous* random variables, which we call the Laplace transform. Today we will define the Laplace transform and show how to differentiate it to get moments of the corresponding random variable. We will show that the properties we saw regarding addition and conditioning for z-transforms carry over the Laplace transforms as well. Finally, we will use the Laplace transform to provide a proof of the Central Limit Theorem.

1 Definition of Laplace Transform

Definition 1 *The* **Laplace Transform** *of a continuous random variable, X with density function, $f_X(t)$, $t \geq 0$, is defined as:*

$$\widetilde{X}(s) = \int_0^\infty e^{-st} f_X(t)dt$$

You can think of s as just being some parameter, where the Laplace transform is a function of s. Observe that:

$$\widetilde{X}(s) = \mathbf{E}\left\{e^{-sX}\right\}$$

Question: Derive the Laplace transform of $X \sim \text{Exp}(\lambda)$

Answer:

$$\begin{aligned}
\widetilde{X}(s) &= \int_0^\infty e^{-st}\lambda e^{-\lambda t}dt \\
&= \lambda \int_0^\infty e^{-(\lambda+s)t}dt \\
&= \frac{\lambda}{\lambda + s}
\end{aligned}$$

Question: Derive the Laplace transform of $X = a$, where a is some constant

Answer:

$$\widetilde{X}(s) = e^{-sa}$$

Question: Derive the Laplace transform of $X \sim \text{Uniform}(a, b)$, $a, b \geq 0$

Answer:

$$\begin{aligned}
\widetilde{X}(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_a^b e^{-st}\frac{1}{b-a}dt \\
&= \left(\frac{-e^{-sb}}{s} + \frac{e^{-sa}}{s}\right)\frac{1}{b-a} \\
&= \frac{e^{-sa} - e^{-sb}}{s(b-a)}
\end{aligned}$$

(Observe that here $f(t)$ is defined to be 0 when it is outside the (a, b) range.)

Question: How do we know that the Laplace transform of random variable X necessarily converges?

Answer: It does if the density function of X is $f(t)$, $t \geq 0$ and $s \geq 0$. To see this observe that

$$e^{-t} \leq 1$$

for all non-negative values of t. Thus

$$e^{-st} = \left(e^{-t}\right)^{s} \leq 1$$

assuming that s is non-negative. Thus:

$$\widetilde{X}(s) = \int_0^{\infty} e^{-st} f(t)dt \leq \int_0^{\infty} 1 \cdot f(t)dt = 1$$

2 Getting moments from transforms: peeling the onion

Theorem 2 *Let X be a continuous r.v. with p.d.f. $f(t)$, $t \geq 0$. Then*

$$\mathbf{E}\{X^n\} = (-1)^n \frac{d^n \widetilde{X}(s)}{ds}\bigg|_{s=0}$$

Proof:

$$\begin{aligned}
e^{-st} &= 1 - (st) + \frac{(st)^2}{2!} - \frac{(st)^3}{3!} + \ldots \\
e^{-st} f(t) &= f(t) - (st)f(t) + \frac{(st)^2}{2!} f(t) - \frac{(st)^3}{3!} f(t) + \ldots \\
\widetilde{X}(s) = \int_0^{\infty} e^{-st} f(t)dt &= \int_0^{\infty} f(t)dt - \int_0^{\infty} (st)f(t)dt + \int_0^{\infty} \frac{(st)^2}{2!} f(t)dt - \int_0^{\infty} \frac{(st)^3}{3!} f(t)dt + \ldots \\
&= 1 - s\mathbf{E}\{X\} + \frac{s^2}{2!}\mathbf{E}\left\{X^2\right\} - \frac{s^3}{3!}\mathbf{E}\left\{X^3\right\} + \ldots \\
\frac{d\widetilde{X}(s)}{ds} &= -\mathbf{E}\{X\} + s\mathbf{E}\left\{X^2\right\} - 3\frac{s^2}{3!}\mathbf{E}\left\{X^3\right\} + \ldots \\
\frac{d\widetilde{X}(s)}{ds}\bigg|_{s=0} &= -\mathbf{E}\{X\} \\
\frac{d^2\widetilde{X}(s)}{ds} &= \mathbf{E}\left\{X^2\right\} - s\mathbf{E}\left\{X^3\right\} + \ldots \\
\frac{d^2\widetilde{X}(s)}{ds}\bigg|_{s=0} &= \mathbf{E}\left\{X^2\right\}
\end{aligned}$$

Continuing along these lines we can see from the original Taylor series expansion that each time we take another derivative, we get a higher moment, with alternating sign. ■

Question: Compute the 1st and 2nd moments of $X \sim \text{Exp}(\lambda)$

Answer:

$$\begin{aligned}
\widetilde{X}(s) &= \frac{\lambda}{\lambda + s} \\
\mathbf{E}\{X\} &= -\frac{d\widetilde{X}(s)}{ds}\bigg|_{s=0} \\
&= -\frac{-\lambda}{(\lambda + s)^2}\bigg|_{s=0} \\
&= \frac{1}{\lambda} \\
\mathbf{E}\left\{X^2\right\} &= (-1)^2 \frac{d^2\widetilde{X}(s)}{ds}\bigg|_{s=0} \\
&= \frac{d}{ds}\left(\frac{-\lambda}{(\lambda + s)^2}\right)\bigg|_{s=0} \\
&= \frac{2\lambda}{(\lambda + s)^3}\bigg|_{s=0} \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

3 Linearity of transforms

Theorem 3 *Let X and Y be continuous independent random variables with p.d.f. $x(t)$, $t \geq 0$, and $y(t)$, $t \geq 0$, respectively. Let $Z = X + Y$. Then the Laplace transform of Z is given by*

$$\widetilde{Z}(s) = \widetilde{X}(s) \cdot \widetilde{Y}(s) \tag{1}$$

Proof:

$$\begin{aligned}
\widetilde{Z}(s) &= \mathbf{E}\left\{e^{-sZ}\right\} \\
&= \mathbf{E}\left\{e^{-s(X+Y)}\right\} \\
&= \mathbf{E}\left\{e^{-sX} \cdot e^{-sY}\right\} \\
&= \mathbf{E}\left\{e^{-sX}\right\} \cdot \mathbf{E}\left\{e^{-sY}\right\} \qquad \text{(because } X \perp Y\text{)} \\
&= \widetilde{X}(s) \cdot \widetilde{Y}(s)
\end{aligned}$$

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4 Conditioning

Theorem 4 *Let X, A, and B be continuous random variables where*

$$X = \begin{cases} A & \textit{with probability } p \\ B & \textit{with probability } 1 - p \end{cases}$$

Then

$$\widetilde{X}(s) = p \cdot \widetilde{A}(s) + (1 - p) \cdot \widetilde{B}(s)$$

Proof:

$$\begin{aligned}
\widetilde{X}(s) &= \mathbf{E}\left\{e^{-sX}\right\} \\
&= \mathbf{E}\left\{e^{-sX} \mid X = A\right\} \cdot p + \mathbf{E}\left\{e^{-sX} \mid X = B\right\} \cdot (1 - p) \\
&= p\mathbf{E}\left\{e^{-sA}\right\} + (1 - p)\mathbf{E}\left\{e^{-sB}\right\} \\
&= p\widetilde{A}(s) + (1 - p)\widetilde{B}(s)
\end{aligned}$$

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5 Applying Laplace Transforms to Prove the Central Limit Theorem

We start by deriving the Laplace transform for the Normal distribution, and then we use it to prove the Central Limit Theorem (CLT).

5.1 Two-sided Laplace transform for Normal Distribution

In the case where a random variable X can take on negative values, we define the Laplace-transform of X with density function $f_X(t)$, $-\infty < t < \infty$ by:

$$\widetilde{X}(s) = \int_{-\infty}^{\infty} e^{-st} f_X(t) dt$$

Question: Let $X \sim \text{Normal}(0, 1)$ be the standard Normal. What is $\widetilde{X}(s)$?

Answer:

$$\begin{aligned}
\widetilde{X}(s) &= \int_{-\infty}^{\infty} e^{-st} f_X(t) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-st} e^{-t^2/2} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-t^2-2st)} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-(t+s)^2+s^2)} dt \\
&= e^{\frac{s^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+s)^2} dt \\
& \quad \text{(using a change of variables, where } x = t + s \text{ and } dx = dt\text{)} \\
&= e^{\frac{s^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dt \\
&= e^{\frac{s^2}{2}} \cdot 1
\end{aligned}$$

Thus:

$$\widetilde{X}(s) = e^{\frac{s^2}{2}} \qquad (2)$$

We can see that the moments of X check out:

$$\begin{aligned}
\widetilde{X}'(s) &= se^{\frac{s^2}{2}} \\
\mathbf{E}\{X\} = -\widetilde{X}'(0) &= 0\surd \\
\widetilde{X}''(s) &= 1 \cdot e^{\frac{s^2}{2}} + s \cdot se^{\frac{s^2}{2}} \\
\mathbf{E}\left\{X^2\right\} = \widetilde{X}''(0) &= 1\surd
\end{aligned}$$

5.2 Laplace transform of CLT (approximately)

Recall the Central Limit Theorem (CLT):

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each with mean μ and finite variance σ^2. CLT says that the distribution of

$$S = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \tag{3}$$

tends to the standard Normal as $n \to \infty$. Specifically:

$$\mathbf{P}\left\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx, \text{ as } n \to \infty$$

This is a pretty amazing fact, considering that the X_i's can come from any distribution, so long as they are i.i.d.

There are many difficult proofs of the CLT, but there is only one simple proof, and it involves Laplace transforms. The high-level idea of the proof is to show that the Laplace transform of S in (3) converges (roughly) to the Laplace transform of the standard Normal, given in (2).

We will need to make use of two identities for e^x that you learned in calculus:

1. $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
2. $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$

Question: Do you remember why these hold?

Answer: If you don't know the answer, you should be attending class, where we ask and answer these kinds of questions...

We start with the case where $\mu = 0$ and $\sigma^2 = 1$:

Here

$$S = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \tag{4}$$

$$\begin{aligned}
\widetilde{S}(s) &= \left(\mathbf{E}\left\{e^{-\frac{sX}{\sqrt{n}}}\right\}\right)^n \\
&\approx \left(\mathbf{E}\left\{1 - \frac{sX}{\sqrt{n}} + \frac{s^2X^2}{2\cdot n} - \cdots\right\}\right)^n \\
&\approx \left(1 - \frac{s\mathbf{E}\{X\}}{\sqrt{n}} + \frac{s^2\mathbf{E}\left\{X^2\right\}}{2\cdot n}\right)^n \\
&\approx \left(1 - \frac{s\cdot 0}{\sqrt{n}} + \frac{s^2\cdot 1}{2\cdot n}\right)^n \\
&= \left(1 + \frac{s^2}{2\cdot n}\right)^n \\
&\to e^{s^2/2} \text{ as } n \to \infty
\end{aligned}$$

This is exactly the Laplace transform of the standard Normal given in (2).

We now generalize our solution to arbitrary μ and σ:

This follows immediately by defining

$$Y_i = \frac{X_i - \mu}{\sigma}$$

Now the Y_i's have mean zero and standard deviation 1, so we can apply the result in part (a) to see that the sum of the Y_i's divided by $\sqrt{n}$ converges to the standard Normal. Hence:

$$\mathbf{P}\left\{\frac{Y_1 + Y_2 + \cdots + Y_n}{\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx, \text{ as } n \to \infty$$

so

$$\mathbf{P}\left\{\frac{X_1 - \mu + X_2 - \mu + \cdots + X_n - \mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx, \text{ as } n \to \infty$$

and thus:

$$\mathbf{P}\left\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx, \text{ as } n \to \infty$$