21-640 **Intro to Functional Analysis** **Spring 2013**

Assignment 6

Due on Wednesday, April 24

Please turn in solutions to all 9 problems.

1. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{K}$ and define $T : l^2 \to l^2$ by
$$(Tx)_n = \alpha_n x_n \text{ for all } x \in l^2.$$
Under what conditions on $\{\alpha_n\}_{n=1}^{\infty}$ is T compact?

2. Prove or disprove: Let $T \in \mathcal{L}(l^2; l^2)$ be given and assume that
$$Te^{(n)} \to 0 \text{ (strongly) as } n \to \infty.$$
Then T is compact.

3. Let $\mathbb{K} = \mathbb{R}$. Give an example of an operator $T \in \mathcal{L}(l^2; l^2)$ and a sequence $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{L}(l^2; l^2)$ such that $T_n \to T$ in the strong operator topology as $n \to \infty$ but the sequence $\{T_n^*\}_{n=1}^{\infty}$ fails to converge to T^* in the strong operator topology.

4. Let X, Y be NLS and let $\{T_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{L}(X; Y)$. Assume that for every $x \in X, y^* \in Y^*$, the sequence $\{y^*(T_n x)\}_{n=1}^{\infty}$ is convergent in $\mathbb{K}$. Add whatever assumptions you think are appropriate and prove that there exists $T \in \mathcal{L}(X; Y)$ such that $T_n \to T$ in the weak operator topology as $n \to \infty$.

5. (a) Give an example of two infinite-dimensional Banach spaces X and Y such that $\mathcal{L}(X; Y) = \mathcal{C}(X; Y)$.
   (b) Do there exist infinite-dimensional Banach spaces X and Y such that both X and Y are reflexive and $\mathcal{L}(X; Y) = \mathcal{C}(X; Y)$? Explain.

6. (a) Prove or Disprove: Let X and Y be infinite-dimensional Banach spaces and let $T \in \mathcal{C}(X; Y)$ be given. Then T is not surjective.
   (b) Prove or Disprove: Let X and Y be infinite-dimensional normed linear spaces and let $T \in \mathcal{C}(X; Y)$ be given. Then T is not surjective.

7. (a) Does there exist a continuous linear surjection $T : l^2 \to l^1$? Explain.
   (b) Does there exist a continuous linear surjection $L : c_0 \to l^2$? Explain.

Def: A normed linear space $(X, \|\cdot\|)$ is said to be

(i) *strictly convex* provided that for all $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ we have $\|tx + (1-t)y\| < 1$ for all $t \in (0, 1)$.

(ii) *uniformly convex* provided that for every pair of sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ in X such that $\|x_n\| = \|y_n\| = 1$ for all $n \in \mathbb{N}$ and $\|x_n + y_n\| \to 2$ as $n \to \infty$ we have $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Remark: We note for future reference that every uniformly convex Banach space is reflexive. Strictly convex and uniformly convex normed linear spaces need not be complete, so it is necessary to assume completeness in the statement that uniform convexity implies reflexivity.

8. Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, $\{x_n\}_{n=1}^{\infty}$ be a sequence in X and $x \in X$. Assume that $x_n \rightharpoonup x$ (weakly) as $n \to \infty$ and that $\|x_n\| \to \|x\|$ as $n \to \infty$. Show that $x_n \to x$ (strongly) as $n \to \infty$.

9. (a) Let $(X, \|\cdot\|)$ be a Banach space. Show that if $(X, \|\cdot\|)$ is uniformly convex, then it is strictly convex.

   (b) Give an example of a Banach space $(X, \|\cdot\|)$ that is strictly convex, but not uniformly convex. Try to find an example such that X is reflexive and separable. (You do not need to prove reflexivity and separability with all the details – just explain why your example has these properties.)