Extra Credit Project 3

21-260 Differential Equations

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Section 8.1, Problem 23

(a) By definition of E_n and E_{n+1}, equation (20) gives

$$E_{n+1} = E_n + h(f(t_n, \phi(t_n)) - f(t_n, y_n)) + \frac{1}{2}\phi(\bar{t}_n)h^2.$$

Then, since $h > 0$, the triangle inequality then gives

$$|E_{n+1}| \leq |E_n| + h|f(t_n, \phi(t_n)) - f(t_n, y_n)| + \frac{1}{2}h^2|\phi(\bar{t}_n)|.$$

Since f is Lipschitz in its second argument with Lipschitz constant L, $|f(t_n, \phi(t_n)) - f(t_n, y_n)| \leq L|\phi(t_n) - y_n| = L|E_n|$, so that

$$|E_n| + h|f(t_n, \phi(t_n)) - f(t_n, y_n)| \leq |E_n| + hL|E_n| = \alpha|E_n|. \tag{1}$$

Since, $\frac{1}{2}h^2 \geq 0$ and, by definition of β, $|\phi''(\bar{t}_n)| \leq \beta$,

$$\frac{1}{2}h^2|\phi(\bar{t}_n)| \leq \beta h^2. \tag{2}$$

Adding equations (1) and (2) gives

$$|E_n| + h|f(t_n, \phi(t_n)) - f(t_n, y_n)| + \frac{1}{2}h^2|\phi(\bar{t}_n)| \leq \alpha|E_n| + \beta h^2,$$

so that, as desired,

$$|E_{n+1}| \leq |E_n| + h|f(t_n, \phi(t_n)) - f(t_n, y_n)| + \frac{1}{2}h^2|\phi(\bar{t}_n)| \leq \alpha|E_n| + \beta h^2. \quad \blacksquare$$

(b) By definition of α,

$$|E_n| \leq \beta h^2 \frac{\alpha^n - 1}{\alpha - 1} = \beta h^2 \frac{(1 + hL)^n - 1}{(1 + hL) - 1} = \beta h \frac{(1 + hL)^n - 1}{L}. \quad \blacksquare$$

(c) Since, $\forall x \in \mathbb{R}$, $1 + x \leq e^x$ (this can be shown for $x \geq 0$ by noting that the first derivative of $e^x - 1$ is positive and $e^0 - 1 = 0$, and can be shown for $x < 0$ by noting that e^x is everywhere positive), $1 + hL \leq e^{hl}$. Thus, $(1 + hL)^n \leq e^{nhl}$. $\blacksquare$