21-373, Algebraic Structures, Department of Mathematical Sciences, Carnegie Mellon University
Fall 2011: (Math Studies Section) Monday, Wednesday, Friday, 10:30 am, Porter Hall 226B.
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Assignment 4 - Saturday October 1, 2011. Due Friday October 7

Exercise 22: Let p be a prime, and for $n \geq 1$ let $E_{p^n} = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (with n factors). How many subgroups of order p are there in E_{p^n}?

Exercise 23: For a group G, the *exponent of* G is the smallest positive integer n such that $g^n = e$ for all $g \in G$, and the exponent is ∞ if no such n exists.

Show that every finite group has a finite exponent n, without necessarily having an element of order n.

Give an example of an infinite group having a finite exponent.

Exercise 24: For a group G (written multiplicatively) and p a prime, one writes $G^p = \{g^p \mid g \in G\}$ and $G_p = \{x \in G \mid x^p = e\}$.

Show that G^p and G_p are subgroups of G.

If for groups H, K one has $G = H \times K$, show that $G^p = H^p \times K^p$ and $G_p = H_p \times K_p$.

Exercise 25: Notation of Exercice 24.

If $G = \mathbb{Z}_n$, what are G^p and G_p?

Show that for any finite Abelian group G, one has $G/G^p \simeq G_p$, and that the number of subgroups of G of order p is equal to the number of subgroups of G of index p.

Exercise 26: In S_5 one considers the 5-cycle $a = (1\,2\,3\,4\,5)$ and the 4-cycle $b = (1\,2\,4\,3)$. Show that $b\,a = a^2 b$, and that the group $\langle a, b\rangle$ generated by a and b has order 20, with elements $a^\alpha b^\beta$ with $0 \leq \alpha \leq 4, 0 \leq \beta \leq 3$, and that $(a^\alpha b^\beta)\,(a^\gamma b^\delta) = a^\epsilon b^\zeta$, where $\epsilon = \alpha + 2^\beta \gamma \pmod 5$ and $\zeta = \beta + \delta \pmod 4$.

Exercise 27: Let $p < q < r$ be primes.

Show that no group G of order $p\,q\,r$ is simple.

Show that no group G of order $p^2 q$ is simple.

Exercise 28: Show that a simple group G of order 90 must contain 60 elements of order 9. Deduce that no such simple group exists.