Homework 2

21-621 Introduction to Lebesgue Integration
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Chapter 1, Problems 5,7,11,13,16,17, provide the details of Property 3 on page 29

The following lemma is used in the solutions to problems 5 and 13:

Lemma 1: If $C \subseteq \mathbb{R}^d$ is closed, $x \in C$ if and only if $d(x, C) = 0$.

Proof of Lemma 1: It is clear that, if $x \in C$, then $d(x, C) = 0$, since $\|x - x\| = 0$. Suppose that $x \notin C$. Let $y \in C$, and let $B = B(x, \|x - y\|)$ by the ball of radius $\|x - y\|$ centered at x. Let $K := \overline{B \cap C}$, the closure of the intersection of B and C. Since K is compact, the continuous function taking $y \in K$ to $\|x - y\|$ achieves a minimum on K, so that there must exist $y \in K$ with $\|y - x\| = d(x, K)$. Thus, if $x \notin K$, $d(x, K) > 0$. From the definition of K, it is clear that $d(x, K) = d(x, C)$, so that, if $x \notin C$, $d(x, C) > 0$. ■

Lemma 2: $F \subseteq \mathbb{R}^d$ is a G_δ if and only if $R^d \backslash F$ is an F_σ.

Proof of Lemma 2: If $F = \bigcap_{i=1}^\infty U_i$ with each U_i open, then

$$\mathbb{R}^d - F = \mathbb{R}^d - \bigcap_{i=1}^\infty U_i = \bigcup_{i=1}^\infty \mathbb{R}^d - U_i,$$

so that, since each $\mathbb{R}^d - U_i$ is closed, $\mathbb{R}^d - F$ is an F_σ.

If $\mathbb{R}^d - F = \bigcup_{i=1}^\infty C_i$, with each C_i closed, then

$$F = \mathbb{R}^d - (\mathbb{R}^d - F) = \mathbb{R}^d - \bigcup_{i=1}^\infty C_i = \bigcap_{i=1}^\infty \mathbb{R}^d - C_i,$$

so that, since $\mathbb{R}^d - C_i$ is open, F is a G_δ. ■

Chapter 1, Problem 5

(a) Suppose E is compact. $\forall \in \mathbb{N}$, since $\mathcal{O}_n$ is open, $\mathcal{O}_n$ is measurable. Since E is compact, it is bounded by some ball B of radius r centered at the origin. Then, clearly, $\mathcal{O}_1$ is bounded (by the ball of radius $r + 1$ centered at the origin), so that $\mu_*(\mathcal{O}_n) < \infty$.

Thus, by part (ii) of Corollary 3.3, it suffices to show that $\mathcal{O}_n \searrow E$ as $n \to \infty$. Clearly, $\forall n \in \mathbb{N}$, $\mathcal{O}_n \supseteq \mathcal{O}_{n+1}$. It is also apparent that $E \subseteq \mathcal{O}_n, \forall n \in \mathbb{N}$, so that $E \subseteq \bigcap_{i=1}^\infty \mathcal{O}_n$. If $x \in \mathbb{R}^d$, then, by Lemma 1 above, $d(x, E) > 0$, so that, for some $n \in \mathbb{N}$, $x \notin \mathcal{O}_n$, and so $E \supseteq \bigcap_{i=1}^\infty \mathcal{O}_n$. ■

(b) Let $C = \{(x, 0) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$, so that C is closed but unbounded. $\forall \epsilon > 0$, we can cover C with the countably many almost disjoint rectangles of dimension $1 \times \frac{\epsilon}{2^i}$, each centered at

$(\pm i, 0)$. Thus, by monotonicity of μ_*,

$$\mu_*(C) \le 2 \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon < \infty.$$

On the other hand, $\forall n \in \mathbb{N}$, if $\mathcal{O}_n = \{x \in \mathbb{R}^2 : d(x, C) < 1/n\}$, then $\mathcal{O}_n$ is the union of countably many almost disjoint rectangles of dimension $1 \times 1/n$, each centered at $(\pm i, 0)$, so that

$$\mu_*(\mathcal{O}_n) = \sum_{i=1}^{\infty} 1/n = \infty.$$

Therefore, $\mu_*(\mathcal{O}_n)$ does not converge to $\mu_*(C)$ as $n \to \infty$. ∎

Chapter 1, Problem 7

Let $\mathcal{R}$ be the set of rectangles in $\mathbb{R}^d$. Clearly, $\{R_1, R_2, \dots\} \subseteq \mathcal{R}$ is a countable cover of E if and only if $\{\delta R_1, \delta R_2, \dots\} \subseteq \mathcal{R}$ is a countable cover of δE. Thus, since, $\forall i \in \mathbb{N}$, it is clear that $\mu_*(\delta R_i) = \mu_*(R_i) \prod_{i=1}^{d} \delta_i$,

$$\begin{aligned}
\mu_*(\delta E) &= \inf \left\{ \sum_{i=1}^{\infty} \mu_*(\delta R_i) \,\middle|\, \delta E \subseteq \bigcup_{i=1}^{\infty} \delta R_i, \delta R_i \in \mathcal{R} \right\} \\
&= \inf \left\{ \left(\sum_{i=1}^{\infty} \mu_*(\delta R_i) \right) \prod_{i=1}^{d} \delta_i \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} R_i, R_i \in \mathcal{R} \right\} \\
&= \left(\prod_{i=1}^{d} \delta_i \right) \inf \left\{ \sum_{i=1}^{\infty} \mu_*(\delta R_i) \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} R_i, R_i \in \mathcal{R} \right\} = \mu_*(E) \prod_{i=1}^{d} \delta_i.
\end{aligned}$$

Thus, it remains only to show that δE is measurable, so that

$$\mu(\delta E) = \mu_*(\delta E) = \mu_*(E) \prod_{i=1}^{d} \delta_i = \mu(E) \prod_{i=1}^{d} \delta_i.$$

Let $\epsilon > 0$. Since E is measurable, there is an open set $U \supseteq E$ with $\mu_*(U - E) < \frac{\epsilon}{\prod_{i=1}^{d} \delta_i}$. Then,

$$\mu_*(\delta U - \delta E) = \mu_*(\delta(U - E)) = \mu_*(U - E) \prod_{i=1}^{d} \delta_i < \epsilon,$$

so that, since δU is clearly open, by Theorem 3.4, δE is measurable. ∎

Chapter 1, Problem 11

Chapter 1, Problem 13

(a) Let $C \subseteq \mathbb{R}^d$ be closed, and, $\forall n \in \mathbb{N}$, let $\mathcal{O}_n := \{x \in \mathbb{R}^d : d(x, C) < 1/n\}$. Clearly, $\forall n \in \mathbb{N}$, $C \subseteq \mathcal{O}_n$, so that $C \subseteq \bigcap_{i=1}^{\infty} \mathcal{O}_n$. $C \subseteq$. It follows from Lemma 1 above that, if $x \notin C$, then $d(x, C) > 0$, so that, for some $n \in \mathbb{N}$, $x \notin \mathcal{O}_n$. Therefore, $\bigcap_{i=1}^{\infty} \mathcal{O}_n \subseteq C$, so that C is a G_δ. ■

Let $U \subseteq \mathbb{R}^d$ be open, so that $C := \mathbb{R}^d - U$ is closed. Then, as shown above, C is a G_δ, so that, by Lemma 2 above, U is an F_σ. ■

(b) Let f be a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Then, $\mathbb{Q} = \bigcup_{i=1}^{\infty} \{f(i)\}$, so that, since each $\{f(i)\}$ is closed, $\mathbb{Q}$ is an F_σ. Thus, by Lemma 2 above, $\mathbb{R} - \mathbb{Q}$ is a G_δ. However, $\mathbb{Q}$ is not a G_δ, since this would contradict the Baire category theorem, as $\emptyset = \mathbb{Q} \cap (\mathbb{R} - \mathbb{Q})$ would be the countable intersection of open dense sets (as any set with a dense subset is dense). ■

(c) By the same argument as in part (b) $(\mathbb{Q} \cap [1, 2])$ is not a G_δ and thus, by Lemma 2, $(\mathbb{R} - \mathbb{Q}) \cap [-2, -1]$ is not an F_σ (as, clearly, any translation of an F_σ is, itself an F_σ), $S := (\mathbb{Q} \cap [1, 2]) \cup ((\mathbb{R} - \mathbb{Q}) \cap [-2, -1])$ is neither a G_δ nor an F_σ.

Since $\mathbb{Q}$ is an F_σ and $\mathbb{R} - \mathbb{Q}$ is a G_δ, and $[1, 2]$ and $[-2, -1]$ are closed (and thus, all these sets are Borel), since a σ-algebra is closed under finite unions and intersections, S is Borel. ■

Chapter 1, Problem 16

Note that $E = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j$, since x is in the latter if and only if $x \in E_k$ for infinitely many k.

(a) Since the set of Lebesgue measurable sets is a σ-algebra, and thus closed under countable unions and intersections, it follows immediately from the above characterization of E that E is Lebesgue measurable, since each E_k is measurable. ■

(b) Since $\forall k \in \mathbb{N}$, $\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j \subseteq \bigcup_{j=k}^{\infty} E_j$, it follows from monotonicity and countable subadditivity of μ, given the above characterization of E, that

$$\mu(E) = \mu\left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j\right) = \mu\left(\bigcup_{j=k}^{\infty} E_j\right) \leq \sum_{j=k}^{\infty} \mu(E_j).$$

Since $\sum_{i=1}^{\infty} \mu(E_j) < \infty$,

$$\mu(E) = \lim_{k \to \infty} \mu(E) \leq \lim_{k \to \infty} \sum_{j=k}^{\infty} \mu(E_j) = 0. \quad ■$$

Chapter 1, Problem 17

$\forall n \in \mathbb{N}$, since f_n is finite almost everywhere, $\exists c_n \in \mathbb{R}$ such that f_n is bounded by c_n/n except on a set of measure less than 2^{-n}, so that, for $E_n := \{x : |f_n(x)/c_n| > 1/n\}$, $m(E_n) < 2^{-n}$. Then,

$$\sum_{i=1}^{\infty} \mu(E_i) < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 < \infty.$$

Since each $(c_n/n, \infty)$ and f_n are measurable, each $S_n = (|f_n|)^{-1}((c_n/n, \infty))$ is also measurable. Thus, for $E := \limsup_{i \to \infty}(E_i)$, by the Borel-Cantelli lemma, $\mu(E) = 0$.

Since the set of points where $\frac{f_n(x)}{c_n}$ does converge to 0 is precisely E, $\frac{f_n(x)}{c_n} \to 0$ as $n \to \infty$ a.e. ■

Details of Proof of Property 3

Since each f_n is measurable, $\forall a \in \mathbb{R}$, if, $\forall n \in \mathbb{N}$, $E_n = \{x \in \mathbb{R} : f_n(x) > a\}$, E_n is measurable. Note that $\sup_n f_n(x) > a$ if and only if $\exists n \in \mathbb{N}$ such that $f_n(x) > a$, so that $E := \{x \in \mathbb{R} : \sup_n f_n(x) > a\} = \bigcup_{i=1}^{\infty} E_i$. Thus, since the set of Lebesuge measurable sets is a σ-algebra, and thus closed under countable unions, E is measurable. Then, since this holds $\forall a \in \mathbb{R}$, the function $x \mapsto \sup_n f(x)$ is measurable. The proof for the pointwise infimum of countably many measurable functions is analagous. ■

Since each f_n is measurable, as shown above, $\forall k \in \mathbb{N}$, the functions $x \mapsto \sup_n f_n(x)$ is measurable. Thus, since the pointwise infimum of countable many measurable functions is measurable, the function

$$x \mapsto \limsup_{n \to \infty} f_n(x) = \inf_k \{\sup_{n \geq k} f_n(x) : k \in \mathbb{N}\}$$

is measurable. The proof for the pointwise limit inferior of countable many measurable functions is analagous. ■