Homework 2

21-484A Graph Theory
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Problem 1

Let a be the number of vertices in T of degree 1, let b be the number of vertices in T of degree 3, and let $e = |E(T)|$. Since T is a tree, $e = n - 1$. Furthermore,

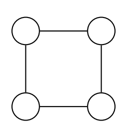
$$2e = \sum_{v \in V(T)} \deg(v) = a + 3b.$$

Finally, $a + b = n$. Solving this system of three linear equations in three variables gives $a = \frac{n+2}{2}$, $b = \frac{n-2}{2}$. Thus, the number of leaves in T is

$$a = \boxed{\frac{n+2}{2}.}$$

Problem 2

Suppose G is a graph with at least 4 vertices, such that the subgraph induced by any 3 vertices in G is a tree. Suppose, for sake of contradiction, that G contains at least $|V(G)| \geq 5$. Then, let $v_1, v_2, \dots, v_5$ be distinct vertices in G, and let H be the subgraph of G induced by $v_1, v_2, \dots, v_5$. Then, H also has the property that, the subgraph of H induced by any 3 vertices in H is a tree. For any distinct $t, u, v \in \{v_1, v_2, \dots, v_5\}$, let $T_{t,u,v}$ be the subgraph of H induced by t, u, v. Since T_{v_1,v_2,v_3} is a tree, it contains $3 - 1 = 2$ edges; without loss of generality, v_1v_2 and v_2v_3 are in H. If v_2v_5 were in H, then neither v_1v_5 nor v_3v_5 could be in H (since, if either were the case, then either T_{v_1,v_2,v_5} or T_{v_3,v_2,v_5} would have 3 edges and thus not be a tree); therefore, v_2v_5 is not an edge in H. A similar argument shows that v_2v_4 is not an edge in H. Then, however, T_{v_2,v_4,v_5} contains at most 1 edge, contradicting the hypothesis that T_{v_2,v_4,v_5} is a tree. Thus, G contains at most 4 vertices, and so it contains exactly 4 vertices. Thus, by inspection of the 11 distinct unlabeled graphs on 4 vertices, it can be seen that G must be of the form of the following graph:



■

Problem 3

Let G be a connected graph, and suppose some $e \in E(G)$ is a bridge. Let v_1, v_2 be the endpoints of e. Then, by definition of bridge, all walks from v_1 to v_2 contain e (since, if e is removed from G, there are no walks from v_1 to v_2). Suppose T is a spanning tree of G. Then, T must contain a walk from v_1 to v_2. Therefore, T must contain e.

Suppose, on the other hand, that, for some edge e in G, every spanning tree T of G contains e. Suppose, for sake of contradiction, that e is not a bridge. Then, by definition of bridge, letting H be the graph resulting from removing e from G, H is connected. Thus, H has a spanning tree T. Since $V(H) = V(G)$, T is also a spanning tree of G. However, since T is a spanning tree of H, e is not in T, contradicting the given that e is in every spanning tree of G.

Therefore, an edge e in a connected graph G is a bridge if and only if e is in every spanning tree of G. ■

Problem 4

Let G be a connected graph, let $n = |V(G)|$, and let T and T' be two spanning trees of G. Let k be the number of edges that are in T' but not in T (i.e., $k = |E(T') \backslash E(T)|$). We proceed by induction on k.

If $k = 0$, then, since $|E(T')| = n - 1 = |E(T)|$, $E(T') = E(T)$, so that $T = T'$, and the sequence T fulfills the desired properties.

Suppose that, for some $k \in \mathbb{N}$, $\forall$ spanning trees T' of G such that $|E(T') \backslash E(T)| \leq k$, there exists a sequence $T = T_1, T_2, \ldots, T_k = T'$ such that

$$|E(T_i) \cap E(T_{i+1})| \geq n - 2, \forall i \in \mathbb{N}, 0 \leq i \leq k - 1.$$

Suppose that, for some T', $|E(T') \backslash E(T)| \leq k + 1$. Then, $\exists e_1 \in E(T')$ such that $e_1 \notin E(T)$. Let H be the graph resulting from adding e_1 to T. Since T is a tree, H contains a cycle. Thus, since T' is a tree, there is some edge e_2 in this cycle such that $e_2 \notin T'$. Let T_1 be the graph resulting from removing e_2 from H. Then, since T_1 is a connected graph on n vertices with $n - 1$ edges, T_2 is a tree. Furthermore, $|E(T_1) \cap E(T)| \geq n - 2$, and $|E(T') \backslash E(T_1)| \leq k$. Then, by the inductive hypothesis, there exists a sequence $T_1, T_2, \ldots, T_m = T'$ such that, $|E(T_i) \cap E(T_{i+1})| \geq n - 2$, $\forall i \in \mathbb{N}$ with $1 \leq i \leq m - 1$, so that the sequence $T = T_0, T_1, T_2, \ldots, T_m = T'$ has the desired properties. Thus, by the Principle of Mathematical Induction, the claim in question holds $\forall k \in \mathbb{N}$. ■

Problem 5

The number labeled trees on n vertices is the same as the number of spanning trees of K_n, the complete graph on n vertices, since any spanning tree of K_n is by definition a tree on n vertices, and every tree on n vertices is a subgraph of K_n. By definition,

$$L_{K_n} = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix},$$

where L_{K_n} is of size $n \times n$. The $(n-1)$ vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{bmatrix}, \in \mathbb{R}^n$$

are eigenvectors of L_{K_n}, each of eigenvalue n. Thus, by Kirchoff's Theorem, the number of spanning trees of K_n is $\frac{1}{n}n^{n-1} = n^{n-2}$, proving Cayley's formula. ■

Problem 6

Let G be a graph, $n = |V(G)|$, $e = |E(G)|$. Order the vertices of G as $v_1, v_2, \ldots, v_n$, and the edges of G $e_1, e_2, \ldots, e_m$, so that, letting $M = E_G$, for $i \in [n]$, $j \in [m]$, M_{v_i,e_j}, $M_{i,j} = -1$ if the source of e_j is v_i, $M_{i,j} = 1$ if the destination of e_i is v_i, and $M_{i,j} = 0$ otherwise. Let $L = L_G$. Then, $\forall i \in [n]$, $\left(MM^T\right)_{i,i}$ is the sum of the squares of the elements of the i^{th} row of M; since there is a 1 or a (-1) in this row for each incoming or outgoing edge of v_i, $\left(MM^T\right)_{i,i} = \deg(v_i)$, so that $\left(MM^T\right)_{i,i} = L_{i,i}$.

$\forall i,j \in [n]$ with $i \neq j$, $\left(MM^T\right)_{i,j}$ is -1 if, for some $k \in [m]$, the k^{th} entry in both the i^{th} and j^{th} rows of M are non-zero (i.e., one is 1 and the other is (-1)). By definition of M, this occurs precisely when the e_k has v_i and v_j as endpoints, so that this happens precisely when there is an edge between v_i and v_j. Therefore, by definition of the L, $\left(MM^T\right)_{i,j} = L_{i,j}$.

Thus, $\forall i,j \in [n]$, $\left(MM^T\right)_{i,j} = L_{i,j}$, so that $MM^T = L$. ■

Problem 7

For $k = 0$, $A^k = I_n$ (the $n \times n$ identity matrix. Since, for $(i,j) \in [n] \times [n]$, $I_{i,j} = 1$ if and only if $i = j$, and there exists a (necessarily unique) walk of length 0 from vertex i to vertex j in G, if and only if $i = j$, the claim in question holds for $k = 0$.

Suppose that, for some $k \in \mathbb{N}$, $\forall (i,j) \in [n] \times [n]$, $A^k_{i,j}$ gives the number of walks of length k from vertex i to vertex j in G. $\forall (v,u) \in [n] \times [n]$, let $e_{v,j} = 1$ if and only if v and j are adjacent in G

($e_{v,j} = 0$ otherwise). $\forall v \in [n]$ the number of walks of length $(k+1)$ from i to j whose last edge is vj is the same as the number of walks of length k from i to v if v is adjacent to j in G, and 0 otherwise. Thus, partitioning the walks of length $(k+1)$ from i to j in G by their last edge shows that the number of such walks is given by

$$\sum_{v \in V(G)} A^k_{v,j} e_{v,j} = \sum_{v=1}^{n} A^k_{v,j} e_{v,j}.$$

$\forall (i,j) \in [n] \times [n]$, by construction of $e_{i,j}$ and the definition of the adjacency matrix, $A_{i,j} = e_{i,j}$. Thus, matrix multiplication gives

$$A^{k+1}_{i,j} = \left(A^k A\right)_{i,j} = \sum_{i=1} A^k_{i,j} A_{i,j} = \sum_{i=1} A^k_{i,j} e_{i,j},$$

so that $A^{k+1}_{i,j}$ is the number of walks of length $(k+1)$ from i to j in G. Thus, by the Principle of Mathematical Induction, $\forall k \in \mathbb{N}$, $\forall (i,j) \in [n] \times [n]$, $A^k_{i,j}$ gives the number of walks of length k from i to j in G. $\blacksquare$

Problem 8

By Kirchoff's Theorem, the following matlab code gives the desired quantity as output (`m` is defined as L_G):

```
>> m = [[6,0,-1,-1,-1,-1,-1,-1];
        [0,6,-1,-1,-1,-1,-1,-1];
        [-1,-1,6,0,-1,-1,-1,-1];
        [-1,-1,0,6,-1,-1,-1,-1];
        [-1,-1,-1,-1,6,0,-1,-1];
        [-1,-1,-1,-1,0,6,-1,-1];
        [-1,-1,-1,-1,-1,-1,6,0];
        [-1,-1,-1,-1,-1,-1,0,6]];
>> eigenvalues = eig(m);
>> n = 8;
>> prod(eigenvalues(2:8))/n            %the first eigenvalue is 0

ans =

      82944
```

Thus, the number of spanning trees of G is $\boxed{82944.}$