1 Preliminaries

A. Uniform Convergence

Definition 1.1. *Let I be an interval and let*

$$f_n : I \to \mathbb{R}^N$$

for each $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Let $f : I \to \mathbb{R}^N$.

1. *$f_n \to f$ pointwise on I if $f_n(x) \to f(x)$ for each $x \in I$, i.e., $\forall x \in I$ and $\varepsilon > 0 \ \exists \ N(x, \varepsilon)$ s.t. $n > N(x, \varepsilon) \Rightarrow |f_n(x) - f(x)| < \varepsilon$.*

2. *$f_n \to f$ uniformly on I if $\forall \varepsilon > 0 \ \exists \ N(\varepsilon)$ s.t. $n > N(\varepsilon)$ and $x \in I \Rightarrow |f_n(x) - f(x)| < \varepsilon$.*

Comment: If $f_n \to f$ uniformly on I then $f_n \to f$ pointwise on I.

Examples

1. $I = \mathbb{R}$

$$f_n(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ nx & \text{if} \quad 0 < x < \dfrac{1}{n} \\ 1 & \text{if} \quad \dfrac{1}{n} \leq x. \end{cases}$$

$f_n \to f$ pointwise on $\mathbb{R}$ where

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ 1 & \text{if} \quad 0 < x. \end{cases}$$

$f_n \not\to f$ uniformly on I: Suppose it did, then $\exists N = N\left(\dfrac{1}{2}\right)$ s.t.

$$n > N \text{ and } x \in \mathbb{R} \Rightarrow |f_n(x) - f(x)| < \frac{1}{2}.$$

Let $n > N$ and $x = \dfrac{1}{2n}$, then $\dfrac{1}{2} > |f_n(x) - f(x)| = \left|\dfrac{1}{2} - 1\right| = \dfrac{1}{2}$.

Contradiction.

Note: $f_n \to f$ pointwise, f_n is continuous $\forall n$, but f is discontinuous.

2. $I = \mathbb{R}$

$$f_n(x) = 1 + \sum_{k=1}^{n} \frac{x^k}{k!} \text{ and } f(x) = e^x.$$

$f_n \to f$ pointwise on $\mathbb{R}$. Let $B > 0$, then $f_n \to f$ uniformly on $[-B, B]$: for any $x \in [-B, B]$

$$|f_n(x) - f(x)| = \left|\sum_{k=n+1}^{\infty} \frac{x^k}{k!}\right| \leq \sum_{n+1}^{\infty} \frac{|x|^k}{k!} \leq \sum_{n+1}^{\infty} \frac{B^k}{k!}.$$

Let $\varepsilon > 0$. $\sum_{1}^{\infty} \dfrac{B^k}{k!}$ converges so $\exists\ N$ s.t. $n > N \Rightarrow \sum_{n+1}^{\infty} \dfrac{B^k}{k!} < \varepsilon$. Now

$$x \in [-B, B] \text{ and } n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

$f_n \not\to f$ on $\mathbb{R}$: Suppose it did, then $\exists N = N(1)$ s.t.

$$n > N \text{ and } x \in \mathbb{R} \Rightarrow |f_n(x) - f(x)| < 1.$$

Let $n > N$ and $x = [(n+1)!]^{\frac{1}{n+1}}$, then

$$1 > |f_n(x) - f(x)| = \sum_{n+1}^{\infty} \frac{x^k}{k!} > \frac{x^{n+1}}{(n+1)!} = 1.$$

Contradiction.

Theorem 1.1. *Let $f_n \to f$ uniformly on I and assume f_n is continuous on I $\forall n \in \mathbb{N}$. Then f is continuous on I. Also $\forall a, b \in I$*

$$\int_a^b f_n dx \to \int_a^b f\, dx.$$

Corollary 1.1. *Let $f_n \to f$ uniformly on $[0, B]$ for each $B > 0$ and assume f_n is continuous on $[0, \infty)$. Then f is continuous on $[0, \infty)$.*

Proof. Consider any $x \in [0, \infty)$. Let $B = x + 1$. $f_n \to f$ uniformly on $[0, B]$ and f_n is continuous on $[0, B]$ $\forall n$, so f is continuous on $[0, B]$. Since $x \in [0, B]$ f is continuous at x. Since $x \in [0, \infty)$ was arbitrary, f is continuous on $[0, \infty)$. $\square$

B. Two Examples

1. $$\begin{cases} \dot{X}(t) = X^2(t) \\ X(0) = \dfrac{1}{\varepsilon} \end{cases} \qquad \varepsilon > 0$$

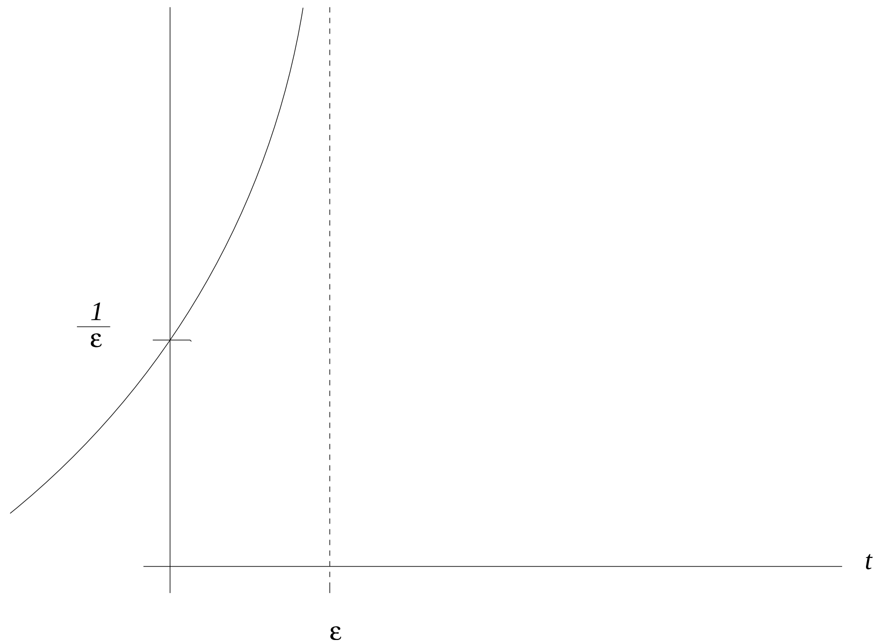
$$\frac{\dot{X}}{X^2} = \frac{d}{dt}(-X^{-1}) = 1$$

$$-X^{-1} = t + C$$

$$-\varepsilon = -X^{-1}(0) = 0 + C$$

$$-X^{-1} = t - \varepsilon$$

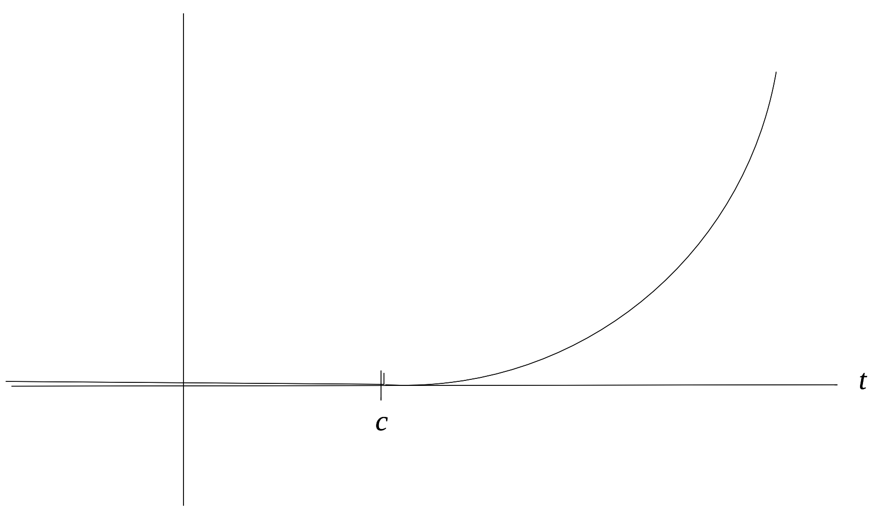
$$X = \frac{1}{\varepsilon - t} \qquad \text{for } t \in (-\infty, \varepsilon)$$



2. $$\begin{cases} \dot{X} = X^{\frac{1}{3}} \\ X(0) = 0 \end{cases}$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{3}{2}X^{\frac{2}{3}}\right) &= X^{-\frac{1}{3}}\dot{X} = 1 && \text{if } X \neq 0 \\
\frac{3}{2}X^{\frac{2}{3}} &= t + C \\
X &= \left(\frac{2}{3}t + \tilde{C}\right)^{\frac{3}{2}} \\
X(0) &= 0 = \tilde{C}^{\frac{3}{2}} \\
X(t) &= \left(\frac{2}{3}t\right)^{\frac{3}{2}} && \text{for } t \geq 0 \text{ is a solution} \\
X(t) &= 0 && \text{is another} \\
X(t) &= \begin{cases} 0 & t \leq c \\ \left[\frac{2}{3}(t-c)\right]^{\frac{3}{2}} & c < t \end{cases}
\end{aligned}
$$

is a solution for any $c \geq 0$.



C. Reduction to First Order

A differential equation is a relationship among the derivatives of a function. The order of the differential equation is the order of the highest order derivative that appears. We'll assume it can be solved for that derivative:

$$X^{(N)}(t) = f\left(t, X(t), \ldots, X^{(N-1)}(t)\right). \tag{1}$$

Any scalar equation may be reduced to a first order system: Let

$$Y = \begin{pmatrix} X \\ X' \\ \vdots \\ X^{(N-1)} \end{pmatrix}$$

then (1) holds if, and only if,

$$\dot{Y} = \begin{pmatrix} Y_2 \\ \vdots \\ Y_N \\ f(t, Y_1, \ldots, Y_N) \end{pmatrix}.$$

Example

$$\begin{cases} \dot{X} &= XY \\ \ddot{Y} &= X\dot{Y} + t^2 \end{cases} \tag{2}$$

Let

$$Z = \begin{pmatrix} X \\ Y \\ \dot{Y} \end{pmatrix}$$

then (2) holds if, and only if,

$$\dot{Z} = \begin{pmatrix} Z_1 \, Z_2 & \\ \\ Z_3 & \\ \\ Z_1 \, Z_3 & +t^2 \end{pmatrix}.$$

Comment: We'll consider

$$\dot{X} = f(t, X)$$

where

$$X : \mathbb{R} \to \mathbb{R}^N \text{ and } f : \mathbb{R}^{N+1} \to \mathbb{R}^N.$$

D. Lipschitz and Hölder Conditions

Consider $f : D \to \mathbb{R}^N$ where $D \subset \mathbb{R} \times \mathbb{R}^N$.

Definition 1.2. *We say f satisfies a Lipschitz condition in x (on D) if there is a constant $C > 0$ s.t.*

$$|f(t,x) - f(t,y)| \leq C|x - y|$$

for all (t, x) and $(t, y) \in D$.

Comment This says nothing about how f depends on t. If

$$|f(t,x) - f(s,y)| \leq C\sqrt{(t-s)^2 + |x-y|^2}$$

held on D, we would say f is Lipschitz in x **and** t.

Examples

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(t,x) = \begin{cases} x^2 & \text{if} \quad t \geq 1 \\ \\ 0 & \text{if} \quad t < 1. \end{cases}$$

   Note f is not continuous in t.

(a) Let $B > 0$ and $D = \{(t, x) : t \in \mathbb{R} \text{ and } |x| \leq B\}$, then f satisfies a Lipschitz condition in x on D : For any $x, y, \in [-B, B]$

$$|f(t,x) - f(t,y)| \leq |x^2 - y^2| \leq (|x| + |y|)|x - y| \leq 2B|x - y|.$$

(b) f does not satisfy a Lipschitz condition in x on $D = \mathbb{R}^2$: If it did, then $\exists\ C > 0$ s.t.

$$|f(t,x) - f(t,y)| \leq C|x - y|$$

$\forall t, x, y$. Take $t = 2$ and $y = 0$:

$$x^2 \leq C|x|$$

$\forall x$. Taking $x = 2C$ yields

$$\begin{aligned} 4C^2 &\leq C(2C), \\ 4 &\leq 2, \end{aligned}$$

contradiction.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t,x) = x^{\frac{1}{3}}.$$

(a) Let $\varepsilon > 0$ and $D = \mathbb{R} \times [\varepsilon, \infty)$, then f satisfies a Lipschitz condition in x on D:

$$\begin{aligned} |f(t,x) - f(t,y)| &= |x^{\frac{1}{3}} - y^{\frac{1}{3}}| \\ &= \frac{|x - y|}{x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}} \leq \frac{|x - y|}{3\varepsilon^{\frac{2}{3}}} \end{aligned}$$

for $x, y \geq \varepsilon$.

(b) f does not satisfy a Lipschitz condition in x on $D = \mathbb{R} \times [0, \infty)$: if it did, then (taking $y = 0$)

$$Cx = C|x - y| \geq |f(t, x) - f(t, y)| = x^{\frac{1}{3}}$$

$\forall x \geq 0$. Taking $x = \frac{1}{2} C^{-\frac{3}{2}}$ yields

$$\begin{aligned} C \frac{1}{2} C^{-\frac{3}{2}} &\geq \left(\frac{1}{2} C^{-\frac{3}{2}} \right)^{\frac{1}{3}}, \\ \frac{1}{2} &\geq \left(\frac{1}{2} \right)^{\frac{1}{3}}, \ \left(\frac{1}{2} \right)^{\frac{2}{3}} \geq 1, \ \frac{1}{2} \geq 1, \end{aligned}$$

contradiction.

Proposition 1.1. *Let* $x_0 \in \mathbb{R}^N, \ t_0 < t_1, \ R > 0$ *and*

$$D = \{(t, x) : t \in [t_0, t_1] \text{ and } |x - x_0| \leq R\}.$$

Assume $f : D \to \mathbb{R}^m$ *and that* $\frac{\partial f_i}{\partial x_k}$ *exists and is continuous on* D *for* $i = 1, \ldots, m$ *and* $k = 1, \ldots, N$. *Then* f *satisfies a Lipschitz condition* x *on* D.

Proof. D is compact and $\frac{\partial f_i}{\partial x_k}$ is continuous on D so we may define

$$C_{ik} = \max_D \left| \frac{\partial f_i}{\partial x_k} \right|$$

and

$$C = \max \{C_{ik} : 1 \leq i \leq m \text{ and } 1 \leq k \leq N\}.$$

Then $\forall (t, x)$ and $(t, y) \in D$

$$
\begin{aligned}
& |f_i(t,x) - f_i(t,y)| \\
= & \left| \int_0^1 \frac{d}{ds} \left[f_i(t, x + s(y-x)) \right] ds \right| \\
= & \left| \int_0^1 \sum_{k=1}^N \frac{\partial f_i}{\partial x_k}(t, x + s(y-x))(y-x)_k ds \right| \\
\leq & \int_0^1 \sum_{k=1}^N C_{ik} |y_k - x_k| ds \leq \sum_{k=1}^N C|y-x| \\
= & CN|y-x|
\end{aligned}
$$

and hence

$$
\begin{aligned}
& |f(t,x) - f(t,y)| \\
= & \sqrt{\sum_{i=1}^m (f_i(t,x) - f_i(t,y))^2} \\
\leq & \sqrt{\sum_{i=1}^m (CN|y-x|)^2} \\
= & CN\sqrt{m}\ |y-x|.
\end{aligned}
$$

$\square$

Example

$f(t,x) = |x|$ satisfies a Lipschitz condition in x on $\mathbb{R} \times \mathbb{R}$, but the proposition does not apply since $\dfrac{\partial f}{\partial x}$ is discontinuous.

Definition 1.3. *Let $\alpha \in (0,1)$. We say f satisfies a Hölder condition in x with exponent α (on D) if $\exists C > 0$ s.t.*

$$|f(t,x) - f(t,y)| \leq C|x-y|^\alpha$$

$\forall (t,x)$ and $(t,y) \in D$.

Comments

Let $\delta > 0,\ \varepsilon > 0,\ t_0 \in \mathbb{R},\ x_0 \in \mathbb{R}^N$, and

$$D = \{(t, x) : |t - t_0| < \delta \text{ and } |x - x_0| < \varepsilon\}.$$

1.
$$f \text{ satisfies Lipschitz condition in } x$$
$$\Rightarrow \quad f \text{ satisfies Hölder condition in } x$$
with exponent α for every $\alpha \in (0, 1)$.

2. f satisfies Hölder condition in x with exponent $\alpha \in (0, 1) \Rightarrow f$ satisfies Hölder condition in x with exponent $\beta \,\forall \beta \in (0, \alpha]$.

3. Consider
$$\begin{cases} \dot{X} &= f(t, X) \\ X(t_0) &= x_0. \end{cases}$$
We'll prove later that if f satisfies a Lipschitz condition in x then there is at most one solution. If f satisfies a Hölder condition in x, but not a Lipschitz condition, then uniqueness may fail.

2 Existence

A. Iteration

Assume f is continuous and consider

$$\begin{cases} \dot{X} &= f(t, X) \\ X(t_0) &= x_0 \end{cases} \tag{3}$$

and

$$X(t) = x_0 + \int_{t_0}^{t} f(s, X(s))ds. \tag{4}$$

Note that $X \in C^1$ and (3) $\Leftrightarrow X \in C^0$ and (4) holds.

Define $X^{(n)}$ by

$$X^{(0)}(t) = x_0$$

and

$$X^{(n+1)}(t) = x_0 + \int_{t_0}^{t} f(s, X^{(n)}(s))ds$$

for $n \geq 0$.

Comments

1. Same as

$$\begin{cases} \dot{X}^{(n+1)} & = & f(t, X^{(n)}) \\ X^{(n)}(t_0) & = & x_0. \end{cases}$$

2. If $X^{(n)} \to X$ uniformly then

$$\begin{array}{ccc} X^{(n+1)}(t) & = & x_0 + \int_{t_0}^{t} f(s, X^{(n)}(s))ds \\ \downarrow & & \downarrow \\ X(t) & = & x_0 + \int_{t_0}^{t} f(s, X(s))ds, \end{array}$$

hence X satisfies (4) and hence $X \in C^1$ satisfies (3).

3. For numerical computation Euler's method is **much** better than iteration: Let $\Delta t > 0$ and define X_k by

$$\begin{cases} X_0 &= x_0, \\ \dfrac{X_{k+1} - X_k}{\Delta t} &= f(t_0 + k\Delta t, X_k) \; k \geq 0. \end{cases}$$

Then

$$X(t_0 + k\Delta t) \approx X_k.$$

Example

$$\begin{cases} \dot{X} &= -X \\ X(0) &= 1 \end{cases} \qquad \text{solution } X(t) = e^{-t}$$

$$\begin{aligned} X^{(0)}(t) &= 1 \\ X^{(1)}(t) &= 1 + \int_0^t \left(-X^{(0)}(s)\right) ds = 1 - t \\ X^{(2)}(t) &= 1 + \int_0^t \left(-X^{(1)}(s)\right) ds = 1 - \int_0^t (1-s) ds \\ &= 1 - t + \frac{1}{2}t^2 \\ X^{(n)}(t) &= 1 - t + \frac{1}{2!}t^2 - \ldots + \frac{1}{n!}(-t)^n. \end{aligned}$$

$\forall T > 0 \; X^{(n)}(t) \to e^{-t}$ uniformly on $[-T, T]$.

B. Contraction Mapping
General Discussion
Definitions

1. $\mathcal{C}_B = \mathcal{C}[t_0, t_1] = \{X : [t_0, t_1] \to B \text{ s.t. } X \text{ is continuous}\}$ where $B \subset \mathbb{R}^N$.
2. $\|X\|_{\mathcal{C}} = \sup\{|X(t)| : t \in [t_0, t_1]\}$.
3. $\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_{\mathbb{R}^N}$ is a contraction if $\exists\, C \in (0, 1)$ s.t.

$$\|\mathcal{F}(X) - \mathcal{F}(Y)\|_{\mathcal{C}} \le C\|X - Y\|_{\mathcal{C}}$$

   $\forall X, Y \in \mathcal{C}_B$.

Contraction Mapping Theorem Let $B \subset \mathbb{R}^N$ be closed and $\mathcal{F} : \mathcal{C}_B \to \mathcal{C}_B$ be a contraction. Then there is exactly one $X \in \mathcal{C}_B$ s.t.

$$\mathcal{F}(X) = X.$$

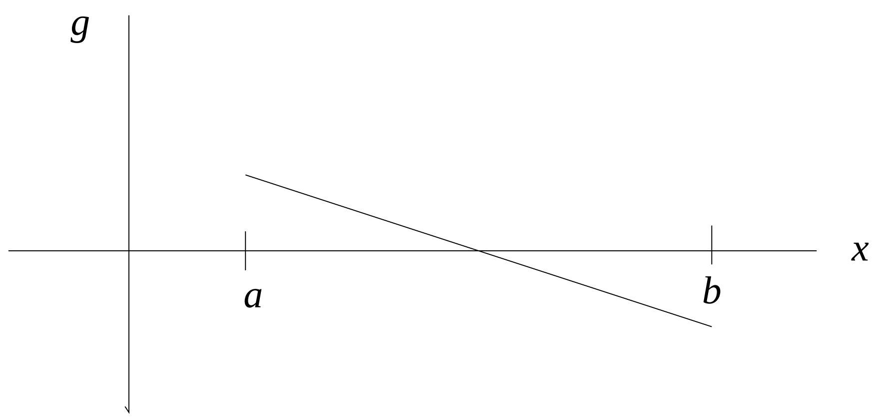
Comments

1. The theorem works for complete metric spaces.
2. We'll use

$$\mathcal{F}(X)\big|_t = x_0 + \int_{t_0}^{t} f(s, X(s))ds.$$

Examples

0. Consider solving $g(x) = 0$ where



Let $f(x) = x + \varepsilon g(x)$. For $\varepsilon > 0$ sufficiently small $f : [a,b] \to [a,b]$. Note that

$$g(x) = 0 \Leftrightarrow f(x) = x.$$

1. Let $T > 0$ and $g : [0,T] \to \mathbb{R}$ be continuous. Consider

$$X(t) = g(t) + \int_0^T (t-\tau)X(\tau)d\tau.$$

Define $\mathcal{F} : \mathcal{C}_{\mathbb{R}}[0,T] \to \mathcal{C}_{\mathbb{R}}[0,T]$ by

$$\mathcal{F}(X)|_t = g(t) + \int_0^T (t-\tau)X(\tau)d\tau.$$

Then

$$
\begin{aligned}
& \left|\mathcal{F}(X)|_t - \mathcal{F}(Y)|_t\right| \\
= & \left|\int_0^T (t-\tau)(X(\tau)-Y(\tau))d\tau\right| \\
\leq & \int_0^T |t-\tau|\|X-Y\|_{\mathcal{C}} d\tau \\
= & \frac{t^2+(T-t)^2}{2}\|X-Y\|_{\mathcal{C}} \leq \frac{T^2}{2}\|X-Y\|_{\mathcal{C}}.
\end{aligned}
$$

If $T < \sqrt{2}$ $\mathcal{F}$ is a contraction and has a unique fixed point. If $T \geq \sqrt{2}$ the above yields no conclusion.

2. Consider

$$X(t) = g(t) + \int_0^1 (t-\tau)X^2(\tau)d\tau.$$

Define

$$\mathcal{F}(X)|_t = g(t) + \int_0^1 (t-\tau)X^2(\tau)d\tau.$$

Then

$$
\begin{aligned}
\left|\mathcal{F}(X)|_t - \mathcal{F}(Y)|_t\right| & \leq \int_0^1 |t-\tau| \left|X^2(\tau)-Y^2(\tau)\right| d\tau \\
& \leq \int_0^1 |t-\tau| \left(|X|+|Y|\right) |X-Y| d\tau \\
& \leq \int_0^1 |t-\tau| \left(\|X\|_{\mathcal{C}}+\|y\|_{\mathcal{C}}\right) \|X-Y\|_{\mathcal{C}}\ d\tau \\
& \leq \frac{1}{2}\left(\|X\|_{\mathcal{C}}+\|Y\|_{\mathcal{C}}\right)\|X-Y\|_{\mathcal{C}}
\end{aligned}
$$

so $\mathcal{F}$ is a contraction on $\mathcal{C}_{[-.9,.9]}[0,1]$. We also need $\mathcal{F}:\mathcal{C}_{[-.9,.9]}[0,1]\to\mathcal{C}_{[-.9,.9]}[0,1]:$ for $\|X\|_{\mathcal{C}}\le .9$

$$|\mathcal{F}(X)|_t| \le \|g\|_{\mathcal{C}} + \int_0^1 |t-\tau|\|X\|_{\mathcal{C}}^2 d\tau \le \|g\|_{\mathcal{C}} + \frac{1}{2}(.9)^2$$

so

$$\|\mathcal{F}(X)\|_{\mathcal{C}} \le \|g\|_{\mathcal{C}} + \frac{(.9)^2}{2}.$$

But $\|g\|_{\mathcal{C}}+\frac{(.9)^2}{2} \le .9 \Leftrightarrow \|g\|_{\mathcal{C}} \le .9-\frac{(.9)^2}{2} = .495$, so for $\|q\|_{\mathcal{C}} \le .495$, $\mathcal{F}$ has a unique fixed point in $\mathcal{C}_{[-.9,.9]}[0,1]$; there could be other fixed points in $\mathcal{C}_{\mathbb{R}}[0,1]$.

Comment If a sequence, $X^{(n)}$, is Cauchy in $\|\cdot\|_{\mathcal{C}}$, then it's convergent in $\|\cdot\|_{\mathcal{C}}$.

Proof. Assume $X^{(n)}$ is Cauchy, i.e., $\forall\varepsilon>0$
$\exists N$ s.t. $k,n>M \Rightarrow \|X^{(n)}-X^{(k)}\|_{\mathcal{C}}<\varepsilon$.

$$\forall t\in[t_0,t_1] k,n>M \Rightarrow \left|X^{(n)}(t)-X^{(k)}(t)\right|_{\mathcal{C}}<\varepsilon.$$

$X^{(n)}(t)$ is Cauchy (in $\mathbb{R}^N$) hence convergent, let

$$X(t)=\lim_{n\to\infty}X^{(n)}(t).$$

Now $k>M \Rightarrow \left|X^{(k)}(t)-X(t)\right| = \lim_{n\to\infty}\left|X^{(k)}(t)-X^{(n)}(t)\right| \le \varepsilon$ $\forall t\in[t_0,t_1]$. Hence

$$k>M \Rightarrow \|X^{(k)}-X\|_{\mathcal{C}} \le \varepsilon$$

and $X^{(n)}\to X$ in $\|\cdot\|_{\mathcal{C}}$ □

Proof of Contraction Mapping Thorem

Suppose $\mathcal{F}(X) = X$ and $\mathcal{F}(Y) = Y$ with $X, Y \in \mathcal{C}_B$. Then

$$\begin{aligned} \|X - Y\|_{\mathcal{C}} &= \|\mathcal{F}(X) - \mathcal{F}(Y)\|_{\mathcal{C}} \\ &\leq C\|X - Y\|_{\mathcal{C}} \end{aligned}$$

so

$$(1 - C)\|X - Y\|_{\mathcal{C}} \leq 0.$$

Since $C < 1$, $\|X - Y\|_{\mathcal{C}} = 0$ and $X = Y$.

Let $X^{(0)} \in \mathcal{C}_B$ and define $X^{(n)}$ by

$$X^{(n+1)} = \mathcal{F}(X^{(n)}) \quad \forall n \geq 0.$$

Then

$$\begin{aligned} \|X^{(n+1)} - X^{(n)}\| &= \|\mathcal{F}(X^{(n)}) - \mathcal{F}(X^{(n-1)})\|_{\mathcal{C}} \\ &\leq C\|X^{(n)} - X^{(n-1)}\|_{\mathcal{C}} \\ &\leq C^2\|X^{(n-1)} - X^{(n-2)}\|_{\mathcal{C}} \\ &\leq \cdots \leq C^n\|X^{(1)} - X^{(0)}\|_{\mathcal{C}} \end{aligned}$$

and hence

$$\begin{aligned}
& \|X^{(n+k)} - X^{(n)}\|_{\mathcal{C}} \\
\leq & \sum_{\ell=n}^{n+k-1} \|X^{(\ell+1)} - X^{(\ell)}\|_{\mathcal{C}} \\
\leq & \sum_{\ell=n}^{n+k-1} C^{\ell} \|X^{(1)} - X^{(0)}\|_{\mathcal{C}} \\
= & \|X^{(1)} - X^{(0)}\|_{\mathcal{C}} C^n \sum_{j=0}^{k-1} C^j \\
= & \|X^{(1)} - X^{(0)}\|_{\mathcal{C}} C^n \ \frac{1 - C^k}{1 - C} \\
\leq & \frac{\|X^{(1)} - X^{(0)}\|_{\mathcal{C}}}{1 - C} C^n \longrightarrow \underset{n\to\infty}{0}.
\end{aligned}$$

Thus, $X^{(n)}$ is Cauchy in $\|\cdot\|_{\mathcal{C}}$, say $\|X^{(n)} - X\|_{\mathcal{C}} \to 0$. Then

$$\begin{aligned}
\|X^{(n)} - \mathcal{F}(X)\|_{\mathcal{C}} &= \|\mathcal{F}(X^{(n-1)}) - \mathcal{F}(X)\|_{\mathcal{C}} \\
&\leq C\|X^{(n-1)} - X\|_{\mathcal{C}} \to 0.
\end{aligned}$$

So $X^{(n)} \to X$ and $X^{(n)} \to \mathcal{F}(X)$. Thus

$$\mathcal{F}(X) = X.$$

Comment The assumption that B is closed is needed so that

$$X(t) = \lim_{n\to\infty} X^{(n)}(t) \in B$$

and hence

$$X \in \mathcal{C}_B.$$

Application to ODE's

Assume f is continuous on

$$D = \{(t,x) : t_0 \leq t \leq t_0 + \delta_0,\ |x - x_0| \leq \varepsilon_0\}$$

with $\delta_0, \varepsilon_0 > 0$. Assume $\exists\ L > 0$ s.t.

$$|f(t,x) - f(t,y)| \leq L|x-y| \text{ on } D.$$

Comments

1. Let

$$M = \max_D |f|.$$

2. Define

$$B = \left\{x \in \mathbb{R}^N : |x - x_0| \leq \varepsilon_0\right\}$$

and

$$\mathcal{F}(X)\big|_t = x_0 + \int_{t_0}^{t} f(s, X(s))ds.$$

We'll construct $\delta \in (0, \delta_0]$ s.t.

$$\mathcal{F} : \mathcal{C}_B[t_0, t_0+\delta] \to \mathcal{C}_B[t_0, t_0+\delta]$$

is a contraction.

Lemma 2.1. *Let* $\delta_1 = \min\left(\delta_0, \dfrac{\varepsilon_0}{M}\right)$ *and* $\delta \in (0, \delta_0]$, *then*

$$\mathcal{F} : \mathcal{C}_B[t_0, t_0+\delta] \to \mathcal{C}_B[t_0, t_0+\delta].$$

Proof. Let $X \in \mathcal{C}_B[t_0, t_0+\delta]$ and $t \in [t_0, t_0+\delta]$, then

$$\begin{aligned} \left|\mathcal{F}(X)|_t - x_0\right| &= \left|\int_{t_0}^t f(s, X(s))ds\right| \\ &\le \int_{t_0}^t |f(s, X(s))|\, ds \le \int_{t_0}^t M\ ds \\ &= M(t-t_0) \le M\delta \le \varepsilon_0. \end{aligned}$$

Hence $\mathcal{F}(X) \in \mathcal{C}_B$. $\square$

Next we further restrict δ to get $\mathcal{F}$ to be a contraction. Let $X, Y \in \mathcal{C}_B$ and $\delta \in (0, \delta_1]$. For $t \in [t_0, t_0+\delta]$

$$\begin{aligned} &\left|\mathcal{F}(X)|_t - \mathcal{F}(Y)|_t\right| \\ \le\ & \int_{t_0}^t |f(s, X(s)) - f(s, Y(s))|\, ds \\ \le\ & \int_{t_0}^t L\, |X(s) - Y(s)|\, ds \\ \le\ & \int_{t_0}^t L\|X-Y\|_{\mathcal{C}}\ ds \le L\ \delta \|X-Y\|_{\mathcal{C}}. \end{aligned}$$

Taking $\delta = \min(\delta_1,\ .99L^{-1})$ yields

$$\|\mathcal{F}(X) - \mathcal{F}(Y)\|_{\mathcal{C}} \le .99\|X-Y\|_{\mathcal{C}}.$$

By the contraction mapping theorem we've shown:

Cauchy Lipschitz Theorem If f is continuous and satisfies a Lipschitz condition in x on

$$D = \{(t, x) : t_0 \le t \le t_0 + \delta_0,\ |x - x_0| \le \varepsilon_0\},$$

then there exists $\delta > 0$ s.t.

$$\begin{cases} \dot{X} &= f(t, X) \\ X(t_0) &= x_0 \end{cases}$$

has a solution on $[t_0, t_0 + \delta]$.

Comment We may solve on $[t_0 - \tilde{\delta}, t_0]$ by applying the theorem to

$$\begin{cases} \dot{Y} &= -f(2t_0 - t, Y(t)) \\ Y(t_0) &= x_0 \end{cases}$$

and taking $X(t) = Y(2t_0 - t)$.

C. Compactness

The goal of this section is to prove:

Cauchy-Peano Theorem Assume f is continuous on

$$D = \{(t, x) : t_0 \le t \le t_0 + \delta_0,\ |x - x_0| \le \varepsilon_0\}.$$

Then $\exists \delta \in (0, \delta_0]$ s.t.

$$\begin{cases} \dot{X} &= f(t, X) \\ X(t_0) &= x_0 \end{cases}$$

has at least one solution on $[t_0, t_0 + \delta]$.

Comments

1. A Lipschitz condition is not required so the theorem applies to examples like

$$\begin{cases} \dot{X} &= X^{\frac{1}{3}} \\ X(0) &= 0 \end{cases}$$

where uniqueness fails.

2. We'll consider a sequence of approximate solutions. We'll show it has a uniformly convergent subsequence and then that the limit is a solution.

Example

Let

$$f(t,x) = \begin{cases} 1 & \text{if} \quad x \le 0 \\ \\ -1 & \text{if} \quad 0 < x. \end{cases}$$

Suppose $X : [0, \delta] \to \mathbb{R}$ is differentiable with

$$\begin{cases} \dot{X} &= f(t,x) \\ \\ X(0) &= 0. \end{cases}$$

Then $\dot{X}(0) = f(0,0) = 1$ and $\exists \delta_1 \in (0, \delta]$ s.t. $X(t) > X(0) = 0 \ \forall t \in (0, \delta_1]$. By the mean value theorem $\exists \xi \in (0, \delta_1)$ s.t.

$$0 < \frac{X(\delta_1) - X(0)}{\delta_1} = \dot{X}(\xi) = f(\xi, X(\xi)) = -1.$$

Contradiction.

Euler's Method

Picard iteration does not work here, see problem 2, assignment 2.
Let $\Delta t > 0$, $t_k = t_0 + k\Delta t$, and define $X_k^{\Delta t} \approx X(t_k)$ by

$$X_0^{\Delta t} = x_0,$$

$$\frac{X_{k+1}^{\Delta t} - X_k^{\Delta t}}{\Delta t} = f(t_k, X_k^{\Delta t}) \quad k \ge 0.$$

Further define

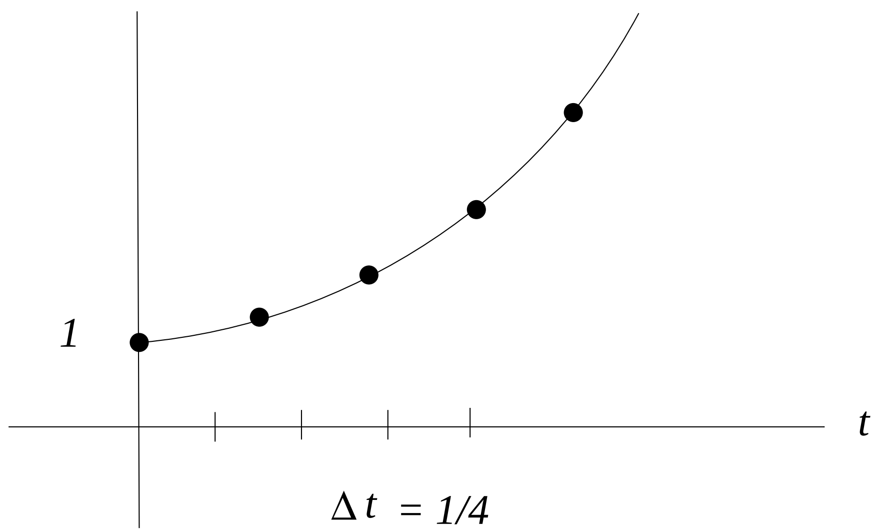
$$X^{\Delta t}(t) = \frac{t_{k+1} - t}{\Delta t} X_k^{\Delta t} + \frac{t - t_k}{\Delta t} X_{k+1}^{\Delta t}$$

for $t \in [t_k, t_{k+1}]$. We'll seek a convergent subsequence of $X^{\frac{1}{n}}$.

Example

$$\begin{cases} \dot{X} &= f(t,X) := X \\ X(0) &= x_0 := 1 \end{cases} \Rightarrow X(t) = e^t.$$

$$\begin{aligned} X_0^{\Delta t} &= 1, \\ \frac{X_{k+1}^{\Delta t} - X_k^{\Delta t}}{\Delta t} &= f(t_k, X_k^{\Delta t}) = X_k^{\Delta t}, \\ X_{k+1}^{\Delta t} &= (1+\Delta t) X_k^{\Delta t}, \\ X_k^{\Delta t} &= (1+\Delta t)^k X_0^{\Delta t} = (1+\Delta t)^k. \end{aligned}$$



$$X^{\Delta t}(t_k) = X_k^{\Delta t} = (1+\Delta t)^k = \left[(1+\Delta t)^{\frac{1}{\Delta t}}\right]^{t_k}$$

and

$$\lim_{\Delta t \to 0^+} (1+\Delta t)^{\frac{1}{\Delta t}} = e.$$

The Arzela Ascoli: Theorem

Definitions

Let $\left\{X^{(n)}\right\}$ be a sequence of functions from $[t_0, t_1] \to \mathbb{R}^N$.

1. $\{X^{(n)}\}$ is pointwise bounded on $[t_0, t_1]$ if for each $t \in [t_0, t_1]$, $\{X^{(n)}(t)\}$ is bounded (the bound can depend on t).

2. $\{X^{(n)}\}$ is uniformly bounded on $[t_0, t_1]$ if $\exists\ C > 0$ s.t.

$$\left|X^{(n)}(t)\right| \leq C\ \forall t \in [t_0, t_1],\ \forall n \in \mathbb{N}.$$

3. $\{X^{(n)}\}$ is equicontinuous on $[t_0, t_1]$ if $\forall \varepsilon > 0\ \exists\ \delta > 0$ s.t.

$$\left|X^{(n)}(t) - X^{(n)}(s)\right| < \varepsilon \quad \forall n \in \mathbb{N},$$

$$\forall s, t \in [t_0, t_1] \text{ s.t. } |s - t| < \delta.$$

Arzela Ascoli Theorem Let $\{X^{(n)}\}$ be a sequence in $\mathcal{C}_{\mathbb{R}^N}[t_0, t_1]$ that is pointwise bounded and equicontinuous. Then $\{X^{(n)}\}$ is uniformly bounded and has a uniformly convergent subsequence.

Examples

1. Let

$$X^{(n)}(t) = \begin{cases} n & \text{if} \quad 0 < t < \dfrac{1}{n} \\[2ex] \dfrac{1}{t} & \text{if} \quad \dfrac{1}{n} < t. \end{cases}$$

$\{X^{(n)}\}$ is pointwise bounded but not uniformly bounded on $(0, \infty)$.

Proof.

$$\begin{aligned}
\left|X^{(n)}(t)\right| &\le \frac{1}{t}\forall n,\ t. && \text{If} \\
\left|X^{(n)}(t)\right| &\le C\ \forall n,\ t && \textit{then} \\
n &= \left|X^{(n)}\left(\frac{1}{n}\right)\right| \le C\ \forall n
\end{aligned}$$

which is false. ∎

2 Let

$$X^{(n)}(t) = \begin{cases} 1 & t < 0 \\ 1 - nt & 0 \le t \le \dfrac{1}{n} \\ 0 & \dfrac{1}{n} < t. \end{cases}$$

$0 \le X^{(n)}_{(t)} \le 1 \quad \forall n, t.$ $\left\{X^{(n)}\right\}$ is not equicontinuous. If it were then $\exists\ \delta > 0$ s.t.

$$\left|X^{(n)}(t) - X^{(n)}(s)\right| < 1\ \ \forall n,\ \text{if } |s-t| < \delta.$$

Choose $n > \dfrac{1}{\delta}$ then $\delta > \dfrac{1}{n} = \left|\dfrac{1}{n} - 0\right|$ but

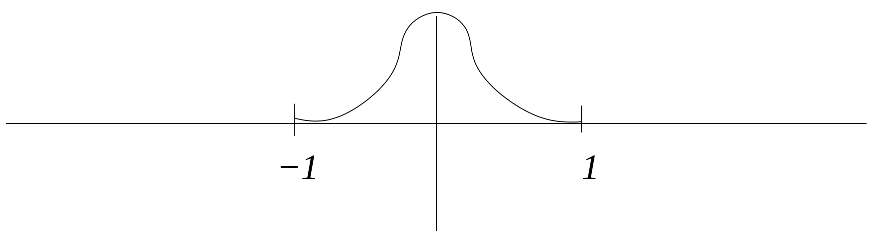
$$\left|X^{(n)}\left(\frac{1}{n}\right) - X^{(n)}(0)\right| = |0-1| = 1.$$

Contradiction.

Note: $X^{(n)}(t) \to \begin{cases} 1 & \text{if} \quad t \le 0 \\ 0 & \text{if} \quad t > 0. \end{cases}$

If $\left\{X^{(k_n)}\right\}$ were a uniformly convergent subsequence, it's limit would have to be continuous. But it would have to have the same limit as $X^{(n)}$ which is discontinuous. Hence, there is no uniformly convergent subsequence.

3. Let $f : \mathbb{R} \to \mathbb{R}$ have the graph:



Define $X^{(n)}(t) = f(t-n)$. $\{X^{(n)}\}$ is uniformly bounded and equicontinuous. On any bounded interval $X^{(n)}$ converges uniformly to 0. On $\mathbb{R}$ no subsequence of $X^{(n)}$ converges uniformly.

4. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Define

$$X^{(n)}(t) = a_n + b_n t$$

for $t \in [0,1]$. $\{X^{(n)}\}$ has a uniformly convergent subsequence.

Direct Proof. $\{a_n\}$ is bounded so it has a convergent subsequence, $\{a_{k_n}\}$. Let $A = \lim a_{k_n}$. $\{b_{k_n}\}$ is bounded so it has a convergent subsequence, $\{b_{\ell_{k_n}}\}$. Let $B = \lim b_{\ell_{k_n}}$. Then $\forall t \in [0,1]$

$$\begin{aligned} & \left|X^{(\ell_{k_n})}(t) - (A+Bt)\right| \\ = \quad & \left|a_{\ell_{k_n}} - A + \left(b_{\ell_{k_n}} - B\right) t\right| \\ \leq \quad & \left|a_{\ell_{k_n}} - A\right| + \left|b_{\ell_{k_n}} - B\right| \to 0. \end{aligned}$$

Hence, $X^{(\ell_{k_n})}$ converges uniformly.

Proof Using Arzela Ascoli

Choose C_a and C_b s.t.

$$|a_n| \leq C_a \text{ and } |b_n| \leq C_b \ \ \forall n.$$

Then

$$|X^{(n)}(t)| \leq |a_n| + |b_n| \leq C_a + C_b$$

$\forall n \in \mathbb{N}$ and $t \in [0,1]$. Also

$$|X^{(n)}(t) - X^{(n)}(s)| = |b_n(t-s)| \leq C_b|t-s|$$

so $\forall \varepsilon > 0$

$$|t-s| < \frac{\varepsilon}{C_b} \Rightarrow |X^{(n)}(t) - X^{(n)}(s)| < \varepsilon \ \ \forall n.$$

By Arzela Ascoli $\{X^{(n)}\}$ has a uniformly convergent subsequence.

Lemma 2.2. *$\forall n \in \mathbb{N}$ let $X^{(n)} : D \to \mathbb{R}^N$ where $D \subset \mathbb{R}$. Let E be a countable subset of D and assume that*

$$\forall \ e \in E, \ \{X^{(n)}(e)\} \text{ is bounded.}$$

Then $\{X^{(n)}\}$ has a subsequence which converges pointwise on E.

Proof. Write $E = \bigcup_{\ell=1}^{\infty}\{e_\ell\}$. $\{X^{(n)}(e_1)\}$ is bounded so $\exists k : \mathbb{N} \to \mathbb{N}$ increasing s.t. $\{X^{(k_n^1)}(e_1)\}$ converges. $\{X^{(k_n^1)}(e_2)\}$ is bounded so $\exists \tilde{k}^2 : \mathbb{N} \to \mathbb{N}$ increasing s.t. $\{X^{(k^1_{\tilde{k}_n^2})}(e_2)\}$ converges. Let $k_n^2 = k^1_{\tilde{k}_n^2}$ and note that $\{X^{(k_n^2)}(e_1)\}$ and $\{X^{(k_n^2)}(e_2)\}$ both converge. Continue and obtain $\forall \ell \geq 2$ $k^\ell : \mathbb{N} \to \mathbb{N}$ increasing s.t. $\{X^{(k_n^\ell)}\}$ is a subsequence of $\{X^{(k_n^{\ell-1})}\}$ and $\{X^{(k_n^\ell)}(e_1)\}, \ldots, \{X^{(k_n^\ell)}(e_\ell)\}$ converge.

Define $P_n = k_n^n$.

Illustration

k_n^1	2	4	6	8	10	12	14
k_n^2	4	8	12	16	20	24	28
k_n^3	8	16	24	32	40	48	56
k_n^4	16	32	48	64	80	96	112

$P_n^n = k_n^n : 2,\ 8,\ 24,\ 64, \ldots$

$\{X^{(P_n)}\}$ is a subsequence of $\{X^{(n)}\}$

$\{X^{(P_n)}\}$ is a subsequence of $\{X^{(k_n^1)}\}$

so $\{X^{(P_n)}(e_1)\}$ converges.

$\{X^{(P_n)}\}_{n=2}^{\infty}$ is a subsequence of $\{X^{(k_n^2)}\}$ so $\{X^{(P_n)}(e_2)\}$ converges.

$\{X^{(P_n)}\}_{n=\ell}^{\infty}$ is a subsequence of $\{X^{(k_n^\ell)}\}$ so $\{X^{(P_n)}(e_\ell)\}$ converges

$\forall \ell \in \mathbb{N}$. $\square$

Example

Let $X^{(n)}(t) = \sin(n\pi t)$ and $E = \bigcup_{\ell=1}^{\infty}\{e_\ell\}$ where $e_\ell = \dfrac{1}{\ell}$.

Take $k_n^1 = n : X^{(k_n^1)}(e_1) = \sin(n\pi e_1) = 0$.

Take $k_n^2 = 2n : X^{(k_n^2)}(e_2) = \sin(2n\pi e_2) = \sin(n\pi) = 0$.

Take $k_n^3 = 3!n : X^{(k_n^3)}(e_3) = \sin(3!n\pi e_3) = \sin(2n\pi) = 0$.

Take $k_n^\ell = \ell!n : X^{(k_n^\ell)}(e_\ell) = \sin(\ell!n\pi \dfrac{1}{\ell}) = 0$.

Let $P_n = k_n^n = n!n$. Then

$$X^{(P_n)}(e_\ell) = \sin(n!n\pi\frac{1}{\ell}) = 0 \text{ if } n \geq \ell$$

so

$$X^{(P_n)}(e_\ell) \to 0 \quad \forall \ell.$$

Proof of Arzela Ascoli

Assume $\forall t \in [t_0, t_1] \ \exists \ B(t)$ s.t.

$$|X^{(n)}(t)| \leq B(t) \quad \forall n$$

and $\forall \varepsilon > 0 \ \exists \ \delta(\varepsilon) > 0$ s.t.

$$s, t \in [t_0, t_1] \text{ and } |s - t| < \delta(\varepsilon) \Rightarrow |X^{(n)}(s) - X^{(n)}(t)| < \varepsilon \quad \forall n.$$

To show $\{X^{(n)}\}$ is uniformly bounded choose $M > \dfrac{t_1 - t_0}{\delta(1)}$. Then $\forall t \in [t_0, t_1]$

$$\begin{aligned} |X^{(n)}(t)| &\leq |X^{(n)}(t_0)| \\ &\quad + \sum_{k=1}^{M} \left| X^{(n)}\left(t_0 + k\frac{t - t_0}{M}\right) - X^{(n)}\left(t_0 + [k-1]\frac{t - t_0}{M}\right) \right| \\ &\leq B(t_0) + \sum_{k=1}^{M} 1 = B(t_0) + M. \end{aligned}$$

This bound is uniform in t (and n).

Let $E = [t_0, t_1] \bigcap \mathbb{Q}$. Since E is countable, there is a subsequence, $\{X^{(P_n)}\}$, which converges pointwise on E. Claim $\{X^{(P_n)}\}$ converges uniformly on $[t_0, t_1]$.

Let $\varepsilon > 0$. Choose $e_1, e_2, \ldots, e_L \in E$ s.t.

$$[t_0, t_1] \subset \bigcup_{\ell=1}^{L} \left(e_\ell - \delta\left(\frac{\varepsilon}{3}\right), e_\ell + \delta\left(\frac{\varepsilon}{3}\right)\right).$$

$\forall \ell \in \{1, \ldots, L\}$, $\{X^{(P_n)}(e_\ell)\}$ converges so $\exists M_\ell$ s.t.

$$n, m > M_\ell \Rightarrow |X^{(P_n)}(e_\ell) - X^{(P_m)}(e_\ell)| < \frac{\varepsilon}{3}.$$

Let $M = \max\{M_1, M_2, \ldots, M_L\}$. Consider any $t \in [t_0, t_1]$. $\exists\ \ell \in \{1, \ldots, L\}$ s.t.

$$t \in \left(e_\ell - \delta\left(\frac{\varepsilon}{3}\right),\ e_\ell + \delta\left(\frac{\varepsilon}{3}\right)\right)$$

so

$$|t - e_\ell| < \delta\left(\frac{\varepsilon}{3}\right)$$

and hence $n, m > M \Rightarrow$

$$\begin{aligned}
\left|X^{(P_n)}(t) - X^{(P_m)}(t)\right| &\leq \left|X^{(P_n)}(t) - X^{(P_n)}(e_\ell)\right| \\
&\quad + \left|X^{(P_n)}(e_\ell) - X^{(P_m)}(e_\ell)\right| + \left|X^{(P_m)}(e_\ell) - X^{(P_m)}(t)\right| \\
&< \frac{\varepsilon}{3} + \left|X^{(P_n)}(e_\ell) - X^{(P_m)}(e_\ell)\right| + \frac{\varepsilon}{3} < \varepsilon.
\end{aligned}$$

Therefore, $\left\{X^{(P_n)}\right\}$ is uniformly Cauchy and hence uniformly convergent. □

Proof of Cauchy Peano

Assume f is continuous on

$$D = \{(t, x) : t_0 \leq t \leq t_0 + \delta_0, |x - x_0| \leq \varepsilon_0\}$$

and let

$$M = \max_D |f|$$

and

$$\delta = \min\left(\delta_0, \frac{\varepsilon_0}{M}\right).$$

$\forall n \in \mathbb{N}$ let $\Delta t = \dfrac{\delta}{n}$, $t_k = t_0 + k\Delta t$ $(k = 0, 1, \ldots, n)$, and define

$$X_0^{(n)} = x_0$$

and

$$\frac{X_{k+1}^{(n)} - X_k^{(n)}}{\Delta t} = f\left(t_k, X_k^{(n)}\right)$$

as long as $|X_k^{(n)} - x_0| \leq \varepsilon_0$ (and $k \leq n$).

Note that if $k < n$ and $|X_k^{(n)} - x_0| \leq \varepsilon_0$ then

$$\begin{aligned} |X_{k+1}^{(n)} - x_0| &= |X_k^{(n)} - x_0 + \Delta t f(t_k, X_k^{(n)})| \\ &\leq |X_k^{(n)} - x_0| + M\Delta t. \end{aligned}$$

By induction it follows that

$$\begin{aligned} |X_k^{(n)} - x_0| &\leq |X_{k-1}^{(n)} - x_0| + M\Delta t \\ &\leq |X_{k-2}^{(n)} - x_0| + 2M\Delta t \\ &\leq \ldots \leq |X_0^{(n)} - x_0| + kM\Delta t = kM\frac{\delta}{n} \\ &\leq M\delta \leq \varepsilon_0 \end{aligned}$$

for $k = 0, 1, \ldots, n$.

Further define

$$X^{(n)}(t) = \frac{t_{k+1} - t}{\Delta t} X_k^{(n)} + \frac{t - t_k}{\Delta t} X_{k+1}^{(n)}$$

for $t_k \leq t \leq t_{k+1}$. For $t_k < t < t_{k+1}$

$$\left|\frac{dX^{(n)}}{dt}\right| = \left|\frac{X_{k+1}^{(n)} - X_k^{(n)}}{\Delta t}\right| = \left|f(t_k, X_k^{(n)})\right| \leq M.$$

Since $X^{(n)}$ is continuous on $[t_0, t_0 + \delta]$ it follows that

$$\left|X^{(n)}(s) - X^{(n)}(t)\right| = \left|\int_s^t f(\tau, X^{(n)}(\tau)) d\tau\right| \leq M|t - s|.$$

$\forall \varepsilon > 0$

$$|t-s| < \frac{\varepsilon}{M} \Rightarrow |X^{(n)}(s) - X^{(n)}(t)| < \varepsilon$$

$\forall n$ and $\forall s, t \in [t_0, t_0 + \delta]$. This is equicontinuity. By Arzela Ascoli there is a uniformly convergent subsequence,

$$X^{(P_n)} \to X.$$

Define

$$t^{(n)}(s) = t_k \text{ if } t_k \le s < t_{k+1}.$$

Then

$$|t^{(n)}(s) - s| \le \Delta t = \frac{\delta}{n}$$

and

$$|X^{(n)}(t^{(n)}(s)) - X^{(n)}(s)| \le M\Delta t = \frac{M\delta}{n}$$

so $t^{(P_n)}(s) \to s$ and $X^{(P_n)}(t^{(P_n)}(s)) \to X(s)$ uniformly on $[t_0, t_0+\delta]$. It follows that $f(t^{(P_n)}(s), X^{(P_n)}(t^{(P_n)}(s))) \to f(s, X(s))$ uniformly and hence

$$\begin{aligned} X(t) &= \lim_{n\to\infty} X^{(P_n)}(t) \\ &= \lim_{n\to\infty} \left(x_0 + \int_{t_0}^t \dot{X}^{(P_n)}(s) ds \right) \\ &= \lim_{n\to\infty} \left(x_0 + \int_{t_0}^t f(t^{(P_n)}(s), X^{(P_n)}(t^{(P_n)}(s))) ds \right) \\ &= x_0 + \int_{t_0}^t f(s, X(s)) ds. \end{aligned}$$

The theorem now follows. $\square$

Comment For f continuous on

$$D = \{(t,x) : t_0 \le t \le t_0 + \delta_0,\ |x - x_0| \le \varepsilon_0\}$$

we got a solution on $[t_0, t_0 + \delta]$ where $\delta = \min\left(\delta_0, \dfrac{\varepsilon_0}{M}\right)$ and $M = \max\limits_{D}|f|$.

D. Continuation

Definitions

1. Suppose

$$\begin{aligned} \dot{X} &= f(t, X) \quad \forall t \in I_x \\ \dot{Y} &= f(t, Y) \quad \forall t \in I_y. \end{aligned}$$

   If $I_x \subset I_y$ and $t \in I_x \Rightarrow Y(t) = X(t)$ then we say Y is an extension of X.

2. Suppose

$$\dot{X} = f(t, X) \quad \forall t \in [t_0, T).$$

   We say X is a right maximal solution if there is no extension to $[t_0, \tilde{T})$ with $\tilde{T} > T$.

Example

Let

$$f(t, x) = \begin{cases} x^{\frac{1}{3}} & \text{if} \quad x \leq 1 \\ x^2 & \text{if} \quad 1 < x, \end{cases}$$

$$X(t) = 0 \quad \text{if} \quad 0 \leq t \leq 2,$$

$$Y(t) = 0 \quad \text{if} \quad 0 \leq t,$$

$$Z(t) = \begin{cases} 0 & 0 \le t \le 2 \\ \left[\dfrac{2}{3}(t-2)\right]^{\frac{3}{2}} & 2 < t \le \dfrac{7}{2} \\ \left(\dfrac{9}{2} - t\right)^{-1} & \dfrac{7}{2} < t < \dfrac{9}{2}. \end{cases}$$

Y and Z are right maximal extensions of X.

Extension Theorem Let $D = I_t \times D_x$ with I_t an open interval and $D_x \subset \mathbb{R}^N$ open. Let $f : D \to \mathbb{R}^N$ be continuous. Let $(t_0, x_0) \in D$.

1. Any solution of

$$\begin{cases} \dot{X} &= f(t, X) \\ X(t_0) &= x_0 \end{cases}$$

   may be extended to be right maximal.

2. If X is a right maximal solution on $[t_0, T)$ and $T \in I_t$ then $\forall$ compact set $S \subset D_x$ $\exists$ $t \in [t_0, T)$ s.t.

$$X(t) \notin S.$$

Comments

1. Interpret the theorem as f is undefined outside of D so $(t, X(t)) \in D$ is required.

2. Consider the case $D = \mathbb{R} \times \mathbb{R}^N$:

   (a) Suppose X is a right maximal solution on $[t_0, T)$ and X is bounded. Then $T = +\infty$.

Proof. Suppose T is finite. Choose B s.t.

$$|X(t)| \leq B \quad \forall t \in [t_0, T).$$

$S = \{x \in \mathbb{R}^N : |x| \leq B\}$ is compact so $\exists\, t \in [t_0, T)$ s.t.

$$X(t) \notin S.$$

□

Contradiction.

(b) Suppose f is continuous and bounded on $D = \mathbb{R} \times \mathbb{R}^N$. Then every solution may be extended to all of $\mathbb{R}$.

Proof. Consider any solution. It may be extended to be right maximal; say X is a solution on $[t_0, T)$. Choose M s.t.

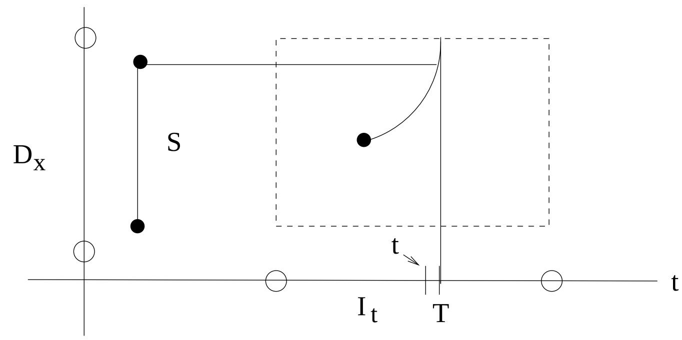
$$|f(t,x)| \leq M \quad \forall\, t, x$$

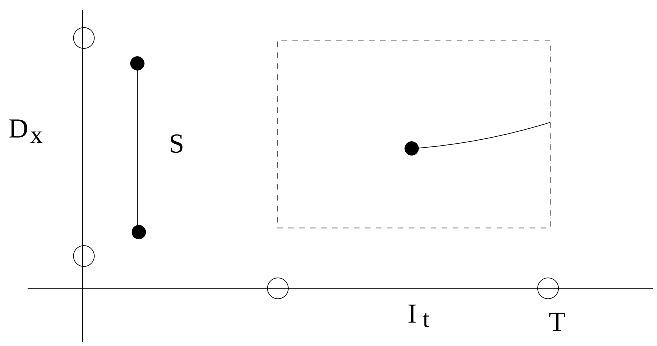
then

$$\begin{aligned} |X(t)| &= \left| x_0 + \int_{t_0}^{t} f(s, X(s))ds \right| \\ &\leq |x_0| + M(t - t_0) \quad \forall\, t \in [t_0, T). \end{aligned}$$

If T were finite then X is bounded by $|x_0| + M(T - t_0)$ and by part (a), T would be infinite; contradiction. $T = +\infty$.
Similarly, X may be extended to $(-\infty, t_0]$. □

3.





Note: $T \notin I_t$ here.

Sketch of Extension to Maximal Solution

Let $X^{(1)}$ be a solution on $[t_0, t)$. Let

$$T^{(1)} = \sup \left\{ T \geq t_1 : \exists \text{ an extension of } X^{(1)} \text{ to } [t_0, T) \right\}.$$

If $T^{(1)} = +\infty$ choose $X^{(2)}$ to be an extension of $X^{(1)}$ to $[t_0, t_2)$ with $t_2 \geq t_1 + 1$. If $T^{(1)}$ is finite choose $X^{(2)}$ to be an extension on $[t_0, t_2)$ with $t_2 \geq t_1 + \dfrac{1}{2}\left(T^{(1)} - t_1\right)$. Extend $X^{(2)}$ to $X^{(3)}$ on $[t_0, t_3)$ in the same way and so on. Define

$$\begin{aligned} t_\infty &= \lim_{n \to \infty} t_n \\ X(t) &= X^{(n)}(t) \ \text{ if } t_n > t \end{aligned}$$

$\forall t \in [t_0, t_\infty)$. Then X is a right maximal extension.

Lemma 2.3. *If X is uniformly continuous on $[t_0, T)$ with $T \in \mathbb{R}$ then*

$$\lim_{t \to T^-} X(t)$$

exists and is finite.

Proof. Assume that $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ s.t.

$$s, t \in [t_0, T) \text{ and } |s - t| < \delta(\varepsilon) \Rightarrow |X(s) - X(t)| < \varepsilon.$$

Let $t_n = T - \dfrac{T - t_0}{n}$. Then $n, k > \dfrac{T - t_0}{\delta(\varepsilon)} \Rightarrow |t_n - t_k| = (T - t_0)|\dfrac{1}{n} - \dfrac{1}{k}| < \delta(\varepsilon) \Rightarrow |X(t_n) - X(t_k)| < \varepsilon$. Thus $X(t_n)$ is Cauchy and hence convergent. Let $L = \lim X(t_n)$.

Let $t \in (T - \delta(\varepsilon), T) \bigcap [t_0, t)$. Take $n > (T - t_0)/\delta(\varepsilon)$. Then $|X(t_n) - L| \leq \varepsilon$. Also $|t - t_n| < \delta(\varepsilon)$ so $|X(t) - X(t_n)| < \varepsilon$. Therefore

$$|X(t) - L| \leq |X(t) - X(t_n)| + |X(t_n) - L| < 2\varepsilon.$$

It follows that $\lim\limits_{t \to T^-} X(t) = L$. $\square$

Proof of Extension Theorem

Assume X is a right maximal solution on $[t_0, T)$ and that $T \in I_t$. Let $S \subset D_x$ be compact. Suppose $X(t) \in S\ \forall t \in [t_0, T)$ and seek a contradiction.

f is continuous on the compact set $[t_0, T] \times S$; let (since $T \in I_t$)

$$M = \max_{[t_0,T] \times S} |f(t, x)|.$$

$\forall s, t \in [t_0, T)$ with $|s - t| < \dfrac{\varepsilon}{M}$ we have

$$|X(t) - X(s)| = \left| \int_s^t f(\tau, X(\tau)) d\tau \right| \leq M|t - s| < \varepsilon,$$

so X is uniformly continuous. By the lemma we may define

$$X(T) = \lim_{t \to T^-} X(t).$$

Choose $\delta_0 > 0,\ \varepsilon_0 > 0$ s.t.

$$\{(t, x) : T \leq t \leq T + \delta_0, |x - X(T)| \leq \varepsilon_0\} \subset D.$$

Applying the Cauchy Peano theorem to

$$\begin{cases} \tilde{X}(T) &= X(T) \\ \dot{\tilde{X}} &= f(t, \tilde{X}) \end{cases}$$

we may extend X to $[t_0, \tilde{T})$ with $\tilde{T} > T$.
Contradiction. $\square$

3 Uniqueness

A. Gronwall's Inequality

Simple Version Assume $A \in \mathbb{R},\ B \geq 0$, and X is continuous on $I = [t_0, t]$ or $[t_0, \infty)$ with

$$X(t) \leq A + B \int_{t_0}^{t} X(s)ds$$

$\forall t \in I$. Then

$$X(t) \leq Ae^{B(t-t_0)} \quad \forall t \in I.$$

Proof. Let $R(t) = A + B \displaystyle\int_{t_0}^{t} X(s)ds$ and note that X is C^1, $X(t) \leq R(t)$, and since $B \geq 0$

$$\dot{R}(t) = BX(t) \leq BR(t)$$

so

$$\begin{aligned} &\frac{d}{dt}\left(e^{-B(t-t_0)}R(t)\right) \\ = \; & e^{-B(t-t_0)}\left(\dot{R}(t) - BR(t)\right) \leq 0. \end{aligned}$$

Hence

$$e^{-B(t-t_0)}R(t) \leq R(t_0) = A$$

and

$$X(t) \leq R(t) \leq A\ e^{B(t-t_0)}.$$

$\square$

Gronwall's Inequality Full Version

Let $a, b \in \mathcal{C}_{\mathbb{R}}(I)$ where $I = [t_0, t_1]$ or $[t_0, \infty)$ and $b(t) \geq 0 \;\; \forall t \in I$. Assume $X \in \mathcal{C}_{\mathbb{R}}(I)$ with

$$X(t) \leq a(t) + \int_{t_0}^{t} b(s)X(s)ds$$

$\forall t \in I$, then

$$X(t) \leq a(t) + \int_{t_0}^{t} a(s)b(s)e^{\int_s^t b(\tau)d\tau}ds$$

$\forall t \in I$.

Example

$$\begin{cases} \dot{X} = (1+t^2)^{-1}X(t)\sin^2(X(t)) \\ \\ X(0) \text{ given} \end{cases}$$

For $t \geq 0$

$$\begin{aligned} |X(t)| &\leq |X(0)| + \int_0^t \left|(1+s^2)^{-1}X(s)\sin^2(X(s))\right| ds \\ &\leq |X(0)| + \int_0^t (1+s^2)^{-1}|X(s)|ds \end{aligned}$$

so $\forall t \geq 0$

$$
\begin{aligned}
|X(t)| &\leq |X(0)| + \int_0^t |X(0)|(1+s^2)^{-1} e^{\int_s^t (1+\tau^2)^{-1} d\tau} ds \\
&= |X(0)| \left(1 + \int_0^t (1+s^2)^{-1} e^{\tan^{-1}(t) - \tan^{-1}(s)} ds \right) \\
&\leq |X(0)| \left(1 + \int_0^t (1+s^2)^{-1} e^{\frac{\pi}{2}} ds \right) \\
&= |X(0)| \left(1 + e^{\frac{\pi}{2}} \tan^{-1}(t) \right) \leq |X(0)| \left(1 + e^{\frac{\pi}{2}} \frac{\pi}{2} \right).
\end{aligned}
$$

Comment Suppose $a(t) = A$ and $b(t) = B$ are constant. Then

$$
\begin{aligned}
X(t) &\leq a(t) + \int_{t_0}^t a(s) b(s) e^{\int_s^t b(\tau) d\tau} ds \\
&= A + A \int_{t_0}^t B e^{B(t-s)} ds \\
&= A \left(1 + \int_{t_0}^t \frac{d}{ds} \left(-e^{B(t-s)} \right) ds \right) \\
&= A \left(1 - \left[e^{B(t-t)} - e^{B(t-t_0)} \right] \right) \\
&= A e^{B(t-t_0)}
\end{aligned}
$$

as before.

Proof. Let $R(t) = \displaystyle\int_{t_0}^t b(s) X(s) ds$ and note that R is C^1 with (since $b(t) \geq 0$)

$$R'(t) = b(t) X(t) \leq b(t)(a(t) + R(t)).$$

Let $\beta(t) = \displaystyle\int_{t_0}^t b(s) ds$ then

$$\begin{aligned}
\frac{d}{dt}\left(e^{-\beta(t)}R(t)\right) &= e^{-\beta(t)}\left(R'(t)-\beta'(t)R(t)\right) \\
&= e^{-\beta(t)}\left(R'(t)-b(t)R(t)\right) \\
&\le e^{-\beta(t)}(a(t)b(t))
\end{aligned}$$

so

$$\begin{aligned}
e^{-\beta(t)}R(t) &= e^{-\beta(t)}R(t)-e^{-\beta(t_0)}R(t_0) \\
&\le \int_{t_0}^{t} e^{-\beta(s)}a(s)b(s)ds
\end{aligned}$$

and

$$\begin{aligned}
X(t) &\le a(t)+R(t) \\
&\le a(t)+\int_{t_0}^{t} a(s)b(s)e^{\beta(t)-\beta(s)}ds.
\end{aligned}$$

Since

$$\beta(t)-\beta(s)=\int_{s}^{t} b(\tau)d\tau.$$

The proof is complete. □

Examples

1. Suppose

   $$\dot{X}=f(t,X) \text{ and } \dot{Y}=f(t,Y)$$

   where

   $$f(t,x)=t+\frac{x}{1+x^2}.$$

   Note that

$$\left|\frac{\partial f}{\partial x}(t,x)\right| = \left|\frac{(1+x^2)-x2x}{(1+x^2)^2}\right| = \frac{|1-x^2|}{(1+x^2)^2} \leq 1$$

so

$$|f(t,x)-f(t,y)| \leq |x-y| \quad \forall t,x,y.$$

So $\forall t \geq 0$

$$\begin{aligned} |X(t)-Y(t)| &= \left|X(0)-Y(0)+\int_0^t (f(s,X)-f(s,Y))ds\right| \\ &\leq |X(0)-Y(0)|+\int_0^t |X(s)-Y(s)|ds. \end{aligned}$$

By Gronwall $\forall t \geq 0$

$$|X(t)-Y(t)| \leq |X(0)-Y(0)|e^t.$$

Suppose $T>0$ and $\varepsilon>0$. Then

$$\begin{aligned} |X(0)-Y(0)| &< e^{-T}\varepsilon \Rightarrow \\ |X(t)-Y(t)| &\leq |X(0)-Y(0)|e^t \\ &< e^{-T}\varepsilon\ e^t \leq \varepsilon \end{aligned}$$

$\forall t \in [0,T]$.

2. Suppose $\dot{X} = X^{\frac{1}{3}}, \dot{Y} = Y^{\frac{1}{3}}, X(0)=Y(0)$. This holds for $X \equiv 0$ and

$$Y(t) = \left(\frac{2}{3}t\right)^{\frac{3}{2}}$$

so $\forall t > 0$

$$0 < |X(t)-Y(t)| \not\leq |X(0)-Y(0)|e^{\text{any power}} = 0.$$

B. Uniqueness

Theorem 3.1. *Assume f is continuous and satisfies a Lipschitz condition in x on*

$$D = \{(t, x) : t_0 \le t \le t_0 + \delta_0,\ |x - x_0| \le \varepsilon_0\}$$

for some $\delta_0 > 0, \varepsilon_0 > 0$. Suppose

$$\begin{aligned} \dot{X} &= f(t, X) \quad \text{on} \quad [t_0, t_0 + \delta_1] \\ \dot{Y} &= f(t, X) \quad \text{on} \quad [t_0, t_0 + \delta_1] \end{aligned}$$

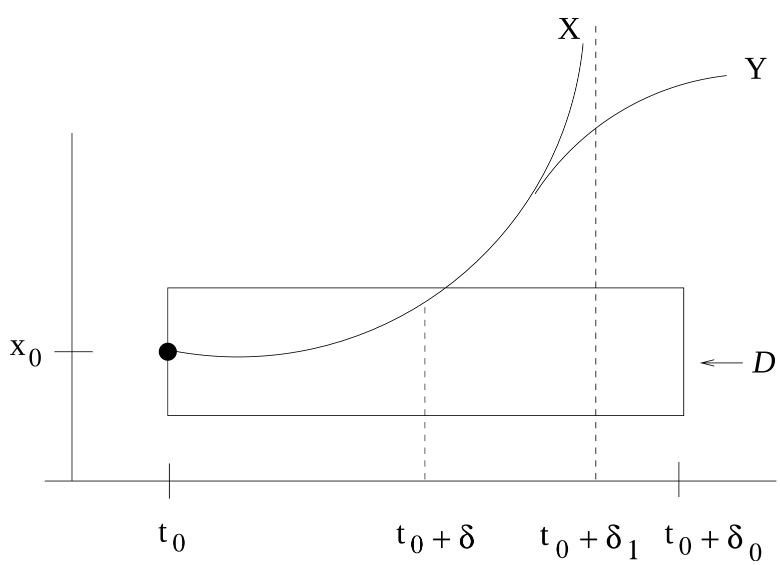
for some $\delta_1 \in (0, \delta_0]$ and

$$X(t_0) = Y(t_0) = x_0.$$

Then $\exists \delta \in (0, \delta]$ s.t. $X(t) = Y(t)$ $\forall t \in [t_0, t_0 + \delta]$.

Comments

1.



2. Solutions may "branch" only at a point (t_0, x_0) where f fails to satisfy a Lipschitz condition in x on

$$\{(t,x) : t_0 \leq t \leq t_0 + \delta_0,\ |x - x_0| \leq \varepsilon_0\}$$

$\forall \delta_0, \varepsilon_0 > 0$.

3. If $\dfrac{\partial f_k}{\partial x_\ell}$ is continuous in a neighborhood of (t_0, x_0) then f satisfies a Lipschitz condition in x on

$$\{(t,x) : t_0 \leq t \leq t_0 + \delta_0,\ |x - x_0| \leq \varepsilon_0\}$$

for some $\delta_0, \varepsilon_0 > 0$ and a solution cannot "branch" at this point.

Proof. By continuity of X at $t_0 \exists\ \delta_2 > 0$ s.t.

$$|X(t) - x_0| \leq \varepsilon_0 \quad \forall\ t \in [t_0, t_0 + \delta_2].$$

Similarly for $Y\ \exists\ \delta_3 > 0$ s.t.

$$|Y(t) - x_0| \leq \varepsilon_0 \quad \forall t \in [t_0, t_0 + \delta_3].$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$ and choose $L > 0$ s.t.

$$|f(t,x) - f(t,y)| \leq L|x - y| \quad \forall (t,x), (t,y) \in D.$$

Then $\forall\ t \in [t_0, t_0 + \delta]$

$$\begin{aligned} |X(t) - Y(t)| &\leq |X(t_0) - Y(t_0)| + \int_{t_0}^{t} |f(s,X) - f(s,Y)|\, ds \\ &\leq |X(t_0) - Y(t_0)| + L \int_{t_0}^{t} |X(s) - Y(s)|\, ds. \end{aligned}$$

By Gronwall (simple version)

$$|X(t) - Y(t)| \leq |X(t_0) - Y(t_0)| e^{Lt}$$

$\forall t \in [t_0, t_0 + \delta]$. Since $X(t_0) = Y(t_0) = x_0$ the theorem follows. $\square$

Proposition 3.1. *Assume*

$$(f(t,x) - f(t,y)) \cdot (x-y) \le 0$$

for $(t,x) \in D = \{(t,x) : t_0 \le t \le t_0 + \delta, |x - x_0| \le \varepsilon_0\}$ *and*

$$\left.\begin{array}{rcl} \dot{X} &=& f(t,X) \\ \\ \dot{Y} &=& f(t,Y) \end{array}\right\} \text{ on } t_0 \le t \le t_0 + \delta_1$$

with $0 < \delta_1 \le \delta_0$ *and* $X(t_0) = Y(t_0) = x_0$. *Then* $\exists\ \delta \in (0, \delta_1]$ *s.t.* $X(t) = Y(t) \ \ \forall\ t \in [t_0, t_0 + \delta]$.

Example

If $f(t,x) = -x^{\frac{1}{3}}$ then

$$\begin{array}{rcl} (f(t,x) - f(t,y))(x-y) &=& -(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x-y) \\ \\ &=& -(x^{\frac{1}{3}} - y^{\frac{1}{3}})^2 (x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}}) \le 0 \end{array}$$

since

$$\begin{array}{rcl} x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}} &\ge& x^{\frac{2}{3}} - 2|x|^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}} \\ \\ &=& (|x|^{\frac{1}{3}} - |y|^{\frac{1}{3}})^2 \ge 0. \end{array}$$

Hence, $X(t) = 0$ is the only solution of

$$\left\{\begin{array}{rcl} \dot{X} &=& -X^{\frac{1}{3}} \\ \\ X(t_0) &=& 0. \end{array}\right.$$

Proof. Choose δ as in the previous proof. For $t \in (t_0, t_0 + \delta)$

$$\frac{d}{dt}|X(t) - Y(t)|^2 = 2(X(t) - Y(t)) \cdot (f(t,X) - F(t,Y)) \le 0$$

so

$$|X(t) - (Y(t)|^2 \le |X(t_0) - Y(t_0)|^2 = 0.$$

$\square$

Comment Assume existence and uniqueness and define $X(t, t_0, x_0)$ by

$$\begin{cases} \dot{X} &= f(t, X) \\ X(t_0, t_0, x_0) &= x_0. \end{cases}$$

Then

(SG) $$X(t, t_1, X(t_1, t_0, x_0)) = X(t, t_0, x_0).$$

To show this note that both sides of the equation are solutions and that

$$X(t_1, t_1, X(t_1, t_0, x_0)) = X(t_1, t_0, x_0).$$

By uniqueness, (SG) follows.

C. Continuity with respect to Initial Conditions

Assume existence and uniqueness and define $X(t, t_0, x_0)$ by

$$\begin{cases} \dfrac{dX}{dt} &= f(t, X) \\ X(t_0, t_0, x_0) &= x_0. \end{cases}$$

Consider the continuity of

$$x_0 \mapsto X(t, t_0, x_0).$$

Examples

1. $f(t, x) = x \Rightarrow X(t, t_0, x_0) = x_0\ e^{t-t_0}$.
2. Let $f(t, x) = M(t)x$ where $M : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N$ is continuous. Claim $x_0 \mapsto X(t, t_0, x_0)$ is linear.

Proof. Let $a, b \in \mathbb{R},\ x_0, y_0 \in \mathbb{R}^N$, and

$$\begin{aligned} Y(t) &= X(t, t_0, ax_0 + by_0) \\ Z(t) &= aX(t, t_0, x_0) + bX(t, t_0, y_0) \end{aligned}$$

and show $Y(t) = Z(t)$. Note that

$$\begin{aligned} y(t_0) &= ax_0 + by_0 = Z(t_0), \\ \dot{Y} &= f(t, Y) = M(t)Y \end{aligned}$$

and

$$\begin{aligned} \dot{Z} &= a\dot{X}(t, t_0, x_0) + b\dot{X}(t, t_0, y_0) \\ &= aM(t)X(t, t_0, x_0) + bM(t)X(t, t_0, y_0) \\ &= M(t)(aX(t, t_0, x_0) + bX(t, t_0, y_0)) \\ &= M(t)Z. \end{aligned}$$

By uniqueness $Z = Y$. $\square$

Theorem 3.2. *Let $D \subset \mathbb{R} \times \mathbb{R}^N$ be open and $f : D \to \mathbb{R}$ be continuous. Assume f satisfies a Lipschitz condition in x on every compact subset of D. Assume that $(t_0, \overline{x}_0) \in D$ and*

$$\begin{aligned} \dot{\overline{X}}(t) &= f(t, \overline{X}(t)) \quad \text{for } t \in [t_0, t_1] \\ \overline{X}(t_0) &= \overline{x}_0 \end{aligned}$$

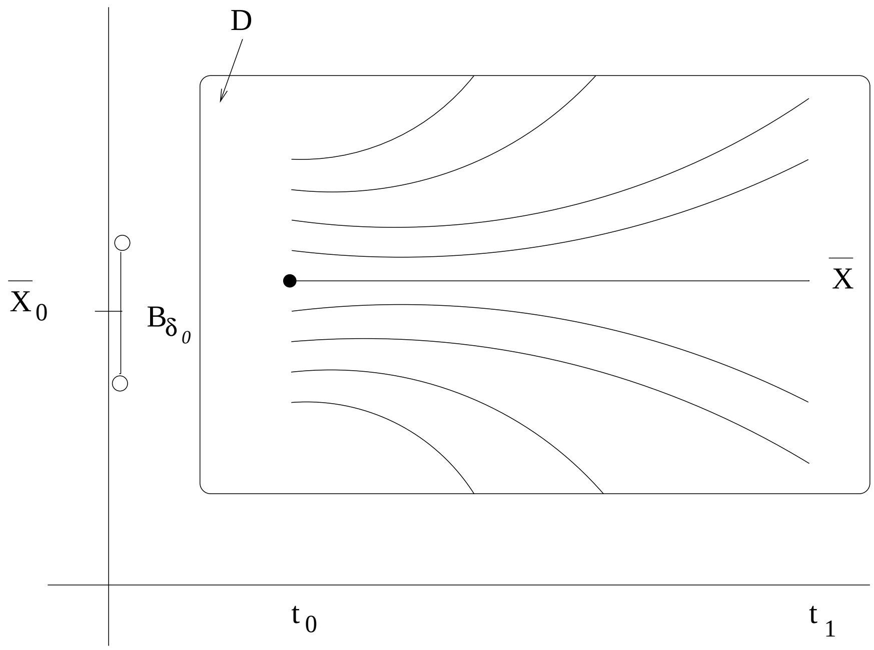
with $(t, \overline{X}(t)) \in D \;\; \forall t \in [t_0, t_1]$. Then $\exists\ \delta_0 > 0$ s.t. $\forall\ x_0 \in B_{\delta_0}(\overline{x}_0)$

$$\begin{cases} \dot{X} = f(t, X) \\ X(t_0) = x_0 \end{cases}$$

has a solution on $[t_0, t_1]$. Moreover, $\forall t \in [t_0, t_1]$

$$x_0 \mapsto X(t, t_0, x_0)$$

is continuous on $B_{\delta_0}(\overline{x}_0)$.



Example

$f(t,x) = x^2$ satisfies a Lipschitz condition in x on every compact subset of $\mathbb{R} \times \mathbb{R}$, but not on $\mathbb{R} \times \mathbb{R}$ itself.

If $t_0 = 0$ and $\overline{x}_0 = 1$ then

$$\overline{X}(t) = \frac{1}{1-t} \quad \text{for} \quad 0 \le t < 1.$$

We may apply the theorem on $[0, t_1]$ $\forall t_1$ but not $t_1 = 1$. In fact, $x_0 > 1 \Rightarrow X(t, 0, x_0)$ does not exist on all of $[0, 1)$.

Proof. Choose $\varepsilon_0 > 0$ s.t.

$$S = \left\{ (t,x) : t_0 \le t \le t_1, \ \left| x - \overline{X}(t) \right| \le \varepsilon_0 \right\} \subset D.$$

Choose L s.t.

$$|f(t,x) - f(t,y)| \le L|x-y|$$

$\forall (t,x) \in S, \ (t,y) \in S.$

Define $X(t, x_0)$ by

$$\begin{cases} \dot{X} &= f(t, X) \\ X(t_0, x_0) &= x_0 \end{cases}$$

and take X to be right maximal. Consider $x_0 \in B_{\varepsilon_0}(\overline{x}_0)$ and define

$$\tau = \tau(x_0) = \sup \{t \in [t_0, t_1] : (s, X(s, x_0)) \in S \ \forall x \in [t_0, t]\}.$$

Then for $t_0 \leq t \leq \tau(x_0)$

$$\begin{aligned} &|X(t, x_0) - \overline{X}(t)| \\ = &\left| x_0 - \overline{x}_0 + \int_{t_0}^{t} (f(s, X(s, x_0)) - f(s, \overline{X}(s))) ds \right| \\ \leq &|x_0 - \overline{x}_0| + \int_{t_0}^{t} L|X(s, x_0) - \overline{X}(s)| ds \end{aligned}$$

and by Gronwall

$$\begin{aligned} |X(t, x_0) - \overline{X}(t)| &\leq |x_0 - \overline{x}_0| e^{L(t - t_0)} \\ &\leq |x_0 - \overline{x}_0| e^{L(t_1 - t_0)}. \end{aligned}$$

Let

$$\delta_0 = \frac{1}{2} e^{-L(t_1 - t_0)} \varepsilon_0$$

and take $x_0 \in B_{\delta_0}(\overline{x}_0)$. Then

$$|X(\tau(x_0), x_0) - \overline{X}(\tau(x_0))| < \delta \ e^{L(t_1 - t_0)} = \frac{1}{2} \varepsilon_0$$

so $\exists \ \eta > 0$ so that

$$|X(t, x_0) - \overline{X}(t)| < \varepsilon_0 \ \ \forall \ t \in [t_0, t_1 + \eta].$$

If $\tau(x_0) \neq t_1$ this contradicts the definition of τ, so $x_0 \in B_{\delta_0}(\overline{x}_0) \Rightarrow \tau(x_0) = t_1$.

Consider $x_0, y_0 \in B_{\delta_0}(\overline{x}_0)$. $X(t, x_0) \in S$ and $X(t, y_0) \in S \ \ \forall t \in [t_0, t_1]$ so

$$
\begin{aligned}
& |X(t,x_0)-X(t,y_0)| \\
\leq\ & |x_0-y_0|+\int_{t_0}^{t}|f(s,X(s,x_0))-f(s,X(s,y_0))|ds \\
\leq\ & |x_0-y_0|+\int_{t_0}^{t}L|X(s,x_0)-X(s,y_0)|ds
\end{aligned}
$$

and

$$
|X(t,x_0)-X(t,y_0)|\leq|x_0-y_0|e^{L(t_1-t_0)}
$$

$\forall t\in[t_0,t_1]$. Thus

$$
x_0\mapsto X(t,x_0)
$$

is Lipschitz continuous on $B_{\delta_0}(\overline{x}_0)$. $\square$

Theorem 3.3. *Let $D\subset\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^M$ be open and $f:D\to\mathbb{R}^N$ be continuous. Assume that*

$$
\frac{\partial f_i}{\partial x_j}\text{ and }\frac{\partial f_i}{\partial\lambda_k}
$$

exist and are continuous of D for $i,j\in\{1,\dots,N\}$ and $k\in\{1,\dots,M\}$. Define $X(t,t_0,x,\lambda)$ by

$$
\begin{cases}
\dot{X} &= f(t,X,\lambda)\\
X(t_0,t_0,x_0,\lambda) &= x_0.
\end{cases}
$$

Then X is C^1 in all its arguments.

Comments

0. $X_i(t,t_0,x_0,\lambda) \;=\; (x_0)_i+\int_{t_0}^{t}f_i(s,X(s,t_0,x_0,\lambda),\lambda)ds\ .$

1. $$\frac{\partial X_i}{\partial (x_0)_j} = \delta_{ij} + \int_{t_0}^{t} \sum_{\ell=1}^{N} \frac{\partial f_i}{\partial x_\ell}(s, X, \lambda)\ \frac{\partial X_\ell}{\partial (x_0)_j}\ ds \text{ where } \delta_{ij}$$
$$= 1 \text{ if } i = j \text{ and } 0 \text{ otherwise.}$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial X_i}{\partial (x_0)_j}\right) = \sum_{\ell=1}^{N} \dfrac{\partial f_i}{\partial x_\ell}\ (t, X, \lambda)\ \dfrac{\partial X_\ell}{\partial (x_0)_j} \\ \dfrac{\partial X_i}{\partial (x_0)_j}(t_0, t_0, x_0, \lambda) = \delta_{ij} \end{cases}$$

2. $$\frac{\partial X_i}{\partial \lambda_k} = \int_{t_0}^{t} \left(\sum_{\ell=1}^{N} \frac{\partial f_i}{\partial x_\ell}\ (s, X, \lambda)\ \frac{\partial X_\ell}{\partial \lambda_k} + \frac{\partial f_i}{\partial \lambda_k}(s, X, \lambda) \right) ds$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial X_i}{\partial \lambda_k}\right) = \sum_{\ell=1}^{N} \dfrac{\partial f_i}{\partial x_\ell}\ (t, X, \lambda)\ \dfrac{\partial X_\ell}{\partial k_k} + \dfrac{\partial f_i}{\partial \lambda_k}\ (t, X, \lambda) \\ \dfrac{\partial X_i}{\partial \lambda_k}(t_0, t_0, x_0, \lambda) = 0 \end{cases}$$

3. $$\frac{\partial X_i}{\partial t_0} = \int_{t_0}^{t} \sum_{\ell=1}^{N} \frac{\partial f_i}{\partial x_\ell}\ (s, X, \lambda)\ \frac{\partial X_\ell}{\partial t_0}\ ds - f(t_0, x_0, \lambda)$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial X_i}{\partial t_0}\right) = \sum_{\ell=1}^{N} \dfrac{\partial f_i}{\partial x_\ell}\ (t, X, \lambda)\ \dfrac{\partial X_\ell}{\partial t_0} \\ \dfrac{\partial X_i}{\partial t_0}\ (t_0, t_0, x_0, \lambda) = -f(t_0, x_0, \lambda). \end{cases}$$

Comment on 1st Order PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + f(t, x)\dfrac{\partial u}{\partial x}(t, x) = 0 \\ u(0, x) = g(x) \end{cases}$$

with f and g given C^1 functions. Define $X(t, t_0, x_0)$ by

$$\begin{cases} \dfrac{dX}{dt} = f(t,X) \\ X(t_0,t_0,x_0) = x_0. \end{cases}$$

Then

$$\begin{aligned} &\frac{d}{dt}\left[u(t,X(t,t_0,x_0))\right] \\ &= \frac{\partial u}{\partial t}(t,X) + \frac{\partial u}{\partial x}(t,X)\ \frac{dX}{dt} \\ &= \frac{\partial u}{\partial t}(t,X) + f(t,X)\ \frac{\partial u}{\partial x}\ (t,X) = 0 \end{aligned}$$

so

$$u(t_0,x_0) = u(s,X(s,t_0,x_0)) \quad \forall s$$

and

$$\begin{aligned} u(t_0,x_0) &= u(0,X(0,t_0,x_0)) \\ &= g(X(0,t_0,x_0)). \end{aligned}$$

Proof. We'll take $N = 1$ and drop λ and show $X(t,t_0,x_0)$ is C^1 in x_0. For a complete proof see Hartman.

Let $Z(t)$ be the solution of

$$\begin{cases} \dot{Z} = \dfrac{\partial f}{\partial x}(t,X(t,t_0,x_0))Z \\ Z(t_0) = 1. \end{cases}$$

We'll show that

$$(*) \qquad \lim_{\Delta x_0 \to 0} \frac{X(t,t_0,x_0+\Delta x_0) - X(t,t_0,x_0)}{\Delta x_0} = Z(t).$$

Let $Z_\Delta(t) = \dfrac{X(t,t_0,x_0+\Delta) - X(t,t_0,x_0)}{\Delta}$.

Choose $\varepsilon_0 > 0$ s.t.

$$S = \{(s, x) : t_0 \le s \le t \text{ and } |x - X(s, t_0, x_0)| \le \varepsilon_0\} \subset D$$

and let $L = \max_S \left|\frac{\partial f}{\partial x}\right|$. By the previous theorem we may consider $|\Delta|$ small enough that $(s, X(s, t_0, x_0 + \Delta)) \in S \ \ \forall s \in [t_0, t]$. By the mean value theorem $\exists\ \xi(\tau)$ between $X(\tau, t_0, x_0)$ and $X(\tau, t_0, x_0 + \Delta)$ s.t.

$$\begin{aligned}
& |Z_\Delta(s) - Z(s)| \\
= & \left|\Delta^{-1}\left[x_0 + \Delta + \int_{t_0}^{s} f(\tau, X(\tau, t_0, x_0 + \Delta))d\tau\right.\right. \\
& \left. -x_0 - \int_{t_0}^{s} f(\tau, X(\tau, t_0, x_0))d\tau\right] \\
& \left. -1 - \int_{t_0}^{s} \frac{\partial f}{\partial x}\ (\tau, X(\tau, t_0, x_0))Z(\tau)d\tau\right| \\
= & \left|\int_{t_0}^{s}\left(\frac{f(\tau, X(\tau, t_0, x_0 + \Delta)) - f(\tau, X(\tau, t_0, x_0))}{\Delta} - \frac{\partial f}{\partial x}(\tau, X(\tau, t_0, x_0))Z(\tau)\right) d\tau\right| \\
= & \left|\int_{t_0}^{s}\left(\frac{\partial f}{\partial x}(\tau, \xi(\tau))Z_\Delta(\tau) - \frac{\partial f}{\partial x}(\tau, X(\tau, t_0, x_0))Z(\tau)\right) d\tau\right| \\
\le & \int_{t_0}^{s}\left(\left|\frac{\partial f}{\partial x}(\tau, \xi(\tau))\right| |Z_\Delta(\tau) - Z(\tau)| + \left|\frac{\partial f}{\partial x}(\tau, \xi(\tau)) - \frac{\partial f}{\partial x}(\tau, X(\tau, t_0, x_0))\right| |Z(\tau)|\right) d\tau \\
\le & A_\Delta + L\int_{t_0}^{s} |Z_\Delta(\tau) - Z(\tau)|d\tau
\end{aligned}$$

where

$$A_\Delta = (t - t_0) \sup_{[t_0,t]} \left|\frac{\partial f}{\partial x}\ (\tau, \xi(\tau)) - \frac{\partial f}{\partial x}\ (\tau, X(\tau, t_0, x_0))\right| |Z(\tau)|.$$

By Gronwall

$$|Z_\Delta(s) - Z(s)| \leq A_\Delta e^{L(s-t_0)} \leq A_\Delta e^{L(t-t_0)}$$

$\forall s \in [t_0, t]$. Once we show $A_\Delta \to 0$ as $\Delta \to 0$, (*) follows.

To show $A_\Delta \to 0$ consider $\varepsilon > 0$. f is continuous on the compact set S so it is uniformly continuous there; choose $\eta > 0$ s.t. $(\tau, x), (\tau, y) \in S$ and

$$|x - y| < \eta \Rightarrow \left|\frac{\partial f}{\partial x}(\tau, x) - \frac{\partial f}{\partial x}(\tau, y)\right| < \frac{\varepsilon}{(t - t_0)\max\limits_{[t_0,t]}|Z|}.$$

By the previous theorem $\exists\, \delta > 0$ s.t. $|\Delta| < \delta$ and $\tau \in [t_0, t] \Rightarrow$

$$|X(\tau, t_0, x_0 + \Delta) - X(\tau, t_0, x_0)| < \eta$$

$\Rightarrow$

$$|\xi(\tau) - X(\tau, t_0, x_0)| < \eta$$

$\Rightarrow$

$$\left|\frac{\partial f}{\partial x}(\tau, \xi(\tau)) - \frac{\partial f}{\partial x}(\tau, X(\tau, t_0, x_0))\right| < \frac{\varepsilon}{(t - t_0)\max |Z|}$$

$\Rightarrow$

$$A_\Delta < \varepsilon.$$

It remains to show that $x_0 \mapsto \dfrac{\partial X}{\partial x_0}(t, t_0, x_0)$ is continuous. Define $Y(t)$ by

$$\begin{cases} \dot{Y} &= \dfrac{\partial f}{\partial x}(t, X(t, t_0, y_0))Y(t) \\ Y(t_0) &= 1 \end{cases}$$

and note that

$$
\begin{aligned}
& \left|\frac{\partial X}{\partial x_0}(t,t_0,y_0) - \frac{\partial X}{\partial x_0}(t,t_0,x_0)\right| \\
= \; & |Y(t) - Z(t)| \\
= \; & \left|\int_{t_0}^{t} \left(\frac{\partial f}{\partial x}(s, X(s,t_0,y_0))Y(s) - \frac{\partial f}{\partial x}(s, X(s,t_0,x_0))Z(s)\right) ds\right| \\
\leq \; & \int_{t_0}^{t} \left[\left|\frac{\partial f}{\partial x}(s, X(s,t_0,y_0)) - \frac{\partial f}{\partial x}(s, X(s,t_0,x_0))\right| |Y(s)| \right. \\
& \left. + \left|\frac{\partial f}{\partial x}(s, X(s,t_0,x_0))\right| |Y(s) - Z(s)|\right] \, ds
\end{aligned}
$$

and proceed as before. $\square$

Comments

1. If f is C^k with $k \geq 1$ then X is C^k also.
2. If f is C^k with $k \geq 1$ and satisfies a Hölder condition in x with exponent $p \in (0,1)$ then X does too.
3. If f is C^0 and satisfies a Hölder condition in x with exponent $p \in (0,1)$ then X can be discontinuous.

Examples

1. Let

$$
f(t,x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x > 0. \end{cases}
$$

Then f satisfies a Lipschitz condition in x but is not C^1. Let's compute $X(t, t_0, x_0)$. If $x_0 \leq 0$ then $X(t, t_0, x_0) = x_0$ for all t and t_0. If $x_0 > 0$ then

$$X(t, t_0, x_0) = e^{t-t_0} x_0.$$

Then

$$\begin{aligned}
&|X(t, t_0, x_0) - X(t, t_0, x_1)| \\
= &\begin{cases} 0 & \text{if} \quad x_0, x_1 \leq 0 \\[2ex] e^{t-t_0}|x_1 - x_0| & \text{if} \quad x_0, x_1 > 0 \\[2ex] |e^{t-t_0} x_1 - x_0| & \text{if} \quad x_0 \leq 0 < x_1 \\[2ex] |e^{t-t_0} x_0 - x_1| & \text{if} \quad x_1 \leq 0 < x_0 \end{cases} \\
\leq \quad & e^{t-t_0}|x_1 - x_0|.
\end{aligned}$$

But

$$\frac{\partial X}{\partial x_0}(t, t_0, x_0) = \begin{cases} 1 & \text{if} \quad x_0 < 0 \\[2ex] e^{t-t_0} & \text{if} \quad 0 < x_0 \end{cases}$$

is not continuous. Same as f.

2. Let

$$f(t, x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\[2ex] x^p & \text{if} \quad x > 0. \end{cases}$$

Then f satisfies a Hölder condition in x but not a Lipschitz condition. As before, $x_0 < 0 \Rightarrow X(t, t_0, x_0) = x_0$. Consider $x_0 > 0 : X(t, t_0, x_0) > 0$ for $t > t_0$

$$\begin{aligned}
\frac{dX}{dt} &= X^p \\
\frac{d}{dt}X^{1-p} &= (1-p)X^{-p}\dot{X} = 1-p \\
X^{1-p} &= (1-p)t + \text{ constant} \\
x_0^{1-p} &= X^{1-p}(t_0,t_0,x_0) = (1-p)t_0 + \text{ constant} \\
X^{1-p}(t,t_0,x_0) &= (1-p)(t-t_0) + x_0^{1-P} \\
X(t,t_0,x_0) &= \left[(1-p)(t-t_0) + x_0^{1-p}\right]^{\frac{1}{1-p}}
\end{aligned}$$

For $t > t_0$, $\lim\limits_{x_0\to 0^+} X(t,t_0,x_0) = [(1-p)(t-t_0)]^{\frac{1}{1-p}} \neq 0 = \lim\limits_{x_0\to 0^-} X(t,t_0,x_0)$ so X is discontinuous.

4 Linear Equations

A. Fundamental Matrices

Consider

$$\text{(H)} \qquad \dot{X} = A(t)X$$

where $A : I \to \mathbb{R}^{N\times N}$ is continuous and $I \subset \mathbb{R}$ is an interval. We've already shown solutions are unique and may be extended to all of I.

Theorem 4.1. *$V = \{X : X$ satisifies (H) on $I\}$ is a vector space of dimension N.*

Proof. Let $X, Y \in V$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned}
\frac{d}{dt}(\alpha X + \beta Y) &= \alpha\dot{X} + \beta\dot{Y} = \alpha A(t)X + \beta A(t)Y \\
&= A(t)(\alpha X + \beta Y)
\end{aligned}$$

so $\alpha X + \beta Y \in V$.

Let $t_0 \in I$. $\forall k \in \{1, \ldots, N\}$ define $\Phi^{(k)}$ to be the solution of (H) that satisfies

$$\Phi_i^{(k)}(t_0) = \delta_{ik} \qquad i = 1, \ldots, N$$

Claim $\{\Phi^{(1)}, \ldots, \Phi^{(N)}\}$ is a basis of V.

Suppose $\sum_1^N C_k \Phi^{(k)} = 0$ then

$$0 = \sum_1^N C_k \Phi^{(k)}(t_0) = \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix}.$$

Thus $\Phi^{(1)}, \ldots, \Phi^{(N)}$ are linearly independent.

Let $X \in V$. Define

$$\overline{X} = \sum_1^N X_k(t_0)\Phi^{(k)}.$$

Then $X, \overline{X} \in V$ and $X(t_0) = \overline{X}(t_0)$. By uniqueness

$$X = \overline{X} \in \operatorname{span}\left(\Phi^{(1)}, \ldots, \Phi^{(N)}\right).$$

$\square$

Definition and Notation Let $\{\Phi^{(1)}, \ldots, \Phi^{(N)}\}$ be a basis of V. $\{\Phi^{(1)}, \ldots, \Phi^{(N)}\}$ is called a fundamental set of solutions and

$$\begin{aligned} \Phi &= \left(\Phi^{(1)}, \cdots, \Phi^{(N)}\right) \\ &= \begin{pmatrix} \Phi_1^{(1)} & \cdots & \Phi_1^{(N)} \\ \vdots & & \vdots \\ \Phi_N^{(1)} & \cdots & \Phi_N^{(N)} \end{pmatrix} = \begin{pmatrix} \Phi_{11} & & \Phi_{1N} \\ & & \\ \Phi_{N1} & & \Phi_{NN} \end{pmatrix} \end{aligned}$$

is called a fundamental matrix.

Comments

1.
$$\begin{aligned}\Phi\vec{C} &= \left(\Phi^{(1)},\dots,\Phi^{(N)}\right)\begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix} \\ &= C_1\Phi^{(1)}+\ldots+C_N\Phi^{(N)}\end{aligned}$$

uniquely represents every solution of (H).

2.
$$\begin{aligned}\frac{d}{dt}\Phi &= \left(\dot{\Phi}^{(1)},\dots,\dot{\Phi}^{(N)}\right) \\ &= \left(A(t)\Phi^{(1)},\dots,A(t)\Phi^{(N)}\right) = A(t)\Phi.\end{aligned}$$

Theorem 4.2. *(Abel/Liouville) If $\psi(t)$ is a matrix solution of (H) then*

$$\det(\psi(t)) = \det(\psi(t_0))e^{\int_{t_0}^{t} \text{trace}\,(A(s))ds}$$

$\forall t_0, t \in I$.

Proof. Claim that

$$\frac{d}{dt}\det(\psi(t)) = \det\begin{pmatrix} \dot{\psi}_{11} & \cdots & \dot{\psi}_{1N} \\ \psi_{21} & \cdots & \psi_{2N} \\ \vdots & & \\ \psi_{N1} & \cdots & \psi_{NN} \end{pmatrix} + \ldots + \det\begin{pmatrix} \psi_{11} & \cdots & \psi_{1N} \\ \vdots & & \\ \psi_{N-11} & \cdots & \psi_{N-1N} \\ \dot{\psi}_{N1} & \cdots & \dot{\psi}_{NN} \end{pmatrix}.$$

Note that, for example,

$$\det\begin{pmatrix} \psi_{11} & \cdots & \psi_{1N} \\ \vdots & & \vdots \\ \psi_{N-11} & \cdots & \psi_{N-1N} \\ \dot{\psi}_{N1} & \cdots & \dot{\psi}_{NN} \end{pmatrix} = \det\begin{pmatrix} \psi_{11} & \cdots & \psi_{1N} \\ \vdots & & \vdots \\ \psi_{N-11} & \cdots & \psi_{N-1N} \\ \sum_k A_{Nk}\psi_{k1} & \cdots & \sum_k A_{Nk}\psi_{kN} \end{pmatrix}$$

$$= \det\begin{pmatrix} \psi_{11}\cdots & \psi_{1N} & \\ \vdots & & \\ \psi_{N-11} & \cdots & \psi_{N-1N} \\ A_{NN}\psi_{N1} & \cdots & A_{NN}\psi_{NN} \end{pmatrix} = A_{NN}\det\psi.$$

Using the claim

$$\begin{aligned}\frac{d}{dt}\det\psi(t) &= \sum_k A_{kk}\det\psi \\ &= \text{trace }(A)\det\psi\end{aligned}$$

and hence

$$\begin{aligned}&\frac{d}{dt}\left(\det(\psi(t))e^{-\int_{t_0}^t \text{trace }A(s)ds}\right)\\ =\;& e^{-\int_{t_0}^t \text{trace}\,A(s)ds}\left(\frac{d}{dt}\left(\det\psi(t)\right)-\text{trace}\,(A(t))\det\psi(t)\right)\\ =\;& 0.\end{aligned}$$

The theorem follows. □

Proof of Claim

Recall that

$$\det\psi = \sum_{\sigma\in\text{ perm}}\text{sgn}(\sigma)\psi_{1\sigma(1)}\cdots\psi_{N\sigma(N)}$$

where perm is the group of permutations on $\{1,\ldots,N\}$. Then

$$\begin{aligned}\frac{d}{dt}(\det\psi) &= \sum_{\sigma\in\text{ perm}}\text{sgn}(\sigma)\Big[\dot\psi_{1\sigma(1)}\psi_{2\sigma(2)}\cdots\psi_{N\sigma(N)}\\ &\quad+\cdots+\psi_{1\sigma(1)}\cdots\psi_{N-1\sigma(N-1)}\dot\psi_{N\sigma(N)}\Big]\\ &=\det\begin{pmatrix}\dot\psi_{11} & \cdots & \dot\psi_{1N}\\ \psi_{21} & \cdots & \psi_{2N}\\ \vdots & & \vdots\\ \psi_{N1} & \cdots & \psi_{NN}\end{pmatrix}+\cdots+\det\begin{pmatrix}\psi_{11} & \cdots & \psi_{1N}\\ \vdots & & \vdots\\ \psi_{N-11} & \cdots & \psi_{N-1N}\\ \dot\psi_{N1} & \cdots & \dot\psi_{NN}\end{pmatrix}.\end{aligned}$$

Corollary 4.1. *Let $\psi(t)$ be a matrix solution of (H) on I. Assume $A(t)$ is continuous. Then the following are equivalent:*

1. *ψ is a fundamental matrix*

2. $\det \psi(t) \neq 0 \quad \forall t \in I$,

3. *$\exists\, t \in I$ s.t.* $\det \psi(t) \neq 0$.

Example

$$\begin{aligned}
\dot{X} &= t^{-1} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} X \\
\psi(t) &= \begin{pmatrix} t & t^3 \\ -t & t^3 \end{pmatrix}
\end{aligned}$$

is a matrix solution. $\det \psi(t) = 2t^4$ is 0 if $t = 0$ and > 0 if $t \neq 0$. Note that A is not continuous at 0. In fact, for $t_0, t > 0$

$$\begin{aligned}
\det(\psi(t_0)) e^{\int_{t_0}^{t} \text{ trace } (A(s)) ds} &= 2t_0^4 e^{\int_{t_0}^{t} 4s^{-1} ds} \\
&= 2t_0^4 e^{4 \ln \frac{t}{t_0}} = 2t_0^4 \left(\frac{t}{t_0}\right)^4 \\
&= 2t^4 = \det(\psi(t)).
\end{aligned}$$

Proof. To show $3 \Rightarrow 1$ assume $\exists\, t_0 \in I$ s.t. $\det \psi(t_0) \neq 0$. Let

$$\sum_{1}^{N} C_k \psi^{(k)} \equiv 0$$

then

$$\psi(t_0) \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det \psi(t_0) \neq 0, \quad C_1 = \cdots = C_N = 0$. Thus $\psi^{(1)}, \ldots, \psi^{(N)}$ are linearly independent and hence a basis.

Assume 1 holds. Let $x_0 \in \mathbb{R}^N$ and $t_0 \in I$ and let $Y(t)$ be the solution of (H) with

$$Y(t_0) = x_0.$$

$\exists\ C_1, \ldots, C_N$ s.t. $Y \equiv C_1\psi^{(1)} + \cdots + C_N\psi^N$. Hence

$$x_0 = C_1\psi^{(1)}(t_0) + \cdots + \psi^{(N)}(t_0) = \psi(t_0)\begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix}.$$

Since this equation has a solution $\forall\ x_0$, $\det\psi(t_0) \neq 0$ follows $\forall\ t_0$. This is 2. $2 \Rightarrow 3$ is immediate. $\square$

Comment Suppose $\psi(t)$ is a fundamental matrix and let

$$\Phi(t) = \psi(t)\psi^{-1}(t_0).$$

Then

$$\begin{aligned} \dot{\Phi}(t) &= \dot{\psi}(t)\psi^{-1}(t_0) = (A(t)\psi(t))\psi^{-1}(t_0) \\ &= A(t)(\psi(t)\psi^{-1}(t_0)) = A(t)\Phi(t) \end{aligned}$$

and

$$\Phi(t_0) = I.$$

B. Nonhomogeneous Equations

Theorem 4.3. *(Variation of Parameters) Let I be an interval and $A : I \to \mathbb{R}^{N\times N}$ and $b : \mathbb{R}^N \to \mathbb{R}^N$ be continuous. Let Φ be a fundamental matrix solution for (H). Then $\forall\ t_0 \in I$ and $x_0 \in \mathbb{R}^N$ the solution of*

(NH)
$$\begin{cases} \dot{X} = A(t)X + b(t) \\ X(t_0) = x_0 \end{cases}$$

is

$$X(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)ds.$$

Proof. Let

$$\mathcal{H}(t) = \Phi(t)\Phi^{-1}(t_0)x_0$$

and

$$\mathcal{N}(t) = \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)ds.$$

Then

$$\begin{cases} \dot{\mathcal{H}} &= A(t)\mathcal{H} \\ \mathcal{H}(t_0) &= x_0. \end{cases}$$

Also

$$\begin{aligned} \dot{\mathcal{N}}(t) &= \Phi(t)\Phi^{-1}(t)b(t) + \int_{t_0}^{t} \Phi'(t)\Phi^{-1}(s)b(s)ds \\ &= b(t) + \int_{t_0}^{t} A(t)\Phi(t)\Phi^{-1}(s)ds \\ &= A(t)\mathcal{N}(t) + b(t) \end{aligned}$$

and

$$\mathcal{N}(t_0) = 0.$$

Hence

$$\begin{aligned} \dot{X} &= \dot{\mathcal{H}} + \dot{\mathcal{N}} = A\mathcal{H} + A\mathcal{N} + b \\ &= A(\mathcal{H} + \mathcal{N}) + b = AX + b \end{aligned}$$

and

$$X(t_0) = x_0 + 0 = x_0.$$

□

Comment If $A(t) = A$ is independent of t then

$$\Phi(t)\Phi^{-1}(s) = \Phi(t-s).$$

Proof. For s fixed let

$$\begin{aligned}\mathcal{L}(t) &= \Phi(t)\Phi^{-1}(s) \\ \mathcal{R}(t) &= \Phi(t-s).\end{aligned}$$

Then

$$\begin{aligned}\dot{\mathcal{L}}(t) &= A(t)\Phi(t)\Phi^{-1}(s) = A(t)\mathcal{L}(t), \\ \dot{\mathcal{R}}(t) &= \Phi'(t-s) = A(t-s)\Phi(t-s) \\ &= A\Phi(t-s),\end{aligned}$$

and

$$\mathcal{R}(s) = I = \Phi(s)\Phi^{-1}(s) = \mathcal{L}(s).$$

By uniqueness $\mathcal{L} \equiv \mathcal{R}$. □

A Derivation Using $\delta(t)$

Definition 4.1.

$$\begin{aligned}\delta(t) &= 0 \text{ if } t \neq 0 \\ \int_a^b \delta(t)dt &= 1 \text{ if } a < 0 < b.\end{aligned}$$

1. Let $t_1 > t_0$ and $b \in \mathbb{R}^N$ and solve

$$\begin{cases} \dot{X} = A(t)X + b\,\delta(t-t_1) \\ X(t_0) = 0. \end{cases}$$

For $t < t_1,\ X(t) = 0$. For $\varepsilon > 0$

$$
\begin{aligned}
X(t_1+\varepsilon) &= \int_{t_0}^{t_1+\varepsilon} \dot{X}(s)ds \\
&= \int_{t_0}^{t_1+\varepsilon} (A(s)X(s) + b\delta(s-t_1))ds \\
&= \int_{t_1}^{t_1+\varepsilon} A(s)X(s)ds + b \\
&\underset{\varepsilon\to 0^+}{\longrightarrow} b.
\end{aligned}
$$

For $t > t_1,\ b\ \delta(t-t_1) = 0$ so

$$X(t) = \Phi(t)\Phi^{-1}(t_1)b.$$

Thus

$$X(t) = \begin{cases} 0 & t < t_1 \\ \Phi(t)\Phi^{-1}(t_1)b & t_1 < t. \end{cases}$$

2. Let $t_0 < t_1 < \cdots < t_M,\ b^{(1)}, \dots, b^{(M)} \in \mathbb{R}^N$ and solve

$$(*) \qquad \begin{cases} \dot{X} &= A(t)X + \sum\limits_{k=1}^{M} b^{(k)}\delta(t-t_k) \\ X(t_0) &= 0. \end{cases}$$

We'll use the principle of superposition:
If

$$\begin{cases} \dot{X}^{(k)} &= A(t)X^{(k)} + \beta^{(k)}(t) \\ X^{(k)}(t_0) &= 0 \end{cases}$$

then $X = \sum\limits_{1}^{M} X^{(k)}$ satisfies

$$\begin{cases} \dot{X} &= \sum_1^M \left(A(t)X^{(k)} + \beta^{(k)}(t)\right) = A(t)X + \sum_1^M \beta^{(k)}(t) \\ X(t_0) &= 0. \end{cases}$$

Using 1 and superposition, the solution of (*) is

$$X(t) = \sum_{\substack{k=1 \\ t_k<t}}^{M} \Phi(t)\Phi^{-1}(t_k)b^{(k)}.$$

3. Consider

$$\begin{cases} \dot{X} &= A(t)X + b(t) \\ X(t_0) &= 0. \end{cases}$$

Let $\Delta t > 0,\ t_k = t_0 + k\Delta t$ for $k = 1, \dots, M$ and approximate

$$b(t) \approx \sum_{k=1}^{M} b(t_k)\Delta t\ \delta(t - t_k).$$

Then

$$\begin{aligned} X(t) &\approx \sum_{\substack{k=1 \\ t<t_k}}^{M} \Phi(t)\Phi^{-1}(t_k)b(t_k)\Delta t \\ &\approx \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)ds. \end{aligned}$$

Compare this with variation of parameters when $x_0 = 0$.

Example

Consider

$$\begin{cases} \dot{X} = -X + f(t, X) \\ X(0) = x_0 \end{cases}$$

where $N = 1,\ x_0 \approx 0$, and

$$|f(t,x)| \le C_f x^2 \quad \forall t, x.$$

Picard iteration (that we used before) is

$$X^{(n+1)}(t) = x_0 + \int_{t_0}^{t} (-X^{(n)}(s) + f(s, X^{(n)}(s)))ds.$$

This gets the wrong behavior as $t \to \infty$, e.g. if $X^{(0)} \equiv 0$ then

$$\begin{aligned} X^{(1)}(t) &= x_0 \\ X^{(2)}(t) &= x_0 + \int_{t_0}^{t} (-x_0 + f(s, x_0))ds. \end{aligned}$$

If we use

$$(*) \qquad X^{(n+1)}(t) = x_0 + \int_{t_0}^{t} (-X^{(n+1)}(s) + f(s, X^{(n)}(s)))ds$$

then we get $\lim\limits_{t \to +\infty} X^{(n)}(t) = 0$ for $x_0 \approx 0$. Note that (*) is

$$\begin{cases} \dot{X}^{(n+1)} = -X^{(n+1)} + f(t, X^{(n)}) \\ X(0) = x_0 \end{cases}$$

so using $\Phi(t) = e^{-t}$

$$\begin{aligned} X^{(n+1)}(t) &= \Phi(t)\Phi^{-1}(0)x_0 \\ &\quad + \int_0^t \Phi(t)\Phi^{-1}(s) f(s, X(s))ds \\ &= e^{-t}x_0 + \int_0^t e^{s-t}\ f(s, X^{(n)}(s))ds. \end{aligned}$$

Suppose $|x_0| < \dfrac{1}{4C_f}$ and

$$|X^{(n)}(t)| \leq 2|x_0|e^{-t} \quad \forall t \geq 0.$$

Then

$$\begin{aligned}
|X^{(n+1)}(t)| &\leq e^{-t}|x_0| + \int_0^t e^{-(t-s)} C_f \left(2|x_0|e^{-s}\right)^2 ds \\
&= e^{-t}|x_0| + 4C_f \; x_0^2 \; e^{-t} \int_0^t e^{-s} ds \\
&= e^{-t}|x_0| + (4C_f|x_0|)\,|x_0|e^{-t}e^{2t_0}\left(-e^{-t}+1\right) \\
&< e^{-t}|x_0| + |x_0|e^{-t} \\
&= 2|x_0|e^{-t}.
\end{aligned}$$

C. The Constant Coefficient Case

Definition 4.2. *Let* $A \in \mathbb{R}^{N \times N}$,

$$|A| = \max\left\{|Ax| : |x| = 1\right\}.$$

Example

$$A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_N \end{pmatrix} \qquad A_{ij} = a_i \delta_{ij}$$

For $|x| = 1$

$$\begin{aligned} |Ax| = \sqrt{\sum_1^N (a_i x_i)^2} &\leq \sqrt{\max_i |a_i|^2 \sum_1^N x_i^2} \\ &= \max_i |a_i| \end{aligned}$$

so

$$|A| \leq \max_i |a_i|.$$

Choose $m \in \{1, \dots, N\}$ s.t. $|a_m| = \max_i |a_i|$. Let $x_k = \delta_{km}$ then $|x| = 1$ so

$$|A| \geq |Ax| = \left| \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_N \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} = |a_m|.$$

Hence, $|A| = \max_i |a_i|$.

Comments

1. Let $|A|_\infty = \max\{|A_{ij}| : 1 \leq i, j \leq N\}$.

   (a) For $|x| = 1$

$$
\begin{aligned}
(Ax)_i^2 &= \left(\sum_{j=1}^{N} A_{ij}x_j\right)^2 \le \left(\sum_{j=1}^{N} A_{ij}^2\right)\left(\sum_{j=1}^{N} x_j^2\right) \\
&= \sum_{j=1}^{N} A_{ij}^2 \le |A|_\infty^2 N
\end{aligned}
$$

so

$$
|Ax| = \sqrt{\sum_{i=1}^{N}(Ax)_i^2} \le \sqrt{|A|_\infty^2 N^2} = |A|_\infty N
$$

and

$$
|A| \le |A|_\infty N.
$$

(b) Choose α, β s.t. $|A_{\alpha\beta}| = |A|_\infty$. Define x by

$$
x_i = \delta_{i\beta}.
$$

Then $|x| = 1$ and

$$
\begin{aligned}
|A| &\ge |Ax| \ge |(Ax)_\alpha| = \left|\sum_{j=1}^{N} A_{\alpha j}x_j\right| \\
&= \left|\sum_{j=1}^{N} A_{\alpha j}\delta_{j\beta}\right| = |A_{\alpha\beta}| = |A|_\infty.
\end{aligned}
$$

Thus

$$
|A|_\infty \le |A| \le N|A|_\infty
$$

$\forall \;\; A \in \mathbb{R}^N \times \mathbb{R}^N$.

2. $\forall \lambda \in \mathbb{R}, \; x \in \mathbb{R}^N$

$$
|\lambda Ax| = |\lambda||Ax|
$$

so

$$|\lambda A| = |\lambda||A|.$$

3. $\forall\ x \in \mathbb{R}^N \backslash \{0\}$

$$|A| \geq |A(|x|^{-1}x)| = |x|^{-1}|Ax|$$

so $\forall\ \ x \in \mathbb{R}^N$

$$|A||x| \geq |Ax|.$$

4. For $|x| = 1$

$$|(A+B)x| = |Ax + Bx| \leq |Ax| + |Bx| \leq |A| + |B|,$$

hence

$$|A+B| \leq |A| + |B|.$$

5. For $|x| = 1$

$$|ABx| = |A(Bx)| \leq |A||Bx| \leq |A||B|$$

so

$$|AB| \leq |A||B|$$

(a) This can be strict inequality: consider $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then

$$|AB| = \left| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right| = 0 < |A||B| = 1.$$

(b) If $A = B = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$ then

$$|AB|_\infty = \left| \begin{pmatrix} N & \cdots & N \\ \vdots & & \vdots \\ N & \cdots & N \end{pmatrix} \right|_\infty = N \not\leq 1 = |A|_\infty |B_\infty|.$$

6. $|A^k| \leq |A|^k$.

Definition 4.3. $S_L(t) = I + \sum_{k=1}^{L} \frac{1}{k!}\ t^k A^k = \sum_{k=0}^{L} \frac{1}{k!}\ (tA)^k$

Comments

1.

$$\begin{aligned} S'_L(t) &= \sum_{k=1}^{L} \frac{1}{k!}\ kt^{k-1}A^k = A\sum_{k=1}^{L} \frac{1}{(k-1)!}\ t^{k-1}A^{k-1} \\ &= A\sum_{k=0}^{L-1} \frac{1}{k!}\ t^k A^k = A\, S_{L-1}(t) \end{aligned}$$

2. For $M > L$

$$\begin{aligned} |S_M(t) - S_L(t)| &= \left| \sum_{k=L+1}^{M} \frac{1}{k!}\ t^k A^k \right| \\ &\leq \sum_{k=L+1}^{M} \frac{1}{k!}\ |t^k A^k| = \sum_{L+1}^{M} \frac{1}{k!}\ |t|^k |A^k| \\ &\leq \sum_{L+1}^{M} \frac{1}{k!}\ |t|^k |A|^k < \sum_{L+1}^{\infty} \frac{1}{k!}\ (|t||A|)^k \\ &\underset{L\to\infty}{\longrightarrow}\ 0. \end{aligned}$$

Definition 4.4. $e^{At} = \sum_0^\infty \frac{1}{k!}\, t^k A^k$

Comments

1. On any bounded interval, $t \in [-B_0, B_0]$:

$$\begin{aligned} |e^{At} - S_L(t)| &\leq \sum_{L+1}^\infty \frac{1}{k!}\,(|t||A|)^k \\ &\leq \sum_{L+1}^\infty \frac{1}{k!}\,(B_0|A|)^k \to 0 \text{ as } L \to \infty \end{aligned}$$

   so $S_L(t) \to e^{At}$ uniformly on $[-B_0, B_0]$ and e^{At} is continuous on $[-B_0, B_0]$. Since B_0 is arbitrary, e^{At} is continuous on $\mathbb{R}$.

2. $S_L'(t) = AS_{L-1}(t) \to Ae^{At}$ uniformly on compact sets. Hence e^{At} is differentiable on $\mathbb{R}$ and

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

3. $e^{At}\big|_0 = I + \sum_{k=1}^\infty \frac{1}{k!}\, 0^k A^k = I$

4. $e^{As}e^{A} = e^{A(t+s)}$

   Proof. Fix s and let

$$L(t) = e^{At}e^{As} \text{ and } R(t) = e^{A(t+s)}.$$

   Then

$$\begin{aligned} L(0) &= e^{As} = R(0), \\ L'(t) &= (Ae^{At})e^{As} = A(e^{At}e^{As}) = AL(t), \\ R'(t) &= Ae^{A(t+s)} = AR(t). \end{aligned}$$

   By uniqueness $L \equiv R$. □

5. $$\begin{aligned} e^{At}e^{A(-t)} &= e^{A(t-t)} = I \text{ so} \\ (e^{At})^{-1} &= e^{-At}. \end{aligned}$$

6. Assume $AB = BA$:

   (a) $A^kB = A^{k-1}\ BA = \cdots = BA^k$

   (b) $$\begin{aligned} Be^{At} &= B\sum_0^\infty \frac{1}{k!}\ t^kA^k = \sum_0^\infty \frac{1}{k!}\ t^kBA^k \\ &= \sum_{k=0}^\infty \frac{1}{k!}\ t^kA^kB = e^{At}B \end{aligned}$$

   (c) $$\begin{aligned} e^{Bt}e^{At} &= \left(\sum_0^\infty \frac{1}{k!}\ t^kB^k\right)e^{At} = \sum_0^\infty \frac{1}{k!} t^k(B^ke^{At}) \\ &= \sum_0^\infty \frac{1}{k!}\ t^k(e^{At}B^k) = e^{At}e^{Bt} \end{aligned}$$

Examples

1. Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$ then

$$\begin{aligned} e^{Dt} &= \sum_0^\infty \frac{t^k}{k!}\ D^k = \sum_0^\infty \frac{t^k}{k!}\ \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_N^k \end{pmatrix} \\ &= \begin{pmatrix} \sum_0^\infty \frac{t^k}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_0^\infty \frac{t^k}{k!}\lambda_N^k \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_N t} \end{pmatrix} \end{aligned}$$

2. $A = PBP^{-1}$:

$$A^k = (PBP^{-1}) \cdots (PBP^{-1}) = PB^kP^{-1}$$

so

$$\begin{aligned} e^{At} &= \sum_0^\infty \frac{t^k}{k!} A^k = \sum_0^\infty \frac{t^k}{k!} PB^kP^{-1} \\ &= P\left(\sum_0^\infty \frac{t^k}{k!} B^k\right) P^{-1} = Pe^{Bt}P^{-1} \end{aligned}$$

3.

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}:$$

By induction

$$A^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix}$$

$(i, j) \in \{(2, 1), (3, 1), (3, 2)\} \Rightarrow (e^{At})_{ij} = 0.$

$$(e^{At})_{ii} = \sum_0^\infty \frac{1}{k!} t^k \lambda^k = e^{\lambda t}$$

For $(i, j) \in \{(1, 2), (2, 3)\}$

$$\begin{aligned} (e^{At})_{ij} &= \sum_0^\infty \frac{t^k}{k!} k\lambda^{k-1} = t \sum_{k=1}^\infty \frac{(t\lambda)^{k-1}}{(k-1)!} \\ &= t \sum_0^\infty \frac{(t\lambda)^k}{k!} = te^{\lambda t}. \end{aligned}$$

Finally,

$$
\begin{aligned}
(e^{At})_{13} &= \sum_0^\infty \frac{t^k}{k!}\,\frac{1}{2}k(k-1)\lambda^{k-2} \\
&= t^2\frac{1}{2}\sum_2^\infty \frac{t^{k-2}}{(k-2)!}\,\lambda^{k-2} = \frac{1}{2}t^2e^{\lambda t}.
\end{aligned}
$$

So

$$
e^{At} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.
$$

4. Let

$$
A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 \ddots & \\ 0 & & \ddots & \lambda \end{pmatrix},
$$

claim

$$
e^{At} = e^{\lambda t}\begin{pmatrix} 1 & t & & \cdots & \dfrac{t^{n-1}}{(N-1)!} \\ & & \ddots & \ddots & \vdots \\ & & & & t \\ 0 & & & & 1 \end{pmatrix}:
$$

Proof. Let $L(t) = e^{At}$ and

$$
R(t) = e^{\lambda t}\begin{pmatrix} 1 & t & \cdots & \dfrac{t^{N-1}}{(N-1)!} \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}.
$$

Then $L(0) = I = R(0)$, $L'(t) = AL(t)$, and

$$
\begin{aligned}
R'(t) &= \lambda R(t) + e^{\lambda t} \begin{pmatrix} 0 & 1 & \ddots & \cdots & \dfrac{t^{N-2}}{(N-2)!} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix} \\
&= \left[\lambda I + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix} \right] R(t) = AR(t).
\end{aligned}
$$

By uniqueness $L \equiv R$. □

Jordan Canonical Form

Definition 4.5. *v is a generalized eigenvector of rank k (of A) associated with the eigenvalue λ if*

$$(A-\lambda I)^k v = 0 \quad \text{and} \quad (A-\lambda I)^{k-1} v \neq 0.$$

Comments

1. Let v, λ, k be as above. Let

$$
\begin{aligned}
v^{(k)} &= v \\
v^{(k-1)} &= (A-\lambda I)v \\
&\vdots \\
v^{(1)} &= (A-\lambda I)^{k-1} v.
\end{aligned}
$$

$\{v^{(1)}, \ldots, v^{(k)}\}$ is called a chain.

$v^{(1)}, \ldots, v^{(k)}$ are linearly independent:

Suppose $\sum\limits_{\ell=1}^{k} C_\ell v^{(\ell)} = 0$, then

$$
\begin{aligned}
0 &= (A-\lambda I)^{k-1} \sum_{1}^{k} C_\ell \; v^{(\ell)} = C_k v^{(1)} \text{ so } C_k = 0 \\
0 &= (A-\lambda I)^{k-2} \sum_{1}^{k-1} C_\ell \; v^{(\ell)} = C_{k-1} v^{(1)} \text{ so } C_{k-1} = 0
\end{aligned}
$$

etc.

2. Let v and w be generalized eigenvectors of rank k and ℓ respectively, associated with λ. Let

$$
\begin{aligned}
v^{(k-i)} &= (A-\lambda I)^i v \quad i = 0, \ldots, k-1 \\
w^{(\ell-j)} &= (A-\lambda I)^j w \quad j = 0, \ldots, \ell-1.
\end{aligned}
$$

If $v^{(1)}$ and $w^{(1)}$ are linearly independent then $v^{(1)}, \ldots, v^{(k)}, w^{(1)}, \ldots, w^{(\ell)}$ are linearly independent. The proof is similar to 1.

3. Let v and w be generalized eigenvectors of rank k and ℓ, respectively, associated with λ and β, respectively. Let

$$
\begin{aligned}
v^{(k-i)} &= (A-\lambda I)^i v \qquad i = 0, \ldots, k-1 \\
w^{(\ell-j)} &= (A-\beta I)^j w \qquad j = 0, \ldots, \ell-1.
\end{aligned}
$$

If $\lambda \neq \beta$ then $v^{(1)}, \ldots, v^{(k)}, w^{(1)}, \ldots w^{(\ell)}$ are linearly independent:

Suppose

$$\sum_{i=1}^{k} C_i v^{(i)} + \sum_{j=1}^{\ell} D_j w^{(j)} = 0,$$

then

$$
\begin{aligned}
0 &= (A-\lambda I)^{k-1}(A-\beta I)^{\ell}\left(\sum_{i=1}^{k} C_i v^{(i)} + \sum_{j=1}^{\ell} C_j w^{(j)}\right) \\
&= (A-\beta I)^{\ell}\left(\sum_{i=1}^{k} C_i (A-\lambda I)^{k-1} v^{(i)}\right) \\
&= (A-\beta I)^{\ell} C_k v^{(i)} = C_k (A-\beta I)^{\ell-1}(\lambda-\beta) v \\
&= \cdots = C_k(\lambda-\beta)^{\ell} v^{(1)} \text{ so } C_k = 0.
\end{aligned}
$$

Apply $(A-\lambda I)^{k-2}(A-\beta I)^{\ell}$ to show $C_{k-1}=0$, etc.

Theorem 4.4. *Let $A \in \mathbb{C}^{N\times N}$, $\exists P, J \in \mathbb{C}^{N\times N}$ with P invertible,*

$$A = PJP^{-1}$$

and

$$J = \begin{pmatrix} J_0 & & & 0 \\ & J_1 & & \\ & & \ddots & \\ 0 & & & J_s \end{pmatrix} \text{ or } J = \begin{pmatrix} J_1 & & 0 \\ & \cdots & \\ 0 & & J_s \end{pmatrix}$$

where J_0 is diagonal and $J_1, \ldots, J_s$ are of the form

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}.$$

Outline of Construction

1. $\det(A-\lambda I) = (-1)^N(\lambda-\lambda_1)^{P_1}\cdots(\lambda-\lambda_m)^{P_m}$ with $i \neq j \Rightarrow \lambda_i \neq \lambda_j$. Note that

$$P_1 + \cdots + P_m = N.$$

2. Find a generalized eigenvector, v, associated with λ_1 of largest possible rank, k, and define

$$v^{(k-i)} = (A - \lambda_1 I)^i v \qquad i = 0, \dots, k-1.$$

If $k < P_1$, consider the chains associated with λ_1 that are independent with all previous choices; choose one of largest possible rank. Continue until sum of ranks $= P_1$.

3. Repeat for $\lambda_2, \dots, \lambda_m$.

4. Reorder the basis so that diagonal part comes first.

Examples

1.

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 1 & 3-\lambda & -1 \\ 0 & 1 & 1 & 1-\lambda \end{pmatrix} \\ &= (2-\lambda)\left[(2-\lambda)(3-\lambda)(1-\lambda) + (2-\lambda)\right] \\ &= (\lambda-2)^2(3 - 4\lambda + \lambda^2 + 1) = (\lambda - 2)^4. \end{aligned}$$

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \text{ and } (A - 2I)^2 = 0$$

Let

$$v^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad v^{(1)} = (A - 2I)v^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$v^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v^{(3)} = (A - 2I)v^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{array}{rcl} Av^{(1)} & = & 2v^{(1)} \\ Av^{(2)} & = & 2v^{(2)} + v^{(1)} \\ Av^{(3)} & = & 2v^{(3)} \\ Av^{(4)} & = & 2v^{(4)} + v^{(3)} \end{array} \qquad J = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

$$P = \left(v^{(1)},\, v^{(2)},\, v^{(3)},\, v^{(4)}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

then $A = PJP^{-1}$. For example,

$$\begin{array}{rcl} PJP^{-1}v^{(2)} & = & PJ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = P \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) = v^{(1)} + 2v^{(2)} \\ & = & Av^{(2)}. \end{array}$$

2. For A 4 by 4 with $\det(A - \lambda I) = (\lambda - 2)^4$:
   (a) If $A - 2I = 0 : A = J = 2I,\ P = I$
   (b) If $(A - 2I)^2 = 0$ and $A - 2I \neq 0$:

$$J = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right) \quad \text{or} \quad J = \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

(c) If $(A-2I)^3 = 0$ and $(A-2I)^2 \neq 0$:

$$J = \left(\begin{array}{c|ccc} 2 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

(d) If $(A-2I)^4 = 0$ and $(A-2I)^3 \neq 0$:

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Comments

1. Let B be k by k, C be $N-k$ by $N-k$ and

$$A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right).$$

Claim that

$$e^{At} = \left(\begin{array}{c|c} e^{Bt} & 0 \\ \hline 0 & e^{Ct} \end{array}\right).$$

Proof. Let $L(t) = e^{At}$ and

$$R(t) = \left(\begin{array}{c|c} e^{Bt} & 0 \\ \hline 0 & e^{Ct} \end{array}\right).$$

Then $L(0) = I = R(0)$, $L'(t) = AL(t)$, and

$$
\begin{aligned}
R'(t) &= \left(\begin{array}{c|c} Be^{Bt} & 0 \\ \hline 0 & Ce^{Ct} \end{array}\right) = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right) \left(\begin{array}{c|c} e^{Bt} & 0 \\ \hline 0 & e^{Ct} \end{array}\right) \\
&= AR(t).
\end{aligned}
$$

By uniqueness $L \equiv R$ $\square$

2. If

$$
J = \left(\begin{array}{c|c|c} J_0 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & J_m \end{array}\right)
$$

then

$$
e^{Jt} = \left(\begin{array}{c|c|c} e^{J_0 t} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & e^{J_m t} \end{array}\right).
$$

3. $e^{(PJP^{-1})t} = e^{P(Jt)P^{-1}} = Pe^{Jt}P^{-1}$

Example

$$
\begin{array}{ccccc}
\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} & = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} & \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\
A & = & P & J & P^{-1}
\end{array}
$$

$$
J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \; e^{J_1 t} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}
$$

$$
e^{At} = Pe^{Jt}P^{-1} = Pe^{2t} \left(\begin{array}{cc|cc} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{array}\right) P^{-1}
$$

Theorem 4.5. *Let A be N by N and*

$$\sigma = \max\left\{\text{real}(\lambda) : \lambda \text{ is an eigenvalue of } A\right\}.$$

Then $\exists\ C > 0$ s.t.

$$|e^{At}| \leq Ce^{\sigma t}(1 + t^{N-1}) \quad \forall t \geq 0.$$

Example

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A^2 = 0$ so

$$e^{At} = \sum_0^\infty \frac{t^k}{k!}\, A^k = I + tA = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Note that $|e^{At}|_\infty = \max(1, t) \forall t \geq 0$ and recall that

$$|e^{At}|_\infty \leq |e^{At}| \leq N|e^{At}|_\infty$$

so

$$\max(1, t) \leq |e^{At}| \leq 2\max(1, t).$$

Hence,

$$\frac{1}{2}(1 + t) \leq |e^{At}| \leq 2(1 + t) \quad \forall t \geq 0.$$

Corollary 4.2. $\forall \varepsilon > 0 \exists\ C_\varepsilon > 0$ *s.t.*

$$|e^{At}| \leq C_\varepsilon e^{(\sigma+\varepsilon)t} \quad \forall t \geq 0.$$

Proof. Let $\varepsilon > 0$. Choose $C > 0$ s.t.

$$|e^{At}| \leq Ce^{\sigma t}(1 + t^{N-1}) \quad \forall t \geq 0.$$

Let $C_\varepsilon = C \max\left\{(1 + t^{N-1})e^{-\varepsilon t} : t \geq 0\right\}$. Then

$$|e^{At}| \leq Ce^{(\sigma+\varepsilon)t}(1 + t^{N-1})e^{-\varepsilon t} \leq C_\varepsilon e^{(\sigma+\varepsilon)t}.$$

$\square$

Proof of Theorem Write the Jordan form, $A = PJP^{-1}$ so that

$$e^{At} = Pe^{Jt}P^{-1}$$

and

$$\begin{aligned} |e^{At}| &\leq |P||e^{Jt}||P^{-1}| \\ &\leq (N|P||P^{-1}|)|e^{Jt}|_\infty. \end{aligned}$$

J is composed of blocks of the form

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

so e^{Jt} is composed of blocks of the form

$$e^{\lambda t}\begin{pmatrix} 1 & \cdots & \dfrac{t^{k-1}}{(k-1)!} \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} |e^{\text{Jordan Block } t}|_\infty &\leq |e^{\lambda t}| \max(1, t^{N-1}) \\ &\leq e^{\text{real}(\lambda)t}(1 + t^{N-1}) \end{aligned}$$

$\forall t \geq 0$. Hence

$$|e^{At}| \leq (N|P||P^{-1}|)\ e^{\sigma t}(1 + t^{N-1}).$$

5 Stability

Definitions Consider

$$\begin{cases} \dot{X}(t, t_0, x_0) = f(t, X(t, t_0, x_0)) \\ X(t_0, t_0, x_0) = x_0. \end{cases}$$

1. $\overline{x} \in \mathbb{R}^N$ is an equilibrium (critical, stationary, singular) point at time T if $t \geq T \Rightarrow f(t, \overline{x}) = 0$. We'll take $T = 0$. It's isolated if $\exists\ R > 0$ s.t. $|x - \overline{x}| < R$ and x an equilibrium point $\Rightarrow x = \overline{x}$.

2. An equilibrium point, $\overline{x}$, is stable if $\forall\ \varepsilon > 0$ and $\forall t_0 \geq 0\ \ \exists \delta(\varepsilon, t_0) > 0$ s.t.

   $$|x - \overline{x}| < \delta(\varepsilon, t_0) \text{ and } t \geq t_0 \Rightarrow |X(t, t_0, x_0) - \overline{x}| < \varepsilon.$$

   If there's a choice of δ that is independent of t_0 we say x_0 is uniformly stable.

3. An equilibrium point is unstable if it is not stable , i.e., $\exists\ t_0 \geq 0$ and $\varepsilon > 0$ and $x^{(k)} \to \overline{x}$ with

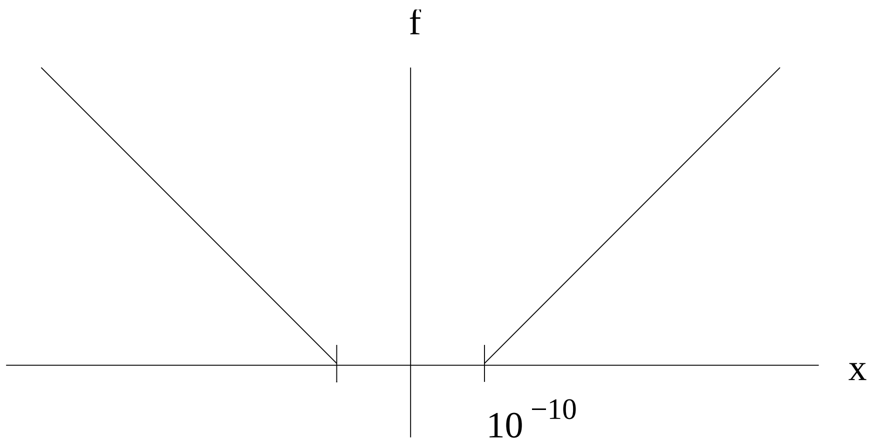
   $$\sup_{t \geq t_0} |X(t, t_0, x^{(k)}) - \overline{x}| \geq \varepsilon_0 \quad \forall k.$$

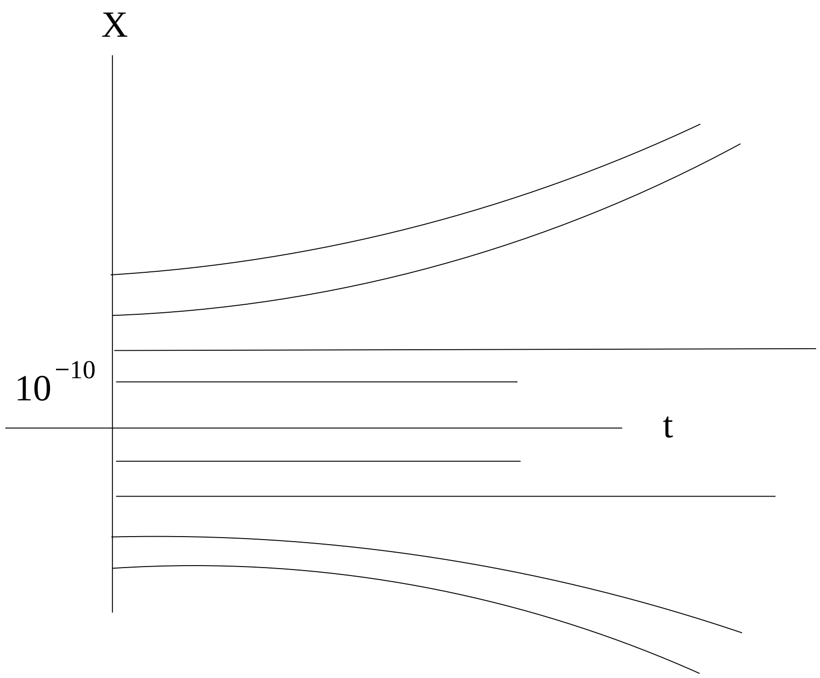
4. An equilibrium point, $\overline{x}$, is asymptotically stable if it is stable and $\forall t_0 > 0\ \exists\ \eta(t_0) > 0$ s.t.

   $$|x - \overline{x}| < \eta(t_0) \Rightarrow \lim_{t \to +\infty} X(t, t_0, x_0) = \overline{x}.$$

Example

$$\dot{X} = f(X) \text{ where } f(x) = \begin{cases} 0 & \text{if } \ |x| \leq 10^{-10} \\ |x| - 10^{-10} & \text{if } \ |x| > 10^{-10} \end{cases}$$





(a) $\overline{x} \in (-10^{-10}, 10^{-10})$ is an equilibrium. It's stable: let $\varepsilon > 0$, take

$$\delta = \min(\varepsilon, 10^{-10} - \overline{x}, \overline{x} + 10^{-10}).$$

Then

$$
\begin{aligned}
|x_0 - \overline{x}| < \delta \;\; &\Rightarrow \;\; X(t, t_0, x_0) = x_0 \;\; \forall t \\
&\Rightarrow \;\; |X(t, t_0, x_0) - \overline{x}| = |x_0 - \overline{x}| < \delta \leq \varepsilon \;\; \forall t.
\end{aligned}
$$

(b) $\overline{x} = 10^{-10}$ is unstable: $\forall\ x_0 > \overline{x}$

$$
\sup_{t \geq t_0} |X(t, t_0, x_0) - \overline{x}| = \sup_{t \geq t_0} |x_0 e^t - \overline{x}| = +\infty.
$$

Theorem 5.1. *Let A be a constant N by N matrix with Jordan form*

$$
J = \begin{pmatrix} J_0 & & & 0 \\ & J_1 & & \\ & & \ddots & \\ 0 & & & J_m \end{pmatrix}
$$

(J_0 diagonal). Note that 0 is an equilibrium point of $\dot{X} = AX$.

1. 0 is stable if, and only if, the eigenvalues associated with $J_1, \ldots, J_m$ have strictly negative real part and the eigenvalues associated with J_0 have real part ≤ 0.

2. 0 is asymptotically stable if, and only if, every eigenvalue has strictly negative real part.

Examples

1.
$$
A = \begin{pmatrix} -1+i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}
$$

0 is stable but not asymptotically.

2.
$$A = \begin{pmatrix} -1+i & 0 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{pmatrix}$$

0 is unstable.

Comment. If $f(t,x) = f(x)$ it suffices to consider $t_0 = 0$: suppose $\forall \varepsilon > 0 \; \exists \; \delta(\varepsilon) > 0$ s.t.

$$|x - \overline{x}| < \delta(\varepsilon) \Rightarrow |X(t, 0, x_0) - \overline{x}| < \varepsilon \;\; \forall \; t \geq 0.$$

Let $t_0 \geq 0$.

$$\begin{aligned} |x_0 - \overline{x}| < \delta(\varepsilon) \;\; &\Rightarrow \;\; |X(t, t_0, x_0) - \overline{x}| \\ &= \;\; |X(t - t_0, 0, x_0) - \overline{x}| < \varepsilon \;\; \forall \; t \geq t_0. \end{aligned}$$

ex check that

$$X(t, t_0, x_0) = X(t - t_0, 0, x_0).$$

Comment. 0 is stable for $\dot{X} = PMP^{-1}X$ if, and only if, it is for $\dot{Y} = MY$. Similarly for unstable and asymptotically stable.

Proof of Comment. Suppose 0 is stable for $\dot{Y} = MY$. $\forall \varepsilon > 0 \; \exists \; \delta(\varepsilon)$ s.t.

$$|Y(0)| < \delta(\varepsilon) \Rightarrow |Y(t)| < \varepsilon \forall \; t \geq 0.$$

Suppose $\dot{X} = PMP^{-1}X$ and $|X(0)| < \dfrac{\delta(\frac{\varepsilon}{|P|})}{|P^{-1}|}$.
Let $Y = P^{-1}X$ and note that $\dot{Y} = MY$ and

$$|Y(0)| \leq |P^{-1}||X(0)| < \delta\left(\frac{\varepsilon}{|P|}\right)$$

so

$$|Y(t)| < \frac{\varepsilon}{|P|} \quad \forall t \geq 0$$

and

$$|X(t)| = |PY(t)| \le |P||Y(t)| < \varepsilon \quad \forall t \ge 0.$$

Thus 0 is stable for $\dot{X} = PMP^{-1}X$.

Suppose 0 is stable for $\dot{X} = PMP^{-1}X$. Then from above 0 is stable for

$$\dot{Z} = (P^{-1})(PMP^{-1})(P^{-1})^{-1}Z = MZ.$$

Proof of Part 1 of Theorem Let $A = PJP^{-1}$ with

$$J = \begin{pmatrix} J_0 & & & 0 \\ & J_1 & & \\ & & \ddots & \\ 0 & & & J_m \end{pmatrix}$$

the Jordan form of A. Suppose

$$J_0 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_\ell \end{pmatrix}.$$

Assume real$(\lambda_i) \le 0$ for $i = 1, \ldots, \ell$ and all eigenvalues associated with $J_1, \ldots, J_m$ have strictly negative real part. Then

$$\begin{aligned} |e^{J_0 t}| &= \left| \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_\ell t} \end{pmatrix} \right| \\ &= \max\left\{ |e^{\lambda_1 t}|, \ldots, |e^{\lambda_\ell t}| \right\} \le 1. \end{aligned}$$

Let

$$\tilde{J} = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_m \end{pmatrix}$$

and

$$\tilde{\sigma} = \max\left\{ \text{real}(\lambda) : \lambda \text{ is an eigenvalue of } \tilde{J} \right\}.$$

Then $\tilde{\sigma} < 0$ and $\forall t \ge 0$

$$\left|e^{\tilde{J}t}\right| \le Ce^{\tilde{\sigma}t}(1+t^{N-1}) \le C_1.$$

Now for $|x| = 1$

$$\begin{aligned}
\left|e^{Jt}x\right| &= \left| \left(\begin{array}{c|c} e^{J_0 t} & 0 \\ \hline 0 & e^{\tilde{J}t} \end{array} \right) \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \right| \\
&\le \left|e^{J_0 t}\right| \sqrt{x_1^2 + \cdots x_\ell^2} + \left|e^{\tilde{J}t}\right| \sqrt{x_{\ell+1}^2 + \cdots + x_N^2} \\
&\le 1 + C_1.
\end{aligned}$$

Let $\varepsilon > 0$:

$$\begin{aligned}
|Y(t_0)| &< \frac{\varepsilon}{1+C_1} \Rightarrow |Y(t)| = \left|e^{J(t-t_0)}Y(t_0)\right| \\
&\le \left|e^{J(t-t_0)}\right| |Y(t_0)| < (C_1+1)\frac{\varepsilon}{C_1+1} = \varepsilon
\end{aligned}$$

so 0 is stable.

Suppose an eigenvector associated with $J_1, \ldots, J_m$ has non-negative real part; without loss of generality suppose

$$J_1 = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & 1 \\ 0 & & \ddots & \lambda \end{pmatrix}$$

with real$(\lambda) \ge 0$. $\forall \delta > 0$

$$\left| e^{J_1 t} \begin{pmatrix} 0 \\ \delta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| = \left| e^{\lambda t} \begin{pmatrix} 1 & \cdots & \frac{t^{k-1}}{(k-1)!} \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \delta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|$$

$$= e^{\text{real}(\lambda)t} \delta \left| \begin{pmatrix} t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| = e^{\text{real}(\lambda)t} \delta \sqrt{1+t^2}$$

$$\geq \delta \sqrt{t^2+1} \to +\infty \text{ as } t \Rightarrow +\infty.$$

Hence, 0 is unstable. Similarly if J_0 has an eigenvalue with positive real part.

B. Comparison with Linear Systems

Theorem 5.2. *Assume $f(t,x)$ is continuous and C^1 in x. Assume $\overline{x} \in \mathbb{R}^N$ s.t.*

$$f(t, \overline{x}) = 0 \qquad \forall t \geq 0.$$

Define A by

$$A_{ij} = \frac{\partial f_i}{\partial x_j}(t, \overline{x})$$

and assume this is independent of t. Assume all eigenvalues of A have negative real part and

$$\lim_{x \to \overline{x}} \frac{|f(t,x) - A(x - \overline{x})|}{|x - \overline{x}|} = 0$$

uniformly in $t \geq 0$. Then $\overline{x}$ is asymptotically stable for $\dot{X} = f(t, X)$.

Examples

1. $f(x) = \begin{pmatrix} x_1^2 + x_2 \\ \\ x_1 - x_2 \end{pmatrix}$

$$\text{equilibria} \quad \begin{array}{ll} x_1^2 + x_2 = 0 & x_1^2 + x_1 = 0 = x_1(x_1+1) \\ x_1 - x_2 = 0 & \\ & x_2 = x_1 = 0 \text{ or } -1 \end{array}$$

$$Df = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 1 \\ 1 & -1 \end{pmatrix}$$

(a) at $\begin{pmatrix} 0 \\ 0 \end{pmatrix} : A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$\det(A-\lambda I) = (-\lambda)(-\lambda-1)-1 = \lambda^2+\lambda-1 = 0 \Leftrightarrow \lambda = \frac{1}{2}(-1\pm\sqrt{5})$

$\dfrac{\sqrt{5}-1}{2} \geq 0$ so this theorem does not apply.

(b) at $\begin{pmatrix} -1 \\ -1 \end{pmatrix} : A = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$

$$\det(A - \lambda I) = (-2-\lambda)(-1-\lambda) - 1 = \lambda^2 + 3\lambda + 1$$

$$= \quad 0 \Leftrightarrow \lambda = \frac{1}{2}(-3 \pm \sqrt{5}) < 0$$

$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ is asymptotically stable.

2. $\dot{X} = f(t, X)$ where $f(t,x) = -x + g(t)x^2$

$\overline{x} = 0 : f(t,0) = 0 \quad \forall t$

$$A = \frac{\partial f}{\partial x}(t,0) = -1 + 2g(t)x\Big|_0 = -1 < 0$$

$$\frac{|f(t,\overline{x}) - A(x-\overline{x})|}{|x-\overline{x}|} = \frac{|-x+g(t)x^2 - (-1)x|}{|x|} = |g(t)|\ |x|$$

$\to 0$ as $x \to \overline{x} = 0$ uniformly in $t \Leftrightarrow g$ is bounded. So g bounded implies 0 is asymptotically stable.

Note: consider $g(t) = e^t$ then

$$\frac{d}{dt}(e^t X) = e^t(\dot{X} + X) = e^t g(t) X^2 = (e^t X)^2$$

so

$$e^t X(t) = \frac{1}{\frac{1}{X(0)} - t} \text{ if } X(0) \neq 0$$

and

$$X(0) > 0 \Rightarrow X(t) \to +\infty \text{ as } t \to \frac{1}{X(0)}.$$

Hence 0 is unstable.

3. (a) $\dot{X} = -X^3$ 0 is asymptotically stable.
   (b) $\dot{X} = \ \ X^3$ 0 is unstable.

   In both (a) and (b)

$$A = \frac{\partial f}{\partial x}\Big|_0 = 0.$$

Proof. Let

$$\sigma = \max\{\text{real}(\lambda) :\ \lambda \text{ is an eigenvalue of } A\}$$

then $\exists C > 0$ s.t.

$$|e^{At}| \le Ce^{(\sigma+\frac{1}{2}|\sigma|)t} = Ce^{-\frac{1}{2}|\sigma|t}$$

$\forall t \ge 0$. Let $\dot{X} = f(t,X),\ X(t_0) = x_0$, then

$$\frac{d}{dt}(X-\overline{x}) = A(X-\overline{x}) + (f(t,X) - A(X-\overline{x}))$$

so

$$\begin{aligned}
|X(t)-\overline{x}| &= \left| e^{A(t-t_0)}(x_0-\overline{x}) + \int_{t_0}^{t} e^{A(t-s)}\left[f(s,X(s)) - A(X(s)-\overline{x})\right] ds \right| \\
&\le \left|e^{A(t-t_0)}\right| |x_0-\overline{x}| + \int_{t_0}^{t} \left|e^{A(t-s)}\right| |f(s,X(s)) - A(X(s)-\overline{x})|\, ds \\
&\le Ce^{-\frac{1}{2}|\sigma|(t-t_0)}|x_0-\overline{x}| + \int_{t_0}^{t} Ce^{-\frac{1}{2}|\sigma|(t-s)}\, |f(s,X(s)) - A(X(s)-\overline{x})|\, ds.
\end{aligned}$$

Choose $\eta > 0$ s.t.

$$0 < |x-\overline{x}| \le \eta \text{ and } t \ge 0 \Rightarrow \frac{|f(t,x) - A(x-\overline{x})|}{|x-\overline{x}|} < \frac{|\sigma|}{4C}.$$

Consider $|x_0-\overline{x}| < \dfrac{\eta}{C+1}$ and let

$$T = \sup\left\{t > t_0 : |X(s)-\overline{x}| \le \eta \text{ on } [t_0,t]\right\}.$$

For $t_0 \le t < T$ and for $t = T$ if T is finite

$$e^{\frac{1}{2}|\sigma|(t-t_0)}\, |X(t)-\overline{x}| \le C|x_0-\overline{x}| + C\int_{t_0}^{t} e^{\frac{1}{2}|\sigma|(s-t_0)}\frac{\sigma}{4C}\, |X(s)-\overline{x}|\, ds$$

so by Gronwall

$$e^{\frac{1}{2}|\sigma|(t-t_0)}\, |X(t)-\overline{x}| \le C|x_0-\overline{x}|e^{\frac{|\sigma|}{4}(t-t_0)}$$

and

(*) $\qquad |X(t)-\overline{x}| \le C\, |x_0-\overline{x}|\, e^{-\frac{1}{4}|\sigma|(t-t_0)}.$

If T is finite then

$$|X(T) - \overline{x}| \leq C\frac{\eta}{C+1}e^{-\frac{1}{4}|\sigma|(T-t_0)} < \eta$$

which (using X continuous) contradicts the definition of T; hence, $T = +\infty$ and (*) holds $\forall t \geq t_0$.

Let $\varepsilon > 0$. Take $\delta = \dfrac{\min(\eta, \varepsilon)}{C+1}$. For $|x_0 - \overline{x}| < \delta,\ X(t) \to \overline{x}$ as $t \to +\infty$ and

$$|X(t) - \overline{x}| \leq C|x_0 - \overline{x}| < C\frac{\varepsilon}{C+1} < \varepsilon$$

$\forall t \geq t_0$. Therefore $\overline{x}$ is asymptotically stable. $\square$

Theorem 5.3. *Assume $f : B_r(\overline{x}) \to \mathbb{R}^N$ is C^2 for some $\overline{x} \in \mathbb{R}^N$ and $r > 0$ with*

$$f(\overline{x}) = 0.$$

If $A = Df(\overline{x})$ has an eigenvalue with positive real part then 0 is unstable.

Lemma 5.1. *Let $a > 0,\ b > 0,\ \sigma > 0$ and $S_0 : [0, \infty) \to \mathbb{R}$ be continuous and satisfy*

$$0 \leq S_0(t) \leq a + \int_0^t be^{\sigma s}S_0^2(s)\,ds \quad \forall t \geq 0.$$

Then

$$\frac{ab}{\sigma}e^{\sigma t} < 1 \Rightarrow S_0(t) \leq \frac{a}{1 - \frac{ab}{\sigma}e^{\sigma t}}$$

and

$$\frac{ab}{\sigma}e^{\sigma t} < \frac{1}{2} \Rightarrow S_0(t) \leq 2a.$$

Proof. Let

$$R(t) = a + \int_0^t be^{\sigma s}S_0^2(s)\,ds$$

then

$$\dot{R}(t) = be^{\sigma t} S_0^2(t) \le b\; e^{\sigma t} R^2(t)$$

so

$$\frac{d}{dt} R^{-1}(t) = \frac{-\dot{R}(t)}{R^2(t)} \ge -be^{\sigma t}$$

and

$$\begin{aligned} R^{-1}(t) \ge R^{-1}(0) - \int_0^t b\; e^{\sigma s} ds &= a^{-1} - \frac{b}{\sigma}\;(e^{\sigma t} - 1) \\ &\ge a^{-1} - \sigma^{-1} b\; e^{\sigma t}. \end{aligned}$$

For $ab\; \sigma^{-1} e^{\sigma t} < 1$,

$$a^{-1} - \sigma^{-1} b\; e^{\sigma t} > 0$$

so

$$S_0(t) \le R(t) \le \frac{1}{a^{-1} - \sigma^{-1} b\, e^{\sigma t}} = \frac{a}{1 - \frac{ab}{\sigma} e^{\sigma t}}$$

and the lemma follows. $\square$

Restricted Proof of Theorem

We'll assume A is diagonalizable and that

$$|f(x) - A(x - \overline{x})| \le C_0 |x - \overline{x}|^2 \qquad \forall x \in \mathbb{R}^N.$$

Since f is independent of t it is sufficient to consider $t_0 = 0$. Choose P invertible s.t. $A = PDP^{-1}$ where

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

and

$$r\,e(\lambda_1) \ge r\,e(\lambda_i) \qquad \forall i.$$

Note that

$$\begin{aligned} |e^{Dt}| &= e^{r\,e(\lambda_1)t}, \\ |e^{At}| &\leq |P|\,|P^{-1}|e^{r\,e(\lambda_1)t}, \end{aligned}$$

and $\exists v \in \mathbb{R}^N$ with $|v| = 1$ and $Av = \lambda_1 v$ so

$$|e^{At}| \geq |e^{At}v| = |e^{\lambda_1 t}v| = e^{r\,e(\lambda_1)t}.$$

Let $\sigma = r\,e(\lambda_1)$.

Consider

$$\begin{aligned} \dot{Y} &= f(Y) & Y(0) &= \overline{x} + \delta v \\ \dot{Z} &= A(Z - \overline{x}) & Z(0) &= Y(0). \end{aligned}$$

Let $S = Y - Z$. We'll bound $|S|$ using the lemma and use

$$\begin{aligned} |Y(t) - \overline{x}| &\geq |Z(t) - \overline{x}| - |S(t)| \\ (*) \qquad &= |e^{\lambda_1 t}\delta v| - |S(t)| \\ &= \delta\, e^{\sigma t} - |S(t)| \end{aligned}$$

to show $\exists C > 0$ s.t.

$$\sup_{t\geq 0}|Y(t) - \overline{x}| \geq C$$

for all δ near 0.

Bounding $|S(t)|$

$$\frac{d}{dt}(Y - \overline{x}) = f(Y) = A(Y - \overline{x}) + (f(Y) - A(Y - \overline{x}))$$

so

$$
\begin{aligned}
|S(t)| &= |(Y(t)-\overline{x})-(Z(t)-\overline{x})| \\
&= |e^{At}(Y(0)-\overline{x})+\int_0^t e^{A(t-s)}(f(Y(s))-A(Y(s)-\overline{x}))\,ds-e^{At}(Z(0)-\overline{x})| \\
&\leq \int_0^t |e^{A(t-s)}||f(Y(s))-A(Y(s)-\overline{x})|\,ds \\
&\leq \int_0^t |P|\;|P^{-1}|e^{\sigma(t-s)}C_0|Y(s)-\overline{x}|^2\,ds.
\end{aligned}
$$

But

$$
\begin{aligned}
|Y-\overline{x}|^2 &= |Z+S-\overline{x}|^2 \leq (|Z-\overline{x}|+|S|)^2 \\
&= |Z-\overline{x}|^2+2|Z-\overline{x}||S|+|S|^2 \\
&\leq 2|Z-\overline{x}|^2+2|S|^2 \\
&= 2e^{2\sigma s}|Z(0)-\overline{x}|^2+2|S|^2 \\
&= 2e^{2\sigma s}\delta^2+2|S|^2
\end{aligned}
$$

so

$$
|S(t)| \leq 2|P|\,|P^{-1}|C_0\int_0^t e^{\sigma(t-s)}\left(e^{2\sigma s}\delta^2+|S(s)|^2\right)\,ds.
$$

Let

$$
C_1 = 2|P|\,|P^{-1}|C_0\sigma^{-1}
$$

then

$$
\begin{aligned}
e^{-\sigma t}|S(t)| &\leq C_1\sigma\int_0^t \left(\delta^2 e^{\sigma s}+e^{-\sigma s}|S(s)|^2\right)\,ds \\
&= C_1\left[\delta^2(e^{\sigma t}-1)+\sigma\int_0^t e^{\sigma s}\left|e^{-\sigma s}S(s)\right|^2\,ds\right].
\end{aligned}
$$

Hence, for $0 \le t \le T$

$$e^{-\sigma t}|S(t)| \le C_1 \delta^2 e^{\sigma T} + C_1 \sigma \int_0^t e^{\sigma s} |e^{-\sigma s} S(s)|^2 \, ds$$

and by the lemma

$$e^{-\sigma t}|S(t)| \le 2C_1 \delta^2 e^{\sigma T}$$

on $[0, T)$ if

$$(C_1 \delta^2 e^{\sigma T})(C_1 \sigma)\sigma^{-1} e^{\sigma T} \le \frac{1}{2},$$

i.e.,

$$\begin{aligned} (C_1 \delta e^{\sigma T})^2 &\le \frac{1}{2}, \\ C_1 \delta e^{\sigma T} &\le \frac{1}{\sqrt{2}}. \end{aligned}$$

Collect the Parts

If $C_1 \delta \, e^{\sigma T} = \dfrac{1}{3}$ then

$$|S(T)| \le 2C_1 \delta^2 e^{2\sigma T} = \frac{2}{9C_1}$$

and by (*)

$$\begin{aligned} |Y(T) - \overline{x}| &\ge \delta \, e^{\sigma T} - |S(T)| \\ &\ge \frac{1}{3C_1} - \frac{2}{9C_1} = \frac{1}{9C_1}. \end{aligned}$$

So for $0 < \delta < \dfrac{1}{3C_1}, \exists T > 0$ s.t.

$$C_1 \delta \, e^{\sigma T} = \frac{1}{3}$$

(namely, $T = \sigma^{-1} \ln \left(\dfrac{1}{3C_1 \delta} \right)$) and hence,

$$|Y(T) - \overline{x}| \geq \frac{1}{9C_1}.$$

Lyapunov Functions
A Class of Examples

$$\ddot{X} = -U'(X)$$

or letting $Y = \begin{pmatrix} X \\ \dot{X} \end{pmatrix}$ and $f(y) = \begin{pmatrix} y_2 \\ -U'(y_1) \end{pmatrix}$

$$\dot{Y} = f(Y).$$

Note that

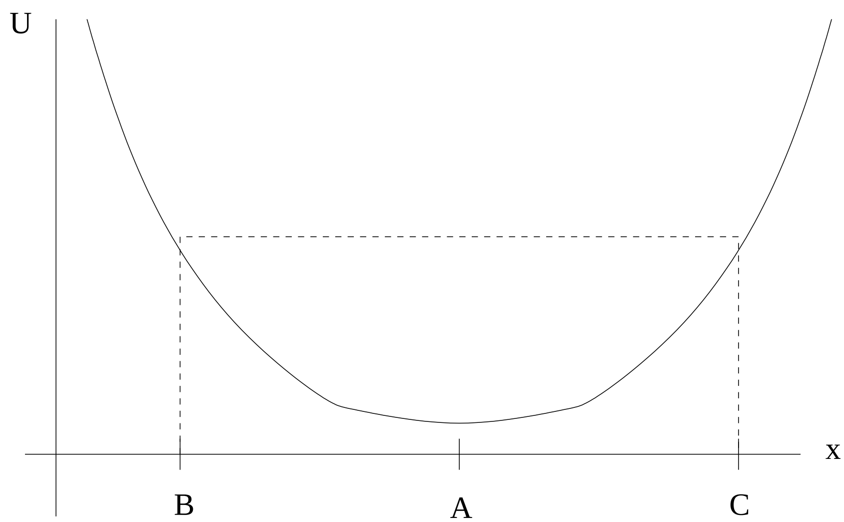
$$\frac{d}{dt}\left[\frac{1}{2}\dot{X}^2 + U(X)\right] = \dot{X}\ddot{X} + U'(X)\dot{X} = 0$$

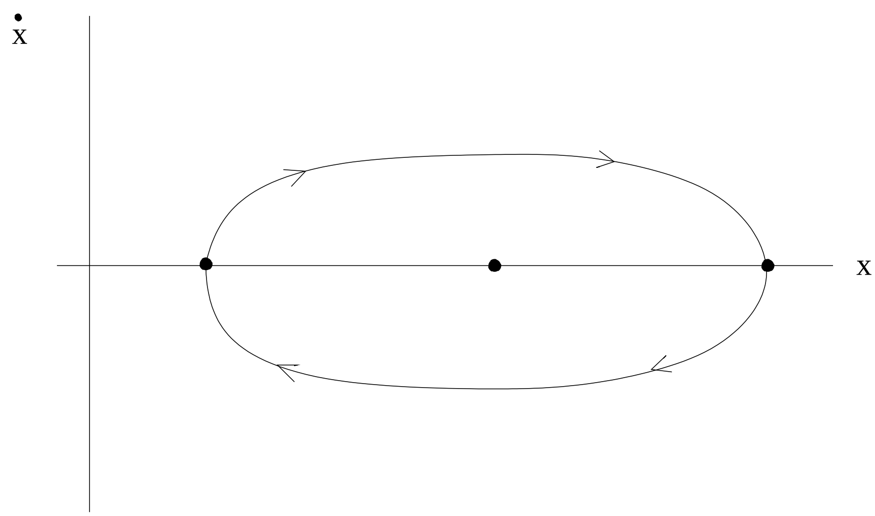
so

$$\left.\left[\frac{1}{2}\dot{X}^2 + U(X)\right]\right|_t = \left.\left[\frac{1}{2}\dot{X}^2 + U(X)\right]\right|_0.$$

Comment: $\overline{y} = \begin{pmatrix} \overline{y}_1 \\ \overline{y}_2 \end{pmatrix}$ is an equilibrium point iff $\overline{y}_2 = 0$ and $U'(\overline{y}_1) = 0$.

Examples

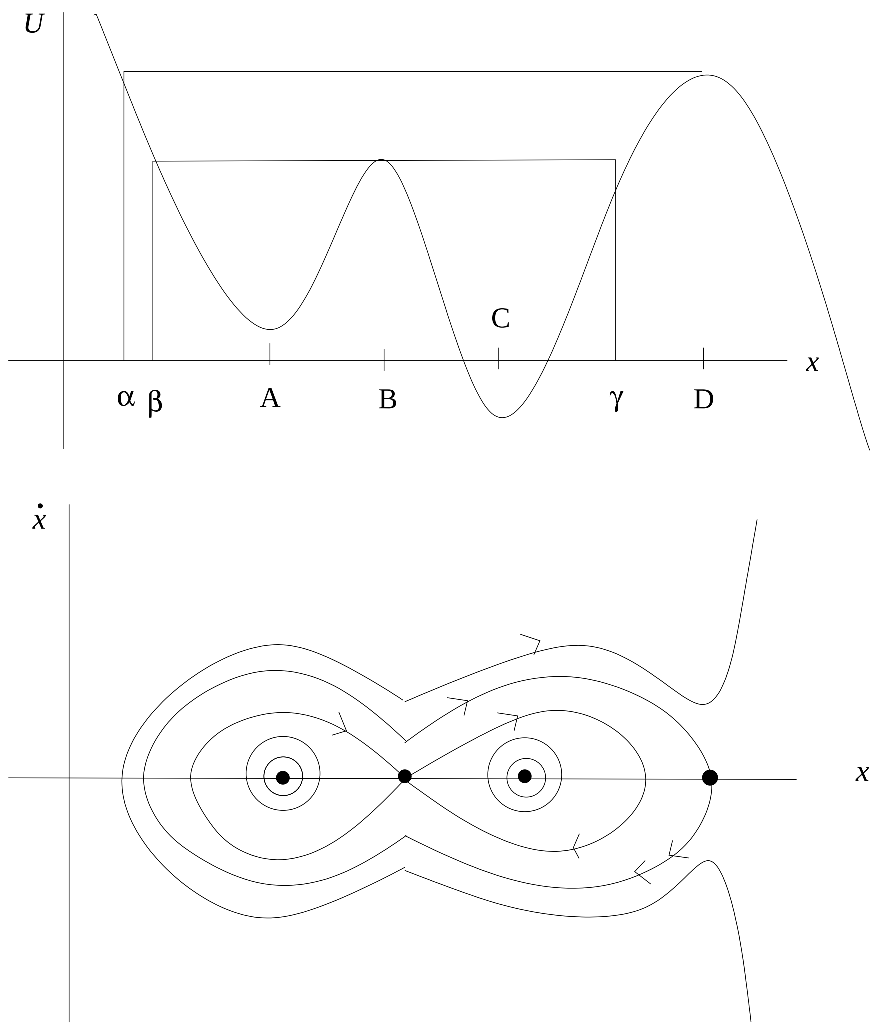




Suppose $X(0) = B$ and $\dot{X}(0) = 0$, then

$$\frac{1}{2}\dot{X}^2(t) + U(X(t)) = U(B) \quad \forall t.$$

Note that $\begin{pmatrix} A \\ 0 \end{pmatrix}$ is stable.



Definitions

Let $R \in (0,\infty], B_R = \left\{x \in \mathbb{R}^N : |x| < R\right\},$ and $w : B_R \to \mathbb{R}$ be continuous.

1. w is positive definite if $w(0) = 0$ and $\exists r > 0$ s.t. $0 < |x| < r \Rightarrow w(x) > 0$.

2. w is positive semidefinite if $w(0) = 0$ and $\exists r > 0$ s.t. $0 < |x| < r \Rightarrow w(x) \geq 0$.

3. w is negative definite if $-w$ is positive definite. Similarly for negative semidefinite.

4. Suppose $w(0) = 0$. w is indefinite of $\forall r > 0\ \exists x_r, y_r$ with $|x_r| < r,\ |y_r| < r$, and

$$w(x_r) < 0 < w(y_r).$$

Comment. A real N by N matrix, A, is positive definite if and only if

$$x \mapsto x^T \forall x$$

is positive definite.

Examples

1. $$\begin{aligned} w(x_1, x_2) &= 100^{-100} x_1^6 - 1000 x_1^7 + 10^{10} |x_2|^{\frac{1}{2}} \\ &= 100^{-100} x_1^6 (1 - 100^{100} 1000 x_1) + 10^{10} |x_2|^{\frac{1}{2}} \quad \text{is positive definite.} \end{aligned}$$

2. $w(x_1, x_2) = (x_1 + x_2)^2 - (x_1 + x_2)^4$ is positive semidefinite but not positive definite since

$$w(x_1, -x_1) = 0 \qquad \forall x_1.$$

3. $w(x_1, x_2) = (x_1 + x_2)^2 - x_1^4$ is indefinite since $\forall x_2 \neq 0$

$$w(-x_2, x_2) = -x_2^4 < 0 < x_2^2 = w(0, x_2).$$

Definitions. Let B_R be as before and let $v : [0, \infty) \times B_R \to \mathbb{R}$ be continuous.

v is positive definite if $v(t, 0) = 0 \quad \forall t \geq 0$, and $\exists w : B_R \to \mathbb{R}$ that is positive definite s.t.

$$v(t, x) \geq w(x) \quad \forall t, x.$$

Similarly for positive semidefinite, negative definite, and negative semidefinite.

Example

$$v(t,x) = \frac{x^2}{1+t}$$

is positive semidefinite since $v \geq 0$ but is not positive definite since

$$\frac{x^2}{1+t} = v(t,x) \geq w(x) \quad \forall t, x$$

implies

$$0 = \lim_{t \to +\infty} \frac{x^2}{1+t} \geq w(x) \quad \forall x.$$

Comments

1. Suppose $v(t,x)$ is positive definite. Choose $w(x)$ positive definite with

$$v(t,x) \geq w(x) \quad \forall t \geq 0, x.$$

Choose $r > 0$ s.t. $0 < |x| \leq r \Rightarrow w(x) > 0$. Define

$$\overline{w}(\rho) = \min \{ w(x) : \rho \leq |x| \leq r \}.$$

Then $\overline{w}(0) = 0$, $\overline{w}$ is continuous and nondecreasing, and

$$0 < \overline{w}(|x|) \leq w(x) \leq v(t,x) \text{ if } 0 < |x| \leq r.$$

2. Let $v : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ be C^1 and

$$\dot{X}(t) = f(t, X(t)).$$

Then

$$\begin{aligned}
\frac{d}{dt}\left[v(t,X(t))\right] &= \frac{\partial v}{\partial t}(t,X(t)) \\
&\quad + \sum_{i=1}^{N} \frac{\partial v}{\partial x_i}(t,X(t))\frac{dX_i}{dt} \\
&= \frac{\partial v}{\partial t}(t,X(t)) + \nabla_x v(t,X(t)) \cdot \dot{X}(t) \\
&= \left(\frac{\partial v}{\partial t} + f \cdot \nabla_x v\right)\Big|_{(t,X(t))}.
\end{aligned}$$

Definition 5.1. *Given f*

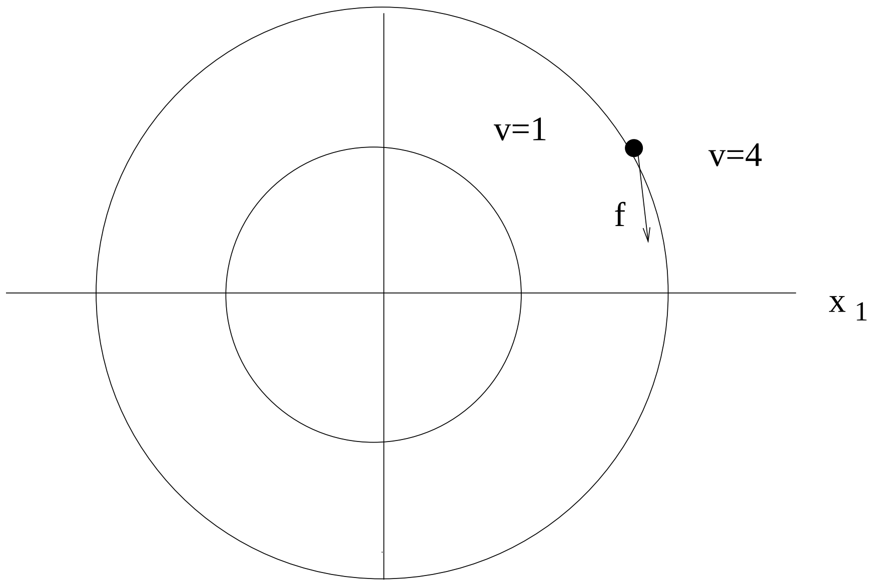
$$D_* v = \frac{\partial v}{\partial t} + f \cdot \nabla_x v.$$

Example

$$f(t,x) = \begin{pmatrix} x_2 \\ -x_1 - x_2 \end{pmatrix} \qquad v(t,x) = x_1^2 + x_2^2$$

(a)

$$\begin{aligned}
D_* v &= 0 + \begin{pmatrix} x_2 \\ -x_1 - x_2 \end{pmatrix} \cdot \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = x_2 2x_1 \\
&\quad + (-x_1 - x_2)2x_2 = -2x_2^2
\end{aligned}$$



(b) If $\dot{X} = f(t, X)$ then

$$\begin{aligned}
\frac{d}{dt} v(t, X(t)) &= \frac{d}{dt}\left(X_1^2 + X_2^2\right) \\
&= 2X_1\dot{X}_1 + 2X_2\dot{X}_2 \\
&= 2X_1X_2 + 2X_2(-X_1 - X_2) = -2X_2^2 \\
&= D_* v(t, X(t)).
\end{aligned}$$

Theorem 5.4. *Assume f is continuous with $f(t, 0) = 0 \ \ \forall t \geq 0$. If there is $v(t, x)$ which is positive definite, C^1, with D_*v negative semidefinite, then 0 is stable.*

Examples

1. $\dfrac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_2 \\ -X_1 - X_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

   Let $v(t, x) = x_1^2 + x_2^2$. v is positive definite and C^1.

$$D_*v = \nabla v \cdot f = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ -x_1 - x_2 \end{pmatrix} = -2x_2^2$$

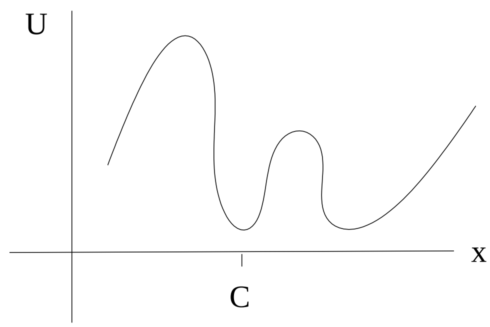
is negative semidefinite. By the theorem $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is stable.

In fact

$$\det\left(\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} - \lambda I\right) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 + \lambda + 1 = 0$$

$$\Leftrightarrow \quad \lambda = \frac{-1 \pm \sqrt{3}i}{2}$$

so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable.



2. $\ddot{X} = -U'(X)$ Claim $\begin{pmatrix} C \\ 0 \end{pmatrix}$ is stable:

Let $Y = \begin{pmatrix} X - C \\ \dot{X} \end{pmatrix}$ then:

$$\dot{Y} = f(Y) \text{ where } f(y) = \begin{pmatrix} y_2 \\ -U'(y_1 + C) \end{pmatrix}.$$

Let $v(t, y) = \frac{1}{2}y_2^2 + U(y_1 + C) - U(C), v$ is positive definite and

$$D_* v = \begin{pmatrix} U'(y_1) \\ y_2 \end{pmatrix} \cdot \begin{pmatrix} y_2 \\ -U'(y_1) \end{pmatrix} = 0.$$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is stable for $\dot{Y} = f(Y)$, $\begin{pmatrix} C \\ 0 \end{pmatrix}$ is stable for $\ddot{X} = -U'(X)$.

Proof. Choose $r > 0$ and $\overline{w} : [0, r] \to [0, \infty)$ continuous and nondecreasing with

$$0 < \overline{w}(|x|) \leq v(t, x) \quad \forall t \geq 0 \text{ and } 0 < |x| \leq r$$

and

$$D_* v(t,x) \leq 0 \quad \forall t \geq 0 \text{ and } |x| \leq r.$$

If

$$\dot{X} = f(t,X)$$

then

$$|X(t)| \leq r \Rightarrow \frac{d}{dt}\left[v(t,X(t))\right] = D_* v(t,X(t)) \leq 0.$$

Hence,

$$|X(s)| \leq r \forall s \in [t_0,t] \Rightarrow v(t_0,X(t_0)) \geq v(t,X(t)) \geq \overline{w}(|X(t)|).$$

Let $\varepsilon > 0$, without loss of generality take $\varepsilon < r$. Choose $\delta \in (0,\varepsilon)$ s.t.

$$|x| < \delta \Rightarrow v(t_0,x) = |v(t_0,x) - v(t_0,0)| < \overline{w}(\varepsilon).$$

Let $|X(t_0)| < \delta$ and define

$$T = \sup\left\{t \geq t_0 : |X(t)| < \varepsilon\right\}.$$

For $t \in [t_0,T)$ and $t = T$ if T is finite

$$\overline{w}(\varepsilon) > v(t_0,X(t_0)) \geq \overline{w}(|X(t)|)$$

so

$$\varepsilon > |X(t)|.$$

If T is finite

$$\varepsilon > |X(T)|$$

which yields a contradiction (by using $|X(t)|$ continuous and the definition of T). Hence, $T = +\infty$. Therefore 0 is stable. □

Lemma 5.2. *Let $w : \overline{B_r}(0) \to [0, \infty)$ and $X : [t_0, \infty) \to \mathbb{R}^N$ be continuous. Assume*

$$\begin{aligned} w(0) &= 0, \\ 0 < |x| \leq r &\Rightarrow 0 < w(x), \\ |X(t)| &\leq r \quad \forall t \geq t_0. \end{aligned}$$

Then $w(X(t))$ is bounded away from 0 if, and only if, $|X(t)|$ is too.

Proof. Suppose $|X(t)| \geq C_1 > 0 \quad \forall t \geq t_0$. Let

$$C_2 = \min \{ w(x) : C_1 \leq |x| \leq r \},$$

then

$$w(X(t)) \geq C_2 > 0 \quad \forall t \geq t_0.$$

Suppose

$$w(X(t)) \geq C_2 > 0 \quad \forall t \geq t_0.$$

Suppose $|X(t)|$ is not bounded away from 0 then $\exists t_k$ with $|X(t_k)| \to 0$. Then

$$C_2 \leq w(X(t_k)) \to 0,$$

contradiction. □

Theorem 5.5. *Assume f is continuous with $f(t, 0) = 0 \quad \forall t \geq 0$. Assume $v(t, x)$ is C^1 and positive definite with $D_* v$ negative definite. Also assume $\exists b(x)$ positive definite and $r > 0$ s.t.*

$$v(t, x) \leq b(x) \quad \forall t \geq 0, |x| < r.$$

Then 0 is asymptotically stable.

Examples

1.

$$\begin{aligned}\dot{X}_1 &= -X_1 + X_2 \\ \dot{X}_2 &= -X_1 - C_2 X_2\end{aligned}$$

$v(x) = x_1^2 + x_2^2$ is C^1 and positive definite

$$\begin{aligned}D_* v &= \begin{pmatrix} -x_1 + x_2 \\ -x_1 - C_2 x_2 \end{pmatrix} \cdot \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \\ &= -2x_1^2 + 2x_1 x_2 - 2x_2 x_1 - 2C_2 x_2^2 \\ &= -2x_1^2 - 2C_2 x_2^2\end{aligned}$$

is negative definite if $C_2 > 0$. So

$$C_2 > 0 \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is asymptotically stable}$$

and

$$C_2 = 0 \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is stable (by the previous theorem.)}$$

In fact, when $C_2 = 0$

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\det\begin{pmatrix} -1-\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda(\lambda+1) + 1 = 0 \Leftrightarrow \lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

so $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is actually asymptotically stable.

2.

$$\begin{aligned}\dot{X}_1 &= -X_1^3 + X_1 X_2^4 \\ \dot{X}_2 &= -X_1^4 X_2 - X_2^3 \\ \frac{d}{dt}\,\frac{X_1^4 + X_2^4}{4} &= X_1^3 \dot{X}_1 + X_2^3 \dot{X}_2 = -X_1^6 - X_2^6.\end{aligned}$$

Taking $v = \frac{1}{4}\left(x_1^4 + x_2^4\right)$ shows $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable.

Note: $f(x) = \begin{pmatrix} -x_1^3 + x_1 x_2^4 \\ \\ -x_1^4 x_2 - x_2^3 \end{pmatrix}$

so

$$Df(x) = \begin{pmatrix} -3x_1^2 + x_2^4 & 4x_1 x_2^3 \\ \\ -4x_1^3 x_2 & -x_1^4 - 3x_2^2 \end{pmatrix}$$

and

$$Df\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So comparison with a linear system yields no conclusion.

Proof. Choose $r > 0$ and w_2, w_3 s.t. $0 < |x| \le r$ and $t \ge 0 \Rightarrow$

$$\begin{aligned} b(x) \;\ge\;& v(t,x) \ge w_2(x) > 0 \\ & -D_* v(t,x) \ge w_3(x) > 0. \end{aligned}$$

By the previous theorem 0 is stable so $\forall t_0 \ge 0 \exists \delta > 0$ s.t. $|X(t_0)| < \delta \Rightarrow |X(t)| < r \;\; \forall t \ge t_0$. Take $|X(t_0)| < \delta$ and show

$$X(t) \to 0 \text{ as } t \to +\infty.$$

Note that

$$\frac{d}{dt}v(t, X(t)) = D_* v(t, X(t)) \leq 0$$

so $v(t, X(t))$ is decreasing and nonnegative. Let

$$L = \lim_{t \to +\infty} v(t, X(t))$$

and note that $L \geq 0$. Claim $L = 0$; suppose $L > 0$. Then

$$b(X(t)) \geq v(t, X(t)) \geq L > 0.$$

$b(0) = 0$ so $X(t)$ is bounded away from 0. Thus

$$\frac{d}{dt}v(t, X(t)) = D_* v(t, X(t)) \leq -w_3(X(t))$$

is bounded away from 0. This implies $v(t, X(t)) \to -\infty$, which is a contradiction. Thus $L = 0$. But

$$L = 0 \leftarrow v(t, X(t)) \geq w_2(X(t)) \geq 0$$

so

$$w_2(X(t)) \to 0$$

and hence,

$$X(t) \to 0$$

as $t \to +\infty$. $\square$

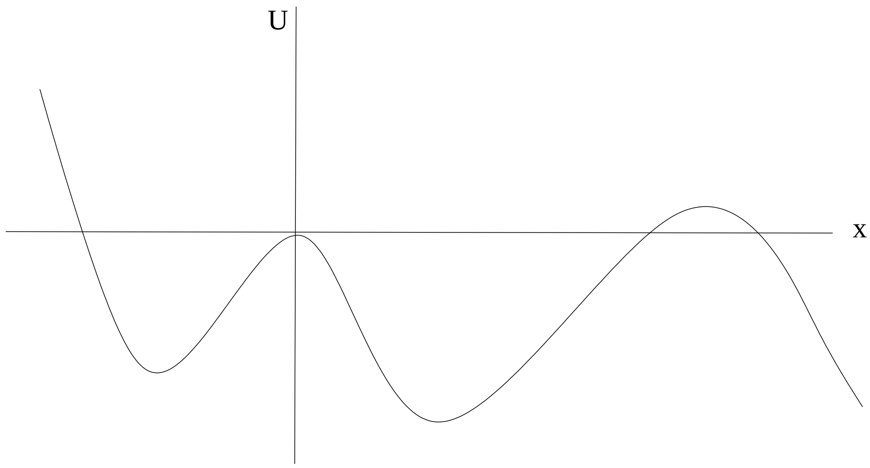
Theorem 5.6. Instability Theorem. *Assume f is continuous with $f(t, 0) = 0 \quad \forall t \geq 0$. Assume $v(t, x)$ is C^1 with $v(t, 0) = 0 \quad \forall t \geq 0$, $D_* v$ positive definite, and that every neighborhood of 0 has a point where $v(0, \cdot)$ is positive. Further assume there is $w(x)$, positive definite, with*

$$w(x) \geq v(t, x) \qquad \forall t \geq 0, x.$$

Then 0 is unstable.

Examples

1. Assume $U(x)$ is negative definite and C^1; note that $U(0) = U^{'}(0) = 0$.



Consider $\ddot{X} = -U^{'}(X)$

$$\frac{d}{dt}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Y_2 \\ -U'(Y_1) \end{pmatrix}.$$

Let

$$v(t, y) = y_1 y_2$$

then

$$\begin{aligned} D_* v &= \frac{\partial v}{\partial t} + f \cdot \nabla_y v = 0 + \begin{pmatrix} y_2 \\ -U'(y_1) \end{pmatrix} \cdot \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \\ &= y_2^2 - y_1 U'(y_1) \end{aligned}$$

is positive definite. Also

$$w(y) = y_1^2 + y_2^2 \geq v(t, y) \quad \forall t, y$$

and

$$v(0, y_1, y_1) = y_1^2 > 0 \text{ if } y_1 \neq 0.$$

Hence, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unstable.

2.
$$\begin{aligned} \dot{X}_1 &= C_1 X_1^3 + X_1 X_2 \\ \dot{X}_2 &= X_1^2 - C_2 X_2^3 \end{aligned} \qquad C_1, C_2 > 0$$

$$\begin{aligned} X_1 \dot{X}_1 - X_2 \dot{X}_2 &= C_1 X_1^4 + X_1^2 X_2 - X_2 X_1^2 + C_2 X_2^4 \\ \frac{d}{dt} \frac{1}{2} (X_1^2 - X_2^2) &= C_1 X_1^4 + C_2 X_2^4 \end{aligned}$$

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is unstable.

3. $\dot{X} = -X \qquad X(t) = X(t_0) e^{-(t-t_0)}$

0 is asymptotically stable.

Let

$$v(t, x) = e^{4t} x^2$$

then

$$D_* v = 4e^{4t} x^2 + e^{4t} 2x(-x) = 2e^{4t} x^2$$

is positive definite. The theorem does not apply since $\nexists$ $w(x)$ with

$$w(x) \geq v(t, x) \qquad \forall t \geq 0, x.$$

Proof. Choose $r > 0$ and $w_2(x)$ s.t. $w_2(0) = 0$ and

$$\begin{array}{ll} w(x) \geq v(t, x) & \text{if} \quad t \geq 0, |x| \leq r \\ D_* v(t, x) \geq w_2(x) > 0 & \text{if} \quad t \geq 0, \ 0 < |x| \leq r. \end{array}$$

Suppose 0 is stable. Choose $\delta > 0$ s.t. $|X(0)| < \delta \Rightarrow |X(t)| < r \ \ \forall t \geq 0$. Choose $X(0)$ with

$$0 < |X(0)| < \delta \text{ and } v(0, X(0)) > 0.$$

Then $\forall t \geq 0$

$$\frac{d}{dt} v(t, X(t)) = D_* v(t, X(t)) \geq w_2(X(t)) \geq 0$$

so

$$w(X(t)) \geq v(t, X(t)) \geq v(0, X(0)) > 0.$$

Since $w(0) = 0$ and w is continuous $X(t)$ must be bounded away from 0. Therefore,

$$\frac{d}{dt} v(t, X(t)) \geq w_2(X(t))$$

is bounded away from 0 and $v(t, X(t)) \to +\infty$ as $t \to +\infty$. But

$$\max_{|x| \leq r} w(x) \geq w(X(t)) \geq v(t, X(t)) \qquad \forall t$$

so this is a contradiction. Therefore, 0 is unstable. $\square$

Digression on the Two Body Problem

$$\begin{aligned} m_1 \ddot{X} &= m_1 m_2 G \frac{Y - X}{|Y - X|^3} \\ m_2 \ddot{Y} &= m_1 m_2 G \frac{X - Y}{|X - Y|^3}. \end{aligned}$$

Note that

$$C(t) = \frac{m_1 X + m_2 Y}{m_1 + m_2}$$

satisfies

$$\ddot{C} = 0$$

so

$$C(t) = C(0) + \dot{C}(0)t.$$

Define

$$S(t) = X(t) - Y(t)$$

and note that

$$\begin{aligned} C + \frac{m_2}{m_1+m_2}S &= \frac{m_1X + m_2Y + m_2(X-Y)}{m_1+m_2} = X \\ C - \frac{m_1}{m_1+m_2}S &= \frac{m_1X + m_2Y - m_1(X-Y)}{m_1+m_2} = Y. \end{aligned}$$

Also

$$\begin{aligned} \ddot{S} &= m_2G\frac{Y-X}{|Y-X|^3} - m_1G\frac{X-Y}{|X-Y|^3} \\ &= m_2G\frac{(-S)}{|S|^3} - m_1G\frac{S}{|S|^3} = -M\frac{S}{|S|^3} \end{aligned}$$

where

$$M := G(m_1 + m_2).$$

Choose coordinates so that

$$S_3(0) = \dot{S}_3(0) = 0.$$

Then

$$\ddot{S}_3 = -M\frac{S_3}{|S|^3}$$

so $S_3 \equiv 0$ by uniqueness.

Write

$$\langle S_1, S_2 \rangle = r(t)\langle \cos\theta(t), \sin\theta(t) \rangle$$

then

$$
\begin{aligned}
\ddot{S}_1 &= \dot{r}\cos\theta - r\sin\theta\dot{\theta}, \\
\ddot{S}_1 &= \ddot{r}\cos\theta - 2\dot{r}\sin\theta\dot{\theta} - r\cos\theta\dot{\theta}^2 - r\sin\theta\ddot{\theta}, \\
\dot{S}_2 &= \dot{r}\sin\theta + r\cos\theta\dot{\theta}, \\
\ddot{S}_2 &= \ddot{r}\sin\theta + 2\dot{r}\cos\theta\dot{\theta} - r\sin\theta\dot{\theta}^2 + r\cos\theta\ddot{\theta}
\end{aligned}
$$

so

$$
\begin{aligned}
\langle \ddot{S}_1, \ddot{S}_2 \rangle &= \left(\ddot{r} - r\dot{\theta}^2\right)\langle \cos\theta, \sin\theta\rangle + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\langle -\sin\theta, \cos\theta\rangle \\
&= -M\frac{\langle S_1, S_2\rangle}{|S|^3} = -Mr^{-2}\langle \cos\theta, \sin\theta\rangle.
\end{aligned}
$$

Hence,

$$\ddot{r} - r\dot{\theta}^2 = -Mr^{-2}$$

and

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

Note that

$$\frac{d}{dt}r^2\dot{\theta} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$$

so

$$L := r^2\dot{\theta} = \text{ constant (angular momentum).}$$

Next

$$r\dot{\theta}^2 = r^{-3}(r^2\dot{\theta})^2 = L^2 r^{-3}$$

so

$$\ddot{r} = L^2 r^{-3} - Mr^{-2}.$$

Comment. Let

$$U(\rho) = \frac{1}{2}L^2\rho^{-2} - M\rho^{-1}$$

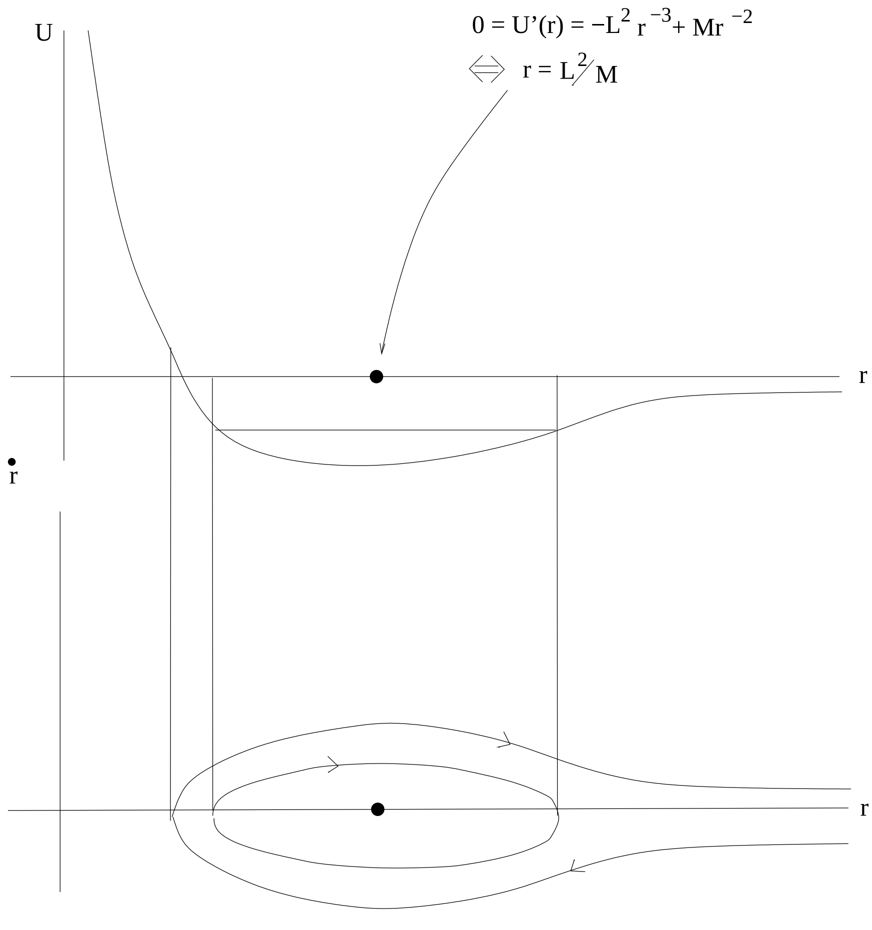
then

$$\ddot{r} = -U^{'}(r)$$

so

$$\frac{1}{2}\dot{r}^2 + U(r) = \text{ constant.}$$

Hence,



Conic Sections

Define R by $R(\theta(t)) = r(t)$:

$$\begin{aligned} \dot{r} &= R'(\theta)\dot{\theta} \\ \ddot{r} &= R''(\theta)\dot{\theta}^2 + R'(\theta)\ddot{\theta} \end{aligned}$$

Also

$$L = r^2\dot{\theta}$$

so

$$\dot{\theta} = Lr^{-2} = LR^{-2}(\theta)$$

and

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

so

$$\begin{aligned} \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} &= \frac{-2(R'(\theta)\dot{\theta})\dot{\theta}}{R(\theta)} \\ &= \frac{-2R'(\theta)}{R(\theta)}(LR^{-2}(\theta))^2 = \frac{-2L^2R'(\theta)}{R^5(\theta)}. \end{aligned}$$

Hence,

$$L^2R^{-3}(\theta) - MR^{-2}(\theta) = \ddot{r}$$

$$\begin{aligned} &= R''(\theta)\dot{\theta}^2 + R'(\theta)\ddot{\theta} \\ (*) \qquad &= R''(\theta)(LR^{-2}(\theta))^2 + R'(\theta)\left(\frac{-2L^2R'(\theta)}{R^5(\theta)}\right) \\ &= L^2R^{-4}(\theta)\left(R''(\theta) - 2\frac{(R'(\theta))^2}{R(\theta)}\right). \end{aligned}$$

In fact

$$R(\theta) = L^2M^{-1}(1 + A\cos(\theta - \theta_0))^{-1}$$

satisfies (*) $\forall A$ and θ_0.

$$\begin{aligned} A = 0 &\quad \text{Circle} \\ 0 < A < 1 &\quad \text{Ellipse} \\ A = 1 &\quad \text{Parabola} \\ 1 < A &\quad \text{Hyperbola.} \end{aligned}$$

Comment

Suppose $f(t,x)$ is continuous and $\overline{x} \in \mathbb{R}^N$ with $f(t,\overline{x}) = 0 \ \ \forall t \geq 0$. We may extend the previous three theorems by translation. We'll say $v(t,x)$ or $w(x)$ are positive definite about $\overline{x}$ if $v(t, x-\overline{x})$ or $w(x-\overline{x})$ is positive definite. Then the stability theorem becomes:

Theorem 5.7. *Assume $v(t,x)$ is C^1 with v positive definite about $\overline{x}$ and D_*v negative semidefinite about $\overline{x}$. Then $\overline{x}$ is stable.*

To prove this apply the previous theorem to

$$\begin{aligned} f^T(t,x) &:= f(t, x-\overline{x}) \\ v^T(t,x) &:= v(t, x-\overline{x}). \end{aligned}$$

D. Invariance Theory

Consider the autonomous equation

$$\dot{X} = f(X)$$

with $f \in C^1(\mathbb{R}^N)$. Note if X is a solution then $X(t\text{-constant})$ is too.

Definitions

1. Define $Y(t, x_0)$ by

$$\begin{cases} \dot{Y} = f(t, Y) \\ Y(0, x_0) = x_0. \end{cases}$$

Assume $Y(t, x_0)$ exists for all $t \geq 0$.

2. $C^+(x_0) = C^+ = \{Y(t, x_0) : t \geq 0\}$ is a positive semitrajectory. Similarly

$$C(x_0) = \{Y(t, x_0) : t \in \mathbb{R}\}.$$

3. Let $S \subset \mathbb{R}^N$. S is positively invariant if $x_0 \in S \Rightarrow Y(t, x_0) \in S \ \ \forall t \geq 0$. S is invariant if $x_0 \in S \Rightarrow Y(t, x_0) \in S \ \ \forall t \in \mathbb{R}$. Note $Y(t, x_0)$ must exist to satisfy $Y(t, x_0) \in S$.

4. The positive limit set is

$$\Omega(x_0) = \{\overline{x} : \exists \text{ a sequence } t_k \geq 0, \text{ with } t_k \to +\infty \text{ and } Y(t_k, x_0) \to \overline{x}\}.$$

Examples

1. For $\ddot{X} + X = 0$

$$C^+ = \left\{ \begin{pmatrix} X(0)\cos t + \dot{X}(0)\sin t \\ -X(0)\sin t + \dot{X}(0)\cos t \end{pmatrix} : t \geq 0 \right\} = \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : x^2 + v^2 = X^2(0) + \dot{X}^2(0) \right\}$$

$\Omega = C^+$

2. For $\ddot{X} + \varepsilon\dot{X} + X = 0$, with $\varepsilon > 0$, all solutions $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to +\infty$ so

$$\Omega = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Also

$$S = \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : x^2 + v^2 \leq C^2 \right\}$$

is positively invariant since

$$
\begin{aligned}
\frac{d}{dt}X^2 + \dot{X}^2 &= 2X\dot{X} + 2\dot{X}(-X - \varepsilon\dot{X}) \\
&= -2\varepsilon\dot{X}^2 \leq 0.
\end{aligned}
$$

Lemma 5.3. $\Omega(x_0)$ *is positively invariant.*

Proof. Let $\overline{x} \in \Omega(x_0)$ and $T \geq 0$, we must show $Y(T, \overline{x}) \in \Omega(x_0)$. $\overline{x} \in \Omega(x_0)$ so there is a sequence, $t_k \geq 0$, with $t_k \to +\infty$ and

$$Y(t_k, x_0) \to \overline{x}.$$

By continuity with respect to initial conditions

$$Y(T, Y(t_k, x_0)) \to Y(T, \overline{x}).$$

But

$$Y(T + t_k, x_0) = Y(T, Y(t_k, x_0)) \to Y(T, \overline{x})$$

so

$$Y(T, \overline{x}) \in \Omega(x_0).$$

□

Lemma 5.4. $\Omega(x_0)$ *is closed.*

Proof. Let $\{\overline{x}^{(\ell)}\}$ be a sequence of points in $\Omega(x_0)$ that converge to $\overline{x}$. We must show $\overline{x} \in \Omega(x_0)$. For each $\ell \in \mathbb{N}$ $\overline{x}^{(\ell)} \in \Omega(x_0)$, so $\exists$ a sequence, $\{t_k^{(\ell)}\}_{k=1}^{\infty}$, s.t. $t_k^{(\ell)} \geq 0 \ \ \forall k,\ t_k^{(\ell)} \to +\infty$ as $k \to \infty$, and $Y(t_k^{(\ell)}, x_0) \to \overline{x}^{(\ell)}$ as $k \to \infty$.

Choose k_1 s.t.

$$\left| Y(t_{k_1}^{(1)}, x_0) - \overline{x}^{(1)} \right| < 1.$$

Given $t_{k_\ell}^{(\ell)}$ choose $k_{\ell+1}$ s.t.

$$t_{k_{\ell+1}}^{(\ell+1)} > t_{k_\ell}^{(\ell)} + 1$$

and

$$\left|Y(t^{(\ell+1)}_{k_{\ell+1}}, x_0) - \overline{x}^{(\ell+1)}\right| < \frac{1}{\ell+1}.$$

Now $t^{(\ell)}_{k_\ell} \to +\infty$ and

$$\begin{aligned} & Y(t^{(\ell)}_{k_\ell}, x_0) \\ = \;& \overline{x}^{(\ell)} + \Big(Y(t^{(\ell)}_{k_\ell}, x_0) - \overline{x}^{(\ell)}\Big) \\ \to \;& \overline{x} + 0 = \overline{x}, \end{aligned}$$

so $\overline{x} \in \Omega(x_0)$. $\square$

Lemma 5.5. *If $C^+(x_0)$ is bounded then $\Omega(x_0)$ is nonempty, compact, and connected. Also*

$$\operatorname{dist}(Y(t, x_0), \Omega(x_0)) \to 0 \text{ as } t \to +\infty.$$

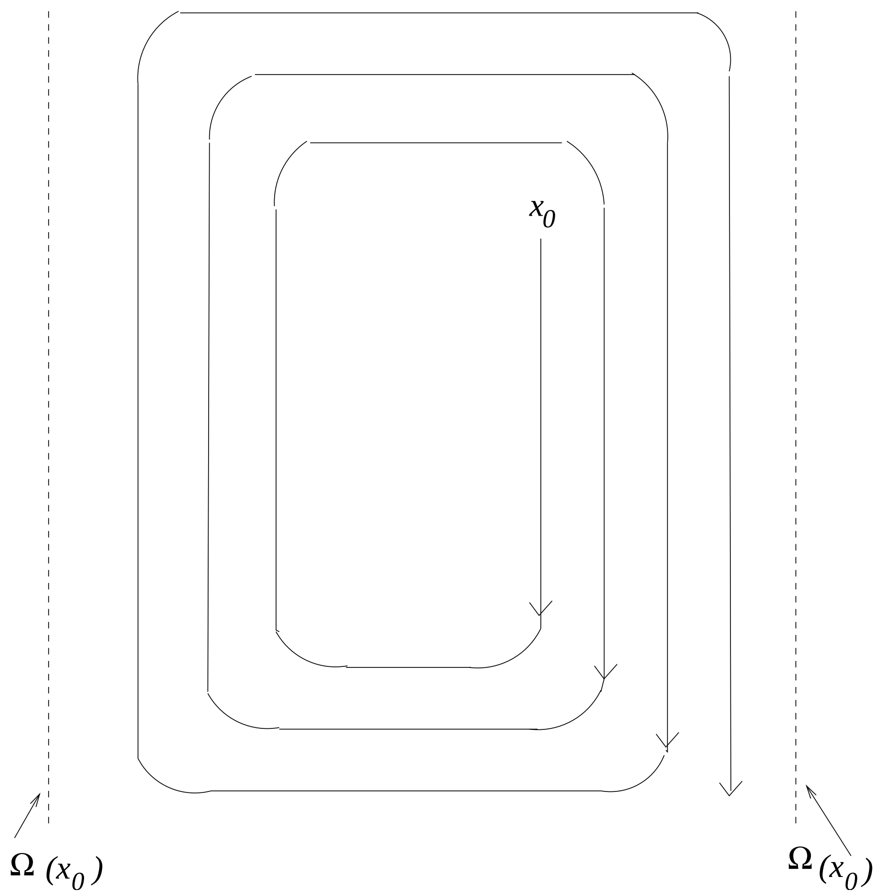
Note: for $x \in \mathbb{R}^N$ and $S \subset \mathbb{R}^N$ non-empty

$$\operatorname{dist}(x, S) := \inf\{|x - s| : s \in S\}.$$

Examples

1. $\dot{X} = 1 \qquad X(t) = X(0) + t \qquad \Omega(x_0) = \phi$

2.



$\Omega(x_0)$ is disconnected.

Comment.

Consider

$$\begin{aligned}\dot{X} &= f(X,Y)\\ \dot{Y} &= g(X,Y)\end{aligned}$$

Choose $r(t), \theta(t)$ s.t. $X = r\cos\theta,\ Y = r\sin\theta$.
Then

$$
\begin{aligned}
\dot{X} &= \dot{r}\cos\theta - r\sin\theta\dot{\theta} = f(r\cos\theta,\ r\sin\theta) \\
\dot{Y} &= \dot{r}\sin\theta + r\cos\theta\dot{\theta} = g(r\cos\theta,\ r\sin\theta)
\end{aligned}
$$

so

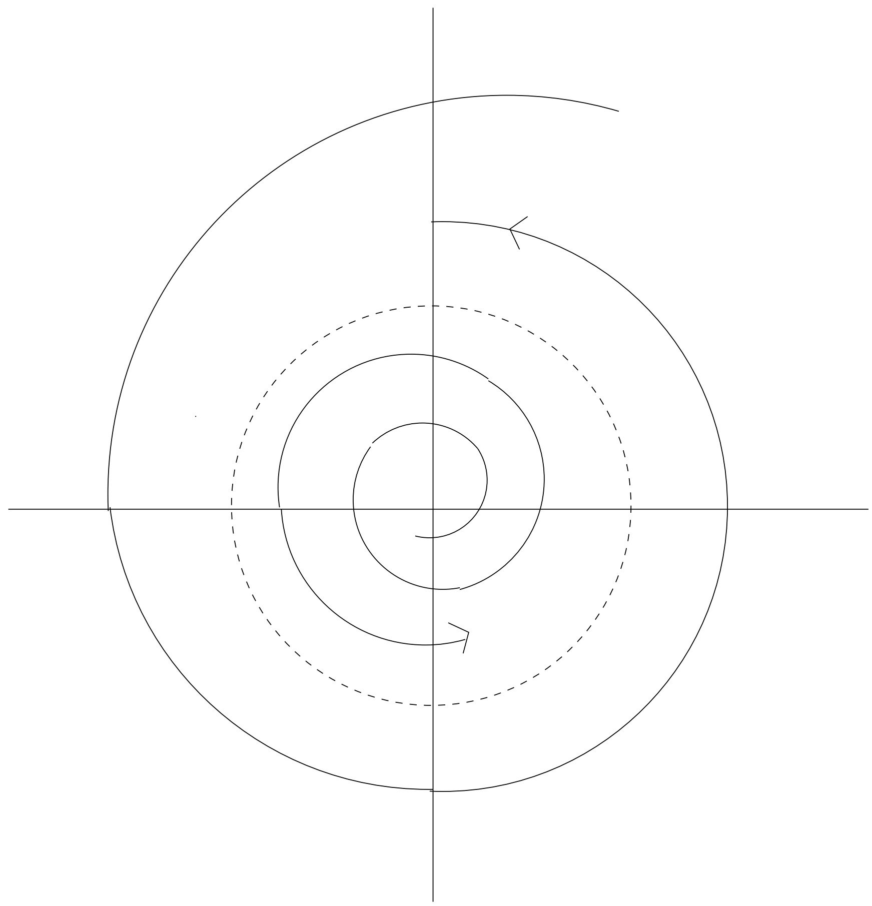
$$
\begin{aligned}
\dot{r} &= \cos\theta f(r\cos\theta,\ r\sin\theta) + \sin\theta g(r\cos\theta,\ r\sin\theta) \\
\dot{\theta} &= \frac{\cos\theta}{r} g(r\cos\theta,\ r\sin\theta) - \frac{\sin\theta}{r} f(r\cos\theta,\ r\sin\theta).
\end{aligned}
$$

Example

$$
\begin{aligned}
f(x,y) &= \frac{x}{x^2+y^2} - x - y \\
g(x,y) &= \frac{y}{x^2+y^2} + x - y
\end{aligned}
$$

leads to

$$
\begin{aligned}
\dot{r} &= \cos\theta\left(\frac{r\cos\theta}{r^2} - r\cos\theta - r\sin\theta\right) \\
&\quad + \sin\theta\left(\frac{r\sin\theta}{r^2} + r\cos\theta - r\sin\theta\right) = r^{-1} - r \\
\dot{\theta} &= \frac{\cos\theta}{r}\left(\frac{r\sin\theta}{r^2} + r\cos\theta - r\sin\theta\right) \\
&\quad - \frac{\sin\theta}{r}\left(\frac{r\cos\theta}{r^2} - r\cos\theta - r\sin\theta\right) = 1
\end{aligned}
$$



$$\Omega(x_0) = \left\{ \left(\begin{array}{c} y \\ z \end{array} \right) : y^2 + z^2 = 1 \right\} \text{ if } x_0 \neq 0.$$

Proof. $C^+(x_0) \supset \{Y(k, x_0) : k \in \mathbb{N}\}$ is bounded so $Y(k, x_0)$ has a convergent subsequence. It's limit is an element of $\Omega(x_0)$ so $\Omega(x_0) \neq \phi$.

Choose R s.t. $C^+(x_0) \subset \{x : |x| \leq R\}$. If $Y(t_k, x_0) \to \overline{x} \in \Omega(x_0)$ then

$$|Y(t_k, x_0)| \leq R \quad \forall k$$

so

$$|\overline{x}| \leq R.$$

Hence, $\Omega(x_0)$ is bounded. Since $\Omega(x_0)$ is closed, it's also compact.

Suppose $\Omega(x_0)$ is disconnected, then $\exists\, O_1, O_2 \subset \mathbb{R}^N$ open with

$$\begin{aligned} \Omega(x_0) &\subset O_1 \bigcup 0_2, \\ O_1 \bigcap O_2 &= \phi, \\ O_1 \bigcap \Omega(x_0) &\neq \phi, \\ O_2 \bigcap \Omega(x_0) &\neq \phi. \end{aligned}$$

We seek a contradiction. Choose $t_k \to \infty$ and $\tau_k \to \infty$ with

$$\begin{aligned} Y(t_k, x_0) &\to \overline{a} \in O_1 \bigcap \Omega(x_0) \\ Y(\tau_k, x_0) &\to \overline{b} \in O_2 \bigcap \Omega(x_0). \end{aligned}$$

Choose k_1 s.t.

$$Y(t_{k_1}, x_0) \in O_1.$$

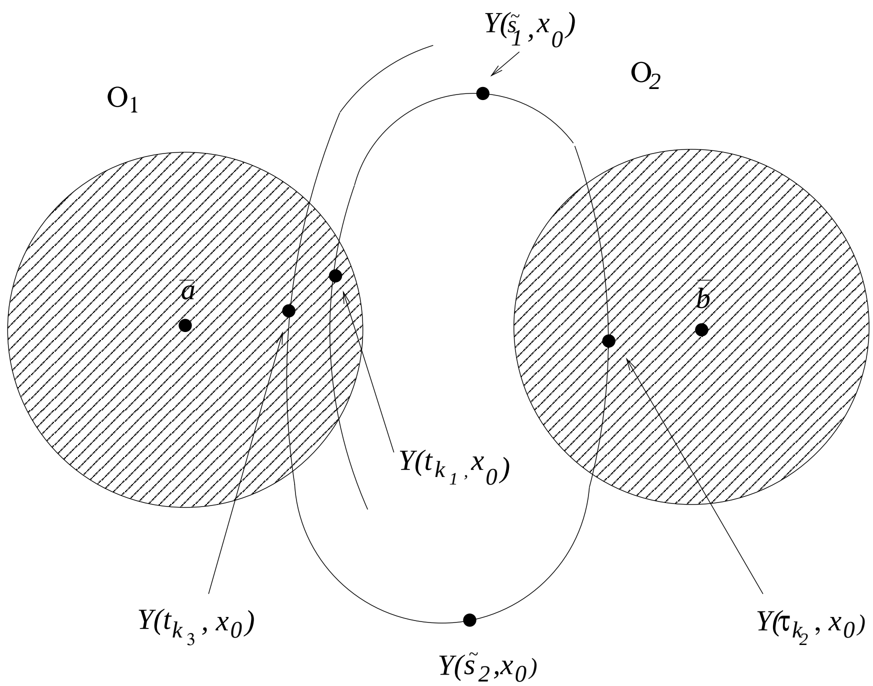
Choose k_2 s.t. $\tau_{k_2} > t_{k_1} + 1$ and

$$Y(\tau_{k_2}, x_0) \in O_2.$$

Choose k_3 s.t. $t_{k_3} > \tau_{k_2} + 1$ and

$$Y(t_{k_3}, x_0) \in O_1.$$

Continue.



Let $\tilde{s}_1 = t_{k_1}, s_2 = \tau_{k_2}, s_3 = t_{k_3}, \ldots \forall k \in \mathbb{N} \exists \tilde{s}_k \in (s_k, s_{k+1})$ s.t.

$$Y(\tilde{s}_k, x_0) \notin O_1 \bigcup O_2.$$

$Y(\tilde{s}_k, x_0)$ is bounded so it has a convergent subsequence. It's limit is in

$$\Omega(x_0) \bigcap \left(\mathbb{R}^N \backslash (O_1 \bigcup O_2) \right)$$

contradicting $\Omega(x_0) \subset O_1 \bigcup O_2$.

Finally, suppose

$$\inf \{ |Y(t, x_0) - \overline{x}| : \overline{x} \in \Omega(x_0) \} \not\to 0 \text{ as } t \to \infty$$

and seek a contradiction. $\exists \varepsilon > 0$ and $t_k \to +\infty$ s.t.

$$\inf \{ |Y(t_k, x_0) - \overline{x}| : \overline{x} \in \Omega(x_0) \} \geq \varepsilon \quad \forall k.$$

Then $\forall \overline{x} \in \Omega(x_0)$ and $\forall k$

$$(*) \qquad |Y(t_k, x_0) - \overline{x}| \geq \varepsilon.$$

$Y(t_k, x_0)$ is bounded so it has a convergent subsequence, $Y(t_{n_k}, x_0)$. Let

$$L = \lim Y(t_{n_k}, x_0),$$

then $L \in \Omega(x_0)$. But, by (*)

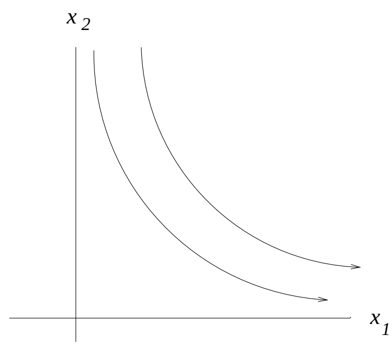
$$|Y(t_{n_k}, x_0) - L| \geq \varepsilon \quad \forall k,$$

contradition. $\square$

Definition 5.2. *For $S \subset \mathbb{R}^N$ define*

$$M = M_s = \{x_0 \in S : Y(t, x_0) \in S \quad \forall t \geq 0\}.$$

Example



$$\begin{aligned} \dot{X}_1 &= X_1 \\ \dot{X}_2 &= -X_2 \end{aligned}$$

$$\begin{aligned} S &= \{(x_1, x_2) : x_1 > 0 \text{ and } x_2 > 0\} \Rightarrow M = S \\ S &= \{(x_1, x_2) : x_1 > 0 \text{ and } x_2 > \varepsilon\} \text{ and } \varepsilon > 0 \Rightarrow M = \phi \end{aligned}$$

Comments

1. M is positively invariant:
   Let $x_0 \in M$, then $Y(t, x_0) \in S \;\; \forall t \geq 0$.
   Consider $T \geq 0$, then

$$Y(T + t, x_0) \in S \quad \forall t \geq -T,$$

$$Y(T + t, x_0) = Y(t, Y(T, x_0)) \in S \quad \forall t \geq 0,$$

so

$$Y(T, x_0) \in M.$$

2. Suppose $\tilde{M} \subset S$ is positively invariant, then $\tilde{M} \subset M$:

   Let $x_0 \in \tilde{M}$ then $Y(t, x_0) \in \tilde{M} \subset S \;\; \forall t \geq 0$. Hence, $x_0 \in M$.

3. M_s is the largest positively invariant subset of S.

Theorem 5.8. *Let S be open with $0 \in S$ and let $w \in C^1(S)$ with $w(0) = 0$ and*

$$D_* w \leq 0 \text{ on } S.$$

Let $\eta \geq 0$ and let H_η be the connected component of $\{x : w(x) \leq \eta\}$ that contains 0. Let M be the largest positively invariant subset of $H_\eta \bigcap \{x \in S : D_* w(x) = 0\}$. Assume H_η is bounded and that H_η is a closed subset of S, then $\forall x_0 \in H_\eta$ $\text{dist}(Y(t, x_0), M) \to 0$ as $t \to \infty$.

Examples

1. $$\begin{aligned} \dot{X}_1 &= -X_1 X_2^2 \\ \dot{X}_2 &= -X_1^2 X_2 \end{aligned}$$

   Let $w(x) = x_1^2 + x_2^2$ on $S = \mathbb{R}^2$, then

   $$D_* w = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \cdot \begin{pmatrix} -x_1 x_2^2 \\ -x_1^2 x_2 \end{pmatrix} = -4x_1^2 x_2^2 \leq 0.$$

   The set $\{x : w(x) \leq \eta\}$ is connected so

   $$H_\eta = \left\{x : x_1^2 + x_2^2 \leq \eta\right\}.$$

   The set $\{x : D_* w(x) = 0\} = \{x : x_1 = 0 \text{ or } x_2 = 0\}$ is invariant so

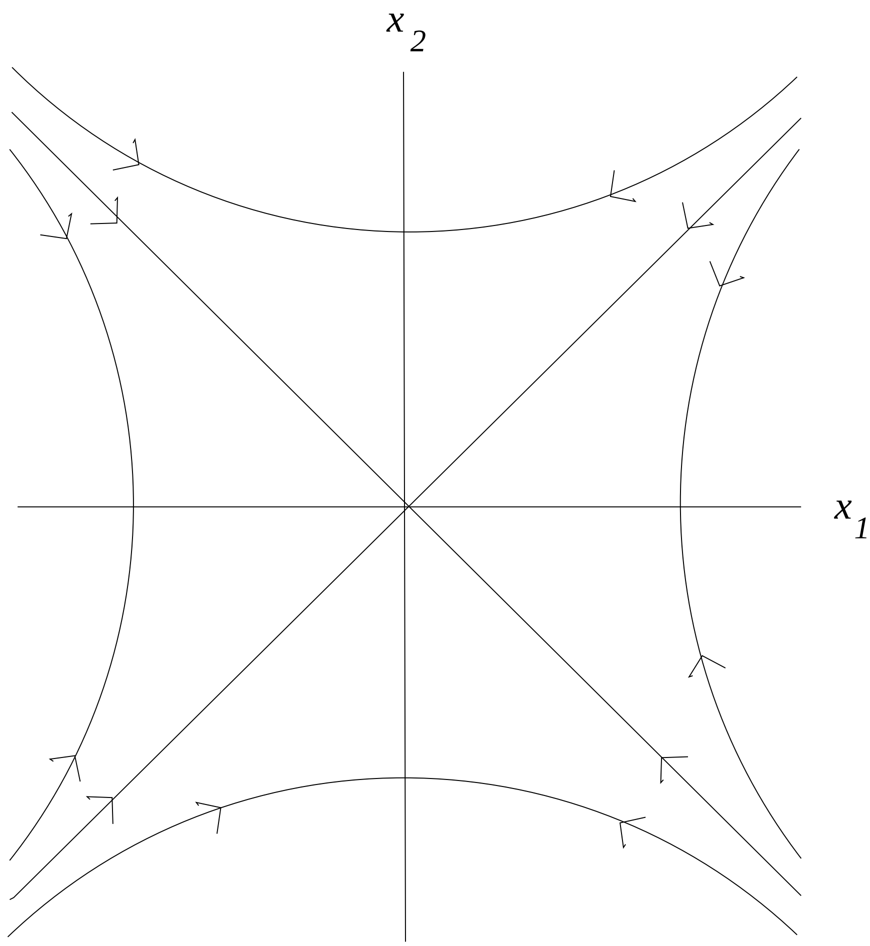
   $$M = H_\eta \bigcap \{x : x_1 = 0 \text{ or } x_2 = 0\}.$$

If $x_0 \in H_\eta$ then

$$\operatorname{dist}(Y(t, x_0), M) \to 0 \text{ as } t \to \infty.$$

In fact

$$\frac{d}{dt}\left(X_1^2 - X_2^2\right) = 0.$$



2. $\ddot{X} + (X + \dot{X})^2 \dot{X} + X = 0$
   Let $V = \dot{X}$ then

$$\frac{d}{dt}\begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} V \\ -(X+V)^2V - X \end{pmatrix}.$$

Comment: We may prove $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is stable:

$V(t,x,v) = x^2 + v^2$ is positive definite

$$\begin{aligned} D_*V(t,x,v) &= \begin{pmatrix} 2x \\ 2v \end{pmatrix} \cdot \begin{pmatrix} v \\ -(x+v)^2v - x \end{pmatrix} \\ &= 2xv - 2(x+v)^2v^2 - 2xv = -2(x+v)^2v^2 \end{aligned}$$

is negative semi-definite.

We may show $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable by using the theorem of this section:

Let $w(x,v) = x^2 + v^2$ then

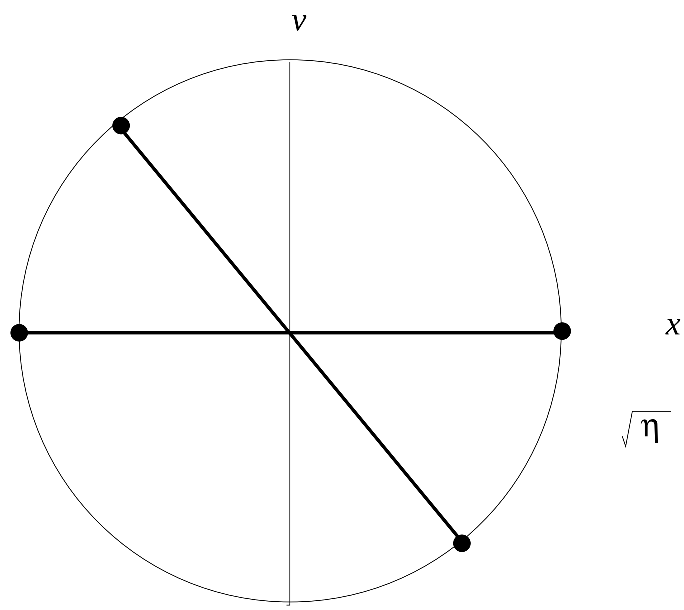
$$D_*w(x,v) = -2(x+v)^2v^2 \leq 0.$$

Let $\eta \geq 0$. The set $\left\{ \begin{pmatrix} x \\ v \end{pmatrix} : w(x,v) \leq \eta \right\}$ is connected so

$$H_\eta = \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : x^2 + v^2 \leq \eta \right\}.$$

Let M be the largest positively invariant subset of

$$\begin{aligned} & H_\eta \bigcap \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : D_*w(x,v) = 0 \right\} \\ = & \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : x^2 + v^2 \leq \eta \text{ and } (v = 0 \text{ or } x + v = 0) \right\}. \end{aligned}$$



Claim that $M = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$. Once this is known then

$$\operatorname{dist}\left(\begin{pmatrix} X \\ V \end{pmatrix}, M \right) = \sqrt{X^2 + V^2} \to 0 \text{ as } t \to +\infty$$

if $\begin{pmatrix} X(0) \\ V(0) \end{pmatrix} \in H_\eta$. Since $\eta \geq 0$ is arbitrary, it follows that all solutions $\to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable.

Proof. Suppose $\begin{pmatrix} X(0) \\ V(0) \end{pmatrix} \in M$, then $\begin{pmatrix} X(t) \\ V(t) \end{pmatrix} \in M \subset S \;\; \forall t \geq 0$. Suppose $V(t_0) \neq 0$ for some $t_0 \geq 0$ then $\exists \delta > 0$ s.t. $V(t) \neq 0$ on $t \in (t_0, t_0 + \delta)$. So for

$$X + V = 0 \qquad \text{on } (t_0, t_0 + \delta),$$

$$0 = \dot{X} + \dot{V} = V - (X+V)^2 V - X = X - V \quad \text{on } (t_0, t_0 + \delta),$$

and

$$X(t) = V(t) = 0 \text{ on } (t_0, t_0 + \delta).$$

Contradiction, hence, $V \equiv 0$. But now

$$0 = \dot{V} + (X+V)^2 V + X = X \qquad \forall t$$

so

$$X \equiv V \equiv 0.$$

$\square$

Lemma 5.6. *Let w be C^1 on an open set S with*

$$D_* w \leq 0 \text{ on } S.$$

Assume $C^+(x_0)$ is bounded with

$$C^+(x_0) \subset S \text{ and } \Omega(x_0) \subset S.$$

Then $\forall \overline{x} \in \Omega(x_0)$

$$D_* w(\overline{x}) = 0.$$

Proof. Let $\overline{x} \in \Omega(x_0)$ and choose $t_k \to +\infty$ with

$$Y(t_k, x_0) \to \overline{x}.$$

Claim that

$$w(Y(\tau, \overline{x})) = w(\overline{x}) \qquad \forall \tau \geq 0.$$

From this it follows that

$$0 = \frac{d}{d\tau} w(Y(\tau, \overline{x})) = D_* w(Y(\tau, \overline{x})) \qquad \forall \tau \geq 0$$

and taking $\tau = 0$

$$0 = D_* w(\overline{x}).$$

Note that since $D_* w \leq 0$,

$$w(\overline{x}) = w(Y(0, \overline{x})) \geq w(Y(\tau, \overline{x})) \qquad \forall \tau \geq 0$$

Next note that

$$w(Y(t_k+\tau,x_0)) = w(Y(\tau, Y(t_k,x_0))) \to w(Y(\tau,\overline{x})) \quad \forall \tau \geq 0.$$

$\forall k\ \exists \ell_k$ s.t. $t_{\ell_k} > t_k + \tau$ so

$$w(Y(t_k+\tau,x_0)) \geq w(Y(t_{\ell_k},x_0)).$$

But

$$w(Y(t_{\ell_k},x_0)) \to w(\overline{x})$$

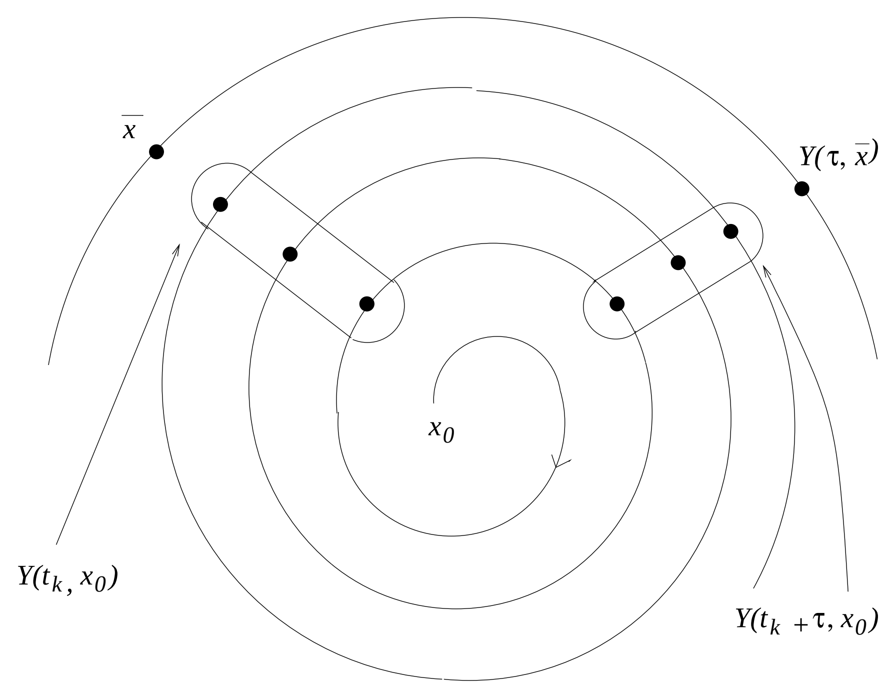
so

$$w(Y(\tau,\overline{x})) \geq w(\overline{x}).$$

Hence,

$$w(Y(\tau,\overline{x})) = w(\overline{x})$$

as claimed. □



Proof. First show that H_η is positively invariant: Let $x_0 \in H_\eta$ and

$$T = \sup \{t \geq 0 : Y(s, x_0) \in H_\eta \quad \forall s \in [0, t]\}.$$

Suppose T is finite. Since H_η is closed $Y(T, x_0) \in H_\eta \subset S$. Since S is open $\exists \delta > 0$ s.t. $Y(t, x_0)) \in S \quad \forall t \in [t_0, T + \delta]$. But now

$$w(Y(t, x_0)) \leq w(x_0) \leq \eta \quad \forall t \in [t_0, T + \delta]$$

and

$$Y(t, x_0) \in H_\eta \quad \forall t \in [t_0, T + \delta]$$

contradicting the definition of T. Hence, $T = +\infty$ and H_η is positively invariant. Note also that $\Omega(x_0) \subset H_\eta$ since H_η is closed.

By Lemma 5.3 $\Omega(x_0)$ is positively invariant. By Lemma 5.5 (and since H_η is bounded) $x_0 \in H_\eta \Rightarrow$

$$\text{dist}(Y(t, x_0), \Omega(x_0)) \to 0 \text{ as } t \to +\infty.$$

By Lemma 5.6 $D_* w = 0$ on $\Omega(x_0)$ so

$$\Omega(x_0) \subset M.$$

Hence

$$0 \leq \text{dist}(Y(t, x_0), M) \leq \text{dist}(Y(t, x_0), \Omega(x_0))$$

and hence,

$$\text{dist}(Y(t, x_0), M) \to 0.$$

□

Another Example

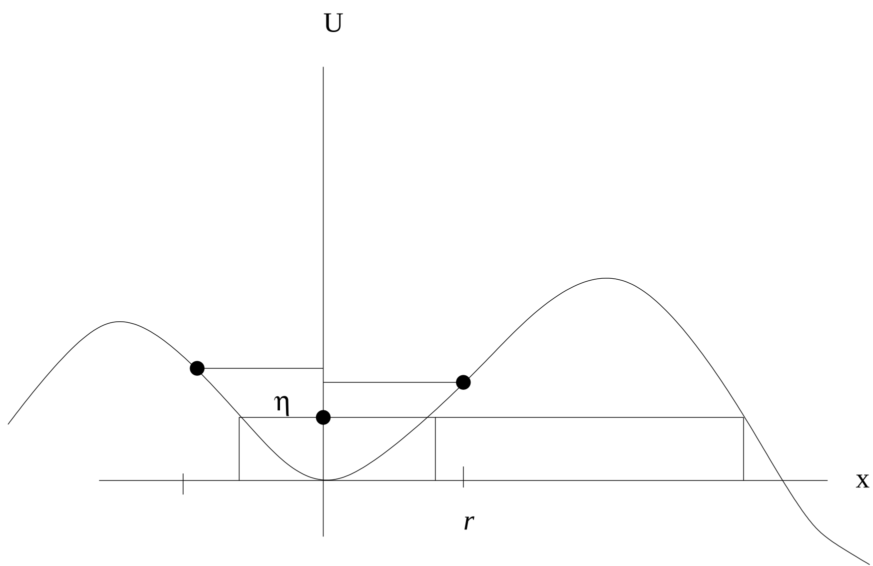
$$\ddot{X} + \sigma(X)\dot{X} + U'(X) = 0$$

Assume σ, U, U' are continuous,

$$\sigma(0) = U(0) = U'(0) = 0,$$

$\exists r > 0$ s.t.

$$0 < |x| < r \Rightarrow \sigma(x) > 0,\ U(x) > 0,\ xU'(x) > 0.$$



Let $w(x,v) = \dfrac{1}{2}v^2 + U(x)$ and $S = \{(x,v) : |x| < r\}$.
Then

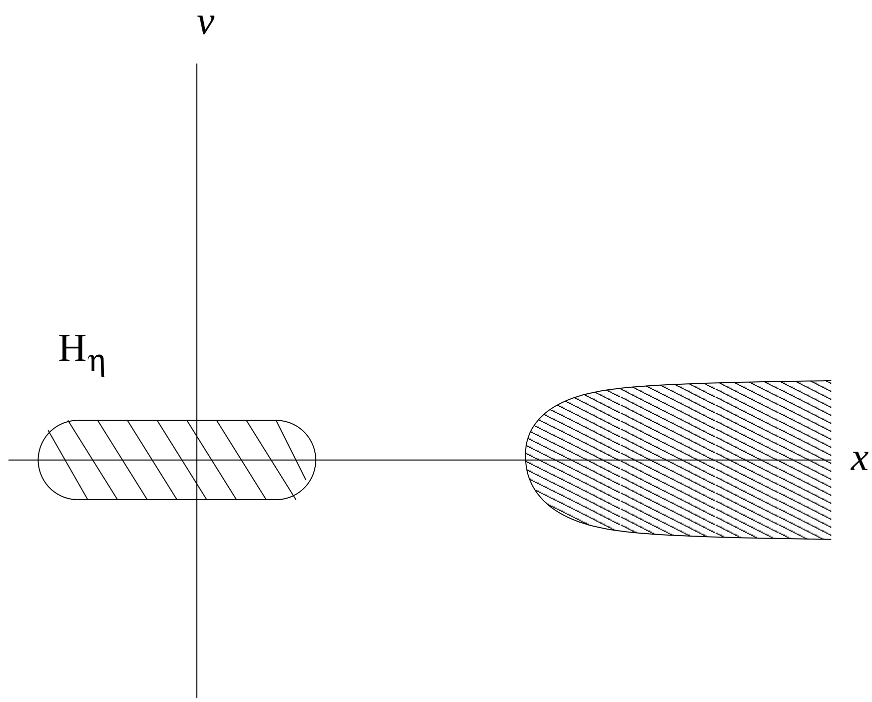
$$D_* w = v(-\sigma(x)v - U'(x)) + U'(x)v = -\sigma(x)v^2$$

which is negative semidefinite.

Let $0 < \eta < \min(U(r), U(-r))$. Note that $\left\{ \begin{pmatrix} x \\ v \end{pmatrix} : w(x,v) \le \eta \right\}$ could be unbounded but that

$$H_\eta \subset \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : |x| \le r \text{ and } \frac{1}{2}v^2 \le \eta \right\}$$

is bounded.



Let M be the largest positively invariant subset of

$$\begin{aligned} & H_\eta \bigcap \left\{ \begin{pmatrix} x \\ v \end{pmatrix} : D_* w(x,v) = 0 \right\} \\ = \ & H_\eta \bigcap \{(x,v) : x = 0 \text{ or } v = 0\}. \end{aligned}$$

Claim $M = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$. Once this is known $\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \in H_\eta \Rightarrow$

$$\operatorname{dist}(Y(t,x_0,v_0), M) = |Y(t,x_0,v_0)| \to 0$$

and it follows that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is asymptotically stable.

To show $M = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ let $\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \in M$, then $Y(t,x_0,v_0) \in M \quad \forall t \geq 0$. Write

$$Y(t,x_0,v_0) = \begin{pmatrix} X(t) \\ V(t) \end{pmatrix}.$$

If $\exists t_o$ s.t. $V(t_0) \neq 0$ $\exists t_1 > t_0$ to s.t.

$$V(t) \neq 0 \text{ on } (t_0, t_1)$$

so

$$X(t) = 0 \text{ on } (t_0, t_1)$$

and hence,

$$V(t) = \dot{X}(t) = 0 \text{ on } (t_0, t_1).$$

Contradiction, so $V \equiv 0$. Hence,

$$0 \equiv \dot{V} + \sigma(X)V + U'(X) \equiv U'(X).$$

Since $\begin{pmatrix} X \\ V \end{pmatrix} \in H_\eta$, $X \equiv 0$ follows.

Question: Assume $\sigma(t, x)$ is continuous and

$$\sigma(t, x) \geq C > 0 \quad \forall t, x.$$

Is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ asymptotically stable for

$$\ddot{X} + \sigma(t, X)V + X = 0?$$

Answer: Not in general, e.g., if

$$\sigma(t, x) = 2 + e^t$$

then $\forall \varepsilon > 0$

$$X(t) = \varepsilon(1 + e^{-t})$$

is a solution:

$$\begin{aligned} \ddot{X} + \sigma\dot{X} + X &= (\varepsilon e^{-t}) + (2 + e^t)(-\varepsilon e^{-t}) + \varepsilon(1 + e^{-t}) \\ &= \varepsilon e^{-t} - 2\varepsilon e^{-t} - \varepsilon + \varepsilon + \varepsilon e^{-t} = 0. \end{aligned}$$

But

$$\lim_{t \to +\infty} X(t) = \varepsilon \neq 0.$$

6 Two Dimensional Systems

Comment: Suppose $f : \mathbb{R}^N \to \mathbb{R}^N$ is C^1,

$$\dot{X} = f(X),$$

$T > 0$, and

$$X(T) = X(0).$$

Then

$$X(t+T) = X(t) \quad \forall t \geq 0$$

(both sides are solutions of the same initial value problem). Also

$$\Omega(X(0)) = C^+(X(0)).$$

See the homework.

A. The Poincaré Bendixson Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 and let X be a bounded solution of

$$\dot{X} = f(X) \quad \forall t \geq 0.$$

If $\Omega(X(0))$ contains no critical points then either

1. X is periodic and $\Omega(X(0)) = C^+(X(0))$ or
2. $\exists$ a periodic solution, $\tilde{X}$, s.t.

$$\Omega(X(0)) = C^+(\tilde{X}(0)).$$

Examples

1. Let $\varepsilon \geq 0$ and

$$\begin{aligned} \dot{X} &= X(1 - X^2 - Y^2) - \varepsilon Y \\ \dot{Y} &= Y(1 - X^2 - Y^2) + \varepsilon X. \end{aligned}$$

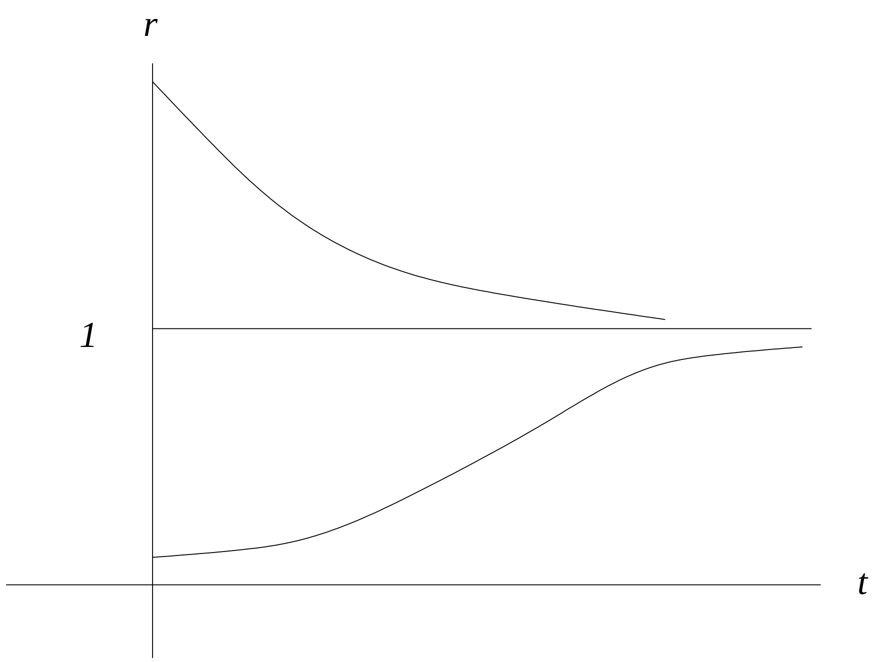
Take $X = r\cos\theta,\ Y = r\sin\theta$, then

$$\begin{aligned}\dot r &= \cos\theta\,[r\cos\theta(1-r^2) - \varepsilon r\sin\theta] + \sin\theta\,[r\sin\theta(1-r^2) + \varepsilon r\cos\theta] \\ &= r(1-r^2)\end{aligned}$$

and

$$\dot\theta = \frac{\cos\theta}{r}\left[r\sin\theta(1-r^2) + \varepsilon r\cos\theta\right] - \frac{sin\theta}{r}\left[r\cos\theta(1-r^2) - \varepsilon r\sin\theta\right] = \varepsilon.$$

Note that $(0,0)$ is a critical point and that $(X(0),Y(0)) \neq (0,0) \Rightarrow r \to 1$ as $t \to +\infty$.



(a) Consider $\varepsilon > 0$. If $r(0) = 1$ then (X,Y) is periodic with period $\dfrac{2\pi}{\varepsilon}$ and

$$
\begin{aligned}
C^+(X(0), Y(0)) &= \Omega(X(0), Y(0)) \\
&= \{(x, y) : x^2 + y^2 = 1\}.
\end{aligned}
$$

If $r(0) \in (0, \infty)\backslash\{1\}$ then

$$\Omega(X(0), Y(0)) = \left\{(x, y) : x^2 + y^2 = 1\right\}.$$

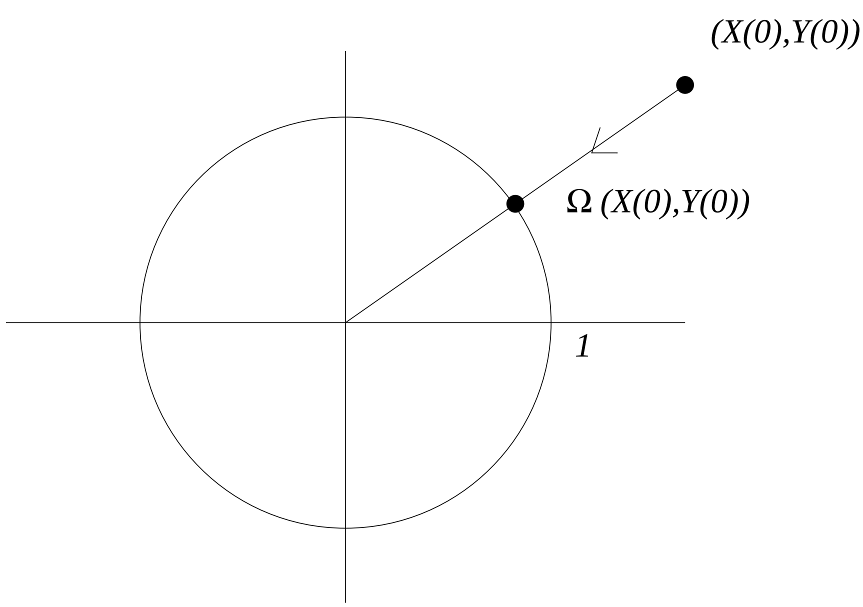
(b) Consider $\varepsilon = 0$. Note that the set of critical points is

$$\{(0, 0)\} \bigcup \left\{(x, y) : x^2 + y^2 = 1\right\}.$$

If $r(0) > 0$ then

$$\Omega(X(0), Y(0)) = \left\{\left(\frac{X(0)}{r(0)}, \frac{Y(0)}{r(0)}\right)\right\}.$$

The theorem does not apply since $\Omega(X(0), Y(0))$ contains a critical point.



2. The theorem does not hold in $\mathbb{R}^3$:

$$\begin{aligned}\dot{X} &= \frac{-XZ}{\sqrt{X^2+Y^2}} - \pi Y \\ \dot{Y} &= \frac{-YZ}{\sqrt{X^2+Y^2}} + \pi X \\ \dot{Z} &= \sqrt{X^2+Y^2} - 2\end{aligned}$$

Use $X = r\cos\theta,\ Y = r\sin\theta$:

$$\begin{aligned}\dot{r} &= -Z \\ \dot{\theta} &= \pi \\ \dot{Z} &= r-2.\end{aligned}$$

$\dot{\theta} = \pi$ so there are no critical points. Suppose there is a solution with period T. Since $\dot{\theta} = \pi,\ \theta(T) - \theta(0) = n2\pi$ for some $n \in \mathbb{N}$, and

$$T = 2n.$$

Also

$$\ddot{r} + r = -\dot{Z} + r = 2.$$

Hence, $\exists k \in \mathbb{N}$ s.t. (if r is not $\equiv 2$)

$$T = 2k\pi.$$

Thus,

$$\pi = \frac{T}{2k} = \frac{2n}{2k} \in \mathbb{Q},$$

contradiction. Thus there is no periodic solution with $r \not\equiv 2$.

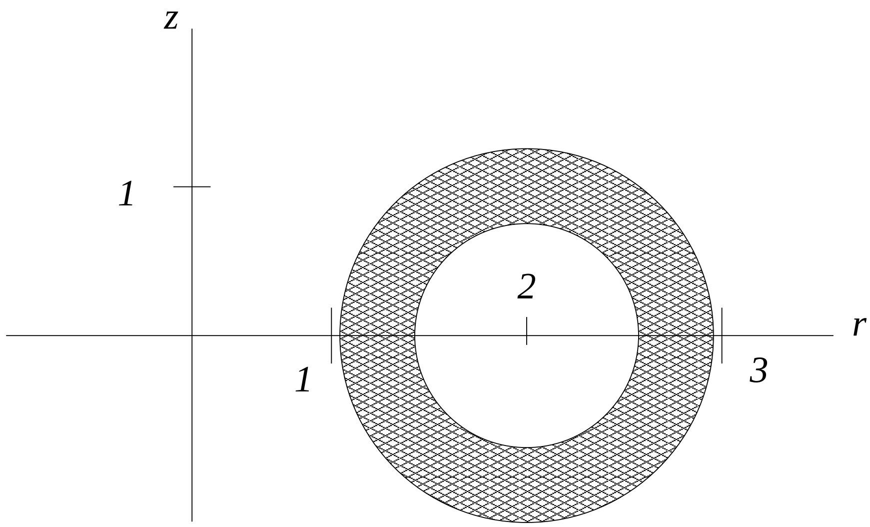
Note

$$\frac{d}{dt}\left[(r-2)^2+Z^2\right]=2(r-2)(-Z)+2Z(r-2)=0.$$

Thus,

$$\left\{(x,y,z):\frac{1}{4}\le(r-2)^2+z^2\le 1\right\}$$

is invariant.



Preliminaries

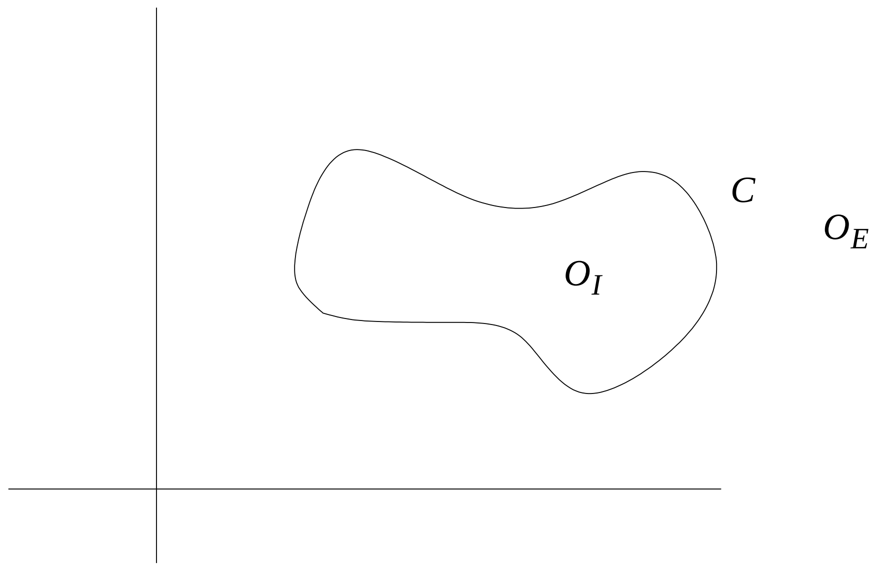
Definition 6.1. *$C\subset\mathbb{R}^2$ is a Jordan curve if $\exists\psi:S^1\to C$ ($S^1:=\{(x,y):x^2+y^2=1\}$) with ψ bijective and continuous.*

Jordan Curve Theorem Let C be a Jordan curve, then $\exists$ open sets, O_I and O_E, that are disjoint and pathwise connected with

$$\mathbb{R}^2\backslash C=O_I\bigcup O_E,$$

O_I bounded

O_E unbounded



Definitions

1. A line segment is defined by two nonequal points, x and y:

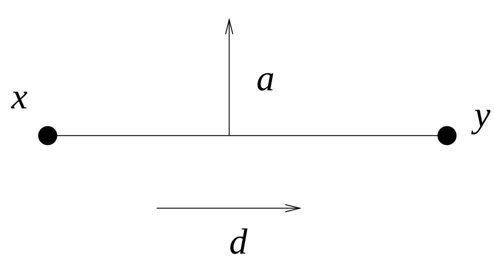
$$L = \{x + t(y - x) : 0 \leq t \leq 1\}.$$

Its direction is

$$d = \frac{y - x}{|y - x|}.$$

2.

Let $|a| = 1$ with $a \cdot d = 0$.
A continuous map
$\phi : (\alpha, \beta) \to \mathbb{R}^2$
crosses L at $t_0 \in (\alpha, \beta)$ if
$\phi(t_0) \in L$



and $\exists \delta > 0$ s.t. either

$$\begin{cases} t_0 - \delta < t < t_0 \quad \Rightarrow \quad (\phi(t) - \phi(t_0)) \cdot a < 0 \\ t_0 < t < t_0 + \delta \quad \Rightarrow \quad (\phi(t) - \phi(t_0)) \cdot a > 0 \end{cases}$$

or

$$\begin{cases} t_0 - \delta < t < t_0 & \Rightarrow \quad (\phi(t) - \phi(t_0)) \cdot a > 0 \\ \\ t_0 < t < t_0 + \delta & \Rightarrow \quad (\phi(t) - \phi(t_0)) \cdot a < 0. \end{cases}$$

Two maps that cross at t_0 either cross in the same or opposite directions.

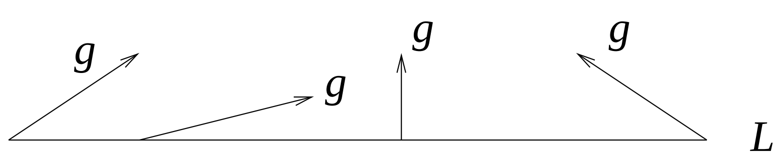
3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. A line segment, L, defined by x and y is a transversal if $\forall z \in L$

$$f(z) \neq 0$$

and

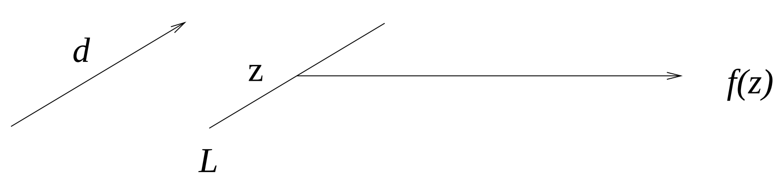
$$f(z) \text{ and } d = \frac{y - x}{|y - x|}$$

are not parallel.



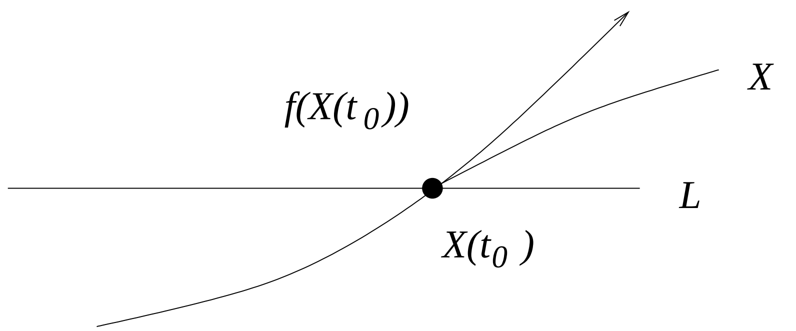
Comments

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous and let $z \in \mathbb{R}^2$ with $f(z) \neq 0$. Let $|d| = 1$ with $f(z)$ and d not parallel. Then $\exists$ a transversal, L, through z with direction d (where z is the center of L).

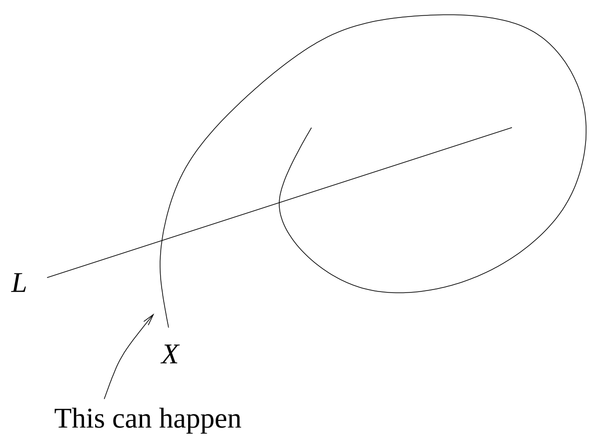


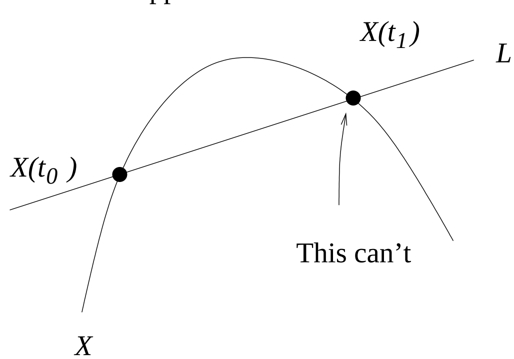
2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous and let L be a transversal. Let $\dot{X} = f(X)$ with $X(t_0) \in L$.

   (a) X crosses L at t_0:



   (b) If X crosses L again, it must be in the same direction:





   It would force f to be 0 or parallel to L at some point on L between $X(t_0)$ and $X(t_1)$.

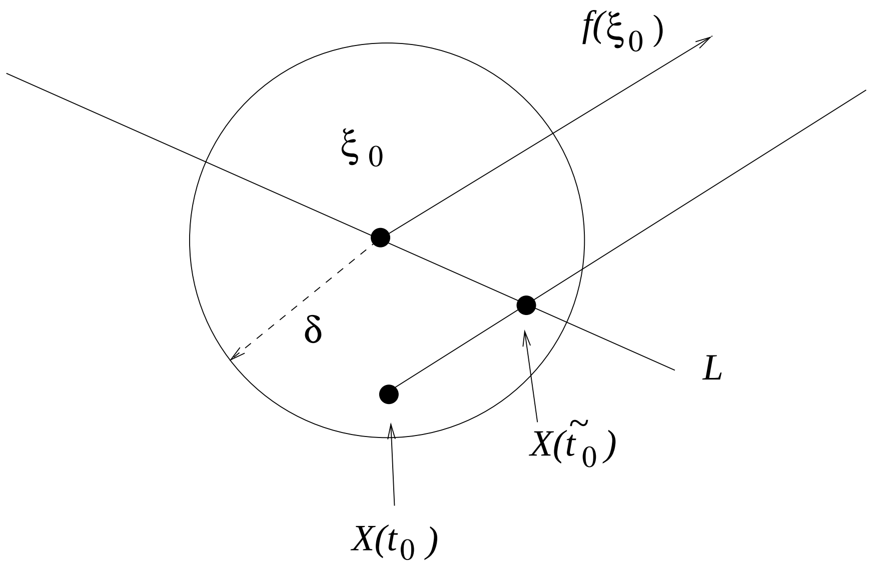
(c) If $\dot{Y} = f(Y)$ and Y crosses L, it must cross in the same direction as X.

Lemma 6.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous and let L be a transversal. Let $\xi_0 \in L$ (not an end point). $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon) > 0$ s.t. if*

$$\dot{X} = f(X) \text{ and } |X(t_0) - \xi_0| < \delta$$

then X crosses L at a time $\tilde{t}_0$ with

$$|t_0 - \tilde{t}_0| < \varepsilon \text{ and } |X(\tilde{t}_0) - \xi_0| < \varepsilon.$$



Corollary 6.1. *Let f and L be as above and $\overline{x} \in \Omega(X(0)) \bigcap (L \setminus$ {end points}) Then $\exists s_k \to +\infty$ with $X(s_k) \in L \forall k$ and $X(s_k) \to \overline{x}$ and $s_{k+1} > s_k$ $\forall k$.*

Proof. Choose $t_k \to +\infty$ with $X(t_k) \to \overline{x}$.

Choose k_1 s.t. $|X(t_{k_1}) - \overline{x}| < \delta(1)$ so X crosses L at $\tilde{t}_{k_1}$ with

$$|t_{k_1} - \tilde{t}_{k_1}| < 1 \text{ and } |X(\tilde{t}_{k_1}) - \overline{x}| < 1.$$

Given k_ℓ choose $k_{\ell+1}$ s.t.

$$t_{k_{\ell+1}} > t_{k_\ell} + 3 \text{ and } |X(t_{\ell+1}) - \overline{x}| < \frac{1}{\ell + 1}$$

so that X crosses L at $\tilde{t}_{k_{\ell+1}}$ with

$$|t_{k_{\ell+1}} - \tilde{t}_{k_{\ell+1}}| < \frac{1}{\ell+1} \text{ and } |X(\tilde{t}_{k_{\ell+1}}) - \overline{x}| < \frac{1}{\ell+1}.$$

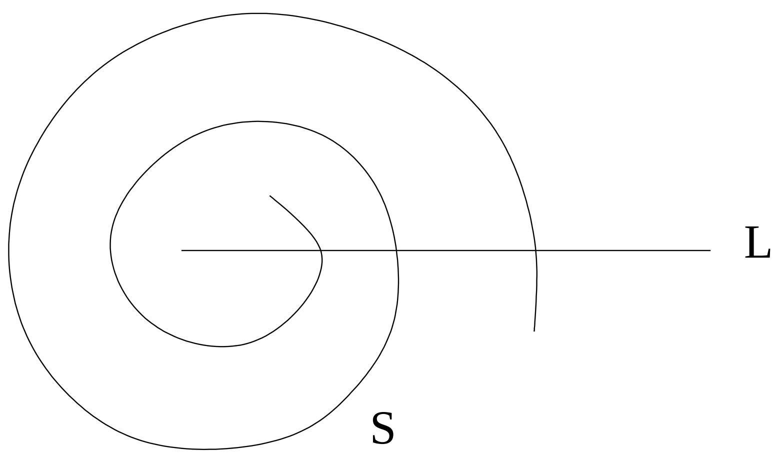
Note that

$$\begin{aligned} \tilde{t}_{k_{\ell+1}} - \tilde{t}_{k_\ell} &\geq \left(t_{k_{\ell+1}} - \frac{1}{\ell+1}\right) - \left(t_{k_\ell} - \frac{1}{\ell}\right) \\ &> 3 - \frac{1}{\ell+1} - \frac{1}{\ell} > 1 \end{aligned}$$

so $\tilde{t}_{k_\ell} \to +\infty$ and $X(\tilde{t}_{k_\ell}) \in L \ \ \forall \ell$ with $X(\tilde{t}_{k_\ell}) \to \overline{x}$.

$\square$

Lemma 6.2. *Let $\dot{X} = f(X)$ and $S = \{X(t) : t \in [\alpha, \beta]\}$. Let L be a transversal Then $L \bigcap S$ is finite. If $L \bigcap S \neq \phi$ then the points of $L \bigcap S$ are monotone with respect to t.*



Proof. Suppose $L \bigcap S$ is infinite; choose $t_k \in [\alpha, \beta] \ \ \forall k \in \mathbb{N}$ s.t.

$$X(t_k) \in L \bigcap S \text{ and } i \neq j \Rightarrow X(t_i) \neq X(t_j).$$

Choose a convergent subsequence,

$$t_{n_k} \to \tau \in [\alpha, \beta].$$

$L \bigcap S$ is closed so $X(\tau) \in L \bigcap S$.

Note that t_{n_k} could equal τ at most once. Now

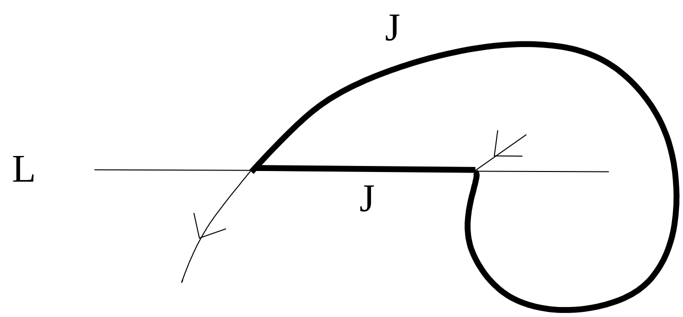
$$f(X(\tau)) = \lim_{k\to\infty} \frac{X(t_{n_k}) - X(\tau)}{t_{n_k} - \tau}.$$

But $X(t_{n_k}) - X(\tau)$ is parallel to L so $f'(X(\tau))$ is too. This contradicts L being a transversal.

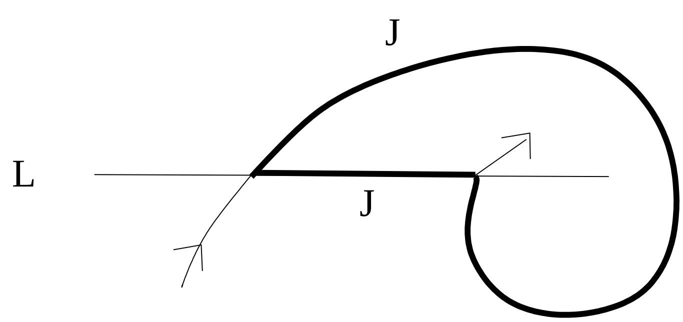
Suppose $X(t_1)$ and $X(t_2) \in L \bigcap S$ with $\alpha \leq t_1 < t_2 \leq \beta$ and $X(t) \notin L$ for $t_1 < t < t_2$.

$$J = \{X(t) : t_1 \leq t \leq t_2\} \bigcup \{\theta X(t_1) + (1-\theta)X(t_2) : 0 \leq \theta \leq 1\}$$

is a Jordan curve so $\exists O_I, O_E$ open with $\mathbb{R}^2 =$ disjoint union of J, O_i, O_E. Also O_I and O_E connected.
Either $\alpha \leq t < t_1 \Rightarrow X(t) \in O_I$ and $t_2 < t \leq \beta \Rightarrow X(t) \in O_E$



or $\alpha \leq t < t_1 \Rightarrow X(t) \in O_E$ and $t_2 < t \leq \beta \Rightarrow X(t) \in O_I$



$\square$

Comment: If $S \bigcap L$ has two or more points then X is not periodic.

Lemma 6.3. *Let L be a transversal and X a solution with $\{X(t) : t \geq 0\}$ bounded. Then*

$$(L \backslash \text{ end points}) \bigcap \Omega(X(0))$$

can have at most one element.

Proof. Suppose

$$\overline{x}, \overline{y} \in (L \backslash \text{ end points}) \bigcap \Omega(X(0))$$

with $\overline{x} \neq \overline{y}$, seek contradiction.

Using the corollary to Lemma 6.1 choose $\tilde{t}_k \to \infty$ and $\tilde{\tau}_k \to \infty$ with

$$X(\tilde{t}_k),\ X(\tilde{\tau}_k) \in L,$$

$$X(\tilde{t}_k) \to \overline{x},\ X(\tilde{\tau}_k) \to \overline{y}.$$

Choose $\mathcal{K}$ s.t. $k > \mathcal{K} \Rightarrow$

$$|X(\tilde{t}_k) - \overline{x}| < \frac{1}{2}|\overline{x} - \overline{y}|$$

and

$$|X(\tilde{\tau}_k) - \overline{y}| < \frac{1}{2}|\overline{x} - \overline{y}|.$$

Choose $k_1 > \mathcal{K}, k_2$ s.t. $\tilde{\tau}_{k_2} > \tilde{t}_{k_1},\ k_3$ s.t. $\tilde{t}_{k_3} > \tilde{\tau}_{k_2}$. Then $X(\tilde{t}_{k_1}), X(\tilde{\tau}_{k_2}), X(\tilde{t}_{k_3})$ contradict the monotonicity of Lemma 6.2.

∎

Lemma 6.4. *Let $f \in C^1(\mathbb{R}^2)$ and let X be a nonconstant solution of $\dot{X} = f(X)$ with $C^+(X(0))$ bounded. Let Y be a nonconstant periodic solution with*

$$C^+(Y(0)) \subset \Omega(X(0))$$

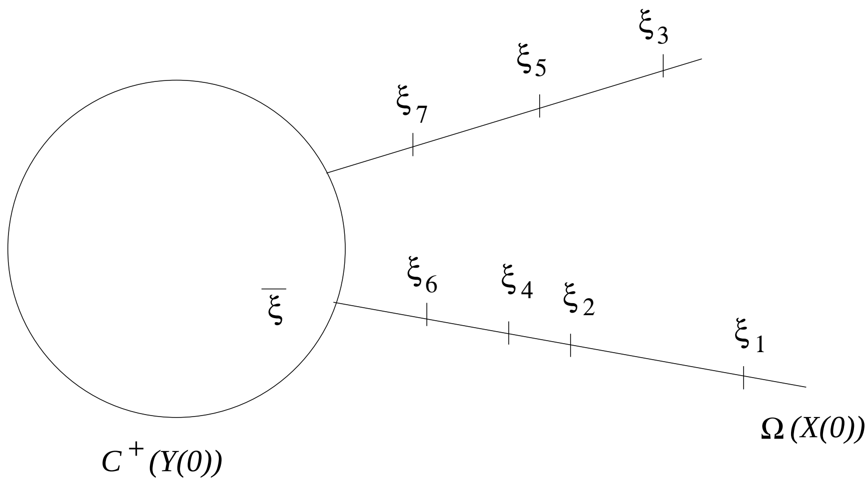
then

$$C^+(Y(0)) = \Omega(X(0)).$$

Proof. Suppose

$$\xi_1 \in \Omega(X(0)) \backslash C^+(Y(0))$$

and seek a contradiction. Since $\Omega(X(0))$ is connected, $\forall k \geq 2 \; \exists \xi_k \in \Omega(X(0)) \backslash C^+(Y(0))$ with $\mathrm{dist}(\xi_k, C^+(Y(0)) < \dfrac{1}{k}$.



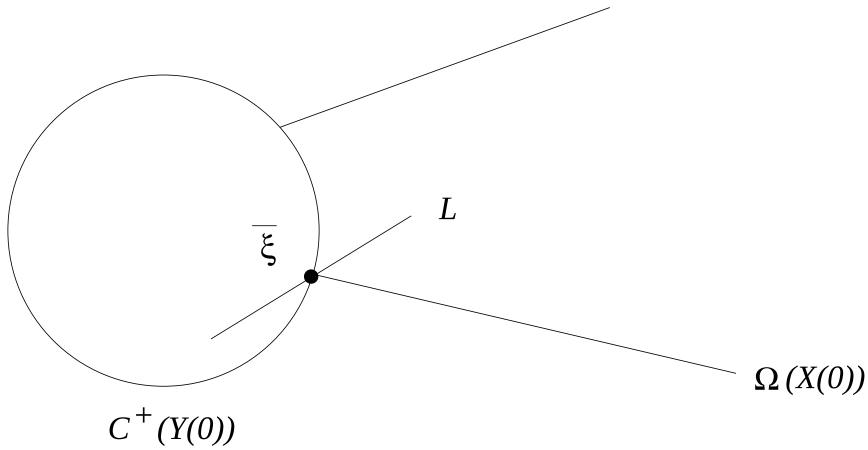
$\Omega(X(0))$ is bounded so ξ_k has a convergent subsequence,

$$\xi_{n_k} \to \overline{\xi}.$$

$\mathrm{dist}(\overline{\xi}, C^+(Y(0)) = \lim(\xi_{n_k}, C^+(Y(0)) = 0$ and $C^+(Y(0))$ is closed so

$$\overline{\xi} \in C^+(Y(0)).$$

$f(\overline{\xi}) \neq 0$ so $\exists$ a transversal whose center is $\overline{\xi}$.



By Lemma 6.3

$$L \bigcap \Omega(X(0)) = \{\overline{\xi}\}. \tag{*}$$

By Lemma 6.1 for k large the solution through ξ_{n_k} must cross L. But $\Omega(X(0))$ is positively invariant so by (*) the only place it can cross L is at $\overline{\xi}$. By uniqueness $\xi_{n_k} \in C^+(Y(0))$ for k large, contradiction. $\square$

Proof of Poincaré Bendixson

Assume X is not periodic. $\Omega(X(0))$ is nonempty, compact, positively invariant, and contains no critical points. Choose $Y(0) \in \Omega(X(0))$ then

$$C^+(Y(0)) \subset \Omega(X(0)).$$

Since $\Omega(X(0))$ is closed

$$\Omega(Y(0)) \subset \Omega(X(0)).$$

Note that Y is nonconstant and bounded.

Choose $\overline{x} \in \Omega(Y(0))$. $f(\overline{x}) \neq 0$ so $\exists$ a transversal, L, whose center is $\overline{x}$. By the corollary to Lemma 6.1 $\exists \tilde{t}_k \to \infty$ s.t.

$$Y(\tilde{t}_k) \in L \text{ and } Y(\tilde{t}_k) \to \overline{x}.$$

$\Omega(X(0))$ is positively invariant so $\forall k$

$$Y(\tilde{t}_k) \in L \bigcap \Omega(X(0)).$$

But, by Lemma 3

$$L \bigcap \Omega(X(0)) = \{\overline{x}\}$$

so $\forall k$

$$Y(\tilde{t}_k) = \overline{x}.$$

Hence, Y is periodic. By Lemma 4

$$\Omega(X(0)) = C^+(Y(0)).$$

Example

$$\begin{aligned}
\dot{X} &= f(X,Y) = X - Y + XY - X(X^2+Y^2) \\
\dot{Y} &= g(X,Y) = X + Y + Y^2 - Y(X^2+Y^2)
\end{aligned}$$

Set

$$\begin{aligned}
f(x,y) &= g(x,y) = 0 \\
xy(x^2+y^2) &= xy - y^2 + xy^2 \\
&= x^2 + xy + xy^2 \\
x^2 = -y^2 & \qquad x = y = 0
\end{aligned}$$

$(0,0)$ is the only critical point

$$\begin{aligned}
\frac{d}{dt}\,\frac{1}{2}(X^2+Y^2) &= X\dot{X} + Y\dot{Y} \\
&= X^2 - XY + X^2Y - X^2(X^2+Y^2) \\
&\quad +XY + Y^2 + Y^3 - Y^2(X^2+Y^2) \\
&= X^2 + Y^2 + Y(X^2+Y^2) - (X^2+Y^2)^2
\end{aligned}$$

$$
\begin{aligned}
X^2+Y^2 &= 2^2 \Rightarrow \frac{d}{dt}\,\frac{1}{2}(X^2+Y^2) \le 2^3+2(2)^2-2^4=-4 \\
X^2+Y^2 &= \left(\frac{1}{2}\right)^2 \Rightarrow \frac{d}{dt}\,\frac{1}{2}(X^2+Y^2) \ge \left(\frac{1}{2}\right)^2-\frac{1}{2}\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^4=\frac{1}{16}
\end{aligned}
$$

so

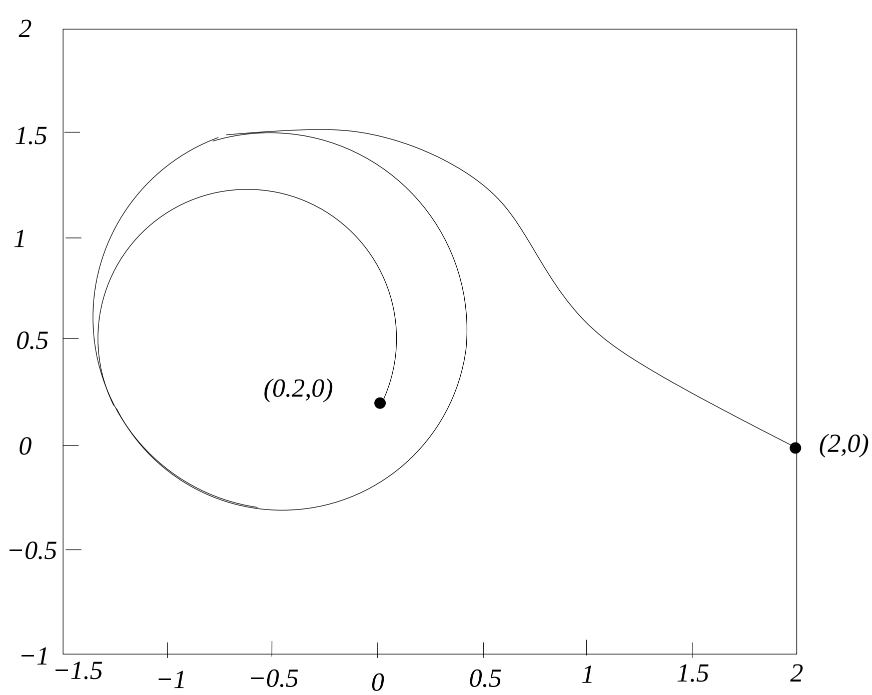
$$
S := \left\{(x,y) : \frac{1}{4} \le x^2+y^2 \le 4\right\}
$$

is positively invariant.

By Poincaré Bendixson $(X(0),Y(0)) \in S \Rightarrow \Omega(X(0),Y(0))$ is the orbit of a periodic solution.

Comment: Note that the periodic orbit encloses a critical point.

$$
\begin{aligned}
\dot{X} &= X - Y + XY - X(X^2+Y^2) \\
\dot{Y} &= X + Y + Y^2 - Y(X^2+Y^2)
\end{aligned}
$$



Theorem 6.1. *Assume Y is periodic and*

$$\mathbb{R}^2 = C^+(Y(0)) \bigcup O_I \bigcup O_E$$

with O_I and O_E disjoint open sets with O_I bounded. Then O_I contains a critical point.

Lemma 6.5. *With the same assumptions as above O_I contains either a critical point or a periodic orbit.*

Proof. (Lemma) Let $S = O_I \bigcup C^+(Y(0))$. S is compact and invariant. Suppose O_I contains no critical points and no periodic orbits. Then $\forall$ solution with $X(0) \in O_I$, Poincaré Bendixson implies $\Omega(X(0)) = C^+(Y(0))$.

Let $z \in C^+(Y(0))$. $f(z) \neq 0$ so $\exists$ a transversal, L, with center z. By the corollary to Lemma 6.1 $\exists \tilde{t}_n \to +\infty$ s.t.

$$X(\tilde{t}_n) \in L \text{ and } X(\tilde{t}_n) \to z.$$

By applying the above to $\dot{\tilde{X}} = -f(\tilde{X})$ we concluded $\exists \tilde{s}_n \to -\infty$ s.t. $X(\tilde{s}_n) \in L$ and $X(\tilde{s}_n) \to z$. This violates the monotonicity of Lemma 6.2. $\square$

.

Proof. (Theorem) Assume O_I contains no critical points. Let

$$\begin{aligned} A &= \inf \{\text{area enclosed by } C^+(X(0)) : \\ &\qquad X \text{ is periodic and } C^+(X(0)) \subset O_I\} \end{aligned}$$

(note that this set is not empty by the lemma). Choose periodic solutions X_n, with

$$C^+(X_n(0)) \subset O_I$$

and

$$A_n := \text{ area enclosed by } C^+(X_n(0)) \to A.$$

Since $O_I \bigcup C^+(Y(0))$ is compact, $X_n(0)$ has a convergent subsequence, $X_{k_n}(0)$. Let X be the solution with

$$X(0) = \lim_{n\to\infty} X_{k_n}(0).$$

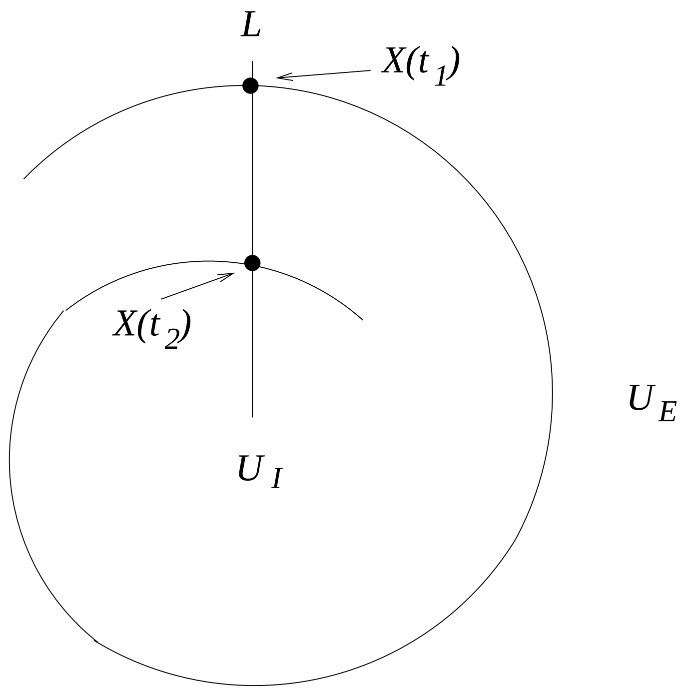
Claim that X is periodic and the area enclosed by $C^+(X(0))$ is A. $A = 0 \Rightarrow X(t) \equiv X(0)$ which can't happen since O_I contains no critical point; hence, $A > 0$. No periodic solution or critical point is enclosed by $C^+(X(0)$, contradicting the lemma. □

Proof of Claim

Proof. Suppose X is not periodic. By Poincaré Bendixson $\exists$ a periodic solution, Z, s.t.

$$\Omega(X(0)) = C^+(Z(0)).$$

$Z(0)$ is not a critical point, so $\exists$ a transversal, L, with center $Z(0)$. By Lemma 6.1 X crosses L infinitely many times. Choose t_1, t_2 with $0 < t_1 < t_2$, $X(t_1),\ X(t_2) \in L$ and $t \in (t_1, t_2) \Rightarrow X(t) \notin L$.



Let $J = \{X(t) : t_1 \leq t \leq t_2\} \bigcup \{\theta X(t_1) + (1-\theta)X(t_2) : \theta \in [0,1]\}$ and let U_I and U_E be the related open sets.

Consider $X(0) \in U_E$. By continuity with respect to initial conditions

$$X_{k_n}(0) \in U_E \qquad X_{k_n}(t_2+1) \in U_I$$

for n sufficiently large. Now

$$t > t_2 + 1 \Rightarrow X_{k_n}(t) \in U_I$$

which contradicts X_{k_n} periodic.

$X(0) \in U_I$ leads to a contradiction similarly, so X must be periodic.

Choose $T > 0$ s.t. $X(T) = X(0)$. $X_{k_n} \to X$ uniformly on $[0, 2T]$ and it follows that $A_n \to A$. □

Theorem 6.2. *Let Y be a nonconstant periodic solution and choose O_I and O_E as before. Then*

$$\iint_{O_I} \operatorname{div} f \, dy \, dx = 0.$$

Proof. By the divergence theorem

$$\iint_{O_I} \operatorname{div} f \, dy \, dx = \oint_{C^+(Y(0))} f \cdot n \, ds$$

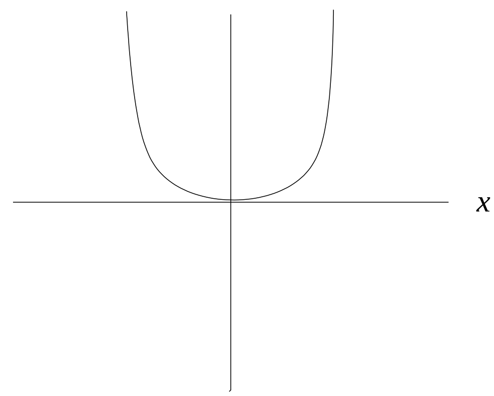
where n is the outward unit normal. But $\dot{Y}(t)$ is tangent to $C^+(Y(0))$ so

$$0 = \dot{Y}(t) \cdot n\Big|_{Y(t)} = f(Y(t)) \cdot n|_{Y(t)} \, .$$

□

B. Orbital Stability

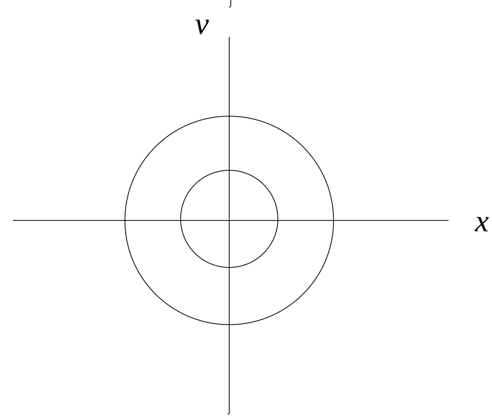
Example



$$\ddot{X} + 2X^3 = 0$$

$$U(x) = \frac{1}{2}x^4$$

$$\mathcal{E} = \frac{1}{2}\dot{X}^2 + U(X)$$



All nonzero solutions are periodic.

$$\text{Say} \begin{cases} X(0) &= x_0 > 0 \\ \dot{X}(0) &= 0 \end{cases}$$

and let P_{x_0} be the period, then $\exists C > 0$ s.t.

$$P_{x_0} = \frac{C}{x_0}.$$

Compare X and $\tilde{X}$ where

$$X(0) = 1 \quad \dot{X}(0) = 0$$

$$\tilde{X}(0) = 1 + \frac{1}{n} \quad \dot{\tilde{X}}(0) = 0.$$

Then

$$\sup_{t\geq 0}\left|X(t)-\tilde{X}(t)\right|\geq 1 \quad \forall n.$$

Definition 6.2. *Let Y be a periodic solution of $\dot{Y}=f(Y)$. Y is orbitally stable if $\forall\varepsilon>0\ \exists\delta>0$ s.t. if $\dot{X}=f(X)$,*

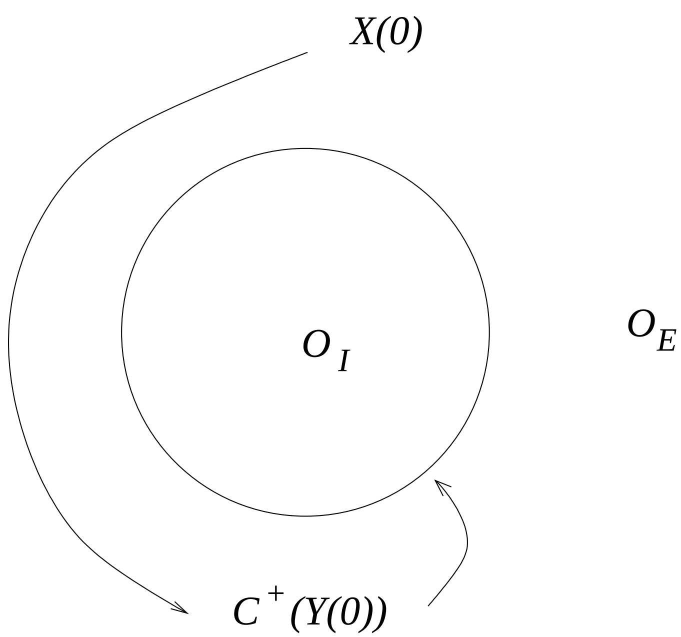
$$\operatorname{dist}(X(0),C^+(Y(0)))<\delta \ \Rightarrow$$

$$\operatorname{dist}(X(t),C^+(Y(0)))<\varepsilon \qquad \forall t>0.$$

Theorem 6.3. *Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be C^1, Y a periodic (nonconstant) solution, and X a solution with $w(X(0))=C^+(Y(0))$. Let O_I be the bounded open set with boundary $(C^+(Y(0))$ and let O_E be the unbounded open set with boundary $C^+(Y(0))$. Suppose $X(0)\in O_E$. Then $\exists\delta>0$ s.t. $\dot{Z}=f(Z),\ Z(0)\in O_E$*

$$\operatorname{dist}(Z(0),C^+(Y(0)))<\delta \quad \Rightarrow$$

$$\operatorname{dist}(Z(t),C^+(Y(0))) \ \to 0.$$



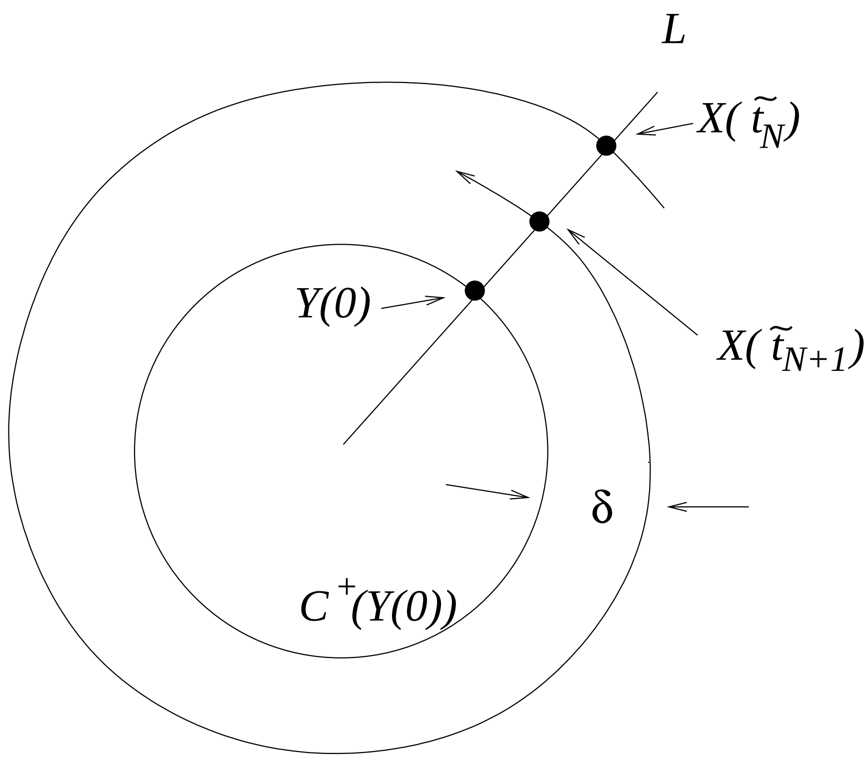
Comment: There's a similar statement for $X(0)\in O_I$.

Proof. Since $C^+(Y(0))$ is compact and $f \neq 0$ on $C^+(Y(0))$ $\exists\, \delta_0 > 0$ s.t.

$$\mathrm{dist}(x, C^+(Y(0)) < \delta_0 \Rightarrow f(x) \neq 0.$$

$f(Y(0)) \neq 0$ so $\exists$ a transversal, L, whose center is $Y(0)$. By the corollary to Lemma 6.1 $\exists\, \tilde{t}_n \to +\infty$ with $X(\tilde{t}_n) \to Y(0)$ and $X(\tilde{t}_n)$. Choose N s.t.

$$\mathrm{dist}(X(t), C^+(Y(0))) < \delta_0 \ \ \forall t \in \left[\tilde{t}_N,\ \tilde{t}_{N+1}\right].$$



Choose $\delta \in (0, \delta_0)$ s.t.

$$\delta < \min\left\{\mathrm{dist}(X(t),\ C^+(Y(0))) : \tilde{t}_N \leq t \leq \tilde{t}_{N+1}\right\}$$

and consider $Z(0) \in O_E$

$$\mathrm{dist}(Z(0),\ C^+(Y(0)) < \delta.$$

By Poincaré Bendixson $\exists\, \tilde{Y}$ periodic with

$$w(Z(0)) = C^+(\tilde{Y}(0)).$$

$C^+(\tilde{Y}(0))$ must enclose a critical point and hence must enclose $C^+(Y(0))$. But

$$\text{dist}(X(t), w(X(0))) = \text{ dist}(X(t), C^+(Y(0)) \to 0$$

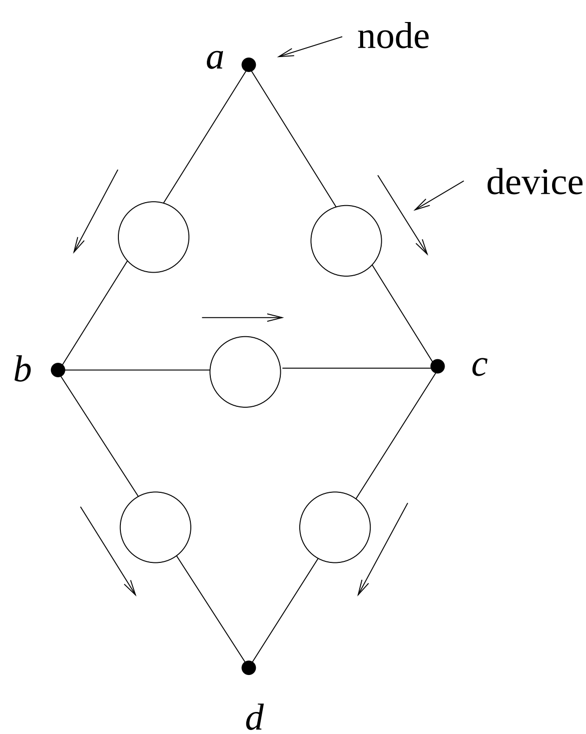
so $C^+(Y(0))$ must enclose $C^+(\tilde{Y}(0))$. Thus $C^+(Y(0)) = C^+(\tilde{Y}(0))$. Finally,

$$\text{dist}(Z(t), w(Z(0))) = \text{ dist}(Z(t), C^+(Y(0))) \to 0 \text{ as } t \to +\infty.$$

$\square$

C. Applications

1. Circuit Theory



$$\begin{aligned}
i_{ab} &= \text{current from } a \text{ to } b \\
v_{ab} &= \text{voltage at } a - \text{ voltage at } b = v(a) - v(b)
\end{aligned}$$

Kirchoff's current law: sum of currents into a node is zero, e.g.

$$i_{ab} - i_{bc} - i_{bd} = 0.$$

Kirchoff's voltage law: sum of voltage drops around a loop $= 0$, e.g.

$$v_{ab} + v_{bc} - v_{ac} = 0.$$

Comment: For the network drawn above there are 5 currents and 5 voltages drops but KCL imposes

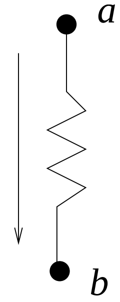
$$\left.\begin{array}{l}
-i_{ab} - i_{ac} = 0 \\
i_{ab} - i_{bc} - i_{bd} = 0 \\
i_{bc} + i_{ac} - i_{cd} = 0 \\
i_{bd} + i_{cd} = 0
\end{array}\right\} \quad \text{3 independent equations}$$

and KVL imposes

$$\begin{aligned}
v_{ab} + v_{bc} - v_{ac} &= 0 \\
v_{bd} - v_{cd} - v_{bc} &= 0
\end{aligned}$$

so there are two independent currents and three independent voltage drops.

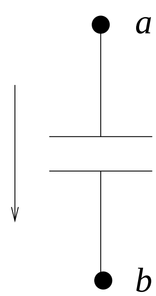
Resistors:



$$v_{ab} = f(i_{ab})$$

$f(i) = ki$ is Ohm's law.

Capacitors:

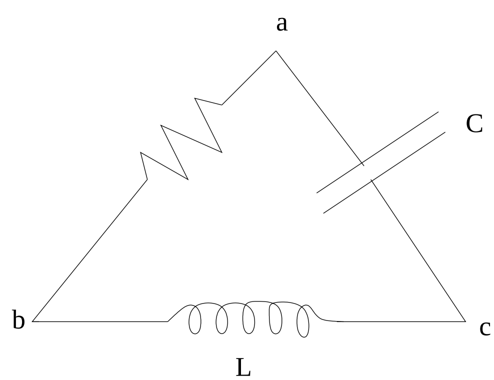


$$C\frac{dv_{ab}}{dt} = i_{ab}$$

Inductors:

$$L\frac{di_{ab}}{dt} = v_{ab}$$

Example



$$\begin{cases} i_{ab} = i_{bc} &= -i_{ac} \\ v_{ab} + v_{bc} - v_{ac} &= 0 \\ v_{ab} &= f(i_{ab}) \\ L\dfrac{di_{bc}}{dt} &= v_{bc} \\ C\dfrac{dv_{ac}}{dt} &= i_{ac} \end{cases}$$

Use $x = i_{ab}$ and $y = v_{ac}$:

$$C\dot{y} = \quad i_{ac} = -x$$

$$\left.\begin{aligned} L\dot{x} &= v_{bc} \\ v_{ab} &= f(x) \end{aligned}\right\} y = v_{ab} + v_{bc} = L\dot{x} + f(x)$$

$$\begin{aligned} L\dot{x} &= y - f(x) \\ C\dot{y} &= -x \end{aligned}$$

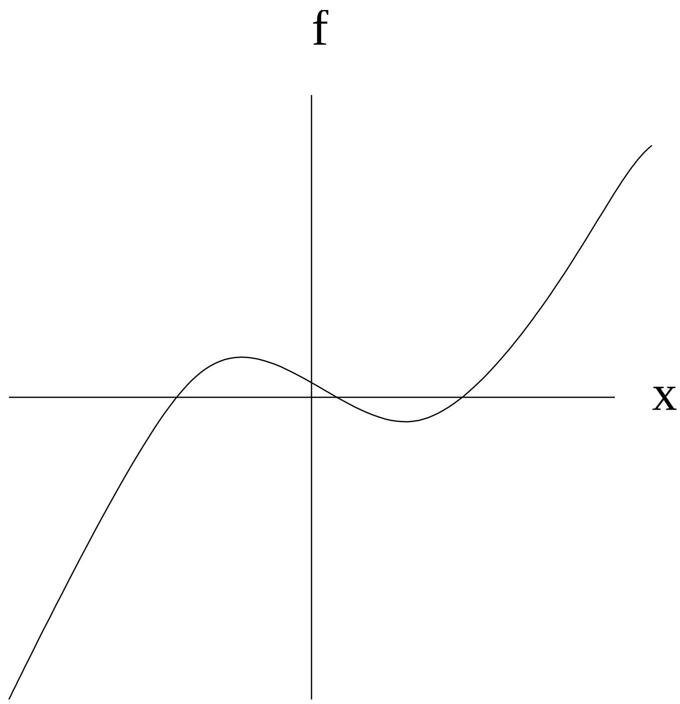
We'll take $L = C = 1$.

Linear Resistor $f(x) = kx$

$$
\begin{aligned}
\dot{x} &= y - kx \\
\dot{y} &= -x \\
\ddot{y} &= -\dot{x} = -y + kx = -y - k\dot{y} \\
\ddot{y} &+ k\dot{y} + y = 0
\end{aligned}
$$

All solutions $\to 0$ as $t \to \infty$.

Tunnel Diode $f(x) = x^3 - \mu x \quad \mu \geq 0$



$$\begin{aligned} \dot{x} &= y - x^3 + \mu x \\ \dot{y} &= -x \end{aligned}$$

$\mu = 1$ gives

van der Pol system

Find critical points: $\begin{cases} y - x^3 + \mu x = 0 \\ -x = 0 \end{cases}$

$(0,0)$ only one.

Linearize: $F(x,y) = \begin{pmatrix} y & - & x^3 + \mu x \\ & - & x \end{pmatrix}$

$$
\begin{aligned}
DF &= \begin{pmatrix} -3x^2+\mu & 1 \\ \\ -1 & 0 \end{pmatrix} \\
DF(0,0) &= \begin{pmatrix} \mu & 1 \\ \\ -1 & 0 \end{pmatrix} \\
(\mu-\lambda)(-\lambda)+1 &= \lambda^2-\mu\lambda+1=0 \Leftrightarrow \\
\lambda &= \frac{\mu \pm \sqrt{\mu^2-4}}{2}
\end{aligned}
$$

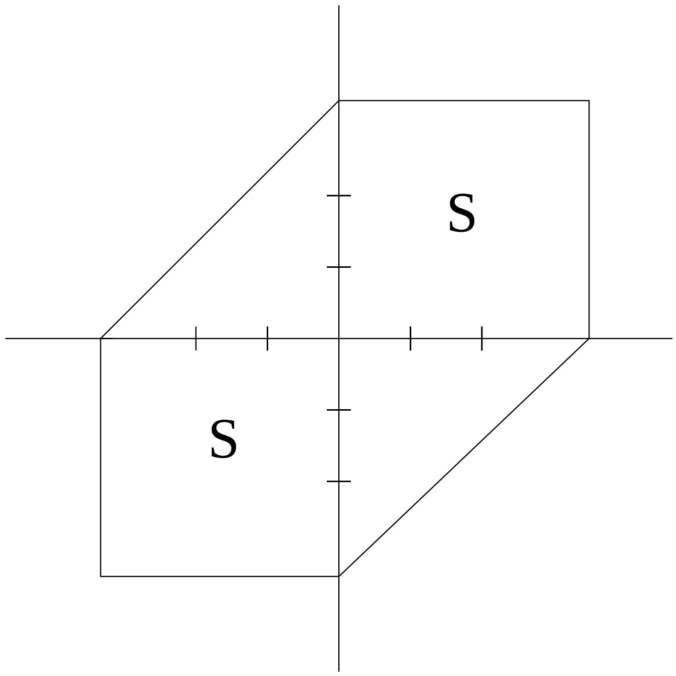
Asymptotically stable if $\mu < 0$.
Unstable if $\mu > 0$ (both eigenvalues have positive real part)

$$
\begin{aligned}
\frac{d}{dt}\left(X^2+Y^2\right) &= 2X\left(Y-X^3+\mu X\right)+2Y(-X) \\
&= -2X^4+2\mu X^2
\end{aligned}
$$

is indefinite.
Consider $0 < \mu < 1$: Claim that

$$
\begin{aligned}
S &= \{(x,y) : -3 \le x \le 0 \text{ and } -3 \le y \le 3+x\} \\
&\bigcup \{(x,y) : 0 \le x \le 3 \text{ and } -3+x \le y \le 3\}
\end{aligned}
$$

is positively invariant:



If $x = 3$ and $0 < y < 3$, $n = \langle -1, 0 \rangle$

$$\begin{aligned} F(3, y) \cdot n &= \langle y - 3^3 + 3\mu, -3 \rangle \cdot \langle -1, 0 \rangle \\ &= -y + 27 - 3\mu \geq -3 + 27 - 3 > 0. \end{aligned}$$

Also

$$\begin{aligned} F(3, 3) &= \langle 3 - 3^3 + 3\mu, -3 \rangle \\ F(3, 0) &= \langle -3^3 + 3\mu, -3 \rangle. \end{aligned}$$

If $0 < x < 3$ and $y = -3 + x$, $n = \langle -1, 1 \rangle$

$$F(x, -3 + x) \cdot n = \langle -3 + x - x^3 + \mu x, -x \rangle \cdot \langle -1, 1 \rangle$$

$$= \quad x^3 - (2 + \mu)x + 3 \geq x^3 - 3x + 3 = 1 + (x - 1)^2(x + 2) \geq 1.$$

Also if $(X(0), Y(0)) = (0, -3)$ then

$$\begin{aligned}\left(\dot{X}(0), \dot{Y}(0)\right) &= F(0,-3) = \langle -3, 0\rangle \\ \ddot{Y}(0) &= -\dot{X}(0) = 3.\end{aligned}$$

If $-3 < x < 0$ and $y = -3, \quad n = \langle 0, 1\rangle$

$$\begin{aligned}F(x,-3)\cdot n &= \langle 3 - x^3 + \mu x, -x\rangle \cdot \langle 0,1\rangle \\ &= -x > 0.\end{aligned}$$

The rest follows by symmetry.

Also,

$$S_I = \left\{(x,y) : x^2 + y^2 \geq \mu\right\}$$

is positively invariant since

$$\frac{d}{dt}\left(X^2 + Y^2\right) = 2X^2(\mu - X^2) \geq 0$$

on a neighborhood of $x^2 + y^2 = \mu$. Hence $S \bigcap S_I$ is positively invariant. $(X(0), Y(0)) \in S \bigcap S_I \Rightarrow$

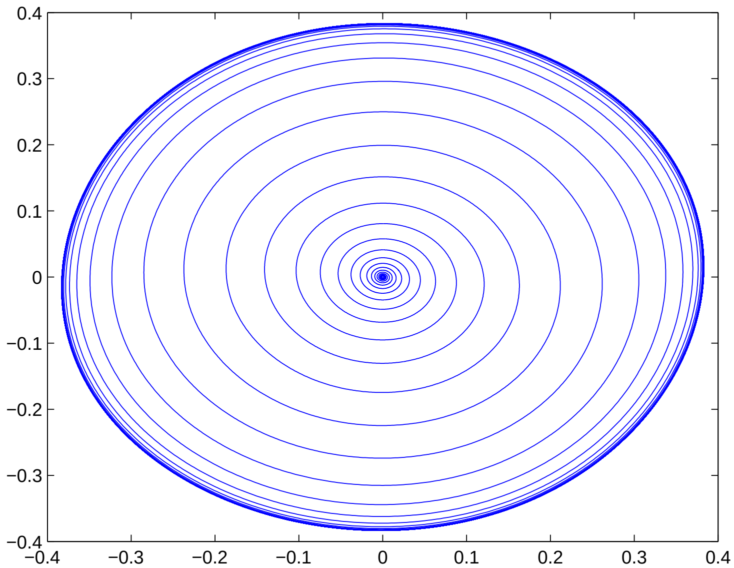
$$\Omega(X(0), Y(0)) \subset S \bigcap S_I$$

and contains no critical points. By Poincaré Bendixson

$$\Omega(X(0), Y(0)) = \text{ orbit of a periodic solution.}$$

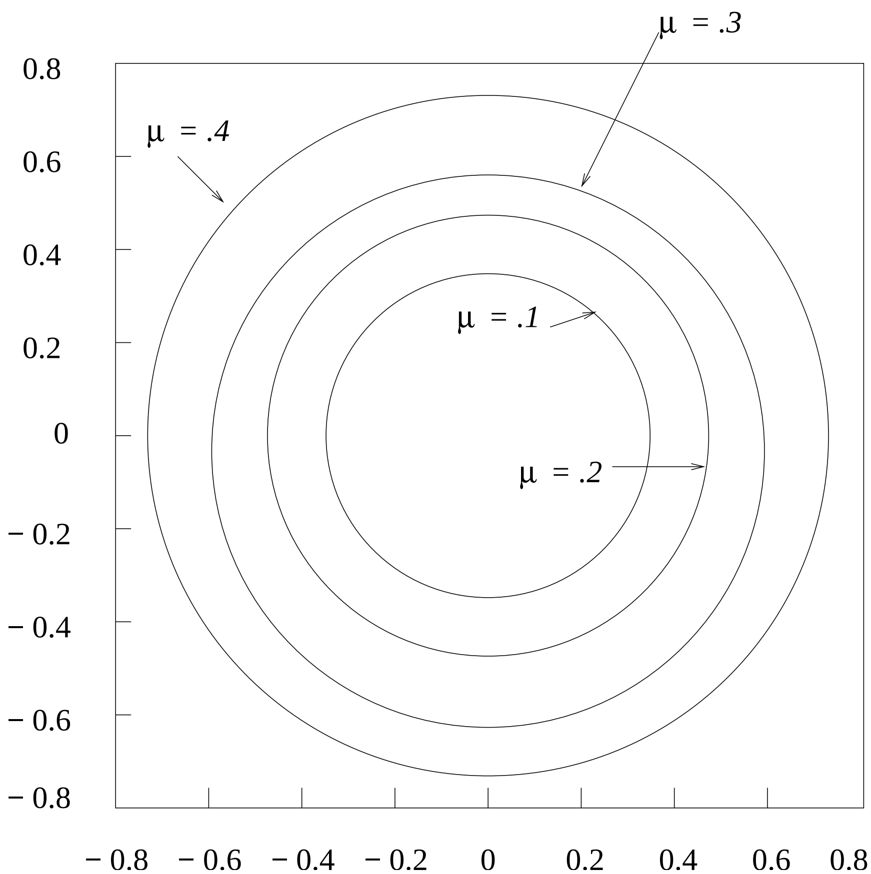
Comment: $\mu < 0$: stable critical point at $(0,0)$. $1 > \mu > 0 : (0,0)$ is unstable and there is a periodic solution. This is called a Hopf bifurcation.

$$
\begin{aligned}
\dot{X} &= Y - X^3 + 0.1X & \qquad X(0) &= .001 \\
\dot{Y} &= -X & \qquad Y(0) &= 0
\end{aligned}
$$



Periodic Solutions of

$$
\begin{aligned}
\dot{X} &= Y - X^3 + \mu X \\
\dot{Y} &= -X
\end{aligned}
$$



2. Predator Prey Model

$X(t) =$ prey population
$Y(t) =$ predator population

$$\dot{X} = aX - bXY \qquad a, b, c, d > 0$$

$$\dot{Y} = -cY + dXY$$

Note

$$ax - bxy = 0 \iff x = 0 \text{ or } y = \frac{a}{b}$$

$$-cy + dxy = 0 \iff y = 0 \text{ or } x = \frac{c}{d}$$

so there are two equilibria : $(0,0)$ and $\left(\frac{c}{d}, \frac{a}{b}\right)$.

Consider $X(0) \geq 0$ and $Y(0) \geq 0$. If $X(0) = 0$ then $X(t) = 0\ \forall t$ and $Y(t) = Y(0)e^{-ct}$. If $Y(0) = 0$, $Y(t) = 0$ and $X(t) = X(0)e^{at}$. $(0,0)$ is unstable.

Consider $X(0) > 0,\ Y(0) > 0$.

$$\begin{aligned}
\dot{X} &= (a - bY)X \\
\dot{Y} &= (dX - c)Y
\end{aligned}$$

so $X(t) > 0,\ Y(t) > 0\ \forall t$. Linearization yields no conclusion at $\left(\frac{c}{d}, \frac{a}{b}\right)$ so we seek a Lyapunov function. Note that

$$\frac{d}{dt} f(X) = f'(X)X(a - bY)$$

and

$$\frac{d}{dt}\ g(Y) = g'(Y)Y(dX - c).$$

If we set these equal:

$$\begin{aligned}
f'(x)x(a - by) &= g'(y)y(dx - c), \\
\frac{f'(x)x}{dx - c} &= \frac{g'(y)y}{a - by} = \text{ constant.}
\end{aligned}$$

Take the constant to be 1:

$$\begin{aligned}
f'(x) &= \frac{dx - c}{x} = d - cx^{-1}, \\
g'(y) &= \frac{a - by}{y} = ay^{-1} - b, \\
f(x) &= dx - c \ln x, \\
g(y) &= a \ln y - by.
\end{aligned}$$

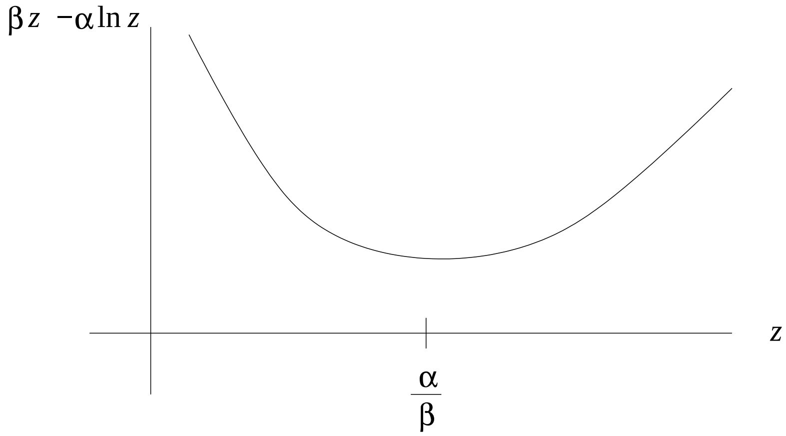
Hence, $D_*(f(x) - g(y)) = 0$ and

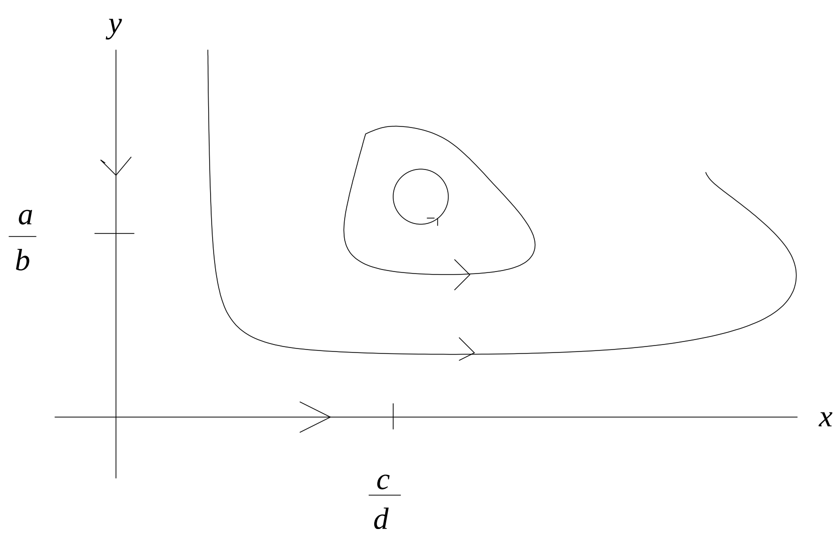
$$h(x,y) := f(x) - g(y) = dx - c\ln x + by - a\ln y.$$

Note that $f''(x) = \dfrac{c}{x^2} > 0,\ g''(y) = \dfrac{a}{y^2} > 0,\ f'\left(\dfrac{c}{d}\right) = g'\left(\dfrac{a}{b}\right) = 0$, so

$$h(x,y) - h\left(\frac{c}{d}, \frac{a}{b}\right)$$

is positive definite.





Every solution with $X(0) > 0,\ Y(0) > 0$ is periodic.

Rigid Body Motion

Let a rigid body occupy a volume V and have density $\rho(x)$. Take coordinates with origin at the center of mass and coordinate axes along the "principal axes" of the body. Then

$$I_1 := \int_V \rho(x)\left(x_2^2 + x_3^2\right) dV$$

is the moment of inertia about the 1st axis. Similarly for I_2 and I_3.

Assume no forces are applied to the body. The center of mass moves with constant velocity; we'll use the coordinates described above so the center of mass is at 0. Let

$\omega_k =$ rate of rotation about kth axis.

Then

$$\begin{aligned} \dot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\ \dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\ \dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2. \end{aligned}$$

Without loss of generality $I_1 \leq I_2 \leq I_3$. Consider $I_1 < I_2 < I_3$. Then

$$\begin{aligned} \dot{\omega}_1 &= -C_1\omega_2\omega_3 & C_1 &= \frac{I_3 - I_2}{I_1} > 0 \\ \dot{\omega}_2 &= C_2\omega_1\omega_3 & C_2 &= \frac{I_3 - I_1}{I_2} > 0 \\ \dot{\omega}_3 &= -C_3\omega_1\omega_2 & C_3 &= \frac{I_2 - I_1}{I_3} > 0. \end{aligned}$$

The set of equilibria are

$$\{\omega : \omega_1 = \omega_2 = 0 \text{ or } \omega_1 = \omega_3 = 0 \text{ or } \omega_2 = \omega_3 = 0\}.$$

Letting

$$f(\omega) = \begin{pmatrix} -C_1\omega_2\omega_3 \\ \\ C_2\omega_1\omega_3 \\ \\ -C_3\omega_1\omega_2 \end{pmatrix}$$

we have

$$Df(\omega) = \begin{pmatrix} 0 & -C_1\omega_3 & -C_1\omega_2 \\ \\ C_2\omega_3 & 0 & C_2\omega_1 \\ \\ -C_3\omega_2 & -C_3\omega_1 & 0 \end{pmatrix}.$$

Consider $\omega = (0, \omega_2, 0)$ with $\omega_2 \neq 0$:

$$\det(Df(\omega) - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & -C_1\omega_2 \\ \\ 0 & -\lambda & 0 \\ \\ -C_3\omega_2 & 0 & -\lambda \end{pmatrix}$$

$= -\lambda^3 + C_1C_3\omega_2^2\lambda = -\lambda\left(\lambda^2 - C_1C_3\omega_2^2\right).$
This has a positive root, $\sqrt{C_1C_3}\ |\omega_2|$, so $(0, \omega_2, 0)$ is unstable.

For $(\omega_1, 0, 0)$

$$\begin{aligned} \det(Df - \lambda I) &= \det \begin{pmatrix} -\lambda & 0 & 0 \\ \\ 0 & -\lambda & C_2\omega_1 \\ \\ 0 & -C_3\omega_1 & -\lambda \end{pmatrix} \\ &= -\lambda^3 - \lambda C_2C_3\omega_1^2 \\ &= -\lambda(\lambda^2 + C_2C_3\omega_1^2). \end{aligned}$$

The roots are 0 and $\pm\, i\, \sqrt{C_2C_3}\omega_l$ so linearization yields no conclusion. Similarly for $(0, 0, \omega_3)$.

Note that

$$
\begin{aligned}
\omega_1\dot{\omega}_1 &= -C_1\omega_1\omega_2\omega_3 \\
\omega_2\dot{\omega}_2 &= C_2\omega_1\omega_2\omega_3 \\
\omega_3\dot{\omega}_3 &= -C_3\omega_1\omega_2\omega_3
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{d}{dt}\left(C_2\omega_1^2 + C_1\omega_2^2\right) &= 0 \\
\frac{d}{dt}\left(C_2\omega_3^2 + C_3\omega_2^2\right) &= 0.
\end{aligned}
$$

Suppose $\Omega_1 > 0$ and

$$
|\omega(0) - (\Omega_1, 0, 0)| < \delta,
$$

then

$$
\begin{aligned}
C_2\omega_3^2(t) + C_3\omega_2^2(t) &= C_2\omega_3^2(0) + C_3\omega_2^2(0) \\
&\leq (C_2 + C_3)(\omega_2^2(0) + \omega_3^2(0)) < (C_2 + C_3)\delta^2,
\end{aligned}
$$

so

$$
\omega_2^2(t) + \omega_3^2(t) < D\delta^2
$$

where

$$
D = \frac{C_2 + C_3}{\min(C_2, C_3)}.
$$

Also

$$
\begin{aligned}
|C_2(\omega_1^2(t) - \Omega_1^2)| &= |C_2(\omega_1^2(0) - \Omega_1^2) + C_1(\omega_2^2(0) - \omega_2^2(t))| \\
&\leq C_2|\omega_1^2(0) - \Omega_1^2| + C_1(\omega_2^2(0) + \omega_2^2(t)) \\
&\leq C_2|\omega_1(0) - \Omega_1|\ |2\Omega_1 + \omega_1(0) - \Omega_1| + C_1(\delta^2 + D\delta^2) \\
&\leq C_2\delta(2\Omega_1 + \delta) + C_1(1 + D)\delta^2.
\end{aligned}
$$

Taking δ small enough that the right hand side is $< C_2\Omega_1^2$ forces $\omega_1(t) > 0\ \forall t$ and then

$$\begin{aligned}
C_1\Omega_1|\omega_1(t) - \Omega_1| &\leq C_1(\omega_1(t) + \Omega_1)|\omega_1(t) - \Omega_1| \\
&= |C_2(\omega_1^2(t) - \Omega_1^2)| \\
&\leq C_2(2\Omega_1 + \delta)\delta + C_1(D+1)\delta^2.
\end{aligned}$$

It follows that $(\Omega_1, 0, 0)$ is stable. Similarly for $(0, 0, \Omega_3)$.

7 Boundary Value Problems

A. Preliminary Points

Standing Assumption Let p, p', q, f be continuous on $[a, b]$ with

$$p(x) > 0 \quad \forall x \in [a, b]$$

and

$$q(x) \geq 0 \quad \forall x \in [a, b].$$

Theorem 7.1. *Let U and $W \in C^2[a, b]$ with*

$$\begin{aligned}
(pU')' - qU &= f \\
(pW')' - qW &= f
\end{aligned}$$

on $[a, b]$.

(a) $U(a) = W(a)$ and $U(b) = W(b) \Rightarrow U \equiv W$ on $[a, b]$.

(b) $U'(a) = W'(a)$ and $U(b) = W(b) \Rightarrow U \equiv W$.

(c) $U(a) = W(a)$ and $U'(b) = W'(b) \Rightarrow U \equiv W$.

If we further assume that $\exists x \in [a, b]$ s.t. $q(x) > 0$ then

(d) $U'(a) = W'(a)$ and $U'(b) = W'(b) \Rightarrow U \equiv W$.

$$
\begin{aligned}
\textit{(e)} \quad U(a) &= U(b),\ U'(a) = U'(b),\ W(a) = W(b), \\
W'(a) &= W'(b),\ p(a) = p(b) \Rightarrow U \equiv W.
\end{aligned}
$$

Proof. Let $S = U - W$ and note that

$$(pS')' - qS = f - f = 0$$

and so

$$
\begin{aligned}
0 &= -\int_a^b (0)(S)dx = \int_a^b ((pS')' - qS)'(-S)dx \\
&= \int_a^b (qS^2 - (pSS')' + p(S')^2)dx
\end{aligned}
$$

so

(*) $$\int_a^b (p(S')^2 + qS^2)dx = pSS'\Big|_a^b$$

In case (b)

$$S'(a) = 0 \quad \text{and} \quad S(b) = 0$$

so

(**) $$pSS'|_a^b = 0.$$

Similarly (**) holds in cases (a), (c), and (d).
In case (e)

$$S(a) = U(a) - W(a) = U(b) - W(b) = S(b)$$

and

$$S'(a) = S'(b)$$

so (**) still holds. By (*)

$$\int_a^b (p(S')^2 + qS^2)dx = 0.$$

Since $p > 0$, $S' \equiv 0$ and $S =$ constant follows. In cases (a), (b), (c) $S \equiv 0$ follows. Assume $\exists x_0 \in [a, b]$ s.t. $q(x_0) > 0$.

$$S(x_0) \neq 0 \Rightarrow \int_a^b qS^2 \, dx > 0$$

so $S(x_0) = 0$. Hence $S \equiv 0$ in cases (d) and (e) also. $\square$

Examples

1. $u'' = 0 \quad u'(0) = u'(1) = 0$ has infinitely many solutions, $u =$ constant.
2. $u'' = 0 \quad u(0) = u(1) \quad u'(0) = u'(1)$ does too.
3. $(p = 1, q = -1)$. Consider $(b > 0)$

$$\begin{cases} (pu')' - qu = u'' + u = 0 \quad \text{on} \quad [0, b] \\ u(0) = u(b) = 0. \end{cases}$$

$$u = C_1 \cos x + C_2 \sin x$$

$$\begin{aligned} 0 = u(0) &= C_1 \\ u &= C_2 \sin x \end{aligned}$$

   (a) If $b \notin \{k\pi : k \in \mathbb{N}\}$ then

$$0 = u(b) = C_2 \sin b \Rightarrow C_2 = 0 \Rightarrow u \equiv 0.$$

   (b) If $b = k\pi$ then

$$u = C_2 \sin x$$

   is a solution $\forall C_2 \in \mathbb{R}$.

4. $u'' + u = 0 \quad u(0) = 0 \quad u(\pi) = 1$ has no solution.

Comment

If

$$(pU_1')' - qU_1 = f$$

$$U_1(a) = 0 \quad U_1'(b) = 0$$

and

$$(pU_2')' - qU_2 = 0$$

$$U_2(a) = A \;\; U_2'(b) = B$$

then $U := U_1 + U_2$ satisfies

$$(pU')' - qU = f$$

$$U(a) = A \;\; U'(b) = B$$

We'll focus on U_1.

B. Green's Functions

Idea

Define $G(x, y)$ by

$$\begin{cases} \dfrac{d}{dx}\left(p(x)\dfrac{dG}{dx}\right) - q(x)G(x, y) = \delta(x - y) \\ \\ G(a, y) = 0 \quad \dfrac{dG}{dx}(b, y) = 0. \end{cases}$$

Now consider

$$\begin{cases} (pU')' - qU = f \\ \\ U(a) = U(b) = 0. \end{cases}$$

Approximate f : Let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$,

$$f(x) \approx \sum_{i=1}^{n-1} f(x_i)\Delta x \delta(x - x_i) =: \tilde{f}$$

(so that $\int_a^b f(x)dx \approx \sum_{i=1}^{n-1} f(x_i)\Delta x$). But we may check that

$$\tilde{U} := \sum_{i=1}^{n-1} f(x_i)\Delta x G(x, x_i)$$

satisfies

$$\begin{aligned}
&(p\tilde{U}')' - q\tilde{U} \\
= \ & \sum_{i=1}^{n-1} f(x_i)\Delta x \left(\left(p\frac{dG}{dx}(x, x_i) \right)' - qG(x, x_i) \right) \\
= \ & \sum_{i=1}^{n-1} f(x_i)\Delta x \delta(x - x_i) = \tilde{f},
\end{aligned}$$

$$\begin{aligned}
\tilde{U}(a) &= \sum_{i=1}^{n-1} f(x_i)\Delta x G(a, x_i) = 0, \\
\tilde{U}(b) &= \sum_{i=1}^{n-1} f(x_i)\Delta x \frac{dG}{dx}(b, x_i) = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
U(x) &\approx \tilde{U}(x) = \sum_{i=1}^{n-1} f(x_i)\Delta x G(x, x_i) \\
&\approx \int_a^b f(y)G(x, y)dy
\end{aligned}$$

and

$$U(x) = \int_a^b f(y)G(x, y)dy.$$

Example

$p = 1 \quad q = 0 \quad a = 0 \quad b = 1$

$$\begin{cases} \dfrac{d^2G}{dx^2} &= \delta(x-y) \\ \\ G(0,y) &= \dfrac{dG}{dx}(1,y) = 0 \end{cases}$$

For $0 \le x < y$

$$\frac{d^2G}{dx^2} = 0$$

so

$$G(x,y) = C_0(y) + C_1(y)x.$$

Also

$$0 = G(0,y) = C_0(y).$$

Similarly for $x \in (y,1]$.

$$G(x,y) = C_2(y) + C_3(y)x$$

and

$$0 = \frac{dG}{dx}(1,y) = C_3(y).$$

So

$$G(x,y) = \begin{cases} C_1(y)x & \text{if} \quad 0 \le x < y \\ \\ C_2(y) & \text{if} \quad y < x \le 1. \end{cases}$$

Next for $y \in (0,1)$ and ε sufficiently small

$$
\begin{aligned}
1 &= \int_{y-\varepsilon}^{y+\varepsilon} \delta(x-y)dx \\
&= \int_{y-\varepsilon}^{y+\varepsilon} \frac{d^2G}{dx^2}(x,y)dx \\
&= \frac{dG}{dx}(y+\varepsilon,y) - \frac{dG}{dx}(y-\varepsilon,y) \\
&= 0 - C_1(y)
\end{aligned}
$$

so

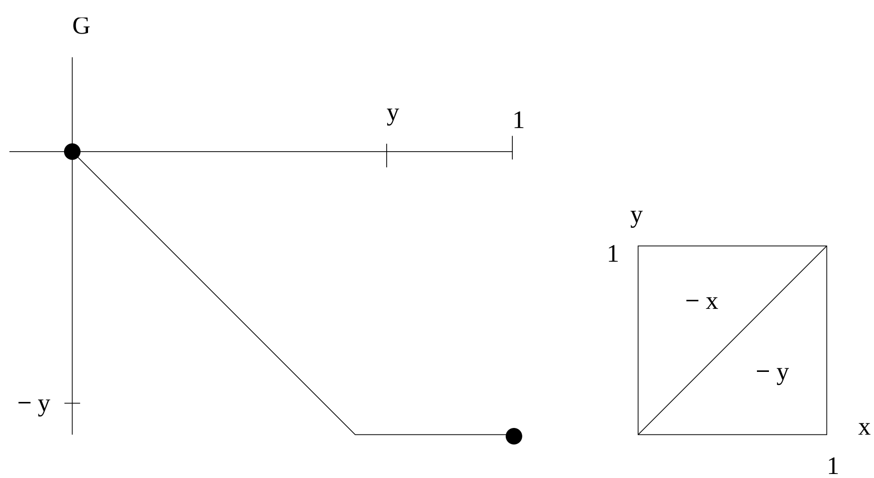
$$C_1(y) = -1.$$

Finally, $\dfrac{dG}{dx}$ has a jump discontinuity so G is continuous and hence

$$\lim_{x\to y^-} G(x,y) = C_1(y)y = -y = \lim_{x\to y^+} G(x,y) = C_2(y).$$

Finally

$$G(x,y) = \begin{cases} -x & \text{if} \quad x \le y \\ -y & \text{if} \quad y \le x. \end{cases}$$



Comments

1. $$G(x,y)=\begin{cases} -y & \text{if} \quad y \le x \\ -x & \text{if} \quad x \le y \end{cases}$$

2. $$\begin{aligned} U(x) &= \int_0^1 f(y)G(x,y)dy \\ &= \int_0^x (-y)f(y)dy + \int_x^1 (-x)f(y)dy \end{aligned}$$

is the solution of

$$\begin{cases} U'' &= f \\ U(0) &= U'(1) = 0. \end{cases}$$

3. $$\begin{aligned} G(y,x) &= \begin{cases} -x & \text{if} \quad x \le y \\ -y & \text{if} \quad y \le x \end{cases} \\ &= G(x,y) \end{aligned}$$

Constructing G

Consider

$$\begin{cases} \dfrac{d}{dx}(p(x)\dfrac{dG}{dx}) - q(x)G &= \delta(x-y) \\ G(a,y) = \dfrac{dG}{dx}(b,y) &= 0. \end{cases}$$

Let $\mathcal{L}(x)$ be the solution of

$$\begin{cases} (p\mathcal{L}')' - q\mathcal{L} = 0 \\ \mathcal{L}(a) = 0 \quad \mathcal{L}'(a) = 1 \end{cases}$$

and $\mathcal{R}(x)$ be the solution of

$$(p\mathcal{R}')' - q\mathcal{R} = 0$$

$$\mathcal{R}(b) = 1 \quad \mathcal{R}'(b) = 0.$$

Then

$$G(x,y) = \begin{cases} C_1(y)\mathcal{L}(x) & \text{if} \quad x < y \\ \\ C_2(y)\mathcal{R}(x) & \text{if} \quad y < x. \end{cases}$$

We require

$$\begin{aligned} 1 &= \int_{y-\varepsilon}^{y+\varepsilon} \delta(x-y)dx = \int_{y-\varepsilon}^{y+\varepsilon} \left(\left(p\frac{dG}{dx} \right)' - qG \right) dx \\ &= p(x)\frac{dG}{dx}(x,y)\bigg|_{x=y-\varepsilon}^{y+\varepsilon} - \int_{y-\varepsilon}^{y+\varepsilon} qG\ dx \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. Letting $\varepsilon \to 0^+$:

$$\begin{aligned} 1 &= p(y)\left(\lim_{x\to y^+} \frac{dG}{dx}(x,y) - \lim_{x \to y^-} \frac{dG}{dx}(x,y) \right) \\ &= p(y)(C_2(y)\mathcal{R}'(y) - C_1(y)\mathcal{L}'(y)). \end{aligned}$$

Also, G is continuous at $x = y$ so

$$C_1(y)\mathcal{L}(y) = C_2(y)\mathcal{R}(y).$$

Thus

$$\begin{pmatrix} \mathcal{L}(y) & -\mathcal{R}(y) \\ \\ -\mathcal{L}'(y) & \mathcal{R}'(y) \end{pmatrix} \begin{pmatrix} C_1(y) \\ \\ C_2(y) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{p(y)} \end{pmatrix}.$$

We may solve uniquely for $C_1(y)$ and $C_2(y)$ provided

$$\begin{aligned} 0 &\neq \det \begin{pmatrix} \mathcal{L}(y) & -\mathcal{R}(y) \\ \\ -\mathcal{L}'(y) & \mathcal{R}'(y) \end{pmatrix} \\ &= \mathcal{L}(y)\mathcal{R}'(y) - \mathcal{L}'(y)\mathcal{R}(y). \end{aligned}$$

Let $W = \mathcal{L}\mathcal{R}' - \mathcal{L}'\mathcal{R}$. Then

$$\begin{aligned}
(pW)' &= (\mathcal{L}(p\mathcal{R}') - \mathcal{R}(p\mathcal{L}'))' \\
&= \mathcal{L}'(p\mathcal{R}') + \mathcal{L}q\mathcal{R} \\
&\quad -\mathcal{R}'(p\mathcal{L}') - \mathcal{R}(q\mathcal{L}) = 0
\end{aligned}$$

so

$$\begin{aligned}
\frac{p(y)}{p(a)}W(y) &= W(a) = \mathcal{L}(a)\mathcal{R}'(a) - \mathcal{L}'(a)\mathcal{R}(a) \\
&= -\mathcal{R}(a).
\end{aligned}$$

If $W(y) = 0$ then $\mathcal{R}(y)$ is a nonzero solution of

$$\begin{aligned}
(pU')' - qU &= 0 \\
U(a) = U'(b) &= 0,
\end{aligned}$$

contradicting the uniqueness theorem of the previous section. Thus

$$G(x, y) = \begin{cases} C_1(y)\mathcal{L}(x) & \text{if} \quad x \le y \\ C_2(y)\mathcal{R}(x) & \text{if} \quad y \le x \end{cases}$$

where

$$\begin{aligned}
C_2(y)\mathcal{R}'(y) - C_1(y)\mathcal{L}'(y) &= \frac{1}{p(y)} \\
C_1(y)\mathcal{L}(y) &= C_2(y)\mathcal{R}(y).
\end{aligned}$$

Example

$$p = 1 \quad q = 0 \quad a = 0 \quad b = 1$$

$$\mathcal{L}(x) = x \qquad \mathcal{R}(x) = 1$$

$$\begin{aligned} 0 - C_1(y)(1) &= 1 \\ C_1(y)y &= C_2(y) \\ C_1(y) &= -1 \\ C_2(y) &= -y \end{aligned}$$

$$G(x,y) = \begin{cases} -x & \text{if} \quad x \leq y \\ -y & \text{if} \quad y \leq x \end{cases}$$

as before.

Theorem 7.2. *Let*

$$U(x) = \int_a^b f(y)G(x,y)dy$$

then

$$\begin{aligned} (pU')' - qU &= f \\ U(a) = U'(b) &= 0. \end{aligned}$$

Proof.

$$U(x) = \int_a^x C_2(y)\mathcal{R}(x)f(y)dy + \int_x^b C_1(y)\mathcal{L}(x)f(y)dy.$$

So

$$\begin{aligned} U(a) &= \int_a^a C_2(y)\mathcal{R}(x)f(y)dy + \int_a^b C_1(y)\mathcal{L}(a)f(y)dy \\ &= 0 \ \text{ since } \ \mathcal{L}(a) = 0. \end{aligned}$$

Next

$$
\begin{aligned}
U'(x) &= C_2(x)\mathcal{R}(x)f(x) + \int_a^x C_2(y)\mathcal{R}'(x)f(y)dy \\
&\quad -C_1(x)\mathcal{L}(x)f(x) + \int_x^b C_1(y)\mathcal{L}'(x)f(y)dy \\
&= \int_a^x C_2(y)f(y)dy\mathcal{R}'(x) + \int_x^b C_1(y)f(y)dy\mathcal{L}'(x)
\end{aligned}
$$

and

$$U'(b) = 0 \quad \text{since} \quad \mathcal{R}'(b) = 0.$$

Lastly,

$$
\begin{aligned}
& (pU')' - qU \\
&= (p\mathcal{R}' \int_a^x C_2 f\, dy + p\mathcal{L}' \int_x^b C_1 f\, dy)' \\
&\quad -q\mathcal{R} \int_a^x C_2 f\, dy - q\mathcal{L} \int_x^b C_1 f \\
&= ((p\mathcal{R}')' - q\mathcal{R}) \int_a^x C_2 f\, dy \\
&\quad +((p\mathcal{L}')' - q\mathcal{L}) \int_x^b C_1 f\, dy \\
&\quad +p\mathcal{R}'C_2 f - p\mathcal{L}'C_1 f \\
&= p(x)f(x)(\mathcal{R}'(x)C_2(x) - \mathcal{L}'(x)C_1(x)) \\
&= p(x)f(x)\frac{1}{p(x)} = f.
\end{aligned}
$$

□

Comment

$$\begin{cases} U''(t) &= f(t) \\ U(0) &= U'(0) = 0 \end{cases}$$

$$\Rightarrow U(t) = \int_0^t (t-\tau) f(\tau) d\tau$$

U at time t depends on $f(\tau)$ for $0 \le \tau < t$ and not for $\tau > t$.

$$\begin{cases} U''(x) &= f(x) \\ U(0) &= U'(1) = 0 \end{cases}$$

$$\Rightarrow U(x) = \int_0^1 f(y) G(x,y) dy$$

depends on $f(y)$ for all $y \in [0,1]$.

C. Convolutions
Example

$$\begin{cases} u''(x) - u(x) = f(x) \quad & x \in \mathbb{R} \\ u \to 0 \quad \text{as} \quad |x| \to \infty \end{cases}$$

Comment This problem is translation invariant in the following sense: if $U(x) = u(x - x_0)$ then

$$\begin{cases} U''(x) - U(x) &= u''(x-x_0) - u(x-x_0) \\ &= f(x - x_0) \\ U \to 0 \quad \text{as} &\quad |x| \to \infty. \end{cases}$$

Comment

$$\begin{cases} u'' - u &= f \\ u(0) = u(1) &= 0 \end{cases}$$

is not translation invariant since

$$U(x) = u(x - x_0)$$

vanishes at x_0 and $1 + x_0$, not at 0 and 1.

Comment

$$\begin{cases} u'' - q(x)u = f \\ \\ u \to 0 \quad \text{as} \quad |x| \to \infty \end{cases}$$

is not translation invariant if q is not constant since

$$\begin{aligned} & U''(x) - q(x)U(x) \\ = \;& u''(x - x_0) - q(x)u(x - x_0) \\ \neq \;& u''(x - x_0) - q(x - x_0)u(x - x_0) = 0. \end{aligned}$$

Comment: Define F by

$$\begin{cases} F'' - F = \delta(x) \\ \\ F \to 0 \quad \text{as} \quad |x| \to \infty. \end{cases}$$

Then $G(x, x_0) = F(x - x_0)$ satisfies

$$\begin{cases} \dfrac{d^2G}{dx^2} - G & = & \delta(x - x_0) \\ \\ \quad G \to 0 & & \text{as } |x| \to \infty. \end{cases}$$

Comment $\begin{cases} u'' - u & = & f(x) \\ \\ u \to 0 & & 0 = |x| \to \infty \end{cases}$

$\Rightarrow$

$$\begin{aligned} u(x) &= \int_{-\infty}^{\infty} G(x, x_0) f(x_0) dx_0 \\ &= \int_{-\infty}^{\infty} F(x - x_0) f(x_0) dx_0 \end{aligned}$$

Definition Integrals of the form

$$\int_{-\infty}^{\infty} F(x - x_0) f(x_0) dx_0$$

and of the form

$$\int_0^t F(t-\tau) f(\tau) d\tau$$

are both called convolutions.

Comment Recall that

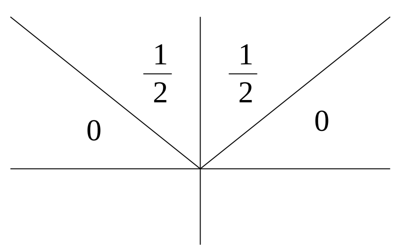
$$\dot{X} = AX + f(t) \Rightarrow X(t) = e^{At} X(0) + \int_0^t e^{A(t-\tau)} f(\tau) d\tau$$

for A constant.

Examples

1. $$\begin{aligned} \frac{\partial^z u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= f(t,x) \\ u(0,x) &= \frac{\partial u}{\partial t}(0,x) = 0 \\ u(t,x) &= \frac{1}{2} \int_0^t \int_{x-(t-\tau)}^{x+(t+\tau)} f(\tau, y)) dy\, d\tau \end{aligned}$$

Let



$$F(t,x) = \begin{cases} \frac{1}{2} & \text{if} \quad |x| < t \\ 0 & \quad \text{else} \end{cases}$$

then

$$\int_0^t \int_{-\infty}^{\infty} F(t-\tau, x-y) f(\tau, y) dy\, d\tau = u(t,x).$$

2. $$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(t,x) \\ u(0,x) &= 0 \\ u(t,x) &= \int_0^t \int F(t-\tau, x-y) f(\tau, y) dy\, d\tau \end{aligned}$$

where

$$F(t,x) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{|x|^2}{4t}}.$$

3. $$\begin{cases} u'' - u = f \\ u \to 0 \quad \text{as} \quad |x| \to \infty \end{cases}$$

Solve

$$\begin{cases} F'' - F = \delta(x) \\ f \to 0 \qquad\qquad : \end{cases}$$

$$F(x) = \begin{cases} Ae^{x} & x < 0 \\ Be^{-x} & 0 < x. \end{cases}$$

$$\begin{aligned} 1 &= \lim_{x \to 0^+} F'(x) - \lim_{x \to 0^-} F'(x) \\ &= \lim_{x \to 0^+} (-Be^{-x}) - \lim_{x \to 0^-} (Ae^{x}) = -B - A \end{aligned}$$

and

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^-} F(x) = A = B$$

so

$$A = B = -\frac{1}{2}$$

and

$$F(x) = -\frac{1}{2}e^{-|x|}.$$

Hence

$$\begin{aligned} u(x) &= \int F(x-y)f(y)dy \\ &= -\frac{1}{2}\int e^{-|x-y|}f(y)dy. \end{aligned}$$

Comment

$$\begin{cases} u'' = \delta(x) \\ u \to 0 \ \text{ as } \ |x| \to \infty \end{cases}$$

has no solution. If we take

$$F(x) = \frac{1}{2}|x|$$

and

$$\begin{aligned} u(x) &= \int_{-\infty}^{\infty} F(x-y)f(y)dy \\ &= \frac{1}{2}\int_{-\infty}^{x}(x-y)f(y)dy + \frac{1}{2}\int_{x}^{\infty}(y-x)f(y)dy \end{aligned}$$

(assuming f decays at infinity) then

$$u'(x) = \frac{1}{2}\int_{-\infty}^{x} f(y)dy - \frac{1}{2}\int_{x}^{\infty} f(y)dy$$

and

$$u''(x) = \frac{1}{2}f(x) - \frac{1}{2}f(x)(-1) = f(x).$$

If we also assume that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} xf(x)dx = 0$$

then

$$\begin{aligned} u(x) &= u(x) - \frac{1}{2}\int_{-\infty}^{\infty}(x-y)f(y)dy \\ &= \int_{x}^{\infty}(y-x)f(y)dy \to 0 \quad \text{as} \quad x \to +\infty. \end{aligned}$$

Similarly

$$u \to 0 \quad \text{as} \quad x \to -\infty.$$

D. Numerics

Consider

(IVP) $$\begin{cases} U'' &= F(U) + f(x) \quad x > 0 \\ U(0) &= U'(0) = 0 \end{cases}$$

and

(BVP) $$\begin{cases} U'' &= F(U) + f(x) \quad 0 < x < 1 \\ U(0) &= U(1) = 0 \qquad . \end{cases}$$

Let $\Delta x = \dfrac{1}{N}$, $x_k = k\Delta x$, and use

$$\begin{aligned} U''(x_k) &\approx \frac{\frac{U(x_{k+1})-U(x_k)}{\Delta x} - \frac{U(x_k)-U(x_{k-1})}{\Delta x}}{\Delta x} \\ &= \frac{U(x_{k+1}) - 2U(x_k) + U(x_{k-1})}{\Delta x^2}. \end{aligned}$$

For (IVP) $U(x_k) \approx U_k$ where

$$
(\tilde{\text{IVP}}) \quad \begin{cases} U_0 = 0 & \\ U_1 = 0 & \begin{aligned} U_1 &\approx U(\Delta x) \\ &\approx U(0) + U'(0)\Delta x \end{aligned} \\ \dfrac{U_{k+1} - 2U_k + U_{k-1}}{\Delta x^2} = F(U_k) + f(x_k) \quad k \geq 1 \end{cases}
$$

Solving this is just iteration:

$$
U_{k+1} = 2U_k - U_{k-1} + \Delta x^2 (F(U_k) + f(x_k)).
$$

For (BVP) $U(x_k) \approx U_k$ where

$$
(\tilde{\text{BVP}}) \quad \begin{cases} U_0 = 0 \\ U_N = 0 \\ \dfrac{U_{k+1} - 2U_k + U_{k-1}}{\Delta x^2} = F(U_k) + f(x_k) \quad 1 \leq k \leq N-1 \end{cases}
$$

It's not clear when there is a solution.

Linear Case $F(u) = Qu$

$$
U_{k+1} - (2 + \Delta x^2 Q) U_k + U_{k-1} = \Delta x^2 f(x_k)
$$

$$
U_2 - (2 + \Delta x^2 Q) U_1 = \Delta x^2 f(x_1)
$$

$$
- \quad (2 + x^2 Q) U_{N-1} + U_{N-2} = \Delta x^2 f(x_{N-1})
$$

Let

$$
\begin{aligned}
\vec{U} &= (U_1, \ldots, U_{N-1})^T \\
\vec{f} &= \Delta x^2 (f(x_1), \ldots, f(x_{N-1}))^T
\end{aligned}
$$

and

$$A = \begin{pmatrix} -(2+\Delta x^2 Q) & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -(2+\Delta x^2 Q) \end{pmatrix}$$

then

$$A\vec{U} = \vec{f}.$$

Comments

1. If A is invertible

$$\vec{U} = A^{-1}\vec{f}$$

is the unique solution of $(\tilde{\text{BVP}})$ (no matter what $f(x)$ is). A^{-1} is related to the Green's function.

2. Recall

$$\begin{cases} (pU')' - q(x)U &= f \\ U(0) = U(1) &= 0 \end{cases}$$

has a unique solution when $p > 1$ and $q \geq 0$. Consider $p(x) = 1 > 0$ and $q(x) = Q \geq 0$. Claim A is invertible. It suffices to show that

$$(\tilde{\text{BVP}}) \qquad \begin{cases} U_0 = U_N = 0 \\ \dfrac{U_{k+1} - 2U_k + U_{k-1}}{\Delta x^2} = QU_k \qquad k = 2, \ldots, N-1 \end{cases}$$

$$\Rightarrow U_k = 0 \qquad k = 0, \ldots, N.$$

$$Q\sum_{k=1}^{N-1}U_k^2 = \Delta x^{-2}\sum_{k=1}^{N-1}U_k((U_{k+1}-U_k)-(U_k-U_{k-1})),$$

$$\begin{aligned}
0 \le \Delta x^2 Q\sum_1^{N-1}U_k^2 &= \sum_1^{N-1}U_k(U_{k+1}-U_k) - \sum_0^{N-2}U_{k+1}(U_{k+1}-U_k) \\
&= \sum_0^{N-1}U_k(U_{k+1}-U_k) - \sum_0^{N-1}U_{k+1}(U_{k+1}-U_k) \\
&= -\sum_0^{N-1}(U_{k+1}-U_k)^2
\end{aligned}$$

so

$$\begin{aligned}
\sum_0^{N-1}(U_{k+1}-U_k)^2 &= 0, \\
U_{k+1}-U_k = 0 & \qquad k = 0,\ldots,N-1, \\
0 = U_0 = U_1 = \ldots &= U_N.
\end{aligned}$$

E. A Nonlinear Problem

$$\text{(NBVP)} \qquad \begin{cases} U'' = -2|U|^3 & 0 < x < 1 \\ U(0) = U(1) = 0 \end{cases}$$

Shooting Method: $W(x,\lambda)$

$$\begin{cases} \dfrac{d^2W}{dx^2} = -2|W|^3 & x > 0 \\ W(0,\lambda) = 0 \\ \dfrac{dW}{dx}(0,\lambda) = \lambda \end{cases}$$

Seek λ s.t. $W(1,\lambda) = 0$. Note that $\lambda = 0$ gives a solution. For $\lambda < 0$, $W(x,\lambda) < 0 \ \forall x > 0$. Consider $\lambda > 0 : \exists L(\lambda) > 0$ s.t.

$$\begin{cases} \dfrac{dW}{dx}(x,\lambda) & > & 0 \quad \text{if} \quad 0 \leq x < L(\lambda) \\ \dfrac{dW}{dx}(L(\lambda),\lambda) & = & 0. \end{cases}$$

Note that

$$W(L(\lambda)+x,\lambda) = W(L(\lambda)-x,\lambda)$$

(argue this by uniqueness) and hence

$$W(2L(\lambda),\lambda) = W(0,\lambda) = 0.$$

Thus we seek $\lambda > 0$ s.t. $L(\lambda) = \dfrac{1}{2}$. For $0 \leq x \leq L(\lambda)$

$$\frac{d^2W}{dx^x} + 2W^3 = 0$$

so

$$\left(\frac{dW}{dx}\right)^2 + W^4 = \text{ Constant } = \lambda^2 + 0^4.$$

Evaluating at $x = L(\lambda)$ yields

$$0^2 + W^4(L(\lambda),\lambda) = \lambda^2$$

so

$$W(L(\lambda),\lambda) = \sqrt{\lambda}.$$

Now for $0 \leq x \leq L(\lambda)$

$$
\begin{aligned}
\frac{dW}{dx}(x,\lambda) &= \sqrt{\lambda^2 - W^4} \\
L &= \int_0^L \frac{W'(x,\lambda)dx}{\sqrt{\lambda^2 - W^4(x,\lambda)}} \\
&= \lambda^{-1}\int_0^L \frac{W'(x,\lambda)dx}{\sqrt{1-\lambda^{-2}W^4(x,\lambda)}} \\
&\underset{\substack{v=\frac{W(x,\lambda)}{\sqrt{\lambda}} \\ x=L\Rightarrow v=1}}{=} \lambda^{-1}\int_0^1 \frac{\sqrt{\lambda}\,dv}{\sqrt{1-v^4}} = C\lambda^{-\frac{1}{2}}
\end{aligned}
$$

where

$$
C = \int_0^1 \frac{dv}{\sqrt{1-v^4}}.
$$

$\exists!\lambda > 0$ s.t. $L(\lambda) = \dfrac{1}{2}$.

$\exists!$ positive solution of (NBVP).

$\exists$ exactly 2 solutions of (NBVP).

Comment

For

$$
\text{(LBVP)} \qquad \begin{cases} U'' + a(x)U' + b(x)U = 0 & 0 < x < 1 \\ U(0) = U(1) = 0 \end{cases}
$$

there is either one solution ($U \equiv 0$) or infinitely many (if $U \neq 0$ is a solution then CU is $\forall C \in \mathbb{R}$).