VIETNAM NATIONAL UNIVERSITY, HO CHI MINH CITY
UNIVERSITY OF TECHNOLOGY
FACULTY OF COMPUTER SCIENCE AND ENGINEERING

DISCRETE STRUCTURES FOR COMPUTING (CO1007)

Assignment

Relation - Counting - Probability and Graph

Advisor: Fullname
Students: Fullname of Student 1 - Student 1 ID numbers.
Fullname of Student 2 - Student 2 ID numbers.

HO CHI MINH CITY, SEPTEMBER 2020

Contents

1 Member list & Workload

No.	Fullname	Student ID	Problems	Percentage of work
1	Lưu Quốc Bình	2033009	- Exercise 1 Bonus: 1, 2, 3. - Probability: 1, 2, 3.	30%
2	Nguyễn Văn B	19181717	- Relation & Counting: 4, 5, 6 Bonus: 4, 5, 6. - Graph: 1, 2, 3, Bonus: 1, 2, 3.	20%
1	Nguyễn Văn A	19181716	- Relation & Counting: 1, 2, 3 Bonus: 1, 2, 3. - Probability: 1, 2, 3.	30%
1	Nguyễn Văn A	19181716	- Relation & Counting: 1, 2, 3 Bonus: 1, 2, 3. - Probability: 1, 2, 3.	30%

2 Relation & Counting

2.1 Problem 1

Write on the report a very detailed introduction to the IVPs Sys. (3) and the formulae of its exact solutions for general a, b, c, and d and initial condition R0 and J0. Then complete Tab. 2 for all possible cases of eigenvalues of general 2 × 2 matrix A

2.1.1 Method of solving system of differential equations

Consider system:

$$\begin{cases} R' = aR + bJ \\ J' = cR + dJ \\ R(0) = R_0 \, J(0) = J_0 \end{cases} \tag{1}$$

written in vector form

$$\vec{x'} = A\vec{X}$$

will have solution form

$$\vec{X} = \vec{\eta} e^{\lambda t}$$

where λ and $\vec{\eta}$ are eigenvalues and eigenvectors of the matrix A , and

$$\vec{x'} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \vec{X}(0) = \begin{pmatrix} R_0 \\ J_0 \end{pmatrix}$$

We are going to be looking for two solutions $\vec{X_1}(t)$ and $\vec{X_2}(t)$ where the determinant of the matrix

$$X = \begin{pmatrix} \vec{X_1} & \vec{X_2} \end{pmatrix}$$

So, the first thing that we need to do is find the eigenvalues for the matrix.

$$det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

simplifying to

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

we have

$$\Delta = [-(a+d)]^2 - 4(ad - bc)$$

general solution in this case will be

$$\vec{X}(t) = C_1 e^{\lambda_1 t} \vec{\eta}^{(1)} + C_2 e^{\lambda_2 t} \vec{\eta}^{(2)} \tag{2}$$

2.1.1.a $\Delta > 0$, *Real Eigenvalues*, (4) *have* 2 *solution*λ_1, λ_2

With λ_1, we'll need to solve,

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} (a-\lambda_1)\eta_1 + b\eta_2 \\ (d-\lambda_1)\eta_2 + c\eta_1 \end{pmatrix} 0 \Rightarrow \eta_1 = \frac{-b}{a-\lambda_1}\eta_2 = \frac{d-\lambda_1}{-c}\eta_2 \tag{3}$$

The eigenvector in this case is,

$$\eta^{\vec{(1)}} = \begin{pmatrix} \dfrac{-b}{a-\lambda_1}\eta_2 \\ \eta_2 \end{pmatrix}, \eta_2 \ \in \ R \tag{4}$$

and the same with λ_2

$$\eta^{\vec{(2)}} = \begin{pmatrix} \dfrac{-b}{a-\lambda_2}\eta_2' \\ \eta_2' \end{pmatrix}, \eta_2' \ \in \ R$$

$$\Leftrightarrow \vec{X}(t) = C_1 e^{\lambda_1 t} \begin{pmatrix} \dfrac{-b}{a-\lambda_1}\eta_2 \\ \eta_2 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} \dfrac{-b}{a-\lambda_2}\eta_2' \\ \eta_2' \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} R_0 \\ J_0 \end{pmatrix} = \vec{X}(0) = C_1 e^{\lambda_1 t} \begin{pmatrix} \dfrac{-b}{a-\lambda_1}\eta_2 \\ \eta_2 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} \dfrac{-b}{a-\lambda_2}\eta_2' \\ \eta_2' \end{pmatrix}$$

All we need to do now is multiply the constants through and we then get two equations (one for each row) that we can solve for the constants. This gives C1, and C2.

$$\left.\begin{matrix} \dfrac{-b}{a-\lambda_1}\eta_2 C_1 + \dfrac{-b}{a-\lambda_2}\eta_2' C_2 = R_0 \\ \eta_2 C1 + \eta_2' C2 = J_0 \end{matrix}\right\} \Rightarrow C1, C2$$

2.1.1.b $\Delta < 0$, *Complex Eigenvalues*, $\lambda_{1,2}$= **p** $\pm qi$

Following (3) using second equation, with $\lambda = p + qi$ (choose negative or positive, in this case I choose positive.), we have

$$\eta_1 = \frac{-b}{a-(p+qi)}\eta_2 = \frac{d-(p+qi)}{c}\eta_2 \tag{5}$$

So, the first eigenvector is,

$$\vec{\eta^{(1)}} = \begin{pmatrix} \frac{d-(p+qi)}{-c}\eta_2 \\ \eta_2 \end{pmatrix}, \eta_2 \in R$$

The solution corresponding to this eigenvalue and eigenvector is

$$\vec{X_1}(t) = e^{(p+qi)t} \begin{pmatrix} \frac{d-(p+qi)}{-c}\eta_2 \\ \eta_2 \end{pmatrix}, \eta_2 \in R$$

$$\vec{X_1}(t) = e^{pt}e^{qit} \begin{pmatrix} \frac{d-(p+qi)}{-c}\eta_2 \\ \eta_2 \end{pmatrix}$$

Apply Euler's formula (https://tutorial.math.lamar.edu/Classes/DE/ComplexRoots.aspxEulerFormula),

$$\vec{X_1}(t) = e^{pt}(cos(qt)+isin(qt)) \begin{pmatrix} \frac{d-(p+qi)}{-c}\eta_2 \\ \eta_2 \end{pmatrix}$$

Withdraw η_2,

$$\vec{X_1}(t) = e^{pt}\eta_2(cos(qt)+isin(qt)) \begin{pmatrix} \frac{d-(p+qi)}{-c} \\ 1 \end{pmatrix}$$

Transformation steps,

$$\vec{X_1}(t) = e^{pt}\eta_2(cos(qt)+isin(qt)) \begin{pmatrix} \frac{d-p-qi}{-c} \\ 1 \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt}\eta_2(cos(qt)+isin(qt)) \begin{pmatrix} \frac{(d-p)-qi}{-c} \\ 1 \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt}\eta_2 \begin{pmatrix} \frac{[cos(qt)+isin(qt)][(d-p)-qi]}{-c} \\ (cos(qt)+isin(qt)) \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt}\eta_2 \begin{pmatrix} \frac{cos(qt)(d-p)+isin(qt)(d-p)-qicos(qt)-qiisin(qt))]}{-c} \\ (cos(qt)+isin(qt)) \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt}\eta_2 \begin{pmatrix} \frac{(d-p)cos(qt)+i(d-p)sin(qt)-qicos(qt)+qsin(qt))]}{-c} \\ (cos(qt)+isin(qt)) \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt}\eta_2 \begin{pmatrix} \frac{[(d-p)cos(qt)+qsin(qt)]+[i(d-p)sin(qt)-qicos(qt))]}{-c} \\ (cos(qt)+isin(qt)) \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt}\eta_2 \begin{pmatrix} \frac{[(d-p)cos(qt)+qsin(qt)]+i[(d-p)sin(qt)-qcos(qt))]}{-c} \\ (cos(qt)+isin(qt)) \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt}\eta_2 \begin{pmatrix} \frac{[(d-p)cos(qt)+qsin(qt)]}{-c} \\ cos(qt) \end{pmatrix} + e^{pt}\eta_2 i \begin{pmatrix} \frac{[(d-p)sin(qt)-qcos(qt))]}{-c} \\ sin(qt) \end{pmatrix}$$

$$\vec{X_1}(t) = e^{pt} \begin{pmatrix} \frac{\eta_2[(d-p)cos(qt)+qsin(qt)]}{-c} \\ \eta_2 cos(qt) \end{pmatrix} + ie^{pt} \begin{pmatrix} \frac{\eta_2[(d-p)sin(qt)-qcos(qt))]}{-c} \\ \eta_2 sin(qt) \end{pmatrix}$$

$$\vec{X_1}(t) = \vec{u}(t) + i\vec{v}(t)$$

The general solution to this system then,

$$\boxed{\vec{X_1}(t) = C_1 e^{pt} \begin{pmatrix} \frac{\eta_2[(d-p)cos(qt)+qsin(qt)]}{-c} \\ \eta_2 cos(qt) \end{pmatrix} + C_2 e^{pt} \begin{pmatrix} \frac{\eta_2[(d-p)sin(qt)-qcos(qt))]}{-c} \\ \eta_2 sin(qt) \end{pmatrix} \quad (6)}$$

Now apply the initial condition and find the constants.

$$\boxed{\begin{pmatrix} R_0 \\ J_0 \end{pmatrix} = (0) = \mathrm{C}_1 \begin{pmatrix} \frac{\eta_2(d-p)}{-c} \\ \eta_2 \end{pmatrix} + C_2 \begin{pmatrix} \frac{\eta_2(-q)}{-c}\eta_2 \\ 0 \end{pmatrix}}$$

$$\left.\begin{aligned} &\frac{\eta_2(d-p)}{-c}C_1 + \frac{\eta_2(-q)}{-c}\eta_2 C_2 = R_0 \\ &\eta_2 C1 = J_0 \end{aligned}\right\} \Rightarrow C1, C2$$

2.1.1.c $\Delta = 0$, *Repeated Eigenvalues*

This is the final case that we need to take a look at. In this section we are going to look at solutions to the system,

$$\vec{x'} = A\vec{X}$$

where the eigenvalues are repeated eigenvalues. Since we are going to be working with systems in which A is a 2×2 matrix we will make that assumption from the start. So, the system will have a double eigenvalue, λ.

This presents us with a problem. We want two linearly independent solutions so that we can form a general solution. However, with a double eigenvalue we will have only one. Following this article, I have find form to find solution fo this case. First solution is,

$$\vec{X_1} = \vec{\eta}e^{\lambda t} = \begin{pmatrix} \frac{-b}{a-\lambda_1}\eta_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{d-\lambda_1}{-c}\eta_2 \\ \eta_2 \end{pmatrix}, (\forall \eta_2 \in R), because (3), (4) \quad (7)$$

And second solution will be

$$\vec{X_2} = te^{\lambda t}\vec{\eta} + e^{\lambda t}\vec{\rho} \quad (8)$$

will be a solution to the system provided $\vec{\rho}$ is a solution to

$$(A - \lambda I)\vec{\rho} = \vec{\eta} \quad (9)$$

Transform (9),

$$(9) \Rightarrow \begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

dựa vào (4), va thực hiện biến đổi tương tự, ta có

$$\Leftrightarrow \begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix} \begin{pmatrix} \vec{\rho_1} \\ \vec{\rho_2} \end{pmatrix} = \begin{pmatrix} \frac{-b}{a-\lambda_1}\eta_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{d-\lambda_1}{-c}\eta_2 \\ \eta_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} (a-\lambda_1)\rho_1 + b\rho_2 \\ (d-\lambda_1)\rho_2 + c\rho_1 \end{pmatrix} = \begin{pmatrix} \frac{-b}{a-\lambda_1}\eta_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \frac{d-\lambda_1}{-c}\eta_2 \\ \eta_2 \end{pmatrix}$$

$$\Rightarrow \rho_1 = (\frac{\frac{-b}{a-\lambda_1}\eta_2 - b\rho_2}{a-\lambda_1}) = (\frac{\frac{d-\lambda_1}{-c}\eta_2 - b\rho_2}{a-\lambda_1}), and \rho_2 \in R$$

hoặc

$$\rho_1 = (\frac{\frac{-b}{a-\lambda_1}\eta_2 - (d-\lambda_1)\rho_2}{c}) = (\frac{\frac{d-\lambda_1}{-c}\eta_2 - (d-\lambda_1)\rho_2}{c}), and \rho_2 \in R$$

Công thức nghiệm $\vec{X}$ của hệ sẽ là

$$\vec{X} = C_1\vec{X_1} + C_2\vec{X_2}$$

$$\Leftrightarrow \vec{X} = C_1\vec{\eta}e^{\lambda t} + C_2(te^{\lambda t}\vec{\eta} + e^{\lambda t}\vec{\rho})$$

$$\Leftrightarrow \begin{pmatrix} R \\ J \end{pmatrix} = C_1 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} e^{\lambda t} + C_2(te^{\lambda t} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix})$$

$$\Leftrightarrow \vec{X} = C_1 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} e^{\lambda t} + C_2 e^{\lambda t}(t \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix})$$

$$\boxed{\Leftrightarrow \vec{X} = C_1 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} e^{\lambda t} + C_2 e^{\lambda t} \begin{pmatrix} t\eta_1 + \rho_1 \\ t\eta_2 + \rho_2 \end{pmatrix}}$$

Now apply the initial condition and find the constants.

$$\Leftrightarrow \begin{pmatrix} R_0 \\ J_0 \end{pmatrix} = C_1 \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} e^{\lambda t} + C_2 \begin{pmatrix} t\eta_1 + \rho_1 \\ t\eta_2 + \rho_2 \end{pmatrix}$$

$$\boxed{\Rightarrow \left.\begin{matrix} \eta_1 C_1 + \rho_1 C_2 = R_0 \\ \eta_2 C_1 + \rho_2 C_2 = J_0 \end{matrix}\right\} \Rightarrow C1, C2}$$

2.2 Problem 2

...

2.3 Bonus exercises

...

3 Probabilty

3.1 Problem 1

...

3.2 Problem 2

...

3.3 Bonus exercises

...

4 Graph

4.1 Problem 1

...

4.2 Problem 2

...

4.3 Bonus exercises

...

References

[1] ...

[2] ...