Mathematical Programming 53 (1992) 199-211
North-Holland

Traveling salesman games

Jos A.M. Potters

Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, Netherlands

Imma J. Curiel

Department of Mathematics and Statistics, University of Maryland, Baltimore County, Catonsville, MD 21228, USA

Stef H. Tijs

Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, Netherlands

Received 18 March 1988
Revised manuscript received October 1988 and July 1989

In this paper we discuss the problem of how to divide the total cost of a round trip along several institutes among the institutes visited. We introduce two types of cooperative games — fixed-route traveling salesman games and traveling salesman games — as a tool to attack this problem. Under very mild conditions we prove that fixed-route traveling salesman games have non-empty cores if the fixed route is a solution of the classical traveling salesman problem. Core elements provide us with fair cost allocations. A traveling salesman game may have an empty core, even if the cost matrix satisfies the triangle inequality. In this paper we introduce a class of matrices defining TS-games with non-empty cores.

AMS 1980 Subject Classifications: Primary 90D12; Secondary 90C08.

Key words: Traveling salesman problem, game theory.

1. Introduction

Fishburn and Pollak [3] investigated the following problem. A person, sponsored by several sponsors, has visited starting from his home city t_0, the residences of the sponsors $t_1, \ldots, t_n$ and has returned to his home city. The total cost of the trip should be paid by the sponsors. One may also think of a speaker invited by several universities for a lecture. He makes a round-trip along the universities and the total cost should be paid by the universities which invited him. The problem is to find a "fair" rule for the distribution of the total cost among the sponsors (universities).

Fishburn and Pollak propose three conditions which a division rule should satisfy. The first condition defines in a certain sense the problem to be solved:

(a) The sum of the contributions of the sponsors sum up to the total travel cost ("efficiency").

The second condition restricts the contribution which the sponsors are willing to pay:

(b) Each sponsor is willing to pay at most the cost of a direct trip from the traveler's home city t_0 to the sponsor's city and back ("individual rationality").

The third condition describes the role played by the marginal cost in determining the contribution of each sponsor to the total cost. Here the *marginal cost of the ith sponsor* is the cost of the total tour minus the cost of the tour obtained by skipping the *i*th sponsor's residence from the tour but letting the order of the other cities unchanged. These marginal costs play a somewhat ambiguous role in [3]. Initially, Fishburn and Pollak are inclined to consider the marginal cost of each sponsor as a minimal contribution of the sponsor to the total cost. But, since the sum of the marginals can exceed the total cost — this can only happen if the fixed route is *not* a minimal cost route — they adopt another point of view and take as third condition for a fair cost allocation:

(c) If the marginal cost of sponsor i does not exceed the marginal cost of sponsor j, then the contribution of sponsor i does not exceed the contribution of sponsor j.

Condition (c) seems to be meant as a kind of fairness condition. But, apart from the fact that, in general, the conditions (a), (b) and (c) cannot be satisfied simultaneously (as Fishburn and Pollak show) one may wonder whether condition (c) does not lay too much weight on marginal cost. If, for example, the cities $t_0, t_1, \ldots, t_n$ lie in this order equally spaced on a circle of radius 100 and the travel cost between cities is equal to the distance of the cities, then each sponsor has the same marginal cost and has to pay, by condition (c), the same contribution to the total cost. This means that the proposed cost allocation doesn't give any account for the differences in the willingness to pay of the sponsors (cf. condition (b)).

Another peculiarity of [3] is that the authors try to find a cost division rule for every route and not only for routes with minimal total cost. In particular, in the example they gave in order to show that the conditions (a), (b) and (c) are incompatible, they took a fixed route which is far from being the cheapest one. But one may argue that the sponsors will refuse to pay if the traveler wants to deviate from a cheapest route and if one or more sponsors want a deviation from a minimal cost route, then these sponsors have to pay the increase of cost. Of course, there remains the problem of how to divide the increase of cost caused by deviation from a minimal cost tour, if several sponsors and/or the traveler want to deviate. This, however, is another problem which we will not discuss here.

Let us now reconsider the example of Fishburn and Pollak (see Figure 1). (In the figure and in the rest of the paper we will denote the cities by i instead of t_i.)

The route 0-1-2-3-0 has total cost 1500 and marginal costs (640, 670, 640). This is the route which Fishburn and Pollak use to show the incompatibility of the conditions (a), (b) and (c). A minimal cost route is 0-2-1-3-0 with total cost 890 and marginal costs (30, 60, 30). First we remark that the incompatibility of the conditions (a), (b) and (c) also holds for this minimal cost route. There is, however, another more interesting property of the marginal costs we will emphasize: The sum of the marginal costs of the minimal cost tour (=120) does not exceed the total cost (=890). For non-minimal cost tours this is no longer true as Fishburn and Pollak's

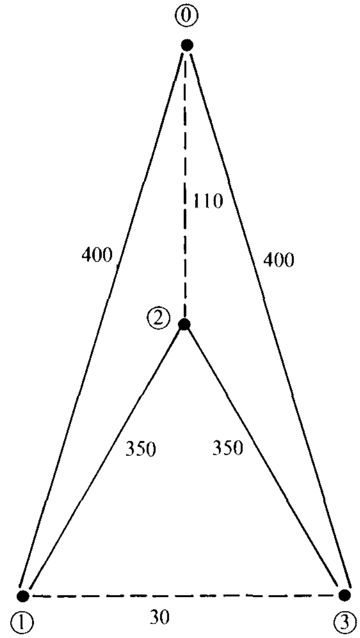

Fig. 1.

route show $(640+670+640>1500)$. This fact (which holds in general for minimal cost tours when the cost matrix satisfies the triangle inequalities (see (2) below)) casts a new light on marginal costs as minimal contribution of each sponsor.

One of the problems we will investigate in this paper is the possibility of a division rule of the total cost of a *minimal cost route* which satisfies the following conditions:

(a) The contributions of the sponsors sum up to the total cost ("efficiency").

(b) No sponsor pays more than the cost of a direct trip from the traveler's home city to the sponsor's residence and back ("individual rationality").

(c*) Each sponsor pays at least his own marginal cost ("minimal obligation").

The methods we shall use, are taken from cooperative game theory and have been fruitfully used in many other division problems. The recent books of Lucas [5] and of Young [14] are completely devoted to the application of game theoretical methods to cost allocation. We should also mention the airport fee problem as discussed in Owen's book on game theory [7], the papers of Granot and Huberman ([2] a.o.) and the paper of Tijs and Driessen [13]. A first indication of the kind of problems we discuss in this paper, can be found in Shapley's comment in [11, p. 136].

Before we can give the results of this paper, we have to introduce some terminoogy and have to agree upon some notational conventions.

Let $N_0=\{0,1,2,\ldots,n\}$ denote the set of cities and let K be an $(n+1)\times(n+1)$-matrix where K_{ij} denotes the travel cost of a trip from city i to city j $(i,j\in N_0)$. We always assume that

$$K_{ii}=0 \quad \text{for all } i\in N_0. \tag{1}$$

We frequently suppose that

$$K_{ij} + K_{jk} \geqslant K_{ik} \quad \text{for all } i, j, k \in N_0 \quad \text{(triangle inequalities).} \tag{2}$$

A *traveling salesman tour along* N_0 is a tour which visits each city of N_0 exactly once. It will be described by a *cyclic* permutation σ where $\sigma(i)$ is the city visited immediately after city i for all $i \in N_0$.

An *almost traveling salesman tour* is a tour along all the cities of N_0 which may visit only the home city 0 more than once. For each coalition $S \subset N := \{1, 2, \ldots, n\}$, each cyclic permutation σ and each element $i \in S_0 := S \cup \{0\}$ we define

$$p(S, i) := \min\{p \in \{1, 2, \ldots, n\} \mid \sigma^p(i) \in S_0\}$$

where σ^p is the composition of σ with itself p times. Then the permutation

$$i \in S_0 \to \sigma^{p(S,i)}(i) \in S_0$$

is a *cyclic* permutation of S_0, denoted by $\sigma | S_0$. The tour along S_0 defined by the permutation $\sigma | S_0$ is the tour obtained from the tour fixed by σ by skipping the cities of $N \backslash S$ and leaving the order of the remaining cities unchanged. For each coalition $S \subset N$ we define

$$c_{K,\sigma}(S) := \sum_{i \in S_0} K_{i,\sigma|S_0(i)}.$$

In this way we obtain a map $c_{K,\sigma}: 2^N \to \mathbb{R}$ with $c_{K,\sigma}(\emptyset) = 0$ (by (1)). Such a map assigning real numbers to coalitions (and zero to the empty set) is called a *cooperative (cost) game with side payments.* The game $c_{K,\sigma}$ is called a *fixed-route traveling salesman game.*

An element $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$ is called a *core allocation* of the game $c_{K,\sigma}$ if

$$x(N) := \sum_{i \in N} x_i = c_{K,\sigma}(N) \quad \text{and} \quad x(S) := \sum_{i \in S} x_i \leqslant c_{K,\sigma}(S) \quad \text{for all } S \subset N.$$

A core allocation is a stable cost allocation in the sense that it gives no coalition $S \subset N$ an incentive to split off.

The first theorem we shall prove, states the existence of core allocations of the game $c_{K,\sigma}$ if the matrix K has the properties (1) and (2) and the cyclic permutation σ determines a *minimal cost* route along N_0.

Theorem 1. *If the* $(n+1) \times (n+1)$*-matrix* K *has the properties* (1) *and* (2) *and* σ *is a cyclic permutation of* N_0 *such that*

$$\sum_{i \in N_0} K_{i,\sigma(i)} = \min\left\{ \sum_{i \in N_0} K_{i,\tau(i)} \,\middle|\, \tau \text{ is cyclic} \right\},$$

then the game $c_{K,\sigma}$ *has a core allocation.*

Note that Theorem 1 implies that there are cost allocations which satisfy the conditions (a), (b) and (c*) if the conditions of the theorem hold. For, if $x \in \mathbb{R}^N$

is a core allocation of $c_{K,\sigma}$, then the cost allocation is efficient (condition (a)), $x_i \leqslant c_{K,\sigma}(i) = K_{0i} + K_{i0}$ for all $i \in N$ (condition (b)) and $x_i = x(N) - x(N \backslash i) \geqslant c_{K,\sigma}(N) - c_{K,\sigma}(N \backslash i)$ for all $i \in N$ (condition (c*)) because $c_{K,\sigma}(N) - c_{K,\sigma}(N \backslash i)$ is, by definition, the marginal cost of sponsor i. (To avoid a too clumsy notation we write $N \backslash i$ instead of $N \backslash \{i\}$ and $v(1, 2, 5)$ instead of $v(\{1, 2, 5\})$.) Furthermore, it is interesting that no symmetry of the cost matrix K is supposed.

If the cost matrix K satisfies property (1) and (2), we can even give a cost allocation *rule* for minimal cost tours which satisfies the conditions (a), (b) and (c*) by

$$C_i = \lambda c_{K,\sigma}(i) + (1-\lambda)(c_{K,\sigma}(N) - c_{K,\sigma}(N \backslash i))$$

where $\lambda \in [0, 1]$ is chosen such that $\sum_{i \in N} C_i = c_{K,\sigma}(N)$. In general, this cost allocation does not give a core allocation of the game $c_{K,\sigma}$.

The route along S_0 obtained by skipping the cities of $N \backslash S$ from a minimal cost route along N_0 (i.e. the transition from σ to $\sigma | S_0$) may not give a minimal cost route for S_0. This fact raises the following natural question: Is there a cost allocation which is efficient (condition (a)) and also satisfies the following new condition:

(b*) No coalition $S \subset N$ is paying more than the cost of a minimal cost tour along the cities of S_0.

In other words: Is the core of the cooperative cost game c_K defined by

$$c_K(S) := \min \left\{ \sum_{i \in S_0} K_{i,\sigma(i)} \,\middle|\, \sigma \text{ is a cyclic permutation of } S_0 \right\} \tag{3}$$

empty or not?

Therefore, for every $(n+1) \times (n+1)$-matrix K satisfying (1) we define the *traveling salesman game* c_K by (3) and ask for the non-emptiness of the core of c_K. Note that for each coalition $S \subset N$ the calculation of $c_K(S)$ requires the solution of a traveling salesman problem. Because $c_K \leqslant c_{K,\sigma}$ and $c_K(N) = c_{K,\sigma}(N)$ if σ is a minimal cost tour along N_0, the last question is more difficult than the problem solved by Theorem 1. Further Theorem 1 seems to indicate that the matrix K should satisfy at least some of the triangle inequalities (2) in order to have a non-empty core. The most simple non-trivial example ($n = 2$) confirms this intuition.

For, if $N_0 = \{0, 1, 2\}$ and $c_K(N) = K_{01} + K_{12} + K_{20}$, then a core allocation has to satisfy the conditions

$$x_1 + x_2 = c_K(N), \qquad x_1 \leqslant K_{01} + K_{10} \quad \text{and} \quad x_2 \leqslant K_{02} + K_{20}.$$

Hence,

$$K_{01} + K_{12} + K_{20} \leqslant K_{01} + K_{10} + K_{02} + K_{20} \quad \text{or} \quad K_{12} \leqslant K_{10} + K_{02},$$

i.e. one of the triangle inequalities (2). One might think that the triangle inequalities (2) would be sufficient for the existence of core elements but the second example below and the recent results of Tamir [12] show that this idea is false.

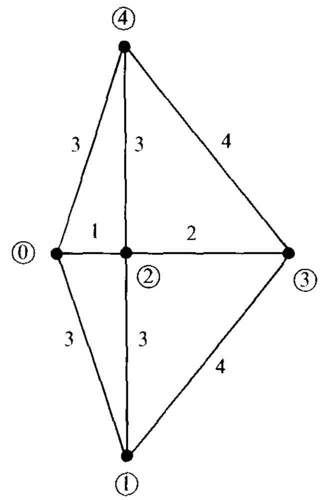

Fig. 2.

Example 1. $N_0 = \{0, 1, 2, 3, 4\}$ and

$$K = \begin{pmatrix} 0 & 3 & 1 & 3 & 3 \\ 3 & 0 & 3 & 4 & 6 \\ 1 & 3 & 0 & 2 & 3 \\ 3 & 4 & 2 & 0 & 4 \\ 3 & 6 & 3 & 4 & 0 \end{pmatrix} \text{ satisfies (1) and (2).}$$

A minimal cost tour is 0-1-2-3-4-0 (see Figure 2). The values of the game c_K are shown in Table 1. Here $x = (5, 1, 4, 5)$ is a core allocation.

Example 2. $N_0 = \{0, 1, 2, 3, 4\}$ as before and

$$K = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 & 0 \end{pmatrix} \text{ satisfies (1) and (2).}$$

$c_K(N) = 6$ (with minimal tour 0-1-2-3-4-0), Further $c_K(1, 2, 3) = 4 = K_{01} + K_{12} + K_{23} + K_{30}$, $c_K(1, 2, 4) = 4 = K_{04} + K_{42} + K_{21} + K_{10}$ and $c_K(3, 4) = 3 = K_{04} + K_{43} + K_{30}$. If $x = (x_1, \ldots, x_4)$ would be a core allocation, then

$$6 = x_1 + x_2 + x_3 + x_4 = \tfrac{1}{2}(x_1 + x_2 + x_3) + \tfrac{1}{2}(x_1 + x_2 + x_4) + \tfrac{1}{2}(x_3 + x_4) \leq \tfrac{1}{2}(4 + 4 + 3).$$

Table 1

S:	(1)	(2)	(3)	(4)	(12)	(13)	(14)	(23)	(24)	(34)	(123)	(124)	(134)	(234)	N
$c_K(S)$:	6	2	6	6	7	10	12	6	7	10	10	12	14	10	15

Hence the core of c_K is empty although the conditions (1) and (2) are satisfied.

In a recent paper Tamir [12] gives several examples of traveling salesman games c_K with an empty core while the cost matrix K is *symmetric* and satisfies the triangle inequalities (2).

In the remainder of this paper we prove two positive results about the existence of core allocations. In the first theorem we show that we get core allocations if we disturb the cost matrix in a very simple way. Let K be any $(n+1)\times(n+1)$-matrix satisfying (1). For each pair of vectors $a, b \in \mathbb{R}^{N_0}$ we define the $(n+1)\times(n+1)$-matrix $L(a, b)$ by

$$L(a,b)_{ij} = \begin{cases} a_i + b_j & \text{if } i, j \in N_0 \text{ and } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

We may think of the matrix $L(a, b)$ as a cost matrix where the numbers b_i and a_i are the entry tax and the exit tax which one has to pay when one is entering or leaving city i. The next theorem describes the influence which adding $L(a, b)$ to K has on the existence of core allocation and the triangle inequalities.

Theorem 2. *Let K be an $(n+1)\times(n+1)$-matrix satisfying condition* (1), $a, b \in \mathbb{R}^{N_0}$ *and $K' = K + L(a, b)$. Then the following statements hold:*

(i) *If $a_0 + b_0 = 0$, then the games c_K and $c_{K'}$ have both a non-empty core or both an empty core.*

(ii) *For all $j \in N$ the matrix K' satisfies the triangle inequalities*

$$K'_{ij} + K'_{jk} \geq K'_{ik} \quad \textit{for all } i, k \in N_0$$

if $a_j + b_j$ is large enough.

(iii) *There are real numbers $\alpha(K)$ and $\beta(K)$ such that:*

(a) *K' satisfies the triangle inequalities $K'_{i0} + K'_{0j} \geq K'_{ij}$ for all $i, j \in N_0$ if and only if $a_0 + b_0 \geq \alpha(K)$.*

(b) *The game $c_{K'}$ has a non-empty core if and only if $a_0 + b_0 \geq \beta(K)$.*

From Theorem 2(i) and (ii) we can see that only the triangle inequalities

$$\Delta_{i0j} := K_{i0} + K_{0j} - K_{ij} \geq 0 \quad \text{for all } i, j \in N \tag{4}$$

may give an indication for the existence of core allocations. The last part of Theorem 2 says that if $a_0 + b_0$ is large enough the matrix K' satisfies the triangle inequalities (4) and the game $c_{K'}$ has a non-empty core.

In the last part of the paper we give an admittedly small set of cost matrices K such that $c_K = c_{K,\sigma}$ where σ determines a minimal cost route for the matrix K. For those matrices the core of the game c_K is non-empty by Theorem 1.

The definition of this set of cost matrices is as follows. Let τ be a cyclic permutation of N_0. A non-empty subset $S \subset N$ is called a τ-interval if $S = \{i, \tau(i), \tau^2(i), \ldots, \tau^p(i)\}$

for some $i \in N$ and $p \geq 0$. Let $\mathcal{T}(\tau)$ be the collection of all τ-intervals. For each τ-interval T we define the cost matrices E_T^+ and E_T^- by

$$(E_T^+)_{ij} = \begin{cases} 1 & \text{if } i \notin T \text{ and } j \in T, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad (E_T^-)_{ij} = \begin{cases} 1 & \text{if } i \in T \text{ and } j \notin T, \\ 0 & \text{else.} \end{cases}$$

If we understand the matrices E_T^+ and E_T^- as cost matrices, we may have the following associations. Suppose the cities of T lie on an island and the cities of N_0/T are on the shore. If traveling is costless unless we go from the shore to the island (E_T^+) or from the island to the shore (E_T^-), then the matrices $E_T^{\pm}$ are the cost matrices for this situation. It is easy to check that the matrices $E_T^{\pm}$ satisfy (1) and (2) and that the matrices of the positive cone $\mathcal{M}(\tau)$ generated by $\{E_T^+, E_T^- \mid T \in \mathcal{T}(\tau)\}$ satisfy (1) and (2) as well. Now we can formulate the following theorem:

Theorem 3. *If K is an $(n+1)\times(n+1)$-matrix of $\mathcal{M}(\tau)$ for some cyclic permutation τ, then τ determines a minimal cost route of K and $c_K = c_{K,\tau}$. Furthermore, the core of c_K is non-empty.* □

We give no proof of this theorem since in [8] one of the authors gives a larger class of matrices $\bar{\mathcal{M}}(\tau) \subset \mathcal{M}(\tau)$ with the property that $c_K = c_{K,\tau}$ for all $K \in \bar{\mathcal{M}}(\tau)$ and a polynomially bounded algorithm to decide whether a given matrix K belongs to one of the sets $\bar{\mathcal{M}}(\tau)$ or not.

2. Proof of Theorems 1 and 2

The proof of Theorem 1 consists of three parts. First we define another cooperative game $\bar{c}_{K,\sigma}$ as solutions of linear programs. This game is a linear production game in the sense of Owen [6]. Games of this type have non-empty cores as Owen has proved. Secondly we prove that $\bar{c}_{K,\sigma}(S) \leq c_{K,\sigma}(S)$ for all $S \subset N$. Finally we show that for the grand coalition N we have $\bar{c}_{K,\sigma}(N) = c_{K,\sigma}(N)$. From these three facts we infer that

$$\emptyset \neq \mathrm{Core}(\bar{c}_{K,\sigma}) \subset \mathrm{Core}(c_{K,\sigma}).$$

Proof of Theorem 1.[1] Let K be an $(n+1)\times(n+1)$-matrix which satisfies the triangle inequalities

$$K_{i0} + K_{0j} \geq K_{ij} \quad \text{for all } i, j \in N$$

and σ be a cyclic permutation on N_0 determining an optimal TS-tour for K. After renumbering of the cities we may assume that σ gives the optimal tour $0\text{–}1\text{–}2\text{–}\cdots\text{–}n\text{–}0$.

Let V be the vector space

$$V = \{X = (X_{ij})_{i,j=0,\ldots,n} \mid X_{ij} = 0 \text{ for } 1 \leq j \leq i \leq n\}.$$

[1] We are indebted to the referee for the idea of this proof.

For each coalition $S \subset N$ we define

$$\Pi(S) := \left\{ X \in V \,\middle|\, X \geqslant 0, \sum_{j=0}^{n} X_{ij} = \sum_{k=0}^{n} X_{ki} = \delta_{i \in S} \text{ for all } i \in N \right\}$$

where $\delta_{i \in S} = 1$ if $i \in S$ and zero else. We define $\bar{c}_{K,\sigma}(S)$ as the minimal value of the goal function

$$\sum_{i,j=0,\ldots,n} K_{ij} X_{ij}$$

on the set $\Pi(S)$. This means that $\bar{c}_{K,\sigma}(S)$ is the solution of an LP-problem. We shall denote this problem with $P(S)$.

From the main result of Owen [6] we may infer immediately that the core of $\bar{c}_{K,\sigma}$ is non-empty. In fact, if $(\hat{y}_i, \hat{z}_i)_{i \in N}$ is an optimal solution of the dual problem of $P(N)$, then $(\hat{y}_i + \hat{z}_i)_{i \in N}$ is a core allocation of $\bar{c}_{K,\sigma}$. Furthermore, from the fact that the permutation matrix belonging to $\sigma | S_0$ satisfies the constraints of $P(S)$ follows that $\bar{c}_{K,\sigma}(S) \leqslant c_{K,\sigma}(S)$ for all $S \subset N$. So, we are left to prove that for the grand coalition N we have $\bar{c}_{K,\sigma}(N) \geqslant c_{K,\sigma}(N)$. In order to prove this statement we need the following proposition.

Proposition 4. *The extreme points of $\Pi(N)$ are integer-valued.*

Proof. Let $\bar{\Pi}(N)$ be defined by the same relations as $\Pi(N)$ except "$X \in V$". Then the matrix defining $\bar{\Pi}(N)$ is a submatrix of the *unimodular* matrix defining the assignment problem (cf. Schrijver [9]). Further the extreme points of $\Pi(N)$ are also extreme in $\bar{\Pi}(N)$. □

Proof of Theorem 1 (continued). Since the minimal value of the LP-problem $P(N)$ is attained in an extreme point of $\Pi(N)$, we find by the proposition that the optimal value is attained in an integer-valued point of $\Pi(N)$ i.e. an almost TS-tour of which each subtour visits the cities in an increasing order. If the almost TS-tour consists of more than one subtour, we can skip the city 0 when we visit the city 0 for the second, third ... time. By the triangle inequalities (4) the cost of the tour is not increased. Hence, there is a TS-tour with total cost $\leqslant \bar{c}_{K\sigma}(N)$ and because of the optimality of the tour σ we find $c_{K,\sigma}(N) \leqslant \bar{c}_{K,\sigma}(N)$. The core of $\bar{c}_{K,\sigma}$ is contained in the core of $c_{K,\sigma}$ and the last one is therefore non-empty. □

Proof of Theorem 2. The well-known result of Bondareva [1] and Shapley [10] is the following:

A cost game c has a non-empty core if and only if for every non-negative solution of the equation $\sum_{S \subset N} \lambda_S e_S = e_N$ there is the inequality

$$\sum_{S \subset N} \lambda_S c(S) \geqslant c(N).$$

Remark. The set of non-negative solutions of the equations $\sum_{S\subset N}\lambda_S e_S = e_N$ is a compact convex set in the real number space $\mathbb{R}^{2^N}$. If we have a solution with $\lambda_N > 0$, then we can write this solution as a convex combination of a solution with $\lambda_N = 0$ and the trivial solution $1\cdot e_N = e_N$ which obviously satisfies the Bondareva–Shapley condition.

Proof of Theorem 2 (continued). Let K be an $(n+1)\times(n+1)$-matrix and K' the matrix $K+L(a, b)$. Then for every coalition $S\subset N$ and every cyclic permutation σ_0 we have

$$\sum_{i\in S_0} K'_{i,\sigma(i)} = \sum_{i\in S_0} K_{i,\sigma(i)} + a(S_0) + b(S_0).$$

This means that the same tours are optimal for K and K' and

$$c_{K'}(S) = c_K(S) + a(S_0) + b(S_0).$$

Therefore, if $\sum_{S\subset N}\lambda_S e_S = e_N$ and $\lambda_S \geqslant 0$ for all $S\subset N$ and $\lambda_N = 0$,

$$\begin{aligned}
&\sum_{S\subset N}\lambda_S c_{K'}(S) - c_{K'}(N)\\
&\quad= \sum_{S\subset N}\lambda_S c_K(S) - c_K(N) + \sum_{S\subset N}\lambda_S(a(S_0)+b(S_0)) - a(N_0) - b(N_0)\\
&\quad= \sum_{S\subset N}\lambda_S c_K(S) - c_K(N) + \Big(\sum_{S\subset N}\lambda_S - 1\Big)(a_0+b_0)
\end{aligned}$$

where the last equality follows from

$$\begin{aligned}
\sum_{S\subset N}\lambda_S(a(S)+b(S)) &= \sum_{i\in N}\sum_{S:i\in S}\lambda_S(a_i+b_i) = \sum_{i\in N}(a_i+b_i)\\
&= a(N)+b(N).
\end{aligned}$$

For the triangles we find

$$K'_{ij} + K'_{jk} - K'_{ik} = K_{ij} + K_{jk} - K_{ik} + a_j + b_j$$

and therefore by taking $a_j + b_j$ large enough for all $j\neq 0$ we can make K' satisfy the triangle inequalities with $j\neq 0$. This shows that these triangle inequalities are immaterial for the existence of core elements. Finally, by taking $a_0+b_0 \geqslant \alpha(K) := -\min_{p\neq q}(\Delta K)_{p0q}$ where $(\Delta K)_{p0q} := K_{p0} + K_{0q} - K_{pq}$, the matrix K' satisfies the triangle inequalities (4). If we take

$$a_0+b_0 \geqslant \beta(K) = -\min\left\{\Big(\sum_{S\subset N}\lambda_S - 1\Big)^{-1}\Big(\sum_{S\subset N}\lambda_S c_K(S) - c_K(N)\Big)\right\}$$

where the minimum is taken over the non-negative solutions of $\sum_{S\subset N\subset}\lambda_S e_S = e_N$ with $\lambda_N = 0$, then the core of $c_{K'}$ is non-empty. This minimum exists because the set of non-negative solutions of the equation $\sum_{S\subset N}\lambda_S e_S = e_N$ with $\lambda_N = 0$ is a compact (and convex) set in $\mathbb{R}^{2^N}$ and the function to be minimized is continuous on this set because $\sum_{S\subset N}\lambda_S \geqslant n/(n-1)$ for every solution. □

3. Traveling salesman games with non-empty core

First we prove the following:

Theorem 5. *Cost games c_K have a non-empty core if $n \leqslant 3$ and*

$$(\Delta K)_{p0q} \geqslant 0 \quad \textit{for all } p, q \in N.$$

Proof. If $\{S_1, S_2, \dots, S_m\}$ is a partition of N, then we always have by the triangle inequalities (4),

$$c_K(S_1) + c_K(S_2) + \cdots + c_K(S_m) \geqslant c_K(N).$$

If $n = 2$, there is in fact only one (non-trivial) non-negative solution of $\sum_{S \subset N} \lambda_S e_S = e_N$ and this is a partition. So the core is non-empty. For $n = 3$ we only have to check the inequality

$$\tfrac{1}{2}(c_K(1,2) + c_K(1,3) + c_K(2,3)) \geqslant c_K(N).$$

Up to renumbering of the cities there are two cases to be considered:

(a) In the minimal cost tours of $\{0, 1, 2\}$, $\{0, 1, 3\}$ and $\{0, 2, 3\}$ there is an arc $0 \to i$ occurring twice.

(b) In the minimal cost tours above the arcs $0 \to 1$, $0 \to 2$ and $0 \to 3$ occur once.

In the first case we may assume that the arc $0 \to 1$ occurs twice i.e. 0–1–2–0 and 0–1–3–0 are minimal cost tours for the coalitions $\{1, 2\}$ and $\{1, 3\}$. Up to now the cities 2 and 3 occur symmetricaly and we may assume without loss of generality that 0–2–3–0 is a minimal cost tour for coalition $\{2, 3\}$. Then we can rearrange the occurring arcs as follows:

$$\sigma\colon 0\text{–}1\text{–}2\text{–}3\text{–}0, \qquad \tau\colon 0\text{–}1\text{–}3\text{–}0\text{–}2\text{–}0.$$

Then we find using the triangle inequality $(\Delta K)_{3,0,2} \geqslant 0$,

$$c_K(1,2) + c_K(1,3) + c_K(2,3) = c_K(\sigma) + c_K(\tau) \geqslant c_K(\sigma) + c_K(\tau')$$

where τ' is the tour 0–1–3–2–0. From this relation follows the inequality to be proved.

In case (b) we rearrange the arcs to 0–1–2–3–0 and 0–3–1–0–2–0 and follow the same argument. □

Note that in Example 2 of the introduction the minimal cost tours for the coalitions $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{3, 4\}$ are

0–1–2–3–0, 0–4–2–1–0 and 0–4–3–0,

and these arcs cannot be rearranged in the way described above.

Summary

In this paper we investigate the existence of a division rule of the cost of a fixed minimum cost route among the cities visited which satisfies the following properties:

(a) The contributions of the cities sum up to the total cost.

(b) The contribution of no city exceeds the cost of the direct trip from the visitor's home city to the city considered and back.

(c) Each city pays at least his marginal cost.

It turns out that there exists such a division rule if the cost matrix satisfies the conditions (1) and (2). A division rule of this kind is provided by

$$x_i = \lambda(K_{0i} + K_{i0}) + (1-\lambda)\mathrm{MC}_i \quad \text{for all } i \in \{1, 2, \ldots, n\}$$

where MC_i is the marginal cost of city i and $\lambda \in [0, 1]$ is chosen such that $x = (x_1, \ldots, x_n)$ satisfies condition (a). By inspecting the proof of this statement one can see that only the triangle inequalities (4) are actually needed.

Further we examine the harder problem to find a core allocation for the cooperative cost game c_K where

$$c_K(S) := \min\left\{ \sum_{i \in S_0} K_{i,\tau(i)} \,\middle|\, \tau \text{ is a cyclic permutation of } S_0 \right\}.$$

For $|N| \leq 3$, the triangle inequalities (4) are sufficient to guarantee the existence of core allocations. For $|N| = 4$ we find a non-symmetric couter example. In a recent paper of Tamir [12] there are several examples of traveling salesman games with a symmetric cost matrix which have an empty core. Moreover, that paper contains graph oriented conditions which ensure the existence of a core point.

In the last part of this paper we describe a subset of matrices which generate traveling salesman games with non-empty cores.

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