SOLVING THE MULTI-DEPOT LOCATION-ROUTING PROBLEM WITH LAGRANGIAN RELAXATION

Zeynep Özyurt[1] and Deniz Aksen[2]
[1]Industrial Engineering Deparment, Koç University; [2]College of Administrative Sciences and Economics, Koç University, Rumelifeneri yolu 34450 Sariyer, Istanbul, Türkiye

Abstract: Multi-depot Location-Routing Problem (MDLRP) is about finding the optimal number and locations of depots while allocating customers to depots and determining vehicle routes to visit all customers. In this study we propose a nested Lagrangian relaxation-based method for the discrete uncapacitated MDLRP. An outer Lagrangian relaxation embedded in subgradient optimization decomposes the parent problem into two subproblems. The first subproblem is a facility location-like problem. It is solved to optimality with Cplex 9.0. The second one resembles a capacitated and degree constrained minimum spanning forest problem, which is tackled with an augmented Lagrangian relaxation. The solution of the first subproblem reveals a depot location plan. As soon as a new distinct location plan is found in the course of the subgradient iterations, a tabu search algorithm is triggered to solve the multi-depot vehicle routing problem associated with that plan, and a feasible solution to the parent problem is obtained. Its objective value is checked against the current upper bound on the parent problem's true optimal objective value. The performance of the proposed method has been observed on a number of test problems, and the results have been tabulated.

Key words: location routing; Lagrangian relaxation; heuristics; tabu search.

1. INTRODUCTION AND LITERATURE SURVEY

Location-Routing Problem (LRP) involves finding the optimal number and locations of depots while allocating customers to depots and determining vehicle routes to visit all customers. Another problem which establishes depots considering demand and location data of customer nodes is the classical location/allocation problem (LAP). The main difference of the LRP from the LAP is that, once the facilities have been placed, the LRP requires the visitation of demand nodes through tours, whereas the latter assumes straight-line or radial trips between the facilities and respective customers. The LRP considers three main decisions of different levels simultaneously: location of depots - strategic level; allocation of customers to depots - tactical level and the routes to visit these customers - operational level. The interdependence between these decisions has been noticed by researchers long ago. The effect of ignoring routes when locating depots has also been stressed by Salhi and Rand (1989). However, due to the complexity of both location and routing problems, these two have been traditionally solved separately. In the literature there exist heuristic solution methods – for example; Tüzün and Burke (1999), Wu et al. (2002), Albareda-Sambola et al. (2005) – as well as exact methods – for example; Laporte et al. (1988) – proposed for solving the LRP as a whole.

The newest annotated literature review of the LRP and its extensions is currently due to Ahipasaoglu et al. (2004). A complete synthesis and survey of the LRP was accomplished earlier by Min et al. (1998) who propose a classification scheme for LRPs. The authors argue that sequential methods consisting of decomposition for the LRP have their limitation. They recommend solving the subproblems of the LRP concurrently in order to be able to analyze the tradeoffs between location and routing factors at the same level of decision hierarchy. Among the exact solution methods developed for the LRP is Laporte et al.'s (1988) method which transforms the Multi-Depot Vehicle Routing Problem (MDVRP) and LRP into a constrained assignment problem solved by branch and bound. Ambrosino and Scutellà (2005) attribute the significance of strategic decisions like facility locations, transportation and inventory levels in the distribution network design problem (DNDP) to an article by Crainic and Laporte (1997), along with an extensive literature review of the LRP and DNDP. Ambrosino and Scutellà adopt Laporte's (1988) classification of LRPs. Their proposed integrated DNDP can be formulated as an LRP of category 4/R/T/T involving facility, warehousing and transportation as well as inventory decisions. This category label means distribution networks are made up of four layers, with routes of type replenishment (direct shipments) and type tour (vehicle routes). Jacobsen and Madsen (1980) model a newspaper delivery system as a three layer LRP and suggest three heuristic methods for the problem. Another study which introduces three heuristics for the LRP is due to Srivastava (1993). Tüzün and Burke (1999) propose a two-phase tabu search architecture for the solution of the standard 2-layer multi-depot LRP where depots have unlimited throughput capacity. Wu et al. (2002) decompose the standard LRP with capacitated depots into a facility location-allocation problem and a vehicle routing problem, and try then to solve both subproblems using simulated annealing. For the same class of the LRP, Albareda-Sambola et al. (2005) apply a method that generates first a lower bound either from the linear relaxation of the given problem or from the solutions of a pair of ad hoc knapsack and asymmetric traveling salesman problems. This lower bound is then used as a starting point of a tabu search heuristic. Lastly, Melechovský et al. (2005) address an LRP with nonlinear depot costs that grow with the total demand satisfied by the depots. They present a hybrid metaheuristic method consisting of tabu search and variable neighborhood search heuristics. We are aware of one study by Aksen and Altınkemer (2005) on Lagrangian relaxation for the LRP. They propose a 3-layer distribution logistics model for the conversion from brick-and-mortar to click-and-mortar retailing. A static one-period optimization model is built and solved using Lagrangian relaxation.

In this paper, we solve a 2-layer multi-depot location-routing problem (MDLRP) where transportation follows directly from depots to customers. There exist two kinds of depots: present depots and candidate depots. Present depots are already operating facilities that can be preserved or closed. If a present depot is closed, a fixed closing cost is incurred. This cost may turn out to be a gain since the closure of a depot usually brings about savings in overhead costs. Candidate depot locations are potential sites in which new depots can be opened. For each new depot to be opened, a fixed opening cost is incurred. In addition, there exist fixed operating costs which are charged for each preserved or newly opened depot. Customers are visited by a homogenous fleet of capacitated vehicles. For each of them, a vehicle acquisition cost is charged. Each customer has a deterministic demand which should be satisfied by the single visit of a vehicle. There is no capacity constraint on depots. The sum of depot opening-closing and operating, vehicle acquisition and traveling costs is minimized subject to the vehicle capacity in the problem.

The remainder of the paper is organized as follows. The problem description and its mathematical model are given in Section 2. In Section 3, detailed explanations of the Lagrangian relaxation scheme and the solution methods for the subproblems are provided. The heuristic method that is used to obtain upper bounds on the true optimal solution is detailed in Section 4. Section 5 presents the computational experiments, results and comparisons. Finally, Section 6 comprises a summary with concluding remarks.

2. MATHEMATICAL MODEL FOR THE MDLRP

In the MDLRP, the sum of depot opening-closing, vehicle acquisition and traveling costs is minimized subject to vehicle capacity constraints. According to Laporte's (1988) classification of LRPs, our problem is 2/T. It means there exist 2 layers; namely, depots and customers where the transportation between these layers is realized via tours (vehicle routes). The objective function and constraints of the model can be stated as follows:

$$\mathbf{P}: \ \text{Min} \sum_{k\in ID} OC_k y_k + \sum_{k\in ID_{cand}} FC_k y_k + \sum_{k\in ID_{pres}} FC_k(1-y_k) + \sum_{i\in ID}\sum_{j\in IC} VC_i x_{iji}$$

$$+ \sum_{k\in ID}\sum_{i\in I}\sum_{\substack{j\in I\\ i\neq j}} c_{ij} x_{ijk} \tag{1}$$

$$\text{s.t.:} \sum_{k\in ID} \delta_{ik} = 1 \qquad \forall i\in IC \tag{2}$$

$$\sum_{\substack{j\in IC\cup\{k\}\\ j\neq i}} x_{ijk} = \delta_{ik} \qquad \forall i\in IC,\ \forall k\in ID \tag{3}$$

$$\sum_{\substack{j\in IC\cup\{k\}\\ j\neq i}} x_{jik} = \delta_{ik} \qquad \forall i\in IC,\ \forall k\in ID \tag{4}$$

$$\sum_{i\in IC} x_{ikk} = \sum_{i\in IC} x_{kik} \qquad \forall i\in ID, \forall k\in ID \tag{5}$$

$$\sum_{k\in ID}\sum_{i\in IC} x_{kik} + \sum_{k\in ID}\sum_{i\in IC}\sum_{\substack{j\in IC\\ i\neq j}} x_{ijk} = |IC| \tag{6}$$

$$\sum_{k\in ID}\sum_{i\in S}\sum_{\substack{j\in S\\ j\neq i}} x_{ijk} \le |S| - L(S) \qquad \forall S\subseteq IC,\ |S|\ge 2 \tag{7}$$

$$\sum_{i\in IC} \delta_{ik} \le |IC|\, y_k \qquad \forall k\in ID \tag{8}$$

$$x_{ijk} \in \{0,1\} \qquad \forall i\in I, \forall j\in I, \forall k\in ID \tag{9}$$

$$\delta_{ik}, y_k \in \{0,1\} \qquad \forall i\in I, \forall k\in ID \tag{10}$$

IC, *ID* and *I* in the model denote the set of customer nodes, the set of depot nodes and the union of these two sets, respectively. *ID* consists of ID_{pres} and ID_{cand}. The former is the set of already existing depots, and the latter is the set of candidate depots that can be opened. There are three sets of binary decision variables: x_{ijk} is equal to 1 if node j is visited after node i on a route originating from depot k. The variable y_k is equal to 1 if depot k is opened for $k \in ID_{cand}$, or if it is preserved for $k \in ID_{pres}$. The binary variable δ_{ik} is equal to 1 if customer i is assigned to depot k. FC_k is the fixed cost of having depot k in the solution. If depot k is not already present, then FC_k will have a relatively large positive value. Otherwise, it will denote the cost (gain) of closing depot k and will possibly have a negative value. OC_k is the depot operating cost. VC_k denotes the unit vehicle acquisition cost at depot k. Parameter c_{ij} denotes the traveling cost of one vehicle from node i to node j. M denotes a big number and Q is the uniform vehicle capacity. $L(S)$ in Eq.(7) is the optimal solution to the one-dimensional bin packing problem where the bin length is equal to the vehicle capacity Q, and demand values d_i ($i \in IC$) are the sizes of items to be packed into the bins.

The objective of **P**, shown in Eq. (1), is a combination of objectives of a facility location-allocation problem (FLAP) and a multi-depot vehicle routing problem (MDVRP). The constraints are comprised of pure FLAP constraints, pure MDVRP constraints and coupling constraints linking routing decisions with location decisions. Equation (2) assigns each customer to a depot. Equations (3)–(4) are flow conservation constraints which ensure all customers be visited exactly once on a route originating from the assigned depot. Equation (5) ensures that the numbers of incoming and outgoing arcs at each depot are equal. Equation (6) is identical to the sum of the constraints in Eq. (4). In order to obtain the second subproblem as a minimum spanning forest like problem after the Lagrangian relaxation, we add this redundant constraint to the model. Equation (7) is the well-known subtour elimination constraints which ensure all routes will start and end at a depot. The assignment of a customer to a closed or unopened depot, and routes originating from such a depot are avoided by Eq. (8). Finally, Eqs. (9)–(10) are integrality constraints.

3. LAGRANGIAN RELAXATION FOR THE MDLRP

Lagrangian relaxation is a decomposition method used for a variety of NP-hard optimization problems (see Geoffrion 1974). In this method, the true optimal objective value of the problem (Z^*_P) is bracketed between a lower and an upper bound $[Z_{lb}, Z_{ub}]$. In case of minimization, a good feasible solution constitutes an upper bound for the optimal solution of the minimization problem while its lower bound is obtained by solving the Lagrangian relaxed problem. The quality of the solution is assessed based on the gap between these two bounds. When Lagrangian relaxation is applied to the MDLRP in Eqs. (1)–(10), the coupling constraints in Eqs. (3)–(4) are relaxed. The left-hand sides of the constraints are subtracted from their right-hand sides. The differences are multiplied by the Lagrange multipliers λ and μ, respectively, which are unrestricted in sign. The terms are then augmented into the original objective function and the new objective function $ZLR(\lambda, \mu)$ in Eq. (11) is obtained.

$$ZLR(\lambda,\mu) = \sum_{k\in ID} OC_k y_k + \sum_{k\in ID_{cand}} FC_k y_k + \sum_{k\in ID_{pres}} FC_k(1-y_k) + \sum_{i\in ID}\sum_{j\in IC} VC_i x_{iji}$$
$$+ \sum_{k\in ID}\sum_{i\in I}\sum_{\substack{j\in I\\ j\neq i}} c_{ij} x_{ijk} + \sum_{k\in ID}\sum_{i\in IC}\sum_{\substack{j\in IC\cup\{k\}\\ j\neq i}} (\delta_{ik} - x_{ijk})\lambda_{ik}$$
$$+ \sum_{k\in ID}\sum_{j\in IC}\sum_{\substack{i\in IC\cup\{k\}\\ i\neq j}} (\delta_{jk} - x_{ijk})\mu_{jk} \qquad (11)$$

The resulting Lagrangian relaxed problem **LR**(λ, μ) can be partitioned into two independent subproblems. The first subproblem resembles an uncapacitated FLAP (**SubP1**) and can be solved with Cplex 9.0 in reasonable time. The second one is similar to a degree constrained minimum spanning forest problem (DCMSF) (**SubP2**). However, **SubP2** is still an NP-hard problem, which is tackled with an augmented Lagrangian relaxation by relaxing the degree constraints. The relaxed **SubP2** becomes a minimum spanning forest problem with a minimum number of outgoing arcs at root nodes (depots). It is solved with a modified version of Prim's minimum spanning tree algorithm. Figure 1 displays the flow chart of the iterative subgradient optimization procedure with the Lagrangian relaxation scheme applied to the parent problem **P**. The flow chart's segment in the box shows the inner augmented Lagrangian relaxation which is applied to the second subproblem **SubP2**. The structure of the Lagrangian relaxed problem LR(λ, μ) is presented below in plain English.

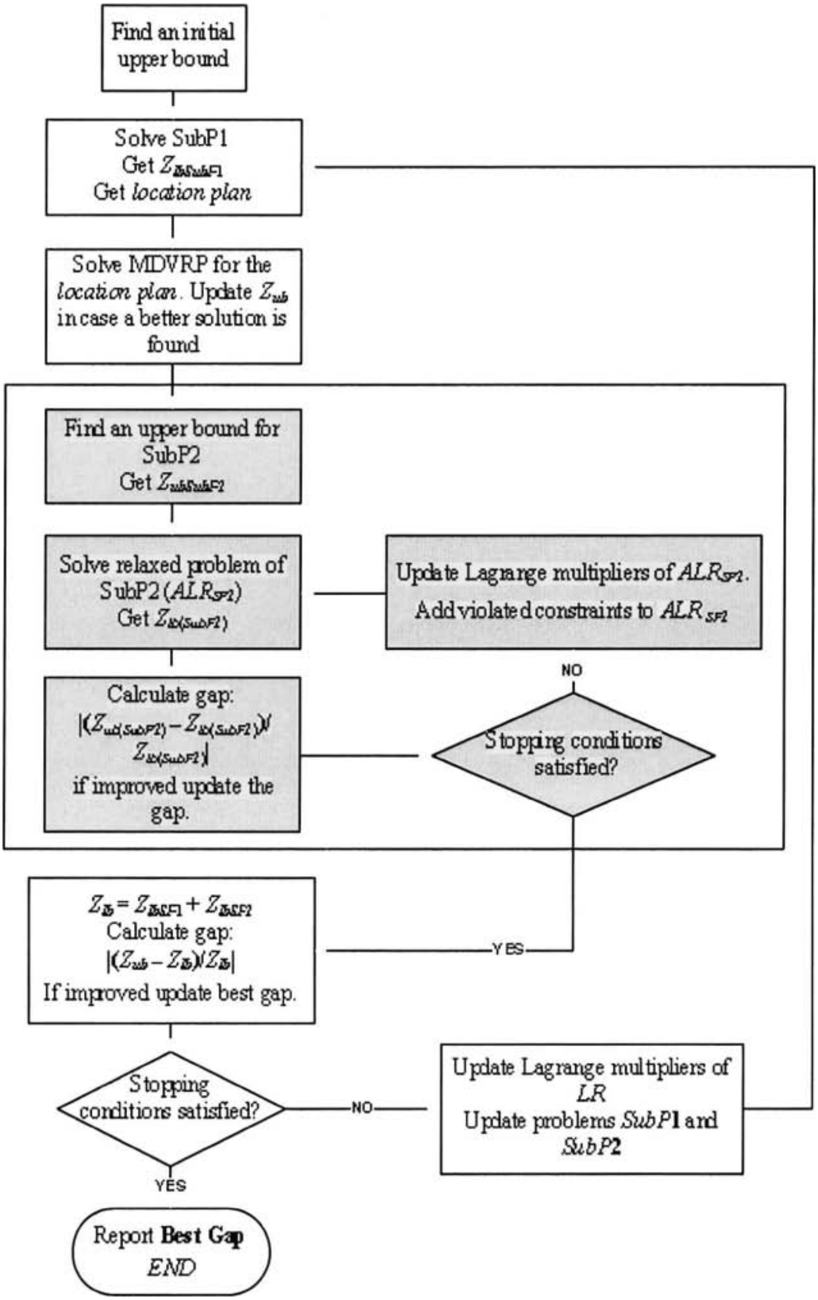
LR(λ, μ): Minimize *ZLR*(λ, μ) = Σ*Augmented* FLAP *objectives*
+ Σ*Augmented* MDVRP *objectives*

subject to: i. Pure FLAP constraints (2)–(8)
ii. Pure MDVRP constraints (5)–(7)
iii. Nonnegativity and integrality constraints (9)–(10)

3.1 The Lagrangian relaxed problem LR

ZLR(λ, μ) which is the objective function of the Lagrangian relaxed problem turns out to be separable into two as FLAP and MDVRP objectives. In order to obtain two independent components, *ZLR*(λ, μ) needs to be rearranged. One part of the relaxed constraints that are augmented into the objective function can be separated as shown in Eq. (12). The Lagrange multipliers in this FLAP objective component represent pseudo costs of allocating customers to depots.

$$\sum_{i\in IC}\sum_{k\in ID} \delta_{ik}(\mu_{ik} + \lambda_{ik}) \qquad (12)$$


Find an initial upper bound
Solve SubP1
Get $Z_{lbSubP1}$
Get *location plan*
Solve MDVRP for the *location plan*. Update Z_{ub} in case a better solution is found
Find an upper bound for SubP2
Get $Z_{ubSubP2}$
Solve relaxed problem of SubP2 (ALR_{SP2})
Get $Z_{lb(SubP2)}$
Update Lagrange multipliers of ALR_{SP2}.
Add violated constraints to ALR_{SP2}
NO
Calculate gap:
$|(Z_{ub(SubP2)} - Z_{lb(SubP2)})/Z_{lb(SubP2)}|$
if improved update the gap.
Stopping conditions satisfied?
$Z_{lb} = Z_{lbSP1} + Z_{lbSP2}$
Calculate gap:
$|(Z_{ub} - Z_{lb})/Z_{lb}|$
If improved update best gap.
YES
Stopping conditions satisfied?
NO
Update Lagrange multipliers of *LR*
Update problems *SubP***1** and *SubP***2**
YES
Report **Best Gap**
END

By reordering the remaining terms in Eq. (11), we derive the 3-dimensional asymmetric and depot dependent traveling cost matrix $C_{new}= [(c_{ijk})^{new}]$. In this cost structure, the cost of traveling from node i to node j by a vehicle not only depends on the distance between i and j, but also on the depot k which sends off that vehicle. Let $G(I,A)$ denote the complete weighted and directed graph of customers and depots, i.e. $A = \{(i,j) \in (I \times I),\ i\ ?\ j\}$. Let $(c_{ijk})^{new}$ denote the cost of arc (i, j), if it is traversed with a vehicle dispatched from depot $k \in ID$. Arc costs in G are then defined as follows:

1. $(i,j) \in IC \times IC, i\ ?\ j, k \in ID$: $(c_{ijk})^{new} = c_{ij} - \lambda_{ik} - \mu_{jk}$
2. $(i,j) \in IC \times ID$: $(c_{ijj})^{new} = c_{ij} - \lambda_{ij}$
3. $(i,j) \in ID \times IC$: $(c_{iji})^{new} = c_{ij} - \mu_{ji} + VC_i$
4. $(i,j) \in ID \times IC, k \in ID, i\ ?\ k$: $(c_{ijk})^{new} = +8$
5. $(i,j) \in IC \times ID, k \in ID, j\ ?\ k$: $(c_{ijk})^{new} = +8$

The last two cost assignments avoid illegal arc definitions. An arc that is emanating from a depot or entering a depot cannot be defined on a route which originates from a different depot. That means, if i or j is a depot node and k is another depot node, than x_{ijk} cannot be 1. For this, c_{ijk} is assigned to infinity in these cases. After the relevant rearrangements, the Lagrangian relaxed problem **LR**(λ, μ) can be stated as follows.

$$\text{Min} \quad ZLR(\lambda,\mu) = \sum_{k \in ID} OC_k y_k + \sum_{k \in ID_{cand}} FC_k y_k + \sum_{k \in ID_{pres}} FC_k (1 - y_k)$$

$$+ \sum_{k \in ID} \sum_{i \in IC} \delta_{ik}(\lambda_{ik} + \mu_{ik}) + \sum_{k \in ID} \sum_{i \in I} \sum_{\substack{j \in I \\ j \neq i}} (c_{ijk})^{new} x_{ijk} \tag{13}$$

subject to : (2), (5)–(10)

We use a tabu search heuristic to find a good feasible solution whose objective value will be an upper bound on Z^*_P, the true optimal objective value of the problem **P**. This upper bound is updated throughout the subgradient iterations of the Lagrangian relaxation. The upper bound generation and updating method are explained in Section 4.

3.2 Subgradient optimization

Let SG^q denote the subgradient vector of the problem LR(λ, μ) at iteration q of the subgradient optimization procedure. Step size s^q is then derived from the norm square of SG^q and the gap between the current best objective Z_{ub} (upper bound on Z^*_P) and current Lagrangian objective Z^q_{LR}. It is multiplied also by a scalar $?_q$ whose first value $?_1$ is 2.0 by convention (see Fisher, 1981). This scalar is halved whenever the objective Z_{LR} does not improve for a specified number of consecutive iterations. At the beginning, we set all Lagrange multipliers to the initial value zero.

Formulae of the subgradient optimization routine for the Lagrangian relaxation of problem **P** are given below.

$$(SG_{ik}^{\lambda})^q = (\delta_{ik})^q - \sum_{\substack{j \in IC \cup \{k\} \\ j \neq i}} (x_{ijk})^q \qquad \forall i \in IC, \forall k \in ID \tag{14}$$

$$(SG_{ik}^{\mu})^q = (\delta_{ik})^q - \sum_{\substack{j \in IC \cup \{k\} \\ j \neq i}} (x_{jik})^q \qquad \forall i \in IC, \forall k \in ID \tag{15}$$

$$\left\|SG\right\|^2 = \left\|SG^{\lambda}\right\|^2 + \left\|SG^{\mu}\right\|^2 \qquad s^q = \Lambda^q \frac{Z_{ub}^q - Z_{LR}^q(\lambda, \mu)}{\left\|SG\right\|^2}$$

$$(\lambda_{ik})^{q+1} = (\lambda_{ik})^q + s^q (SG_{ik})^q \qquad \forall i \in IC, \forall k \in ID \tag{16}$$

$$(\mu_{ik})^{q+1} = (\mu_{ik})^q + s^q (SG_{ik})^q \qquad \forall i \in IC, \forall k \in ID \tag{17}$$

3.3 FLAP-like problem SubP1

The first of the two subproblems comprising **LR**(λ, μ) is the FLAP like problem **SubP1**. The formulation of **SubP1** can be written as follows.

$$\text{Min } Z_{SubP1} = \sum_{k \in ID} OC_k y_k + \sum_{k \in ID_{cand}} FC_k y_k + \sum_{k \in ID_{pres}} FC_k (1 - y_k) + \sum_{k \in ID} \sum_{i \in IC} \delta_{ik} (\lambda_{ik} + \mu_{ik}) \tag{18}$$

subject to : (2), (8), (10)

Since the technological coefficients matrix of **SubP1** is unimodular, we can define δ_{ik}'s as positive continuous variables between 0 and 1 instead of binary variables. Furthermore, the constraints in (8) should be disaggregated as $\delta_{ik} \leq y_k \quad \forall i \in IC, \ \forall k \in ID$. As a result, we get a formulation with less integer variables, which in turn yields significantly better solution times for **SubP1**. The depots to be preserved or opened are determined, and customers are allocated to these. Each depot has its operating (OC_k) and opening cost (FC_k). FC_k is incurred when a candidate depot is opened or when a present depot is closed. FC_k of the present depots is generally negative corresponding to the savings in overhead costs and to the salvage value obtained by closing the depot. The Lagrange multipliers λ and μ represent allocation costs of customers to depots. These costs change as the Lagrange multipliers get updated during the subgradient optimization iterations on the

Lagrangian relaxed problem. At the beginning of each subgradient iteration, allocation costs are plugged in and **SubP1** is solved with Cplex to optimality. The solution times are generally reasonable. A problem instance with 20 depots and 1000 customers takes 2.84 seconds on a present-day desktop PC.

3.4 Minimum spanning forest-like subproblem SubP2

The second subproblem of **LR(λ, μ)** is **SubP2** which resembles a degree and capacity constrained minimum spanning forest problem (CMSF). The cost matrix C_{new} comprises the coefficients in the objective function of the subproblem. Since the Lagrange multipliers λ and μ are embedded in this matrix, C_{new} changes as the multipliers change at each subgradient iteration. The mathematical formulation of **SubP2** can be stated as follows:

$$\text{Min } Z_{SubP2} = \sum_{k\in ID}\sum_{i\in I}\sum_{\substack{j\in I\\ j\neq i}} (c_{ijk})^{new} x_{ijk} \tag{19}$$

subject to : (5)–(7), (9)

In **SubP2** depot locations $k \in ID$ represent center nodes, while customers $i \in IC$ are terminal nodes. The terminals should be accessible from one of the center nodes via a subtree rooted at that center. Eq. (5) enforces that the numbers of outgoing and incoming arcs at each center node be equal. This balance-of-in-and-outdegree condition differentiates **SubP2** from the classical MSF. Capacity and subtour elimination constraints are given in Eq. (7). The capacity constraint requires that the total demand on a subtree rooted at a center node do not exceed Q. Equation (6) provides connectivity of the tree, while Eq. (7) avoids the formation of subtrees which are not linked to any of the center nodes. Since the constraints in Eqs. (3)–(4) are relaxed, any node can have more than one offspring nodes.

SubP2 is actually still hard to solve. If balance of degree constraints are discarded, and if the number of depots in ID is dropped to one, **SubP2** would reduce to the capacitated minimum spanning tree (CMST) problem. Papadimitriou (1978) showed that CMST is an NP-hard problem. Hence, **SubP2** also belongs to the NP-hard class. In order to solve **SubP2** we use the method proposed in Aksen and Altinkemer (2005) where the augmented Lagrangian relaxation method of Gavish (1985) is adopted and modified to tackle balance of degree constraints. We relax the subtour elimination constraints in **SubP2**, since this relaxation scheme achieves empirically better lower bounds on Z^*_{SubP2}. First, the constraint set in Eq. (7) is divided into two parts as (7.a) and (7.b), the second of which is relaxed. Secondly, a trivial constraint which sets the minimum number of vehicles required is added to the original formulation as (7.c). This minimum number is calculated by solving the associated bin-packing problem that embraces all demand values d_i, $i \in IC$.

$$\sum_{k\in ID}\sum_{i\in S}\sum_{\substack{j\in S\\ j\neq i}} x_{ijk} \leq |S|-1 \qquad \forall S \subseteq IC \;\; \exists |S| \geq 2 \qquad (7.a)$$

$$\sum_{k\in ID}\sum_{i\in S}\sum_{j\in S} x_{ijk} \leq |S|-\mathrm{L}_S \qquad \forall S \subseteq IC, |S| \geq 2 \qquad (7.b)$$

$$\sum_{k\in ID}\sum_{i\in IC} x_{kik} \geq \mathrm{L}_{IC} \qquad (7.c)$$

The relaxed constraint set (7.b) is multiplied with Lagrange multipliers α, where $\alpha = 0$. Left hand side values are subtracted from their right hand sides and the resulting terms are augmented into the objective function of **SubP2** in Eq. (19). In order to combine the embedded terms with Z_{SubP2} and to get a compact formulation for the objective function of the problem after the Lagrangian relaxation we separate Eq. (19) into three parts as follows:

$$\sum_{k\in ID}\sum_{i\in I}\sum_{\substack{j\in I\\ j\neq i}} (c_{ijk})_{new} x_{ijk} = \sum_{k\in ID}\sum_{i\in IC} \left(c_{kik}\right)^{new} x_{kik} + \sum_{k\in ID}\sum_{i\in IC} \left(c_{ikk}\right)^{new} x_{ikk}$$
$$+ \sum_{k\in ID}\sum_{i\in IC}\sum_{\substack{j\in IC\\ j\neq i}} \left(c_{ijk}\right)^{new} x_{ijk} \qquad (20)$$

After necessary rearrangements, the objective function and constraints of $\mathbf{ALR}^{SubP2}$ (the Lagrangian relaxed **SubP2**) can be stated as follows:

$$\mathrm{Min}\; Z_{LR}^{SubP2}(\alpha) = \sum_{k\in ID}\sum_{i\in IC}\left[\left(c_{kik}\right)^{new}\right] x_{kik} + \sum_{k\in ID}\sum_{i\in IC}\left[\left(c_{ikk}\right)^{new}\right] x_{ikk} +$$
$$\sum_{k\in ID}\sum_{i\in IC}\sum_{\substack{j\in IC\\ j\neq i}}\left[\left(c_{ijk}\right)^{new} - \sum_{s\in G_i}\alpha_s\right] x_{ijk} + \sum_{s\in ?}\left(|S|-\mathrm{L}_S\right)\alpha_s \qquad (21)$$

subject to : (5), (6), (7.a), (7.c), (9)

The last term in Eq. (21) is constant for a given set of Lagrange multipliers α. Since the solution to $\mathbf{ALR}^{SubP2}$ will constitute a lower bound for the optimal solution of **LR**, omitting of the constant term would overestimate or underestimate the lower bound depending on the negativity of the terms. Observe that S in the relaxed constraints represents any unordered subset of IC with a cardinality greater than one, which requires two or more vehicles to deliver orders. The set of such subsets is denoted by $\mathbf{\Psi}$. For each $S \in \mathbf{\Psi}$, there is an associated Lagrange multiplier $\alpha_s = 0$. Let G_{ij} denote the index set of subsets S in $\mathbf{\Psi}$ that contain customer nodes i and j. The augmented Lagrangian relaxation feature is used here, because we do not explicitly generate all constraints in Eq. (7.b). Therefore, we do not compute the entire multiplier vector α, either. The augmented Lagrangian relaxed problem $\mathbf{ALR}^{SubP2}$ is equivalent to an MSF problem without capacity constraints where the cost matrix C_{new} is dependent on the center node of departure. However, there are two distinct restrictions in this MSF problem:

- The sum of outgoing degrees of all center nodes has to be equal to or greater than L_{IC} as required by the constraints in Eq. (7.c).
- At each center node, incoming and outgoing degrees should be equal as required by constraints in Eq. (5).

The solution of the problem $\mathbf{ALR}^{SubP2}$ is checked against the violation of constraints in Eq. (7.b) in **SubP2**. If any violated constraint is detected, it is added together with its associated Lagrange multiplier to the set of active constraints and multipliers. The objective function is augmented with the product of the difference between the violated constraint's right- and left-hand side values and the associated Lagrange multiplier's initial value. We do not remove previously augmented constraints from the set of active constraints in the Lagrangian problem; neither do we generate any such constraint for a second time. Gavish explains a further technique to generate a tight Lagrangian objective function by finding an initial multiplier value for every augmented constraint while maintaining the optimality property of the Lagrangian solution before that constraint. We adopted this technique into our augmented Lagrangian relaxation of **SubP2**. Finally, the degree balance constraints in Eq. (5) and the minimum sum constraint in Eq. (7.c) on the center nodes' outgoing degrees should be reckoned with.

The closest version of $\mathbf{ALR}^{SubP2}$ is the degree-constrained minimum spanning tree problem (DCMST). Garey and Johnson (1979) prove that the DCMST with arbitrary degree constraints on nodes other than the center is NP-hard. In spite of copious methods and algorithms developed for the DCMST in the literature, we cannot use any of them as is. First of all, $\mathbf{ALR}^{SubP2}$ displays a forest structure with asymmetrical and center-node dependent costs. Secondly, the degree constraints that appear in $\mathbf{ALR}^{SubP2}$ relate to the balance of incoming and outgoing degrees at the center nodes only. There exists also a lower bound on the sum of outgoing degrees at those centers. From this perspective, $\mathbf{ALR}^{SubP2}$ is conceivably easier to solve than a general DCMST problem. Aksen and Altinkemer (2005) develop a polynomial-time procedure called [MSF-ALR] which is largely an adaptation of Prim's MST algorithm. We take on their solution method for solving the problem $\mathbf{ALR}^{SubP2}$.

3.5 Subgradient optimization in the augmented Lagrangian relaxation

The subgradient vector **Y** is calculated according to the formulae given below. The cardinality of the subgradient vector increases as the number of violated constraints goes up. In the formulae, G^q denotes the index set of those subtour elimination and capacity constraints in Eq. (7.b) which have been violated and thus generated either in the current iteration q or in a previous iteration. Each index r in G^q corresponds to some subtree of customer nodes whose indices comprise a particular subset S in $\mathbf{\Psi}$ as explained in Section 3.4. There are as many as $|G^q|$ constraints from Eq. (7.b) relaxed and augmented into $\mathbf{ALR}^{SubP2}$. In Eq. (23), s^q_{ALR} denotes the step size of the subgradient optimization, ? $^q_{ALR}$ is a scalar with the initial value 2.0, $Z^q_{ub(SubP2)}$ is an upper bound on the true optimal objective value of **SubP2**, and finally $Z^q_{ALR(SubP2)}$ is the current augmented Lagrangian objective value. The scalar ? $_{ALR}$ is halved whenever $Z^q_{ALR(SubP2)}$ does not increase for a specified number of consecutive iterations. S_r in Eq. (22) indicates the r^{th} subset of customers in $\mathbf{\Psi}$ which are spanned by the same subtree.

$$\left(\Upsilon_r\right)^q = \left(|S_r| - L_{S_r}\right) - \sum_{k\in ID}\sum_{i\in S_r}\sum_{\substack{j\in S_r\\ j\neq i}}\left(x_{ijk}\right)^q \qquad \forall r\in G^q \tag{22}$$

$$^{q}_{ALR} = \Lambda^q_{ALR}\,\frac{Z^q_{ub(SubP2)} - Z^q_{ALR(SubP2)}(\alpha)}{\|\Upsilon\|^2} \tag{23}$$

$$\left(\alpha_r\right)^{q+1} = \min\left\{0, \left(\alpha_r\right)^q + s^q_{ALR}\left(\left(\Upsilon^\alpha_r\right)^q\right)\right\} \qquad \forall r\in G^q \tag{24}$$

4. GENERATING UPPER BOUNDS FOR P

At each subgradient iteration of the outer Lagrangian relaxation of **P**, the solution obtained for **SubP1** reveals which depots are preserved and which ones are opened. Once this information is provided, the remainder of the problem becomes a MDVRP any feasible solution of which constitutes an upper bound to **P**. Each time a new depot location plan is obtained by solving **SubP1**, a tabu search (TS) heuristic is triggered in the hope of achieving a better upper bound for **P**. When the Lagrangian iterations terminate, a greedy method called Add-Drop heuristic starts in case the final gap is greater than 2%. First, dosed or unopened depots are added to the solution one by one; then, currently open depots are dropped from the solution in a similar decremental fashion. An MDVRP is solved with respect to each of these scenarios. If a better feasible solution is realized, the new depot location plan is adopted, and Z_{ub} is updated.

TS is an meta-heuristic algorithm that guides the local search to prevent it from being trapped in premature local optima or in cycling. It starts with an initial solution. At each iteration of the TS, a neighborhood of solutions is generated for the current solution. The best one from this neighborhood is picked as the current solution depending on a number of criteria. Certain attributes of previous solutions are kept in a tabu list which is updated at the end of each iteration. The selection of the best solution in the neighborhood is done such that it does not attain any of the tabu attributes. The best feasible solution so far (incumbent) is updated if the current solution is both feasible and better than the incumbent. The procedure continues until one or more stopping criteria are fulfilled. In our study, we adopted the same tabu search procedure as proposed by Aksen et al. (2006) for the open vehicle routing problem with fixed driver nodes. We tailored the procedure for the MDVRP, and also enriched it with additional neighborhood generation moves.

4.1 An initial solution for P

In order to generate an initial solution for our TS, we make use of the constructive heuristic [PFIH-NN] proposed by Aksen and Altinkemer (2003). It is a hybrid of Push Forward Insertion and Nearest Neighborhood methods where customers are first assigned to the nearest depot. They are placed in an array sorted in the non-decreasing order of a special cost coefficient. This coefficient is calculated for each customer based on his distance to the assigned depot. The customer with the lowest cost coefficient is appended to a route. The remaining customers in the array are then chosen one at a time, and inserted into this first route according to the cheapest insertion principle. When the next to-be-inserted customer's demand exceeds the spare capacity on the current route, a new route is initiated.

4.2 Evaluation of solutions

For a given location plan the objective of the problem is to minimize the vehicle acquisition and total traveling cost. In our tabu search method, we apply strategic oscillation by admitting infeasible solutions where infeasible solutions are penalized in proportion to the violation of capacity constraints. The penalty terms are added to the objective value of an infeasible solution. Penalty coefficients are updated every 10 iterations based on the number of feasible and infeasible solutions visited. The objective value for a solution is given by $\sum_{k\in I}\sum_{D \notin I}\sum_{\substack{\in I \\ j\neq i}} c_{ijk} + \sum_{r\in R} p_c V_c(r)$ where the first term is the total traveling cost, R is the set of all routes, $V_c(r)$ denotes the overload (total demand of customers in route r minus vehicle capacity Q), and p_c denotes the penalty coefficient for overload on a route.

4.3 Neighborhood structure and tabu attributes

We use four move operators to create a neighborhood for the current solution. A pictorial description of the first three can be found in the paper by Tarantilis and Kiranoudis (2002). Each move involves two pilot nodes:

1-0 *move* : One of the pilot nodes is taken from its current position and inserted after the other.

1-1 *exchange* : Two pilot nodes are swapped.

2-*Opt move* : For two pilot nodes in the same route, the arcs emanating from these are removed. Two arcs are added one of which connects the pilot nodes, and the other connects their successor nodes. If the pilot nodes are in different routes, then the route segments following them are swapped preserving the order of nodes succeeding the pilots on each segment.

2-2 *exchange* : One of the pilot nodes and its successor are swapped with the other pilot node and its successor.

The size of the neighborhood generated in each iteration depends on the number of operating depots and the number of customer nodes in the problem. Besides neighborhood generation, we incorporate also a local search with these moves into the tabu search as a tool of local post optimization (LPO). A series of LPO operations are to be applied to the initial solution, to the current solution at the end of every 100 iterations if it is feasible, and also to the incumbent (current best solution) whenever it is updated. This helps the intensification of tabu search on the given MDVRP instance. We determine the sequence of LPO operations empirically, according to the results of extensive experimentation. In the application of LPO, all customers are set one by one as the first pilot node. For a given pilot node, the second one is chosen such that the related move yields the highest improvement in total distance without causing any infeasibility.

The tabu list is updated at the end of each iteration. Tabu attributes of a solution generated by a move can be stated as follows.

1-0 *move* : If node i is inserted after node j, the position of i cannot be changed by the same move while it is tabu-active.

1-1 *exchange* : If nodes i and j are swapped, they cannot be swapped again while they are tabu-active.

2-*Opt move* : If 2-Opt move is applied to nodes i and j, the move cannot be applied again to the same nodes while they are tabu-active.

2-2 Exchange: If nodes i and $(i+1)$ are swapped with nodes j and $(j+1)$, these cannot be swapped again while they are tabu active.

At each iteration, the tabu tenure is selected randomly between 5 and 15 iterations. In some cases, namely if the so-called aspiration criterion is satisfied, a move can be executed although its attributes are tabu-active. Aspiration criterion is considered to be satisfied if the total distance resulting from the move is better than the incumbent's objective value.

4.4 Stopping criteria

Tabu search terminates when any one of two stopping criteria is satisfied. The first criterion is the total number of iterations performed. The second criterion is the maximum permissible number of iterations during which the best feasible or best infeasible solution does not improve. Both values are determined based on the number of customers and on the number of operating depots found in the solution of **SubP1**.

5. COMPUTATIONAL RESULTS

The code of the proposed method is written in ANSI C language, compiled in Visual C++ .NET and executed on a 3.20 GHz Intel Xeon processor with 2 GB RAM. The algorithm is tested with 44 problems which consist of two parts. The first part includes 20 randomly generated small size test problems with 15 up to 35 customers and 2 up to 6 depots. The second part comprises 24 problems solved in Tüzün and Burke (1999) which have 100 up to 150 customers and 10 up to 20 depots. The problems in the first part are also solved by Cplex 9.0 with a time limit of five hours. These small size problems constitute benchmarks for the upper bounds obtained by our method. Upper bounds to the problems in the second part are compared with the solutions found in Tüzün and Burke (1999).

The stopping conditions of the Lagrangian relaxation have been fine-tuned by extensive experimentation on 16 test problems. Since the solution times of the larger problems are not practical for such experimentation, 10 of these problems have been selected from the ones in the first part. The mutually exclusive stopping conditions of the subgradient optimization for the outer Lagrangian relaxation are fixed as follows. If the number of subgradient iterations performed exceeds 300, or if the number of consecutive subgradient iterations during which the Lagrangian gap does not improve reaches 100, or finally if the amount of absolute increment in the Lagrange multipliers is not greater than 1.0e–7, the subgradient optimization procedure for the problem **P** stops. The stopping conditions in case of the augmented Lagrangian relaxation applied to **SubP2** are satisfied if the predefined limit on one of following parameters is reached: 150 subgradient iterations performed in the augmented Lagrangian relaxation, the step size or the gap between $Z_{lb(SubP2)}$ and $Z_{ub(SubP2)}$ dropping below 1.0e–5, and finally 75 consecutive iterations during which the gap does not improve.

Table 1. Results for 20 randomly generated test problems with *nc* between 15 and 35

Nc	*npd*	*ncd*	Z_{Cplex}	%GAP2	Z_{lb}	Z_{ub}	%GAP1	CPU(s)
15	1	2	1127.84	0.00%	1075.58	1127.84	4.86%	54.69
15	1	2	994.92	0.00%	994.92	994.92	0.00%	27.56
15	-	2	1024.19	0.29%	975.28	1027.14	5.32%	53.49
15	-	2	1032.08	4.13%	1031.51	1074.68	4.19%	48.46
20	1	3	1136.52	1.07%	1128.51	1148.72	1.79%	38.15
20	1	3	1285.05	2.56%	1262.14	1317.96	4.42%	96.66
20	-	3	1442.47	0.00%	1435.11	1442.48	0.51%	155.10
20	-	3	1022.49	0.00%	953.04	1022.49	7.29%	74.44
25	1	2	1407.29	–0.34%	1321.13	1402.44	6.15%	261.56
25	1	2	1271.85	1.18%	1244.53	1286.85	3.40%	161.76
25	1	4	1424.57	–0.45%	1370.51	1418.18	3.48%	241.80
25	1	4	1368.62	0.14%	1367.03	1370.47	0.25%	241.31
30	1	4	1629.90	–6.43%	1471.51	1525.03	3.64%	356.78
30	1	4	1432.56	0.65%	1348.73	1441.94	6.91%	640.78
30	-	4	1599.46	–3.48%	1511.43	1543.86	2.15%	232.63
30	-	4	1619.42	–0.47%	1555.81	1611.87	3.60%	482.01
35	1	4	1909.93	–5.08%	1735.65	1812.81	4.45%	945.08
35	1	4	1408.74	–1.61%	1362.82	1386.02	1.70%	285.20
35	-	6	1844.70	–2.35%	1658.61	1801.37	8.61%	682.30
35	-	6	1730.64	–5.55%	1556.13	1634.60	5.04%	582.43
Averages			**1385.66**	**–0.79%**	**1318.00**	**1369.58**	**3.89%**	**283.11**

Table 2. Results for Tüzün and Burke's instances

prob id	*nc*	*ncd*	Z_{Cplex}	%GAP3	Z_{lb}	Z_{ub}	%GAP1	CPU(s)
P111112	100	10	1556.64	−8.95%	1283.09	1417.30	10.46%	19875.27
P111122	100	20	1531.88	−7.95%	1178.19	1410.04	19.68%	10554.93
P111212	100	10	1443.43	−2.57%	1140.54	1406.33	23.30%	9562.77
P111222	100	20	1511.39	−3.08%	1186.54	1464.84	23.45%	16420.19
P112112	100	10	1231.11	−1.72%	1079.16	1209.88	12.11%	14443.91
P112122	100	20	1132.02	−9.95%	925.16	1019.44	10.19%	18333.10
P112212	100	10	825.12	−11.95%	627.05	726.48	15.86%	7158.19
P112222	100	20	740.64	−0.31%	541.66	738.34	36.31%	15391.94
P113112	100	10	1316.98	−1.59%	1069.98	1296.04	21.13%	16432.57
P113122	100	20	1274.50	−8.98%	1055.33	1160.09	9.93%	12327.16
P113212	100	10	920.75	−1.30%	753.37	908.79	20.63%	6190.90
P113222	100	20	1042.21	−10.84%	780.93	929.22	18.99%	11696.95
P131112	150	10	2000.97	−6.57%	1561.25	1869.43	19.74%	52546.65
P131122	150	20	1892.84	0.35%	1465.80	1899.42	29.58%	54043.24
P131212	150	10	2022.11	3.83%	1589.11	2099.50	32.12%	43472.18
P131222	150	20	1854.97	−2.55%	1438.10	1807.63	25.70%	55900.30
P132112	150	10	1555.82	−4.34%	1151.67	1488.29	29.23%	42149.14
P132122	150	20	1478.80	1.58%	1144.07	1502.16	31.30%	59226.08
P132212	150	10	1231.34	0.26%	959.29	1234.50	28.69%	26122.60
P132222	150	20	948.28	−1.06%	742.16	938.22	26.42%	69757.69
P133112	150	10	1762.45	−5.38%	1232.78	1667.65	35.28%	10469.41
P133122	150	20	1488.34	−2.38%	1051.04	1452.97	38.24%	32540.27
P133212	150	10	1264.63	−7.22%	930.82	1173.29	26.05%	55394.52
P133222	150	20	1182.28	0.61%	973.35	1189.44	22.20%	26393.21
Averages			**1383.73**	**−3.84%**	**1077.52**	**1333.72**	**23.61%**	**28600.13**

For all of the small size problems, %GAP1 is under 10% and the average %GAP1 of these problems is 3.89%. For 10 out of the 20 problems Z_{ub} outperforms Z_{Cplex}, while for three of them the proposed method finds the same solution as Cplex. For seven of the problems Cplex does better than the proposed method; yet the maximum gap between Z_{ub} and Z_{Cplex} is 4.13%. The quality of %GAP1 diminishes in the problems of the second part. Although, there is no indication of a continuous increase in %GAP1 as the number of customers in the problem increases, we observe that the average %GAP1 of the problems with 150 customers is higher than that of the problems with 100 customers. The upper bounds found for Tüzün and Burke instances update 19 out of 24 solutions given in their study, while an average improvement of 3.84% is obtained. The solution times of the problems with more than 100 customers are significantly long which makes the revision of the implementation of the procedure imperative.

6. SUMMARY AND CONCLUSIONS

In this study, an uncapacitated multi-depot location routing problem (MDLRP) is solved using Lagrangian relaxation. Two subproblems emerge from the relaxation of the coupling constraints in the MDLRP model. The first of them has a structure similar to a facility location-allocation problem (FLAP), and is solved with Cplex 9.0 to optimality in negligible amount of time. The second one is a capacity and degree constrained minimum spanning forest-like problem which is still an NP-hard problem. To tackle it, an augmented Lagrangian relaxation is applied. The nested Lagrangian relaxation-based solution method is tested on 44 MDLRP instances which consist of 20 randomly generated problems and 24 problems solved in Tüzün and Burke (1999). For the problems in the first part, gaps are below 10%. In most of the small size problems, the final upper bounds are better than the corresponding Cplex solutions. For problems in the second part, gaps are higher with an average of 23.61%, while the upper bounds for these improve most of the solutions given in Tüzün and Burke (1999). The experimental results not only assess the performance of the proposed procedure, but also point to new research directions. The next step would be solving the MDLRP with time windows. This type of time restrictions is a crucial quality of service (QoS) guarantee promised more and more often to customers in distribution logistics. Finally, long solution times especially for problems with more than 100 customers are a critical disadvantage of the proposed method. This might be overcome by a new implementation of the modified Prim's algorithm which is used to solve the Lagrangian relaxed subproblem $\mathbf{ALR}^{SubP2}$.

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