Lagrangian Relaxation for the Vehicle Routing Problem with time windows

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***Abstract*— In this paper, an optimization algorithm based on lagrangian relaxation is proposed to solve the vehicle routing problem with time windows (VRPTW). The VRPTW is the generalization of the Vehicle Routing Problem, with a service time interval constraint available with every customer node. Here, an approach is used which relaxes the constraint set that justify that each customer should be serviced. The problem consists of introducing optimal lagrangian multipliers and then the subgradient method is used to find the appropriate multipliers. Here the problem focuses on, compared to the problems in the literature, not only the minimization of the transportation cost but also the used number of vehicles. The method has been implemented and tested on a series of well known benchmark test problems. The algorithm results to be very competitive and the results are better than the optimal solution.**

Keywords—Lagrangian relaxation; Vehicle Routing Problem with time windows; subgradient; lagrangian multipliers; lower bound.

I. INTRODUCTION

The assignment of designing delivery or pickup service in the area of transportation and supply chain is termed as Vehicle Routing Problem in literature. VRP was first stated in the work of Dantzig and Ramser [11] titled “Truck Dispatching Problem” which aims at designing optimal routes for gasoline delivery trucks providing service between single distribution terminal and number of service stations. Vehicle Routing Problem refers to the situation where distinct routes are determined for each vehicles starting and ending their journey at one or more depots to service various consumers delivering goods and services. VRP comes under combinatorial problem. The objective of VRP is to serve customers with known demands with minimum cost (or distance travelled) while satisfying some requirements or constraints. The VRP has played a key role in distribution and logistics. Nowadays, a viable logistics network can have great impact for the industries and firms having transportation activities as their main domain.

The Vehicle Routing Problem with time windows (VRPTW) is the specialization of the classical VRP, being very applicable in real world problems. It involves a fleet of vehicles starting from a depot to serve a number of customers, diversely located, with fixed demand request and within particular time interval and returning back to the depot ultimately. Because of the great applicability of the VRPTW in real life, the research has always paid more attraction in this area. VRPTW is considered to be NP-hard problem, which suggests that it cannot be optimally solved for larger instances. Here we consider a lagrangian relaxation based approach for the problem considering two objectives- the minimization of transportation cost and the used number of vehicles. Here Fig. 1 gives the illustration of the vehicle routing problem with time windows. In fig, different colored lines shows the routes taken by the three vehicles among 10 customer nodes.

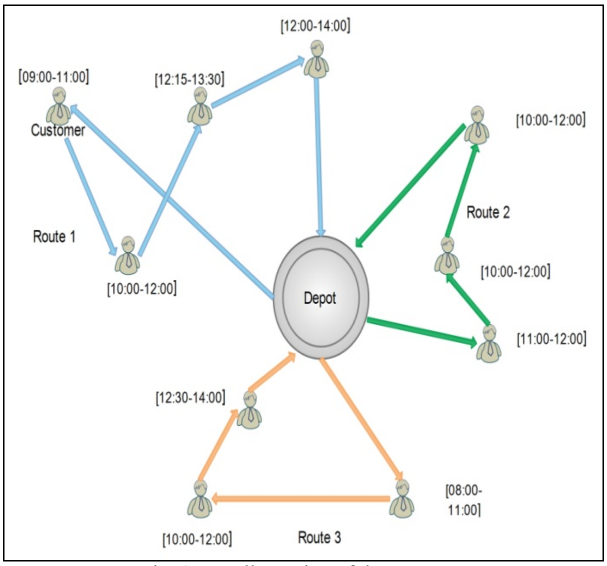

Fig. 1. Illustration of the VRPTW

II. RELATED WORK

VRPTW is considered NP-hard because of the NP-hardness of VRP. The VRPTW is the expansion of the VRP with addition of time windows to provide the service at each customer within a specific time interval. Various researches have been done in the field of VRPTW which can be used to specify many real-world problems. The early works on the VRPTW include case studies [3-5]. Earlier studies on VRPTW include both optimization algorithms and heuristic approaches. However, the focus of current research is heuristic and Metaheuristic approaches. Various surveys have been done in the field of VRPTW [6-8].

The various effective approaches found in the literature consist of constrained shortest path relaxations [1] [2]. Column generation [1] and lagrangian decomposition [2] are the similar methods. In both the methods the complete problem is partitioned into two sub problems- the master problem and the sub problem. The sub problem is the shortest path problem satisfying the time and capacity constraints. There are various papers with little variation in approaches for Lagrange relaxation such as variable splitting accompanied by Lagrange relaxation, k-tree procedure along with lagrangian relaxation[9][10] .[13] suggested an approach solving shortest path with side constraints which is followed by Lagrange relaxation. This involves relaxing the constraint ensuring that every customer should be served exactly once and adding to objective function.

Various optimization and heuristics approaches can be found in the literature for the vehicle routing problem with time windows. Many exact approaches can be found in the literature. Various sophisticated approaches based on lagrangian relaxation, which now can solve large instance of problem to optimality have been introduced. [14] proposed a relax-and-cut algorithm which combines the lagrangian based approach with comb and multi star inequalities. This approach improves the quality of lagrangian bound. [15] proposed a cutting plane algorithm to solve the Lagrangian dual. Cutting planes are generated by solving the subproblem and then they are introduced in a master problem to impose bounds on the dual variables. Imposing bounds will ensure the stability in between the iterations. [16] proposed the acceleration of lagrangian relaxation for the vehicle routing problem with time windows. Here analytical center cutting plane method is used for the acceleration. The results are then compared with basic cutting plane algorithm applied with column generation which suggests the improved results in terms of computational time and quality of lower bounds.

In this paper, the problem consists of introducing lagrangian multipliers for the relaxed problem and subgradient method is used for finding the values of the multipliers. The upper bound is computed by finding the feasible solutions by satisfying the constraints. Here, the problem is bi-objective, minimizing the transportation cost and use of number of vehicles.

III. MATHEMATICAL FORMULATION

The VRPTW is defined on a graph (C', E) which is to be traversed by a fleet of vehicles V. Generally, the vehicles are considered homogeneous, i.e., every vehicle has same capacity.

A. Sets and notation

C	Set of all customers $(1, 2, \ldots, n)$
C'	Set of all nodes $C' = C \cup 0 \cup n+1$
Q	Vehicle capacity
V	Number of vehicles
c_{ij}	Cost of edge $(i, j) \in E$
t_{ij}	Travel time of edge
r_i	Demand to service customer i
(e_i, l_i)	Time window for customer i

Here, the model includes two types of decision variables X and S. The decision variable X_{ijk} is defined $\forall (i, j) \in E, k \in V$ as

$$X_{ijk} = \begin{cases} 1; & \text{if edge from } i \text{ to } j \text{ is traversed by vehicle } k \\ 0; & \text{otherwise} \end{cases}$$

The decision variable S_{ik} defined $\forall i \in C', \forall k \in V$ as

S_{ik} time at which vehicle k service customer i

Also assumed, $S_{0k} = 0$ and $S_{(n+1)k}$ denotes the time at which vehicles arrives at returning depot.

The objective is to design a set of minimal cost routes, one for each vehicle, such that all customers are serviced exactly once while minimizing the number of vehicles so here optimization is bi-objective. Hence, split deliveries are not allowed. The routes must be feasible with respect to the capacity of the vehicles and the time windows of the customers serviced. The VRPTW can be stated mathematically as:

$$Minimise \sum_{k \in C'} \sum_{(i,j) \in E} c_{ij} X_{ijk} \tag{1}$$

Subject to

$$\sum_{k \in V} \sum_{j \in C'} X_{ijk} = 1, \quad \forall i \in C \tag{2}$$

$$\sum_{i \in C} r_i \sum_{j \in C'} X_{ijk} \le Q, \quad \forall k \in V \tag{3}$$

$$\sum_{j \in C'} X_{ojk} = 1, \quad \forall k \in V \tag{4}$$

$$\sum_{i \in C'} X_{ipk} - \sum_{j \in C'} X_{hjk} = 0, \quad \forall p \in C, \forall k \in V \tag{5}$$

$$\sum_{i\in C'} X_{i,n+1,k} = 1, \quad \forall k \in V \tag{6}$$

$$X_{ijk}(S_{ik} + t_{ij} - S_{jk}) \le 0, \ \ \forall (i,j)\in E, \forall k \in V \tag{7}$$

$$e_i \le S_{ik} \le l_i, \qquad \forall i \in C', \forall k \in V \tag{8}$$

$$X_{ijk} \in \{0,1\} \qquad \forall (i,j)\in E, \forall k \in V \tag{9}$$

The objective function (1) states that the total cost should be minimized. Constraint (2) suggests that only one vehicle should service one node while constraint (3) ensures that the capacity constraint is fulfilled. Here no vehicle can serve the customer demands beyond its capacity permits. Constraints (4)-(6) fulfill the flow constraint for the path traversed by vehicle k. Constraint (7) makes sure that vehicle $k, k \in V$ should not arrive at customer j before $S_{ik} + t_{ij}$ if travelling from node i to j. Constraint (8) checks for the satisfaction of time constraint. An unused vehicle is represented by an empty route from node 0 to $n+1$.

Another constraint for minimization of number of vehicles can be added as

$$\sum_{k\in V}\sum_{j\in C'} x_{0jk} \le |V|, \ \ \forall k \in V, \forall j \in C' \tag{10}$$

Note that constraint (7) may result in non-convex optimization as it involves quadratic terms so it must be linearized as

$$S_{ik} + t_{ij} - M_{ij}(1 - X_{ijk}) \le S_{jk} \quad \forall i,j \in C', \forall k \in V \tag{11}$$

Here M_{ij} is described as large constant which can be replaced by $\max(l_i + t_{ij} - e_i, 0), \ (i,j)\in E$.

IV. Solution Approach

Here, we present a Lagrangian relaxation-based approach for solving the above stated problem. The main approach includes the relaxation of the integrity constraints which gives the relaxed problem which is easier to solve and it gives the lower bound on the original problem. Then a feasible solution to the problem is constructed which gives the upper bound. We have used the sub gradient method here for the non differentiable constraints which help to adjust the Lagrange multipliers so as to reduce the constraint violation, and this procedure will update the Lagrange multipliers. This process is repeated until some stopping condition is not met.

A. Steps of Algorithm

1. Initialize parameters Lower Bound (LB) = -∞, Upper Bound (UB) = +∞, lagrangian multiplier $\lambda_i = 0, \forall i$ constraints. Primal is used to obtain better UB.
2. **Repeat**
3. Lower Bound is evaluated
   Z_{LB}=getObjectiveValue()
   If (LB<Z_{LB})
   Modify LB=Z_{LB}
   Z_{LB} is the lower bound on the optimal objective value. It is obtained by computing the model relaxing the required constraint which is marked as "hard".
4. If lower bound is improved in previous step apply primal heuristics else goto step5
5. Set UB=primal heuristic solution, Compute the UB
   Z_{UB}=getObjectiveValue()
   If (UB>Z_{UB})
   UB=Z_{UB}
6. If UB< (LB + tolerance), terminate algorithm, found optimum else go to Step 6.
7. Apply gradient search λ_i
8. If norm=0, terminate found optimum(LB is optimum in this case)
9. **Until** the iteration count is reached
10. Output the results.

Fig. 2 suggests the proposed approach. It depicts how the flow should work in the solution approach. The steps of algorithm are followed.

B. Computation of Lower Bound

The technique used to find linear bounds is linear programming relaxation. Here we take the mixed integer programming formulation of the problem and relax the integrity requirements on the variables. Here the constraint set is relaxed. We relax constraint (2) which satisfies that each customer must be serviced with the lagrangian multiplier, thus obtaining the corresponding Lagrangian relaxation problem.

$$Minimise \sum_{k\in C'}\sum_{(i,j)\in E} c_{ij}X_{ijk} + \sum_{i\in C}\lambda_i\,(1 - \sum_{k\in V}\sum_{j\in C'} X_{ijk}) \tag{12}$$

We rewrite the above equation as:

$$Minimise \sum_{k\in C'}\sum_{(i,j)\in E}(c_{ij} - \lambda_j)X_{ijk} + \sum_{i\in V}\lambda_i \tag{13}$$

C. Computation of Upper Bound

At each iteration, feasible solution to the problem is calculated by finding feasible solution by satisfying the constraint of the problem. It depends on the results of the relaxed problem generated by relaxing the required constraint.

D. Sub gradient method

It is an iterative method which involves updation of multipliers in some systematic manner. This procedure helps to maximize the lower bound obtained from the relaxed problem.

The basic steps are

1. Let π be the user defined parameter with value in range (0, 2]. Here, in our problem we set this parameter value equal to 1.
2. Solve the relaxed problem for the lower bound to get solution Z_{LB}.
3. Evaluate subgradients for the constraint (2)

$$G_i = 1 - \sum_{k \in V} \sum_{j \in C'} x_{ijk}, \ \forall i \in C \qquad (14)$$

4. Calculate step size

$$step = \pi(Z_{UB} - Z_{LB}) / \sum_{i=1}^{m} (G_i)^2 \qquad (15)$$

This step size depends upon the gap between the upper bound and lower bound.

5. Update the multipliers λ_i

$$\lambda_i = \max(0, \lambda_i + step \times G_i) \qquad (16)$$

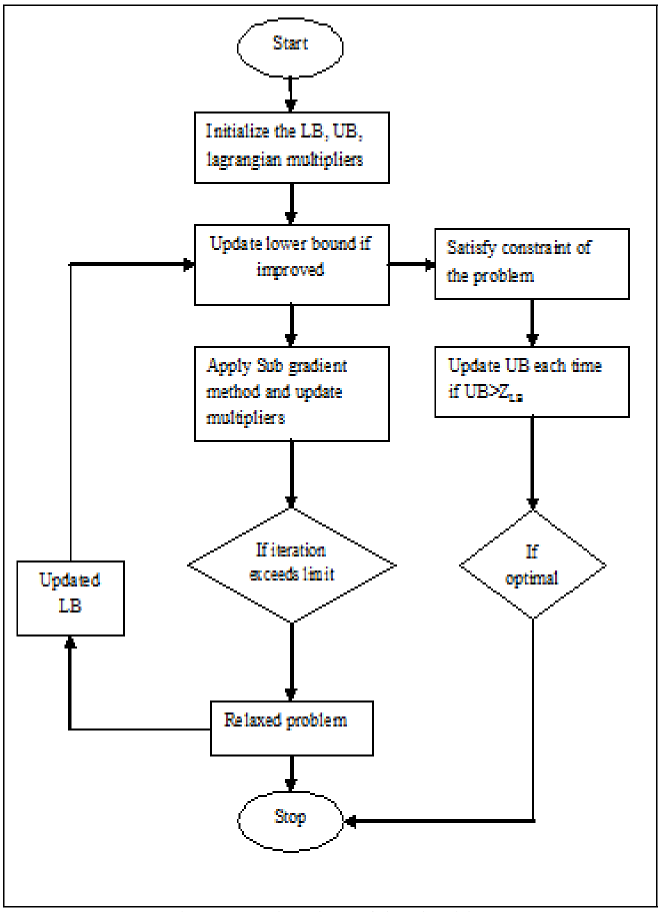

Fig. 2. Flowchart of the algorithm

E. Stopping criterion

The algorithm will terminate in case of any of the condition as follows:

- Optimality Gap: This happens when gap between the lower and upper bound gets close to zero
- Iteration limit: When the limit of the iteration is reached to the specified value. Here the iteration limit is generally set to be 100.
- Step size: When the constant step size gets close to zero.

V. EXPERIMENTAL RESULTS

A. Dataset used

The model specified in Section 3 is developed and implemented in CPLEX 12.6 on a PC clocked at 2.2 GHz under the operating system Windows 7. Here, the model is tested on series of benchmark problems R101-R112 from the Solomon instances [12] which belong to the class of randomly generated customers. Here, Table 1 shows the computational results obtained from R1 class instances. The objective of the problem is minimizing the transportation cost and to minimize number of vehicle should be used from 25 vehicles, each vehicle having 200 capacity.

It is observed from Table 1 that first column gives the instance considered, second column records the Lower bound objective value, third columns gives the optimal solution value, fourth column records the number of vehicle used of the total vehicles, fifth column calculates the gap to the optimal solution and iterations of the problem. Sixth column and seventh column records iterations and computation time of the problem.

$$Gap = \frac{Optimal\ \ Solution - Lower\ Bound value}{Optimal\ \ Solution} \qquad (17)$$

Fig. 3 shows the statistic graph of the algorithm on R101 for 25 customer nodes.

TABLE I. RESULTS OF THE ALGORITHM

Problem Instance	**Lower Bound (LB)**	**Optimal Solution**	**Vehicle**	**Gap (%)**	**Iterations**	**Time**
R101.25	618.34	617.1	6	-0.20	42	1.24
R101.50	1046.2	1044.0	11	-0.210	70	1.98
R101.100	-	1637.7	-	-	220	4.34
R102.25	547.8	547.1	7	-0.12	63	1.25

R102.50	909	909	11	0	137	1.34
R102.100	1468.2	1466.6	17	-0.10	240	20.87
R103.25	453.2	454.6	4	0.30	58	0.87
R103.50	770.8	772.9	8	0.27	270	7.50
R103.100	1207.9	1208.7	13	0.06	660	8.39
R104.25	416.9	416.9	4	0	71	1.11
R104.50	625.4	625.4	5	0	689	10.83
R104.100	971.5	971.5	11	0	1243	10.87
R105.25	530.5	530.5	5	0	45	3.21
R105.50	899.3	899.3	9	0	143	4.54
R105.100	1355.3	1355.3	15	0	520	9.59
R106.25	460.2	465.4	3	1.11	77	1.11
R106.50	790.87	793	5	0.26	128	1.53
R106.100	1226.33	1234.6	12	0.66	2983	12.35
R107.25	425.2	424.3	4	-0.21	70	1.25
R107.50	713.8	711.1	7	-0.37	376	1.34
R107.100	1068.3	1064.6	11	-0.34	6580	20.8
R108.25	396.66	397.3	4	0.16	115	1.11
R108.50	615.34	617.7	6	0.38	15643	1.53
R108.100	-	-	-	-		-
R109.25	441.1	441.3	5	0.045	23	0.98
R109.50	777.33	786.8	8	1.20	567	5.21
R109.100	1140.32	1146.9	13	0.57	10945	8.45
R110.25	444.1	444.1	4	0	119	1.3
R110.50	694.80	697.0	7	0.315	110	3.7
R110.100	-	1068	-	-	-	-
R111.25	425.43	428.8	5	0.78	77	1.11
R111.50	706.2	707.2	7	0.14	998	1.53
R111.100	-	1048.7	12	-	-	-
R112.25	390.5	393	4	0.63	123	2.8
R112.50	620.8	630.2	6	1.49	15673	20.87
R112.100		-	-	-	-	

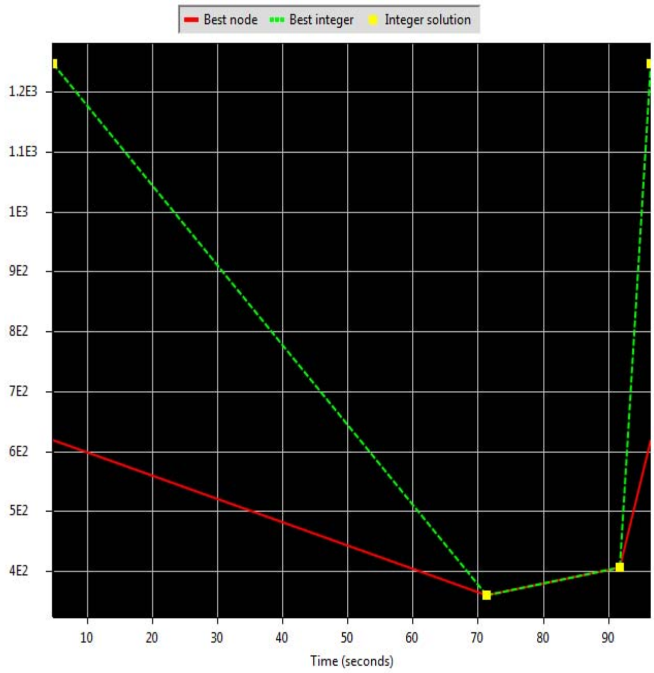

Fig. 3. Statistics graph for thr R101 instance with 25 customers

The results generated are quite encouraging and better than optimal solutions in most of the cases. In some cases, the results are not computable and shown by (-). Here number of vehicle is also reduced. The results are obtained in short amount of time. Here optimality gap is calculated so to indicate the quality of solutions. Positive value of optimality gap shows the better values of solution than the optimal values. Optimality gap is used to find tighter bounds.

VI. CONCLUSION

Here we have presented a lagrangian based relaxation approach for the VRPTW which relaxes the constraint set visiting all the customers. Here the algorithm has been tested on well known Solomon benchmark problems. The instances are solved to 100 customer nodes. For some of the instances, the algorithm does not give results. However, the results are better than the optimal solution in some cases and respective gap has been recorded.

REFERENCES

[1] M. Desrochers, J. Desrochers and M. Soloman, “A New Optimization Algorithm for the Vehicle Routing Problem with Time Windows” Opns. Res., vol. 40, 1992, pp. 342-354.

[2] K. Halse, “Modelling and Solving Complex Vehicle Routing Problems” Ph.D. Dissertation No. 60, Institute of Mathematical Statistics and Operations Research, Technical University of Denmark, DK-2800 Lyngby, Denmark, 1992.

[3] H. Pullen and M. Webb, “A computer application to a transport scheduling problem,” Computer Journal, vol. 10, pp.10–13, 1967.

[4] O. B. G. Madsen, “Optimal scheduling of trucks - A routing problem with tight due times for delivery,” In H. Strobel, R. Genser and M. Etschmaier, “Optimization applied to transportation systems,” IIASA, International Institute for Applied System Analysis, Laxenburgh, 1976, pp.126–136.

[5] K. Knight and J. Hofer, “Vehicle scheduling with timed and connected calls: A case study,” Operational Research Quarterly, vol. 19, 1968, pp 299–310.

[6] B. L. Golden and A. A. Assad, “Perspectives on Vehicle Routing: Exciting New Developments,” Operations Research, vol. 34, 1986, pp. 803-809.

[7] B. L. Golden and A. A. Assad, “Vehicle Routing: Methods and Studies,” Elsevier Science Publishers, 1988.

[8] M. Desrochers, J. K. Lenstra, M.W.P Savelsbergh, and F. Soumis, “Vehicle Routing with Time Windows: Optimization and Approximation,” In: B. Golden and A. Assad (eds.), “Vehicle Routing: Methods and Studies,” Elsevier Science Publishers, 1988, pp. 65-84.

[9] M. Fisher, “Optimal solution of vehicle routing problem using minimum k-trees,” Operations Research, vol. 4, 1994, pp. 626–642.

[10] H. D Holland, “Adaptation in Natural and Artificial Systems,” The University of Michigan Press, Ann Arbor, MI, 1975.

[11] G. B. Dantzig and J. H. Ramser, “The Truck Dispatching problem,” Management Science, vol. 6, 1959, pp. 80-92.

[12] M. M. Solomon, “Algorithms for the Vehicle Routing and Scheduling Problems with Time Window Constraints,” Operations Research, vol. 35, 1987, pp. 254-265.

[13] N. Kohl and O. B. G. Madsen, “An Optimization Algorithm for the Vehicle Routing Problem with Time Windows based on Lagrangean Relaxation,” Operations Research, vol. 45, 1997, pp. 395 -406.

[14] C. Martinhon, A. Lucena, N. Maculan, “A relax and cut algorithm for the vehicle routing problem,” Technical Report RT-05/00, Universidade Federal Fluminense, Niterói, Brasil, 2000.

[15] B. Kallehauge, J. Larsen, O.B.G. Madsen, “Lagrangean duality applied to the vehicle routing with time window,” Computers & Operations Research, vol. 33, 2006, pp. 1464–1487.

[16] H. Karimi and A. Seifi, “ Acceleration of lagrangian method for the Vehicle Routing Problem with time windows, International Journal of Industrial Engineering & Production Research, vol. 23, 2012, pp. 309-315.