

Solutions

Chapter 2 Foundations of Probability

2.1 (COMPOSING RANDOM ELEMENTS) Show that if f is \mathcal{F}/\mathcal{G} -measurable and g is \mathcal{G}/\mathcal{H} -measurable for sigma algebras \mathcal{F}, \mathcal{G} and \mathcal{H} over appropriate spaces, then their composition, $g \circ f$ (defined the usual way: $(g \circ f)(\omega) = g(f(\omega)), \omega \in \Omega$), is \mathcal{F}/\mathcal{H} -measurable.

Proof. Since g is \mathcal{G}/\mathcal{H} -measurable, therefore $\forall C \in \mathcal{H}, \exists B = g^{-1}(C) \in \mathcal{G}$. Similarly, since f is \mathcal{F}/\mathcal{G} -measurable, $\forall B \in \mathcal{G}, \exists A = f^{-1}(B) \in \mathcal{F}$. Thus $\forall C \in \mathcal{H}, \exists A = f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C) \in \mathcal{F}$ and the proof is complete. □

2.2 Let X_1, \dots, X_n be random variables on (Ω, \mathcal{F}) . Prove that $X = (X_1, \dots, X_n)$ is a random vector.

Proof. Since X_i is a random variable ($\forall i = 1, 2, \dots, n$), it holds that X_i is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable, which means that $\forall B \in \mathcal{B}(\mathbb{R}), X_i^{-1}(B) \in \mathcal{F}$. We first prove that X is $\mathcal{F}/(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}))$ -measurable (totally n $\mathcal{B}(\mathbb{R})$ s). $\forall A = A_1 \times A_2 \times \dots \times A_n \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}), X^{-1}(A) = X_1^{-1}(A_1) \cap X_2^{-1}(A_2) \cap \dots \cap X_n^{-1}(A_n) \in \mathcal{F}$, which holds since $X_i^{-1}(A_i) \in \mathcal{F}, \forall i = 1, 2, \dots, n$ and \mathcal{F} is a σ -algebra. Thus we conclude that X is $\mathcal{F}/(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}))$ -measurable.

By definition $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}))$ (totally n $\mathcal{B}(\mathbb{R})$ s). And according to the property in 2.5(b), we can get that X is $\mathcal{F}/\mathcal{B}(\mathbb{R}^n)$ -measurable, thus it is a random vector. □

2.3 (RANDOM VARIABLE INDUCED σ -ALGEBRA) Let \mathcal{U} be an arbitrary set and (\mathcal{V}, Σ) a measurable space and $X : \mathcal{U} \rightarrow \mathcal{V}$ an arbitrary function. Show that $\Sigma_X = \{X^{-1}(A) : A \in \Sigma\}$ is a σ -algebra over \mathcal{U} .

Proof. (i) We need to show that Σ_X is closed under countable union. Let $U_i = X^{-1}(A_i), A_i \in \Sigma, i \in \mathbb{N}$.

It follows that $\bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} X^{-1}(A_i) = X^{-1}(\bigcup_{i=1}^{\infty} A_i)$. Since $\bigcup_{i=1}^{\infty} A_i \in \Sigma, \bigcup_{i=1}^{\infty} U_i \in \Sigma_X$.

(ii) We need to show that Σ_X is closed under set subtraction $-$. $\forall U_1, U_2 \in \Sigma_X, U_1 - U_2 = X^{-1}(A_1) - X^{-1}(A_2) = X^{-1}(A_1 - A_2)$. Since $A_1 - A_2 \in \Sigma, U_1 - U_2 \in \Sigma_X$.

(iii) We need to show that Σ_X is closed to \mathcal{U} itself. Since $\mathcal{U} = X^{-1}(\mathcal{V})$ and $\mathcal{V} \in \Sigma$, it follows that $\mathcal{U} \in \Sigma_X$. □

2.4 Let (Ω, \mathcal{F}) be a measurable space and $A \subseteq \Omega$ and $\mathcal{F}|_A = \{A \cap B : B \in \mathcal{F}\}$.

Proof. (a) (i) We need to show that $\mathcal{F}|_A$ is closed under countable union. Let $X_1 = A \cap B_1, X_2 = A \cap B_2, \dots$ and $X' = \bigcup_{i=1}^{\infty} X_i$ and $B' = \bigcup_{i=1}^{\infty} B_i$ where $B_1, B_2, \dots \in \mathcal{F}$. Since \mathcal{F} is sigma algebra, $B' \in \mathcal{F}$. Furthermore, since $X' = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} A \cap B_i = A \cap \left(\bigcup_{i=1}^{\infty} B_i \right) = A \cap B'$, we can see that $X' \in \mathcal{F}|_A$.

(ii) We need to show that $\mathcal{F}|_A$ is closed under set subtraction $-$. $\forall X_1, X_2 \in \mathcal{F}|_A, X_1 - X_2 = (A \cap B_1) - (A \cap B_2) = A \cap (B_1 - B_2)$. Since $B_1 - B_2 \in \mathcal{F}$, it follows that $X_1 - X_2 \in \mathcal{F}|_A$.

- (iii) We need to show that $\mathcal{F}|_A$ is closed to A itself. Since $\emptyset \in \mathcal{F}$, we have $\emptyset = A \cap \emptyset \in \mathcal{F}|_A$ and $A = \emptyset^C \in \mathcal{F}|_A$.
- (b) Let $P = \{A \cap B : B \in \mathcal{F}\}, Q = \{B : B \subset A, B \in \mathcal{F}\}$.
 - (i) We claim that $P \subset Q$. Let $X = A \cap B, B \in \mathcal{F}$. Since $A \in \mathcal{F}, X = A \cap B \in \mathcal{F}$. Furthermore, $X \in Q = \{B : B \subset A, B \in \mathcal{F}\}$.
 - (ii) We claim that $Q \subset P$. $\forall X \in Q$, we have $X \subset A$ and $X \in \mathcal{F}$, which means that $X = X \cap A$ and $X \in \mathcal{F}$. It follows that $X \in P$.
 - (iii) Take both (i)(ii) into consideration, we can see that $P = Q$.

□

2.5 Let $\mathcal{G} \subseteq 2^\Omega$ be a non-empty collection of sets and define $\sigma(\mathcal{G})$ as the smallest σ -algebra that contains \mathcal{G} . By ‘smallest’ we mean that $\mathcal{F} \in 2^\Omega$ is smaller than $\mathcal{F}' \in 2^\Omega$ if $\mathcal{F} \subset \mathcal{F}'$.

- (a) Show that $\sigma(\mathcal{G})$ exists and contains exactly those sets A that are in every σ -algebra that contains \mathcal{G} .
- (b) Suppose (Ω', \mathcal{F}) is a measurable space and $X : \Omega' \rightarrow \Omega$ be \mathcal{F}/\mathcal{G} -measurable. Show that X is also $\mathcal{F}/\sigma(\mathcal{G})$ -measurable. (We often use this result to simplify the job of checking whether a random variable satisfies some measurability property).
- (c) Prove that if $A \in \mathcal{F}$ where \mathcal{F} is a σ -algebra, then $\mathbb{I}\{A\}$ is \mathcal{F} -measurable.

Proof. (a) Let $\mathcal{K} = \{\mathcal{F} | \mathcal{F} \text{ is a } \sigma\text{-algebra and contains } \mathcal{G}\}$, It holds obviously that \mathcal{K} is not an empty set since it contains 2^Ω .

Then $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ contains exactly those sets that are in every σ -algebra that contains \mathcal{G} . Given its existence, we only need to prove that $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ is the smallest σ -algebra that contains \mathcal{G} .

First we show $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ is a σ -algebra. Since \mathcal{F} is a σ -algebra and therefore $\Omega \in \mathcal{F}$ for all $\mathcal{F} \in \mathcal{K}$, it follows that $\Omega \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$. Next, for any $A \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$, $A^c \in \mathcal{F}$ for all $\mathcal{F} \in \mathcal{K}$. Since they are all σ -algebras, $A^c \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$. Hence $A^c \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$. Finally, for any $\{A_i\}_i \subset \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$, $\{A_i\}_i \subset \mathcal{F}$ for all $\mathcal{F} \in \mathcal{K}$. Since they are all σ -algebras, $\bigcup_i A_i \in \mathcal{F}$ for all $\mathcal{F} \in \mathcal{K}$. Hence $\bigcup_i A_i \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$.

Next we want to prove $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ is the smallest σ -algebra that contains \mathcal{G} . It is quite obvious that $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F} \subseteq \mathcal{F}'$ for all $\mathcal{F}' \in \mathcal{K}$.

Above all, we have $\sigma(\mathcal{G}) = \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$.

- (b) Define $\mathcal{H} = \{A : X^{-1}(A) \in \mathcal{F}\}$. To show X is $\mathcal{F}/\sigma(\mathcal{G})$ -measurable, it is sufficient to prove $\sigma(\mathcal{G}) \subseteq \mathcal{H}$.

First we prove that \mathcal{H} is a σ -algebra. It holds that $\Omega \in \mathcal{H}$ since $X^{-1}(\Omega) = \Omega' \in \mathcal{F}$. For any $A \in \mathcal{H}$, we have $X^{-1}(A) \in \mathcal{F}$, thus $X^{-1}(A^c) = X^{-1}(A)^c \in \mathcal{F}$, which holds since \mathcal{F} is a σ -algebra. Thus $A^c \in \mathcal{H}$. For any $A_i \in \mathcal{H}$, $i = 1, 2, \dots$, $X^{-1}(A_i) \in \mathcal{F}$, $X^{-1}(\bigcup_i A_i) = \bigcup_i X^{-1}(A_i) \in \mathcal{F}$. We can then conclude $\bigcup_i A_i \in \mathcal{H}$ and \mathcal{H} is a σ -algebra.

Also, since X is \mathcal{F}/\mathcal{G} -measurable, we have $\mathcal{G} \subseteq \mathcal{H}$. Thus \mathcal{H} is σ -algebra that contains \mathcal{G} . By applying the result of (a), we have $\sigma(\mathcal{G}) \subseteq \mathcal{H}$, which completes the proof.

- (c) The idea is to show $\forall B \in \mathfrak{B}(\mathbb{R}), \mathbb{I}\{A\}^{-1}(B) \in \mathcal{F}$.

If $\{0, 1\} \in B$, $\mathbb{I}\{A\}^{-1}(B) = \Omega \in \mathcal{F}$. If $\{0\} \in B$, $\mathbb{I}\{A\}^{-1}(B) = A^c \in \mathcal{F}$. If $\{1\} \in B$, $\mathbb{I}\{A\}^{-1}(B) = A \in \mathcal{F}$. If $\{0, 1\} \cap B = \emptyset$, $\mathbb{I}\{A\}^{-1}(B) = \emptyset \in \mathcal{F}$.

□

2.6 (KNOWLEDGE AND σ -ALGEBRAS: A PATHOLOGICAL EXAMPLE) In the context of Lemma 2.5, show an example where $Y = X$ and yet Y is not $\sigma(X)$ measurable.

HINT As suggested after the lemma, this can be arranged by choosing $\Omega = \mathcal{Y} = \mathcal{X} = \mathbb{R}, X(\omega) = Y(\omega) = \omega, \mathcal{F} = \mathcal{H} = \mathfrak{B}(\mathbb{R})$ and $\mathcal{G} = \{\emptyset, \mathbb{R}\}$ to be the trivial σ -algebra.

Proof. As the hint suggests, Let $\Omega = \mathcal{Y} = \mathcal{X} = \mathbb{R}$, $X(\omega) = Y(\omega) = \omega$, $\mathcal{F} = \mathcal{H} = \mathfrak{B}(\mathbb{R})$. In this case, $\sigma(X) = \{X^{-1}(A) : A \in \mathcal{G}\} = \{\emptyset, \mathbb{R}\}$, we can find that $Y^{-1}((0, 1)) = (0, 1) \notin \sigma(X)$, thus Y is not $\sigma(X)$ -measurable. \square

2.7 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $B \in \mathcal{F}$ be such that $\mathbb{P}(B) > 0$. Prove that $A \mapsto \mathbb{P}(A|B)$ is a probability measure over (Ω, \mathcal{F}) .

Proof. First we have $\mathbb{P}(\Omega | B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$. Then, for any $A \in \mathcal{F}$, $\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \geq 0$. Next, for any $A \in \mathcal{F}$, $\mathbb{P}(A^c | B) = \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}((\Omega - A) \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 1 - \mathbb{P}(A | B)$. Finally, for all countable collections of disjoint sets $\{A_i\}_i$ with $A_i \in \mathcal{F}$ for all i , we have $\mathbb{P}(\bigcup_i A_i | B) = \frac{\mathbb{P}((\bigcup_i A_i) \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\bigcup_i (A_i \cap B))}{\mathbb{P}(B)} = \sum_i \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_i \mathbb{P}(A_i | B)$. \square

2.8 (BAYES LAW) Verify (2.2).

Proof. With the definition of conditional probability, we have $\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$. \square

2.9 Consider the standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ generated by two standard, unbiased, six-sided dice that are thrown independently of each other. Thus, $\Omega = \{1, \dots, 6\}^2$, $\mathcal{F} = 2^\Omega$ and $\mathbb{P}(A) = |A|/6^2$ for any $A \in \mathcal{F}$ so that $X_i(\omega) = \omega_i$ represents the outcome of throwing dice $i \in \{1, 2\}$.

- (a) Show that the events ' $X_1 < 2$ ' and ' X_2 is even' are independent of each other.
- (b) More generally, show that for any two events, $A \in \sigma(X_1)$ and $B \in \sigma(X_2)$, are independent of each other.

Proof. (a) The event $\{X_1 < 2\} = \{1\} \times \{1, 2, 3, 4, 5, 6\}$, $\{X_2 \text{ is even}\} = \{1, 2, 3, 4, 5, 6\} \times \{2, 4, 6\}$, $\{X_1 < 2, X_2 \text{ is even}\} = \{(1, 2), (1, 4), (1, 6)\}$.

Thus $\mathbb{P}(X_1 < 2) = \frac{6}{36} = \frac{1}{6}$, $\mathbb{P}(X_2 \text{ is even}) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(X_1 < 2, X_2 \text{ is even}) = \frac{3}{36} = \frac{1}{12}$, which satisfies $\mathbb{P}(X_1 < 2, X_2 \text{ is even}) = \mathbb{P}(X_1 < 2) \times \mathbb{P}(X_2 \text{ is even})$. These two events are independent of each other.

- (b) $\sigma(X_1) = \{X_1^{-1}(A'), A' \subseteq [6]\} = \{A' \times [6] : A' \subseteq [6]\}$, $\sigma(X_2) = \{X_2^{-1}(B'), B' \subseteq [6]\} = \{[6] \times B' : B' \subseteq [6]\}$. Thus $\forall A \in \sigma(X_1), B \in \sigma(X_2)$, $\mathbb{P}(A) = \frac{|A'| \times 6}{36} = \frac{|A'|}{6}$, $\mathbb{P}(B) = \frac{6 \times |B'|}{36} = \frac{|B'|}{6}$ and $\mathbb{P}(A \cap B) = \frac{|A'| \times |B'|}{36} = \mathbb{P}(A) \times \mathbb{P}(B)$. So A and B are independent of each other. \square

2.10 (SERENDIPITOUS INDEPENDENCE) The point of this exercise is to understand independence more deeply. Solve the following problems:

- (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that \emptyset and Ω (which are events) are independent of any other event. What is the intuitive meaning of this?
- (b) Continuing the previous part, show that any event $A \in \mathcal{F}$ with $\mathbb{P}(A) \in \{0, 1\}$ is independent of any other event.
- (c) What can we conclude about an event $A \in \mathcal{F}$ that is independent of its complement, $A^c = \Omega \setminus A$? Does your conclusion make intuitive sense?
- (d) What can we conclude about an event $A \in \mathcal{F}$ that is independent of itself? Does your conclusion make intuitive sense?
- (e) Consider the probability space generated by two independent flips of unbiased coins with the smallest possible σ -algebra. Enumerate all pairs of events A, B such that A and B are independent of each other.

- (f) Consider the probability space generated by the independent rolls of two unbiased three-sided dice. Call the possible outcomes of the individual dice rolls 1, 2 and 3. Let X_i be the random variable that corresponds to the outcome of the i th dice roll ($i \in \{1, 2\}$). Show that the events $\{X_1 \leq 2\}$ and $\{X_1 = X_2\}$ are independent of each other.
- (g) The probability space of the previous example is an example when the probability measure is uniform on a finite outcome space (which happens to have a product structure). Now consider any n -element, finite outcome space with the uniform measure. Show that A and B are independent of each other if and only if the cardinalities $|A|$, $|B|$, $|A \cap B|$ satisfy $n|A \cap B| = |A| \cdot |B|$.
- (h) Continuing with the previous problem, show that if n is prime, then no non-trivial events are independent (an event A is **trivial** if $\mathbb{P}(A) \in \{0, 1\}$).
- (i) Construct an example showing that pairwise independence does not imply mutual independence.
- (j) Is it true or not that A, B, C are mutually independent if and only if $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$? Prove your claim.

Proof. (a) Empty sets and complete sets are independent of any event:

$$P(A \cap \Omega) = P(A) = 1 \times P(A) = P(\Omega) \times P(A)$$

$$P(A \cap \emptyset) = P(\emptyset) = 0 = P(\emptyset) \times P(A)$$

- (b) For any $B \in \Omega$ and $P(A) \in \{0, 1\}$:
 when $P(A) = 1, P(A^c \cap B) \leq P(A^c) = 1 - P(A) = 0$, we have $P(A \cap B) = P(A \cap B) + P(A^c \cap B) = P(B) = P(A)P(B)$; when $P(A) = 0$, we have $P(A \cap B) \leq P(A) = 0 = P(A)P(B)$
- (c) $P(A^c \cap A) = P(A)P(A^c)$, we have $0 = P(A)(1 - P(A)) \Rightarrow P(A) \in \{0, 1\}$
- (d) $P(A \cap A) = P(A)P(A)$, we have $P(A) = \{0, 1\}$
- (e) $\Omega = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$. $A, B \subseteq \Omega$ denote the events.

First of all, if either A or B is trivial, then A and B are independent of each other.

Then, we only need to enumerate $A, B \notin \Omega, \emptyset$ satisfied that $P(A \cap B) = P(A)P(B)$. Since $P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{|A \cap B|}{4}$ and $P(A)P(B) = \frac{|A||B|}{16}$, we can conclude that $|A| = 2$, $|B| = 2$ and $|A \cap B| = 1$ is the only situation satisfying the condition.

Thus, besides trivial A or B , all A, B satisfying $|A| = 2$, $|B| = 2$ and $|A \cap B| = 1$ are the solution.

- (f) $P(X_1 \leq 2) = 2/3$
 $P(X_1 = X_2) = 3/9 = 1/3$
 $P(X_1 \leq 2, X_1 = X_2) = P(X_1 = X_2 = 1) + P(X_1 = X_2 = 2) = 1/9 + 1/9 = 2/9$
 So, $P(X_1 \leq 2, X_1 = X_2) = P(X_1 = X_2)P(X_1 \leq 2)$
- (g) Necessity : $\frac{|A \cap B|}{n} = P(A \cap B) = P(A)P(B) = \frac{|A|}{n} \frac{|B|}{n}$
 $\Rightarrow |A \cap B| \times n = |A||B|$
 Sufficiency : $|A \cap B| \times n = |A||B| \Rightarrow \frac{|A|}{n} \frac{|B|}{n} = \frac{|A \cap B|}{n}$
 $\Rightarrow P(A \cap B) = P(A)P(B)$
- (h) If A, B are two non-trivial events independent to each other, $|A \cap B| \times n = |A||B| \Rightarrow n \mid (|A||B|) \Rightarrow n \mid (|A|)$ or $n \mid (|B|) \Rightarrow |A| = n$ or $|B| = n$, contradictory to non-trivial assumption.
- (i) Let $\Omega = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$. A, B, C are pairwise independent but $P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$.

- (j) Consider rolling a dice and set $A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$, $C = \{1, 4, 5, 6\}$. Then $P(A \cap B \cap C) = \frac{1}{6} = (1/2) * (1/2) * (2/3) = P(A)P(B)P(C)$, however $P(A \cap B) = 1/3 \neq \frac{1}{2} * \frac{1}{2} = P(A)P(B)$. Thus $P(A \cap B \cap C) = P(A)P(B)P(C)$ does not mean mutual independence. \square

2.11

- (a) $X: \Omega \rightarrow x$

Because X, Y are independent equivalent to $\sigma(X), \sigma(Y)$ are independent;

For any $A \in \sigma(Y)$,

$$P(\phi \cap A) = P(\phi) = 0 = P(\phi)P(A)$$

$$P(\Omega \cap A) = P(A) = P(\Omega)P(A)$$

- (b) We know that $P(X = x) = 1$

$$P(X = x|Y) = \frac{P((X = x) \cap Y)}{P(Y)} = 1 = P(X = x)$$

$$P(X \neq x|Y) = 1 - P(X = x|Y) = 0 = P(X \neq x)$$

- (c) Notice the relation: $P(A) = P(X(A) = 1)$

$$P(B) = P(X(B) = 1)$$

$$P(A \cap B) = P(X(A \cap B) = 1)$$

The first two formulas follow the definition. Let's prove the third equation:

$$P(X(A \cap B) = 1) = P(X(A) + X(B) - X(A \cup B) = 1)$$

Let's discuss $X(A), X(B), X(A \cup B)$:

$X(A)$	$X(B)$	$X(A \cup B)$	$X(A) + X(B) - X(A \cup B)$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

We can see that, $P(X(A \cap B) = 1) = P(X(A) + X(B) - X(A \cup B) = 1)$, this is only one case of the first row of the table.

$$\text{that is } P(X(A \cap B) = 1) = P(X(A) = 1, X(B) = 1) = P(A \cap B)$$

$$\text{that is } P(X(A \cap B) = 1) = P(A \cap B)$$

$$\text{So, } P(A \cap B) = P(A)P(B) \text{ is equivalent to } P(X(A \cap B) = 1) = P(X(A) = 1)P(X(B) = 1)$$

- (d) A_i pairwise i $\Leftrightarrow I\{A_i\}$ pairwise i

$$\text{mutual i} \Leftrightarrow P(\bigcap_i A_i) = \prod_i P(A_i)$$

$$\Leftrightarrow P(\bigcap_{i \in K^I} A_i \cap \bigcap_{i \in K^I} A_i^c)$$

$$= \prod_{i \in K^I} P(A_i) \prod_{i \in K^I} P(A_i^c)$$

$$\Leftrightarrow \{\phi, \Omega, A_i, A_i^c\} \text{ mutual independent}$$

$$\Leftrightarrow \sigma(I\{w \in A_i\}) \text{ mutual i}$$

$$\Leftrightarrow I\{w \in A_i\} \text{ mutual i}$$

2.12 X integrable $|X|$ integrable

- (a) For any $A \in \mathcal{B}(R) \Rightarrow A$ is open,

so, $f^{-1}(A)$ is open, so $f^{-1}(A) \in \mathcal{B}(R)$

(b) X is known to be a random variable, $f(x) = |x|$ continuous.

r.v. X is $\mathbf{F}/\mathbf{B}(R)$ -measurable

$\Rightarrow |X|$ is $\mathbf{B}(R)/\mathbf{B}(R)$ -measurable

$\Rightarrow |X|$ is $\mathbf{F}/\mathbf{B}(R)$ -measurable

$\Rightarrow |X|$ is r.v.

From (a)(b), X integrable $\Leftrightarrow |X|$ integrable.

2.14

(a) Assume $\forall i, X_i$ is simple function.

$$\begin{aligned}
 \mathbb{E}[X] &= \mathbb{E} \left[\sum_{i=1}^n X_i \right] \\
 &= \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \mathbb{I}_{A_{i,j}} \{\omega\} \right] \\
 &= \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \mathbb{I}_{A_{i,j}} \{\omega\} d\mathbb{P}(\omega) \\
 &= \sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \int_{\Omega} \mathbb{I}_{A_{i,j}} \{\omega\} d\mathbb{P}(\omega) \\
 &= \sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \mathbb{P}(A_{i,j}) \\
 &= \sum_{i=1}^n \mathbb{E}[X_i]
 \end{aligned}$$

(b) Assume $\forall i, X_i$ is non-negative random variable.

$$\begin{aligned}
 \mathbb{E}[X] &= \mathbb{E} \left[\sum_{i=1}^n X_i \right] \\
 &= \sup \left\{ \int_{\Omega} h d\mathbb{P} : h \text{ is simple and } 0 \leq h \leq X = \sum_{i=1}^n X_i \right\} \\
 &= \sum_{i=1}^n \sup \left\{ \int_{\Omega} h_i d\mathbb{P} : h_i \text{ is simple and } 0 \leq h_i \leq X_i \right\} \\
 &= \sum_{i=1}^n \mathbb{E}[X_i]
 \end{aligned}$$

(c) Assume $\forall i, X_i$ is arbitrary random variable.

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E} \left[\sum_{i=1}^n X_i \right] \\
&= \mathbb{E} \left[\sum_{i=1}^n (X_i^+ - X_i^-) \right] \\
&= \mathbb{E} \left[\sum_{i=1}^n X_i^+ \right] - \mathbb{E} \left[\sum_{i=1}^n X_i^- \right] \\
&= \sum_{i=1}^n \mathbb{E}[X_i^+] - \sum_{i=1}^n \mathbb{E}[X_i^-] \\
&= \sum_{i=1}^n (\mathbb{E}[X_i^+] - \mathbb{E}[X_i^-]) \\
&= \sum_{i=1}^n \mathbb{E}[X_i]
\end{aligned}$$

2.15

(a) Assume X is simple function.

$$\begin{aligned}
\mathbb{E}[cX] &= \mathbb{E} \left[c \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\} \right] \\
&= \int_{\Omega} c \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\} d\mathbb{P}(\omega) \\
&= c \int_{\Omega} \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\} d\mathbb{P}(\omega) \\
&= c\mathbb{E}[X]
\end{aligned}$$

(b) Assume X is non-negative random variable.

$$\begin{aligned}
\mathbb{E}[cX] &= \sup \left\{ \int_{\Omega} h d\mathbb{P} : h \text{ is simple and } 0 \leq h \leq cX \right\} \\
&= c \sup \left\{ \int_{\Omega} h' d\mathbb{P} : h' \text{ is simple and } 0 \leq h' \leq X \right\} \\
&= c\mathbb{E}[X]
\end{aligned}$$

(c) Assume X is arbitrary random variable.

(i) $c \geq 0$

$$\begin{aligned}
\mathbb{E}[cX] &= \mathbb{E}[(cX)^+] - \mathbb{E}[(cX)^-] \\
&= \mathbb{E}[c(X)^+] - \mathbb{E}[c(X)^-] \\
&= c\mathbb{E}[(X)^+] - c\mathbb{E}[(X)^-] \\
&= c\mathbb{E}[X]
\end{aligned}$$

(ii) $c < 0$

By definition, we have

$$\begin{aligned}
(cX)^+ &= cX\mathbb{I}\{cX > 0\} \\
&= cX\mathbb{I}\{x < 0\} \text{ (since } c \downarrow 0) \\
&= (-c)(-X)\mathbb{I}\{X < 0\} \\
&= (-c)(X)^-
\end{aligned}$$

Along the similar line, we have

$$\begin{aligned}
(cX)^- &= -cX\mathbb{I}\{cX < 0\} \\
&= -cX\mathbb{I}\{X > 0\} \\
&= -c(X)^+
\end{aligned}$$

Now we can see that

$$\begin{aligned}
\mathbb{E}[cX] &= \mathbb{E}[(cX)^+] - \mathbb{E}[(cX)^-] \\
&= \mathbb{E}[(-c)(X)^-] - \mathbb{E}[-c(X)^+] \\
&= -c\mathbb{E}[(X)^-] + c\mathbb{E}[(X)^+] \\
&= c\mathbb{E}[X]
\end{aligned}$$

2.16

(a) Assume $X = \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\}$, $Y = \sum_{j=1}^m \beta_j \mathbb{I}_{B_j} \{\omega\}$ are simple functions.

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{I}_{A_i} \{\omega\} \mathbb{I}_{B_j} \{\omega\}\right] \\
&= \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{I}_{A_i} \{\omega\} \mathbb{I}_{B_j} \{\omega\} d\mathbb{P}(\omega) \\
&= \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{P}(A_i \cap B_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{P}(A_i) \mathbb{P}(B_j) \text{ (by the definition of independence)} \\
&= \left(\sum_{i=1}^n \alpha_i \mathbb{P}(A_i)\right) \left(\sum_{j=1}^m \beta_j \mathbb{P}(B_j)\right) \\
&= \mathbb{E}[X] \mathbb{E}[Y]
\end{aligned}$$

(b) Assume X, Y are non-negative random variables.

$$\begin{aligned}
\mathbb{E}[XY] &= \sup \{\mathbb{E}[h] : h \text{ is simple and } 0 \leq h \leq XY\} \\
&= \sup \{\mathbb{E}[h_1 h_2] : h_1, h_2 \text{ are simple and } 0 \leq h_1 \leq X, 0 \leq h_2 \leq Y\} \\
&= \sup \{\mathbb{E}[h_1] \mathbb{E}[h_2] : h_1, h_2 \text{ are simple and } 0 \leq h_1 \leq X, 0 \leq h_2 \leq Y\} \\
&= \sup \{\mathbb{E}[h_1] : h_1 \text{ is simple and } 0 \leq h_1 \leq X\} \cdot \sup \{\mathbb{E}[h_2] : h_2 \text{ is simple and } 0 \leq h_2 \leq Y\} \\
&= \mathbb{E}[X] \mathbb{E}[Y]
\end{aligned}$$

(c) Assume X, Y are arbitrary random variables.

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[(X^+ - X^-)(Y^+ - Y^-)] \\
&= \mathbb{E}[X^+Y^+ - X^+Y^- - X^-Y^+ + X^-Y^-] \\
&= \mathbb{E}[X^+]\mathbb{E}[Y^+] - \mathbb{E}[X^+]\mathbb{E}[Y^-] - \mathbb{E}[X^-]\mathbb{E}[Y^+] + \mathbb{E}[X^-]\mathbb{E}[Y^-] \\
&= (\mathbb{E}[X^+] - \mathbb{E}[X^-])(\mathbb{E}[Y^+] - \mathbb{E}[Y^-]) \\
&= \mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}$$

2.17 Before proving Ex.2.17, we need to make minor changes to the definition of conditional expectation and give a small lemma.

Definition 1. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. $\mathcal{G} \subset \mathcal{F}$ is a sub- σ -algebra of \mathcal{F} . $X : \Omega \rightarrow \mathbb{R}$ is a random variable. The conditional expectation of X given \mathcal{G} is denoted by any random variable Y which satisfies the following 2 properties:

- Y is \mathcal{G} -measurable
- $\forall A \in \mathcal{G}$,

$$\int_A Y d\mathbb{P} = \int_A X d\mathbb{P}$$

Formally, we denoted Y by notation $\mathbb{E}[X|\mathcal{G}]$.

Lemma 1. If X is \mathcal{G} -measurable, then $\mathbb{E}[X|\mathcal{G}] = X$ holds a.s.

Proof. Since X is \mathcal{G} -measurable, property1 holds. And property2 holds trivially. \square

We can now handily prove Ex.2.17. Since $\mathbb{E}[X|\mathcal{G}_1]$ is \mathcal{G}_1 -measurable and $\mathcal{G}_1 \subset \mathcal{G}_2$, we can see that $\mathbb{E}[X|\mathcal{G}_1]$ is \mathcal{G}_2 -measurable. By Lemma 1, $\mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]|\mathcal{G}_2] = \mathbb{E}[X|\mathcal{G}_1]$ holds almost surely.

2.18 Suppose $X = Y$ with $\mathbb{V}[X] \neq 0$. Then, we have $\mathbb{E}[XY] = \mathbb{E}[X^2] = \mathbb{V}[X] + \mathbb{E}[X]^2 \neq \mathbb{E}[X]^2 = \mathbb{E}[X]\mathbb{E}[Y]$.

2.19 As the hint suggests, $X(\omega) = \int_{[0, \infty)} \mathbb{I}\{[0, X(\omega)]\}(x)dx$. Hence, we have

$$\begin{aligned}
\mathbb{E}[X(\omega)] &= \mathbb{E}\left[\int_{[0, \infty)} \mathbb{I}\{[0, X(\omega)]\}(x)dx\right] \\
&= \int_{[0, \infty)} \mathbb{E}[\mathbb{I}\{[0, X(\omega)]\}(x)]dx \\
&= \int_{[0, \infty)} P(X(\omega) > x)dx
\end{aligned} \tag{1}$$

where the second equality is given by Fubini–Tonell theorem.

2.20 We prove the following properties all by contradiction (for the sake of rigor).

- (1) Let $G = \{\omega : \mathbb{E}[X | \mathcal{G}](\omega) < 0\}$. Then $G \in \mathcal{G}$ since $\mathbb{E}[X | \mathcal{G}]$ is \mathcal{G} -measurable by definition. Now suppose $\mathbb{P}(G) > 0$, then

$$\begin{aligned}
\int_G X d\mathbb{P} &= \int_G \mathbb{E}(X | \mathcal{G}) d\mathbb{P} \\
&< 0
\end{aligned} \tag{2}$$

where the equality holds by the definition of conditional expectation. Now we can find it contradictory as $X \geq 0$. Therefore $\mathbb{P}(G) = 0$, and $\mathbb{E}[X | \mathcal{G}] \geq 0$ a.s.

- (2) Let $G = \{\omega : \mathbb{E}[1 \mid \mathcal{G}](\omega) \neq 1\}$. Then $G \in \mathcal{G}$ since $\mathbb{E}[1 \mid \mathcal{G}]$ is \mathcal{G} -measurable by definition. Now suppose $\mathbb{P}(G) > 0$, then

$$\int_G 1 d\mathbb{P} = \int_G \mathbb{E}(1 \mid \mathcal{G}) d\mathbb{P} \neq 1 \quad (3)$$

where the equality holds by the definition of conditional expectation. Now we can find it contradictory as $\int_G 1 d\mathbb{P} = 1$. Therefore $\mathbb{P}(G) = 0$, and $\mathbb{E}[1 \mid \mathcal{G}] = 1$ a.s.

- (3) Let $G = \{\omega : \mathbb{E}[X + Y \mid \mathcal{G}](\omega) \neq \mathbb{E}[X \mid \mathcal{G}](\omega) + \mathbb{E}[Y \mid \mathcal{G}](\omega)\}$. Then $G \in \mathcal{G}$ since $\mathbb{E}[X + Y \mid \mathcal{G}]$, $\mathbb{E}[X \mid \mathcal{G}]$, and $\mathbb{E}[Y \mid \mathcal{G}]$ are all \mathcal{G} -measurable by definition. Now suppose $\mathbb{P}(G) > 0$, then

$$\begin{aligned} \int_G (X + Y) d\mathbb{P} &= \int_G \mathbb{E}(X + Y \mid \mathcal{G}) d\mathbb{P} \\ &\neq \int_G [\mathbb{E}(X \mid \mathcal{G}) + \mathbb{E}(Y \mid \mathcal{G})] d\mathbb{P} \\ &= \int_G \mathbb{E}(X \mid \mathcal{G}) d\mathbb{P} + \int_G \mathbb{E}(Y \mid \mathcal{G}) d\mathbb{P} \\ &= \int_G X d\mathbb{P} + \int_G Y d\mathbb{P} \end{aligned} \quad (4)$$

where the first equality and the last one hold by the definition of conditional expectation. It contradicts the linearity of expectation in that $\int_G (X + Y) d\mathbb{P} \neq \int_G X d\mathbb{P} + \int_G Y d\mathbb{P}$. Therefore $\mathbb{P}(G) = 0$, and $\mathbb{E}(X + Y \mid \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G}) + \mathbb{E}(Y \mid \mathcal{G})$ a.s.

- (4) Let $G = \{\omega : \mathbb{E}[XY \mid \mathcal{G}](\omega) \neq Y(\omega)\mathbb{E}[X \mid \mathcal{G}](\omega)\}$. Then $G \in \mathcal{G}$ since $\mathbb{E}[XY \mid \mathcal{G}]$, Y , and $\mathbb{E}[X \mid \mathcal{G}]$ are all \mathcal{G} -measurable by definition. Now suppose $\mathbb{P}(G) > 0$, then

$$\begin{aligned} \int_G XY d\mathbb{P} &= \int_G \mathbb{E}(XY \mid \mathcal{G}) d\mathbb{P} \\ &\neq \int_G Y \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} \end{aligned} \quad (5)$$

Now our target is to show it is contradictory. This is a bit tricky, so we start from the simplest case and then generalize it step by step.

- a. Suppose $Y = \mathbb{I}_A$ for some $A \in \mathcal{G}$. Then

$$\int_G XY d\mathbb{P} = \int_{G \cap A} X d\mathbb{P} \quad (6)$$

and

$$\begin{aligned} \int_G Y \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} &= \int_{G \cap A} \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} \\ &= \int_{G \cap A} X d\mathbb{P} \end{aligned} \quad (7)$$

Hence it holds that $\int_G XY d\mathbb{P} = \int_G Y \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P}$.

- b. Suppose Y is non-negative and let $\{Y_n\}$ be sequence of non-negative simple functions converging to Y from below. Then by linearity, it holds that

$$\int_G X^+ Y_n d\mathbb{P} = \int_G Y_n \mathbb{E}[X^+ \mid \mathcal{G}] d\mathbb{P} \quad (8)$$

and

$$\int_G X^- Y_n d\mathbb{P} = \int_G Y_n \mathbb{E}[X^- | \mathcal{G}] d\mathbb{P} \quad (9)$$

Applying the monotone convergence we end up with

$$\int_G X^+ Y d\mathbb{P} = \int_G Y \mathbb{E}[X^+ | \mathcal{G}] d\mathbb{P} \quad (10)$$

and

$$\int_G X^- Y d\mathbb{P} = \int_G Y \mathbb{E}[X^- | \mathcal{G}] d\mathbb{P} \quad (11)$$

Hence,

$$\begin{aligned} \int_G XY d\mathbb{P} &= \int_G X^+ Y d\mathbb{P} - \int_G X^- Y d\mathbb{P} \\ &= \int_G Y (\mathbb{E}[X^+ | \mathcal{G}] - \mathbb{E}[X^- | \mathcal{G}]) d\mathbb{P} \\ &= \int_G Y \mathbb{E}[X^+ - X^- | \mathcal{G}] d\mathbb{P} \\ &= \int_G Y \mathbb{E}[X | \mathcal{G}] d\mathbb{P} \end{aligned} \quad (12)$$

- c. Finally, for arbitrary Y , we can separate $Y = Y^+ - Y^-$ and the contradiction still holds by linearity of expectation.

Therefore, in any case Eq.5 is contradictory. So $\mathbb{P}(G) = 0$, and $\mathbb{E}[XY | \mathcal{G}] = Y \mathbb{E}[X | \mathcal{G}]$ a.s.

- (5) Let $G = \{\omega : \mathbb{E}[X | \mathcal{G}_1](\omega) \neq \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1](\omega)\}$. Then $G \in \mathcal{G}_1$ since both $\mathbb{E}[X | \mathcal{G}_1]$ and $\mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1]$ are \mathcal{G}_1 -measurable by definition. Now suppose $\mathbb{P}(G) > 0$, then

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}(X | \mathcal{G}_1) d\mathbb{P} \\ &\neq \int_G \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1] d\mathbb{P} \\ &= \int_G \mathbb{E}(X | \mathcal{G}_2) d\mathbb{P} \\ &= \int_G X d\mathbb{P} \end{aligned} \quad (13)$$

The last equality stands since $G \in \mathcal{G}_1$ and $\mathcal{G}_1 \subset \mathcal{G}_2$, which suggests $G \in \mathcal{G}_2$. Now we can find it contradictory. Therefore $\mathbb{P}(G) = 0$, and $\mathbb{E}[X | \mathcal{G}_1] = \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1]$ a.s.

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}(X | \mathcal{G}_1) d\mathbb{P} \\ &\neq \int_G \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1] d\mathbb{P} \\ &= \int_G \mathbb{E}(X | \mathcal{G}_2) d\mathbb{P} \\ &= \int_G X d\mathbb{P} \end{aligned} \quad (14)$$

- (6) Let $G = \{\omega : \mathbb{E}[X \mid \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)](\omega) \neq \mathbb{E}[X \mid \mathcal{G}_1](\omega)\}$. Notice that $\mathbb{E}[X \mid \mathcal{G}_1]$ is not only \mathcal{G}_1 -measurable but also $\sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$ -measurable. Thus we have $G \in \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$. Now suppose $\mathbb{P}(G) > 0$, then

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}[X \mid \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)] d\mathbb{P} \\ &\neq \int_G \mathbb{E}[X \mid \mathcal{G}_1] d\mathbb{P} \end{aligned} \quad (15)$$

To show it is contradictory, we want to prove that $\forall G \in \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$,

$$\int_G X d\mathbb{P} = \int_G \mathbb{E}[X \mid \mathcal{G}_1] d\mathbb{P} \quad (16)$$

The following techniques are closely related to ‘Dynkin system’, which is beyond my knowledge. The main idea is that if we assume X is non-negative, which can be generalized by linearity, it is enough to establish Eq.16 for some π -system that generates $\sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$.

One possibility is $\mathcal{H} = \{G_1 \cap G_2 : G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2\}$. Then, $\forall G_1 \cap G_2 \in \mathcal{H}$,

$$\begin{aligned} \int_{G_1 \cap G_2} \mathbb{E}[X \mid \mathcal{G}_1] d\mathbb{P} &= \int_{\Omega} \mathbb{E}[X \mid \mathcal{G}_1] \mathbb{I}_{G_1} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{\Omega} \mathbb{E}[X \mid \mathcal{G}_1] \mathbb{I}_{G_1} d\mathbb{P} \int_{\Omega} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{\Omega} X \mathbb{I}_{G_1} d\mathbb{P} \int_{\Omega} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{\Omega} X \mathbb{I}_{G_1} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{G_1 \cap G_2} X d\mathbb{P} \end{aligned} \quad (17)$$

where the second and fourth equality holds due to independence between $\sigma(X)$ and \mathcal{G}_2 given \mathcal{G}_1 .

Hence, we find it contradictory. So $\mathbb{P}(G) = 0$ and $\mathbb{E}[X \mid \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)] = \mathbb{E}[X \mid \mathcal{G}_1]$ a.s.

- (7) Let $G = \{\omega : \mathbb{E}[X \mid \mathcal{G}](\omega) \neq \mathbb{E}[X]\}$. Then $G \in \mathcal{G}$ since $\mathbb{E}[X \mid \mathcal{G}]$ is \mathcal{G} -measurable by definition. And because \mathcal{G} is trivial, $G = \emptyset$ or $G = \Omega$.

- a. If $G = \emptyset$, $P(G) = 0$ for sure.
- b. If $G = \Omega$, which suggests $\mathbb{E}[X \mid \mathcal{G}] \neq \mathbb{E}[X]$ always holds, we have

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} \\ &\neq \int_G \mathbb{E}[X] d\mathbb{P} \\ &= \int_{\Omega} \mathbb{E}[X] d\mathbb{P} \\ &= \mathbb{E}[X] \end{aligned} \quad (18)$$

which is obviously contradictory since $\int_G X d\mathbb{P} = \int_{\Omega} X d\mathbb{P} = \mathbb{E}[X]$.

Therefore, $P(G) = 0$ and hence $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$ a.s.

Chapter 3 Stochastic Processes and Markov Chains

3.1

- (a) On $([0, 1], \mathcal{B}, \lambda)$, for any $x \in [0, 1]$

Let $F_1(x), F_2(x), F_3(x), \dots$ be the binary expansion of x .

$$F_t(x) = \begin{cases} 1, & A \\ 0, & \bar{A} \end{cases} \quad (\bar{A} \text{ is the opposite case of } A)$$

$F_t(x)$ is Bernoulli random variable.

$$(b) \begin{cases} F_1 = 0 : 0 \leq x < 0.5 \\ F_1 = 1 : 0.5 \leq x < 1 \\ \dots \\ \begin{cases} F_2 = 0 : 0 \leq x' < 0.5 \\ F_2 = 1 : 0.5 \leq x' < 1 \end{cases} & x' = 2x - 1 \\ \dots \\ \begin{cases} F_t = 0 : 0 \leq x^t < 0.5 \Rightarrow \mathbb{P}(F_t = 0) = \frac{1}{2} \\ F_t = 1 : 0.5 \leq x^t < 1 \Rightarrow \mathbb{P}(F_t = 1) = \frac{1}{2} \end{cases} \end{cases}$$

- (c) It is obviously that $(F_t)_{t=1}^\infty$ are independent. It satisfies independent equation: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

- (d) $(X_{m,t})_{t=1}^\infty$ is a subsequence of $(F_t)_{t=1}^\infty$ and $(X_{m,t})_{t=1}^\infty$ are mutually exclusive.

- (e) Such as (d).

- (f) Such as (d).

3.2

- (a) $S_t = \sum_{s=1}^t X_s 2^{s-1}$

X_t is a \mathcal{F} -adapted martingale.

$$(1) \mathbb{E}[X_t | \mathcal{F}_{t-1}] = X_{t-1}.$$

$$(2) X_t \text{ is integrable} \Rightarrow S_t \text{ is integrable.}$$

$$\begin{aligned} \mathbb{E}[S_t | \mathcal{F}_{t-1}] &= \mathbb{E}[S_{t-1} + X_t 2^{t-1} | \mathcal{F}_{t-1}] \\ &= S_{t-1} + \mathbb{E}[X_t 2^{t-1} | \mathcal{F}_{t-1}] \\ &= S_{t-1} + 2^t \times (1) \times \frac{1}{2} + 2^t \times (-1) \times \frac{1}{2} \\ &= S_{t-1} \end{aligned}$$

$$\Rightarrow (S_t)_{t=1}^\infty$$

- (b) $t=1$, if $S_t \neq 1 \Rightarrow X_1 = -1, S_t = -1$
 $t=2$, if $S_t \neq 1 \Rightarrow X_1 = -1, S_t = -3$
 $t=3$, if $S_t \neq 1 \Rightarrow X_1 = -1, S_t = -7$

...

If avoid $S_t = 1$, the X_s sequence must be -1 .

$$\tau = \min\{t : S_t = 1\} = \min\{t : X_T = 1\}$$

$$\Rightarrow \mathbb{P}(\tau < n) = 1 - \mathbb{P}(\tau \geq n) = 1 - \frac{1}{2^n}$$

$$\Rightarrow \mathbb{P}(\tau < \infty) = 1 - \lim_{n \rightarrow \infty} \mathbb{P}(\tau \geq n) = 1 - \frac{1}{2^n} = 1 - \lim_{n \rightarrow \infty} \frac{1}{2^n}$$

- (c) If $t = \tau$, then $S_t = 1$, so $S_\tau \equiv 1$

$$\Rightarrow \mathbb{E}[S_\tau] = 1$$

- (d) Doob's(a) can be proved by 3.2(b)

$$\tau = 1 \Rightarrow X_1 = 1 \Rightarrow \mathbb{P}(\tau = 1) = \frac{1}{2}$$

$$\tau = 2 \Rightarrow X_1 = -1, X_2 = 1 \Rightarrow \mathbb{P}(\tau = 1) = \frac{1}{4}$$

$$\tau = 3 \Rightarrow X_1 = -1, X_2 = -1, X_3 = 1 \Rightarrow \mathbb{P}(\tau = 1) = \frac{1}{8}$$

...

$$\mathbb{P}(\tau < \infty) = \mathbb{P}(\tau = 1) + \mathbb{P}(\tau = 2) + \mathbb{P}(\tau = 3) + \dots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$

because of $n \neq \infty$, $\mathbb{P}(\tau = n) = \frac{1}{n^2} \neq 0$.

Doob's(b)(c) can also be proved by 3.2(b)

$$t=1 \text{ , if } S_t \neq 1 \Rightarrow X_1 = -1, S_t = -1$$

$$t=2 \text{ , if } S_t \neq 1 \Rightarrow X_1 = -1, S_t = -3$$

$$t=3 \text{ , if } S_t \neq 1 \Rightarrow X_1 = -1, S_t = -7$$

...

It can be concluded that $-S_t-$ and $-S_{t-1}-$ can not be bounded, so $\mathbb{E}[|X_{t+1}| \mathcal{F}]$ and $|4X_{t \wedge \tau}|$ can not be bounded neither.

3.4 If $X_t \geq 0$ is dropped, $\mathbb{E}[X_\tau | \{\tau \leq n\}] \geq \mathbb{E}[\varepsilon | \{\tau \leq n\}]$ not always true.

Chapter 4 Stochastic Bandits

4.1 By definition

$$\begin{aligned}
 R_n(\pi, v) &= n\mu^*(v) - \mathbb{E}\left[\sum_{t=1}^n X_t\right] \\
 &= \sum_{t=1}^n \mu^*(v) - \sum_{t=1}^n \mathbb{E}[X_t] \\
 &= \sum_{t=1}^n [\mu^* - \mu_{A_t}]
 \end{aligned}$$

(a) $\mu^* = \max \mu_a \geq \mu_{A_t} \Rightarrow R_n(\pi, v) = \sum_{t=1}^n [\mu^* - \mu_{A_t}] \geq 0.$

(b) If π choose $A_t \in \arg \max_a \mu_a$ for all $t \in [n] \Rightarrow \sum_{t=1}^n [\mu^* - \mu_{A_t}] = 0.$

(c) If $R_n(\pi, v) = 0$ for some policy π , then $A_t \in \arg \max_a \mu_a \Rightarrow \mathbb{P}(\mu_{A_t} = \mu^*) = 1.$

4.3 Denote $h_t = a_1, x_1, \dots, a_t, x_t.$

(a) According to the definition of conditional probability and marginal distribution, we have

$$\begin{aligned}
 p_{v\pi}(a_n \mid h_{n-1}) &= \frac{p_{v\pi}(h_{n-1}, a_n)}{p_{v\pi}(h_{n-1})} \\
 &= \frac{\int_{\mathbb{R}} p_{v\pi}(h_n) dx_n}{p_{v\pi}(h_{n-1})} \\
 &= \frac{\int_{\mathbb{R}} \prod_{t=1}^n \pi(a_t \mid h_{t-1}) p_{a_t}(x_t) dx_n}{p_{v\pi}(h_{n-1})} \\
 &= \frac{\prod_{t=1}^{n-1} \pi(a_t \mid h_{t-1}) p_{a_t}(x_t)}{p_{v\pi}(h_{n-1})} \int_{\mathbb{R}} \pi(a_n \mid h_{n-1}) p_{a_n}(x_n) dx_n \\
 &= \pi(a_n \mid h_{n-1}) \int_{\mathbb{R}} p_{a_n}(x_n) dx_n \\
 &= \pi(a_n \mid h_{n-1})
 \end{aligned}$$

(b) According to the definition of conditional probability and marginal distribution, we have

$$\begin{aligned}
p_{v\pi}(x_n \mid h_{n-1}, a_n) &= \frac{p_{v\pi}(h_n)}{p_{v\pi}(h_{n-1}, a_n)} \\
&= \frac{p_{v\pi}(h_n)}{\int_{\mathbb{R}} p_{v\pi}(h_n) dx_n} \\
&= \frac{p_{v\pi}(h_n)}{\int_{\mathbb{R}} [\prod_{t=1}^n \pi(a_t \mid h_{t-1}) p_{a_t}(x_t)] dx_n} \\
&= \frac{p_{v\pi}(h_n)}{\prod_{t=1}^{n-1} \pi(a_t \mid h_{t-1}) p_{a_t}(x_t)} \frac{1}{\int_{\mathbb{R}} \pi(a_n \mid h_{n-1}) p_{a_n}(x_n) dx_n} \\
&= \pi(a_n \mid h_{n-1}) p_{a_n}(x_n) \frac{1}{\pi(a_n \mid h_{n-1})} \\
&= p_{a_n}(x_n)
\end{aligned}$$

4.4 Denote $h_t = a_1, x_1, \dots, a_t, x_t$. The policy that mixes the policies can be defined as

$$\pi_t^\circ(a_t \mid h_{t-1}) = \frac{\sum_{\pi \in \Pi} p(\pi) \prod_{s=1}^t \pi_s(a_s \mid h_{s-1})}{\sum_{\pi \in \Pi} p(\pi) \prod_{s=1}^{t-1} \pi_s(a_s \mid h_{s-1})}$$

By the definition of the canonical probability space and the product of probability kernels,

$$\begin{aligned}
\mathbb{P}_{v\pi^\circ}(B) &= \sum_{a_1=1}^k \int_{\mathbb{R}} \cdots \sum_{a_n=1}^k \int_{\mathbb{R}} \mathbb{I}_B(h_n) v_{a_n}(dx_n) \pi_n^\circ(a_n \mid h_{n-1}) \cdots v_{a_1}(dx_1) \pi_1^\circ(a_1) \\
&= \sum_{\pi \in \Pi} p(\pi) \sum_{a_1=1}^k \int_{\mathbb{R}} \cdots \sum_{a_n=1}^k \int_{\mathbb{R}} \mathbb{I}_B(h_n) v_{a_n}(dx_n) \pi_n(a_n \mid h_{n-1}) \cdots v_{a_1}(dx_1) \pi_1(a_1) \\
&= \sum_{\pi \in \Pi} p(\pi) \mathbb{P}_{v\pi}(B),
\end{aligned}$$

where the second equality follows by substituting the definition of π_n° and induction.

Chapter 5 Concentration of Measure

5.1

$$V(\hat{\mu}) = E((\hat{\mu} - \mu)^2) = E\left(\left(\frac{1}{n} \sum_{t=1}^n X_t - \mu\right)^2\right) = E\left(\frac{1}{n^2} \sum_{t=1}^n (X_t - \mu)^2\right) = \frac{1}{n^2} \sum_{t=1}^n E(X_t - \mu)^2 = \frac{1}{n^2} \sum_{t=1}^n \sigma^2 = \frac{\sigma^2}{n} \quad (19)$$

5.4

(a)

$$P(|X| \geq \varepsilon) = P(X \geq \varepsilon)I\{X \geq 0\} + P(X \leq -\varepsilon)I\{X < 0\} = \int_{\varepsilon}^{\infty} \frac{x}{2} \exp\left\{-\frac{x^2}{2}\right\} dx + \int_{-\infty}^{-\varepsilon} \frac{-x}{2} \exp\left\{-\frac{x^2}{2}\right\} dx \quad (20)$$

Calculate the above formula and get the result ,

$$P(|X| \geq \varepsilon) = \frac{1}{2} \exp\left\{-\frac{\varepsilon^2}{2}\right\} + \frac{1}{2} \exp\left\{-\frac{\varepsilon^2}{2}\right\} \\ = \exp\left\{-\frac{\varepsilon^2}{2}\right\}$$

(b)

Let's start with a lemma:

If X is σ -subgaussian, then $P(|X| > t) \leq \exp\{-b\varepsilon^2\}$, where $b = \exp\{-\sigma^2\}$

The proof of lemma is omitted.

It can be seen from the first question , $P(|X| \geq \varepsilon) = \exp\left\{-\frac{\varepsilon^2}{2}\right\}$

The comparison of the two formulas shows that , $0 < b \leq 1/2$. That is, $\sigma \geq \sqrt{\ln 2}$

By topic condition , $\sigma = \sqrt{2 - \varepsilon}$

Hence , $\varepsilon \leq 2 - \ln 2$, this is in contradiction with the arbitrariness of ε

5.7

(a) If X is σ -subgaussian , then $E(X) = 0, E(X^2) \leq \sigma^2$

proof:

$$E(e^{\lambda X}) = \sum_{n=0}^{\infty} \frac{\lambda^n E(X^n)}{n!} = 1 + \lambda E(X) + \frac{\lambda^2 E(X^2)}{2} + O(\lambda^2) \quad (21)$$

By definition ,

$$E(e^{\lambda X}) \leq e^{\frac{\lambda^2 \sigma^2}{2}} = 1 + \frac{\lambda^2 \sigma^2}{2} + O(\lambda^2) \quad (22)$$

By comparing the above two formulas and discussing the case that λ approaches to 0 from above and below 0, we get the conclusion that ,

$$E(X) = 0, E(X^2) \leq \sigma^2$$

(b)

If X is σ -subgaussian, then $E(X) = 0, E(X^2) \leq \sigma^2$.

$$\begin{aligned} E(e^{c\lambda x}) &= 1 + \lambda E(cx) + \frac{\lambda^2 E(c^2 x^2)}{2} + O(\lambda^2) \\ &\leq 1 + c\lambda E(x) + \frac{\lambda^2 c^2}{2} E(x^2) + O(\lambda^2) \\ &\leq 1 + \frac{\lambda^2 c^2 \sigma^2}{2} + O(\lambda^2) \\ &\leq e^{\frac{\lambda^2 c^2 \sigma^2}{2}} \end{aligned}$$

Hence, cX is $|c|\sigma$ -subgaussian.

(c)

If X_1 is σ_1 -subgaussian, X_2 is σ_2 -subgaussian

$$\begin{aligned} \text{then } E(X_1) &= 0, E(X_1^2) \leq \sigma_1^2, E(X_2) = 0, E(X_2^2) \leq \sigma_2^2 \\ E(e^{\lambda(x_1+x_2)}) &= 1 + \lambda E(x_1+x_2) + \frac{\lambda^2 E((x_1+x_2)^2)}{2} + O(\lambda^2) \\ &= 1 + \frac{\lambda^2}{2} \text{Var}(x_1+x_2) + O(\lambda^2) \\ &= 1 + \frac{\lambda^2}{2} (\text{var}(x_1) + \text{var}(x_2) + 2\text{cov}(x_1, x_2)) + O(\lambda^2) \\ \text{Because } x_1, x_2 &\text{ are independent,} \\ &= 1 + \frac{\lambda^2}{2} (E(x_1^2) + E(x_2^2)) + O(\lambda^2) \\ &\leq 1 + \frac{\lambda^2}{2} (\sigma_1^2 + \sigma_2^2) + O(\lambda^2) \\ &\leq e^{\frac{\lambda^2(\sigma_1^2 + \sigma_2^2)}{2}} \end{aligned}$$

Hence, $X_1 + X_2$ is $\sqrt{\sigma_1^2 + \sigma_2^2}$ -subgaussian.

5.11

(a)

$$E(e^{\lambda X}) = 1 + \lambda E(X) + \frac{\lambda^2 E(X^2)}{2} + O(\lambda^2) = 1 + \frac{\lambda^2 E(X^2)}{2} + O(\lambda^2) \quad (23)$$

If the conclusion is true, then the above formula satisfies

$$\leq 1 + \frac{\lambda^2}{2} \left(\frac{(b-a)^2}{4} \right) + O(\lambda^2)$$

So just prove:

$$E(x^2) \leq \left(\frac{b-a}{2} \right)^2$$

$$E(x^2) = \text{var}(x) = E(x - \bar{x})^2$$

However, $(x - \bar{x})^2 \leq \left(\frac{b-a}{2} \right)^2$. The conclusion is proved.

(b)

The proof of Hoeffding's Inequality:

Let $X_i = Z_i - E(Z_i)$, $\bar{X} = \frac{1}{m} \sum_{i=1}^m X_i$

By Markov inequality, for all $\lambda > 0$, $\varepsilon > 0$,

$$P(\bar{X} \geq \varepsilon) = P(e^{\lambda \bar{X}} \geq e^{\lambda \varepsilon}) \leq \frac{E(e^{\lambda \bar{X}})}{e^{\lambda \varepsilon}}$$

Z_1, \dots, Z_m iid.r.v.

$$\text{So, } E(e^{\lambda \bar{X}}) = \prod_{i=1}^m E(e^{\frac{\lambda X_i}{m}})$$

By Hoeffding's lemma,

$$E(e^{\frac{\lambda X_i}{m}}) \leq e^{\frac{\lambda^2 (b-a)^2}{8m^2}}$$

$$\text{So, } P(\bar{X} \geq \varepsilon) \leq e^{-\lambda \varepsilon} \prod_{i=1}^m E(e^{\frac{\lambda X_i}{m}})$$

$$\leq e^{-\lambda \varepsilon} e^{\frac{\lambda^2 (b-a)^2}{8m}}$$

$$\leq e^{-\lambda \varepsilon + \frac{\lambda^2 (b-a)^2}{8m}}$$

$$\text{Let } \lambda = \frac{4m\varepsilon}{(b-a)^2}, \text{ then } P(\bar{X} \geq \varepsilon) \leq e^{\frac{-2m\varepsilon^2}{(b-a)^2}}$$

Similarly, we can prove the other side of the inequality.

5.16 By assumption $Pr(X_t \leq x) \leq x$, which means that for $\lambda < 1$,

$$\mathbb{E} \left[\exp(\lambda \log(\frac{1}{x_t})) \right] = \int_0^\infty P(\exp(\lambda \log(\frac{1}{x_t})) \geq x) dx = 1 + \int_1^\infty P(X_t \leq x^{-\frac{1}{\lambda}}) dx \quad (24)$$

Applying the Cramer-Chernoff method,

$$P \left(\sum_{t=1}^n \log(\frac{1}{X_t}) \geq \epsilon \right) = P \left(\exp(\lambda \sum_{t=1}^n \log(\frac{1}{X_t})) \geq \exp(\lambda \epsilon) \right) \leq \left(\frac{1}{1-\lambda} \right)^n \exp(-\lambda \epsilon)$$

choosing $\lambda = \frac{\epsilon - n}{\epsilon}$ completes the claim.

5.18(a) Let $\lambda > 0$. Then,

$$\exp(\lambda \mathbb{E}[Z]) \leq \mathbb{E}[\exp(\lambda Z)] \leq \sum_{t=1}^n \mathbb{E}[\exp(\lambda X_t)] \leq n \exp\left(\frac{\lambda^2 \sigma^2}{2}\right)$$

Rearranging shows that,

$$\mathbb{E}(Z) \leq \frac{\log(n)}{\lambda} + \frac{\lambda \sigma^2}{2}$$

Choosing $\lambda = \frac{1}{\sigma} \sqrt{2 \log(n)}$ shows that $\mathbb{E}(Z) \leq \sqrt{2 \sigma^2 \log(n)}$

Chapter 11 The Exp3 Algorithm

11.2 Let π be a deterministic policy, and we define $x_{ti} = 0$ if $A_t = i$ otherwise $x_{ti} = 1$. The deterministic policy collects zero rewards all time,

$$\max_{i \in [k]} \sum_{t=1}^n x_{ti} \geq \frac{1}{k} \sum_{t=1}^n \sum_{i=1}^k x_{ti} = \frac{n(k-1)}{k}$$

11.5 Let P be a probability vector with nonzero components and let $A \sim P$. Suppose \hat{X} is a function such that for all $x \in \mathbb{R}^k$,

$$\mathbb{E} [\hat{X}(A, x_A)] = \sum_{i=1}^k P_i \hat{X}(i, x_i) = x_1$$

Show that there exists an $a \in \mathbb{R}^k$ such that $\langle a, P \rangle = 0$ and for all i and z in their respective domains, $\hat{X}(i, z) = a_i + \frac{\mathbb{I}\{i=1\}z}{P_1}$

Proof. Let x, x' be arbitrary but agree on the first component $x_1 = x'_1$. Let $f(x) = \sum_{i=1}^k P_i \hat{X}(i, x_i)$ Note that,

$$0 = f(x) - f(x') = \sum_{i=j}^k P_j \hat{X}(j, x_j)$$

for all $j > 1$. Since x, x' are arbitrary, $\hat{X}(j, \cdot) = \text{const.}$ Let a_j equal to $\hat{X}(j, \cdot)$.

Further, let $a_1 = \hat{X}(1, 0)$ and then given any $x_1 \in \mathbb{R}$, $\hat{X}(1, x_1) = a_1 + x_1/P_1$.

Finally, let x be such that $x_1 = 0$. Then $0 = f(x) = \sum_i P_i a_i$. □

11.7 First, note that if $G = -\log(-\log(U))$ then $\mathbb{P}(G \leq g) = e^{-\exp(-g)}$.

$$\begin{aligned} \mathbb{P} \left(\log a_i + G_i \geq \max_{j \in [k]} \log a_j + G_j \right) &= \mathbb{E} \left[\prod_{j \neq i} \mathbb{P}(\log a_j + G_j \leq \log a_i + G_i \mid G_i) \right] \\ &= \mathbb{E} \left[\prod_{j \neq i} \exp \left(-\frac{a_j}{a_i} \exp(-G_i) \right) \right] \\ &= \mathbb{E} \left[U_i^{\sum_{j \neq i} \frac{a_j}{a_i}} \right] \\ &= \frac{1}{1 + \sum_{j \neq i} \frac{a_j}{a_i}} \\ &= \frac{a_i}{\sum_{j=1}^k a_j} \end{aligned}$$

11.8 Let Z_{ti} be a standard Gambel. The follow-the-perturbed-leader algorithm chooses

$$A_t = \operatorname{argmax}_{i \in [k]} \left(Z_{ti} - \eta \sum_{s=1}^{t-1} \hat{Y}_{si} \right)$$

is the same as EXP3. Given (11.7)

$$\mathbb{P} \left(\log(a_i) + G_i = \max_{j \in [k]} (\log(a_j) + G_j) \right) = \frac{a_i}{\sum_{j=1}^k a_j}$$

Just simply take a_i as $-\eta \sum_{s=1}^{t-1} \hat{Y}_{si}$, then the form is identical.

Chapter 18

18.1

(a) By Jensen's inequality,

$$\begin{aligned}\sum_{c \in \mathcal{C}} \sqrt{\sum_{t=1}^n \mathcal{I}\{c_t = c\}} &= \|C\| \sum_{c \in \mathcal{C}} \frac{1}{\|C\|} \sqrt{\sum_{t=1}^n \mathcal{I}\{c_t = c\}} \\ &\leq \|C\| \sqrt{\sum_{c \in \mathcal{C}} \frac{1}{\|C\|} \sum_{t=1}^n \mathcal{I}\{c_t = c\}} \\ &= \sqrt{\|C\|n}\end{aligned}$$

(b) When each context occurs $\frac{n}{\|\mathcal{C}\|}$ times we have

$$\sum_{c \in \mathcal{C}} \sqrt{\sum_{t=1}^n \mathcal{I}\{c_t = c\}} = \sqrt{n\|C\|}$$