

## Solutions



# Chapter 2 Foundations of Probability

**2.1 (COMPOSING RANDOM ELEMENTS)** Show that if  $f$  is  $\mathcal{F}/\mathcal{G}$ -measurable and  $g$  is  $\mathcal{G}/\mathcal{H}$ -measurable for sigma algebras  $\mathcal{F}, \mathcal{G}$  and  $\mathcal{H}$  over appropriate spaces, then their composition,  $g \circ f$  (defined the usual way:  $(g \circ f)(\omega) = g(f(\omega)), \omega \in \Omega$ ), is  $\mathcal{F}/\mathcal{H}$ -measurable.

*Proof.* Since  $g$  is  $\mathcal{G}/\mathcal{H}$ -measurable, therefore  $\forall C \in \mathcal{H}, \exists B = g^{-1}(C) \in \mathcal{G}$ . Similarly, since  $f$  is  $\mathcal{F}/\mathcal{G}$ -measurable,  $\forall B \in \mathcal{G}, \exists A = f^{-1}(B) \in \mathcal{F}$ . Thus  $\forall C \in \mathcal{H}, \exists A = f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C) \in \mathcal{F}$  and the proof is complete. □

**2.2** Let  $X_1, \dots, X_n$  be random variables on  $(\Omega, \mathcal{F})$ . Prove that  $X = (X_1, \dots, X_n)$  is a random vector.

*Proof.* Since  $X_i$  is a random variable ( $\forall i = 1, 2, \dots, n$ ), it holds that  $X_i$  is  $\mathcal{F}/\mathcal{B}(\mathbb{R})$ -measurable, which means that  $\forall B \in \mathcal{B}(\mathbb{R}), X_i^{-1}(B) \in \mathcal{F}$ . We first prove that  $X$  is  $\mathcal{F}/(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}))$ -measurable (totally  $n$   $\mathcal{B}(\mathbb{R})$ s).  $\forall A = A_1 \times A_2 \times \dots \times A_n \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}), X^{-1}(A) = X_1^{-1}(A_1) \cap X_2^{-1}(A_2) \cap \dots \cap X_n^{-1}(A_n) \in \mathcal{F}$ , which holds since  $X_i^{-1}(A_i) \in \mathcal{F}, \forall i = 1, 2, \dots, n$  and  $\mathcal{F}$  is a  $\sigma$ -algebra. Thus we conclude that  $X$  is  $\mathcal{F}/(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}))$ -measurable.

By definition  $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R}))$  (totally  $n$   $\mathcal{B}(\mathbb{R})$ s). And according to the property in 2.5(b), we can get that  $X$  is  $\mathcal{F}/\mathcal{B}(\mathbb{R}^n)$ -measurable, thus it is a random vector. □

**2.3 (RANDOM VARIABLE INDUCED  $\sigma$ -ALGEBRA)** Let  $\mathcal{U}$  be an arbitrary set and  $(\mathcal{V}, \Sigma)$  a measurable space and  $X : \mathcal{U} \rightarrow \mathcal{V}$  an arbitrary function. Show that  $\Sigma_X = \{X^{-1}(A) : A \in \Sigma\}$  is a  $\sigma$ -algebra over  $\mathcal{U}$ .

*Proof.* (i) We need to show that  $\Sigma_X$  is closed under countable union. Let  $U_i = X^{-1}(A_i), A_i \in \Sigma, i \in \mathbb{N}$ .

It follows that  $\bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} X^{-1}(A_i) = X^{-1}(\bigcup_{i=1}^{\infty} A_i)$ . Since  $\bigcup_{i=1}^{\infty} A_i \in \Sigma, \bigcup_{i=1}^{\infty} U_i \in \Sigma_X$ .

(ii) We need to show that  $\Sigma_X$  is closed under set subtraction  $-$ .  $\forall U_1, U_2 \in \Sigma_X, U_1 - U_2 = X^{-1}(A_1) - X^{-1}(A_2) = X^{-1}(A_1 - A_2)$ . Since  $A_1 - A_2 \in \Sigma, U_1 - U_2 \in \Sigma_X$ .

(iii) We need to show that  $\Sigma_X$  is closed to  $\mathcal{U}$  itself. Since  $\mathcal{U} = X^{-1}(\mathcal{V})$  and  $\mathcal{V} \in \Sigma$ , it follows that  $\mathcal{U} \in \Sigma_X$ . □

**2.4** Let  $(\Omega, \mathcal{F})$  be a measurable space and  $A \subseteq \Omega$  and  $\mathcal{F}|_A = \{A \cap B : B \in \mathcal{F}\}$ .

*Proof.* (a) (i) We need to show that  $\mathcal{F}|_A$  is closed under countable union. Let  $X_1 = A \cap B_1, X_2 = A \cap B_2, \dots$  and  $X' = \bigcup_{i=1}^{\infty} X_i$  and  $B' = \bigcup_{i=1}^{\infty} B_i$  where  $B_1, B_2, \dots \in \mathcal{F}$ . Since  $\mathcal{F}$  is sigma algebra,  $B' \in \mathcal{F}$ . Furthermore, since  $X' = \bigcup_{i=1}^{\infty} X_i = \bigcup_{i=1}^{\infty} A \cap B_i = A \cap \left( \bigcup_{i=1}^{\infty} B_i \right) = A \cap B'$ , we can see that  $X' \in \mathcal{F}|_A$ .

(ii) We need to show that  $\mathcal{F}|_A$  is closed under set subtraction  $-$ .  $\forall X_1, X_2 \in \mathcal{F}|_A, X_1 - X_2 = (A \cap B_1) - (A \cap B_2) = A \cap (B_1 - B_2)$ . Since  $B_1 - B_2 \in \mathcal{F}$ , it follows that  $X_1 - X_2 \in \mathcal{F}|_A$ .

- (iii) We need to show that  $\mathcal{F}|_A$  is closed to  $A$  itself. Since  $\emptyset \in \mathcal{F}$ , we have  $\emptyset = A \cap \emptyset \in \mathcal{F}|_A$  and  $A = \emptyset^C \in \mathcal{F}|_A$ .
- (b) Let  $P = \{A \cap B : B \in \mathcal{F}\}, Q = \{B : B \subset A, B \in \mathcal{F}\}$ .
  - (i) We claim that  $P \subset Q$ . Let  $X = A \cap B, B \in \mathcal{F}$ . Since  $A \in \mathcal{F}, X = A \cap B \in \mathcal{F}$ . Furthermore,  $X \in Q = \{B : B \subset A, B \in \mathcal{F}\}$ .
  - (ii) We claim that  $Q \subset P$ .  $\forall X \in Q$ , we have  $X \subset A$  and  $X \in \mathcal{F}$ , which means that  $X = X \cap A$  and  $X \in \mathcal{F}$ . It follows that  $X \in P$ .
  - (iii) Take both (i)(ii) into consideration, we can see that  $P = Q$ .

□

**2.5** Let  $\mathcal{G} \subseteq 2^\Omega$  be a non-empty collection of sets and define  $\sigma(\mathcal{G})$  as the smallest  $\sigma$ -algebra that contains  $\mathcal{G}$ . By ‘smallest’ we mean that  $\mathcal{F} \in 2^\Omega$  is smaller than  $\mathcal{F}' \in 2^\Omega$  if  $\mathcal{F} \subset \mathcal{F}'$ .

- (a) Show that  $\sigma(\mathcal{G})$  exists and contains exactly those sets  $A$  that are in every  $\sigma$ -algebra that contains  $\mathcal{G}$ .
- (b) Suppose  $(\Omega', \mathcal{F})$  is a measurable space and  $X : \Omega' \rightarrow \Omega$  be  $\mathcal{F}/\mathcal{G}$ -measurable. Show that  $X$  is also  $\mathcal{F}/\sigma(\mathcal{G})$ -measurable. (We often use this result to simplify the job of checking whether a random variable satisfies some measurability property).
- (c) Prove that if  $A \in \mathcal{F}$  where  $\mathcal{F}$  is a  $\sigma$ -algebra, then  $\mathbb{I}\{A\}$  is  $\mathcal{F}$ -measurable.

*Proof.* (a) Let  $\mathcal{K} = \{\mathcal{F} | \mathcal{F} \text{ is a } \sigma\text{-algebra and contains } \mathcal{G}\}$ , It holds obviously that  $\mathcal{K}$  is not an empty set since it contains  $2^\Omega$ .

Then  $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$  contains exactly those sets that are in every  $\sigma$ -algebra that contains  $\mathcal{G}$ . Given its existence, we only need to prove that  $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$  is the smallest  $\sigma$ -algebra that contains  $\mathcal{G}$ .

First we show  $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$  is a  $\sigma$ -algebra. Since  $\mathcal{F}$  is a  $\sigma$ -algebra and therefore  $\Omega \in \mathcal{F}$  for all  $\mathcal{F} \in \mathcal{K}$ , it follows that  $\Omega \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ . Next, for any  $A \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ ,  $A^c \in \mathcal{F}$  for all  $\mathcal{F} \in \mathcal{K}$ . Since they are all  $\sigma$ -algebras,  $A^c \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ . Hence  $A^c \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ . Finally, for any  $\{A_i\}_i \subset \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ ,  $\{A_i\}_i \subset \mathcal{F}$  for all  $\mathcal{F} \in \mathcal{K}$ . Since they are all  $\sigma$ -algebras,  $\bigcup_i A_i \in \mathcal{F}$  for all  $\mathcal{F} \in \mathcal{K}$ . Hence  $\bigcup_i A_i \in \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ .

Next we want to prove  $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$  is the smallest  $\sigma$ -algebra that contains  $\mathcal{G}$ . It is quite obvious that  $\bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F} \subseteq \mathcal{F}'$  for all  $\mathcal{F}' \in \mathcal{K}$ .

Above all, we have  $\sigma(\mathcal{G}) = \bigcap_{\mathcal{F} \in \mathcal{K}} \mathcal{F}$ .

- (b) Define  $\mathcal{H} = \{A : X^{-1}(A) \in \mathcal{F}\}$ . To show  $X$  is  $\mathcal{F}/\sigma(\mathcal{G})$ -measurable, it is sufficient to prove  $\sigma(\mathcal{G}) \subseteq \mathcal{H}$ .

First we prove that  $\mathcal{H}$  is a  $\sigma$ -algebra. It holds that  $\Omega \in \mathcal{H}$  since  $X^{-1}(\Omega) = \Omega' \in \mathcal{F}$ . For any  $A \in \mathcal{H}$ , we have  $X^{-1}(A) \in \mathcal{F}$ , thus  $X^{-1}(A^c) = X^{-1}(A)^c \in \mathcal{F}$ , which holds since  $\mathcal{F}$  is a  $\sigma$ -algebra. Thus  $A^c \in \mathcal{H}$ . For any  $A_i \in \mathcal{H}$ ,  $i = 1, 2, \dots$ ,  $X^{-1}(A_i) \in \mathcal{F}$ ,  $X^{-1}(\bigcup_i A_i) = \bigcup_i X^{-1}(A_i) \in \mathcal{F}$ . We can then conclude  $\bigcup_i A_i \in \mathcal{H}$  and  $\mathcal{H}$  is a  $\sigma$ -algebra.

Also, since  $X$  is  $\mathcal{F}/\mathcal{G}$ -measurable, we have  $\mathcal{G} \subseteq \mathcal{H}$ . Thus  $\mathcal{H}$  is  $\sigma$ -algebra that contains  $\mathcal{G}$ . By applying the result of (a), we have  $\sigma(\mathcal{G}) \subseteq \mathcal{H}$ , which completes the proof.

- (c) The idea is to show  $\forall B \in \mathfrak{B}(\mathbb{R}), \mathbb{I}\{A\}^{-1}(B) \in \mathcal{F}$ .

If  $\{0, 1\} \in B$ ,  $\mathbb{I}\{A\}^{-1}(B) = \Omega \in \mathcal{F}$ . If  $\{0\} \in B$ ,  $\mathbb{I}\{A\}^{-1}(B) = A^c \in \mathcal{F}$ . If  $\{1\} \in B$ ,  $\mathbb{I}\{A\}^{-1}(B) = A \in \mathcal{F}$ . If  $\{0, 1\} \cap B = \emptyset$ ,  $\mathbb{I}\{A\}^{-1}(B) = \emptyset \in \mathcal{F}$ .

□

**2.6** (KNOWLEDGE AND  $\sigma$ -ALGEBRAS: A PATHOLOGICAL EXAMPLE) In the context of Lemma 2.5, show an example where  $Y = X$  and yet  $Y$  is not  $\sigma(X)$  measurable.

**HINT** As suggested after the lemma, this can be arranged by choosing  $\Omega = \mathcal{Y} = \mathcal{X} = \mathbb{R}, X(\omega) = Y(\omega) = \omega, \mathcal{F} = \mathcal{H} = \mathfrak{B}(\mathbb{R})$  and  $\mathcal{G} = \{\emptyset, \mathbb{R}\}$  to be the trivial  $\sigma$ -algebra.

*Proof.* As the hint suggests, Let  $\Omega = \mathcal{Y} = \mathcal{X} = \mathbb{R}$ ,  $X(\omega) = Y(\omega) = \omega$ ,  $\mathcal{F} = \mathcal{H} = \mathfrak{B}(\mathbb{R})$ . In this case,  $\sigma(X) = \{X^{-1}(A) : A \in \mathcal{G}\} = \{\emptyset, \mathbb{R}\}$ , we can find that  $Y^{-1}((0, 1)) = (0, 1) \notin \sigma(X)$ , thus  $Y$  is not  $\sigma(X)$ -measurable.  $\square$

**2.7** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $B \in \mathcal{F}$  be such that  $\mathbb{P}(B) > 0$ . Prove that  $A \mapsto \mathbb{P}(A|B)$  is a probability measure over  $(\Omega, \mathcal{F})$ .

*Proof.* First we have  $\mathbb{P}(\Omega | B) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$ . Then, for any  $A \in \mathcal{F}$ ,  $\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \geq 0$ . Next, for any  $A \in \mathcal{F}$ ,  $\mathbb{P}(A^c | B) = \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}((\Omega - A) \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)} = 1 - \mathbb{P}(A | B)$ . Finally, for all countable collections of disjoint sets  $\{A_i\}_i$  with  $A_i \in \mathcal{F}$  for all  $i$ , we have  $\mathbb{P}(\bigcup_i A_i | B) = \frac{\mathbb{P}((\bigcup_i A_i) \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\bigcup_i (A_i \cap B))}{\mathbb{P}(B)} = \sum_i \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_i \mathbb{P}(A_i | B)$ .  $\square$

**2.8** (BAYES LAW) Verify (2.2).

*Proof.* With the definition of conditional probability, we have  $\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$ .  $\square$

**2.9** Consider the standard probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  generated by two standard, unbiased, six-sided dice that are thrown independently of each other. Thus,  $\Omega = \{1, \dots, 6\}^2$ ,  $\mathcal{F} = 2^\Omega$  and  $\mathbb{P}(A) = |A|/6^2$  for any  $A \in \mathcal{F}$  so that  $X_i(\omega) = \omega_i$  represents the outcome of throwing dice  $i \in \{1, 2\}$ .

- (a) Show that the events ' $X_1 < 2$ ' and ' $X_2$  is even' are independent of each other.
- (b) More generally, show that for any two events,  $A \in \sigma(X_1)$  and  $B \in \sigma(X_2)$ , are independent of each other.

*Proof.* (a) The event  $\{X_1 < 2\} = \{1\} \times \{1, 2, 3, 4, 5, 6\}$ ,  $\{X_2 \text{ is even}\} = \{1, 2, 3, 4, 5, 6\} \times \{2, 4, 6\}$ ,  $\{X_1 < 2, X_2 \text{ is even}\} = \{(1, 2), (1, 4), (1, 6)\}$ .

Thus  $\mathbb{P}(X_1 < 2) = \frac{6}{36} = \frac{1}{6}$ ,  $\mathbb{P}(X_2 \text{ is even}) = \frac{18}{36} = \frac{1}{2}$ ,  $\mathbb{P}(X_1 < 2, X_2 \text{ is even}) = \frac{3}{36} = \frac{1}{12}$ , which satisfies  $\mathbb{P}(X_1 < 2, X_2 \text{ is even}) = \mathbb{P}(X_1 < 2) \times \mathbb{P}(X_2 \text{ is even})$ . These two events are independent of each other.

- (b)  $\sigma(X_1) = \{X_1^{-1}(A'), A' \subseteq [6]\} = \{A' \times [6] : A' \subseteq [6]\}$ ,  $\sigma(X_2) = \{X_2^{-1}(B'), B' \subseteq [6]\} = \{[6] \times B' : B' \subseteq [6]\}$ .  $\square$

**2.10** (SERENDIPITOUS INDEPENDENCE) The point of this exercise is to understand independence more deeply. Solve the following problems:

- (a) Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Show that  $\emptyset$  and  $\Omega$  (which are events) are independent of any other event. What is the intuitive meaning of this?
- (b) Continuing the previous part, show that any event  $A \in \mathcal{F}$  with  $\mathbb{P}(A) \in \{0, 1\}$  is independent of any other event.
- (c) What can we conclude about an event  $A \in \mathcal{F}$  that is independent of its complement,  $A^c = \Omega \setminus A$ ? Does your conclusion make intuitive sense?
- (d) What can we conclude about an event  $A \in \mathcal{F}$  that is independent of itself? Does your conclusion make intuitive sense?
- (e) Consider the probability space generated by two independent flips of unbiased coins with the smallest possible  $\sigma$ -algebra. Enumerate all pairs of events  $A, B$  such that  $A$  and  $B$  are independent of each other.
- (f) Consider the probability space generated by the independent rolls of two unbiased three-sided dice. Call the possible outcomes of the individual dice rolls 1, 2 and 3. Let  $X_i$  be the random variable that corresponds to the outcome of the  $i$ th dice roll ( $i \in \{1, 2\}$ ). Show that the events  $\{X_1 \leq 2\}$  and  $\{X_1 = X_2\}$  are independent of each other.

- (g) The probability space of the previous example is an example when the probability measure is uniform on a finite outcome space (which happens to have a product structure). Now consider any  $n$ -element, finite outcome space with the uniform measure. Show that  $A$  and  $B$  are independent of each other if and only if the cardinalities  $|A|$ ,  $|B|$ ,  $|A \cap B|$  satisfy  $n|A \cap B| = |A| \cdot |B|$ .
- (h) Continuing with the previous problem, show that if  $n$  is prime, then no non-trivial events are independent (an event  $A$  is **trivial** if  $\mathbb{P}(A) \in \{0, 1\}$ ).
- (i) Construct an example showing that pairwise independence does not imply mutual independence.
- (j) Is it true or not that  $A, B, C$  are mutually independent if and only if  $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$ ? Prove your claim.

*Proof.* (a) Empty sets and complete sets are independent of any event:

$$P(A \cap \Omega) = P(A) = 1 * P(A) = P(\Omega) * P(A)$$

$$P(A \cap \phi) = P(\phi) = 0 = P(\phi) * P(A)$$

- (b) Prove when  $P(A) = 0$  or  $1$ ,  $A$  is independent of any event: for any  $B \in \Omega$

$$P(A) \in \{0, 1\}$$

$$\text{When } P(A) = 1, P(A^c \cap B) \leq P(A^c) = 1 - P(A) = 0,$$

$$\text{we have } P(A \cap B) = P(A \cap B) + P(A^c \cap B) = P(B) = P(A)P(B)$$

$$\text{When } P(A) = 0, \text{ we have } P(A \cap B) \leq P(A) = 0 = P(A)P(B)$$

- (c)  $P(A^c \cap A) = P(A)P(A^c)$

$$\text{we have } 0 = P(A)(1 - P(A)) \Rightarrow P(A) \in \{0, 1\}$$

- (d)  $P(A \cap A) = P(A)P(A), P(A) = 0, 1$

- (e)  $\Omega = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$ .  $A, B \subseteq \Omega$  denote the events.

First of all, if either  $A$  or  $B$  is trivial, then  $A$  and  $B$  are independent to each other.

Then, we only need to enumerate  $A, B \notin \Omega, \emptyset$  satisfied that  $P(A \cap B) = P(A)P(B)$ . Since  $P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{|A \cap B|}{4}$  and  $P(A)P(B) = \frac{|A||B|}{16}$ , we can conclude that  $|A| = 2$ ,  $|B| = 2$  and  $|A \cap B| = 1$  is the only situation satisfying the condition.

Thus, besides trivial  $A$  or  $B$ , all  $A, B$  satisfying  $|A| = 2$ ,  $|B| = 2$  and  $|A \cap B| = 1$  are the solution.

- (f)  $P(X_1 \leq 2) = 2/3$

$$P(X_1 = X_2) = 3/9 = 1/3$$

$$P(X_1 \leq 2, X_1 = X_2) = P(X_1 = X_2 = 1) + P(X_1 = X_2 = 2) = 1/9 + 1/9 = 2/9$$

$$\text{So, } P(X_1 \leq 2, X_1 = X_2) = P(X_1 = X_2)P(X_1 \leq 2)$$

- (g) Necessity :  $\frac{|A \cap B|}{n} = P(A \cap B) = P(A)P(B) = \frac{|A|}{n} \frac{|B|}{n}$

$$\Rightarrow |A \cap B| * n = |A||B|$$

$$\text{Sufficiency : } |A \cap B| * n = |A||B| \Rightarrow \frac{|A|}{n} \frac{|B|}{n} = \frac{|A \cap B|}{n}$$

$$\Rightarrow P(A \cap B) = P(A)P(B)$$

- (h) If  $A, B$  are two non-trivial events independent to each other,  $|A \cap B| * n = |A||B| \Rightarrow n \mid |A||B| \Rightarrow n \mid |A|$  or  $n \mid |B| \Rightarrow |A| = n$  or  $|B| = n$ , contradictory to non-trivial assumption.

- (i) Let  $\Omega = \{1, 2, 3, 4\}$ ,  $A = \{1, 2\}$ ,  $B = \{1, 3\}$ ,  $C = \{1, 4\}$ .  $A, B, C$  are pairwise independent but  $P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$ .

- (j) Let's take a counter example: roll a die and set the A event to  $\{1, 2, 3\}$ , B event is set to  $\{1, 2, 4\}$ , C event is set to  $\{1, 4, 5, 6\}$

$$P(X_1 X_2 X_3) = \frac{1}{6}$$

$$P(X_1)P(X_2)P(X_3) = (1/2) * (1/2) * (2/3) = 1/6$$

$$\text{while } P(X_1 \cap X_2) = 1/3 \neq \frac{1}{2} * \frac{1}{2}$$

□

## 2.11

- (a)  $X: \Omega \rightarrow x$

Because X, Y are independent equivalent to  $\sigma(X), \sigma(Y)$  are independent;

For any  $A \in \sigma(Y)$ ,

$$P(\phi \cap A) = P(\phi) = 0 = P(\phi)P(A)$$

$$P(\Omega \cap A) = P(A) = P(\Omega)P(A)$$

- (b) We know that  $P(X = x) = 1$

$$P(X = x|Y) = \frac{P((X = x) \cap Y)}{P(Y)} = 1 = P(X = x)$$

$$P(X \neq x|Y) = 1 - P(X = x|Y) = 0 = P(X \neq x)$$

- (c) Notice the relation:  $P(A) = P(X(A) = 1)$

$$P(B) = P(X(B) = 1)$$

$$P(A \cap B) = P(X(A \cap B) = 1)$$

The first two formulas follow the definition. Let's prove the third equation:

$$P(X(A \cap B) = 1) = P(X(A) + X(B) - X(A \cup B) = 1)$$

Let's discuss  $X(A), X(B), X(A \cup B)$ :

$X(A)$	$X(B)$	$X(A \cup B)$	$X(A) + X(B) - X(A \cup B)$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

We can see that,  $P(X(A \cap B) = 1) = P(X(A) + X(B) - X(A \cup B) = 1)$ , this is only one case of the first row of the table.

$$\text{that is } P(X(A \cap B) = 1) = P(X(A) = 1, X(B) = 1) = P(A \cap B)$$

$$\text{that is } P(X(A \cap B) = 1) = P(A \cap B)$$

$$\text{So, } P(A \cap B) = P(A)P(B) \text{ is equivalent to } P(X(A \cap B) = 1) = P(X(A) = 1)P(X(B) = 1)$$

- (d)  $A_i$  pairwise i  $\Leftrightarrow I\{A_i\}$  pairwise i

$$\text{mutual i} \Leftrightarrow P(\bigcap_i A_i) = \prod_i P(A_i)$$

$$\Leftrightarrow P(\bigcap_{i \in K^I} A_i \cap \bigcap_{i \in K^I} A_i^c)$$

$$= \prod_{i \in K^I} P(A_i) \prod_{i \in K^I} P(A_i^c)$$

$$\Leftrightarrow \{\phi, \Omega, A_i, A_i^c\} \text{ mutual independent}$$

$$\Leftrightarrow \sigma(I\{w \in A_i\}) \text{ mutual i}$$

$$\Leftrightarrow I\{w \in A_i\} \text{ mutual i}$$

## 2.12 X integrable $|X|$ integrable

(a) For any  $A \in \mathcal{B}(R) \Rightarrow A$  is open,

so,  $f^{-1}(A)$  is open, so  $f^{-1}(A) \in \mathcal{B}(R)$

(b)  $X$  is known to be a random variable,  $f(x) = |x|$  continuous.

r.v.  $X$  is  $\mathbf{F}/\mathbf{B}(R)$ -measurable

$\Rightarrow |X|$  is  $\mathbf{B}(R)/\mathbf{B}(R)$ -measurable

$\Rightarrow |X|$  is  $\mathbf{F}/\mathbf{B}(R)$ -measurable

$\Rightarrow |X|$  is r.v.

From (a)(b),  $X$  integrable  $\Leftrightarrow |X|$  integrable.

## 2.14

(a) Assume  $\forall i, X_i$  is simple function.

$$\begin{aligned}
 \mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^n X_i\right] \\
 &= \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \mathbb{I}_{A_{i,j}}\{\omega\}\right] \\
 &= \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \mathbb{I}_{A_{i,j}}\{\omega\} d\mathbb{P}(\omega) \\
 &= \sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \int_{\Omega} \mathbb{I}_{A_{i,j}}\{\omega\} d\mathbb{P}(\omega) \\
 &= \sum_{i=1}^n \sum_{j=1}^m \alpha_{i,j} \mathbb{P}(A_{i,j}) \\
 &= \sum_{i=1}^n \mathbb{E}[X_i]
 \end{aligned}$$

(b) Assume  $\forall i, X_i$  is non-negative random variable.

$$\begin{aligned}
 \mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^n X_i\right] \\
 &= \sup \left\{ \int_{\Omega} h d\mathbb{P} : h \text{ is simple and } 0 \leq h \leq X = \sum_{i=1}^n X_i \right\} \\
 &= \sum_{i=1}^n \sup \left\{ \int_{\Omega} h_i d\mathbb{P} : h_i \text{ is simple and } 0 \leq h_i \leq X_i \right\} \\
 &= \sum_{i=1}^n \mathbb{E}[X_i]
 \end{aligned}$$



(c) Assume  $\forall i, X_i$  is arbitrary random variable.

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E} \left[ \sum_{i=1}^n X_i \right] \\
&= \mathbb{E} \left[ \sum_{i=1}^n (X_i^+ - X_i^-) \right] \\
&= \mathbb{E} \left[ \sum_{i=1}^n X_i^+ \right] - \mathbb{E} \left[ \sum_{i=1}^n X_i^- \right] \\
&= \sum_{i=1}^n \mathbb{E} [X_i^+] - \sum_{i=1}^n \mathbb{E} [X_i^-] \\
&= \sum_{i=1}^n (\mathbb{E}[X_i^+] - \mathbb{E}[X_i^-]) \\
&= \sum_{i=1}^n \mathbb{E}[X_i]
\end{aligned}$$

## 2.15

(a) Assume  $X$  is simple function.

$$\begin{aligned}
\mathbb{E}[cX] &= \mathbb{E} \left[ c \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\} \right] \\
&= \int_{\Omega} c \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\} d\mathbb{P}(\omega) \\
&= c \int_{\Omega} \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\} d\mathbb{P}(\omega) \\
&= c\mathbb{E}[X]
\end{aligned}$$

(b) Assume  $X$  is non-negative random variable.

$$\begin{aligned}
\mathbb{E}[cX] &= \sup \left\{ \int_{\Omega} h d\mathbb{P} : h \text{ is simple and } 0 \leq h \leq cX \right\} \\
&= c \sup \left\{ \int_{\Omega} h' d\mathbb{P} : h' \text{ is simple and } 0 \leq h' \leq X \right\} \\
&= c\mathbb{E}[X]
\end{aligned}$$

(c) Assume  $X$  is arbitrary random variable.

(i)  $c \geq 0$

$$\begin{aligned}
\mathbb{E}[cX] &= \mathbb{E}[(cX)^+] - \mathbb{E}[(cX)^-] \\
&= \mathbb{E}[c(X)^+] - \mathbb{E}[c(X)^-] \\
&= c\mathbb{E}[(X)^+] - c\mathbb{E}[(X)^-] \\
&= c\mathbb{E}[X]
\end{aligned}$$

(ii)  $c < 0$

By definition, we have

$$\begin{aligned}
(cX)^+ &= cX\mathbb{I}\{cX > 0\} \\
&= cX\mathbb{I}\{x < 0\} \text{ (since } c \downarrow 0) \\
&= (-c)(-X)\mathbb{I}\{X < 0\} \\
&= (-c)(X)^-
\end{aligned}$$

Along the similar line, we have

$$\begin{aligned}
(cX)^- &= -cX\mathbb{I}\{cX < 0\} \\
&= -cX\mathbb{I}\{X > 0\} \\
&= -c(X)^+
\end{aligned}$$

Now we can see that

$$\begin{aligned}
\mathbb{E}[cX] &= \mathbb{E}[(cX)^+] - \mathbb{E}[(cX)^-] \\
&= \mathbb{E}[(-c)(X)^-] - \mathbb{E}[-c(X)^+] \\
&= -c\mathbb{E}[(X)^-] + c\mathbb{E}[(X)^+] \\
&= c\mathbb{E}[X]
\end{aligned}$$

## 2.16

(a) Assume  $X = \sum_{i=1}^n \alpha_i \mathbb{I}_{A_i} \{\omega\}$ ,  $Y = \sum_{j=1}^m \beta_j \mathbb{I}_{B_j} \{\omega\}$  are simple functions.

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{I}_{A_i} \{\omega\} \mathbb{I}_{B_j} \{\omega\}\right] \\
&= \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{I}_{A_i} \{\omega\} \mathbb{I}_{B_j} \{\omega\} d\mathbb{P}(\omega) \\
&= \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{P}(A_i \cap B_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \mathbb{P}(A_i) \mathbb{P}(B_j) \text{ (by the definition of independence)} \\
&= \left(\sum_{i=1}^n \alpha_i \mathbb{P}(A_i)\right) \left(\sum_{j=1}^m \beta_j \mathbb{P}(B_j)\right) \\
&= \mathbb{E}[X] \mathbb{E}[Y]
\end{aligned}$$

(b) Assume  $X, Y$  are non-negative random variables.

$$\begin{aligned}
\mathbb{E}[XY] &= \sup \{\mathbb{E}[h] : h \text{ is simple and } 0 \leq h \leq XY\} \\
&= \sup \{\mathbb{E}[h_1 h_2] : h_1, h_2 \text{ are simple and } 0 \leq h_1 \leq X, 0 \leq h_2 \leq Y\} \\
&= \sup \{\mathbb{E}[h_1] \mathbb{E}[h_2] : h_1, h_2 \text{ are simple and } 0 \leq h_1 \leq X, 0 \leq h_2 \leq Y\} \\
&= \sup \{\mathbb{E}[h_1] : h_1 \text{ is simple and } 0 \leq h_1 \leq X\} \cdot \sup \{\mathbb{E}[h_2] : h_2 \text{ is simple and } 0 \leq h_2 \leq Y\} \\
&= \mathbb{E}[X] \mathbb{E}[Y]
\end{aligned}$$

(c) Assume  $X, Y$  are arbitrary random variables.

$$\begin{aligned}
 \mathbb{E}[XY] &= \mathbb{E}[(X^+ - X^-)(Y^+ - Y^-)] \\
 &= \mathbb{E}[X^+Y^+ - X^+Y^- - X^-Y^+ + X^-Y^-] \\
 &= \mathbb{E}[X^+]\mathbb{E}[Y^+] - \mathbb{E}[X^+]\mathbb{E}[Y^-] - \mathbb{E}[X^-]\mathbb{E}[Y^+] + \mathbb{E}[X^-]\mathbb{E}[Y^-] \\
 &= (\mathbb{E}[X^+] - \mathbb{E}[X^-])(\mathbb{E}[Y^+] - \mathbb{E}[Y^-]) \\
 &= \mathbb{E}[X]\mathbb{E}[Y]
 \end{aligned}$$

**2.17** Before proving Ex.2.17, we need to make minor changes to the definition of conditional expectation and give a small lemma.

**Definition 1.** Assume  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space.  $\mathcal{G} \subset \mathcal{F}$  is a sub- $\sigma$ -algebra of  $\mathcal{F}$ .  $X : \Omega \rightarrow \mathbb{R}$  is a random variable. The conditional expectation of  $X$  given  $\mathcal{G}$  is denoted by any random variable  $Y$  which satisfies the following 2 properties:

- $Y$  is  $\mathcal{G}$ -measurable
- $\forall A \in \mathcal{G}$ ,

$$\int_A Y d\mathbb{P} = \int_A X d\mathbb{P}$$

Formally, we denote  $Y$  by notation  $\mathbb{E}[X|\mathcal{G}]$ .

**Lemma 1.** If  $X$  is  $\mathcal{G}$ -measurable, then  $\mathbb{E}[X|\mathcal{G}] = X$  holds a.s.

*Proof.* Since  $X$  is  $\mathcal{G}$ -measurable, property1 holds. And property2 holds trivially.  $\square$

We can now handily prove Ex.2.17. Since  $\mathbb{E}[X|\mathcal{G}_1]$  is  $\mathcal{G}_1$ -measurable and  $\mathcal{G}_1 \subset \mathcal{G}_2$ , we can see that  $\mathbb{E}[X|\mathcal{G}_1]$  is  $\mathcal{G}_2$ -measurable. By Lemma 1,  $\mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]|\mathcal{G}_2] = \mathbb{E}[X|\mathcal{G}_1]$  holds almost surely.

**2.18** Suppose  $X = Y$  with  $\mathbb{V}[X] \neq 0$ . Then, we have  $\mathbb{E}[XY] = \mathbb{E}[X^2] = \mathbb{V}[X] + \mathbb{E}[X]^2 \neq \mathbb{E}[X]^2 = \mathbb{E}[X]\mathbb{E}[Y]$ .

**2.19** As the hint suggests,  $X(\omega) = \int_{[0, \infty)} \mathbb{I}\{[0, X(\omega)]\}(x) dx$ . Hence, we have

$$\begin{aligned}
 \mathbb{E}[X(\omega)] &= \mathbb{E}\left[\int_{[0, \infty)} \mathbb{I}\{[0, X(\omega)]\}(x) dx\right] \\
 &= \int_{[0, \infty)} \mathbb{E}[\mathbb{I}\{[0, X(\omega)]\}(x)] dx \\
 &= \int_{[0, \infty)} P(X(\omega) > x) dx
 \end{aligned} \tag{1}$$

where the second equality is given by Fubini–Tonell theorem.

**2.20** We prove the following properties all by contradiction (for the sake of rigor).

- (1) Let  $G = \{\omega : \mathbb{E}[X | \mathcal{G}](\omega) < 0\}$ . Then  $G \in \mathcal{G}$  since  $\mathbb{E}[X | \mathcal{G}]$  is  $\mathcal{G}$ -measurable by definition. Now suppose  $\mathbb{P}(G) > 0$ , then

$$\begin{aligned}
 \int_G X d\mathbb{P} &= \int_G \mathbb{E}(X | \mathcal{G}) d\mathbb{P} \\
 &< 0
 \end{aligned} \tag{2}$$

where the equality holds by the definition of conditional expectation. Now we can find it contradictory as  $X \geq 0$ . Therefore  $\mathbb{P}(G) = 0$ , and  $\mathbb{E}[X | \mathcal{G}] \geq 0$  a.s.

- (2) Let  $G = \{\omega : \mathbb{E}[1 \mid \mathcal{G}](\omega) \neq 1\}$ . Then  $G \in \mathcal{G}$  since  $\mathbb{E}[1 \mid \mathcal{G}]$  is  $\mathcal{G}$ -measurable by definition. Now suppose  $\mathbb{P}(G) > 0$ , then

$$\int_G 1 d\mathbb{P} = \int_G \mathbb{E}(1 \mid \mathcal{G}) d\mathbb{P} \neq 1 \quad (3)$$

where the equality holds by the definition of conditional expectation. Now we can find it contradictory as  $\int_G 1 d\mathbb{P} = 1$ . Therefore  $\mathbb{P}(G) = 0$ , and  $\mathbb{E}[1 \mid \mathcal{G}] = 1$  a.s.

- (3) Let  $G = \{\omega : \mathbb{E}[X + Y \mid \mathcal{G}](\omega) \neq \mathbb{E}[X \mid \mathcal{G}](\omega) + \mathbb{E}[Y \mid \mathcal{G}](\omega)\}$ . Then  $G \in \mathcal{G}$  since  $\mathbb{E}[X + Y \mid \mathcal{G}]$ ,  $\mathbb{E}[X \mid \mathcal{G}]$ , and  $\mathbb{E}[Y \mid \mathcal{G}]$  are all  $\mathcal{G}$ -measurable by definition. Now suppose  $\mathbb{P}(G) > 0$ , then

$$\begin{aligned} \int_G (X + Y) d\mathbb{P} &= \int_G \mathbb{E}(X + Y \mid \mathcal{G}) d\mathbb{P} \\ &\neq \int_G [\mathbb{E}(X \mid \mathcal{G}) + \mathbb{E}(Y \mid \mathcal{G})] d\mathbb{P} \\ &= \int_G \mathbb{E}(X \mid \mathcal{G}) d\mathbb{P} + \int_G \mathbb{E}(Y \mid \mathcal{G}) d\mathbb{P} \\ &= \int_G X d\mathbb{P} + \int_G Y d\mathbb{P} \end{aligned} \quad (4)$$

where the first equality and the last one hold by the definition of conditional expectation. It contradicts the linearity of expectation in that  $\int_G (X + Y) d\mathbb{P} \neq \int_G X d\mathbb{P} + \int_G Y d\mathbb{P}$ . Therefore  $\mathbb{P}(G) = 0$ , and  $\mathbb{E}(X + Y \mid \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G}) + \mathbb{E}(Y \mid \mathcal{G})$  a.s.

- (4) Let  $G = \{\omega : \mathbb{E}[XY \mid \mathcal{G}](\omega) \neq Y(\omega)\mathbb{E}[X \mid \mathcal{G}](\omega)\}$ . Then  $G \in \mathcal{G}$  since  $\mathbb{E}[XY \mid \mathcal{G}]$ ,  $Y$ , and  $\mathbb{E}[X \mid \mathcal{G}]$  are all  $\mathcal{G}$ -measurable by definition. Now suppose  $\mathbb{P}(G) > 0$ , then

$$\begin{aligned} \int_G XY d\mathbb{P} &= \int_G \mathbb{E}(XY \mid \mathcal{G}) d\mathbb{P} \\ &\neq \int_G Y \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} \end{aligned} \quad (5)$$

Now our target is to show it is contradictory. This is a bit tricky, so we start from the simplest case and then generalize it step by step.

- a. Suppose  $Y = \mathbb{I}_A$  for some  $A \in \mathcal{G}$ . Then

$$\int_G XY d\mathbb{P} = \int_{G \cap A} X d\mathbb{P} \quad (6)$$

and

$$\begin{aligned} \int_G Y \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} &= \int_{G \cap A} \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} \\ &= \int_{G \cap A} X d\mathbb{P} \end{aligned} \quad (7)$$

Hence it holds that  $\int_G XY d\mathbb{P} = \int_G Y \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P}$ .

- b. Suppose  $Y$  is non-negative and let  $\{Y_n\}$  be sequence of non-negative simple functions converging to  $Y$  from below. Then by linearity, it holds that

$$\int_G X^+ Y_n d\mathbb{P} = \int_G Y_n \mathbb{E}[X^+ \mid \mathcal{G}] d\mathbb{P} \quad (8)$$

and

$$\int_G X^- Y_n d\mathbb{P} = \int_G Y_n \mathbb{E}[X^- | \mathcal{G}] d\mathbb{P} \quad (9)$$

Applying the monotone convergence we end up with

$$\int_G X^+ Y d\mathbb{P} = \int_G Y \mathbb{E}[X^+ | \mathcal{G}] d\mathbb{P} \quad (10)$$

and

$$\int_G X^- Y d\mathbb{P} = \int_G Y \mathbb{E}[X^- | \mathcal{G}] d\mathbb{P} \quad (11)$$

Hence,

$$\begin{aligned} \int_G XY d\mathbb{P} &= \int_G X^+ Y d\mathbb{P} - \int_G X^- Y d\mathbb{P} \\ &= \int_G Y (\mathbb{E}[X^+ | \mathcal{G}] - \mathbb{E}[X^- | \mathcal{G}]) d\mathbb{P} \\ &= \int_G Y \mathbb{E}[X^+ - X^- | \mathcal{G}] d\mathbb{P} \\ &= \int_G Y \mathbb{E}[X | \mathcal{G}] d\mathbb{P} \end{aligned} \quad (12)$$

c. Finally, for arbitrary  $Y$ , we can separate  $Y = Y^+ - Y^-$  and the contradiction still holds by linearity of expectation.

Therefore, in any case Eq.5 is contradictory. So  $\mathbb{P}(G) = 0$ , and  $\mathbb{E}[XY | \mathcal{G}] = Y \mathbb{E}[X | \mathcal{G}]$  a.s.

(5) Let  $G = \{\omega : \mathbb{E}[X | \mathcal{G}_1](\omega) \neq \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1](\omega)\}$ . Then  $G \in \mathcal{G}_1$  since both  $\mathbb{E}[X | \mathcal{G}_1]$  and  $\mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1]$  are  $\mathcal{G}_1$ -measurable by definition. Now suppose  $\mathbb{P}(G) > 0$ , then

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}(X | \mathcal{G}_1) d\mathbb{P} \\ &\neq \int_G \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1] d\mathbb{P} \\ &= \int_G \mathbb{E}(X | \mathcal{G}_2) d\mathbb{P} \\ &= \int_G X d\mathbb{P} \end{aligned} \quad (13)$$

The last equality stands since  $G \in \mathcal{G}_1$  and  $\mathcal{G}_1 \subset \mathcal{G}_2$ , which suggests  $G \in \mathcal{G}_2$ . Now we can find it contradictory. Therefore  $\mathbb{P}(G) = 0$ , and  $\mathbb{E}[X | \mathcal{G}_1] = \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1]$  a.s.

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}(X | \mathcal{G}_1) d\mathbb{P} \\ &\neq \int_G \mathbb{E}[\mathbb{E}[X | \mathcal{G}_2] | \mathcal{G}_1] d\mathbb{P} \\ &= \int_G \mathbb{E}(X | \mathcal{G}_2) d\mathbb{P} \\ &= \int_G X d\mathbb{P} \end{aligned} \quad (14)$$

- (6) Let  $G = \{\omega : \mathbb{E}[X \mid \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)](\omega) \neq \mathbb{E}[X \mid \mathcal{G}_1](\omega)\}$ . Notice that  $\mathbb{E}[X \mid \mathcal{G}_1]$  is not only  $\mathcal{G}_1$ -measurable but also  $\sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$ -measurable. Thus we have  $G \in \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$ . Now suppose  $\mathbb{P}(G) > 0$ , then

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}[X \mid \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)] d\mathbb{P} \\ &\neq \int_G \mathbb{E}[X \mid \mathcal{G}_1] d\mathbb{P} \end{aligned} \quad (15)$$

To show it is contradictory, we want to prove that  $\forall G \in \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$ ,

$$\int_G X d\mathbb{P} = \int_G \mathbb{E}[X \mid \mathcal{G}_1] d\mathbb{P} \quad (16)$$

The following techniques are closely related to ‘Dynkin system’, which is beyond my knowledge. The main idea is that if we assume  $X$  is non-negative, which can be generalized by linearity, it is enough to establish Eq.16 for some  $\pi$ -system that generates  $\sigma(\mathcal{G}_1 \cup \mathcal{G}_2)$ .

One possibility is  $\mathcal{H} = \{G_1 \cap G_2 : G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2\}$ . Then,  $\forall G_1 \cap G_2 \in \mathcal{H}$ ,

$$\begin{aligned} \int_{G_1 \cap G_2} \mathbb{E}[X \mid \mathcal{G}_1] d\mathbb{P} &= \int_{\Omega} \mathbb{E}[X \mid \mathcal{G}_1] \mathbb{I}_{G_1} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{\Omega} \mathbb{E}[X \mid \mathcal{G}_1] \mathbb{I}_{G_1} d\mathbb{P} \int_{\Omega} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{\Omega} X \mathbb{I}_{G_1} d\mathbb{P} \int_{\Omega} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{\Omega} X \mathbb{I}_{G_1} \mathbb{I}_{G_2} d\mathbb{P} \\ &= \int_{G_1 \cap G_2} X d\mathbb{P} \end{aligned} \quad (17)$$

where the second and fourth equality holds due to independence between  $\sigma(X)$  and  $\mathcal{G}_2$  given  $\mathcal{G}_1$ .

Hence, we find it contradictory. So  $\mathbb{P}(G) = 0$  and  $\mathbb{E}[X \mid \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)] = \mathbb{E}[X \mid \mathcal{G}_1]$  a.s.

- (7) Let  $G = \{\omega : \mathbb{E}[X \mid \mathcal{G}](\omega) \neq \mathbb{E}[X]\}$ . Then  $G \in \mathcal{G}$  since  $\mathbb{E}[X \mid \mathcal{G}]$  is  $\mathcal{G}$ -measurable by definition. And because  $\mathcal{G}$  is trivial,  $G = \emptyset$  or  $G = \Omega$ .

- a. If  $G = \emptyset$ ,  $P(G) = 0$  for sure.
- b. If  $G = \Omega$ , which suggests  $\mathbb{E}[X \mid \mathcal{G}] \neq \mathbb{E}[X]$  always holds, we have

$$\begin{aligned} \int_G X d\mathbb{P} &= \int_G \mathbb{E}[X \mid \mathcal{G}] d\mathbb{P} \\ &\neq \int_G \mathbb{E}[X] d\mathbb{P} \\ &= \int_{\Omega} \mathbb{E}[X] d\mathbb{P} \\ &= \mathbb{E}[X] \end{aligned} \quad (18)$$

which is obviously contradictory since  $\int_G X d\mathbb{P} = \int_{\Omega} X d\mathbb{P} = \mathbb{E}[X]$ .

Therefore,  $P(G) = 0$  and hence  $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$  a.s.

# Chapter 3 Stochastic Processes and Markov Chains

## 3.1

- (a) On  $([0, 1], \mathcal{B}, \lambda)$ , for any  $x \in [0, 1]$

Let  $F_1(x), F_2(x), F_3(x), \dots$  be the binary expansion of  $x$ .

$$F_t(x) = \begin{cases} 1, & A \\ 0, & \bar{A} \end{cases} \quad (\bar{A} \text{ is the opposite case of } A)$$

$F_t(x)$  is Bernoulli random variable.

- (b) 
$$\begin{cases} F_1 = 0 : 0 \leq x < 0.5 \\ F_1 = 1 : 0.5 \leq x < 1 \end{cases}$$
  

$$\begin{cases} F_2 = 0 : 0 \leq x' < 0.5 \\ F_2 = 1 : 0.5 \leq x' < 1 \end{cases} \quad x' = 2x - 1$$
  

$$\dots$$
  

$$\begin{cases} F_t = 0 : 0 \leq x^t < 0.5 \Rightarrow \mathbb{P}(F_t = 0) = \frac{1}{2} \\ F_t = 1 : 0.5 \leq x^t < 1 \Rightarrow \mathbb{P}(F_t = 1) = \frac{1}{2} \end{cases}$$

- (c) It is obviously that  $(F_t)_{t=1}^\infty$  are independent. It satisfies independent equation:  $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ .
- (d)  $(X_{m,t})_{t=1}^\infty$  is a subsequence of  $(F_t)_{t=1}^\infty$  and  $(X_{m,t})_{t=1}^\infty$  are mutually exclusive.
- (e) Such as (d).
- (f) Such as (d).

## 3.2

- (a)  $S_t = \sum_{s=1}^t X_s 2^{s-1}$

$X_t$  is a  $\mathcal{F}$ -adapted martingale.

(1)  $\mathbb{E}[X_t | \mathcal{F}_{t-1}] = X_{t-1}$ .

(2)  $X_t$  is integrable  $\Rightarrow S_t$  is integrable.

$$\begin{aligned} \mathbb{E}[S_t | \mathcal{F}_{t-1}] &= \mathbb{E}[S_{t-1} + X_t 2^{t-1} | \mathcal{F}_{t-1}] \\ &= S_{t-1} + \mathbb{E}[X_t 2^{t-1} | \mathcal{F}_{t-1}] \\ &= S_{t-1} + 2^t \times (1) \times \frac{1}{2} + 2^t \times (-1) \times \frac{1}{2} \\ &= S_{t-1} \end{aligned}$$

$$\Rightarrow (S_t)_{t=1}^\infty$$

- (b)  $t=1$  , if  $S_t \neq 1 \Rightarrow X_1 = -1, S_t = -1$   
 $t=2$  , if  $S_t \neq 1 \Rightarrow X_1 = -1, S_t = -3$   
 $t=3$  , if  $S_t \neq 1 \Rightarrow X_1 = -1, S_t = -7$

...

If avoid  $S_t=1$  , the  $X_s$  sequence must be  $-1$ .

$$\tau = \min\{t : S_t = 1\} = \min\{t : X_T = 1\}$$

$$\Rightarrow \mathbb{P}(\tau < n) = 1 - \mathbb{P}(\tau \geq n) = 1 - \frac{1}{2^n}$$

$$\Rightarrow \mathbb{P}(\tau < \infty) = 1 - \lim_{n \rightarrow \infty} \mathbb{P}(\tau \geq n) = 1 - \frac{1}{2^n} = 1 - \lim_{n \rightarrow \infty} \frac{1}{2^n}$$

- (c) If  $t=\tau$  , then  $S_t=1$  , so  $S_\tau \equiv 1$

$$\Rightarrow \mathbb{E}[S_\tau] = 1$$

- (d) Doob's(a)can be proved by 3.2(b)

$$\tau = 1 \Rightarrow X_1 = 1 \Rightarrow \mathbb{P}(\tau = 1) = \frac{1}{2}$$

$$\tau = 2 \Rightarrow X_1 = -1, X_2 = 1 \Rightarrow \mathbb{P}(\tau = 1) = \frac{1}{4}$$

$$\tau = 3 \Rightarrow X_1 = -1, X_2 = -1, X_3 = 1 \Rightarrow \mathbb{P}(\tau = 1) = \frac{1}{8}$$

...

$$\mathbb{P}(\tau < \infty) = \mathbb{P}(\tau = 1) + \mathbb{P}(\tau = 2) + \mathbb{P}(\tau = 3) + \dots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$

because of  $n \neq \infty, \mathbb{P}(\tau = n) = \frac{1}{n^2} \neq 0$ .

Doob's(b)(c)can also be proved by 3.2(b)

$$t=1 \text{ , if } S_t \neq 1 \Rightarrow X_1 = -1, S_t = -1$$

$$t=2 \text{ , if } S_t \neq 1 \Rightarrow X_1 = -1, S_t = -3$$

$$t=3 \text{ , if } S_t \neq 1 \Rightarrow X_1 = -1, S_t = -7$$

...

It can be concluded that  $-S_t-$  and  $-S_{t-1}-$  can not be bounded, so  $\mathbb{E}[|X_{t+1}|\mathcal{F}]$  and  $|4X_{t \wedge \tau}|$  can not be bounded neither.

**3.4** If  $X_t \geq 0$  is dropped,  $\mathbb{E}[X_\tau | \{\tau \leq n\}] \geq \mathbb{E}[\varepsilon | \{\tau \leq n\}]$  not always true.



# Chapter 4 Stochastic Bandits

4.1 By definition

$$\begin{aligned}
 R_n(\pi, v) &= n\mu^*(v) - \mathbb{E}\left[\sum_{t=1}^n X_t\right] \\
 &= \sum_{t=1}^n \mu^*(v) - \sum_{t=1}^n \mathbb{E}[X_t] \\
 &= \sum_{t=1}^n [\mu^* - \mu_{A_t}]
 \end{aligned}$$

(a)  $\mu^* = \max_a \mu_a \geq \mu_{A_t} \Rightarrow R_n(\pi, v) = \sum_{t=1}^n [\mu^* - \mu_{A_t}] \geq 0.$

(b) If  $\pi$  choose  $A_t \in \arg \max_a \mu_a$  for all  $t \in [n] \Rightarrow \sum_{t=1}^n [\mu^* - \mu_{A_t}] = 0.$

(c) If  $R_n(\pi, v) = 0$  for some policy  $\pi$ , then  $A_t \in \arg \max_a \mu_a \Rightarrow \mathbb{P}(\mu_{A_t} = \mu^*) = 1.$

4.3 Denote  $h_t = a_1, x_1, \dots, a_t, x_t.$

(a) According to the definition of conditional probability and marginal distribution, we have

$$\begin{aligned}
 p_{v\pi}(a_n \mid h_{n-1}) &= \frac{p_{v\pi}(h_{n-1}, a_n)}{p_{v\pi}(h_{n-1})} \\
 &= \frac{\int_{\mathbb{R}} p_{v\pi}(h_n) dx_n}{p_{v\pi}(h_{n-1})} \\
 &= \frac{\int_{\mathbb{R}} \prod_{t=1}^n \pi(a_t \mid h_{t-1}) p_{a_t}(x_t) dx_n}{p_{v\pi}(h_{n-1})} \\
 &= \frac{\prod_{t=1}^{n-1} \pi(a_t \mid h_{t-1}) p_{a_t}(x_t)}{p_{v\pi}(h_{n-1})} \int_{\mathbb{R}} \pi(a_n \mid h_{n-1}) p_{a_n}(x_n) dx_n \\
 &= \pi(a_n \mid h_{n-1}) \int_{\mathbb{R}} p_{a_n}(x_n) dx_n \\
 &= \pi(a_n \mid h_{n-1})
 \end{aligned}$$

(b) According to the definition of conditional probability and marginal distribution, we have

$$\begin{aligned}
p_{v\pi}(x_n \mid h_{n-1}, a_n) &= \frac{p_{v\pi}(h_n)}{p_{v\pi}(h_{n-1}, a_n)} \\
&= \frac{p_{v\pi}(h_n)}{\int_{\mathbb{R}} p_{v\pi}(h_n) dx_n} \\
&= \frac{p_{v\pi}(h_n)}{\int_{\mathbb{R}} [\prod_{t=1}^n \pi(a_t \mid h_{t-1}) p_{a_t}(x_t)] dx_n} \\
&= \frac{p_{v\pi}(h_n)}{\prod_{t=1}^{n-1} \pi(a_t \mid h_{t-1}) p_{a_t}(x_t)} \frac{1}{\int_{\mathbb{R}} \pi(a_n \mid h_{n-1}) p_{a_n}(x_n) dx_n} \\
&= \pi(a_n \mid h_{n-1}) p_{a_n}(x_n) \frac{1}{\pi(a_n \mid h_{n-1})} \\
&= p_{a_n}(x_n)
\end{aligned}$$

**4.4** Denote  $h_t = a_1, x_1, \dots, a_t, x_t$ . The policy that mixes the policies can be defined as

$$\pi_t^\circ(a_t \mid h_{t-1}) = \frac{\sum_{\pi \in \Pi} p(\pi) \prod_{s=1}^t \pi_s(a_s \mid h_{s-1})}{\sum_{\pi \in \Pi} p(\pi) \prod_{s=1}^{t-1} \pi_s(a_s \mid h_{s-1})}$$

By the definition of the canonical probability space and the product of probability kernels,

$$\begin{aligned}
\mathbb{P}_{v\pi^\circ}(B) &= \sum_{a_1=1}^k \int_{\mathbb{R}} \cdots \sum_{a_n=1}^k \int_{\mathbb{R}} \mathbb{I}_B(h_n) v_{a_n}(dx_n) \pi_n^\circ(a_n \mid h_{n-1}) \cdots v_{a_1}(dx_1) \pi_1^\circ(a_1) \\
&= \sum_{\pi \in \Pi} p(\pi) \sum_{a_1=1}^k \int_{\mathbb{R}} \cdots \sum_{a_n=1}^k \int_{\mathbb{R}} \mathbb{I}_B(h_n) v_{a_n}(dx_n) \pi_n(a_n \mid h_{n-1}) \cdots v_{a_1}(dx_1) \pi_1(a_1) \\
&= \sum_{\pi \in \Pi} p(\pi) \mathbb{P}_{v\pi}(B),
\end{aligned}$$

where the second equality follows by substituting the definition of  $\pi_n^\circ$  and induction.

# Chapter 5 Concentration of Measure

## 5.1

$$V(\hat{\mu}) = E((\hat{\mu} - \mu)^2) = E\left(\left(\frac{1}{n} \sum_{t=1}^n X_t - \mu\right)^2\right) = E\left(\frac{1}{n^2} \sum_{t=1}^n (X_t - \mu)^2\right) = \frac{1}{n^2} \sum_{t=1}^n E(X_t - \mu)^2 = \frac{1}{n^2} \sum_{t=1}^n \sigma^2 = \frac{\sigma^2}{n} \quad (19)$$

## 5.4

(a)

$$P(|X| \geq \varepsilon) = P(X \geq \varepsilon)I\{X \geq 0\} + P(X \leq -\varepsilon)I\{X < 0\} = \int_{\varepsilon}^{\infty} \frac{x}{2} \exp\left\{-\frac{x^2}{2}\right\} dx + \int_{-\infty}^{-\varepsilon} \frac{-x}{2} \exp\left\{-\frac{x^2}{2}\right\} dx \quad (20)$$

Calculate the above formula and get the result ,

$$P(|X| \geq \varepsilon) = \frac{1}{2} \exp\left\{-\frac{\varepsilon^2}{2}\right\} + \frac{1}{2} \exp\left\{-\frac{\varepsilon^2}{2}\right\} \\ = \exp\left\{-\frac{\varepsilon^2}{2}\right\}$$

(b)

Let's start with a lemma:

If  $X$  is  $\sigma$ -subgaussian, then  $P(|X| > t) \leq \exp\{-b\varepsilon^2\}$  , where  $b = \exp\{-\sigma^2\}$

The proof of lemma is omitted.

It can be seen from the first question ,  $P(|X| \geq \varepsilon) = \exp\left\{-\frac{\varepsilon^2}{2}\right\}$

The comparison of the two formulas shows that ,  $0 < b \leq 1/2$  . That is,  $\sigma \geq \sqrt{\ln 2}$

By topic condition ,  $\sigma = \sqrt{2 - \varepsilon}$

Hence ,  $\varepsilon \leq 2 - \ln 2$  , this is in contradiction with the arbitrariness of  $\varepsilon$

## 5.7

(a) If  $X$  is  $\sigma$ -subgaussian , then  $E(X) = 0, E(X^2) \leq \sigma^2$

proof:

$$E(e^{\lambda X}) = \sum_{n=0}^{\infty} \frac{\lambda^n E(X^n)}{n!} = 1 + \lambda E(X) + \frac{\lambda^2 E(X^2)}{2} + O(\lambda^2) \quad (21)$$

By definition ,

$$E(e^{\lambda X}) \leq e^{\frac{\lambda^2 \sigma^2}{2}} = 1 + \frac{\lambda^2 \sigma^2}{2} + O(\lambda^2) \quad (22)$$

By comparing the above two formulas and discussing the case that  $\lambda$  approaches to 0 from above and below 0, we get the conclusion that ,

$$E(X) = 0, E(X^2) \leq \sigma^2$$

(b)

If  $X$  is  $\sigma$ -subgaussian, then  $E(X) = 0, E(X^2) \leq \sigma^2$ .

$$\begin{aligned} E(e^{c\lambda x}) &= 1 + \lambda E(cx) + \frac{\lambda^2 E(c^2 x^2)}{2} + O(\lambda^2) \\ &\leq 1 + c\lambda E(x) + \frac{\lambda^2 c^2}{2} E(x^2) + O(\lambda^2) \\ &\leq 1 + \frac{\lambda^2 c^2 \sigma^2}{2} + O(\lambda^2) \\ &\leq e^{\frac{\lambda^2 c^2 \sigma^2}{2}} \end{aligned}$$

Hence,  $cX$  is  $|c|\sigma$ -subgaussian.

(c)

If  $X_1$  is  $\sigma_1$ -subgaussian,  $X_2$  is  $\sigma_2$ -subgaussian

$$\begin{aligned} \text{then } E(X_1) &= 0, E(X_1^2) \leq \sigma_1^2, E(X_2) = 0, E(X_2^2) \leq \sigma_2^2 \\ E(e^{\lambda(x_1+x_2)}) &= 1 + \lambda E(x_1+x_2) + \frac{\lambda^2 E((x_1+x_2)^2)}{2} + O(\lambda^2) \\ &= 1 + \frac{\lambda^2}{2} \text{Var}(x_1+x_2) + O(\lambda^2) \\ &= 1 + \frac{\lambda^2}{2} (\text{var}(x_1) + \text{var}(x_2) + 2\text{cov}(x_1, x_2)) + O(\lambda^2) \\ \text{Because } x_1, x_2 &\text{ are independent,} \\ &= 1 + \frac{\lambda^2}{2} (E(x_1^2) + E(x_2^2)) + O(\lambda^2) \\ &\leq 1 + \frac{\lambda^2}{2} (\sigma_1^2 + \sigma_2^2) + O(\lambda^2) \\ &\leq e^{\frac{\lambda^2 (\sigma_1^2 + \sigma_2^2)}{2}} \end{aligned}$$

Hence,  $X_1 + X_2$  is  $\sqrt{\sigma_1^2 + \sigma_2^2}$ -subgaussian.

### 5.11

(a)

$$E(e^{\lambda X}) = 1 + \lambda E(X) + \frac{\lambda^2 E(X^2)}{2} + O(\lambda^2) = 1 + \frac{\lambda^2 E(X^2)}{2} + O(\lambda^2) \quad (23)$$

If the conclusion is true, then the above formula satisfies

$$\leq 1 + \frac{\lambda^2}{2} \left( \frac{(b-a)^2}{4} \right) + O(\lambda^2)$$

So just prove:

$$E(x^2) \leq \left( \frac{b-a}{2} \right)^2$$

$$E(x^2) = \text{var}(x) = E(x - \bar{x})^2$$

However,  $(x - \bar{x})^2 \leq \left( \frac{b-a}{2} \right)^2$ . The conclusion is proved.

(b)

The proof of Hoeffding's Inequality:

Let  $X_i = Z_i - E(Z_i)$ ,  $\bar{X} = \frac{1}{m} \sum_{i=1}^m X_i$

By Markov inequality, for all  $\lambda > 0$ ,  $\varepsilon > 0$ ,

$$P(\bar{X} \geq \varepsilon) = P(e^{\lambda \bar{X}} \geq e^{\lambda \varepsilon}) \leq \frac{E(e^{\lambda \bar{X}})}{e^{\lambda \varepsilon}}$$

$Z_1, \dots, Z_m$  iid.r.v.

$$\text{So, } E(e^{\lambda \bar{X}}) = \prod_{i=1}^m E(e^{\frac{\lambda X_i}{m}})$$

By Hoeffding's lemma,

$$E(e^{\frac{\lambda X_i}{m}}) \leq e^{\frac{\lambda^2 (b-a)^2}{8m^2}}$$

$$\text{So, } P(\bar{X} \geq \varepsilon) \leq e^{-\lambda \varepsilon} \prod_{i=1}^m E(e^{\frac{\lambda X_i}{m}})$$

$$\leq e^{-\lambda \varepsilon} e^{\frac{\lambda^2 (b-a)^2}{8m}}$$

$$\leq e^{-\lambda \varepsilon + \frac{\lambda^2 (b-a)^2}{8m}}$$

$$\text{Let } \lambda = \frac{4m\varepsilon}{(b-a)^2}, \text{ then } P(\bar{X} \geq \varepsilon) \leq e^{\frac{-2m\varepsilon^2}{(b-a)^2}}$$

Similarly, we can prove the other side of the inequality.

**5.16** By assumption  $Pr(X_t \leq x) \leq x$ , which means that for  $\lambda < 1$ ,

$$\mathbb{E} \left[ \exp(\lambda \log(\frac{1}{x_t})) \right] = \int_0^\infty P(\exp(\lambda \log(\frac{1}{x_t})) \geq x) dx = 1 + \int_1^\infty P(X_t \leq x^{-\frac{1}{\lambda}}) dx \quad (24)$$

Applying the Cramer-Chernoff method,

$$P \left( \sum_{t=1}^n \log(\frac{1}{X_t}) \geq \epsilon \right) = P \left( \exp(\lambda \sum_{t=1}^n \log(\frac{1}{X_t})) \geq \exp(\lambda \epsilon) \right) \leq \left( \frac{1}{1-\lambda} \right)^n \exp(-\lambda \epsilon)$$

choosing  $\lambda = \frac{\epsilon - n}{\epsilon}$  completes the claim.

**5.18(a)** Let  $\lambda > 0$ . Then,

$$\exp(\lambda \mathbb{E}[Z]) \leq \mathbb{E}[\exp(\lambda Z)] \leq \sum_{t=1}^n \mathbb{E}[\exp(\lambda X_t)] \leq n \exp\left(\frac{\lambda^2 \sigma^2}{2}\right)$$

Rearranging shows that,

$$\mathbb{E}(Z) \leq \frac{\log(n)}{\lambda} + \frac{\lambda \sigma^2}{2}$$

Choosing  $\lambda = \frac{1}{\sigma} \sqrt{2 \log(n)}$  shows that  $\mathbb{E}(Z) \leq \sqrt{2 \sigma^2 \log(n)}$



# Chapter 11 The Exp3 Algorithm

**11.2** Let  $\pi$  be a deterministic policy, and we define  $x_{ti} = 0$  if  $A_t = i$  otherwise  $x_{ti} = 1$ . The deterministic policy collects zero rewards all time,

$$\max_{i \in [k]} \sum_{t=1}^n x_{ti} \geq \frac{1}{k} \sum_{t=1}^n \sum_{i=1}^k x_{ti} = \frac{n(k-1)}{k}$$

**11.5** Let  $P$  be a probability vector with nonzero components and let  $A \sim P$ . Suppose  $\hat{X}$  is a function such that for all  $x \in \mathbb{R}^k$ ,

$$\mathbb{E} [\hat{X}(A, x_A)] = \sum_{i=1}^k P_i \hat{X}(i, x_i) = x_1$$

Show that there exists an  $a \in \mathbb{R}^k$  such that  $\langle a, P \rangle = 0$  and for all  $i$  and  $z$  in their respective domains,  $\hat{X}(i, z) = a_i + \frac{\mathbb{I}\{i=1\}z}{P_1}$

*Proof.* Let  $x, x'$  be arbitrary but agree on the first component  $x_1 = x'_1$ . Let  $f(x) = \sum_{i=1}^k P_i \hat{X}(i, x_i)$  Note that,

$$0 = f(x) - f(x') = \sum_{i=j}^k P_j \hat{X}(j, x_j)$$

for all  $j > 1$ . Since  $x, x'$  are arbitrary,  $\hat{X}(j, \cdot) = \text{const.}$  Let  $a_j$  equal to  $\hat{X}(j, \cdot)$ .

Further, let  $a_1 = \hat{X}(1, 0)$  and then given any  $x_1 \in \mathbb{R}$ ,  $\hat{X}(1, x_1) = a_1 + x_1/P_1$ .

Finally, let  $x$  be such that  $x_1 = 0$ . Then  $0 = f(x) = \sum_i P_i a_i$ . □

**11.7** First, note that if  $G = -\log(-\log(U))$  then  $\mathbb{P}(G \leq g) = e^{-\exp(-g)}$ .

$$\begin{aligned} \mathbb{P} \left( \log a_i + G_i \geq \max_{j \in [k]} \log a_j + G_j \right) &= \mathbb{E} \left[ \prod_{j \neq i} \mathbb{P}(\log a_j + G_j \leq \log a_i + G_i \mid G_i) \right] \\ &= \mathbb{E} \left[ \prod_{j \neq i} \exp \left( -\frac{a_j}{a_i} \exp(-G_i) \right) \right] \\ &= \mathbb{E} \left[ U_i^{\sum_{j \neq i} \frac{a_j}{a_i}} \right] \\ &= \frac{1}{1 + \sum_{j \neq i} \frac{a_j}{a_i}} \\ &= \frac{a_i}{\sum_{j=1}^k a_j} \end{aligned}$$

**11.8** Let  $Z_{ti}$  be a standard Gambel. The follow-the-perturbed-leader algorithm chooses

$$A_t = \operatorname{argmax}_{i \in [k]} \left( Z_{ti} - \eta \sum_{s=1}^{t-1} \hat{Y}_{si} \right)$$

is the same as EXP3. Given (11.7)

$$\mathbb{P} \left( \log(a_i) + G_i = \max_{j \in [k]} (\log(a_j) + G_j) \right) = \frac{a_i}{\sum_{j=1}^k a_j}$$

Just simply take  $a_i$  as  $-\eta \sum_{s=1}^{t-1} \hat{Y}_{si}$ , then the form is identical.



# Chapter 18

## 18.1

(a) By Jensen's inequality,

$$\begin{aligned}\sum_{c \in \mathcal{C}} \sqrt{\sum_{t=1}^n \mathcal{I}\{c_t = c\}} &= \|C\| \sum_{c \in \mathcal{C}} \frac{1}{\|C\|} \sqrt{\sum_{t=1}^n \mathcal{I}\{c_t = c\}} \\ &\leq \|C\| \sqrt{\sum_{c \in \mathcal{C}} \frac{1}{\|C\|} \sum_{t=1}^n \mathcal{I}\{c_t = c\}} \\ &= \sqrt{\|C\|n}\end{aligned}$$

(b) When each context occurs  $\frac{n}{\|\mathcal{C}\|}$  times we have

$$\sum_{c \in \mathcal{C}} \sqrt{\sum_{t=1}^n \mathcal{I}\{c_t = c\}} = \sqrt{n\|C\|}$$