Ex 4.1 Prove Lemma 4.4

Let ν be a stochastic bandit environment, then

(1) $R_n(\pi, \nu) \geq 0$ for all policy π

(2) The policy π choosing $A_t \in \arg\max_a \mu_a$ for all t statisfies $R_n(\pi, \nu) = 0$

(3) If $R_n(\pi, \nu) = 0$ for some policy π, then $P(\mu_{A_t} = \mu^*) = 1$ for all $t \in [n]$.

Proof: By definition

$$R_n(\pi, \nu) = n\mu^*(\nu) - E[\sum_{t=1}^n X_t]$$

$$= \sum_{t=1}^n \mu^*(\nu) - \sum_{t=1}^n \underbrace{E[X_t]}_{\Rightarrow \mu_{A_t}}$$

$$= \sum_{t=1}^n [\mu^* - \mu_{A_t}]$$

~~$\mu^* = \arg\max$~~ $\boxed{\mu^* = \max_{a \in A} \mu_a}$

(1) $\mu^* = \max_{a \in A} \mu_a \geq \mu_{A_t} \Rightarrow R_n(\pi, \nu) = \sum_{t=1}^n [\mu^* - \mu_{A_t}] \geq 0$

(2) If π choose $A_t \in \arg\max_a \mu_a$ for all $t \in [n]$

$\Downarrow$

$\mu_{A_t} = \mu^* \Rightarrow R_n(\pi, \nu) = \sum_{t=1}^n (\mu^* - \mu_{A_t}) = 0$

(3) If $R_n(\pi, \nu) = 0$; for $t \in [n]$ $\mu^* = \mu_{A_t} \Rightarrow P(\mu_{A_t} = \mu^*) = 1$ for $t \in [n]$

Ex 2.5

$\mathcal{G} \subseteq 2^\Omega$ define $\sigma(\mathcal{G})$ as the smallest σ-algebra that containing $\mathcal{G}$

Smallest means: $\mathcal{F} \in 2^\Omega$ is smaller than $\mathcal{F}' \in 2^\Omega$ if $\mathcal{F} \subset \mathcal{F}'$

(a) Show $\sigma(\mathcal{G})$ exists and contains exactly those sets A that are in every σ-algebra that contains $\mathcal{G}$

meaning: exists a ~~to~~ smallest σ-algebra that containing $\mathcal{G}$ and this smaller σ-algebra contains all sharing sets thar are in every σ-algebra that contains $\mathcal{G}$

idea: ① construt a σ-algebra containing $\underset{=}{\overset{\sigma(\mathcal{G})}{\mathcal{G}}}$ & all of other sharing sets in those σ-algebras

② prove this σ-algebra is the smallest ($\forall \mathcal{H}$ containing $\mathcal{G}$, $\mathcal{H} \subset \sigma(\mathcal{G})$)

now proof: $\left\{\begin{array}{l} ① \ \mathcal{G} \in \sigma(\mathcal{G}) \\ ② \text{ all of other sharing set } \in \sigma(\mathcal{G}) \end{array}\right.$ ~~may infinite~~

可能有无限多个，不影响证明

(1) construct $\sigma(\mathcal{G}) = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3 \cap \cdots \mathcal{G}_n$; Let $\mathcal{G}_i$ be σ-algebra containing $\mathcal{G}$

we need to prove $\sigma(\mathcal{G})$ is a σ-algebra $\left\{\begin{array}{l} ① \ \Omega \\ ② \ A \in \sigma(\mathcal{G}),\ A^c \in \sigma(\mathcal{G}) \\ ③ \ A_i \in \sigma(\mathcal{G}) \quad \bigcup A_i \in \sigma(\mathcal{G}) \end{array}\right.$

① $\mathcal{G}_i$ is the σ-algebra. $\Omega \in \mathcal{G}_i$ for $i \in [n]$

$\Omega \in \sigma(\mathcal{G}) = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \cdots \mathcal{G}_n$

② $A \in \sigma(\mathcal{G})$, $A \in \mathcal{G}_1 \cap \mathcal{G}_2 \cap \cdots \mathcal{G}_n$, $A^c \in \sigma(\mathcal{G}) = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3 \cdots \mathcal{G}_n$

③ $A_i \in \sigma(\mathcal{G})$ $A_i \in \mathcal{G}_1 \cap \mathcal{G}_2 \cap \cdots \mathcal{G}_n$. $\bigcup_i A_i \in \sigma(\mathcal{G}) = \mathcal{G}_1 \cap \mathcal{G}_2 \cdots \mathcal{G}_n$

(2) $\sigma(\mathcal{G})$ is the smallest

~~$\forall \mathcal{G}_i$~~ for $\forall$ σ-algebra containing $\mathcal{G}$ $(\mathcal{G}_i)$: $\sigma(\mathcal{G}) \subseteq \mathcal{G}_i$

(b) $(\Omega', \mathcal{F})$ $X: \Omega' \to \Omega$ is $\mathcal{F}/\mathcal{G}$-measurable ; show X is $\mathcal{F}/\sigma(\mathcal{G})$-measurable

$\forall A \in \mathcal{G}$, $X^{-1}(A) \in \mathcal{F}$; ~~construct~~ $\forall A \in \underbrace{\sigma(\mathcal{G})}_{\mathcal{H}}$, $X^{-1}(A) \in \mathcal{F}$

idea: construct σ-algebra $\mathcal{H}$, $\forall A \in \mathcal{H}$, $X^{-1}(A) \in \mathcal{F}$; prove $\sigma(\mathcal{G}) \subseteq \mathcal{H}$ → contains $\mathcal{G}$

$\sigma(X) = \{X^{-1}(A) : A \in \mathcal{G}\}$ | $\mathcal{H} = \{A : X^{-1}(A) \in \mathcal{F}\}$ $\mathcal{G} \subseteq \mathcal{H}$, $\sigma(\mathcal{G}) \subseteq \mathcal{H}$

→ by definition $\forall A \in \mathcal{H}$, $X^{-1}(A) \in \mathcal{F}$

$\sigma(\mathcal{G}) \subseteq \mathcal{H} \Rightarrow \forall A \in \sigma(\mathcal{G})\ X^{-1}(A) \in \mathcal{F}$

~~now proof~~ now we need to prove $\mathcal{H}$ is a σ-algebra.

① Ω: $X: \underset{\mathcal{F}}{\Omega'} \to \underset{\mathcal{G}}{\Omega}$ ~~$\mathcal{G}$ is a σ-algebra on Ω $\Omega \in \mathcal{G}$ $\Rightarrow$ $\forall A \in \mathcal{G}, X^{-1}(A) \in \mathcal{F}$~~

~~$\mathcal{F}$ is a σ-algebra on Ω'~~ $\Omega' \in \mathcal{F}$

~~$X^{-1}(\Omega) \in \mathcal{F}$, $\Omega \in \mathcal{H}$~~ $X^{-1}(\Omega) = \Omega' \in \mathcal{F}$ $\therefore \Omega \in \mathcal{H}$

② $(\forall A \in \mathcal{H}.\ A^c \in \mathcal{H})$

$\forall A \in \mathcal{H}$, $X^{-1}(A) \in \mathcal{F}$, $\mathcal{F}$ is a σ-algebra, $X^{-1}(A^c) = (X^{-1}(A))^c \in \mathcal{F}$

$A^c \in \mathcal{H}$

③ $\forall A_i \in \mathcal{H}$, $X^{-1}(A_i) \in \mathcal{F}$. $X^{-1}(\bigcup_i A_i) = \bigcup_i (X^{-1}(A_i)) \in \mathcal{F}$

$\bigcup_i A_i \in \mathcal{H}$

$\mathcal{H}$ is a σ-algebra, ~~$X^{-1}(\mathcal{G}) \in \mathcal{F}$~~ $\mathcal{G} \subseteq \mathcal{H}$ (对应(1)中的 $\mathcal{G}_i$)

$\sigma(\mathcal{G}) \subseteq \mathcal{H}$, $\forall A \in \sigma(\mathcal{G})$, $X^{-1}(A) \in \mathcal{F}$

(c) if $A \in \mathcal{F}$, ($\mathcal{F}$ is a σ-algebra) then $\mathbb{1}\{A\}$ is a $\mathcal{F}$-measurable ($\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable)

$$\mathbb{1}\{A\} = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{else} \end{cases}$$

$\forall B \in \mathcal{B}(\mathbb{R})$, $\mathbb{1}\{A\}^{-1}(B) \in \mathcal{F}$

if $\{0,1\} \in B$, $\mathbb{1}\{A\}^{-1}(B) = \Omega \in \mathcal{F}$

if $\{0\} \in B$, $\mathbb{1}\{A\}^{-1}(B) = A^c \in \mathcal{F}$

if $\{1\} \in B$. $\mathbb{1}\{A\}^{-1}(B) = A \in \mathcal{F}$

if $\{0,1\} \cap B = \emptyset$, $\mathbb{1}\{A\}^{-1}(B) = \emptyset = \Omega^c \in \mathcal{F}$

thus we prove that for $\forall B \in \mathcal{B}(\mathbb{R})$, $\mathbb{1}\{A\}^{-1}(B) \in \mathcal{F}$; $\mathbb{1}\{A\}$ is $\mathcal{F}$-measurable.

$\boxed{\text{Ex 2.6}}$ In the context of lemma 2.5 ; show an example where $Y = X$, Y is not $\sigma(X)$-measurable

Lemma 2.5 Assume $(\mathcal{Y}, \mathcal{H})$ is a Borel space, Then Y is $\sigma(X)$-measurable if and only if there exists a $\mathcal{G}/\mathcal{H}$-measure map $f: \mathcal{X} \to \mathcal{Y}$ such that $Y = f \circ X$

$$\begin{tikzcd} (\Omega, \mathcal{F}) \arrow[r, "X"] \arrow[dr, "Y"'] & (\mathcal{X}, \mathcal{G}) \arrow[d, "f"] \\ & (\underset{\mathbb{R}}{\mathcal{Y}}, \mathcal{H}) \end{tikzcd}$$

$(\sigma(Y) \subseteq \sigma(X))$: $\sigma(X) = \{X^{-1}(A) : A \in \mathcal{G}\}$

$\sigma(Y) = \{X^{-1}(A) : A \in \mathcal{H}\}$ $\quad Y$: $\mathcal{F}/\sigma(Y)$-measurable

solutions: Trivially, $\sigma(X) = \{\emptyset, \mathbb{R}\}$

Y is not $\sigma(X)/\mathcal{B}(\mathbb{R})$-measurable because $Y^{-1}([0,1]) = [0,1] \notin \sigma(X)$

$$\begin{tikzcd} (\Omega,\mathcal{F}) \arrow[r, "X"] \arrow[dr, "Y"'] & (\mathcal{X},\mathcal{G}) \arrow[d, "f"] \\ & (\mathcal{Y},\mathcal{H}) \end{tikzcd}$$

$(\mathcal{Y},\mathcal{H})$ - Borel space

Y is $\sigma(X)$-measureble if and only if there exists a $\mathcal{G}/\mathcal{H}$-measurable map $f: \mathcal{X}\to\mathcal{Y}$ such that $Y = f\circ X$

To prove $Y=X$, Y is not $\sigma(X)$-measurable

we need to prove we couldn't find a $\mathcal{G}/\mathcal{H}$-measureble map $f:\mathcal{X}\to\mathcal{Y}$ such that $Y=f\circ X$

choose
$$\begin{cases} \Omega=\mathcal{X}=\mathcal{Y}=\mathbb{R}\ ; \\ X(\omega)=Y(\omega)=\omega \\ \mathcal{F}=\mathcal{H}=\mathbb{B}(\mathbb{R}) \\ \mathcal{G}=\{\emptyset,\mathbb{R}\} \end{cases}$$

$f: \mathcal{X}\to\mathcal{Y}$ such that $Y=f\circ X$

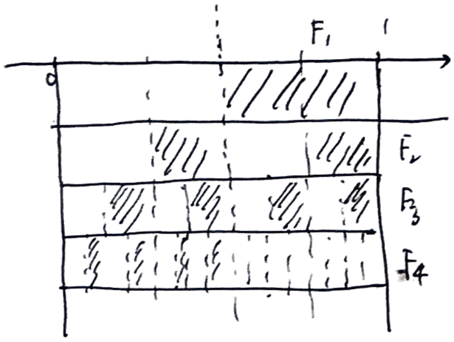
f: $\underline{\mathcal{G}/\mathcal{H}\text{-measurable}}$ $\boxed{f(\omega)=\omega}$

$\forall A\in\mathcal{H},\ f^{-1}(A)\in\mathcal{G}=\{\emptyset,\mathbb{R}\}$

$A=[0,1]\quad f^{-1}([0,1]) = [0,1]\notin\mathcal{G}$

3.1 (a) Prove $F_t \in \{0,1\}$ is a Bernoulli random variable for $t \geq 1$

$F_t(x)$ is the binary expansion of x



$$F_t = \mathbb{1}\{U_t\} = \mathbb{1}\left\{x \in \{1\} \sqcup \bigsqcup_{1 \leq S \leq 2^{t-1}} \left[\frac{2S-1}{2^t}, \frac{2S}{2^t}\right]\right\}$$

$\downarrow$

$\{0,1\}$

F_t: $\mathcal{B}([0,1]) / \mathcal{B}(\mathbb{R})$ - measurable

According to 2.5 (c) $U_t \in \mathcal{B}([0,1])$ $\mathbb{1}\{U_t\}$ is $\mathcal{B}([0,1]) / \mathcal{B}(\mathbb{R})$ - measurable

(b) show for ~~ay~~ any $t \geq 0$, $P(F_t = 0) = P(F_t = 1) = \frac{1}{2}$

(c) $\{F_t\}_{t=1}^{\infty}$ are independent $P(A \cap B) = P(A)P(B)$

看图：公共面积

(d) show that $\{X_{m,t}\}_{t=1}^{\infty}$ is an independent sequence of Bernoully random variable that are uniformly distrib

已经证明 F_t is independent / Bernoulli random / uniform distribution

~~$X_{m,t} =$~~ $X_{m,1} = F_1, F_2, F_4, F_7 \cdots$ $\boxed{X_{m,t} \text{ is a subsequence of } F_t}$

Since satisfies the property of F_t

(e) Show that $X_t = \sum_{t=1}^{\infty} X_{m,t} 2^{-t}$ is uniformly distributed on $[0,1]$

$Y(x) = \sum_{t=1}^{\infty} F_t(x) 2^{-t}$ X_t satisfies the same property of Y $\boxed{Y(x) = x}$

(f) F_t are independent

$X_t = \sum_{t=1}^{\infty} X_{m,t} 2^{-t}$ $\{X_{m,t}\}_{m=1}^{\infty}$ 是 F_t 中不相交的子集合

$\therefore$ ~~X_m~~ X_t 也是 independent

Ex 2.1 – 2.4

(2.1) Show: if f is $\mathcal{F}/\mathcal{G}$-measurable, g is $\mathcal{G}/\mathcal{H}$-measurable for sigmal algebras $\mathcal{F}, \mathcal{G}, \mathcal{H}$

then $(g\circ f)(\omega) = g(f(\omega))$ is $\mathcal{F}/\mathcal{H}$-measurable.

By definition f is $\mathcal{F}/\mathcal{G}$-measurable $\quad \forall A \in \mathcal{G},\ f^{-1}(A) \in \mathcal{F}$

g is $\mathcal{G}/\mathcal{H}$-measurable $\quad \forall B \in \mathcal{H},\ g^{-1}(B) \in \mathcal{G}$

To show $(g\circ f)($ is $\mathcal{F}/\mathcal{H}$-measurable, we need to prove $\forall C \in \mathcal{H},\ (g\circ f)^{-1}(C) \in \mathcal{F}$

$= \cancel{g(f(\omega))}\ f^{-1}(g^{-1}(\omega))$

$\forall C \in \mathcal{H},\ g^{-1}(C) \in \mathcal{G},\ f^{-1}(g^{-1}(C)) \in \mathcal{F}$

$(g\circ f)^{-1}(C) \in \mathcal{F}$. Hence $g\circ f$ is $\mathcal{F}/\mathcal{H}$-measurable.

(2.2) $X_1, X_2 \ldots X_n$ be random variables on $(\Omega, \mathcal{F})$

Prove $(X_1, X_2 \ldots X_n)$ is a random vector.

$(\Omega, \mathcal{F})$

random variable X: $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

$\mathcal{F}/\sigma(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \ldots)$ random vector $X: \Omega \to \mathbb{R}^k$ $\mathcal{F}/\mathcal{B}(\mathbb{R}^k)$-measurable

$\mathcal{F}/\sigma(\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \mathcal{G}_n)$

~~$\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$~~ $\mathcal{F}/\mathcal{G} \Rightarrow \mathcal{F}/\sigma(\mathcal{G})$-measurable

$\mathcal{F}/\mathcal{G}$-measurable $\quad \mathcal{F}/\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \mathcal{G}_n$

$\forall A \in \mathcal{G}_1 \times \mathcal{G}_2 \cdots \mathcal{G}_n \quad A = A_1 \times A_2 \times \cdots A_n \quad A_i \in \mathcal{G}_i$

$X^{-1}(A) = X^{-1}(A_1 \times A_2 \times \cdots A_n)$

$= X^{-1}(A_1) \cap X^{-1}(A_2) \cdots X^{-1}(A_n) \quad A_i \in \mathcal{B}(\mathbb{R}) \quad X^{-1}(A_i) \in \mathcal{F}$

$\in \mathcal{F}$

$X = (X_1, X_2 \cdots X_n)$ is $\mathcal{F}/\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \mathcal{G}_n$-measurble

$\mathcal{F}/\sigma(\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \mathcal{G}_n) \to \mathcal{F}/\mathcal{B}(\mathbb{R}^n)$-measurable.

(2.3) $\mathcal{U} \to \mathcal{V}$

Σ

show $\Sigma_X = \{X^{-1}(A) : A \in \Sigma\}$ is a 6-algebra over $\mathcal{U}$

definition $\begin{cases} ① \ \mathcal{U} \in \Sigma_X \\ ② \ \forall C \in \Sigma_X, \ C^c \in \Sigma_X \\ ③ \ C_i \in \Sigma_X \quad \bigcup_{i=1}^{\infty} C_i \in \Sigma_X \end{cases}$

① $X: \mathcal{U} \to V$ is an arbitary funtion, means

$\forall u \in U, \ X(u) \in V \quad ; \quad X^{-1}(V) = \mathcal{U} \in \Sigma_X$

② $C \in \Sigma_X, \quad \exists B \in \Sigma, \quad X^{-1}(B) = C$

$B^c \in \Sigma, \ C^c = (X^{-1}(B))^c = X^{-1}(B^c) \in \Sigma_X$

③ $C_i \in \Sigma_X, \quad \exists B_i \in \Sigma, \quad X^{-1}(B_i) = C_i$

$\bigcup_i^{\infty} B_i \in \Sigma, \quad \bigcup_{i=1}^{\infty} C_i = \bigcup_i^{\infty} X^{-1}(B_i) = X^{-1}(\bigcup_i^{\infty} B_i) \in \Sigma_X$

(2.4) $(\Omega, \mathcal{F}) \quad A \subseteq \Omega \quad \mathcal{F}_{|A} = \{A \cap B : B \in \mathcal{F}\}$

(a) show $(A, \mathcal{F}_{|A})$ is a measurable space

$\mathcal{F}_{|A}$ is a 6-algebra on A

$\begin{cases} A \in \mathcal{F}_{|A} \\ \forall C \in \mathcal{F}_{|A}, \ C^c \in \mathcal{F}_{|A} \\ C_i \in \mathcal{F}_{|A} \quad \bigcup_i^{\infty} C_i \in \mathcal{F}_{|A} \end{cases}$

$\mathcal{F}$ is a 6-algebra on Ω : $\Omega \in \mathcal{F}$

① $A \cap \Omega = A \in \mathcal{F}_{|A}$

② $A \cap B \in \mathcal{F}_{|A}, \ B \in \mathcal{F}$: $(A \cap B)^{c_A} = A^{c_A} \cup B^{c_A} = \emptyset \cup B^{c_A} = A \cap \underset{\in \mathcal{F}}{B^c} \in \mathcal{F}_{|A}$

③ $A \cap B_i \in \mathcal{F}_{|A}, \ B_i \in \mathcal{F} \to \in \mathcal{F}$

$\bigcup_i^{\infty} (A \cap B_i) = A \cap (\bigcup_i^{\infty} B_i) \in \mathcal{F}_{|A}$

(b) if $A \in \mathcal{F}$, then $\mathcal{F}_{|A} = \{B : B \in \mathcal{F}, B \subseteq A\}$ $A \in \mathcal{F}$

证明2个集合等价 $\quad \overset{F_1}{\{A \cap B : B \in \mathcal{F}\}} \Leftrightarrow \overset{F_2}{\{B : B \in \mathcal{F}, B \subseteq A\}}$

① $\subseteq \quad \forall A \cap B \in F_1 \quad A \in \mathcal{F}, B \in \mathcal{F}$, ~~$B \subseteq A$, $A \cap B = B$~~ $A \cap B \in \mathcal{F}$ ① $A \cap B \subseteq A$ ✓.

$A \cap B \in F_2$

② $\supseteq \quad \forall B \in F_2, \quad B \in \mathcal{F}, \ B \subseteq A$

$B = B \cap A \in F_1$.

$F_1 = F_2$