Chromatic Numbers and Hom Complexes

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Introduction

- In 1852, Francis Guthrie postulated the four color conjecture.
- Guthrie's brother passed on the question to his mathematics teacher Augustus de Morgan at University College, who mentioned it in a letter to William Hamilton in 1852.
- Arthur Cayley raised the problem at a meeting of the London Mathematical Society in 1879.
- The same year, Alfred Kempe published a paper that claimed to establish the result.

Introduction

- In 1890, Heawood pointed out that Kempe's argument was wrong and he proved the five color theorem.
- In 1976, the four color theorem was finally proved by Kenneth Appel and Wolfgang Haken.
- Hadwiger has stated his conjecture (Hadwiger conjecture is equivalent to the four-color theorem when $\chi(G) = 5$) in 1943.
- The Kneser conjecture was posed in 1955 and solved in 1978 by L. Lovász.

Chromatic Numbers

Definition

Let G be a graph. A vertex-coloring of G is a set map

$$c : V(G) \to S$$

such that $(x, y) \in E(G)$ implies $c(x) \neq c(y)$.

A vertex coloring exists if and only if G has no loops.

Chromatic Numbers

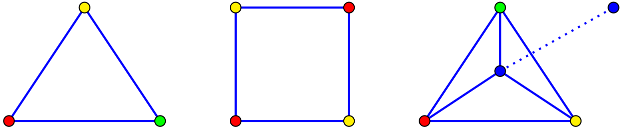
Definition

The chromatic number of G, $\chi(G)$, is the minimal cardinality of a set S such that there exists a vertex-coloring $c : V(G) \to S$.

If G has loops, we use the convention $\chi(G) = \infty$.

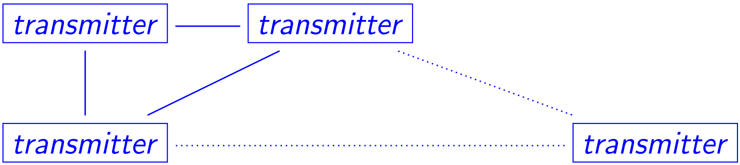
Chromatic Numbers

Example

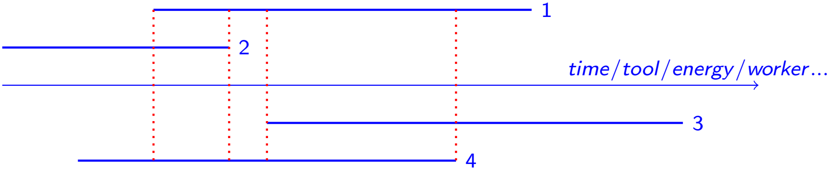


Chromatic Numbers

1. Frequency assignment problem



2. Task scheduling problem



Chromatic Numbers

The problem of computing the chromatic number of a graph is NP-complete.

Two examples are the problem of deciding whether a given planar graph is 3-colorable, and the problem of finding a coloring with 4 colors for a 3-colorable graph.

The Hadwiger Conjecture

Theorem (Euler-Poincaré formula)

Let G be a nonempty finite connected graph, drawn in a plane without self-intersections. Let $R(G)$ denote the set of regions into which the plane is divided. Then we have the following formula:

$$|V(G)| + |R(G)| - |E(G)| = 2.$$

The Hadwiger Conjecture

Example

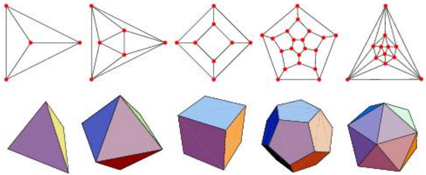


Figure 1: Regular polyhedron

A point connects n edges, we have $nV = 2E$. A face connects m edges, we have $mR = 2E$. Recall that $R + V - E = 2$. We have

The Hadwiger Conjecture

Example

1. A point connects n edges, we have $nV = 2E$.
2. A face connects m edges, we have $mR = 2E$.
3. Recall that $R + V - E = 2$.

We have

$$V = \frac{4n}{2m + 2n - mn}, R = \frac{4m}{2m + 2n - mn}, E = \frac{2mn}{2m + 2n - mn}.$$

Since $2m + 2n - mn > 0$ and $m \geq 3$, we have

$$\frac{1}{n} > \frac{1}{2} - \frac{1}{m} \geq \frac{1}{6}.$$

This shows that $n < 6$.

The Hadwiger Conjecture

Example

- $n = 3$, $m < 6$ implies $m = 3, 4, 5$.
  $E = 4, 8, 12$.
- $n = 4$, $m < 4$ implies $m = 3$.
  $E = 6$.
- $n = 3$, $m \leq 3$ implies $m = 3$.
  $E = 20$.

The Hadwiger Conjecture

Example

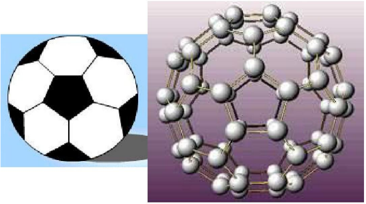


Figure 2: Buckyballs C_{60}

The Hadwiger Conjecture

The question of computing $\chi(G)$ for the planar graph G has a long history. The question was formulated in 1852 by F. Guthrie. The first time this question appeared in print was in a paper by Cayley, after which it became known as the four-color problem, one of the most famous questions in graph theory, as well as a popular brain-teaser.

The Hadwiger Conjecture

Theorem (Five-color theorem)

Every loopless planar graph is five-colorable.

Theorem (Four-color theorem)

Every loopless planar graph is four-colorable.

The Hadwiger Conjecture

The four-color theorem was reduced to the analysis of the large but finite set of "unavoidable" configurations. Following that, the original conjecture was proved 1976 by Appel & Haken, using extensive computer computations.

A new, shorter, and more structural proof (though still relying on computers) was obtained only rather recently, in 1997, by Robertson, Sanders, Seymour & Thomas.

The Hadwiger Conjecture

In 1943, Hadwiger stated a conjecture closely related to the four-color theorem. A graph H is called a minor of another graph G if H can be obtained from a subgraph of G by a sequence of edge-contractions.

The Hadwiger Conjecture

Hadwiger Conjecture

For every positive integer t, if a loopless graph has no K_{t+1} minor, then it has a t-coloring, in other words, every graph G has $K_{\chi(G)}$ as its minor.

It was shown in 1937 by Wagner, that the case $\chi(G) = 5$ of the Hadwiger conjecture is equivalent to the four-color theorem.

The Hadwiger Conjecture

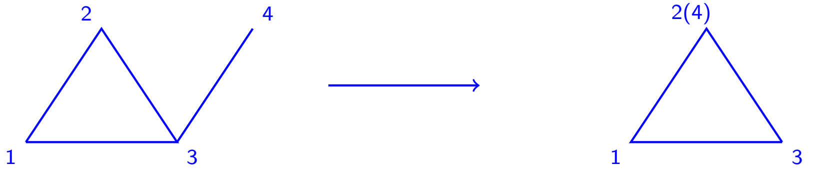
- The Hadwiger conjecture is trivial for $\chi(G) = 1$, since K_1 is a minor of any graph.
- For $\chi(G) = 2$, it just says that K_2 is a minor of an arbitrary graph containing an edge.
- If $\chi(G) = 3$, then G contains an odd cycle, and in particular it has K_3 as a minor.
- The case $\chi(G) = 4$ is reasonably easy, and was shown by Hadwiger and Dirac.
- Currently, the Hadwiger conjecture has been proved for $\chi(G) \leq 5$.

Variations of the Chromatic Number

Definition (A bridge between chromatic numbers and graph homomorphisms)

The chromatic number of G, $\chi(G)$, is the minimal positive integer n such that there exists a graph homomorphism $\phi : G \to K_n$.

Example



Variations of the Chromatic Number

Corollary

If there exists a graph homomorphism $\phi : T \to G$, then $\chi(T) \leq \chi(G)$.

View:
For a given graph G, we choose a family of graphs F_n with graph homomorphisms

$$G \to F_n, \quad n \in \mathbb{N}^+.$$

We call F_n the state graphs. And we have $\chi(G) \leq \chi(F_n)$.

Variations of the Chromatic Number

State graphs:

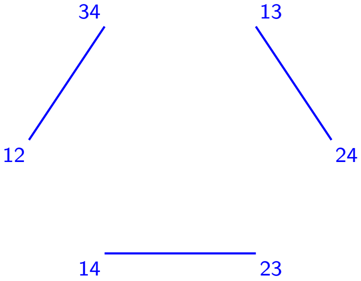
1. The complete graph $\{K_n\}_{n\geq 1}$.
2. The Kneser graphs $\{KG_{n,k}\}_{n\geq 2k}$.
3. Circular graphs $\{R_r\}_{r\in\mathbb{R}^+}$.

Definition

Let n, k be positive integers, $n \geq 2k$. The Kneser graph $KG_{n,k}$ is defined to be the graph whose set of vertices is the set of all k-subsets of $[n]$, and the set of edges is the set of all pairs of disjoint k-subsets.

Variations of the Chromatic Number

- $KG_{2k,k}$ is a matching on $\binom{2k}{k}$ vertices.



$KG_{4,2}$

- $KG_{n,1}$ is a the unlooped complete graph K_n.

Variations of the Chromatic Number

Definition

Let G be a graph. The fractional chromatic number of G, $\chi_f(G)$, is defined by

$$\chi_f(G) = \inf_{(n,k)} \frac{n}{k},$$

where the infimum is taken over all pairs (n, k) such that there exists a graph homomorphism from G to $KG_{n,k}$.

Variations of the Chromatic Number

Definition

Let G be a graph. The number $\tilde{\chi}_f(G)$ is defined by

$$\tilde{\chi}_f(G) = \inf_{(n,k)} \frac{n}{k},$$

where the infimum is taken over all covers of $V(G)$ by n independent sets $I_1, \ldots, I_n$ such that each vertex is covered at least k times, i.e., $|\{i | v \in I_i\}| \geq k$ for all $v \in V(G)$.

Variations of the Chromatic Number

Definition

The fractional coloring is a function $f : I(G) \to \mathbb{R}_{\geq 0}$ such that for every $v \in V(G)$ we have

$$\sum_{I \in I(G), v \in I} f(I) \geq 1,$$

here $I(G)$ denotes the collection of all the independent sets of G. The weight of a fractional coloring is defined to be the sum of the values of f over all independent sets: $weight(f) = \sum_{I \in I(G)} f(I)$. The number $\hat{\chi}_f(G)$ is defined to be the infimum of the weight, taken over the set of all fractional colorings.

Variations of the Chromatic Number

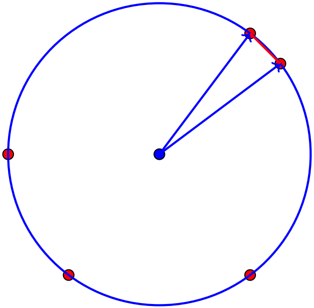
Proposition

$\chi_f(G) = \tilde{\chi}_f(G) = \hat{\chi}_f(G)$.

The Circular Chromatic Number

Definition

Let r be a real number, $r \geq 2$. Then R_r is defined to be the graph whose set of vertices is the set of unit vectors in the plane pointing from the origin, and two vertices x and y are connected by an edge if and only if $2\pi/r \leq \alpha$, where α is the sharper of the two angles between x and y.



The Circular Chromatic Number

Definition

Let G be a graph. The circular chromatic number of G is

$$\chi_c(G) = \inf r,$$

where the infimum is taken over all positive reals r such that there exists a graph homomorphism from G to R_r .

The Circular Chromatic Number

Definition

Let n, k be positive integers, $n \geq 2k$. Then $R_{n,k}$ is defined to be the graph whose set of vertices is $[n]$, and two vertices $x, y \in [n]$ are connected by an edge if and only if

$$k \leq |x - y| \leq n - k.$$

The Circular Chromatic Number

Proposition

Let G be a graph. We have the equality

$$\chi_c(G) = \inf_{(n,k)} \frac{n}{k},$$

where the infimum is taken over all pairs (n, k) such that there exists a graph homomorphism from G to $R_{n,k}$.

We remark that for any graph G, we have

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G),$$

or we write $\chi(G) = \lceil \chi_c(G) \rceil$.

Kneser Conjecture

It is known that $\chi(KG_{n,k}) \leq n - 2k + 2$. The Kneser conjecture states that in fact equality holds. This was proved in 1978 by L. Lovász, who used geometric obstructions of Borsuk-Ulam type to show the nonexistence of certain graph colorings.

Theorem (Kneser-Lovász)

For arbitrary positive integers n, k such that $n \geq 2k$, we have $\chi(KG_{n,k}) = n - 2k + 2$.

For example, $KG_{4,2}$.

Kneser Conjecture

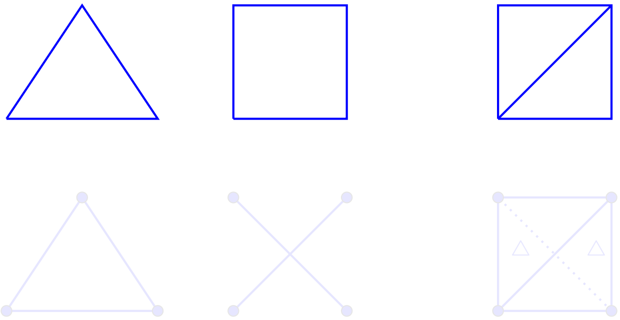
Lovász has introduced the neighborhood complex $\mathcal{N}(G)$ as a part of his topological approach to the resolution of the Kneser conjecture.

Definition

Let G be a graph. The neighborhood complex of G is the abstract simplicial complex $\mathcal{N}(G)$ defined as follows: its vertices are all nonisolated vertices of G, and its simplices are all the subsets of $V(G)$ that have a common neighbor.

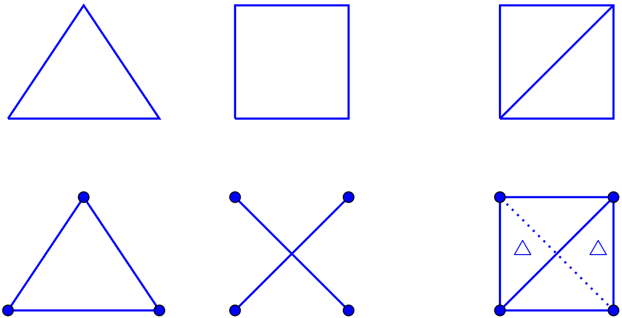
Kneser Conjecture

Example



Kneser Conjecture

Example



Kneser Conjecture

Theorem

Let G be a graph such that $\mathcal{N}(G)$ is k-connected for some $k \in \mathbb{Z}, k \geq -1$. Then $\chi(G) \geq k + 3$.

Proposition

For arbitrary positive integers n and k such that $n \geq 2k$, the abstract simplicial complex $\mathcal{N}(KG_{n,k})$ is homotopy equivalent to a wedge of spheres of dimension $n - 2k$. In particular, the complex $\mathcal{N}(KG_{n,k})$ is $(n - 2k - 1)$-connected.

Kneser Conjecture

The stable Kneser graph an induced subgraph of $KG_{n,k}$.

Definition

Let n, k be positive integers, $n \geq 2k$. The stable Kneser graph $KG^{stab}_{n,k}$ is defined to be the graph whose set of vertices is the set of all k-subsets S of $[n]$ such that if $i \in S$, then $i + 1 \notin S$, and if $n \in S$, then $1 \notin S$.

Kneser Conjecture

Theorem

The graph $KG_{n,k}^{stab}$ is a vertex-critical subgraph of $KG_{n,k}$, i.e., $KG_{n,k}^{stab}$ is a vertex-critical graph, and $\chi(KG_{n,k}^{stab}) = n - 2k + 2$.

Kneser Conjecture

Definition

For a hypergraph $\mathcal{H}$, the chromatic number $\chi(\mathcal{H})$ is, by definition, the minimal number of colors needed to color the vertices of $\mathcal{H}$ so that no hyperedge is monochromatic.

Definition

Let n, k, r be positive integers such that $r \geq 2$ and $n \geq rk$. The Kneser r-hypergraph $KG^r_{n,k}$ is the r-uniform hypergraph whose ground set consists of all k-subsets of $[n]$, and the set of hyperedges consists of all r-tuples of disjoint k-subsets.

Kneser Conjecture

Theorem

For arbitrary positive integers n, k, r such that $r \geq 2$ and $n \geq rk$, we have

$$\chi(KG^r_{n,k}) = \lceil \frac{n - rk + r}{r - 1} \rceil.$$

Motivation to introduce Hom complexes

Test for graph colorings:

$$\begin{tikzcd}
\text{Graph } G \arrow[r] \arrow[d, dotted] & \text{Topological space } X(G) \arrow[d] \\
\text{Combinatorial properties of } G & \text{Topological properties of } X(G) \arrow[l]
\end{tikzcd}$$

Motivation I

Different topological spaces $X(G)$:

- Lovász: neighborhood complex $\mathcal{N}(G)$
- Box complexes...

All these constructions are avatars of the same object. In contrast to that, the Hom complexes have been shown to have an intricate and interesting behavior, going substantially beyond the original Lovász complexes.

Motivation II

Theorem

The complexes $\mathcal{N}(G)$ and $\mathrm{Bip}(G)$ have the same simple homotopy type.

This fact leads one to consider the family of Hom complexes as a natural context in which to look for further obstructions to the existence of graph homomorphisms.

Stiefel-Whitney Characteristic Classes and Test Graphs

Definition

The element $\varpi_1(X) \in H^1(X/\mathbb{Z}_2; \mathbb{Z}_2)$ is called the Stiefel-Whitney class of the $\mathbb{Z}_2$-space X.

The Stiefel-Whitney classes can be used to determine the nonexistence of certain $\mathbb{Z}_2$-maps.

Stiefel - Whitney Characteristic Classes and Test Graphs

The following theorem is an example.

Theorem (Borsuk-Ulam Theorem)

Let n and m be nonnegative integers. If there exists a $\mathbb{Z}_2$-map $\varphi : \mathbb{S}_a^n \to \mathbb{S}_a^m$, then $n \leq m$. Here $\mathbb{S}_a^n$ denotes a $\mathbb{Z}_2$-sphere.

Proposition

Let X be a $\mathbb{Z}_2$-space, and assume that X is $(k-1)$-connected for some $k \geq 0$. Then there exists a $\mathbb{Z}_2$-map $\varphi : \mathbb{S}_a^k \to X$. In particular, $\varpi_1^k(X) \neq 0$.

Stiefel - Whitney Characteristic Classes and Test Graphs

Theorem

Let T and G be two arbitrary graphs such that T has a $\mathbb{Z}_2$-action that flips some edge in T, whereas G has no loops. Assume that $\varpi_1^k(\mathrm{Hom}(T,G)) \neq 0$, and that $\varpi_1^k(\mathrm{Hom}(T,K_m)) = 0$, for some integers $k \geq 0$, $m \geq 1$. Then we can conclude that $\chi(G) \geq m+1$.

Stiefel - Whitney Characteristic Classes and Test Graphs

Stiefel-Whitney Test Graphs are most useful when it comes to looking for characteristic class obstructions to graph colorings.

Definition

The Stiefel-Whitney height of X (or simply the height of X), denoted by $h(X)$, is defined to be the maximal nonnegative integer h such that $\varpi_1^h(X) \neq 0$. If no such h exists, then the space X is said to have infinite height.

If $\phi : X \to Y$ is a $\mathbb{Z}_2$-map, then $h(X) \leq h(Y)$.

Stiefel - Whitney Characteristic Classes and Test Graphs

Definition

Let T be a graph with a $\mathbb{Z}_2$-action that flips an edge. Then T is called a Stiefel-Whitney n-test graph if we have

$$h(\mathrm{Hom}(T, K_n)) = n - \chi(T).$$

Furthermore, T is called a Stiefel-Whitney test graph if it is a Stiefel-Whitney n-test graph for every integer $n \geq \chi(T)$.

Stiefel - Whitney Characteristic Classes and Test Graphs

Corollary

Assume that T is a Stiefel-Whitney test graph. Then for an arbitrary graph G, we have

$$\chi(G) \geq \chi(T) + h(\mathrm{Hom}(T, G)).$$

Stiefel - Whitney Characteristic Classes and Test Graphs

Definition

A graph T is called a homotopy test graph if for an arbitrary graph G, the following equation is satisfied:

$$\chi(G) > \chi(T) + \mathrm{conn}(\mathrm{Hom}(T, G)).$$

Note that $h(X) \geq \mathrm{conn} X + 1$ for an arbitrary $\mathbb{Z}_2$-space. Therefore, comparing the two definitions, we see that if a graph T is a Stiefel-Whitney test graph, then, it is also a homotopy test graph.

Stiefel – Whitney Characteristic Classes and Test Graphs

1. Let $A \to G \to B$ be $\mathbb{Z}_2$-equivariant maps, and A, B are Stiefel-Whitney test graphs, then G is a Stiefel-Whitney test graph.
2. Any connected bipartite graph T with a $\mathbb{Z}_2$-action that flips an edge is a Stiefel-Whitney test graph. Since we have $\mathbb{Z}_2$-equivariant maps
$$K_2 \to T \to K_2.$$

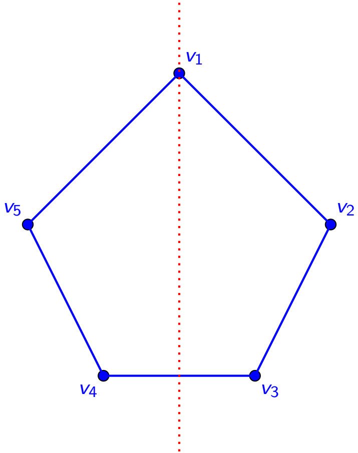
Stiefel - Whitney Characteristic Classes and Test Graphs

Theorem

Let G be a graph, and let $n, k \in \mathbb{Z}$ such that $n > 2, k \geq -1$. If $\varpi_1^k(\mathrm{Hom}(T, K_n)) \neq 0$, then $\chi(G) \geq n + k$.

Odd Cycles as Stiefel - Whitney Test Graphs

Recall that for $r \in \mathbb{N}$, we let C_{2r+1} denote both the cyclic graph with $2r+1$ vertices and the additive cyclic group with $2r+1$ elements.



Odd Cycles as Stiefel - Whitney Test Graphs

Theorem (Babson-Kozlov conjecture)

For all integers $r \geq 1$ and $n \geq 3$, we have

$$\varpi_1^{n-2}(\mathrm{Hom}(C_{2r+1}, K_n)) = 0.$$

Odd Cycles as Stiefel - Whitney Test Graphs

Theorem (Lovász conjecture)

For any graph G and any integers $r \geq 1$, we have

$$\chi(G) \geq \operatorname{conn}(\operatorname{Hom}(C_{2r+1}, G)) + 4.$$

Homology Tests for Graph Colorings

Theorem

Assume that T is a graph with a $\mathbb{Z}_2$-action that flips an edge such that additionally,

(1) T is a Stiefel-Whitney Test Graph,

(2) $\widetilde{H}_i(\mathrm{Hom}(T, G); \mathbb{Z}_2) = 0$ for $i \leq d$.

Then $\chi(G) \geq d + 1 + \chi(T)$.

What is the Hom complex?

Let G be a graph, and let T be a test graph. The complex $\mathrm{Hom}(T, G)$ can be understood as a gap/obstruction/relation from T to G.

A rough understanding:

$$\boxed{G \approx T + \mathrm{Hom}(T, G)}$$

What is the Hom complex?

Let $A, B \subset V(G)$ and $A, B \neq \emptyset$. We call A, B a complete bipartite subgraph of G if for any $x \in A, y \in B$, we have $(x, y) \in E(G)$, i.e., $A \times B \in E(G)$.

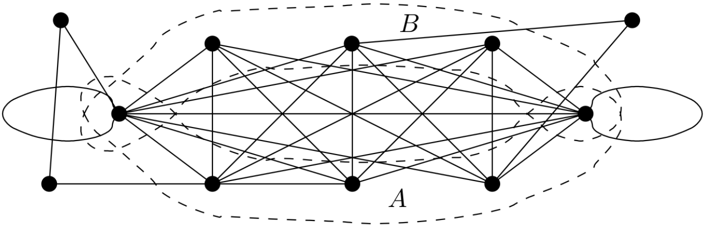


Figure 3: A complete bipartite subgraph

What is the Hom complex?

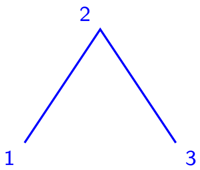
Let G be a finite graph. Recall that $\Delta^{V(G)}$ is a simplex whose set of vertices is $V(G)$; in particular, the simplices of $\Delta^{V(G)}$ can be identified with the subsets of $V(G)$.

Definition

The (prodsimplicial) complex $\mathrm{Bip}(G)$ is the subcomplex of $\Delta^{V(G)} \times \Delta^{V(G)}$ defined by the following condition: $\sigma \times \tau \in \mathrm{Bip}(G)$ if and only if (σ, τ) a complete bipartite subgraph of G.

What is the Hom complex?

1. $\mathrm{Bip}(G)$ is a complex.
2. L_3 denotes the string graph on 3 vertices.



$$\mathrm{Bip}(L_3) = (13, 2), (1, 2), (3, 2), (2, 13), (2, 1), (2, 3).$$

What is the Hom complex?

Definition

The box complex of a graph G is a $\mathbb{Z}_2$-poset

$$B(G) = \{(\sigma, \tau) | \sigma, \tau \in 2^{V(G)} \setminus \{\emptyset\}, \sigma \times \tau \subset E(G)\}$$

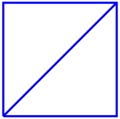
ordered by the product of the inclusion orderings.

In fact, by definition we have

$$B(G) = \mathrm{Bip}(G).$$

Idea

Is there a neighborhood hypergraph or box hypergraph?



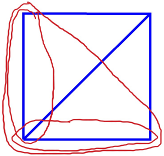


Figure 4: A neighborhood hypergraph given by connectivity

What is the Hom complex?

Definition

The complex $\mathrm{Hom}(G,H)$ is the subcomplex of $\prod_{x\in V(G)} \Delta^{V(H)}$ defined by the following condition:

$$\sigma = \prod_{x\in V(G)} \sigma_x \in \mathrm{Hom}(G,H)$$

if and only if for any $x, y \in V(G)$, if $(x,y) \in E(G)$, then (σ_x, σ_y) is a complete bipartite subgraph of H.

What is the Hom complex?

A multi-homomorphism from G to H is a map

$$\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$$

such that $(v, w) \in E(G)$ implies $\eta(v) \times \eta(w) \in E(H)$. For a pair of multi-homomorphisms η and η', we write $\eta \le \eta'$ to mean that $\eta(v) \le \eta'(v)$ for every vertex v of G.

Definition

The Hom complex from G to H is the poset of the multi-homomorphisms from G to H, and denoted by $\mathrm{Hom}(G, H)$.

The homotopy type of Hom complex

- The above two definitions coincide.
- The prodsimplicial complex $\mathrm{Bip}(G)$ is our first example of Hom complexes, namely, it is isomorphic to $\mathrm{Hom}(K_2, G)$ as $\mathbb{Z}_2$-space.
- Graph homomorphisms $f, g : G \to H$ are $\times$-homotopic if they belong to the same connected component of $\mathrm{Hom}(G, H)$.

The homotopy type of Hom complex

For a pair of graphs T and G, the singular complex is the simplicial set $Sing(T, G)$ whose n-simplices are the graph homomorphisms from $T \times K_n$ to G, i.e. $Sing(T, G)_n = Mor(T \times K_n, G)$.

The face and degeneracy maps are defined in an obvious way. A 0-simplex of $Sing(T, G)$ is identified with a graph homomorphism from T to G.

The homotopy type of Hom complex

Theorem (Matsushita)

There is a homotopy equivalence

$$\Phi : |\mathit{Sing}(T, G)| \to |\mathrm{Hom}(T, G)|$$

which is natural with respect to both T and G.

The homotopy type of Hom complex

Γ is a finite group. If T is a right Γ-graph, then the Hom complex $\mathrm{Hom}(T, G)$ becomes a left -space and a graph homomorphism $f : G_1 \to G_2$ induces a Γ-map $f_* : \mathrm{Hom}(T, G_1) \to \mathrm{Hom}(T, G_2)$. Since an n-coloring of a graph G is identified with a graph homomorphism from G to K_n, we have that if there is no Γ-map from $Hom(T, G)$ to $\mathrm{Hom}(T, K_n)$ then we have $\chi(G) \geq n$.

The homotopy type of Hom complex

Let T be a Γ-graph. The functor $G \to \mathrm{Hom}_T(G) = \mathrm{Hom}(T, G)$ has neither a left nor a right adjoint, and hence it is not a Quillen functor. So we use the singular complex functor $Sing_T(G) = Sing(T, G)$. It is known that the functor

$$Sing_T : \mathcal{G} \to sSet^{\Gamma}, \quad G \to Sing(T, G)$$

is a right adjoint functor.

Recent researches of Hom complex

- Takahiro Matsushita: Homotopy types of the Hom complexes of graphs
- Kouyemon Iriye, Daisuke Kishimoto: Hom complexes and hypergraph colorings
- Dmitry N Kozlov: Forman's Discrete Morse Theory and Hom complexes
- Csorba and Lutz: Try to connect the Hom-complexes to manifolds, specially PL manifolds.

Some questions of Hom complex

Hom complexes might be easily computed, but certainly their connectivity is not.

In this matter the Stiefel-Whitney classes come in handy, because they are probably much more computable.

Some questions of Hom complex

- Is there other computable Hom complex besides the Hom complex $\mathrm{Hom}(K_2, G)$? For example,

$$\mathrm{Hom}(C_m, K_n).$$

- Hom complex of hypergraphs and digraphs.
- Is there a Hom hypergraph?
- Find topological descriptions of other questions in graph theory.

Thank you !