Notes on Topology

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This note is used just for study. The note is wriiten to be a 'dictionary' rather than a textbook. So I'll try to include as many notions, propostions, theorems as possible and use my logic to connect them. I'll also try to give a brief but clear and intuitive but intrinsic explaination on important notions and theorems. But the proof of some propositions and theorems will not be given. The ultimate version of this note will be very long.

Several typical topology textbooks are used, including:

- Lecture Notes in Basic topology, ChengYe You.
- Topology, Munkres.
- Basic Topology, Armstrong.
- Topology without tears, Sidney.

The first one is written in chinese while the others are written in english. I mainly focuse on the former two textbooks. The latter two textbooks are used to be additions. These textbooks can be found online, or you can visit here: `https://liuyisi238.github.io//teaching/`. You can find these textbooks in 'Louis's Library'.

I have very limited knowledge and bad English writting skills. There might be many mistakes in this note. It would be highly appreciated if you could find them and tell me.

Thanks for reading!

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1 Introduction

This section will be finished when there is free time. My main attention will be focused on following sections.

1.1 Introduction

1.2 Outline of This Note

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2 Topological Space

Topology is a generalization of 'open set'. So it can be claimed that to study topology is to learn how to define 'open set' and topology is the set of open set.

2.1 Topological Space

Definition 2.1.1 *X is a nonempty set. $\tau \subset 2^X$ is a family of some subsets of X. τ is called a **topology** on X, if*

(1) X and ϕ are in τ, i.e. $X, \phi \in \tau$.

(2) Union of any elements in τ is still in τ, i.e. $\bigcup\limits_{\alpha \in I} A_\alpha \in \tau$, for $A_\alpha \in \tau$ and $\alpha \in I$, I is an indexing set.

(3) Intersection of finitely many elements in τ is still in τ, i.e. $\bigcap\limits_{i=1}^{n} A_i \in \tau$, for $A_i \in \tau, i = 1, 2, ..., n$.

*(X, τ) is called a **topological space**. Sometimes it can be abbreviated to X if it won't cause confusion. Elements of a topology τ are called **open sets**. This definition is also called **three axioms of topology.***

$2^X := \{A \subset X\}$ is called the **power set** of X. It can be verified that both $\{X, \phi\}$ and 2^X are topology. The former is called **trivial topology**, as the latter is called **discrete topology**. Besides, there are many other examples of topology:

- **cofinite topology** Given X an infinite set. The set consisting of complements of finite subsets of X and empty set is a topology, in other words, $\tau_f = \{A^c | A \subset X, A \text{ is finite}\} \bigcup \{\phi\}$ is a topology of X. Such a topology is called cofinite topology.

- **cocountable topology** Give X an uncountable and infinite set. The set consisting of complements of countable sets and empty set is a topology, i.e. $\tau_c = \{A^c | A \subset X, A \text{ is countable}\}$ is a topology. Such a topology is called cocountable topology.

- **Euclidean topology** The set consisting of union of any open sets in $\mathbf{R}$ is a topology, denoted by τ_e. So it's clear that $\phi, X \in \tau_e$. Such a topology is called euclidean topology. Set $E^1 = (\mathbf{R}, \tau_e)$.

And there is a special topology called **metric topology**. The name of this kind of topology implies that **A metric can determine a topology**.

Recall that $d : X \times X \to \mathbf{R}$ is said to be a metric if (i)It is positively defined: $d(x, x) \geq 0, \forall x \in X, d(x, y) = 0 \Leftrightarrow x = y$; (ii)It is symmetric: $d(x, y) = d(y, x), \forall x, y \in X$; (iii)The triangle inequality holds true: $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$. (X, d) is called a metric space. With metric, one can define notions like open ball(spherical neighborhood): $B(x, r) \triangleq \{y | d(x, y) < r\}$. Notice that *the intersection of two spherical neighborhood in a metric space is the union of some spherical neighborhood.* So it can be verified that ***the set consisting of the union of any spherical neighborhood** $\tau_d = \{U | U = \bigcup\limits_{\alpha} U_\alpha\}$ **is a topology on metric space.*** The proof is easy to obtain, so the work is left to readers. Such a topology is called metric topology.

As we have mentioned above, a metric can determine a topology, whence a metric space can determine a topological space. However, the converse of this conclusion doesnot always hold true. If a topological space can be interpreted as a metric space, saying $(X, \tau) = (X, \tau_d)$ if there is a metric d, this metric space is called **metrizable**.

Let τ_1, τ_2 are topologies on X. τ_2 is said to be **bigger** or **finer** than τ_1, if $\tau_1 \subset \tau_2$. Discrete topology is finer than all topologies, while trivial topology is smaller than all topologies. On $\mathbf{R}$, we have defined five topologies: discrete topology τ_t, trivial topology τ_s, cofinite topology τ_f, cocountable topology τ_c, euclidean topology τ_e. It can be proved that $\tau_s \subset \tau_f \subset \tau_c, \tau_e \subset \tau_t$. However, the relation between τ_c and τ_e is unclear.

2.2 Basic Topological Concepts

These basic concepts in topological space all comes from the concepts of open set and closed set in $\mathbf{R}^n$. Here we gives a more abstract and original definition of these concepts.

Definition 2.2.1 *Given topological space X. Set $A \subset X$ is said to be **closed**, if A^c is open.*

From this definition we know that 'open' and 'closed' are two complementary notions. And from the definition of topology, we have following proposition.

Proposition 2.2.1

(1) ϕ, X are closed. Moreover, they are both open and closed.

(2) Intersection of any closed set is closed, i.e. $\bigcap\limits_{\alpha} A_\alpha$ is closed, for all closed set A_α, where $\alpha \in \Lambda$ is an indexing set.

(3) Union of finitely many closed set is closed, i.e. $\bigcup\limits_{i=1}^{n} A_i$ is closed, for closed set A_i.

Proof. It will be finished later.

This proposition is highly similar with the definition of topology. This similarity comes from the complementarity of closed set and open set. Actually the definition of topology can be replaced with this proposition.

It should be mentioned that 'open set' is a relative notion, which means that a set can be open in a topological space but in another topological space it becomes closed. And there exist neither open nor closed sets, which means the family of topological space does not consist of open sets and closed sets. For instance, the set $[0, 1)$ is neither closed nor open in $\mathbf{R}$, but closed in $[0, 2)$.

Definition 2.2.2 *Given topological space X and subset $A \subset X$. A point $x_0 \in X$ is called an **interior point** of A, if there exists an open set $U \in \tau$, such that $x_0 \in U \subset A$. In this case, the set A is called a **neighborhood** of x_0. A point $x_0 \in X$ is called an **exterior point** of A, if there exists an open set $U \in \tau$, such that $x_0 \in U \subset A^c$. A point $x_0 \in X$ is called an **boundary point** of A, if for all open set $U \in \tau$, $U \bigcap A \neq \phi, U \bigcap A^c \neq \phi$. The set of all interior points of A is denoted by A^0. The set of all boundary points of A is denoted by ∂A.*

The sets of these three kinds of points respectively are a partition of topological space, which means points in X are either interoior points, or exterior points, or boundary points. Interior points of a set belong to this set. Exterior points of a set donot belong to this set. Boundary points of a set do not necessarily belong to this set.

If neighborhood A is open(closed), we will say it's an open(closed) neighborhood. The neighborhood used in definition 2.2.2 can be replaced by open neighborhood.

Proposition 2.2.2 *Given topological space X and subsets $A \subset X, B \subset X$.*

(1) If $A \subset B$, then $A^0 \subset B^0$.

(2) A^0 is the union of all open set contained in A. So A^0 is also said to be the maximal open set contained in A.

(3) A is open, if and only if $A = A^0$, if and only if $A \subset A^0$. Thus all points in open set are interior points, and a set is open, if and only if all points in this set in interior points.

(4) $(A \bigcap B)^0 = A^0 \bigcap B^0$.

(5) $(A \bigcup B)^0 \supset A^0 \bigcup B^0$.

Proof. It will be finished later.

Proposition 2.2.2(4) can be extend to finitely many situation by induction, While it doesnot hold true in the infinitly many situation. Try to give an example to show it. '$\supset$' cannot be replaced with '=' in proposition2.2.2(5). Try to give an example to show it.

Definition 2.2.3 *Given topological space X and subset $A \subset X$. A point $x_0 \in X$ is called a **limit point** of A, if for all neighborhoods U of x_0, $U \bigcap A \backslash \{x_0\} \neq \phi$. $x_0 \in A$ is called an **isolated point** of A if there exists a neighborhood U of x_0 such that $U \cap A \backslash \{x_0\} = \phi$. The set of all limit points of A is denoted by A'. The set $\overline{A} := A \bigcup A'$ is called the **closure** of A.*

Limit points donot need to belong to A but isolated points do belong to A. Isolated points are boundary points. Limit points are either interior points or boundary points. Interior points are limit points, but boundary points are not necessarily limit points and also not necessarily isolated points.

We should notice that notions of interior points, exterior points, boundary points and limit points, isolated points are all relative conceptions. We cannot discuss them without set A. Interior points, exterior points and boundary points must belong to topological space X and are defined relative to A. Isolated points and limit points also must belong to topological space X and are also defined relative to A.

Proposition 2.2.3 *Given topological space X and its subsets A, B. If $A = B^c$, then $\overline{A} = (B^0)^c$.*

Proof. It will be finished later.

Proposition 2.2.4 *Given topological space X and subsets $A \subset X, B \subset X$.*

(1) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(2) $\overline{A}$ is intersection of all closed sets containing A. Thus $\overline{A}$ is the minimal closed set containing A.

(3) A is closed, if and only if $A = \overline{A}$, if and only if $A' \subset A$. Thus all limit points of a closed set belong to this set, and the set contains all its limit points, if and only if this set is closed.

(4) $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$.

(5) $\overline{A \bigcap B} \subset \overline{A} \bigcup \overline{B}$.

(6) $\overline{A} = A^0 \bigcup \partial A$.

Proof. It will be finished later.

Proposition 2.2.3(4) can also be extend to finitely many situation. Try to give examples to show that (4) doesnot hold true for infinitely many situation and that in (5) we cannot replace '$\subset$' with '='.

Definition 2.2.4 *Given topological space X. The subset $A \subset X$ is **dense**, if $\overline{A} = X$. X is said to be **separable** if X has a countable dense subset A.*

Definition 2.2.5 *convergence*

2.3 Subspace and Topological Basis

Like linear space and group theory, the notion subspace is also considered in topological space. However, the introduction of subspace in topology is more natural and direct. As we do in linear space and group, we firstly consider the subset of tological space and then try to find what properties should the subset have to be a topological space. And we find the there naturally exists a topology on the subset, seen in the following definition.

Definition 2.3.1 *Given topological space (X, τ) and subset A. $\tau_A := \{A \bigcap U | U \in \tau\}$. τ_A is a topology on A and (A, τ_A) is a topological space. In this sense, A is called the subspace of X.*

Proposition 2.3.1 *Show that τ_A is a topology on A in the definition 2.3.1.*

Proof. Clearly, τ_A is a family of subsets of A and $\phi = A \bigcap \phi, A = A \bigcap X$ are both in τ_A. For $A_\alpha \in \tau_A$, $\bigcup\limits_\alpha A_\alpha = \bigcup\limits_\alpha (A \bigcap U_\alpha) = A \bigcap (\bigcup\limits_\alpha U_\alpha)$ with $U_\alpha \in \tau, \alpha \in \Lambda$. Union of any elements in τ_A is still in τ_A. For $B_i \in \tau_A$, $\bigcap\limits_{i=1}^n B_i = \bigcap\limits_{i=1}^n (A \bigcap V_i) = A \bigcap (\bigcap\limits_{i=1}^n V_i)$ with $V_i \in \tau$. Intersection of finitely many elements in τ_A is still in τ_A. So τ_A is a topology on A.

Proposition 2.3.1 tells us **a subset of topological space has a 'succeeded' topology, and thus is naturally a topological space and subspace.** In other words, all subsets of topological space are topological spaces and subspaces, which is quite different with linear space and group. So the union of any subspace must be a topological space and the intersection of any subspace must be a topological space as long as this intersection is noempty.

Proposition 2.3.2 *Given topological space X and subsets $B \subset A \subset X$. B is the closed set in A, if and only if B is the intersection of A and a closed set in X.*

Proof. It will be finished later.

Proposition 2.3.3 *Given topological space X and subsets $B \subset A \subset X$.*

(1) If B is closed(open) set in X, then B is the closed(open) set in A

(2) If A is the closed(open) set in X and B is the closed(open) set in A, then B is the closed(open) set in X.

Proof. It will be finished later.

Definition 2.3.2 *Basis for a topology;Basis for the topology*

2.4 Countinous Mapping

countinous at a point

countinous everywhere

2.5 Homeomorphism

2.6 Product Space

3 Important Topological Properties

As we have defined, topology and topological space are general enough to contain most cases. However, its generality arises the defect that it has not good enough properties. For the purpose of getting some better properties, mathematicians add some conditions of the pure topology we define above, which are called axioms. Based on topology and these axioms, we can get more important and more useful conclusions to help us solve mathematical problems.

In this section, several axioms and important topological properties will be introduced. Properties like separability, countability, connectivity and compactness will be carefully discussed. Other properties such as inheritance and multiplicativity will just be given a short discussion. Urysohn lemma and its applications are also discussed here.

3.1 Separation Axioms

We start from the separability of a topological space. Thanks to the hard work of mathematicians, We have at least ten separation axioms that suits different situations respectively. Here four common and useful separation axioms are discussed carefully and other separation axioms are introducted at the end of this subsection.

Definition 3.1.1(Separation Axioms) *Given a topological space X.*

(1) $\mathbf{T_1}$ ***axiom.*** *For each pair consisting of two different points, there exists a neighborhood containing only one points. In other words, for $x, y \in X, x \neq y$, x has a neighborhood A excluding y(i.e.$x \in A, y \notin A$), and y has a neighborhood B excluding x(i.e.$y \in B, x \notin B$).*

(2) $\mathbf{T_2}$ ***axiom.*** *For each pair consisting of two different points, there exist disjoint neighborhoods containing these two points respectively. In other words, for $x, y \in X, x \neq y$, $\exists$ neighborhood A of x and neighborhood B of y, such that $A \cap B = \phi$.*

(3) $\mathbf{T_3}$ ***axiom.*** *For each pair consisting of a point and closed set excluding this point, there exist disjoint neighborhoods containing this point and closed set respectively. In other words, for $x \in X$ and closed set $U \subset X$, $\exists$ neighborhood A of x and neighborhood B of U, such that $A \cap B = \phi$.*

(4) $\mathbf{T_4}$ ***axiom.*** *For each pair consisting of two disjoint closed sets, there exist disjoint neighborhoods containing these closed sets respectively. In other words, for closed sets $U, V \subset X$, $\exists$ neighborhood A of U and neighborhood B of V, such that $A \cap B = \phi$.*

The topological spaces satisfying T_i axiom are called $\mathbf{T_i}$ **space** respectively. Especially, T_2, T_3, T_4 spaces are called **Hausdorff space**, **regular space** and **normal space** respectively.

Proposition 3.1.1 *Topological space satisfies T_1 axiom if and only if any finite subset of this topological space is closed.*

Proof. Finish it later.

Corollary 3.1.1 *Given topological space X satisfying T_i axiom and a subset $A \subset X$. If x is a limit point of A, then A and any neighborhood of x intersects at a infinite set.*

Proof. Finish it later.

Proposition 3.1.2 *In Hausdorff space, a sequence (of points) will never converge into two or more points.*

Proof. Finish it later.

Proposition 3.1.3

(1) Hausdorff space is a T_1 space. The converse doesnot hold true. In other words, $T_2 \Rightarrow T_1$ but $T_1 \nRightarrow T_2$.

(2) Normal space as well as T_1 space is a regular space. Regular space as well as T_1 space is a Hausdorff space. Thus normal space as well as T_1 space is also a Hausdorff space. In other words, $T_4 \overset{T_1}{\Rightarrow} T_3 \overset{T_1}{\Rightarrow} T_2$.

Proof. Finish it later.

Proposition 3.1.4 *Given topological space X.*

(1) X is regular if and only if for any point x and its open neighborhood W, there exists open neighborhood U of x such that $\overline{U} \subset W$.

(2) X is normal if and only if for any closed set A and its open neighborhood W, there exists open neighborhood U of A such that $\overline{U} \subset W$.

Proof. Finish it later.

Proposition 3.1.5 *Metric space satisfies T_i axiom($i = 1, 2, 3, 4$).*

Proof. Finish it later.

3.2 Countability Axioms

There are mainly two coutability axioms in mathematics. Combination of separation axioms and countability axioms can generate vast power in topology, which creates many good properties and results.

Before introducing countability axioms, several concepts should be defined first.

Definition 3.2.1 *Given topological space X.*

Definition 3.2.2(Countability Axioms) *Given topological space X.*

(1) $\mathbf{C_1}$ ***axiom.***

(2) $\mathbf{C_2}$ ***axiom.***

3.3 Urysohn Lemma

3.4 Compactness

3.5 Connectivity