高等数学

Advanced Mathematics

Samuel S . Watson 编著

前　言

教科书通常会包含大量的材料，而让您离开找出重点。尽管有令人信服的理由每个部分都提供了许多示例，这样的方法可以对于那些认为他们需要更精简的人而言并不令人满意资源。本文旨在支持不同的工作流程：（i）阅读本书，以简约的方式介绍核心思想。（ii）在需要时寻求其他示例的替代资源。一世希望您能够仔细阅读每个部分并进行所有工作例子和练习，从而不遗余力地找出来跳过什么。我建议以下来源进行更多练习，讨论或示例：

1. 附录 **??**，其中列出了其他练习部分。其中一些是原创的，其他是从练习中修改詹姆斯·斯图尔特（James Stewart）的多变量微积分或 Susan J. Colley 的微分向量。
2. 教材书籍何在网站 `communitycalculus.org` 上获取，这是一个免费开放的学习平台，上面有大量的的知识练习。
3. 有许多多元微分知识的网站 MathInsight (`mathinsight.org`)，有诸多的程序代码为本课题的教学提供教学的便利。
4. 标准的多变量微积分教科书。
5. 3Blue1Brown，数学视频创作者，在线性代数。不幸的是，他没有做过多元微积分但是，这些视频仅可用于矢量主题。

文本中的以下空白注解图标是可单击的：

1. links to a CoCalc worksheet with a relevant calculation (see Appendix **??** for more discussion)
2. links to a relevant 3Blue1Brown YouTube video.
3. links to a relevant page at `mathinsight.org`

此 PDF 中的所有 3D 图形可能都是交互式的操作，但是该功能要求使用 Adobe 的免费 Acrobat Reader 来进行阅读。(`https://get.adobe.com/reader/`).

在使用本书过程中发现有任何错误欢迎通过邮件的方式与我们 (`sswatson@brown.edu`) 进行联系。

目　录

1 欧氏空间中的线性变换

1.1 n-维欧氏空间

如图 1.1 所示，我们可以将实数用数轴（给定定义的实数轴）进行表示出来 $\mathbb{R}$ 轴。

图 1.1 实数数轴

x 轴上标记的每一个点表示这个点到数轴上的点 0，的距离，对于负数而言，这表示这个数的相反数。

* 一般情况下，我们只取定一维、二维和三维空间上的图像进行可视化。

我们取两个实数，记为 $(x, y)(x \in \mathbb{R}, y \in \mathbb{R})$，这样的所有数对构成的集合是一个平面，如图 1.2，我们通常将其称为 xOy 平面。其中横轴我们称为 x 轴，纵轴我们称为 y 轴。构成的二维平面我们可以记为 $\mathbb{R}^2$ 空间，$\mathbb{R}^2$ 在几何图形上构成一个平面。

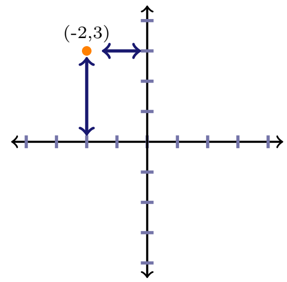


图 1.2 $\mathbb{R}^2$ 二维平面

* 在没有特殊情况下，我们在绘制三维空间的时候，通常选取的是由 x 轴正半轴、y 轴正半轴和 z 轴正半轴构成的空间区域

三维空间 $\mathbb{R}^3$ 是由所有的 (x, y, z) 其中 $(x, y, z \in \mathbb{R})$ 构成的集合，对于三维空间上的点，每个点（的位置）与每个数对 (x, y, z) 唯一的对应，我们称由所有 $(x, y, 0)$ 构成的集合形成的平面为 xy-平面，由所有 $(x, 0, z)$ 构成的集合形成的平面为 xz-平面，由所有 $(0, y, z)$ 构成的集合形成的平面为 yz-平面（如图 1.3）(Figure).，

类似地，我们通过确定不同个数的实数对形成的集合来进行定义 $\mathbb{R}^4$ 维空间、$\mathbb{R}^5$ 维空间，以及 $\mathbb{R}^6$ 维空间，这样，确定正整数 n（有限），$\mathbb{R}^n$ 我们称之为欧氏空间。

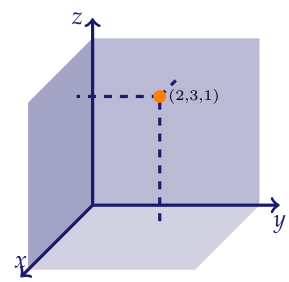


图 1.3 $\mathbb{R}^3$ 三维空间上的点

Exercise 1.1.1

分别写出在 $\mathbb{R}^2$ 平面上和 $\mathbb{R}^3$ 空间上两个不同的点 A 和 B 的距离的数学公式.

!!!

我们都知道，在 $\mathbb{R}^n$ 空间中，满足数学表达式方程的所有的点构成了该表达式在 $\mathbb{R}^n$ 空间中的曲线，例如，在二维平面 $\mathbb{R}^2$ 上，满足表达式 $x + y = 1$ 的所有的点 (x, y) 构成的集合是一条直线如图 1.4。在三维空间 $\mathbb{R}^3$ 上，满足数学表达式 $x^2 + y^2 = 1$ 的所有点的集合构成的是一个圆筒如图 1.5。我们不难看出，该圆筒包含的变量为平面 xOx 上的单位圆上的所有点而非所有的变量。

Exercise 1.1.2

现在，我们来思考下面的问题：

(a) 在数轴绘制满足方程：$x(x-1)(x+1) = 0$ 的点.

(b) 在二维空间 $\mathbb{R}^2$ 上绘制满足方程 $x(x-1)(x+1) = 0$ 的点集合形成的图像.

(c) 在三维空间 $\mathbb{R}^3$ 上绘制满足方程 $x(x-1)(x+1) = 0$ 的点集合形成的图像.

如图所示，这样我们就能更进一步的对函数的可视化有了解。接下来，我们进一步考虑 $\mathbb{R}^n$ 上的函数。

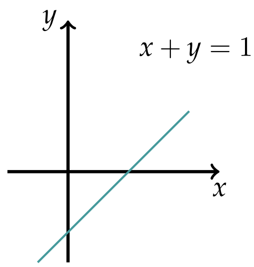


图 **1.4** The level sets of $f(x) = x-1$ are concentric circles in $\mathbb{R}^2$. They are the projections onto the x-line of intersections of the graph of f with "z = constant" planes, as shown

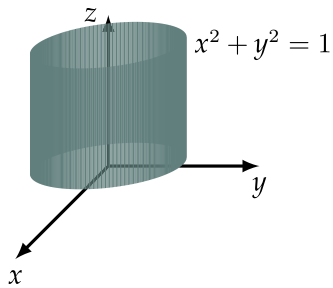


图 **1.5** The level sets of $x^2+y^2 = 1$ are surfaces in $\mathbb{R}^3$. The graph of f can't be visualized, but the level surfaces contain some geometric information about f

1.2 $\mathbb{R}^n$ 到 $\mathbb{R}^m$ 的函数

1.2.1 函数的可视化

我们定义在实数域上的函数 $f(x)$: $\mathbb{R}^1 \to \mathbb{R}^1$，输入端自变量是一个实数，而（输出端）因变量也是一个实数，通过建立平面直角坐标系，将横轴看成是自变量 x 轴，纵轴定为因变量的实数轴，那么，因变量 y 随着 x 的变化而变化，这样对应地，所有的 (x,y) 构成的集合形成了该函数 $f(x)$ 的图像，如图 1.6所示，函数 $f(x)=x^2$: $\mathbb{R}^1 \to \mathbb{R}^1$，更一般的来说，我们引用映射的范畴，函数作为映射的一种特殊情形，函数在可视化方面，能够给我们更加直观、便于理解的优势去学习和深入的研究。

* 函数 $y=f(x)$ 作为 $\mathbb{R}^1 \to \mathbb{R}^1$ 的一种特殊映射，也即是说，每个因变量 y 都对应一个或者多个自变量 x，一个自变量唯一地映射出一个 y。

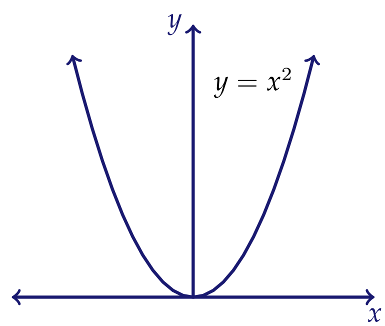


图 **1.6** The graph of a function from $\mathbb{R}^1$ to $\mathbb{R}^1$

* 特别地，函数 $f(x)=x^2$，是一个偶函数，即 $f(x)=x^2$ 的图像关于 y 轴对称，也就是 $f(x)=f(-x)$ 对 $\forall x \in \mathbb{R}^1$ 都成立。

现在，我们考虑这样函数 $f(x)$: $\mathbb{R}^2 \to \mathbb{R}^1$，如函数 $f(x)=\mathrm{e}^{-x^2-y^2}$ 定义了由二维空间 $\mathbb{R}^2 \to \mathbb{R}^1$ 的映射，我们确定区间 $[-2,2]\times[-2,2]$，对该函数的函数图像绘制出来，对于 $\mathbb{R}^2 \to \mathbb{R}^1$ 的函数。我们建立空间直角坐标系 $O-xyz$，建立如图所示，我们绘制区域 $[-2,2]\times[-2,2]$ 上 $f(x)=\mathrm{e}^{-x^2-y^2}$ 的部分图像如图 1.7所示。

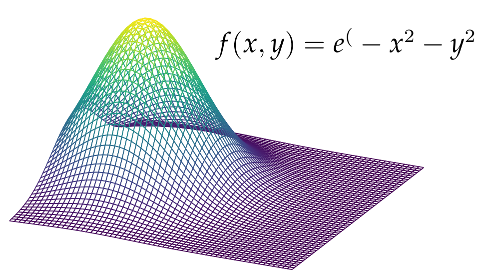


图 **1.7** A graph of a function from $\mathbb{R}^2$ to $\mathbb{R}^1$

在函数的可视化中，对于 $f(x,y):\mathbb{R}^2 \to \mathbb{R}^1$ 的函数，我们可将其可视化，即该函数的图像为满足 $z=f(x,y)$ 的函数表达式的做有点的集合 (x,y,z)，其中，$\forall x,y,z \in \mathbb{R}^1$，另外，x 称为自变量 x，y 称为自变量 y，这样 $f(x,y)$ 就定义了一个

二元函数，自变量 x,y 构成了 xy-自变量区域，z 是因变量，满足 $z = f(x,y)$ 的所有点 (x,y,z) 构成了函数 $z = f(x,y)$ 的函数图像。例如，函数 $z = f(x,y) = \mathrm{e}^{x^2+y^2}$ 是 $\mathbb{R}^2 \to \mathbb{R}^1$ 的函数，我们取定自变量区域 $[0,4] \times [-1,3]$ 的自变量区域，最后我们得到的绘制图像如图所示

现在，我们再来看函数 $f(x,y) = (u(x,y), v(x,y))$ 由二维空间 $\mathbb{R}^2 \to$ 二维空间 $\mathbb{R}^2$ 的函数的图像，和前面的类似，我们考虑自变量区域 $[a,b] \times [c,d] (\subset \mathbb{R}^2 \times \mathbb{R}^2) \to [A,B] \times [C,D] (\subset \mathbb{R}^2 \times \mathbb{R}^2)$ 的函数，考虑其图像的绘制，我们取定函数 $u(x,y): \mathbb{R}^2 \to \mathbb{R}^1$ 以及 $v(x,y): \mathbb{R}^2 \to \mathbb{R}^1$ 的函数，那么由所有的 $(x,y) \in [a,b] \times [c,d]$ 经过函数 $u(x,y)$、$v(x,y)$ 后，得到的所有的点 $(u(x,y), v(x,y))$ 的集合就构成了其函数图像。如函数 $f(x,y) = (x + y + \frac{x^3}{100} + 1, x - y)$，图像如图 1.8所示。

* 我们在可视化一个图变成另一个图时最好的方法就是，将图分开来进行绘制以便于对比

通常情况下，我们称 $\mathbb{R}^2$ 到 $\mathbb{R}^2$ 的函数称为变换，这是 $\mathbb{R}^2$ 到 $\mathbb{R}^2$ 的函数的一种特殊代名词，主要是在可视化方面具有特殊的意义。类似地，我们也将称 $\mathbb{R}^3$ 到 $\mathbb{R}^3$ 的函数称为变换，更准确的来说，称之为 R^3 空间上的变换，它的可视化和 $\mathbb{R}^3$ 到 $\mathbb{R}^3$ 的函数相同。现在，我们把目光聚焦在 $\mathbb{R}^3$ 到 $\mathbb{R}^3$ 的函数可视化上。

* 特别地，对于变换，我们最初是从最简单的线性变换来研究，这就会涉及到代数学习过程中的知识内容，如矩阵、行列式等内容，这将在后续的章节里面呈现出来

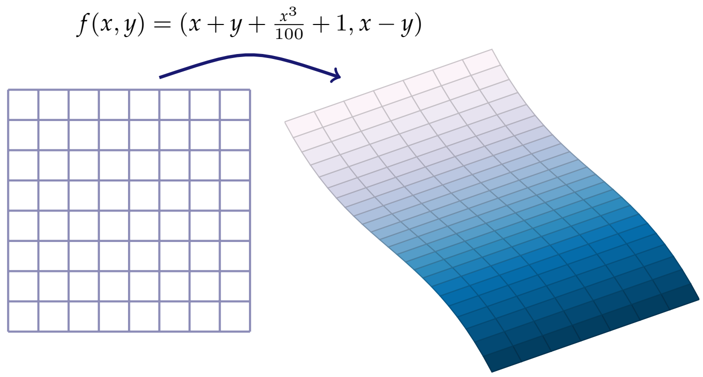


图 1.8 A transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

函数的等值面是指三维空间 $\mathbb{R}^3$ 上函数，我们令 $z =$ 某个常数之后形成的图像。为了更加方面的了解，我们先来看下列的例子，我们考虑函数 $f(x,y) = x^2 + y^2$，它的水平切面是诸多的同心圆，如图 1.9所示，而函数 $f(x,y,z) = x^2 + y^2 - z^2$ 的则是一个表平面，如图 1.10所示。在可视化多元函数过程中，我们把函数置相等的点形成的集合称为等高线，等高线即是函数图像中令 $z = A$(A 是常数) 与函数图像 $z = f(x,y)$ 的图像相交后所得到的交线。

* This means that Figure 1.9 involves an abuse of notation: the level sets should be points in $\mathbb{R}^2$ rather than points in the xy-plane in $\mathbb{R}^3$. However, it is common practice to make such implicit use of the natural association $(x,y) \leftrightarrow (x,y,0)$ b...

从 $\mathbb{R}^2$ 到 $\mathbb{R}^1$ 的函数可以使用其图形或使用等高线图，而函数图从 $\mathbb{R}^3$ 到 $\mathbb{R}^1$ 无法在空间上可视化。因此，轮廓图提供了一个重要的可视化工具，用于实现三个变量的功能。

等高线图有一些缺点：除非输出值使用标签或标签标识与每个级别集相对应的着色方案，有关函数值的信息是图片中缺少。换句话说，绘制函数图输入和输出成单个图形，同时绘制水平集仅功能的域 *。

1.2.2 线性变换

on linear transformations

微分学的关键思想之一是使用线性函数以曲线近似函数图。这种近似是有用的，因为（i）所有可微当您放大某个点时，功能看起来越来越线性，并且（ii）线性函数非常简单。我们将应用相同的原理，并使用线性变换近似非线性的就像您了解线性在学习单变量演算之前的函数，我们将探索讨论微分之前的线性变换概括为多变量设置。

* Actually, this view is ubiquitous in mathematics, because the essence of linearity is the pair of properties $f(x+y) = f(x) + f(y)$ and $f(ax) = af(x)$, and we only get these if we insist $b = 0$

那么从 $\mathbb{R}^2$ 到 $\mathbb{R}^2$ 的线性函数是什么？形式为 $f(x) = mx + b$ 的单变量函数，其中 m 和 b 是常数，通常称为线性。但是，我们将采取要求 $b = 0$ 的视图略有不同，因此仅函数形式 $f(x)mx$ 被认为是线性的。高维的线性相似：仅形式为 “恒定时间变量”：

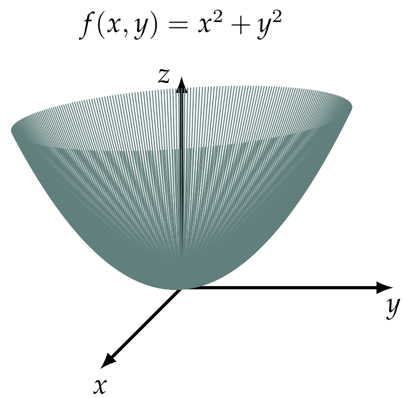


图 1.9 The level sets of $f(x,y) = x^2 + y^2$ are concentric circles in $\mathbb{R}^2$. They are the projections onto the xy-plane of intersections of the graph of f with "z = constant" planes, as shown

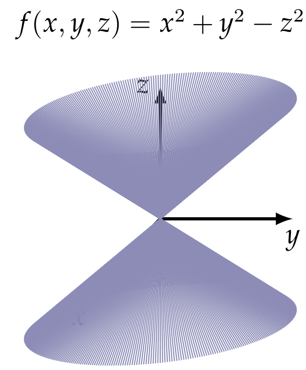


图 1.10 The level sets of $f(x,y,z) = x^2 + y^2 - z^2$ are surfaces in $\mathbb{R}^3$. The graph of f can't be visualized, but the level surfaces contain some geometric information about f

Definition 1.2.1: 线性变换的定义

A function f from $\mathbb{R}^2$ to $\mathbb{R}^2$ is **linear** if there exist* $a, b, c, d \in \mathbb{R}$ so that

$$f(x,y) = (ax + by, cx + dy).$$

* This notation means that a, b, c, d are in the set of real numbers

图 1.11 显示从 $\mathbb{R}^2$ 到 $\mathbb{R}^2$ 线性变换的四个例子。的转变图 1.11 *scale*，*shear*，*旋转/缩放*和 *project*。未显示是 *reflection*，例如 $f\ x,y\ =\ x,-y$ 。这是从 R^2 到 $\mathbb{R}^2$ 可以写成这些变换的组合基本类型。

基于图 1.11我们进行猜想，线性转换将等距线映射到等距线，其中重合线等距间隔为零（如上例所示）。这几乎是准确的：有时也一样等距的线可以映射到等距的点（锻炼 1.2.2）。以下定理给出了线性的几何特征。

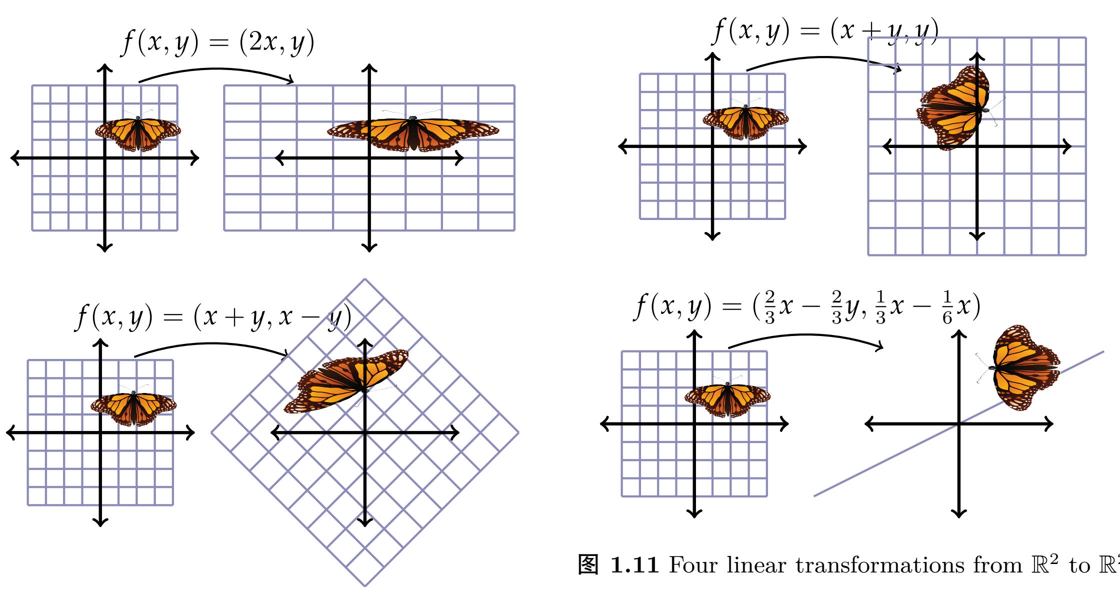


图 1.11 Four linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$

Theorem 1.2.1: 线性变换的等价定理

A function from $\mathbb{R}^2$ to $\mathbb{R}^2$ is linear if and only if it maps the origin to the origin and equally spaced lines* to equally spaced lines or points.

* *Equally spaced lines* are lines which are parallel and for which the distances between consecutive lines are the same

但是，我们还没有完成。请注意，我们假设从原点到 a, c 的线段为从原点到 b, d 的线段顺时针旋转。要是我们切换这些线段，同样的理由给我们公式 $bc - ad$。我们可以说所有这些区域转换的因子为 $\boxed{|ad - bc|}$。

* 反转三个点的方向 A，B 和 C 表示如果这些点在三角形周围以逆时针顺序给出 ABC，则它们的图像在三角形周围按顺时针顺序它们形成

Exercise 1.2.1: 线性变换定理的应用练习

Use Theorem 1.2.1 to show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ rotates every point counterclockwise about the origin by $30°$, there necessarily exist $a, b, c, d \in \mathbb{R}$ such that $f(x, y) = (ax + by, cx + dy)$ for all $(x, y) \in \mathbb{R}^2$.

我们可以按以下方式解释 1.3.1的结果：$ad - bc$ 告诉我们 $f(x, y) = (ax + by\ cx + dy)$ 如何转换区域（通过其绝对值）以及是否将 f 应用于这三个点 $(0, 0)$，$(1, 0)$ 和 $(0, 1)$ 会反转其方向 *（通过标志）。这个想法很重要，值得它自己的名字。至为了简化定义，我们将长度称为 1 维和面积作为二维体积。

Exercise 1.2.2: 线性变换定理的应用

Show that the linear function $f(x, y) = (2x, 0)$ maps any collection of equally spaced vertical lines to a collection of equally spaced points.

on the determinant

1.3 行列式

线性函数从 $\mathbb{R}^1$ 到 $\mathbb{R}^1$ 的斜率它如何扭曲长度。例如，函数 $f\ x\ = 3x$ 映射任何间隔 $[a, b]$ 到间隔 $[3a, 3b]$ 的三倍长。函数 $g\ x\ = -\frac{1}{2}x$ 将任何间隔映射到间隔的一半，它也会翻转间隔周围。可以说线性的斜率的绝对值函数是长度转换所依据的因素，并且斜率的符号告诉我们函数是否反转实数数字线。

那么，从 $\mathbb{R}^2$ 到 $\mathbb{R}^2$。我们可以看一下线性变换吗？计算变换扭曲的因数地区？答案是肯定的!

对于图 1.11中的每个线性变换，图片图像侧的四边形都是全等。这表明线性变换确实可以以相同的因子变换每个区域。根据这个事实，我们考虑一个正方形的图像就足够了将取为 $[0, 1]^2$，这是两个点的坐标集介于 0 和 1 之间。

Example 1.3.1

Find the area of the image of the unit square $[0, 1]^2$ under the transformation

$$f(x, y) = (ax + by, cx + dy).$$

Solution

The area of the unit square can be calculated by filling in some triangles to get a complete rectangle, as follows:

The area of the larger rectangle is $(a+b)(c+d)$, and the total area of the triangles we added is $2 \cdot \frac{1}{2}(a+b)(c) + 2 \cdot \frac{1}{2}(c+d)(b)$. Subtracting, we get that the area of the parallelogram is $ad - bc$.

We are not quite finished, however. Note that we assumed in our diagram that the line segment from the origin to (a, c) is clockwise from the line segment from the origin to (b, d). If we switched these line segments around, the same reasoning would have given us the formula $bc - ad$. We can put this all together by saying that the factor by which areas are transformed is $\boxed{|ad - bc|}$.

We can interpret the result of Example 1.3.1 as follows: $ad - bc$ tells us how $f(x, y) = (ax + by, cx + dy)$ transforms areas (via its absolute value) and whether applying f to the three points $(0, 0)$, $(1, 0)$, and $(0, 1)$ reverses their orientation* (via its sign). This idea is important enough to deserve its own name. To simplify the definition, we refer to length as 1-dimensional volume and area as 2-dimensional volume.

* Reversing the orientation of three points A, B, and C means that if these points are given in counterclockwise order around the triangle ABC, then their images are in clockwise order around the triangle they form.

Definition 1.3.1: Determinant

The **determinant** of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is the signed factor by which it transforms n-dimensional volumes.

We have already figured out that the determinant of $f(x) = mx$ from $\mathbb{R}^1$ to $\mathbb{R}^1$ is the slope m, and the determinant of a function $f(x, y) = (ax + by, cx + dy)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is given by the formula $ad - bc$.

For convenience, we sometimes represent a linear function by arranging its coefficients into a grid of numbers called a *matrix*. By convention, rows correspond to coordinates of the output of the function, and columns correspond to the variables. So, for example, $f(x, y) = (ax + by, cx + dy)$ is represented by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So we have

$$det[m] = m, \text{ and } \qquad det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Exercise 1.3.1

Find the determinant of each of the following matrices, and draw the image of the unit square under the corresponding linear transformations to see that the value of the determinant you computed makes sense.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

We define the determinant of a linear transformation

$$f(x,y,z) = (ax+by+cz,\ dx+ey+fz,\ gx+hy+iz)$$

to be the product of two quantities:

(i) the volume of the three-dimensional shape, called a *parallelepiped*, whose vertices are the images under f of the vertices of the unit cube $[0,1]^3$ (Figure 1.12).

(ii) a factor of ± 1 which is equal to -1 if and only if the orientation of a small loop drawn on a face is reversed (see Figure 1.12), from the point of view of a small person standing on the solid with their head pointing toward the outside.

图 **1.12** A linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$

The formula for the determinant turns out to be*

$$det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei - afh - bdi + bfg + cdh - ceg.$$

Unlike the formula $ad-bc$ for the 2×2 matrix, this formula is not easy to memorize. Let's abbreviate $det[\ ]$ to $|\ |$ and write this formula as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

This formula is *still* not easy to memorize, so let's break it down: each term on the right-hand side includes three factors: (i) a $+1$ or -1 which alternates starting with $+1$, (ii) an entry from the top row (going from left to right), and (iii) the determinant of the matrix you get when you remove the row and column of that entry from the original matrix. These smaller matrices are called *minors*, and this method of calculating the determinant is called **expansion by minors** along the first row. You can also expand by minors along any row or column (see Exercise 1.3.3 below), but if it's an even-numbered row or column, then the signs start with -1 instead of $+1$.

* The *Geometry of Determinants* section of the free online book *Linear Algebra* by Jim Hefferon includes a lengthy discussion of $n\times n$ determinants

Exercise 1.3.2

Calculate each determinant.

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} -4 & 2 & 1 \\ 5 & 0 & 3 \\ -2 & 1 & 3 \end{vmatrix}$

Exercise 1.3.3

Expand by minors along the first *column* of this matrix, and show that you get the same result as when you expand by minors along the first row.

$$\begin{vmatrix} -2 & 1 & 4 \\ 1 & 1 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$

Exercise 1.3.4

Find the values of t for which the determinant of the following matrix is zero.

$$\begin{vmatrix} -2 & t^2 & 4 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$

Exercise 1.3.5

The **transpose** A^T of a matrix A is obtained by swapping rows and columns. In other words,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Show that $det A = det A^T$.

Exercise 1.3.6

Show that if two rows of a 3×3 matrix A are the same, then $det A = 0$.

2 欧氏向量空间

on vectors

2.1 向量空间

A **vector** in $\mathbb{R}^n$ is an arrow from one point in $\mathbb{R}^n$ (the *tail*) to another (the *head*). See Figure 2.1). The **length*** of a vector is the distance from the head to the tail. Two vectors are considered equivalent if they have the same length and the same direction.

* The length of a vector is also called the **norm** or the **magnitude** of the vector

图 **2.1** A vector in $\mathbb{R}^2$

The **components** of a vector are the coordinates of its head when it is translated so that its tail is at the origin. In other words, to find the components of a vector, we subtract each coordinate of its tail from the corresponding coordinate of the head. The components of the vector in Figure 2.1 are $\langle \frac{3}{2}, 1\rangle$—note that we use the pointy brackets to distinguish components of a vector from coordinates of a point.* We can calculate the components by subtracting the coordinates of the tail from the coordinates of the head. Two vectors are equivalent if and only if their components are the same.

* Actually, vectors and points kind of *are* the same, since they are both specified by an ordered tuple of real numbers. The distinction is in how we use them and visualize them, although you should be prepared to play fast and loose with this distinction at times

The main things we will do with vectors are (i) add two of them together and (ii) multiply a vector by a real number (which is called a **scalar** in this context). These are defined componentwise:

$$\langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle, \text{ and}$$
$$c\langle u_1, u_2\rangle = \langle cu_1, cu_2\rangle.$$

These definitions of vector addition and scalar multiplication lead to natural geometric interpretations, as shown in Figures 2.2 and 2.3.

图 **2.2** Vector addition: $\langle 3, 1\rangle + \langle 1, 2\rangle = \langle 4, 3\rangle$

图 **2.3** Scalar multiplication: $2\langle 2, 1\rangle = \langle 4, 2\rangle$

We typically assign names for vectors which are lowercase boldface letters, like $\mathbf{u}$ or $\mathbf{v}$. Looking at Figure 2.3, we make the following observation.

!!!

Observation 2.1.1: Parallel vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = c\,\mathbf{v}$ for some scalar c.

Vector operations satisfy several properties suggested by the notation,* such as commutativity ($\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$), associativity ($(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$), the distributive property of scalar multiplication across vector addition (Exercise 2.1.1) and so on. One simple strategy for proving such property-verification exercises is to write out what each side of the equation means in terms of components and then simplify both sides until it's clear that they are equal.

* That is, we use the same notation as we use for multiplication/addition of real numbers because these operations satisfy many of the same properties

Exercise 2.1.1

Show that scalar multiplication distributes across vector addition. In other words, show that for all $c \in \mathbb{R}$ and vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we have

$$c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}.$$

Exercise 2.1.2

Show that scalar multiplication distributes across scalar addition. In other words, show that for all $c \in \mathbb{R}$, $d \in \mathbb{R}$, and vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we have

$$(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}.$$

Exercise 2.1.3

Choose two vectors $\mathbf{u}$ and $\mathbf{v}$ with small integer coordinates and draw a figure to show how $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$ fit together to form a triangle.

Exercise 2.1.4

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and suppose that $\mathbf{w} = c\mathbf{u} + d\mathbf{v}$, where c and d are both in $[0,1]$. If the tail of $\mathbf{w}$ is at the origin, then what is the set of possible locations for the head of $\mathbf{w}$?

The following example shows how vector ideas can be applied to geometry problems.

Example 2.1.1

Use vectors to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Solution

Define $\mathbf{u}$ and $\mathbf{v}$ to be two vectors with a common tail at one vertex O of the triangle and heads at the other two vertices A and B as shown. Then the vectors from O to the midpoints of OA and OB are $\frac{1}{2}\mathbf{u}$ and $\frac{1}{2}\mathbf{v}$, since the midpoint of a line segment is defined to be the point which is halfway between the endpoints.

Therefore, the vector $\mathbf{w}$ from one midpoint to another is $\frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$. By the distributive property, this is equal to $\frac{1}{2}(\mathbf{v}-\mathbf{u})$. The vector from A to B is $\mathbf{v}-\mathbf{u}$. Therefore, $\mathbf{w}$ has the same direction as the vector from A to B (by Observation 2.1.1) and is half as long.

Exercise 2.1.5

Use vectors to show that the diagonals of a parallelogram bisect one another.

Exercise 2.1.6

A median of a triangle is a line segment from a vertex of the triangle to the midpoint of the opposite side. Use vectors to show that for any triangle, there is a point on all three medians. (Hint: This point will split each median into two segments, one of which is twice as long as the other.)

on the dot product

2.2 向量的点乘

The fundamental vector operations of scalar multiplication and vector addition are not sufficient to capture information about a really important geometric concept: *angle*. So we introduce a new vector operation.

Definition 2.2.1: Dot product

The **dot product** of two three-dimensional vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The dot product distributes across vector addition, and it is closely related to length, as shown in the following exercise. We denote by $|\mathbf{u}|$ the length of $\mathbf{u}$.

Exercise 2.2.1

Verify that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ and that $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.

Now we establish the relationship between the dot product and angle.

Example 2.2.1

Use the law of cosines to show that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution

We apply the law of cosines to the triangle with sides $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$. We get

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos\theta$$

The left-hand side works out to

$$|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,\mathbf{u} \cdot \mathbf{v}.$$

Subtracting these equations yields $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$.

Particularly noteworthy is the case where θ is a right angle: We say that two vectors are **perpendicular** or **orthogonal** or **normal** if they meet at a right angle.

Observation 2.2.1: Perpendicular vectors

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

!!!

The following example shows how we can use the relation $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ to find an angle when information about coordinates is available.

Example 2.2.2

Find the angle between the diagonal of a cube and a diagonal of one of its faces.

Solution

The vector from the origin to the opposite corner of the cube is $\langle 1, 1, 1\rangle$. The vector from the origin to the opposite corner of the bottom face is $\langle 1, 1, 0\rangle$. Therefore, the angle is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{1+1+0}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+0^2}}\right) = \boxed{\cos^{-1}\left(\frac{2}{\sqrt{6}}\right)}.$$

Exercise 2.2.2

Sketch the vectors $\mathbf{u} = \langle 4, 2\rangle$ and $\mathbf{v} = \langle -1, 2\rangle$ and show geometrically that they are perpendicular.

Then verify that the coordinate formula for dot product indeed gives $\mathbf{u} \cdot \mathbf{v} = 0$ for these two vectors.

We conclude this section by addressing the out-of-nowhere step where we defined the formula for the dot product. We could have begun with the geometric formula $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ as the *definition* of the dot product, derived the distributive property of the dot product across vector addition using geometry, and then obtained the formula for the dot product in the following way: define $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$. Then a vector $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ can be written as

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k},$$

and similarly for $\mathbf{v}$. Then

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k})(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \cdot \mathbf{i} + u_1v_2\mathbf{i} \cdot \mathbf{j} + u_1v_3\mathbf{i} \cdot \mathbf{k} + \\
&\quad u_2v_1\mathbf{j} \cdot \mathbf{i} + u_2v_2\mathbf{j} \cdot \mathbf{j} + u_2v_3\mathbf{j} \cdot \mathbf{k} + \\
&\quad u_3v_1\mathbf{k} \cdot \mathbf{i} + u_3v_2\mathbf{k} \cdot \mathbf{j} + u_3v_3\mathbf{k} \cdot \mathbf{k},
\end{aligned}$$

by the distributive property. This looks like a mess, but since $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are perpendicular, six of these nine terms are zero. Furthermore, since $\mathbf{i} \cdot \mathbf{i} = 1$ and similarly for $\mathbf{j}$ and $\mathbf{k}$, we end up with $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$, as desired.

Exercise 2.2.3

Show that the vectors $\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$ and $\mathbf{v} = \langle -\sin\theta, \cos\theta\rangle$ are unit vectors (meaning that the length of each is 1). Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Exercise 2.2.4: Orthogonal Projection

Suppose vectors $\mathbf{u}$ and $\mathbf{v}$ are given with $|\mathbf{u}| = 1$. If $\mathbf{v} = \mathbf{v}_1 + \mathbf{v_2}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular and $\mathbf{v}_1$ is parallel to $\mathbf{u}$, then $\mathbf{v}_1$ is called the *orthogonal projection* of $\mathbf{v}$ onto $\mathbf{u}$. The magnitude of $\mathbf{v}_1$ is called the *component* of $\mathbf{v}$ in the $\mathbf{u}$ direction.

(a) Draw a figure illustrating the relationship between $\mathbf{u}$, $\mathbf{v}$, $\mathbf{v}_1$, and $\mathbf{v}_2$.

(b) Use right-triangle trigonometry to find a formula for the length of $\mathbf{v}_1$ in terms of the angle θ between $\mathbf{u}$ and $\mathbf{v}$, and then use dot products to find a formula for $\mathbf{v}_1$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

(c) Use your findings to support the statement *dotting with a unit vector* $\mathbf{u}$ *gives the component in the* $\mathbf{u}$ *direction.*

on the cross product

2.3 向量的叉乘

In the last section we introduced a vector product which reveals information about *angle*; in this section we'll see a new vector product which gives us information about *area*.

* We put *determinant* in scare quotes because the matrix entries are not num-

A beautiful explanation of this formula

The **cross product** of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined by expanding the following 'determinant' by minors along the first row:*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

图 **2.4** The cross product of $\mathbf{u}$ and $\mathbf{v}$, whose length is equal to the area of the parallelogram shown

Note that the dot product of two vectors is a scalar, while the cross product of two vectors is another vector. It turns out that this vector is orthogonal to *both* of the first two.

Example 2.3.1

Confirm that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and to $\mathbf{v}$.

Solution

We compute $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ as follows:

$$\langle u_1, u_2, u_3 \rangle \cdot \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle =$$
$$(u_2v_3 - u_3v_2)u_1 - (u_1v_3 - u_3v_1)u_2 + (u_1v_2 - u_2v_1)u_3 = 0.$$

This implies that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$. Swapping out $\mathbf{u}$ for $\mathbf{v}$, we see that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$ too.

Alternatively, we could note that

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

and conclude using Exercise 1.3.6 that this determinant equals zero.

The following exercise provides the advertised connection to area. Recall from geometry that the area of a parallelogram with sides of length a and b meeting at an angle θ is equal to $ab\sin\theta$.*

* Proof:

Exercise 2.3.1

Verify that $|\mathbf{u}\times\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2$. Use this fact to show that

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

So to sum up: $\mathbf{u}\times\mathbf{v}$ is a vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and whose length is equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. Note that there are only two vectors satisfying both of these conditions. To determine which one is $\mathbf{u}\times\mathbf{v}$, we use the *right-hand rule*: imagine orienting your right hand so that you can curl your fingers from $\mathbf{u}$ towards $\mathbf{v}$. The direction of your thumb (if it's orthogonal to your fingers) is the direction of $\mathbf{u}\times\mathbf{v}$.

Exercise 2.3.2

Find the volume of the parallelepiped spanned by $\langle 3,4,1\rangle$, $\langle -2,4,0\rangle$, and $\langle -5,5,2\rangle$. (Hint: first find the area of the base, then figure out how to use dot products to find the height.)

Exercise 2.3.3: Cross product distributive property

Show that $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors.

3 R^3 空间上的几何

3.1 线与平面

There are various ways to describe a line in 2D space using an equation, including point-slope form and y-intercept form. In this section we will learn the 3D analogue: equation descriptions of lines and planes in space. We begin with an example.

Example 3.1.1

Describe the line L in $\mathbb{R}^3$ passing through the points $A = (3, -4, 1)$ and $B = (2, -1, 4)$.

Solution

We can tell whether a given point (x, y, z) in $\mathbb{R}^3$ is on the line L using vectors: (x, y, z) is on L if and only if the vector from $(3, -4, 1)$ to (x, y, z) is a scalar multiple of the vector from $(3, -4, 1)$ to $(2, -1, 4)$ (see Figure 3.1). We can turn this into an equation: a point (x, y, z) is on L if and only if there exists $t \in \mathbb{R}$ such that

$$t\langle 2-3, -1-(-4), 4-1\rangle = \langle x-3, y-(-4), z-1\rangle.$$

Setting components equal, we find that (x, y, z) is on L if and only if there exists t so that

$$\begin{aligned} x &= 3 - t, \\ y &= -4 + 3t, \text{ and} \\ z &= 1 + 3t. \end{aligned} \tag{3.1.1}$$

Note that the solution above involves a new variable t; this is called a *parameter*, and the form we gave as an answer is called **parametric form**. You can imagine drawing the line by starting with $t = 0$, so that your pen begins at A, and then sweeping t through the values from 0 to 1, changing the location of your pen according to the parametric equations (3.1.1). Your pen will sweep out the line segment from A and B. Then you can let t vary beyond 1 to get the rest of the ray past B, and you can let t vary over the negative numbers to get the part of the line on the other side of A.

图 **3.1** Checking whether (x, y, z) is on the line through A and B

If we didn't want to involve t, note that we could solve for t in one equation and substitute into the other two, thereby obtaining *two* equations involving x, y, and z. This makes sense: starting from the plane, imposing one equation on x and y cuts the dimension down by one and gives a line. However, starting from 3D space, we need to reduce the dimension by *two*. So we need two equations.

The procedure developed in Example 3.1.1 works in general: the line through $A = (a, b, c)$ and B has parametric form*

$$x = a + v_1 t,$$
$$y = b + v_2 t$$
$$z = c + v_3 t,$$

where $\langle v_1, v_2, v_3 \rangle = \overrightarrow{AB}$ is the vector from A to B.

* Note that this representation is not unique, since we could've used B (or any other point on the line) in place of A

Example 3.1.2

Describe the plane P passing through the points $A = (1, 0, 0)$, $B = (0, 1, 1)$, and $C = (0, 0, 2)$.

Solution

We can tell whether (x, y, z) is on P using vectors. Define $\mathbf{u}$ and $\mathbf{v}$ to be the vectors from A to B and from A to C, respectively.

If we can find a vector $\mathbf{n}$ which is orthogonal to P, then we can say (x, y, z) is on P if and only if the vector from A to (x, y, z) is orthogonal to $\mathbf{n}$. But we can take $\mathbf{n} = \mathbf{u} \times \mathbf{v}$, since the cross product of two vectors is orthogonal to both of them. So

$$\mathbf{n} = \langle -1, 1, 1 \rangle \times \langle -1, 0, 2 \rangle = \langle 2, 1, 1 \rangle.$$

Now we can say that (x, y, z) is on P if and only if*

$$\mathbf{n} \cdot \langle x - 1, y - 0, z - 0 \rangle = 0,$$

which simplifies to $\boxed{2x + y + z = 2}$.

* Note that we could use B or C instead $A = (1, 0, 0)$ here and we'd get the same equation for the plane

Observation 3.1.1: Vector normal to a plane

A vector $\mathbf{n}$ normal to the plane $ax + by + cz = d$ can be read off from the coefficients:

$$\mathbf{n} = \langle a, b, c \rangle.$$

!!!

One important 3D geometry problem is to find distances between points, lines, and planes.* We define the distance between two sets to be the *minimum* distance between any pair of points from the respective sets.

* We can ask about the distance from a point to a point, a point to a line, a point to a plane, a line to a line, a line to a plane, or a plane to a plane

Example 3.1.3

Consider the line ℓ given by the parametric equation $(x, y, z) = (1 - 2t, 3, t)$. Find the distance from ℓ to the line m which is parallel to ℓ and which passes through the point $(9, 4, 1)$.

Solution

The parametric equations give us convenient access to a point on each of the two lines as well as a vector $\mathbf{v}$ which is parallel to both lines. So we make a figure with this information.

If we define $\mathbf{u}$ to be the vector from connecting the two given points, we can see by applying right-triangle trigonometry to the figure that the desired distance d is equal to $|\mathbf{u}|\sin\theta$. Therefore,

$$d = \frac{|\mathbf{u}||\mathbf{v}|\sin\theta}{|\mathbf{v}|} = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{105}}{\sqrt{5}} = \boxed{\sqrt{21}}.$$

Note the basic strategy: (i) draw a figure containing the information that the problem gives us (a schematic diagram suffices; there is no need to make it particularly precise), (ii) use right triangle trigonometry to express the desired distance terms of vectors we have, and (iii) use vector formulas to calculate the desired quantity using a dot or cross product.

A second important operation is finding points of intersection.

Example 3.1.4

Find the point where the line $\langle 3+t, -2t, 3\rangle$ intersects the plane $x+y+z=7$.

Solution

A point is on the intersection of two graphs if and only if it's on both graphs. Therefore, we're looking for a point (x, y, z) that satisfies $x+y+z=7$ and has the property that there exists $t \in \mathbb{R}$ such that $x = 3+t, y = -2t$, and $z = 3$. Thus we have four equations and four unknowns (x, y, z, t), and we can substitute the last three equations into the first to find that $6-t=7$, which implies $t=-1$. Therefore, the point of intersection is $(2, 2, 3)$.

Exercise 3.1.1

Find the equation of the plane passing through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Find the distance from that plane to the origin.

Exercise 3.1.2

Find the distance between the planes $x+y-2z=3$ and $x+y-2z=0$.

Exercise 3.1.3

Find the distance between the lines $(x, y, z) = (2t, 1-t, 4)$ and $(x, y, z) = (1+t, -2t, -1-t)$. Hint: these lines are *skew*, meaning that they are not parallel but do not intersect. Begin by using a cross product to find a vector which is perpendicular to both lines.

Exercise 3.1.4

Describe parametrically the intersection of the planes $2x + z = 3$ and $x + y - 2z = 4$.

3.2 向量值函数

3.2.1 空间上的路径

Consider a particle moving along the number line in such a way that its position at time t is given by $r(t)$. Then the velocity of the particle at time t is given by the first derivative $v(t) = r'(t)$. The velocity specifies the *speed* of the particle as well as its *direction* (left if negative, right if positive).

The same is true of a particle moving in 2D or 3D space: its location is specified by a function customarily denoted $\mathbf{r}(t)$ from $\mathbb{R}$ to either $\mathbb{R}^2$ or $\mathbb{R}^3$, and its derivative* $\mathbf{v}(t) = \mathbf{r}'(t)$ at time t tells us the speed of the particle at that time (via its length) as well as the direction. We call $\mathbf{r}$ a *path**.

* The derivative of a path $\mathbf{r}$ is the vector obtained by taking the derivative of each component

* For a function from $\mathbb{R}^1$ to $\mathbb{R}^2$ or $\mathbb{R}^3$ to count as a path, we require that each of its components be continuous. For example, $\mathbf{r}(t) = (e^t, t^2 \sin t)$ is a path

Example 3.2.1

Consider a bug which crawls counterclockwise around a cylinder of radius 1 meter from $(1, 0, 0)$ to $(1, 0, 1)$ as shown:

Assuming the bug moves at constant speed and makes the whole journey in one second, find a formula for the position and velocity of the bug at time t.

Solution

We can see that the z-coordinate of the bug's position increases at a constant rate from 0 to 1 as t goes from 0 to 1, so the z-coordinate of $\mathbf{r}(t)$ is t.

For the x and y coordinates, we need a pair of functions $(x(t), y(t))$ that traces out the unit circle in one second. Recall that cosine and sine are defined to be the functions that trace out the unit circle according to angle, so we can scale them so they make it around in 1 second instead of 2π seconds:

$$(x(t), y(t)) = (\cos 2\pi t, \sin 2\pi t)\,.$$

So all together we have

$$\boxed{\mathbf{r}(t) = \langle \cos 2\pi t, \sin 2\pi t, t\rangle},$$

which means that

$$\boxed{\mathbf{v}(t) = \langle -2\pi \sin 2\pi t, 2\pi \cos 2\pi t, 1\rangle}.$$

Exercise 3.2.1

Find the times $t \geq 0$ when a particle whose location at time t is $\mathbf{r}(t) = \langle t^3 - t^2, t \rangle$ is at its slowest.

Exercise 3.2.2

The acceleration $\mathbf{a}(t)$ of a particle whose position at time t is given by $\mathbf{r}(t)$ is defined to be $\mathbf{v}'(t) = \mathbf{r}''(t)$. Suppose that the acceleration $\mathbf{a}(t)$ of a particle is given by $\mathbf{a}(t) = \langle 2t, t^2 \rangle$. If $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \langle 1, 2 \rangle$, then find a formula for $\mathbf{r}(t)$.

Exercise 3.2.3

An astronaut is using a rope to move in space in such a way that his position at time t is given by $\mathbf{r}(t) = (2+t)\mathbf{i} + (2+\ln t)\mathbf{j} + \left(7 - \frac{4}{t^2+1}\right)\mathbf{k}$. The coordinates of the space station doorway are $(5,4,9)$. When should the astronaut let go of the rope so as to drift into the doorway?

Given a curve in $\mathbb{R}^n$, a path $\mathbf{r}$ that traces it out is called a parametric equation for the curve.

Example 3.2.2

Find a parametric equation describing the intersection of the sphere

$$x^2 + y^2 + z^2 = 1$$

and the plane

$$y + z = 1.$$

Solution

The main idea is to reduce the 3D problem to a 2D problem by finding a shadow of the curve that we can parameterize. Looking at the figure, it appears that the shadow of this curve in the xz plane is an ellipse. If we substitute $y = 1 - z$ into $x^2 + y^2 + z^2 = 1$, we see that every point on the curve indeed satisfies* $x^2 + 2z^2 - 2z = 0$.

* This equation describes an elliptical *cylinder*, because it is the set of all points whose shadow in the xz-plane is in the red ellipse shown in the figure

Completing the square to get this equation in standard form, we get

$$2x^2 + 4\left(z - \tfrac{1}{2}\right)^2 = 1.$$

The intersection of this surface with the xz-plane is an ellipse which can be parameterized as

$$\left(\tfrac{1}{\sqrt{2}}\cos t, 0, \tfrac{1}{2} + \tfrac{1}{2}\sin t\right),$$

where t ranges over $[0, 2\pi]$. Using the equation $y = 1 - z$, we see that the point on the curve whose shadow is $\left(\frac{1}{\sqrt{2}}\cos t, 0, \frac{1}{2} + \frac{1}{2}\sin t\right)$ is $\left(\frac{1}{\sqrt{2}}\cos t, \frac{1}{2} - \frac{1}{2}\sin t, \frac{1}{2} + \frac{1}{2}\sin t\right)$. As t ranges from 0 to 2π, this point goes all the way around the cylinder, so this formula indeed parameterizes the curve.

Exercise 3.2.4

The intersection of the cylinder of unit radius centered along the x-axis and the cylinder of unit radius centered along the y-axis consists of four curves connecting the points $(0,0,1)$ and $(0,0,-1)$. Choose one of them and parametrize it.

Example 3.2.3

Consider a wheel of unit radius centered at $(0,1)$ in a coordinate plane. At time $t=0$, the wheel begins rolling (without slipping) along the x-axis to the right at a rate of one unit per second. Find the location $\mathbf{r}(t)$ of the point on the wheel which was originally located at the origin.

Solution

Because the wheel is moving at a unit rate of speed, the location of its center at time t is $(t,1)$. After t seconds, the wheel has rotated t units along its perimeter, which corresponds to rotating t radians. Therefore, the angle of the point on the wheel originally at the origin starts at $\frac{3\pi}{2}$ and advances in the clockwise direction at a rate of one radian per second. Thus the vector from the wheel's center to the desired point is $\langle \cos\left(\frac{3\pi}{2}-t\right), \sin\left(\frac{3\pi}{2}-t\right)\rangle$, which simplifies to $\langle -\sin t, -\cos t\rangle$. So the location of the desired point is the sum of the point $(t,1)$ and the vector $\langle -\sin t, -\cos t\rangle$:

$$\boxed{\mathbf{r}(t) = (t-\sin t, 1-\cos t)}.$$

3.2.2 曲线的弧长 *

Example 3.2.4

Find the speed of the bug in Example 3.2.1.

Solution

Since the speed of the bug is constant and the whole journey takes a second, the speed in meters per second is equal to the length of the bug's path divided by 1 second.

Consider any time $t \in [0,1)$ and a very small time Δt. Over the interval $[t, t+\Delta t]$, the bug's speed is approximately $|\mathbf{v}(t)|$, so the distance it moves is approximately equal to $|\mathbf{v}(t)|\Delta t$. Fixing a large integer n and splitting up the interval $[0,1]$ into n intervals of width $\Delta t = 1/n$, total distance is approximately

$$\overbrace{|\mathbf{v}(0)|\Delta t}^{\text{over }[0,\Delta t]} + \overbrace{|\mathbf{v}(\Delta t)|\Delta t}^{\text{over }[\Delta t, 2\Delta t]} + \overbrace{|\mathbf{v}(2\Delta t)|\Delta t}^{\text{over }[2\Delta t, 3\Delta t]} + \cdots + \overbrace{|\mathbf{v}(1-\Delta t)|\Delta t}^{\text{over }[1-\Delta t, 1]}.$$

This expression is a Riemann sum approximating the integral $\int_0^1 |\mathbf{v}(t)|\,dt$, so as $\Delta t \to 0$, it converges to

$$\int_0^1 |\mathbf{v}(t)|\,dt = \int_0^1 \sqrt{4\pi^2\sin^2 2\pi t + 4\pi^2\cos^2 2\pi t + 1}\,dt = \boxed{\sqrt{4\pi^2+1}}.$$

Although we arrived at this answer using approximations that we weren't so careful about, this is indeed the same result we would've gotten if we'd cut a slit in the cylinder and unrolled it to find that the bug's path is equal in length to the hypotenuse of a rectangle with side lengths 1 (the height of the cylinder) and 2π (the circumference of the cylinder).

The method developed in Example 3.2.4 leads to the following definition. A function from an interval to $\mathbb{R}$ is **piecewise differentiable** if its domain can be subdivided into finitely many intervals such that the function is differentiable on each of them. A function from an interval to $\mathbb{R}^n$ is defined to be piecewise differentiable if all of its components are.

Definition 3.2.1: Arclength

The **arclength** of a piecewise differentiable path $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is defined to be

$$\int_a^b |\mathbf{r}'(t)|\, dt,$$

assuming this integral exists.

One important property of this definition is that it should not depend on the rate at which the curve is traced out.* In other words, suppose person A draws a curve with a consistent pace, while person B rushes through the first part of the curve and slows down towards the end. As long as the curves they draw are the same, the lengths should be the same. Let's try an example.*

* A property of a path which depends only on the curve it traces out is called **parameterization independent**

* See Appendix ?? for an explanation of why this is true in general

Example 3.2.5: Parameterization independence

Use the arclength formula to find the length of the portion of the unit circle in the first quadrant in two ways: using $\mathbf{r}_1(t) = (\cos t, \sin t)$, and using $\mathbf{r}_2(t) = (t, \sqrt{1 - t^2})$.

Solution

For $\mathbf{r}_1$, we have

$$\int_0^{\pi/2} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \boxed{\frac{\pi}{2}},$$

and for $\mathbf{r}_2$, we have

$$\int_0^1 \sqrt{1^2 + \left(\frac{-2t}{2\sqrt{1-t^2}}\right)^2}\, dt = \int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt = \arcsin(1) - \arcsin(0) = \boxed{\frac{\pi}{2}},$$

where we performed the last integral by recalling that the derivative of the inverse sine function is $\frac{1}{\sqrt{1-t^2}}$.

Exercise 3.2.5

Sketch the path $\mathbf{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t \rangle$. Show that the length of this path over the interval $[0, a]$ converges as $a \to \infty$. What does this say about the length of the path over the interval $[0, \infty)$?

3.2.3 曲线的曲率 *

The curvature of a path at a point on the path is a measure of how curvy the path is at that point. For example, the path shaped like the letter "U" has a small curvature (zero, in fact) near its endpoints and a larger curvature at points along the bend.

One natural way to distinguish points on a path where the path has large curvature from points where the path's curvature is small is to ascertain *how rapidly the direction of the velocity vector changes, per unit of arclength.* This leads to the following definition.

Definition 3.2.2: Curvature

(i) The **unit tangent vector** of a differentiable path $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

(ii) The **arclength function** $s(t)$ is defined to be the length of $\mathbf{r}$ over the interval $[a, t]$.

(iii) The **curvature** $\kappa(t)$ of $\mathbf{r}$ at time t is defined by

$$\kappa(t) = \frac{|d\mathbf{T}/dt|}{ds/dt}.$$

Example 3.2.6

Find the curvature of a circle of radius a using Definition 3.2.2.

Solution

We parameterize the circle as $(a \cos t, a \sin t)$ where t ranges over $[0, 2\pi]$, and we determine that

$$\mathbf{T} = \frac{\langle -a \sin t, a \cos t \rangle}{|\langle -a \sin t, a \cos t \rangle|} = \langle -\sin t, \cos t \rangle.$$

The arclength function is* $s(t) = displaystyle \int_0^t |\langle -a \sin t, a \cos t \rangle| \, d\tau = at$.

Therefore, the curvature is $displaystyle \kappa(t) = \frac{|\langle -\cos t, -\sin t \rangle|}{|d(at)/dt|} = \boxed{\frac{1}{a}}$.

This formula makes sense, because a very large circle looks quite flat (consider standing on the equator as opposed to standing on a latitude line a few feet from the North Pole).

* We'll use τ as the variable of integration, since we're already using t as a limit of integration

Exercise 3.2.6

Suppose that $\mathbf{r}$ is a differentiable path and that $\mathbf{T}$ is its unit tangent vector. Use the fact that the length of $\mathbf{T}(t)$ is constant to show that $\mathbf{T}(t)$ is always orthogonal to $\mathbf{T}'(t)$.

Exercise 3.2.7

Find the curvature at each point on the graph of the function $f : (0, \pi) \to \mathbb{R}$ defined by $f(x) = \ln(\sin x)$.

3.3 二次曲面

The *graph* of an equation involving the variables x, y, z is the set of points (x, y, z) in $\mathbb{R}^3$ which satisfy the equation. For example, we have seen that the graph of $x + y + z = 1$ is a plane in $\mathbb{R}^3$. More generally, the graph of any linear equation in $\mathbb{R}^3$ is a plane. So let's step it up a notch and consider *quadratic* equations.* A graph of a quadratic equation in the variables x, y, z is called a *quadric surface*.

* The 2D analogues are *conic sections*: parabolas, ellipses, and hyperbolas. These are graphs of various quadratic equations in two variables

Perhaps the simplest quadratic equation to reason about is $x^2 + y^2 + z^2 = 1$. The left-hand side has a geometric interpretation as *the squared distance from* (x, y, z) *to the origin*. Therefore, a point (x, y, z) satisfies this equation if and only if its squared distance to the origin is 1. We have a name for the set of such points: the *sphere* of radius 1, centered at the origin.

The situation is not always so simple. So here's a key idea for tackling 3D geometry problems: **slice it up**. Consider planes of the form $z =$ constant, $y =$ constant, or $z =$ constant and see what your graph looks like in these planes. Here's an archetypal example.

Example 3.3.1

Figure out what the graph of $x^2 + y^2 - z^2 = 1$ looks like.

Solution

We begin by finding all the points which satisfy this equation and the equation $z = 0$. If (x, y, z) satisfies this equation and $z = 0$, then that means that $x^2 + y^2 = 1$. Furthermore, if $x^2 + y^2 = 1$ and $z = 0$, then (x, y, z) satisfies the equation $x^2 + y^2 - z^2 = 1$. This means that the intersection of the desired graph and the line $z = 0$ is the circle of radius 1 centered at the origin.
Similarly, the intersection of the desired graph and the plane $z = 1$ is a circle which is centered at $(0, 0, 1)$ and has radius $\sqrt{2}$. Drawing in several more of these traces*, we get a picture that looks like the figure above. This is already a pretty clear picture of what the graph looks like: it's rotationally symmetric about the z-axis and "flares out" as you move away from the xy-plane. This graph is called a *one-sheeted hyperboloid*.

* A *trace* of a figure is an intersection of that figure with a plane

Exercise 3.3.1

Sketch $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, where $a = b = c = 1$. This is called an elliptic paraboloid.

Exercise 3.3.2

Sketch $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, where $a = b = c = 1$. This is called an elliptic cone.

Exercise 3.3.3

Sketch the graph of $x^2 + y^2 - z^2 = -1$. This is called a two-sheeted hyperboloid.

Exercise 3.3.4

Show that the graph of $z = y^2 - x^2$ looks like the figure shown. This is called a hyperbolic paraboloid.

3.4 （$\mathbf{R}^3$）极坐标，圆柱坐标和球坐标

A coordinate system is a way of identifying locations using pairs or triples of real numbers. Rectangular coordinates—the ones commonly denoted (x, y) or (x, y, z)—have some nice properties, but some tasks are much more convenient in other coordinate systems.

For example, a captain at sea wishing to communicate the location of a nearby pirate ship would probably describe its location in terms of the distance r between the two ships and an angle θ (which might be given with reference to the ship's orientation or as a cardinal direction). The captain is using *polar coordinates*.

Given a point in the plane, we define r to be its distance from the origin and θ to be the signed angle formed between the positive horizontal axis and the vector from the origin to the point. The word *signed* means that an angle measured clockwise from the positive x-axis counts as negative. The values r and θ are called the radial and angular polar coordinates of the point, respectively.*

* So a coordinate is a function from a set of points to $\mathbb{R}$

Exercise 3.4.1

Show that if the polar coordinates of a point (x, y) are r and θ, then we have

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta.$$

* Try coming up with geometric descriptions of the coordinates r, θ, and z in 3D space *before* looking at the answer below

Correspondingly, we can coordinatize three-dimensional space by replacing either one or two spatial coordinates with an angular coordinate. Perhaps the simplest way to do this is leave z the same and replace (x, y) with polar coordinates (r, θ). In other words, we define* for each point P in $\mathbb{R}^3$:

$$\begin{aligned} r(P) &= \text{distance from } P \text{ to the } z\text{-axis} \\ \theta(P) &= \text{signed angle* of the } P\text{-containing half-plane whose boundary is along the } z\text{-axis} \\ z(P) &= \text{signed distance from } P \text{ to the } xy\text{-plane.} \end{aligned}$$

* …measured counterclockwise with respect to the positive x-axis

Then we can describe a point by its **cylindrical coordinates** (r, θ, z) rather than its rectangular coordinates. We may also describe a solid in $\mathbb{R}^3$ by giving inequalities in the variables r, θ, and z such that a point with cylindrical coordinates (r, θ, z) satisfies the inequalities if and only if the point is in the given solid.

Example 3.4.1

Graph* the system of cylindrical coordinate inequalities $r \leq 4$, $\quad 0 \leq \theta \leq \pi/3$, $\quad 0 \leq z \leq 2$. Find the volume of the resulting region.

* Familiarity with coordinate slices (Table **??** in Appendix **??**) is helpful for graphing inequalities

Solution

The problem is asking us to find the points whose cylindrical coordinates satisfy all of the given inequalities. Such a point is less than or equal to 4 units from the z-axis, lies between the half-planes $\theta = 0$ and $\theta = \pi/3$, and is above the xy-plane and less than two units away from it. The set of such points is shown to the right.

This region is one-sixth of a cylinder whose volume is $\pi r^2 h = 32\pi$, so its volume is $\boxed{\dfrac{16\pi}{3}}$.

Exercise 3.4.2

Graph the system of inequalities $0 \leq r \leq z, \quad \pi \leq \theta \leq 2\pi$.

Cylindrical coordinates have two distance coordinates and one angular coordinate. How can we specify a point in space using one distance coordinate and two angular coordinates? The most natural candidate for the distance coordinate is the distance from the origin. In other words, we define $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. We call this coordinate ρ instead of r to distinguish it from the radial polar coordinate.

As for the angular coordinates, let's use the cylindrical coordinate θ for one of them. For the other, we measure the angle ϕ between the positive z-axis and the vector from the origin to (x, y, z). This pair of angular coordinates might be familiar: we use them to describe locations on the surface of the earth. In that context, the angle θ is called longitude and the angle ϕ is called latitude.

Note that θ varies from 0 to 2π as one loops around the z-axis. However, the angle ϕ varies only from 0 to π as one goes from the north pole to the south pole. Thus the angles θ and ϕ do not play symmetric roles.*

* This is because ϕ measures the angle required to rotate a vector *freely* so as to align with the positive z-axis, while $-\theta$ measures the signed angle needed to rotate the vector *about the z-axis* to get to the positive half of the xz-plane

Example 3.4.2

Graph the system of inequalities $\frac{1}{2} \leq \rho \leq 1, \quad 0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq \phi \leq \frac{\pi}{2}$.

Solution

The set of points with $\rho \leq 1$ is the set of points on or inside of the sphere of radius 1 centered at the origin. Imposing the additional constraint $\rho \geq \frac{1}{2}$ removes the sphere of radius $\frac{1}{2}$ centered at the origin. Then the angular constraints carve out a portion of this hollowed out sphere, as shown.

Exercise 3.4.3

Find a system of inequalities in spherical coordinates to describe the portion of the unit ball* above the plane $z = \frac{1}{2}$.

* The unit ball is the set of points satisfying $x^2 + y^2 + z^2 \leq 1$

Exercise 3.4.4

Use the given figure to show that

$$
\begin{aligned}
x &= \rho\cos\theta\sin\phi \\
y &= \rho\sin\theta\sin\phi \\
z &= \rho\cos\phi.
\end{aligned}
$$

Hint: use right-triangle trigonometry to write $\sqrt{x^2+y^2}$ and z in terms of ρ and ϕ, and then use a different right triangle to write $(x, y, 0)$ in terms of ρ, ϕ, and θ.

Exercise 3.4.5

Determine the graph of the spherical-coordinate equation $\rho = 2\cos\phi$. (Hint: multiply both sides by ρ and then switch to rectangular coordinates.)

Exercise 3.4.6

Determine the graph of $\rho = \sin\phi\sin\theta$.

Exercise 3.4.7

Sketch the set of points satisfying $1 < \rho < 2$ and $\phi < \pi/4$.

4 多元微分

In this chapter, we will be considering functions from $\mathbb{R}^n$ to $\mathbb{R}^1$, where $n \geq 2$. The main objectives will be to extend various important notions in single-variable calculus to the higher-dimensional setting.

4.1 多元函数极限

Limits are useful for discussing a function's output values *near*—but not *at*—a point in its domain. For example, if $f : \mathbb{R} \to \mathbb{R}$ maps 0 to 3 and every nonzero number to 7, then we would say that the limit of $f(x)$ as $x \to 0$ is equal to 7.

We begin with the notion of a limit at 0 for a *monotone* function from* $(0, 1)$ to $\mathbb{R}$. We say that a is a **lower bound** for a function f if its range is a subset of $[a, \infty)$. A function is **bounded below** if it has a lower bound. Likewise, b is an **upper bound** of f if the range of f is a subset of $(-\infty, b]$, and a function is **bounded above** if it has an upper bound.*

* The right endpoint of the interval $(0, 1)$ is immaterial here, since we will be interested in the limit at 0

* For example, $f(x) = x$ is neither bounded above nor below, while $g(x) = e^x$ is bounded below but not above, and $h(x) = \sin x$ is bounded above and below (where f, g, and h are from $\mathbb{R}$ to $\mathbb{R}$)

Definition 4.1.1: Limit of a monotone function on $(0, 1)$

Suppose that $f : (0, 1) \to \mathbb{R}$ is bounded below and has the property that $f(r)$ decreases as r decreases — in other words, we have $f(r) \leq f(s)$ for all $0 \leq r \leq s \leq 1$. Then we define L to be the greatest real number such that $L \leq f(r)$ for all $r \in (0, 1)$.

Similarly, if f is bounded above and has the property that $f(r)$ increases as r decreases, then we define L to be the least real number such that $L \geq f(r)$ for all $r \in (0, 1)$.

In either case, we say that the limit as $r \to 0$ of $f(r)$ exists and is equal to L, that is, $displaystyle \lim_{r \to 0} f(r) = L$.

Example 4.1.1

(a) Find the limit as $r \to 0$ of the function $f(r) = 1 + r^2$ defined on $(0, 1)$.

(b) Find the limit as $r \to 0$ of the function $g(r) = \cos r$ defined on $(0, 1)$.

Solution

(a) Since $f(r)$ decreases as r decreases and f is bounded below, Definition 4.1.1 says that the desired limit exists and is equal to the greatest lower bound of the range of f. Since the range of f is the interval $(1, 2)$, we see that the greatest lower bound is $\boxed{1}$.

(b) Similarly, since $g(r)$ increases as r decreases and is bounded above, Definition 4.1.1 says that the desired limit exists and is equal to the least upper bound of the range of f. We can see from the geometric definition of cosine (see Section A.1.2) that 1 is an upper bound for the range of cos and that no smaller number is an upper bound. Therefore, the limit is equal to $\boxed{1}$.

Definition 4.1.1 has many restrictions: it only applies to monotone, single-variable functions, and it is fundamentally one-sided.* We will drop all these restrictions in one swoop:

* This means that only values of r to the right of zero are considered. Often the notation $\lim_{r\to 0+}$ is used to communicate the exclusion of values to the left, but this notation is not necessary here since the do- mai... clud... tive...

Definition 4.1.2: Limit of a function of function of multiple variables

Let $n \geq 1$. Suppose that $D \subset \mathbb{R}^n$, that $\mathbf{a} \in \mathbb{R}^n$ and that $f : D \to \mathbb{R}$ is a function. For each $r \in (0,1)$, we consider the **punctured ball** $B^\star(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^n : 0 < |\mathbf{x} - \mathbf{a}| \leq r\}$ of radius r centered at $\mathbf{a}$. Suppose that $B^\star(\mathbf{a}, r) \cap D$ is non-empty for all $r > 0$.

We define* $m(r)$ and $M(r)$ so that $[m(r), M(r)]$ is the smallest closed interval which contains the image of $B^\star(\mathbf{a}, r) \cap D$ under f. Then we say that the limit of $f(\mathbf{x})$ as $\mathbf{x} \to \mathbf{a}$ exists if $m(r)$ and $M(r)$ converge to a common value L as $r \to 0$. In that case, we say $displaystyle\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$; otherwise, we say that the limit does not exist.

* Put another way, $m(r)$ is the greatest lower bound and $M(r)$ the least upper bound of the range of f restricted to $B^\star(\mathbf{a}, r) \cap D$

To illustrate this definition, let's look at a function defined on $\mathbb{R}^1$ whose limit at the origin doesn't exist.

Example 4.1.2

Show that the limit of $\cos(1/x)$ does not exist as $x \to 0$.

Solution

For any $r > 0$, the function $\cos(1/x)$ has a maximum value of 1 and a minimum value of -1 in the punctured interval $B^\star(0, r)$, because $\cos(1/x) = 1$ whenever $x = \frac{1}{\pi k}$ for some even integer k, and $\cos(1/x) = -1$ whenever $x = \frac{1}{\pi k}$ for some odd integer k.

Therefore, M is the constant function 1 and m is the constant function -1. These functions converge to different values, so the limit does not exist.

The limit in Example 4.1.2 fails to exist because f is *oscillatory* at the origin. Limit existence failures for functions of multiple variables can look quite different:

Example 4.1.3

Show that* $f(x, y) = -\frac{xy}{x^2+y^2}$ does not have a limit as $(x, y) \to (0, 0)$.

* See Figure 4.2 for a graph

* We use t instead of r since we're already using r as the radius of the punctured ball

Solution

Let's represent the point (x, y) as* $(t\cos\theta, t\sin\theta)$. For all $t > 0$, we have

$$f(t\cos\theta, t\sin\theta) = -\frac{t^2\sin\theta\cos\theta}{t^2(\cos^2\theta + \sin^2\theta)} = -\tfrac{1}{2}\sin 2\theta.$$

Therefore, for any $r > 0$, we have $m(r) = -\frac{1}{2}$ and $M(r) = \frac{1}{2}$. Since these functions converge to unequal values as $r \to 0$, it follows that the limit does not exist.

* See Figure 4.1 for a graph

Example 4.1.4

Use Definition 4.1.2 to show directly that* $displaystyle\lim_{(x,y)\to(0,0)}[3 + x^2 - y^2] = 3$.

Solution

Writing $f(x, y) := 3 + x^2 - y^2$ in terms of polar coordinates as $3 + t^2(\cos^2\theta - \sin^2\theta) = 3 + t^2\cos 2\theta$, we see that the least and greatest values of $f(x, y)$ for $(x, y) \in B^\star((0, 0), r)$ are $m(r) = 3 - r^2$ and $M(r) = 3 + r^2$, respectively. These functions converge as $r \to 0$ to a common value of 3, by Definition 4.1.1. Therefore, $displaystyle\lim_{(x,y)\to(0,0)}[3 + x^2 - y^2] = 3$.

图 **4.1** The image under f of the ball of radius r is the interval $[3 - r^2, 3 + r^2]$, which shrinks down around 3 as $r \to 0$. So the limit exists and equals 3

图 **4.2** The image under f of the ball of radius r is $\left[-\frac{1}{2}, \frac{1}{2}\right]$ no matter how small r is. So the limit does not exist

Exercise 4.1.1

Use Definition 4.1.2 to show that $f(x, y) = \sin\left(\frac{1}{x^2+y^2}\right)$ does not have a limit as $(x, y) \to (0, 0)$.

* Indeed, these values are all equal to $-\frac{1}{2}$

In Example 4.1.3, there are two directions of approach along which f has different limits. Along the ray $\theta = \frac{\pi}{4}$, the values of $f(x, y)$ approach* $-\frac{1}{2}$. Along the ray $\theta = -\frac{\pi}{4}$, $f(x, y)$ converges to $\frac{1}{2}$. This is always an obstruction to the existence of a limit:

Theorem 4.1.1

!!!

Suppose that $\mathbf{r}_1$ and $\mathbf{r}_2$ are paths in the plane with the property that $\mathbf{r}_1(0) = (a, b)$ and $\mathbf{r}_2(0) = (a, b)$. If $\lim_{t\to 0} f(\mathbf{r}_1(t))$ and $\lim_{t\to 0} f(\mathbf{r}_2(t))$ exist and are unequal, show that $displaystyle\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Proof

Let's define $L_i = \lim_{t\to 0} f(\mathbf{r}_i(t))$ for $i = 1$ and $i = 2$. Note that $m(r) \le \min(L_1, L_2)$ for all $r > 0$ since any number larger than $\min(L_1, L_2)$ is not a lower bound for the values of the function on $B^\star((a,b), r)$. Similarly, $M(r) \ge \max(L_1, L_2)$ for all $r > 0$. Therefore, $\lim_{r\to 0} m(r) \le \min(L_1, L_2)$ while $\lim_{r\to 0} M(r) \ge \max(L_1, L_2)$. So the limit of $f(x, y)$ does not exist as $(x, y) \to (a, b)$.

With the notion of a multidimensional limit in hand, we can define continuity the same way we did for the one-dimensional case.

Definition 4.1.3

Suppose $n \ge 2$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at a point in $\mathbb{R}^n$ if and only if the limit of f exists at that point and equals the value of the function there.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be continuous if it is continuous at every point in $\mathbb{R}^n$.

More generally, a function $f : D \to \mathbb{R}$—where $D \subset \mathbb{R}^n$—is said to be continuous if it is continuous at each point in its domain D. The following theorem gives us some tools for establishing continuity.

Theorem 4.1.2: Continuous functions

1. If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then* $g \circ f : \mathbb{R}^n \to \mathbb{R}$ is continuous.
2. A sum or product of continuous functions is continuous.
3. The "coordinate-extracting" functions $f(x, y, z) = x$, $f(x, y, z) = y$, etc., are continuous.

$g \circ f$ denotes the *composition* of g and f. See Appendix A.1.1

Example 4.1.5

Show that $displaystyle \lim_{(x,y,z)\to(0,0,0)} \left(\mathrm{e}^{\sin x} + \frac{xyz}{1+x^2z^2}\right) = 1$.

Solution

We begin by showing that $\mathrm{e}^{\sin x} + \frac{xyz}{1+x^2z^2}$ is continuous. Note that $\mathrm{e}^{\sin x}$ is a composition of continuous functions:

$$(x, y, z) \mapsto x \mapsto \sin x \mapsto \mathrm{e}^{\sin x},$$

Therefore, it's continuous by Theorem 4.1.2. Similarly, $\frac{xyz}{1+x^2z^2}$ is continuous wherever $1 + x^2z^2 \ne 0$, which is everywhere since $(xz)^2 \ge 0$. Finally, the sum of two continuous functions is continuous, so $\mathrm{e}^{\sin x} + \frac{xyz}{1+x^2z^2}$ is continuous.

Since the function above is continuous, its limit at each point is equal to its value at that point. So we substitute $x = y = z = 0$ and find that the value of the function at the origin is $\mathrm{e}^0 + \frac{0}{1+0} = 1$.

Suppose we know that the limits of $f(x, y)$ along every line passing through the origin exist and that they are all equal to some common value L. Does this imply that $\lim_{(x,y)\to(0,0)} f(x, y) = L$? Perhaps it seems that it should, since we've accounted for every possible angle of approach. Remarkably, this turns out not to be the case:

Example 4.1.6

Show that $\lim_{(x,y)\to(0,0)} \frac{-x^2y}{x^4+y^2}$ does not exist even though the limits along every line through the origin exist and are equal.

Solution

We begin by checking the limit along the line $\mathbf{r}(t) = (t\cos\theta, t\sin\theta)$ (which is the line passing through the origin as well as the point on the unit circle whose angle with respect to the positive x-axis is θ). We find If

$$\begin{aligned} f(t\cos\theta, t\sin\theta) &= \frac{-t^3\cos^2\theta\sin\theta}{t^4\cos^4\theta + t^2\sin^2\theta} \\ &= \frac{-t\cos^2\theta\sin\theta}{t^2\cos^4\theta + \sin^2\theta}. \end{aligned}$$

we consider the limit of this expression as $t \to 0$ with θ fixed, then the $\cos\theta$ and $\sin\theta$ factors are constants. So we see that the numerator converges to 0 and the denominator converges to $\sin^2\theta$. Therefore, as long as $\sin\theta \neq 0$, we have $\lim_{t\to 0} f(t\cos\theta, t\sin\theta) = 0/\sin^2\theta = 0$. However, if $\sin\theta = 0$, then $f(t\cos\theta, t\sin\theta) = 0$ for all t, so $\lim_{t\to 0} f(t\cos\theta, t\sin\theta) = 0$ in that case too.

However, note that if we consider the limit along the parabolic path $\mathbf{r}(t) = (t, -t^2)$, we get

$$f(t, -t^2) = -\frac{t^2(-t^2)}{t^4 + (-t^2)^2} = \frac{1}{2}.$$

Therefore, the limit along this path is equal to $\frac{1}{2}$. Thus there are two paths (this one, as well as any straight-line path through the origin) along which f has different limits. Therefore, the limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist.

Note: this makes sense graphically, because this function also has a crease along the z-axis. But now we have to follow a parabolic path to travel along the top "ridge" and realize a limiting value other than zero.

Exercise 4.1.2

Show that $displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$.

Exercise 4.1.3

Consider the function f defined by $f(x,y) = \frac{x-y}{x^3-y}$ whenever $y \neq x^3$, and $f(x,y) = 1$ when $y = x^3$. Show that f is not continuous at $(1,1)$. Evaluate the limits along $x = 1$ and along $y = 1$.

on partial derivatives

4.2 多元函数的偏导数

Suppose f is a function from $\mathbb{R}$ to $\mathbb{R}$. The derivative f' of f is the answer to the question "how does $f(x)$ change when x changes just a little?" More precisely, if $a \in \mathbb{R}$, we define

$$f'(a) = \lim_{h \to 0} \frac{\overbrace{f(a+h) - f(a)}^{\text{how much } f \text{ changes}}}{\underbrace{h}_{\text{how much the input changes}}}$$

This means that if we know $f'(a)$, then we can estimate $f(a+h) - f(a)$ for h very small:

$$f(a+h) - f(a) \approx h f'(a).$$

So the derivative measures **how sensitive $f(x)$ is to small changes in x.**

What is the most natural corresponding idea for the derivative at some point (a, b) of a function f from $\mathbb{R}^2$ to $\mathbb{R}$? We were only able to adjust a value $x \in \mathbb{R}$ by increasing or decreasing it a little. A point in $\mathbb{R}^2$, by contrast, can be moved in any direction. Two directions are particularly easy to study: (i) move x a little while holding y fixed, and (ii) move y a little while holding x fixed. Accordingly, we define **partial derivatives***

$$(\partial_x f)(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}, \text{ and}$$
$$(\partial_y f)(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}.$$

* ∂_x is read "partial x". Also, the role of x here is purely as a label that means "with respect to the first coordinate". It does not represent a number, as the symbol x us

Calculating partial derivatives of elementary functions *isn't actually a new skill*: since one of the two variables is being held constant, we are effectively taking a derivative of a single-variable function.

Example 4.2.1

Find the partial derivatives* f_x and f_y of $f(x, y) = e^x \sin(xy)$ at $(x, y) = (1, 0)$.

* f_x is an alter nate notation for $\partial_x f$, and similarly for y

Solution

We can find the partial derivative with respect to x at *any* point (x, y) by treating y as constant and applying single-variable differentiation rules:*

$$(\partial_x f)(x, y) = e^x y \cos(xy) + e^x \sin(xy)$$
$$(\partial_y f)(x, y) = x \cos(xy) \, e^x$$

So the partial derivatives at $(1, 0)$ with respect to x and y are 0 and e, respectively.

* If you have difficulty getting used to holding a variable constant, consider replacing it with some number like 17; then substitute bacl at the end

Example 4.2.2

Consider the function f whose graph is shown. Determine the sign of $(\partial_x f)(1, 1)$ and the sign of $(\partial_y f)(1, 1)$.

Solution

If we increase x a little while holding y constant, then f decreases. Therefore, $(\partial_x f)(1,1) < 0$. If we increase y a little while holding x constant, then f increases. Therefore, $(\partial_y f)(1,1) > 0$.

Graphically, the partial derivative with respect to x at a point is equal to the slope of the trace of the graph in the "$y =$ constant" plane passing through that point. Similarly, the partial derivative with respect to y at a point is equal to the slope of the trace of the graph in the "$x =$ constant" plane passing through that point.*

* Thus we can think of partial derivatives as an application of our "slice it up" strategy for understanding three dimensional objects through two dimensional traces

Exercise 4.2.1

The following three graphs represent a function f and its two partial derivatives $\partial_x f$ and $\partial_y f$, in some order. Which is which?

The following theorem says that order doesn't matter when successively taking partial derivatives.

Theorem 4.2.1: Clairaut's theorem

Suppose $f : D \to \mathbb{R}$, where D is a disk in $\mathbb{R}^2$. If $\partial_x\partial_y f$ and $\partial_y\partial_x f$ exist and are continuous, then $\partial_x\partial_y f = \partial_y\partial_x f$ throughout D.

Exercise 4.2.2

Verify the conclusion of Clairaut's theorem for $f(x,y) = \mathrm{e}^{xy}\sin y$.

Exercise 4.2.3

Suppose that $f(4.2, 6.8) = 2$, $f(4.3, 6.8) = 3$, $f(4.2, 6.9) = 4$, and $f(4.3, 6.9) = 6$. Justify the approximation $\partial_x\partial_y f(4.2, 6.8) \approx 100$. Apply similar reasoning to obtain the approximation $\partial_y\partial_x f(4.2, 6.8) \approx 100$.

4.3 函数的（线性）逼近

mi on linear approximation

The following example shows that partial derivatives don't tell the whole story when it comes to differentiating functions of multiple variables.

Example 4.3.1

Consider the function f for which $f(0,0) = 0$ and $f(x,y) = -\frac{xy}{x^2+y^2}$ for all $(x,y) \neq (0,0)$. Show that both partial derivatives of f at the origin are equal to zero.

Solution

If we move x a little from $x = 0$ while holding $y = 0$ fixed, the value of f doesn't change at all. Therefore,

$$(\partial_x f)(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

The same is true for the partial derivative with respect to y.

However, recall from Example 4.1.3 that the function in Example 4.3.1 isn't even continuous at the origin! We haven't said yet what is required for a function of two variables to be considered differentiable, but whatever the definition, we surely cannot allow functions which aren't continuous to be deemed differentiable. This shouldn't be surprising: the partial derivatives only look at the behavior of the function along two slices. A good definition of differentiability at (a,b) should account for how the function behaves all around (a,b).

Another perspective on differentiability in the single-variable context is that *differentiable functions are the ones which are well-approximated by linear functions*:

Theorem 4.3.1

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if and only if there exists a linear function $L(x) = c_0 + c_1(x-a)$ such that*

$$\lim_{x\to a} \frac{f(x) - L(x)}{|x-a|} = 0.$$

* This equation says that L approximates f so well that the difference between f and L, *even after being divided by the tiny number* $|x-a|$ still goes to 0 as $x \to a$

This perspective on differentiability turns out to generalize very nicely to functions of multiple variables. Let's make it a definition.

Definition 4.3.1: Differentiability for a function of two variables

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(a,b) \in \mathbb{R}^2$ if and only if there exists a linear function $L(x,y) = c_0 + c_1(x-a) + c_2(y-b)$ such that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Lots of functions are differentiable. The following theorem establishes a handy way to check differentiability.

Theorem 4.3.2: Criterion for differentiability

If the partial derivatives $\partial_x f$ and $\partial_y f$ exist in some disk centered at (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

The most common situation is that partial derivatives exist and are continuous everywhere, in which case Theorem 4.3.2 implies that f is differentiable everywhere.

Example 4.3.2

Show that $f(x, y) = \mathrm{e}^{xy} \sin(x^2 + y^2)$ is differentiable at every point in $\mathbb{R}^2$.

Solution

We can take partial derivatives of f with respect to both x and y and get functions which are built from x and y using addition/multiplication as well as the continuous functions $x \mapsto \mathrm{e}^x$, $x \mapsto \sin x$, and $x \mapsto \cos x$. Therefore, the partial derivatives exist and are continuous everywhere. Thus Theorem 4.3.2 implies that f is differentiable everywhere.

In the denominator we replaced $|x - a|$, whose geometric meaning is the distance from x to a on the number line, with the formula for the distance from (x, y) to (a, b) in the plane.

Graphically, Definition 4.3.1 says that a function is differentiable at (a, b) if we can draw a plane which is tangent* to the graph of f at the point $(a, b, f(a, b))$.

* To interpret this statement as a theorem, we would need to first say what it means for the plane to be *tangent* on some basis other than Definition 4.3.1 (to avoid circular reasoning). So take this as intuition only

In Theorem 4.3.1, the coefficients of the approximating function L are the value of the function f at a and the derivative of f at a. The coefficients in Definition 4.3.1 are also quantities that we have names for: as suggested by Figure 4.3, c_1 is the value of the function at (a, b) and c_1 and c_2 are the two partial derivatives at (a, b).

图 4.3 A plane tangent to the graph of a function f

Theorem 4.3.3: Linear Approximation

!!!

If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(a, b) \in \mathbb{R}^2$, then

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - \overbrace{\left[f(a, b) + (\partial_x f)(a, b)(x - a) + (\partial_y f)(a, b)(y - b)\right]}^{L(x,y)}}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

Let's see how this theorem can be used numerically.

Example 4.3.3

Consider the function $f(x,y) = \frac{e^{xy}}{e(1+x^2)}$. Use a tangent plane to approximate $f(0.99, 0.98)$.

Solution

Noticing that $(0.99, 0.98)$ is very close to $(1,1)$, we differentiate $f(x,y)$ with respect to x and with respect to y and find that*

$$(\partial_x f)(1,1) = \left(\frac{y e^{xy}}{e(x^2+1)} - \frac{2\,x e^{xy}}{e\,(x^2+1)^2} \right)\Bigg|_{(x,y)=(1,1)} = 0.$$

$$(\partial_y f)(1,1) = \frac{x e^{xy}}{e(x^2+1)}\Bigg|_{(x,y)=(1,1)} = \frac{1}{2}.$$

Therefore, $f(0.99, 0.98) \approx f(1,1) + 0(0.99-1) + \frac{1}{2}(0.98-1) = \frac{1}{2} + \frac{1}{2}\cdot(-\frac{1}{50}) = 0.49$.

* The bar notation means "substitute"

The actual value is $0.490197\ldots$

4.4 泰勒公式定理 *

图 **4.4** The first few Taylor polynomials for the exponential function $\exp(x) = e^x$

图 **4.5** The linear Taylor polynomial of a function f centered at the origin

图 **4.6** The quadratic Taylor polynomial of a function f centered at the origin

Let us briefly recall the Taylor* series story for functions of a single variable. The **kth order Taylor polynomial** of an n-times-differentiable function $f : \mathbb{R} \to \mathbb{R}$ centered at a point $a \in \mathbb{R}$ is defined to be

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

* A Taylor polynomial centered at the origin is called a *Maclaurin* polynomial

The terms on the right-hand side are motivated by the idea that as many derivatives of P_k as possible should match those of f at the point a. Then P_k is an excellent approximation of f for values of x near a:

Theorem 4.4.1: Taylor's theorem, functions of a single variable

If I is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is differentiable k times, then

$$\lim_{x \to a} \frac{f(x) - P_k(x)}{(x-a)^k} = 0.$$

The idea in higher dimensions is entirely analogous: given a function $f : \mathbb{R}^n \to \mathbb{R}$, we define the kth order Taylor polynomial of f at a point $\mathbf{a} \in \mathbb{R}^n$ to be the polynomial p such that all of p's *mixed* partial derivatives of order k and lower match those of f at $\mathbf{a}$. For simplicity, we state the theorem for $\mathbb{R}^2$-valued functions.

Theorem 4.4.2: Taylor's theorem, functions of multiple variables

If U is an open set* in $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is differentiable n times, then we define

$$P_k(x,y) = f(a,b) + (\partial_x f)(a,b)(x-a) + (\partial_y f)(a,b)(y-b) \\ + \frac{1}{2!0!}(\partial_x^2 f)(a,b)(x-a)^2 + \frac{1}{1!1!}(\partial_x\partial_y f)(a,b)(x-a)(y-b) + \frac{1}{0!2!}(\partial_y^2 f)(a,b)(y-b)^2 + \cdots,$$

where we continue until we have all terms of the form $\frac{1}{i!j!}(\partial_x^i\partial_y^j f)(a,b)(x-a)^i(y-b)^j$ where $i+j \le k$. Then

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - P_k(x,y)}{|\langle x,y\rangle - \langle a,b\rangle|^k} = 0.$$

* An open set is one that contains a small ball around each of its points—in other words, a set that contains none of the points on its boundary

Exercise 4.4.1

How many terms of degree k appear in $P_k(x,y)$? Write out all the terms of degree 3, and show that all the third-order partial derivatives of f and P_3 match at (a,b).

Exercise 4.4.2

Find the Taylor polynomials P_1 and P_2 for the function $\frac{1}{x^2+y^2+1}$ from $\mathbb{R}^2$ to $\mathbb{R}$ centered at $(0,0)$.

4.5 多变量优化问题

on local extrema

The following problem is a typical example of a single-variable optimization problem.

Example 4.5.1

Find the maximum and minimum of $f(x) = |(x-1)(3-x)|$ over the interval $[0,3]$.

Solution

Since f is continuous over the closed and bounded interval $[0,3]$, we know by the extreme value theorem* that it has a maximum and a minimum value over $[0,3]$. Furthermore, these extrema must be realized at either a *critical point*, at which f is either not differentiable or has derivative zero, or else at an endpoint of the interval.

We check that f is not differentiable at 1 or 3 (see the graph). Also, we can solve $f'(x) = 0$ to find that f has a horizontal tangent line at $x = 2$.

Finally, we can check the values of f at the endpoints 0 and 3, as well as the critical points strictly between them, namely 1 and 2. We find that the maximum value is $f(0) = \boxed{3}$, and the minimum value is $\boxed{0}$, which occurs at $x = 1$ and at $x = 3$.

* The extreme value theorem for single-variable functions says that a continuous function on a closed interval $[a,b]$ achieves a max and a min

How does this story change when we consider a function of multiple variables? For concreteness, let's suppose $D = [0,1]^2$ and that $f : D \to \mathbb{R}$ is a continuous function. Consider the graph of the function

$$f(x,y) = -x^2 - y^2 + x + \frac{2}{3}y + \frac{23}{36},$$

shown in Example 4.5.2. As in the single-variable case, f does indeed have a maximum value and a minimum value. This is ensured by the following multivariable generalization of the single-variable extreme value theorem.

Theorem 4.5.1: Extreme value theorem

Suppose that $D \subset \mathbb{R}^n$. We say that D is **closed** if it contains all of its boundary points. We say that D is **bounded** if it is contained in a ball of radius R for some $R < \infty$.

If D is closed and bounded and if $f : D \to \mathbb{R}$ is a continuous function, then f achieves a maximum value and a minimum value on D. In other words, there exists $\mathbf{a} \in D$ such that $f(\mathbf{a}) \geq f(\mathbf{b})$ for all $\mathbf{b} \in D$.

Furthermore, if f is differentiable at a point $\mathbf{a}$ on the inside of D and $(\partial_x f)(\mathbf{a}) > 0$ or $(\partial_x f)(\mathbf{a}) < 0$, then f cannot have a maximum or minimum value at $\mathbf{a}$ since we can increase the function's output value by slightly adjusting the first coordinate of $\mathbf{a}$ in one direction, and we can decrease it by adjusting it in the opposite direction. So $\partial_x f = 0$ at any value where f has an extremum, and similarly for $\partial_y f$.

Theorem 4.5.2: Critical points

If $f : D \to \mathbb{R}$ achieves its maximum or minimum value at $\mathbf{a} \in D$, then

(i) $(\partial_x f)(\mathbf{a}) = (\partial_y f)(\mathbf{a}) = 0$, or

(ii) f is not differentiable at $\mathbf{a}$, or

(iii) $\mathbf{a}$ is on the boundary of D.

These theorems give rise to a strategy for finding the extrema of a function $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$: (i) set both partial derivatives of f equal to 0 and solve to find critical points inside D (also include any points where f is not differentiable), and (ii) find the extreme values of f on* ∂D. Let's do an example.

* The notation ∂D means "the boundary of D", which is the set of points $p \in \mathbb{R}^2$ such that any small ball centered at p includes points in D and points not in D

Example 4.5.2

Find the extreme values of the function

$$f(x,y) = -x^2 - y^2 + x + \frac{2}{3}y + \frac{23}{36}$$

over the square $[0,1]^2$.

Solution

We begin by finding the critical points inside the square. We find

$$(\partial_x f)(x,y) = -2x + 1$$
$$(\partial_y f)(x,y) = -2y + \frac{2}{3}.$$

These quantities are both equal to zero only when $x = \frac{1}{2}$ and $y = \frac{1}{3}$. So $(1/2, 1/3)$ is the only critical point inside the square.

To optimize f along the $x = 0$ side, we look at

$$f(0, y) = -y^2 + \frac{2}{3}y + \frac{23}{36},$$

which has a critical point at $y = 1/3$. So $(0, 1/3)$ is a **boundary critical point**, and we should also check the two endpoints $(0, 0)$ and $(0, 1)$. Similarly, for the other three sides, we identify the points $(1, 1/3)$, $(1/2, 0)$, and $(1/2, 1)$, as boundary critical points as well as the other two corners $(1, 1)$ and $(1, 0)$. So, all together:

(x, y)	$(0, 0)$	$(1, 0)$	$(0, 1)$	$(1, 1)$	$(0, 1/3)$	$(1, 1/3)$	$(1/2, 0)$	$(1/2, 1)$	$(1/2, 1/3)$
$f(x, y)$	23/36	23/36	11/36	11/36	3/4	3/4	8/9	5/9	1
$f(x, y)$	0.64	0.64	0.31	0.31	0.75	0.75	0.89	0.56	1

So the maximum value is $\boxed{1}$ and the minimum value is $\boxed{\frac{11}{36}}$.

Exercise 4.5.1

Find the maximum value of $f(x, y) = 10x^2y - x$ over the closed triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

4.6 二阶导数

* A function f has a local minimum at a point $\mathbf{a}$ if there is a small neighborhood of $\mathbf{a}$ throughout which the function's values are no smaller than $f(\mathbf{a})$

Recall from single-variable calculus that the second derivative of a twice-differentiable function can—if it is nonzero—be used to ascertain whether a function has a local minimum* or a local maximum at a given critical point. This is because the convexity of a twice-differentiable function indicates whether the graph of the function is shaped like $\smile$ or $\frown$.

The situation in higher dimensions is more subtle. Archetypal examples are

(a) $x^2 + y^2$, which has a bowl-shaped graph like , and so a local minimum at the origin,

(b) $-x^2 - y^2$, which has an umbrella-shaped graph like , and so a local maximum at the origin, and

(c) $x^2 - y^2$, which has a saddle-shaped graph like , and so neither a local min nor a local max at the origin.

The following theorem gives us a direct way to distinguish these cases.

Theorem 4.6.1: Second derivative test

Suppose that U is an open set in $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is a twice-differentiable function with a critical point at (a, b). We define $D = (\partial_x^2 f\, \partial_y^2 f - [\partial_x \partial_y f]^2)(a, b)$. Then*

(a) if $D > 0$ and $(\partial_x^2 f)(a, b) > 0$, then f has a local minimum at (a, b),

(b) if $D > 0$ and $(\partial_x^2 f)(a, b) < 0$, then f has a local maximum at (a, b), and

(c) if $D < 0$, then f has a saddle point at (a, b).

* When $D = 0$ this theorem doesn't tell us anything

A proof of the second derivative test is given in Appendix A.2.3.

图 **4.7** If every vertical slice of the graph of f (passing through the origin) is convex, then f has a local minimum at the origin

图 **4.8** If every slice is concave, then f has local maximum

图 **4.9** If there are both concave and convex slices, then f has a saddle point

Example 4.6.1

Find the critical points of the function $f(x, y) = (2x^2 + 3y^2)\, \mathrm{e}^{-x^2-y^2}$ defined on $\mathbb{R}^2$, and classify each critical point as a local minimum, a local maximum, or a saddle point.

Solution

We have

$$\begin{aligned} \partial_x f &= 2x\left(-2x^2 - 3y^2 + 2\right)\mathrm{e}^{-x^2-y^2}, \text{ and} \\ \partial_y f &= 2y\left(-2x^2 - 3y^2 + 3\right)\mathrm{e}^{-x^2-y^2}. \end{aligned}$$

To find the critical points of f, we look for all pairs (x, y) for which both of these expressions are equal to zero. We see that $x = 0$ implies $\partial_x f = 0$. In that case, we have $\partial_y f = 0$ if and only if $y \in \{-1, 0, 1\}$. Similarly, $y = 0$ implies $\partial_y f = 0$, and in that case we have $\partial_x f = 0$ only if $x \in \{-1, 0, 1\}$. If neither x nor y is zero, then we may divide the two equations by x and y, respectively, and we arrive at a contradiction since the resulting left-hand sides are unequal and therefore cannot both be equal to zero. Therefore, we have found all the critical points.

To apply the second derivative test, we work out that*

Definitely want to use some computational assistance here

$$D = \partial_x^2 f \partial_y^2 f - (\partial_x \partial_y f)^2 = \Big(-32x^6 - 128x^4y^2 + 128x^4 - 168x^2y^4 + 328x^2y^2 - 136x^2 - 72y^6 + 228y^4 - 156y^2 + 24\Big)\mathrm{e}^{-2x^2-2y^2}.$$

Then we check the sign of D at each critical point:

(x,y)	$(0,0)$	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$
$De^{2x^2+2y^2}$	24	-16	-16	24	24

We see that the points $(1,0)$ and $(-1,0)$ are saddle points of f, while f has a local extremum at the origin and at the points $(0,0)$, $(0,1)$ and $(0,-1)$. To classify these extrema, we check that $\partial_x^2 f$ at these points is equal to 4, $-2/e$, and $-2/e$, respectively. Thus f has a local minimum at the origin and a local maximum at $(0,1)$ and $(0,-1)$.
These classifications accord with the graph of f, shown in Figure 4.10.

图 4.10 The graph of f

Exercise 4.6.1

Find all the critical points of $f(x,y) = x(x+y)(y+y^2)$ and apply the second derivative test to classify as many as possible as a local minimum, a local maximum, or a saddle point.

Exercise 4.6.2

Use the examples x^4+y^4 and $-x^4-y^4$, which have a critical point at the origin, to show that the second derivative test must be inconclusive when $D=0$.

Exercise 4.6.3

Show that in parts (a) and (b) of the second derivative test, we have $(\partial_x^2 f)(a,b) > 0$ if and only if $(\partial_y^2 f)(a,b) > 0$.

4.7 多元函数的方向导数和梯度

mi on directional derivatives and the gradient

* That is: up/down, left-/right

The two partial derivatives of a function $f:\mathbb{R}^2 \to \mathbb{R}$ tell us how f changes when (x,y) is wiggled a bit, but only in the four cardinal directions.* What about all the other directions? Suppose that $\mathbf{u}$ is a **unit vector** in $\mathbb{R}^2$, meaning that its length is 1. (see Figure 4.11).

Definition 4.7.1: Directional derivative

The **directional derivative** of a function $f:\mathbb{R}^2 \to \mathbb{R}$ in the direction $\mathbf{u} \in \mathbb{R}^2$ is defined by

$$D_{\mathbf{u}}(f)(a,b) = \lim_{h\to 0} \frac{f((a,b)+h\mathbf{u}) - f(a,b)}{h}.$$

In other words, move (x,y) a small distance h in the $\mathbf{u}$ direction, measure how much f changed, and then divide by h.

图 4.11 The derivative of f in the direction $\mathbf{u}$

If f is differentiable at (a,b), then it is well-approximated by a linear function $L(x,y) = c_0 + c_1(x-a) + c_2(y-b)$ around (a,b). This linear function has the property that its slope in the $\mathbf{u}$ direction can be worked out by separating the $h\mathbf{u}$-step into a $\langle hu_1, 0\rangle$ step followed by a $\langle 0, hu_2\rangle$ step. The value of L changes by hc_1u_1 over the $\langle hu_1, 0\rangle$ step and by hc_2u_2 over the $\langle 0, hu_2\rangle$ step. Altogether, the change of L is equal to $(c_1u_1 + c_2u_2)h$. This leads to the following theorem.

Theorem 4.7.1: Directional derivative formula

If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) and $\mathbf{u} \in \mathbb{R}^2$, then

$$D_{\mathbf{u}} f(a,b) = (\partial_x f)(a,b)u_1 + (\partial_y f)(a,b)u_2 = (\nabla f)(a,b) \cdot \mathbf{u},$$

where $(\nabla f)(a,b) = \langle (\partial_x f)(a,b), (\partial_y f)(a,b)\rangle$.

The quantity $(\nabla f)(a,b)$ introduced in Theorem 4.7.1—the vector of partial derivatives of f at (a,b)—is called the **gradient** of f at (a,b). Observe that the directional derivative of f at (a,b) in the $\mathbf{u}$ direction is equal to $\nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$, where θ is the angle between ∇f and $\mathbf{u}$. Since $\cos\theta$ is maximized when $\theta = 0$, we see that **the gradient of f at (a,b) is f's direction of maximum increase at (a,b)**. Furthermore, the direction opposite to the gradient is the direction of maximum decrease, and f has zero derivative in any direction orthogonal to the gradient. !!!

The function from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps (a,b) to $(\nabla f)(a,b)$ is called the gradient of f and is denoted ∇f. Thus the *gradient of f at a point* is a vector, while the *gradient of f* is a vector-valued function defined on $\mathbb{R}^2$.

Example 4.7.1

Find $(\nabla f)(3,4)$, where $f(x,y) = x^2 + xy + y^2$. Find all possible values of $D_{\mathbf{u}} f(3,4)$, where $\mathbf{u}$ is any unit vector.

Solution

We have $(\partial_x f)(x,y) = 2x + y$ and $(\partial_y f)(x,y) = 2y + x$, so

$$\nabla f = \langle 2x + y, 2y + x\rangle.$$

Therefore, the gradient evaluated at $(3,4)$ is equal to $\boxed{\langle 10, 11\rangle}$.

Since $D_{\mathbf{u}} f(3,4) = \langle 10, 11\rangle \cdot \mathbf{u} = \sqrt{221}\cos\theta$, where θ is the angle between $\langle 10, 11\rangle$ and $\mathbf{u}$, we see that as $\mathbf{u}$

ranges over the unit circle, the directional derivative of $\mathbf{f}$ in the $\mathbf{u}$ direction ranges over $\boxed{[-\sqrt{221}, \sqrt{221}]}$.

Example 4.7.2

Some of the level curves of a function $g(x, y)$ are shown. Sketch the direction of the gradient at the marked point.

Solution

!!!

The key idea is that a function neither increases or decreases along its level curve. Therefore, g has directional derivative equal to 0 in the direction of any line tangent the level curve passing through a given point. This means that the **gradient of g is orthogonal to g's level curve** at any given point. So the gradient looks like the figure shown (zoomed in).

The gradient of a function from $\mathbb{R}^3$ to $\mathbb{R}$ is likewise defined to be the vector of its partial derivatives: $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$. The formula $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ holds for all differentiable functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^3$.

Example 4.7.3

Find the equation of a plane tangent to the ellipsoid $x^2 + y^2 + 2z^2 = 4$ at the point $(1, 1, 1)$.

Solution

The ellipsoid is a level set of the function $g(x, y, z) = x^2 + y^2 + 2z^2$. The direction vectors $\mathbf{u}$ contained in the plane of tangency at $(1, 1, 1)$ are characterized by the fact that g is unchanging in the $\mathbf{u}$ direction at $(1, 1, 1)$. Since $D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u}$, this means that* $\mathbf{u}$ is orthogonal to ∇g. So we calculate

$$(\nabla g)(1, 1, 1) = \langle 2x, 2y, 4z \rangle|_{(x,y,z)=(1,1,1)} = \langle 2, 2, 4 \rangle,$$

and see that the desired plane is orthogonal to $\langle 2, 2, 4 \rangle$ and passes through $(1, 1, 1)$. So the equation of the plane is

$$\langle 2, 2, 4 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0 \implies \boxed{x + y + 2z = 4}.$$

* All together: **the gradient of a function at a point is orthogonal to the function's level set through that point**

Exercise 4.7.1

(a) Confirm that the gradient of $x^2 + y^2$ is orthogonal to the level curves of $x^2 + y^2$ at each point.

(b) Confirm that the gradient of $x^2 + y^2 + z^2$ is orthogonal to the level surfaces of $x^2 + y^2 + z^2$ at each point.

mi on the chain rule

4.8 多变量链式法则

The basic idea of the chain rule is that when considering how $f(g(t))$ changes when we increase t by some small amount h, we can note that $g(t)$ changes by approximately $hg'(t)$, and that change in the input to f induces a change of

$$f(g(t+h)) - f(g(t)) \approx \left(\overbrace{hg'(t)}^{\text{change in input to } f} \right) \left(\overbrace{f'(g(t))}^{\text{sensitivity of } f \text{ to change in input}} \right)$$

in the value of $f(g(t))$. Thus $\frac{d}{dt} f(g(t)) = g'(t) f'(g(t))$.

The simplest multivariable generalization of this idea is to define a function from $\mathbb{R}$ to $\mathbb{R}$ by composing a differentiable function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ with a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. Let's look at an example.

Example 4.8.1

Suppose $f(x, y) = \sin xy \cos y$ and $\mathbf{r}(t) = (\mathrm{e}^t, t^2)$. Find the derivative of $f \circ \mathbf{r}$.

Solution

We can calculate directly

$$(f \circ \mathbf{r})(t) = f(\mathbf{r}(t)) = \sin(t^2 \mathrm{e}^t) \cos t^2.$$

So the desired derivative is

$$\begin{aligned} &\cos(t^2 \mathrm{e}^t) \left[t^2 \mathrm{e}^t + 2t \mathrm{e}^t \right] \cos t^2 - \sin(t^2 \mathrm{e}^t) 2t \sin t^2 \\ &\qquad = t^2 \mathrm{e}^t \cos(t^2 \mathrm{e}^t) \cos t^2 + 2t \mathrm{e}^t \cos(t^2 \mathrm{e}^t) \cos t^2 - 2t \sin(t^2 \mathrm{e}^t) \sin t^2. \end{aligned}$$

In our solution to Example 4.8.1, we composed the given functions before differentiating. The **multivariable chain rule** is an alternate approach which describes the general relationship between $(f \circ \mathbf{r})'(t)$ and the derivatives of f and $\mathbf{r}$. Let's write $\mathbf{r}(t) = \langle r_1(t), r_2(t) \rangle$. When we change t by h, the value of $f(\mathbf{r}(t))$ changes as follows:

$$f\left(\overbrace{r_1(t)}^{\text{changes by } hr_1'(t)}, \overbrace{r_2(t)}^{\text{changes by } hr_2'(t)} \right)$$

The change of $hr_1'(t)$ in the first argument induces a change of $hr_1'(t)(\partial_x f)(\mathbf{r}(t))$ in the value of f, while the change of $hr_2'(t)$ in the second argument induces a change of $hr_2'(t)(\partial_y f)(\mathbf{r}(t))$. By linear approximation, the overall change is the *sum* of these two changes.

图 4.12 The composition $f \circ \mathbf{r}$ can be visualized using a slice of the graph of f along the path $\mathbf{r}$. The graph of the composition $f \circ \mathbf{r}$ is shown in the red inset figure.

More precisely, if we define L to be the linear approximation of f at $\mathbf{r}(t)$, and then we have*

$$\begin{aligned}(f\circ\mathbf{r})'(t) &= \lim_{h\to 0}\frac{f(\mathbf{r}(t+h))-f(\mathbf{r}(t))}{h}\\ &= \lim_{h\to 0}\frac{L(\mathbf{r}(t+h))-L(\mathbf{r}(t+h))+f(\mathbf{r}(t+h))-f(\mathbf{r}(t))}{h}\\ &= \lim_{h\to 0}\left[\frac{L(\mathbf{r}(t+h))-f(\mathbf{r}(t))}{h}+\frac{f(\mathbf{r}(t+h))-L(\mathbf{r}(t+h))}{h}\right].\end{aligned}$$

* In the first step we're essentially swapping out f for L, the idea being that they're essentially the same when zoomed way in around

The second term goes to 0 as $h\to 0$ since f is differentiable, so we're left with*

$$\lim_{h\to 0}\frac{L(\mathbf{r}(t+h))-f(\mathbf{r}(t))}{h}=\lim_{h\to 0}\frac{(\partial_x f)(a,b)(r_1(t+h)-r_1(t))+(\partial_y f)(a,b)(r_2(t+h)-r_2(t))}{h}.$$

* The change in L is x-slope times x-change plus y-slope times y-change.

Taking $h\to 0$ gives us a factor of $r_1'(t)$ in the first term and $r_2'(t)$ in the second term, yielding the following theorem.

Theorem 4.8.1: Multivariable chain rule

If $f:\mathbb{R}^2\to\mathbb{R}$ and $\mathbf{r}=\langle r_1,r_2\rangle:\mathbb{R}\to\mathbb{R}^2$, are differentiable, then

$$(f\circ\mathbf{r})'(t)=(\partial_x f)(\mathbf{r}(t))r_1'(t)+(\partial_y f)(\mathbf{r}(t))r_2'(t)=(\nabla f)(\mathbf{r}(t))\cdot\mathbf{r}'(t). \tag{4.8.1}$$

The chain rule (4.8.1) can be written with the more suggestive notation*

* $\frac{\partial f}{\partial x}$ means $\partial_x f$

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt},$$

where x and y represent r_1 and r_2. Although this formula is more memorable, it does involve some abuse of notation: the symbols x and y are being used* as independent variables (in the partial derivative expressions) and as function names (in dx/dt and dy/dt). Also, on the left-hand side f is being treated as a function of a single variable; actually this instance of f is shorthand for the single-variable function $f\circ\mathbf{r}$.

* using the same symbol to mean different things (relying on context to distinguish) is called *overloading* the symbol

Exercise 4.8.1

Verify that applying the multivariable chain rule to Example 4.8.1 gives the same result we found by calculating that derivative directly.

Exercise 4.8.2

Find the derivative with respect to t of the function $g(t)=t^t$ by writing the function as $f(x(t),y(t))$ where $f(x,y)=x^y$ and $x(t)=t$ and $y(t)=t$.

4.9 拉格朗日乘数

Consider the function

$$f(x,y)=-x^2-y^2+x+\frac{2}{3}y+\frac{23}{36},$$

which we optimized over the square $[0,1]^2$ in Example 4.5.2. In that case, we identified possible extreme values on the boundary of the square by doing a single-variable optimization along each edge of the square. But suppose that we want to find the maximum and minimum values of f over a disk D (see Figure 4.13)? Let's consider the disk D of radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

* To *parameterize* a curve means to find a path which traces it out

The minimum is $0.\overline{5}$

We could take a similar approach to this problem. We can parameterize* the boundary of the disk as

$$\mathbf{r}(t) = \left\langle \tfrac{1}{2} + \tfrac{1}{2}\cos t, \tfrac{1}{2} + \tfrac{1}{2}\sin t \right\rangle, \quad 0 \le t < 2\pi.$$

Then the single-variable function $t \mapsto f(\mathbf{r}(t))$ can be optimized over $[0, 2\pi]$ using the standard single-variable technique (as in Example 4.5.1).

图 **4.13** The graph of a function f defined on a disk D

However, this approach is limited because it requires a parameterization of the boundary of D, which is not always convenient. Suppose that ∂D is specified as a level set of some function $g : \mathbb{R}^2 \to \mathbb{R}$. For example, the circle in Figure 4.13 is a level set $\left\{(x, y) : g(x, y) = \frac{1}{2}\right\}$ of the function

$$g(x, y) = \left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2.$$

Let's derive an approach to finding the extreme values on the boundary which begins with the functions f (the *objective* function) and g (the *constraint* function).

Imagine a bug moving around on the edge of the graph in Figure 4.13. How can it tell that it is *not* at a maximum or minimum? One approach is to calculate the gradient of f at its location. If the gradient of f is not orthogonal to ∂D, then the value of the function can be increased by sliding a bit in one direction* and can be decreased by sliding a bit in the opposite direction. So, for example, in Figure 4.14, a bug at the point p could increase the value of f at its location by moving slightly clockwise and decrease the value of f by moving slightly counterclockwise around ∂D.

图 **4.14** $\mathbf{q}$ is a boundary critical point and $\mathbf{p}$ is not

* Specifically, the direction where ∇f is leaning (that is, the direction whose dot product with ∇f is positive)

Therefore, if the gradient of f at a point is *not* orthogonal to ∂D, then f does not have an extreme value there. So to find points where f might have an extreme value on ∂D, we can restrict our attention to the points where ∂f is orthogonal to ∂D.

We can simplify this idea further: recall that the gradient of g at each point is orthogonal to the level set of g passing through that point. It follows that if ∂D is a level set of g and $p \in \partial D$ is a point where f has an extreme value, then ∇g and ∇f are both orthogonal to ∂D. If $\nabla g \neq \mathbf{0}$, this means that they are parallel! By Observation 2.1.1, then, there exists a scalar λ such that $\nabla f = \lambda \nabla g$.

* So far we've been considering the restriction of f to the boundary of a region D, but any level set of a differential function g will do

Theorem 4.9.1: Method of Lagrange multipliers

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are differentiable functions and $c \in \mathbb{R}$. If the restriction of f to the level set* $\{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) = c\}$ has a local extremum at $\mathbf{x} \in \mathbb{R}^n$, then either $\nabla g(\mathbf{x}) = \mathbf{0}$ or

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}).$$

!!!

Theorem 4.9.1 implies that we can optimize $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = c$ by solving the system of **Lagrange equations**

$$\begin{cases} \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) = c, \end{cases}$$

with the only caveat that we have to watch out for the possibility that $\nabla g = \mathbf{0}$. Let's see how this works out for the example from the beginning of the section.

Example 4.9.1

Find the maximum and minimum values of

$$f(x, y) = -x^2 - y^2 + x + \frac{2}{3}y + \frac{23}{36}$$

over the disk of radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Solution

The only interior critical point is $\left(\frac{1}{2}, \frac{1}{3}\right)$, as in Example 4.5.2. To find boundary critical points, we use $g(x, y) = \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2$ as our constraint function and set up the Lagrange equations

$$\begin{aligned} \partial_x f &= \lambda \, \partial_x g \\ \partial_y f &= \lambda \, \partial_y g \implies \\ -2x + 1 &= \lambda(2x - 1) && (4.9.1) \\ -2y + \tfrac{2}{3} &= \lambda(2y - 1). && (4.9.2) \end{aligned}$$

We're looking for pairs (x, y) which satisfy both of these equations **and** the equation

$$g(x, y) = \left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4} \qquad (4.9.3)$$

since (x, y) must be on ∂D. So we have three equations and three variables: x, y, and λ. The first equation implies that either $\lambda = -1$ or $x = \frac{1}{2}$.

In the case $x = \frac{1}{2}$, we can use the constraint equation to conclude that $y = 0$ or $y = 1$. In either case, we can substitute into (4.9.2) to get a value for λ so that all three equations are satisfied. So, we have $(x, y) = (1/2, 0)$ and $(1/2, 1)$ as boundary critical points.

If $x \neq \frac{1}{2}$, then $\lambda = -1$. Substituting into (4.9.2) gives a contradiction, which means that we've already found all the boundary critical points.

Finally, we evaluate f at the interior critical point and the two boundary critical points:

(x, y)	$\left(\frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, \frac{1}{3}\right)$
$f(x, y)$	$\frac{8}{9}$	$\frac{5}{9}$	1

So the maximum of f over D is 1, and the minimum is $\frac{5}{9}$.

The following example is a 3D application of Lagrange multipliers.

Example 4.9.2

Find the maximum possible volume of a box made with 72 square centimeters of cardboard and having sides and a bottom but no top.

Solution

Denote by x, y, and z the dimensions (in centimeters) of the cardboard. Then the amount of cardboard used is

$$g(x, y, z) = 2yz + 2xz + xy = 72,$$

while the objective function is the volume $f(x, y, z) = xyz$. Setting up the Lagrange equations, we get $yz = \lambda(2z + y)$ $xz = \lambda(2z + x)$, $xy = \lambda(2x + 2y)$, and $2yz + 2xz + xy = 72$, where the last one is the constraint equation. Multiplying the first two equations by x and y, respectively, and setting the resulting right-hand sides equal implies that either $\lambda = 0$ or $z = 0$ or $x = y$. Since $\lambda = 0$ or $z = 0$ clearly give zero volume (and thus not the maximum volume), it follows that $x = y$. Substituting y for x in the third equation gives*

$$x^2 = 4\lambda x \implies \lambda = \tfrac{x}{4}.$$

Substituting this into the second equation and simplifying, we get $x = 2z$. Finally, substituting into the constraint equation gives $z = \sqrt{6}$, which in turn implies $x = y = 2\sqrt{6}$.

* Once again, we can divide by x because we know that $x = 0$ wouldn't make sense for the maximum volume

Exercise 4.9.1

Find the set of all critical points of $f(x, y, z) = 3 - x^2 - 2y^2 - z^2$ subject to the constraint $2x + y + z = 2$.

Exercise 4.9.2

Find the points on the ellipse $\left(\frac{x-1}{2}\right)^2 + (y - 2)^2 = 1$ which are nearest and farthest from the origin. Hint: for the objective function, use *squared* distance rather than distance.

Exercise 4.9.3

Find the maximum value of $f(x, y) = y$ for any point (x, y) on the curve $y^2 - 4x^3 + 4x^4 = 0$ in two ways: (i) using Lagrange multipliers, and (ii) writing the upper half of the given curve as the graph of a function and maximizing that function using standard single-variable techniques.

Exercise 4.9.4

Suppose that we want to maximize a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ subject to the constraint $g(x, y) = c$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $c \in \mathbb{R}$. Consider the *Lagrangian* function

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

Show that the equation $\nabla\mathcal{L} = 0$ is equivalent to the Lagrange equations.

5 重积分

To find the area under the graph of a continuous function f over the unit interval $[0,1]$, we first approximate the area by splitting $[0,1]$ into many short intervals and sum up the areas of rectangles approximating the area under the graph over each short interval:

This approximation converges to the actual area under the graph as $n \to \infty$.

In this section we will work out how to generalize this concept to integrate of functions of multiple variables over regions in $\mathbb{R}^2$ or $\mathbb{R}^3$.*

* See Appendix **??** for a more in-depth discussion of integration

5.1 二重积分

mi on double integrals

We can state the definition of the integral, described above, more informally and generally: split the region of integration into many tiny pieces, multiply the volume* of each piece by the value of the function at some point on that piece*, and add up the results. If we take the number of pieces to ∞ and the piece size to zero, then this sum should converge to a number, and if it does then we declare that number to be the value of the integral.

* Recall our convention that 1D volume is length and 2D volume

* It doesn't ultimately matter where we evaluate the function, since the piece is very small and the function is continuous. If we want an overestimate/underestimate of the integral, we can use imu mum func each

Stated at this level of generality, the idea of the integral definition applies to a function $f : \mathbb{R}^2 \to \mathbb{R}$ over a bounded region $D \subset \mathbb{R}^2$. See Figure 5.1 for an illustration, and see Appendix **??** for a more general definition.

Definition 5.1.1: Integral over a 2D region

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and that D is a bounded region in $\mathbb{R}^2$ such that ∂D has zero area. Then the limit*

$$\lim_{n\to\infty} \sum_{(i,j)\,:\,\left(\frac{i}{n},\,\frac{j}{n}\right)\in D} f\left(\frac{i}{n}, \frac{j}{n}\right) \overbrace{\frac{1}{n^2}}^{\Delta A} \tag{5.1.1}$$

exists. We define the integral of f over D, denoted $\iint_D f\,dA$ or $\int_D f\,dA$, to be the value of this limit.

* The notation means that the sum includes one term for each integer pair (i,j) such that $(i/n, j/n)$ is in the region D

The sums appearing on the right-hand side of (5.1.1) are called **Riemann sums**. As in the single-variable case, the Riemann-sum definition is not generally practical for exact evaluation of integrals. The fundamental theorem of

图 5.1 The integral of f over a disk D, defined as a limit of sums of volumes of narrow boxes

* The fundamental theorem of calculus says that the integral of f from a to b is equal to $F(b) - F(a)$ where F is an antiderivative of f

calculus* is the primary tool for evaluating integrals in single-variable calculus, and fortunately we can bootstrap our way up from 1D integration to 2D integration by applying our primary strategy for tackling higher dimensional problems: slicing. Let's start by considering integrals over rectangular regions D.

Example 5.1.1

Find the integral of $f(x, y) = y\sin(\pi xy)$ over the square $[0, 1]^2$.

Solution

Let's slice up the desired solid using many 'y = constant' cuts, producing many thin slices like the one shown. The volume of one of these slices, situated at a particular y-value, is given by* the thickness Δy times the area $A(y)$ under the graph of the single-variable function $x \mapsto f(x, y)$. So we can use the fundamental theorem of calculus to compute

* ...ignoring an error, having to do with the top of the slice not being flat—this error tends to zero as the number of slices tends to infinity

$$\begin{aligned} A(y) &= \int_0^1 y\sin(\pi xy)\,dx = -\frac{\cos(\pi xy)}{\pi}\bigg|_0^1 \\ &= \frac{1-\cos \pi y}{\pi}. \end{aligned}$$

Once we have each area $A(y)\Delta y$, we can add them all up and take $\Delta y \to 0$ (as the number of slices goes to ∞) to find that the desired volume is

$$\sum_{\text{all slices}} A(y)\Delta y \to \int_0^1 A(y)\,dy.$$

We can again evaluate this integral using the fundamental theorem to get a final answer of $\boxed{\frac{1}{\pi}}$.

For some confidence that our answer is reasonable, we can calculate a Riemann sum for this integrand.

We can express this process more succinctly as

$$\int_0^1 \int_0^1 y\sin(\pi xy)\,dx\,dy = \int_0^1 \frac{1-\cos \pi y}{\pi}\,dy = \frac{1}{\pi}. \tag{5.1.2}$$

The first expression in (5.1.2) is called an **iterated integral**, since it expresses an integral over a 2D region in terms of two successive single-variable integrals.

Let's see how this works over a non-rectangular region.

Example 5.1.2

Find the integral over the triangle T with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$ of the function $f(x, y) = x^2y$, by first finding the area under each 'y = constant' slice.

Solution

As in the previous example, we slice up the desired volume making many '$y =$ constant' cuts of thickness Δy, yielding thin slices such that each one has volume (very close to) $A(y)\Delta y$, where y is the slice's signed distance from the xz-plane and $A(y)$ is the area of the cross-section (see figure). Since this cross section is an area under a curve, we can find it by integrating $x \mapsto f(x, y)$ over the set of relevant x-values:*

$$A(y) = \int_0^{2-\frac{2}{3}y} f(x, y)\, dx.$$

Thus $A(y) = \frac{1}{3}\left(2 - \frac{2}{3}y\right)^3 y$. Finally, adding up all these areas and taking $\Delta y \to 0$ gives the result

$$\int_0^3 A(y)\, dy = \int_0^3 \left(-\frac{8}{81}y^4 + \frac{8}{9}y^3 - \frac{8}{3}y^2 + \frac{8}{3}y\right) dy = \boxed{\frac{6}{5}}.$$

* We can find the formula $2 - \frac{2}{3}y$ by writing an equation for the line connecting $(2, 0)$ to $(0, 3)$ and solving for x

Let's summarize what we figured out in Example 5.1.2.

Theorem 5.1.1: Iterated integrals for two-variable functions

Suppose that

- D is a region in $\mathbb{R}^2$,
- $f : D \to \mathbb{R}$ is a continuous function, and
- for all $y \in \mathbb{R}$, the intersection of D and horizontal line through $(0, y)$ is a segment $[c(y), d(y)] \times \{y\}$.

Then

$$\iint_D f\, dA = \int_a^b \int_{c(y)}^{d(y)} f(x, y)\, dx\, dy.$$

In light of Theorem 5.1.1, we sometimes write the area differential* as $dA = dx\, dy$. We can describe the procedure in Theorem 5.1.1 less formally:

* Think of dA merely as a reminder that the positive quantity ΔA involved in the corresponding Riemann sums represents an area

Observation 5.1.1: Limits of integration over a 2D region

To set up an iterated integral to evaluate $\iint_D f\, dA$ (where f is continuous and D is a region such that the intersection of every horizontal line with D is a segment):*

1. Find the least and greatest y values for any point in D. These are your **outer limits** of integration.
2. For each fixed horizontal line which intersects D, identify the least and greatest values of x for any point which is in D *and on that line*, expressed in terms of the vertical position y of the line. These are the **inner limits** of integration, and they may depend on y.

* You should be prepared for these steps to spawn geometric subproblems that you might need to solve on the side

The role of x and y in Observation 5.1.1 can be reversed (in which case we have vertical rather than horizontal lines in Step 2). The following exercise shows how this can be useful.

Exercise 5.1.1

Find

$$\int_0^{1/2} \int_{2y}^{1} 4e^{x^2}\, dx\, dy$$

by first rewriting it as an integral over a 2D region and then reversing the order of integration.

on triple integrals

5.2 三重积分

We interpret the integral of a single-variable function as an area and the integral of a two-variable function as a volume. So how should we interpret the integral of a function of *three* variables over a region D in $\mathbb{R}^3$? *Four-dimensional volume* is a reasonable answer, but of course this is unsatisfactory from a visualization point of view, since we don't have access to four spatial dimensions with which to visualize.

Therefore, let's consider a physics interpretation of integration which permits a visualization *not* involving the graph of the function being integrated.

Example 5.2.1

Consider a square plate occupying the square $[1,3]^2$ whose density at each point is $\sigma(x,y) = xy$ kilograms per square unit.* Find the mass of the plate.

* See the figure in the solution, where darker color indicates a denser portion of the plate

Solution

Let's imagine physically cutting the plate into small squares, computing the mass of each one, and adding up the resulting masses. The mass of a small plate of area $\Delta x \Delta y$ containing the point (x,y) is approximately the area density times the area: $\sigma(x,y)\,\Delta x\,\Delta y$. The sum of these masses is a Riemann sum (see Definition 5.1.1) which converges as the number of small squares goes to ∞ to the integral

$$\int_1^3 \int_1^3 xy\, dx\, dy = \boxed{16} \text{ kilograms.}$$

Let's do a three-dimensional example.

Example 5.2.2

Consider a cubical block occupying $D = [1,2]^3$ whose density at each point is $delta(x,y,z) = x^2 + y^2 + z^2$ kilograms per cubic unit. Find the mass of the block.

Solution

We cut the cube into n^3 small cubes, where n is a large integer. The mass of one of these cubes with bottom, back* corner (x, y, z) is approximately equal to the product of its volume $\frac{1}{n^3}$ and the approximate density $delta(x, y, z)$ throughout the small cube. So the approximate volume is

$$\sum_{\text{all cubes}} delta\,(x, y, z)\,\frac{1}{n^3}.$$

Intuitively, this sum should converge to a limit as $n \to \infty$, and if so, then we should define the limiting value to be the integral of $delta$ over D. Let's state this idea for any continuous function $f : \mathbb{R}^3 \to \mathbb{R}$: we define the integral of f over D by

$$\iiint_D f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{(i,j,k)\,:\,\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)\in D} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)\frac{1}{n^3}.$$

We can calculate the integral by slicing up the region of integration into thin slabs along 'z = constant' slices, and then performing double integrals to find the area of each slab. This works the same as double iterated integration, but with one extra step. Rather than writing Δz and then taking a limit to turn Δz into dz, we'll skip to the limit and work directly with dz*

$$\begin{aligned}
\text{mass} &= \int_1^2 \overbrace{\int_1^2 \int_1^2 (x^2 + y^2 + z^2)\, dx\, dy\, dz}^{\text{mass of slice from } z \text{ to } z+dz} \\
&= \int_1^2 \int_1^2 \left(y^2 + z^2 + \frac{7}{3}\right) dy\, dz \\
&= \int_1^2 \left(z^2 + \frac{14}{3}\right) dz \\
&= \boxed{7}\ \text{kilograms.}
\end{aligned}$$

* Any corner, or indeed any point in the cube, would give the same result—we choose the bottom, back corner (the point nearest the origin) for concreteness

* This is a general theme: we contract the following two steps into a single step (by writing dz instead of Δz from the outset): (i) reason about sums involving a small but positive quantity Δz, and (ii) replace the sum with an integral over the relevant z values and replace Δz with dz

The following theorem summarizes the idea of integrating in 3D by breaking down the 3D region of integration into 2D slices.

Theorem 5.2.1: Iterated integrals for three-variable functions

Suppose f is a continuous function over a region D which is bounded between the planes $z = a$ and $z = b$. For each $z \in (a, b)$, define $D_z \subset \mathbb{R}^2$ to be the region*

$$D_z = \{(x, y) \in \mathbb{R}^2 : (x, y, z) \in D\}.$$

Then

$$\iiint_D f\, dV = \int_a^b \left[\iint_{D_z} f(x, y, z)\, dx\, dy\right] dz,$$

* D_z is the region obtained by intersecting D with the plane which is z units from the xy-plane and then dropping off the third coordinate

Let's break this theorem down into a simple algorithm (the following observation is the 3D analogue of Observation 5.1.1):

Observation 5.2.1: Limits of integration over a 3D region

To set up an iterated integral to evaluate $\iiint_D f\,dV$ (where f is continuous and D is a region which intersects every line parallel to the x-axis in an interval):

1. Find the least and greatest z values for any point in D. These are your **outer limits** of integration.
2. For each fixed '$z =$ constant' plane which intersects D, identify the least and greatest values of y for any point which is in D *and on that plane*, expressed in terms of the vertical position z of the plane. These are the **middle limits** of integration, and they may depend on z.
3. For each line of the form '$z =$ constant and $y =$ constant' which intersects D, find the least and greatest values of x for any point which is in D *and on that line*. These are your **inner limits** of integration, and they may depend on both z and y.

Example 5.2.3

Find the volume of the tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,3,0)$, and $(0,0,4)$ using a triple integral.

Solution

The volume of a region is equal to the integral of the constant function 1 over that region:

$$\text{volume}(D) = \iiint_D 1\,dV,$$

because $\iiint_D 1\,dV$ is equal to the mass of a solid occupying the region D and having density 1 at every point. But if a solid has a constant mass density of 1, then its mass is equal to its volume.*

* Alternatively, note that calculating the Riemann sums approximating $\iiint_D 1\,dV$ amounts to splitting D into small pieces and summing their volumes

So we set up our iterated integral: the least and greatest values of z are 0 and 4, so those are our outer limits. For a fixed value of z, the least and greatest values of y for a point in D are 0 and $3 - \frac{3}{4}z$, respectively. Finally, for fixed y and z, the least and greatest values of x for a point in D are 0 and the point on the plane $6x + 4y + 3z = 12$ with the given values of y and z (see figure).
So we get

$$\begin{aligned}\text{volume}(D) &= \int_0^4 \int_0^{3-\frac{3}{4}z} \int_0^{2-\frac{2}{3}y-\frac{1}{2}z} 1\,dx\,dy\,dz \\ &= \int_0^4 \int_0^{3-\frac{3}{4}z} \left(2 - \frac{2}{3}y - \frac{1}{2}z\right) dy\,dz \\ &= \int_0^4 \frac{3}{16}(z-4)^2\,dz \\ &= \boxed{4}.\end{aligned}$$

There is nothing special about the order $dx\,dy\,dz$—any way of slicing up the region gives the same result. The region of integration for the following exercise can be sliced up six different ways, and you can check that the integral is the same with respect to all the different orders of integration.

Exercise 5.2.1

Write the iterated integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx.$$

as an integral over a 3D region. Then sketch that region and use your figure to rewrite the integral in five other ways, using the five other permutations of (x, y, z).

5.3 极坐标、柱坐标和球坐标下的多元积分

Some regions in $\mathbb{R}^2$ are more conveniently described in polar coordinates than rectangular coordinates. If we are integrating a function $f : \mathbb{R}^2 \to \mathbb{R}$ over such a region, it is helpful to work directly in polar coordinates. Let's do an example.

Example 5.3.1

(i) Find the area of the region D enclosed by the solution set of the polar coordinate equation $r = 1 + \cos\theta$.
(ii) Integrate the function $f(x, y) = x + y$ over D.

Solution

(i) Let's slice D into small pieces using equally spaced cuts along rays and circles of the form r = constant and θ = constant, as shown. This decomposes D into a set of **coordinate patches**. The figure suggests that these pieces farther away from the origin are larger than the ones that are close to the origin, which leads us to investigate the area of each patch.

To find the area of the set of points with radial polar coordinate between r and $r + \Delta r$ and angular polar coordinate between θ and $\theta + \Delta\theta$, we note that this region is approximately a rectangle. The straight side length is Δr, and the curvy side length is $r\Delta\theta$, because the perimeter of the circle of radius r is $2\pi r$, and the angle represents $\frac{\theta}{2\pi}$ of the whole circle. So the area is approximately $r\Delta r\Delta\theta$.

Now, for fixed θ, we can add up all the coordinate patches between θ and $\theta + \Delta\theta$, and this sum of areas is approximately equal to the product of $\Delta\theta$ and the integral

$$\int_0^{1+\cos\theta} r\,dr.$$

Adding up these areas over all θ from 0 to 2π, we get

$$\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta = \boxed{\frac{3\pi}{2}}.$$

(ii) We can find this integral using the same procedure as above, except that at the step where we calculate the area of a patch, we also need to multiply it by the value of the function at some point in the patch. Since our function is defined in terms of x and y, we need to substitute $x = r\cos\theta$ and $y = r\sin\theta$ to discover the value of f at the point whose polar coordinates are (r, θ). So we get

This one is tedious if done by hand, so we use computer algebra assistance

$$\int_0^{2\pi}\int_0^{1+\cos\theta} (r\cos\theta + r\sin\theta)r\,dr\,d\theta = \boxed{\frac{5\pi}{4}}.$$

We can see from this example that the ideas for setting up an iterated polar integral are similar to those for rectangular integration:

Observation 5.3.1: Iterated polar integration

To write the integral of a function f on a region D in $\mathbb{R}^2$ as a double iterated integral, we may

(i) find the least and greatest values of θ for any point in the region of integration,

(ii) for each fixed value of θ, find the least and greatest values of r for any point which is in D and on the ray of angle θ,

(iii) include the area differential* $dA = r\,dr\,d\theta$, and

(iv) plug $x = r\cos\theta$ and $y = r\sin\theta$ into f, that your integrand varies appropriately as r and θ vary

* Don't forget the extra factor of r in the polar area differential!

This same basic idea can be carried out in cylindrical and spherical coordinates. The ingredients we need are (i) the volume differential dV expressed in terms of cylindrical and spherical coordinates, and (ii) the formulas for x, y, z in terms of r, θ, z and in terms of ρ, θ, ϕ. This information is listed in Appendix **??**. The only surprising entry in the tables of that appendix is the spherical coordinate volume differential $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

Example 5.3.2: Spherical coordinate volume differential

Explain why the volume differential in spherical coordinates is $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$.

Solution

The volume differential arises from slicing up the region of integration into small coordinate "patches", each of which consists of those points whose three spherical coordinates lie in the intervals* $[\rho, \rho + \Delta\rho]$, $[\theta, \theta + \Delta\theta]$, and $[\phi, \phi + \Delta\phi]$, respectively (see the top figure, where a wedge has been decomposed into spherical coordinate patches). Thus we must calculate the approximate volume of one such patch.

* Here (ρ, θ, ϕ) are the spherical coordinates of one of the corners of the patch

When $\Delta\rho$, $\Delta\theta$, and $\Delta\phi$ are all very small, the coordinate patch is approximately a rectangular prism. The dimensions of this rectangular prism, as marked in the lower figure, are $\Delta\rho$, $\rho\,\Delta\phi$, and $\rho\sin\phi\,\Delta\theta$.
To see why the top edge length is $\rho\sin\phi\Delta\theta$, note that the dashed circle in the figure has radius $\rho\sin\phi$, since the *cylindrical* radial coordinate r satisfies the equation $r = \rho\sin\phi$. Thus the volume of the patch is approximately $\rho^2 \sin\phi \Delta\rho\,\Delta\phi\,\Delta\theta$.

Example 5.3.3

Consider a solid whose density at each point (x, y, z) is $delta(x, y, z) = \frac{1}{x^2+y^2+z^2}$ and which occupies the region enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$. Find the mass of the solid.

Solution

Let's set up the iterated integral with the order $d\rho\, d\phi\, d\theta$. The solid has points with θ values as small as 0 and as large as 2π, so the outer limits will be 0 and 2π.

For any given value of θ, there are points with that θ value whose ϕ value is as small as zero (for the points on the positive z-axis) and as large as $\frac{\pi}{4}$ (for the points on the cone $z = \sqrt{x^2+y^2}$). So the middle limits are 0 and $\frac{\pi}{4}$.

Finally, for any given ϕ and θ, the solid contains points with z as small as 0 and as large as $\frac{1}{\cos\phi}$ (by right-triangle trigonometry; see figure).

Help with calculating this integral

For the integrand, we should substitute the spherical coordinate formulas for x, y, and z. However, we know that it will simplify to $\frac{1}{\rho^2}$, since $\rho^2 = x^2+y^2+z^2$. So we get

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^{-2}\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \sec\phi \sin\phi\, d\phi\, d\theta = (2\pi)(\tfrac{1}{2}\ln 2) = \boxed{\pi \ln 2}.$$

Exercise 5.3.1

What proportion of the volume of the unit sphere lies above the plane $z = \frac{1}{2}$?

Exercise 5.3.2

Find

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$$

by writing it as integral over a 3D region and then rewriting that integral using cylindrical coordinates.

on change of variables

5.4 不同坐标系下的积分

Sometimes we want to integrate a function over a region which is not conveniently described using any of the standard coordinate systems. In this section we will develop a general program for integrating with respect to a custom-tailored coordinate system.

Example 5.4.1

Find $\iint_D y^2\, dA$, where D is the region in the first quadrant bounded by the lines through the origin of slope $\frac{1}{2}$ and 2, as well as the hyperbolas $xy = 1$ and $xy = 3$.

Solution

Looking at the shape of the region (in the axes on the right above), it makes sense to slice it up along lines of the form $y = ux$ where u ranges over equally spaced values between $\frac{1}{2}$ and 2, and along hyperbolas of the form $xy = v$ where v ranges over $[1, 3]$.

Note that each point (x, y) in D can be identified by its u and v values.* In other words, u and v provide a coordinate system for the first quadrant. We can represent the relationship between (u, v) and (x, y) as a transformation T that maps each (u, v) pair to its corresponding (x, y) pair, as shown in the figure. If we want to find a formula for this map, we can solve the system $y = ux$ and $xy = v$ for x and y to find that $y = \sqrt{uv}$ and $x = \sqrt{v/u}$.

Now, to integrate $f(x, y) = y^2$ over D, we want to find the area of each of the small pieces we sliced D into, multiply each such area by the value of f somewhere on the piece, and sum the resulting products. Each patch is the image under T of a rectangle of the form $[u, u + \Delta u] \times [v, v + \Delta v]$. The area of this rectangle is $\Delta u \Delta v$, and the transformation distorts its area by an amount that we can approximate by treating the transformation as linear around (u, v) and using the fact that area distortion is measured by the determinant. Writing $T(u, v) = (\sqrt{v/u}, \sqrt{uv}) = (g(u, v), h(u, v))$, we linearly approximate g at a point (u, v) as the function L_g defined by

$$L_g(\widetilde{u}, \widetilde{v}) = g(u, v) + \partial_u g(u, v)(\widetilde{u} - u) + \partial_v g(u, v)(\widetilde{v} - v),$$

and similarly for the linear approximation L_h of h. Thus for small Δu and Δv, T behaves like

$$T(u + \Delta u, v + \Delta v) \approx T(u, v) + \big(\partial_u g(u, v)\Delta u + \partial_v g(u, v)\Delta v, \partial_u h(u, v)\Delta u + \partial_v h(u, v)\Delta v\big). \tag{5.4.1}$$

So the area of the image of $[u, u + \Delta u] \times [v, v + \Delta v]$ under T is approximately*

$$\left| det \begin{bmatrix} \partial_u g & \partial_v g \\ \partial_u h & \partial_v h \end{bmatrix} \right| \Delta u \Delta v, = \left| det \begin{bmatrix} -\frac{1}{2}\sqrt{v}u^{-3/2} & \frac{1}{2\sqrt{uv}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{bmatrix} \right| \Delta u \Delta v = \frac{1}{2u}\Delta u \Delta v.$$

So the contribution of each piece to the value of the integral is $uv\frac{1}{2u}\Delta u\,\Delta v$. Then summing and taking $(\Delta u, \Delta v) \to (0, 0)$ yields $displaystyle\int_1^3 \int_{1/2}^2 uv\frac{1}{2u} du\,dv = \boxed{3}$.

* These specify, respectively, which line through the origin and which "$xy =$ constant" hyperbola the point is on

* The four entries of the matrix below come from the coefficients of Δu and Δv in (5.4.1), or equivalently from the coefficients of $\widetilde{u}$ and $\widetilde{v}$ in L_g and L_h

The matrix $\begin{bmatrix} \partial_u g & \partial_v g \\ \partial_u h & \partial_v h \end{bmatrix}$ is called the *Jacobian matrix*, and its determinant is called the *Jacobian determinant*. Often we just say "Jacobian", relying on context to distinguish. It can be written as $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ for short.*

* Note that in this new notation, the symbols g and h are replaced by x and y. Thus we are abusing notation by regarding x and y as functions of u and v, even though they also represent independent variables

The following theorem summarizes the technique we developed in Example 5.4.1.

Theorem 5.4.1: Multivariable change of variables

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable transformation that maps a region R one-to-one onto a region D. Then for any continuous function f, we have

$$\iint_D f(x, y)\,dx\,dy = \iint_R f(T^{-1}(x, y)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv.$$

The biggest challenge in custom coordinate problems typically lies in choosing suitable coordinates. The strategy of

* If all the boundary arcs are level sets of a *single* function u, then you'll have some flexibility in choosing the other; often $v = x$ or $v = y$ works

Example 5.4.1 is pretty broadly useful: try to express all boundary arcs of D as level sets of at most two functions of x and y. These two functions* are good choices for the coordinates u and v.

Exercise 5.4.1

Find the integral of $\frac{(x-y)^2}{x+y+2}$ over the square whose vertices are the four points of intersection between the axes and the unit circle.

Exercise 5.4.2

Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region above the x-axis bounded by the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$.

5.5 积分的应用 *

5.5.1 均值

The most basic example of averaging is summing a list of numbers and dividing by the number of entries in the list. While this concept is usually used without reference to functions, one way to express it is to say that the average of a function $f : \{1, 2, \ldots, n\} \to \mathbb{R}$ is defined to be

$$\text{average}(f) = \frac{f(1) + f(2) + \cdots + f(n)}{n}. \tag{5.5.1}$$

Now suppose that the domain of f is a region $R \subset \mathbb{R}^n$. The most natural way to adapt (5.5.1) to this setting is to replace the sum in the numerator with an integral, and replace the "size" of the domain n in the denominator with the volume of R. We will state the definition for a function defined on a region in $\mathbb{R}^3$:

Definition 5.5.1: Average value of a function over a solid

If $R \subset \mathbb{R}^3$ is a region and $f : R \to \mathbb{R}$, then the **average value** of f is

$$\text{average}(f) = \frac{displaystyle\iiint_R f\,dV}{displaystyle\iiint_R 1\,dV}.$$

Let's explain the connection between Definition 5.5.1 and (5.5.1) more carefully: intuitively, the average value of a continuously varying function $f : R \to \mathbb{R}$ should be very close to the result of overlaying R with a fine grid, evaluating the function at all the corners of the cells of that grid, and averaging the results. By (5.5.1), this latter average is equal to the quotient* of (1) the sum of all the function's values at the grid points and (2) the number of grid points. If we multiply numerator and denominator of this quotient by the volume of a single small grid cell, then the numerator becomes a Riemann sum approximating $\iiint_R f\,dV$, and the denominator becomes a Riemann sum approximating the volume $\iiint_R 1\,dV$ of R. Therefore, the limit of this quotient as the mesh of the grid tends to zero is equal to the formula given in Definition 5.5.1.

* We're sticking with verbal descriptions of the quantities involved here to avoid the somewhat unwieldy notation associated with Riemann sums

Exercise 5.5.1

(a) Find the average squared distance from the origin to a point in the unit sphere.

(b) Find the average distance from the origin to a point in the unit sphere.

(c) Is the square of the average distance equal to the average of the squared distance?

If we want some of the numbers being averaged to count more than others, we can replace (5.5.1) with a **weighted average**. If $f : \{1, 2, \dots, n\} \to \mathbb{R}$ and $w : \{1, 2, \dots, n\} \to \mathbb{R}$, then the w-weighted average of f is defined to be

$$\text{average}_w(f) = \frac{f(1)w(1) + f(2)w(2) + \cdots + f(n)w(n)}{w(1) + w(2) \cdots + w(n)}. \tag{5.5.2}$$

For example, if your scores on 3 exams are $f(1) = 90, f(2) = 90$, $f(3) = 100$ with weights $w(1) = 30$, $w(2) = 30$, $w(3) = 40$, then your weighted average is $\frac{90 \cdot 30 + 90 \cdot 30 + 100 \cdot 40}{30 + 30 + 40} = 94$. We can adapt (5.5.2) for continuously varying functions over a domain in $\mathbb{R}^3$ as we did in Definition 5.5.1.

Definition 5.5.2: Weighted average of a function over a solid

If $R \subset \mathbb{R}^3$ is a region, and that $f : R \to \mathbb{R}$ and $w : R \to [0, \infty)$ are functions. Then the **w-weighted average** of f is

$$\text{average}_w(f) = \frac{\mathit{displaystyle} \iiint_R f w \, dV}{\mathit{displaystyle} \iiint_R w \, dV}.$$

Exercise 5.5.2

Define $w : [0, 1]^3 \to \mathbb{R}$ via $w(x, y, z) = x + y + z$. (a) Find the w-weighted average of the constant function $f(x, y, z) = 1$ on $[0, 1]^3$. (b) Find the w-weighted average of $f(x, y, z) = x$ on $[0, 1]^3$.

Exercise 5.5.3

Identify the changes to the paragraph following Definition 5.5.1 necessary to obtain the corresponding explanation of Definition 5.5.2.

5.5.2 重心

The center of mass of a physical rod is the point along the length of the rod where it balances. The center of mass of a rod with constant mass density is the point halfway between the rod's ends. But what if the rod has a varying mass density?

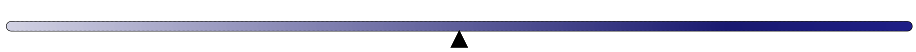


图 **5.2** A rod with mass density indicated by color

* See Section 5.5.3 for more discussion on torque

Clearly the rod in Figure 5.2 will tip to the right if we support it in the middle. To find its center of mass, recall from physics that *torque** applied by a force to an object rotating about a fixed pivot is equal to the product of the force applied* and the distance to the pivot. The rod is balanced if the torque applied by gravity on the two sides is equal. If the rod is situated along the interval $[a, b]$, has density $delta : [a, b] \to \mathbb{R}$, and is supported at the point x_0, then the clockwise torque is equal to

* Actually, we include only the component of the force in the direction perpendicular to the vector from the pivot to the point where the force is being applied

$$\overbrace{g}^{\text{gravitational acceleration}} \int_{x_0}^{b} \overbrace{(x - x_0)}^{\text{length of moment arm}} \underbrace{\overbrace{delta(x)}^{\text{density of small piece } [x, x+dx]} \overbrace{dx}^{\text{length of small piece}}}_{\text{mass of small piece}}.$$

Similarly, the counterclockwise torque is

$$g \int_{a}^{x_0} (x_0 - x) delta(x)\, dx.$$

Setting these two integrals equal and collecting terms we get

$$0 = \int_{x_0}^{b} (x - x_0) delta(x)\, dx + \int_{a}^{x_0} (x - x_0) delta(x)\, dx = \int_{a}^{b} (x - x_0) delta(x)\, dx.$$

Now we can distribute the $delta(x)$ factor, apply linearity of the integral, and solve for x_0 to find

$$x_0 = \frac{displaystyle \int_a^b x delta(x)\, dx}{displaystyle \int_a^b delta(x)\, dx}.$$

In other words, the location of the center of mass is equal to **the density-weighted average value of the coordinate function x**.

We can apply the same idea (one axis at a time) to a **lamina**, which is a plate* occupying a region $L \subset \mathbb{R}^2$ and having mass density $\sigma : L \to [0, \infty)$. And similarly for a solid in $\mathbb{R}^3$ with a mass density function $delta$. In all cases, the coordinates of the center of mass are equal to the density-weighted averages of the corresponding coordinate functions:

* By slight abuse of notation, we may use L to refer to either the lamina or the region in $\mathbb{R}^2$ that it occupies

Definition 5.5.3: Center of mass

The center of mass of a lamina L with mass density $\sigma : L \to [0, \infty)$ is

$$\left(\frac{displaystyle \iint_L x\, \sigma(x, y)\, dx\, dy}{displaystyle \iint_L \sigma(x, y)\, dx\, dy}, \frac{displaystyle \iint_L y\, \sigma(x, y)\, dx\, dy}{displaystyle \iint_L \sigma(x, y)\, dx\, dy} \right).$$

The center of mass of a solid occupying a region R with mass density $delta : R \to [0, \infty)$ is

$$\left(\frac{displaystyle \iiint_R x\, delta(x, y, z)\, dx\, dy\, dz}{displaystyle \iiint_R delta(x, y, z)\, dx\, dy\, dz}, \frac{displaystyle \iiint_R y\, delta(x, y, z)\, dx\, dy\, dz}{displaystyle \iiint_R delta(x, y, z)\, dx\, dy\, dz}, \frac{displaystyle \iiint_R z\, delta(x, y, z)\, dx\, dy\, dz}{displaystyle \iiint_R delta(x, y, z)\, dx\, dy\, dz} \right).$$

Example 5.5.1

Find the center of mass of the triangle T with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Assume constant density.*

* Convention dictates that leaving the density function unstated should be taken to imply constant density

Solution

If the density is k, then by Definition 5.5.3, the x-coordinate of the center of mass is

$$\frac{\iint_T kx\, dx\, dy}{\iint_T k\, dx\, dy} = \frac{k/6}{k/2} = \frac{1}{3}.$$

By symmetry, the y-coordinate of the center of mass is also $\frac{1}{3}$. Therefore, the center of mass of the lamina is $\boxed{\left(\frac{1}{3}, \frac{1}{3}\right)}$.

Exercise 5.5.4

Find the center of mass of a tetrahedron with vertices $\mathbf{0}$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Exercise 5.5.5

Show that the ρ-value of the center of a mass of an origin-centered sphere is **not** equal to the average ρ value of a point in the sphere.

5.5.3 惯性矩

Newton's second law says that

$$\mathbf{F} = m\mathbf{a}, \tag{5.5.3}$$

where $\mathbf{F}$, m, and $\mathbf{a}$ are the net force acting upon a particle, its mass, and its resulting acceleration, respectively. The rotational analogue of Newton's second law says that, for any line ℓ in space,

$$\tau = I\alpha \tag{5.5.4}$$

where τ is the torque about ℓ being applied to the particle, I is the particle's **moment of inertia** about ℓ, and α is the angular acceleration of the particle about ℓ. Thus the moment of inertia is the rotational analogue of mass.

Let's derive (5.5.4) from (5.5.3) as context for providing mathematical definitions for the quantities in (5.5.4). Denote by $\mathbf{r}$ the vector from the nearest point on ℓ to the location of the particle. Then crossing $\mathbf{r}$ with both sides of (5.5.3), we find that

$$\mathbf{r} \times \mathbf{F} = m\mathbf{r} \times \mathbf{a}.$$

We define the torque about ℓ to be $\tau := \mathbf{r} \times \mathbf{F}$. Letting $r = |\mathbf{r}|$ be the distance from ℓ to the particle, we define the angular acceleration of the particle about ℓ to be $\alpha = \frac{\mathbf{r}}{r} \times \frac{\mathbf{a}}{r}$. Then we obtain (5.5.4) with $I = mr^2$.

图 5.3 The vectors τ, $\mathbf{a}$, and $\mathbf{r}$

Exercise 5.5.6: Angular acceleration

Show that $\left|\frac{\mathbf{r}}{r} \times \frac{\mathbf{a}}{r}\right|$ is the tangential component of the acceleration vector $\mathbf{a}$.

Moment of inertia, like mass, is additive. That is, the moment of inertia of an ensemble of particles about a line ℓ is equal to the sum of the moments of inertia about ℓ of the particles. Therefore, the moment of inertia of a solid with continuously varying mass density may be approximated by subdividing the solid into small pieces and approximating the moment of inertia of each piece as its mass times the squared distance from some point in the small piece to ℓ. Summing the results and taking the size of the pieces to zero leads to the following formula:

Definition 5.5.4: Moment of inertia

The moment of inertia about a line ℓ of a solid occupying a region $R \subset \mathbb{R}^3$ and having density $delta : \mathbb{R}^3 \to [0, \infty)$ is

$$I = \iiint_R r(x,y,z)^2\, delta(x,y,z)\, dx\, dy\, dz,$$

where $r(x,y,z)$ is the distance from (x,y,z) to ℓ.

Example 5.5.2

Find the moment of inertia of the unit-density, unit sphere about the z-axis.

Solution

The distance from the z-axis to (x,y,z) is $r = \rho \sin\phi$. Therefore, the moment of inertia is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \overbrace{(\rho\sin\phi)^2}^{r^2}\, \overbrace{1}^{delta}\, \overbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}^{dV} = \left(\frac{1}{5}\right)\left(\frac{4}{3}\right)(2\pi) = \boxed{\frac{8\pi}{15}}.$$

Exercise 5.5.7

Find the moment of inertia about the z-axis of the cylinder described by the inequalities $(x-1)^2 + y^2 \le 1$ and $0 \le z \le 1$. Repeat with the unit cube $[0,1]^3$. Which is harder to rotate about the z-axis?

Exercise 5.5.8: Parallel axis theorem

Suppose that I_k is the moment of inertia of a solid of mass M around a line ℓ_k, for $k \in \{1, 2\}$. Show that if ℓ_1 passes through the solid's center of mass and ℓ_1 and ℓ_2 are parallel, then

$$I_2 = I_1 + md^2,$$

where d is the distance between the lines ℓ_1 and ℓ_2.

5.5.4 概率

A dart thrown at a dartboard $D \subset \mathbb{R}^2$ strikes a *random* point P in D. We model this state of affairs by describing a **probability density function*** $f : D \to [0, \infty)$ with the property that the probability that P lies in any region A given by integrating f over A.

* or pdf, for short

Example 5.5.3

Suppose that the probability density function for the random point where your dart hits the dartboard* $D = \mathbb{R}^2$ is given by

$$f(x,y) = \frac{1}{\pi}\mathrm{e}^{-x^2-y^2},$$

where the origin is situated at the dartboard's bull's eye, and where x and y are measured in inches. Find the probability of scoring triple 20 on your next throw.

Note: the triple 20 region is the smaller of the two thin red strips in the sector labeled "20". The inner and outer radii of this thin strip are 3.85 inches and 4.2 inches, respectively.

* This function is positive everywhere in $\mathbb{R}^2$, so the "dartboard" includes the disk shown as well as the (infinite) wall it i mounted on—this is realistic insofar as one can indeed hit the wall with a dart throw

Solution

The region in question is described most easily in polar coordinates: it is the set of points whose polar coordinates (r,θ) satisfy $r_\mathrm{i} \le r \le r_\mathrm{o}$ and* $81^\circ \le \theta \le 99^\circ$, where $r_\mathrm{i} = 3.85$ and $r_\mathrm{o} = 4.2$.

Therefore, we can obtain the probability of hitting the triple 20 by expressing the density function in polar coordinates and integrating

$$\int_{9\pi/20}^{11\pi/20}\int_{r_\mathrm{i}}^{r_\mathrm{o}} \frac{1}{\pi}\mathrm{e}^{-r^2} r\,dr\,d\theta = \left(\frac{\pi}{10}\right)\left(\frac{1}{\pi}\right)\left(-\tfrac{1}{2}\mathrm{e}^{-r_\mathrm{o}^2} - \left(-\tfrac{1}{2}\mathrm{e}^{-r_\mathrm{i}^2}\right)\right).$$

Substituting the given values of r_o and r_i yields a probability of approximately $\boxed{1.717 \times 10^{-8}}$ of scoring 60 on a single throw.

* Note that the width of each sector is $360^\circ/20 = 18^\circ$, so the angles of the rays bounding the sector labeled 20 are $90^\circ \pm \frac{18^\circ}{2}$

The integral of a probability density function over its whole domain must be equal to 1, since that is the probability that the random point P is *somewhere* in D.

Exercise 5.5.9

Verify that $f(x,y) = \frac{1}{\pi}\mathrm{e}^{-x^2-y^2}$ defined on $\mathbb{R}^2$ is a valid probability density function (that is, integrating it over its domain yields the value 1).

Exercise 5.5.10

Suppose that f is a probability density function for a random point P in a domain $D \subset \mathbb{R}^2$. Is it possible that there exists a point $(x,y) \in D$ such that $f(x,y) = 100$? Explain why or why not. (Hint: is it possible for a solid whose total mass is 1 kilogram to have a density of 100 kilograms per cubic centimeter at a particular point in the solid?)

The analogy between probability density and mass density suggested in Exercise 5.5.10 is a broadly useful conceptual tool: you can imagine probability as some kind of abstract dust lying around on D, and regions which contain more of it are more likely to include P. To figure out exactly how much is in a given region, we integrate over that region.

图 **5.4** The pdf $\frac{1}{\pi}\mathrm{e}^{-x^2-y^2}$

Exercise 5.5.11

Explain why the answer found in Example 5.5.3 was unrealistically low for a typical dart thrower. If you were actually good enough that the pdf given in that problem describes your accuracy well, where should you consider aiming?

Note that the green bullseye is worth 25 points and the red bullseye is worth 50 points.

Given a function $g : D \to \mathbb{R}$, the value of $g(P)$ depends on the value of P and is therefore random. We call $g(P)$ a **random variable** and define its **expected value** to be the probability-weighted integral of g.

Example 5.5.4

Find the average distance from the origin to a point P selected uniformly at random from the triangle T with vertices at $(0,0)$, $(1,0)$, and $(1,1)$.

Solution

The word "uniform" means that P's pdf is a constant function. To determine the value of this constant, we use the fact that the pdf integrates to 1. If $f(x,y) = k$ for all $(x,y) \in T$, then

$$\iint_T k\,dA = k\,\mathrm{area}(T) = 1,$$

which implies that $k = 2$. We are calculating the expected value of $g(P)$, where $g(x,y) = \sqrt{x^2+y^2}$. So the expected value of $g(P)$ is given by $\iint_T g\,2\,dA$. Because the integrand is conveniently expressed in polar coordinates, we represent T in polar coordinates as the set of points where θ is between 0 and $\pi/4$ and where r is between 0 and $\sec\theta$. Then we have

$$\iint_T g\,2\,dA = \int_0^{\pi/4}\int_0^{\sec\theta} r\,2\,r\,dr\,d\theta = \frac{2}{3}\int_0^{\pi/4}\sec^3\theta\,d\theta.$$

We can use integration by parts to work out $\int \sec^3\theta\,d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)$. Substitution yields an expected value of

$$\boxed{\frac{1}{3}\left(\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right) \approx 0.765}.$$

Exercise 5.5.12

Explain why the answer to Example 5.5.4 is also the average distance from the origin to a point selected uniformly at random from the unit square $[0,1]^2$.

Example 5.5.5

Find the expected value of the latitude (as measured from the north pole) of a point chosen uniformly at random from the upper half of the earth (the part above the plane passing through the equator). (Model the earth as a sphere.)

Solution

The answer doesn't change if we change the radius of the sphere, so we might as well choose unit radius. The pdf is some constant k, and the probability of the randomly selected point lying somewhere in the upper half ball H is

$$\int_H k\,dV = k\,\text{vol}(H) = \frac{2\pi k}{3}.$$

Since the pdf must integrate to 1, we have $k = \frac{3}{2\pi}$.

Now the function on H that we want to calculate the expected value of is the coordinate function ϕ. So we have

$$\int_H k\phi\,dV = \frac{3}{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \phi\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 1,$$

where we've used the integration-by-parts side-problem $displaystyle\int_0^{\pi/2}\phi\sin\phi\,d\phi = 1$. Therefore, the expected value of the latitude is $\boxed{\text{1 radian}}$.

Exercise 5.5.13

Find the expected value of $g(P)$ where $g(x,y) = xy$ and P is a random point in $[0,1]^2$ whose pdf is given by $f(x,y) = xy(1-x)(1-y)$.

Exercise 5.5.14

Suppose that X and Y represent your score and your friend's score on an upcoming exam. If the pair of scores $P = (X,Y)$ has pdf given by $f(x,y) = \frac{363}{310} - \frac{30}{31}\left(x - y - \frac{1}{10}\right)^2$ on $[0,1]^2$, then what is the probability that your friend scores higher than you on the exam?*

help with calculating thi integral

6 向量微分

6.1 向量场与线积分

on vector fields

So far we have been considering functions from $\mathbb{R}^n$ to $\mathbb{R}^1$ (where n is 2 or 3). In this chapter we work with functions from $\mathbb{R}^n$ to $\mathbb{R}^2$ or $\mathbb{R}^3$. How should such functions be visualized? Let's begin by considering how they arise in applications.

图 **6.1** A gravitational vector field

To describe the gravitational force* felt by a particle, we would use a function with a three-dimensional input (to specify the particle's location) as well as a three-dimensional output, to specify the direction and magnitude of the force. It is natural to represent this function by drawing a small arrow indicating the output vector at several points in space, because this makes it easy to imagine how the force changes as the particle moves around (see Figure 6.1).

* Many other physical phenomena, such as electromagnetic forces, heat flow, fluid flow are modeled as vector fields

This picture suggests the term **vector field** for a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, where $n > 1$. The gravitational vector field plotted in Figure 6.1 is

$$\mathbf{F}(x,y,z) = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}} \langle x-1, y-1, z-1 \rangle . \tag{6.1.1}$$

where G is the gravitational constant and Mm is the product of the masses of particle and the attracting body at $(1,1,1)$.

Exercise 6.1.1

The vector plot shown represents the velocity of water on the surface of a river. The water is flowing due east, and it is flowing faster near the south end of the river than the north. Come up with a vector field $\mathbf{F}$ whose vector plot looks approximately like the one shown.

on line integrals

Now suppose that rather than remaining stationary, our particle moves along a path in the presence of a force field (see Figure 6.2).* Sometimes the particle is moving with the force field and getting a boost from it, whereas other times it's working against the force field. How much net work does it take to move along the path?

* A vector field in which the vectors represent a physical force

If the force field were constant and the path were straight, then physics tells us that work is equal to the product of the magnitude F of the force, the distance d traveled, and the cosine of the angle θ between the force and the path. Alternatively, we may interpret the force and distance as vectors $\mathbf{F}$ and $\mathbf{d}$ and write the work as a dot product:

$$W = Fd\cos\theta = \mathbf{F}\cdot\mathbf{d}.$$

图 **6.2** The path of a particle moving through a vector field

So how do we bootstrap our way from constant force and a straight path to varying force and a curvy path? We can cut up the path into small pieces, handle each small piece by treating the force as approximately constant and the path as approximately straight, and then add up the amount of work for each small piece. We will assume that our path $\mathbf{r}(t)$ is differentiable.*

* The curve could alternatively be *piecewise* differentiable, meaning that the curve is non-differentiable at only finitely many points

Suppose we have a path C parameterized as $\mathbf{r}(t)$ where t ranges from a to b. Over the time interval $[t, t + \Delta t]$, the particle is displaced by* the vector $\mathbf{r}'(t)\,\Delta t$, and the force it feels over that time is* $\mathbf{F}(\mathbf{r}(t))$. Therefore, the contribution from the time period $[t, t + \Delta t]$ is equal to

* * Approximately, with an error that vanishes as $\Delta t \to 0$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\Delta t.$$

Summing all these contributions and taking $\Delta t \to 0$, we arrive at the formula

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where the last expression is an abbreviation for the middle expression. We call an integral of the form $displaystyle \int_C \mathbf{F} \cdot d\mathbf{r}$ a **line integral**.

Example 6.1.1

Find the line integral of $\mathbf{F}(x, y) = \langle xy, y \rangle$ along the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$.

Solution

Let's parameterize the parabola using the x-coordinate as the parameter:

$$\mathbf{r}(t) = \langle t, t^2 \rangle.$$

Note that the point $(2, 4)$ is visited at time $t = 2$, while the origin is visited at time $t = 0$. Therefore,

$$W = \int_0^2 \langle t(t^2), t^2 \rangle \cdot \langle 1, 2t \rangle\,dt = \int_0^2 (t^3 + 2t^3)\,dt = \boxed{12}.$$

The following theorem states that the choice of parameterization of a curve doesn't matter when computing a line integral. This makes sense physically, since the formula $W = Fd$ does not involve time, and our derivation of the line integral formula was based on $W = Fd$. The role of the parameterization was merely to provide a convenient way to split up the path into short pieces. Exercise 6.1.2 below gives an example.

Theorem 6.1.1: Independence of parameterization

If C is a curve parameterized by $\mathbf{r}_1$ over $[a, b]$ and also by $\mathbf{r}_2$ over $[c, d]$, then

$$\int_a^b \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\,dt = \int_c^d \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t)\,dt.$$

In other words, $displaystyle \int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the curve C, not the choice of parameterization.

Exercise 6.1.2

(i) Compute the line integral of $\mathbf{F} = \langle x^2, -xy \rangle$ over the portion of the unit circle in the first quadrant, using the parameterization $\mathbf{r}(t) = \langle \sin t, \cos t \rangle$.

(ii) Perform the same line integral using the parameterization $\mathbf{r}(t) = \langle t, \sqrt{1-t^2} \rangle$.

Exercise 6.1.3

Consider the vector field $\mathbf{F}$ and path C shown in Figure 6.2. Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive or negative?

6.2 微分基本定理

on the gradient theorem for line integrals

In general, the line integral of $\mathbf{F}$ over a path between two points depends on the path, not just the starting and ending points. For example, in Figure 6.3, the line integral along the blue (top) path is positive, while the line integral along the orange (bottom) path is negative.

图 **6.3** The vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$ and two semicircular paths

However, there is an important class of vector fields which are path-independent, meaning that the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the starting and ending points of C. These are the vector fields which can be written as the gradient of a function from $\mathbb{R}^n$ to $\mathbb{R}^1$. For example, $\mathbf{F}(x, y, z) = \langle -2x, -2y, z \rangle$ is the gradient of the function

$$f(x, y, z) = -x^2 - y^2 + \tfrac{1}{2}z^2.$$

Such vector fields are called **conservative**.

If we calculate the line integral of ∇f along a curve C parameterized by $\mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$, then the contribution from the portion of the curve from $\mathbf{r}(t)$ to $\mathbf{r}(t + \Delta t)$ is*

$$\langle \partial_x f, \partial_y f, \partial_z f \rangle \cdot \langle r_1'(t), r_2'(t), r_3'(t) \rangle \, \Delta t,$$

which by the chain rule is* the change in $f(\mathbf{r}(t))$ over that interval. Therefore, the line integral of ∇f along a path is equal to the change in f from the beginning to the end of the path.

* approximately, with an error that vanishes as $\Delta t \to 0$

Theorem 6.2.1: Fundamental theorem for line integrals

If C is a path from $\mathbf{a}$ to $\mathbf{b}$ and f is a differentiable function, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}).$$

Example 6.2.1

Suppose $\mathbf{F}(x,y,z) = \langle 2xy^3z, 3x^2y^2z + y, x^2y^3\rangle$ and that C is the circular arc from the origin to the point $(1,1,1)$ and passing through the point $(1/2,1/2,1)$. Find $\int_C \mathbf{F}\cdot d\mathbf{r}$.

Solution

Finding a parameterization for C seems computationally messy. However, if $\mathbf{F}$ is conservative, then we can use Theorem 6.2.1. Integrating $2xy^3z$ with respect to x, we see that if $\mathbf{F} = \nabla f$ for some function f, then we would have

$$f(x,y,z) = x^2y^3z + C_1(y,z),$$

where $C_1(y,z)$ denotes a function not depending on x. Similarly, we can integrate the second and third components with respect to y and z to find that

$$f(x,y,z) = x^2y^3z + \frac{1}{2}y^2 + C_2(x,z)$$
$$f(x,y,z) = x^2y^3z + C_3(x,y).$$

We see that these three conditions are simultaneously satisfied by the function $f(x,y,z) = x^2y^3z + \frac{1}{2}y^2$. So the desired line integral is equal to

$$f(1,1,1) - f(0,0,0) = \frac{3}{2} - 0 = \boxed{\frac{3}{2}}.$$

The following theorem provides a convenient way to check whether a two-dimensional vector field is conservative.

Theorem 6.2.2

A vector field $\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ which is differentiable on* $\mathbb{R}^2$ is conservative if and only if

$$\partial_x N = \partial_y M. \tag{6.2.1}$$

* Here it is important that $\mathbf{F}$ is differentiable on all of $\mathbb{R}^2$, rather than an arbitrary subset thereof—see Exercise 6.2.2

To see where (6.2.1) comes from, note that this equation follows directly from Clairaut's theorem for conservative fields $\mathbf{F}$. So the more interesting aspect of Theorem 6.2.2 is the converse direction: merely checking $\partial_x N = \partial_y M$ establishes existence or nonexistence of a gradient function.

Exercise 6.2.1

Show that the gravitational force in (6.1.1) is conservative.

Exercise 6.2.2

(i) Try to apply Theorem 6.2.2 to the vector field

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle.$$

(ii) Show by plotting this vector field that it is not conservative. How does this square with Theorem 6.2.2?

on Green's theorem

6.3 Green's 公式定理

Is it possible to engineer a simple mechanical device that measures the area bounded by whatever curve it traces out on paper? This seems surprising, since computing the area would seem to require some inspection of the region inside the curve. However, the *planimeter** can record the area of a region based on the motion of its wheels as its tip traverses the boundary of the region. The design of the planimeter takes advantage of the following beautiful relationship between line integrals along the boundary of a curve and double integrals over the enclosed region.

* Invented in 1854

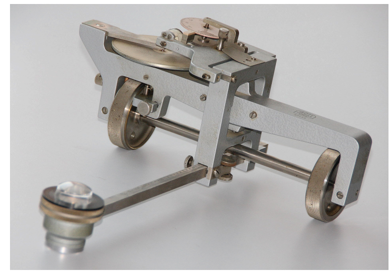


图 **6.4** A planimeter

Theorem 6.3.1: Green's theorem

If $\mathbf{F} = \langle M, N\rangle$ is a vector field* on $\mathbb{R}^2$ with continuous partial derivatives, and if D is a region bounded by a simple, counterclockwise-oriented, piecewise smooth curve C, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\partial_x N - \partial_y M\right)\, dA.$$

* $\mathbf{F} = \mathbf{F}(x, y)$ and similarly for M and N; we will begin using these abbreviations for notational convenience; bear in mind that $\mathbf{F}$ always represents a vector field which implicitly depends on (x, y) or (x, y, z)

Example 6.3.1

Verify Green's theorem in the case where D is the unit disk and $\mathbf{F}(x, y) = \langle 0, x\rangle$.

Solution

We parameterize the unit disk trigonometrically as $(\cos t, \sin t)$, and we calculate the line integral

$$\int_0^{2\pi} \langle 0, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \int_0^{2\pi} \cos^2 t\, dt = \pi.$$

This last integral can be done with a trick: note that

$$\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

However, the contributions of $\int_0^{2\pi} \cos^2 t\, dt$ and $\int_0^{2\pi} \sin^2 t\, dt$ are equal, since their graphs over the region of integration are the same up to a shift. So each is equal to π.

The integrand for the double integral is* $\partial_x N - \partial_y M = 1 - 0 = 1$, so the value of the double integral is the area of the unit disk, which is equal to π. Thus the conclusion of Green's theorem is satisfied.

* Using $\mathbf{F} = \langle 0, x\rangle$ gives an integrand of 1 on the right-hand side of Green's theorem, but: (i) $\langle 0, x\rangle$ isn't the only vector field with this property, and (ii) Green's theorem is also useful when the integrand is non-constant

Proving Green's theorem

The idea of the proof of Green's theorem is to cut D into small rectangles along grid lines (shown with small gaps for visual clarity). Green's theorem holds approximately on each small rectangle $R = [x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2}] \times [y - \frac{\Delta y}{2}, y + \frac{\Delta y}{2}]$, because the left and right sides of R contribute to $\int_R \mathbf{F} \cdot d\mathbf{r}$ approximately

$$\overbrace{N\left(x + \frac{\Delta x}{2}, y\right)\Delta y}^{\mathbf{F}\text{ at midpoint dotted with }\langle 0, \Delta y\rangle\text{ step}} \quad - \quad \overbrace{N\left(x - \frac{\Delta x}{2}, y\right)\Delta y}^{\mathbf{F}\text{ at midpoint dotted with }\langle 0, -\Delta y\rangle\text{ step}} \quad \approx (\partial_x N)(x, y)\,\Delta x\,\Delta y.$$

Similarly, the contribution of the top and bottom sides is $-(\partial_y M)(x, y)\,\Delta x\,\Delta y$. So all together, the **circulation*** of $\mathbf{F}$ around R is $(\partial_x N - \partial_y M)\,\Delta x\,\Delta y$. This is also approximately equal to the integral of $(\partial_x N - \partial_y M)$ over R, since $\partial_x N - \partial_y M$ is approximately constant over R.

The line integrals of $\mathbf{F}$ over the small rectangles sum to the line integral of $\mathbf{F}$ around the boundary* of D, because each interior segment is integrated along twice (once for each adjoining rectangle) and in opposite directions. These contributions sum to zero, leaving only the integrals along the outer edges. Since these outer edges fit together to form ∂D, the line integrals along them sum to the line integral along ∂D.

Since the integral of $\partial_x N - \partial_y M$ over D is also equal to the sum of the integrals of $\partial_x N - \partial_y M$ over the small rectangles, the (approximate) Green's theorem for the small rectangles implies (approximate) Green's theorem for D. Letting the sizes of the rectangles tend to 0, this approximation becomes exact and yields Green's theorem.

* *Circulation* is a synonym of *line integral* which is used only when curve starts and ends at the same

* In other words, the circulation around the boundary of a region is **additive**

Exercise 6.3.1

Use Green's theorem to find the line integral of $\mathbf{F} = \langle \sqrt{x^2 + 1}, \arctan x \rangle$ along a counterclockwise traversal of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

Exercise 6.3.2

Use Green's theorem to find the area under each arch of the cycloid shown below.

6.4 面积分与微分流行

6.4.1 Surface integrals

on surface integrals

What is the average temperature on the surface of the earth? Let's look past the scientific challenges of this problem and imagine that the earth is a sphere S and that we have a reading of the temperature T at every point on its surface at a particular point in time. The average temperature at that moment should then be the integral of T over S divided by the surface area of S. But what does it mean to integrate over a surface?

We can use the same approach we use throughout calculus when studying quantities which are additive and continuously varying: split S into tiny patches over which T may be treated as constant, multiply the area of each patch by the value of T somewhere on that patch, and sum the resulting products. As the size of the patches tends to zero, we expect this sum to converge to some limiting value, and we can declare that limit to be the value of the **surface integral*** of f over S, denoted $\iint_S f\,dA$.

* Or scalar surface integral, to distinguish from vector surface integral introduced in the next subsection

Example 6.4.1

Find the surface integral of $f(x,y,z) = 2x^2 + 2y^2 + 2z^2$ over the unit sphere S. You may assume that the surface area of a sphere of radius R is $4\pi R^2$.

Solution

This function is equal to 2 everywhere on the unit sphere, so if we split S into many small pieces and consider one of them, then the product of the value of f somewhere on the piece and the area of the piece is just twice the area of the piece. When we sum these contributions over all the pieces, we will end up with twice the total surface area of the sphere. In other words, we have

$$\iint_S 2\,dA = 2\iint_S 1\,dA = 2 \times \text{surface area}(S) = 8\pi.$$

Example 6.4.1 was special because the function happened to be constant over the surface. Another special situation which simplifies the process of finding a surface integral is when we're integrating over a surface which is contained in a plane.

Example 6.4.2

Find the surface integral of $f(x,y,z) = xyz$ over the rectangular prism $[0,1] \times [0,2] \times [0,3]$.

Solution

The value of the function f at every point in a coordinate plane is zero. So the contribution from these faces is zero.

For the top face, the value of the function at each point $(x, y, 3)$ is $3xy$. Therefore, the integral of f over this face is what you get when you split the rectangle $[0, 1] \times [0, 2]$ into many small rectangles, multiply the value of $3xy$ by the area of each one (where x and y are the first and second coordinates of some point in the small rectangle), sum the results, and take the rectangle size to zero. In other words, the contribution from the top face is

$$\int_0^1 \int_0^2 3xy \, dy \, dx = 3.$$

Likewise, the contributions from the other two faces are

$$\int_0^3 \int_0^2 1yz \, dy \, dz = 9, \text{ and}$$
$$\int_0^3 \int_0^1 2xz \, dx \, dz = \frac{9}{2}$$

So altogether the surface integral is equal to $displaystyle 3 + 9 + \frac{9}{2} = \boxed{\frac{33}{2}}$.

Let's develop a general-purpose method for evaluating surface integrals. Recall that a parameterization of a curve C in $\mathbb{R}^3$ is a function $\mathbf{r} : [a, b] \to \mathbb{R}^3$ for which $\mathbf{r}(t)$ traces out the points of C as t goes from a to b. Likewise, a **parameterization** of a surface S in $\mathbb{R}^3$ is a function $\mathbf{r}$ from some planar domain $D \subset \mathbb{R}^2$ to $\mathbb{R}^3$ such that $\mathbf{r}(u, v)$ sweeps out* the surface S as (u, v) varies over D.

* More precisely, a parameterization $\mathbf{r}$ is a bijective map from D to S

Example 6.4.3

Find a parameterization of the surface consisting of the points $(x, y, z) \in \mathbb{R}^3$ such that $y = \sin 7z$, $0 \le x \le 1$, and $0 \le z \le \frac{\pi}{2}$.

Solution

One way to map a pair of points (u, v) to a triple of points (x, y, z) satisfying the given equations is to use $u = z$ and $v = x$ as our two parameters. Then $\mathbf{r}(u, v) = (v, \sin(7u), u)$ maps the rectangle $[0, \frac{\pi}{2}] \times [0, 1]$ to the desired surface, as shown.

Example 6.4.4

Find a parameterization of the portion of the unit sphere in the first octant.

Solution

A parameterization is a way of using two numbers to identify points on the surface. We have a standard way of doing this on the sphere we live on: latitude and longitude. So we can parametrize a sphere or a portion thereof by using θ and ϕ as parameters.* As we discovered in Exercise 3.4.4, the point on the sphere with spherical coordinates 1, θ, and ϕ is

$$\mathbf{r}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

* We could define $u = \theta$ and $v = \phi$, but since we already have standard names for these parameters, we'll just switch to using θ and ϕ instead of u and v

This function $\mathbf{r}$ maps $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ onto the portion of the sphere in the first octant:

The lesson of Examples 6.4.3 and 6.4.4 is that **coordinates make good parameters**. It might be convenient to use Cartesian coordinates, cylindrical or spherical coordinates, or even custom coordinates. But if we find two coordinate functions whose values conveniently specify a location (x, y, z) on the surface, then we can define $\mathbf{r}$ to be the map which sends each pair of coordinate values to the triple (x, y, z).

Exercise 6.4.1

(i) Parametrize the portion of the plane $x + 2y + 3z = 6$ in the first octant.

(ii) Sketch and parametrize the set of points (x, y, z) satisfying $0 \le z \le x$ and $x^2 + y^2 = 1$.

If we can parametrize a surface, then we can split S into many small pieces by splitting D into many small rectangles. Each small rectangle $[u, u + \Delta u] \times [v, v + \Delta v]$ maps under the parameterization to an approximate parallelogram spanned by

$$\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v) \approx \partial_u \mathbf{r}(u, v)\Delta u \quad \text{and} \quad \mathbf{r}(u, v + \Delta v) - \mathbf{r}(u, v) \approx \partial_v \mathbf{r}(u, v)\Delta v.$$

The area spanned by this small parallelogram is the cross product of these two vectors. Multiplying this area by the value of f somewhere in the parallelogram and summing over all the tiny rectangles, we get the following theorem.

on surface parameterization

Theorem 6.4.1: (Surface integral formula)

If $\mathbf{r} : D \to \mathbb{R}^3$ is a parameterization of a surface S, then

$$\iint_S f \, dA = \iint_D f(\mathbf{r}(u, v))|\partial_u \mathbf{r} \times \partial_v \mathbf{r}| \, du \, dv. \tag{6.4.1}$$

Example 6.4.5

Find the average value of the function $f(x, y, z) = z$ on the upper unit half-sphere.

Solution

Let's use the parametrization

$$\mathbf{r}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi),$$

as θ ranges from 0 to 2π and ϕ ranges from 0 to $\frac{\pi}{2}$. Then

$$|\partial_\theta \mathbf{r} \times \partial_\phi \mathbf{r}|^2 = \left(-\sin\phi \sin^2\theta \cos\phi - \sin\phi \cos\phi \cos^2\theta\right)^2 + \sin^4\phi \sin^2\theta + \sin^4\phi \cos^2\theta = \sin^2\phi.$$

Theorem 6.4.1 gives

$$\int_S z \, dA = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \overbrace{(\cos\phi)}^{z} \; \overbrace{\sin\phi}^{|\partial_\theta \mathbf{r} \times \partial_\phi \mathbf{r}|} \; d\phi \, d\theta = \pi.$$

The surface area of the upper half-sphere is*

$$\int_S 1\,dA = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \overbrace{1}^{1}\ \overbrace{\sin\phi}^{|\partial_\theta \mathbf{r}\times\partial_\phi \mathbf{r}|}\ d\phi\,d\theta = 2\pi.$$

Therefore, the average value of z is $displaystyle\frac{\pi}{2\pi} = \boxed{\frac{1}{2}}$.

* We could also use the formula for the surface area of the sphere, but let's do it from scratch

Exercise 6.4.2

Suppose that S is the surface consisting of the points $(x, y, z) \in \mathbb{R}^3$ satisfying $y^2 + z^2 = 4$ and $-1 \le x \le 2$. Equip S with the density function $\sigma(x, y, z) = y + 2$, and find the resulting center of mass of S.

Exercise 6.4.3

(i) Find the surface area of the portion of the origin-centered unit sphere lying above the line $z = \frac{1}{2}$.

(ii) Find the value z_0 such that half of the sphere lies above the plane $z = z_0$.

Exercise 6.4.4

Suppose that S is the portion of the graph of the function $f(x, y) = 3 - x - y$ lying above the disk $(x-2)^2 + (y-1)^2 \le 1$. Suppose that a point P is selected uniformly at random from S. Find the expected value of $g(P)$, where $g(x, y, z) = 2x + z$.

Exercise 6.4.5

Show that if S is a planar domain, then the formula (6.4.2) agrees with the custom-coordinate integration formula (Theorem 5.4.1).

6.4.2 流行理论

on surface integrals of a vector field

Example 6.4.6

Consider a constant-velocity river flowing through a net as shown. Find the volume of water flowing through the net per unit time, in terms of the area A of the net, the velocity v, and the angle θ between the direction of the river's flow and a vector normal to the plane of the net.

Solution

Imagine letting the water flow for one time unit and then taking a snapshot. The locations of all the water molecules which have flowed through the net during this period occupy a parallelepiped, as shown. The base area of this parallelepiped is A, while its height is equal to* $v\cos\theta$. Therefore, the volume of water passing through the net per unit of time is $\boxed{Av\cos\theta}$.

* This is a right-triangle trigonometry exercise

Let's define the vector $\mathbf{A}$ whose length is equal to A and whose direction is orthogonal to the net. Then the **flow** $Av\cos\theta$ can be written as

$$\text{flow} = \mathbf{A}\cdot\mathbf{v},$$

where $\mathbf{v}$ is the river's velocity vector.

Example 6.4.7

Suppose that the velocity field of a body of water is given by $\mathbf{F} = \langle -2y, 4z, x\rangle$ (in meters per second) and that a rectangular net is positioned in the water with corners at $(1,1,3)$, $(1,4,3)$, $(1,4,5)$, and $(1,1,5)$ (in meters). Find the volume of water flowing through the frame of the net per second.

Solution

Since the velocity field isn't constant, we divide the net into small patches and treat the velocity as constant on each one. Since the rectangle is contained in the plane $x = 1$, the vector $\langle 1,0,0\rangle$ is normal to the rectangle. Therefore, the flow through a small patch of area ΔA located at (x,y,z) is approximately equal to

$$\mathbf{A}\cdot\mathbf{v} = \big\langle \Delta A, 0, 0\big\rangle \cdot \big\langle -2y, 4z, x\big\rangle = -2y\,\Delta A.$$

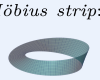
If we sum the flow through each patch across the whole rectangular region occupied by the net, we get a Riemann sum that converges as $\Delta A \to 0$ to

$$\int_1^4 \int_3^5 (-2y)\,dz\,dy = -30.$$

Therefore, the volume of water is $\boxed{30}$ cubic meters per second, and the net flow is in the direction *towards* the side facing the yz-plane.

The ideas in Example 6.4.7 yield the following definition, which develops the notion of vector field integration over surfaces in terms of the scalar surface integral.

A surface with a distinguished side is called an *oriented surface*, and not every surface is orientable. For example, the *Möbius strip*:



Definition 6.4.1

The **flow** of a vector field $\mathbf{F}$ through a surface S from one side $\mathfrak{s}$ to the other side $\mathfrak{t}$ is defined by

$$\iint_S \mathbf{F}\cdot d\mathbf{A} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dA,$$

where $\mathbf{n} = \mathbf{n}(x,y,z)$ is a unit vector which is orthogonal to S at each point (x,y,z) and points in the direction from $\mathfrak{s}$ to $\mathfrak{t}$.

Exercise 6.4.6

Find the flow of the vector field $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$ from the inside to the outside of the unit sphere.

We can write down an integral formula for calculating flow through an arbitrary parametrized surface S. Since both $\partial_u \mathbf{r}$ and $\partial_v \mathbf{r}$ are tangent to the surface S, the cross product of $\partial_u \mathbf{r}$ and $\partial_v \mathbf{r}$ is orthogonal to S. When we divide this cross product by its length and substitute into Theorem 6.4.1, we get a cancellation of the $|\partial_u \mathbf{r} \times \partial_v \mathbf{r}|$ factors. In other words, $d\mathbf{A} = \partial_u \mathbf{r} \times \partial_v \mathbf{r} \, du \, dv$, so to find the flow we can dot $\mathbf{F}$ with $\partial_u \mathbf{r} \times \partial_v \mathbf{r}$ and integrate over D:

Theorem 6.4.2: (Flow integral formula)

mi on surface parameterization

If $\mathbf{r} : D \to \mathbb{R}^3$ is a parameterization of a surface S such that at each point of S, the vector $\partial_u \mathbf{r} \times \partial_v \mathbf{r}$ points from one side $\mathfrak{s}$ of S to the other side $\mathfrak{t}$, then the flow of $\mathbf{F}$ through S from $\mathfrak{s}$ to $\mathfrak{t}$ is

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\partial_u \mathbf{r} \times \partial_v \mathbf{r}) \, du \, dv. \tag{6.4.2}$$

Example 6.4.8

Find the flow of the vector field $\mathbf{F} = \langle y^2, 0, z \rangle$ up through the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane.

Solution

Let's parametrize the surface using the cylindrical coordinates $u = r$ and $v = \theta$, so that

$$\mathbf{r}(u,v) = \langle u \cos v, u \sin v, 4 - u^2 \rangle,$$

where (u,v) ranges over $[0,2] \times [0,2\pi]$. Then

$$\partial_u \mathbf{r}(u,v) = \langle \cos v, \sin v, -2u \rangle \quad \text{and} \quad \partial_v \mathbf{r}(u,v) = \langle -u \sin v, u \cos v, 0 \rangle,$$

which yields

$$\partial_u \mathbf{r} \times \partial_v \mathbf{r} = \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle.$$

This vector indeed points from the bottom side of the surface to the top, so Theorem 6.4.2 tells us that the desired flow is

$$\int_0^{2\pi} \int_0^2 \langle u^2 \sin^2 v, 0, 4 - u^2 \rangle \cdot \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle \, du \, dv,$$

which simplifies to

$$\int_0^{2\pi} \int_0^2 (2u^4 \sin^2 v \cos v + 4u - u^3) du \, dv = \boxed{8\pi}.$$

Exercise 6.4.7

Find the flow of the vector field $\mathbf{F} = \langle -y, x, z^{3e^{\sin xyz}} \rangle$ from the inside to the outside of the cylinder $\{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le 3 \text{ and } x^2 + y^2 = 1\}$. Explain why your answer makes sense physically.

6.5 卷积理论

* Highly recommended

As suggested by the title of the classic book *div, grad, curl, and all that* by H.M. Schey*, the gradient is one of the main characters in the vector calculus story. In this section we will develop the two other fundamental vector calculus derivative operators: *divergence* and *curl*. Like Schey, we will emphasize the underlying physical intuition.

on divergence

6.5.1 发散

The **divergence** of a vector field $\mathbf{F} = \langle M, N, P\rangle$ is the function whose value at a point (x, y, z) is equal to the **net flow density** of $\mathbf{F}$ out of a small region located at (x, y, z). In other words, if we put a small box of dimensions $\Delta x \times \Delta y \times \Delta z$ centered at (x, y, z), then the ratio of the net flow of $\mathbf{F}$ out of the box to the volume of the box converges to $\operatorname{div}\mathbf{F}(x, y, z)$ as $(\Delta x, \Delta y, \Delta z) \to 0$.

* The normal vector for the top is $\langle 0, 0, 1\rangle$, and that dots with $\mathbf{F}$ to give P.

Let's work out a formula for $\operatorname{div}\mathbf{F}$. The net flow through the top of the box of dimensions $\Delta x \times \Delta y \times \Delta z$ centered at (x, y, z) is approximately* $P(x, y, z + \Delta z/2)\Delta x\Delta y$, and the net flow through the bottom of the box is approximately $P(x, y, z - \Delta z/2)\,\Delta x\,\Delta y$. Thus the difference is approximately $(\partial_z P)(x, y, z)\,\Delta x\,\Delta y\,\Delta z$, and the difference per unit volume is $\partial_z P(x, y, z)$. Similarly, the front/back and left/right sides contribute $\partial_x M(x, y, z)$ and $\partial_y N(x, y, z)$ to the net flow density out of the box. Therefore, $\operatorname{div}\mathbf{F} = \partial_x M + \partial_y N + \partial_z P$. This formula suggests the alternate notation $\nabla\cdot\mathbf{F}$ for the divergence of $\mathbf{F}$.

Definition 6.5.1: Divergence

The **divergence** of a vector field $\mathbf{F} = \langle M, N, P\rangle$ is a scalar function defined by

$$\nabla\cdot\mathbf{F} = \partial_x M + \partial_y N + \partial_z P.$$

For example, the divergence of $\langle x^2, xy, z\rangle$ is $2x + x + 1 = 3x + 1$.

Example 6.5.1

Figure out where $\nabla\cdot\mathbf{F}$ is positive for the vector field $\mathbf{F}$ shown.

Solution

We can see that in the top left of the diagram that there is more flow into each small region than out of it, since the vectors are downward-pointing and longer than the vectors below them. Therefore, the divergence is negative in the top left.

By similar reasoning, we see that the divergence is positive in the bottom-right part of the figure. The dividing line between regions of positive and negative divergence is $y = x$, since points along that line have vectors of equal length pointing towards and away from them.

Exercise 6.5.1

Find the divergence of the gravitational vector field in (6.1.1).

Exercise 6.5.2

Look at a vector plot* to figure out where $\nabla \cdot \mathbf{F} > 0$, using the approach of Example 6.5.1, for the vector field $\mathbf{F} = \langle xy, y^2 \rangle$. Then evaluate $\nabla \cdot \mathbf{F}$ and find where $\nabla \cdot \mathbf{F} > 0$ algebraically.

for drawing a vector plot

6.5.2 卷积

on curl

The **curl** of a vector field $\mathbf{F} = \langle M, N, P \rangle$ is the vector field whose value at a point (x, y, z) describes the **circulation density** of $\mathbf{F}$ around (x, y, z). Specifically, consider a curve ℓ centered at a point (x, y, z) looping counterclockwise around the vector $\langle 0, 0, 1 \rangle$. We define the *third* component of curl $\mathbf{F}(x, y, z)$ to be the limit as the size of the loop tends to zero of the ratio of $\int_\ell \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ to the area enclosed by ℓ. We define the first two components of curl $\mathbf{F}(x, y, z)$ similarly, with ℓ running counterclockwise around $\langle 1, 0, 0 \rangle$ or $\langle 0, 1, 0 \rangle$, respectively, instead of $\langle 0, 0, 1 \rangle$.

By Green's theorem, third component of the curl is equal to* $\partial_x N - \partial_y M$. If we swap out (x, y) for (y, z) or (z, x) to get similar formulas for the first two coordinates, we find that curl $\mathbf{F} = \left\langle \partial_y P - \partial_z N,\ -\partial_x P + \partial_z M,\ \partial_x N - \partial_y M \right\rangle$. This formula suggests the notation $\nabla \times \mathbf{F}$ for the curl of $\mathbf{F}$.

* Since Green's theorem says that circulation equals $\partial_x N - \partial_y M$ integrated, it can be interpreted as the assertion "circulation density is given by $\partial_x N - \partial_y M$".

Definition 6.5.2: Curl

The **curl** of a vector field $\mathbf{F} = \langle M, N, P \rangle$ is a vector field on $\mathbb{R}^3$ defined by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \left\langle \partial_y P - \partial_z N,\ -\partial_x P + \partial_z M,\ \partial_x N - \partial_y M \right\rangle.$$

Another way to visualize curl physically is to interpret the vector field as a fluid velocity field and place a small paddle wheel (with an axis of rotation in a coordinate direction) at a particular point in this field. The corresponding component of the curl measures how rapidly and in which direction this paddle wheel turns.

Example 6.5.2

Consider the vector field $\mathbf{F}$ shown. Find the sign of the z-component of the curl of $\mathbf{F}$ at any point in the xy-plane.

Solution

We can see that if we place a small paddle wheel at a point of interest (as shown in the figure above) it will rotate in the counterclockwise direction. This is because the vectors on the right (meaning the side where x is larger) push harder than the vectors on the left. Therefore, the z-component of the curl is $\boxed{\text{positive}}$.

When we studied gradients, we learned that directional derivatives of a function in the coordinate directions actaully determine its directional derivatives in all directions: the derivative in the $\langle u_1, u_2 \rangle$ direction is equal to a linear combination with weights u_1 and u_2 of the derivatives in the coordinate directions. The same idea holds for the curl: if $\mathbf{u}$ is a unit vector, let's define $\operatorname{curl}_{\mathbf{u}} \mathbf{F}$ to be the circulation density of $\mathbf{F}$ in the counterclockwise direction around $\mathbf{u}$. More precisely, if ℓ is a small loop which is perpendicular to $\mathbf{u}$ and oriented counterclockwise as viewed from the head of $\mathbf{u}$, then the ratio of $\int_\ell \mathbf{F} \cdot d\mathbf{r}$ to the area enclosed by ℓ converges to $\operatorname{curl}_{\mathbf{u}} \mathbf{F}$ as the size of ℓ tends to zero.

Theorem 6.5.1

!!!

The circulation density $\operatorname{curl}_{\mathbf{u}} \mathbf{F}$ of a vector field $\mathbf{F}$ around a unit vector $\mathbf{u}$ is equal to $(\nabla \times \mathbf{F}) \cdot \mathbf{u}$.

Thus $\nabla \times \mathbf{F}$ is a vector field whose **main purpose is to be dotted with unit vectors to compute circulation**. We will revisit this idea in Section 6.7 when we discuss Stokes' theorem.

Example 6.5.3

Find the orientation for a paddle wheel at the point $(1, 1, 1)$ in the velocity field $\langle xyz, x^2 - y, z \rangle$ which will maximize how fast it spins.

Solution

We calculate the curl: $\nabla \times \mathbf{F} = \langle 0, xy, -xz + 2x \rangle$, which at the point $(1, 1, 1)$ is equal to $\langle 0, 1, 1 \rangle$. Since the dot product of a fixed vector $\mathbf{v}$ with a unit vector is maximized when the unit vector is aligned with $\mathbf{v}$, we see that the paddle wheel should be oriented so that its axis is in the direction $\boxed{\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle}$.

Exercise 6.5.3

Calculate $\nabla \times \mathbf{F}$, where $\mathbf{F} = \langle e^{\sin \log x} + y^2, -2z, y^3 + \cos z \rangle$.

Exercise 6.5.4

Show that the curl of a conservative vector field is zero.

6.6 发散理论

on the divergence theorem

图 **6.5** The sum of the flows of **F** (not shown) out of each cell is equal to the flow out of the whole box

Suppose that $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field. Just as integrating mass density yields total mass, or integrating charge density yields total charge, integrating flow density yields total flow. This fact is called the *divergence theorem*.

The property of physical mass which makes it compatible with the idea of computing a total by integrating a density function is **additivity**: any way you divide up a solid, its mass is equal to the sum of the masses of its parts. Flow density works similarly: the net flow out of a region D is equal to the sum of the net flows out any set of regions in a subdivision of D. This is because any surface connecting adjoining regions contributes two opposite terms to the sum (see Figure 6.5).

Theorem 6.6.1: Divergence theorem

If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field with continuous partial derivatives and D is a region in $\mathbb{R}^3$ bounded by a piecewise smooth surface $S = \partial D$, then

$$\overbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}^{\text{net flow density integrated over } D} = \overbrace{\iint_{\partial D} \mathbf{F} \cdot d\mathbf{A}}^{\text{total flow out through } \partial D}.$$

Example 6.6.1

Verify that the divergence theorem holds in the case where $\mathbf{F} = \langle x^2, 3z^2, 2z^2 + y^2 \rangle$ and $D = [0,1]^3$.

Solution

The divergence of $\mathbf{F}$ is $2x + 4z$, so the divergence theorem asserts that

$$\iiint_D (2x + 4z)\, dV = \iint_{\partial D} \langle x^2, 3z^2, 2z^2 + y^2 \rangle \cdot d\mathbf{A}.$$

The left-hand side equals

$$\int_0^1 \int_0^1 \int_0^1 (2x + 4z)\, dx\, dy\, dz = 3.$$

To evaluate the right-hand side directly, we split the boundary of the cube into its six square faces. For the top face, we get

$$\iint_{\text{top face}} \mathbf{F} \cdot d\mathbf{A} = \iint_{\text{top face}} \langle x^2, 3z^2, 2z^2 + y^2 \rangle \cdot \langle 0, 0, 1 \rangle dA = \int_0^1 \int_0^1 (2 + y^2)\, dx\, dy = \frac{7}{3},$$

where we've substituted 1 for z since $z = 1$ for every point in the top face. Likewise, the integral over the bottom face is

$$-\int_0^1 \int_0^1 (0 + y^2)\, dx\, dy = -\frac{1}{3},$$

where the negative sign comes from the fact that the outward-pointing normal on the bottom face is $\langle 0, 0, -1 \rangle$.

Similarly, the integral over the $x = 1$ face is

$$\int_0^1 \int_0^1 1\,dy\,dz = 1,$$

while the $x = 0$ face contributes 0. The $y = 1$ face yields

$$\int_0^1 \int_0^1 3z^2\,dx\,dz = 1,$$

and the $y = 0$ face gives $-\int_0^1 \int_0^1 3z^2\,dx\,dz = -1$. Indeed,

$$3 \overset{\checkmark}{=} \frac{7}{3} + \left(-\frac{1}{3}\right) + 1 + 0 + 1 + (-1).$$

Exercise 6.6.1

Verify the divergence theorem in the case where $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$ and S is the unit sphere.

Exercise 6.6.2

Consider the vector field $\mathbf{F} = \langle x^3, xz, 1 - 3zx^2 \rangle$. Verify that the divergence of $\mathbf{F}$ is zero everywhere. Then use the divergence theorem to calculate the flow of $\mathbf{F}$ through the surface S shown. Note that this is not a closed surface: it excludes the square $[0, 1]^2 \times \{0\}$ on the bottom.

6.7 Stokes' 公式

on Stokes' theorem

图 6.6 The circulation of $\mathbf{F}$ (not shown) around each patch sums to the circulation around the boundary of the surface

The planar domain D in Green's theorem can be thought of as a surface S, in which case the conclusion of Green's theorem can be written as

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

The argument for Green's theorem then applies even if S doesn't lie in a plane, because Theorem 6.5.1 tells us that $\nabla \times \mathbf{F} \cdot \mathbf{n}\, dA = \nabla \times \mathbf{F} \cdot d\mathbf{A}$ yields the circulation around a small patch of the surface S. As we discussed for Green's theorem, circulation is additive: if you divide a surface into many small patches and sum the circulations around all of them, you get the circulation around the boundary of the surface. Thus we arrive at a generalization of Green's theorem known as *Stokes' theorem*.*

* If you take a differential geometry course, you will learn a far more general result of the same name which implies the divergence theorem and Theorem 6.7.1 as sp

Theorem 6.7.1: Stokes' theorem

If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field with continuous partial derivatives and S is a surface in $\mathbb{R}^3$, then*

$$\overbrace{\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A}}^{\text{circulation density integrated over } S} = \overbrace{\int_{\partial S} \mathbf{F} \cdot d\mathbf{r}}^{\text{circulation around } \partial S}.$$

* The orientation used for the surface integral must be compatible with the orientation on ∂S used for the line integral, a in Figure 6.6. See also Exam ple 6.7.2

Example 6.7.1

Let $\mathbf{F} = \langle x\sin(\pi y), \mathrm{e}^x, -\cos(\pi z)\rangle$. Find the flow of $\nabla \times \mathbf{F}$ up through the surface shown in Exercise 6.6.2.

Solution

Stokes' theorem tells us that the flow of $\nabla \times \mathbf{F}$ upwards through S is equal to the line integral of $\mathbf{F}$ around ∂S in the counterclockwise direction (see Figure 6.6). Since ∂S consists of four line segments, we calculate $\int \mathbf{F} \cdot d\mathbf{r}$ along each edge and sum the results. Integrating from $(0,0,0)$ to $(1,0,0)$, we get

$$\int_0^1 \mathbf{F}(x,0,0) \cdot \langle 1,0,0\rangle\, dx = 0.$$

From $(1,0,0)$ to $(1,1,0)$, we get

$$\int_0^1 \mathbf{F}(1,y,0) \cdot \langle 0,1,0\rangle\, dy = e.$$

From $(1,1,0)$ to $(0,1,0)$, we get

$$\int_0^1 \mathbf{F}(x,1,0) \cdot \langle -1,0,0\rangle\, dx = 0.$$

And finally from $(0,1,0)$ back to the origin, we get

$$\int_0^1 \mathbf{F}(0,y,0) \cdot \langle 0,-1,0\rangle\, dy = -1.$$

So altogether the circulation of $\mathbf{F}$ around the boundary of S is $\boxed{e-1}$.

Theorem 6.7.1 implies that for any vector field $\mathbf{F}$, a surface can be deformed without changing the flow of $\nabla \times \mathbf{F}$ through it, as long as it is deformed in such a way that its boundary is preserved:

Observation 6.7.1

In the context of Stokes' theorem, if S_1 and S_2 are surfaces whose boundaries are the same, then

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}.$$

Exercise 6.7.1

Redo Example 6.7.1 using Observation 6.7.1 in place of the calculation of the line integral around ∂S.

Example 6.7.2

Verify Stokes' theorem for the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 4$ and the vector field $\mathbf{F} = \langle x^2 y, z, y \rangle$.

Solution

We parametrize the cone as $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle$ as (u, v) ranges over $[1, 4] \times [0, 2\pi]$. We also calculate $\nabla \times \mathbf{F} = \langle 0, 0, -x^2 \rangle$. Then applying the parametrization formula for flow (Theorem 6.4.2), we get

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A} = \int_0^{2\pi} \int_1^4 \langle 0, 0, -u^2 \cos^2 v \rangle \cdot \langle -u \cos v, -u \sin v, u \rangle \, du \, dv = -\frac{255\pi}{4}.$$

Since the normal vector $\langle -u\cos v, -u\sin v, u\rangle$ points in/up for all $(u,v) \in [1,4]\times[0,2\pi]$, the value $-\frac{255\pi}{4}$ represents the flow of the curl $\mathbf{F}$ from the *outside* of the cone to the *inside*. We should be careful to orient the boundary arcs in a manner which is compatible with this orientation. We can do this by following the derivation of Stokes' theorem: we subdivide the surface into many small patches, and we orient each little boundary in the counterclockwise direction around the normal vector used in the surface integral (in this case, in/up). Then the boundary of S is obtained by putting together the oriented segments whose contributions aren't canceled (see the small oriented red loop in Figure 6.7). We can see from this procedure that the top edge of the surface is oriented counterclockwise when viewed from above.

图 6.7 The normal vectors for the surface integral and the corresponding orientations of the boundary arcs in Stokes' theorem

Since the bottom edge of each patch is oriented *clockwise* when viewed from above, the bottom edge of the surface should be oriented clockwise. So we can parametrize the top edge as $\langle 4\cos t, 4\sin t, 4\rangle$ and the bottom edge as* $\langle \cos t, -\sin t, 1\rangle$, in both cases as t ranges from 0 to 2π. We get

$$\int_{\partial S}\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(4\cos t, 4\sin t, 4)\cdot\langle -4\sin t, 4\cos t, 0\rangle\,dt + \int_0^{2\pi}\mathbf{F}(\cos t, -\sin t, 1)\cdot\langle -\sin t, -\cos t, 0\rangle\,dt$$

which works out to

$$\int_0^{2\pi}\left[(4\cos t)^2(4\sin t)(-4\sin t) + (4)(4\cos t) + 0\right]dt + \int_0^{2\pi}\left[\cos^2 t(-\sin t)^2 + (1)(-\cos t) + 0\right]dt$$
$$= -64\pi + \frac{\pi}{4} = -\frac{255\pi}{4},$$

*We reverse the orientation by replacing t with $-t$, and then we can use the identities $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$

as desired.

The following consequence of Stokes' theorem tells us that the flow of the curl of a vector field through a closed surface (such as a sphere, a donut, or a rectangular prism) is zero:

Observation 6.7.2

If $\mathbf{G}$ is a vector field which is equal to the curl of some vector field with continuous partial derivatives and if S is a piecewise smooth closed surface (in other words, a surface with no boundary), then the flow of $\mathbf{G}$ through S is zero.

Exercise 6.7.2

Suppose that S is the surface consisting of the points on the sphere $x^2 + y^2 + z^2 = 1$ which are not inside the sphere $x^2 + y^2 + (z+1)^2 = 1$. Find $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$, where $\mathbf{F} = \langle yz, x, \mathrm{e}^{xyz} \rangle$.

Exercise 6.7.3

Suppose $\mathbf{F} = \langle xy, y, xz \rangle$. Find $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$ where S is the portion of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Colophon

This text was typeset with LuaLaTeX, using `tcolorbox` and a version of the `mathpazo` package's Palatino fonts which was modified to borrow Greek symbols from Utopia and blackboard bold symbols from Computer Modern. The cover art was rendered using Ti*k*Z.

The figures are all produced in Asymptote and are included using the `asymptote` LaTeX package. All the files necessary to produce this document are available at `github.com/sswatson`.

A 附　录

A.1 知识回顾

A.1.1 几何与函数

A **set** is a collection of elements. These elements can be numbers, points, shapes, vectors, other sets, whatever. For example,

$$A = \{1, 4, 9\}$$

is the set consisting of the positive, single-digit perfect squares. The main thing you can do with a set is check whether a particular element is in it. For example, we say that $1 \in A$ (read "1 is an element of A"), while $2 \notin A$ ("2 is not an element of A").

Some sets with standard and specially typeset names include

- $\mathbb{R}$, the set of real numbers,
- $\mathbb{Q}$, the set of rational numbers,
- $\mathbb{Z}$, the set of integers, and
- $\mathbb{N}$, the set of natural numbers.

Subsets and set equality

We say that $A \subset B$ (read "A is a **subset** of B") if every element of A is an element of B. For example,

$$\{1, 4, 9\} \subset \{1, 2, 3, 4, 9, 10\}.$$

We say that two sets A and B are **equal** if $A \subset B$ and $B \subset A$. Note that

$$\{1, 1, 2\} = \{1, 2\} = \{2, 1\}.$$

since each element of each set is in the others. Thus we can see that sets "don't care" about repeated elements or order. All that matters is what is in and what is not. It is customary to writesets with repeats omitted, for clarity.

Intersections and unions

We write $A \cap B$, the **intersection** of A and B, for the set of all the elements that are in both A and B. So, for example,

$$\{1, 4, 9\} \cap \{x \in \mathbb{R} : x^2 > 15\} = \{4, 9\}.$$

That second set on the left-hand side, which is written in *set-builder* notation, is read as "the set of all real numbers x such that the square of x is greater than 15".

We write $A \cup B$, the **union** of A and B, for the set of all the elements that are in either A or B. So, for example,

$$\{1, 4, 9\} \cup \{1, 9, 25\} = \{1, 4, 9, 25\}.$$

Functions

If A and B are sets, then a function $f : A \to B$ is a rule that assigns a single element of B to each element of A. The set A is called the **domain** of f and B is called the **codomain** of f. Given a subset A' of A, we define the **image** $f(A')$ to be

$$f(A') = \{b \in B : \text{there exists } a \in A' \text{ so that } f(a) = b\}. \tag{A.1.1}$$

This is the set of all elements of B that get mapped to from some element of A'. The **range** of f is defined to be the set $f(A)$, which contains all the elements of B that get mapped to at least once.

Similarly, if $B' \subset B$, then the **preimage** $f^{-1}(B')$ of B' is defined by

$$f^{-1}(B') = \{a \in A : f(a) \in B'\}.$$

This is the subset of A consisting of every element of A that maps to some element of B'.

A function f is **injective** if no two elements in the domain map to the same element in the codomain; in other words if $f(a) = f(a')$ implies $a = a'$.

A function f is **surjective** if the range of f is equal to the codomain of f; in other words, if $b \in B$ implies that there exists $a \in A$ with $f(a) = b$.

A function f is **bijective** if it is both injective and surjective. This means that for every $b \in B$, there is exactly one $a \in A$ such that $f(a) \in b$. If f is bijective, then the function from B to A that maps $b \in B$ to the element $a \in A$ that satisfies $f(a) = b$ is called the **inverse** of f.

If $f : A \to B$ and $A' \subset A$, then the **restriction** of f to A is the function $f|_{A'} : A' \to B$ defined by $f|_{A'}(x) = f(x)$ for all $x \in A'$.

If $f : A \to B$ and $g : B \to C$, then the function $g \circ f$ which maps $x \in A$ to $g(f(x)) \in C$ is called the **composition** of g and f.

If the rule defining a function is sufficiently simple, we can describe the function using **anonymous function notation**. For example, $x \in \mathbb{R} \mapsto x^2 \in \mathbb{R}$, or $x \mapsto x^2$ for short, is the squaring function from $\mathbb{R}$ to $\mathbb{R}$. Note that bar on the left edge of the arrow, which distinguishes the arrow in anonymous function notation from the arrow between the domain and codomain of a named function.

Cartesian product

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all pairs (a, b) where $a \in A$ and $b \in B$. For example, $[0, 3] \times [0, 2]$ is a rectangle in the plane. We sometimes use exponents for a Cartesian product of a set with itself. Thus $[0, 1]^2$ is a unit square in $\mathbb{R}^2$, and $[0, 1]^3$ is a unit cube in $\mathbb{R}^3$.

A.1.2 Trig review

This appendix provides a streamlined presentation of trig which is intended to provide enough starting off points to recover everything else you need.

Trig Review

1. **Cosine and sine**. The basic trig functions are $\cos\theta$ and $\sin\theta$. The most important definition of these functions is the following: the cosine of an angle θ is equal to the **x-coordinate** of the point obtained by rotating $(1,0)$ an angle of θ about the origin. Sine is the same, but with the y-coordinate instead of x.

   This idea bears repeating: the point on the unit circle obtained by rotating $(1,0)$ an angle θ about the origin is equal to $(\cos\theta, \sin\theta)$, **by definition of cosine and sine**. !!!

2. **The other ones**. The other four trig functions are simply abbreviations for various combinations of sine and cosine:

$$\begin{aligned} \sin\theta &= \sin\theta & \sec\theta &= \tfrac{1}{\cos\theta} \\ \cos\theta &= \cos\theta & \csc\theta &= \tfrac{1}{\sin\theta} \\ \tan\theta &= \tfrac{\sin\theta}{\cos\theta} & \cot\theta &= \tfrac{\cos\theta}{\sin\theta} \end{aligned}$$

3. **Special right triangles**. The following two triangles, each half of a regular polygon, can be handy for evaluating trig functions at special angles.

4. **Pythagorean identities**. The famous identity $\sin^2\theta + \cos^2\theta = 1$ follows from the definition of sine and cosine combined with the Pythagorean theorem. Dividing both sides of this equation by $\sin^2\theta$ or $\cos^2\theta$, we get

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{and} \qquad 1 + \cot^2\theta = \csc^2\theta.$$

5. **Sum-angle formulas**. The sine sum-angle formula is worth memorizing: for all α and β, we have

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha}\,.$$

   The cosine sum-angle formula is worth memorizing too, although it can be derived fairly easily from the sine formula by substituting $\frac{\pi}{2} - \alpha$ for α and $-\beta$ for β. We get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

From the above identities, we can derive many others. For example, setting $\alpha = \beta$ in the cosine sum-angle formula, we get

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$$

Substituting $\cos^2\alpha = 1 - \sin^2\alpha$, we find that

$$\cos 2\alpha = 1 - 2\sin^2\alpha.$$

which can be solved to express $\sin^2\alpha$ in terms of $\cos 2\alpha$.

A.1.3 Summation notation

Some shorthand is essential for writing sums with many terms. Perhaps the most common approach is to use ellipses:

$$1 + 2 + 3 + \cdots + 99 + 100 = 5050.$$

However, this approach is not ideal because the reader is left to infer the pattern.

When more precision is required, we would like to specify a formula for the kth term as well as a starting and ending value. For example, the sum

$$1 + 4 + 9 + 16 + \cdots + 100$$

can be written as "the sum of k^2 as k ranges from 1 to 10". The math notation that has been adopted for abbreviating this English phrase is the following:

$$\sum_{k=1}^{10} k^2$$

The variable k is called a *dummy variable*, since it is only there as a way to specify a formula for generating the terms. We could change each k to a different symbol without changing the essential meaning, which is "sum the first 10 positive perfect squares".

Exercise A.1.1

Find $displaystyle\sum_{k=1}^{5} \frac{1}{k(k+1)}$.

Exercise A.1.2

Express $\frac{1}{1} + \frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots$ using summation notation. Hint: your upper limit will be ∞.

Exercise A.1.3

Find $displaystyle\sum_{k=1}^{5} \sum_{j=1}^{k} j$.

A.2 Reference

A.2.1 Visualizing functions

A.2.2 Polar, cylindrical, and spherical coordinate reference

A.2.3 Proof of the second derivative test