IN4320: Machine Learning Assignment 3

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1 Exercise 1

1.1 a

For strategy A, we consider the expert with smallest cumulative loss, then for each step I calculated the cumulative loss by equation 1, and the result is in table 1. Based on table 1, I obtain the p_t table 2 (the first column is defined initially, $p_t = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$).

$$L_{t-1}^i = \sum_{s=1}^{t-1} z_s^i \tag{1}$$

For strategy B, it applies the Aggregating Algorithm, it holds the same cumulative loss as strategy A as equation 1, and the p_t is defined as equation 2.

$$p_t^i = \frac{e^{-L_{t-1}^i}}{C_{t-1}} \tag{2}$$

where $C_{t-1} = \sum_{d}^{j=1} e^{-L_{t-1}^i}$ ensures $\sum_i p_t^i = 1$. Based on equation 1 and results in table 2, I calculated the p_t for strategy B manually.

$p_2 = (\frac{e^0}{e^0+e^{-0.1}+e^{-0.2}}, \frac{e^{-0.1}}{e^0+e^{-0.1}+e^{-0.2}}, \frac{e^{-0.2}}{e^0+e^{-0.1}+e^{-0.2}}) = (0.37, 0.33, 0.30)$

$p_3 = (\frac{e^0}{e^0+e^{-0.1}+e^{-0.3}}, \frac{e^{-0.1}}{e^0+e^{-0.1}+e^{-0.3}}, \frac{e^{-0.3}}{e^0+e^{-0.1}+e^{-0.3}}) = (0.38, 0.34, 0.28)$

$p_4 = (\frac{e^{-1}}{e^{-1}+e^{-0.1}+e^{-0.3}}, \frac{e^{-0.1}}{e^{-1}+e^{-0.1}+e^{-0.3}}, \frac{e^{-0.3}}{e^{-1}+e^{-0.1}+e^{-0.3}}) = (0.18, 0.45, 0.37)$

The results are shown in table 3 ($p_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is set initially as strategy A).

t	e_1	e_2	e_3
2	0	0.1	0.2
3	0	0.1	0.3
4	1	0.1	0.3

Table 1: cumulative loss for $t = 1, ..., 4$

	p_1	p_2	p_3	p_4
e_1	0.33	1	1	0
e_2	0.33	0	0	1
e_3	0.33	0	0	0

Table 2: the selected action p_t for $t = 1, ..., 4$ for strategy A

	p_1	p_2	p_3	p_4
e_1	0.33	0.37	0.38	0.18
e_2	0.33	0.33	0.34	0.45
e_3	0.33	0.30	0.28	0.37

Table 3: the selected action p_t for $t = 1, ...4$ for strategy B

1.2 b

From (a), I have calculated p_t for both strategy A and B. Then apply the mix loss equation 3 to calculate the mix loss.

$$l_m(p_t, z_t) = -ln(\sum_{i=1}^{d} p_t^i e^{z_t^i}) \tag{3}$$

For strategy A:

$l_m(p_1, z_1) = -ln(0.33 \times e^0 + 0.33 \times e^-0.1 + 0.33 \times e^-0.2) = 0.0967$
$l_m(p_2, z_2) = -ln(1 \times e^0) = 0$
$l_m(p_3, z_3) = -ln(1 \times e^-1) = 1$
$l_m(p_4, z_4) = -ln(1 \times e^-0.9) = 0.9$
the **total mix loss for A is 1.9967**.
For strategy B:
$l_m(p_1, z_1) = -ln(0.33 \times e^0 + 0.33 \times e^-0.1 + 0.33 \times e^-0.2) = 0.0967$
$l_m(p_2, z_2) = -ln(0.37 \times e^0 + 0.33 \times e^0 + 0.30 \times e^-0.1) = 0.0290$
$l_m(p_3, z_3) = -ln(0.38 \times e^-1 + 0.34 \times e^-0 + 0.28 \times e^0) = 0.2730$
$l_m(p_4, z_4) = -ln(0.18 \times e^0 + 0.45 \times e^-0.9 + 0.37 \times e^0) = 0.3102$
the **total mix loss for B is 0.7089**.

1.3 c

The equation 19 is to calculate the regret.

$$\begin{aligned} R_n^E &= \sum_{t-1}^{n} l(p_t, z_t) - min_i \sum_{t=1}^{n} l(e_i, z_t) \\ &= \sum_{t-1}^{n} l(p_t, z_t) - min_i \sum_{t=1}^{n} z_t^i \end{aligned} \tag{4}$$

From given table it is clear that expert 3 is currently the best expert, and the loss of expert 3 for $t = 1, ...4$ is 0.3. So **regret for A and B** are $R_4^A = 1.9967 - 0.3 =$ **1.6967** and $R_4^B = 0.7089 - 0.3 =$ **0.4089** respectively.

1.4 d

From equation 19, we have: $R_n^E = \sum_{t-1}^{n} l(p_t, z_t) - min_i \sum_{t=1}^{n} z_t^i$,
also we have the theoretical guarantee that $R_n^E \leq log(d)$,
thus $\sum_{t-1}^{n} l(p_t, z_t) - min_i \sum_{t=1}^{n} z_t^i \leq log(d)$,
it is $\sum_{1}^{n} l(p_t, z_t) \leq log(d) + min_i \sum_{t=1}^{n} z_t^i$.
So, the **formula of C** should be $C = log(d) + min_i \sum_{t=1}^{n} z_t^i$,
in this setting, $n = 4, d = 3$, $C = log(3) + min_i \sum_{t=1}^{4} z_t^i$.

1.5 e

As I have mentioned in (c), the best expert applies the value of 0.3, thus $min_i \sum_{t=1}^{4} z_t^i = 0.3$, so the **numerical value of C** is $C = log(3) + 0.3 =$ **1.3986**.

1.6 f

It is obvious that for strategy A $1.9967 > C$ and for B $0.7089 < C$. I conclude the reason is that C is calculated from Aggregating Algorithm, so B is limited by this bound but A is not, that is why mix loss of B is lower than C but A's is larger.

2 Exercise 2

2.1 a

By equation 19, and we have $n = 1$ and $d = 2$, $p_1 = (a, 1 - a)$, where $0 \leq a \leq 1$, write regret function as follow:

$$R_1^E = -ln(ae^{-z_1^1} + (1 - a)e^{-z_1^2}) - minz_1^1, z_1^2 \tag{5}$$

for equation 5, the term $minminz_1^1, z_1^2$ has three conditions.
(1) when $z_1^1 > z_1^2$

$$\begin{aligned} R_1^E &= -ln(ae^{-z_1^1} + (1 - a)e^{-z_1^2}) - z_1^2 \\ &= -ln(ae^{-z_1^1} + (1 - a)e^{-z_1^2}) - ln(e^{z_1^2}) \\ &= -ln(ae^{z_1^2 - z_1^1} + (1 - a)) \end{aligned} \tag{6}$$

(2) when $z_1^1 < z_1^2$

$$\begin{aligned} R_1^E &= -ln(ae^{-z_1^1} + (1 - a)e^{-z_1^2}) - z_1^1 \\ &= -ln(ae^{-z_1^1} + (1 - a)e^{-z_1^2}) - ln(e^{z_1^1}) \\ &= -ln(a + (1 - a)e^{z_1^1 - z_1^2}) \end{aligned} \tag{7}$$

(3) when $z_1^1 = z_1^2$

$$
\begin{aligned}
R_1^E &= -ln(ae^{-z_1^1} + (1-a)e^{-z_1^2}) - z_1^2 \\
&= -ln(ae^{-z_1^1} + (1-a)e^{-z_1^2}) - ln(e^{z_1^2}) \\
&= -ln(ae^{z_1^1 - z_1^2} + (1-a)e^{z_1^1 - z_1^2}) \\
&= -ln(ae^0 + (1-a)e^0) = 0
\end{aligned}
\tag{8}
$$

it is intuitively obvious that if $z_1^1 = z_1^2$, it does not matter which expert we would trust more because whoever we choose would give us the same result, also the best result as well, and the regret is also 0.

Based on the relation from equation 6,7 and 8, now try to fix z_1^1 and z_1^2. for (1), when $z_1^1 > z_1^2$, let $z_1^1 = \infty$, $z_1^2 = -\infty$, then have:

$$
\begin{aligned}
R_1^E &= -ln(ae^{-\infty} + (1-a)) \\
&= -ln(1-a)
\end{aligned}
\tag{9}
$$

for (2), when $z_1^1 < z_1^2$, let $z_1^1 = -\infty$, $z_1^2 = \infty$, then have:

$$
\begin{aligned}
R_1^E &= -ln(a + (1-a)e^{-\infty}) \\
&= -ln(a)
\end{aligned}
\tag{10}
$$

to maximise regret, we should pick the larger value of $-ln(1-a)$ and $-ln(a)$. Let $-ln(1-a) = -ln(a)$, $a = \frac{1}{2}$, thus:
when (1) $0.5 < a < 1$, $z_1^1 > z_1^2$, could be $z_1^1 = \infty$ and $z_1^2 = -\infty$;
when (2) $0 < a < 0.5$, $z_1^1 < z_1^2$, could be $z_1^1 = -\infty$ and $z_1^2 = \infty$;
when (3) $a = 0.5$, $z_1^1 = z_1^2$, could be $z_1^1 = \infty$ and $z_1^2 = -\infty$ or $z_1^1 = \infty$ and $z_1^2 = -\infty$, but here $\frac{z_1(+\infty)}{|z_1(-\infty)|} = \infty$, say the $+\infty$ should be higher-order infinite compared with the $|-\infty|$ to ensure to maximise the regret.

2.2 b

To find values of a that offer a regret equal or larger than $ln(d)$, here, $d = 2$, $ln(2)$ it is. Solve:

$$
\begin{cases}
-ln(1-a) \leq ln(2) & (11) \\
-ln(a) \leq ln(2) & (12)
\end{cases}
$$

I got $a = \frac{1}{2}$, so when $a = \frac{1}{2}$, there is an adversary move such that the regret is larger or equal to $ln(d)$.

2.3 c

When $d > 2$, write mix loss function in equation 3 and the regret function obtained in equation 5, I got the regret function under the certain requirement:

$$\begin{aligned} R_1^E &= -ln(\sum_{i=1}^{d} p_1^i e^{-z_1^i}) - minz_1 \\ &= -ln(\sum_{i=1}^{d} p_1^i e^{-z_1^i}) - ln(e^{minz_1}) \\ &= -ln(\sum_{i=1}^{d} p_1^i e^{min(z_1)-z_1^i}) \end{aligned} \tag{13}$$

Set when the kth expert has the lowest value and all the other experts' loss is way higher than z_1^k. Thus, for any $j \neq k$, $z_1^k - z_1^j = -\infty$, for $e^{min(z_1)-z_1^k} = 1$ and all $j \neq k$, $e^{min(z_1)-z_1^j} = 0$. Then, the regret couod be expressed as follow:

$$\begin{aligned} R_1^E &= -ln(\sum_{i=1}^{d} p_1^i e^{min(z_1)-z_1^i}) \\ &= -ln(p_1^k \times 1 + \sum_{i=1, i\neq k}^{d} p_1^i \times 0) \quad = -ln(p_1^k) \end{aligned} \tag{14}$$

this result applies the result obtained in (b), where which expert has lowest loss, its probability is assigned to the regret function.

Similarly in (c), solve:

$$\begin{cases} -ln(p_1^1) \leq ln(d) & (15) \\ -ln(p_1^2) \leq ln(d) & (16) \\ ... & (17) \\ -ln(p_1^d) \leq ln(d) & (18) \end{cases}$$

getting that the boundary is $p_1^i = p_1^j = \frac{1}{d}$ for any i, j. Thus, $R_1^E = -ln(\frac{1}{d}) = ln(d)$, so there is an adversary move such that the regret is larger or equal to $ln(d)$.

Compare (a), (b), (c) and (d), clearly, (c) and (d) is the generalisation situation of (a) and (b).

2.4 d

As the question is asked to prove there is a set of $z_1, ..., z_n$ fulfils the condition. So, set in following steps ($n > 2$), for any i, j, $z_2^i = z_2^j = c_2$, $z_3^i = z_3^j = c_3$, ...,$z_n^i = z_n^j = c_n$, where $c_2, c_3, ..., c_n$ is constants, say in a certain step of all latter steps, the loss of each expert is the same, we have $min(z_j) - z_j^i = 0$, then $e^{min(z_j)-z_j^i} = 1$ So,

$$\begin{aligned}
R_n^E &= \sum_{t-1}^{n} l(p_t, z_t) - min_i \sum_{t=1}^{n} l(e_i, z_t) \\
&= \sum_{t-1}^{n} l(p_t, z_t) - min_i \sum_{t=1}^{n} z_t^i \\
&= \sum_{t=1}^{n} -ln(\sum_{i=1}^{d} p_t^i e^{-z_1^i}) - min_i \sum_{t=1}^{n} z_t^i \\
&= -ln(\sum_{i=1}^{d} p_1^i e^{min(z_1)-z_1^i}) + ... + -ln(\sum_{i=1}^{d} p_n^i e^{min(z_n)-z_n^i}) \\
&= R_1^E + -ln(\sum_{i=1}^{d} p_2^i \times 1) + ... + -ln(\sum_{i=1}^{d} p_n^i \times 1) \\
&= R_1^E + -ln(1) + ... + -ln(1) \\
&= R_1^E
\end{aligned} \tag{19}$$

Now, I give out an idea that fulfils the requirement, which could transfer the $t = 1, 2, .., n(n > 2)$ condition into $t = 1$, and its generalisation situation has been proved in (c) and (d).

3 Exercise 3

As the exercise 3 mentioned, we consider dot loss for both A and B strategies. the theoretical guarantee for any $n > 0$, if l is the dot loss l_d, we have for the Exp Strategy that

$$R_n^E \leq n\frac{\eta}{8} + \frac{ln(d)}{\eta} \tag{20}$$

Also, we have Exp strategy with learning rate for expert A is $\eta_A = \sqrt{4log(d)}$ and for expert B is $\eta_B = \sqrt{2log(d)}$. Thus,

3.1 a

when $n = 2$, the regret for A is $R_n^A \leq 2\frac{\sqrt{4ln(d)}}{8} + \frac{ln(d)}{\sqrt{4ln(d)}} = \sqrt{ln(d)}$, so $C_n^A = \sqrt{ln(d)}$. Similarly, substitute $\eta_B = \sqrt{2ln(d)}$ into equation20, I have $R_n^B \leq 2\frac{\sqrt{2ln(d)}}{8} + \frac{ln(d)}{\sqrt{2ln(d)}} = \frac{3\sqrt{2}}{4}\sqrt{ln(d)}$, so $C_n^B = \frac{3\sqrt{2}}{4}\sqrt{ln(d)}$

3.2 b

From (a), I calculated C_n^A and C_n^B. The fraction of them is:

$$\frac{C_n^A}{C_n^A} = \frac{\frac{3}{2}\sqrt{ln(d)}}{\sqrt{2}\sqrt{ln(d)}} = \frac{3\sqrt{2}}{4} \tag{21}$$

3.3 c

From (b), it is obvious that C_n^A is smaller than C_B^n since $\frac{2\sqrt{2}}{3}$ is smaller than 1, in other word, the bound for strategy A is tighter.

3.4 d

Similarly as (a), substitute n=4 into the equation, I got the new bound for A and B as follow: $C_n^A = \frac{3}{2}\sqrt{ln(d)}$ and $C_n^B = \sqrt{2}\sqrt{ln(d)}$

3.5 e

From (d), I calculated R_n^A and R_n^B. The fraction of them is:

$$\frac{C_n^A}{C_n^A} = \frac{\frac{3}{2}\sqrt{ln(d)}}{\sqrt{2}\sqrt{ln(d)}} = \frac{3\sqrt{2}}{4} \tag{22}$$

3.6 f

From (e), it is clear that $\frac{3\sqrt{2}}{4}$ is larger than 1, so C_n^A is larger and C_n^B is smaller, thus when $n = 4$ the bound for B is tighter.

3.7 g

Firstly, give my answer that it does not necessarily hold. Secondly, from (a)-(f), intuitively, there is bound for strategy B is tighter but there is situation that there exists n that makes a tighter bound for A as well. Third, I will give an inferance for my idea.

$$\begin{aligned}
&C_n^B < C_n^A \\
&\Rightarrow n\frac{\eta_B}{8} + \frac{ln(d)}{\eta_B} < n\frac{\eta_A}{8} + \frac{ln(d)}{\eta_A} \\
&\Rightarrow n\frac{\sqrt{2ln(d)}}{8} + \frac{ln(d)}{\sqrt{2ln(d)}} < n\frac{\sqrt{4ln(d)}}{8} + \frac{ln(d)}{\sqrt{4ln(d)}} \\
&\Rightarrow n > 2\sqrt{2} \approx 2.8
\end{aligned} \tag{23}$$

so, meaning when n is 3 or higher value B has a tighter bound, then for any adversary moves z_t for $t = 1, ..., n$ not necessarily hold B has tighter bound, when $n = 1, 2$ are the counter example, in this situation, A has tighter bound, thus there exists $R_n^B > R_n^A$.

3.8 h

For dot loss strategy, we have:

$$p_t^i = \frac{e^{-\eta L_{t-1}^i}}{\sum_{j=1}^d e^{-\eta L_{t-1}^i}} \tag{24}$$

where $L_{t-1}^i = \sum_{s=1}^{t-1} z_s^i$, the equation 24 could be written as:

$$\begin{aligned} p_t^i &= \frac{e^{-\eta \sum_{s=1}^{t-1} z_s^i}}{\sum_{j=1}^d e^{-\eta \sum_{s=1}^{t-1} z_s^i}} \\ &= \frac{e^{-\eta \sum_{s=1}^{t-1} z_s^i}}{e^{-\eta \sum_{s=1}^{t-1} z_s^1} + e^{-\eta \sum_{s=1}^{t-1} z_s^2} + ... + e^{-\eta \sum_{s=1}^{t-1} z_s^i} + ... + e^{-\eta \sum_{s=1}^{t-1} z_s^d}} \\ &= \frac{1}{e^{-\eta \sum_{s=1}^{t-1}(z_s^1 - z_s^i)} + e^{-\eta \sum_{s=1}^{t-1}(z_s^2 - z_s^i)} + ... + 1 + ... + e^{-\eta \sum_{s=1}^{t-1}(z_s^1 - z_s^i)}} \end{aligned} \tag{25}$$

My answer is that there are three conditions for strategy p_t change.

When an expert has smallest summation term i on previous steps, for all $j \neq i$, $z_s^j > z_s^i$, $e^{-\eta \sum_{s=1}^{t-1}(z_s^j - z_s^i)}$ decrease with increasing of η, thus the ratio (weight) of term i increases. On the contrary, for the largest summation term i on previous steps, for all $j \neq i$, $z_s^j < z_s^i$, $e^{-\eta \sum_{s=1}^{t-1}(z_s^j - z_s^i)}$ increase with increasing of η, thus the ratio (weight) of term i decreases. For the other terms, the results (increase or decrease) are not determined.

3.9 i

If we set the learning rate extremely high ($\eta = \infty$), from equation 25 we can see that the smallest the summation of previous steps a expert has, the highest weight it could obtain, which is 1 (For $e^{-\eta \sum_{s=1}^{t-1} z_s^i}$ except the smallest one i, all $z_s^j > z_s^i$, meaning for all $j \neq i$, $e^{-\eta \sum_{s=1}^{t-1}(z_s^j - z_s^i)}$...goes to 0, thus the smallest summmation term has weight 1).

4 Exercise 4

4.1 a

We have equation 26, 28 and 27,

$$W_{t+1} = p_t^T r_t W_t \tag{26}$$

$$r_t^i = \frac{x_{t+1}^i}{x_t^i} \tag{27}$$

$$z_t^i = -ln(r_t^i) \tag{28}$$

The equality is shown below 29:

$$
\begin{aligned}
\sum_{t=1}^{n} -ln(\sum_{i=1}^{d} p_t^i e^{-z_t^i}) &= \sum_{t=1}^{n} -ln(\sum_{i=1}^{d} p_t^i e^{-ln(r_t^i)}) \\
&= \sum_{t=1}^{n} -ln(\sum_{i=1}^{d} p_t^i r_t^i) \\
&= \sum_{t=1}^{n} -ln(p_t^T r_t) \\
&= \sum_{t=1}^{n} -ln(\frac{W_{t+1}}{W_t}) \\
&= -ln(\prod_{t=1}^{n} \frac{W_{t+1}}{W_t}) \\
&= -ln(W_n/W_1)
\end{aligned}
\tag{29}
$$

4.2 b

The equation 29 above illustrates that the wealth we would obtain after n days. I claim that the mix loss appropriate to minimise the investment setting. For a small mix loss, the term $\sum_{i=1}^{d} p_t^i e^{-z_t^i}$ is relatively large, then by $-ln()$ function the result then become small, which means the term $-ln(W_n/W_1)$ is small, thus the fraction of W_n/W_1 is large, it means that our total wealth after n days grows. The lower the mix loss is, the larger our wealth increases. If we want our wealth increase as much as possible, then the mix loss should be as small as possible.

4.3 c,d

Done! I ran the code "z = zeros(213,5); z = -log(r);" to implement the equation about rate and loss. I obtained a 213 × 5 matrix.

$$z_t^i = -ln(r_t^i) \tag{30}$$

4.4 e

Because all 5 experts take pure strategy, meaning they choose one and only one kind of bitcoin and never change. Thus, as we already have the loss for each kind of bitcoin at each time step, to calculate the expert loss is simply to be done by calculate the summation of the loss matrix by column. Only one line of Matlab code is needed: "loss_expert = sum(z)", which applies the equation:

$$\sum_{t=1}^{n} l(e_i, z_t) = \sum_{t=1}^{n} z_t^i \tag{31}$$

	Exp_BCH	Exp_BTC	Exp_ETH	Exp_LTC	Exp_XRP
loss	-1.1530	-1.3649	-1.3249	-1.5774	-1.6543

Table 4: The expert loss on 5 different kinds of bitcoins

The result is shown in **table 4**.

4.5 f

I implemented AA, which is shown in appendix.

4.6 g

By applying the loss function in equation 32 and the AA strategy in equation 33 and 34, which is shown in exercise 1, I calculated the loss of AA, the result is **-1.4311**.

$$l_m(p_t, z_t) = -ln(\sum_{i=1}^{d} p_t^i e^{-z_t^i}) \tag{32}$$

$$L_t^i = \sum_{s=1}^{t} z_s^i \tag{33}$$

$$p_t^i = \frac{e^- L_{t-1}^i}{C_{t-1}} \tag{34}$$

4.7 h

The equation to calculate regret is given in equation 19. I encode to achieve the best expert. The result is **0.2232**.

4.8 i

I compared the experts' loss and the loss from AA. For the experts who invest BCH, BTC and ETH, they have higher loss (-1.1530, -1.3649 and -1.3249 respective) than our AA strategy loss (-1.4311), for those who invest LTC and XRP, the loss is lower than our AA's, which is -1.5774 and -1.6543 respectively.

4.9 j

The expert regret of AA is smaller than the guaranteed expert regret. The relation is shown in the inequality 35.

$$R_n^E = 0.2232 < ln(5) \approx 1.6094 \tag{35}$$

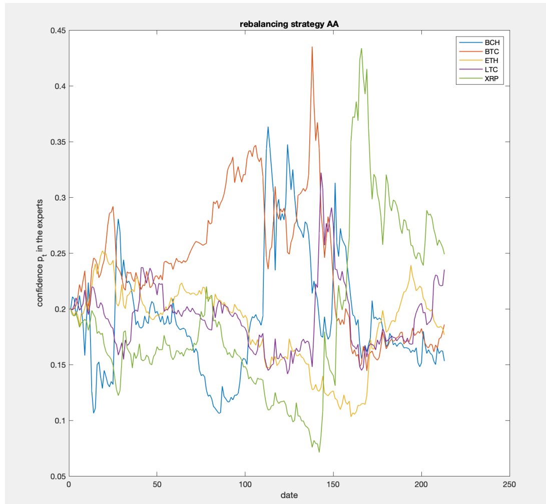

Figure 1: Plot of re-balance strategy of AA

4.10 k

I think there is not difficult data. I understand that here "difficult" means that data is really difficult to be predicted, in a way say it makes result bad. But our 5 experts' loss are all smaller than 0, shown in table 4, meaning actually they get positive profit. Then I claim that there is no "difficult" data in adversary.

4.11 l

The visualisation result of re-balance strategy of AA p_t vs t is shown in figure 1 and the result of the values of the coins, x_t vs t is shown in figure 2.

4.12 m

The zoomed plot is shown in figure 3 and 4. I analyse the reason that a coin raise in value, however the AA strategy will invest less in that coin in the next time step **because that although this coin's value is increased, at the same time, other coins value increases as well and the increasing rate is higher than the certain focused coin, comparably and relatively, the weight of the increasing value coin has a lower weight in next step**.

4.13 n

By calculating the total value we own and then minus the start wealth we have, I obtained the wealth have increased. The value is approximately $\mathbf{1.7\times10^3}$**.**

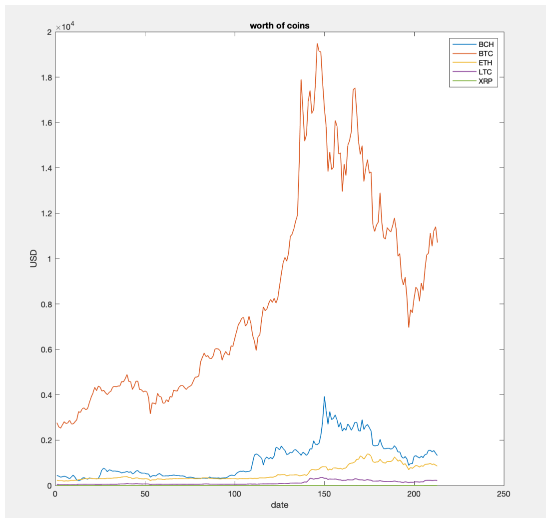

Figure 2: Plot of values of the coins

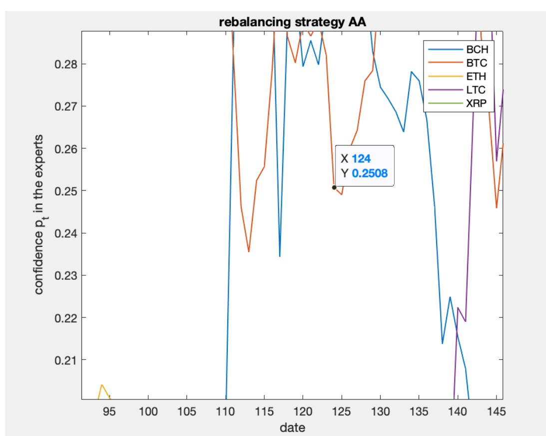

Figure 3: Zoomed information of plot of re-balance strategy of AA

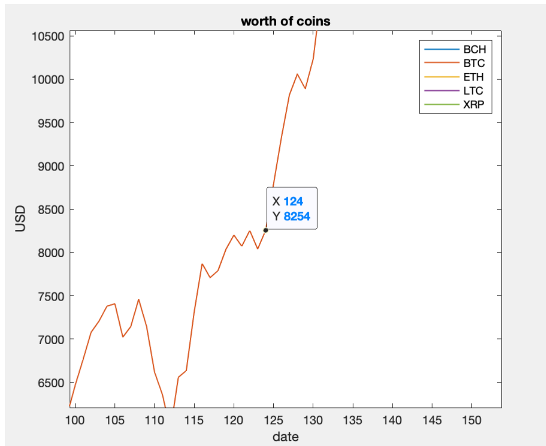

Figure 4: Zoomed information of plot of value of coin

4.14 o

In reality in cost-profit perspective, the profit is not only considered from the bitcoin profit itself, but also as a "miner", the cost you purchase for the "mining machine" and even when you run the machine the electricity fee you consume should be considered.

In investment and bitcoin perspective, the total value of bitcoin in circulation and the number of companies using bitcoin are still very small compared to their possible size. Therefore, relatively small events, transactions or business activities can significantly affect their prices. In other word, currently, the expert could not be really perfect.

In the end, even laws published for administrating bitcoin or some other political reason could influence the value of bitcoin a lot, which is not predictable by expert.

5 Exercise 5

5.1 a

Firstly, consider that the optimal learning rate. As the lecture slide shows, online gradient descent strategy with learning rate η has a regret R_n that can be bounded by:

$$R_n \leq \frac{R^2}{2\eta} + \frac{\eta G^2 n}{2} \tag{36}$$

on the other word, it could be considered that the right part of the inequality as a function of η, notate it as $f(\eta)$,m then the inequality

36 could be regarded in this way:

$$\begin{aligned} &R_n \leq f(\eta) \\ \Rightarrow &R_n \leq f_{min}(\eta) \end{aligned} \tag{37}$$

thus the question now could be considered to solve the minimum value of function $f(\eta)$. Take the derivative:

$$\begin{aligned} &\frac{f(\eta)}{d\eta} = (\frac{R^2}{2\eta} + \frac{\eta G^2 n}{2})'_\eta \\ \Rightarrow &f^{'}(\eta) = -\frac{R^2}{2\eta^2} + \frac{G^2 n}{2} \end{aligned} \tag{38}$$

set to zero:

$$\begin{aligned} &-\frac{R^2}{2\eta^2} + \frac{G^2 n}{2} = 0 \\ \Rightarrow &\frac{R^2}{2\eta^2} = \frac{G^2 n}{2} \\ \Rightarrow &\eta^2 = \frac{R^2}{G^2 n} \\ \Rightarrow &\eta = \frac{R}{G\sqrt{n}} \end{aligned} \tag{39}$$

Solve! So, the minimum (smallest) value of function $f(\eta)$ is when $\eta = \frac{R}{G\sqrt{n}}$, say when $\eta = \frac{R}{G\sqrt{n}}$ R_n has the tightest bound, so the optimal value of learning rate is $\eta_n = \frac{R}{G\sqrt{n}}$. After having optimal learning rate, substitute the η_n into the inequality 36,

$$\begin{aligned} &R_n \leq \frac{R^2}{2\eta_n} + \frac{\eta_n G^2 n}{2} \\ \Rightarrow &R_n \leq \frac{R^2}{2\frac{R}{G\sqrt{n}}} + \frac{\frac{R}{G\sqrt{n}} G^2 n}{2} \\ \Rightarrow &R_n \leq RG\sqrt{n} \end{aligned} \tag{40}$$

So, the optimal choice that $R_n \leq RG\sqrt{n}$

5.2 b

For this question, firstly set some parameters will be used later.

W_0**: the original total stocks value;**

W_t**: the stocks value at time step t;**

W_{t+1}**: the stocks value at time step $t+1$, the latter step of W_t;**

$p_t = (p_t^1, ..., p_t^d)$**: distribution of the wealth at time t on different d assets;**

x_t^i**: value of asset i at time step t.**

Constant rebalanced portfolio can have exponential wealth increase:

5.3 c

6 Exercise 6

A Appendix

Listing 1: Exercise 4 AA.m

```
% Exercise: Aggregating Algorithm (AA)
clear all;
load coin_data;
d = 5;
n = 213;
% (d) compute adversary movez z_t
z = zeros(213,5);
z = -log(r);
% (e) compute losses of experts
loss_expert = sum(z);
% compute strategy p_t (see slides)
p(1,:) = [0.2, 0.2, 0.2, 0.2, 0.2];
p(2,:) = exp(-z(1, :)) ./ sum(exp(-z(1, :)));
for i = 3: n
    cum_loss = sum(z(1:i-1, :));
    summation = sum(exp(-cum_loss));
    p(i,:) = exp(-cum_loss) ./ summation;
end
% compute loss of strategy p_t
% (g) AA loss (our loss)
p_z = sum((p .* exp(-z)), 2);
% calculate loss on each step
loss_our = 0;
for i = 1: n
    loss(i, :) = -log(p_z(i, :));
    loss_our = loss(i, :) + loss_our;
end
% pick out best expert
loss_expert_min = min(loss_expert);
% (j) compute regret
regret = loss_our - loss_expert_min;
% (n) compute total gain of investing with strategy p_t
increase = p(n, :) * s(n, :)' - p(1, :) * s0(1, :)';


%% plot of the strategy p and the coin data
% if you store the strategy in the matrix p (size n * d)
% this piece of code will visualize your strategy
figure
subplot(1, 2, 1);
plot(p, 'LineWidth', 1)
legend(symbols_str)
title('rebalancing strategy AA')
xlabel('date')
ylabel('confidence p_t in the experts')
subplot(1, 2, 2);
```

```
plot(s,'LineWidth', 1)
legend(symbols_str)
title('worth of coins')
xlabel('date')
ylabel('USD')
```