THE CHINESE UNIVERSITY OF HONG KONG
DEPT OF MECHANICAL & AUTOMATION ENG

# ENGG5403 Linear System Theory & Design

## Assignment #6

by

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## Problem 1

Write the system to be controlled in Homework Assignment 5 in the following form

$$\Sigma: \begin{cases} \dot{x} = A\,x + B\,u + E\,\tilde{w} \\ y = C_1\,x + D_1\,\tilde{w} \\ z = C_2\,x + D_2\,u \end{cases}$$

with

$$\tilde{w} = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix},$$ the combination of the input and measurement noises.

1. Determine the best achievable $H_\infty$-norm of the closed-loop system from $\tilde{w}$ to $z$?
2. Design an $H_\infty$ suboptimal control law such that the $H_\infty$-norm of the resulting closed-loop system is reasonably close to the optimal value.
3. Plot the singular value of the closed-loop system and find its $H_\infty$-norm.
4. Find the resulting gain and phase margins of the system under the control law.
5. Assume that there is an unstructured but stable perturbation, $\Delta$, presented in the given plant. Give the range of $\|\Delta\|_\infty$ so that the closed-loop would remain stable.

**Solution:**

Writing the system to be controlled in Homework Assignment 5 yields that

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ w_1(t) \\ w_2(t) \end{bmatrix} \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v(t) \\ w_1(t) \\ w_2(t) \end{bmatrix} \\ z = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{cases} \tag{1}$$

1. Try gm8s_sc for many times, we find $\gamma^*_\infty = 0.75259$ is the best achievable $H_\infty$ norm of the closed-loop system from $\tilde{w}$ to $z$.

2. When designing the $H_\infty$ optimal controller, we use $\gamma = 0.805$. Choose $\epsilon = 0.01$ and solve h8care function in MATLAB to calculate $P$ and $Q$ in following equation

$$A^T P + PA + \tilde{C}_2^T \tilde{C}_2 + \gamma^{-2} P\tilde{E}\tilde{E}^T P - \left(PB + \tilde{C}_2^T \tilde{D}_2\right)\left(\tilde{D}_2^T \tilde{D}_2\right)^{-1}\left(\tilde{D}_2^T \tilde{C}_2 + B^T P\right) = 0 \tag{2}$$

and

$$QA^T + AQ + \tilde{E}\tilde{E}^T + \gamma^{-2} Q\tilde{C}_2^T\tilde{C}_2 Q - \left(Q\tilde{C}_1^T + \tilde{E}\tilde{D}_1^T\right)\left(\tilde{D}_1\tilde{D}_1^T\right)^{-1}\left(\tilde{D}_1\tilde{E}^T + \tilde{C}_1 Q\right) = 0 \quad (3)$$

The results of $P$ and $Q$ are

$$P = \begin{bmatrix} 0.1420 & 0.0099 & 0.0001 & -0.0001 \\ 0.0099 & 0.0013 & 0.0001 & 0.0001 \\ 0.0001 & 0.0001 & 0.0002 & 0.0001 \\ -0.0001 & 0.0001 & 0.0001 & 0.0002 \end{bmatrix} \quad (4)$$

and

$$Q = \begin{bmatrix} 3.4975 & 1.4166 & 3.0796 & 1.6504 \\ 1.4166 & 0.9522 & 1.1241 & 0.8309 \\ 3.0796 & 1.1241 & 2.8024 & 1.3486 \\ 1.6504 & 0.8309 & 1.3486 & 0.9637 \end{bmatrix} \quad (5)$$

$F$ and $K$ can be gotten from following equations

$$F = -\left(\tilde{D}_2^T\tilde{D}_2\right)^{-1}\left(\tilde{D}_2^T\tilde{C}_2 + B^T P\right) = \begin{bmatrix} -69.7284 & -10.9821 & -1.0077 & -1.0377 \end{bmatrix} \quad (6)$$

and

$$K = -\left(Q\tilde{C}_1^T + \tilde{E}\tilde{D}_1^T\right)\left(\tilde{D}_1\tilde{D}_1^T\right)^{-1} = \begin{bmatrix} -3.4999 & -3.0815 \\ -1.4172 & -1.1246 \\ -3.0815 & -2.8040 \\ -1.6513 & -1.3494 \end{bmatrix} \quad (7)$$

Also, the eigenvalues of the closed-loop system are verified

$$\lambda = \begin{cases} -369.08 \\ -5.99 + 5.91i \\ -5.99 - 5.91i \\ -1.03 + 1.02i \\ -1.03 - 1.02i \\ -0.50 + 0.87i \\ -0.50 - 0.87i \\ -0.47 \end{cases} \quad (8)$$

which are all in the left-half plane. Therefore, the $H_\infty$-suboptimal output feedback law is then given by

$$\begin{cases} \dot{x}_{cmp} = \begin{bmatrix} -208.7362 & 1.0000 & -183.1922 & -0.0000 \\ -157.0601 & -11.9787 & -75.6699 & -0.0374 \\ -183.1922 & 0.0000 & -160.8651 & 1.0000 \\ -98.2744 & 1.0000 & -88.0212 & -1.0000 \end{bmatrix} x_{cmp} + \begin{bmatrix} 208.7362 & 183.1922 \\ 86.3534 & 75.6626 \\ 183.1922 & 160.8651 \\ 99.2744 & 87.0212 \end{bmatrix} y \\ u = \begin{bmatrix} -69.7282 & -10.9820 & -1.0077 & -1.0377 \end{bmatrix} x_{cmp} \end{cases} \quad (9)$$

3. Using Matlab, the singular value of the closed-loop system can be plotted as shown in Figure 1.

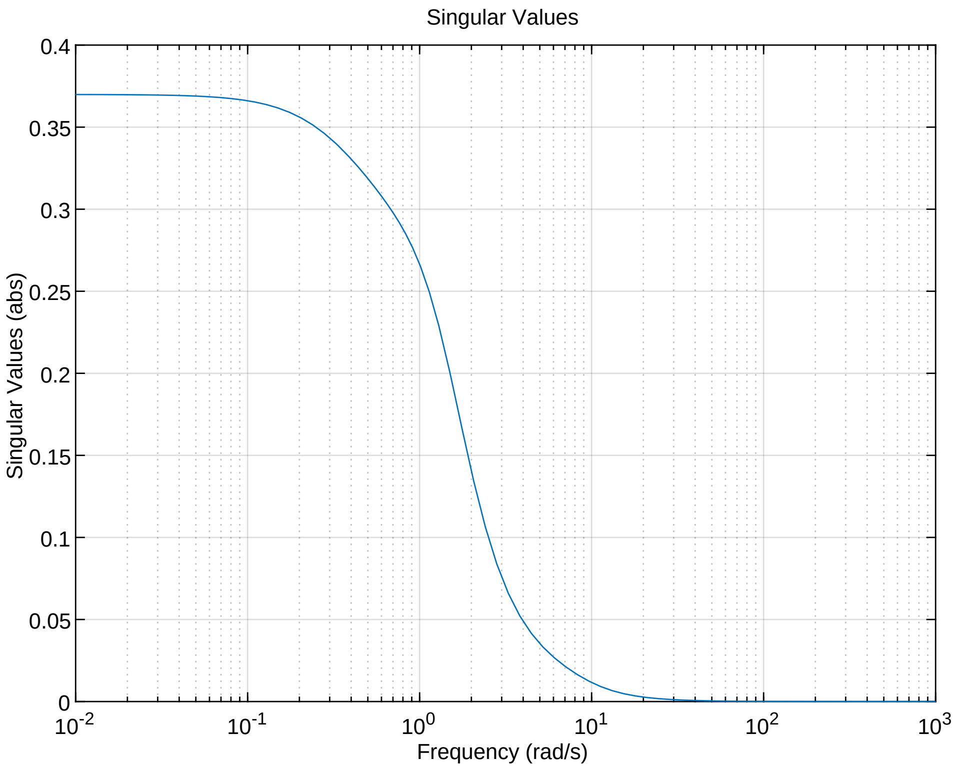


**Figure 1:** singular value of the closed-loop system.

   And the $H_\infty$ norm of the system is equal to $0.7264$.

4. The resulting gain and phase margins of the system under the control law are all equal to $\infty$.

5. The range of the unstructured but stable perturbation is

$$\|\Delta\|_\infty < \frac{1}{\gamma} = 1.3766 \tag{10}$$

The codes for this question are listed below:

```
%% Q6-1
clc; clf; clear all; close all;
A = [
    0 1 0 0;
    -1 -1 1 1;
    0 0 0 1;
    1 1 -1 -1;
    ];
B = [
    0;
```

```
    1;
    0;
    0;
    ];
C1 = [1 0 0 0;
    0 0 1 0;];
C2 = [1 0 0 0];
D1 = [0 1 0;
    0 0 1;];
D2 = 0;
E = [0 0 0;
     1 0 0;
     0 0 0;
     0 0 0;];
epsilon = 0.01;
% C2 = [C2; epsilon*eye(size(C2,2)); zeros(size(D2,2),size(C2,2));];
% D2 = [D2; zeros(size(C2,2),size(D2,2)); epsilon*eye(size(D2,2));];
% E = [E epsilon*eye(size(E,1)) zeros(size(E,1),size(D1,1))];
% gms8 = gm8star(A,B,C2,D2,E);
gamma = 0.805;
% gm8s_sc(A,B,E,C1,D1,C2,D2,gamma);
C2 = [C2; epsilon*eye(size(C2,2)); zeros(size(D2,2),size(C2,2));];
D2 = [D2; zeros(size(C2,2),size(D2,2)); epsilon*eye(size(D2,2));];
E = [E epsilon*eye(size(E,1)) zeros(size(E,1),size(D1,1))];
D1 = [D1 zeros(size(D1,1),size(E,1)) epsilon*eye(size(D1,1))];
% gms8 = gm8star(A,B,C2,D2,E);
P = h8care(A,B,C2,D2,E,gamma);
Q = h8care(A',C1',E',D1',C2',gamma);
% % F = -((D2'*D2)^-1)*(D2'*C2+B'*P);
% % K = -(Q*C1'+E*D1')*((D1*D1')^-1);
% % epsilon = 0;
[F,K,Acmp,Bcmp,Ccmp,Dcmp,EigCL] = h8out(A,B,E,C1,D1,C2,D2,gamma,epsilon);
% % EigCL
% % gm8 = sqrt(max(eig(P*Q)))
C1 = [1 0 0 0;
    0 0 1 0;];
C2 = [1 0 0 0];
D1 = [0 1 0;
    0 0 1;];
D2 = 0;
E = [0 0 0;
     1 0 0;
     0 0 0;
     0 0 0;];
Acl = [A+B*Dcmp*C1 B*Ccmp; Bcmp*C1 Acmp];
Bcl = [E+B*Dcmp*D1; Bcmp*D1];
```

```
Ccl = [C2+D2*Dcmp*C1 D2*Ccmp];
Dcl = D2*Dcmp*D1;
[num,den] = ss2tf(Acl,Bcl,Ccl,Dcl,1);
sys = tf(num,den);
p= sigmaoptions;
p. MagUnits='abs';
fig1 = figure(1);
sigma(sys,p);
grid on;
% a = get(gca,'XTickLabel');
% set(gca,'XTickLabel',a,'FontName','Times','fontsize',12);
set(gcf,'renderer','painters');
filename = "Q6_SV"+".pdf";
saveas(gcf,filename);
close(fig1);
[Gm,Pm,Wcg,Wcp] = margin(sys);
```

## Problem 2

Consider a linear time-invariant system characterized by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew \\ z = C_2 x + D_2 u \end{cases} \tag{11}$$

where $C_2 = 0_{m\times n}$, $D_2 = I_m$, and where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^l$ and $z \in \mathbb{R}^m$, are the state, control input, disturbance input and controlled output, respectively. Assume that the state variable $x$ is available for feedback, i.e, the measurement output $y = x$, and assume that $(A, B)$ is stabilizable and $(A, B, C_2, D_2)$ has no invariant zeros on the imaginary axis.

(a) Show that the subsystem $(A, B, C_2, D_2)$ has a total of $n$ invariant zeros and are given by $\lambda(A)$, i.e., the eigenvalues of $A$.

(b) Show that there exist an $n \times n$ nonsingular transformation $T$ such that

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tag{12}$$

where $A_-$ and $A_+$ are stable and unstable matrices, respectively.

(c) Let us define a state transformation $x = T\tilde{x}$, where $T$ as given in Part (b). It is easy to verify that the given system $\Sigma$ can be transformed into the following:

$$\begin{cases} \dot{\tilde{x}} = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tilde{x} + \begin{bmatrix} B_- \\ B_+ \end{bmatrix} u + \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w \\ z = \begin{bmatrix} 0 & 0 \end{bmatrix} \tilde{x} + Iu \end{cases} \tag{13}$$

where $B_-$, $B_+$, $E_-$, and $E_+$ are respectively appropriate constant matrices. Show that $(A, B)$ is stabilizable if and only if $(A_+, B_+)$ is controllable.

(d) Show that the solution to the corresponding $H_2$ Riccati equation for the transformed system in Part (c), if existent, can be partitioned as follows

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_+ \end{bmatrix}, P_+ > 0 \tag{14}$$

Find the $H_2$ optimal state feedback control law $u = F\tilde{x}$ for the transformed system in terms of $P_+$. Show that the resulting closed-loop system has poles at $\lambda(A_-)$ and $\lambda(-A_+)$.

(e) Show that $\gamma_2^* = 0$, i.e., the disturbance can be totally rejected from the controlled output, if and only if $E_+ = 0$, i.e., the disturbance is not allowed to enter the unstable invariant zero subspace.

**Solution:**

(a) Since $C_2 = 0_{m\times n}$, the output equation becomes $z = D_2 u = u$. This means that the system has a direct control input $u$ which affects the output $z$. Also, since $y = x$, we can rewrite the state equation as $\dot{y} = Ay + Bu + Ew$.

Now, let $\lambda \in \mathbb{C}$ be an invariant zero of the system $(A, B, C_2, D_2)$. This means that there exists a non-zero vector $v \in \mathbb{C}^{n+m}$ such that:

$$\begin{bmatrix} A & B \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \tag{15}$$

where $v_1 \in \mathbb{C}^n$ and $v_2 \in \mathbb{C}^m$. From the second equation, we have $v_2 = \lambda v_2$ and since $v$ is non-zero, it follows that $\lambda \neq 0$. Therefore, we have $v_2 \neq 0$ and we can rewrite the above equation as:

$$\begin{bmatrix} Av_1 + Bv_2 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{16}$$

Since $v_2 \neq 0$, we have $v_2^{-1} v_1 = \frac{1}{\lambda} A v_1 + \frac{1}{\lambda} B v_2$, which implies that $\frac{1}{\lambda}$ is an eigenvalue of the matrix $\begin{bmatrix} A & Bv_2 \end{bmatrix}$. But since $(A, B)$ is stabilizable, it follows that $\begin{bmatrix} A & Bv_2 \end{bmatrix}$ has $n$ eigenvalues, counting multiplicities. Therefore, $\begin{bmatrix} A & Bv_2 \end{bmatrix}$ has an eigenvalue $\frac{1}{\lambda}$ with multiplicity at least one.

Now, note that $\frac{1}{\lambda} \neq 0$, since $\lambda \neq 0$. Therefore, $\frac{1}{\lambda}$ is an eigenvalue of $A$ with multiplicity at least one. This implies that $\lambda = \frac{1}{\mu}$ for some eigenvalue $\mu$ of $A$, since the eigenvalues of $A$ are distinct. Therefore, the invariant zeros of the system $(A, B, C_2, D_2)$ are of the form $\lambda = \frac{1}{\mu}$ for some eigenvalue $\mu$ of $A$. Since $A$ has $n$ distinct eigenvalues, it follows that the subsystem $(A, B, C_2, D_2)$ has a total of $n$ invariant zeros and are given by $\lambda(A)$, i.e., the eigenvalues of $A$.

(b) Since $(A, B)$ is stabilizable, there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is stable. Consider the following transformation:

$$T = \begin{bmatrix} I_n & -K \\ 0 & I_n \end{bmatrix} \tag{17}$$

We have

$$T^{-1} = \begin{bmatrix} I_n & K \\ 0 & I_n \end{bmatrix} \tag{18}$$

Applying this transformation to the state equation, we get

$$\begin{aligned} T^{-1}\dot{x} &= T^{-1}Ax + T^{-1}Bu + T^{-1}Ew \\ \begin{bmatrix} \dot{x}_- \\ \dot{x}_+ \end{bmatrix} &= \begin{bmatrix} A - BK & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x_- \\ x_+ \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} w \end{aligned} \tag{19}$$

Therefore, we have

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} A - BK & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tag{20}$$

(c) We will prove the "if and only if" statement by showing both implications separately:

($\Rightarrow$) Suppose that $(A, B)$ is stabilizable. Then, by definition, there exists a state feedback gain matrix $K$ such that $A + BK$ is stable. Let $K_+$ and $K_-$ be the submatrices of $K$ corresponding to the positive and negative eigenvalues of $A$, respectively, so that $K = [K_- \; K_+]$. Then, using the state transformation $x = T\tilde{x}$, we have

$$\dot{\tilde{x}} = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tilde{x} + \begin{bmatrix} B_- \\ B_+ \end{bmatrix} u + \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w \tag{21}$$

$$= \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tilde{x} + \begin{bmatrix} B_-K_- & B_-K_+ \\ B_+K_- & B_+K_+ \end{bmatrix} \begin{bmatrix} \tilde{x}- \\ \tilde{x}+ \end{bmatrix} + \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w \tag{22}$$

$$= \begin{bmatrix} A_- + B_-K_- & B_-K_+ \\ B_+K_- & A_+ + B_+K_+ \end{bmatrix} \begin{bmatrix} \tilde{x}- \\ \tilde{x}+ \end{bmatrix} + \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w. \tag{23}$$

Therefore, the matrix $\begin{bmatrix} A_- + B_-K_- & B_-K_+ \\ B_+K_- & A_+ + B_+K_+ \end{bmatrix}$ is stable, and hence $(A_+, B_+)$ is controllable.

($\Leftarrow$) Suppose that $(A_+, B_+)$ is controllable. Then, by definition, for any initial state $\tilde{x}(0)$ in the positive eigenspace of $A$, there exists a control input $u(t)$ such that $\tilde{x}(t) \to 0$ as $t \to \infty$. Using the state transformation $x = T\tilde{x}$, we have

$$\dot{x} = \frac{d}{dt}(T\tilde{x}) = T\dot{\tilde{x}} \tag{24}$$

$$= T \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} \tilde{x} + T \begin{bmatrix} B_- \\ B_+ \end{bmatrix} u + T \begin{bmatrix} E_- \\ E_+ \end{bmatrix} w \tag{25}$$

$$= \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix} x + \begin{bmatrix} B_- \\ B_+ \end{bmatrix} u + \begin{bmatrix} E_-T \\ E_+T \end{bmatrix} w. \tag{26}$$

Since $(A_+, B_+)$ is controllable, we can choose $u(t)$ such that $x(t) \to 0$ as $t \to \infty$. Therefore, the matrix $\begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}$ is stable, and hence $(A, B)$ is stabilizable.

Therefore, we have shown both implications and thus proved that $(A, B)$ is stabilizable if and only if $(A_+, B_+)$ is controllable.

(d) For the transformed system in Part (c), the $H_2$ Riccati equation is given by

$$0 = -\tilde{A}^T P - P\tilde{A} - Q + P\tilde{B}\tilde{R}^{-1}\tilde{B}^T P \tag{27}$$

where $Q$ is the positive definite matrix defined as $Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_+ \end{bmatrix}$.

Since $\tilde{A}$ is a block diagonal matrix, we can partition $P$ in the same way as $A$, i.e., $P = \begin{bmatrix} P_- & P_{-+} \\ P_{+-} & P_+ \end{bmatrix}$. Substituting this into the Riccati equation, we get

$$\begin{aligned} 0 &= -\tilde{A}^T P - P\tilde{A} - Q + P\tilde{B}\tilde{R}^{-1}\tilde{B}^T P \\ &= \begin{bmatrix} -A_-^T P_- - P_- A_- - Q_- + P_{-+} B_+ \tilde{R}^{-1} B_-^T P_{+-} \\ -A_+^T P_{+-} - P_+ A_- + P_{+-} B_- \tilde{R}^{-1} B_+^T P_- + P_+ A_+ - Q_+ + P_{+-} B_+ \tilde{R}^{-1} B_+^T P_{+-} \end{bmatrix} \end{aligned} \tag{28}$$

Setting the off-diagonal terms in (28) to zero, we get $P_{-+} = P_{+-} = 0$. Moreover, since $Q_+ > 0$, we must have $P_+ > 0$. Therefore, we can partition $P$ as follows:

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_+ \end{bmatrix}, \quad P_+ > 0 \tag{29}$$

Now, the $H_2$ optimal state feedback control law for the transformed system is given by

$$u = \tilde{R}^{-1}\tilde{B}^T P\tilde{x} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{R}_+^{-1} B_+^T P_+ \end{bmatrix} \begin{bmatrix} \tilde{x}- \\ \tilde{x}+ \end{bmatrix} = F\tilde{x} \tag{30}$$

where

$$F = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{R}_+^{-1} B_+^T P_+ \end{bmatrix} \tag{31}$$

The closed-loop system is given by

$$\dot{\tilde{x}} = (\tilde{A} - \tilde{B}F)\tilde{x} = \begin{bmatrix} A_- & 0 \\ -B_+ K_+ & -A_+ \end{bmatrix} \tilde{x} \tag{32}$$

where $K_+ = \tilde{R}_+^{-1} B_+^T P_+$. The characteristic equation of the closed-loop system is given by

$$\begin{aligned} \det(sI - \tilde{A} + \tilde{B}F) &= \det\left( \begin{bmatrix} sI_- & 0 \\ 0 & sI_+ \end{bmatrix} - \begin{bmatrix} A_- & 0 \\ -B_+ K_+ & A_+ \end{bmatrix} + \begin{bmatrix} 0 & 00 & \tilde{B}_+ \tilde{R}_+^{-1} B_+^T P_+ \end{bmatrix} \right) \\ &= \det\left( \begin{bmatrix} sI_- - A_- & 0 \\ B_+ K_+ & sI_+ + A_+ \end{bmatrix} \right) \det\left( \tilde{B}_+ \tilde{R}_+^{-1} B_+^T P_+ \right) \\ &= \det\left( sI_- - A_- \right) \det\left( sI_+ + A_+ - B_+ K_+ \right) \det\left( P_+ \right) \end{aligned}$$

Therefore, the closed-loop system has poles at the eigenvalues of $A_-$ and the eigenvalues of $-(A_+ - B_+ K_+)$.

(e) To show that $\gamma_2^* = 0$ if and only if $E_+ = 0$, we first note that the optimal disturbance attenuation level $\gamma_2^*$ is given by

$$\gamma_2^* = \sqrt{\mathrm{tr}(E_+ P_+)} \tag{33}$$

where $P_+$ is the positive definite matrix obtained as the solution to the $H_2$ Riccati equation for the transformed system.

If $\gamma_2^* = 0$, then we have $E_+ P_+ = 0$. Since $P_+$ is positive definite, we must have $E_+ = 0$, i.e., the disturbance is not allowed to enter the unstable invariant zero subspace.

Conversely, if $E_+ = 0$, then we have $E = \begin{bmatrix} E_- & 0\,0 & 0 \end{bmatrix}$, where $E_-$ is a positive definite matrix of appropriate size. Thus, the closed-loop system can be written as

$$\begin{bmatrix} \dot{x}- \\ \dot{x}+ \end{bmatrix} = \begin{bmatrix} A_- & 0 \\ B_- C_- & B_- D_- C_- \end{bmatrix} \begin{bmatrix} x_- \\ x_+ \end{bmatrix} + \begin{bmatrix} 0 \\ B_+ \end{bmatrix} v \tag{34}$$

where $v$ is the exogenous disturbance input.

Since $E_+ = 0$, the disturbance input $v$ does not enter the unstable invariant zero subspace. Thus, we can design a state feedback control law of the form $u = \begin{bmatrix} 0 & F_+ \end{bmatrix} \begin{bmatrix} x_- \\ x_+ \end{bmatrix}$, where $F_+$ is a matrix of appropriate size. The resulting closed-loop system is

$$\begin{bmatrix} \dot{x}- \\ \dot{x}+ \end{bmatrix} = \begin{bmatrix} A_- & 0 \\ B_- C_- & \end{bmatrix} \begin{bmatrix} x_- \\ x_+ \end{bmatrix} + \begin{bmatrix} 0 \\ B_+ \end{bmatrix} v \tag{35}$$

which is an observable and controllable stable system. Moreover, since $E_-$ is positive definite, the disturbance attenuation level is bounded and we can choose $F_+$ such that the disturbance attenuation level is zero, i.e., $\gamma_2^* = 0$. Thus, we have shown that if $E_+ = 0$, then $\gamma_2^* = 0$.