Problems that I discussed with Professor last time

1. Can Inhomogeneous Mathieu equation be rearranged into homogeneous Mathieu equation if the forced part is constant of time?

Considering a originally coupled differential sets of equation of motion depicting each ions dynamics inside a Paul trap. We expand Coulomb interaction to the second order around its equilibrium points and have Paul trapping potential intact which means physically we take micromotion amplitude into account. The sets of equations can be decoupled by finding the orthogonal transformation of the system "coupling matrix" which is not in diagonal form. Denoting into motional mode, a new inhomogeneous Mathieu equation with only mode indices for each dynamics equation arrives:

$$\frac{d^2 n_i}{dt^2} + (a_i + 2q_i \cos(\Omega t))n_i = f_i$$

,where a_i, q_i, f_i being constant of time.
Now, if we shift $n_i \rightarrow n_i + n_i^{(0)}$, $n_i^{(0)}$is also constant of time (since we are trying to absorb f_i to the LHS). the equation becomes:

$$\frac{d^2(n_i + n_i^{(0)})}{dt^2} + (a_i + 2q_i \cos(\Omega t))(n_i + n_i^{(0)}) = f_i$$

If we turn off the RF field, $q_i = 0$, the original dynamics equation is

$$\frac{d^2 n_i}{dt^2} + a_i n_i = f_i$$

By shifting $n_i \rightarrow n_i + n_i^{(0)}$we get $\frac{d^2(n_i + n_i^{(0)})}{dt^2} + a_i(n_i + n_i^{(0)}) = f_i$, $a_i = \frac{f_i}{n_i^{(0)}}$, the equation becomes $\frac{d^2 n_i}{dt^2} + a_i n_i = 0$ characterzing a harmonic oscillator behavior. Hence we can claim that a constant force shift the equilibrium not its dynamical behavior. However, if $q_i \neq 0$, namely turn on the RF field. shifting $n_i \rightarrow n_i + n_i^{(0)}$ gives us homogeneous Mathieu equation if $f_i = (a_i + 2q_i \cos(\Omega t))n_i^{(0)}$ which contradicts our requirement that f_i being constant of time. Hence, we may say a constant force provide a rather different behavior of such Mathieu system.

2.Physical interpretations of micromotion in Quadrupole trap

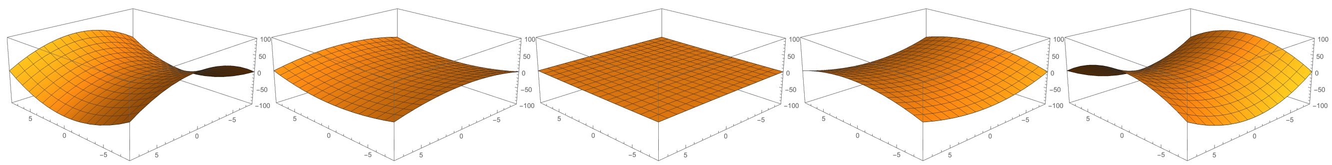
Ideally,we seek to find a harmonic potential:

$$\Phi = \Phi_0(\alpha x^2 + \beta y^2 + \gamma z^2)$$

with α,β same sign. However, it violates the fundamental Guass's law for free space: $\nabla^2 \Phi = 0 \rightarrow \alpha + \beta + \gamma = 0$
A time oscillating RF frequency can attain a rather similar potential to the above harmonic potential in x-y plane in Paul trap:

$$\Phi = V_{DC}(x^2 + y^2 - 2z^2) + U_{AC}(x^2 + y^2 - 2z^2)cos(\Omega t)$$

with $\Omega >> \omega$, which is the frequency that ion oscillate back and forth in the trap. Precisly, $\omega_i = \frac{\beta_i}{2}\Omega \approx \frac{1}{2}\sqrt{a_i + \frac{q_i^2}{2}}$which can be viewed as a psuedo harmonic motion triggered by V_{DC} and U_{AC} together, where $(a_x, a_y, a_z) \propto$



Simply plot y-z plane or x-z plane (they are same) potential with vertical direction being RF potential energy , it is a saddle potential well behavior most of the time but at particular time $t = \frac{\pi}{2\Omega}n$, a flat surface (no potential), not a potential well.

$\frac{V_{DC}}{\Omega^2}(1, 1, -1), (q_x, q_y, q_z) \propto \frac{U_{AC}}{\Omega^2}(1, 1, -2)$

We take $V_{DC}, U_{DC} > 0$: The first term of the above trap potential Φ is a perfectly harmonic potential well in x-y plane, but a harmonic potential hill along the z direction. It means due to the static field, the ions are confined in x-y plane but tend to escape from the center along z direction. What occurs at the same time is a time oscillating field produced by RF field characterized by the second term which can be viewed as a flapping potential in both y-z plane and x-z plane like the figure below:

Noted that a gravitational rotating-saddle potential is usually used as an analogy for intepreting RF-electricquadrupole ion traps while some slight difference may arises just as above mentioned, since gravitational rotating-saddle potential do not have such a specific time with no potential well at all.
To sum up, the net effect of both two potentials combined are a perfectly potential well in x-y plane and a perfectly potential hill in z plane combined with a time oscillating flapping harmonic potential. In fact, this do not gurantee a stable behavior for ion in all directions, and it leads to the choice of parameters a_i, q_i. It should be reasonable that a stable behavior occurs in x,y direction since there is a potential well in the x-y plane and we might lose our ions in z direction due to the potential hill.
If we look at the equation we have above:

$$\omega_i = \frac{\beta_i}{2}\Omega \approx \frac{1}{2}\sqrt{a_i + \frac{q_i^2}{2}}$$

as long as $a_i > 0$, the secular frequency is real, indicating a stable harmonic behavior. However, if $a_i < -\frac{q_i^2}{2}$, the frequency is imaginary, indicating an unbound motion of ions in the trap. Hence, the choice of parameters $(a_x, a_y, a_z) \propto \frac{V_{DC}}{\Omega^2}(1, 1, -1), (q_x, q_y, q_z) \propto \frac{U_{AC}}{\Omega^2}(1, 1, -2)$ with the secular frequency formula show the behavior in three directions. Moreover, secular frequency in z direction is lower than in x,y direction form above formula which correlates with the well-hill analogy.
The effects of RF field not only contribute to the secular frequency motion but also give rise to a small amplitude oscillation called micromotion.
If we simply take one of the direction in the potential of the time RF term based on the independence of motion, say x direction we can predict such micromotion phenomenon:

$$\Phi_x = U_{AC}x^2\cos(\Omega t)$$

We may visualize micromotion by simplify our flapping model as $cU_{AC}x^2$, $-cU_{AC}x^2$, $cU_{AC}x^2$,$-cU_{AC}x^2$,... every half period. Then the ions motion analogy from gravitational force is a switching sequence of hill and valley with frequency $\frac{2\pi}{\Omega}$, like a potential roller coaster. As expected, if the ion is at the middle of the flapping potential. no such hill and valley switching, but a flat surface only. For ions away from the center, the ion is taking a potential roller coaster. As the distance farer away from the center, the roller coaster is even steeper. The slope also gives insights into micromotion amplitude, since for the same time period (magnitude $\frac{1}{\Omega}$), the ions move farer in each period. Qualitatively, we may infer a larger micromotion amplitude at the edge of the ions and no micromotion effects if the ion motion is considered on the axis of the potential.

Mathematically, The ions move in x direction governing by the equation of motion:

$$\frac{d^2}{dt^2}x = -\frac{\partial \Phi_x}{\partial x} = -2xU_{AC}\cos(\Omega t)$$

Since for trapped ion system, we work in stable region where we choose $a_i << q_i << 1$. This results in a fast oscillating RF frequency $\Omega >> \omega$ contained in a rather slowly secular frequency motion of the ions. Hence, if we only take the minor motion around a particular fixed equilibrium position into account $x = x^{(0)}(\omega) + x^{(1)}(\Omega)$, the above equations becomes, $x^{(0)}$ will be treated time dependently, since we are working in a small time scale $\frac{1}{\Omega}$ including the integral inside a small period of time compared to $\frac{1}{\omega}$.

$$\frac{d^2}{dt^2}(x^{(0)}(\omega) + x^{(1)}(\Omega)) = -2(x^{(0)}(\omega) + x^{(1)}(\Omega))U_{AC}\cos(\Omega t)$$

$$\rightarrow \frac{d^2}{dt^2}x^{(1)}(\Omega) \simeq -2(x^{(0)}(\omega))U_{AC}\cos(\Omega t)$$

$$\rightarrow x^{(1)}(\Omega) \approx -2(x^{(0)}(\omega))U_{AC}\frac{1}{\Omega^2}\cos(\Omega t)$$

The above equation is exactly the first order micromotion $x^{(1)}(\Omega)$ amplitude as ascertained in numerous paper. It indeed shows the micromotion effect is enhanced when the ions are located far away from the trapping center. Also, the small amplitude can be shown in term $\frac{1}{\Omega^2}$. Above discuusion also shows why the micromotion along the transverse direction of the planar system can be neglected, since the displacement along z direction is small to form a planar.

3.The meaning of equilibrium position $r^{(0)}$ in a decoupled Mathieu equation:

As elaborated in section 1. the equatiton of motion for planar or chain ion system is in the form of Decoupled Mathieu equation:

$$\frac{d^2 n_s}{dt^2} + (a_s + 2q_s\cos(\Omega t))n_s = f_s$$

s denoting different motional modes, f_i is the driving force, essentially contains the expansion of Coulomb interaction when plugged in equilibrium positions from the psuedo potential limit.

It is said that the micromotion induced by driving force cannot be laser cooled. Such micromotion is called excess micromotion and it is often beyond lamb-Dicke limit. To elucidate, we recall that in section 2 micromotion comes from RF field with a ions displaced away from the potential axis. The reason I thought that it cannot be laser cooled is because such displacement comes from Coulomb interaction. Hence, no matter how cold we cool the ion, the Coulomb interaction still displace each ion in the whole trapping field. For a linear chain, such excess micromotion only occurs in breathing mode, or in other words, breathing mode is the mode that are used to characterize displacement from Coulomb interaction.

For the instrinsic micromotion that can be laser cooled arises from the secular frequency that carries ions back and forth on the nodal line of RF potential. Such periodic journey inevitably experienced the same micromotion force explained in section 2. However, if we keep cooling the ions, the harmonic oscillator motion with a lower kinetic energy will have smaller vibrating amplitude, thus resulting in a smaller micromotion.

Driving force f_i lead to a purely displacement for SHO if no RF field ($q_i = 0$), just like a vertically spring under a constant gravity. From section 1, we found for trapped ion, such analogy can still be used, but result in not only the shift of equilibrium but also the accompanied effects of micromotion from the additional RF field. Hence, to assume a RF periodic Fourier series $\overline{n}_s(t) = f_s \sum_0^n c_s^n \cos(n\Omega t)$ is reasonable to depict such "shifting" effects from driving force(Coulomb interaction) in trapped ion system. Noted we may calculate the equilibrium positions from the first order $\vec{r_d} = Q\vec{n}^{(0)}$,and often $\vec{n}_s$ is the magnitude beyond lamb dicke regime. If we transform the shifting effects back from the normal modes, the higher order term correspond to the excess micromotion that cannot be laser

cooled.n
The whole story does not ends since we have not treated the homogeneous part, which is the dynamics of the ions. In addition, as expected, it should be a secular frequency harmonic behavior with intrinsic micromotion correction. This standard Mathieu equation for each mode has the quantized form:

$$n_s^m(t) = n_s^0(v^*(t)\hat{a}_s + v(t)a_s^\dagger)$$

with $\hat{a}_s = \sqrt{\frac{m}{2\hbar\omega}} i(v(t)\dot{\hat{u}} - \dot{v}(t)\hat{u})$, not being the standard annilation operator, $\hat{u}, v$ are the quantum and classical solution of the Mathieu equation with certain boundary conditions.
The ions motion are then the combination of the displacement part and the motional mode dynamics part of reference operator:

$$n_s^{(total)} = (n_s^{(m)}(t) + \overline{n}_s(t))$$

$$\rightarrow \overrightarrow{r} = Q\overrightarrow{n}^{total}$$

If we can have n_s^0 is way smaller than $f_s c_s^n$, then $\overrightarrow{r} \approx Q\overrightarrow{n}^{(0)} = \overrightarrow{r}^{(0)} + \overrightarrow{r}^{(1)}\cos(\Omega t) + \overrightarrow{r}^{(2)}\cos(2\Omega t)$ is applicable. 3