Bosonic Quantum Error Correction

Chen-Yu Liu, Hui-Hang Chen, and Jens N. Watty
Department of Physics, National Taiwan University, Taipei 10617, Taiwan

Quantum Error Correction (QEC) is needed to correct errors that occur during the execution of quantum protocols due to imperfections in hardware as well as unintended interactions with the environment. This is generally done by encoding the quantum information of a single logical qubit in an enlarged Hilbert space and periodically performing syndrome measurements that do not disturb the quantum information but extract the error from it. This enlarged Hilbert space can, for instance, be that of multiple physical qubits or that of an optical mode.

In this paper, we give a short introduction to quantum error correction and the error correction condition. We then examine methods that encode the two-dimensional qubit Hilbert space into the Hilbert space of a harmonic oscillator. After introducing cat codes, binomial codes, and GKP codes, we verify that these methods satisfy the quantum error correction criterion for their respective error models and compare and contrast their advantages and disadvantages.

I. INTRODUCTION

We are currently in the so-called NISQ era of Quantum Computing, where quantum hardware exists but it is **N**oisy and **I**ntermediate **S**cale **Q**uantum hardware [1]. Aside from hardware improvements to scale up and purify quantum devices, Quantum Error Correction (QEC) plays a crucial role in achieving a transition to fault-tolerant quantum computing.

The basic idea of QEC is to encode the two-dimensional qubit Hilbert space into a larger Hilbert space, such as that of multiple physical qubits or that of a harmonic oscillator. We then define a set of stabilizers $\{S_i\}$, which are operators on the enlarged Hilbert space, whose eigenspaces define the code space and various error spaces. Typically, the code space is the joint +1-eigenspace of all the stabilizers. By measuring these stabilizers, the superpositions within the code space, which define our logical quantum information, are unaffected. If the measurement of any stabilizer S_i yields a result other than +1, we know that an error occurred, and the knowledge of the stabilizer that produced this outcome tells us what that error was. This allows us to subsequently correct this error [2].

The simplest idea for an error-correcting code is the 3-qubit repetition code. This code encodes the logical 0-state as $|0_L\rangle = |000\rangle$ and the logical 1-state as $|1_L\rangle = |111\rangle$. The stabilizers associated with this code are the parity measurements $\{Z_1Z_2, Z_1Z_3, Z_2Z_3\}$. These observables yield measurement results of +1 if both qubits are in the same state and −1 if they are in different states. When a bit flip error occurs on one of the qubits, the two stabilizers associated with this qubit will yield −1, while the other one will still give +1. This way we can detect X-errors on the physical qubits and apply corrections accordingly.

We note that this code only protects against single qubit bit flips, as bit flips on two qubits (e.g. the first and second) will be indistinguishable from a single bit flip on the third qubit. This notion of distinguishability of errors can be formalized as the error correction condition [3]

$$PE_i^\dagger E_j P = \alpha_{ij} P\,, \tag{1}$$

where $\{E_i\}$ is the set of errors to be corrected, $P = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ is the projector onto the code space and α is a Hermitian matrix. Sometimes, this condition is given in the equivalent form

$$\begin{aligned} \langle 0_L| E_i^\dagger E_j |0_L\rangle &= \langle 1_L| E_i^\dagger E_j |1_L\rangle\,, \\ \langle 0_L| E_i^\dagger E_j |1_L\rangle &= \langle 1_L| E_i^\dagger E_j |0_L\rangle = 0\,. \end{aligned} \tag{2}$$

The first of these equations ensures that the error spaces are orthogonal and no dephasing between the two logical states occurs. The second equation ensures that errors that occur cannot lead to logical bit flips.

In the following, we will focus on error-correcting codes designed for photonic systems that encode the logical qubit into the Hilbert space of a harmonic oscillator. The dominant error in these physical systems is the lowering operator $\hat{a}$, which corresponds to photon loss. There is a wide variety of codes, such as cat codes, binomial codes, and GKP codes [4]. We will go over various different codes, discuss their advantages and disadvantages, physical implementations, and practical relevance.

II. METHODS

Considering that the dominant error source in our case is the single photon loss $\hat{a}$, it is natural to try to exploit the fact that coherent states are right eigenstates of $\hat{a}$. At first it might be tempting to use the encoding $|0_L\rangle = |\alpha\rangle$, $|1_L\rangle = |-\alpha\rangle$. However, these states are only approximately orthogonal if α has a large magnitude. Additionally, a single photon loss causes a logical Z-error in this code. Since the error maps one state within the code space to another state within the code space, this error cannot be detected. Therefore, this code cannot be used for error correction.

A. Cat codes

Cat states are superpositions of coherent states with opposite signs. We define the odd and even cat states

$$|C^{\pm}_{\alpha}\rangle = \mathcal{N}(|\alpha\rangle \pm |{-\alpha}\rangle)\,, \tag{3}$$

where $\mathcal{N}$ is a normalisation constant. We note that $\mathcal{N}$ will be used as a generic normalisation constant in the following. Different occurrences of $\mathcal{N}$ may refer to different normalisation constants.

Cat codes were the first codes to surpass the break-even point [5].

1. *Two orthogonal cat states*

Using two cat states that are rotated by 90° relative to each other in phase space we can define [6]

$$|0_L\rangle = |C^{+}_{\alpha}\rangle\ , |1_L\rangle = |C^{+}_{\mathrm{i}\alpha}\rangle \tag{4}$$

as our computational basis states. As with the previous attempt at a code, these basis states are not exactly orthogonal but the overlap decays exponentially in the magnitude of α.

This code space has one stabiliser $S = (-1)^{\hat{n}}$. This can be easily seen by the fact that the even cat states are superpositions of exclusively even Fock states. A single photon loss maps an arbitrary logical state

$$c\,|0_L\rangle + d\,|1_L\rangle \xrightarrow{\hat{a}} c\,|C^{-}_{\alpha}\rangle + \mathrm{i}d\,|C^{-}_{\mathrm{i}\alpha}\rangle \tag{5}$$

which has odd parity and can therefore be detected, without disturbing the quantum information. A subsequent second photon loss maps the logical qubit back to the even parity code space but the relative phase factor introduced by the photon losses gives a logical Z-error. After 2 additional photon losses, the qubit returns to its original state. This means that instead of active error correction, it is also possible to simply track the number of errors that occur and correct at the end [4]. This cycle is visualised in Fig. 1.

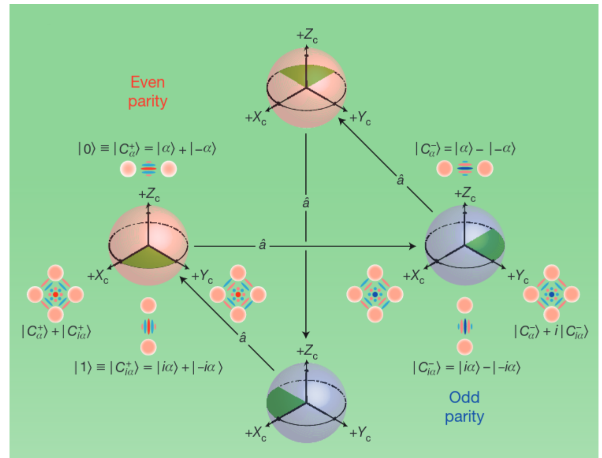

FIG. 1. The error correction cycle of the cat code. Adapted from Cai *et al.* [4].

Another very similar code to the one defined in (4) takes the computational basis states to be

$$= \mathcal{N}\left(|C^{+}_{\alpha}\rangle \pm |C^{+}_{\mathrm{i}\alpha}\rangle\right)\,, \tag{6}$$

which has the advantage that they are exactly orthogonal. In this case each basis state individually undergoes the cycle described before. This leads to a logical X-error occurring after two photon losses instead of a logical Z-error. Aside from this fact, the procedure is the same.

Something that both these codes have in common is that they cannot detect two-photon losses, since these map the code word back into the code space, introducing a logical error. Cat codes can be made resistant against multiple photon losses by generalising the code from (6) to [7]

$$|w_L\rangle \propto \sum_{k=1}^{n} e^{\mathrm{i}w4k\pi/n}\,|\alpha e^{\mathrm{i}2k\pi/n}\rangle\,, \tag{7}$$

which can correct up to $n/2-1$ errors, and where $w = 0, 1$. However, due to the fact that these states are closer to each other in phase space, the magnitude of α needs to be chosen bigger to ensure that the states are sufficiently orthogonal to each other [4].

2. *Cat qubit with biased noise*

Thinking back to the naive approach of using two coherent states with opposing signs as the computational basis states, we might notice that, while a photon loss causes a logical Z-error, logical X-errors can only occur due to the exponentially small overlap between the two states. This property is known as *biased noise* and qubits with this property have many useful applications.

In fact, a different encoding, the so called cat qubit (as opposed to the cat code), can be used to make the two basis states exactly orthogonal and preserve the biased noise property

$$|(0/1)_L\rangle = \frac{1}{\sqrt{2}}\left(|C^{+}_{\alpha}\rangle \pm |C^{-}_{\alpha}\rangle\right) = |\pm\alpha\rangle + O\left(e^{-2|\alpha|^2}\right)\,, \tag{8}$$

For this qubit, a single photon loss results in the transformation

$$|(0/1)_L\rangle \xrightarrow{\hat{a}} 2^{-1/2}\left(\mathcal{N}_{\mp}/\mathcal{N}_{\pm}\,|C^{+}_{\alpha}\rangle \pm \mathcal{N}_{\pm}/\mathcal{N}_{\mp}\,|C^{-}_{\alpha}\rangle\right)\,, \tag{9}$$

where $\mathcal{N}_{+}$ ($\mathcal{N}_{-}$) is the normalisation constant associated with $|C^{+}_{\alpha}\rangle$ ($|C^{-}_{\alpha}\rangle$) and $\frac{\mathcal{N}_{+}}{\mathcal{N}_{-}} = 1 + O\left(e^{-2|\alpha|^2}\right)$. In this error space, the two states again have an overlap of the order $O\left(e^{-2|\alpha|^2}\right)$.

These qubits can, for example, be used for a surface code that has particularly large error thresholds [8]. They have also been shown numerically to perform better than standard qubits in some NISQ applications without error correction, when the logical error (which is self-inverse) commutes through part of the circuit [9].

B. Binomial codes

We have seen that cat codes can be used to correct single photon loss errors. However, we might encounter two-photon loss errors or even higher orders of photon loss error $\hat{a}$. To be able to handle these higher order errors, we can use so-called binomial codes.

Binomial codes are based on finite superpositions of Fock states. And, by the properties of Fock states, it is obvious that binomial codes are also able to correct photon gain errors $\hat{a}^\dagger$ and dephasing errors $\hat{n}$. Thus, the error set is

$$\mathcal{E} = \{\mathbb{1}, \hat{a}, \hat{a}^2, \ldots, \hat{a}^L, \hat{a}^\dagger, (\hat{a}^\dagger)^2, \ldots, (\hat{a}^\dagger)^G, \hat{n}, \hat{n}^2, \ldots, \hat{n}^D\}, \tag{10}$$

where L, G, and D are arbitrary integers. The logical basis states are

$$|(0/1)_L\rangle = \frac{1}{2^{N/2}} \sum_{p\ (\text{even/odd})}^{N+1} \sqrt{\binom{N+1}{p}}\, |p(S+1)\rangle\ , \tag{11}$$

where $S = L + G$, $N = \max\{L, G, 2D\}$.

Firstly, since these two logical states do not share the same Fock states, they are exactly orthogonal. Secondly, they have the same average photon numbers. Hence, binomial codes satisfy the error correction condition in (2). Now, we take the lowest-order binomial states as an example. In this case

$$|0_L\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |4\rangle) \quad \text{and} \quad |1_L\rangle = |2\rangle\ , \tag{12}$$

which have an average photon number of 2 and are resistant against a single photon loss error, or equivalently, a single photon gain error (but not both at the same time).

For binomial codes, the stabilizer is the generalised photon number parity $e^{\mathrm{i}\frac{2\pi}{S+1}\hat{n}}$, where $S = 1$ for this specific code.

Binomial codes have been shown to almost reach the break-even point [10].

C. GKP code

GKP codes can protect a logical qubit from most noise sources. Following the process from [11], we show a direct and intuitive way to develop the GKP codes. In order to define the logical code basis, we start by finding the eigenspaces with eigenvalue +1 for our stabilizers defined as

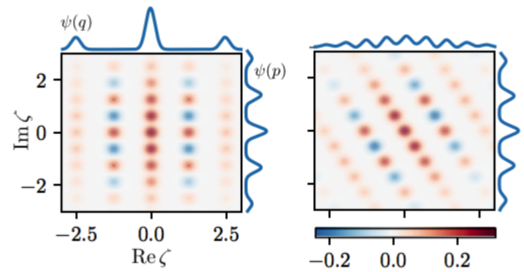

FIG. 2. The state shown is the logical zero state. The logical one state lies in the gaps between the columns. **Left:** The square code Wigner function. **Right:** The hexagonal code Wigner function. Adapted from Grimsmo and Puri [11].

$$\hat{S}_X = \hat{D}(2\alpha)\,, \quad \hat{S}_Z = \hat{D}(2\beta)\,, \tag{13}$$

with $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^\star\hat{a}}$ being the usual displacement operator on the phase space and $\beta\alpha^\star - \beta^\star\alpha = i\pi$.

The corresponding computational basis states are superpositions of eigenstates of the position operator $\hat{q}$ that form a regular lattice. Explicitly, they are

$$|0_L\rangle = \sum_{j=-\infty}^{\infty} |2j\sqrt{\pi}\rangle_{\hat{q}}\ , \tag{14}$$
$$|1_L\rangle = \sum_{j=-\infty}^{\infty} |(2j+1)\sqrt{\pi}\rangle_{\hat{q}}\ ,$$

which can be easily verified to lie in the +1 eigenspace of the stabilizers.

We observe that the state $|0_L\rangle$ can also be expressed as $|0_L\rangle \propto \sum_{i,j=-\infty}^{\infty} \left(\sqrt{\hat{S}_Z}\right)^i \left(\hat{S}_X\right)^j |0\rangle$. The reason is that $\sqrt{\hat{S}_Z}\hat{S}_X$, $\sqrt{\hat{S}_Z}$, and $\hat{S}_X$ commute with the stabilisers based on the relation between α and β, and therefore share an eigenbasis with them. Here, the square roots of the stablisers are chosen to anticommute with each other to satify the above conditions. An alternative expression for the GKP logical code basis can then be given by

$$|0_L\rangle \propto \sum_{k,l=-\infty}^{\infty} e^{-i\pi kl}\, |2k\alpha + l\beta\rangle\ , \tag{15}$$
$$|1_L\rangle \propto \sum_{k,l=-\infty}^{\infty} e^{-i\pi l(k+\frac{1}{2})}\, |(2k+1)\alpha + l\beta\rangle\ ,$$

where the terms in the sums are now coherent states that form a lattice structure. Unfortunately, GKP codes defined in this way cannot be realised in practice due to the requirement of infinite photon numbers extending over the whole phase space. Hence, our strategy is to suppress states with a large energy with a Gaussian function, such that

$$|\bar{w}_L\rangle \propto e^{-\Delta^2\hat{a}^\dagger\hat{a}}\, |w_L\rangle\ , \tag{16}$$

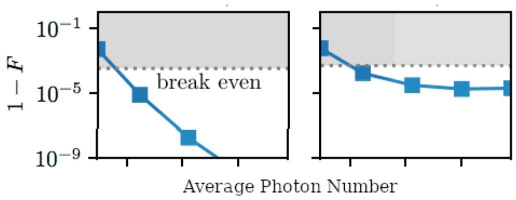

FIG. 3. **Left:** When only photon loss errors occur, we get a low infidelity. **Right:** When both photon losses and dephasing errors occur, the infidelity increases. The horizontal axis is the mean photon number used in the GKP code basis. Noted that only its qualitative performance is displayed here. Adapted from Grimsmo and Puri [11].

where $w = 0, 1$. It is noted that under certain selections for phase space displacement α, β, different GKP codes can be defined. However, the constraint on α and β given above preserves the area of an elementary cell of the lattice at a value of $\frac{\pi}{2}$.

The Wigner function of the logical basis states is shown in Fig. 2, where we can see the coherent states suppressed by the Gaussian function fading away as they get further away from the origin. As $\Delta \to 0$, we retrieve the original undamped lattice.

Previously, we focused on photon loss as the error, which is also where GKP codes outperform cat codes and binomial codes [4]. As can be seen from the stabilisers of the GKP code, the error that this code is resistant against is not the single photon loss but the displacement operator $\hat{D}(\epsilon)$, which can be decomposed into a product of the two stabilizers. However, since displacement operators form a complete basis, any error channel $\mathcal{E}(\rho)$ can be expressed in terms of displacement operators

$$\mathcal{E}(\rho) = \int \mathrm{d}^2\epsilon \mathrm{d}^2\epsilon' M(\epsilon, \epsilon') \hat{D}(\epsilon) \rho \hat{D}^\dagger(\epsilon)\,, \tag{17}$$

where $M(\epsilon, \epsilon')$ is the expansion coefficient and the integration is over the complex plane. Since

$$D(\epsilon) = e^{\frac{iab}{2}} \hat{S}_X^{\frac{a}{2\sqrt{\pi}}} \hat{S}_Z^{\frac{b}{2\sqrt{\pi}}}\,, \tag{18}$$

the error correction condition is fulfilled for errors of the form $D(\epsilon)$ if ϵ is sufficiently small (the displacement needs to be within the Wigner-Seitz cell of the lattice). Therefore, we can correct errors if their corresponding $M(\epsilon, \epsilon')$ is sufficiently concentrated around zero. Then, the error shifting of the grid in phase space is within the Wigner-Seitz cell and can hence be corrected [4]. However, for most realistic noise models, $M(\epsilon, \epsilon')$ has support outside of this area, which means that the quantum error correction condition is only approximately satisfied in those cases [11].

GKP codes do not perform as well under dephasing noise since dephasing errors are associated with rotations in phase space, which lead to large displacements far away from the origin [11]. Numerical performance of GKP codes under different error models can be seen in Fig. 3.

GKP codes have been shown to surpass the break-even point by a factor of more than 2 [12].

III. RESULTS

Table I summarises some of the main features of the different error correction codes. Cat codes are the only codes that have a non-orthogonality error. As was mentioned before, in practice the GKP code performs best against photon loss errors [4]. All of the codes can at least come close to the break-even point [5, 10, 12].

Code	$\langle 0_L \vert\, 1_L \rangle$	Correctable errors
Cat code	$O\left(e^{-2\vert\alpha\vert^2}\right)$	$\hat{a}$ (extension: $\hat{a}^k$)
Binomial code	0	$\hat{a}^L, (\hat{a}^\dagger)^G, \hat{n}^D$
GKP code	0	$\hat{D}(\epsilon)$ for small ϵ

TABLE I. Comparison of the three different types of codes. Exponents in the "Correctable errors" column are meant as maximums, i.e., E^k being correctable implies that E^{k-1} is also correctable for $k \geq 1$.

IV. CONCLUSION

In this paper, we introduced the basic concept of quantum error correction and presented three different bosonic error correcting codes. These codes have very different characteristics that we attempted to contrast with each other. In the end, all of these codes have practical relevance, each in their own way.

A high-level comparison such as the one we performed can hardly scratch the surface of all the advantages and disadvantages of the different codes. One major aspect that we neglected is the physical implementation of these schemes and the generation of these states. Also, we did not consider how easy or difficult it is to implement logical gates and the errors associated with imperfect gate implementations.

V. CONTRIBUTIONS OF EACH MEMBER

Chen-Yu Liu and Jens Watty wrote the Abstract and Introduction, all group members wrote the Results and Conclusion together. Jens Watty contributed the section about cat codes, Hui-Hang Chen wrote about binomial codes and Chen-Yu Liu wrote the GKP section. Jens Watty reviewed the grammar and wording of the paper.

[1] J. Preskill, Quantum computing in the nisq era and beyond, Quantum **2**, 79 (2018).
[2] D. Gottesman, *Stabilizer codes and quantum error correction* (California Institute of Technology, 1997).
[3] M. A. Nielsen and I. Chuang, Quantum computation and quantum information (2002).
[4] W. Cai, Y. Ma, W. Wang, C.-L. Zou, and L. Sun, Bosonic quantum error correction codes in superconducting quantum circuits, Fundamental Research **1**, 50 (2021).
[5] N. Ofek, A. Petrenko, R. Heeres, P. Reinhold, Z. Leghtas, B. Vlastakis, Y. Liu, L. Frunzio, S. Girvin, L. Jiang, *et al.*, Extending the lifetime of a quantum bit with error correction in superconducting circuits, Nature **536**, 441 (2016).
[6] Z. Leghtas, G. Kirchmair, B. Vlastakis, R. J. Schoelkopf, M. H. Devoret, and M. Mirrahimi, Hardware-efficient autonomous quantum memory protection, Physical Review Letters **111**, 120501 (2013).
[7] L. Li, C.-L. Zou, V. V. Albert, S. Muralidharan, S. Girvin, and L. Jiang, Cat codes with optimal decoherence suppression for a lossy bosonic channel, Physical Review Letters **119**, 10.1103/physrevlett.119.030502 (2017).
[8] D. K. Tuckett, S. D. Bartlett, and S. T. Flammia, Ultrahigh error threshold for surface codes with biased noise, Physical Review Letters **120**, 10.1103/physrevlett.120.050505 (2018).
[9] P. Vikstål, L. García-Álvarez, S. Puri, and G. Ferrini, Quantum approximate optimization algorithm with cat qubits (2023), arXiv:2305.05556 [quant-ph].
[10] Y. Xu, Y. Ma, W. Cai, X. Mu, W. Dai, W. Wang, L. Hu, X. Li, J. Han, H. Wang, *et al.*, Demonstration of controlled-phase gates between two error-correctable photonic qubits, Physical review letters **124**, 120501 (2020).
[11] A. L. Grimsmo and S. Puri, Quantum error correction with the gottesman-kitaev-preskill code, PRX Quantum **2**, 020101 (2021).
[12] V. V. Sivak, A. Eickbusch, B. Royer, S. Singh, I. Tsioutsios, S. Ganjam, A. Miano, B. L. Brock, A. Z. Ding, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Real-time quantum error correction beyond break-even, Nature **616**, 50 (2023).