Useful Inequalities $\{x^2 \geqslant 0\}$ v0.27b · August 19, 2015

Cauchy-Schwarz
$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Minkowski
$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} \quad \text{for } p \geq 1.$$

Hölder
$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}\left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q} \quad \text{for } p,q > 1, \ \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Bernoulli

$(1+x)^r \geq 1 + rx \quad \text{for } x \geq -1, \ r \in \mathbb{R} \setminus (0,1).$ Reverse for $r \in [0,1]$.

$(1+x)^r \leq 1 + (2^r - 1)x \quad \text{for } x \in [0,1], \ r \in \mathbb{R} \setminus (0,1).$

$(1+x)^n \leq \frac{1}{1-nx} \quad \text{for } x \in [-1,0], \ n \in \mathbb{N}.$

$(1+x)^r \leq 1 + \frac{rx}{1-(r-1)x} \quad \text{for } x \in [-1, \frac{1}{r-1}), \ r > 1.$

$(1+nx)^{n+1} \geq (1+(n+1)x)^n \quad \text{for } x \in \mathbb{R}, \ n \in \mathbb{N}.$

$(a+b)^n \leq a^n + nb(a+b)^{n-1} \quad \text{for } a,b \geq 0, \ n \in \mathbb{N}.$

$\left(1+\frac{x}{p}\right)^p \geq \left(1+\frac{x}{q}\right)^q \quad \text{for } (i) \ x > 0, \ p > q > 0,$

$(ii) \ -p < -q < x < 0, \ (iii) \ -q > -p > x > 0.$ Reverse for:

$(iv) \ q < 0 < p, \ -q > x > 0, \ (v) \ q < 0 < p, \ -p < x < 0.$

exponential

$e^x \geq \left(1+\frac{x}{n}\right)^n \geq 1+x, \quad \left(1+\frac{x}{n}\right)^n \geq e^x\left(1-\frac{x^2}{n}\right) \quad \text{for } n > 1, \ |x| \leq n.$

$e^x \geq x^e \quad \text{for } x \in \mathbb{R}, \ \text{and} \quad \frac{x^n}{n!} + 1 \leq e^x \leq \left(1+\frac{x}{n}\right)^{n+x/2} \quad \text{for } x,n > 0.$

$e^x \geq 1 + x + \frac{x^2}{2} \quad \text{for } x \geq 0, \text{ reverse for } x \leq 0.$

$e^{-x} \leq 1 - \frac{x}{2} \quad \text{for } x \in [0, \sim 1.59] \ \text{ and } \ 2^{-x} \leq 1 - \frac{x}{2} \quad \text{for } x \in [0,1].$

$\frac{1}{2-x} < x^x < x^2 - x + 1 \quad \text{for } x \in (0,1).$

$x^{1/r}(x-1) \leq rx(x^{1/r} - 1) \quad \text{for } x, r \geq 1.$

$x^y + y^x > 1 \quad \text{and} \quad e^x > \left(1+\frac{x}{y}\right)^y > e^{\frac{xy}{x+y}} \quad \text{for } x,y > 0.$

$2 - y - e^{-x-y} \leq 1 + x \leq y + e^{x-y}, \ \text{and} \ e^x \leq x + e^{x^2} \quad \text{for } x,y \in \mathbb{R}.$

logarithm

$\frac{x-1}{x} \leq \ln(x) \leq \frac{x^2-1}{2x} \leq x - 1, \quad \ln(x) \leq n(x^{\frac{1}{n}} - 1) \text{ for } x, n > 0.$

$\frac{2x}{2+x} \leq \ln(1+x) \leq \frac{x}{\sqrt{x+1}} \quad \text{for } x \geq 0, \text{ reverse for } x \in (-1,0].$

$\ln(n+1) < \ln(n) + \frac{1}{n} \leq \sum_{i=1}^{n} \frac{1}{i} \leq \ln(n) + 1$

$\ln(1+x) \geq \frac{x}{2} \quad \text{for } x \in [0, \sim 2.51], \text{ reverse elsewhere.}$

$\ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{4} \quad \text{for } x \in [0, \sim 0.45], \text{ reverse elsewhere.}$

$\ln(1-x) \geq -x - \frac{x^2}{2} - \frac{x^3}{2} \quad \text{for } x \in [0, \sim 0.43], \text{ reverse elsewhere.}$

trigonometric

$x - \frac{x^3}{2} \leq x \cos x \leq \frac{x \cos x}{1 - x^2/3} \leq x\sqrt[3]{\cos x} \leq x - x^3/6 \leq x \cos \frac{x}{\sqrt{3}} \leq \sin x,$

hyperbolic

$x \cos x \leq \frac{x^3}{\sinh^2 x} \leq x \cos^2(x/2) \leq \sin x \leq (x \cos x + 2x)/3 \leq \frac{x^2}{\sinh x},$

$\frac{2}{\pi} x \leq \sin x \leq x \cos(x/2) \leq x \leq x + \frac{x^3}{3} \leq \tan x \quad \text{all for } x \in \left[0, \frac{\pi}{2}\right].$

$\cosh(x) + \alpha \sinh(x) \leq e^{x(\alpha + x/2)} \quad \text{for } x \in \mathbb{R}, \ \alpha \in [-1,1].$

binomial

$\max\left\{\frac{n^k}{k^k}, \frac{(n-k+1)^k}{k!}\right\} \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \frac{(en)^k}{k^k} \quad \text{and} \quad \binom{n}{k} \leq \frac{n^n}{k^k(n-k)^{n-k}} \leq 2^n.$

$\frac{n^k}{4k!} \leq \binom{n}{k} \quad \text{for } \sqrt{n} \geq k \geq 0 \ \text{ and } \ \frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{8n}\right) \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{9n}\right).$

$\binom{n_1}{k_1}\binom{n_2}{k_2} \leq \binom{n_1+n_2}{k_1+k_2} \quad \text{for } n_1 \geq k_1 \geq 0, \ n_2 \geq k_2 \geq 0.$

$\frac{\sqrt{\pi}}{2} G \leq \binom{n}{\alpha n} \leq G \quad \text{for } G = \frac{2^{nH(\alpha)}}{\sqrt{2\pi n \alpha(1-\alpha)}}, \ H(x) = -\log_2(x^x(1-x)^{1-x}).$

$\sum_{i=0}^{d} \binom{n}{i} \leq n^d + 1 \quad \text{and} \quad \sum_{i=0}^{d} \binom{n}{i} \leq 2^n \quad \text{for } n \geq d \geq 0.$

$\sum_{i=0}^{d} \binom{n}{i} \leq \left(\frac{en}{d}\right)^d \quad \text{for } n \geq d \geq 1.$

$\sum_{i=0}^{d} \binom{n}{i} \leq \binom{n}{d}\left(1 + \frac{d}{n-2d+1}\right) \quad \text{for } \frac{n}{2} \geq d \geq 0.$

$\binom{n}{\alpha n} \leq \sum_{i=0}^{\alpha n} \binom{n}{i} \leq \frac{1-\alpha}{1-2\alpha}\binom{n}{\alpha n} \quad \text{for } \alpha \in (0, \frac{1}{2}).$

square root

$2\sqrt{x+1} - 2\sqrt{x} < \frac{1}{\sqrt{x}} < \sqrt{x+1} - \sqrt{x-1} < 2\sqrt{x} - 2\sqrt{x-1} \quad \text{for } x \geq 1.$

Stirling

$e\left(\frac{n}{e}\right)^n \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n} \leq en\left(\frac{n}{e}\right)^n$

means

$\min\{x_i\} \leq \frac{n}{\sum_i x_i^{-1}} \leq \left(\prod_i x_i\right)^{1/n} \leq \frac{1}{n}\sum_i x_i \leq \sqrt{\frac{1}{n}\sum_i x_i^2} \leq \max\{x_i\}$

power means

$M_p \leq M_q \quad \text{for } p \leq q, \text{ where } M_p = \left(\sum_i w_i |x_i|^p\right)^{1/p}, \ w_i \geq 0, \ \sum_i w_i = 1.$

In the limit $M_0 = \prod_i |x_i|^{w_i}, \ M_{-\infty} = \min_i\{x_i\}, \ M_\infty = \max_i\{x_i\}.$

Lehmer

$\frac{\sum_i w_i |x_i|^p}{\sum_i w_i |x_i|^{p-1}} \leq \frac{\sum_i w_i |x_i|^q}{\sum_i w_i |x_i|^{q-1}} \quad \text{for } p \leq q, \ w_i \geq 0.$

log mean

$\sqrt{xy} \leq \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)(xy)^{\frac{1}{4}} \leq \frac{x-y}{\ln(x)-\ln(y)} \leq \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \leq \frac{x+y}{2} \ \text{ for } x,y > 0.$

Heinz

$\sqrt{xy} \leq \frac{x^{1-\alpha}y^\alpha + x^\alpha y^{1-\alpha}}{2} \leq \frac{x+y}{2} \quad \text{for } x,y > 0, \ \alpha \in [0,1].$

Maclaurin-Newton

$S_k^2 \geq S_{k-1} S_{k+1} \quad \text{and} \quad \sqrt[k]{S_k} \geq \sqrt[(k+1)]{S_{k+1}} \quad \text{for } 1 \leq k < n,$

$S_k = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} a_{i_2} \cdots a_{i_k}, \quad \text{and} \quad a_i \geq 0.$

Jensen

$\varphi\left(\sum_i p_i x_i\right) \leq \sum_i p_i \varphi(x_i) \quad \text{where } p_i \geq 0, \ \sum p_i = 1, \text{ and } \varphi \text{ convex.}$

Alternatively: $\varphi(\mathrm{E}[X]) \leq \mathrm{E}[\varphi(X)].$ For concave φ the reverse holds.

Chebyshev

$$\sum_{i=1}^{n} f(a_i)g(b_i)p_i \geq \left(\sum_{i=1}^{n} f(a_i)p_i\right)\left(\sum_{i=1}^{n} g(b_i)p_i\right) \geq \sum_{i=1}^{n} f(a_i)g(b_{n-i+1})p_i$$

for $a_1 \leq \cdots \leq a_n, \ b_1 \leq \cdots \leq b_n$ and f,g nondecreasing, $p_i \geq 0, \ \sum p_i = 1.$

Alternatively: $\mathrm{E}[f(X)g(X)] \geq \mathrm{E}[f(X)]\mathrm{E}[g(X)].$

rearrangement

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\pi(i)} \geq \sum_{i=1}^{n} a_i b_{n-i+1} \quad \text{for } a_1 \leq \cdots \leq a_n,$$

$b_1 \leq \cdots \leq b_n$ and π a permutation of $[n]$. More generally:

$$\sum_{i=1}^{n} f_i(b_i) \geq \sum_{i=1}^{n} f_i(b_{\pi(i)}) \geq \sum_{i=1}^{n} f_i(b_{n-i+1})$$

with $\left(f_{i+1}(x) - f_i(x)\right)$ nondecreasing for all $1 \leq i < n.$

Weierstrass $\prod_i (1-x_i)^{w_i} \geq 1 - \sum_i w_i x_i$ where $x_i \leq 1$, and either $w_i \geq 1$ (for all i) or $w_i \leq 0$ (for all i).
If $w_i \in [0,1]$, $\sum w_i \leq 1$ and $x_i \leq 1$, the reverse holds.

Young $\left(\frac{1}{px^p} + \frac{1}{qx^q}\right)^{-1} \leq xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y \geq 0$, $p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

Kantorovich $\left(\sum_i x_i{}^2\right)\left(\sum_i y_i{}^2\right) \leq \left(\frac{A}{G}\right)^2 \left(\sum_i x_i y_i\right)^2$ for $x_i, y_i > 0$,
$0 < m \leq \frac{x_i}{y_i} \leq M < \infty$, $A = (m+M)/2$, $G = \sqrt{mM}$.

sum-integral $\int_{L-1}^{U} f(x)\, dx \leq \sum_{i=L}^{U} f(i) \leq \int_{L}^{U+1} f(x)\, dx$ for f nondecreasing.

Cauchy $\varphi'(a) \leq \frac{f(b)-f(a)}{b-a} \leq \varphi'(b)$ where $a < b$, and φ convex.

Hermite $\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(x)\, dx \leq \frac{\varphi(a)+\varphi(b)}{2}$ for φ convex.

Chong $\sum_{i=1}^n \frac{a_i}{a_{\pi(i)}} \geq n$ and $\prod_{i=1}^n a_i{}^{a_i} \geq \prod_{i=1}^n a_i{}^{a_{\pi(i)}}$ for $a_i > 0$.

Gibbs $\sum_i a_i \log \frac{a_i}{b_i} \geq a \log \frac{a}{b}$ for $a_i, b_i \geq 0$, or more generally:
$\sum_i a_i \varphi\left(\frac{b_i}{a_i}\right) \leq a\, \varphi\left(\frac{b}{a}\right)$ for φ concave, and $a := \sum a_i$, $b := \sum b_i$.

Shapiro $\sum_{i=1}^n \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2}$ where $x_i > 0$, $(x_{n+1}, x_{n+2}) := (x_1, x_2)$,
and $n \leq 12$ if even, $n \leq 23$ if odd.

Schur $x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geq 0$
where $x, y, z \geq 0$, $t > 0$

Hadamard $(\det A)^2 \leq \prod_{i=1}^n \sum_{j=1}^n A_{ij}^2$ where A is an $n \times n$ matrix.

Schur $\sum_{i=1}^n \lambda_i^2 \leq \sum_{i,j=1}^n A_{ij}^2$ and $\sum_{i=1}^k d_i \leq \sum_{i=1}^k \lambda_i$ for $1 \leq k \leq n$.
A is an $n \times n$ matrix. For the second inequality A is symmetric.
$\lambda_1 \geq \cdots \geq \lambda_n$ the eigenvalues, $d_1 \geq \cdots \geq d_n$ the diagonal elements.

Ky Fan $\frac{\prod_{i=1}^n x_i{}^{a_i}}{\prod_{i=1}^n (1-x_i)^{a_i}} \leq \frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i(1-x_i)}$ for $x_i \in [0, \frac{1}{2}]$, $a_i \in [0,1]$, $\sum a_i = 1$.

Aczél $\left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2 \geq \left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right)$
given that $a_1^2 > \sum_{i=2}^n a_i^2$ or $b_1^2 > \sum_{i=2}^n b_i^2$.

Mahler $\prod_{i=1}^n (x_i + y_i)^{1/n} \geq \prod_{i=1}^n x_i^{1/n} + \prod_{i=1}^n y_i^{1/n}$ where $x_i, y_i > 0$.

Abel $b_1 \min_k \sum_{i=1}^k a_i \leq \sum_{i=1}^n a_i b_i \leq b_1 \max_k \sum_{i=1}^k a_i$ for $b_1 \geq \cdots \geq b_n \geq 0$.

Milne $\left(\sum_{i=1}^n (a_i + b_i)\right)\left(\sum_{i=1}^n \frac{a_i b_i}{a_i + b_i}\right) \leq \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)$

Carleman $\sum_{k=1}^n \left(\prod_{i=1}^k |a_i|\right)^{1/k} \leq e \sum_{k=1}^n |a_k|$

sum & product $\sum_{j=1}^m \prod_{i=1}^n a_{ij} \geq \sum_{j=1}^m \prod_{i=1}^n a_{i\pi(j)}$ and $\prod_{j=1}^m \sum_{i=1}^n a_{ij} \leq \prod_{j=1}^m \sum_{i=1}^n a_{i\pi(j)}$
where $0 \leq a_{i1} \leq \cdots \leq a_{im}$ for $i = 1, \ldots, n$ and π is a permutation of $[n]$.

$\left|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\right| \leq \sum_{i=1}^n |a_i - b_i|$ for $|a_i|, |b_i| \leq 1$.

$\prod_{i=1}^n (\alpha + a_i) \geq (1 + \alpha)^n$, where $\prod_{i=1}^n a_i \geq 1$, $a_i > 0$, $\alpha > 0$.

Callebaut $\left(\sum_i a_i^{1+x} b_i^{1-x}\right)\left(\sum_i a_i^{1-x} b_i^{1+x}\right) \geq \left(\sum_i a_i^{1+y} b_i^{1-y}\right)\left(\sum_i a_i^{1-y} b_i^{1+y}\right)$
for $1 \geq x \geq y \geq 0$, and $i = 1, \ldots, n$.

Karamata $\sum_{i=1}^n \varphi(a_i) \geq \sum_{i=1}^n \varphi(b_i)$ for $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$,
and $\{a_i\} \succeq \{b_i\}$ (majorization), i.e. $\sum_{i=1}^t a_i \geq \sum_{i=1}^t b_i$ for all $1 \leq t \leq n$,
with equality for $t = n$ and φ is convex (for concave φ the reverse holds).

Muirhead $\frac{1}{n!}\sum_\pi x_{\pi(1)}^{a_1} \cdots x_{\pi(n)}^{a_n} \geq \frac{1}{n!}\sum_\pi x_{\pi(1)}^{b_1} \cdots x_{\pi(n)}^{b_n}$
where $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ and $\{a_k\} \succeq \{b_k\}$,
$x_i \geq 0$ and the sums extend over all permutations π of $[n]$.

Hilbert $\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^\infty a_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty b_n^2\right)^{\frac{1}{2}}$ for $a_m, b_n \in \mathbb{R}$.
With $\max\{m, n\}$ instead of $m + n$, we have 4 instead of π.

Hardy $\sum_{n=1}^\infty \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p$ for $a_n \geq 0$, $p > 1$.

Carlson $\left(\sum_{n=1}^\infty a_n\right)^4 \leq \pi^2 \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty n^2 a_n^2$ for $a_n \in \mathbb{R}$.

Mathieu $\frac{1}{c^2 + 1/2} < \sum_{n=1}^\infty \frac{2n}{(n^2 + c^2)^2} < \frac{1}{c^2}$ for $c \neq 0$.

Copson $\sum_{n=1}^\infty \left(\sum_{k \geq n} \frac{a_k}{k}\right)^p \leq p^p \sum_{n=1}^\infty a_n{}^p$ for $a_n \geq 0$, $p > 1$, reverse if $p \in (0,1)$.

Kraft $\sum 2^{-c(i)} \leq 1$ for $c(i)$ depth of leaf i of binary tree, sum over all leaves.

LYM $\sum_{X \in \mathcal{A}} \binom{n}{|X|}^{-1} \leq 1$, $\mathcal{A} \subset 2^{[n]}$, no set in $\mathcal{A}$ is subset of another set in $\mathcal{A}$.

Sauer-Shelah $|\mathcal{A}| \leq |\text{str}(\mathcal{A})| \leq \sum_{i=0}^{\text{vc}(\mathcal{A})} \binom{n}{i}$ for $\mathcal{A} \subseteq 2^{[n]}$, and
$\text{str}(\mathcal{A}) = \{X \subseteq [n] : X \text{ shattered by } \mathcal{A}\}$, $\text{vc}(\mathcal{A}) = \max\{|X| : X \in \text{str}(\mathcal{A})\}$.

Bonferroni $\Pr\left[\bigvee_{i=1}^n A_i\right] \leq \sum_{j=1}^k (-1)^{j-1} S_j$ for $1 \leq k \leq n$, k odd,
$\Pr\left[\bigvee_{i=1}^n A_i\right] \geq \sum_{j=1}^k (-1)^{j-1} S_j$ for $2 \leq k \leq n$, k even.
$S_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Pr\left[A_{i_1} \wedge \cdots \wedge A_{i_k}\right]$ where A_i are events.

Bhatia-Davis $\quad \operatorname{Var}[X] \leq \big(M - \operatorname{E}[X]\big)\big(\operatorname{E}[X] - m\big) \quad$ where $X \in [m, M]$.

Samuelson $\quad \mu - \sigma\sqrt{n-1} \leq x_i \leq \mu + \sigma\sqrt{n-1} \quad$ for $i = 1, \ldots, n$.

Where $\mu = \sum x_i/n$, $\sigma^2 = \sum (x_i - \mu)^2/n$.

Markov $\quad \Pr\big[|X| \geq a\big] \leq \operatorname{E}[|X|]/a \quad$ where X is a random variable (r.v.), $a > 0$.

$\Pr\big[X \leq c\big] \leq (1 - \operatorname{E}[X])/(1 - c) \quad$ for $X \in [0, 1]$ and $c \in \big[0, \operatorname{E}[X]\big]$.

$\Pr\big[X \in S\big] \leq \operatorname{E}[f(X)]/s \quad$ for $f \geq 0$, and $f(x) \geq s > 0$ for all $x \in S$.

Chebyshev $\quad \Pr\big[|X - \operatorname{E}[X]| \geq t\big] \leq \operatorname{Var}[X]/t^2 \quad$ where $t > 0$.

$\Pr\big[X - \operatorname{E}[X] \geq t\big] \leq \operatorname{Var}[X]/(\operatorname{Var}[X] + t^2) \quad$ where $t > 0$.

2nd moment $\quad \Pr\big[X > 0\big] \geq (\operatorname{E}[X])^2/(\operatorname{E}[X^2]) \quad$ where $\operatorname{E}[X] \geq 0$.

$\Pr\big[X = 0\big] \leq \operatorname{Var}[X]/(\operatorname{E}[X^2]) \quad$ where $\operatorname{E}[X^2] \neq 0$.

k^{th} moment $\quad \Pr\big[|X - \mu| \geq t\big] \leq \dfrac{\operatorname{E}\big[(X - \mu)^k\big]}{t^k} \quad$ and

$\Pr\big[|X - \mu| \geq t\big] \leq C_k \left(\dfrac{nk}{t^2}\right)^{k/2} \quad$ for $X_i \in [0, 1]$ k-wise indep. r.v.,

$X = \sum X_i, \ i = 1, \ldots, n, \ \mu = \operatorname{E}[X], \ C_k = 2\sqrt{\pi k} e^{1/6k} \leq 1.0004, \ k$ even.

Chernoff $\quad \Pr[X \geq t] \leq F(a)/a^t \quad$ for X r.v., $\Pr[X = k] = p_k$,

$F(z) = \sum_k p_k z^k$ probability gen. func., and $a \geq 1$.

$\Pr\big[X \geq (1 + \delta)\mu\big] \leq \left(\dfrac{e^\delta}{(1 + \delta)^{(1+\delta)}}\right)^\mu \leq \exp\left(\dfrac{-\mu\delta^2}{3}\right)$

for X_i i.r.v. from $[0,1]$, $X = \sum X_i, \ \mu = \operatorname{E}[X], \ \delta \geq 0$ resp. $\delta \in [0, 1)$.

$\Pr\big[X \leq (1 - \delta)\mu\big] \leq \left(\dfrac{e^{-\delta}}{(1 - \delta)^{(1-\delta)}}\right)^\mu \leq \exp\left(\dfrac{-\mu\delta^2}{2}\right)$ for $\delta \in [0, 1)$.

Further from the mean: $\Pr\big[X \geq R\big] \leq 2^{-R} \quad$ for $R \geq 2e\mu \ (\approx 5.44\mu)$.

$\Pr\big[X \geq t\big] \leq \dfrac{\binom{n}{k}p^k}{\binom{t}{k}} \quad$ for $X_i \in \{0, 1\}$ k-wise i.r.v., $\operatorname{E}[X_i] = p, \ X = \sum X_i$.

$\Pr\big[X \geq (1 + \delta)\mu\big] \leq \binom{n}{\hat k}p^{\hat k}/\binom{(1+\delta)\mu}{\hat k} \quad$ for $X_i \in [0, 1]$ k-wise i.r.v.,

$k \geq \hat k = \lceil \mu\delta/(1 - p) \rceil, \ \operatorname{E}[X_i] = p_i, \ X = \sum X_i, \ \mu = \operatorname{E}[X], \ p = \frac{\mu}{n}, \ \delta > 0$.

Hoeffding $\quad \Pr\big[|X - \operatorname{E}[X]| \geq \delta\big] \leq 2\exp\left(\dfrac{-2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right) \quad$ for X_i i.r.v.,

$X_i \in [a_i, b_i]$ (w. prob. 1), $X = \sum X_i, \ \delta \geq 0$.

A related lemma, assuming $\operatorname{E}[X] = 0, \ X \in [a, b]$ (w. prob. 1) and $\lambda \in \mathbb{R}$:

$\operatorname{E}\big[e^{\lambda X}\big] \leq \exp\left(\dfrac{\lambda^2(b - a)^2}{8}\right)$

Kolmogorov $\quad \Pr\big[\max_k |S_k| \geq \varepsilon\big] \leq \frac{1}{\varepsilon^2}\operatorname{Var}[S_n] = \frac{1}{\varepsilon^2}\sum_i \operatorname{Var}[X_i]$

where $X_1, \ldots, X_n$ are i.r.v., $\operatorname{E}[X_i] = 0$,

$\operatorname{Var}[X_i] < \infty$ for all i, $S_k = \sum_{i=1}^k X_i$ and $\varepsilon > 0$.

Paley-Zygmund $\quad \Pr\big[X \geq \mu \operatorname{E}[X]\big] \geq 1 - \dfrac{\operatorname{Var}[X]}{(1 - \mu)^2 (\operatorname{E}[X])^2 + \operatorname{Var}[X]} \quad$ for $X \geq 0$,

$\operatorname{Var}[X] < \infty$, and $\mu \in (0, 1)$.

Vysochanskij- $\quad \Pr\big[|X - \operatorname{E}[X]| \geq \lambda\sigma\big] \leq \frac{4}{9\lambda^2} \quad$ if $\lambda \geq \sqrt{\frac{8}{3}}$,

Petunin-Gauss $\quad \Pr\big[|X - m| \geq \varepsilon\big] \leq \frac{4\tau^2}{9\varepsilon^2} \quad$ if $\varepsilon \geq \frac{2\tau}{\sqrt{3}}$,

$\Pr\big[|X - m| \geq \varepsilon\big] \leq 1 - \frac{\varepsilon}{\sqrt{3}\tau} \quad$ if $\varepsilon \leq \frac{2\tau}{\sqrt{3}}$.

Where X is a unimodal r.v. with mode m,

$\sigma^2 = \operatorname{Var}[X] < \infty, \ \tau^2 = \operatorname{Var}[X] + (\operatorname{E}[X] - m)^2 = \operatorname{E}[(X - m)^2]$.

Etemadi $\quad \Pr\big[\max_{1 \leq k \leq n} |S_k| \geq 3\alpha\big] \leq 3 \max_{1 \leq k \leq n} \big(\Pr\big[|S_k| \geq \alpha\big]\big)$

where X_i are i.r.v., $S_k = \sum_{i=1}^k X_i, \ \alpha \geq 0$.

Doob $\quad \Pr\big[\max_{1 \leq k \leq n} |X_k| \geq \varepsilon\big] \leq \operatorname{E}\big[|X_n|\big]/\varepsilon \quad$ for martingale (X_k) and $\varepsilon > 0$.

Bennett $\quad \Pr\big[\sum_{i=1}^n X_i \geq \varepsilon\big] \leq \exp\left(-\dfrac{n\sigma^2}{M^2}\theta\big(\dfrac{M\varepsilon}{n\sigma^2}\big)\right) \quad$ where X_i i.r.v.,

$\operatorname{E}[X_i] = 0, \ \sigma^2 = \frac{1}{n}\sum \operatorname{Var}[X_i], \ |X_i| \leq M$ (w. prob. 1), $\varepsilon \geq 0$,

$\theta(u) = (1 + u)\log(1 + u) - u$.

Bernstein $\quad \Pr\big[\sum_{i=1}^n X_i \geq \varepsilon\big] \leq \exp\left(\dfrac{-\varepsilon^2}{2(n\sigma^2 + M\varepsilon/3)}\right) \quad$ for X_i i.r.v.,

$\operatorname{E}[X_i] = 0, \ |X_i| < M$ (w. prob. 1) for all i, $\sigma^2 = \frac{1}{n}\sum \operatorname{Var}[X_i], \ \varepsilon \geq 0$.

Azuma $\quad \Pr\big[|X_n - X_0| \geq \delta\big] \leq 2\exp\left(\dfrac{-\delta^2}{2\sum_{i=1}^n c_i^2}\right) \quad$ for martingale (X_k) s.t.

$|X_i - X_{i-1}| < c_i$ (w. prob. 1), for $i = 1, \ldots, n, \ \delta \geq 0$.

Efron-Stein $\quad \operatorname{Var}[Z] \leq \frac{1}{2}\operatorname{E}\left[\sum_{i=1}^n \big(Z - Z^{(i)}\big)^2\right] \quad$ for $X_i, X_i' \in \mathcal{X}$ i.r.v.,

$f : \mathcal{X}^n \to \mathbb{R}, \ Z = f(X_1, \ldots, X_n), \ Z^{(i)} = f(X_1, \ldots, X_i', \ldots, X_n)$.

McDiarmid $\quad \Pr\big[|Z - \operatorname{E}[Z]| \geq \delta\big] \leq 2\exp\left(\dfrac{-2\delta^2}{\sum_{i=1}^n c_i^2}\right) \quad$ for $X_i, X_i' \in \mathcal{X}$ i.r.v.,

$Z, Z^{(i)}$ as before, s.t. $|Z - Z^{(i)}| \leq c_i$ for all i, and $\delta \geq 0$.

Janson $\quad M \leq \Pr\big[\bigwedge \overline{B_i}\big] \leq M\exp\left(\dfrac{\Delta}{2 - 2\varepsilon}\right) \quad$ where $\Pr[B_i] \leq \varepsilon$ for all i,

$M = \prod(1 - \Pr[B_i]), \ \Delta = \sum\limits_{i \neq j, B_i \sim B_j} \Pr[B_i \wedge B_j]$.

Lovász $\quad \Pr\big[\bigwedge \overline{B_i}\big] \geq \prod(1 - x_i) > 0 \quad$ where $\Pr[B_i] \leq x_i \cdot \prod\limits_{(i,j) \in D} (1 - x_j)$,

for $x_i \in [0, 1)$ for all $i = 1, \ldots, n$ and D the dependency graph.

If each B_i mutually indep. of the set of all other events, exc. at most d,

$\Pr[B_i] \leq p$ for all $i = 1, \ldots, n$, then if $ep(d + 1) \leq 1$ then $\Pr\big[\bigwedge \overline{B_i}\big] > 0$.