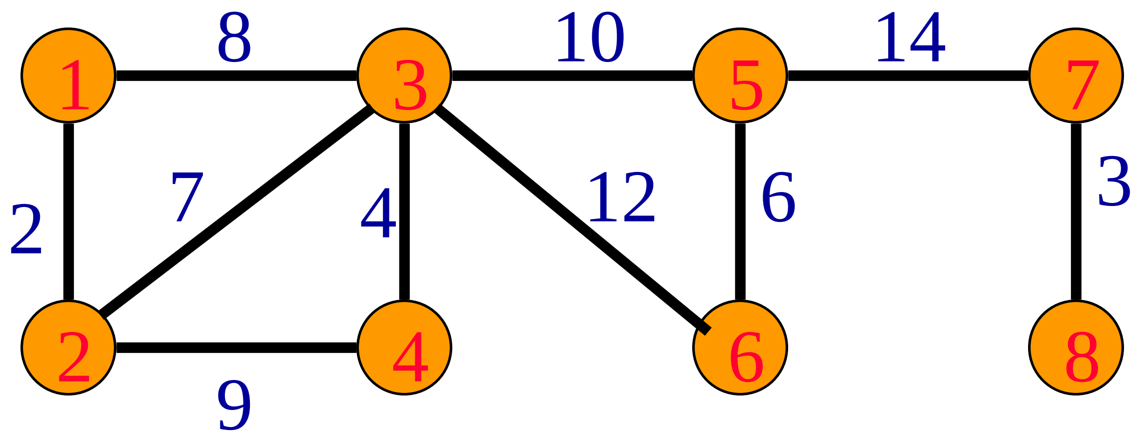
Minimum-Cost Spanning Tree

- weighted connected undirected graph
- spanning tree
- cost of spanning tree is sum of edge costs
- find spanning tree that has minimum cost

Example



- Network has 10 edges.
- Spanning tree has only n - 1 = 7 edges.
- Need to either select 7 edges or discard 3.

Edge Selection Strategies

- Start with an n-vertex 0-edge forest. Consider edges in ascending order of cost. Select edge if it does not form a cycle together with already selected edges.
  - Kruskal's method.
- Start with a 1-vertex tree and grow it into an n-vertex tree by repeatedly adding a vertex and an edge. When there is a choice, add a least cost edge.
  - Prim's method.

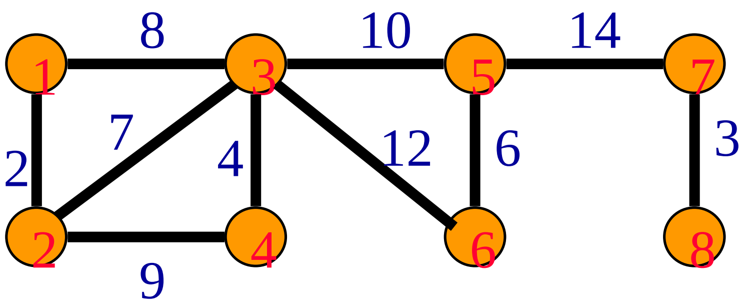
Edge Selection Strategies

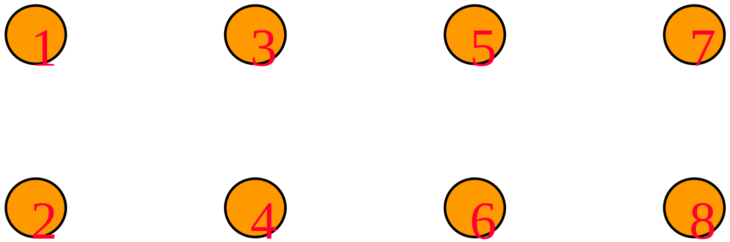
- Start with an n-vertex forest. Each component/tree selects a least cost edge to connect to another component/tree. Eliminate duplicate selections and possible cycles. Repeat until only 1 component/tree is left.
  - Sollin’s method.

Edge Rejection Strategies

- Start with the connected graph. Repeatedly find a cycle and eliminate the highest cost edge on this cycle. Stop when no cycles remain.
- Consider edges in descending order of cost. Eliminate an edge provided this leaves behind a connected graph.

Kruskal's Method

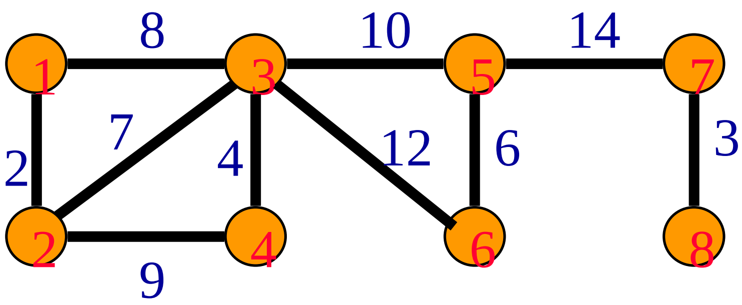


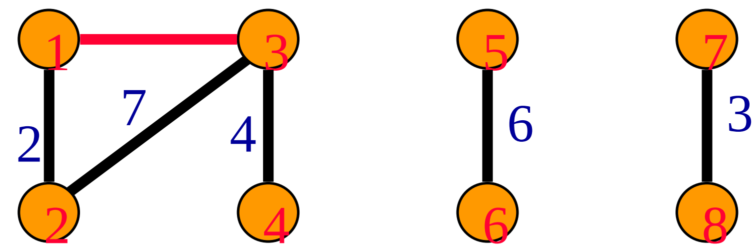


- Start with a forest that has no edges.

- Consider edges in ascending order of cost.
- Edge (1,2) is considered first and added to the forest.

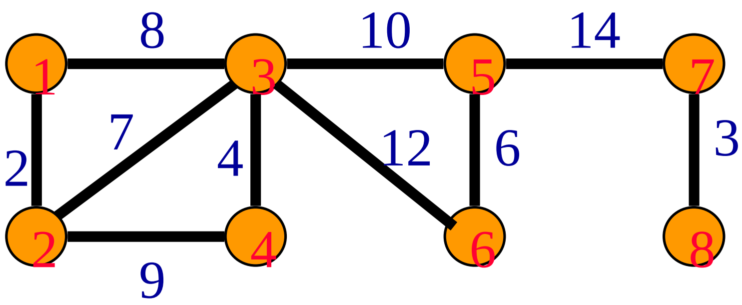
Kruskal's Method

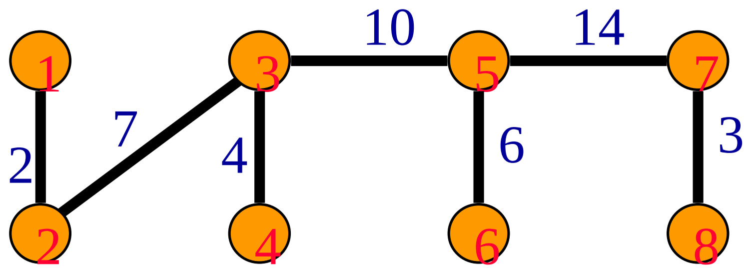




- Edge (7,8) is considered next and added.
- Edge (3,4) is considered next and added.
- Edge (5,6) is considered next and added.
- Edge (2,3) is considered next and added.
- Edge (1,3) is considered next and rejected because it creates a cycle.

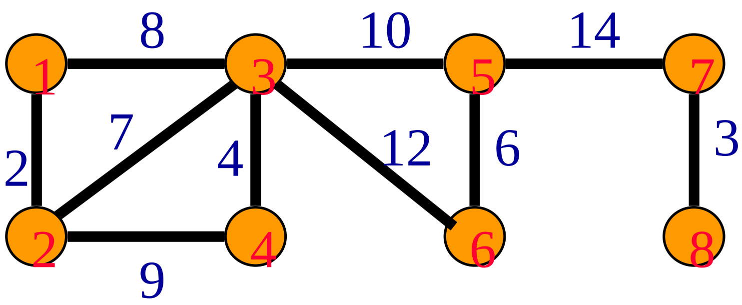
Kruskal's Method

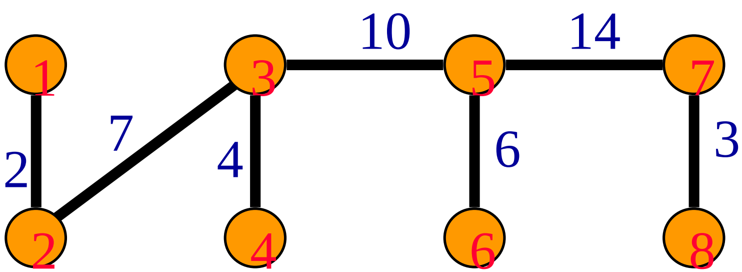




- Edge (2,4) is considered next and rejected because it creates a cycle.
- Edge (3,5) is considered next and added.
- Edge (3,6) is considered next and rejected.
- Edge (5,7) is considered next and added.

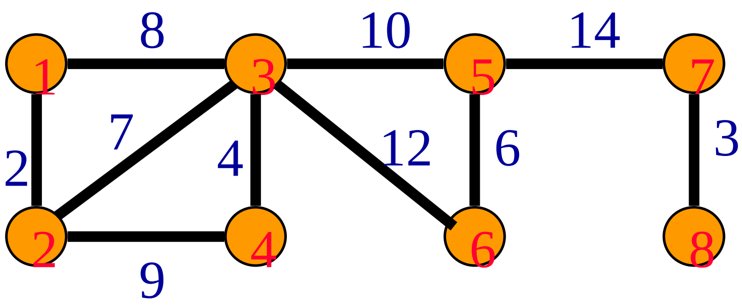
Kruskal's Method

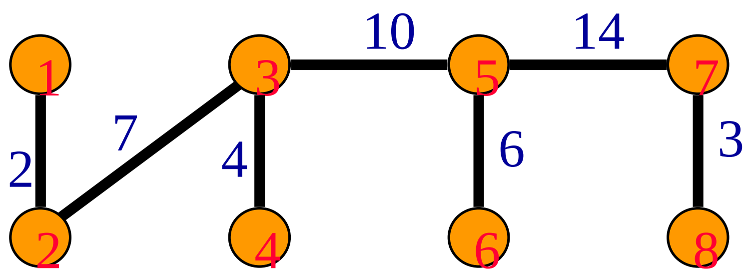




- n - 1 edges have been selected and no cycle formed.
- So we must have a spanning tree.
- Cost is 46.
- Min-cost spanning tree is unique when all edge costs are different.

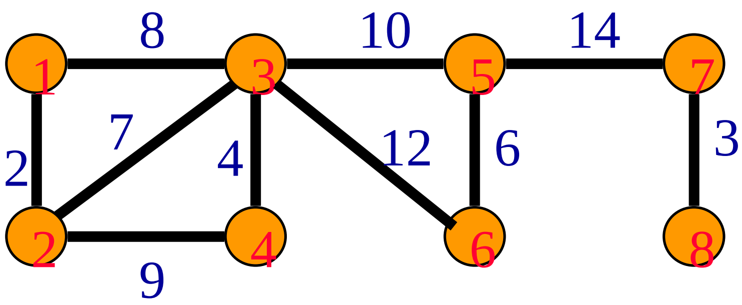
Prim's Method

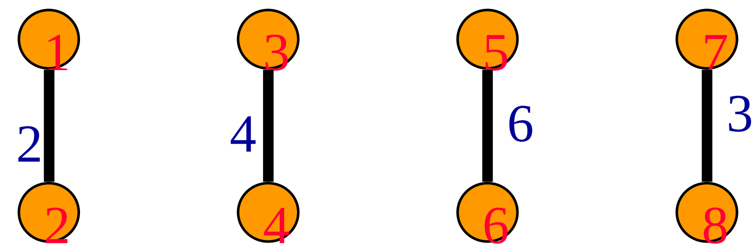




- Start with any single vertex tree.
- Get a 2-vertex tree by adding a cheapest edge.
- Get a 3-vertex tree by adding a cheapest edge.
- Grow the tree one edge at a time until the tree has n - 1 edges (and hence has all n vertices).

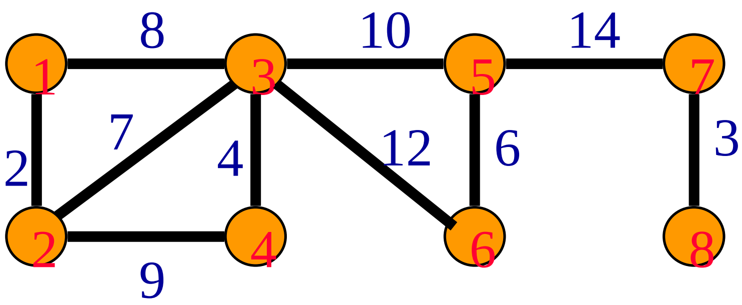
Sollin’s Method

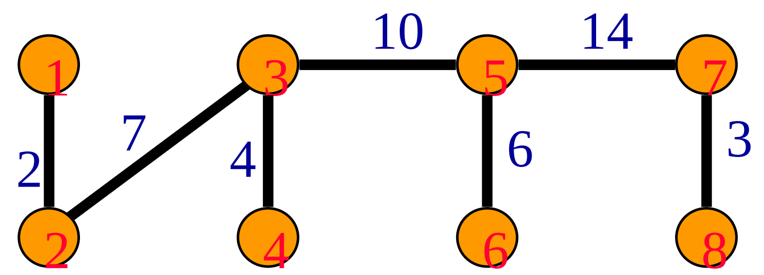




- Start with a forest that has no edges.
- Each component selects a least cost edge with which to connect to another component.
- Duplicate selections are eliminated.
- Cycles are possible when the graph has some edges that have the same cost.

Sollin's Method





- Each component that remains selects a least cost edge with which to connect to another component.
- Beware of duplicate selections and cycles.

Minimum-Cost Spanning Tree Methods

- Can prove that all stated edge selection/rejection result in a minimum-cost spanning tree.
- Prim's method is fastest.
  - $O(n^2)$ using an implementation similar to that of Dijkstra's shortest-path algorithm.
  - $O(e + n \log n)$ using a Fibonacci heap.
- Kruskal's uses union-find trees to run in $O(n + e \log e)$ time.

Pseudocode For Kruskal's Method

```
Start with an empty set T of edges.
while (E is not empty && |T| != n-1)
{
    Let (u,v) be a least-cost edge in E.
    E = E - {(u,v)}. // delete edge from E
    if ((u,v) does not create a cycle in T)
      Add edge (u,v) to T.
}
if (| T | == n-1) T is a min-cost spanning tree.
else Network has no spanning tree.
```

Data Structures For Kruskal's Method

Edge set E.

Operations are:

- Is E empty?
- Select and remove a least-cost edge.

Use a min heap of edges.

- Initialize. O(e) time.
- Remove and return least-cost edge. O(log e) time.

Data Structures For Kruskal's Method

Set of selected edges T.

Operations are:

- Does T have n - 1 edges?
- Does the addition of an edge (u, v) to T result in a cycle?
- Add an edge to T.

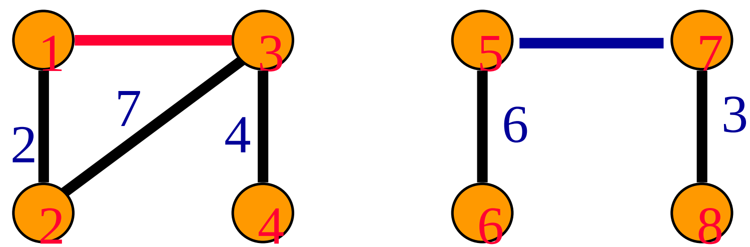
Data Structures For Kruskal's Method

Use an array for the edges of T.

- Does T have n - 1 edges?
  - Check number of edges in array. O(1) time.
- Does the addition of an edge (u, v) to T result in a cycle?
  - Not easy.
- Add an edge to T.
  - Add at right end of edges in array. O(1) time.

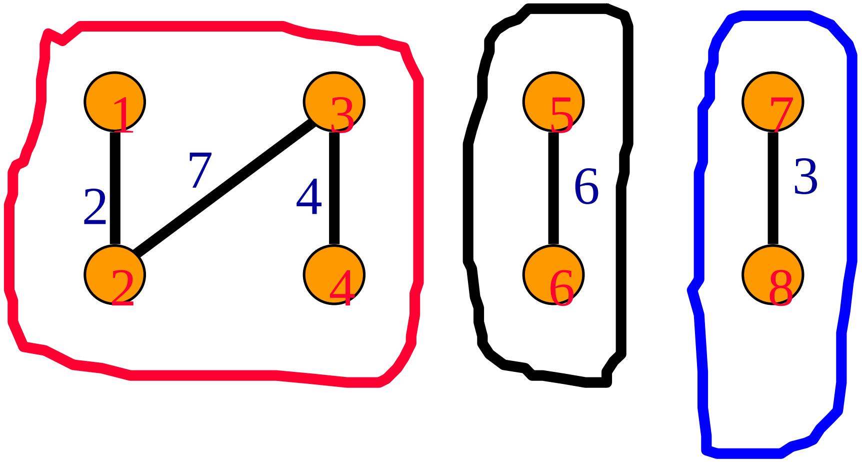
Data Structures For Kruskal's Method

Does the addition of an edge (u, v) to T result in a cycle?



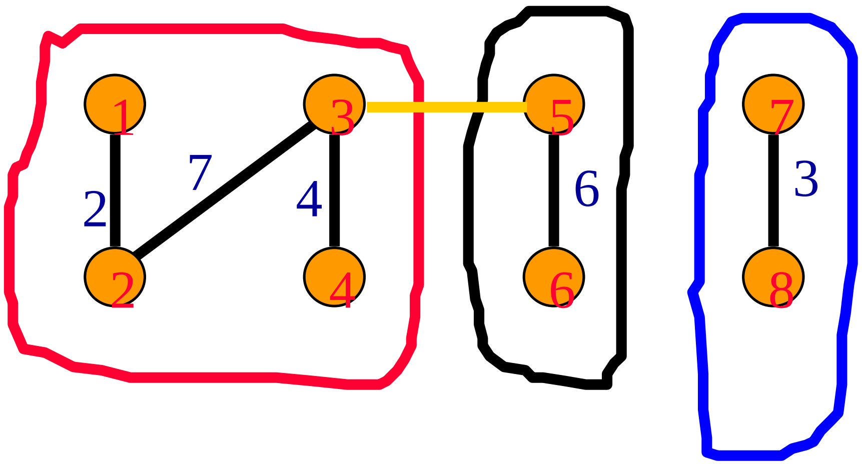
- Each component of T is a tree.
- When u and v are in the same component, the addition of the edge (u,v) creates a cycle.
- When u and v are in the different components, the addition of the edge (u,v) does not create a cycle.

Data Structures For Kruskal's Method



- Each component of T is defined by the vertices in the component.
- Represent each component as a set of vertices.
  - {1, 2, 3, 4}, {5, 6}, {7, 8}
- Two vertices are in the same component iff they are in the same set of vertices.

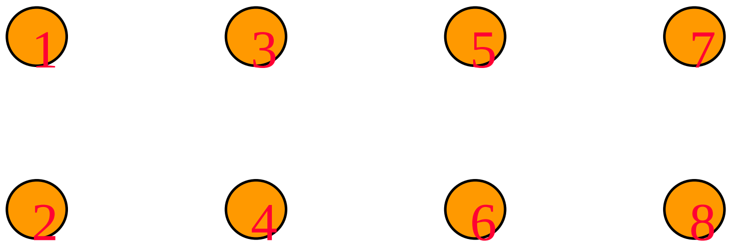
Data Structures For Kruskal's Method



- When an edge (u, v) is added to T, the two components that have vertices u and v combine to become a single component.
- In our set representation of components, the set that has vertex u and the set that has vertex v are united.
  - {1, 2, 3, 4} + {5, 6} => {1, 2, 3, 4, 5, 6}

Data Structures For Kruskal's Method

- Initially, T is empty.



- Initial sets are:
  - {1} {2} {3} {4} {5} {6} {7} {8}
- Does the addition of an edge (u, v) to T result in a cycle? If not, add edge to T.

```
s1 = Find(u); s2 = Find(v);

if (s1 != s2) Union(s1, s2);
```

Data Structures For Kruskal's Method

- Use fast solution for disjoint sets.
- Initialize.
  - $O(n)$ time.
- At most $2e$ finds and $n-1$ unions.
  - Very close to $O(n + e)$.
- Min heap operations to get edges in increasing order of cost take $O(e \log e)$.
- Overall complexity of Kruskal's method is $O(n + e \log e)$.