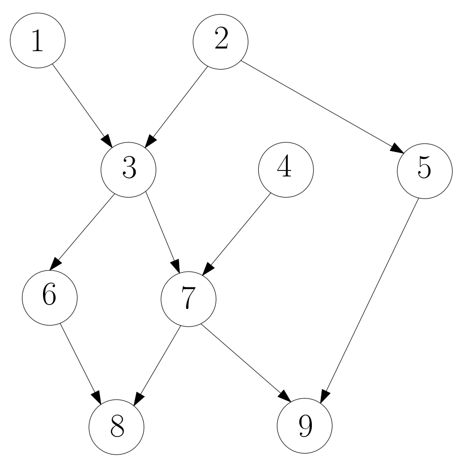
Data Mining 2025
Exercises Bayesian Networks

Exercise 1: Independence Properties of Bayesian Networks

Consider the following directed independence graph.



42A-P.22

(a) Give the factorization of $P(X_1, X_2, \ldots, X_9)$ corresponding to this independence graph.

Construct the appropriate moral graphs to check whether the following conditional independencies hold: 42A-P.28-33

(b) $6 \perp\!\!\!\perp 7$

(c) $6 \perp\!\!\!\perp 7 \mid 3$

(d) $6 \perp\!\!\!\perp 7 \mid 8$

(e) $2 \perp\!\!\!\perp 9 \mid \{5, 7\}$

(f) $2 \perp\!\!\!\perp 9 \mid \{3, 5\}$

(g) $5 \perp\!\!\!\perp 8$

(h) $5 \perp\!\!\!\perp 8 \mid 3$

Exercise 2: Learning Bayesian Networks

In structure learning of Bayesian networks one often uses a score function to determine the quality of a network structure, in combination with a hill-climbing local search strategy. One popular score function is BIC (Bayesian Information Criterion):

$$\mathrm{BIC}(M) = \mathcal{L}(M) - \frac{\ln n}{2}\,\dim(M),$$

where $\mathcal{L}(M)$ denotes the value of the loglikelihood function of model M evaluated at the maximum (also called the loglikelihood score), $\dim(M)$ denotes the number of parameters of model M, and n denotes the number of observations in the data set.
We want to construct a model on the following data set on 3 binary variables:

	X_1	X_2	X_3
1	1	1	0
2	1	0	0
3	1	0	0
4	1	0	0
5	0	0	0
6	0	1	1
7	1	1	1
8	0	1	1
9	0	0	1
10	0	0	1

The initial model in the search is the mutual independence model (corresponding to the empty graph).

42B-P.6 (a) Give the maximum likelihood estimates of the parameters of the mutual independence model.

42B-P.7 (b) Compute the loglikelihood score of the mutual independence model. The loglikelihood score is the value of the loglikehood function evaluated in the maximum. Use the *natural* logarithm in your computations.

(c) Give all neighbours of the current model, assuming a neighbour can be obtained by either: adding an edge, removing an edge, or reversing an edge. Which of these neighbour models are equivalent? **Note:** Define the skeleton of a directed graph as the undirected graph obtained by dropping the directions of the edges. Two models are equivalent if and only if they have the same skeleton and the same v-structures.

42B-P.44

(d) Would adding an edge from X_1 to X_2 (or vice versa) improve the BIC score? Explain.

42B-P.12-22 (example) (e) Consider the neighbour model obtained by adding an edge from X_1 to X_3. Is this model preferred to the initial model on the basis of the BIC-score? Explain.

42B P.35 counting dim(M)

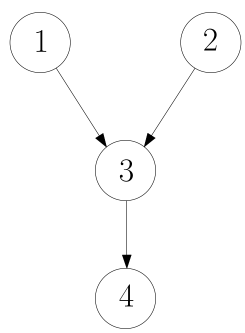
Exercise 3: Learning Bayesian Networks

This exercise is similar to exercise 2; it just gives you more practice.

We are constructing a model on the following data set on 4 binary variables:

	X_1	X_2	X_3	X_4
1	1	1	0	0
2	1	0	0	1
3	1	0	0	0
4	1	0	0	1
5	0	1	0	1
6	1	1	1	1
7	1	1	1	0
8	0	1	1	0
9	0	0	1	0
10	0	0	1	0

Suppose the current model in the search has the following structure:



(a) Give the maximum likelihood estimates of the model parameters.

(b) Compute the loglikelihood score for the given model and data set. Use the *natural* logarithm in your computations.

(c) Compute the BIC score of this model on the given data set.

(d) Give all neighbours of the current model, assuming a neighbour can be obtained by either: adding an edge, removing an edge, or reversing an edge. Which of these neighbour models are equivalent?

(e) Consider the neighbour model obtained by adding an edge from X_1 to X_4. Is this model preferred to the current model? Explain.

Exercise 4: Essential Graph

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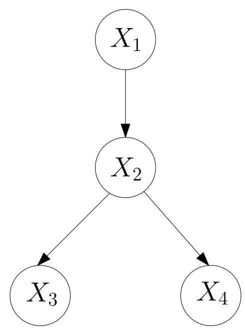
Construct a graph from the DAG below as follows: orient all edges whose direction is fixed in the equivalence class that the DAG belongs to, and make edges bi-directional if there are two members in the equivalence class which have edges in opposite directions. The resulting graph is called the *essential* graph. Recall that two DAGs belong to the same equivalence class iff they have the same skeleton and the same immoralities (v-structures). Hint: it doesn't suffice to check if you remain in the same equivalence class if you turn a single edge around!



Exercise 5: Structure Learning

We perform a greedy hill-climbing search to find a good Bayesian network structure on 5 variables denoted A, B, C, D, and E. Neighbour models are obtained by adding, deleting, or reversing an edge. We start our search from the empty graph. In step 1 of the search we find that adding the edge $A \to D$ gives the biggest improvement in the BIC score. Which Δ scores do we need to compute in step 2?

Exercise 6: Maximum Likelihood Estimation

The loglikelihood function of a Bayesian network is given by:

$$\mathcal{L} = \sum_{i=1}^{k} \left\{ \sum_{x_i, x_{pa(i)}} n(x_i, x_{pa(i)}) \log p(x_i \mid x_{pa(i)}) \right\}$$

To simplify matters somewhat, we assume all variables are binary, so that we can write:

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^{k} \left\{ \sum_{x_{pa(i)}} n(x_i = 0, x_{pa(i)}) \log p(x_i = 0 \mid x_{pa(i)}) + n(x_i = 1, x_{pa(i)}) \log p(x_i = 1 \mid x_{pa(i)}) \right\} \\ &= \sum_{i=1}^{k} \left\{ \sum_{x_{pa(i)}} n(x_i = 0, x_{pa(i)}) \log p(x_i = 0 \mid x_{pa(i)}) + n(x_i = 1, x_{pa(i)}) \log(1 - p(x_i = 0 \mid x_{pa(i)})) \right\}\end{aligned}$$

(a) Determine

$$\frac{\partial \mathcal{L}}{\partial p(x_j = 0 \mid x_{pa(j)})},$$

that is, the partial derivative of the loglikelihood function with respect to $p(x_j = 0 \mid x_{pa(j)})$ for arbitrary $j \in \{1, \ldots, k\}$ and arbitrary parent configuration $x_{pa(j)} \in \{0,1\}^{|pa(j)|}$.
Verify that this partial derivative doesn't depend on any unknown parameter, except for $p(x_j = 0 \mid x_{pa(j)})$ itself.

(b) Equate the answer you obtained under (a) to zero, and solve for $p(x_j = 0 \mid x_{pa(j)})$. You should get the solution

$$p(x_j = 0 \mid x_{pa(j)}) = \frac{n(x_j = 0, x_{pa(j)})}{n(x_j = 0, x_{pa(j)}) + n(x_j = 1, x_{pa(j)})} = \frac{n(x_j = 0, x_{pa(j)})}{n(x_{pa(j)})}$$

Verify that this solution coincides with the general formula given for the maximum likelihood parameter estimates of a Bayesian network.