600.645 CIS HOMEWORK 4

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Question 1

A. It should be noticed that the pose is calculated through POE so that a_k remains constant regardless of θ. First we consider one matrix case:

$$R_{rob}(\theta + \Delta\theta) = \Delta R_{left} R(\theta) = R(\theta)\Delta R_{right}$$

For each revolute joint we have,

$$R(\Delta\theta) = Rot(a, \Delta\theta)$$

Using these formulas we will have

$$\Delta R_{left} = Rot(a_1, \Delta\theta_1) R_1 Rot(a_2, \Delta\theta_2) R_2 Rot(a_3, \Delta\theta_3) R_3 Rot(a_4, \Delta\theta_4) R_3^T R_2^T R_1^T$$

$$\Delta R_{right} = R_4^T R_3^T R_2^T Rot(a_1, \Delta\theta_1) R_2 Rot(a_2, \Delta\theta_2) R_3 Rot(a_3, \Delta\theta_3) R_4 Rot(a_4, \Delta\theta_4)$$

B. Considering the fact that

$$R Rot(a, \theta) R^T = Rot(Ra, \theta)$$

Also, we could represent the delta term in following way,

$$Rot(a, \Delta\theta) = I + sk(a)\Delta\theta$$

we can simplify the expressions derived in part A as,

$$I + sk(\alpha_{left}) = I + sk(a_1)\Delta\theta_1 + R_1 sk(a_2)\Delta\theta_2 R_1^T + R_1 R_2 sk(a_3)\Delta\theta_3 R_2^T R_1^T + R_1 R_2 R_3 sk(a_4)\Delta\theta_4 R_3^T R_2^T R_1^T$$

$$sk(\alpha_{left}) = sk(a_1)\Delta\theta_1 + sk(R_1 a_2)\Delta\theta_2 + sk(R_1 R_2 a_3)\Delta\theta_3 + sk(R_1 R_2 R_3 a_4)\Delta\theta_4$$

so that we could derive A_{left} to relate α_{left} and $\Delta\theta$ as,

$$\alpha_{left} = A_{left}\Delta\theta = \begin{bmatrix} a_1 & R_1 a_2 & R_1 R_2 a_3 & R_1 R_2 R_3 a_4 \end{bmatrix} \Delta\theta$$

Perform the same process for right hand side,

$$I + sk(\alpha_{right}) = R_4^T R_3^T R_2^T sk(a_1)\Delta\theta_1 R_2 R_3 R_4 + R_4^T R_3^T sk(a_2)\Delta\theta_2 R_2 R_3 + R_4^T sk(a_3)\Delta\theta_3 R_4 + sk(a_4)\Delta\theta_4 + I$$

$$sk(\alpha_{right}) = sk(R_4^T R_3^T R_2^T a_1)\Delta\theta_1 + sk(R_4^T R_3^T a_2)\Delta\theta_2 + sk(R_4^T a_3)\Delta\theta_3 + sk(a_4)\Delta\theta_4$$

so that we could derive A_{left} to relate α_{right} and $\Delta\theta$ as,

$$\alpha_{right} = A_{right}\Delta\theta = \begin{bmatrix} R_4^T R_3^T R_2^T a_1 & R_4^T R_3^T a_2 & R_4^T a_3 & a_4 \end{bmatrix} \Delta\theta$$

C. We could derive the delta transformation term in both hand sides as,
Left Hand Side,

$$\Delta F_{left} = F(q + \Delta q) F^{-1}(q)$$

so that, in homogeneous form we have

$$\Delta F_{left} = \begin{bmatrix} \Delta R_{left}R & p+\Delta p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Delta R_{left} & p+\Delta p - \Delta R_{left}p \\ 0 & 1 \end{bmatrix}$$

Knowing that,

$$F_{rob} = \begin{bmatrix} R_{rob} & p_{cart} + R_{rob}p_{cr} \end{bmatrix}$$

then we have, for left hand side,

$$\Delta F_{left} = \begin{bmatrix} \Delta R_{left} & p_{cart} + \Delta p_{cart} - \Delta R_{left}p_{cart} \\ 0 & 1 \end{bmatrix}$$

and Right Hand Side,

$$\Delta F_{right} = F^{-1}(q)F(q+\Delta q)$$

so that, in homogeneous form we have,

$$\Delta F_{right} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R\Delta R_{right} & p+\Delta p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Delta R_{right} & R^T\Delta p \\ 0 & 1 \end{bmatrix}$$

then we have,

$$\Delta F_{right} = \begin{bmatrix} \Delta R_{right} & R^T\Delta p_{cart} + \Delta R_{right}p_{cr} - p_{cr} \\ 0 & 1 \end{bmatrix}$$

D. We have already derived the relation between ΔR and $\Delta\theta$ in both hand sides, recorded as,

$$\alpha_{left} = A_{left}\Delta\theta$$
$$\alpha_{right} = A_{right}\Delta\theta$$

In part C, we derived the relation of the translation part Δp.
In Left Hand Side,

$$\varepsilon_{left} = \Delta p_{cart} - \Delta R_{left}p_{cart} = \Delta p_{cart} + sk(p_{cart})A_{left}\Delta\theta$$

So that combine two relation together, we have,

$$\delta_{left} = \begin{bmatrix} \alpha_{left} \\ \varepsilon_{left} \end{bmatrix} = \begin{bmatrix} A_{left} & 0_{3\times3} \\ sk(p_{cart})A_{left} & I_{3\times3} \end{bmatrix} \Delta\theta = J_{left}\Delta\theta$$

In Right Hand Side,

$$\varepsilon_{right} = R^T\Delta p_{cart} + \Delta R_{right}p_{cr}$$

So that combine two relation together, we have,

$$\delta_{right} = \begin{bmatrix} \alpha_{right} \\ \varepsilon_{right} \end{bmatrix} = \begin{bmatrix} A_{right} & 0_{3\times3} \\ -sk(p_{cr})A_{right} & R^T \end{bmatrix} \Delta\theta = J_{right}\Delta\theta$$

E. Firstly, we use left hand side formula here to present the delta transformation term, so that we have

$$\Delta F_{left} = F^{goal}_{rob}F^{-1}_{rob}$$

In homogeneous form we have,

$$\begin{bmatrix} \Delta R_{left} \\ \Delta p_{left} \end{bmatrix} = \begin{bmatrix} R_{rob}^{goal} & p_{rob}^{goal} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{rob}^{goal} R^T & p_{rob}^{goal} - R_{rob}^{goal} R^T p \\ 0 & 1 \end{bmatrix}$$

Then we could derive the expression for α_{left} and ε_{left}

$$I + \alpha_{left} = R_{rob}^{goal} R^T$$

$$\varepsilon_{left} = p_{rob}^{goal} - R_{rob}^{goal} R^T p = p_{cart}^{goal} - R_{rob}^{goal} R^T p_{cart}$$

Applying Jacobian matrix derive above,

$$\begin{bmatrix} \alpha_{left} \\ \varepsilon_{left} \end{bmatrix} = J_{left} \Delta_q$$

Then we perform the same process for right hand side,

$$\Delta F_{right} = F_{rob}^{-1} F_{rob}^{goal}$$

In homogeneous form we have,

$$\begin{bmatrix} \Delta R_{right} \\ \Delta p_{right} \end{bmatrix} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{rob}^{goal} & p_{rob}^{goal} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T R_{rob}^{goal} & -R^T p + R^T p_{rob}^{goal} \\ 0 & 1 \end{bmatrix}$$

Then we could derive the expression for α_{left} and ε_{left}

$$I + \alpha_{right} = R^T R_{rob}^{goal}$$

$$\varepsilon_{right} = -R^T p + R^T p_{rob}^{goal} = R^T (p_{cart}^{goal} - p_{cart}) + (R_{goal} - R) p_{cr}$$

Applying Jacobian matrix derive above,

$$\begin{bmatrix} \alpha_{right} \\ \varepsilon_{right} \end{bmatrix} = J_{right} \Delta q$$

To move the robot to a nearby pose, we could derive Δq by solving an optimization problem. The normal error term for this optimization problem is to combine both hand sides.

$$\Delta q = \underset{\Delta q}{argmin} \left(\left|\left| J_{left} \Delta q - \begin{bmatrix} \alpha_{left} \\ \varepsilon_{left} \end{bmatrix} \right|\right| + \left|\left| J_{right} \Delta q - \begin{bmatrix} \alpha_{right} \\ \varepsilon_{right} \end{bmatrix} \right|\right| \right)$$

The left hand side expression and right hand side expression should be the same considering the correspondence. Relating the expression for $\Delta\theta$ and Δp we could rewrite a more elegant expression to solve the optimization problem.

$$\Delta\theta = \underset{\Delta\theta}{argmin} \left|\left| A_{left} \Delta\theta - (R_{rob}^{goal} R^T - I) \right|\right|$$

or,

$$\Delta\theta = \underset{\Delta\theta}{argmin} \left|\left| A_{right} \Delta\theta - (R^T R_{rob}^{goal} - I) \right|\right|$$

and,

$$\Delta p = p_{cart}^{goal} - p_{cart}$$

F. The redundancy of this robot shows in the rotation part, where,

$$\Delta q = A_{3\times 4}\Delta\theta$$

Let the A be written in column form,

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \end{bmatrix}$$

Know that each column is a vector of size 3 by 1, then A_4 could be linearly represented by other three column vector, as

$$0 = \alpha A_1 + \beta A_2 + \gamma A_3 + A_4 = A \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ -1 \end{bmatrix}$$

From part E, let the optimization problem gives out a feasible solution $\Delta\theta$, since θ is 4 degrees of freedom, which is redundant, we could always derive a feasible set of solution by introduction a coefficient k, so that,

$$\Delta q = A \left(\Delta\theta + k \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ -1 \end{bmatrix} \right)$$

Besides the optimization above, here we introduce another one for the constraint of mid-points.

$$k = \underset{k}{argmin} \left|\left| \theta + \Delta\theta + k \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ -1 \end{bmatrix} - \theta_{mid} \right|\right|$$

where,

$$\theta_{mid} = \theta_{upper} + \theta_{lower}$$

G. The optimization problem for this motion could be divided in several parts.

(1) move towards the goal pose
(2) joint positions as close to their midpoints
(3) move along a straight path as a uniform velocity
(4) rotate about some axis at a uniform angular velocity

The first and second constraints have already been derived in previous part, as,

move towards the goal pose

$$\Delta\theta = \underset{\Delta\theta}{argmin} \left|\left| A_{left}\Delta\theta - (R^{goal}_{rob} R^T - I) \right|\right|$$

or,

$$\Delta\theta = \underset{\Delta\theta}{argmin} \left|\left| A_{right}\Delta\theta - (R^T R^{goal}_{rob} - I) \right|\right|$$

and,

$$\Delta p = p_{rob}^{goal} - p$$

close to their midpoints

$$k = \underset{k}{argmin}\left|\left|\theta + \Delta\theta + k\begin{bmatrix}\alpha \\ \beta \\ \gamma \\ -1\end{bmatrix} - \theta_{mid}\right|\right|$$

move along a line with uniform velocity

Let the uniform velocity as v, the orientation of this line should be $p_{rob}^{goal} - p$. And given any motion, the orientation of this motion should be Δq. So that, we could derive the optimization problem as,

$$\Delta q = \underset{\Delta q}{argmin}[(p_{rob}^{goal} - p) - (p_{rob}^{goal} - p) \cdot \Delta q)\Delta q]$$

and the velocity constraint gives,

$$\dot{\Delta q} = v$$

rotate along a line with uniform velocity

Let the uniform angular velocity as w, the orientation of this line should can be derived as,

$$Rot(\alpha, \theta) = R^{goal}R^T = U$$

where $||\alpha|| = 1$, based on some robotics formula, we have,

$$\theta_{goal} = cos^{-1}(\frac{trace(U) - 1}{2})$$

$$\alpha_{goal} = \frac{1}{2sin\theta}\begin{bmatrix}U_{32} - U_{23} \\ U_{13} - U_{31} \\ U_{21} - U_{12}\end{bmatrix}$$

The actual rotation line of given $\Delta\theta$ should be,

$$\alpha = A_{left}\Delta\theta$$

$$\Delta\theta = \underset{\Delta\theta}{argmin}[(\alpha_{goal} - \alpha(\alpha_{goal} \cdot \alpha)) \times \alpha_{goal}]$$

and the velocity constraint gives,

$$\dot{\Delta\theta} = w$$

Question 2

A. We apply the composition rule to compute the transformation F_{rc} between the robot base coordinate system and the CT coordinate system.

$$F_{rc} = F_{r1}F_1^{-1}F_2F_{2c}$$

B. Present the desired tool tip position with respect to robot base, we have,

$$F_{rob}^{goal}F_{rt} = F_{rc}F_{tool}^{goal}$$

So that, the corresponding robot pose with respect to its base can be derived as,

$$F_{rob}^{goal} = F_{rc} F_{tool}^{goal} F_{rt}^{-1}$$

C. this part introduces some constraints for the optimization problem.

(1) the tool tip moves at speed s
(2) move in a straight line
(3) rotate about a uniform axis
(4) joint positions should stay as close to their midrange

To obtain a stable translational and rotational velocity, firstly of all we compute the uniform path from F to F_{goal}. Let the rotational axis be α with magnitude θ

$$F_{rob}^{goal} = R(\alpha, \theta) F_{rob}$$

Knowing that the moving speed is at s, we have the knowledge about the time.

$$T = |p_{tip}^{goal} - p_{tip}|/s$$

Then the vector form of linear velocity and angular velocity $\omega, v \in \mathcal{R}^3$ should be,

$$\omega = \theta/T, \ v = (p_{tip}^{goal} - p_{tip})/T$$

For small interval of time, we have,

$$\Delta p_{tip}/\Delta t = \omega \hat{\alpha} R(p_{tip} + p_{cr}) + \dot{p}_{cart}$$

which could be rewritten as,

$$\Delta p_{tip}/\Delta t = \begin{bmatrix} \hat{\alpha} R(p_{tip} + p_{cr}) & I_{3\times 3} \end{bmatrix} \dot{q} = v$$

Eventually we could give out the optimization with constraints for a small motion in Δt as,

$$\dot{q} = \underset{\dot{q}}{argmin} ||q + \dot{q}\Delta t - q_{mid}||^2$$

with constraints,
angular velocity:

$$\begin{bmatrix} A_{left} 0_{3\times 3} \end{bmatrix} \dot{q} = \omega \alpha$$

linear velocity:

$$\begin{bmatrix} \hat{\alpha} R(p_{tip} + p_{cr}) & I_{3\times 3} \end{bmatrix} \dot{q} = v$$

joint velocity:

$$\dot{q}_{min} \leq \dot{q} \leq \dot{q}_{max}$$

D. Besides the kinematics constraints listed above, we take forces exerted by the surgeons hand into account, which is given as,

$$\begin{bmatrix} \alpha_{right} \\ \varepsilon_{right} \end{bmatrix} = A \begin{bmatrix} f \\ \tau \end{bmatrix}$$

Relating the Jacobian matrix derived above, then the optimization problem can be written as,

$$\dot{q} = \underset{\dot{q}}{argmin} \left|\left| A \begin{bmatrix} f \\ \tau \end{bmatrix} - J_{right}\dot{q} \right|\right|^2$$

To sum up, the final expression should be,

$$\dot{q} = \underset{\dot{q}}{argmin} \zeta \left|\left| A \begin{bmatrix} f \\ \tau \end{bmatrix} - J_{right}\dot{q} \right|\right|^2 + \eta \left|\left| q + \dot{q}\Delta t - q_{mid} \right|\right|^2$$

where,

$$\zeta + \eta = 1$$

E. As the problem stated, the coordinates of the shape to be cut are bounded with respect to CT frame as,

$$x^2 + y^2 \leq \rho^2$$

$$-\psi \leq z \leq 0$$

The transformation from tool to the cylindrical shape can be derived as,

$$F_{tool}^{cyl} = F_{cyl}^{-1} F_{rc}^{-1} F_{rob} F_{tip}$$

Still we give out the optimization problem by following,

$$\dot{q} = \underset{\dot{q}}{argmin} \zeta \left|\left| A \begin{bmatrix} f \\ \tau \end{bmatrix} - J_{right}\dot{q} \right|\right|^2 + \eta \left|\left| q + \dot{q}\Delta t - q_{mid} \right|\right|^2$$

with,

$$\zeta + \eta = 1$$

We present the limitation of boundary by the constraints,
in direction x and y,

$$\left|\left| \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{cyl}^{-1} F_{rc}^{-1} p_{tip} \right|\right| \leq \rho^2$$

in direction z,

$$\left|\left| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{cyl}^{-1} F_{rc}^{-1} p_{tip} \right|\right| \geq -\psi$$

where, if we consider a small motion, we could present p_{tip} as,

$$p_{tip} = p_{tool} + v\Delta t$$

and according to what we have derived above, we could represent v by $\dot{q}$ as,

$$\begin{bmatrix} \hat{\alpha}R(p_{tip} + p_{cr}) & I_{3\times3} \end{bmatrix} \dot{q} = v$$

Besides, the tool axis should be aligned within 30 degrees of the axis of the cylinder, which means in geometrical description we should have,

$$p_z > |p|cos(\pi/6)$$

Relating the pose we derived above we have,

$$\left\| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{cyl}^{-1} F_{rc}^{-1} p_{tip} \right\| \geq |p_{tip}|cos(\pi/6)$$

F. To impose a tight bound on the shape cut, we should adjust the error bound in the pose constraint, so that each small interval the bound will get close to the upper bound.

$$\left\| \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} F_{cyl}^{-1} F_{rc}^{-1} p_{tip} \right\| \leq (\rho - e^{-k}\Delta\rho)^2$$

where, $\Delta\rho$ is a constant small term, and the sum should satisfy such equation,

$$\lim_{k\to\infty} (\rho - e^{-k}\Delta\rho) = \rho$$

G. We could realize the resistance within distance η by introducing a term in optimization expression.

$$\dot{q} = \underset{\dot{q}}{argmin} z_m(z_m - |z_m|)$$

where m refers to margin, if the margin of z coordinate is positive which means the surgeon's motion never reaches within the distance, the expression should have zero sum and has no resistance to the surgeon, while if the surgeon moves beyond the distance the sum is positive then we could see the effect of resistance by the optimization process.

Following we present how to get the expression of z_m.

$$z_m = z + \psi - \eta$$