Notes on flow analysis of heavy ion collision

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Abstract

Notes of how to exactly deduce flow observables.

1 $\mathbf{v}_n\{k\}$

In non-central collisions, the initial anisotropy of geometry can be convert into anisotropy in the momentum space of the final particle. The final one particle distribution can be decomposed with Fourier series, which is

$$\frac{EdN}{dp^3} \propto 1 + \sum v_n \cos(n(\phi - \psi_n)) \tag{1}$$

where v_n are expansion coefficients, ϕ are particle angle and ψ_n are n-order reference plane angle.

It's easy to deduce v_n, use orthogonality of trigonometric functions, we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \delta_{m,n}\pi \tag{2}$$

The probability that one final particle in ϕ_i bin $\propto 1 + \sum v_n \cos(n(\phi_i - \psi_n))$ .If we have m final particles in an event and the j-th particle has its angle ϕ_j, In $n - th$ Fourier order sum over all these partcles, we have

$$\sum_{j=1}^{m} \cos(n(\phi_j - \psi_n))(1 + \sum v_n \cos(n(\phi_j - \psi_n))) = \int_{-\pi}^{\pi} \cos(n(\phi - \psi_n))(1 + \sum v_n \cos(n(\phi - \psi_n))) \tag{3}$$

In Eq.(3) we ignore the size of ϕ bin which can be absorbed and assume that particles number is enough to change summation to integration. Use orthogonality in Eq.(4) and absorb π factor and divided by the number of particles to eliminate these factor's effect($N = \int \frac{EdN}{dp^3} dp^3$ and hence we can eliminate the proportional coefficient),we can get $v_n =< \cos(n(\phi - \psi_n)) >$ and here $< O >$ means particles average. And later we may use this notation to express event average. We can also define $V_n = v_n \exp(i\psi_n)$ and you can easily show that $V_n =< \cos n\phi >$.

It is often difficult to determine the position of the reaction plane in practices. So we usually use mult-particle correlation function to calculate v_n, which is $v_n\{k\}$ where k denote k-particle correlation.

We define $< 2 >=< \cos(n(\phi_1 - \phi_2)) >$ where ϕ_1 and ϕ_2 are different particles(thus 2 particle correlation). It can also be written into $< e^{in(\phi_1 - \phi_2)} >$ and more clearly

$$< 2 >=< e^{in(\phi_1 - \phi_2)} >= \frac{\sum_{i,j,i\neq j}^{n} e^{in(\phi_i - \phi_j)}}{\sum_{i,j,i\neq j}^{n} 1} = \frac{\sum_{i,j,i\neq j}^{n} e^{in(\phi_i - \phi_j)}}{n(n-1)} \tag{4}$$

The $\sin n(\phi_i - \phi_j)$ terms are eliminated in summation because it is an odd function and summation is performed on all particles. The condition that $i \neq j$ is aimed to Remove auto-selfcorrelation effects, which is important to do in heavy ion collision because we wants to research collective behavior rather than some specific particle.

Here<> means take average by using multiparticle joint distribution function, we can simply assume that $f(\phi_1, \phi_2, ..., \phi_n) = f(\phi_1)f(\phi_2)...f(\phi_n)$, i.e., independent identically distributed.

To generalized to K particles, let's first introduce some useful notion. First, we define

$$< m >_{n_1,n_2,...n_m} =< e^{i(n_{i1}\phi_{i1}+n_{i2}\phi_{i2}+...+n_{im}\phi_{im})} >= \frac{\sum_{i_1,i_2...i_m=1,i_1\neq i_2\neq ...\neq i_m}^{n} w_{i_1}w_{i_2}...w_{i_m}e^{i(n_{i_1}\phi_{i_1}+n_{i_2}\phi_{i_2}+...+n_{i_m}\phi_{i_m}}}{\sum_{i_1,i_2...i_m=1,i_1\neq i_2\neq ...\neq i_m}^{n} w_{i_1}w_{i_2}...w_{i_m}} \tag{5}$$

Here m mean m different particle used to do this correlation(here m in a number and in $< O >$ O is observable), n_p means the Fourier order of $p-th$ particle and w_p is the weight of $p-th$ particle(usually we use 1 like Eq.(4) does).

I'll show you an example. Imagine that we have total 10 particles and we want to calculate $< 3 >_{5,7,-9}$, using former equation, we have $< 3 >_{5,7,-9}= \frac{\sum_{i,j,k=1,ijk}^{10} e^{i(5\phi_i+7\phi_j-9\phi_k)}}{3*2*1}$.

On the other hand, we can connect $< m >_{n_1,n_2,...,n_m}$ to V_n. Using that $f(\phi_1, \phi_2, ..., \phi_n) = f(\phi_1)f(\phi_2)...f(\phi_n)$, we have $< m >_{n_1,n_2,...,n_m} =< e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)} >$ and

$$< e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)} >= \frac{e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)}f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m}{f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m} \tag{6}$$

at same time,

$$\frac{e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)}f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m}{f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m} = \frac{e^{in_1\phi_1}f(\phi_1)d\phi_1...e^{in_m\phi_m}f(\phi_m)d\phi_m}{f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m} \tag{7}$$

and

$$e^{in\phi}f(\phi)d\phi \propto e^{in\phi}(1 + \sum_n v_n \cos n(\phi - \psi_n))d\phi = v_n e^{in\psi_n} = V_n \tag{8}$$

So we have

$$< m >_{n_1,n_2,...n_m} = V_{n_1}V_{n_2}...V_{n_m} = v_{n_1}v_{n_2}...v_{n_m}e^{i(n_1\psi_{n_1}+n_2\psi_{n_2}+...+n_m\psi_{n_m})} \tag{9}$$

and we have $V_{-n_i} = V_{n_i}^*$ from V_n definition.

Once we have multiparticle correlation, we can further compute $v_n\{k\}$, it is event average of multiparticle correlation. We define

$$c_n\{2\} =<< 2 >_{n,-n}> \tag{10}$$

$$c_n\{4\} =<< 4 >_{n,-n}> -2* << 2 >_{n,-n}>^2 \tag{11}$$

$$v_n\{2\} = \sqrt{c_n\{2\}} \tag{12}$$

$$v_n\{4\} = \sqrt[4]{-c_n\{4\}} \tag{13}$$

There are two <> in Eq.(10) and Eq.(11). The inner one means particle average like Eq.(4) does, it just taking average of choosen particles in one event, the outside one is event average which means take average of all events once we get $< m >_{n_1,n_2,...,n_m}$ of each event.

To calculate $< m >_{n_1,n_2,...,n_m}$ we usually use Q-cumulants method which define

$$q_n = \sum_{i=1}^{k} e^{in\phi_k} \tag{14}$$

Then $< m >_{n_1,n_2,...,n_m}$ is just some of their product subtract the self-associated value. For example, $< 2 >_{n,-n}$

$$< 2 >_{n,-n} = \frac{q_n^* q_n - M}{M(M-1)} \tag{15}$$

where M is the particle number we used to calculate $< 2 >_{n,-n}$, and $q_n^* q_n = \sum_{i,j=1}^{n} e^{in(\phi_i-\phi_j)}$, $M = \sum_{i,j=1,i=j}^{n} e^{in(\phi_i-\phi_j)}$, you can thus check Eq.(15) equals to Eq.(4). One of the advantage to use Q-cumulants method is that you don't need to do loop calculation.

knowledge on how to calculate flow by Q-cumulants, taking reference of [1][2] and python package[3][4].

2 $SC(m, n)$

Let's first define what is cumulants and know the difference between cumulants and correlation and moments.

For a random variable X, we can define its $n - order$ moment $< X^n >= \int x^n f(x)dx$ where x is the value of X and $f(x)$ is the distribution function. As for correlation function, it is defined on different random variable(or a random variable defined in different point, like two point correlation function in QFT $< \phi(x)\phi(y) >$ or spin correltaion function $< S(x)S(0) >$), one of these correlation function is covariance $cov(X, Y) =< \delta X\delta Y >$.

Cumulants is another thing. Consider a random variable X whose distribution function is $f(x)$. This function can be used to get characteristic Function $\psi(x)$. We define that $\ln \psi(x) = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} < x^n >_c$ and $< x^n >_c$ is $n - order$ cumulants,$< X >_c$ is so-called cumulants average(You need'nt konw how it been calculated, we'll calculate cumulants in other way). We can also define cumulants in different random variables thus we can deduce relation between cumulants and correlation function[5]. Here we use notion in[6], We have

$$Cum(\{n\}) = \sum_{l=1}^{n}(l-1)!(-1)^{l-1} \sum_{\sum_{i=1}^{l}\{m_i\}=\{n\}} \Pi_{i=1}^{l} Corr(m_i) \tag{16}$$

here $Cum(\{n\}) =< X_1X_2...X_n >_c$ where X_i is random variable. l denote the portion we split these n variables into and it is trivial from 1 to n. After we make sure how many portion we want to split them into, there are many ways to do that and the second summation is to sum up these ways and in each way we need to multiple their correlation, which is the final product need to do. For example $< X_1X_2X_3 >_c$, when $l = 1$, we have $< X_1X_2X_3 >$, when $l = 2$, we have $- < X_1 >< X_2X_3 >$,$- < X_2 >< X_1X_3 >$ and $- < X_3 >< X_1X_2 >$, when $l = 3$, we have $2 < X_1 >< X_2 >< X_3 >$, thus

<X$_1X_2X_3 >_c =< X_1X_2X_3 > - < X_1 >< X_2X_3 > - < X_2 >< X_1X_3 > - < X_3 >< X_1X_2 > +2 < X_1 >< X_2 >< X_3 >$ (17)

We can thus define

$$cum(n_1, n_2, ..., n_m) =<< e^{i(n_1\phi_1+n_2\phi_2+...+n_m\phi_m)} >>_c \tag{18}$$

For 4-particles[6], we have

$$cum(n_1, n_2, n_3, n_4) =<< e^{i(n_1\phi_1+n_2\phi_2+n_3\phi_3+n_4\phi_4)} >>_c \tag{19}$$

expand it, You can get

$$<< e^{i(n_1\phi_1+n_2\phi_2+n_3\phi_3+n_4\phi_4)} >> \tag{20}$$

and

$$- << e^{i(n_1\phi_1+n_2\phi_2+n_3\phi_3)} >><< e^{in_4\phi_4} >> -(\text{rotate} \quad \text{term}) \tag{21}$$

and

$$- << e^{i(n_1\phi_1+n_2\phi_2)} >><< e^{in_3\phi_3+n_4\phi_4} >> -(\text{rotate} \quad \text{term}) \tag{22}$$

and

$$2* << e^{in_1\phi_1)} >><< e^{in_2\phi_2} >><< e^{i(n_3\phi_3+n_4\phi_4)} >> +(\text{rotate} \quad \text{term}) \tag{23}$$

and

$$- 6* << e^{in_1\phi_1} >><< e^{in_2\phi_2} >><< e^{in_3\phi_3} >><< e^{in_4\phi_4} >> \tag{24}$$

the inner backet is particles average and outer means event average. We should know that if we take particle average like $< e^{in_1\phi_1} >$ we get $V_{n_1} = v_{n_1}e^{i\psi_{n_1}}$ and ψ_{n_1} is the event plane of this event. However, event plane is random for each event so when we take outer average(event average), it turns to the zeros, i.e., $<< e^{in_1\phi_1} >>= 0$. So Eq.(21),Eq.(23),Eq.(24) vanish. As for $<< e^{i(n_1\phi_1+n_2\phi_2+...+n_m\phi_m)} >>=< v_{n_1}e^{i\psi_{n_1}}v_{n_2}e^{i\psi_{n_2}}...v_{n_m}e^{i\psi_{n_m}} >$ term, the only way to keep it non-zero is to make $n_1 + n_2 + ... + n_m = 0$ which lead the production of exp function turns to 1.

We can define $SC(n, m)$ now

$$SC(n, m) = cum(n, m, -n, -m) \tag{25}$$

the order of these do not make sense and we can get

$$SC(n, m) =<< e^{i(n\phi_1+m\phi_2-n\phi_3-m\phi_4)} >> - << e^{i(n\phi_1-n\phi_3)} >><< e^{i(m\phi_2-m\phi_4)} >>=< v_n^2v_m^2 > - < v_n^2 >< v_m^2 > \tag{26}$$

here we denote $v_n^2 = V_n^*V_n =< e^{i(n\phi_1-n\phi_3)} >$, but remember to subtract self-correlation. It's easy for you to calculate term like $< e^{i(n\phi_1+m\phi_2-n\phi_3-m\phi_4)} >$ or $< e^{i(n\phi_1-n\phi_3)} >$ using Q-method as we showed before.

References

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A Appendix