Notes on flow analysis of heavy ion collision

Qi Liu
Peking University, Beijing
2101110100@stu.pku.edu.cn

Abstract

Notes of how to exactly deduce flow observables.

1 $\mathbf{v}_n\{k\}$

In non-central collisions, the initial anisotropy of geometry can be convert into anisotropy in the momentum space of the final particle. The final one particle distribution can be decomposed with Fourier series, which is

$$\frac{EdN}{dp^3} \propto 1 + \sum v_n \cos(n(\phi - \psi_n)) \tag{1}$$

where v_n are expansion coefficients, ϕ are particle angle and ψ_n are n-order reference plane angle.

It's easy to deduce v_n, use orthogonality of trigonometric functions, we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \delta_{m,n}\pi \tag{2}$$

The probability that one final particle in ϕ_i bin $\propto 1 + \sum v_n \cos(n(\phi_i - \psi_n))$ .If we have m final particles in an event and the j-th particle has its angle ϕ_j, In $n-th$ Fourier order sum over all these partcles, we have

$$\sum_{j=1}^{m} \cos(n(\phi_j - \psi_n))(1 + \sum v_n \cos(n(\phi_j - \psi_n))) = \int_{-\pi}^{\pi} \cos(n(\phi - \psi_n))(1 + \sum v_n \cos(n(\phi - \psi_n))) \tag{3}$$

In Eq.(3) we ignore the size of ϕ bin which can be absorbed and assume that particles number is enough to change summation to integration. Use orthogonality in Eq.(4) and absorb π factor and divided by the number of particles to eliminate these factor's effect($N = \int \frac{EdN}{dp^3} dp^3$ and hence we can eliminate the proportional coefficient),we can get $v_n =< \cos(n(\phi - \psi_n)) >$ and here $< O >$ means particles average. And later we may use this notation to express event average. We can also define $V_n = v_n \exp(i\psi_n)$ and you can easily show that $V_n =< \cos n\phi >$.

It is often difficult to determine the position of the reaction plane in practices. So we usually use mult-particle correlation function to calculate v_n, which is $v_n\{k\}$ where k denote k-particle correlation.

We define $< 2 >=< \cos(n(\phi_1 - \phi_2)) >$ where ϕ_1 and ϕ_2 are different particles(thus 2 particle correlation). It can also be written into $< e^{in(\phi_1 - \phi_2)} >$ and more clearly

$$< 2 >=< e^{in(\phi_1 - \phi_2)} >= \frac{\sum_{i,j,i\neq j}^{n} e^{in(\phi_i - \phi_j)}}{\sum_{i,j,i\neq j}^{n} 1} = \frac{\sum_{i,j,i\neq j}^{n} e^{in(\phi_i - \phi_j)}}{n(n-1)} \tag{4}$$

The $\sin n(\phi_i - \phi_j)$ terms are eliminated in summation because it is an odd function and summation is performed on all particles. The condition that $i \neq j$ is aimed to Remove auto-selfcorrelation effects, which is important to do in heavy ion collision because we wants to research collective behavior rather than some specific particle.

Here$<>$ means take average by using multiparticle joint distribution function, we can simply assume that $f(\phi_1, \phi_2, ..., \phi_n) = f(\phi_1)f(\phi_2)...f(\phi_n)$, i.e., independent identically distributed.

To generalized to K particles, let's first introduce some useful notion. First, we define

$$< m >_{n_1,n_2,...n_m} =< e^{i(n_{i1}\phi_{i1}+n_{i2}\phi_{i2}+...+n_{im}\phi_{im})} >= \frac{\sum_{i_1,i_2...i_m=1,i_1\neq i_2\neq...\neq i_m}^{n} w_{i_1}w_{i_2}...w_{i_m} e^{i(n_{i_1}\phi_{i_1}+n_{i_2}\phi_{i_2}+...+n_{i_m}\phi_{i_m})}}{\sum_{i_1,i_2...i_m=1,i_1\neq i_2\neq...\neq i_m}^{n} w_{i_1}w_{i_2}...w_{i_m}} \tag{5}$$

Here m mean m different particle used to do this correlation(here m in a number and in $< O >$ O is observable), n_p means the Fourier order of $p-th$ particle and w_p is the weight of $p-th$ particle(usually we use 1 like Eq.(4) does).

I'll show you an example. Imagine that we have total 10 particles and we want to calculate $< 3 >_{5,7,-9}$, using former equation, we have $< 3 >_{5,7,-9}= \frac{\sum_{i,j,k=1,ijk}^{10} e^{i(5\phi_i+7\phi_j-9\phi_k)}}{3*2*1}$.

On the other hand, we can connect $< m >_{n_1,n_2,...,n_m}$ to V_n. Using that $f(\phi_1, \phi_2, ..., \phi_n) = f(\phi_1)f(\phi_2)...f(\phi_n)$, we have $< m >_{n_1,n_2,...,n_m} =< e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)} >$ and

$$< e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)} >= \frac{e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)} f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m}{f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m} \tag{6}$$

at same time,

$$\frac{e^{i(n_1\phi_1+n_2\phi_2+...n_m\phi_m)} f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m}{f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m} = \frac{e^{in_1\phi_1} f(\phi_1)d\phi_1...e^{in_m\phi_m} f(\phi_m)d\phi_m}{f(\phi_1, \phi_2, ..., \phi_m)d\phi_1 d\phi_2...d\phi_m} \tag{7}$$

and

$$e^{in\phi} f(\phi)d\phi \propto e^{in\phi}(1 + \sum_n v_n \cos n(\phi - \psi_n))d\phi = v_n e^{in\psi_n} = V_n \tag{8}$$

So we have

$$< m >_{n_1,n_2,...n_m} = V_{n_1}V_{n_2}...V_{n_m} = v_{n_1}v_{n_2}...v_{n_m} e^{i(n_1\psi_{n_1}+n_2\psi_{n_2}+...+n_m\psi_{n_m})} \tag{9}$$

and we have $V_{-n_i} = V_{n_i}^*$ from V_n definition.

Once we have multiparticle correlation, we can further compute $v_n\{k\}$, it is event average of multiparticle correlation. We define

$$c_n\{2\} =<< 2 >_{n,-n}> \tag{10}$$

$$c_n\{4\} =<< 4 >_{n,-n}> -2* << 2 >_{n,-n}>^2 \tag{11}$$

$$v_n\{2\} = \sqrt{c_n\{2\}} \tag{12}$$

$$v_n\{4\} = \sqrt[4]{-c_n\{4\}} \tag{13}$$

There are two $<>$ in Eq.(10) and Eq.(11). The inner one means particle average like Eq.(4) does, it just taking average of choosen particles in one event, the outside one is event average which means take average of all events once we get $< m >_{n_1,n_2,...,n_m}$ of each event.

To calculate $< m >_{n_1,n_2,...,n_m}$ we usually use Q-cumulants method which define

$$q_n = \sum_{i=1}^{k} e^{in\phi_k} \tag{14}$$

Then $< m >_{n_1,n_2,...,n_m}$ is just some of their product subtract the self-associated value. For example, $< 2 >_{n,-n}$

$$< 2 >_{n,-n}= \frac{q_n^* q_n - M}{M(M-1)} \tag{15}$$

where M is the particle number we used to calculate $< 2 >_{n,-n}$, and $q_n^* q_n = \sum_{i,j=1}^{n} e^{in(\phi_i - \phi_j)}$, $M = \sum_{i,j=1,i=j}^{n} e^{in(\phi_i - \phi_j)}$, you can thus check Eq.(15) equals to Eq.(4). One of the advantage to use Q-cumulants method is that you don't need to do loop calculation.

More knowledge on how to calculate flow by Q-cumulants, taking reference of [1][2] and python package[3][4].

References

[1] Ante Bilandzic, Raimond Snellings, and Sergei Voloshin. Flow analysis with cumulants: Direct calculations. *Physical Review C*, 83(4):044913, 2011. 3

[2] Ante Bilandzic. *Anisotropic flow measurements in ALICE at the large hadron collider*. 2012. 3

[3] DukeQCD group. hic. `https://github.com/Duke-QCD/hic`. 3

[4] `https://github.com/Liuqi19990113/HIC-observables/blob/master/HICO/hic_flow.py`. 3

A Appendix