Homework 8 solutions.

1. Minimise $\Delta = \sum_{i=1}^{n} (Y_i - \alpha)^2$

$$\frac{\partial \Delta}{\partial \alpha} = -\sum_{i=1}^{n} 2(Y_i - \alpha) = 0$$

$$\Rightarrow \sum_{i=1}^{n} Y_i - n \cdot \hat{\alpha} = 0.$$

$$\Rightarrow \hat{\alpha} = \frac{\sum Y_i}{n} = \bar{Y}.$$

which is the same as the method of moments.

2. $L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y_i - \hat{y}_i}{\sigma}\right)^2}$

maximising $L(\theta)$ is equivalent to minimising

$$\sum \left(\frac{y_i - \hat{y}_i}{\sigma}\right)^2$$

or minimising $\sum (y_i - \hat{y}_i)^2$

which is the same as least squares.

3. Minimise $\Delta = \sum_{i=1}^{n} (Y_i - \beta x_i)^2$

$$\frac{\partial \Delta}{\partial \beta} = -2 \sum_{i=1}^{n} x_i (y_i - \beta x_i) = 0$$

$$\Rightarrow \quad \sum x_i y_i - \hat{\beta} \sum x_i^2 = 0$$

$$\Rightarrow \quad \hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$E(\hat{\beta}) = \frac{1}{\sum x_i^2} \cdot \sum x_i E(Y_i)$$

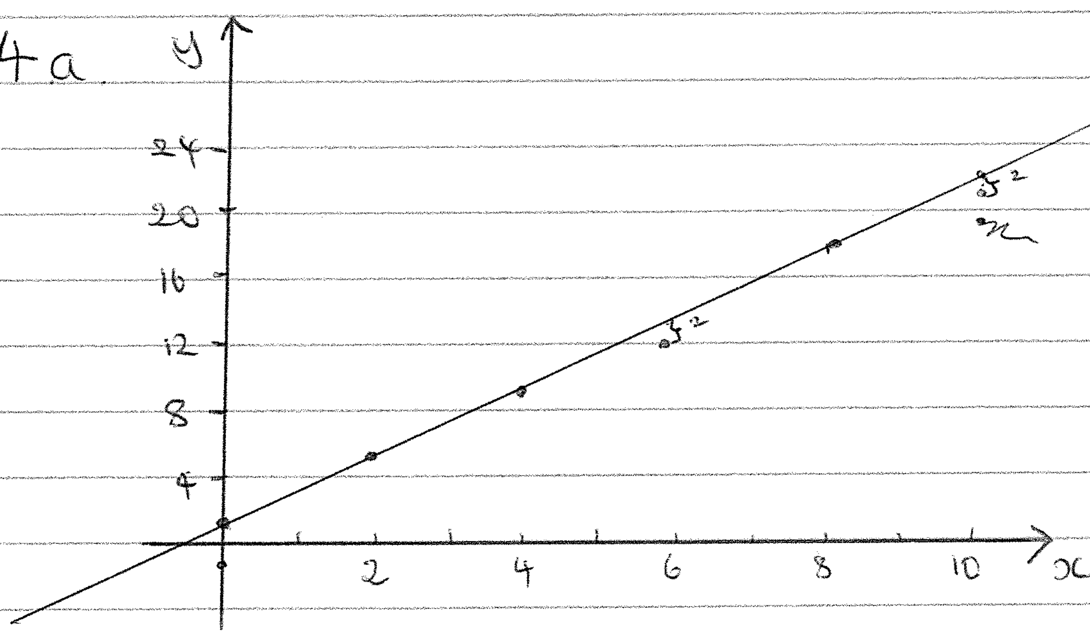
$$= \frac{1}{\sum x_i^2} \sum x_i \cdot x_i \beta$$

$$= \beta. \quad \checkmark$$

$$Var(\hat{\beta}) = \frac{1}{(\sum x_i^2)^2} \cdot \sum x_i^2 \cdot Var(Y_i)$$

$$= \frac{\sigma^2}{\sum x_i^2}.$$

4a



b. note $\bar{x} = 6$

$$\bar{y} = \frac{5+9+12+18+21}{6} = \frac{65}{5} = 13$$

x	2	4	6	8	10
u	-4	-2	0	2	4
y	5	9	12	18	21

$$\hat{\beta} = \frac{\sum u_i y_i}{\sum u_i^2}$$

$$= \frac{-20-18+36+84}{16+4+4+16} = \frac{82}{40}$$

$$= \frac{41}{20}$$

$$\hat{\alpha} = \bar{y} - \beta \bar{x}$$

$$\cancel{= 1 - \frac{40}{24} \cdot 6 \; \frac{41}{20} \cdot 6}$$

$$\cancel{= -\frac{22}{15} \approx -1.47}$$

$$= 13 - \frac{41}{20} \cdot 6$$

$$= 0.7$$

c. squared distances from line to points.

9.6.3. $\hat{\alpha} \pm t_{n-2}^{1-\alpha/2} \cdot \frac{S}{\sqrt{n}}$

$\hat{\beta} \pm t_{n-2}^{1-\alpha/2} \cdot \frac{S^2}{\sum(x_i-\bar{x})^2}$

9.6.4.

a. $\hat{\eta}_0 \overset{a}{\sim} N\left(\alpha+\beta(x_0-\bar{x}),\ \frac{\sigma^2}{n} + (x_0-\bar{x})^2 \cdot \frac{S^2}{\sum(x_i-\bar{x})^2}\right)$

b. $\hat{\eta}_0 \pm t_{n-2}^{1-\alpha/2} \sqrt{\frac{S^2}{n} + (x_0-\bar{x})^2 \cdot \frac{S^2}{\sum(x_i-\bar{x})^2}}$

9.6.6. $\sum\left[Y_i - \alpha_0 - \beta(x_i-\bar{x})\right]^2$

$= \sum\left[Y_i - \hat{\alpha}_0 - \hat{\beta}(x_i-\bar{x}) + (\hat{\alpha}_0-\alpha) + (\hat{\beta}-\beta)(x_i-\bar{x})\right]^2$

$= \sum\left[Y_i - \hat{\alpha}_0 - \hat{\beta}(x_i-\bar{x})\right]^2 + n(\hat{\alpha}_0-\alpha) + (\hat{\beta}-\beta)^2\sum(x_i-\bar{x})^2$

$+ 2\cdot\sum\left[Y_i - \hat{\alpha}_0 - \hat{\beta}(x_i-\bar{x})\right](\hat{\alpha}_0-\alpha)$ — 1

$+ 2\sum\left[Y_i - \hat{\alpha}_0 - \hat{\beta}(x_i-\bar{x})\right](\hat{\beta}-\beta)(x_i-\bar{x})$ — 2

$+ 2\cdot\sum(\hat{\alpha}_0-\alpha)(\hat{\beta}_0-\beta)(x_i-\bar{x})$ — 3

1 $= 2(\hat{\alpha}_0-\alpha)\cdot\sum\left(Y_i - \bar{y} - \hat{\beta}\cdot(x_i-\bar{x})\right)$

note $\sum(Y_i-\bar{y})=0 \quad \sum(x_i-\bar{x})=0.$

$\Rightarrow$ 1 $= 0.$

$$③ = 2\cdot(\hat{\alpha}-\alpha)(\hat{\beta}-\beta)\underbrace{\sum(x_i-\bar{x})}_{=0}$$

$$② = 2\cdot\sum(y_i-\bar{y})(x_i-\bar{x})(\hat{\beta}-\beta)$$
$$-2\sum(\hat{\beta}-\beta)\cdot\hat{\beta}(x_i-\bar{x})^2$$

$$= 2\cdot(\hat{\beta}-\beta)\cdot\sum\left[(y_i-\bar{y})(x_i-\bar{x})-\left[\frac{\sum(x_j-\bar{x})y_j}{\sum(x_j-\bar{x})^2}\right](x_i-\bar{x})^2\right]$$

$$= 2\cdot(\hat{\beta}-\beta)\sum\left[(y_i-\bar{y})(x_i-\bar{x})-y_i(x_i-\bar{x})\right]$$

$$= 2(\hat{\beta}-\beta)\sum-\bar{y}(x_i-\bar{x})$$
$$= 0$$

9.6.10. $$\operatorname{Cov}(\hat{\alpha}_c,\hat{\beta}) = \operatorname{cov}\left(\frac{1}{n}\sum Y_i,\ \frac{1}{\sum u_i^2}\sum u_iY_i\right)$$

$$= \frac{1}{n\sum u_i^2}\operatorname{Cov}\left(\sum Y_i,\sum u_iY_i\right)$$

$$= \frac{1}{n\sum u_i^2}\cdot\sum u_i\operatorname{cov}(Y_i,Y_i)$$

$$= \frac{\sigma^2}{n\sum u_i^2}\cdot\underbrace{\sum u_i}_{=0}$$

$$= 0.$$

9.6.12 $\Delta(a) = \sum (y_i - (a+x_i))^2$

$$= (1-a)^2 + (3-(1+a))^2 + (4-(2+a))^2$$
$$= (1-a)^2 + (2-a)^2 + (2-a)^2$$
$$= 1-2a+a^2 + 4-4a+a^2 + 4-4a+a^2$$
$$= 9 - 10a + 3a^2$$

$$\frac{\partial \Delta(a)}{\partial a} = -10 + 6a = 0.$$

$$\Rightarrow \hat{a} = 10/6 = 5/3$$