Homework 5 solutions

1a First apply inverse transform to Y

$$x = \frac{2}{\pi}\arcsin(\sqrt{y})$$

$$\frac{x\pi}{2} = \arcsin(\sqrt{y})$$

$$y = \left(\sin\frac{x\pi}{2}\right)^2$$

$\Rightarrow$ To simulate Y, use $\left(\sin\left(\frac{U\pi}{2}\right)\right)^2$ where $U \sim R[0,1]$

Now we use AR to generate X.

$$M = \max_x \frac{f_X(x)}{f_Y(x)}$$

$$= \max_{x\in[0,1]} \frac{\dfrac{\sin x}{1.41\sqrt{x(1-x)}}}{\dfrac{1}{\pi\sqrt{x(1-x)}}}$$

$$= \max_{x\in[0,1]} \frac{\pi \sin x}{1.41}$$

$$= \frac{\pi \sin 1}{1.41} \approx 1.87$$

$\Rightarrow$ To generate an obs of X, then simulate Y^* and accept as X^* if

$$\frac{f_X(Y^*)}{M f_Y(Y^*)} \geq U^*, \quad U^* \sim R[0,1]$$

Continue until an acceptance.

b. $P\left(U \le \frac{f_X(Y)}{Mf_Y(Y)}\right)$

$$= \int_0^1 P\left(U \le \frac{f_X(y)}{Mf_Y(y)}\right) \cdot f_Y(y)\,dy \quad \text{(LTP)}$$

$$= \int_0^1 \frac{f_X(y)}{Mf_Y(y)} \cdot f_Y(y)\,dy$$

$$= \frac{1}{M}\int_0^1 f_X(y)\,dy = \frac{1}{M} \approx 0.53.$$

2. If Y is sufficient, then

$$f_X(x_1|\theta) \dots f_{X_n}(x_n|\theta) = k_1(y|\theta)k_2(x_1, \dots, x_n)$$

| let $Z = \phi(Y) \Rightarrow Y = \phi^{-1}(Z)$

$$= k_1(\phi^{-1}(Z)|\theta)k_2(x_1, \dots, x_n)$$

$\Rightarrow$ By factorisation thm as we have a function of Z & θ still, $\Rightarrow Z$ is sufficient.

$\Rightarrow$ Given there are infinite one-to-one functions, hence infinitly many suff stats.

4.8.6. We know $F^{-1}(U) \stackrel{d}{=} X \quad (U \sim R[0,1])$

Need to find F^{-1}.

$$F(x) = \int_0^x 4y^4\, dy$$
$$= x^4$$
$$\Rightarrow F^{-1}(x) = x^{-\frac{1}{4}}$$

$$\Rightarrow U^{1/4} \overset{d}{=} X$$

4.8.9 As usual we seek $F^{-1}(U) \overset{d}{=} X$.

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+y^2)}\, dy$$

$$= \frac{1}{\pi}\left[\arctan x + \frac{\pi}{2}\right]$$

$$y \cdot \pi = \arctan x + \pi/2$$
$$x = \tan(\pi \cdot y - \pi/2)$$

$$\Rightarrow \tan(\pi \cdot U - \pi/2) \overset{d}{=} X.$$

4.8.19. We know how to generate Cauchy from ↑

Just need

$$M = \max_x \frac{f(x)}{g(x)}$$

$$> \max_x \frac{\frac{\Gamma(\frac{r+1}{2})}{\sqrt{(r\pi)}\cdot\Gamma(r/2)} \cdot \left(1 + \frac{x^2}{r}\right)^{-\left(\frac{r+1}{2}\right)}}{\frac{1}{\pi(1+x^2)}}$$

$$= \max_x \, C\,(1+x^2)\left(1+\frac{x^2}{r}\right)^{-\left(\frac{r+1}{2}\right)}$$

max is at $x = \pm 1$

$$\Rightarrow \quad M = C \cdot 2 \cdot \left(1+\frac{1}{r}\right)^{-\left(\frac{r+1}{2}\right)}.$$

$\Rightarrow$ Accept an obs from Y (Cauchy) if

$$U \leq \frac{f(Y)}{Mg(Y)}$$

7.2.1. $f(x_1|\theta), \ldots f(x_n|\theta)$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2}\frac{x_i^2}{\theta}}$$

$$= \underbrace{(2\pi)^{-n/2}}_{k_2} \cdot \underbrace{\theta^{-n/2} e^{-\frac{1}{2}\frac{\sum x_i^2}{\theta}}}_{k_1(\sum x_i^2|\theta)}$$

$\Rightarrow \quad \sum x_i^2$ is sufficient.

7.2.3. $f(x_1|\theta) \ldots f(x_n|\theta)$

$$= \frac{1}{\theta^n} \cdot \mathbb{1}_{\{0 \leq x_1 \leq \theta, \ldots, 0 \leq x_n \leq \theta\}}$$

$$= \frac{1}{\theta^n} \mathbb{1}_{\{x_{(1)} \geq 0\}} \mathbb{1}_{\{x_{(n)} \leq \theta\}}$$

$$= \underbrace{\frac{1}{\theta^n} \mathbb{1}_{\{x_{(n)} \le \theta\}}}_{k_1(x_{(n)}|\theta)} \underbrace{\mathbb{1}_{\{x_{(1)} \ge 0\}}}_{k_2}$$

$\Rightarrow X_{(n)}$ is sufficient.

$f(x_1|\theta) \ldots f(x_n|\theta)$

$$= Q(\theta)^n \prod M(x_i) \cdot \mathbb{1}_{\{x_{(1)} \ge 0\}} \mathbb{1}_{\{x_{(n)} \le \theta\}}$$

$$= \underbrace{Q(\theta)^n \mathbb{1}_{\{x_{(n)} \le \theta\}}}_{k_1(x_{(n)}|\theta)} \underbrace{\prod M(x_i) \cdot \mathbb{1}_{\{x_{(1)} \ge 0\}}}_{k_2}$$

$\Rightarrow X_{(n)}$ still sufficient.

7.24. $f(x_1|\theta) \ldots f(x_n|\theta)$

$$= \underbrace{\frac{1}{\theta^n} \cdot e^{-\sum x_i/\theta}}_{k_1(\sum x_i|\theta)} \underbrace{1}_{k_2}$$

$\Rightarrow \sum X_i$ is suff.

7.2.6. $f(x_1|\theta) \ldots f(x_n|\theta)$

$= \cancel{\Pi \frac{1}{\Gamma(\theta+1)}}$

$= \Pi \frac{1}{\Gamma(\theta+5)} x_i^{\theta+5-1} e^{-x_i}$

$$= \underbrace{\frac{1}{\Gamma(\theta+5)^n} (x_1 \ldots x_n)^{\theta+4}}_{k_1(x_1 \ldots x_n|\theta)} \cdot \underbrace{e^{-\Sigma x_i}}_{k_2}$$

$\Rightarrow$ $X_1 \ldots X_n$ is sufficient.