Homework 9 Solutions

1a. $y = -8.366 + .95x$

or $x = 37.2668 + 0.566y$

b. See attached.

c. $\frac{R\sqrt{(n-2)}}{\sqrt{1-R^2}} \overset{d}{=} t_{n-2}$ under $H_0: \rho = 0$.

$\hat{R} = 0.734 \quad n = 10$

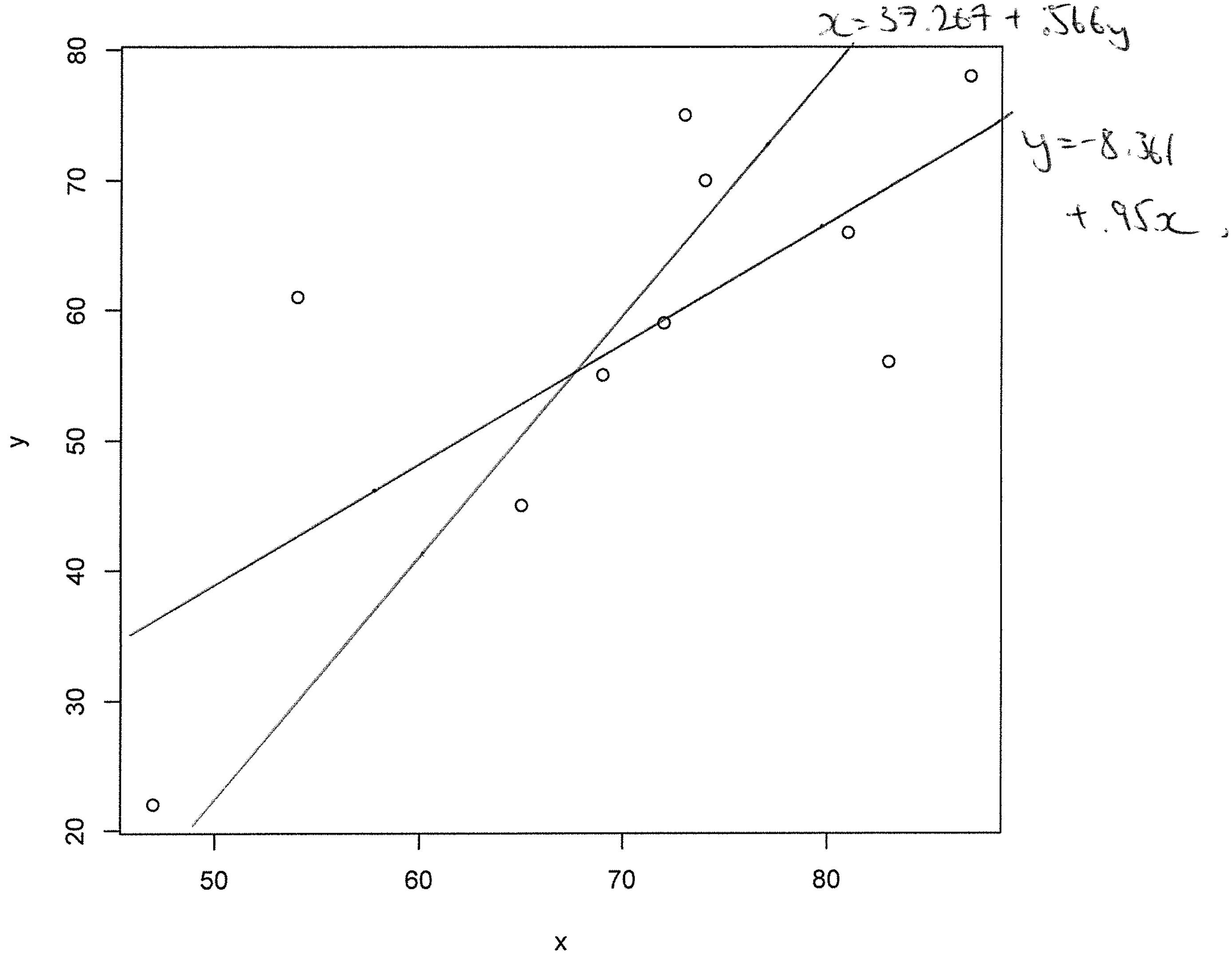
$\frac{\hat{R}\sqrt{n-2}}{\sqrt{1-\hat{R}^2}} = 3.056.$

$\Rightarrow 2P(t_8 > 3.056) = 0.0156$

$\Rightarrow$ Reject H_0.

2a. $\begin{pmatrix} 4 \\ 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \varepsilon$

$A^TA = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 15 \end{pmatrix}$



x = 37.267 + .566y
y = -8.361 + .95x
80
70
60
50
40
30
20
y
50
60
70
80
x

$$(A^T A)^{-1} = \frac{1}{6\cdot 15 - 3\cdot 3} \cdot \begin{pmatrix} 15 & -3 \\ -3 & 6 \end{pmatrix}$$

$$= \frac{1}{27} \cdot \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (A^T A)^{-1} A^T y$$

$$= \frac{1}{27} \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 5 \\ 3 \end{pmatrix}$$

$$= \frac{1}{27} \cdot \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 15 \\ 18 \end{pmatrix}$$

$$= \frac{1}{27} \begin{pmatrix} 57 \\ 21 \end{pmatrix} \approx \begin{pmatrix} 2.11 \\ 0.78 \end{pmatrix}.$$

$$s^2 = \frac{1}{n-p} \cdot \left(y^T y - \hat{\theta}^T A^T y \right)$$

$$= \frac{1}{2} \cdot \left(66 - \begin{pmatrix} 19/9 \\ 7/9 \end{pmatrix}^T \begin{pmatrix} 15 \\ 18 \end{pmatrix} \right)$$

$$= \frac{1}{2} \cdot (66 - 45.67)$$

$$= 10.17$$

b. $E(Y_1) = \alpha + 2\beta$

$$\hat{\alpha} + 2\hat{\beta} = \frac{57 + 42}{27} = 11/3.$$

note: $\mathrm{Var}(\hat{\alpha}+2\hat{\beta}) = \mathrm{Var}(\hat{\alpha}) + 2^2\,\mathrm{Var}(\hat{\beta}) + 4\,\mathrm{Cov}(\hat{\alpha},\hat{\beta})$

recall: $(A^TA)^{-1} = \frac{1}{27}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$

$\Rightarrow \mathrm{Var}(\hat{\alpha}) = 5/27\,\sigma^2 \quad \mathrm{Var}(\hat{\beta}) = 2/27\,\sigma^2$

$\mathrm{Cov}(\hat{\alpha},\hat{\beta}) = -\frac{1}{27}\sigma^2.$

$\Rightarrow$ CI: $11/3 \pm t_2^{.975}\cdot\sqrt{\frac{1}{3}\cdot 10.17}$

$= 11/3 \cdot 4.30 \cdot \sqrt{10.17/3}$

$= (-4.25,\ 11.58)$

3a. $y_{ij} = \mu + \alpha_i + e_{ij}$

$$A = \underbrace{\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ & & \ddots & & \\ 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}}_{k} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} n \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} n \end{array}$$

b. $$A^TA = \begin{pmatrix} 1 & 1 & \cdots & & & & 1 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 & 0 \; \cdots \\ 0 & \cdots & & \cdots & 0 & 1 \cdots 1 & 0 \\ & & & \vdots & & & \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ \vdots & & & & \\ 1 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} nk & n & n & \cdots & n \\ n & n & & & \\ \vdots & & \ddots & & O \\ \vdots & & O & \ddots & \\ n & & & & n \end{pmatrix}$$

not full rank: $R_1 = R_2 + \cdots + R_k$.

Suggest: $\mu = 0$.
or $\sum \alpha_i = 0$
or $\alpha_1 = 0$.
Anything is fine really

c. I'm going to assume $\mu = 0$.

$$\Rightarrow A^TA = \left.\begin{pmatrix} n & & 0 \\ & \ddots & \\ 0 & & n \end{pmatrix}\right\} k$$

$$A^T y = \begin{pmatrix} \sum_j y_{1j} \\ \sum_j y_{2j} \\ \vdots \\ \sum_j y_{Lj} \end{pmatrix}$$

$$\Rightarrow \hat{\underset{\sim}{\theta}} = \begin{pmatrix} \sum_j y_{1j}/n \\ \vdots \\ \sum_j y_{Lj}/n \end{pmatrix}$$

c.

$$\sigma^2 (A^T A)^{-1} = \begin{pmatrix} \sigma^2/n & & 0 \\ & \ddots & \\ 0 & & \sigma^2/n \end{pmatrix}$$

4a.

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix}$$

b.

$$A^T A = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ x_1^2 & \cdots & x_n^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_1^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix}$$

$$= \begin{pmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix}$$

$$A^T y = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

$$\hat{\theta} = \begin{pmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

c. $D(\hat{\theta}) = \sigma^2 (A^T A)^{-1}$

$$= \sigma^2 \cdot \begin{pmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix}^{-1}$$

9.6.11

a. $\hat{\theta} = \underset{\theta}{\mathrm{Argmin}} \; \| Y - \theta \|^2$

$$= \underset{\alpha, \beta}{\mathrm{Arg\,min}} \left(\sum y_i - (\alpha + \beta x_i) \right)^2$$

$\Rightarrow$ Same as LS.

b. $e_i = y_i - \hat{\theta}_i$

$= y_i - (\hat{\alpha} + \hat{\beta}x_i) = y_i - (\hat{\alpha}_0 + \hat{\beta}u_i)$

$= y_i - \hat{y}_i$ ✓

c. want to show that

$$\hat{\theta}^T \hat{e} = 0$$

$\hat{\underset{\sim}{\theta}}^T \hat{\underset{\sim}{e}} = \sum_i (\hat{\alpha} + \hat{\beta}u_i)(y_i - (\hat{\alpha} + \hat{\beta}u_i))$

$= \hat{\alpha} \cdot \sum y_i - n \cdot \hat{\alpha}^2 - \hat{\alpha}\hat{\beta}\sum u_i + \hat{\beta}\sum u_i y_i$

$- \hat{\alpha}\hat{\beta} \cdot \sum u_i - \hat{\beta}^2 \sum u_i^2$

note: $n\hat{\alpha} = \sum y_i$, & $\sum u_i = 0$.

$= \hat{\beta}(\sum u_i y_i - \hat{\beta}\sum u_i^2)$

$= \hat{\beta}(\sum u_i y_i - \frac{\sum u_i y_i}{\sum u_i^2} \cdot \sum u_i^2)$

$= 0.$

d. $\sum e_i = \sum (y_i - (\hat{\alpha} + \hat{\beta} u_i))$

$= \sum (y_i - \bar{y} - \hat{\beta} u_i)$

$= n\cdot\bar{y} - n\cdot\bar{y} - \hat{\beta}\cdot\sum u_i$

$= 0.$

9.6.14.

a. mean matrix : $X \cdot \underset{\sim}{\beta}$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} \beta_1 + \beta_2 + 2\beta_3 \\ \beta_1 - \beta_2 + 2\beta_3 \\ \beta_1 - 3\beta_3 \\ \beta_1 - \beta_3 \end{pmatrix}$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

Cov matrix: $\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{18} \end{pmatrix} \cdot \sigma^2$

b. $\hat{\beta} = (X^T X)^{-1} X^T y$

$$= \begin{pmatrix} \frac{1}{4} & & 0 \\ & \frac{1}{2} & \\ 0 & & \frac{1}{18} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \\ 11 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & & 0 \\ & \frac{1}{2} & \\ 0 & & \frac{1}{18} \end{pmatrix} \begin{pmatrix} 21 \\ 5 \\ -22 \end{pmatrix}$$

$$= \begin{pmatrix} 21/4 \\ 5/2 \\ -22/18 \end{pmatrix}$$

9.7.2. $\dfrac{\sqrt{n-2}\,R}{\sqrt{1-R^2}} \overset{d}{=} t_{n-2}$

$$\frac{\sqrt{n-2}\,r}{\sqrt{1-R^2}} = 3.904$$

$$t_4^{.975} = 2.78 < 3.904$$

$\Rightarrow$ Reject $H_0: \rho = 0$.