494 MSE1 Solutions

1a. $E(\bar{X}) = E\left(\frac{X_1 + \dots + X_n}{n}\right)$

$= \frac{1}{n} E(X_1 + \dots + X_n)$

$= \frac{1}{n} \cdot n \cdot E(X_1) = \mu$

$Var(\bar{X}) = Var\left(\frac{X_1 + \dots + X_n}{n}\right)$

$= \frac{1}{n^2} Var(X_1 + \dots + X_n)$

$= \frac{1}{n^2} \cdot n \cdot Var(X_1) = \frac{\sigma^2}{n}$

b. For a sequence of iid random variables with finite mean & variance

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

c. $M_{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}(t) = E\left(e^{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} t}\right)$

$= \left[E\left(e^{\frac{X_1-\mu}{\sigma\sqrt{n}} t}\right)\right]^n$

$\approx \left[E\left(1 + \frac{X_1-\mu}{\sigma\sqrt{n}} t + \frac{(X_1-\mu)^2}{2\sigma^2 n} t^2 + \text{small}\right)\right]^n$

$\doteq \left(1 + \frac{t^2}{2n}\right)^n$

$\rightarrow e^{t^2/2}$ which is mgf of $N(0,1)$

2a. $\bar{X} \overset{d}{\approx} N\left(\mu, \frac{\sigma^2}{n}\right)$

b. $P\left(\mu - 1.96\frac{\sigma}{\sqrt{n}} \le \bar{X} \le \mu + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$

$\Rightarrow P\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\cdot\frac{\sigma}{\sqrt{n}}\right) = .95$

c. $15 \pm t_{n-1}^{.975} \cdot \frac{\sqrt{10}}{\sqrt{n}}$

d. $15 \pm t_{n-1}^{.975} \sqrt{10} \cdot \sqrt{\frac{1}{n} + 1}$

3a. $L(\theta) = \prod_{i=1}^{n} f_X(x_i \mid \theta)$

The MLE is the value of θ such that $L(\theta)$ is maximised.

b. Let $n_0 = \#0$'s
$n_1 = \#1$'s
$n_2 = \#2$'s

$L(\theta) = (\theta^2)^{n_0} (1-\theta)^{n_1} (\theta(1-\theta))^{n_2}$

$\log L(\theta) = 2n_0 \log\theta + n_1 \log(1-\theta) + n_2 \log(\theta(1-\theta))$.

$\frac{\partial \log L(\theta)}{\partial\theta} = \frac{2n_0}{\theta} + \frac{-n_1}{1-\theta} + \frac{n_2(1-2\theta)}{\theta(1-\theta)}$.

$$\frac{\partial \log L(\theta)}{\partial \theta} = 0$$

$$\Rightarrow \quad 2n_0(1-\theta) - n_1\theta + n_2(1-2\theta) = 0$$

$$\theta(-2n_0 - n_1 - 2n_2) = -2n_0 - n_2$$

$$\hat{\theta} = \frac{2n_0 + n_2}{2n_0 + n_1 + 2n_2}.$$

C. $$D_2 = \frac{\partial^2 \log L(\theta)}{\partial \theta^2} = \frac{-2n_0}{\theta^2} - \frac{n_1}{(1-\theta)^2} + n_2 \frac{(1-2\theta)^2 + 2(\theta)(1-\theta)}{(\theta(1-\theta))^2}$$

now $E(n_0) = n \cdot \theta^2$

$E(n_1) = n(1-\theta)$

$E(n_2) = \theta(1-\theta)$

Substitute into above

Use $$\frac{1}{E(-D_2)} = \frac{\theta(1-\theta)}{\theta(1+\theta)}$$

C.

$$L_1 = \hat{\theta}^{2n_0}(1-\hat{\theta})^{n_1}\left(\hat{\theta}(1-\hat{\theta})\right)^{n_2}$$

$$L_0 = \left(\frac{1}{2}\right)^{2n_0}\left(\frac{1}{2}\right)^{n_1} \cdot \left(\frac{1}{4}\right)^{2n_2}$$

$$\underbrace{\log(L_1) - \log(L_0)}_{\text{Use this as the test statistic.}} \simeq \frac{1}{2}\chi^2_1.$$

Alternatively use assymptotic normality of the MLE.

$$\hat{\Theta} \simeq N\left(\Theta, \frac{1}{I(\Theta)}\right).$$