Math 494 - Mathematical Statistics

Midsemester exam 1

February 11, 2015

Time permitted: 50 minutes

This exam has 4 questions worth a total of 50 marks.

This paper has 1 page including this one.

Full working will be required to achieve full credit.

No calculators are allowed in this examination.

Students are permitted to bring one (double-sided) sheet of notes.

1. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with common mean μ, and common variance σ^2.

   (a) For $\bar{X} = \frac{X_1+X_2+\ldots+X_n}{n}$, showing full working, show that $\mathbb{E}(\bar{X}) = \mu$ and $\mathrm{Var}(\bar{X}) = \sigma^2/n$.

   (b) For an observed sample with $n = 10$, $\bar{x} = 15.2$ and $s^2 = 3.4$, and if we assumed our data came from a normal distribution, give a formula for a 95% confidence interval for μ.

   (c) If we had data from a second population which also followed a normal distribution with mean μ_2, with $n = 15, \bar{y} = 16.2, s_y^2 = 3.6$, give an expression for a 95% confidence interval for $\mu - \mu_2$.

`[15 marks]`

2. (a) Explain what the following three terms mean: statistic, estimator and estimate. In particular explain the differences between them.

   (b) Define what it means for an estimator to be unbiased, consistent and efficient.

`[10 marks]`

3. Consider a sample of n observations on $X \stackrel{d}{=} \mathrm{Geo}(\theta)$ for which the pmf is $p_X(x) = \theta(1-\theta)^x$ for $x = 0, 1, 2, \ldots$. You may wish to use the fact that $\mathbb{E}(X) = \frac{1-\theta}{\theta}$ and $\mathrm{Var}(X) = \frac{1-\theta}{\theta^2}$. Recall our definitions of D_1 and D_2 as the derivatives of the log likelihood function,

$$D_1 = \frac{\partial \log L}{\partial \theta}, \qquad D_2 = \frac{\partial^2 \log L}{\partial \theta^2}.$$

   (a) Confirm that $\mathbb{E}(D_1) = 0$.

   (b) Calculate $\mathbb{E}(D_1^2)$ and hence state the value of $\mathbb{E}(D_2)$.

`[10 marks]`

4. A random sample of n observations is obtained on the random variable Y which has a lognormal distribution, $l\mathrm{N}(\theta, 1)$. The pdf of Y is given by

$$f_Y(y) = \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\log(y)-\theta)^2}. \qquad (y > 0)$$

   (a) Find the maximum likelihood estimator of θ.

   (b) What is the minimum variance bound for an unbiased estimator for θ?

   (c) Hence, give an approximate distribution of $\hat{\theta}$ when n is large, where $\hat{\theta}$ denotes the estimator you derived in part (a).

`[15 marks]`

End of Examination