494 HW4 Solutions

1. $$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2}\left(\frac{x_i-\theta}{\sqrt{\theta}}\right)^2}$$

$$\Rightarrow \frac{L_1}{L_0} = \left(\frac{\theta_1}{\theta_0}\right)^{-n/2} \cdot e^{-\frac{1}{2\theta_1}\sum_{i=1}^{n}(x_i-\theta_1)^2 + \frac{1}{2\theta_0}\sum_{i=1}^{n}(x_i^2-\theta_0)^2}$$

$$= \left(\frac{\theta_1}{\theta_0}\right)^{-n/2} e^{\left\{-\frac{1}{2\theta_1}\sum_{i=1}^{n} x_i^2 + \frac{1}{2\theta_1}\sum x_i \theta_1 2 - \frac{1}{2}\theta_1 n + \frac{1}{2\theta_0}\sum_{i=1}^{n} x_i^2 - \frac{1}{2\theta_0}\sum x_i 2\theta_0 + \frac{1}{2}\theta_0 n\right\}}$$

$$= \left(\frac{\theta_1}{\theta_0}\right)^{-n/2} \cdot e^{-\frac{1}{2}\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\sum_{i=1}^{n} x_i^2 - \frac{1}{2}\theta_1 n + \frac{1}{2}\theta_0 n}$$

$\Rightarrow$ LR test depends on the value of $\sum_{i=1}^{n} x_i^2$.

2a. Overall median $m^* = 690$, $H_0: m_0 = m_1$.

	$> m^*$	$\le m^*$	
Weekday	4	6	10
Weekend	6	4	10

$$\Rightarrow \chi^2 = \frac{1}{5}(1+1+1+1) = 4/5.$$

As $c_{.95}(\chi^2_1) = 3.84$, $\Rightarrow$ do not reject H_0.

b. First we order our data, then assign ranks and note which come from weekdays.

110	115	160	343	392	429	434
1	2	3	4	5	6	7
☆		☆			☆	☆

485	545	683	697	720	732
8	9	10	11	12	13
	☆	☆		☆	☆

757	765	810	928	938	997	1079
14	15	16	17	18	19	20
☆					☆	

The sum of the ☆ ranks

$W = 1+3+6+7+9+10+12+13+14+19 = 94$

Under H_0: $W \approx N\left(\frac{1}{2}(10\times 21), \frac{1}{12}(10^2\times 21)\right)$

$= N(105, 175)$

$P(W < 94) = P\left(Z < \frac{94-105}{\sqrt{175}}\right)$

$= P(Z < -0.83)$

≈ 0.203

$\Rightarrow$ P-value > 0.05

$\Rightarrow$ Do not reject H_0.

3. Need expected table.

Exp	$B=0$	$B=1$	$B\geq 2$	
$M=2$	49.9	51.3	51.8	153
$M=3$	54.1	55.7	56.2	166
	104	107	108	319

$$\Rightarrow \chi^2 = \frac{8.1^2}{49.9} + \frac{0.7^2}{51.3} + \frac{8.8^2}{51.8}$$

$$+ \cancel{\frac{8.1^2}{46}} \frac{8.1^2}{54.1} + \frac{0.7^2}{55.7} + \frac{8.8^2}{56.2}$$

$$= 5.42$$

Compare to $\chi^2_{(r-1)(c-1)} = \chi^2_2$.

$$C_{.95}(\chi^2_2) = 5.99 > 5.42$$

$\Rightarrow$ Do not reject H_0

where H_0: M & B are independent.

6.3.8.

a. $L = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = e^{-\lambda n} \cdot \frac{\lambda^{\Sigma x_i}}{\Pi x_i!}$

$\Rightarrow \frac{L_1}{L_0} = \frac{\hat{\lambda}^{\Sigma x_i}}{\Pi X_i} \cdot \frac{\Pi X_i}{\lambda_0^{\Sigma x_i}}$

$= \left(\frac{\hat{\lambda}}{\lambda}\right)^{\Sigma X_i}$

$\Rightarrow$ LR test is based on $Y = \sum_{i=1}^{n} X_i$.

b. $H_0: \theta_0 = 2$, $n = 5$

$\Rightarrow Y \overset{d}{=} Po(2 \times 5) = Po(10)$

$\alpha = P(Y \leq 4) + P(Y \geq 17)$

$= 0.029 + 0.027$ (using R)

$= 0.056$

6.3.9.

a. $L = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}$

$\Rightarrow \frac{L_1}{L_0} = \frac{\hat{\theta}^{\Sigma x_i}(1-\hat{\theta})^{n-\Sigma x_i}}{\theta_0^{\Sigma x_i}(1-\theta_0)^{n-\Sigma x_i}}$

$$= \left(\frac{\hat{\theta}(1-\theta_0)}{\theta_0(1-\hat{\theta})} \right)^{\sum x_i} \left(\frac{1-\hat{\theta}}{1-\theta_0} \right)^n$$

$\Rightarrow$ LR test is based upon $Y = \sum_{i=1}^{n} X_i$

$\Rightarrow$ Under H_0: $Y \stackrel{d}{=} Bin(n, \theta_0)$

b. Under H_0, $Y \stackrel{d}{=} Bin(100, 0.5)$

$$\approx N(50, 25)$$

If $P(C_1 \leq Y \leq C_2) \approx 0.05$

$$\Rightarrow C_1 = 50 - 1.96 \cdot 5$$
$$= 40.2$$
$$\approx 40.$$

Similarly $C_2 \approx 60.$

4.7.3.

$$\chi^2 = \frac{(20-b)^2}{20} + \frac{(40-b-20)^2}{20.}$$

$$= \frac{(20-b)^2}{10.}$$

$$C_{0.975}(\chi^2_5) = \cancel{11.07}\ 12.83$$

$$\text{So} \quad \frac{(20-b)^2}{10} = \cancel{\text{max}}\ 12.83$$

$$(20-b)^2 = \cancel{\text{max}}\ 128.3$$

$$b = \cancel{9.48}\ 8.67$$

$$\Rightarrow \text{need } b \le 8$$

$$\text{or } b \ge 32.$$

4.7.4.

Obs	86	35	26	13
Exp	90	30	30	10

$$\chi^2 = \frac{4^2}{90} + \frac{5^2}{30} + \frac{4^2}{30} + \frac{3^2}{10}$$

$$= 2.44$$

$$C_{0.99}(\chi^2_3) = 11.34 > 2.44$$

$\Rightarrow$ Do not reject H_0.

~~4.7.6.~~

~~Exp~~	~~B_1~~	~~B_2~~	~~B_3~~	~~B_4~~	
~~A_1~~	~~7.02~~	~~17.42~~	~~16.38~~	~~11.18~~	~~52~~
~~A_2~~	~~9.72~~	~~24.12~~	~~22.68~~	~~15.48~~	~~72~~
~~A_3~~	~~10.26~~	~~25.46~~	~~23.94~~	~~16.34~~	~~76~~
	~~27~~	~~67~~	~~63~~	~~43~~	

~~So $\chi^2 = \frac{3.98^2}{7.02} + \frac{3.68^2}{7.42} + \dots$~~

4.7.9

a. We fit $\hat{\lambda} = \bar{x}$.

$$\hat{\lambda} = \frac{40 + 2 \cdot 16 + 3 \cdot 18 + 4 \cdot 6}{100}$$

$$= 1.5.$$

x	0	1	2	3	>3
freq	20	40	16	18	6
exp	22.31	33.47	25.10	12.55	6.56

$$\chi^2 = \frac{2.31^2}{22.31} + \frac{6.53^2}{33.47} + \frac{9.1^2}{25.10} + \frac{5.45^2}{12.55} + \frac{.56^2}{6.56}$$

$$= 7.23.$$

b. $r = 5 - 2 = 3$.

c. $C_{.95}(\chi^2_3) = 7.81 > 7.23$

$\Rightarrow$ do not reject H_0.

10.2.3.

a. $X' \overset{d}{=} Bin(25, 0.5)$

$P(X' \geq 16) = 0.115.$

b. $P(X > 0) = P(Z > \frac{0.5}{1})$

$= 0.69.$

$\Rightarrow P(Y \geq 16) = 0.78 = \text{Power}$

where $Y \overset{d}{=} Bin(25, 0.69)$

c. Want k such that

$P(\frac{\overline{X}}{1/\sqrt{25}} \geq k) = 0.115$

$= P(Z \geq k) = 0.115$

$\Rightarrow k = 1.20$

Under H_1 : $\overline{X} \overset{d}{=} N(0.5, \frac{1}{5})$.

$\Rightarrow \text{Power} = P(\frac{\overline{X}}{1/\sqrt{25}} \geq 1.20)$

$= P(\overline{X} \geq 0.24)$

$= P(Z \geq \frac{0.24 - 0.5}{1/5})$

$= P(Z \geq -1.3)$

$= 0.903.$