Homework 7 Solutions

1a. $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$
$\leq P(A_1) + P(A_2)$ ✓

Assume $P(A_1 \cup A_2 \cup \dots \cup A_k) \leq P(A_1) + \dots + P(A_k)$

$P(A_1 \cup \dots \cup A_k \cup A_{k+1}) = P(A_1 \cup \dots \cup A_k) + P(A_{k+1}) - P(A_1 \cup \dots \cup A_k \cap A_k$
$\leq P(A_1 \cup \dots \cup A_k) + P(A_{k+1})$
$\leq \sum_{i=1}^{k+1} P(A_i)$ (by assumption)

$\square$

b. Set A_i as the event that for comparison i, the null hypothesis is rejected

~~P(A = family size = P(~~

Let α_F be the family size.

$\alpha_F = P(A_1 \cup A_2 \dots \cup A_k)$
$\leq P(A_1) + \dots + P(A_k)$
$= k \cdot \alpha/k$
$= \alpha.$

$\Rightarrow$ family size is at most α.

	df	SS	MS	F	P
2a. group	2	126.17	63.08	21.42	0.00038.
residuals	9	26.5	2.94		

b. $\hat{\mu}_1 = 3.75$ $\hat{\mu}_2 = 10.5$ $\hat{\mu}_3 = 10.75$

~~width~~ ~~with~~ width of each CI: $t_9^{.945} \cdot 1.716 \cdot \sqrt{\frac{1}{4} + \frac{1}{4}}$

$= 2.74.$

$\hat{\mu}_1 - \hat{\mu}_2$: $3.75 - 10.5 \pm 2.74$
$= (-9.49, -4.01)$

$\hat{\mu}_2 - \hat{\mu}_3$: $10.5 - 10.75 \pm 2.74$
$= (-2.99, 2.49)$

$\hat{\mu}_3 - \hat{\mu}_1$: $10.75 - 3.75 \pm 2.74$
$= (5.26, 10.74)$

c. widths: $t_9^{s} \cdot 1.716 \cdot \sqrt{1/4 + \frac{1}{4}}$

$s = (1 - \frac{.05}{3 \times 2})$
$\Rightarrow t_9^{s} = 2.93$

$\Rightarrow$ width $= 3.56.$

$\Rightarrow$ CIs

$\mu_1-\mu_2$: $(-10.31, -3.19)$
$\mu_2-\mu_3$: $(-3.81, 3.31)$
$\mu_3-\mu_1$: $(3.44, 10.56)$

d. $\mu_1-\mu_2$: $(-10.14, -3.362)$
$\mu_2-\mu_3$: $(-3.64, 3.138)$
$\mu_3-\mu_1$: $(3.61, 10.39)$

3a. 17.935

b. $17.935 + 10.059 + 11.466 + 17.471$
$= 56.931$

c. $F = 11.054$ $\quad p = 0.002 < .05$
$\Rightarrow$ Reject H_0: interactions $= 0$.

d. No. F-test for B is significant also.

9.5.1. $\sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{i.})^2$

$$= \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..} + \bar{X}_{.j} - \bar{X}_{..})^2$$

$$= \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 + \sum_{j=1}^{b}\sum_{i=1}^{a}(\bar{X}_{.j} - \bar{X}_{..})^2$$

$$+ 2\sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})(\bar{X}_{.j} - \bar{X}_{..})$$

$$= \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2 + a\sum_{j=1}^{b}(\bar{X}_{.j} - \bar{X}_{..})^2$$

$$+ 2(\sim)$$

note: 2(~)

$$= 2 \cdot \sum_{j=1}^{b} (\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) \cdot \sum_{i=1}^{a} (X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})$$

$$= 2 \cdot \sum_{j=1}^{b} (\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot}) \underbrace{\left(a \cdot (\bar{X}_{\cdot\cdot} - \bar{X}_{\cdot j}) + \underbrace{\sum_{i=1}^{a} (X_{ij} - \bar{X}_{i\cdot})}_{= a \cdot (\bar{X}_{\cdot j} - \bar{\bar{X}}_{\cdot\cdot})} \right)}_{= 0.}$$

$\square$

9.5.6 $\log L(\theta) = C \cdot \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} (X_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_{ij}))^2$

$$\frac{\partial \log L(\theta)}{\partial \mu} = C^* \sum\sum\sum (X_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_{ij})).$$

| recall: $\sum \alpha_i = \sum \beta_j = \sum \gamma_{ij} = 0$

$$= C^* \cdot \sum\sum\sum (X_{ijk} - \mu). = 0.$$

$$\Rightarrow \hat{\mu} = \bar{X}_{\cdots}$$

$$= \frac{\sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{c} X_{ijk}}{abc}$$

$$\frac{\partial \log L(\theta)}{\partial \alpha_i} = c\sum_{j=1}^{b}\sum_{k=1}^{c}\left(X_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_{ij})\right)$$

$$= c\sum_{j=1}^{b}\sum_{k=1}^{c}\left(X_{ijk} - (\mu + \alpha_i)\right).$$

$$\Rightarrow \hat{\alpha}_i = \bar{X}_{i..} - \hat{\mu}$$

$$= \bar{X}_{i..} - \bar{X}_{...}$$

& Similarly for $\hat{\beta}_i = \bar{X}_{.j.} - \bar{X}_{...}$

$$\frac{\partial \log L}{\partial \gamma_{ij}} = c \cdot \sum_{k=1}^{c}\left(X_{ijk} - (\mu + \alpha_i + \beta_j + \gamma_{ij})\right)$$

$$\Rightarrow \hat{\gamma}_{ij} = \bar{X}_{ij.} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\mu}$$

$$= \bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{...}.$$

now: $E(\bar{X}_{...}) = E(\sum\sum\sum X_{ijk}) \cdot \frac{1}{abc}$

$$= \frac{\sum\sum\sum(\mu + \alpha_i + \beta_j + \gamma_{ijk})}{abc}$$

$$= \mu$$

$\left(\text{as } \sum_i \alpha_i = \sum_j \beta_j = \sum_i \gamma_{ij} = 0\right)$

$$E(\hat{\alpha}_i) = E(\bar{X}_{i..} - \bar{X}_{...})$$

$$= \frac{1}{bc} \cdot E\left(\sum_j \sum_k X_{ijk}\right) - \mu$$

$$= \frac{1}{bc} \sum_j \sum_k (\mu + \beta_j + \gamma_{ij} + \alpha_i) - \mu$$

$$= \frac{bc\mu + bc\alpha_i}{bc} - \mu$$

$$= \alpha_i \quad ✓$$

Similarly for β_j

$$E(\hat{\gamma}_{ijk}) = E\left[\frac{1}{c} \cdot \sum_{k=1}^{c} (X_{ijk} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\mu}\right)$$

$$= \frac{1}{c} (\mu + \alpha_i + \beta_j + \gamma_{ij} - \alpha_i - \beta_j - \mu) \cdot c$$

$$= \gamma_{ij}$$

9.5.7

	df	SS	MS	F	p
row	2	7.30	3.65	30.73	<.05
col	3	8.13	2.71	22.82	<.05
r:c	6	3.41	0.57	4.79	<.05
resid	12	1.43	0.11		

All p-values <.05

⇒ Reject all null hypotheses.

9.5.9.

	df	SS	MS	F	P
batch	4	363.6	90.9	15.54	.00076
treat	2	89.2	44.6	7.62	.014
resid	8	46.8	5.85		

a. $p = .014 < .05$

$\Rightarrow$ Reject H_A.

b. $p = .00076 < .05$

$\Rightarrow$ Reject H_B.