494 Homework 2 solutions

1. $\bar{x} = 38.41$

$s = 20.62$

95% CI $\quad 38.41 \pm t_9^{.975} \frac{20.62}{\sqrt{10}}$

$= (23.66, 53.16)$.

95% PI $\quad 38.41 \pm t_9^{.975}\, 20.62 \sqrt{(1 + \frac{1}{10})}$

$= (-10.51, 87.33)$.

(A negative weight is a bit sus though)

2. $\bar{x}_1 = 34.7 \quad s_1 = 4.86 \quad n_1 = 20$

$\bar{x}_2 = 37.2 \quad s_2 = 4.66 \quad n_2 = 10$.

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \stackrel{d}{=} F_{19,9}$$

$C_{.025}(F_{19,9}) = .347$

$C_{.975}(F_{19,9}) = 3.68$.

95% CI for $\frac{\sigma_1^2}{\sigma_2^2} \quad \left(\frac{1}{3.68} \cdot \frac{4.86^2}{4.66^2}, \frac{1}{.347} \cdot \frac{4.86^2}{4.66^2}\right)$

$= (.296, 3.135)$

Assuming equal variances (unequal also okay...)

$$S_p^2 = \frac{19 \cdot 4.86^2 + 9 \cdot 4.66^2}{28}$$

$$= 23.01$$

95% CI for $\mu_1 - \mu_2$ $\quad 34.7 - 37.2 \pm t_{28}^{.975} \sqrt{23.01\left(\frac{1}{10} + \frac{1}{20}\right)}$

$$= (-6.31, 1.31)$$

3.a. $E(X) = \int_0^\infty \theta x e^{-x} + 2(1-\theta) x e^{-2x} \, dx$

$$= \theta \cdot \left[x e^{-x} + e^{-x} \right]_0^\infty + 2(1-\theta)\left[-\tfrac{1}{2} x e^{-2x} - \tfrac{1}{4} e^{-2x} \right]_0^\infty$$

$$= \theta + 2(1-\theta) \cdot \tfrac{1}{4}$$

$$\Rightarrow \quad \bar{x} = \bar{\theta} + \tfrac{1}{2}(1-\bar{\theta})$$

$$\bar{\theta} = (\bar{x} - \tfrac{1}{2}) \cdot 2 = 2\bar{x} - 1$$

$$\Rightarrow \quad 2\bar{X} - 1 \overset{d}{\approx} N\left(\theta, \tfrac{4}{n}\sigma^2\right)$$

$$EX^2 = \int_0^\infty \theta x^2 e^{-x} + 2(1-\theta) x^2 e^{-2x} \, dx$$

$$= \theta \left[-e^{-x}(x^2 + 2x + 2) \right]_0^\infty + 2(1-\theta)\left[-\tfrac{1}{4} e^{-2x}(2x^2 + 2x + 1) \right]_0^\infty$$

$$= 2\theta + 2(1-\theta)\tfrac{1}{4} = \tfrac{3}{2}\theta + \tfrac{1}{2}$$

$$\Rightarrow Var(x) = 3/2\,\theta + \tfrac{1}{2} - \left(\tfrac{1}{2} + \tfrac{1}{2}\theta\right)^2$$
$$= -1/4\,\theta^2 + \theta + \tfrac{1}{4}$$

$$\Rightarrow \bar{\theta} \overset{d}{\approx} N\left(\theta, \tfrac{1}{n}(-\theta^2 + 4\theta + 1)\right)$$

$$\Rightarrow CI: \quad \bar{\theta} \pm 1.96\sqrt{\tfrac{1}{n}(-\bar{\theta}^2 + 4\bar{\theta} + 1)}$$

b. $$L(\theta) = \prod_{i=1}^{n} \left(\theta e^{-x_i} + 2(1-\theta)e^{-2x_i}\right)$$

4. $$E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right)$$

$$= \frac{1}{n}\cdot E(\sum X_i) = \frac{1}{n}\cdot n \cdot E(X) = \lambda.$$

$$E(S^2) = \frac{1}{n-1} E\left[\sum X_i^2 - \frac{(\sum X_i)^2}{n}\right]$$

$$= \frac{1}{n-1} E\left[\sum X_i^2 - n\cdot \bar{X}^2\right]$$

$$= \frac{1}{n-1}\left[n\cdot(\lambda + \lambda^2) - n\cdot\left(\frac{\lambda}{n} + \lambda^2\right)\right]$$

$$= \lambda$$

$$Var(\bar{X}) = \frac{\sigma^2}{n} = \frac{\lambda}{n}.$$

Now Recall from slide 18

$$Var(S^2) = \frac{\mu_4}{n} - \frac{\sigma^4(n-3)}{n(n-1)}$$

Noting that $\sigma^2 = \lambda$, $r_4 = \lambda + 3\lambda^2$

$$Var(S^2) = \frac{\lambda + 3\lambda^2}{n} - \frac{\lambda^2(n-3)}{n(n-1)}$$

$$= \frac{\lambda}{n} + \frac{3\lambda^2(n-1) - \lambda^2(n-3)}{n(n-1)}$$

$$= \frac{\lambda}{n} + \frac{2\lambda^2 n}{n(n-1)}$$

$$= \frac{\lambda}{n} + \frac{2\lambda^2}{n-1} > Var(\bar{X}).$$

4.1.3 a. $L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!}$

$$\log L(\theta) = \sum_{i=1}^{n} (-\theta + x_i \log\theta - \log(x_i!))$$

$$\frac{\partial \log L(\theta)}{\partial\theta} = \sum_{i=1}^{n}\left(-1 + \frac{x_i}{\theta}\right)$$

$$\Rightarrow \quad -n + \frac{\sum x_i}{\theta} = 0.$$

$$\Rightarrow \quad \hat{\theta} = \frac{\sum x_i}{n}.$$

$$E(\theta) = \frac{1}{n} \cdot E(\sum X_i)$$
$$= \frac{1}{n} \cdot n \cdot \theta = \theta.$$

b. $\hat{\theta} = \frac{1}{10}(9+7+9+15+10+13+11+7+2+12)$
$= 9.5$

Between 9 & 10 am, customers arrive at a rate of 9.5 an hour.

(Or on average, we expect 9.5 customers per hour)

4.2.11. We know $\bar{X} - X_{n+1}$ follows the normal distribution

$$E(\bar{X} - X_{n+1}) = \mu - \mu = 0.$$
$$Var(\bar{X} - X_{n+1}) = \frac{\sigma^2}{n} + \sigma^2 = \sigma^2(1+\frac{1}{n})$$

Also: $\frac{(n-1)S^2}{\sigma^2} \overset{d}{=} \chi^2_{n-1}.$

and if $Z \sim N(0,1)$, $U \sim \chi^2_{n-1}$

$$Z/\sqrt{U/_{n-1}} \overset{d}{=} t_{n-1}.$$

Hence
$$\frac{(\bar{X} - X_{n+1}) \cdot \frac{1}{\sigma\sqrt{1+\frac{1}{n}}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2} \cdot \frac{1}{n-1}}} \overset{d}{=} t_{n-1}$$

$$\Rightarrow \sqrt{\frac{n}{n+1}} \cdot \left(\frac{\bar{X} - X_{n+1}}{S} \right) \overset{d}{=} t_{n-1}$$

$$\Rightarrow C = \sqrt{\frac{n}{n+1}}$$

now $t_7^{.90} = 1.415$

$$\Rightarrow k = 1.415 \cdot \sqrt{1 + \frac{1}{n}}, \quad n=8$$
$$= 1.501$$

4.2.17 $\bar{X} \overset{d}{=} Po(\mu)$

$$\Rightarrow \bar{X} \approx N\left(\mu, \frac{\mu}{n}\right)$$

$$\Rightarrow 90\% \text{ CI}: \quad 3.4 \pm 1.645 \sqrt{\frac{3.4}{200}}$$
$$= (3.19, 3.61)$$

4.2.21 Assuming equal variances

$$S_p^2 = \frac{9 \cdot 8.64 + 9 \cdot 7.88}{18}$$

$$= 8.26.$$

$$95\% \text{ CI}. \quad 4.8 - 5.6 \pm t_{18}^{.975} \sqrt{8.26\left(\tfrac{1}{10} + \tfrac{1}{10}\right)}$$

$$= (-3.50, 1.90)$$

4.225 Firstly we know

$$\bar{X}_1 - \bar{X}_2 \overset{d}{=} N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{9} + \frac{\sigma_2^2}{12}\right)$$

$$= N\left(\mu_1 - \mu_2, \frac{3\sigma_2^2}{9} + \frac{\sigma_2^2}{12}\right)$$

$$= N\left(\mu_1 - \mu_2, \frac{5\sigma_2^2}{9}\right)$$

now recall if $Z \overset{d}{=} N(0,1)$
& $U \overset{d}{=} \chi^2_n$

$$t_n \overset{d}{=} \frac{Z}{\sqrt{U/n}}$$

We also know that

$$\frac{8S_1^2}{\sigma_1^2} + \frac{11S_2^2}{\sigma_2^2} \overset{d}{=} \chi^2_{19}$$

$$\Rightarrow \frac{8S_1^2 + 33S_2^2}{3\sigma_2^2} \overset{d}{=} \chi^2_{19}$$

$$\Rightarrow \frac{\dfrac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{5\sigma_2^2/12}}}{\sqrt{\dfrac{8S_1^2 + 33S_2^2}{3\sigma_2^2} \Big/ 19}} \overset{d}{=} t_{19}$$

$$\Rightarrow \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{5(8S_1^2 + 33S_2^2)}{12 \times 19 \times 3}}} \overset{d}{=} t_{19}$$

6.1.2

a. $f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0$

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

$$\log L(\theta) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log(x_i)$$

$$\frac{\partial \log L(\theta)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(x_i)$$

$$\Rightarrow \quad \hat{\theta} = \cancel{\frac{-\sum_{i=1}^{n}\log(x_i)}{n}} \quad \frac{-n}{\sum_{i=1}^{n}\log(x_i)}$$

b. $f(x|\theta) = e^{-(x-\theta)} \quad \theta \le x < \infty \quad \theta \in \mathbb{R}$

$$L(\theta) = \begin{cases} \prod_{i=1}^{n} e^{-(x_i-\theta)} & \theta \le x_1, \theta \le x_2, \ldots, \theta \le x_n \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-\sum x_i} \cdot e^{n\theta} & \theta \le \min(x_1, \ldots, x_n) \\ 0 & \text{otherwise} \end{cases}$$

$L(\theta)$ is increasing in θ, $\Rightarrow$ want to choose the largest θ, but $\theta \le \min\{x_1, \ldots, x_n\}$

$$\Rightarrow \quad \hat{\theta} = \min\{X_1, \ldots, X_n\}$$