494 HW3 Solutions

1. a. $2p + 4p' = 1$

$$\Rightarrow p' = \frac{1-2p}{4}$$

b. Let X = result of one dice roll

$$E(X) = (1+6)\cdot p + (2+3+4+5)p'$$

$$= 7p + \frac{14(1-2p)}{4}$$

$$= 3.5$$

$$\overset{?}{=} \bar{X} \quad ?$$

This seems unhelpful... Lets use S^2 instead.

$$EX^2 = (1+6^2)p + (2^2+3^2+4^2+5^2)\cdot\frac{(1-2p)}{4}$$

$$= 37p + 54\,\frac{(1-2p)}{4}$$

$$= 10p + 54/4$$

$$\Rightarrow \operatorname{Var} X = 10p + 54/4 - (7/2)^2$$

$$= 10p + 5/4$$

$$\Rightarrow \quad \bar{p} = (S^2 - 5/4)\cdot\frac{1}{10}$$

(Alternatively could construct $X = \#$ 1's or 2's and go from there.)

c. $L(p) = C \cdot p^{f_1+f_6} \left(\left(\frac{1-2p}{4}\right)\right)^{n-f_1-f_6}$

$\hookrightarrow$ some combinatorial constant.

$$\log L(p) = \log(C) + (f_1+f_2)\log p + (n-f_1-f_6)\log\left(\frac{1-2p}{4}\right)$$

$$\frac{\partial \log L(p)}{\partial p} = \frac{f_1+f_2}{p} + (n-f_1-f_6)\cdot\frac{4}{1-2p}\left(-\frac{1}{2}\right)$$

$$= \frac{f_1+f_6}{p} - \frac{2(n-f_1-f_6)}{1-2p}$$

$$\Rightarrow (f_1+f_6)(1-2p) - 2(n-f_1-f_6)p = 0.$$

$$\Rightarrow \hat{p} = \frac{f_1+f_6}{2n}$$

d. $n = 22+11+13+15+15+24 = 100$

$$\Rightarrow \hat{p} = \frac{22+24}{200} = 0.23$$

$\Rightarrow$ Expected frequencies

23, 13.5, 13.5, 13.5, 13.5, 23.

2 a. $L(\theta) = (\theta^2)^{n_0}(\theta(1-\theta))^{n_1}(1-\theta)^{n_2}$

$$= \theta^{2n_0+n_1}(1-\theta)^{n_1+n_2}$$

b. $\cancel{\partial^2 \log L(\theta)}$ $\log L(\theta) = (2n_0+n_1)\log\theta + (n_1+n_2)\log(1-\theta)$

$$\frac{\partial \log L}{\partial\theta} = \frac{2n_0+n_1}{\theta} - \frac{n_1+n_2}{1-\theta}$$

$$\Rightarrow (2n_0+n_1)(1-\theta) - (n_1+n_2)\theta = 0.$$

$$\Rightarrow \hat{\theta} = \frac{2n_0+n_1}{2n_0+2n_1+n_2}.$$

b. $$\frac{\partial^2 \log L(\theta)}{\partial\theta^2} = -\frac{(2n_0+n_1)}{\theta^2} - \frac{n_1+n_2}{(1-\theta)^2}$$

now $E(n_0) = n\theta^2$ $E(n_1) = n\theta(1-\theta)$ $E(n_2) = n(1-\theta)$

$$E\left[\frac{\partial \log L(\theta)}{\partial\theta^2}\right] = n\left(\frac{-2\theta^2-\theta(1-\theta)}{\theta^2} - \frac{\theta(1-\theta)+(1-\theta)}{(1-\theta)^2}\right)$$

$$= n\left(\frac{-2\theta-(1-\theta)}{\theta} - \frac{\theta+\overset{+1}{\cancel{(1-\theta)}}}{1-\theta}\right)$$

$$= n\left(\frac{-1-\theta}{\theta} - \frac{\theta+1}{1-\theta}\right)$$

$$= n\left(\frac{-1+\theta-\theta+\theta^2-\theta^2-\theta}{\theta(1-\theta)}\right)$$

$$= \frac{n(-1-\theta)}{\theta(1-\theta)}$$

$$\Rightarrow \quad Var(\hat{\theta}) \approx \frac{\theta(1-\theta)}{n(1+\theta)}$$

c. $\quad$ 95% CI : $\hat{\theta} \pm 1.96 \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n(1+\hat{\theta})}}$

$$= 0.5 \pm 1.96 \sqrt{\frac{0.5^2}{200 \times 1.5}}$$

$$= (0.443, \; 0.557)$$

3. a. $\quad L(\theta) = \theta^{x}(1-\theta)^{m-x}\,\theta^{2y}(1-\theta^2)^{n-y}$

$$\log L(\theta) = \cancel{\frac{x+2y}{\theta}}$$

$$= (x+2y)\log\theta + (m-x)\log(1-\theta) + (n-y)\log(1-\theta^2)$$

$$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{x+2y}{\theta} - \frac{(m-x)}{1-\theta} + \frac{(n-y)2\theta}{1-\theta^2}$$

If we put these all on the denominator $(1-\theta^2)\theta$, the top would be a quadratic in θ.

$\rightarrow \hat{\theta}$ is a root of a quadratic.

b.

$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{-(x+2y)}{\theta^2} - \frac{m-x}{(1-\theta)^2} - \frac{n-y}{(1-\theta)^2} - \frac{n-y}{(1+\theta)^2}$$

now: $E(X) = m\theta$, $E(Y) = n \cdot \theta^2$

$$E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{m+2n\theta}{\theta} + \frac{m(1-\theta)+n(1-\theta^2)}{(1-\theta)^2} + \frac{n(1-\theta^2)}{(1+\theta)^2}$$

$$= \frac{m+2n\theta}{\theta} + \frac{m}{1-\theta} + \frac{n(1+\theta)}{1-\theta} + \frac{n(1-\theta)}{1+\theta}$$

$$= \frac{(m+2n\theta)(1-\theta^2) + (m+n(1+\theta))\,\theta(1+\theta) + n(1-\theta)\cdot\theta(1-\theta)}{\theta(1-\theta)(1+\theta)}$$

Honestly simplification isn't important...

$$= \ldots$$

$$= \frac{m}{\theta(1-\theta)} + \frac{4n}{1-\theta^2} = I(\theta)$$

$$\Rightarrow Var(\hat{\theta}) = \frac{1}{I(\theta)}$$

4. $H_0: \lambda = 0.6$, $H_1: \lambda \neq 0.6$.

$P = 1 - P(X \leq 9) - P(X \geq 15)$

where $X \overset{d}{=} Po(12)$

$P = 0.4 \Rightarrow$ Do not reject H_0.

Alternatively: $P = 2 \cdot P(X \geq 15) = 0.456$

6.2.1. We know that:

$$E(\bar{X}) = \mu = \theta$$
$$Var(\bar{X}) = \frac{\sigma^2}{n}$$

$$L(\theta) = \prod \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\theta}{\sigma}\right)^2}$$

$$\log L(\theta) = n\log(\sqrt{2\pi}\sigma) - \frac{1}{2} \cdot \sum \left(\frac{x_i-\theta}{\sigma}\right)^2$$

$$\frac{\partial \log L(\theta)}{\partial \theta} = + \sum \left(\frac{x_i-\theta}{\sigma^2}\right)$$

$$\frac{\partial^2 \log L(\theta)}{\partial \theta^2} = \frac{-1}{\sigma^2} \cdot n$$

$$\Rightarrow \frac{-1}{\frac{\partial^2 \log L\theta}{\partial \theta^2}} = \frac{\sigma^2}{n}$$

$\Rightarrow \bar{X}$ attains the MVB.

note $\sqrt{\theta} = \sigma$

6.2.8. $L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\theta} e^{-\frac{1}{2}\left(\frac{x_i}{\sqrt{\theta}}\right)^2}$

$$\log L(\theta) = C - \frac{n}{2}\log\theta - \frac{1}{2}\sum \frac{x_i^2}{\theta}$$

$$\frac{\partial \log L(\theta)}{\partial \theta} = -\frac{n}{2\theta} + \frac{\frac{1}{2}\sum x_i^2}{\theta^2}$$

$$\frac{\partial^2 \log L(\theta)}{\partial \theta^2} = \frac{n}{2\theta^2} - \frac{\frac{1}{2}\cdot 2 \cdot \sum x_i^2}{\theta^3}$$

$$E\left(\frac{\partial^2 \log L(\theta)}{\partial \theta}\right) = \frac{n}{2\theta^2} - \frac{n \cdot \theta}{\theta^3}$$

$$= -\frac{n}{2\theta^2}$$

$$\Rightarrow \quad I(\theta) = \frac{n}{2\theta^2} \quad \text{or} \quad \frac{1}{2\theta^2}$$ depending on your defn of $I(\theta)$

b. $$-\frac{n}{2\theta} - \frac{\sum x_i^2}{2\theta^2} = 0.$$

$$\Rightarrow \quad \hat{\theta} = \frac{1}{n}\sum x_i^2$$

noting that $EX^4 = 3\theta^2$

$$\mathrm{Var}(\hat{\theta}) = \frac{1}{n^2} \cdot n \cdot \left(EX^4 - (EX^2)^2\right)$$
$$= \frac{1}{n}(3\theta^2 - \theta^2)$$

$$= \frac{2\theta^2}{n}$$

$$= \mathrm{MVB}.$$

c. We know: $\hat{\theta} \xrightarrow{d} N\left(\theta, \frac{2\theta^2}{n}\right)$

$$\Rightarrow \quad \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, 2\theta^2)$$

4.5.3. If $\theta = 2$

$$\text{Power} = \int_{3/4}^{1} P(X_1 \geq 3/4x_2) \cdot 2x_2 \, dx_2.$$

$$P(X_1 \geq y) = \int_y^1 2x \, dx$$
$$= 1 - y^2$$

$$\Rightarrow \text{Power} = \int_{3/4}^{1} \left(1 - \left(3/4x_2\right)^2\right) \cdot 2x_2 \, dx_2$$

$$= x_2^2 - 9/8 \log x_2 \Big|_{3/4}^{1}$$

$$= 1 - (3/4)^2 + 9/8 \log 3/4$$

$$= 7/16 + 9/8 \log(3/4).$$

4.6.5. $H_0 : \mu = 10.1$
$H_1 : \mu > 10.1$

Under H_0: $\bar{X} \overset{d}{=} N(10.1, \frac{\sigma^2}{16})$

Reject H_0 if $T = \frac{\bar{X} - 10.1}{s/\sqrt{n}} > t_{15}^{.95}$

$$T = \frac{10.4 - 10.1}{0.4/\sqrt{16}}$$
$$= 3.$$

$t_{15}^{.95} = 1.753$ b. $p = P(t_{15} > 3)$

≈ 0.004

$\Rightarrow$ Reject H_0.

4.6.7.

a. $T = \dfrac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{13} + \frac{1}{16}}} \stackrel{d}{=} t_{27}$

Critical region: $P(t_{27} < q) = 0.05$

$\Rightarrow q = -1.703$

$\Rightarrow$ Crit region: $(-\infty, -1.703)$.

b. $T = \dfrac{72.9 - 81.7}{S_p\sqrt{\frac{1}{13} + \frac{1}{16}}}$ $S_p^2 = \dfrac{12 \cdot 25.6^2 + 15 \cdot 28.3^2}{27}$

$S_p^2 = 736.21$

$\Rightarrow T = -0.86 > -1.703$

$\Rightarrow$ Do not reject H_0.