Math 494 - Mathematical Statistics

Midsemester exam 1

February 24, 2015

Time permitted: 50 minutes

This exam has 3 questions worth a total of 50 marks.

This paper has 2 pages including this one.

Full working will be required to achieve full credit.

No calculators are allowed in this examination.

Students are permitted to bring one (double-sided) sheet of notes.

1. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with common mean μ, and common variance σ^2.

   (a) For $\bar{X} = \frac{X_1+X_2+\ldots+X_n}{n}$, showing full working, show that $\mathbb{E}(\bar{X}) = \mu$ and $\mathrm{Var}(\bar{X}) = \sigma^2/n$.

   (b) State the central limit theorem including all of its assumptions.

   (c) Prove the central limit theorem.

`[15 marks]`

2. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed $\mathrm{N}(\mu, \sigma^2)$ random variables.

   (a) What is the distribution of $\bar{X}$?

   (b) Show that $\mathbb{P}\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$.

   (c) If we had a sample of n observations and obtained $\bar{x} = 15$ and $s^2 = 10$, write an expression for a 95% confidence interval for μ.

   (d) If we had a sample of n observations and obtained $\bar{x} = 15$ and $s^2 = 10$, write an expression for a 95% prediction interval for a new observation.

`[15 marks]`

3. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables.

   (a) Define the method of maximum likelihood if X_i has a continuous density function $f_X(x)$.

   (b) A random sample of n observations is obtained on a discrete random variable X which has probability mass function

x	0	1	2
$p(x)$	θ^2	$1-\theta$	$\theta(1-\theta)$

   Find the maximum likelihood estimate for θ.

   (c) Write an expression for the minimum variance bound of an unbiased estimator for θ.

   (d) Suppose we wished to test

$$H_0 : \theta = 0.5$$
$$H_1 : \theta \neq 0.5$$

   If n is large, give an appropriate test statistic and its approximate distribution.

`[20 marks]`

End of Examination