Homework 6 Solutions

1 a.

	df	SS	MS	F
Between	2	32.67	16.33	6.39
within	6	15.33	2.56	
total	8	48		

note: $S^2_{drunk} = 6.33$

$S^2_{very\ drunk} = 1$

$S^2_{plastered} = 0.33$

$\Rightarrow$ Within SS $= 2\cdot(1 + .33 + 6.33)$

$S_T = 6.$

Total SS $= 8 \times 6 = 48$

b. $S^2 = 2.56$

c. $H_0: \mu_2 = \mu_3$

$H_1: \mu_2 \neq \mu_3$

$$t = \frac{\bar{x}_3 - \bar{x}_2}{S\sqrt{\frac{1}{3} + \frac{1}{3}}} = \frac{7/3 - 3}{\sqrt{2/3 \times 2.56}} = -0.51$$

Compare to t_6.

$P(t_6 < -0.51) = 0.314$

$\Rightarrow$ Do not reject H_0 (Can also use CI)

c. $H_0: \mu_0 = \mu_1 = \mu_2$
H_1: at least one mean differs

Test stat $\overset{d}{=} F_{2,6}$

$P(F_{2,6} > 6.39) = 0.03 < .05$

$\Rightarrow$ Reject H_0.

2a.

	df	SS	MS	F
within	17	2.38	.14	6.04
between	2	1.69	.845	
Total	19	4.07		

$$\beta = \sum n(\hat{x}_i - \bar{\bar{x}})^2$$

$$\bar{\bar{x}} = \frac{10 \cdot 4.23 + 5 \cdot 4.42 + 5 \cdot 4.94}{20}$$

$= 4.455$.

$\Rightarrow \beta = 1.69$ Rest follows from table.

b. $H_0: \mu_1 = \mu_2 = \mu_3$
H_1: not H_0.

$C_{.95}(F_{2,17}) = 3.59 < 6.04 \Rightarrow$ Reject H_0

c. $\hat{\mu}_3 = 4.94$.

$$95\% \text{ CI}: \quad 4.94 \pm t_{17}^{.975} \sqrt{\frac{.14}{5}}$$

$$= (4.59,\ 5.29)$$

7.3.1

a. $L(\theta) = \prod \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2}\left(\frac{x_i^2}{\theta}\right)}$

$$= (2\pi\theta)^{-n/2} \cdot e^{-\frac{1}{2\theta}\sum x_i^2}$$

$$\log L(\theta) = \cancel{-n/2 (2\pi\theta)}$$
$$= -n/2 \log(2\pi\theta) - \frac{1}{2\theta} \sum x_i^2$$

$$\frac{\partial \log L(\theta)}{\partial\theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum x_i^2 = 0.$$

$$\sum x_i^2 = n \cdot \theta.$$
$$\Rightarrow \hat{\theta} = \frac{\sum x_i^3}{n}.$$

$Y = \sum x_i^2$ is suff.

$\Rightarrow \hat{\theta}$ is a fn of Y.

b $L(\theta) = \prod \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

$$= e^{-n\lambda} \cdot \lambda^{\sum x_i} \cdot \prod \frac{1}{x_i!}$$

$$\log L(\theta) = -n\lambda + \sum x_i \log\lambda + \log\left(\prod \frac{1}{x_i}\right)$$

$$\frac{\partial \log L(\theta)}{\partial(\theta)} = -n + \frac{\sum x_i}{\lambda} = 0$$

$$\Rightarrow \hat{\lambda} = \frac{\sum x_i}{n}.$$

$Y = \sum X_i$ suff.

$\Rightarrow$ $\hat{\lambda}$ is a fn of Y.

c. $$L(\theta) = \frac{1}{\theta^n} \cdot \mathbb{1}_{\{1 \le x_1, \dots, x_n \le \theta\}}$$

$$= \frac{1}{\theta^n} \mathbb{1}_{\{\min(x_1, \dots, x_n) > 0\}} \mathbb{1}_{\{\max(x_1, \dots, x_n) \le \theta\}}$$

MLE is $\max(x_1, \dots, x_n)$

so is the suff stat.

d. $$L(\theta) = \prod (1-\theta)^x \theta$$

$$= (1-\theta)^{\sum x_i} \theta^n$$

$$\log L(\theta) = \sum x_i \log(1-\theta) + n \log\theta$$

$$\frac{\partial \log L(\theta)}{\partial\theta} = -\frac{\sum x_i}{1-\theta} + \frac{n}{\theta} = 0.$$

~~$\sum x_i(1-\theta)$~~

$$\Rightarrow \; -\theta \cdot \sum x_i + n(1-\theta) = 0$$

$$\hat{\theta} = \frac{n}{\sum x_i + n}$$

$Y = \sum x_i \Rightarrow$ Suff stat is a fn of $\hat{\theta}$.

7.4.3. $f(x_1|\theta) \ldots f(x_n|\theta)$

$$= \prod \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \underbrace{\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}}_{k_1} \cdot \underbrace{1}_{k_2}$$

$\Rightarrow Y = \sum X_i$ is sufficient.

note $Y \stackrel{d}{=} Bin(n, \theta)$

$$E(u(Y_1)) = \sum_{i=0}^{n} u(i) \binom{n}{i} \theta^i (1-\theta)^{n-i}$$

$$= u(0)(1-\theta)^n + u(1)\, n\theta(1-\theta)^{n-1} + \ldots$$

This is a degree n polynomial in θ.
But if $E(u(Y_1)) = 0$ for all $\theta \Rightarrow > n$ roots.
But a degree n polyn. has at most n roots.

$\Rightarrow$ not a degree n polynomial.

$\Rightarrow$ Must be degree 0 polynomial then.

$\Rightarrow \quad u \equiv 0.$

Now: $E(Y) = E \sum X_i$
$= n \cdot \theta.$

$\Rightarrow \quad Y/n$ is MVB estimator for θ.

7.44 a. $E(u(Z)) = \int_0^\theta u(t) \cdot \frac{1}{\theta} \cdot dt.$

if $E(u(Z)) = 0 \quad \forall \theta$

$$0 = \int_0^\theta u(t)\,dt$$
$$= \frac{\partial}{\partial \theta} \int_0^\theta u(t)\,dt.$$

$$= u(\theta) \qquad \text{(Leibniz rule)}$$

$\Rightarrow \quad u \equiv 0.$

b. $\int_0^1 u(t) \cdot \frac{1}{\theta}\,dt = 0.$

$\Rightarrow \quad \int_0^1 u(t)\,dt = 0.$

any non-zero function that integrates to zero is sufficient to show non-completeness.

eg $u(t) = \frac{1}{2} - t.$

7.4.9a. $L(\theta) = \frac{1}{(3\theta)^n} \mathbb{1}_{\{\min\{X_1,\dots,X_n\} > -\theta,\ \max\{X_1,\dots,X_n\} < 2\theta\}}$

want to choose θ as small as possible but still satisfying the indicator function.

$$\Rightarrow \hat{\theta} = \min\left\{ \frac{\max\{X_1,\dots,X_n\}}{2}, \max\{-X_1,\dots,-X_n\} \right\}.$$

b. Yes.

$f(x_1|\theta)\dots f(x_n|\theta)$

$$= \underbrace{\frac{1}{(3\theta)^n} \mathbb{1}_{\{\theta > \min\{\frac{\max\{X_1,\dots,X_n\}}{2}, \max\{-X_1,\dots,X_n\}\}\}}}_{f^n \text{ of } \theta \text{ and suff stat.}}$$

c. Not to be graded.

7.5.1. $f(x|\theta) = \frac{1}{6\theta^4} x^3 e^{-x/\theta} \quad x > 0.$

$$= \exp\{-\log 6\theta^4 + 3\log x - x/\theta\}$$

$p(\theta) = -\frac{1}{\theta}$ $K(x) = x$ $H(x) = 3\log x$ $q(\theta) = -\log 6\theta^4$

$\Rightarrow$ Complete suff stat : $Y = \sum K(X_i)$
$= \sum X_i$

now $E(Y) = n \cdot E(X)$

but this looks hard...?

$$E(Y) = -n \cdot \frac{q'(\theta)}{p'(\theta)}$$

$$= n \cdot \frac{\frac{1}{6\theta^4} 24\theta^3}{\frac{1}{\theta^2}}$$

$$= 4n\theta.$$

$\Rightarrow$ $Y/4n$ is the MVB estimator.

and $\phi(Y)$ is complete sufficient stat also.

7.5.3.

a. $f(x|\theta) = \theta x^{\theta-1}$

$= \exp(\log\theta + (\theta-1)\log x)$

$p(\theta) = \theta - 1$ $K(x) = \log x$ $H(x) = 0$ $q(\theta) = \log\theta.$

$Y = \sum K(X_i)$

$= \sum \log(X_i) = \log(X_1 \dots X_n)$

note $\phi(Y) = e^{Y/n} = (X_1 \dots X_n)^{\frac{1}{n}}$

as ϕ is one-to-one fn,

$\phi(Y)$ is also complete.

b. $L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$

$= \theta^n (x_1 \ldots, x_n)^{\theta-1}$

$\log L(\theta) = n \log \theta + (\theta-1) \log (x_1 \ldots x_n)$

$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{n}{\theta} + \log(x_1, \ldots x_n) = 0$

$\Rightarrow \quad \theta = \frac{-n}{\log(x_1 \ldots x_n)}$

7.5.5. $L = \int \exp(p(\theta) K(x) + H(x) + q(\theta))\, dx = 1$

take derivative wrt θ

$\frac{\partial L}{\partial \theta} = \int \exp(p(\theta) K(x) + H(x) + q(\theta))$
$\times [K(x) p'(\theta) + q'(\theta)]\, dx = 0.$

$\Rightarrow \quad E(K(x) p'(\theta) + q'(\theta)) = 0.$

$\Rightarrow \quad E(K(x)) = \frac{-q'(\theta)}{p'(\theta)}$

$$\frac{\partial^2 L}{\partial \theta^2} = \int \exp(p(\theta)K(x)+H(x)+q(\theta)) \times [K(x)p'(\theta)+q'(\theta)]^2$$

$$+ \exp(p(\theta)K(x)+H(x)+q(\theta)) \times [K(x)p''(\theta)+q''(\theta)]\,dx$$

$$= p'(\theta)^2 E(K(X)^2) + q'(\theta)^2 + 2E(K(X))\cdot p'(\theta)\cdot q'(\theta)$$

$$+ E(K(X))p''(\theta) + q''(\theta) = 0.$$

$$\Rightarrow E(K(X)^2)p'(\theta)^2 = -q'(\theta)^2 + 2q'(\theta)^2 + \frac{q'(\theta)}{p'(\theta)}p''(\theta) - q''(\theta)$$

$$= q'(\theta)^2 + \frac{q'(\theta)}{p'(\theta)}p''(\theta) - q''(\theta)$$

$$\& \quad E(K(X))^2 = \frac{q'(\theta)^2}{p'(\theta)^2}$$

$$\Rightarrow Var(K(X)) = \cancel{\frac{-q'(\theta)^2 p'(\theta) + 2q'(\theta)^2 p'(\theta)^3 + q}{p'(\theta)^3}}$$

$$= \frac{1}{p'(\theta)^3}\Big[q'(\theta)^2 p'(\theta) + q'(\theta)p''(\theta) - q''(\theta)p'(\theta) - p'(\theta)q'(\theta)^2 \Big].$$

$$= \frac{1}{p'(\theta)^3}\Big[q'(\theta)p''(\theta) - q''(\theta)p'(\theta) \Big].$$