494 Mid-term 1 Solutions.

1 a. $E(\bar{X}) = E\left(\frac{X_1 + \cdots + X_n}{n}\right)$

$$= \frac{1}{n} \cdot E(X_1 + \cdots + X_n)$$
$$= \frac{1}{n} \cdot n \cdot \mu = \mu.$$

$$Var(\bar{X}) = Var\left(\frac{X_1 + \cdots + X_n}{n}\right)$$
$$= \frac{1}{n^2} \cdot Var(X_1 + \cdots + X_n)$$
$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 = \sigma^2/n.$$

b. $15.2 \pm t_9^{.975} \cdot \frac{3.4}{\sqrt{10}}$

c. $15.2 - 15 \pm t_{23}^{.975} \cdot S_p \sqrt{\frac{1}{10} + \frac{1}{15}}$

where $S_p^2 = \dfrac{9 \times 3.4 + 14 \cdot 3.6}{23}.$

2a A statistic is a function on ~~random~~ a sequence of random variables.

An estimator is a statistic that is used ~~as~~ for inference on a particular parameter.

An estimate is the realisation of an estimator.

b. An estimator T for a parameter θ is

Unbiased if $E(T)=\theta$.

Consistent if $T \xrightarrow{P} \theta$ as $n \to \infty$

Efficient if $Var(T) = \frac{1}{MVB}$.

3 a. $L(\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i}$

$\log L(\theta) = n\log\theta + \sum_{i=1}^{n} x_i \log(1-\theta)$

$$D_1 = \frac{\partial \log L(\theta)}{\partial \theta} = \frac{n}{\theta} + \frac{-1}{1-\theta} \cdot \sum_{i=1}^{n} X_i$$

$$= \frac{n}{\theta} - \frac{1}{1-\theta} \cdot n \cdot \frac{1-\theta}{\theta} = 0.$$

b $E(D_1^2) = Var(D_1)$

$$= Var\left(\sum_{i=1}^{n} X_i \cdot \frac{1}{1-\theta} \right).$$

$$= \frac{1}{(1-\theta)^2} \cdot n \cdot \frac{1-\theta}{\theta^2}$$

$$= \frac{n}{\theta^2(1-\theta)} \Rightarrow E(D_2) = -\frac{n}{\theta^2(1-\theta)}$$

4a. $L(\theta) = \prod \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\log y_i - \theta)^2}$

$$\log L(\theta) = n\log y - \frac{1}{2}\sum_{i=1}^{n}(\log y_i - \theta)^2 + C.$$

$$\frac{\partial \log L(\theta)}{\partial \theta} = -\frac{1}{2}\sum_{i=1}^{n}(\log y_i - \theta)\cdot(-2)$$

$$= \sum_{i=1}^{n}(\log y_i - \theta)$$

$$\Rightarrow \quad MLE = \frac{\sum_{i=1}^{n}\log y_i}{n}$$

b. $\frac{\partial^2 \log L(\theta)}{\partial \theta^2} = -n.$

$$\Rightarrow \quad MVB = \frac{1}{n}$$

c. $\Rightarrow \quad \hat{\theta} \stackrel{d}{\approx} N(\theta, \frac{1}{n})$