494 HW1 Solutions

1a. $\bar{x} = \dfrac{11.67 + 9.16 + \dots + 15.40}{7}$

$= 13.02$

$\sum_{i=1}^{7} x_i^2 = 11.67^2 + \dots + 15.4^2$
$= 1254.29$

$S^2 = \frac{1}{6} \cdot (1254.29 - 7 \cdot 13.02^2)$
$= 11.27$

2. $E(S^2) = \frac{1}{n-1} E\left[\sum X_i^2 - \frac{(\sum X_i)^2}{n} \right]$

$= \frac{1}{n-1} E\left[\sum X_i^2 - n\bar{X}^2 \right]$

$= \frac{1}{n-1} \left[n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right) \right]$

$= \frac{1}{n-1} \cdot \sigma^2 (n-1) = \sigma^2$

3. $M_{\frac{S_n - n\mu}{\sqrt{n}\,\sigma}}(t) = E\left(e^{\left(\frac{S_n - n\mu}{\sqrt{n}\,\sigma}\right)t} \right)$

$= \left(E\left[e^{t \frac{X_1 - \mu}{\sqrt{n}\,\sigma}} \right] \right)^n$

$= \left[E\left(1 + t \cdot \frac{X_1 - \mu}{\sqrt{n}\,\sigma} + t^2 \frac{(X_1 - \mu)^2}{2n\sigma^2} + \text{small stuff} \right) \right]^n$

$$\approx \left(1 + \frac{t^2 \cdot \sigma^2}{2n\sigma^2}\right)^n$$

$$\to e^{t^2/2}$$

which is the mgf of the standard normal.

4.2.1. $\bar{x} = 81.2, \quad s^2 = 26.5$.

90% CI: $\cancel{81.2 \pm 1.645\sqrt{26.5}}$

$81.2 \pm 1.73\sqrt{\frac{26.5}{20}} = (79.21, 83.19)$

95% CI $\quad 81.2 \pm 2.09\sqrt{\frac{26.5}{20}} = (78.79, 83.61)$

99% CI $\quad 81.2 \pm 2.86\sqrt{\frac{26.5}{20}} = (77.91, 84.49)$

4.2.7 $\quad \bar{X} \overset{d}{=} N(\mu, 9/n)$

$$\Rightarrow P\left(\mu - 1.645 \cdot \frac{3}{\sqrt{n}} \le \bar{X} \le \mu + 1.645\frac{3}{\sqrt{n}}\right) = 0.9$$

$$P\left(-\bar{x} - 1.645\frac{3}{\sqrt{n}} \le -\mu \le -\bar{X} + 1.645\frac{3}{\sqrt{n}}\right) = 0.9$$

$$P\left(\bar{X} - 1.645\frac{3}{\sqrt{n}} \le \mu \le \bar{X} + 1.645\frac{3}{\sqrt{n}}\right) = 0.9$$

So set $1.645 \cdot \frac{3}{\sqrt{n}} = 1$.

$n = 24.35$

$\Rightarrow$ need $n \approx 25$.

4.4.5. $P(Y_{(4)} \geq 3) = P(\max(X_1, \ldots, X_4) \geq 3)$

$= 1 - P(\max(X_1, \ldots, X_4) < 3)$

$= 1 - P(X_1 < 3, X_2 < 3, X_3 < 3, X_4 < 3)$

$= 1 - P(X_1 < 3)^4$

$= 1 - \left[\int_0^3 e^{-x}\,dx\right]^4$

$= \cancel{1 - (e^{-3}+1)}^4$

$= 1 - (1 - e^{-3})^4$

5.2.11. $M_{\frac{Z_n - n}{\sqrt{n}}}(t) = E\left(e^{\frac{Z_n - n}{\sqrt{n}} t}\right)$

$= e^{-\sqrt{n}t} \cdot M_{Z_n}(t/\sqrt{n})$

$= e^{-\sqrt{n}t} \cdot e^{n \cdot (e^{t/\sqrt{n}} - 1)}$

$\approx e^{-\sqrt{n}t}\, e^{n \cdot \left(1 + t/\sqrt{n} + \frac{t^2}{2n} + \text{small} - 1\right)}$

$\approx e^{t^2/2n}$

Which is MGF of $N(0,1)$

$\square$

5.3.2. if $X \sim \text{Gamma}(2, 4)$

$$E(X) = 2 \cdot 4 = 8$$
$$V(X) = 2 \cdot 4^2 = \cancel{16}\ 32$$

$$\Rightarrow \quad E(\bar{X}) = 8 \qquad Var(\bar{X}) = \cancel{32} \cdot 1/4$$
$$\bar{X} \approx N(8, 1/4)$$

$$\Rightarrow \quad P(7 < X < 9)$$
$$\approx P(-2 < Z < 2)$$
$$= \cancel{0.046}\ .954$$

5.3.4 $\quad E(X) = \int_0^1 x \cdot 3x^2 \, dx$

$$= \frac{3x^4}{4}\Big|_0^1 = \frac{3}{4}.$$

$$E(X^2) = \int_0^1 x^2 \cdot 3x^2 \, dx$$

$$= \frac{3x^5}{5}\Big|_0^1 = \frac{3}{5}.$$

$$\Rightarrow \quad Var(X) = 3/5 - (3/4)^2 = 3/80$$

$$\Rightarrow \quad E(\bar{X}) = 3/4 \quad , \quad Var(\bar{X}) = \frac{3}{80 \cdot 15} = 0.0025$$

$$P(0.6 \le \bar{X} \le 0.8)$$
$$\approx P\left(\frac{-0.15}{0.05} \le Z \le \frac{0.05}{0.05}\right)$$

$$= 0.84.$$

5.3.13. $\hat{p} = \frac{X_1 + \cdots + X_n}{n}$

$$E(\hat{p}) = E(X_1) = p.$$

$$Var(\hat{p}) = \frac{1}{n} Var(X_1)$$

$$= \frac{1}{n}(p \cdot (1-p))$$

$\Rightarrow$ By CLT, $\frac{\hat{p} - p}{\sqrt{\frac{1}{n}(\hat{p}(1-\hat{p}))}} \longrightarrow N(0,1)$.